FERMION MODELS ON THE LATTICE
AND IN FIELD THEORY

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"The transformations of particles are the vibrations and wanderings which occur while the signs of creation are being written in the book of being. They are not games of chance and jumbled meaningless motion like the Materialists and Naturalists fancy. Because a particle raises loads infinitely exceeding its strength, like a seed a size of a grain of wheat shouldering a load like a huge pinetree ···

If every particle is not an official of God acting with His permission and under His authority, and if it is not undergoing change within His Knowledge and Power, then every particle must have infinite knowledge and limitless power, it must have eyes that see everything, a face that looks to all things, and authority over all things ··· Indeed, a particle despite being powerless and lifeless by carrying out its important duties consciously and by raising mighty loads it bears decisive witness to the existence of the Necessarily existent One. "

Bediu'zzaman, Treatise on the Transformation of Particles.

CONTENTS

ABSTRACT

This thesis consists of two parts:

The first part deals with lattice approach to field theories. The fermion doubling problems are described. This doubling can be removed if a dual lattice is introduced, as first pointed out by Stacey. His method is developed and in the process a formalism for the construction of a covariant difference lattice operator and thus of a gauge invariant action, is exhibited. It is shown how this formalism relates to the work of Wilson. Problems of gauge invariance can be traced back to the absence of the Leibnitz rule on the lattice. To circumvent this failure the usual notion of the product is replaced by a convolution. The solutions display a complementarity : the more localised the product the more extended is the approximation to the derivative and vice-versa. It is found that the form of the difference operator in the continuous limit dictates the formulation of the full two-dimensional supersymmetric algebra. The construction of the fields necessary to form the Wess-Zumino model follows from the requirement of anticommutativity of the supersymmetric charges.

In the second part, the Skyrme model is reviewed and Bogomolnyi conditions are defined and discussed. It appears that while the Skyrme model has many satisfactory features, it fails to describe the interactions between nucleons correctly. These problems are brought out and the available solutions reviewed.

PREFACE

This thesis represents work carried out in the Department of Mathematical Sciences of the University of Durham between October 1982 and august 1985.

Chapters two, three and four of this thesis are largely original in their approach and formulation. Chapters five and six are largely review except for the discussion of the Bogomol'nyi conditions on Skyrmion models, and their bearing on the question of whether or not Skyrmion models are integrable. Part of chapter two has been submitted for publication in collaboration with D.B. Fairlie.[12]

I should like to express my deep and sincere gratitude to my supervisor Dr. D.B. Fairlie of the department of mathematical sciences in the University of Durham under whose guidance the work presented herein was carried out, for a great deal of help and encouragement. My thanks are also due to Professor Euan Squires who read the first draft, for his valuable suggestions and generous assistance.

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CHAPTER I

INTRODUCTION

As we all know and apparently as it always has been known, matter is made of basic constituents. And it is these constituents combined in varying proportions that give rise to the variety of observed matter.

Now, the human mind naturally seeks simple and economic theories and it is remarkable that simplicity and elegance have indeed proved to be the best way to understand the physical world; indeed this shows a relation between the workings of human mind and the structure of the world. Thus any overabundance of fundamental objects is unsatisfying. It tells us that it is time to revise our ideas about the nature of the basic constituents. And indeed, this is the way things are in nature; any spectral distributions of masses, charges or energy levels is a manifestation of a compound physical system.

Very long ago it was realised that matter which consists of different proportions of chemical elements was made of atoms. As the number of elements was 92 there were therefore 92 kinds of atoms. However, with Mendeleef periodic table it was clear that atoms were not fundamental particles. In the early 1920's Rutherford showed that at the centre of the atom there is a nucleus. Later on, it was realised that in fact atoms consist of nuclei and electrons. Further, nuclei themselves are built up from simpler constituents the protons and neutrons.

Now, particles behave as if they were spinning systems and the amount of spin they possess is measured by the angular momentum. Quantum mechanics allows the angular momentum to take integral and half-odd-integral values only (the Planck constant $\hbar$ is taken to be one).

The electron, the proton, the neutron and the photon all have non-zero spin; the three first ones have spin-$\frac{1}{2}$ and the latter has spin-1.

Another property of particles is statistics. This is intimately tied up with the spin of particles. All particles with integral spin (0, 1, etc...) obey Bose statistics and are called bosons and all particles with half-odd-integral spin obey Fermi statistics and are called fermions.

At that time, except for the photon, the corpuscule of light, all fundamental entities were fermions. Indeed a system of an odd (even) number of fermions is a fermion (Boson), This situation did not change when in 1968 electron scattering experiments (SLAC) gave the first hint that point-like objects existed inside the protons, "the partons". Hence confirming the proposition of Gell-Mann and Zweig (1964) that proton and other "elementary" particles were made from more basic entities, the quarks. Indeed, quarks which are the constituents of hadrons, possess spin-$\frac{1}{2}$. Therefore baryons are made of three quarks whereas mesons are made of two quarks.
Hence fermion fields are basically the fundamental constituents of matter.

Non-abelian gauge theories[84] provide the best description of the empirically observed properties of elementary particles. Indeed, the weak and electromagnetic interactions when unified in a gauge invariant theory (SU(2) x (U(1)), known as the Salam-Weinberg model, are compatible with all known facts and experimental results of particle physics (neutral currents, $W^{\pm}$ and Z° particles, ...). The success of the unification of weak and electromagnetic interactions led to a further unification with strong interactions at the level of quarks. As quarks are not seen in nature, it is believed that they are confined by a non-abelian gauge theory, the most popular candidate being SU(3) Quantum Chromodynamics (QCD).[63] This is an SU(3) colour analogue of charge. Here quarks of spin-½ carry a quantum number colour. They interact with spin-1 coloured particles called gluons which are somehow the analogues of photons. They are gauge particle of spin 1 but they are different in that the non-abelian gauge groups SU(3) forces them to interact with themselves.

However, whereas QED shows up in the experiments as written in the Lagrangrian, QCD does not. In QED for instance the magnetic moments of the electron, positron and muon are predicted from the theory with such an accuracy that they are in agreement with experimental results up to 9 decimals[73] (including QED + weak and QCD corrections). But QCD does not provide a simple description of hadron physics. Particularly for the strong-coupling limit of QCD, this has resulted in the impossibility of making quantitative predictions of the theory which could be compared with experiment. Even the observed confinement of colour has not been established as a property of the QCD Lagrangian. In order to understand confinement, models have been developed which mimic the effect without producing it from a fundamental theory. In such models quarks experience a long-range attractive force, sufficiently strong to keep them confined in hadrons.

If such a force is to arise, it can only be from non-perturbative effects because a perturbative expansion of QCD gives free quarks as for electrons in QED. Thus, techniques which are not based on perturbation theory are needed.

The lattice approximation is the only known gauge invariant regularization which allows the treatment of non-perturbative effects in QCD. Thus to avoid perturbation theory the lattice is introduced. However, the description of quarks on the lattice is plagued with severe difficulties which can be traced back to those with free fermion fields.[33] The most straightforward fermion formulation produces too many fermions. Various procedures to avoid this multiplicity of fermions have been proposed. But none of them succeeded in consistently solving the problem without abandoning some property such as chiral symmetry or locality of gradient operators.[2] One of these methods[3] (SLAC fermions) has manifest chiral invariance, but does not preserve locality. The SLAC derivative couples all lattice sites along the direction of each component of the gradient instead of coupling only nearest-neighbour sites. In another method, Kogut and Susskind[5] put the upper (lower) components of Dirac spinors on even (odd) lattice sites. The gradient operator is local but there is no explicit continuous chiral symmetry on the lattice either. An alternate projection operator technique introduced by Wilson[34] in his lattice action formulation also destroys γ_5-invariance. Chiral symmetry, realised in the Nambu-Goldstone mode,[86] is an important approximate symmetry of the strong interaction, one of the consequences of which is the smallness of the pion mass. Indeed, strong coupling calculations with Susskind's fermions give a high ratio m_π/m_N.[80] However, a low pion mass does not seem a problem with Wilson's fermions.[81] A lack of chiral invariance also implies the non-existence of a neutrino lattice theory. It is therefore crucial to preserve chiral symmetry. Stacey[1] solves the fermion doubling by defining the fermionic wave functions and their derivatives on points of a new dual lattice with sites at the centre of the hypercubes of the first one. Later on, Bender et al[6] proposed a finite element approach which basically amount to averaging the fermi fields on points of the dual lattice as well.

In these last two schemes however, it is not evident how to implement gauge invariance. Bender et al[8] attempted to formulate an abelian gauge invariant theory where the operator Dirac equation is solved by means of finite element method.

Although being the underlying theory of strong interactions, QCD does not provide a simple description of low energy hadron physics. A more appropriate description is given by an effective field theory of mesons and baryons. The pion field behaviour for instance is satisfactorily described by the non-linear chiral sigma model.

t' Hooft[46] and Witten[47] showed separately that in the large-N limit QCD is equivalent to a theory of mesons fields only. In this theory, the meson coupling $1/_N$ is weak and the tree approximation is valid. As for the baryons, they would arise as solitons. If this were to be realised we would have topological particles in a realistic field theory. This idea is very attractive and indeed it has recently attracted much interest.

In fact this remarkable conjecture was made by Skyrme[35] some twenty years ago. He showed that an additional stabilizing term to the chiral Lagrangian supports non-trivial topological solutions.

In this case there exists a hierarchy in hadron physics : mesons are composed of quarks of the underlying theory (QCD) and baryons (fermions) are formed as solitons of the meson-fields theory. Very recently, Sato[82] suggested that this type of hierarchy may not be peculiar to hadron physics only. He then proposes a model in which solitons behave like leptons and quarks.

This work is organised into five sections.

In the second chapter we describe the fermion doubling problem which occurs when fermions are put on the lattice. Then following a suggestion of Stacey[1] that fermion doubling may be intimately related to the transcription of the gradient operator on the lattice, we analyse the Dirac equation on the lattice and establish that a naive transcription is inconsistent in the sense that it does not yield a box operator. This inconsistency is removed if the fermion fields are defined on the dual lattice sites i.e. at the centre of the hypercubes of the ordinary lattice. To proceed further we construct a difference operator on the lattice. The resulting theory has no fermion doubling and preserves chiral invariance ($m = o$). This operator saves us lengthy and tedious calculations and moreover allows us to draw a clear parallel between the lattice and the continuum.

In formulating gauge theories of fermions on the lattice we are faced with problems. One of these problems is the question of how one introduces gauge fields on the lattice so that the theory is gauge invariant. In the third chapter we note that this problem is directly related to the absence of Leibnitz rule[7] on the lattice. To circumvent this failure of the product rule for differentiation we redefine the usual notion of product. The solution displays a complementarity : the more localised the product is the more extended is the approximation to the derivative and vice-versa. Then we observe that the form of the difference operator-defined in the first chapter - in the continuous limit suggests two ways of introducing the gauge field A_μ. The first way is reminiscent to that of Bender et al[8]; it only works in the Abelian case. The second way is in agreement with the Wilson precription.[4] Then, in order to define a gauge action we construct a lattice analogue of the two-form curvature $F_{\mu\nu}$. The resulting action turns out to be exactly the "revised" Wilson action.

Currently there is intense theoretical interest in supersymmetric theories.[83] These are theories possessing a symmetry between fermions and bosons. For several reasons it would be interesting to put supersymmetry on the lattice. However, this is not an easy task because the supersymmetry transformations involve the momentum operator and translation operators none of which survive on the lattice. Thus only very tentative steps have been taken.

In the fourth chapter we propose a formulation of the full two-dimensional supersymmetric algebra. This is again dictated by the form of the difference operator in the continuous limit. We also construct the fields necessary to form the two-dimensional Wess-Zumino model. This formulation offers some hope of leading to a realistic lattice approximation to a continuum model and we think that it is worthy of further study.

In the fifth chapter, the Skyrme model is reviewed and Bogomolnyi-like conditions are defined and discussed. It appears that while the Skyrme model have many satisfactory features it fails to describe nucleon-nucleon interactions correctly. Indeed at intermediate range repulsion persists and there is no sign of attraction. Two solutions of this problem have been suggested. [43,77]
In the last and sixth chapter, the Skyrmion-Skyrmion interactions as well as nucleon-nucleon interactions within the predictions of the Skyrme model are studied and the problems brought out. And finally the available solutions are reviewed.

CHAPTER II

FERMIONS ON THE LATTICE

II.1 Fermion Problems on the Lattice

The lattice approximation[9] allows the treatment of non-perturbative effects in Quantum Chromodynamics(QCD) in both the weak and strong coupling regimes. However, the description of quarks and therefore of fermions, in general, is plagued with difficulties which can be traced back to those with the Dirac equation.[1] These problems are "fermion doubling"[33] and "missing chiral invariance."

Nielsen et al[2] put forward no-go theorems which state that it is impossible to have a lattice formulation with continuous chiral symmetry and without fermion doubling. On the other hand several solutions to this problem have been proposed,[3,5] but unfortunately all of them have difficulties in consistently solving the problem.

Susskind[5] for instance employed a staggered lattice. This means that different components of the Dirac spinor live on alternate lattice sites. In two-dimensional space-time, for instance, there is a one-component fermion field $\phi(n)$ at each site n(n is an integer). $\phi(n)$ satisfies the anticommutation relation

$$\{ \phi^{+}(n), \phi(m) \} = \delta_{n,m} \tag{1-a}$$

and

$$\{ \phi(n), \phi(m) \} = 0 \tag{1-b}$$

A two-components field $\Psi(n)$ is defined in the following way:

$$\Psi = \begin{pmatrix} \Psi_e \\ \Psi_o \end{pmatrix} \tag{2}$$

where

$$\psi_e(n) = \phi(n) \quad n \text{ even}$$
$$\psi_o(n) = \phi(n) \quad n \text{ odd} \qquad (3)$$

In this scheme the discrete form of the gradient ∂_1 connects only even or odd sites

$$\partial_1\phi(n) = \frac{\phi(n+1) - \phi(n-1)}{2h} \qquad (4)$$

Consequently the discrete form of the momentum operator p_1 translates the lattice by $2h$. Then the components of $\psi(n)$ satisfy

$$\partial_1\psi_o = \frac{\Delta\psi e}{\Delta x} \quad , \quad \partial_1\psi_e = \frac{\Delta\psi o}{\Delta x} \qquad (5)$$

where Δ indicates the discrete difference in equation (4). The action of this model is Hermitian and has no fermion doubling. However, it explicitly violates global chiral symmetry although it has a discrete set of chiral invariances which may be promoted to a continuous set in the continuum limit.

Wilson[34] proposed to add to the action a term

$$\frac{\eta}{2h} \quad \Sigma^{h^4}_{n,} \ (\bar{\psi}_n\psi_{n+\mu} + \bar{\psi}_{n+\mu}\psi_n - 2\bar{\psi}_n\psi_n) \qquad (6)$$

It eliminates the extra states by giving them masses proportional to ηh^{-1} leaving only one massless particle. This term therefore removes fermion doubling. However, it kills the chiral invariance of the massless theory. The Wilson scheme is not permissible in lattice versions of chiral theories such as $SU(2) \otimes U(1)$, in which left and right-handed spinors transform according to different representations of the gauge group. Chiral symmetry violating terms like the mass and the Wilson terms are of the form $\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$ and it is not possible to make such terms gauge invariant.

It is clear that in order to understand the problem and obtain a solution one has first to answer where it comes from. Thus it is our aim to analyse the fermion doubling problem. Let us start with the free massless Dirac theory. The Dirac action is

$$S = \int dx \ \frac{1}{2i} \ \bar{\psi}\gamma_\mu\partial_\mu\psi \tag{7}$$

where

$$\bar{\psi} = \psi^T\gamma_o \quad \text{and} \ \{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \quad .$$

There is a symmetry of this action

$$\psi \to e^{i\alpha} \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha} \tag{8}$$

that corresponds to fermion-number conservation. It is possible to extend this invariance to local transformations

$$\psi \to e^{i\alpha(x)}\,\psi$$

by introducing gauge fields with suitable transformation properties. The massless theory has another symmetry, known as chiral symmetry,

$$\psi \to e^{i\beta\gamma^5\psi} \ \text{and} \ \bar{\psi} \to \ \bar{\psi}e^{-i\beta\gamma_5} \tag{9}$$

The exponentials cancel because

$$\{\gamma^5,\gamma^\mu\} = 0 \ ; \quad \gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4$$

Consider the wave function $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ where

$$\psi_L = \frac{1+\gamma^5}{2}\,\psi \ , \quad \bar{\psi}_L = \bar{\psi}\,\frac{1-\gamma^5}{2}$$

and

$$\Psi_R = \frac{1-\gamma^5}{2}\Psi, \quad \overline{\Psi}_R = \overline{\Psi}\frac{1+\gamma^5}{2}$$

describe, respectively, right and left-handed fermions. $(1\pm\gamma_5)/2$ are the projection operators onto states of definite helicity. They project the spin on the direction of motion. The state of the particle with a definite momentum are necessarily helicity states, for the spin component in the direction of motion has a definite value. The positive-energy states of Ψ_R have positive helicity and the positive-energy states of Ψ_L have negative helicity.

Neutrinos occur in nature only in the left-handed form (antineutrinos in the right-handed one). Such a pure separation into exclusive left-handedness is only possible for massless particles so that all other fermions with non-vanishing masses, have both left-handed and right-handed states. However, at very high energies, massive particles are expected to behave as massless.

Particles of spin -½ are described by the free Dirac wave equation

$$(i\gamma^\mu\partial_\mu - m)\,\Psi \equiv (i\not{\partial} - m)\Psi = 0 \qquad (10)$$

The wave function Ψ may be written as a bispinor $\Psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ in terms of two-component spinors φ and χ. They satisfy

$$\begin{aligned} i\,\partial\varphi/_{\partial t} &= m\varphi + \frac{1}{i}\;\underline{\sigma}\cdot\underline{\nabla}\chi \\ i\,\partial\chi/_{\partial t} &= -m\chi + \frac{1}{i}\;\underline{\sigma}\cdot\underline{\nabla}\varphi \end{aligned} \qquad (11)$$

We want to investigate the plane wave solutions in momentum space corresponding to positive and negative energies respectively,

$$\Psi^{(+)}_{(x)} = e^{-ip.x}\,u(p)$$

$$\Psi^{(-)}_{(x)} = e^{ip.x} \; v\,(p) \quad ; \quad po > o$$

To verify the Klein-Gordon equation we must have $p^2 = m^2$. u(p) and v(p) satisfy[13]

$$(\gamma p - m)\; u(p) = o \; ; \; (\gamma p + m)\; v(p) = o$$

Then,

$$u(p) = \begin{pmatrix} \left(\frac{E+m}{2m}\right)^{\frac{1}{2}} \varphi \\ \\ \frac{\sigma . p}{\sqrt{2m(m+E)}} \varphi \end{pmatrix} \quad ; \quad v(p) = \begin{pmatrix} \frac{\sigma . p}{\sqrt{2m(m+E)}} \quad \chi \\ \\ \sqrt{\frac{E+m}{2m}} \quad \chi \end{pmatrix} \tag{12}$$

Now massless fermions are described by a two-component wave function or equivalently by a "four-components" function which is solution of the free massless Dirac equation

$$\gamma p \Psi = o \quad , \quad p\mu = i \; \partial \mu \tag{13}$$

and which obeys the condition

$$\gamma^5 \Psi = \Psi \tag{14}$$

This last condition (14) guarantees that Ψ has actually only two components. Indeed,

$$\left(\frac{1+\gamma_5}{2}\right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} o \\ \chi \end{pmatrix}$$

and (15)

$$(\varphi^* \; , \; \chi^*) \left(\frac{1-\gamma_5}{2}\right) = (\chi^* \; , \; o)$$

Thus (13) and (14) are equivalent to

$$(po + \underline{p} \; . \; \underline{\sigma})\chi = o \tag{16}$$

In the infinite momentum limit, two of the four components of the wave function describing a massive fermion vanish. Thus it is that massive fermion behave as though massless at very high energies. Chiral symmetry is therefore important in the theory of neutrinos and at high energies. Moreover, in QCD, where Ψ refers to quarks, the $m = o$ chiral invariance is also related to the low mass of the pion.

The massless QCD lagrangian is invariant under chiral $SU(3) \otimes SU(3)$ but the ground state is not. In such a situation the vacuum expectation value (VEV) is not merely zero as it would be for a chirally symmetric vacuum but takes a non-zero value. The symmetry is said to be spontaneously broken. There is a theorem, the Goldstone-Nambu theorem,[86] which says that in such a case, there appear as many massless bosons as there are broken generators (i.e. generators corresponding to the violated symmetries). Hence, in the strict chiral limit we expect eight pseudoscalar massless bosons, one of which is the pion.

In fact, experiments tell us that the pion mass is not exactly zero but it is very small, much smaller than that of other mesons. This indicates that the chiral symmetry is not an exact symmetry that is that quarks are not massless but rather that they have small masses.

If we want to have a lattice action with all the formal properties of the continuum fermionic actions, and if we want to be able to put neutrinos on the lattice, it is preferable to retain the $m = o$ chiral invariance.

The simplest way to put this theory on the lattice is to consider the action (7), substitute finite difference approximations for derivatives and replace the space-time integral by a sum over the lattice sites.

The lattice approximation to S is then

$$S = - \Sigma_{n,\mu} h^4 \bar{\Psi}_n \gamma_\mu \frac{1}{2h} (\Psi_{n+\mu} - \Psi_{n-\mu}) \tag{17}$$

$\Psi_{n+\mu} = \Psi(x + h_\mu)$ where h_μ is a unit four-vector and $x = n\,h$; n is an integer. γ_μ are Hermitian Euclidean Dirac matrices satisfying

$$\{\gamma_\mu , \gamma_\nu\} = 2\delta_{\mu\nu} \text{ and } \gamma_\mu^+ = \gamma_\mu$$

Unfortunately, the action (17) suffers from fermion doubling which can be observed in the corresponding propagator. We obtain this propagator by taking the lattice Fourier transform

$$\tilde{\Psi}_p = \Sigma_{\mu,n} \Psi_n e^{inp_\mu h} \tag{18}$$

Thus,

$$\frac{1}{2h}(\Psi_{n+\mu} - \Psi_{n-\mu}) = \Sigma_{n,\mu} \tilde{\Psi}_p \frac{1}{2h}\left(e^{-ihp_\mu(n+1)} - e^{-ihp_\mu(n-1)}\right)$$

$$= \Sigma_{n,\mu} \tilde{\Psi}_p e^{-ihp_\mu n} \left(\frac{-i}{h} \sin hp_\mu\right)$$

$$= \Psi_n \left(\frac{-i \sin hp_\mu}{h}\right) \tag{19}$$

Inserting (19) into (17) we get

$$S = \Sigma_{\mu,n} h^4 \bar{\Psi}_n \left(\gamma_\mu \frac{i \sin hp_\mu}{h}\right) \Psi_n \tag{20}$$

The propagator is then

$$\left(\Sigma_{\mu} \; i\gamma_{\mu} \; \frac{\sin hp_{\mu}}{h}\right)^{-1} \tag{21}$$

where $o < hp_{\mu} < 2\pi$; p_{μ} is the momentum operator.

This propagator possesses sixteen poles in the Brillouin zone p to $p + {}^{2\pi}/_{h}$, at $p_{\mu} = o$, the value corresponding to zero momentum in the continuum and also at $p \sim \frac{\pi}{h}$. Hence the spectrum of states possesses a multiplication of levels not encountered in the continuum theory. The fermion "doubling" phenomenon consists then, essentially in the fact that on a d-dimensional lattice there appear 2^d fermions (on each site) in contradiction with the continuum theory where only one is present.

These sixteen fermions have non-zero amplitudes for being pair produced if they are coupled to a Maxwell gauge field $U_{\mu}(x)$. Their electric charges are moreover all equal.[79] On the other hand a calculation of their couplings to an axial current reveal eight positive charges and eight negative charges. Thus the axial Adler-Bell-Jackiw anomaly is absent. Indeed this may be a reason for the multiple fermions. In any case one fermion has become sixteen and we want to know how this happened.

If we replace the derivatives in (7) by a forward difference, the action is

$$S = -\Sigma_{n,\mu} h^4 \bar{\Psi}_n \gamma_{\mu} \frac{1}{2h} (\Psi_{n+\mu} - \Psi_n) \tag{22}$$

This form of the action does not lead to fermion doubling. However it is not Hermitian and therefore leads to complex valued energy. If the Slac operator

$$\nabla^S = \frac{1}{2\pi} \int_{\pi}^{\pi} dp \; ip \; \tilde{f}(p) \; e^{ipx} \tag{23}$$

is used, no spectrum doubling occurs but the equations of motion involve non-local derivatives.

At this stage it seems that when placing fermions on the lattice, either one has to accept the doubling of fermionic degrees of freedom or to abandon some property such as chiral invariance or locality of gradient operators. But in any case the occurrence of doubling appears to be determined by the choice of finite difference operator.

II.2 Origin and solution of fermion doubling

Stacey[1] was the first to point out that the problem in replacing $\frac{\partial\Psi}{\partial x}$ by

$$\frac{1}{h}\ (\ \Psi(x+h)\ -\ \Psi(x)) \tag{24}$$

is that it defines $\partial\Psi/_{\partial x}$ at $x+h/2$ while $\Psi(x)$ is defined at x. He proposed that another alternative to the usual one

$$\frac{\partial\Psi}{\partial x} \rightarrow \frac{1}{2h}\ (\Psi(x+h)\ -\ \Psi(x-h)) \tag{25}$$

is to work at $x + h/2$ by averaging Ψ itself :

$$\Psi(x + \tfrac{h}{2}) = \tfrac{1}{2}\left(\Psi\ (x) + \Psi(x+h)\right)$$

A massless fermion in two dimensional space-time is described by the Weyl equation

$$i\,\frac{\partial\Psi}{\partial t} = i\alpha\ \frac{\partial\Psi}{\partial x}$$

which, when space is put on a lattice becomes then

$$i\,\frac{\partial}{\partial t}\ \frac{\Psi(x+h) + \Psi(x)}{2} = i\alpha\ \frac{\Psi(x+h) - \Psi(x)}{h}$$

This solution generalizes to any number of dimensions and eliminates fermion doubling.

Later on, Bender et al[6] reached the same conclusion, that of averaging, using the finite element approach. They defined fermion fields at the centre of the square finite element whose vertices are located at the lattice sites (m,n), (m+1,n), (m+1, n+1) and (m, n+1) by

$$\Psi(x,t) = \Psi_{m,n} \left(1-\frac{x}{h}\right)\left(1-\frac{t}{h}\right) + \Psi_{m+1,n} \frac{x}{h}\left(1-\frac{t}{h}\right) + \Psi_{m+n,n+1} \frac{x}{h}\frac{t}{h} + \Psi_{m,m+1}\left(1-\frac{x}{h}\right)\left(\frac{t}{h}\right) \tag{26}$$

where m and n are the space and time indices respectively. $\Psi_{m,n}$ are operators satisfying the cononical equal-time anti commutations relations

$$\begin{aligned} \left\{\Psi^{+}_{k,n} , \Psi_{e,n}\right\}_{+} &= h^{-1}\delta_{k\ell} \\ \left\{\Psi^{+}_{k,n} , \Psi^{+}_{e,n}\right\}_{+} &= \left\{\Psi_{k,n} , \Psi_{e,n}\right\}_{+} = 0 \end{aligned} \tag{27}$$

Provided these anticommutation relations are exactly maintained on the lattice, this formulation preserves global chiral symmetry and Hermiticity and eliminates fermion doubling.

On the lattice operator differential equations become difference equations. The latter are not easy to handle; they entail lengthy and often tedious, calculations. To avoid this we want to construct a difference operator on the lattice whose continuum limit as the lattice spacing h goes to zero would be the differential operator (∂_{μ}).

II.3 Difference operator on the lattice

Consider P, the operator on a discrete one-dimensional lattice of spacing h which translates a field from a point to the next-neighbour one

$$P\,\Psi(x) = \Psi(x+h) \quad .$$

It may be written as a matrix

$$P = \begin{pmatrix} 0\,1\,0\,.\,. & & & & & \\ 0\,0\,1\,0\,.\,. & & & & & \\ .\,0\,0\,1\,.\,.\,. & & & & & \\ .\,0\,0\,0\,1\,0\,.\,. & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & & & & & 1 \\ 1\,0\,.\,.\,.\,.\,. & & & & & 0 \end{pmatrix} \tag{28}$$

where N is the total number of points on the lattice and where we have taken the field Ψ to be periodic :

$$\Psi(x + Nh) = \Psi(x) \quad , \quad P^N = I$$

In a two dimensional lattice we need two such operators

$$P_1\,\Psi(x,y) = \Psi(x+h,y)$$

$$P_2\,\Psi(x,y) = \Psi(x,y+h)$$

P_1 and P_2 can be constructed out of P in the following way

$$P_1 = P \otimes I$$

$$P_2 = I \otimes P$$

This construction can be generalized to any number of dimensions.

Here are some useful properties of P;

$$P_\mu^T \; \Psi(x) \;=\; \Psi(x-h_\mu)$$

$$\bar{\Psi}(x)\, P_\mu^T \;=\; \bar{\Psi}(x+h_\mu) \qquad P^T \text{ is the Hermitian conjuguate of } P\,.$$

$$\bar{\Psi}(x)\, P_\mu \;=\; \bar{\Psi}(x-h_\mu)$$

P_μ is the translation operator in the direction μ.

The differential operator $\partial\mu$ is replaced on the lattice by the difference operator

$$d_\mu \;=\; h^{-1}\,(P_\mu - 1) \tag{29}$$

where,

$$P_\mu\, \Psi(x) \;=\; \Psi(x+h_\mu),$$

Upon expanding in Taylor series

$$\begin{aligned} \Psi(x+h_\mu) &= \Sigma_n \frac{h^n}{n!} \left(\frac{\partial}{\partial x\mu}\right)^n \Psi(x) \\ &= e^{h\partial\mu}\, \Psi(x) \end{aligned} \tag{30}$$

Therefore, in the continuous limit

$$d_\mu \;=\; h^{-1}\,(e^{h\partial\mu} - 1)$$

Clearly d_μ goes to $\partial\mu$ when h goes to zero since

$$e^{h\partial\mu} = 1 + h\partial\mu + O(h^2)\,.$$

II.4 The Dirac equation is inconsistent on the lattice

If we naively approximate the derivative by a forward difference (24) on the lattice

$$\partial\Psi(x) \rightarrow h^{-1} \ (\Psi_{n+1} - \Psi_n)$$

then the two dimensional Dirac equation would be

$$\sigma_1 \otimes h^{-1}(p_i-1)\,\Psi_{ij} + \sigma_2 \otimes h^{-1}(p_j-1)\,\Psi_{ij} = 0 \tag{31}$$

i.e.

$$\begin{aligned} h^{-1}(p_j-1)\,\Psi^1_{ij} + h^{-1}(p_i-1)\,\Psi^2_{ij} &= 0 \\ h^{-1}(p_i-1)\,\Psi^1_{ij} + h^{-1}(p_j-1)\,\Psi^2_{ij} &= 0 \end{aligned} \tag{32}$$

Ψ^1_{ij} and Ψ^2_{ij} are the two components of the spinor Ψ_{ij}.
Equations (31) and (32) are inconsistent in the sense that their iteration does not give the correct approximation of the Klein-Gordon equation for every component of the fermion field.

Indeed, iterating (32) we get

$$h^{-2}(p_1-1)^2 \ \Psi^1_{ij} + h^{-2}\,(p_2-1)^2\,\Psi^1_{ij} = 0$$

i.e.

$$\Psi^1_{i+2,j} + \Psi^1_{i,j+2} - 2\Psi^1_{i+1,j} - 2\Psi^1_{i,j+1} - 2\Psi^1_{ij} = 0 \tag{33}$$

A good approximation of the box operator would be

$$4\phi_{ij} = \phi_{i-1j} + \phi_{i+1j} + \phi_{ij-1} + \phi_{ij+1}$$

$$\text{i.e.} \qquad \Box = (p_1^T-1)(p_1-1) + (p_2^T-1)(p_2-1) \tag{34}$$

Unfortunately, as it is written in (34), $\Box$ cannot be written as the square of the Dirac operator.

To solve this, one may redefine the approximation of the derivative as suggested by Stacey[1] and Bender et al.[6]

$$\partial_x \phi(x,y) \to d_i \phi_{ij} = \frac{1}{2} h^{-1}(\phi_{i+1j} - \phi_{ij} + \phi_{i+1j+1} - \phi_{ij+1})$$

$$= h^{-1}(p_i - 1) \ \frac{1}{2}(p_j + 1) \ \phi_{i,j}$$

i.e. we take the derivative in one direction of the field averaged in the other directions. The derivative $d_i \ \phi_{ij}$ is now defined at the centre of the lattice unit square.

The Dirac equation becomes then,

$$\sigma_1 \otimes h^{-1}(p_i - 1) \ \frac{1}{2}(p_j + 1) \ \Psi_{ij} + \sigma_2 \otimes h^{-1}(p_j - 1) \ \frac{1}{2}(p_i + 1) \ \Psi_{ij} = 0 \qquad (35)$$

and both sides of the equation are now defined at the same point. The Klein-Gordon equation derived from it, is

$$(p_i^2 \ p_j^2 + p_i^2 + p_j^2 + 1) \ \Psi_{ij}^1 = 4 p_i p_j \ \Psi_{ij}^1$$

that is

$$4\Psi_{i+1 \ j+1}^1 = \Psi_{i+2 \ j+2}^1 + \Psi_{i+2j}^1 + \Psi_{ij+2}^1 + \Psi_{ij}^1$$

where Ψ^1_{ij} is one component of the bispinor $\Psi_{ij} = \begin{pmatrix} \Psi^1_{ij} \\ \Psi^2_{ij} \end{pmatrix}$.

Hence the box operator is

$$\Box = \frac{1}{4} h^{-2} \ [(p_i - 1)^2 \ (p_j + 1)^2 + (p_i + 1)^2 \ (p_j - 1)^2] \qquad (36)$$

Separately approximating $\bar{\Psi}$ and the Dirac equation, we obtain the following form for the action,

$$S = h^2 \Sigma_{ij} \bar{\Psi}_{ij} \left[1 \otimes \tfrac{1}{2}(p_i^T+1)\ \tfrac{1}{2}(p_j^T+1)\right] \times$$

$$\times \left[\sigma_1 \otimes h^{-1}(p_i-1)\ \tfrac{1}{2}(p_j+1) + \sigma_2 \otimes h^{-1}(p_j-1)\ \tfrac{1}{2}(p_i+1)\right] \Psi_{ij} \qquad (37)$$

The action (37) is Hermitian (anti Hermitian) if σ_i are anti Hermitian (Hermitian). One can also easily verify that it does not lead to fermion doubling. Indeed, the propagator derived from (37) is

$$\left[\Sigma_\mu \gamma_\mu \frac{1}{h} \tan\left(\frac{hp_\mu}{2}\right)\right]^{-1} \qquad (38)$$

Equation (38) has the correct continuous limit and only one zero in the Brillouin zone. The theory describes only one massless fermion. Moreover, it is invariant under the following chiral transformations

$$\Psi_{ij} \to (e^{i\beta\gamma s} \otimes 1)\Psi_{ij}$$

$$\bar{\Psi}_{ij} \to \bar{\Psi}_{ij} (e^{-i\beta\gamma s} \otimes 1)$$

Therefore we conclude that averaging is indeed a solution to problems that arise when fermions are put on the lattice.

Very recently, Balaban[11] pointed out that renormalization group transformations are usually defined by some averaging operations. He studies these operations for lattice gauge fields and for gauge transformations charactersing some classes of field configurations on which the averaging operations are regular (e.g. analytic) and applies the results to the renormalization group method in lattice gauge theories.

In conclusion, we have seen that it is possible to construct a fermionic lattice action which is Hermitian, which preserves global chiral symmetry in the massless case and which does not exhibit fermion doubling. Moreover, the derivative approximation is local and all components of the fermion field are defined at one lattice site. Also, it is interesting to notice that in this formulation the lagrangian is local but not the Hamiltonian; nonlocality which is seen explicitly in the discontinuity of tan $({}^{hp_\mu}/_2)$ in equation (38) arises because undifferentiated fields appear as averages.[6] Hence, in the Hamiltonian approach, this formulation escapes the no-go theorems of Ref. 2.

CHAPTER III

GAUGE INVARIANCE ON THE LATTICE

III.1 The Leibnitz Rule on the Lattice

All fundamental interactions, electromagnetism (QED), weak (Standard model), strong (QCD) and gravity apparently are best described by local gauge theories.

Ordinary quantum field theories have parameters associated with them which can only be determined experimentally. These parameters are the coupling constants associated with the basic processes of the theory. For example, in quantum electrodynamics (QED) the electric charge is a coupling constant; it determines the strength of the interaction between the electron and photon. From experiment we know that charge is quantised. This means that there are no particles with charges which are not integer multiple of the electron charge. Guage invariance guarantees these relations among charges after the renormalisation of the theory. Thus the renormalised charges which are computed quantities are still related to each other in a similar way. It is therefore crucial to preserve gauge invariance on the lattice.

However, a straightforward and naive approximation of the continuum action II.7 on the lattice, not only leads to fermion doubling but also to an action II-17 which is not gauge invariant for non-zero lattice spacing.

Indeed, on the lattice the quantity $d_\mu + iA_\mu$ is not covariant and this is because the Leibnitz rule does not hold anymore. Therefore, one needs a finite difference operator satisfying the Leibnitz rule $\nabla(fg) = f.\nabla g + \nabla f.g$ and it has been shown that no definition of a gradient operator on the lattice satisfies this rule[7] (except for the trivial and useless case $\nabla f = o$).

However, one may ask if a version to the Leibnitz rule for differentiation of a product

$$\frac{d}{dx}\ (f\,g) = \left(\frac{df}{dx}\right) g + f\left(\frac{d}{dx}\ g\right) \tag{1}$$

can be restored by modifying the definition of a product of two functions defined on a discrete set.[12] In fact what we find is a complementarity : the more localised is the product the more extended is the approximation to the derivative and vice-versa.

Consider a function f in a one-dimensional lattice. It is represented by a column-vector with N entries $f_1, f_2, ----f_{N-1}, f_N$ where N is the number of sites on the lattice. The k-th component of f is therefore the value of f at the k-th point of the lattice, f_K. Let us define at the point i, the product

$$(fg)_i = C_{ijk}\ f_j\ g_k \tag{2}$$

where sums over repeated indices are implied and where f and g are two functions satisfying the following boundary conditions:

$$f_i = f_{N+i} \quad , \quad g_j = g_{N+j} \ \{\ i,j = o,\ ---N-1\ \} \tag{3}$$

Then we define the derivative at i as a linear combination

$$(Df)_i = \Sigma_j\ d_{ij}\ f_j\ . \tag{4}$$

Next we would like to have a realistic definition of the product and the gradient. We therefore demand that the coefficients d_{ik} and C_{kij} satisfy further natural requirements. If f is a constant then its derivative must vanish i.e.

$$(Df_i) = \Sigma_j\ d_{ij}\ f_j = o$$

where all fj are equal to the same value. This implies that

$$\Sigma_j d_{ij} = 0 \tag{5}$$

Also if f = c, where c is constant then

$$(fg)_i = cg_i$$

giving

$$\Sigma_j C_{ijk} = \delta_{ik} \tag{6}$$

The final requirements we want to impose are commutativity and associativity respectively:

$$(fg) = (gf)$$

and

$$((fg)h) = (f(gh))$$

which translates into

$$C_{ijk} = C_{ikj} \tag{7}$$

and

$$\Sigma_k C_{m\ell k} C_{kij} = \Sigma_k C_{mik} C_{k\ell j} \tag{8}$$

The coefficients C_{ijk} are now symmetric in j and k.
Then the Leibnitz rule is

$$\Sigma_\ell d_{i\ell} C_{\ell jk} = \Sigma_\ell C_{i\ell k} d_{\ell j} + \Sigma_\ell C_{ij\ell} d_{\ell k} \tag{9}$$

The usual definition of the product requires that

$$C_{ijk} = \Sigma_r \delta_{ir} \delta_{jr} \delta_{kr} \tag{10}$$

which vanishes unless all indices i,j and k are equal. It is easy to check that with this choice of the product there are no solutions for (9).

Translation invariance requires further constraints :

$$d_{i+1,j+1} = d_{ij} \tag{11}$$

where i + N is replaced by i. There are similar constraints on the C coefficients.

The set of relations (5) to (9) are non-linear and not straighforward to solve. An approach to the problem is to first satisfy (6) the only equations which are inhomogeneous, together with (7). It turns out that it is easier to first find a particular solution and then generalise it later. Consider the ansatz

$$C_{ijk} = \delta_{ij}p_k + \delta_{ik}p_j - (p_j p_k) \text{ for all } i \tag{12}$$

where p satisfies

$$\Sigma_{k=0}^{N-1} \; p_k = 1 \tag{13}$$

It clearly satisfies (6) and (7). It can also be verified that it satisfies the associativity relation (8). Indeed

$$\Sigma_k \, C_{m\ell k} \, C_{kij}$$

$$= \Sigma_k(\delta_{m\ell}p_k + \delta_{mk}p_\ell - p_\ell p_k)(\delta_{ki}p_j + \delta_{kj}p_i - p_i p_j) \tag{14}$$

$$= \Sigma_k \, \delta_{m\ell}p_j p_i + \delta_{mi}p_\ell p_j + \delta_{mj}p_i p_\ell - 2p_i p_\ell p_j \; .$$

As this expression is cyclic in i,ℓ,j it is clear that it satisfies (8). We then insert (12) in equation (10) and we get

$$\Sigma_\ell(d_{i\ell}\,\delta_{\ell j}p_k + d_{i\ell}\,\delta_{\ell k}p_j - p_j p_k\, d_{i\ell})$$

$$= \Sigma_\ell(d_{\ell j}\,\delta_{i\ell}p_k + \delta_{ik}p_\ell d_{\ell j} - p_\ell p_k\, d_{\ell j}) + \Sigma_\ell(d_{\ell k}\,\delta_{ij}p_\ell + d_{\ell k}\,\delta_{i\ell}p_j - d_{\ell k}p_j p_\ell)$$

$$\tag{15}$$

Using (5) the left hand side of (15) becomes

$$\Sigma_\ell(d_{ij}p_k + d_{ik}p_j \quad)$$

Hence equation (9) is satisfied provided that

$$\Sigma_i p_i d_{i\ell} \doteq o \tag{16}$$

This condition requires that $d_{i\ell}$ is a matrix of rank N-1, and together with (5) implies that the general solution for $d_{i\ell}$ is

$$\begin{aligned} d_{i\ell} &= a_{i\ell} - \Sigma_k \frac{p_i p_k}{p^2} a_{k\ell} - \Sigma_k a_{ik} p_\ell \\ &+ \Sigma_k (\Sigma_r p_k a_{kr}) \frac{p_i p_\ell}{p^2} \end{aligned} \tag{17}$$

where $a_{i\ell}$ is an arbitrary N x N matrix.

In particular, the usual definition of the gradient

$$d_{i\ell} = \delta_{i\ell} - \delta_{i\ell+1} \tag{18}$$

does satisfy both (13) and (16) if p is taken to be

$$\underline{p} = \frac{1}{N}\ (1,1,\ ---,\ 1,1) \quad \text{ie} \quad p_k = \frac{1}{N}\ \forall\, k.$$

It gives for a product

$$\begin{aligned} (fg)_i &= \Sigma C_{ijk}\, f_j g_K \\ &= f_i(\Sigma p_k g_k) + (\Sigma p_j f_j) g_i - (\Sigma p_j f_j)(\Sigma p_k g_k) \\ (fg) &= f \frac{1}{N} \Sigma_k g_k + g \frac{1}{N} \Sigma_k f_k - \frac{1}{N^2} (\Sigma_k f_k)(\Sigma_\ell g_\ell) \end{aligned}$$

i.e.

$$(fg) = f < g> + \ g < f> - < f> < g> \tag{19}$$

where $\langle f \rangle$ denotes an average over sites. (18) and (19) satisfy translation invariance requirements. From (19) we can see that this definition of product is highly non-local. We thus require a more appropriate (in the sense of being local) definition than (12).

Consider a second vector q_k, similarly normalised so that its components sum to unity, and the more general ansatz

$$C_{ijk} = \delta_{ij}p_k + \delta_{ik}p_j - p_jp_k + (q_j-p_j)(q_k-p_k)\,\frac{p^2q_i-(q.p)p_i}{(p^2q^2-(q.p)^2)} \qquad (20)$$

where

$$(q.p) = \Sigma_{\ell=0}^{N-1}\; q_\ell p_\ell$$

This again satisfies (6) and (7) and after a little more calculation can be seen to fulfil associativity (8). Then equation (10) is satisfied provided that d_{ij}, in addition to (5) and (16) also satisfies

$$\Sigma_\ell q_\ell d_{\ell j} = d_{i\ell}(p^2q_\ell - (q.p)p_\ell) = 0 \qquad (21)$$

Therefore d_{ij} is now a matrix of rank N-2. It is not as localised as in the former case (17).

In order to generalize the definition of the product more parameters may be introduced in the expression for C. This may be achieved by adding further linearly independent vectors r, s, t,--etc all similarly normalised at the expense of arbitrariness in the choice of d_{ij}. The structure of this family of solutions is already evident with only three vectors p, q and r.

The solution

$$C_{ijk} = \frac{1}{\Delta}\begin{vmatrix} p_i & q_i & r_i & -p_j\delta_{ik} - p_k\,\delta_{ij} + p_jp_k(\text{all } i) \\ p^2 & p.q & p.r & 0 \\ q.p & q & q.r & (p_j-q_j)(p_k-q_k) \\ r.p & r.q & r^2 & (p_j-r_j)(p_k-r_k) \end{vmatrix} \qquad (22)$$

where ,

$$\Delta = \begin{vmatrix} p^2 & p.q & p.r \\ q.p & q^2 & q.r \\ r.p & r.p & r^2 \end{vmatrix} \tag{23}$$

satisfies (6) and (7). To demonstrate that (22) satisfies associativity (8) it is best to proceed by first establishing the lemma

$$\Sigma_i \, V_i \, C_{ijk} = V_j V_k \tag{24}$$

where V is p, q or r.

It is interesting to note here, "en passant" that (24) is reminiscent of an eigenvalue-equation

$$V_j M_{jk} = V_k$$

where M_{jk} is an arbitrary N x N matrix.

Equation (24) is a sort of non-linear eigenvalue equation. It can be further generalized.

$$\Sigma_{i,n} \, V_i \, C_{ijk} \, C_{nkm} = V_j V_k V_m \quad .$$

With the help of lemma (24) and the identity

$$P_j P_k - V_j V_k = (P_j - V_j) P_k + (P_k - V_k) \, P_j - (P_j - V_j)(P_k - V_k) \tag{25}$$

We proceed to calculate the left hand side of (8) :

$$\Sigma_s \, C_{ijs} \, C_{sk\ell} = \frac{1}{\Delta} \begin{vmatrix} p_i & q_i & r_i & \Sigma_s A \, C_{sk\ell} \\ p^2 & p.q & p.r & o \\ q.p & q^2 & q.r & \Sigma_s (p_j - q_j)(p_s - q_s) C_{sk\ell} \\ r.p & r.q & r^2 & \Sigma_s (p_j - r_j)(p_s - r_s) C_{sk\ell} \end{vmatrix} \tag{26}$$

where

$$A = -p_j\delta_{is} - p_s\,\delta_{ij} + (p_jp_s) \text{ all } i \ .$$

First,

$$\Sigma_s\, A\, C_{sk\ell} = -p_j\, C_{ik\ell} - p_kp_\ell\,\delta_{ij} + (p_jp_kp_\ell) \text{ all } i \ .$$

Then

$$\Sigma_s\, (p_j-q_j)(p_s-q_s)\, C_{sk\ell} = (p_j-q_j)(p_kp_\ell - q_kq_\ell)$$

and similarly

$$\Sigma_s(p_j-r_j)(p_s-r_s)\, C_{sk\ell} = (p_j-r_j)(p_kp_\ell-r_kr_\ell)$$

Inserting these relations into (26) and using some simple determinant properties gives

$$\Sigma_s\, C_{ijs}\, C_{sk\ell} = p_j\, C_{ik\ell} + p_k\, C_{i\ell j} + p_\ell\, C_{ijk}$$

$$+\ \frac{1}{\Delta}\begin{vmatrix} p_i & q_i & r_i & p_jp_\ell\delta_{ik}+p_kp_\ell\delta_{ij}+p_kp_j\delta_{i\ell}-(p_jp_kp_\ell)\text{all } i \\ p^2 & p.q & p.r & 0 \\ p.q & q^2 & q.r & (p_j-q_j)(p_k-q_k)(p_\ell-q_\ell) \\ r.p & r.q & r^2 & (p_j-r_j)(p_k-r_k)(p_\ell-r_\ell) \end{vmatrix}$$

which is clearly symmetric in j, k and ℓ and hence satisfies associativity (8). The distributivity condition (10) imposes that p, q, r are left null eigenvectors of d_{ij}, while the column vectors $(1,1,---1)^T$ and $p_jV^2-(p.V)V_j$ are right null eigenvectors of d_{ij} i.e. d is now of rank N-3.

Translation invariance is automatically implemented in (22) because the product formula involves scalar products of f and g with p, q, r but it remains an extra requirement for d_{ij}. This solution can be extended to incorporate N linearly independent vectors. The product becomes effectively more and more localised as the number of parameters increase until with N vectors the usual formula (10) is recovered. This can be illustrated with the case $N = 3$ where C is given by (22). Choosing

$$p_i = \delta_{io} \quad , \quad q_i = \delta_{i1} \quad \text{and} \quad r_i = \delta_{i2} \quad ,$$

the product formula becomes

$$\begin{aligned} (fg)_i &= f_o g_i + f_i g_o - f_o g_o \quad \text{(all i)} \\ &\quad + (f_o - f_1)(g_o - g_1)(\delta_{i2} + \delta_{i1}) \\ &= f_i g_i \text{ (by enumeration of cases).} \end{aligned} \tag{27}$$

where there is no summation over indices.

However, in this case d has rank zero and this vanishes. This is another example of the complementarity mentioned earlier.

A further generalisation of (22) which treats the vectors on the same footing and which satisfies all the equations (5) to (9) may be constructed as follows. Define an NxL matrix $P_{i\alpha}$ ($i = o, \text{---} N-1; \alpha = o, \text{---} L-1$) whose columns are the vectors p, q, r etc..., and introduce an L-component vector λ_α whose components also sum to unit. Then

$$C_{ijk} = \frac{1}{\Delta} \left| \begin{array}{c|c} P_{i\beta} & \Sigma_\beta \lambda_\beta (P_{j\beta} P_{k\beta} - \delta_{ij} P_{k\beta} \delta_{ik} P_{j\beta}) \\ \hline \Sigma_\ell P_{\ell\alpha} P_{\ell\beta} & \Sigma_\beta \lambda_\beta (P_{j\alpha} - P_{j\beta})(P_{k\beta} - P_{k\beta}) \end{array} \right| \tag{28}$$

where $\Delta = \det \Sigma_\ell P_{\ell\alpha} \; P_{\ell\beta}$

It is remarkable that this expression of the product which is a linear combination of terms of form (22) still satisfies the non-linear equation (8), in virtue of the mechanism of equations

(24) and (25). The conditions on d_{ij} which follow from (9) are simply that the columns of P_i are both right and left null eigenvectors of d.

III.2 Gauge invariance of the Dirac equation

Although the modification of the usual notion of product to circumvent the failure of the product rule for differentiation yields remarkable results it does not restore the discrete analogue of the Leibnitz rule. Indeed, we find that the more localised is the product, the more extended is the approximation to the derivative and vice-versa.

Nevertheless, gauge invariance can be restored. To do this, we must define a covariant derivative. Recall the gradient operator

$$d_\mu = h^{-1}(P_\mu - 1)$$

In the continuous limit $P_\mu = e^{h\partial\mu}$. This suggests that the gauge field A_μ should appear in an exponential as does the continuous derivative ∂_μ. Hence, by analogy with the continuum where $D_\mu = \partial_\mu + iA_\mu$, the discrete covariant derivative may either be

$$D_\mu = d_\mu - h^{-1}(e^{-ihA_\mu} - 1) \tag{29.a}$$

or

$$D_\mu = h^{-1}(e^{ihA_\mu}P_\mu - 1) \tag{29.b}$$

where both definitions have the right continuous limit. Let us examine their behaviour under gauge transformations.

The first case $D_\mu = d_\mu - h^{-1}(e^{-ihA_\mu} - 1)$

Consider the gauge transformation

$$\psi \to g(x)\,\psi \tag{30}$$

for elements $g \in G$ the gauge group, near the identity we can write

$$g(x) = 1 + \Lambda^a T^a + O(\Lambda^2)$$

and $\psi(x) \rightarrow \psi(x) + \delta\psi$ with $\delta\psi = \Lambda^a_{(x)} T^a \psi$

In the case of electromagnetism for example, G is the abelian group $U(1)_0$, and we can define the gauge transformation $\left(g = e^{i\Lambda(x)} \right)$

$$\delta\psi = i\Lambda(x)\psi$$

$$\text{and} \qquad \delta A_\mu = -d_\mu \Lambda \tag{31}$$

$$\delta e^{-ihA\mu} = ie^{-ihA\mu}(P_\mu - 1)\Lambda \qquad \text{(at first order in } \Lambda)$$

The Dirac equation $\not{D}_\mu \psi = o$ is invariant under (31).

Indeed,

$$\not{D}'_\mu \psi' = \gamma_\mu \otimes (d_\mu - h^{-1}(e^{-ihA\mu} + \delta e^{-ihA\mu} - 1))\ (\psi + \delta\psi)$$

$$= h^{-1}\gamma_\mu \otimes (P_\mu - e^{-ihA\mu} - i(P_\mu - 1)\Lambda\, e^{-ihA\mu})(\psi + i\Lambda\psi)$$

$$= \not{D}_\mu \psi + \not{D}_\mu (i\Lambda\psi) - i\not{d}_\mu \Lambda\ e^{-ihA\mu}\psi + O(\Lambda^2)$$

$$= \gamma_\mu \otimes (h^{-1}P_\mu(i\Lambda\psi) - i\Lambda e^{-ihA\mu}\psi - id_\mu \Lambda\, e^{-ihA\mu}\psi)$$

$$= i\gamma_\mu \otimes (h^{-1}P_\mu(\Lambda\psi) - \Lambda(h^{-1}P_\mu \psi) - d_\mu \Lambda\ (h^{-1}P_\mu \psi))$$

$$= ih^{-1}\gamma_\mu\ (\Lambda_\mu \psi_\mu - \Lambda\psi_\mu - \Lambda_\mu \psi_\mu + \Lambda\psi_\mu)$$

$$= o\ .$$

With averaging the infinitesimal gauge transformation becomes

$$\psi \rightarrow \psi + i\Lambda\psi$$

$$\tfrac{1}{2}(P_\mu + 1)A_\mu \rightarrow \tfrac{1}{2}(P_\mu + 1)A_\mu + h^{-1}(P_\mu - 1)\Lambda \tag{32}$$

The Dirac equation becomes then

$$(\gamma_\mu \otimes D_\mu)\psi = \gamma_\mu \otimes \left[\frac{1}{2}(P_\nu + 1)h^{-1}(P_\mu - e^{-\frac{i}{2}h(P_\mu + 1)A_\mu})\right]\psi \qquad (33)$$

It is easy to check that again (33) is invariant under (32). Note that this is essentially due to the redefinition of A_μ; the basic quantity is now the averaged vector field $\frac{1}{2}(P_\mu + 1)A_\mu$.

Bender et al[8] constructed a Dirac equation on a finite element lattice which they claim is manifestly gauge invariant. The interaction term I_{mn} is also of the form $(e^{ihA\mu} - 1)$ although it is written in a complicated way,

$$I_{mn} = -\gamma^1 h^{-1} \Sigma_{n'=1}^{n} (-1)^{n+n'} \exp\left[ih \Sigma_{n''=n'+1}^{n} B_{m,n''}\right]\left(e^{ihB_{m,n'}} - 1\right)\left(\psi_{m,n'} + \psi_{m+1,n'}\right)$$

$$- \gamma^1 h^{-1}(-1)^n \exp\left[ih \Sigma_{n''=1}^{n} B_{m,n''}\right]\left(e^{ihB_{m,1}/2} - 1\right)\left(\psi_{m,o} + \psi_{m+1,o}\right)$$

$$-\gamma^2 \frac{h^{-1}}{2} \sec\left(h \Sigma_{m'=1}^{M} C_{m',n}/2\right) \Sigma_{m''=1}^{M} \operatorname{sgn}(m''-m)\, (-1)^{m+m''}$$

$$\times \exp\left[i\frac{h}{2} \Sigma_{m'''=1}^{M} \operatorname{sgn}(m'''-m)\operatorname{sgn}(m'''-m'')\operatorname{sgn}(m'''-m)\, C_{m''',n}\right]$$

$$\times \left(e^{ihC_{m'',n}} - 1\right)\left(\psi_{m'',n} + \psi_{m'',n+1}\right) ,$$

where $B_{m,n} = \frac{1}{2}\left[(A_1)_{m,n} + (A_1)_{m,n-1}\right]$, $C_{m,n} = \frac{1}{2}\left[(A_2)_{m,n} + (A_2)_{m-1,n}\right]$

and sgn (x) is 1 if $x > o$ and -1 if $x \leqslant o$.

However, to obtain this result and to achieve gauge invariance they make use of

$$\delta\left((P_n+1)(P_m+1)\psi_{mn}\right) = \frac{1}{4}(P_m+1)(P_n+1)\Lambda_{mn}(P_n+1)(P_m+1)\psi_{m,n} \qquad (34.a)$$

and

$$\delta\left[(P_n+1)(P_m-1)\psi_{m,n}\right] = \frac{1}{4}(P_n+1)(P_m+1)\Lambda_{mn}\,(P_n+1)(P_m-1)\psi_{m,n}$$

$$+\left(\Sigma^{m}_{m'=1} - \Sigma^{m}_{m'=m+1}\right)(-)^{m+m'}\left[\frac{1}{4}(P_n+1)(P_m+1)(1-P_m^T)\Lambda_{m'n}\right]\left[(P_n+1)\psi_{m',n}\right] \quad (34.b)$$

From (34a) one can see that the following approximation has been used

$$(P_m+1)(P_n+1)(\Lambda_{mn}\psi_{mn}) \simeq \frac{1}{4}(P_m+1)(P_n+1)\Lambda_{mn}(P_m+1)(P_n+1)\psi_{mn}$$

which amounts to the approximation

$$\Lambda_{mn} \simeq \frac{1}{4}(P_m+1)(P_n+1)\Lambda_{m,n} \quad ,$$

and this is not rigorous. Indeed, it is inconsistent with the fact that Λ_{mn} is a function of space and time.

Under non -abelian transformation we require $D_\mu\psi$ to transform covariantly $D'_\mu\psi' = g\,D_\mu\psi$, where g is now an element of a non-abelian gauge groups G, i.e.

$$D'_\mu\psi' = h^{-1}(P_\mu - e^{-ihA'_\mu})g\psi$$

$$= h^{-1}g(g^{-1}P_\mu - g^{-1}e^{-ihA'_\mu})g\psi$$

$$= h^{-1}g(P_\mu - e^{-ihA_\mu})$$

which implies that

$$e^{-ihA'_\mu} = g\,e^{-ihA_\mu}\,g^{-1} - (\partial_\mu g)g_\mu^{-1}\,P_\mu \quad (35)$$

In the limit $h \to o$, $A_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}$ is recovered.

Unfortunately, when taking averaging into account we could not find the behaviour of $e^{-ihA\mu}$ under the gauge transformation (30).

<u>The Second case $D_\mu = h^{-1}(e^{ihA_\mu}P_\mu - 1)$</u>

This definition yields much better results. Indeed $D_\mu\psi$ transforms covariantly provided that

$$e^{ihA_\mu} = g^{-1}\, e^{ihA'_\mu}\, g_\mu \quad . \tag{36}$$

Expanding the exponential and g_μ in (36) to lowest order in h, we get

$$\begin{aligned} 1 + ihA_\mu &= g^{-1}(1+ihA'_\mu)(g+h\partial_\mu g) \\ &= g^{-1}g + ih\, g^{-1}A'_\mu g + h\, g^{-1}\partial_\mu g + O(h^2) \end{aligned}$$

i.e. $$A_\mu = g^{-1}A'_\mu g - i\, g^{-1}\partial_\mu g \quad .$$

Hence $$A_\mu = g\, A_\mu g^{-1} + i\, \partial_\mu g\, g^{-1}$$

is recovered.

As we have seen in Chap. II section 4, the problems that arise when putting fermions on the lattice can be traced back to the inconsistency of the D.E on the lattice. Indeed a naive transcription of the continuum Dirac operator on the lattice yields an operator whose square is not the box operat or. A solution to this difficulty and therefore to the fermion problems on the lattice is averaging. Averaging consists of taking the derivative in one direction of a field averaged in the other directions such that the derivative is defined at the centre of the lattice hypercube. This averaging being essential we will include it in all what follows. Thus, we have now

$$\begin{aligned} d_\mu &= \frac{h^{-1}}{2}(P_\mu - 1)(P_\nu + 1) \\ &= \tfrac{1}{2}h^{-1}(P_\mu P_\nu + P_\mu - P_\nu + 1) \ . \ \text{(in 1+1 dimensions)} \end{aligned}$$

The corresponding covariant derivative is

$$D_\mu = \tfrac{1}{2}h^{-1}\left(e^{ih\bar{A}^\mu} e^{ih\bar{A}^\nu_\mu} P_\mu P_\nu + e^{ih\bar{A}^\mu} P_\mu - e^{ih\bar{A}^\nu} - 1\right) , \tag{37}$$

where $\bar{A}^\mu = \frac{1}{2}(P_\mu+1)A^\mu$, $\bar{A}^\nu_\mu = \frac{1}{2}(P_\nu+1)A^\nu_\mu = \frac{1}{2}(P_\nu+1)P_\mu A^\nu$.

In order to obtain (37), we have replaced in d_μ ,

$$\begin{aligned}
&P_\nu && \text{by} && e^{ih\bar{A}^\nu} P_\nu \\
&P_\mu P_\nu && \text{by} && e^{ih\bar{A}^\mu} e^{h\bar{A}^\nu_\mu} P_\mu P_\nu \\
&P^T_\mu P_\nu && \text{by} && P^T_\mu e^{-ih\bar{A}^\mu} e^{ih\bar{A}^\nu} P_\nu \\
&P^T_\mu P^T_\nu && \text{by} && P^T_\mu P^T_\nu e^{ih\bar{A}^\mu_\nu} e^{-ih\bar{A}^\nu}
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\text{Note that } e^{-ih\bar{A}^\nu} \text{ transforms as } g_\nu e^{-ih\bar{A}^\nu} g^{-1} \\
&\text{and } e^{ih\bar{A}^\nu} \text{ transforms as } g\, e^{ih\bar{A}^\nu} g^{-1}_\nu \, .
\end{aligned} \tag{39}$$

The fermionic action (37) in chapter II transforms covariantly provided that we introduce a vector field A_μ via the prescription (38). The action becomes then,

$$\begin{aligned}
S &= \frac{1}{8} h^{-1}\bar{\psi}\gamma_\mu \otimes \Big[2(e^{ih\bar{A}^\mu} P_\mu - P^T_\mu e^{-ih\bar{A}^\mu}) \\
&\qquad + \left(e^{ih\bar{A}^\mu} e^{ih\bar{A}^\nu_\mu} P_\mu P_\nu - P^T_\mu P^T_\nu e^{-ih\bar{A}^\nu_\mu} e^{-ih\bar{A}^\mu}\right) \\
&\qquad + \left(P^T_\nu e^{-ih\bar{A}^\nu} e^{ih\bar{A}^\mu} P_\mu - P^T_\mu e^{-ih\bar{A}^\mu} e^{ih\bar{A}^\nu} P_\nu\right)\Big] \psi \\
&= \frac{1}{8} h^{-1} \Big[\left(2\bar{\psi}\gamma_\mu e^{ih\bar{A}^\mu} \psi_\mu - 2\bar{\psi}_\mu\gamma_\mu e^{-ih\bar{A}^\mu} \psi\right) \\
&\qquad + \left(\bar{\psi}\gamma_\mu e^{ih\bar{A}^\mu} e^{ih\bar{A}^\nu_\mu} \psi_{\mu+\nu} - \bar{\psi}_{\mu+\nu}\gamma_\mu e^{-ih\bar{A}^\nu_\mu} e^{-ih\bar{A}^\mu} \psi\right) \\
&\qquad + \left(\bar{\psi}_\nu\gamma_\mu e^{-ih\bar{A}^\nu} e^{ih\bar{A}^\mu} \psi_\mu - \bar{\psi}_\mu\gamma_\mu e^{-ih\bar{A}^\mu} e^{ih\bar{A}^\nu} \psi_\nu\right)\Big]
\end{aligned} \tag{40}$$

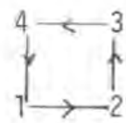
The first two terms in (40) are nothing but the Wilson action - the remaining terms are the result of averaging and vanish all for $h \to o$. Thus in the continuous limit we again recover the Dirac action $\psi\gamma_\mu(\partial_\mu + iA_\mu)$ (see appendix).

The covariant derivative (29.b) is therefore in agreement with Wilson's formulation of gauge invariance. It amounts to replacing a field with values in a Lie algebra (continuum) by a field variable taking its values in a Lie group. On the lattice, symmetries are represented at the group level rather than at the algebra level.

III.3 Gauge Action

A well known action for the gauge field, which is also gauge covariant is the quantity

$$\mathcal{G} \simeq \mathrm{tr}\,(e^{ihA^\mu} e^{ihA^\nu_\mu} e^{-ihA^\mu_\nu} e^{-ihA^\nu}) = \tag{41}$$

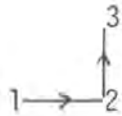
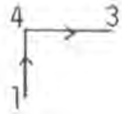


It is the contribution of the simplest closed curve on the lattice, namely a plaquette. This action therefore lives on a square of corners $x_1 = x$, $x_2 = x+n_\mu$, $x_3 = x+n_\mu+n_\nu$, $x_4 = x+n_\nu$ which can be identified with a two-dimensional face of a hypercube on the lattice.

There exists however another alternative which we are going to follow here.

We may define,

$$f^{\mu\nu} = e^{ihA^\mu} e^{ihA^\nu_\mu} - e^{ihA^\nu} e^{ihA^\mu_\nu} \tag{42}$$

$=$  $-$ 

In order to evaluate $f^{\mu\nu}$ to lowest in h, we expand A^{ν}_{μ} and A^{μ}_{ν} and apply the Baker-Hausdorff formula $e^{x}e^{y} = \exp(x+y + \frac{1}{2}[x\ y] + --\ )$. This gives

$$f^{\mu\nu} = \exp(ihA^{\mu} + ihA^{\nu} + ih^{2}\partial_{\mu}A^{\nu} - \frac{h^{2}}{2}[A^{\mu}, A^{\nu}])$$

$$- \exp(ihA^{\nu} + ihA^{\mu} + ih^{2}\partial_{\nu}A\mu - \frac{h^{2}}{2}[A^{\mu}, A^{\nu}])$$

$$= 1 + ih(A^{\mu} + A^{\nu} + i\frac{h}{2}[A^{\mu}, A^{\nu}] + h\,\partial_{\mu}A^{\nu})$$

$$- \frac{h^{2}}{2!}(A^{\mu} + A^{\nu}\, h\,\partial_{\mu}A^{\nu})^{2} + o(h^{3})$$

$$- 1 - ih(A^{\mu} + A^{\nu} + h\,\partial_{\nu}A^{\mu} - i\frac{h}{2}[A^{\mu}, A^{\nu}])$$

$$+ \frac{h^{2}}{2!}(A^{\mu} + A^{\nu} + h\,\partial_{\mu}A^{\mu})^{2} + o(h^{3})$$

$$= ih^{2}\partial_{\mu}A^{\nu} - ih^{2}\partial_{\nu}A^{\mu} - h^{2}[A^{\mu}, A^{\nu}] \tag{43}$$

Hence,

$$f^{\mu\nu} = ih^{2}F^{\mu\nu}_{\text{cont}} = ih^{2}(\partial_{\mu}A^{\nu} - \partial_{\nu}A^{\mu} + i[A^{\mu}, A^{\nu}])$$

We also define the quantity

$$\mathcal{F}^{\mu\nu} = e^{-ihA^{\mu}_{\nu}}\, e^{-ihA^{\nu}} - e^{-ihA^{\nu}_{\mu}}\, e^{-ihA^{\mu}} \tag{44}$$

= 

Again $\mathcal{F}^{\mu\nu} = ih^{2}(\partial_{\mu}A^{\nu} - \partial_{\nu}A^{\mu} + i[A^{\mu}, A^{\nu}]) + o(h^{3})$ (45)

Under a gauge transformation,

$$f^{\mu\nu} \rightarrow g f^{\mu\nu} g^{-1}_{\mu+\nu} \tag{46}$$

$$\mathcal{F}^{\mu\nu} \rightarrow g_{\mu+\nu}\mathcal{F}^{\mu\nu}g^{-1}$$

We expand $g^{-1}_{\mu+\nu}$ and $g_{\mu+\nu}$,

$$g^{-1}_{\mu+\nu} = g^{-1} + h\,\partial_\mu g^{-1} + h\,\partial_\nu g^{-1} + \frac{h^2}{2!}\partial^2_\mu g^{-1} + \frac{h^2}{2!}\partial^2_\nu g^{-1} + h^2\partial_\mu\partial_\nu g^{-1}$$

$$+ \frac{h^3}{2!}\partial^2_\mu\partial_\nu g^{-1} + \frac{h^3}{2!}\partial^2_\nu\partial_\mu g^{-1} + o(h^4)$$

then,

$$g^{-1}_{\mu+\nu}\, g_{\mu+\nu} = g^{-1}g + h(g^{-1}\partial_\mu g + \partial_\mu g^{-1}g + g^{-1}\partial_\nu g + \partial_\nu g^{-1}g)$$

$$+ \frac{h^2}{2!}(g^{-1}\partial^2_\mu g + \partial^2_\mu g^{-1}g + 2\partial_\mu g^{-1}\partial_\mu g + g^{-1}\partial^2_\nu g$$

$$+ \partial^2_\nu g^{-1}g + \partial_\nu g^{-1}\partial_\nu g + g^{-1}\partial_\mu\partial_\nu g + \partial_\mu\partial_\nu g^{-1}g$$

$$+ \partial_\mu g^{-1}\partial_\nu g + \partial_\nu g^{-1}\partial_\mu g) + o(h^3)$$

$$= g^{-1}g = 1$$

since $g^{-1}g = 1$ and $\partial_\mu(g^{-1}g) = \partial_\mu g^{-1}g + g^{-1}\partial_\mu g = o$

$$\partial^2_\mu(g^{-1}g) = \partial^2_\mu g^{-1}g + g^{-1}\partial^2_\mu g + 2\,\partial_\mu g^{-1}\partial_\mu g = o$$

$$\partial_\mu\partial_\nu(g^{-1}g) = \partial_\mu\partial_\nu g^{-1}g + g^{-1}\partial_\mu\partial_\nu g + \partial_\mu g^{-1}\partial_\nu g$$

$$+ \partial_\nu g^{-1}\partial_\mu g = o .$$

Therefore

$$f^{\mu\nu}\mathcal{F}^{\mu\nu} \to g f^{\mu\nu} g^{-1}_{\mu+\nu}\, g_{\mu+\nu}\mathcal{F}^{\mu\nu} g^{-1} = g f^{\mu\nu}\mathcal{F}^{\mu\nu} g^{-1}$$

The quantity $f^{\mu\nu}\mathcal{F}^{\mu\nu}$ is gauge covariant and furthermore it has the right continuum limit, thus we can choose it to be the action.

The dynamical variables live on the links between neighbouring sites i and j. In other words the dynamical state of the system is specified by assigning an element U_{ij} of the gauge group G to every link between i and j, U_{ij} satisfies $U_{ij} = U_{ji}^{-1}$.

Each directed link is associated with a group element $U = e^{ihA\mu}$, the oppositely oriented link is associated with the group inverse element $U_{ij}^{-1} = e^{-ihA\mu}$. $e^{ihA\mu}$ specifies the rotation U in some intrinsic internal symmetry space upon transport between x_μ and $x_\mu + h_\mu(i,j)$. U_{ij}, therefore correspond to the transport along a path of finite length; thus it represents finite (not infinitesimal) group transformations.

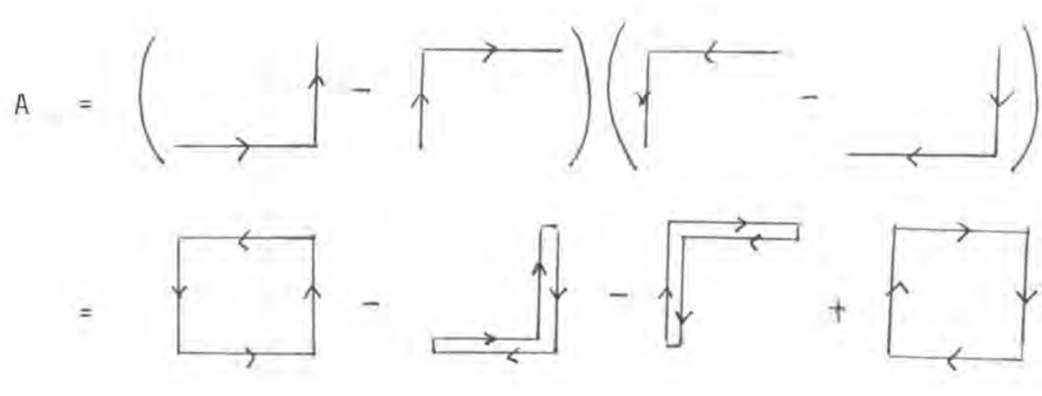
In the same way $e^{ih^2F\mu\nu}$ specifies the rotation in the transport around a closed path e.g. a plaquette.

Gauge transformations are defined by assigning to each site a group element gi ε G. The dynamical variables U_{ij} transform as in (14) i.e.

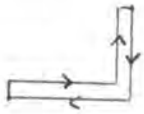
$$e^{ihA\mu} \rightarrow g\, e^{ihA\mu}\, g_{\mu}^{-1}$$

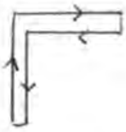
$$\text{or} \qquad U_{ij} \rightarrow g_j U_{ji} g_i^{-1}$$

The diagrammatic representation of this action, $A_\nu \mathrm{tr} f^{\mu\nu} \mathcal{F}_{\mu\nu}$, is



where the close double lines refer to backtracking on lattice links. Their contribution is merely,

 $= \mathrm{tr}\, e^{ihA^\mu} e^{ihA^\nu_\mu} e^{-ihA^\nu_\mu} e^{-ihA^\mu} = \mathrm{tr}\, 1$

and  $= \mathrm{tr}\, e^{ihA^\nu} e^{ihA^\mu_\nu} e^{-ihA^\mu_\nu} e^{-ihA^\nu} = \mathrm{tr}\, 1$

The first term in A is a plaquette (i.e. the usual Wilson action); to lowest order in h it contributes

$$A_1 = \mathrm{tr}\, e^{ihA^\mu} e^{ihA^\nu_\mu} e^{-ihA^\mu_\nu} e^{-ihA^\nu} = \mathrm{tr}\Big[e^{ihA^\mu} e^{ih(A^\nu + h\partial_\mu A^\nu)}$$

$$e^{-ih(A^\mu + h \quad A^\mu)} e^{-ihA^\nu}\Big]$$

$$\simeq \mathrm{tr}\, \exp\Big[ih(A^\mu + A^\nu + h\partial_\mu A^\nu) - \tfrac{1}{2} h^2 \big[A^\mu, A^\nu\big]\Big]$$

$$\exp\Big[-ih\,(A^\mu + A^\nu + h\partial_\nu A^\mu) - \tfrac{1}{2} h^2 \big[A^\mu, A^\nu\big]\Big]$$

$$\simeq \mathrm{tr}\, \exp\,\{\, ih^2(\partial_\mu A^\nu - \partial_\nu A^\mu) - h^2\big[A^\mu, A^\nu\big] \,\}$$

$$\simeq \mathrm{tr}\, \exp\,\{\, ih^2\, F^{\mu\nu} \,\}$$

The Baker-Hausdorff formula has been applied and terms in h^3 have been dropped. Then,

$$A_1 = \mathrm{tr}(\, 1 + ih^2 F_{\mu\nu} - \frac{h^4}{2} F^2_{\mu\nu} - i\,\frac{h^6}{3!} F^3_{\mu\nu} + \text{---}\,)$$

But $\mathrm{tr}\, F_{\mu\nu} = o$ so

$$A_1 = \mathrm{tr}\,(1 - \frac{h^4}{2} F^2_{\mu\nu} - i\,\frac{h^6}{6} F^3_{\mu\nu} + \text{---})$$

The h^4 in $F_{\mu\nu}^2$ is necessary to convert the sum over sites $\sum_\mu$ into an integral $\int d^4x$.

Note that in the limit $h \to o$, the usual Yang-Mills action is recovered up to a constant.

The fourth term in A is also a plaquette oriented oppositely to the first plaquette. It contributes

$$A_2 = \text{tr}\, e^{ihA^\nu} e^{+ihA^\mu_\nu} e^{-ihA^\nu_\mu} e^{-ihA^\mu}$$

$$\simeq \text{tr} \exp \{ -ih^2(\partial_\mu A^\nu - \partial_\nu A^\mu) + h^2 [A^\mu, A^\nu] \}$$

$$= \text{tr} \exp \{ -ih^2 F^{\mu\nu} \}$$

$$= \text{tr} \{ 1 - \frac{h^4}{2} F_{\mu\nu}^2 + i \frac{h^6}{6} F_{\mu\nu}^3 + --- \}$$

The action A is then,

$$A = \beta(A_1 + A_2 - 2 \text{ tr } 1)$$

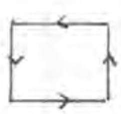
$$= \beta\text{tr} \{ -h^4 F_{\mu\nu}^2 + o(h^8)\} = \beta h^4 \text{tr}(-F_{\mu\nu}^2 + o(h^4))$$

The constants have disappeared as well as the term in $F_{\mu\nu}^3$.

The standard Yang-Mills action has a coefficient of $\frac{1}{4}(F_{\mu\nu})^2$.

Thus we choose $\beta = -\frac{1}{4}$

The action A can be rewritten as

$$A = \beta \Big[\text{tr} \Big($$  $$+ \text{h.c} \Big) -2 \text{ Tr } 1 \Big]$$

This action has the right continuous limit. Moreover the last term cancels with the constants resulting from the expansion of the first two terms. It has not been put in by hand. The terms involving $F^3_{\mu\nu}$, $F^5_{\mu\nu}$, $F^7_{\mu\nu}$ etc. all vanish automatically. After removing a factor h^4 to convert Σ_n into an integral the continuous action is recovered up to the fourth order i.e.

$$A \simeq \mathrm{tr}\ \{ F^2_{\mu\nu} + o(h^4) \}$$

This action is therefore an improved form of the Wilson action.[4]

CHAPTER IV

SUPERSYMMETRY ON THE LATTICE

IV.1 Introduction

In the early seventy's much progress has been made in the extension of invariance considerations in particle physics and field theories, to symmetries in which fermions and bosons play a common role. This has been achieved by introducing a Fermi-Bose symmetry known as supersymmetry, as an extension of the Lie algebra of the Poincaré Group.[20]

Gauge theories with global supersymmetry have been intensively investigated. It is hoped that such theories might help to understand the hierarchy problem and the huge difference between the scale of weak interactions and the Planck scale.[26,21] Furthermore, local sypersymmetry [27,25] allows the unification of gravity with the standard theory of strong and electroweak interactions.

If, indeed the physics of particles and fields is described by supersymmetric theories, the symmetry cannot be exact, it has to be broken since there are no fermions and bosons having the same mass except massless particles (neutrino, photon and graviton).

The breaking of supersymmetry may be explicit i.e soft breaking terms are put in by hand (by choosing appropriate Higgs parameters),or it may be the result of spontaneous breakdown.[22] The last possibility is the most attractive. Also, it is known that if supersymmetry exists on the tree level, it will not be broken to any finite order in pertubation theory.[28] The problem of supersymmetry breaking is therefore inherently non-perturbative. Witten's index theorem[29] states that for a large class of models, supersymmetry will not be broken by nonperturbative effects. But there are important cases[30] which are not covered by this theorem and that are worthy of further study. It is therefore important to develop nonperturbative techniques to investigate the dynamics of supersymmetric theories beyond perturbation theory.

The best candidate is the lattice approach[9] which allows the exploration of the weak and strong coupling regimes non-perturbatively. One would then like to put supersymmetric theories on the lattice hoping that this will help us to fathom their physics and in particular the problem of spontaneous symmetry breaking.

D.M. Scott,[23] gives further reasons for the study of lattice approximations to supersymmetric theories. He sees that super-symmetry would provide further constraints when writing down a gauge invariant action. Moreover, supersymmetry must be broken and the lattice approach allows the study of spontaneous breakdown of symmetry. He also sees that supersymmetry determines uniquely the form of the kinetic term for fermions, given the form of the bosonic term, as supersymmetry relates the two. And this might overcome the difficulties associated with the description of fermions on the lattice.

So far there have been several attempts to put supersymmetry on the lattice but without much success. The problem is that the anti-commutator of two supersymmetry generators is a translation. These infinitesimal space and time translations lose their meaning on the lattice where the Poincaré group reduces to the space group of the lattice. To avoid this difficulty, several attempts such as the Lagrangian[24] or Hamiltonian[14,15] formulation and the Euclidean formulation[17,19,31] have been proposed, all of which are based on modifications of the graded Lie algebra of the super Lie groups.

However, this is not the only possible approach. It has also been pointed out that as the Poincaré group is reduced to a discrete subgroup on the lattice, supersymmetry which is an extension of the Poincaré group may be represented as a discrete subgroup on the superspace lattice.[16] In other words, the lattice is invariant under a subgroup of a graded extension of the Poincaré group. Here we would like to draw attention to another possibility for the formulation of supersymmetry algebra on the lattice.

IV.2 Two-Dimensional Lattice Super-Algebra

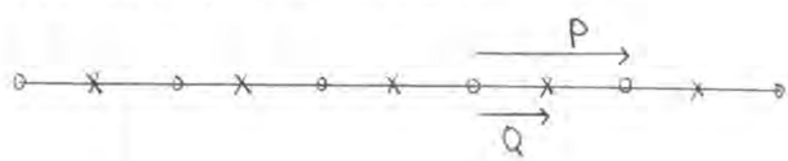
The supersymmetry algebra in its simplest form involves the generators of space-time rotations and translations $M^{\mu\nu}$ and P^{μ} as well as spin-½ generators Q_{α} which turn boson fields into fermion fields and vice versa. They satisfy the following algebra

$$\begin{aligned} [\, Q_{\alpha}, M^{\mu\nu} \,] &= (\sigma^{\mu\nu} Q)_{\alpha} \\ [\, Q_{\alpha}, P^{\mu} \,] &= 0 \\ \{Q_{\alpha}, \bar{Q}_{\beta} \} &= -2(\gamma_{\mu})_{\alpha\beta}\, P^{\mu} \end{aligned} \qquad (1)$$

where $\bar{Q}_{\alpha}$ is the complex conjugate of Q_{α}.

This algebra is called a Graded Lie Algebra.[32] It involves both commutation and anticommutations relations because of the spinorial character of the generators Q_{α}.

First, note that if P is a translation a unit distance on a one dimensional lattice, then $P = Q^2$, where Q is a translation a unit distance to points of the dual lattice.

i.e. 

where o represent points of the ordinary lattice and x points of the dual lattice. Now the continuum N = 1 supersymmetry algebra is

$$\{Q_\rho , Q_\sigma\} = \delta_{\rho\sigma} H + (\gamma^0\gamma^i)_{\rho\sigma} P_i \tag{2}$$

In two dimensions the algebra (2) in a Majorana basis in which $\gamma^0 = \sigma_2$, $\gamma_1 = i\sigma_1$, $\gamma^0\gamma = \sigma_3$ takes the form

$$\begin{aligned} Q_1^2 &= H + P \\ Q_2^2 &= H - P \end{aligned} \tag{3}$$

$$\{ Q_1 , Q_2 \} = 0 \tag{4}$$

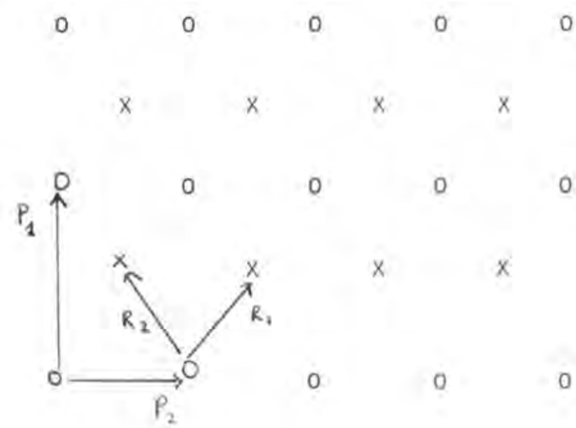
where H is the Hamiltonian and P the momentum.

On the lattice the naive transcription of (3) and (4) is

$$\begin{aligned} Q_1^2 &= P_1 + P_2 \\ Q_2^2 &= P_1 - P_2 \\ \{ Q_1 , Q_2\} &= 0 \end{aligned} \tag{5-a}$$

where P_1 and P_2 are the translation operators in the x_1 and x_2 directions respectively.

However if we defined "diagonal" translations to the dual lattice R_1 and R_2 as below



Then

$$R_1^2 = P_2 P_1 \text{ and } R_2^2 = P_2^T P_1 \tag{6.a}$$

But we know that the continuum limit of P_1 and P_2 is $e^{h\partial_1}$ and $e^{h\partial_2}$ respectively. That is

$$R_1^2 = e^{h(\partial_1 + \partial_2)} \text{ and } R_2^2 = e^{h(\partial_1 - \partial_2)} \tag{6.b}$$

The relations (5.a) can also be written as

$$\begin{aligned} Q_1^2 &= e^{h\partial_1} + e^{h\partial_2} \\ Q_2^2 &= e^{h\partial_2} - e^{h\partial_2} \end{aligned} \tag{5.b}$$

The situation now is remiscent of that in III.2 where we had two definitions for the covariant derivative

$$D\mu = h^{-1}(e^{h\partial\mu} - 1) - h^{-1}(e^{-ihA\mu} - 1)$$

and $$D\mu = h^{-1}(e^{ihA\mu} e^{h\partial\mu} - 1).$$

giving both the right continuum limit. The second definition was retained as it gives better results. Moreover it is in accordance with the fact that the group element corresponding to $\phi_1 + \phi_2$(where ϕ_1 and ϕ_2 are two fields) is $e^{\phi_1 + \phi_2}$and not $e^{\phi_1} + e^{\phi_2}$. Remember that on the lattice symmetries are represented at the group level rather than the algebra level.

For instance the rotation U (in some internal space) upon transport between x^μ and $x^\mu + dx^\mu$ is $U = e^{iA\mu(x)dx\mu}$.
The lattice analogue is given by U_{ij} a group element assigned to the link between neighbouring sites i and j.

The rotation in the transport along a path γ going through the sites i_1, i_2, i_3 --- i_n is then given by

$$U_\gamma = U_{i_n i_{n-1}} \text{ ----- } U_{i_3 i_2} U_{i_2 i_1},$$

the analogue of the Path ordered operator

$$U_\gamma = Pe^{\int_\gamma A\mu \; dx^\mu}$$

The operators R_1 and R_2 translate a field from a point of the ordinary lattice to the next-neighbour point on the dual lattice. We also know that bosonic fields live on the lattice sites whereas fermionic fields live on the dual lattice sites. This is because the Klein-Gordon equation is defined on the lattice points but not the Dirac equation. Indeed the Dirac equation is well defined only on points of the dual lattice (averaging for fermion fields see II.4). Hence, the translations R_1 and R_2 turn a fermion into a boson and vice-versa.

As a result of all these observations we conclude that it is attractive to view the algebra (6) as the appropriate setting for the lattice super-algebra, and to obtain the continuum supersymmetry algebra directly by taking the lattice spacing to zero.
But $[R_1, R_2] = o$. Indeed we know (see II.3) that

$$P_1 = P \otimes 1$$

and $$P_2 = 1 \otimes P$$

i.e. $$R_1^2 = P \otimes P \quad \text{and} \quad R_2^2 = P \otimes P^T$$

Therefore

$$R_1 = \sqrt{P} \otimes \sqrt{P} \quad \text{and} \quad R_2 = \sqrt{P} \otimes \sqrt{P^T} \qquad (7)$$

Any matrix A can be written as $A = \Sigma_i \lambda_i V_i V_i^+$ where λ_i are the eigenvalues of A and V_i the eigenvectors. V_i^+ is the hermitian conjugate of V_i .

let $\quad P = A$ then,

$$P = \lambda_i V_i V_i^+ \tag{8}$$

and

$$\sqrt{P} = \sqrt{\lambda_i} V_i V_i^+$$

Also, since

$$PV_i = \lambda_i V_i$$

we have

$$P^T P V_i = \lambda_i P^T V_i = V_i \text{ (as } P^T P = 1)$$

That is

$$P^T V_i = \lambda_i^{-1} V_i$$

Therefore

$$P^T = \lambda_i^{-1} V_i V_i^+$$

and $$\sqrt{P^T} = \sqrt{\lambda_i}^{-1} V_i V_i^+$$

Thus $\sqrt{P}$ and $\sqrt{P^T}$ commute with each other and so do R_1 and R_2 because of (7). Moreover, this implies that

$$[P,R] = o.$$

So (4) i.e. $\{Q_1, Q_2\} = o$ is not satisfied by R_1 and R_2.

We can get round this by noticing that the Pauli matrices σ_i satisfy

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad i,j = 1, 2, 3$$

and defining

$$Q_1 = \sigma_1 \otimes R_1 \quad \text{and } Q_2 = \sigma_2 \otimes R_2 \tag{9}$$

Thus

$$Q_1^2 = \sigma_1^2 \otimes R_1^2 = 1 \otimes (P_1 P_2) = P_1 P_2$$

$$\text{and} \quad Q_2^2 = \sigma_2^2 \otimes R_2^2 = 1 \otimes (P_1 P_2^T) = P_1 P_2^T$$

$$Q_1 Q_2 + Q_2 Q_1 = \sigma_1 \sigma_2 \otimes R_1 R_2 + \sigma_2 \sigma_1 \otimes R_2 R_1$$

$$= \{\sigma_1 \sigma_2\} \otimes R_1 R_2$$

$$= 0 \qquad \text{since } \{\sigma_1, \sigma_2\} = 0$$

So Q_1 and Q_2 satisfy both (3) and (4) as required. Hence the full algebra is preserved.

Further, the operators Q_1 and Q_2 can be realized explicitly by letting the fields on the lattice sites be "Clifford algebra" valued.

The Clifford algebra can be thought of as multiples of the form

$$C = \{\lambda_0 1 + \lambda_1 \sigma_1 + \lambda_2 \sigma_2 + \lambda_3 \sigma_3 ; \lambda_i \in \mathbb{C}\} \tag{10}$$

Then if Ψ is a field on the lattice with values in C,

$$Q_1 \psi(x) = \sigma_1 \Psi(R_1 x) \tag{11}$$

$$\text{and} \quad Q_2 \psi(x) = \sigma_2 \Psi(R_2 x)$$

is a realization of (9).

In fact, C is Z_2-graded, hence we can define

$$C^+ = \{ \lambda_0 1 + \lambda_3 \sigma_3 \} \subseteq C \tag{12.a}$$

$$C^- = \{ \lambda_1 \sigma_1 + \lambda_2 \sigma_2 \} \subseteq C \tag{12.b}$$

where C^+ is the even part of C and C^- the odd part.

And we can let the fields on the ordinary lattice sites be in C^+, the boson fields, and the fields on the dual lattice sites be in C^-, the fermion fields.

That is

$$\Phi = \phi_0 1 + \phi_3 \sigma_3 \qquad \text{on points o} \tag{13}$$

where ϕ_0 is a scalar and ϕ_3 a pseudoscalar and,

$$\Psi = \psi_1 \sigma_1 + \psi_2 \sigma_2 \qquad \text{on points x} \tag{14}$$

where $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ forms a Weyl spinor.

This gives us the fields necessary to form the Wess-Zumino model.[18]

We can easily check that Q_1 or Q_2 acting on a fermion field $\in C^-$ for instance gives a boson field $\in C^+$.

From (11)

$$\begin{aligned} Q_1 \psi (x) = (\sigma_1 \otimes R_1) \psi (x) &= \sigma_1 \bar{\Psi}(R_1 x) \\ &= \sigma_1 (\sigma_1 \psi_1 + \sigma_2 \psi_2) \\ &= \\ &= 1 \psi_1 + \sigma_3 \psi_2 \end{aligned}$$

The two-dimensional Wess-Zumino model[18] has two fields, one Weyl fermion ψ and one real boson φ. After elimination of the auxiliary field, the Lagrangian[14] is

$$L = \tfrac{1}{2}(\partial_\mu \varphi)^2 + \frac{i}{2} \bar{\psi} \not{\partial} \psi - V'(\varphi) \bar{\psi}\psi - \tfrac{1}{2} V^2(\varphi) \tag{15}$$

That is

$$L = \tfrac{1}{2}\left(\left(\frac{\partial\varphi}{\partial t}\right)^2 + \left(\frac{\partial\varphi}{\partial x}\right)^2\right) + \tfrac{1}{2}(\bar\psi\sigma_1\partial_x\psi + \bar\psi\sigma_2\partial_t\psi)$$
$$- \quad V'(\varphi)\bar\psi\,\psi - \tfrac{1}{2}V^2(\varphi) \; . \qquad (16)$$

To formulate the theory on the lattice we shall discretise the expression for the Lagrangian(16). The derivatives are approximated in the following way:

$$\frac{\partial\varphi}{\partial x} \to \frac{1}{h}\;(\varphi(x+h,t) - \varphi(x,t)) \;= h^{-1}(P_x-1)\varphi(x,t)$$

$$\frac{\partial\psi}{\partial x} \to \frac{1}{2h}\;(P_x-1)(P_t+1)\psi(x,t).$$

where P_x and P_t are defined in chapter II, section 3.

Note that the fermionic derivative is defined in terms of averaging [1,8]. This is essential in order to obtain a supersymmetric latticetheory. It is well known[1,9] that a naive transcription of the derivative on the lattice results in a doubling of fermionic degrees of freedom. And in order to have a supersymmetric theory the number of bosonic degrees of freedom must be equal to the fermionic ones. This can be achieved either by doubling the bosonic degrees[14] or by eliminating fermion doubling (see chap. II).

The Lagrangian is then defined by

$$L = \Sigma_{ij}\;\{[h^{-2}(P_i-1)\varphi_{ij}]^2 + [h^{-2}(P_j-1)\varphi_{ij}]^2$$
$$+ \frac{h^{-1}}{8}\frac{i}{2}\;\bar\psi_{ij}[\,\sigma_1\otimes(P_i^T+1)(P_j^T+1)(P_i-1)(P_j+1) \qquad (17)$$
$$+ \;\sigma_2\otimes(P_i^T+1)(P_j^T+1)(P_j-1)(P_i+1)]\,\psi_{ij} - \tfrac{1}{2}V^2(\varphi)$$
$$-\frac{1V'}{24}(\varphi)\;\bar\psi_{ij}(P_i^T+1)(P_j^T+1)(P_i+1)(P_j+1)\;\psi_{ij}\;\}$$

and all the terms in (17) tend to the right limit in the continuum.

We think that this approach may lead to a realistic lattice approximation to a continuum model and it is therefore worthy of further investigation. It is interesting to ask whether or not this formulation can be extended to four dimensions. To do this one is naturally led to consider the following Clifford algebra

$$C^{-} = \{\lambda_i \gamma_i \ , \quad i = o, 1, 2, 3\} \qquad \text{odd part (fermion)}$$

and $$C^{+} = \{\lambda_o 1 + \lambda_5 \gamma_5 + \lambda \sigma^{\mu\nu}\} \qquad \text{even part (bosons)}$$

where γ_i are the Dirac matrices and $\sigma_{\mu\nu}$ an antisymmetric tensor with six degrees of freedom,

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu , \gamma_\nu\right]$$

and $$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 .$$

However we have not been able to extend this formulation to four dimensions and it remains unclear how it could be done.

This question needs further investigation, though, if it is possible it will be more difficult than the two dimensional case due to the complexity of the four dimensional supersymmetric algebra.

Consider for example the N = 1 supersymmetry

$$\{ Q_\alpha \ , \ Q_\beta \} = \delta_{\alpha\beta} H = (\alpha^i P_i)_{\alpha\beta} \quad \text{where } \alpha^i = \gamma^0 \gamma^i$$

In the 4-dimensional Majorana representation this algebra takes the following form

$$\{ Q_\alpha , Q_\beta\} = \begin{pmatrix} H + P_2 & 0 & -P_3 & -P_1 \\ 0 & H+P_2 & -P_1 & P_3 \\ -P_3 & -P_1 & H-P_2 & 0 \\ -P_1 & P_3 & 0 & H-P_2 \end{pmatrix}$$

i.e.

$$Q_1^2 = Q_2^2 = H + P_2$$

$$Q_3^2 = Q_4^2 = H - P_2$$

$$\{Q_1, Q_2\} = \{Q_3, Q_4\} = 0$$

$$\{Q_2, Q_4\} = -\{Q_1, Q_3\} = P_3$$

$$\{Q_1, Q_4\} = \{Q_2, Q_3\} = -P_1$$

in the Majorana representation, $\gamma^0\gamma^i = \sigma^1 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$

CHAPTER V

THE SKYRME MODEL

V.1 Introduction

QCD is currently believed to be the underlying theory of strong interactions with fundamental quarks and gluons. However, it does not provide a simple description of low energy hadron physics (1 GeV). A more appropriate description is given by an effective field theory of mesons and baryons, or of mesons only.

In the limit of massless quarks, QCD exhibits a global symmetry under the chiral group $G = SU(N_f)_L \otimes SU(N_f)_R$ where N_f is the number of flavours and $SU(N_f)_{L,R}$ act separately on the left and right handed quarks. This global group is spontaneously broken to a vector subgroup $H = SU(N_f)$. The symmetry breakdown $G \rightarrow H$ yields $(N_f^2 - 1)$ Goldstone bosons which are identified with the pseudoscalar mesons π, K, η ..,etc. At low energies, the interactions of these Goldstone modes can be described by a "non-linear" Lagrangian where the fields are valued in the coset space G/H which is topologically the same as $SU(N_f)$. The description of the interactions of pseudoscalar mesons in terms of non-linear Lagrangians was known even before the formulation of QCD.[58]

't Hooft[46] and later on Witten[47] considered the generalisation of QCD from SU(3) to an SU(N) gauge group and assuming confinement they concluded that QCD is then equivalent to a theory of fundamental meson fields in which the meson coupling is $^1/N$. This theory of mesons is very complicated but in the large N-limit the meson coupling $^1/N$ is weak, and the tree approximation to the meson theory is then valid.

It has been pointed out[54] that baryons behave as if they were solitons in the effective meson field theory. Indeed at low energies the large N theory of mesons reduces to a non-linear model of spontaneously broken chiral symmetry. And the solitons of the non-linear model have precisely the quantum numbers of QCD baryons[54] provided that the effects of the Wess-Zumino coupling[59,37] are included. In the large-N limit, the meson theory is weakly coupled and the solitons can be treated semi-classically.

Since we do not know how to derive such an effective theory of mesons with baryons arising as solitons from QCD, we can study model field theories of mesons which admit soliton solutions and which incorporate all the global symmetries of strong interactions and see how accurately these solitons describe baryons.

The description of baryons as chiral solitons was first proposed more than twenty years ago by Skyrme[35] well before the formulation of QCD and chiral symmetry. Skyrme considered the non-linear sigma model which describes the self-interactions of pions[53] (lightest mesons) and showed that if one adds a fourth order term i.e a term which is quartic in derivatives, the model supports non-trivial static solutions. Skyrme made the remarkable conjecture that the topological charge could be identified as baryon number, and that the solitons could be quantized as fermions,[49] the Skyrmions. More recently there has been renewed interest in the Skyrme model.[51] This revival started with the work of Pak and Tze[56] who studied the current algebraic and topological aspects of the chiral model and subsequently with the work of Gipson and Tze[57] who predicted "Skyrme" solitons (Skyrmions) from the weak interaction model. More recently Balachandran et al[36] and Witten[54,37] as well as many other authors have published much work on this model.

As the chiral model is believed to be an effective low-energy limit of QCD and as Skyrme demonstrated that baryon-like solitons do emerge from a mesonic Lagrangian, it is expected that if QCD is correct, the meson theory equivalent to QCD is described by a Skyrme-like model at least in the tree approximation i.e in the large-N limit. Note however that such Skyrme-like Lagrangians have not been derived from QCD yet. In order to test this idea such models have been analyzed from various aspects.[38,75] Using an argument of Goldstone and Wilczeck[52], Balachandran et al[36] demonstrated that if one couples the non-linear sigma model to fermions, the solitons carry the quantum numbers of the fermions. In particular if the model is coupled to quarks, the solitons carry a baryon number equalling the topological charge. Witten[37,54] actually proved that the topological quantum number in the Skyrme model is the baryon number. It has also been shown[45] some time ago that these topological objects i.e. these solitons could, in principle, be fermions. And more recently it has been proved[54,55] that the Skyrmion can be quantized as a fermion (boson) when the number of colours of the underlying theory is odd (even).

The phenomenological implications of the Skyrme model have been investigated thoroughly. In particular, Adkins, Nappi and Witten[38] computed a variety of static properties of the nucleon by approximating the field theory by the quantum mechanics of certain collective degrees of freedom. The parameter values $F_\pi = 64.5$ MeV and $e^2 = 0.00424$ (e is the Skyrme constant) were found to reproduce the nucleon and Δ masses and lead to a skyrmion mass of $m_{B=1} = 0.866$ GeV. The $N\Delta$ mass splitting, the nucleon (N) magnetic moments, the nucleon charge radii and the coupling constant ratio for πNN to $\pi N\Delta$ calculated in the Skyrme model are in reasonable numerical agreement with experiment within 20-30% range. Nappi et al[75,76] have continued this study by introducing the ω-fields in the Lagrangian as well. The origin of the 30% error of the Skyrme and π-ω models reflects the fact that these models are rough approximations to meson physics and may be the use of semi-classical approximations too. Other authors[43] introduced the quarks into the Skyrme model. They consider the quark bag as a defect in the Skyrmion field configuration. The U-field live outside the bag while the quarks are confined to the bag. On the surface, the quark field is subject to some boundary conditions which respect chiral symmetry. It has been shown that the defect does not spoil the interpretation of the topological charge as the baryonic charge, and that the baryonic charges from the inside and the outside of the bag always add up to the topological charge. In this picture, as the radius of the bag goes to zero, the pure Skyrme model is recovered, while when it goes to infinity we approach a pure bag description. This model seems to yield reasonable predictions.

V.2 The Model

The low energy behaviour of pions is described by the non-linear model of chiral symmetry $SU(N_f)_L \otimes SU(N_f)_R$ spontaneously broken to diagonal $SU(N_f)$ where N_f is the number of flavours. To describe the dynamics of the theory, it is convenient to introduce a field $U(x^\alpha)$ that transforms in a so-called non-linear realization of $G = SU(N_f)_L \otimes SU(N)_R$. For each space-time point x^α, $U(x^\alpha)$ is an element of $SU(N_f)$: an $N_f \times N_f$ unitary matrix of determinant one. The chiral group G acts on $U(x\ )$ as follows

$$U \to LUR^{-1}$$

The effective Lagrangian for U is chirally invariant under this transformation and has the smallest possible number of derivatives to describe correctly the low-energy limit; it has the form

$$\mathcal{L}_2 = \frac{1}{16} F_\pi^2 \int d^4x \; Tr(\partial_\mu U \partial_\mu U^+) \qquad (1)$$

where F_π is the pion decay constant, $F_\pi \simeq$ 190 MeV experimentally.

The effective Lagrangian (1) incorporates all relevant symmetries of QCD but it also has an extra symmetry which is not a symmetry of QCD.[37] Indeed Lagrangian (1) is invariant under $U \longleftrightarrow U^T$ this is charge conjugation, it is invariant under $x \longleftrightarrow -x$, $t \longleftrightarrow t$, $U \longleftrightarrow U$, this is "parity" operation P_0 and finally it is invariant under $U \longleftrightarrow U^{-1}$. This last operation counts modulo two the number of bosons; Witten calls it $(-1)^{N_B}$. QCD is parity invariant if the Golstone bosons are pseudoscalars. The parity operation would then be $x \longleftrightarrow -x$, $t \longleftrightarrow t$, $U \longleftrightarrow U^{-1}$ and this is $P_0(-1)^{N_B}$: QCD is invariant under P but not under P_0 or $(-1)^{N_B}$ separately. Witten[37] observed that if the Wess-Zumino action[59] is added to (1), this extra symmetry is violated and the resulting theory correctly describes the low energy limit of strong interactions. In the two flavour case however, the Wess-Zumino term vanishes, so for simplicity we will restrict ourselves to the two flavour ($N_f = 2$) case.

Since the proper large-N effective theory of mesons that is equivalent to QCD is unknown, a rough description in which only pions are present is considered. The simplest such model is the Skyrme model with Lagrangian,

$$\mathcal{L} = \frac{1}{16} F_\pi^2 \, T(\partial_\mu U \partial_\mu U^+) + \frac{1}{32e^2} Tr\left[(\partial_\mu U)U^+, (\partial_\mu U)U^+\right]^2. \qquad (2)$$

where U is now an SU(2) matrix related to the pion field by

$$U = \sigma(x) + i\,\vec{\tau}\,.\,\vec{\pi}(x)$$

with the constraint of the non-linear σ-model $\sigma^2 + \vec{\pi}^2 = 1$. σ and $\vec{\pi}$ are the scalar meson field and the isovector triplet pion field up to a scale factor. The quartic term, $\mathcal{L}_4 = \frac{1}{32e^2} T\,[\partial_\mu UU^+ , \partial_\nu UU^+]^2$ where e is the Skyrme constant, has been introduced by Skyrme in order to stabilise the solitons and prevent them from shrinking to zero size. It is the unique term with four derivatives which leads to a positive Hamiltonian second order in time derivatives. The pion is assumed to be massless.

The energy integral for static fields is

$$E = -\int d^3x\, \mathcal{L}(x)$$

$$= -\int d^3x \left[\frac{1}{16} F_\pi^2 \,\mathrm{Tr}\, \partial_i U \partial_i U^+ + \frac{1}{32e^2} \mathrm{Tr}\,[\partial_i UU^+ , \partial_i UU^+]^2\right] \qquad (3)$$

This energy reaches a minimum and vanishes when

$$\partial_i U = o$$

which amounts to U = constant. The condition for finite E is that the fields approach some E = o configuration at spatial infinity sufficiently fast. That is, that

$$U \to \text{constant} \quad \text{as} \quad |\vec{x}| \to \infty \quad ,$$

without loss of generality we can take this constant to be the unit matrix.

Since U approaches the same value I at infinity the physical coordinate space R_3 is compactified into a spherical surface $S_3^{(phys)}$. Also, since the matrices U form the two-dimensional representation of SU(2), the function U(x) represents a mapping of $S_3^{(phys)}$ into the group space of SU(2). To classify such mappings, let us look at the topology of this group space. By definition of SU(2), the matrices U are the set of all 2 x 2 unimodular matrices. Such matrices can be written uniquely in the form :

$$U = \Sigma^{4}_{\mu=1} a_\mu S_\mu \tag{4}$$

where $S_4 = I$, the unit 2 x 2 matrix ,

$$S_{1,2,3} = i\,\sigma_{1,2,3} ,$$

and a_μ are any four real numbers satisfying

$$\Sigma_\mu a_\mu a_\mu = 1. \tag{5}$$

The group is thus parametrised by these four real variables a_μ subject to the constraint (5). The group space is therefore the three-dimensional surface of a unit sphere in four dimensions, $S_3^{(int)}$. The function U is therefore a mapping of $S_3^{(phys)}$ into $S_3^{(int)}$. The corresponding homolopy group $\Pi(SU(2)) \simeq \Pi_3(S_3)$ is isomorphic to the additive group of integers Z.

Therefore the mappings of $S_3 \to S_3$ can be divided into a discrete infinity of homotopy classes, each characterised by an integer B, known as the Pontryagin index or winding number. This means that our finite-energy configurations can be classified into an infinite number of solitonic sectors characterised by B. The explicit expression for B is

$$B = \frac{1}{24\pi^2} \varepsilon_{ijk} \int d^3x \; Tr(\partial_i UU^+ \partial_j UU^+ \partial_k UU^+) \tag{6}$$

The "charge" B arises from the current

$$J_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} Tr(\partial_\nu UU^+ \partial_\lambda UU^+ \partial_\rho UU^+) \tag{7}$$

which is conserved purely for algebraic reasons (without use of the equations of motions) :

$$\partial_\mu J_\mu = 0$$

This index B can be interpreted as baryon number, in which case the lowest energy state in the $B = 1$ sector should be identified with the nucleon. The lowest classical energy in that sector is attained by the static Skyrme ansatz,

$$U_o(x) = \exp\,(i F(r)\, \underline{\sigma}.\, \hat{\underline{x}})$$

$$= I \cos F(r) + i\, \underline{\sigma}.\, \hat{\underline{x}} \sin F(r) \qquad (8)$$

where $\underline{\sigma}$ are the Pauli matrices, $\hat{\underline{x}} = \frac{\vec{r}}{|r|}$ and $F(r)$ is a function subject to the boundary conditions

$$F(r) = \pi \text{ at } r = o$$

and $\quad F(r) \to o \text{ as } r \to \infty$.

Note that the energy integral can be minimised with respect to functional variation of σ and $\vec{\pi}$. But this leads to intractable Euler-Lagrange equation. To avoid this difficulty Skyrme introduced the hedgehog approximation (8)

If U_o is a soliton solution, then $A(t)U_oA^{-1}(t)$ is also a finite-energy solution; it is isoratating in time but otherwise static. A solution of any given A is not an eigenstate of spin and isospin. However, if we substitute $U = AU_oA^{-1}$ in the Lagrangian[38] we get

$$\mathcal{L} = -E + \lambda T\,[\partial_o A \partial_o A^{-1}] \qquad (9)$$

where E is defined in (3) and $\lambda = \frac{4}{6}\pi\left(\frac{1}{e^3 F_\pi}\right)\Lambda$ with

$$\Lambda = \int \tau^2 \sin^2 F \left[1 + 4\, F'^2 + \frac{\sin^2 F}{\tau^2}\right] d\tau \qquad (10)$$

$A(t)$ is an arbitrary time-dependent SU(2) matrix; it can be written

$$A = a_o + i\underline{a}.\,\underline{\sigma} \quad , \text{ with } a_o{}^2 + \underline{a}^2 = 1 \quad .$$

In terms of the a's , (9) becomes

$$\mathcal{L} = -E + 2\lambda\Sigma_{i=0}^{3} (\partial_t a_i)^2 . \tag{11}$$

Now, we can write the Hamiltonian

$$H = \frac{\partial L}{\partial(\partial_t a_i)} \partial_t a_i - L = 4\lambda \Sigma_i \partial_t a_i \partial_t a_i - L$$

$$= E + 2\lambda \Sigma_i \partial_t a_i \partial_t a_i \tag{12}$$

which can be diagonalised :

$$H = E + \frac{\ell(\ell+1)\hbar^2}{2\Lambda} \tag{13}$$

The corresponding spin and isospin eigenstates are identified with the nucleon and delta. Indeed the Noether charges $\vec{I}$ and $\vec{J}$ corresponding to the isospin and rotational symmetries are

$$\vec{I}^2 = \vec{J}^2 = \ell(\ell+1)\hbar^2 . \tag{14}$$

If the solitons are fermions, then the possible values of ℓ are half-integers [45,49] :

$$\ell = 1/2, \; 3/2, \text{----}$$

The nucleon and delta are identified with $\ell = 1/2$ and $\ell = 3/2$ states respectively. Higher states $\ell = \frac{5}{2}$, ---- are artifacts of the model as they do not have any counterparts in nature.

A topological origin of baryon number is indeed appealing since in contrast to electric charges, the measurement of baryon charges is done through single counting. Thus their law of combination is precisely the same as that of the additivity of the homotopy classes in the homotopy group $\Pi_3(S^3) \simeq Z$. Indeed $B(U)$ is time independent and additive

$$B(U_1U_2) = B(U_1) + B(U_2) \tag{15}$$

This results from the fact that the field $U(x)$ is for any fixed argument x a group element of $SU(2)$. Therefore any solution $U(x)$ to the field equations can be split as

$$U(x) = U_1(x)U_2(x) \tag{16}$$

in accordance with the group composition law. Then the associated topological charge is additive in terms of the contributions due to $U_1(x)$ and $U_2(x)$.

In terms of the topological current we have,

$$\mathcal{J}^{\alpha} = \mathcal{J}^{\alpha}_{(1)} + \mathcal{J}^{\alpha}_{(2)} + \frac{1}{24\pi^2} \partial_\beta \Omega^{\alpha\beta} \tag{17}$$

where $\mathcal{J}^{\alpha}_{(1)}$ and $\mathcal{J}^{\alpha}_{(2)}$ are the densities due to $U_1(x)$ and $U_2(x)$ respectively and $\partial_\beta \Omega^{\alpha\beta}$ a total divergence which vanishes on integration.

Although exact multi-soliton solutions are not usually available except some few exceptions like the Sine-Gordon solitons,[64] it is not so for one-soliton solutions. For the Skyrme model however, no exact analytic, topologically non-trivial solutions have thus far been found.

However, to understand the theory, we may exploit the topological structure of the model in order to study the lowest lying part of the classical soliton spectrum. In other words we want to sandwich the energy as closely as possible by establishing on it lower and upper bounds.

In order to find a lower bound for the energy, we employ a trick due to Belavin and Polyakov.[66] We start with the identity

$$\int d^3x \; Tr \; (\varepsilon_2 U^+\partial_\mu U - \varepsilon_4 \varepsilon_{\mu\nu\rho}\partial_\nu U^+\partial_\rho U)(\varepsilon_2 U^+\partial_\mu U - \varepsilon_4 \varepsilon_{\mu ij}\partial_i U^+\partial_j U) \geqslant 0 \tag{18}$$

After expanding, this becomes

$$\int d^3x \ \mathrm{Tr}(-\varepsilon_2^2 \ \partial_\mu U^+ \partial_\mu U - \varepsilon_2 \varepsilon_4 \varepsilon_{\mu ij} U^+ \ \partial_\mu U \partial_i U^+ \partial_j U - \varepsilon_2 \varepsilon_4 \varepsilon_{\mu\nu\rho} \partial_\nu U^+ \partial_\rho U U^+ \partial_\mu U$$

$$+ \varepsilon_4^2 \ \varepsilon_{\mu\nu\rho} \ \varepsilon_{\mu ij} \ \partial_\nu U^+ \partial_\rho U \partial_i U^+ \partial_j U \ ) \ \geqslant \ 0$$

In a Euclidean metric ,

$$\varepsilon_{\mu ij} \ \varepsilon_{\mu\nu\rho} = \delta_{i\nu} \delta_{j\rho} - \delta_{i\rho} \ \delta_{j\nu} \tag{19}$$

Using this identity (19) and the cyclic property of the trace, we obtain

$$\int d^3x \ \mathrm{Tr}(-\varepsilon_2^2 \ \partial_\mu U^+ \partial_\mu U + \varepsilon_4^2 \ \partial_\nu U \partial_\nu U^+ \partial_\rho U \partial_\rho U^+ - \varepsilon_4^2 \ \partial_\nu U^+ \partial_\rho U \ \partial_\nu U^+ \ \partial_\rho U$$

$$- 2\varepsilon_2 \varepsilon_4 \varepsilon_{\mu\nu\rho} U^+ \partial_\mu U \ \partial_\nu U^+ \ \partial_\rho U \ ) \quad \geqslant \ 0$$

where the first three terms constitute the energy and the second is proportional to the winding number B. That is

$$E \geqslant 2\varepsilon_2 \varepsilon_4 \ \int d^3x \ \varepsilon_{\mu\nu\rho} \ \mathrm{Tr}\, U^+ \ \partial_\mu U \partial_\nu U^+ \partial_\rho U$$

or

$$E \geqslant 2 \ \varepsilon_2 \varepsilon_4 \ 24 \ \pi^2 \ |B|$$

where

$$\varepsilon_2^2 = \frac{F\pi^2}{16} \quad \text{and} \quad \varepsilon_4^2 = \frac{1}{16e^2}$$

Thus

$$E \ \geqslant \ 3\pi^2 \ \frac{F\pi}{e} \ \ |B| \tag{20}$$

To proceed further we need to make specific assumptions about the form of the solution. We consider the static, spherically symmetric Skyrme ansatz given by (8)

Substituting this ansatz into the lagrangian expression we get

$$E = 4\pi \int_0^\infty r^2 dr \left[2\varepsilon_2^2 \left(F'^2 + 2\frac{\sin^2 F}{r^2} \right) + 8\varepsilon_4^2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) \right] \tag{21}$$

where $F' = \partial F/\partial r$; $\varepsilon_2 = \frac{F\pi^2}{16}$ and $\varepsilon_4 = \frac{1}{16e^2}$

Now, we want to find an upper bound on the energy. Thus we consider the Coleman-Fadeev theorem [62,57] :

"Suppose we want to find stationary points of some functional S whose argument varies over a set X. Let H denote a group acting on X which consists of symmetries of S and let X_0 be the fixed points of X under the action of H. Then an extremum of S restricted to X_0 is also an extremum of S over all X."

Here X is the set of all twice differentiable functions, H is the time translations and rotation subgroups of the Poincaré group and X_0 is a subset of X that is rotationally invariant and static. For static solutions the hamiltonian is the negative of the lagrangian and therefore an extremum of one is the extremum of the other.

So we have to write down some function F(r) which satisfies the boundary conditions and depends on some free parameters. The energy is then evaluated and minimized with respect to these parameters.

The Skyrme[35] trial function is

$$\begin{aligned} F(r) &= \pi n \left(1 - \frac{\tau}{\tau_0}\right) , \quad \tau \geqslant \tau_0 \\ F(r) &= 0 \qquad\qquad\qquad , \quad \tau \leqslant \tau_0 \end{aligned} \tag{22}$$

where $r = \frac{2\,\varepsilon_4}{\varepsilon_2}\tau = \frac{2}{eF\pi}\tau$, and τ_o is the parameter to be varied. This function obeys the correct boundary conditions and can be closely approximated by a twice differentiable function.

After inserting (22) into the expression for the energy integral (21) we evaluate the resulting integrals all of which are trivial except

$$\int_0^{\tau_o} d\tau \, \frac{1}{\tau^2} \sin^4\left(\frac{n\pi\tau}{\tau_o}\right) = \frac{n\pi}{\tau_o} \int_0^{n\pi} \frac{dx}{x^2} \sin^4 x$$

where $x = \frac{n\pi\tau}{\tau_o}$; however

$$\frac{n\pi}{\tau_o} \int_0^{n\pi} \frac{dx}{x^2} \sin^4 x < \frac{n\pi}{\tau_o} \int_0^{\infty} \frac{dx}{x^2} \sin^4 x = \frac{n\pi^2}{4\tau_o}$$

So we can write

$$\int_0^{\tau_o} d\tau \, \frac{1}{\tau^2} \sin^4 \frac{n\pi\tau}{\tau_o} = \gamma \frac{n\pi^2}{4\tau_o} \quad , \gamma < 1$$

Hence an upper bound on the energy is

$$E \leqslant 16\pi \, \varepsilon_2 \varepsilon_4 \left\{ \tau_o \left(4 + \frac{(n\pi)^2}{3}\right) + \frac{\pi^2}{2\tau_o}\left(\frac{\gamma n}{2} + n^2\right)\right\} \tag{23}$$

which when minimized,

$$\frac{\partial E}{\partial \tau_o} \leqslant 16\pi \, \varepsilon_2 \varepsilon_4 \left\{ \left(4 + \frac{(n\pi)^2}{3}\right) - \frac{\pi^2}{2\tau_o^2}\left(\frac{\gamma n}{2} + n^2\right)\right\}$$

yields

$$\frac{\partial E}{\partial \tau_o} = 0 \rightarrow \tau_o = \frac{\pi}{2} \left(\frac{\gamma n + 2n^2}{4 + \frac{(n\pi)^2}{3}}\right)^{\frac{1}{2}}$$

Inserting this value of τ_o into (23) gives

$$E \leqslant 16\pi^2 \, \varepsilon_2 \varepsilon_4 \left[\left(4 + \frac{(n\pi)^2}{3}\right)\left(\gamma n + 2n^2\right)\right]^{\frac{1}{2}} \tag{24}$$

When n = o the upper bound is zero; when n = 1 that is for the first excited state, the upper bound is $(\pi^2 F\pi/_e \sqrt{21}) \simeq (4.6)\pi^2 \frac{F\pi}{e}$. The corresponding lower bound is $(3\pi^2 \frac{F\pi}{e})$ which indicates that the Skyrme trial function is quite close to the solution of the equation of motion. For larger values of n, the upper bound grows like n^2 whereas the lower bound is linear in n and this means that we cannot sandwich the energy closely any more. This is not at all surprising, rather it is an indication that the solitons $(n > 1)$ interact.

Another trial function is that of Gipson[64]

$$\cos F(r) = \frac{1 - \left(\frac{r}{r_0}\right)^2}{1 + \left(\frac{r}{r_0}\right)^2} \; ; \; \sin F(r) = \frac{2\frac{r}{r_0}}{1 + \left(\frac{r}{r_0}\right)^2} \tag{25}$$

Using this ansatz and minimizing the energy with respect to r_0 yields the inequality

$$E_{B=1} \leqslant \pi^2 \frac{F\pi}{e} \frac{6}{\sqrt{2}} \tag{26}$$

IV.3 Equations of Motion

Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$$

where

$$\mathcal{L}_2 = \varepsilon_2^2 \; \mathrm{Tr}\, \partial_\mu U^+ \partial_\mu U \quad , \quad \varepsilon_2^2 = \frac{F\pi^2}{16}$$

$$\mathcal{L}_4 = \frac{1}{2} \varepsilon_4^2 \; \mathrm{Tr} \left[\partial_\mu U U^+ , \partial_\nu U U^+ \right]^2 , \quad \varepsilon_4^2 = \frac{1}{16e}$$

$$\mathcal{L}_4 = \varepsilon_4^2 \; \mathrm{Tr} \left[\partial_\mu U \partial_\nu U^+ \partial_\mu U \partial_\nu U^+ - \partial_\mu U \partial_\mu U^+ \partial_\nu U \partial_\nu U^+ \right]$$

Define

$$A_\mu = \frac{1}{2}\ \partial_\mu U U^+ \quad , \quad \text{i.e. half a pure gauge,}$$

hence

$$F_{\mu\nu} = -[A_\mu , A_\nu] \quad .$$

The fourth order term in the lagrangian, $\mathcal{L}_4$ can therefore be rewritten as

$$\mathcal{L}_4 = -\frac{1}{2}\ \varepsilon_4^2 \ \mathrm{Tr}\ F_{\mu\nu}^2$$

Note the resemblance of $\mathcal{L}_4$ to the conventional Yang-Mills Lagrangian density.

The equations of motion are obtained by varying $\mathcal{L}$ with respect to U,

$$\delta\mathcal{L}(U) = 0$$

we find that

$$\begin{aligned}
&\varepsilon_2^2\ (\partial_\mu^2\ U^+ + U^+\ \partial_\mu U \partial_\mu\ U^+) \\
&+\varepsilon_4^2\ \{ -U^+(\partial_\mu U \partial_\mu U^+)(\partial_\nu U \partial_\nu U^+) - \partial_\mu U^+(\partial_\nu U \partial_\nu U^+)\partial_\mu U U^+ - \partial_\mu^2 U^+(\partial_\nu U \partial_\nu U^+) \\
&\quad -2\ \partial_\mu U^+\ \partial_{\mu\nu}^2 U \partial_\nu U^+ + \partial_\mu U^+ \partial_\nu U \partial_{\mu\nu}^2 U^+ + \partial_{\mu\nu}^2 U^+ \partial_\nu U \partial_\mu U^+ \\
&\quad - \partial_\nu U \partial_\nu U^+ \partial_\mu^2 U^+ + 2\ U^+ \partial_\mu U \partial_\nu U^+ \partial_\mu U \partial_\nu U^+ + 2\partial_\nu U^+ \partial_\mu^2 U \partial_\nu U^+ \ \} = 0
\end{aligned} \tag{27}$$

In terms of $A_\mu = \frac{1}{2}\ \partial_\mu U U^+$, these equations become

$$\varepsilon_2^2\ \partial_\mu A_\mu + \varepsilon_4^2\ (\partial_\mu + 2A_\mu)\ [\ A_\nu,\ F_{\mu\nu}\ ] = 0 \tag{28}$$

The equations of motion obtained by varying

$$\mathcal{L} = \varepsilon_2^2\ \mathrm{Tr}\ A_\mu A_\mu^+ \ + \ \frac{1}{2}\ \varepsilon_4^2\ \mathrm{Tr}\ F_{\mu\nu} F_{\mu\nu}$$

with respect to A_μ are

$$\varepsilon_2^2 \; A_\nu \; + \; \varepsilon_4^2\{\partial_\mu F_{\mu\nu} + [\, A_\mu, F_{\mu\nu}] \,\} \;=\; 0 \tag{29}$$

Contracting (29) with respect to ∂_ν gives

$$\varepsilon_2^2 \; \partial_\nu A_\nu + \varepsilon_4^2 \partial_\nu(\partial_\mu F_{\mu\nu}) \;=\; \varepsilon_2^2 \; \partial_\nu A_\nu \;=\; 0$$

where $\mu,\nu \;=\; 1,2,3,4.$

which are the equations of motion derived from $\mathcal{L}_2$ alone.

Let us now look at the analytic properties of the static soliton solutions. We shall start by examining the hedgehog or Skyrme ansatz. In terms of this ansatz, we saw that the expression for the soliton mass is given by (21).

The equations of motion are the Euler-Lagrange equations for the energy,

$$\partial \left(\frac{\partial E}{\partial F'}\right) - \; \frac{\partial E}{\partial F} \;=\; 0$$

which yield

$$(\tfrac{1}{4}\,\tau + 2\,\sin^2 F)F'' + \tfrac{1}{2}\tau F' + \sin 2F\, F'^2 - \tfrac{1}{4}\sin 2F - \frac{\sin^2 F\,\sin^2 2F}{\tau^2} = 0 \tag{30}$$

where τ is a dimensionless variable, $\tau = eF_\pi r$

Changing variables to $t = \ln \tau/\tau_0$, the equations of motion become

$$F''(1 + A_4 e^{-2t}\,\sin^2 F) \;+\; F'(1 - A_4 e^{-2t}\,\sin^2 F)$$

$$- \sin 2F\,(1 - A_4\,[\,F'^2 - \sin^2 F\,]\,) \;=\; 0 \tag{31}$$

where

$$A_4 \;=\; \frac{\varepsilon_4^2}{\varepsilon_2^2\; r_0^2} \;=\; \frac{1}{e^2 F_\pi^2\, r_0^2}$$

At large t (or large r), the first term in the Lagrangian dominates,[65] and equation (29) reduces approximately to

$$F'' + F' - \sin 2F = o$$

which can be linearized to

$$F'' + F' - 2F = o$$

since $F(t) \to o$ when $t \to \infty$.

Choosing the solution which obeys the boundary conditions we find

$$\lim_{\substack{t \to \infty \\ r \to \infty}} F = a\, e^{-2t} = a\, \frac{r_0^2}{\tau^2} \tag{32}$$

Near the origin the second term in the Lagrangian dominates [65] and we have the approximate equation

$$A_4\, e^{-2t} \sin^2 F\, (F'' - F') = o$$

The corresponding regular solution is then,

$$\lim_{\substack{t \to -\infty \\ r \to o}} F = n\pi - \alpha r_0 e^{t} \tag{33}$$

The minimization with respect to r_0 yields

$$E \simeq F\pi/e$$

and equal contributions from $\mathcal{L}_2$ and $\mathcal{L}_4$.

As we have already stated no exact analytic topologically non-trivial solutions have yet been found. These equations, like most coupled non-linear equations in three space-dimensions, are not exactly solvable. We will therefore approach the problem differently.

A lower bound for the energy of any static configuration in a B-sector is given by the inequality (20)

$$E \geqslant 3\pi^2 \frac{F\pi}{e} \mid B \mid$$

The equality is satisfied when

$$\varepsilon_2 \, U^+ \partial_\mu U = \varepsilon_4 \varepsilon_{\mu\nu\rho} \, \partial_\nu U^+ \partial_\rho U \tag{34}$$

where μ,ν,ρ are here space-indices only i.e. $\mu, \nu,\rho = 1,2,3$.

In three space dimensions and for static solutions, the field equation (26) reduces to

$$\varepsilon_2^2 \, \partial_\mu A_\mu + \varepsilon_4^2 \, (\partial_\mu + 2A_\mu) \, [A_\nu, F_{\nu\mu}] = o \tag{35}$$

with $\mu,\nu = 1,2,3$.

Any field configuration that satisfies (34) will minimise the energy E in some B-sector, and will therefore automatically satisfy the extremum condition given by the field equation (35). However, the converse need not be true. One could in principle have solutions of (35) which do not satisfy (34). These would not represent absolute minima of the energy in the corresponding B-sector but some higher valued extrema, such as local minima. The first order differential equations (34) should be easier to solve than the parent field equation (35) which are second order in derivatives.

However writing the equation (34) as

$$\varepsilon_2 \, A_\mu = \tfrac{1}{2}\varepsilon_4 \varepsilon_{\mu\nu\rho} \, [A_\nu, A_\rho]$$

we can see in fact that the only solutions of these equations are gauge equivalent to a constant since we know that the representations of a Lie algebra which are unitary, are finite dimensional constant matrices with no adjustable parameters up to similarity transformations.

This means that the solutions of (34) do not satisfy the boundary conditions for finite energy configurations i.e. $U \to I$ or $A_\mu \to o$ as $|\vec{x}| \to \infty$. Therefore they lie beyond the scope of our interest as they do not belong to any given B-sector. As a result, the lowest bound $E = 3\pi^2 \frac{F\pi}{e} |B|$ cannot be reached. Hence we have to find solutions of the field equation (35) which are not solutions of (34).

Indeed the lowest known classical energy in the B = 1 sector is attained by the static Skyrme approximation $U = e^{iF(r)\underline{\sigma} \cdot \hat{\underline{x}}}$ where F(r) obeys the appropriate boundary conditions. And it does not satisfy the equality $E = 3\pi^2 \frac{F\pi}{e}$ as U is not a solution of (34). This can be verified explicitly

Any SU(2) matrix can be written as

$$U = u_0 I + i \underline{u} \cdot \underline{\sigma} \quad \text{with } u_0^2 + \underline{u}^2 = 1$$

then the Bogomolny´i-like equations (34) become

$$\varepsilon_2 \, i \underline{\sigma} \cdot (\partial_\mu \underline{u} \, u_0 - \underline{u} \partial_\mu u_0 + \underline{u} \wedge \partial_\mu \underline{u})$$

$$= \varepsilon_4 \, i \underline{\sigma} \cdot (2\partial_\nu u_0 \, \partial_\rho \underline{u} + \partial_\nu \underline{u} \wedge \partial_\rho \underline{u})$$

Therefore

$$\partial_\mu \underline{u} \, u_0 - \underline{u} \, \partial_\mu u_0 + \underline{u} \wedge \partial_\mu \underline{u} = \varepsilon_4 \varepsilon_2^{\,1} \, \varepsilon_{\mu\nu\rho} \, (2\partial_\nu \, u_0 \partial_\rho \underline{u} + \partial_\nu \underline{u} \wedge \partial_\rho \underline{u})$$

Let

$$U = e^{iF(r) \, \underline{\sigma} \cdot \hat{\underline{x}}} = \cos F(r) + i \, \underline{\sigma} \cdot \hat{\underline{x}} \sin F(r)$$

where $F(o) = \pi$ and $F(\infty) = o$ and $\hat{\underline{x}} = \frac{\vec{r}}{|\vec{r}|}$; then,

$$\partial_\nu u_o = -\partial_\nu F \sin F$$

and

$$\partial_\rho \underline{u} = \partial_\rho \hat{\underline{x}} \sin F + \partial_\rho F \hat{x} \cos F$$

Thus equation (34) becomes

$$\partial_\mu F \hat{\underline{x}} + \cos F \sin F \, \partial_\mu \hat{\underline{x}} + \hat{\underline{x}} \wedge \partial_\mu \hat{\underline{x}} \sin^2 F$$

$$= \varepsilon_4 \varepsilon_2^{-1} (\sin^2 F \, \partial_\nu \hat{\underline{x}} \wedge \partial_\rho \hat{\underline{x}} + 2\partial_\rho F \cos F \sin F \, \partial_\nu \hat{\underline{x}} \wedge \hat{\underline{x}} + 2\partial_\rho F \sin^2 F \partial_\nu \hat{\underline{x}})$$

However,

$$\varepsilon_{\mu\nu\rho} \, \partial_\nu \hat{\underline{x}} \wedge \partial_\rho \hat{\underline{x}} = 2r^{-3} x_\mu \hat{\underline{x}}$$

$$\varepsilon_{\mu\nu\rho} \, r^{-1} x_\rho \partial_\nu \hat{\underline{x}} = \hat{\underline{x}} \wedge \partial_\mu \underline{x}$$

$$\varepsilon_{\mu\nu\rho} \, r^{-1} x_\rho \partial_\nu \hat{\underline{x}} \wedge \hat{\underline{x}} = \partial_\mu \hat{\underline{x}}$$

On the other hand,

$$\partial_\rho F = \frac{\partial F}{\partial r} \frac{x_\rho}{r} = F' x_\rho r^{-1}$$

Hence,

$$F' \frac{x_\mu}{r} \hat{\underline{x}} + \cos F \sin F \, \partial_\mu \hat{\underline{x}} + \hat{\underline{x}} \wedge \partial_\mu \hat{\underline{x}} \sin^2 F$$

$$= \varepsilon_4 \varepsilon_2^{-1} \left[2 \sin^2 F \frac{x_\mu}{r} \hat{\underline{x}} + 2F' \cos F \sin F \, \partial_\mu \hat{\underline{x}} + 2F' \sin^2 F \, \hat{\underline{x}} \wedge \partial_\mu \hat{\underline{x}} \right]$$

Upon identifying the two sides of this equality term by term we get

$$F' = 2 \varepsilon_4 \varepsilon_2^{-1} \frac{\sin^2 F}{r_2}$$

and

$$2 \varepsilon_4 \varepsilon_2^{-1} F' = 1$$

Taking into account the boundary condition $F(o) = \pi$, the second equality, $2\, \varepsilon_4 \varepsilon_2^{-1} F' = 1$ yields

$$F = \frac{r}{2\varepsilon_4 \varepsilon_2^{-1}} + \pi$$

which is similar to the Skyrme trial function except that it does not have the right limit when r goes to infinity.
We therefore conclude that the Skyrme ansatz is not a solution to the equations (34) as expected.

From another aspect, we can see that for any configuration satisfying (34) we have

$$\varepsilon_2\, \partial_\mu\, (U^+ \partial_\mu U) \;=\; \varepsilon_4 \partial_\mu (\varepsilon_{\mu\nu\rho} \partial_\nu U^+ \partial_\rho U) = o$$

$\mu, \nu, \rho \;= 1,\ 2,\ 3$

which is just the field equation for $\mathcal{L}_2$ alone in three space dimensions and for static solutions. Therefore the minima of the energy in the Skyrme model cannot be solutions of equation (34). This is because the quartic term $\mathcal{L}_4$ has been added to the Lagrangian in order to stabilise the Skyrmion. Hence, any stable solution should be a solution of the full field equation derived from $\mathcal{L}_2 + \mathcal{L}_4$ rather than $\mathcal{L}_2$ alone.

This suggests that if we could find a stabilising term leading to Bogomol'nyi-like conditions which imply the full equations of motion then we would probably find some absolute minima. The obvious term to try is the next order term

$$\mathcal{L}_6 \;=\; \varepsilon_6^2 \;\; \mathrm{Tr}\, B_\mu B_\mu^+$$

where $B_\mu \;= \varepsilon_{\mu ijk}\; \partial_i U \partial_j U^+ \partial_k U.$

It is the unique term with six derivatives that leads to a positive Hamiltonian second order in time derivatives.[65] Also in the absence of $\mathcal{L}_4$, which we will omit as it is phenomenologically small, it stabilises the Skyrmion.

Consider the Lagrangian,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_6 = \varepsilon_2^2 \text{ Tr } \partial_\mu U \partial_\mu U^+ + \varepsilon_6^2 \text{ Tr } B_\mu B_\mu^+ \tag{36}$$

The equations of motion obtained by varying $\mathcal{L}$ with respect to U, are

$$\varepsilon_2^2 \left(\partial_\mu (U^+ \partial_\mu U) + \varepsilon_6^2 \varepsilon_{\mu ijk} \{\partial_k U^+ U (\partial_j B_\mu + U^+ \partial_j U B_\mu) U \partial_i U^+ + \partial_j U^+ \partial_k U (\partial_i B_\mu + \partial_i U^+ \right.$$

$$\left. U B_\mu) + (\partial_k B_\mu + \partial_k U^+ U B_\mu) \partial_i U \partial_j U^+ \} = 0 \tag{37}$$

We proceed as before and we start with the identity

$$\text{Tr } (\varepsilon_2 \partial_\mu U - \varepsilon_6 \varepsilon_{\mu ijk} \partial_i U \partial_j U^+ \partial_k U) (\text{h.c}) \geqslant 0 \tag{38}$$

which leads to the inequality

$$S \geqslant 2 \varepsilon_2 \varepsilon_6 \int \text{Tr } \varepsilon_{\mu ijk} \partial_\mu U^+ \partial_i U \partial_j U^+ \partial_k U \, d^4x$$

i.e. $$S \geqslant 2 \varepsilon_2 \varepsilon_6 \int \text{Tr } \partial_\mu (\varepsilon_{\mu ijk} U^+ \partial_i U \partial_j U^+ \partial_k U) \, d^4x$$

which reduces to

$$S \geqslant 2 \varepsilon_2 \varepsilon_6 \int \text{Tr } (\varepsilon_{\mu ijk} U^+ \partial_i U \partial_j U^+ \partial_k U) \, d^3x$$

by virtue of the Gauss theorem. Thus

$$S \geqslant 2 \varepsilon_2 \varepsilon_6 \; 24\pi^2 \, |B| \tag{39}$$

Fields which extremise the action in any given homotopy sector are solutions of (37) falling in that sector. From equations (38) and

(39) we see that the absolute minimum value of S is attained in any given sector B when

$$\varepsilon_2\, \partial_\mu U = \varepsilon_6\, \varepsilon_{\mu ijk}\, \partial_i U \partial_j U^+ \partial_k U \qquad \mu,i,j,k = 1,2,3,4 \tag{40}$$

Of course, the absolute minima of S need not be its only extrema. But it is probably easier to solve (40) rather than the more general equation (37).

One obvious solution of (40) is $U = b\, x_0 1 + ib\underline{x}.\underline{\sigma}$ (41)

where b depends on ε_2 and ε_6. But it is not clear how to obtain further solutions if there are any. The equation (40) is only invariant under linear transformations. Also a perturbation around the solution (41) does not yield any positive results. We therefore suspect that this is the unique solution to equation (40).

However solution (41) does not obey the boundary conditions that any finite-action configuration must satisfy. Thus the situation is exactly as before; absolute minima cannot be attained. Note that this time also the equation (40) does not imply the field equation (37), although they do not imply the equation of motion for $\mathcal{L}_2$ alone either :

$$\varepsilon_2\, \partial_\mu (\partial_\mu U) = \varepsilon_6\, \varepsilon_{\mu ijk} \partial_\mu (\partial_i U \partial_j U^+ \partial_k U) = 0$$

This is probably due to the fact that the algebra (40) is larger than the Lie algebra (34).

Now, it is well known that a necessary (but not sufficient) condition for a system to be integrable is that it obeys Bogomol'nyi conditions. We therefore conclude that the Skyrmions - as they do not obey Bogomol'nyi conditions - do not form an integrable system.

It is interesting to notice that $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6$ are natural candidates of a hierarchy of σ-models in two, four and six dimensions. Consider the two-dimensional non-linear σ-model

$$\mathcal{L}_2 \sim \mathrm{Tr}\, \partial_\mu U^+ \partial_\mu U$$

As there is a correspondence between the O(3) σ-model and the CP^1 model in two dimensions (they are actually equivalent provided one makes appropriate identifications), there is also a correspondence between $\mathcal{L}_2$ and the O(3) σ-model.[67] In four dimensions the O(5) σ-model corresponds to HP^1, the space of quaternions, in the same fashion that O(3) corresponds to CP^1. It was also realized that HP^{N-1} σ-models[68] provide a convenient setting for generating instanton solutions to four-dimensional Yang-Mills theories. In which case Yang-Mills fields are considered as composites of the more elementary HP^{N-1} fields.

The O(3) model in 2-dimensions has several interesting properties, similar to the Yang-Mills theory in 4-dimensions. Both systems yield instantons of arbitrary size characterised by integer-valued topological indices; in both cases there are inequalities giving a lower bound to the action by a value proportional to the topological index. This leads to first order self-duality equations which are then solved. A straightforward extension of $\mathcal{L}_2$ from two dimensions to four dimensions does not yield instanton solutions.[71] However there is another way of generalising $\mathcal{L}_2$ to any 2k dimensions (k positive integer).

Introduce the Lagrangian

$$\mathcal{L}_{2k} \sim \mathrm{Tr}\, (\varepsilon_{\mu\alpha_1} \;---\; {}_{\alpha k}\, \partial_{\alpha_2} U^+ \;---\; \partial_{\alpha n} U)(h.c) \qquad (42)$$

It represents a class of models in 2k dimensions which for $k = 1$ reduces to CP^N models ($\mathcal{L}_2$) and for $k = 2$ to HP^N models ($\mathcal{L}_4$). In fact this approach leads to CP^N like models when the dimensions of the space-time is $d = 4k + 2$[70] ($k = o, 1, ---$) and to Yang-Mills like models when $d = 4k$.[69] For all values of k it is possible to define a topological charge and derive self-duality relations. Moreover, a simple generalisation of a Bogomolnyi bound shows that solutions of the self-duality equations are local minima of the action. It is then easy to proceed to look for solutions to these self-duality equations.

The lowest examples, $d = 2,4$ respectively, reproduce the CP^1 model and SU(2) Yang-Mills gauge fields. In the six-dimensional space-time i.e. $d = 6$, the same method as in $d = 4$ can be followed. It is possible to construct quaternion-like objects (hexanions!)

$$P = P_i \gamma_i$$

where P_i $(i = 0, 1, 2, ---5)$ are real numbers and γ_i are a set of six matrices satisfying

$$\begin{aligned}
\gamma_0 &= 1 \\
\gamma_{1,2,3} &= i\sigma_2 \otimes \sigma_{1,2,3} \\
\gamma_4 &= -i\,\sigma_1 \otimes 1 \\
\gamma_5 &= -i\,\sigma^3 \otimes 1
\end{aligned}$$

that is $\gamma_i^2 = -1$. The magnitude of P is defined by

$$|P|^2 = \frac{1}{4}\,\mathrm{tr}(pp^+) = P_i P_i \quad .$$

A useful property of the basis γ_i, $i = 0,1$ --- **5** is the following self-duality property.

$$\gamma_i \gamma_j \gamma_k = i\varepsilon_{ijk\ell mn} \gamma_\ell \gamma_m \gamma_n \quad , \quad \gamma_i^+ = -\gamma_i$$

where $\varepsilon_{ijk\ell mn}$ is the totally antisymmetric tensor ($\varepsilon_{012345} = 1$)
Note that in six dimensions self-duality reads

$$\partial_\mu U^+ \partial_\nu U \partial_\rho U^+ = i \varepsilon_{\mu\nu\rho ijk} \partial_i U^+ \partial_j U \partial_k U^+ \tag{43}$$

it is therefore clear that the procedure for solving this equation is parallel to the procedure followed in the four dimensional HP^1 model. For higher values of d see references 70 and 69.

CHAPTER VI

SKYRMION-SKYRMION INTERACTIONS

After investigating the various static properties of skyrmions, it is natural to ask whether the Skyrme model gives a reasonable description of nuclei. We will therefore discuss the application of this model to Skyrmion-Skyrmion interaction and particularly the low energy nucleon-nucleon interaction. One way to apply this model to nuclear physics is the "potential approach". It consists of extracting from the model a Skyrmion-Skyrmion interaction potential. After a suitable identification a nucleon-nucleon interaction potential is obtained.[43, 44]

Single soliton solutions resemble extended particles (e.g. solitons of the Sine-Gordon equation).[61] In fact solitons can be associated with quantum extended particle states : they are quantised. As we have already stated, the skyrmion can be quantised as a fermion when the number of colours in the underlying theory (QCD) is odd.[54,55] There are also some similar resemblance between multi-soliton solutions and a set of particles. In particular the dynamics of systems of two or more solitons can be approximately described in terms of an interaction potential which depends on their relative separation. This is a concept borrowed from particle mechanics.

Exact multi-soliton solutions are not available for the Skyrme model therefore in order to understand more about multi-soliton systems it is necessary to use approximations. In general, since field equations are all non-linear, a superposition of single-soliton functions are not solutions. But if the solitons are far apart their overlap is small and the distortion in each soliton caused by non-linear effects due to the presence of the others is also small. That is, we expect the existence of solutions corresponding to widely separated solitons which retain their individual shape except for small distortions. However, these solutions may not be static because each soliton tend to exert a force on the other and this may accelerate the solitons. In such a case, external forces would have to be added (to the right-hand side of the equations of motion) to maintain the solitons stationary.

Hence we assume that the field configurations for two solitons is a product of the two single soliton fields. Such a product is expected to be correct in the limit of infinite separation between the skyrmions[35] provided that the individual terms describe isolated B = 1 solitons located at r_1 and r_2. This approximation then takes the form[44]

$$U_{B=2}(r) = U_{B=1}(r + \frac{R}{2})\, U_{B=1}(r - \frac{R}{2}) \tag{1}$$

where $R = |r_1 - r_2|$ is the distance between the two solitons.

Although (1) is rigorously valid only in the limit of infinite R, it may be adopted for all separations between the two solitons.[43] Taking $U_{B=1}$ to be the Skyrme ansatz of the hedgehog form (V.8), the energy corresponding to the product ansatz (1) can be computed though $U_{B=2}$ is not a solution of the equations of motion. When the two B = 1 solitons are at the same location i.e. R = o, the energy is almost three times the B = 1 energy[35]

$$E_{B=2} \simeq 3E_{B=1} \tag{2}$$

Naively equation (2) would mean that stable solitons with baryon number two do not exist since the energy is more than twice that of a B = 1 soliton. However, an interaction potential may be defined :

$$V_2(R = o) = E_{B=2} - 2E_{B=1} \simeq E_{B=1} \qquad (3)$$

Equation (3) suggest that there is a short-range repulsion of roughly one soliton mass in magnitude.

Jackson et al[43] claim that this approximation does not introduce significant error in the calculated interaction. According to their calculations, the B = 2 energy obtained with this approximation is only 2.5% higher than the exact B = 2 energy calculated in the quark hedgehog model. This corresponds to an error of about 7.5% in the potential energy for zero separation. However this need not be true in regions where the hedgehogs overlap.

The field configuration given by (1) is spin and isospin degenerate. However, in order to identify the nucleon-nucleon interaction it is necessary to determine the spin and isospin content of the two soliton interaction. This has been done by Jackson et al[43] via the hedgehog quark model. They rotate one soliton with respect to the other in configuration space through Euler angles and compare this angular dependence with that obtained in the quark model under the same rotation. Then they extract the nucleon-nucleon potential as a result of this comparison.

Vinh Mau et al[44] employ another method of lifting the degeneracy. Their method is patterned after the standard treatment of the nucleon given first by Adkins, Nappi, and Witten[38] which has been briefly reviewed in the fifth chapter. They rotate the two solitons independently in isospin space,

$$U_o(r + \tfrac{R}{2}) \rightarrow A\, U_o(r + \tfrac{R}{2})A^+$$
$$U_o(r - \tfrac{R}{2}) \rightarrow B\, U_o(r - \tfrac{R}{2})B^+ \tag{4}$$

where A and B are isospin orientation matrices belonging to SU(2). Under the rotations (4), the field configuration (1) becomes

$$U(r) = A\, U_o(r + \tfrac{R}{2})\, A^+ B U_o(r - \tfrac{R}{2})B^+ \tag{5}$$

Its static energy E(R,C) depends only on the separation R and on the relative isospin orientation $C = A^+B$. As $R \rightarrow \infty$, E(R,C) approaches $2E_{B=1}$, the energy of two isolated skyrmions. The skyrmion-skyrmion interaction potential is then

$$V(R,C) = E(R,C) - 2E_{B=1} \tag{6}$$

$2E_{B=1}$ comes from terms in which the derivatives all act on the same soliton and V(R,C) from mixed derivatives.

A nucleon-nucleon interaction potential can be extracted from (6) by taking a suitable projection over the isospin orientation C.

The potentials resulting from these two methods[43,44] agree in general shape and in sign and reproduce many features of the semi-phenomenological "Paris potential".[79] The nucleon-nucleon potential can be written as[43] :

$$V^{NN}(r) = V_c(r) + (\sigma_1.\sigma_2)(\tau_1.\tau_2)V_{\sigma\tau}(r) + S_{12}(\tau_1.\tau_2)V_{T\tau}(r) \tag{7}$$

where V_c, $V_{\sigma\tau}$ and $V_{T\tau}$ are respectively the central, $(\sigma_1.\sigma_2)(\tau_1.\tau_2)$, and tensor components of the nucleon-nucleon interaction. For more adequate comparison of (7) with the Paris potential it is useful to include a pion mass in the Lagrangian. This is usually taken to be

$$-\frac{m_\pi^2 F_\pi^2}{8}(2 - \mathrm{Tr}\,U)$$

Taking this into account, the comparison of (7) with the Paris potential is made in figs. 1 to 3.
The various asymptotic forms of the Skyrme model interaction is quite good. In figure.1 which shows the purely central potential, both calculations contain a short-range repulsion. However, whereas a central attraction of intermediate range (1 to 2 fm) is present in the Paris potential,[79] it is missing in Jackson's et al Potential (7).

To conclude, the behaviour of V(R,C) at large R provide a correct description of the asymptotic interaction between skyrmions and, after projection, between nucleons in terms of the exchange of a single pion. For small R, however, the two skyrmions overlap and it is difficult to define their isospin orientation and to specify their spin while they are interacting. On account of these ambiguities, the dynamics cannot be accurately described by a potential V(R,C).[35]

Therefore this nucleon-nucleon potential reproduces the one-pion exchange potential at long range and predicts a repulsive force at short range. However, at intermediate range i,e, for inter-nucleon distances between one and two fermi, the repulsion persists. In other words this model suffers the serious defect of not providing any indication of the central, intermediate-range attraction arising from the exchange of an S-wave pair of pions. It is this attractive nuclear force that makes nucleons stick together and remain inside the nuclei. Indeed the electric force between two protons is repulsive and so would blow the nucleus apart were it not overcome by the greater attraction of the nuclear force.

To overcome this difficulty, Jackson et al thought of relaxing the constraint on the function F(r), used in the Skyrme ansatz eq.(V.8),

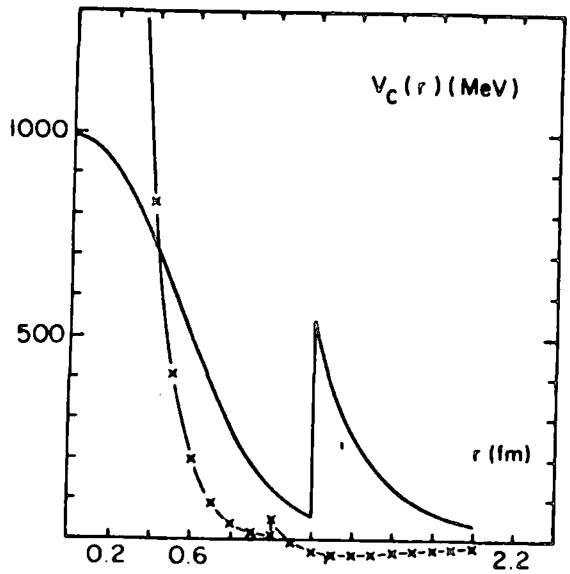


Fig. 1 : The solid line denotes the central potential $V_c(r)$ deduced from the Skyrme model. The crosses indicate the corresponding component of the Paris potential. The break in the curves indicates a tenfold scale change.

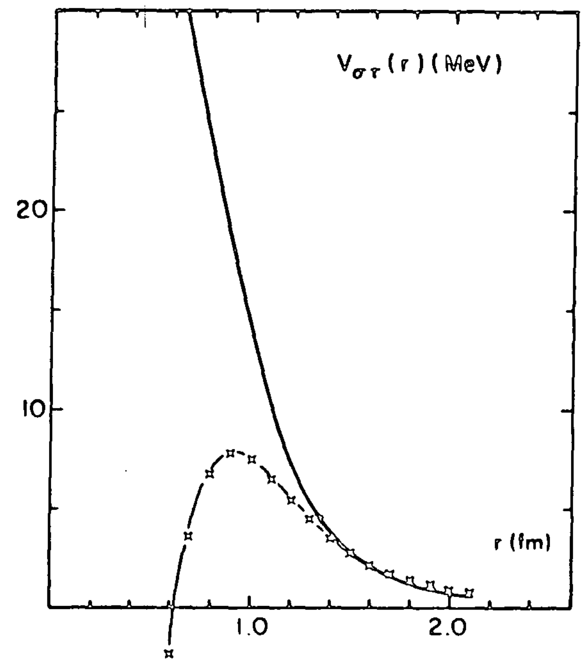


Fig. 2 : The nucleon-nucleon spin-spin potential $V_{\sigma\tau}(r)$.

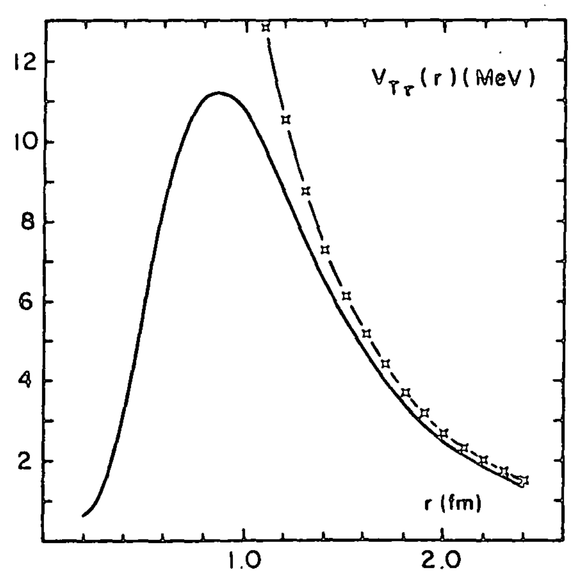


Fig. 3 : The nucleon-nucleon tensor potential $V_{T\tau}(r)$.

by allowing for a localised radial scale change:

$$F(r) = F(\rho(r))$$

where the baryon density $\rho(r)$ depends on some parameters. They then varied these parameters at fixed separation in order to minimize the B = 2 energy. However, the resulting changes were rather small and definitely not sufficient to make the central potential attractive for any soliton separation. For example, at $r = 0.8$ fm V_B is reduced by 12% and for $r = 1.4$ fm it is reduced by 4%. These results seem to justify the use of the hedgehog approximation.

Although it is expected that the exchange of a σ-meson would lead to attraction, Jackson et al[43] pointed out that both analytical and numerical arguments show that the quadractic term in the Skyrme Lagrangian $\mathcal{L}_2$ (which should contain σ-meson effects) does not make any contribution to the central interaction. This suggests that the Skyrme Lagrangian contains a σ-meson of infinite mass. In which case one would have to evaluate the vacuum fluctuation corrections to the classical calculations. Such corrections which would vanish in the large-N limit, can be significant for N = 3 and can give the σ-meson a finite effective mass hence leading to the missing attraction. However it is not clear which corrections should be considered and how they should be handled. Another alternative proposed by Jackson et al[65] is to modify the Skyrme Lagrangian. They suggest a change of sign of the quartic term in the Skyrme model which would turn the previous repulsion into an attraction. As this would also destabilise the soliton they add a sixth order repulsive term ,

$$\mathcal{L}_6 = \varepsilon_6^2 \, \mathrm{Tr}\, B_\mu B_\mu^+$$

with $$B_\mu = \varepsilon_{\mu ijk} \, \partial_i U \partial_j U^+ \partial_k U$$

As we have mentioned in the previous chapter, $\mathcal{L}_6$ is second order in time derivatives. It stabilises the skyrmion and yields short range repulsion in the nucleon-nucleon potential. This modification generates a nucleon-nucleon central potential that is attractive at long range and repulsive at short range but the model thus modified predicts the wrong signs for the pion-pion scattering lengths. The expansion of the quadratic Lagrangian $\mathcal{L}_2$ to fourth order in the pion field yields the Weinberg pion-pion scattering predictions.[60]

Donoghue et al[39] showed that information on pion-pion scattering uniquely determines the form of the quartic terms in the Lagrangian at the tree level. They also showed that the resulting form has a soliton with mass consistent with that of the proton. On the other hand Gasser and Leutwyler[50] observed that there exist only two independent quartic Lagrangians in the limit $m_\pi^2 \to o$. These are the Skyrme quartic term $\mathcal{L}_4$ and another symmetric quartic term[39]

$$\mathcal{L}_{sym} = \frac{\gamma}{8e^2} \left[Tr\,(\partial_\mu U \partial_\mu U^+) \right]^2$$

Lacombe et al[77] showed that the addition of this term $\mathcal{L}_{sym}$ to the Skyrme Lagrangian provides attractive nuclear forces at intermediate range while retaining the correct form for the pion-pion scattering lengths. Using the same method as ref.[44] they define the nucleon-nucleon interaction potential

$$V(R,r,C) = -\frac{\gamma}{8e^2} \int d^3r \left\{ \left[Tr(\partial_i U \partial_i U^+) \right]^2 - m_1 - m_2 \right\}$$

where $C = A^+B$ defined in (4) and (5); $i = 1, 2, 3$. m_1 and m_2 are up to a factor $\gamma/(8e^2)$, the single soliton mass densities.

They then proceed to show that the contribution of $\mathcal{L}_{sym}$ to potential is attractive. This attraction however does not overcome the larger repulsion due to the Skyrme term. But the situation is not hopeless. The parameters γ and e can be adjusted in such a way as to increase the attraction due to $\mathcal{L}_{sym}$ and reduce the repulsion due to $\mathcal{L}_4$ at intermediate range. However, they do not want to spoil the repulsion at short range. To do this they couple the ω-vector meson to the pions via the Lagrangian[75]

$$\mathcal{L}_{\omega} = -\frac{1}{4}(\partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu})^2 + \frac{1}{2}m_{\omega}^2\,\omega_{\mu}^2 + \beta\,\omega_{\mu}\mathcal{B}^{\mu}$$

where ω_{μ} is the ω-vector meson field and,

$$\mathcal{B}_{\mu} = \frac{1}{24\pi^2}\,\varepsilon_{\mu\alpha\beta\gamma}\,(U^{+}\partial_{\alpha}UU^{+}\partial_{\rho}UU^{+}\partial_{\gamma}U)$$

is the baryon current.

The introduction of $\mathcal{L}_{\omega}$ would therefore restore the phenomenological features of the nucleon-nucleon interaction.

Note however, that according to Aitchison et al[74] the fourth order contributions are not sufficient to stabilise the classical soliton. These authors pointed out that the sixth order terms may stabilise the soliton. This suggestion is based on the fact that the evaluation of the total Casimir energy evaluated by Ripka[85] is positive i.e. stabilising and appears to scale roughly as τ^{-3} where τ is the soliton size. And this is how the sixth order terms scale (fourth order terms scale as τ^{-1}).

In conclusion, the soliton picture of baryons appears to be rather satisfactory. The agreement of the model with experiments is rather remarkable in view of the simplicity of the Skyrme model and of the crude approximations involved. After all the aim in adopting this model as an effective Lagrangian for QCD was to see if baryon properties can be deduced from meson properties via semi-classical quantisation of solitons. Note however that neither the Skyrme Lagrangian nor any other effective Lagrangian has yet been deduced from QCD.

APPENDIX

In order to evaluate the Dirac action to second order in h, we expand ψ_μ, ψ_ν, $\psi_{\mu+\nu}$, $\bar\psi_\mu$, $\bar\psi_\nu$, and $\bar\psi_{\mu+\nu}$ in III.40, to third order in h and apply the Baker-Hausdorff formula. This yields the following,

$$\frac{1}{4}h^{-1}\left[\bar\psi\gamma_\mu e^{ihA^\mu}\psi_\mu - \bar\psi_\mu\gamma_\mu e^{-ihA^\mu}\psi\right]$$

$$= \frac{1}{4}h^{-1}\Big[\bar\psi\gamma_\mu\left(1+ihA^\mu-\frac{h^2}{2!}A^{\mu 2}-i\frac{h^3}{3!}A^{\mu 3}\right)\left(\psi+h\partial_\mu\psi+\frac{h^2}{2!}\partial_\mu^2\psi+\frac{h^3}{3!}\partial_\mu^3\psi\right)$$

$$-\left(\bar\psi+h\partial_\mu\bar\psi+\frac{h^2}{2!}\partial_\mu^2\bar\psi+\frac{h^3}{3!}\partial_\mu^3\psi\right)\gamma_\mu\left(1-ihA^\mu-\frac{h^2}{2!}A^{\mu 2}+i\frac{h^3}{3!}A^{\mu 3}\right)\psi\Big]$$

$$= \frac{1}{4}\Big[2i\bar\psi\gamma_\mu A^\mu\psi+\bar\psi\gamma_\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma_\mu\psi+ih\bar\psi\gamma_\mu A^\mu\partial_\mu\psi+ih\partial_\mu\bar\psi\gamma_\mu A^\mu\psi+\frac{h}{2}\bar\psi\gamma_\mu\partial_\mu^2\psi$$

$$-\frac{h}{2}\partial_\mu^2\bar\psi\gamma_\mu\psi-\frac{h^2}{2}\bar\psi\gamma_\mu A^{\mu 2}\partial_\mu\psi+\frac{h^2}{2}\partial_\mu\bar\psi\gamma_\mu A^{\mu 2}\psi+i\frac{h^2}{2}\bar\psi\gamma_\mu A^\mu\partial_\mu^2\psi$$

$$+i\frac{h^2}{2}\partial_\mu^2\bar\psi\gamma_\mu A^\mu\psi+\frac{h^2}{3!}\bar\psi\gamma_\mu\partial_\mu^3\psi-\frac{h^2}{3!}\partial_\mu^3\bar\psi\gamma_\mu\psi-\frac{2i}{3!}\bar\psi\gamma_\mu A^{\mu 3}\psi\Big]$$

Note that we are considering A^μ and not $\bar A^\mu$ as we should. This is because we choose to work in a Lorentz gauge, i.e. $\partial_\mu A^\mu = 0$ and therefore

$$\bar A^\mu = \frac{1}{2}\left(A^\mu + A^\mu_\mu\right) = \frac{1}{2}\left(A^\mu + A^\mu + h\partial_\mu A^\mu + \dots\right)$$
$$= A^\mu$$

when h goes to zero.

$$\frac{1}{8}h^{-1}\left(\bar\psi\gamma_\mu e^{ihA^\mu}e^{ihA^\nu_\mu}\psi_{\mu+\nu} - \bar\psi_{\mu+\nu}\gamma_\mu e^{-ihA^\nu_\mu}e^{-ihA^\mu}\psi\right)$$

$$= \frac{1}{8}\left(i\bar\psi\gamma_\mu A^\mu\psi+2i\bar\psi\gamma_\mu A^\nu\psi+\bar\psi\gamma_\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma_\mu\psi+\bar\psi\gamma_\mu\partial_\nu\psi-\partial_\nu\bar\psi\gamma_\mu\psi\right)$$

$$+\frac{h}{8}\Big(2i\bar\psi\gamma_\mu\partial_\mu A^\nu\psi-\bar\psi\gamma_\mu[A^\mu,A^\nu]\psi+\bar\psi\gamma_\mu\partial_\nu\partial_\mu\psi-\partial_\nu\partial_\mu\bar\psi\gamma_\mu\psi+i\bar\psi\gamma_\mu(A^\mu+A^\nu)\partial_\mu\psi$$

$$+i\partial_\mu\bar\psi\gamma_\mu(A^\mu+A^\nu)\psi+i\bar\psi\gamma_\mu(A^\mu+A^\nu)\partial_\nu\psi+i\partial_\nu\bar\psi\gamma_\mu(A^\mu+A^\nu)\psi\Big)$$

$$+\frac{h^2}{8}\Big(i\bar\psi\gamma_\mu(A^\mu+A^\nu)\left(\frac{1}{2}\partial_\mu^2\psi+\frac{1}{2}\partial_\nu^2\psi+\partial_\mu\partial_\nu\psi\right)+i\left(\frac{1}{2}\partial_\mu^2\bar\psi+\frac{1}{2}\partial_\nu^2\bar\psi+\partial_\mu\partial_\nu\bar\psi\right)\gamma_\mu(A^\mu+A^\nu)\psi+$$

$$
\begin{aligned}
&-\tfrac{1}{2}\bar\Psi\gamma_\mu(A^{\mu 2}+A^{\nu 2})(\partial_\mu\Psi+\partial_\nu\Psi)+\tfrac{1}{2}(\partial_\mu\bar\Psi+\partial_\nu\bar\Psi)\gamma_\mu(A^{\mu 2}+A^{\nu 2})\Psi\\
&-\bar\Psi\gamma_\mu\left(A^\mu\partial_\mu A^\nu+\tfrac{1}{2}A^\mu A^{\nu 2}+\tfrac{1}{2}A^{\mu 2}A^\nu+\tfrac{1}{2}A^\nu\partial_\mu A^\nu+\tfrac{1}{2}\partial_\mu A^\nu A^\nu\right)\Psi\\
&-\bar\Psi\gamma_\mu\left(-\partial_\mu A^\nu A^\mu+\tfrac{1}{2}A^\nu A^{\mu 2}+\tfrac{1}{2}A^{\nu 2}A^\mu-\tfrac{1}{2}\partial_\mu A^\nu A^\nu-\tfrac{1}{2}A^\nu\partial_\mu A^\nu\right)\Psi\\
&+i\bar\Psi\gamma_\mu\,\partial_\mu^2A^\nu\Psi-\tfrac{2}{3!}i\,\bar\Psi\gamma_\mu(A^{\mu 3}+A^{\nu 3})\Psi+\tfrac{1}{3!}\bar\Psi\gamma_\mu(\partial_\mu^3+\partial_\nu^3)\Psi\\
&-\tfrac{1}{3!}(\partial_\mu^3+\partial_\nu^3)\bar\Psi\gamma_\mu\Psi\Big)
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\tfrac{1}{8}h^{-1}\left(\bar\Psi_\nu\gamma_\mu e^{-ihA^\nu}e^{ihA^\mu}\Psi_\mu-\bar\Psi_\mu\gamma_\mu e^{-ihA^\mu}e^{ihA^\nu}\Psi_\nu\right)\\
&=\tfrac{1}{8}\left(2i\,\bar\Psi\gamma_\mu(A^\mu-A^\nu)\Psi+\partial_\nu\bar\Psi\gamma_\mu\Psi-\bar\Psi\gamma_\mu\partial_\nu\Psi+\bar\Psi\gamma_\mu\partial_\mu\Psi-\partial_\mu\bar\Psi\gamma_\mu\Psi\right)\\
&+\tfrac{1}{8}h\Big(\tfrac{1}{2}\bar\Psi\gamma_\mu\partial_\mu^2\Psi+\tfrac{1}{2}\partial_\nu^2\bar\Psi\gamma_\mu\Psi-\tfrac{1}{2}\bar\Psi\gamma_\mu\partial_\nu^2\Psi-\tfrac{1}{2}\partial_\mu^2\bar\Psi\gamma_\mu\Psi\\
&\quad+i\bar\Psi\gamma_\mu(A^\mu-A^\nu)\partial_\mu\Psi+i\bar\Psi\gamma_\mu(A^\mu-A^\nu)\partial_\nu\Psi+i\partial_\nu\bar\Psi\gamma_\mu(A^\mu-A^\nu)\Psi\\
&\quad+i\partial_\mu\bar\Psi\gamma_\mu(A^\mu-A^\nu)\Psi-\bar\Psi\gamma_\mu[A^\mu,A^\nu]\Psi+\partial_\nu\bar\Psi\gamma_\mu\partial_\mu\Psi-\partial_\mu\bar\Psi\gamma_\mu\partial_\nu\Psi\Big)\\
&+\tfrac{h^2}{8}\Big(\tfrac{1}{3!}\bar\Psi\gamma_\mu\partial_\mu^3\Psi+\tfrac{1}{3!}\partial_\nu^3\bar\Psi\gamma_\mu\Psi-\tfrac{2}{3!}i\bar\Psi\gamma_\mu(A^{\mu 3}-A^{\nu 3})\Psi-\tfrac{1}{2}\partial_\nu\bar\Psi\gamma_\mu(A^{\nu 2}+A^{\mu 2})\Psi\\
&-i\bar\Psi\gamma_\mu\left(\tfrac{1}{2}A^{\nu 2}A^\mu-\tfrac{1}{2}A^\nu A^{\mu 2}\right)\Psi+i\partial_\nu^2\bar\Psi\gamma_\mu(A^\mu-A^\nu)\Psi+i\partial_\nu\bar\Psi\gamma_\mu(A^\mu-A^\nu)\partial_\mu\Psi\\
&-\bar\Psi\gamma_\mu\left(\tfrac{1}{2}A^{\nu 2}+\tfrac{1}{2}A^{\mu 2}\right)\partial_\mu\Psi+\tfrac{i}{2}\bar\Psi(A^\mu-A^\nu)\partial_\mu^2\Psi-i\bar\Psi\gamma_\mu\left(-\tfrac{1}{2}A^{\mu 2}A^\nu+\tfrac{1}{2}A^\mu A^{\nu 2}\right)\Psi\\
&+\tfrac{1}{2}\partial_\mu\bar\Psi\gamma_\mu(A^{\mu 2}+A^{\nu 2})\Psi+i\partial_\mu^2\Psi\gamma_\mu(A^\mu-A^\nu)\Psi+i\partial_\mu\bar\Psi\gamma_\mu(A_\mu-A_\nu)\partial_\nu\Psi\\
&+\tfrac{1}{2}\bar\Psi\gamma_\mu(A^{\nu 2}+A^{\mu 2})\partial_\nu\Psi+\tfrac{i}{2}\bar\Psi\gamma_\mu(A^\mu-A^\nu)\partial_\nu^2\Psi-\tfrac{1}{3!}\bar\Psi\gamma_\mu\partial_\nu^3\Psi-\tfrac{i}{3!}\bar\Psi\gamma_\mu(A^{\mu 3}-A^{\nu 3})\Psi\\
&-\tfrac{1}{3!}\partial_\mu^3\bar\Psi\gamma_\mu\Psi\Big)
\end{aligned}
$$

All together, the action III.40 contributes,

$$
\begin{aligned}
S&=i\bar\Psi\gamma_\mu A^\mu\Psi+\tfrac{1}{2}\bar\Psi\gamma_\mu\partial_\mu\Psi-\tfrac{1}{2}\partial_\mu\bar\Psi\gamma_\mu\Psi\\
&\qquad+\text{terms in } h\ (B_1)+\text{terms in } h^2\ (B_2)\\
&=\bar\Psi i\gamma_\mu A^\mu\Psi+\bar\Psi\gamma_\mu\partial_\mu\Psi-\tfrac{1}{2}\partial_\mu(\bar\Psi\gamma_\mu\Psi)+B_1(h)+B_2(h^2).
\end{aligned}
$$

Hence to lowest order in h and up to a total divergence, the continuum action $\psi\, \gamma_\mu\, (\partial_\mu + iA^\mu)\, \psi$ is recovered. The remaining terms are gauge invariant. Indeed,

$$
\begin{aligned}
8h^{-1}B_1(h) = {} & 4i\bar\Psi\gamma_\mu A^\mu \partial_\mu\Psi + 4i\partial_\mu\bar\Psi\gamma_\mu A^\mu\Psi + 2i\partial_\nu\bar\Psi\gamma_\mu A^\mu\Psi \\
& + 2i\bar\Psi\gamma_\mu A^\mu\partial_\nu\Psi - 2\bar\Psi\gamma_\mu[A^\mu, A^\nu]\Psi + 2i\bar\Psi\gamma_\mu\partial_\mu A^\nu\Psi \\
& + \bar\Psi\gamma_\mu\partial_\mu\partial_\nu\Psi + \partial_\nu\bar\Psi\gamma_\mu\partial_\mu\Psi - \partial_\mu\bar\Psi\gamma_\mu\partial_\nu\Psi - \partial_\nu\partial_\mu\bar\Psi\gamma_\mu\Psi \\
& + 2\left(\bar\Psi\gamma_\mu\partial_\mu^2\Psi - \partial_\mu^2\bar\Psi\gamma_\mu\Psi\right) \\
= {} & 4\partial_\mu(\bar\Psi i\gamma_\mu A^\mu\Psi) - 4i\bar\Psi\gamma_\mu\partial_\mu A^\mu\Psi + 2\partial_\nu(\bar\Psi i\gamma_\mu A^\mu\Psi) \\
& - 2i\bar\Psi\gamma_\mu\partial_\nu A^\mu\Psi - 2\bar\Psi\gamma_\mu[A^\mu, A^\nu]\Psi + 2i\bar\Psi\gamma_\mu\partial_\mu A^\nu\Psi \\
& + \partial_\nu(\bar\Psi\gamma_\mu\partial_\mu\Psi) - \partial_\nu(\partial_\mu\bar\Psi\gamma_\mu\Psi) + 2\partial_\mu(\bar\Psi\gamma_\mu\partial_\mu\Psi) \\
& - 2\partial_\mu(\partial_\mu\bar\Psi\gamma_\mu\Psi) \\
= {} & \bar\Psi\gamma_\mu iF^{\mu\nu}\Psi - 4i\bar\Psi\gamma_\mu\partial_\mu A^\mu\Psi + \text{total divergences.}
\end{aligned}
$$

where the first term is gauge invariant, it is a Pauli term (analoguous to the anomalous magnetic moment) and the second term vanishes in the Lorentz gauge.

$$
\begin{aligned}
8h^{-2}B_2(h^2) = {} & \frac{4}{3!}\left(\bar\Psi\gamma_\mu\partial_\mu^3\Psi - \partial_\mu^3\bar\Psi\gamma_\mu\Psi\right) - \frac{8i}{3!}\bar\Psi\gamma_\mu A^{\mu^3}\Psi + 2i\bar\Psi\gamma_\mu A^\mu\partial_\mu^2\Psi \\
& + 2i\partial_\mu^2\bar\Psi\gamma_\mu A^\mu\Psi - 2\bar\Psi\gamma_\mu A^{\mu^2}\partial_\mu\Psi + 2\partial_\mu\bar\Psi\gamma_\mu A^{\mu^2}\Psi \\
& + \bar\Psi i\gamma_\mu(A^\mu + A^\nu)\partial_\mu\partial_\nu\Psi + \partial_\mu\partial_\nu\bar\Psi i\gamma_\mu(A^\mu + A^\nu)\Psi \\
& - i\bar\Psi\gamma_\mu(A^\mu A^{\nu^2} + A^{\nu^2}A^\mu)\Psi - \bar\Psi\gamma_\mu A^{\nu^2}\partial_\mu\Psi + \partial_\mu\bar\Psi\gamma_\mu A^{\nu^2}\Psi_\mu \\
& - \bar\Psi\gamma_\mu(A^\mu\partial_\mu A^\nu - \partial_\mu A^\nu A^\mu)\Psi + i\bar\Psi\gamma_\mu\partial_\mu^2 A^\nu\Psi \\
& + \partial_\nu^2\bar\Psi i\gamma_\mu A^\mu\Psi + \bar\Psi i\gamma_\mu A^\mu\partial_\nu^2\Psi + \partial_\nu\bar\Psi i\gamma_\mu(A^\mu - A^\nu)\partial_\mu\Psi \\
& + \partial_\mu\bar\Psi i\gamma_\mu(A^\mu - A^\nu)\partial_\nu\Psi
\end{aligned}
$$

$$8h^{-2}\mathcal{B}_2(h) = 3i\,\bar{\Psi}\gamma_\mu \partial_\nu^2 A^\mu \Psi - 6\,\partial_\nu\bar{\Psi}\, i\gamma_\mu A^\mu \partial_\nu \Psi$$

+ total divergences + terms that vanish in the lorentz gauge.

In order to estimate $B_2(h^2)$ we also made use of the Dirac equations $\gamma_\mu(\partial_\mu + iA_\mu)\,\psi = 0 = \psi\,\gamma_\mu(\overleftarrow{\partial}_\mu - iA_\mu)$, which hold on the mass shell.

But the term $(\bar{\psi}i\,\gamma_\mu\partial^2 A_\mu\,\psi - 2\partial_\nu\bar{\psi}i\,\gamma_\mu A^\mu\partial_\nu\psi)$ is also gauge invariant up to a total divergence:

$$(A^\mu)' = g\,A^\mu g^{-1} + i\,(\partial_\mu g)\,g^{-1}$$

Therefore,

$$\begin{aligned}(\bar{\Psi}i\gamma_\mu\partial_\nu^2 A^\mu\Psi)' = \bar{\Psi}i\gamma_\mu\big[&g^{-1}\partial_\nu^2 g\,A^\mu + A^\mu\,\partial_\nu^2 g^{-1}g + i g^{-1}\partial_\nu^2\partial_\mu g\\ &+ i g^{-1}\partial_\mu g\,\partial_\nu^2 g^{-1}g + \partial_\nu^2 A^\mu + 2g^{-1}\partial_\nu g\,\partial_\nu A^\mu +\\ &+ 2g^{-1}\partial_\nu g\,A^\mu\partial_\nu g^{-1}g + 2\partial_\nu A^\mu\partial_\nu g^{-1}g + 2i g^{-1}\partial_\mu\partial_\nu g\,\partial_\nu g^{-1}g\big]\Psi\end{aligned}$$

Since $g^{-1}g = 1$,

$$g^{-1}\partial_\nu^2 g = -\partial_\nu^2 g^{-1}g - 2\partial_\nu g^{-1}\partial_\nu g.$$

$$\begin{aligned}g^{-1}\partial_\nu^2\partial_\mu g + g^{-1}\partial_\mu g\,\partial_\nu^2 g^{-1}g = &-\partial_\mu\partial_\nu^2 g^{-1}g - \partial_\nu^2 g^{-1}\partial_\mu g - 2\partial_\mu\partial_\nu g^{-1}\partial_\nu g\\ &- 2\partial_\nu g^{-1}\partial_\mu\partial_\nu g - 2g^{-1}\partial_\mu g\,\partial_\nu g^{-1}\partial_\nu g.\end{aligned}$$

Then,

$$\begin{aligned}(\bar{\Psi}i\gamma_\mu\partial_\nu^2 A_\mu\Psi)' = &\bar{\Psi}i\gamma_\mu\partial_\nu^2 A_\mu\Psi + \bar{\Psi}i\gamma_\mu\big[g^{-1}\partial_\nu^2 g\,A_\mu - A_\mu g^{-1}\partial_\nu^2 g\\ &- 2A_\mu\partial_\nu g^{-1}\partial_\nu g\big]\Psi + \bar{\Psi}i\gamma_\mu\big[i\partial_\mu\partial_\nu^2 g^{-1}g + i\partial_\nu^2 g^{-1}\partial_\mu g\big]\Psi\\ &+ 2\bar{\Psi}i\gamma_\mu\big[i\partial_\mu\partial_\nu g^{-1}\partial_\nu g + i\partial_\nu g^{-1}\partial_\mu\partial_\nu g + i g^{-1}\partial_\mu g\,\partial_\nu g^{-1}\partial_\nu g\big]\Psi\\ &+ 2\bar{\Psi}i\gamma_\mu\big[g^{-1}\partial_\nu g\,\partial_\nu A^\mu + \partial_\nu g^{-1}g\,A^\mu g^{-1}\partial_\nu g - g^{-1}\partial_\mu\partial_\nu g\,\partial_\nu g^{-1}g\big]\Psi\\ &+ 2\bar{\Psi}i\gamma_\mu\big[\partial_\nu A^\mu\,\partial_\nu g^{-1}g\big]\Psi\end{aligned}$$

$$\begin{aligned}
(\bar\Psi i\gamma_\mu \partial_\nu^2 A^\mu \Psi)' = {} & \bar\Psi i\gamma_\mu \partial_\nu^2 A^\mu \Psi + \bar\Psi\, \gamma_\mu (g^{-1}\partial_\nu^2 g)\partial_\mu\Psi + \partial_\mu\bar\Psi\gamma_\mu (g^{-1}\partial_\nu^2 g)\Psi \\
& + \bar\Psi\gamma_\mu\partial_\mu (g^{-1}\partial_\nu^2 g)\Psi + 2\bar\Psi\gamma_\mu\partial_\mu(\partial_\nu g^{-1}\partial_\nu g)\Psi \\
& + 2\bar\Psi i\gamma_\mu\left[- A^\mu \partial_\nu g^{-1}\partial_\nu g + \partial_\mu(\partial_\nu g^{-1}\partial_\nu g) + i g^{-1}\partial_\mu g\partial_\nu g^{-1}\partial_\nu g\right]\Psi \\
& + 2\bar\Psi i\gamma_\mu\left[g^{-1}\partial_\nu g\partial_\nu A^\mu - \partial_\nu A^\mu g^{-1}\partial_\nu g + \partial_\nu g^{-1} g A^\mu g^{-1}\partial_\nu g \right.\\
& \left. - i g^{-1}\partial_\mu\partial_\nu g\,\partial_\nu g^{-1} g\right]\Psi \\
= {} & \bar\Psi i\gamma_\mu \partial_\nu^2 A^\mu\Psi + \partial_\mu(\bar\Psi\gamma_\mu (g^{-1}\partial_\nu^2 g)\Psi) \\
& + 2\bar\Psi i\gamma_\mu\left[-A_\mu\partial_\nu g^{-1}\partial_\nu g + i g^{-1}\partial_\mu g\partial_\nu g^{-1}\partial_\nu g + g^{-1}\partial_\nu g\partial_\nu A^\mu \right.\\
& \left. - \partial_\nu A^\mu g^{-1}\partial_\nu g + \partial_\nu g^{-1} g A^\mu g^{-1}\partial_\nu g - i g^{-1}\partial_\mu\partial_\nu g\partial_\nu g^{-1} g\right]\Psi
\end{aligned}$$

$$\begin{aligned}
-2i(\partial_\nu\bar\Psi\gamma_\mu A^\mu\partial_\nu\Psi)' = {} & -2i(\partial_\nu\bar\Psi g^{-1} + \bar\Psi\partial_\nu g^{-1})\gamma_\mu(g A^\mu g^{-1} + i\partial_\mu g g^{-1})(g\partial_\nu\Psi + \partial_\nu g\Psi) \\
= {} & -2i\,\partial_\nu\bar\Psi\gamma_\mu A^\mu\partial_\nu\Psi + 2\bar\Psi i\gamma_\mu\left[-\partial_\nu g^{-1} g A^\mu g^{-1}\partial_\nu g + i\partial_\nu g^{-1}\partial_\mu g g^{-1}\partial_\nu g\right]\Psi \\
& + 2\partial_\nu\bar\Psi i\gamma_\mu\left[{}^{\mu} g^{-1}\partial_\nu g + i g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\right]\Psi \\
& + 2\bar\Psi i\gamma_\mu\left[-\partial_\nu g^{-1} g A_\mu + i\partial_\nu g^{-1}\partial_\mu g\right]\partial_\nu\Psi \\
& - 2\partial_\nu\bar\Psi\gamma_\mu g^{-1}\partial_\mu g\,\partial_\nu\Psi
\end{aligned}$$

and these two terms added, give, up to total divergences,

$$\bar\Psi i\gamma_\mu\partial_\nu^2 A^\mu\Psi - 2\partial_\nu\bar\Psi i\gamma_\mu A^\mu\Psi$$

This quantity is therefore gauge invariant.

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