The Schwinger-Dyson Equations

and

Confinement

in

Quantum Chromo-Dynamics

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The Schwinger-Dyson Equations

and Confinement in QCD

by A.D.Worrall

Abstract

The Schwinger-Dyson equations for the gluon and quark propagators are investigated in the covariant gauge. The renormalization functions are approximated suitably and the value of the parameters are determined by requiring that the functions be numerically self-consistent solutions over appropriate ranges of momenta.

In the case of the gluon the Schwinger-Dyson equation is truncated by neglecting the the two loop contributions and the triple gluon vertex is approximated by a form proposed by Mandelstam which has the same behaviour as the more complicated longitudinal vertex determined from the Slavnov-Taylor identity. The equation is then closed and the integrals are calculated by dimensional regularization and renormalised to remove a mass term.

In the quark case the dominant part of the quark-gluon vertex is determined from the Ward-Takahashi identity to give, with the gluon, a closed equation. The angular integrals are then calculated by an appropriate choice of coordinate frame. The quark function is approximated by a power series in the non-perturbative regime and the usual perturbative result elsewhere. The radial integrals are then calculated with appropriate regularization and renormalization.

It is found that the gluon propagator has approximately a singularity of the form $1/q^4$ which leads to a roughly linear confining potential. The effect of this enhanced singularity on the quark propagator is to suppress the propagation of quarks at low momenta.

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Contents

" 'Well, in our country,' said Alice, still panting a little, 'you'd generally get somewhere else — if you ran very fast for a long time, as we've been doing.'

'A slow sort of country!' said the Queen. 'Now, *here*, you see, it takes all the running *you* can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!' "

From

"Through the Looking-Glass

and What Alice Found There"

by

Lewis Carroll.

Chaper 1

Confinement

1:1 Historical Background

In 1897 in a classic paper J.J.Thomson [1.1] announced the discovery of the electron, showing that cathode rays were particulate in nature and carried an electric charge. Because of the amount of deflection of these particles it seemed most likely that they were light and carried the basic unit of charge determined from electrolysis and Avogadro's number. That the electron (a name suggested by G.J.Stoney in 1891) was truly a subatomic particle was demonstrated by Thomson by considering the photoelectric effect.

If the electron was a subatomic particle then, since atoms are electrically neutral, there must be some positive charge to balance the electron's negative charge. Fortunately a candidate for this honour had already been found. In 1886 Eugene Goldstein had discovered that by using a perforated cathode, he could produce rays moving in the opposite direction to the cathode rays (which he had previously discovered in 1876), which he called Kanalstrahlen (channel rays). Ernest Rutherford managed to identify these particles as having a positive charge equal and opposite to the electron, and the same mass as the hydrogen ion. He called this particle the proton as it was the first building block of the elements.

In 1906 to 1908 Rutherford performed a series of experiments bombarding thin foils with alpha particles. Most of the particles passed through the foil only slightly deflected, but some were deflected through angles greater than 90°. Since alpha particles could only be deflected through such large angles by an intense electric field, this led Rutherford to suggest [1.2] that all the positive charge resided in a compact body in the centre of the atom. Using this nuclear theory, he was able to account exactly for the observations of Geiger and Marsden in 1913.

This left the problem of the atomic mass since after taking into account the number of protons necessary to give the nuclear charge, there was still a large difference, about a factor of two difference. It was proposed that as well as having electrons around the nucleus, there were some electrons inside the nucleus. These nuclear electrons would then balance the charge from the necessary surfeit of protons. However, it was found that this solution was untenable.

It was not until 1930 that the solution to this problem began to appear when W.Bothe and H.Becker bombarded beryllium with alpha particles and discovered a new penetrating form of radiation. This work was confirmed two years later by Fredric and Irene Joliot-Curie. In the same year James Chadwick performed a series of experiments demonstrating that this new radiation consisted of neutral particles of the same mass as the proton. Chadwick adopted the name for this particle that had already been proposed, the neutron.

The family of subatomic particles now seemed complete, electrons orbiting around a central nucleus, composed of neutrons and protons. The only problem was what held the nucleus together against the Coulomb repulsion of all the protons? Heisenberg in 1932 [1.3] proposed that the neutron and proton were just different facets of the same particle, and that they continually exchanged identity while inside the nucleus. So a proton did not have time to come to terms with its identity as a proton and feel the effect of the Coulomb repulsion before it was not a proton anymore. More formally the neutron and proton could be regarded as having a property called isotopic spin, or isospin, analogous to normal spin, where the proton and neutron are the up and down components of a doublet.

In 1935 Hideki Yukawa [1.4], encouraged by the success of the theory for the Coulomb force in which the photon was exchanged between charged particles, proposed that there should be an exchange particle for the nuclear force. Because of the short range of the nuclear force the particle would have to have a mass between that of the electron and the proton. The particle was called the mesotron or meson for short. To explain the possible types of exchange between two nucleons (proton, neutron), the meson had to come in three forms, positively charged, neutral and negatively charged. N.Kemmer realised that this meant that it had to be an isospin triplet. After a slight misidentification of the muon, which turned out to be a heavy electron, the pi meson was discovered.

At this time, particle accelerators started to become available and experimenters no longer had to rely on natural sources of radioactivity or the vagaries of cosmic rays. Then the number of particles discovered increased dramatically, and the higher the energy of the accelerator the more new particles were produced. Most of these new particles were hadrons, like the proton and pion, as opposed to leptons, like the electron and muon. These hadrons can be split into two groups, the baryons (proton, neutron, etc.) and the mesons (pion, etc.). These two groups are composed of a large number of "stable" particles (decaying electromagnetically in $\sim 10^{-21}$ seconds, or weakly in $\sim 10^{-8}$ seconds) and a plethora of resonances which decay strongly in $\sim 10^{-23}$ seconds. These resonances, when their angular momentum is plotted against the square of their masses, lie on straight lines called Regge trajectories starting with the appropriate stable particle. For the simplest baryons, these trajectories have been extended upto J = 19/2.

In 1960's at the Stanford Linear Accelerator Centre (SLAC) an experiment similar to Rutherford's classic experiment but at much higher energies was perfomed. Protons were bombarded with energetic electrons and it was found that more of the electrons were scattered with large momenta transverse to the beam than had been anticipated [1.5]. This suggested that, within the proton, there are discrete scattering centres and further, the fact that the distribution of the scattered electrons against energy and angle exhibit scale invariance, suggests that the scattering centres are point-like [1.6]. These results were later confirmed by experiments at the Centre for European Nuclear Research (CERN) Intersecting Storage Ring (ISR), where protons were collided head on. These constituents were given the name partons, and for the first time a particle was identified in a bound state before being seen as a free particle. Indeed, even at today's high energies (>100 GeV), none of these partons have been isolated. It is this problem of the confinement of the partons within hadrons that is the concern of this thesis.

1:2 The Parton Model and QCD

In the fifties it became clear that Heisenberg's concept of isospin was not just confined to the nucleon and pion but also applied to the newly discovered strange particles. These strange particles were only produced in pairs and decayed weakly, giving them a much longer life-time than would be expected from their mass. However, the centre of the isospin multiplets for these strange particles did not coincide with those of the nucleon and pion. This led M.Gell-Mann and K.Nishijima [1.7] to propose that the new particles had a property, called strangeness, which was just sufficient to shift the centre of the multiplets to the right place. This idea reached maturity in 1961 when, independently M.Gell-Mann and Y.Ne'eman [1.8] suggested a classification of particles called the Eightfold Way based on the Lie algebra of SU(3). This was basically an extension of isospin, by introducing two new types of spin: U and V spin, which involve changes of the new property of strangeness. While the isospin symmetry is only slightly broken, the neutron and proton having roughly the same mass, the U and V spin symmeteries are more severely broken. The reason for this is now expressed in terms of the mass difference of the associated constituents, although the reason for these mass differences is still not fully understood.

In 1963 M.Gell-Mann and G.Zweig, [1.9] independently, proposed that the new symmetries could be understood if the hadrons were composed of particles called quarks, and that these quarks came in three flavours up, down and strange. Ordinary particles, like the nucleon and pion, are composed of just up and down quarks, and the strange particles contain one or more of the strange quark. The Eightfold Way could then be understood if the quarks are in the fundamental representation of SU(3). Then baryons would contain three quarks $3 \times 3 \times 3 = 1 + 8 + 8 + 10$, and the mesons contain a quark and an anti-quark $3 \times \overline{3} = 1 + 8$, which gives exactly the the observed particle spectrum.

To get the correct spin for the hadrons, the quarks have to be spin 1/2 particles, but this leads to a problem since to make some of the particles some of the quarks have to be in the same state, eg Ω^- which has three strange quarks with the spins aligned, which is not allowed by Fermi-Dirac statistics. O.W.Greenberg in 1964 [1.10] proposed a way round this problem by giving the quarks a new quantum number called colour. All the quarks come in three diferent colours and all the observed hadrons are colourless superpositions of colour, ie. white. This then explains why baryons are made of three quarks, the three primary colours combine together to give white, and mesons a quark and an anti-quark, a primary colour and it's complimentary colour combine to give white. The fact that there are three colours is supported by the experimental evidence of the π^0 decay to two photons and the ratio of the cross-sections of $e^+e^- \rightarrow$ hadrons over $e^+e^- \rightarrow \mu^+\mu^-$. Colour then can be expressed as an exact symmetry of SU(3), as opposed to the broken SU(3) flavour symmetry, where the observed hadrons are colour singlets.

This property of colour must be responsible for the confinement of the quarks within hadrons. It is then natural that there must exist an exchange particle associated with colour which is called the gluon, since it glues the quarks together. Unlike the photon, this gluon has to carry the colour charge, which means that there are eight different coloured gluons coresponding to the eight possible combinations of the primary and complimentary colours. To place this on a mathematical basis it is expressed as a gauge theory in analogy with Quantum Electro-Dynamics (QED) called Quantum Chromo-Dynamics (QCD) [1.11].

The quark fields are then described by the spinors ψ_i, which are in the fundamental representation of SU(3) of colour, $i = 1,2,3$. The gluon fields A_a^μ are in the adjoint representation, $a = 1,\ldots,8$. The SU(3) transformations can be represented by the matrices T^a_{ij} which obey the commutation relation

$$[\, T^a , T^b \,] = i\, f^{abc}\, T^c$$

where the f^{abc} are the structure constants. These generators of this SU(3) Lie algebra can be represented by the Gell-Mann matrices with $T^a = \lambda^a/2$. Then we can construct the Lagrangian density for QCD with massless quarks

$$L = i\, \bar{\psi}_i\, \gamma_\mu\, D^\mu_{ij}\, \psi_j - 1/4\, F^{a\mu\nu}\, F^a_{\mu\nu}$$

where the covariant derivative is

$$D^\mu_{ij} = \delta_{ij}\, \partial^\mu - i\, g\, T^a_{ij}\, A^{a\mu}$$

and the field strength tensor is

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\, f^{abc}\, A^{b\mu} A^{c\nu}$$

with g the bare coupling. This lagrangian is invariant under the set of transformations

$$\psi_i \rightarrow \psi_i - i\, g\, T^a_{ij}\, \psi_j\, \varepsilon^a$$

$$\bar{\psi}_i \rightarrow \bar{\psi}_i + i\, g\, \bar{\psi}_i\, T^a_{ji}\, \varepsilon^a$$

$$A^{a\mu} \rightarrow A^{a\mu} + g\, f^{abc}\, \varepsilon^{b}\, A^{c\mu} - \partial^{\mu}\, \varepsilon^{a}$$

where ε^a is a small quantity dependent upon position. This change is a local gauge transformation and the invariance of the lagrangian under such a transformation is, as we shall see crucial not only in making physical quantites finite, but also in explaining confinement.

This new colour force is then supposed to explain the nuclear force as merely a Van der Waals type residual interaction. Since the nuclear force is characterised by a coupling strength of $g^2_{\pi NN}/4\pi \approx 14$ and the colour force is even stronger, it might seem that we have exchanged a difficult problem for an even harder one. The resolution of this quandary resides in the observation that, from deep inelastic scattering experiments, the partons appear to be free within hadrons. That is, for interactions with a large momentum transfer the coupling between the partons is small. The hope is then, that in calculating such interactions a perturbative expansion, like the one used for QED, will be valid.

In calculating the quantum correction to the classical theory we encounter infinities even in the first order one loop calculations. These infinities have to be removed by regularising, ie. introducing a cutoff in divergent integrals. Then the vertices (Green's functions) for these bare quantities depend upon the regularisers. This dependence can be removed by renormalising

$$\Gamma(p,g,\mu) = Z\, \Gamma_B(p,g_B,\mu)$$

where μ is the scale at which the parameters of the theory are defined. Now the bare Green's function is independent of the scale μ. This means that

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g) \right\} \Gamma(p,g,\mu) = 0$$

where

$$\beta(g) = \mu \frac{\partial}{\partial \mu} g \qquad \text{and} \qquad \gamma(g) = \mu \frac{\partial}{\partial \mu} \ln Z$$

which is the so called renormalization group equation [1.12]. In general μ represents an infinite set of parameters.

We can calculate the β function perturbatively which enables us to write the coupling constant as

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2} \qquad \text{where} \quad \alpha_s = \frac{g^2}{4\pi}$$

and

$$\beta_0 = \frac{11}{3} C_2(A) - \frac{4}{3} N_f \; T_2(F)$$

Now for QCD the colour Casimirs are $C_2(A) = 3$ and $T_2(F) = 1/2$, thus for the number of flavours $N_f < 17$, the β function is negative and the coupling decreases as the momentum Q increases. This phenomenon of asymptotic freedom, first shown by Politzer and Gross and Wilczek, [1.13] is the reason why the parton model works and why we can make a perturbative expansion for QCD.

The success of perturbative QCD in explaining short range phenomena such as scaling violations, jet cross-sections and the like, even upto the energies of the SPS collider (≈600 GeV) is really amazing. This is despite the fact that no calculations have been done beyond two loops. Indeed the only draw back to perturbative QCD is the fact that it fails to explain the behaviour of quarks over large distances. It is fine for calculating hard interactions between the partons but to explain the results in terms of the observed hadrons we have to fall back on ad hoc models of hadronization.

1:3 Confinement

We have seen how the parton model can explain the existence and quantum numbers of hadrons, although not their spectrum in detail. Also how QCD with its property of asymptotic freedom can explain the gamut of short distance processes, in particular deep inelastic scattering. However, we have yet to see how to explain the most fundamental of all experimental observations: how quarks, and gluons, despite their apparent freedom, are confined within hadrons.

In order to exhibit the effect of confinement at its simplest, let us specialise to the case of heavy quark systems. For these a non-relativistic approximation is valid and we can think of the quark and the anti-quark as being in a static potential with their energy levels given by the Schrödinger equation. At short distances, this potential is generated by the one gluon exchange, which gives a Coulomb-like force modified by the logarithmic divergences of asymptotic freedom. Thus when $r \to 0$

$$V(r) = -\frac{\alpha_s(r)}{r}$$

where $\alpha_s(r) = 2\pi/\beta_0 \ln(1/r\Lambda)$ in zeroth order. Such a Coulomb potential is of course not confining. The spectrum of hidden charm $(c\bar{c})$ and hidden beauty $(b\bar{b})$ states confirms that at larger distances the potential becomes increasingly positive. This might lead us to believe that as $r \to \infty$

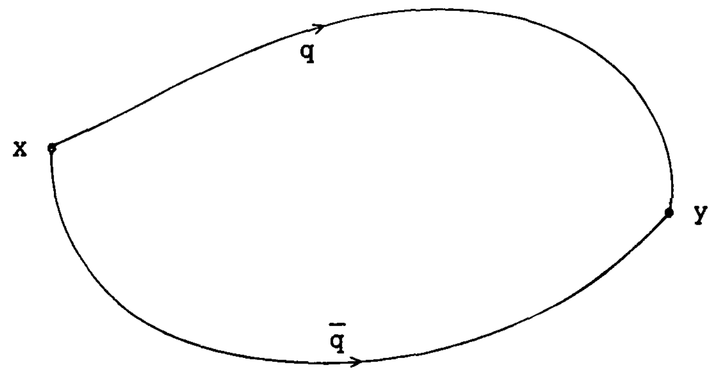
$$V(r) \to \kappa r$$

where κ is called the "string tension". This can be understood in a simple minded picture: imagine that these heavy quarks, Q and $\bar{Q}$ are tied to the two ends of a string, which represents the gluon flux tube. When the quarks are close together, the string is slack and the quarks behave as though they are free. As the quarks move apart the string becomes taut, and an increasing amount of work most be done to separate them further. The amount of energy expended for an infinitesimal stretching dr is just κdr. The question then arises of how to formulate this idea in a field-theoretic manner, in particular for a non-abelian gauge theory.

In a classic paper, Wilson [1.14] gave such a criterion for the confinement of quarks and it is useful to repeat his arguments here. Consider the current-current propagator

$$D_{\mu\nu}(y-x) = \langle \Omega \mid T(J_\mu(y) J_\nu(x)) \mid \Omega \rangle$$

the Fourier transform of which determines the e^+e^- cross-section for annihilation into hadrons. Let us assume that the currents, $J_\mu(y)$, are built from the quark fields and that these interact through the medium of a gauge field. In the Feynman path-integral picture the propagator $D_{\mu\nu}(y-x)$ is given by the weighted integral over all possible quark paths and values of the gauge field. The currents $J_\mu(y)$ and $J_\nu(x)$ are thought of as producing a quark anti-quark pair at the point x, which later annihilate at the point y, and one sums over all possible intermediate paths joining x and y for both the quark and the anti-quark.



Of course the vacuum can also emit and absorb $q\bar{q}$ pairs, so that closed current loops can occur which are unconnected to x and y, and, in principle, these loops must also be summed over. The weight associated with a given path includes a factor

$$\exp \left[i \, g \oint ds^{\mu} A^{a}_{\mu} \lambda^{a} \right]$$

where the λ^{a} are the Gell-Mann matrices. It is the expectation value of this, so-called Wilson loop factor, that will be a measure of confinement. The idea is that confinement should not allow the quark and the anti-quark to separate beyond some finite size, typically of the order of 1 fermi, and this should result in some characteristic behaviour in the loop integrals. To make the problem tractable, Wilson assumes that the vacuum loops are not important.

Then to make it possible to compute the path integrals in an analytic way, Wilson defines the theory at a discrete set of space-time points that make up a lattice. In such a formulation the dynamical gauge variables are associated with oriented links between neighbouring vertices i and j, rather than the vertices themselves, and are finite elements U_{ij} of the gauge group. Under a gauge transformation, the matter fields, φ_i, defined at the lattice site i, is changed by $\varphi_i \to G_i \varphi_i$, whereas the dynamical variables satisfying $U_{ij} = U^{-1}_{ji}$ transform as $U_{ji} \to G_j U_{ji} G^{-1}_i$, so that scalar product, $\varphi^{\dagger}_j U_{ji} \varphi_i$ is gauge invariant.

In the continuum theory, the transport along the path Γ is defined by a path-ordered exponentiated line integral of $A^a_\mu(x)$:

$$U_\Gamma = P \exp \left\{ i g \int_\Gamma dx^\mu A^a_\mu \lambda^a \right\}$$

If on the lattice, the path goes through a sequence of neighbouring sites 1,2,...,N, then the corresponding transport operator is,

$$U_\Gamma = U_{NN-1} \cdots U_{32} U_{12}$$

and under a gauge transformation $U_\Gamma \to G_N U_\Gamma G^{-1}_1$. So for a closed loop, site 1 and site N are the same and then it follows that the quantity $W = \mathrm{Tr}\, U_\Gamma$ is gauge invariant. W is, of course, just the lattice form of the Wilson loop factor.

If we consider the simplest closed path as the perimeter of a square having four sequentially neighbouring sites, 1-4, then we can define the corresponding transport operator around such a plaquette

$$U_P = U_{14} U_{43} U_{32} U_{12}$$

From this operator we can simply define an action on the lattice as a sum over all possible plaquettes, which for SU(2) is,

Confinement

$$S = \frac{4}{g^2} \sum_P \left(1 - 1/2 \, \mathrm{Tr} \, U_P \right)$$

where the factor $4/g^2$ is inserted to make the continuum limit of S just

$$\frac{1}{4} \int d^4x \, F^a_{\mu\nu} \, F^{a\,\mu\nu}$$

Such an S is called the Wilson lattice gauge action.

Confinement is then controlled by the behaviour of the expectation value of the Wilson loop factors, W, over larger and larger loops. Thus we compute:

$$\langle W \rangle = \int d[A] \, W[A] \, \exp(-S) \ / \int d[A] \, \exp(-S)$$

Now imagine computing this over a rectangular path with m sites in the time direction and n site in the space direction. Then if m >> n, ie. the time interval is very long, so we can regard this as the static limit, then $\langle W \rangle$ is related to the potential energy V(r) of a $q\bar{q}$ pair separated by a distance r:

$$\langle W \rangle = \exp\left[- T \, V(r) \right]$$

If a is the lattice spacing, then for a rectangular path T = m a, and r = n a, so that if we imagine that at large distances r, $V(r) \approx \kappa r$, with κ the string tension, then we expect that

$$\langle W \rangle \approx \exp\left(- \kappa A \right)$$

where A is the area enclosed by the rectangular contour, ie. $m \, n \, a^2$.

In fact, this result generalises to any shape of contour. Thus a linear confining potential would be related to an exponential damping of the contribution from large area loops to the Wilson loop function.

To check whether this happens or not, Wilson attempts to calculate $\langle W\rangle$ in the limit of strong coupling, which is appropriate for quark binding. In this limit, Wilson shows that for a given path in the lattice, the lowest non-zero order contribution is a product of contributions from squares which fill the contour. Thus if A is the minimal area enclosed by the path then

$$\langle W\rangle \approx (g^2)^{A/a^2} = \exp [- A/a^2 \ln g^2].$$

So that loops of larger area are indeed exponentially damped, and consequently there is no probability that a quark and an anti-quark become macroscopically separated. For $g \to \infty$, this model confines. This conclusion can be rigorously shown not to be spoiled by higher order corrections.

This seems to be a beautiful result, but unfortunately there is a problem. Hasenfratz has remarked that in the limit $g \to \infty$ this model has little to do with continuum QCD. "The coupling g should be changed towards $g = 0$, where the continuum limit is to be found. It is a long way to go and we might meet surprises. If we want an asymptotically free, confining theory at the end, a deconfining phase transition must not be among them." [1.15]. Thus since the pioneering paper of Wilson, there have been many attempts to understand the relation of the $g \to \infty$ and the $g \to 0$ limits and even to calculate the exponential in the area law and so obtain a precise string tension. We will have occasion to comment briefly on these values after we have obtained our results

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(section 7 of chapter 4)

Though the Wilson loop provides an elegant field theoretic criterion for confinement, it is useful to seek some other condition which may also be amenable to calculation. If we return to the non-relativistic potential approximation we discussed earlier, then we may note that the potential V(r) is just the Fourier transform of the time-time component of the boson propagator in momentum space. Thus if $V(r) \approx r^n$, then $\Delta_{00}(q) \approx q^{-n-3}$. We see immediately that Coulomb's law with $n = -1$ corresponds to a standard $1/q^2$ propagator. This is what we expect to be the behaviour of the gluon at large momentum. In contrast, a confining potential at large distances has $n > 0$, which requires a more singular gluon propagator as q becomes small. Indeed for a linear potential, ie $n = 1$, the gluon propagator must have an enhanced singularity of the form $1/q^4$ at small momenta. With this in mind we will pursue the study of the gluon propagator and the consequences this has for the quark propagator.

Chapter 2

The Schwinger-Dyson Equations

and the

Slavnov-Taylor Identities

2:1 Introduction

In the last section of the first chapter we saw how we expect that, if QCD is to be confining, the gluon propagator should be more singular than the photon, which has a singularity of $1/q^2$. To see how this enhancement of the gluon comes about we need an equation valid at large distances. The Schwinger-Dyson equations [2.1] provide such a non-perturbative method of studying the Green's functions of QCD. Unfortunately these equations come in the form of an infinite set of coupled equations. To reduce them to a finite set, we have to make some simplifying assumptions, and it is here that the Slavnov-Taylor-Ward-Takahashi [2.2] identities come in useful.

As has been found over the past decade, the Green's functions can be adequately described by first order perturbative calculations at short distances of less than about a tenth of a fermi, which corresponds to momenta greater than a few GeV. At longer distances the perturbative approach breaks down, as can be seen by the Landau pole in the running coupling, and it is this region that we wish to investigate. We must not however go to too small a distance for then the energy in the colour field becomes great enough for the creation of real (on shell) quark bound states (hadrons) to take place. That is, couched in terms of the string model, the string breaks. This phenomenon of hadronization is not explicitly included in the Schwinger-Dyson equations and so we must avoid the region of pion formation. We cannot expect to solve the Schwinger-Dyson equations, even in a truncated form, analytically over the entire spectrum of momenta and we are forced to the use of numerical methods. Even so we found that attempts to solve simplified equations by an iterative technique to be numerically unstable because of the non-linearity of the problem. We are then led to the proposal that the gluon and quark propagators be suitably parametrized and the value of the parameters determined by the requirement that the function we put in is then as close as possible to the one we get out, over a range of momenta. It is in this respect of requiring numerical consistency over a finite range momenta that this work differs in principle from that of others, who have been more concerned with the exact analytic form when $q \to 0$.

The Schwinger-Dyson equations for the gluon propagator have been previously studied by Mandelstam [2.3] in the covariant gauge and in a series of papers by Baker, Ball and Zachariasen [2.4] in the axial gauge. We will follow Mandelstam in working in the covariant gauge but we shall work in Euclidean space. Moreover we shall investigate the effect the enhanced gluon propagator has on the propagation of quarks at short distances.

In the rest of this chapter we will consider the form of the Schwinger-Dyson equations considering the gluon equations for the sake of definiteness and what the Slavnov-Taylor identities have to tell us about how we can approximate the equations following the example of Baker, Ball and Zachariasen.

2:2 The Schwinger-Dyson Equation

In this section we consider the Schwinger-Dyson equations and for the sake of definiteness and simplicity we will discuss the equations for the gluon vertices ignoring the effect of any gauge fixing. Physically we can see how the equation for the gluon propagator, or rather it´s inverse, comes about by considering the perturbative expansion

$$\Delta^{\mu\nu} = \Delta_\circ^{\mu\nu} + \Delta_\circ^{\mu\sigma}\, \Sigma_{\sigma\tau}\, \Delta_\circ^{\tau\nu} + \ldots$$

where $\Delta^{\mu\nu}$ and $\Delta_\circ^{\mu\nu}$ are the full and bare propagators respectively and $\Sigma_{\sigma\tau}$ is the sum of the one particle irreducible graphs, the vacuum polarization tensor. This can be resummed as

$$\Delta^{\mu\nu} = \Delta_\circ^{\mu\nu} + \Delta_\circ^{\mu\sigma}\, \Sigma_{\sigma\tau}\, \Delta^{\tau\nu} \tag{2.1}$$

which is illustrated graphically in figure (2.1). We can see how all the possible graphs have been absorbed into the full vertices and propagators. Introducing the inverse of the propagator $\Pi^{\mu\nu}$ defined by

$$\Pi^{\mu\sigma}\, \Delta_{\sigma\nu} = \delta^{\mu}{}_{\nu}$$

we can write equation (2.1) as

$$\Pi^{\mu\nu} = \Pi_\circ^{\mu\nu} - \Sigma^{\mu\nu} \tag{2.2}$$

We can derive this result in a more formal manner by considering the action [2.5]

$$S[A] = \int d^4x \left\{ -\frac{1}{4} F^a_{\mu\nu}(x)\, F^{a\,\mu\nu}(x) - J^a_\mu\, A^{a\,\mu}(x) \right\} \tag{2.3}$$

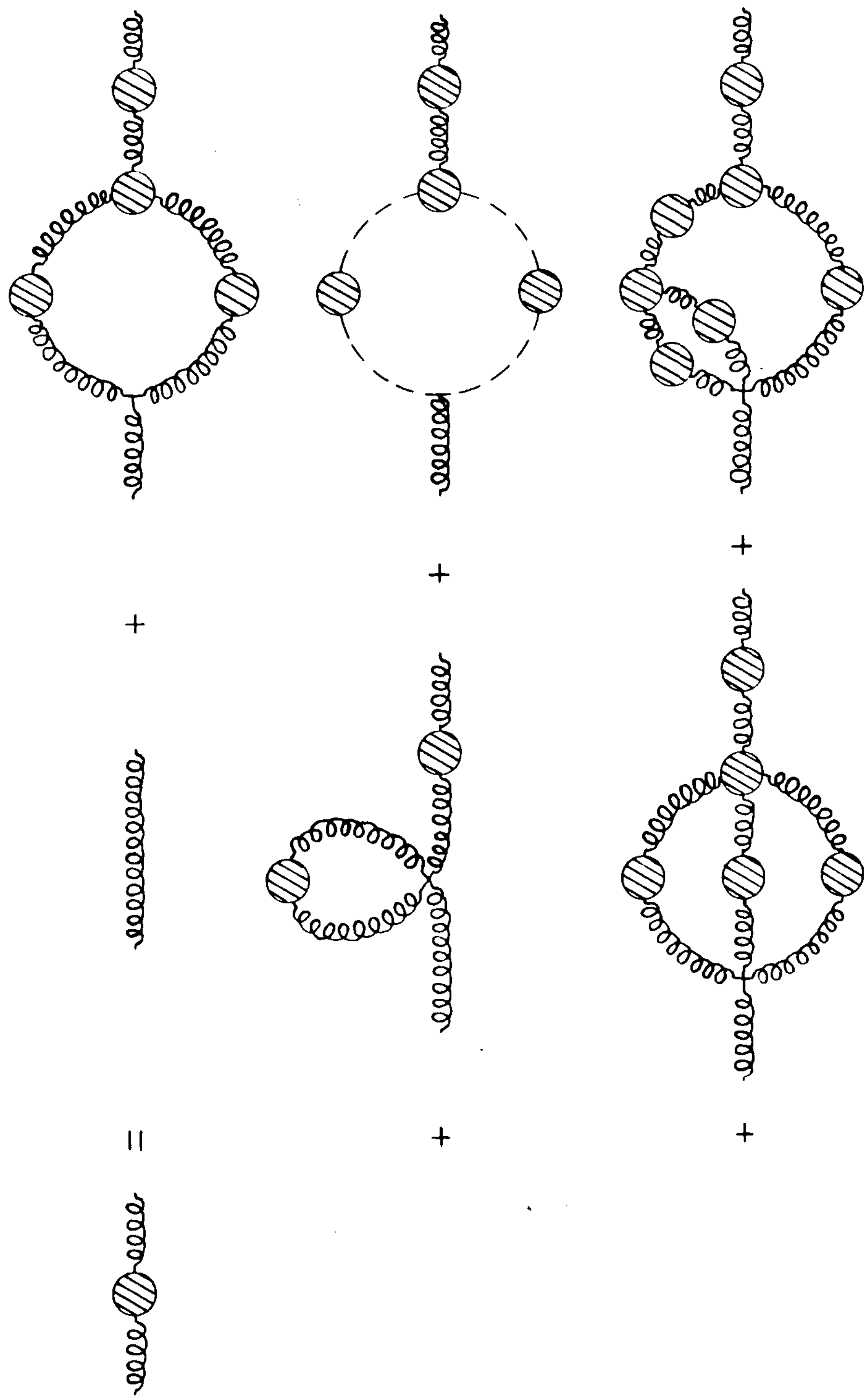

Figure 2.1

The Schwinger-Dyson equation for the gluon propagator in the covariant gauge

Then the Green's functions are the functional averages weighted by the exponential of the action

$$\langle A^{a1}_{\mu 1}(x_1)\ldots A^{aN}_{\mu N}(x_N) \rangle = \frac{1}{Z}\int D[A^a_\mu]\; e^{iS}\; A^{a1}_{\mu 1}(x_1)\ldots A^{aN}_{\mu N}(x_N) \qquad (2.4)$$

where Z is the generating function defined by

$$Z = \int D[A^a_\mu]\; e^{iS} \qquad (2.5)$$

We see then that the presence of the source $J^a_\mu(x)$ in the action allows us to write the Green's functions as functional derivatives of the generating function

$$\langle A^{a1}_{\mu 1}(x_1)\ldots A^{aN}_{\mu N}(x_N) \rangle = \frac{1}{Z}\,\frac{\delta^n\; Z}{\delta(-iJ^{a1}_{\mu 1}(x_1))\ldots\delta(-iJ^{aN}_{\mu N}(x_N))} \qquad (2.6)$$

The two point Green's function is then just the second derivative of the generating function. However, these Green's functions contain disconnected pieces which we can remove by considering the logarithm of the generating function $W[J] = \ln Z[J]$

$$\langle A^{a1}_{\mu 1}(x_1)\ldots A^{aN}_{\mu N}(x_N) \rangle_C = \frac{\delta^n\; W}{\delta(-iJ^{a1}_{\mu 1}(x_1))\ldots\delta(-iJ^{aN}_{\mu N}(x_N))} \qquad (2.7)$$

and hence the propagator is

$$\Delta^{ab}_{\mu\nu}(x,y) = \frac{\delta^2\; W[J]}{\delta(-iJ^a_\mu(x))\;\delta(-iJ^b_\nu(y))}$$

$$= \frac{\delta \langle A^a_\mu(x) \rangle}{\delta(-iJ^b_\nu(y))} \tag{2.8}$$

thus the propagator is related to the Green's function by

$$\langle A^a_\mu(x)\ A^b_\nu(y) \rangle = \Delta^{ab}_{\mu\nu}(x,y) + \langle A^a_\mu(x) \rangle \langle A^b_\nu(y) \rangle \tag{2.9}$$

The three point Green's function can then be written as, using equations (2.6), (2.8) and (2.9),

$$\begin{aligned} \langle A^a_\mu(x)\ A^b_\nu(y)\ A^c_\sigma(z) \rangle = {} & \frac{\delta \Delta^{ab}_{\mu\nu}(x,y)}{\delta(-iJ^c_\sigma(z))} + \Delta^{ab}_{\mu\nu}(x,y) \langle A^c_\sigma(z) \rangle + \\ & + \Delta^{bc}_{\nu\sigma}(y,z) \langle A^a_\mu(x) \rangle + \Delta^{ca}_{\sigma\mu}(z,x) \langle A^b_\nu(y) \rangle + \\ & + \langle A^a_\mu(x) \rangle \langle A^b_\nu(y) \rangle \langle A^c_\sigma(z) \rangle \end{aligned} \tag{2.10}$$

where the first term on the right hand side is the connected three point Green's function (see equations (2.7) and (2.8)). Notice that the connected Green's function is just the Green's function evaluated at $\langle A \rangle = 0$.

Now let us consider a variation in the generating functional (2.5) of $A^a_\mu(x) \to A^a_\mu(x) + \delta A^a_\mu(x)$, then the requirement that the generating function is invariant to first order yields the equation of motion

$$- \langle\ D^{ab}_\nu\ F^b_{\mu\nu}(x)\ \rangle = J^a_\mu(x) \tag{2.11}$$

where $D^{ab}_\nu = \partial_\mu\ \delta^{ab} + g\ f^{abc}\ A^a_\mu(x)$ is the covariant derivative. Expanding the derivative and the field strength tensor we get, using equations (2.9) and (2.10),

$$(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\langle A^a_\nu(x)\rangle - g/2\,\Gamma^{\circ abc}_{\mu\nu\sigma}(x,y,z)\,[\Delta^{bc}_{\nu\sigma}(y,z) + \langle A^b_\nu(y)\rangle\langle A^c_\sigma(z)\rangle]$$

$$-g^2/6\,\Gamma^{\circ abcd}_{\mu\nu\sigma\tau}(x,y,z,w)\left[\frac{\delta\Delta^{bc}_{\nu\sigma}(y,z)}{\delta(-iJ^d_\tau(z))} + 3\,\Delta^{ca}_{\sigma\mu}(z,x)\,\langle A^b_\nu(y)\rangle + \right.$$

$$\left. + \langle A^b_\nu(y)\rangle\,\langle A^c_\sigma(z)\rangle\,\langle A^d_\tau(w)\rangle\right] = J^a_\mu(x) \qquad (2.12)$$

where

$$\Gamma^{\circ abc}_{\mu\nu\sigma}(x,y,z) = f^{abc}\left\{\delta_{\mu\nu}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]_\sigma \delta(x-z)\delta(y-z) + \right.$$

$$\left. + \text{cyclic permutations}\right\}$$

and

$$\Gamma^{\circ abcd}_{\mu\nu\sigma\tau}(x,y,z,w) = \left\{ f^{abe}\,f^{ecd}\,(\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\nu\sigma}\delta_{\mu\tau}) + \right.$$

$$\left. + \text{cyclic permutations}\right\}$$

We can define the response of the current to changes in the vacuum field to be

$$\Pi^{ab}_{\mu\nu}(x,y,\langle A\rangle) = -i\,\frac{\delta J^a_\mu(x)}{\delta\langle A^b_\nu(y)\rangle} \qquad (2.13)$$

which is just the inverse of the gluon propagator (2.8) when evaluated at $\langle A\rangle = 0$. Similarly the truncated N point Green's function (ie. that without external legs) can be obtained by multiplying the connected Green's function by the appropriate number of inverse propagators. Using equation (2.13), it is easy to see that the differentiation of equation (2.12) with respect to the vacuum field $\langle A\rangle$ gives an equation for the inverse propagator.

$$\Pi^{ab}_{\mu\nu}(x,y) = \Pi^{\circ ab}_{\mu\nu}(x,y) - \Sigma^{ab}_{\mu\nu}(x,y) \tag{2.14}$$

and

$$\Sigma^{ab}_{\mu\nu}(x,y) = -\frac{g}{2}\, i\Gamma^{\circ acd}_{\mu\nu\sigma}(x,y,z) \left.\frac{\delta\, \Delta^{cd}_{\nu\sigma}(y,z)}{\delta\langle A^{b}_{\nu}(y)\rangle}\right|_{\langle A\rangle=0}$$

$$- \frac{g^2}{2}\, i\Gamma^{\circ abcd}_{\mu\nu\sigma\tau}(x,y,z,w)\, \Delta^{cd}_{\nu\sigma}(y,z)$$

$$- \frac{g^2}{6}\, i\Gamma^{\circ abcd}_{\mu\nu\sigma\tau}(x,y,z,w) \left.\frac{\delta}{\delta\langle A^{b}_{\nu}(y)\rangle} \frac{\delta\Delta^{bc}_{\nu\sigma}(y,z)}{\delta(-iJ^{d}_{\tau}(z))}\right|_{\langle A\rangle=0}$$

From the definition of the inverse propagator (2.13) and the chain rule we see that this is just the Schwinger-Dyson equation for the gluon propagator. The higher order equations are given by repeated differentiation of (2.12), for example the equation for the three point gluon vertex is obtained from

$$g\, \Gamma^{acd}_{\mu\nu\sigma}(x,y,z) = -i \left.\frac{\delta\, \Pi^{ab}_{\mu\nu}(x,y,\langle A\rangle)}{\delta\langle A^{c}_{\sigma}(z)\rangle}\right|_{\langle A\rangle=0}$$

continuing in this vain we can generate an infinite set of coupled equations for each $\Gamma^{(N)}$ in terms of the Green´s functions upto $\Gamma^{(N+2)}$ which we can write symbolically as

$$\Gamma^{(N)} = G^{(N)}(\, \Gamma^{(2)}, \ldots, \Gamma^{(N+2)}\,).$$

2:3 The Slavnov-Taylor Identites

The gauge invariance of the QCD can be summarised in the equation for current conservation

$$\partial^{\mu} J^{a}_{\mu}(x) + g f^{abc} \langle A^{b}_{\mu}(x)\rangle J^{c\mu}(x) = 0 \tag{3.1}$$

where the current is defined in terms of the vacuum field by the Schwinger-Dyson equation (2.12). If any approximation scheme is to respect the gauge invariance of the theory, equation (3.1) must remain valid. Differentiating equation (3.1) with respect to the vacuum field $\langle A^{b}_{\nu}(y)\rangle$ yields the result

$$\frac{\partial}{\partial x_{\mu}} \Pi^{ab}_{\mu\nu}(x,y) + g f^{acd} \langle A^{c\mu}(x)\rangle \Pi^{db}_{\mu\nu}(x,y) = i g f^{abc} J^{c}_{\nu}(x) \delta(x-y). \tag{3.2}$$

From this equation we can obtain the Slavnov-Taylor identities for the N point Green's function by differentiating N-2 times with respect to the vacuum field $\langle A\rangle$ and evaluating at $\langle A\rangle = 0$. Thus the the Ward identity for the propagator is

$$\frac{\partial}{\partial x_{\mu}} \Pi^{ab}_{\mu\nu}(x,y) \Big|_{\langle A\rangle=0} = 0 \tag{3.3}$$

So transforming into momentum space we have the familiar transversality condition

$$p^{\mu} \Pi^{ab}_{\mu\nu}(p) = 0 \tag{3.4}$$

For the triple gluon vertex we get the result

$$\frac{\partial}{\partial x_\mu} \Gamma^{abc}_{\mu\nu\sigma}(x,y,z) = g\, f^{cdb}\, \Pi^{ad}_{\nu\sigma}(y,z)\, \delta(x-z) + g\, f^{cda}\, \Pi^{db}_{\nu\sigma}(y,z)\, \delta(x-y) \tag{3.5}$$

which in momentum space we write as

$$p^\mu\, \Gamma^{abc}_{\mu\nu\sigma}(p,q,r) = g\, f^{abc}\, (\, \Pi_{\nu\sigma}(q) - \Pi_{\nu\sigma}(r)\,) \tag{3.6}$$

where we have used $\Delta^{ab}_{\nu\sigma} = \delta^{ab}\, \Delta_{\nu\sigma}$. In the covariant gauge this result is complicated by the ghost-gluon vertex which multiplies the propagators. However in the approximation where the ghost takes on its bare value the identity reduces to this form.

The Slavnov-Taylor identities then constrain the N point Green's function to be an antisymmetric linear combination of the N-1 point Green's function when differentiated with respect to one of it's arguments. This means that each vertex has a leading behaviour one order of momentum lower than the previous vertex. So, as Baker, Ball and Zachariasen have pointed out [2.6], if the propagator behaves like $1/q^4$ the six point vertex will be independent of momenta. Thus the six point and all higher vertices will be purely transverse, that is they will vanish when contracted with any of their arguments.

Now the Schwinger-Dyson equation (2.12) satisfies the current conservation equation (3.1) so that the equations resulting from (2.12) will yield a solution for $\Gamma^{(N)}$ which satisfies the appropriate Slavnov-Taylor identities, provided the Green's function appearing on the right hand side satisfy their identities. Moreover, if we neglect the terms in equation (2.12) that contain $\delta\Delta/\delta J$ the resulting equation still satisfies (3.1). This means that the Schwinger-Dyson equations can be written symbolically as

$$\Gamma^{(N)} = G_1^{(N)}(\Gamma^{(2)},\ldots,\Gamma^{(N+1)}) + G_2^{(N)}(\Gamma^{(2)},\ldots,\Gamma^{(N+2)}) \tag{3.7}$$

where all the terms coming from $\delta\Delta/\delta J$ have been collected in to the function G_2. Notice that the only dependence on $\Gamma^{(N+2)}$ is now in G_2. This means that the truncated equation

$$\Gamma^{(N)} = G_1^{(N)}(\Gamma^{(2)},\ldots,\Gamma^{(N+1)}) \tag{3.8}$$

is independent of $\Gamma^{(N+2)}$ and the resulting Green's function will still satisfy the Slavnov-Taylor identity. The above statements are, of course, subject to the proviso that in calculating the Green's function any regularization and renormalization respect the current conservation equation.

2:4 A Consistent Approximation Scheme

In order to calculate the Green's function from the Schwinger-Dyson equations, we must first cast the equations in a closed form. The truncation of the equation to the form (3.8) goes some way to achieving this objective. Just because this truncation yields a solution, which satisfies the Slavnov-Taylor identity, does not mean that it is necessarily a sensible approximation. However, if we choose to renormalise the Schwinger-Dyson equation at some momentum scale, R^2 say, which is large, then the coupling constant there will be small. Thus we can choose an R^2 such that the one loop contribution in G_1 will dominate over the two loop contributions in G_2 in the same way as we do in perturbation theory. Notice that the coupling constant is fixed and depends only upon the renormalization scale R^2. This does not mean to say that the coupling is uniquely specified, as for each vertex $\Gamma^{(N)}$ we have to introduce at least one new parameter R_N^2 to specify the renormalization. Since the number of Schwinger-Dyson equations is infinite the coupling depends upon an infinite set of parameters in the same way as it does in perturbation theory. The only difference between this Schwinger-Dyson approach and the perturbative one is that the contributions have been summed in a more appropriate way to the evaluation of the vertices at small momenta.

Thus if we knew what the Green's function $\Gamma^{(N+1)}$ was in terms of the lower Green's functions, the truncated Schwinger-Dyson equation (3.8), together with its subsidiary equations for $\Gamma^{(2)}$ to $\Gamma^{(N-1)}$, would form a closed set of equations. Unfortunately, we do not know $\Gamma^{(N+1)}$, but the Slavnov-Taylor identities do give us some handle upon its form. The vertex can be split into two parts, a longitudinal part and a transverse part

$$\Gamma^{(N+1)} = \Gamma_L^{(N+1)} + \Gamma_T^{(N+1)}$$

where the longitudinal part satisfies the Slavnov-Taylor identity and the transverse part is unconstrained, as it vanishes when contracted with any one of its momenta. In general, this separation is not unique, since an arbitary amount of the transverse part can be included in the longitudinal part. However, if we demand that the vertex is free of kinematic singularities (1/(p.q), etc.) then the longitudinal part is uniquely determined [2.7]. Furthermore the longitudinal part will dominate in the infra-red region. This is true in QCD and QED because of the spinology of the vertex. For example, if we consider the Boson-Fermion vertex, then the longitudinal part has the tensor form γ^{μ}, whereas the transverse part has the tensor form $q^{\mu}\sigma^{\mu\nu}$ and so is an order of momenta higher than the longitudinal part. Thus, in general, we have that

$$\Gamma^{(N)} \rightarrow \Gamma_{L}^{(N)} \quad \text{as} \quad p_i/p_j \rightarrow 0.$$

Thus in the infra-red limit the vertex $\Gamma^{(N+1)}$ is dominated by its longitudinal part, defined by the Slavnov-Taylor identity and the constraint that it be free of kinematic singularities, which we denote by $F(\Gamma^{(N)})$.

The replacement of the vertex $\Gamma^{(N+1)}$ by $F(\Gamma^{(N)})$ in the truncated Schwinger-Dyson equation (3.8)

$$\Gamma^{(N)} = G_1^{(N)}(\Gamma^{(2)}, \ldots, F(\Gamma^{(N)})) \tag{4.1}$$

makes this equation together with its subsidiary equations form a closed set. The central assumption is then that this set of equations will yield the correct behaviour for $\Gamma^{(N)}$ in the infra-red limit. As we have seen above the longitudinal part of the vertex does indeed dominate in the infra-red limit. But this is not sufficient because the structure of the Schwinger-Dyson equations means that the value of the vertex at any point depends (perhaps only weakly) on its value at all other points. Stated like this it would seem that the situation is hopeless. But if the value of the vertex is dominated by its value at nearby points our assertion may indeed be true. We can also find support in calculations for QED, where this assumption has been investigated and been found to be true [2.8]. In the case of QCD, calculations to one loop have been performed which show that the dominant behaviour in the infra-red limit is indeed given by the longitudinal component of the vertex [2.9].

The Schwinger-Dyson Equations

So we have an equation (4.1) that gives the dominant infra-red behaviour of the vertex $\Gamma^{(N)}$ in terms of the vertices $\Gamma^{(M)}$ $2 \leqslant M \leqslant N$. This allows us to adopt a step by step approach to constructing any vertex $\Gamma^{(N)}$. For instance consider the inverse propagator, then from our approximation we have that

$$\Gamma^{(2)} = G_1^{(2)}(\Gamma^{(2)}, F(\Gamma^{(2)})) \tag{4.2}$$

This first approximation gives the dominant transverse component of the inverse propagator (the roles of the transverse and longitudinal components are swopped over for the propagator as opposed to the vertices). The next approximation would involve solving the coupled equations for the inverse propagator and the triple gluon vertex

$$\begin{aligned} \Gamma^{(2)} &= G_1^{(2)}(\Gamma^{(2)}, \Gamma^{(3)}) + G_2^{(2)}(\Gamma^{(2)}, \Gamma^{(3)}, F(\Gamma^{(3)})) \\ \Gamma^{(3)} &= G_1^{(3)}(\Gamma^{(2)}, \Gamma^{(3)}, F(\Gamma^{(3)})) \end{aligned} \tag{4.3}$$

Where we can now use the full Schwinger-Dyson equation for the inverse propagator, as the longitudinal component of the four point gluon vertex is given by $F(\Gamma^{(3)})$. This gives the corrections to the transverse part of the propagator as well as the previously neglected longitudinal part. This means that in principle we can check the validity of our assumption that the form of the dominant transverse part of the propagator is given by equation (4.2). We can also check the effects of neglecting the two loop contributions in $G_2^{(2)}$.

Before we get carried away, we must realise that we should now have included the effects of the ghost and dynamical quarks. We should then have an intermediate stage between equations (4.2) and (4.3) where we solve the coupled equations

$$\Gamma_g^{(2)} = G_1^{(2)}(\ \Gamma_g^{(2)},\ F(\Gamma_g^{(2)}),\ \Gamma_q^{(2)},\ F(\Gamma_q^{(2)}),\ \Gamma_h^{(2)},\ F(\Gamma_h^{(2)})\ )$$

$$\Gamma_q^{(2)} = Q^{(2)}(\ \Gamma_q^{(2)},\ \Gamma_g^{(2)},\ F(\Gamma_q^{(2)})\ ) \qquad (4.4)$$

$$\Gamma_h^{(2)} = H^{(2)}(\ \Gamma_h^{(2)},\ \Gamma_g^{(2)},\ F(\Gamma_h^{(2)})\ )$$

where $\Gamma_q^{(2)}$ and $\Gamma_h^{(2)}$ are the quark and ghost propagators respectively and $Q^{(2)}$ and $H^{(2)}$ represent the appropriate Schwinger-Dyson equations.

We see that going beyond the first approximation the complexity of the equations increases dramatically. In the rest of this thesis we report on the present status of a continuing programme in which we attempt to find self-consistent solutions to the Schwinger-Dyson equations for the quark and gluon propagators. We shall calculate the infra-red behaviour of the gluon propagator in the general covariant gauge. To do this we shall use an approximation for the triple gluon vertex proposed by S.Mandelstam [2.3] which although it does not satisfy the Slavnov-Taylor identity does have the correct qualitative behaviour and is much simpler than the general solution. Using the result for the gluon propagator, we shall then look at the equation for the quark propagator where we use the form of the quark-gluon vertex determined from the Ward-Takahashi identity. Attempts to go beyond the Mandelstam approximation, in the covariant gauge, have not proved tractable. Moreover, in this thesis, we shall not attempt to solve the quark and the gluon equations as a coupled system.

Chapter 3

The Gluon Propagator

3:1 Introduction

The general belief that QCD gives rise to a linearly confining potential implies that the form of the gluon propagator at low momentum is very different from it's bare value. In fact for the gluon to produce a linear potential means (by Fourier transformation from real space to momentum space) that the propagator must be enhanced so as to have a pole of order $1/q^4$ as opposed to the bare pole of $1/q^2$ which gives rise to a Coulomb potential.

The Schwinger-Dyson equations provide a means of testing whether we can form a propagator exhibiting this enhanced behaviour which is self-consistent in the sense that it satisfies the equation. Unfortunately, the Schwinger-Dyson equation does not come in a closed form as it involves unknown vertex functions. To render the equation closed, as discussed in the previous chapter, we must use a combination of a gauge invariant truncation and the Slavnov-Taylor identities, which constrain the longitudinal part of the gluon vertex.

The first thing we do is to review the normal perturbative calculation of the gluon renormalization function. We do this in Euclidean space in the general covariant gauge. This illustrates the method of dimensional regularization, which we use, and the general structure of the calculation. This calculation also illustrates that though ghost states are essential for ensuring that the vacuum polarization is transverse, they have only a small effect on the numerical result.

We will then go on in the third section to discuss the truncation of the Schwinger-Dyson equation and, illustrating the general discussion of section 2:3, consider what constraints we can obtain from the Slavnov-Taylor identity. We will follow the argument of section 2:4, that if the coupling constant is small, then the terms in the equation involving the four point gluon vertex can be dropped and that this is a gauge invariant truncation. We also find a form of the triple gluon vertex that satisfies the Slavnov-Taylor identities on the assumption that the ghost propagator is unrenormalised. However, this approximation is sufficiently complex as to make the calculation of the vacuum polarization almost intractable. We therefore adopt an approximation proposed by S. Mandelstam that is quite simple, just the bare vertex divided by the gluon function, and has the correct infra red behaviour, though not satisfying the Slavnov-Taylor identity.

Attempts to solve these equations exactly even numerically have proved unstable, so in section 4 we consider a simple parameterization for the gluon renormalization function that exhibits the enhanced behaviour at low momenta and goes to a constant at large momenta with an intermediate term that joins the two regions together. Our aim will then be to see if such a form satisfies the approximate Schwinger-Dyson equation and whether an enhanced behaviour at low momenta is needed. The introduction of an "intermediate" term will lead us to extend the dimensional integrals beyond the standard results for simple powers. To do this we need to introduce the hypergeometric functions in n dimensions. By using a transformation formula, we can expand the hypergeometric functions in terms of a function of momenta keeping only terms up to zeroth order in $\varepsilon = (4 - n)/2$. This can then be resummed to give a fairly simple result.

Finally we can renormalise the vacuum polarization by using the $\overline{MS}$ scheme to remove the pole as $\varepsilon \to 0$ as well as some irrelevant constants.

3:2 The Perturbative Result

Let us calculate the perturbative expansion of the gluon propagator in n dimensions in Euclidean space and use dimensional regularization to calculate the integrals. From the previous chapter we know that the one loop correction to the inverse propagator is given by (2:2.2)

$$\Pi^{\mu\nu} = \Pi_{\circ}^{\mu\nu} - \Sigma^{\mu\nu} \tag{2.1}$$

Then the one loop gluon contribution to the vacuum polarization in the general covariant gauge, is given by the integral,

$$\Sigma_{gl}^{\mu\nu} = \int \frac{d^n k}{(2\pi)^n} \; \Gamma^{\circ\mu\alpha\delta}_{ead}(-p,k,p-k) \; \Delta^{\circ ab}_{\alpha\beta}(k) \; \Gamma^{\circ\beta\nu\gamma}_{bfc}(-k,p,k-p) \; \Delta^{\circ cd}_{\gamma\delta}(k-p) \tag{2.2}$$

where

$$\Gamma^{\circ\mu\nu\sigma}_{abc}(p,q,r) = -i\, g\, \mu^{\varepsilon} f_{abc} \left[(p-q)^{\sigma}\delta^{\mu\nu} + (q-r)^{\mu}\delta^{\nu\sigma} + (r-p)^{\nu}\delta^{\sigma\mu} \right] \tag{2.3}$$

$$\Delta^{\circ\mu\nu}_{ab}(q) = \frac{\delta_{ab}}{q^2} \left\{ \delta^{\mu\nu} - (1 - \xi) \frac{q^{\mu}q^{\nu}}{q^2} \right\} \tag{2.4}$$

and $n = 4 - 2\varepsilon$. Suppressing the colour indices we can write the integral as,

$$\Sigma^{\mu\nu}_{gl} = g^2 \; C_2(A) \; (2\pi\mu)^{2\varepsilon} \int \frac{d^n k}{(2\pi)^4} \frac{1}{k^2 (k - p)^2} \times$$

$$\times \left\{ N_1^{\mu\nu} - (1 - \xi) \left[\frac{N_2^{\mu\nu}}{k^2} + \frac{N_3^{\mu\nu}}{(k - p)^2} \right] + (1 - \xi)^2 \frac{N_4^{\mu\nu}}{k^2 \; (k - p)^2} \right\} \tag{2.5}$$

where

$$C_2(A) \; \delta_{ef} = - f_{eac} \; f_{afc}$$

is the colour Casimir and the numerators are defined as the tensor part of the two triple gluon vertices contracted together by the metric tensor and/or the momenta from the gluon propagators. Thus,

$$\begin{aligned} N_1^{\mu\nu} = & \left[-(p + k)^{\delta} \, \delta^{\mu\alpha} + (2 k - p)^{\mu} \, \delta^{\alpha\delta} + (2 p - k)^{\alpha} \, \delta^{\delta\mu} \right] \delta_{\alpha\beta} \\ & \left[-(p + k)_{\gamma} \, \delta_{\beta}{}^{\nu} + (2 p - k)_{\beta} \, \delta^{\nu}{}_{\gamma} + (2 k - p)^{\nu} \, \delta_{\gamma\beta} \right] \delta_{\gamma\delta} \\ = & \; (5 p^2 - 2 (p.k) + 2 k^2) \, \delta^{\mu\nu} + (n - 6) \, p^{\mu} p^{\nu} \\ + & \; (3 - 2 n) (p^{\mu} k^{\nu} + k^{\mu} p^{\nu}) + (4 n - 6) \, k^{\mu} k^{\nu} \end{aligned} \tag{2.6a}$$

$$\begin{aligned} N_2^{\mu\nu} = & \left[-(p + k)^{\delta} \, \delta^{\mu\alpha} + (2 k - p)^{\mu} \, \delta^{\alpha\delta} + (2 p - k)^{\alpha} \, \delta^{\delta\mu} \right] k_{\alpha} \, k_{\beta} \\ & \left[-(p + k)_{\gamma} \, \delta_{\beta}{}^{\nu} + (2 p - k)_{\beta} \, \delta^{\nu}{}_{\gamma} + (2 k - p)^{\nu} \, \delta_{\gamma\beta} \right] \delta_{\gamma\delta} \\ = & \left[4 \, (p.k)^2 - 4k^2 \, (p.k) + k^4 \right] \delta^{\mu\nu} + k^2 \, p^{\mu} p^{\nu} \\ + & \left[k^2 - 3 \, (p.k) \right] (p^{\mu} k^{\nu} + k^{\mu} p^{\nu}) \\ + & \left[p^2 + 2 \, (p.k) - k^2 \right] k^{\mu} k^{\nu} \end{aligned} \tag{2.6b}$$

$$\begin{aligned} N_3^{\mu\nu} = & \left[-(p + k)^{\delta} \, \delta^{\mu\alpha} + (2 k - p)^{\mu} \, \delta^{\alpha\delta} + (2 p - k)^{\alpha} \, \delta^{\delta\mu} \right] \delta_{\alpha\beta} \\ & \left[-(p + k)_{\gamma} \, \delta_{\beta}{}^{\nu} + (2 p - k)_{\beta} \, \delta^{\nu}{}_{\gamma} + (2 k - p)^{\nu} \, \delta_{\gamma\beta} \right] q_{\gamma} \, q_{\delta} \end{aligned}$$

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$$= (p^2 - k^2)^2 \delta^{\mu\nu} + (2 k^2 - p^2) p^\mu p^\nu$$
$$- (p.k) (p^\mu k^\nu + k^\mu p^\nu) + (2 p^2 - k^2) k^\mu k^\nu \qquad (2.6c)$$

where $q = k - p$, and finally,

$$N_4^{\mu\nu} = [-(p + k)^\delta \delta^{\mu\alpha} + (2 k - p)^\mu \delta^{\alpha\delta} + (2 p - k)^\alpha \delta^{\delta\mu}] k_\alpha k_\beta$$
$$[-(p + k)_\gamma \delta_\beta{}^\nu + (2 p - k)_\beta \delta^\nu{}_\gamma + (2 k - p)^\nu \delta_{\gamma\beta}] q_\gamma q_\delta$$
$$= (p.k)^2 p^\mu p^\nu - p^2 (p.k) (p^\mu k^\nu + k^\mu p^\nu) + p^4 k^\mu k^\nu \qquad (2.6d)$$

Since the integral naturally divides into four parts let us similarly divide the vacuum polarization tensor pulling out the common factors so that,

$$\Sigma_{gl}^{\mu\nu} = \lambda (\Sigma_1^{\mu\nu} + \Sigma_2^{\mu\nu} + \Sigma_3^{\mu\nu} + \Sigma_4^{\mu\nu}) . \qquad (2.7)$$

where the $\Sigma^{\mu\nu}$'s are just the integrals over the tensors $N^{\mu\nu}$'s with the appropriate denominators and,

$$\lambda = \frac{g^2 C_2(A)}{32 \pi^2} . \qquad (2.8)$$

In order to be able to write down the separate parts of the vacuum polarization in a compact form, from which it is simple to calculate the integrals, we denote the standard dimensional integrals in the following manner, see appendix A,

$$A_{ab} = \frac{(2 \pi \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k}{k^{2a} (k - p)^{2b}}$$

$$A_{ab}^\alpha = \frac{(2 \pi \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \; k^\alpha}{k^{2a} (k - p)^{2b}}$$

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$$A^{\alpha\beta}_{ab} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k\, k^\alpha\, k^\beta}{k^{2a}(k-p)^{2b}}$$

$$A^{\alpha\beta\gamma}_{ab} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k\, k^\alpha\, k^\beta\, k^\gamma}{k^{2a}(k-p)^{2b}} \tag{2.9}$$

Using the integrals (2.9), $\Sigma_1^{\mu\nu}$ can be written in the simple form.

$$\Sigma_1^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k\, N_1^{\mu\nu}}{k^2(k-p)^2}$$

$$= (5p^2 A_{11} - 2p_\alpha A^\alpha_{11})\,\delta^{\mu\nu} - 2(1+\varepsilon)\,p^\mu p^\nu A_{11}$$

$$- (5-4\varepsilon)(p^\mu A^\nu_{11} + A^\mu_{11} p^\nu) + 2(5-4\varepsilon) A^{\mu\nu}_{11}$$

It is now just a matter of consulting the table of integrals in appendix A and doing a little algebra to arrive at the result,

$$\Sigma_1^{\mu\nu} = \left\{ \frac{19}{6}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2}\right] + \frac{58}{9}\right\} p^2\,\delta^{\mu\nu}$$

$$- \left\{ \frac{11}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2}\right] + \frac{67}{9}\right\} p^\mu p^\nu$$

By using the $p^\mu p^\nu$ term as a guide, we can separate the above into two parts: a transverse part and a longitudinal part, where the transverse part ($\propto T^{\mu\nu}$) is defined to vanish when contracted with the external momentum p, ie.

$$T^{\mu\nu}(p) = p^2\,\delta^{\mu\nu} - p^\mu p^\nu$$

and

$$p_\mu T^{\mu\nu}(p) = p_\nu T^{\mu\nu}(p) = 0.$$

Obviously this is not a unique separation, as we can add arbitrary amounts of any transverse tensor to the longitudinal part. The reason why we choose this method of using the $p^\mu p^\nu$ as a guide to defining the transverse part of the vacuum polarization will become clear later when we consider the integrals in the Schwinger-Dyson equation. Then the first part of the vacuum polarization becomes,

$$\Sigma_1^{\mu\nu} = \left\{ \frac{11}{3} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + \frac{67}{9} \right\} [\, p^2\delta^{\mu\nu} - p^\mu p^\nu \,]$$

$$- \left\{ \frac{1}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + 1 \right\} p^2 \, \delta^{\mu\nu} \tag{2.10a}$$

We now go on to evaluate the other three terms in the same way. Thus

$$\Sigma_2^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_2^{\mu\nu}}{k^4 (k-p)^2}$$

$$= [\, 4\, p_\alpha\, p_\beta\, A_{21}^{\alpha\beta} - 4\, p_\alpha\, A_{11}^{\alpha} \,]\, \delta^{\mu\nu} + A_{11}\, p^\mu\, p^\nu$$

$$+ 2\, [\, p^\mu\, A_{11}^{\nu} - 3\, p_\alpha\, A_{21}^{\alpha\mu}\, p^\nu \,] + p^2\, A_{21}^{\mu\nu} + 2\, p_\alpha\, A_{21}^{\alpha\mu\nu} - A_{11}^{\mu\nu}$$

$$= - \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} - 2 \right\} [\, p^2\delta^{\mu\nu} - p^\mu p^\nu \,] \tag{2.10b}$$

Similarly,

$$\Sigma_3^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_3^{\mu\nu}}{k^2 (k-p)^4}$$

$$= p^4\, A_{12}\, \delta^{\mu\nu} - p^2\, A_{12}\, p^\mu\, p^\nu - 2\, p_\alpha\, p^\mu\, A_{12}^{\alpha\nu} + 2\, p^2\, A_{12}^{\alpha\nu}$$

$$= - \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} - 2 \right\} [\ p^2 \delta^{\mu\nu} - p^\mu p^\nu\ ] \tag{2.10c}$$

And finally,

$$\Sigma_4^{\mu\nu} = \frac{(\ 2\ \pi\ \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k\ N_4^{\mu\nu}}{k^4 (\ k - p\ )^4}$$

$$= p_\alpha\ p_\beta\ A_{22}^{\alpha\beta}\ p^\mu\ p^\nu - 2\ p^2\ p_\alpha\ p^\mu\ A_{22}^{\alpha\mu} + p^4\ A_{22}^{\mu\nu}$$

$$= [\ p^2 \delta^{\mu\nu} - p^\mu p^\nu\ ]\ /\ 2 \tag{2.10d}$$

We see that these last three term (2.10b), (2.10c) and (2.10d) are automatically transverse, this is guaranteed by the gauge structure of the gluon loop. Combining all the results (2.10) together we get that the one loop gluon contribution to the vacuum polarization tensor is given by,

$$\Sigma_{gl}^{\mu\nu} = \lambda \left\{ \left[\frac{11}{3} + (\ 1 - \xi\ ) \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + \frac{67}{9} - \right.$$

$$\left. - 2\ (\ 1 - \xi\ ) + \frac{(\ 1 - \xi\ )^2}{2} \right\} [\ p^2 \delta^{\mu\nu} - p^\mu p^\nu\ ]$$

$$- \lambda \left\{ \frac{1}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + 1 \right\} p^2\ \delta^{\mu\nu}\ .$$

Doing a little algebra leads to the result,

$$\Sigma_{gl}^{\mu\nu} = \lambda \left\{ \left[\frac{14}{3} - \xi \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + \frac{107}{18} + \xi + \frac{\xi^2}{2} \right\} T^{\mu\nu}$$

$$- \lambda \left\{ \frac{1}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] + 1 \right\} p^2\ \delta^{\mu\nu} \tag{2.11}$$

Gluon Propagator

There are two more graphs that contribute to the same order as the gluon loop, which must be calculated to give a gauge invariant result. Moreover, the tadpole term gives a zero contribution, since the loop integral does not depend upon the external momentum. The ghost loop contribution to the vacuum polarization is given by the integral,

$$\Sigma^{\mu\nu}_{gh} = (-) \int \frac{d^n k}{(2\pi)^n} \Gamma^{\circ\mu\alpha}_{ead} k_\alpha \Delta^{\circ}_{ab}(k) \Gamma^{\circ\nu\beta}_{bfc} (k-p)_\beta \Delta^{\circ}_{cd}(k-p) \quad (2.12)$$

where the minus sign comes about because the ghost is a pseudo-fermion. The ghost-gluon vertex is given by,

$$\Gamma^{\circ\mu\nu}_{abc} = -i\, g\, \mu^{2\varepsilon} \delta^{\mu\nu} f_{abc}$$

and the ghost propagator,

$$\Delta^{\circ}_{ab}(q) = \frac{\delta_{ab}}{q^2} .$$

Substituting these definitions into (2.12), the ghost contribution to the vacuum polarization becomes,

$$\Sigma^{\mu\nu}_{gh} = g^2 C_2(A) \mu^{2\varepsilon} \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu (k-p)^\mu}{k^2 (k-p)^2}$$

$$= 2\lambda (A^{\mu\nu}_{11} - A^{\mu}_{11} p^\nu)$$

$$= \lambda \left\{ -\frac{1}{6} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{8}{3} \right] p^2 \delta^{\mu\nu} \right.$$

$$\left. + \frac{1}{3} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{5}{3} \right] \right\} p^\mu p^\nu .$$

Gluon Propagator

As in the case of the gluon loop we divide this into a transverse part and a longitudinal part using the $p^\mu\, p^\nu$ term as a guide giving,

$$= -\frac{\lambda}{3}\left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2} + \frac{5}{3} \right\} [\, p^2\delta^{\mu\nu} - p^\mu p^\nu \,]$$

$$+ \lambda \left\{ \frac{1}{2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2} \right] + 1 \right\} p^2\, \delta^{\mu\nu}$$

Therefore the one loop correction to the gluon propagator is,

$$\Sigma^{\mu\nu} = \Sigma^{\mu\nu}_{gl} + \Sigma^{\mu\nu}_{gh}$$

$$= \lambda \left\{ \left[\frac{13}{3} - \xi \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2} \right] + \frac{97}{18} + \xi + \frac{\xi^2}{2} \right\} T^{\mu\nu} \qquad (2.13)$$

Notice that longitudinal parts have exactly cancelled each other leaving a result which is totally transverse. By using the $\overline{\mathrm{MS}}$ renormalization scheme to remove the pole as $\varepsilon \to 0$, as well as the Euler-Mascheroni constant γ_E and $\ln 4\pi$, we get,

$$\Sigma^{\mu\nu} = \lambda \left\{ \left[\frac{13}{3} - \xi \right] \ln\frac{\mu^2}{p^2} + \frac{97}{18} + \xi + \frac{\xi^2}{2} \right\} T^{\mu\nu}$$

We see that although the ghost term is essential to make the result transverse its effect upon the coefficient of the transverse part is quite small, less than 10 %.

The inverse gluon propagator to one loop is then by equation (2.1),

$$\Pi^{\mu\nu} = \Pi^{\mu\nu}_{\circ} - \Sigma^{\mu\nu}$$

by writing the full propagator as,

Gluon Propagator

$$\Pi^{\mu\nu} = \frac{T^{\mu\nu}}{G(p^2)} + \frac{p^\mu p^\nu}{\xi} \qquad \text{and} \qquad \Sigma^{\mu\nu} = \Sigma\, T^{\mu\nu}$$

this becomes,

$$\frac{T^{\mu\nu}}{G(p^2)} + \frac{p^\mu p^\nu}{\xi} = T^{\mu\nu} + \frac{p^\mu p^\nu}{\xi} - \Sigma\, T^{\mu\nu}$$

thus,

$$\frac{1}{G(p^2)} = 1 - \Sigma$$

$$= 1 - \lambda \left\{ \left[\frac{13}{3} - \xi \right] \ln \frac{\mu^2}{p^2} + \frac{97}{18} + \xi + \frac{\xi^2}{2} \right\}$$

Inverting this and expanding keeping only the term of order α we get,

$$G(p^2) = 1 + \lambda \left\{ \left[\frac{13}{3} - \xi \right] \ln \frac{\mu^2}{p^2} + \frac{97}{18} + \xi + \frac{\xi^2}{2} \right\}$$

3:3 Schwinger-Dyson equation and the Slavnov-Taylor identities

The Schwinger-Dyson equation for the gluon propagator ignoring dynamical fermions is illustated diagramatically in figure (2.1). This is obviously a very complicated equation involving as it does both the three point and four point vertices. To simplify this, we can make the assumption that the two loop graphs make little contribution to the equation. This is certainly a gauge invariant truncation of the equation as we have seen in the previous chapter. Moreover if the equation is evaluated in a region where the coupling constant is "small" then the two loop graphs will be smaller by a power of α_s.

As was noted in the perturbative case, the ghost term contributes only a small (< 10%) fraction of the final transverse result. We will assume that this is also the case for the full propagator, and so we will neglect the ghost loop.

This leaves the tadpole term, which we have already noticed does not depend upon the external momentum. This means that any contribution from the tadpole must have the tensor form of $\delta^{\mu\nu}$ times some mass scale which might arise in the full propagator and vertex. Thus if we project out the transverse part of the equation, by using the $p^\mu\, p^\nu$ as a guide, the tadpole will not contribute.

We have now reduced the equation to a form involving only the one gluon loop graph, so the Schwinger-Dyson equation can be written as,

$$\Pi^{\mu\nu} \;=\; \Pi_\circ^{\mu\nu} \;-\; \Sigma^{\mu\nu} \tag{3.1}$$

where,

$$\Sigma^{\mu\nu} = \frac{1}{2}\int \frac{d^n k}{(2\pi)^n}\; \Gamma^{\circ\,\mu\alpha\delta}_{\;ead}(-p,k,p-k)\;\; \Delta^{ab}_{\alpha\beta}(k)\;\; \Gamma^{\beta\nu\gamma}_{bfc}(-k,p,k-p)\;\; \Delta^{cd}_{\gamma\delta}(k-p) \tag{3.2}$$

We know from the Ward-Takahaski identity that only the transverse part of the gluon propagator is renormalised. This allows us to write the gluon propagator in the form, suppressing colour indices,

$$\Delta^{\mu\nu}(q) \;=\; \frac{G(q^2)}{q^2}\left\{ \delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right\} + \xi\,\frac{q^\mu q^\nu}{q^4}\,. \tag{3.3}$$

Since $\Delta^{\mu\alpha}\,\Pi_{\alpha\nu} = \delta^\mu{}_\nu$, the inverse propagator is simply,

$$\Pi^{\mu\nu}(p) \;=\; \frac{1}{G(p^2)}\left(\delta^{\mu\nu}\,p^2 - p^\mu\, p^\nu \right) + \frac{1}{\xi}\,p^\mu\, p^\nu\,. \tag{3.4}$$

Putting this in the Schwinger-Dyson equation (3.1) we get an integral

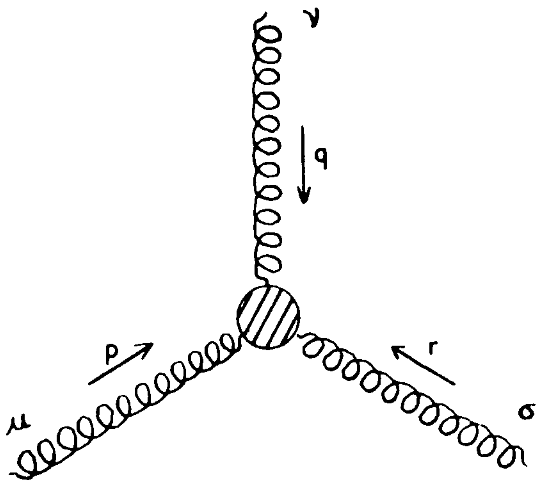

Figure 3.1

The full triple gluon vertex $\Gamma^{\mu\nu\sigma}(p,q,r)$

equation for the gluon renormalization function $G(p^2)$

$$\frac{1}{G(p^2)} = 1 - \Sigma \tag{3.5}$$

where

$$\Sigma^{\mu\nu} = \Sigma\ (\ \delta^{\mu\nu}\ p^2 - p^\mu\ p^\nu\ )$$

The problem is that the vacuum polarization involves the unknown gluon vertex. We can split the vertex (see fig 3.1) into two pieces, a transverse part and a longitudinal part:

$$\Gamma^{\mu\nu\sigma}(p,q,r) = \Gamma_L^{\mu\nu\sigma}(p,q,r) + \Gamma_T^{\mu\nu\sigma}(p,q,r)$$

where the transverse part is defined to vanish when appropriately contracted with any of the external momenta ie.,

$$q_\nu\ \Gamma_T^{\mu\nu\sigma}(p,q,r) = 0.$$

The transverse part of the vertex can be parameterised in the form, [2.9],

$$\Gamma^{T}_{\mu\nu\sigma}(p,q,r) = U(p^2,q^2;r^2)(\ \delta_{\mu\nu}\ p.q - p_\mu\ q_\nu\ )\ V^3_\sigma$$
$$+\ W\ [\ -\delta_{\mu\nu}\ V^3_\sigma + (\ p_\sigma\ q_\mu\ r_\nu - p_\nu\ q_\sigma\ r_\mu\ )\ /\ 3\ ]$$
$$+\ \text{cyclic permutations,}$$

where

$$V^3_\sigma = (\ p_\sigma\ q.r - q_\sigma\ p.r\ )$$

the scalar functions, $U(p^2,q^2;r^2)$ is symmetric in the first two arguments and W is symmetric in p^2, q^2 and r^2.

The longitudinal part of the vertex can be parameterised in the form,

$$\Gamma^{L}_{\mu\nu\sigma}(p,q,r) = A(p^2,q^2;r^2)\ \delta_{\mu\nu}\ (\ p - q\ )_\sigma$$
$$+\ B(p^2,q^2;r^2)\ \delta_{\mu\nu}\ (\ p + q\ )_\sigma$$
$$+\ C(p^2,q^2;r^2)\ (\ p_\nu\ q_\mu - \delta_{\mu\nu}\ p.q\ )\ (\ p - q\ )_\sigma$$
$$+\ S\ (\ p_\nu\ q_\sigma\ r_\mu + p_\sigma\ q_\mu\ r_\nu\ )\ /\ 3 +\ T\ p_\mu\ q_\nu\ r_\sigma\ /\ 3$$
$$+\ \text{cyclic permutations,}$$

where the scalar functions A and C are symmetric in their first two arguments, B is antisymmetric in it's first two arguments and S and T are totally antisymmetric.

The Slavnov-Taylor identity for the triple gluon vertex is,

$$q^\nu\ \Gamma_{\mu\nu\sigma}(p,q,r) = H(q^2)(\ \delta_\mu{}^\nu\ r^2 - r_\mu\ r^\nu\ )\ \Gamma_{\nu\sigma}(p,q;r)\ /\ G(r^2)$$
$$-\ H(q^2)(\ \delta^\nu{}_\sigma\ p^2 - p^\nu\ p_\sigma\ )\ \Gamma_{\nu\mu}(r,q;p)\ /\ G(p^2)$$

where $H(q^2)$ is the ghost renormalization function such that the ghost propagator is given by,

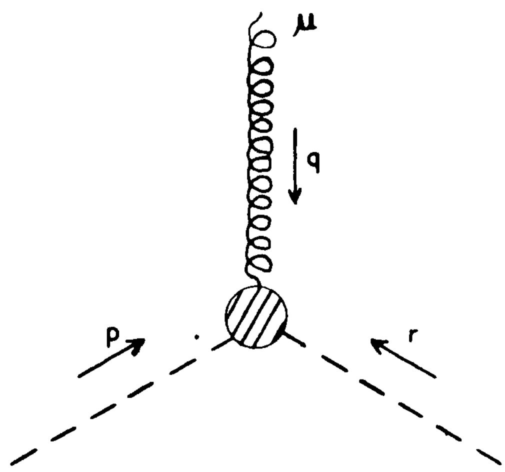

Figure 3.2

The full ghost-gluon vertex $p^{\nu}\,\Gamma_{\nu\sigma}(p,r;q)$

$$\Delta(q) \;=\; i\,\frac{H(q^2)}{q^2}$$

and $p^{\nu}\,\Gamma_{\nu\sigma}(p,r;q)$ is the ghost gluon vertex (see fig 3.2). Since the the ghost-gluon vertex can only depend upon two of the three momenta, by momentum conservation, we can write the vertex in the form,

$$\begin{aligned}\Gamma_{\mu\nu}(p,r;q) \;=\; & D(p^2,r^2,q^2)\delta_{\mu\nu} + E(p^2,r^2,q^2)p_\mu p_\nu + F(p^2,r^2,q^2)p_\mu r_\nu \\ & + F(r^2,p^2,q^2)r_\mu p_\nu + E(r^2,p^2,q^2)r_\mu r_\nu\end{aligned}$$

by making full use of the symmetries of the vertex. The Slavnov-Taylor identity can then be "solved" for the gluon vertex in terms of the gluon function G and the ghost functions D,E,F and H. This leads to a result which is very complicated and, as it involves four unknown functions, not very useful for our purposes. Since we are ignoring the ghost loop, it is consistent to put the ghost functions equal to their bare values, ie.

$H = D = 1$ and $E = F = 0$.

Then the scalar functions in the gluon vertex are given by,

$$A(p^2,q^2,r^2) = \frac{1}{2}\left\{\frac{1}{G(p^2)} + \frac{1}{G(q^2)}\right\}$$

$$B(p^2,q^2,r^2) = \frac{1}{2}\left\{\frac{1}{G(p^2)} - \frac{1}{G(q^2)}\right\}$$

$$C(p^2,q^2,r^2) = \frac{1}{p^2 - q^2}\left\{\frac{1}{G(p^2)} - \frac{1}{G(q^2)}\right\}$$

Even, with this simplification, the result for the vertex is still quite complicated and we look for an even simpler form. Notice that if $G(q^2)$ goes to infinity as q^2 goes to zero, as we may expect, then the longitudinal part of the vertex is inversely proportional to $G(p^2)$. We therefore follow an approximation proposed by Mandelstam [2.3] and suggested by the form of $A(p^2,q^2,r^2)$ above, in which the vertex is just the bare vertex divided by $G(p^2)$. Although at first sight this appears to be a gauge invariant approximation, it is not, because it does not satisfy the Slavnov-Taylor identity except in the trivial limit of $G(p^2)$ going to one. Despite this draw back we will use this approximation because of its nice simple form. The apparent arbitary choice of dividing by $G(p^2)$ instead of $G(r^2)$ is of no concern because the gluon loop is symmeteric in k and k - p, (3.2).

The Schwinger-Dyson equation for the gluon propagator (see fig 3.4) is then given by the following,

$$\Pi^{\mu\nu}(p) = \Pi_{\circ}^{\mu\nu}(p) - \Sigma^{\mu\nu} \tag{3.6}$$

where,

Gluon Propagator

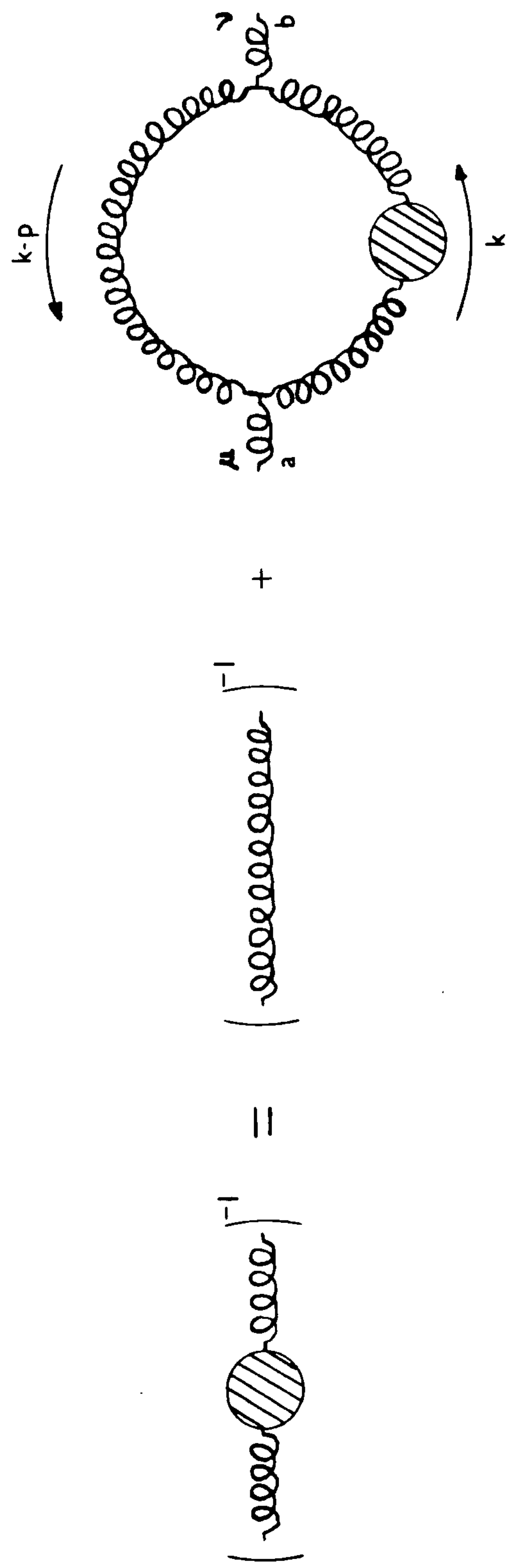

Figure 3.4

The Schwinger-Dyson equation for the gluon propagator in the Mandelstam approximation.

Gluon Propagator

$$\Sigma^{\mu\nu} = \frac{\lambda(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^2 (k - p)^2} \Big\{ G(k^2)\, N_1^{\mu\nu} - (G(k^2) - \xi) \frac{N_2^{\mu\nu}}{k^2}$$

$$- (1 - \xi)\, G(k^2) \frac{N_3^{\mu\nu}}{(k - p)^2} + (1 - \xi) (G(k^2) - \xi) \frac{N_4^{\mu\nu} \, K^{-2}}{(k - p)^2} \Big\}.$$

Splitting off the purely gauge dependent piece,

$$\Sigma^{\mu\nu} = \frac{\lambda(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{G(k^2)}{k^2 (k - p)^2} \Big\{ N_1^{\mu\nu} - \frac{N_2^{\mu\nu}}{k^2}$$

$$- (1 - \xi) \frac{N_3^{\mu\nu}}{(k - p)^2} + (1 - \xi) \frac{N_4^{\mu\nu} \, K^{-2}}{(k - p)^2} \Big\}$$

$$- \xi \frac{\lambda(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^4 (k - p)^2} \Big\{ N_2^{\mu\nu} - (1 - \xi) \frac{N_4^{\mu\nu}}{(k - p)^2} \Big\} . \tag{3.7}$$

where

$\lambda = \alpha_s C_2(A) / 8\pi, \qquad \alpha_s = g^2 / 4\pi$

and the numerator tensor functions are the same as in the perturbative case given earlier.

3:4 Evaluation of the Gluon Integral

In order to do the integrals it is necessary to know the gluon renormalization function $G(p^2)$. Initially we hoped to find this function by an iterative procedure. However studies of a model equation which has the essential features of the gluon equation were found to be unstable. We therefore choose a parameterization of $G(p^2)$ with the properties we expect and see if this can self-consistently satisfy the equation (3.5). With this in mind, we expect that at small momenta the function will behave like $1 / p^2$ as this will give a linear potential at large distances. At large momenta, the function must be approximately constant as is expected from perturbation theory. This gives us two terms, but that is probably not sufficient to reproduce the correct behaviour at intermediate momenta. What we need is some simple term that will not affect the the infra-red and ultra-violet behaviour, but will contribute in the middle region. We can choose this term to be $p^2/(p^2+p_0^2)$, where p_0^2 is some arbitrary mass scale. This function tends to zero as p^2 goes to zero and tends to one as p^2 goes to infinity. We then choose a gluon function, which we hope will provide a sensible approximation in the context of the truncated Schwinger-Dyson equations that we are using, of the form,

$$G(p^2) = A_g \frac{p_0^2}{p^2} + B_g + C_g \frac{p^2}{p^2 + p_0^2} . \tag{4.1}$$

The inclusion of the term that enhances the pole in the gluon propagator at low momenta ($A_g\ p_0^2 / p^2$) does not prejudice the Schwinger-Dyson equation to reproduce such a term, since self consistency may require that A_g be zero. The pole in the intermediate term (C_g) as p^2 tends to $-p_0^2$ is of little concern since we work in Euclidean space and so the pole exists in the "unphysical" region (p_0^2 behaves like a momenta squared and not like a mass under a Wick rotation to Minkowski space). Using this approximation (4.1) for the gluon renormalization function we can break the vacuum polarization tensor into separate pieces. Each piece is multiplied by one of the gluon parameters or the gauge parameter.

$$\Sigma^{\mu\nu} = \lambda (A_g\ p_0^2\ \Sigma_A^{\mu\nu} + B_g\ \Sigma_B^{\mu\nu} + C_g\ \Sigma_C^{\mu\nu} + \xi\ \Sigma_\xi^{\mu\nu}) \tag{4.2}$$

Gluon Propagator

Enhanced Term

Let us consider first the contribution to the vacuum polarization from the enhanced term A_g in (4.2). From equation (3.1) we see that each of these contributions to the vacuum polarization tensor can be further split into four integrals over the same numerator tensors that we used in the perturbative case (2.6). We can then write the integral over $N_1^{\mu\nu}$ as,

$$\Sigma_{A1}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_1^{\mu\nu}}{k^4 (k-p)^2}$$

$$= (5 p^2 A_{11} - 2 p_\alpha A_{21}^\alpha) \delta^{\mu\nu} + (n-6) A_{21}$$
$$+ (3-2n)(p^\mu A_{21}^\nu) + p^\nu A_{21}^\mu + (4n-6) A_{21}^{\mu\nu}$$

We can now project out the transverse part of the integral by using the terms proportional to $p_\mu p_\nu$ as a guide.

$$\Sigma_{A1} = -\frac{2}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} - \frac{3}{2} \right\} \tag{4.3a}$$

where

$$\Sigma_{A1}^{\mu\nu} = \Sigma_{A1} T^{\mu\nu} \quad \text{and} \quad T^{\mu\nu} = p^2 \delta^{\mu\nu} - p^\mu p^\nu,$$

as before. Similarly evaluating the other integrals involved in the enhanced term we obtain,

$$\Sigma_{A2}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_2^{\mu\nu}}{k^6 (k-p)^2}$$

$$= [\, 4 p_\alpha p_\beta A_{31}^{\alpha\beta} - 4 p_\alpha A_{21}^\alpha \,] \delta^{\mu\nu} + A_{21} p^\mu p^\nu$$

$$+ 2 [\, p^\mu A_{21}^\nu - 3 p_\alpha A_{31}^{\alpha\mu} p^\nu \,] + p^2 A_{31}^{\mu\nu} + 2 p_\alpha A_{31}^{\alpha\mu\nu} - A_{21}^{\mu\nu}$$

therefore projecting out the transverse part as before,

$$\Sigma_{A2} = - \frac{1}{2p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4 \pi \mu^2}{p^2} + 3 \right\} \tag{4.3b}$$

The third contribution to the enhanced term is,

$$\Sigma_{A3}^{\mu\nu} = \frac{(2 \pi \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_3^{\mu\nu}}{k^4 (k - p)^4}$$

$$= p^4 A_{22} \delta^{\mu\nu} - p^2 A_{22} p^\mu p^\nu - 2 p_\alpha p^\mu A_{22}^{\alpha\nu} + 2 p^2 A_{22}^{\alpha\nu}$$

projecting out the transverse part we get,

$$\Sigma_{A3} = - \frac{1}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4 \pi \mu^2}{p^2} + \frac{9}{2} \right\} \tag{4.3c}$$

Finally the fourth contribution is,

$$\Sigma_{A4}^{\mu\nu} = \frac{(2 \pi \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_4^{\mu\nu}}{k^6 (k - p)^4}$$

$$= p_\alpha p_\beta A_{32}^{\alpha\beta} p^\mu p^\nu - 2 p^2 p_\alpha p^\mu A_{32}^{\alpha\mu} + p^4 A_{32}^{\mu\nu}$$

which gives a transverse part,

$$\Sigma_{A4} = - \frac{1}{4p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4 \pi \mu^2}{p^2} + 1 \right\} \tag{4.3d}$$

Combining these results (4.3), we get the total contribution from the enhanced term to the vacuum polarization of,

Gluon Propagator

$$\Sigma_A = -\frac{1}{p^2}\left\{\frac{3}{2}\left[1-\frac{(1-\xi)}{2}\right]\left[\frac{1}{\varepsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}\right]-\frac{9}{2}-(1-\xi)\frac{17}{4}\right\}$$

$$= -\frac{1}{p^2}\left\{\frac{3}{4}\left[1-\xi\right]\left[\frac{1}{\varepsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}\right]-\frac{1}{4}(35-17\xi)\right\} \tag{4.4}$$

Constant and Gauge Term

From our calculation of the perturbative result we can immediately write down the contribution from the constant (B_g) term to the vacuum polarization using the equations (2.10) as,

$$\Sigma_B = \left\{\left[\frac{25}{6}+\frac{(1-\xi)}{2}\right]\left[\frac{1}{\varepsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}\right]+\frac{58}{9}-\frac{(1-\xi)}{2}\right\}$$

$$= \left\{\left[\frac{14}{3}-\xi\right]\left[\frac{1}{\varepsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}\right]-\frac{107}{18}+\frac{\xi}{2}\right\} \tag{4.5}$$

The gauge dependent contribution to vacuum polarization is just, using equation (2.10b) and (2.10d),

$$\Sigma_\xi^{\mu\nu} = -\frac{1}{2}\left\{\frac{1}{\varepsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{p^2}-1-\xi\right\} \tag{4.6}$$

Combining the results (4.4), (4.5) and (4.6) we get a total contribution to the vacuum polarization from the enhanced, constant and gauge dependent parts of,

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$$\Sigma = A_g\, p_0^2\, \Sigma_A + B_g\, \Sigma_B + \xi\, \Sigma_\xi$$

$$= A_g \frac{p_0^2}{p^2} \left\{ \frac{3}{4} \left[1 - \xi \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] - \frac{1}{4} (35 - 17 \xi) \right\}$$

$$+ B_g \left\{ \left[\frac{14}{3} - \xi \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] - \frac{107}{18} + \frac{\xi}{2} \right\}$$

$$- \xi \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} - 1 - \xi \right\} \qquad (4.7)$$

3:5 The Integrals in the Intermediate Term

The problem is now to evaluate the integrals containing the intermediate (C_g) term, which we shall see are far from trivial. The simplest of these integrals takes the form,

$$\frac{(4\pi\mu^2)^\varepsilon}{\pi^2} \int d^n k \, \frac{1}{k^2 (k - p)^2 (k^2 + p_0^2)} \qquad (5.1)$$

We could use a Feynman parameterization on this integral straight away. However one of the Feynman integrals is just a round about way of factorising the integrand, so we partial fraction the integral to give,

$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k}{p_0^2} \left[\frac{1}{k^2 (k - p)^2} - \frac{1}{(k - p)^2 (k^2 + p_0^2)} \right]$$

The first part of the integral we already know from the appendix A, so that just leaves the second part. If we now proceed as in the appendix A to derive the dimensional integrals then,

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$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2 p_0^2} \int \frac{d^n k}{(k-p)^2 \, (k^2 + p_0^2)}$$

making a Feynman parameterization (A.5) this becomes,

$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2 p_0^2} \int_0^1 dx \int \frac{d^n k}{[\,(1-x)(k^2+p_0^2) + x(k-p)^2\,]^2}$$

In order to do the angular integrals we make the change of variables $k' = k - x\,p$, so that the denominator is independent of the angular variables.

$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2 p_0^2} \int_0^1 dx \int \frac{d^n k}{[\,k^2 + (1-x)(x\,p^2 + p_0^2)\,]^2}$$

Then performing the angular integals using the result (A.3), we get,

$$\frac{(2\pi\mu^2)^{\varepsilon}}{\Gamma(n/2)p_0^2} \int_0^1 dx \int_0^\infty \frac{k^{n-1}\, dk}{[\,k^2 + (1-x)(x\,p^2 + p_0^2)\,]^2}$$

Finally the radial integral can be done using the result (A.4), to give,

$$\frac{(4\pi\mu^2)^{\varepsilon}}{\Gamma(n/2)p_0^2} \int_0^1 dx \frac{\Gamma(n/2)\,\Gamma(2-n/2)}{[\,(1-x)(x\,p^2+p_0^2)\,]^{2-n/2}}$$

with $n = 4 - 2\,\varepsilon$ this becomes,

$$\Gamma(\varepsilon)\left[\frac{4\pi\mu^2}{p_0^2}\right]^{\varepsilon} \int_0^1 dx\,(1-x)^{-\varepsilon}\left[1 + x\frac{p^2}{p_0^2}\right]^{-\varepsilon}. \qquad (5.2)$$

Now by consulting standard integral tables [3.1] we see that,

$$\int_0^1 x^{a-1} (1 - x)^{b-1} (1 - z x)^{-c} dx = \beta(a,b) \; {}_2F_1(c,a;a+b;z)$$

using this result we can write the integral as,

$$\Gamma(\varepsilon) \left[\frac{4\pi\mu^2}{p_0^2} \right]^{\varepsilon} \beta(1,1-\varepsilon) \; {}_2F_1(\varepsilon,1;2-\varepsilon;-p^2/p_0^2) \tag{5.3}$$

In order to define the integral (5.2) in four dimensions we need to be able to take the limit of (5.3) as ε tends toward zero. To do this we have to be able to express the hypergeometric function ${}_2F_1$ in powers of ε. The standard power series of the hypergeometric function, in terms of it s fourth argument, expanded about zero is,

$${}_2F_1(a,b;c;-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{r=0} \frac{\Gamma(a+r)\Gamma(b+r)}{\Gamma(c+r)} (-z)^r \qquad \text{for } |z| < 1 \tag{5.4}$$

Since this expansion is only valid for $|z| < 1$, which in our case is $p^2 < p_0^2$, this is not of immediate use. However, the hypergeometric function can be written in terms of two other hypergeometric functions of a transformed variable by the relation,

$$\begin{aligned} {}_2F_1(a,b;c;-z) = \; & (1+z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \; {}_2F_1(a,c-b;a-b+1;1/(1+z)) \\ & - (1+z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \; {}_2F_1(b,c-a;b-a+1;1/(1+z)) \end{aligned} \tag{5.5}$$

Now the transformed variable $1/(1+z)$ is less than one for all z so that the hypergeometric funtion in (5.5) can be expanded by use of the power series (5.4). Doing this leads to the somewhat complicated result,

$$
\begin{aligned}
&{}_2F_1(a,b;c;-z)\\
&= (1+z)^{-a}\,\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,\frac{\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)}\sum_{r=0}\frac{\Gamma(a+r)\Gamma(c-b+r)}{\Gamma(a-b+1+r)\Gamma(r+1)}\,(1+z)^{-r}\\
&- (1+z)^{-b}\,\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,\frac{\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)}\sum_{r=0}\frac{\Gamma(b+r)\Gamma(c-a+r)}{\Gamma(b-a+1+r)\Gamma(r+1)}\,(1+z)^{-r}.
\end{aligned}
$$

By using the shift property of the gamma function $\Gamma(n+1) = n\,\Gamma(n)$ we can combine the coefficients of the summations to give,

$$
{}_2F_1(a,b;c;-z) = \frac{\Gamma(c)\Gamma(b-a)\Gamma(a-b+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\times
$$

$$
\times\sum_{r=0}\left\{\frac{\Gamma(a+r)\Gamma(c-b+r)}{\Gamma(a-b+1+r)\Gamma(1+r)}\,(1+z)^{-a-r} - \frac{\Gamma(b+r)\Gamma(c-a+r)}{\Gamma(b-a+1+r)\Gamma(1+r)}\,(1+z)^{-b-r}\right\}
\tag{5.6}
$$

Thus the product of the beta and hypergeometric functions in (5.3) can be expressed as,

$$
\beta(1,1-\varepsilon)\;{}_2F_1(\varepsilon,1;2-\varepsilon;-p^2/p_0^2)
$$

$$
= \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\sum_{r=0}\left\{\frac{\Gamma(1-\varepsilon+r)}{\Gamma(1+r)}\,q^{\varepsilon+r} - \frac{\Gamma(2-2\varepsilon+r)}{\Gamma(2-\varepsilon+r)}\,q^{1+r}\right\}
\tag{5.7}
$$

where

$$
q = 1\,/\,(\,1 + z\,) = 1\,/\,(\,1 + p^2\,/\,p_0^2\,)
\tag{5.8}
$$

Now, having arrived at a more or less simple form for the integral, we can expand the result in powers of ε by using

$$
\Gamma(\,m + n\,\varepsilon\,) = \Gamma(m)\,[\,1 + n\,\varepsilon\,\psi(m) + O(\varepsilon)\,]
\tag{5.9}
$$

where

$$\psi(m) = \frac{d}{dz} \ln \Gamma(z) \Big|_{z=m} = \sum_{k=1}^{m-1} \frac{1}{k} - \gamma_E .$$

expanding the $\Gamma(\varepsilon)$ only after shifting it to $\Gamma(1+\varepsilon)/\varepsilon$, the integral (5.3) then becomes,

$$\frac{1}{\varepsilon} \left[1 + 2\, \varepsilon\, \psi(2) + \varepsilon \ln \frac{4\pi\mu^2}{p_0^2} \right] \times$$

$$\times \sum_{r=0} \left\{ [\, 1 - \varepsilon\, \psi(1+r) + \varepsilon \ln(q)\,]\, q^r - [\, 1 - \varepsilon\, \psi(2+r)\,]\, q^{1+r} \right\}$$

where we have only retained terms of up to zeroth order in ε. This can then be written as,

$$\frac{1}{\varepsilon} \left[1 - 2\, \varepsilon\, \gamma_E + 2 + \varepsilon \ln \frac{4\pi\mu^2}{p_0^2} \right] \left\{ 1 + \varepsilon\, \gamma_E + \varepsilon \ln(q) \sum_{r=0} q^r + \right.$$

$$\left. + \sum_{r=1} [\, 1 - \varepsilon\, \psi(1+r)\,]\, q^r - \sum_{r=0} [\, 1 - \varepsilon\, \psi(2+r)\,]\, q^{r+1} \right\}.$$

By making the change in the index $r \to r + 1$ in the second summation we see that the last two summations cancel, so that the integral (5.3) is just, keeping only terms upto zeroth order in ε

$$\frac{1}{\varepsilon} - \gamma_E + 2 + \ln \frac{4\pi\mu^2}{p_0^2} + \ln(q) \sum_{r=0} q^r$$

$$= \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + 2 + \frac{\ln(q)}{1 - q}$$

$$= \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + 1 + \varphi_{11}(q) \tag{5.10}$$

where we define

$$\varphi_{11}(q) = 1 + \frac{\ln(q)}{1 - q}$$

By using the expansion of the hypergeometric functions (5.6), we can write down the hypergeometric functions in terms of ε as,

$${}_2F_1(\,1 + \varepsilon,\ b;\ b - \varepsilon;\ -z\,) = q + \varepsilon\,\varphi_{0b}(q)$$

$${}_2F_1(\,\varepsilon,\ b;\ b + 1 - \varepsilon;\ -z\,) = 1 + \varepsilon\,\varphi_{11}(q)$$

$${}_2F_1(\,\varepsilon - 1,\ b;\ b + 2 - \varepsilon;\ -z\,) = 1 + \frac{b\,z}{b + 2} + \varepsilon\,\varphi_{2b}(q)$$

where the functions φ can be calculated in a similar manner to the way we calculated $\varphi_{11}(q)$ above. The functions that we need in the evaluation of the intermediate term are,

$$\varphi_{02}(q) = q\left\{\left[\,1 - \frac{q}{1 - q}\ln q\,\right] - 1\right\}.$$

$$\varphi_{04}(q) = q\left\{\left[\,1 - \frac{q^3}{(\,1 - q\,)^3}\ln q\,\right] - \frac{q^2}{2} + \frac{q}{2} - \frac{1}{3} - q^3\left[\,\frac{1}{(\,1 - q\,)^2} + \frac{1}{2\,(\,1 - q\,)}\,\right]\right\}$$

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$$\varphi_{06}(q) = q \left\{ \left[1 - \frac{q^5}{(1-q)^5} \ln q \right] - \frac{q^4}{4} + \frac{q^3}{12} - \frac{q^2}{12} + \frac{q}{4} - \frac{1}{5} \right.$$

$$- q^3 \left[\frac{1}{(1-q)^4} + \frac{1}{2(1-q)^3} + \right.$$

$$\left. \left. + \frac{1}{3(1-q)^2} + \frac{1}{4(1-q)} \right] \right\}$$

$$\varphi_{11}(q) = 1 + \frac{\ln q}{1-q}$$

$$\varphi_{12}(q) = \frac{3}{2} - \frac{1}{1-q} + \left[1 - \frac{q^2}{(1-q)^2} \right] \ln q$$

$$\varphi_{13}(q) = \frac{11}{6} + \frac{1}{(1-q)^2} - \frac{5}{2(1-q)} + \left[1 - \frac{q^3}{(1-q)^3} \right] \ln q$$

$$\varphi_{14}(q) = \frac{25}{12} - \frac{1}{(1-q)^3} + \frac{7}{2(1-q)^2} - \frac{5}{2(1-q)} +$$

$$+ \left[1 - \frac{q^4}{(1-q)^4} \right] \ln q$$

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$$\varphi_{21}(q) = \frac{1}{3q}\left\{ \frac{4}{3} q + \frac{7}{6} + \frac{q^2}{1-q} + \frac{1}{(1-q)^2} \ln q \right\}$$

$$\varphi_{22}(q) = \frac{1}{2q}\left\{ \frac{5}{6} q + \frac{4}{3} - \frac{q^2}{2(1-q)^2} + \frac{1}{2q(1-q)^2} + \frac{2q-1}{q(1-q)^2} \ln q \right\} \tag{5.11}$$

Notice that as $q \rightarrow 1$ ie. $p^2 \rightarrow 0$ all the functions $\varphi(q) \rightarrow 0$, as the apparent singularity is killed due to the presence of the $\ln(q)$ term. This can also be seen by taking the limit $p^2 \rightarrow 0$ in the integral (5.2), then the Feynman integral does not involve the use of the hypergeometric function and the result is easily seen to agree with the limit of equation (5.10) as $p^2 \rightarrow 0$.

3:6 The Intermediate Term Calculated

In the same way as we defined the integrals A_{ab}, we can define the integrals B_{ab} and C_{abc} where,

$$B_{ab} = \frac{(4\pi\mu^2)^{\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{(k^2 + p_0^2)^a \, (k-p)^{2b}}$$

$$C_{abc} = \frac{(4\pi\mu^2)^{\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^{2a} \, (k^2 + p_0^2)^b \, (k-p)^{2c}}$$

$$= \frac{1}{p_0^2} (A_{ab} - B_{bc})$$

etc.

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These integrals can then be expressed in terms of the functions $\varphi(q)$, the results of which are given in the appendix A. Then the contribution to the vacuum polarization from the intermediate term can be readily calculated. Let us split the integral into four parts again, each over the numerator functions (2.6) as we did for the enhanced contribution. Then the integral over the first numerator function is,

$$\Sigma^{\mu\nu}_{C1} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2}\int\frac{d^n k\, N_1^{\mu\nu}}{(k^2+p_0^2)(k-p)^2}$$

$$= (5p^2 B_{11} - 2p_\alpha B^\alpha_{11} + B^{\alpha\beta}_{11}\delta_{\alpha\beta})\delta^{\mu\nu} - 2(1+\varepsilon)p^\mu p^\nu B_{11}$$

$$- (5-4\varepsilon)(p^\mu B^\nu_{11} + B^\mu_{11} p^\nu) + 2(5-4\varepsilon)B^{\mu\nu}_{11}$$

therefore using the $p^\mu p^\nu$ as a guide we can project out the transverse ($\propto T^{\mu\nu}$) part as,

$$-\Sigma_{C1} = -\frac{11}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p_0^2}\right] - \frac{73}{18} - 2\varphi_{11}(q) -$$

$$- 5\varphi_{12}(q) + \frac{10}{3}\varphi_{13}(q) \qquad (6.1a)$$

Similarly for the second term,

$$\Sigma^{\mu\nu}_{C2} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2}\int\frac{d^n k\, N_2^{\mu\nu}}{k^2(k^2+p_0^2)(k-p)^2}$$

$$= [4p_\alpha p_\beta C^{\alpha\beta}_{111} - 4p_\alpha B^\alpha_{11} + B^{\alpha\beta}_{11}\delta_{\alpha\beta}]\delta^{\mu\nu} + B_{11}p^\mu p^\nu$$

$$+ 2[p^\mu B^\nu_{11} - 3p_\alpha C^{\alpha\mu}_{111} p^\nu] + p^2 C^{\mu\nu}_{111} + 2p_\alpha C^{\alpha\mu\nu}_{111} - B^{\mu\nu}_{11}$$

therefore the transverse part is,

$$- \Sigma_{C2} = \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_o^2} \right\} + \frac{1}{4} + \varphi_{11}(q) + \varphi_{12}(q) - \frac{1}{3} \varphi_{13}(q)$$

$$- \frac{3}{2} \varphi_{21}(q) + \frac{1}{3} \varphi_{22}(q)$$

$$+ \frac{p^2}{p_0^2} \left\{ \frac{5}{6} \ln \frac{p^2}{p_0^2} + \frac{1}{72} + \frac{5}{3} \varphi_{13}(q) - \frac{1}{2} \varphi_{14}(q) \right\} \tag{6.1b}$$

The integral over the third term gives,

$$\Sigma^{\mu\nu}_{C3} = \frac{(2 \pi \mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N_3^{\mu\nu}}{(k^2 + p_0^2)(k - p)^4}$$

$$= (p^4 B_{12} - 2 B_{12}^{\alpha\beta} \delta_{\alpha\beta} + B_{12}^{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \delta_{\gamma\delta}) \delta^{\mu\nu} +$$

$$+ (2 B_{12}^{\alpha\beta} \delta_{\alpha\beta} - p^2 B_{12}) p^\mu p^\nu - p_\alpha (p^\mu A_{12}^{\alpha\nu} + p^\nu A_{12}^{\alpha\mu}) +$$

$$+ 2 p^2 A_{12}^{\alpha\nu} - B_{12}^{\alpha\beta\mu\nu} \delta_{\alpha\beta}$$

therefore the transverse part is,

$$- \Sigma_{C3} = \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_o^2} \right\} - \frac{7}{12} + \frac{3}{2} \varphi_{12}(q) - \varphi_{14}(q)$$

$$+ \frac{p^2}{p_0^2} \left\{ - \frac{23}{60} q + \varphi_{02}(q) - 2 \varphi_{04}(q) + \varphi_{06}(q) \right\} \tag{6.1c}$$

and finally the fourth term is

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$$\Sigma^{\mu\nu}_{C4} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int \frac{d^n k \, N^{\mu\nu}_4}{k^2 (k^2 + p_0^2)(k-p)^4}$$

$$= p_\alpha p_\beta C^{\alpha\beta}_{112} p^\mu p^\nu - p^2 p_\alpha (p^\mu C^{\alpha\nu}_{112} + p^\nu C^{\alpha\mu}_{112}) + p^4 C^{\mu\nu}_{112}$$

which gives the transverse part of,

$$\Sigma_{C4} = \frac{p^2}{4p_0^2} \left\{ \ln \frac{p^2}{p_0^2} - \frac{1}{4} + \varphi_{12}(q) \right\} \tag{6.1d}$$

Then collecting the transverse parts (6.1) together we get a total transverse contribution from the intermediate term of,

$$\begin{aligned}
\Sigma_C = & \left\{ \left[\frac{14}{3} - \frac{\xi}{2} \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} \right] + \frac{155}{36} + \right. \\
& + 3\varphi_{11}(q) + \frac{3}{2}(5-\xi)\varphi_{12}(q) - \frac{11}{3}\varphi_{13}(q) - \\
& - (1-\xi)\varphi_{14}(q) - \frac{3}{2}\varphi_{21}(q) + \frac{1}{3}\varphi_{22}(q) + \\
& + \frac{p^2}{p_0^2} \left[\left[\frac{1}{3} + \frac{\xi}{2} \right] \ln \frac{p^2}{p_0^2} + \frac{10}{72} - \frac{\xi}{8} - \frac{23}{60}(1-\xi) q + \right. \\
& \qquad + (1-\xi)(\varphi_{02}(q) - 2\varphi_{04}(q) + \varphi_{06}(q)) - \\
& \qquad \left. \left. - \frac{1-\xi}{4}\varphi_{12}(q) + \frac{5}{3}\varphi_{13}(q) - \frac{1}{2}\varphi_{14}(q) \right] \right\}
\end{aligned} \tag{6.2}$$

Combining (4.7) and (6.2) together we that the integral equation

(3.5) for the gluon renormalization function of the form

$$G(p^2) = A_g \frac{p_0^2}{p^2} + B_g + C_g \frac{p^2}{p^2 + p_0^2}$$

can be written as, using the $\overline{MS}$ renormalization scheme to remove the pole as $\varepsilon \to 0$,

$$\frac{1}{G(p^2)} = 1 - \lambda \left(A_g \, p_0^2 \, \Sigma_A + B_g \, \Sigma_B + C_g \, \Sigma_C + \xi \, \Sigma_\xi \right) \tag{6.3}$$

where

$$\Sigma_A = - \frac{p_0^2}{p^2} \left\{ \frac{3}{4} \left[1 - \xi \right] \ln \frac{\mu^2}{p^2} - \frac{1}{4} (35 - 17 \xi) \right\}$$

$$\Sigma_B = \left\{ \frac{14}{3} - \xi \right\} \ln \frac{\mu^2}{p^2} - \frac{107}{18} + \frac{\xi}{2}$$

$$\Sigma_\xi = - \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4 \pi \mu^2}{p^2} - 1 - \xi \right\}$$

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$$\Sigma_C = \left\{ \left[\frac{14}{3} - \frac{\xi}{2} \right] \ln \frac{\mu^2}{p_0^2} + \frac{155}{36} + \right.$$

$$+ 3\,\varphi_{11}(q) + \frac{3}{2}(5 - \xi)\,\varphi_{12}(q) - \frac{11}{3}\varphi_{13}(q) -$$

$$- (1 - \xi)\,\varphi_{14}(q) - \frac{3}{2}\varphi_{21}(q) + \frac{1}{3}\varphi_{22}(q) +$$

$$+ \frac{p^2}{p_0^2} \left[\left[\frac{1}{3} + \frac{\xi}{2} \right] \ln \frac{p^2}{p_0^2} + \frac{10}{72} - \frac{\xi}{8} - \frac{23}{60}(1 - \xi)\,q + \right.$$

$$+ (1 - \xi)\,(\varphi_{02}(q) - 2\,\varphi_{04}(q) + \varphi_{06}(q)) -$$

$$\left. \left. - \frac{1 - \xi}{4}\varphi_{12}(q) + \frac{5}{3}\varphi_{13}(q) - \frac{1}{2}\varphi_{14}(q) \right] \right\}$$

Chapter 4

The Self-Consistent Gluon

4:1 Introduction

We have seen in the last chapter how to calculate the vacuum polarization in terms of a parameterization of the gluon renormalization function. The problem now is can we find a set of values for which this is a self-consistent solution of the Schwinger-Dyson. From equation (3:6.3) we can see that this equation can not be analytically self-consistent. Because of the presence of logarithms on the right hand side of the Schwinger-Dyson equation one might wonder whether we should have not included logarithms in our approximation. If we did this, then the integrals would give rise to di-logarithms and so we would be in no better position than before. It is clear that powers of, momenta and logarithms of momenta, do not form a complete basis in which we can describe the gluon renormalization function.

Does this fact represent an insurmountable problem in our search for a self-consistent gluon function? To answer this question we must decide what we mean by self-consistent. If we mean that the gluon function we put in to the Schwinger-Dyson equation must exactly equal the result we get out everywhere then the answer is surely yes. However, if we only mean that the the output is approximately equal to the input, upto some error in line with our assumptions, over some range of momenta then the answer is no.

It is clear that we can not expect to obtain a self-consistent answer for the gluon function over macroscopic distances (small momenta) due to the effect of pair creation which is not included in the Schwinger-Dyson equation. Equally we cannot have a self-consistent result for very small distances (large momenta) since we have approximated the longitudinal part of the triple gluon vertex by a form which is only valid for small momenta, and we have completely ignored the unconstrained transverse part of the vertex.

The problem remaining then, other than the determination of the parameters, is to choose over what range of momenta we wish to find a self-consistent result for the gluon renormalization function. Before we address this problem, we must consider the problem of the gluon mass.

4:2 Mass Renormalization

The vacuum polarization we have obtained in the previous chapter must describe a massless gluon because of the gauge invariance of QCD. Now the general form for the inverse propagator for a massive spin one particle is,

$$\Pi^{\mu\nu} = \frac{p^2 - m^2}{p^2\ G(p^2)} (\delta^{\mu\nu} p^2 - p^\mu p^\nu) + \frac{p^2 - \xi m^2}{\xi p^2} p^\mu p^\nu .$$

This means that when multiplied by the momentum squared the inverse propagator should vanish in the limit $p^2 \to 0$ if it is to describe a massless particle, ie.,

$$\lim_{p^2 \to 0} p^2\ \Pi^{\mu\nu}(p^2) \to 0 . \tag{2.1}$$

Since the Schwinger-Dyson equation (3:3.6) is

$$\Pi^{\mu\nu}(p^2) = \Pi^{\circ\mu\nu}(p^2) - \Sigma^{\mu\nu}(p^2) \tag{2.2}$$

equation (2.1) implies a condition on the vacuum polarization for it to describe a massless gluon of,

$$\underset{p^2 \to 0}{\text{Limit}} \; p^2 \, \Sigma^{\mu\nu}(p^2) \to 0 \, . \tag{2.3}$$

From the Ward identity we know that the vacuum polarization is transverse so the tensor structure of the vacuum polarization does not vanish in this limit. Hence we arrive at the result that if the inverse propagator is to describe a massless gluon the scalar vacuum polarization deduced in the last chapter (3:6.3) must vanish in the limit $p^2 \to 0$ when multiplied by p^2. The problem is it doesn't, which is true of all loop calculations, even when calculated in a gauge invariant manner. In order to remedy this state of affairs we must renormalise the vacuum polarization to remove this mass term. The simplest way to do this is just to subtract the non-vanishing term. Thus the scalar vacuum polarization becomes,

$$\Sigma(p^2) \to \Sigma(p^2) + \lambda \, A_g \, \frac{p_0^2}{p^2} \left\{ \frac{3}{4} \left[1 - \xi \right] \ln \frac{\mu^2}{p^2} - \frac{1}{4} (35 - 17 \, \xi) \right\}$$

4:3 Constraints on the Parameters

Using this result the integral equation (3:6.3) is reduced to

$$\frac{1}{G(p^2)} = 1 - \lambda \left(B_g \, \Sigma_B + C_g \, \Sigma_C + \xi \, \Sigma_\xi \right) \tag{3.1}$$

where,

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$$G(p^2) = A_g \frac{p_0^2}{p^2} + B_g + C_g \frac{p^2}{p^2 + p_0^2} \tag{3.2}$$

We can see from these equations that if the parameter A_g is to be non-zero in the limit $p^2 \to 0$ then the right hand side of equation (3.1) must vanish as the momentum goes to zero.

$$\lim_{p^2 \to 0} [\, 1 - \lambda\, (\, B_g\, \Sigma_B + C_g\, \Sigma_C + \xi\, \Sigma_\xi \,)\,] = 0 \tag{3.3}$$

This limit is too strict in the sense that we do not expect the equations to be valid for very small momenta. However, it must be approximately true if we require a non-zero A_g at some small but finite value of p^2. We will therefore impose equation (3.3) in two ways. Firstly as an exact constraint which will lead to what we call a Fourier method, and then in an approximate way using a least squares approach, which we also allow us to check the consistency of a zero value for A_g. First let us impose equation (3.3) exactly.

The parts of the vacuum polarization arising from the gauge term and the constant B_g are, from equations (3:6.4b) and (3:6.4c), just

$$\Sigma_\xi = -\frac{1}{2} \left\{ \ln \frac{\mu^2}{p^2} - 1 - \xi \right\} \tag{3.4}$$

$$\Sigma_B = \left\{ \left[\frac{14}{3} - \frac{\xi}{2} \right] \ln \frac{\mu^2}{p^2} - \frac{107}{18} + \frac{\xi}{2} \right\}. \tag{3.5}$$

In order that the condition (3.3) holds, it is necessary that the coefficients of the logarithms in the parts vacuum polarization (3.4) and (3.5) conspire to give zero. This leads to the constraint on the parameter B_g of,

$$B_g = \frac{3\,\xi}{28 - 3\,\xi}. \tag{3.6}$$

This is sufficient to make the limit (3.3) finite. In order to make it zero we have to consider the part of the vacuum polarization arising from the intermediate term proportional to C_g, (3:6.4d). From equation (3:5.11) we know that the functions $\varphi(q)$ all vanish in the limit $p^2 \to 0$ ie. $q \to 1$. This means that the constant part of this contribution to the vacuum polarization is,

$$C_g \left\{ \left[\frac{14}{3} - \frac{\xi}{2} \right] \ln \frac{\mu^2}{p_0^2} + \frac{155}{36} \right\}. \tag{3.7}$$

Combining this with the constant pieces from equations (3.2) and (3.4) leads to the constraint on the parameter C_g of,

$$C_g = \frac{36/\lambda - (\,214 + 18\,\xi\,)\,B_g - 18\,\xi\,(\,\xi + 1\,)}{(\,168 - 18\,\xi\,)\ln \mu^2/p_0^2 + 155} \tag{3.8}$$

using these constraints on B_g and C_g (3.6) and (3.8) the Schwinger-Dyson equation (2.2) becomes,

$$\frac{1}{G(p^2)} = \lambda\, C_g \Bigg\{ 3\,\varphi_{11}(q) + \frac{3}{2}\,(\,5 - \xi\,)\,\varphi_{12}(q) - \frac{11}{3}\,\varphi_{13}(q) - $$

$$- (\,1 - \xi\,)\,\varphi_{14}(q) - \frac{3}{2}\,\varphi_{21}(q) + \frac{1}{3}\,\varphi_{22}(q) + $$

$$+ \frac{p^2}{p_0^2} \Bigg[\left[\frac{1}{3} + \frac{\xi}{2} \right] \ln \frac{p^2}{p_0^2} + \frac{10}{72} - \frac{\xi}{8} - \frac{23}{60}\,(\,1 - \xi\,)\,q + $$

$$+ (\,1 - \xi\,)\,(\,\varphi_{02}(q) - 2\,\varphi_{04}(q) + \varphi_{06}(q)\,) - $$

$$- \frac{1 - \xi}{4}\,\varphi_{12}(q) + \frac{5}{3}\,\varphi_{13}(q) - \frac{1}{2}\,\varphi_{14}(q) \Bigg] \Bigg\} . \tag{3.10}$$

Notice that the right hand side of the Schwinger-Dyson equation is now totally determined, for a given gauge and on the left hand side the only free parameter is A_g.

4:4 Fourier determination of the Free Parameter

Given the constraints on the parameters B_g (3.6) and C_g (3.8) the right hand side of equation (3.10) can be regarded as a fixed vector in the infinite dimensional space of functions. The left hand side, in contrast, is a variable vector on the subspace defined by the set of functions $1/p^2$, 1, $p^2/(p^2+p_0^2)$. If the left hand side is going to be as close an approximation to the right hand side as possible then the difference must be orthogonal to the subspace.

In order that the term orthogonal has some meaning we have to have some definition of the inner product. In general we can define the inner product to be,

$$\langle f \mid g \rangle = \int_b^a w(y)\, dy\, f(y)\, g(y) \tag{4.1}$$

where w is a weight function and the limits a and b are chosen so that the norm of the functions are finite. For our case the lower limit b must be greater than zero and the upper limit must be not so large as to invalidate the approximations made in solving the Schwinger-Dyson equation. If we choose the weight to be unity the lower end of the integral will dominate and our results will be sensitive to our choice of the lower limit b. For these reasons use a weight of p^2.

If we denote the parametrization of the gluon function by G_{in} and the left hand side of the Schwinger-Dyson equation by $1/G_{out}$ then the orthogonality condition is,

$$\langle (G_{in} - G_{out}) \mid \psi \rangle = 0 \tag{4.2}$$

where ψ is any of the functions in the subspace. But since the parameters B_g and C_g are fixed the only condition of interest is the one when $\psi = 1/p^2$. Expanding G_{in} the orthogonality condition (4.2) determines the parameter A_g to be,

$$A_g = \frac{\langle 1/p^2 \mid G_{out} \rangle - B_g \langle 1/p^2 \mid 1 \rangle - C_g \langle 1/p^2 \mid p^2/(p^2+p_0^2) \rangle}{\langle 1/p^2 \mid 1/p^2 \rangle} \tag{4.3}$$

The inner product between the basis functions is easily calculated analytically and the product containing the function G_{out} can be determined numerically using Simpson's rule. This will then give us the best fit with the definition of the inner product (4.1) using the constraints on B_g (3.6) and C_g (3.8). But we still need to know how good is this fit.

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The relative error between the input and output functions at any point is,

$$\frac{G_{out}(x)}{G_{in}(x)} - 1 .$$

What we want is the average relative error, but we do not want the errors to cancel. For this reason we consider the average of the relative error squared.

$$\sigma^2 = \int_b^a dx \left\{ \frac{G_{out}(x)}{G_{in}(x)} - 1 \right\}^2 \Big/ \int_b^a dx . \tag{4.4}$$

4:5 Least Square Fit

As we have said in the third section the constraints (3.6) and (3.8) on the parameters B_g and C_g are too severe. But without this constraints the fourier techinque will not work as the output function is no longer a fixed vector in the space of functions and the association of the condition (4.2) with the best fit no longer holds. We must therefore look for some other method of determining the parameters for the best fit.

Instead of regarding the output function as a vector in the infinite space of functions we can consider it as an infinite set of data points. In this case the idea that springs to mind when thinking about fitting is a least squares fit. Normally when doing a least squares fit on a finite set of data points you just add up the square of the residuals divided by the error. In our case we have a function which we regard as an infinite set of data points. In that case instead of just adding up the residuals squared, we can integrate over the difference of the function squared with an appropriate weighting (error) function, divided by the width of the bins used in the integration. If we want the relative error between the input and output functions to be the same every where, the obvious choice of weight is the inverse of the input function. (We use the input function to avoid problems of division by zero later when we release the conditions on B_g and C_g and vary them to find a fit). Then the chi squared of the fit can be written as,

$$\chi^2 = \frac{N}{a-b}\int_b^a \frac{dx}{\sigma^2}\left\{\frac{G_{out}(x)}{G_{in}(x)} - 1\right\}^2 \tag{5.1}$$

where N is the number of bins used in the integration and σ is the relative error. For a good fit we require that the chi squared per degree of freedom is unity.

$$\frac{\chi^2}{N-m} = 1 \tag{5.2}$$

where m is the number of parameters used in the fit. Since the number of bins used can be made arbitrarily large we can ignore m. Using (5.2) and (5.1) we find that the relative error is given by

$$\sigma^2 = \frac{1}{a-b}\int_b^a dx\left\{\frac{G_{out}(x)}{G_{in}(x)} - 1\right\}^2 \tag{5.3}$$

This then agrees with our definition of how good a fit we have obtained from our Fourier technique (4.4). But it goes beyond that as by using an appropriate minimization package we can find fits not only with A_g variable but with all the gluon parameters variable.

4:6 Results

We can now proceed to investigate the gluon renormalization function in different covariant gauges using the fourier technique to find a first approximation and then using that result doing a least square fit with all the parameters variable.

Despite the fact that the scale of the momenta is unknown, as we have no physical point to measure. We expect that the momenta p_0 is of the order of a GeV, since that is a typical hadronic scale. All the momenta are measured in terms of p_0, which can effectively be put equal to 1. The question is over what range of momenta should we expect to be able to find a fit. At first it is tempting to make the top limit much greater than p_0. This would be a mistake because the intermediate parameter C_g controls the behaviour of the output at small momenta. This is due to the nature of the gauge structure and mass renormalization of the theory. Another reason why we should not make the upper limit too large is that the Mandelstam approximation for the gluon vertex breaks down as the momenta increase. For this reason we will choose the upper limit to be about the p_0^2, in general 1.05 p_0^2, although we will investigate the effect of changing the upper limit. For the lower limit we choose to use the value of 0.05 p_0^2. This gives us over an order of magnitude range covering the confinement region. We will also choose to use a value for the dimensional regularization parameter of $\mu^2 = 10\ p_0^2$. The results are quite insensitive to any variation of this choice.

First let us consider the results in different gauges. Now the condition (3.6) indicates that as the gauge goes to $\xi = 28/3$ the parameter B_g goes to infinity. This is because in that gauge the coefficient of the logarithm coming from the integral over the constant part of the gluon function vanishes. Thus to compensate for the logarithm from the gauge term B_g has to go to infinity. This is obviously a spurious effect due to our parameterization and method of calculating and we must avoid this gauge. To that end we will confine ourselves to considering the gauge parameter in the range -5 to 5 which is nevertheless a large range for the gauge parameter.

In figures (4.1a-g) we see the results of the Fourier technique for determining the parameter A_g for a number of different gauges. The average relative error in this calculation is really quite good with an average value of only 1 to 2 % and peaking at just under 6 %. We can compare these results with those from the least squared fit illustrated in figures (4.2a-g). Here we can see that using the fit has not substantially changed the results and its main effect is to decrease the average relative error so that now it peaks at 2.5 %. The results for the least squares method were obtained by starting from the values obtained from the Fourier method. Starting the fit with a zero value for A_g does not produce a fit with any where near the accuracy obtained for non-zero value of A_g. The minimiser has great difficulty in finding the minimum with A_g non-zero because it is very narrow in the B_g, C_g parameter space.

The parameters for both these set of results are given in figure (4.3). We see that as the modulus of the gauge parameter increases, A_g increases and the average error decreases, while B_g and C_g stay roughly constant. The increase in A_g and hence the value of the gluon function can be qualititavely explained by considering the perturbative result where the slope of the gluon function at the

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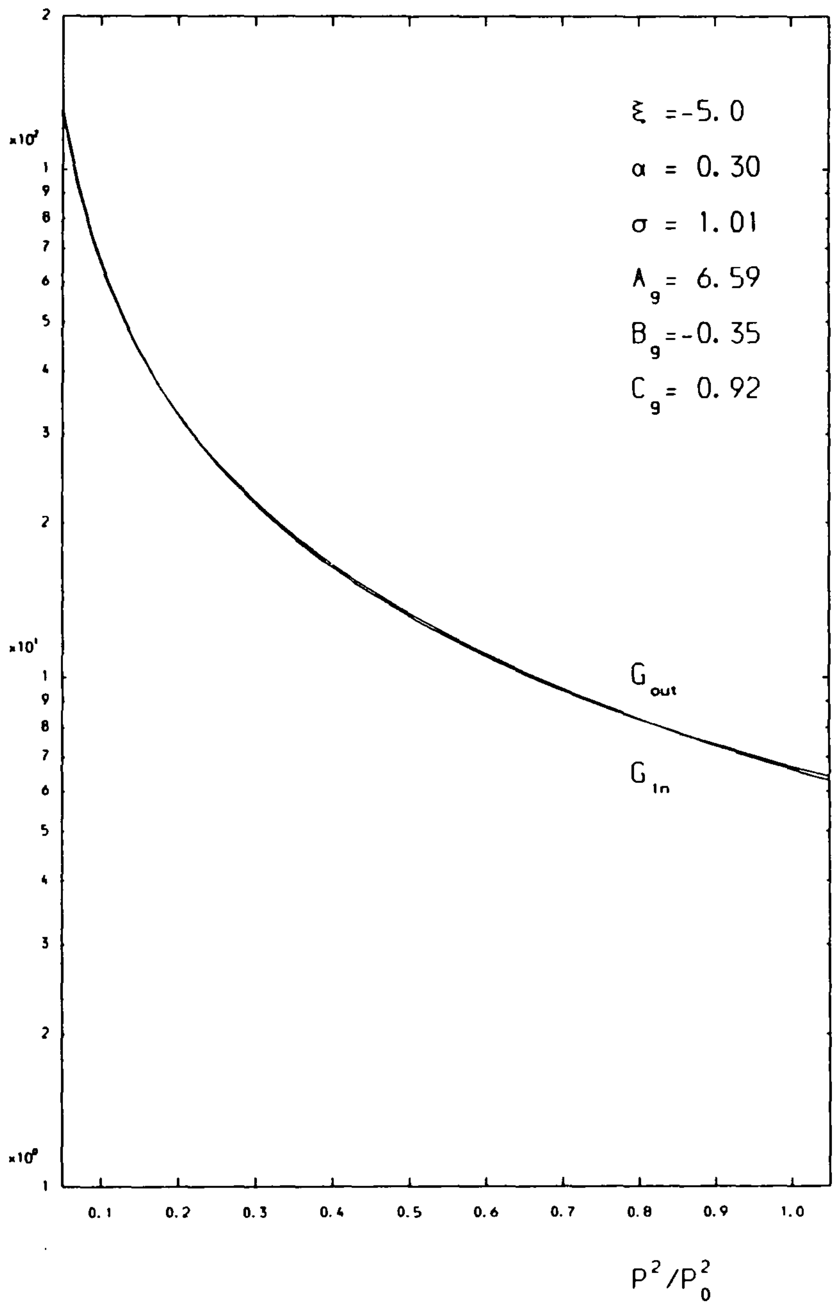

Figure 4.1a

The gluon function determined by the Fourier method in the gauge $\xi = -5$.

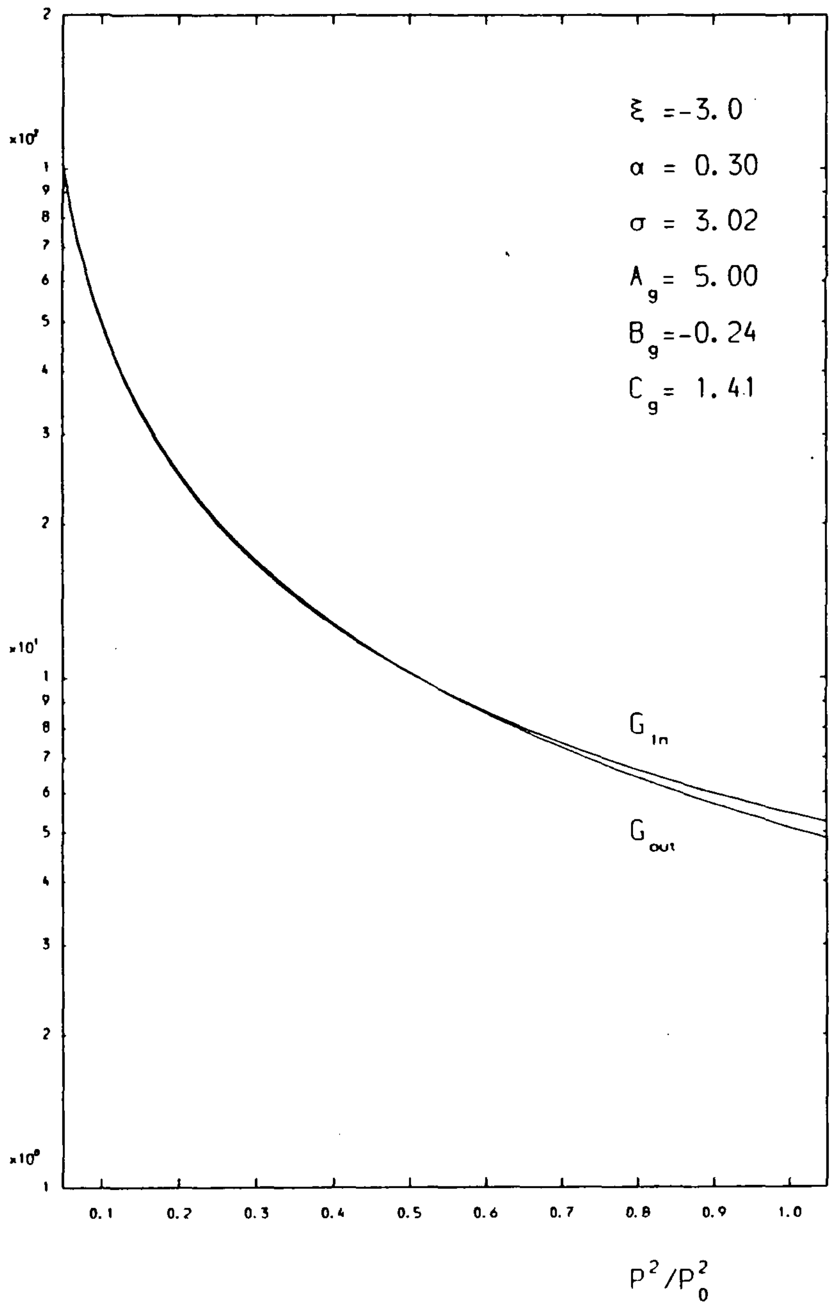

Figure 4.1b

The gluon function determined by the Fourier method in the gauge $\xi = -3$.

Self-Consistent Gluon

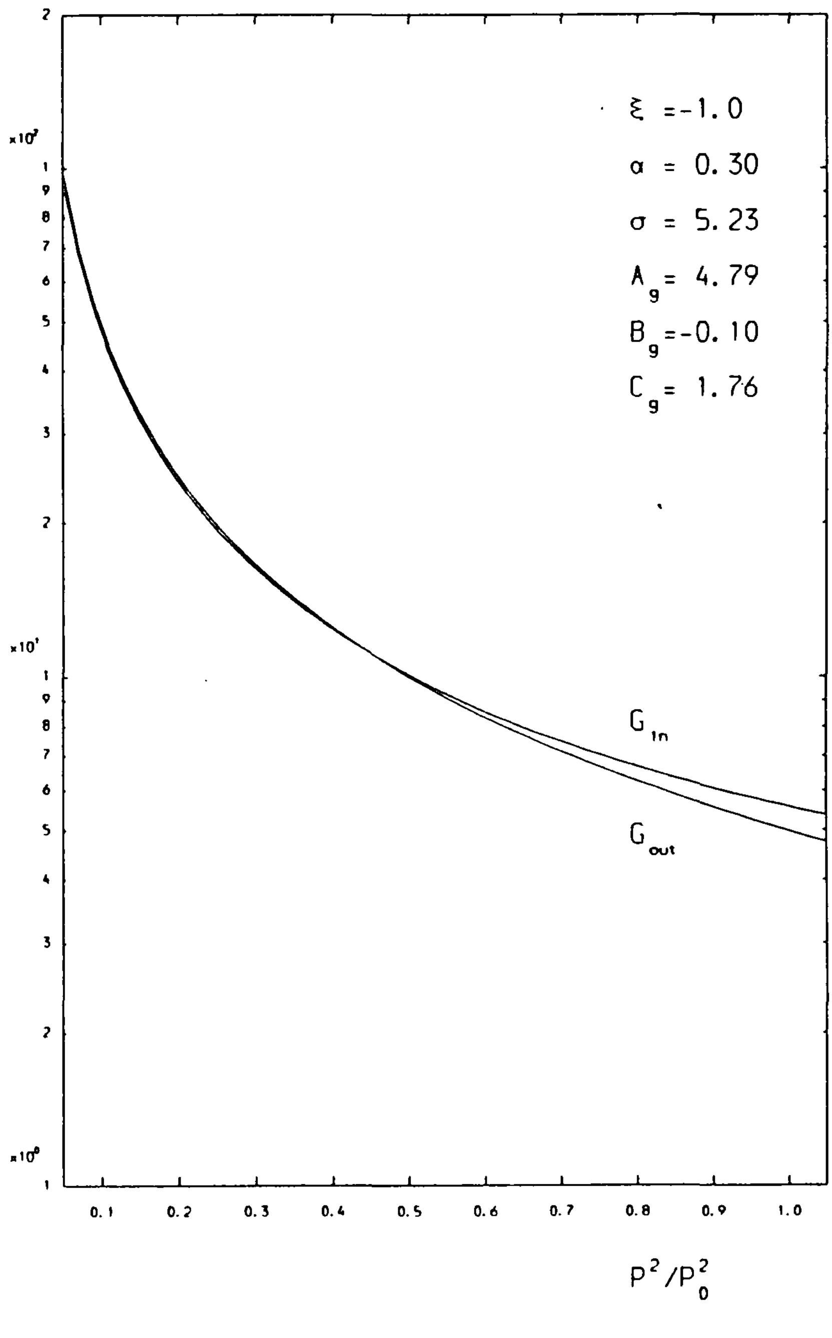

Figure 4.1c

The gluon function determined by the Fourier method in the gauge $\xi = -1$.

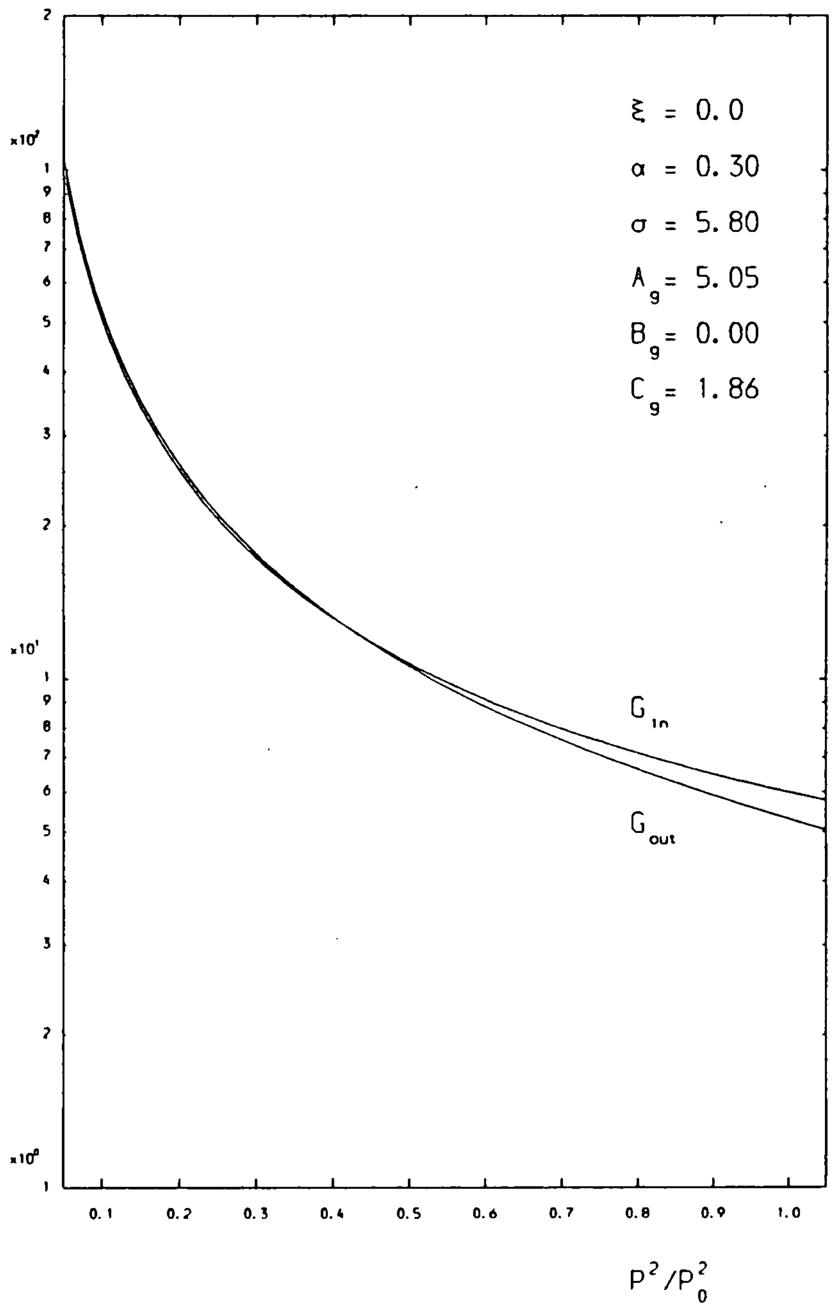

Figure 4.1d

The gluon function determined by the Fourier method in the gauge $\xi = 0$.

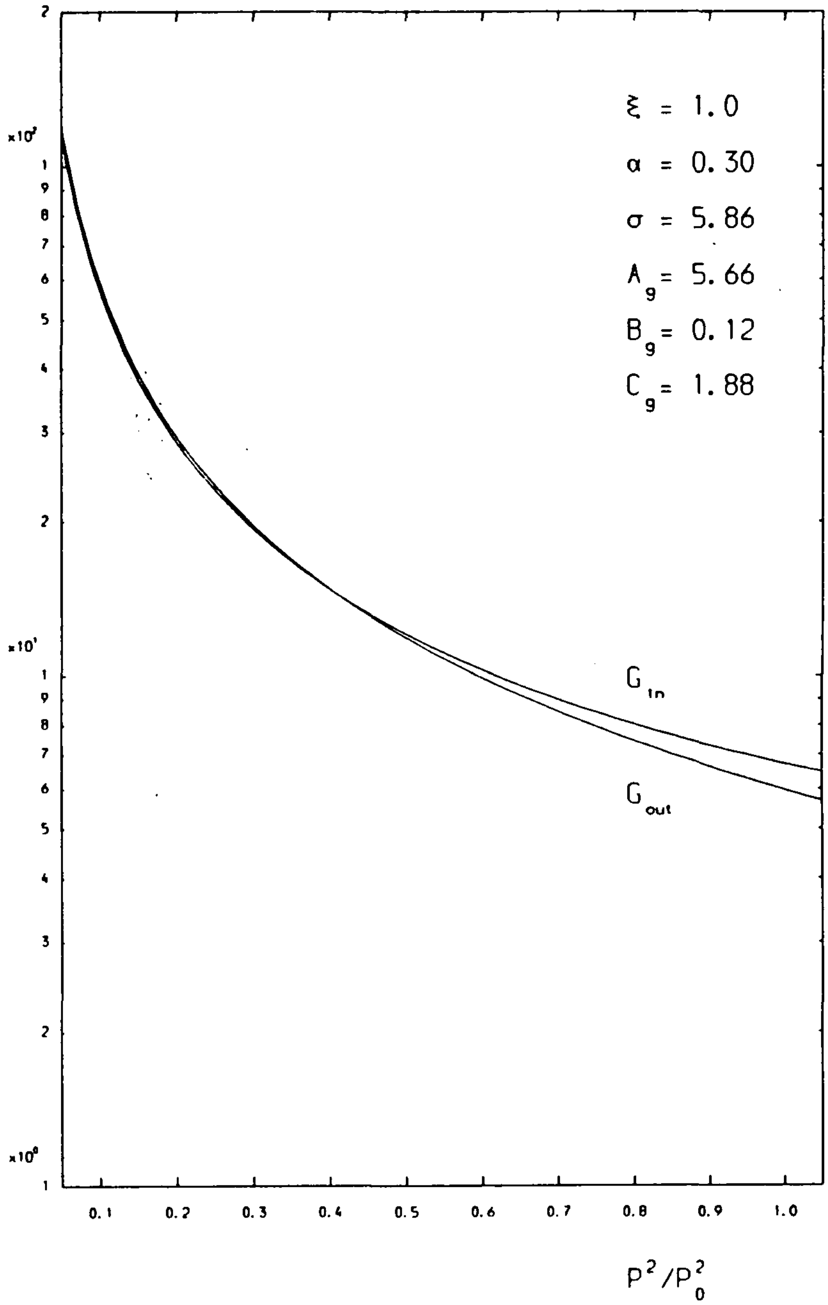

Figure 4.1e

The gluon function determined by the Fourier method in the gauge $\xi = 1$.

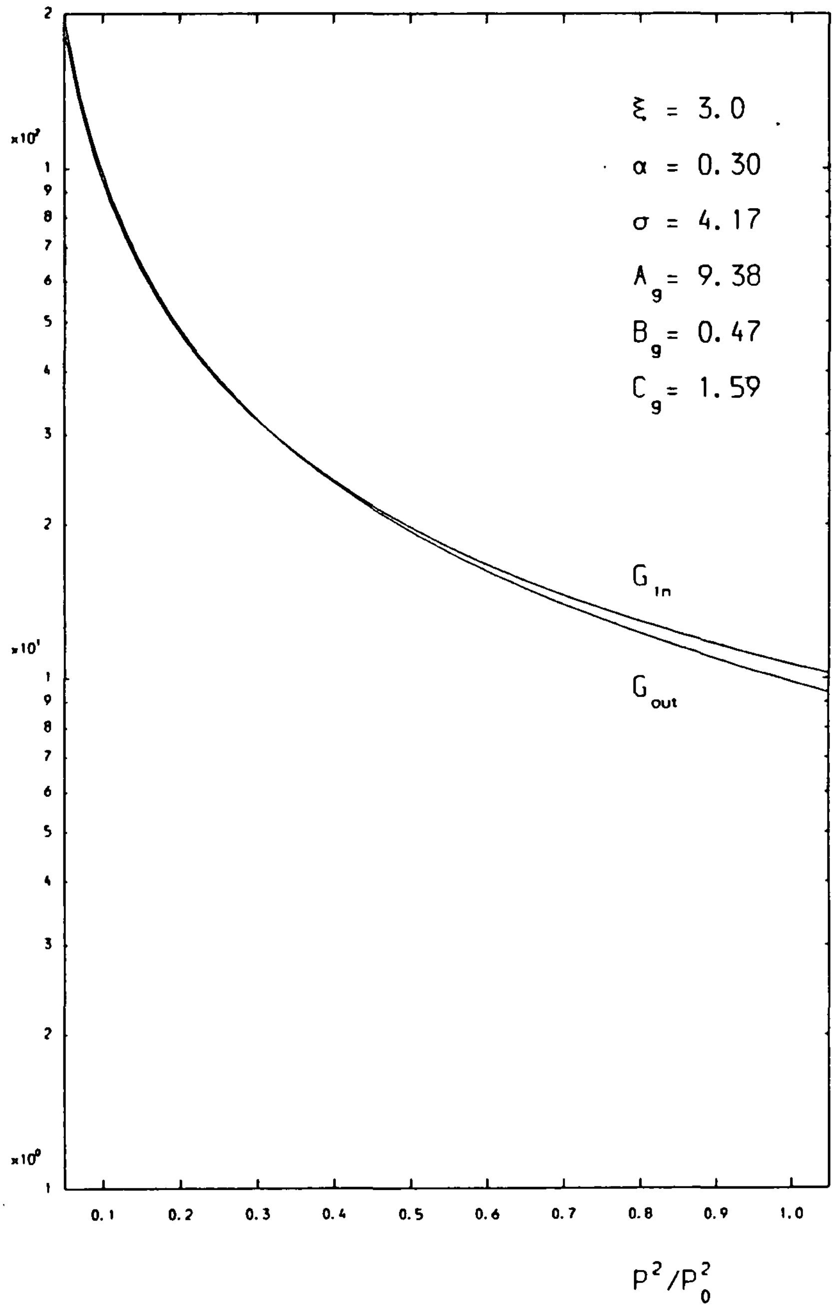

Figure 4.1f

The gluon function determined by the Fourier method in the gauge $\xi = 3$.

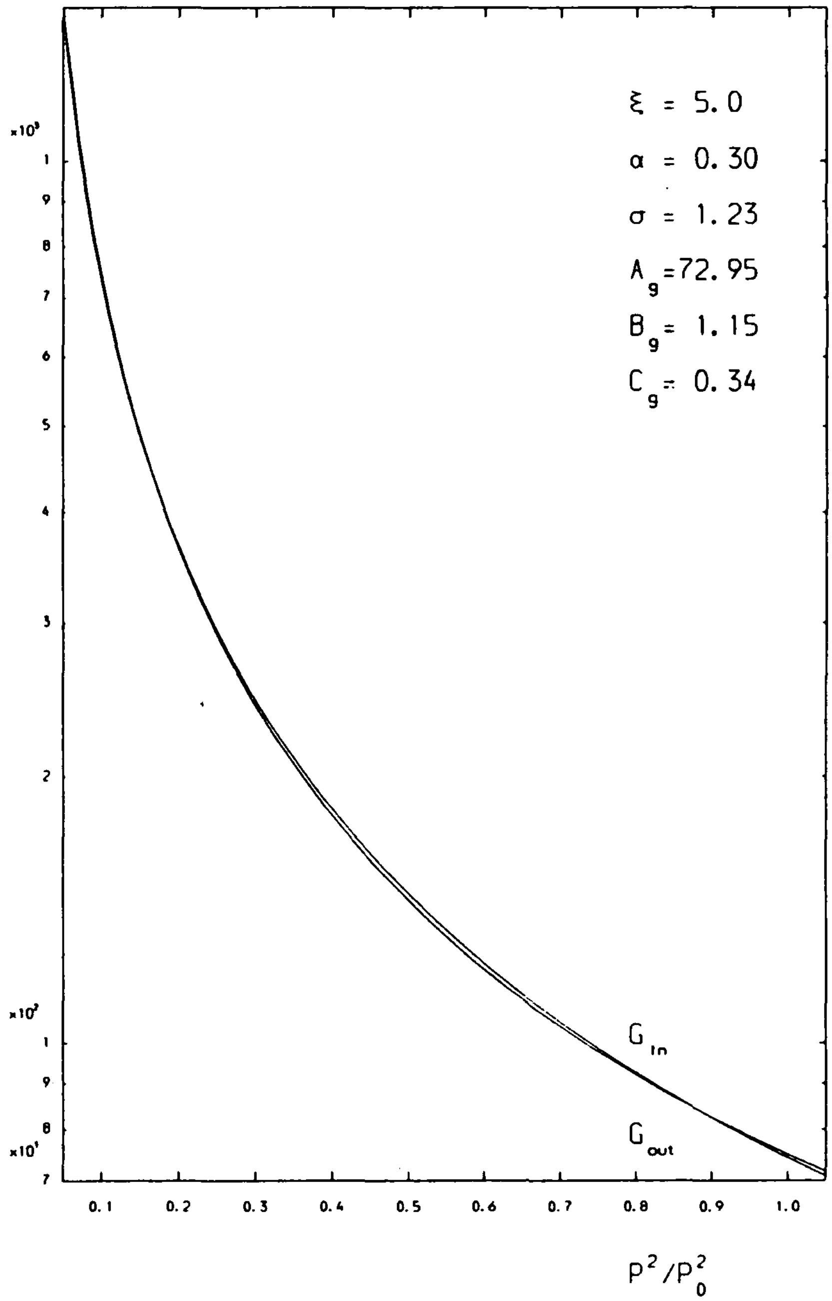

Figure 4.1g

The gluon function determined by the Fourier method in the gauge ξ = 5.

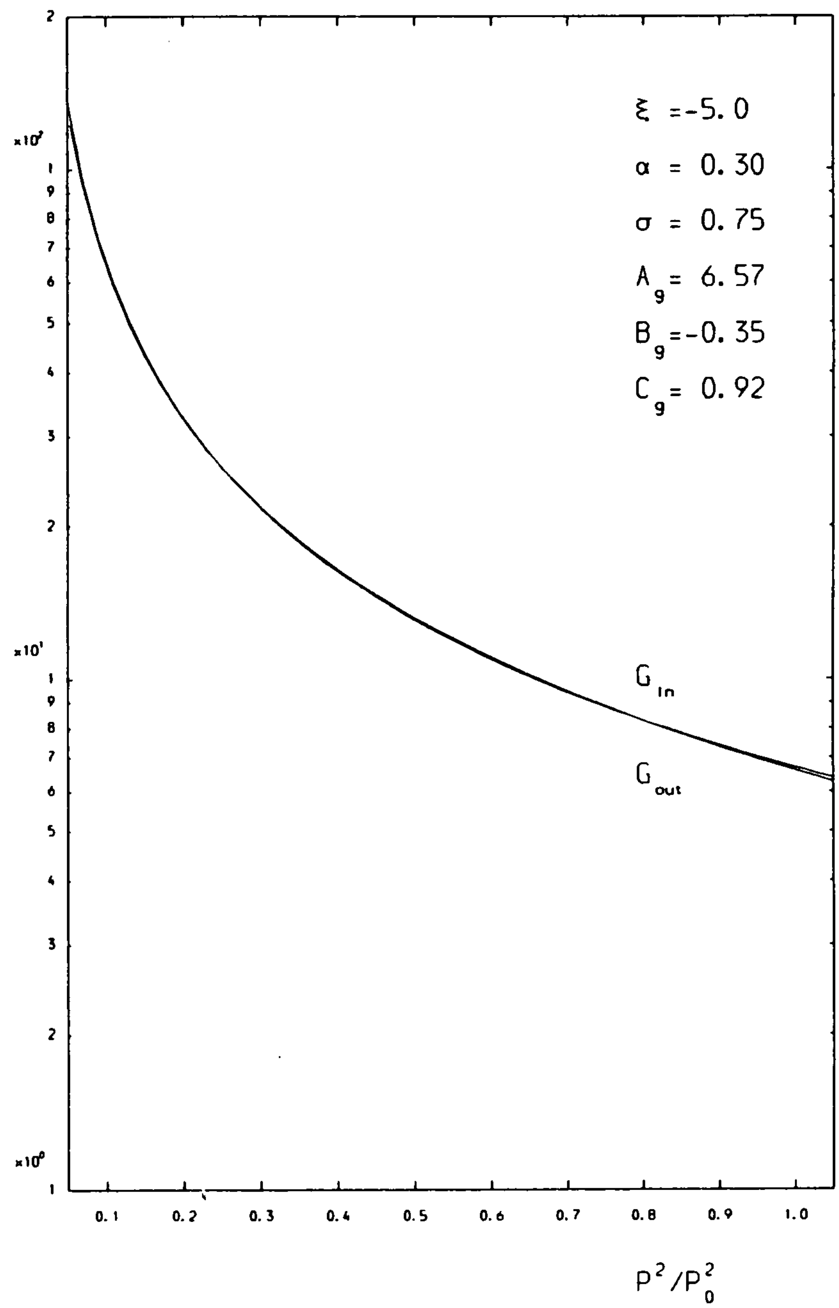

Figure 4.2a

The gluon function determined by the least squares fit method in the gauge $\xi = -5$.

Self-Consistent Gluon

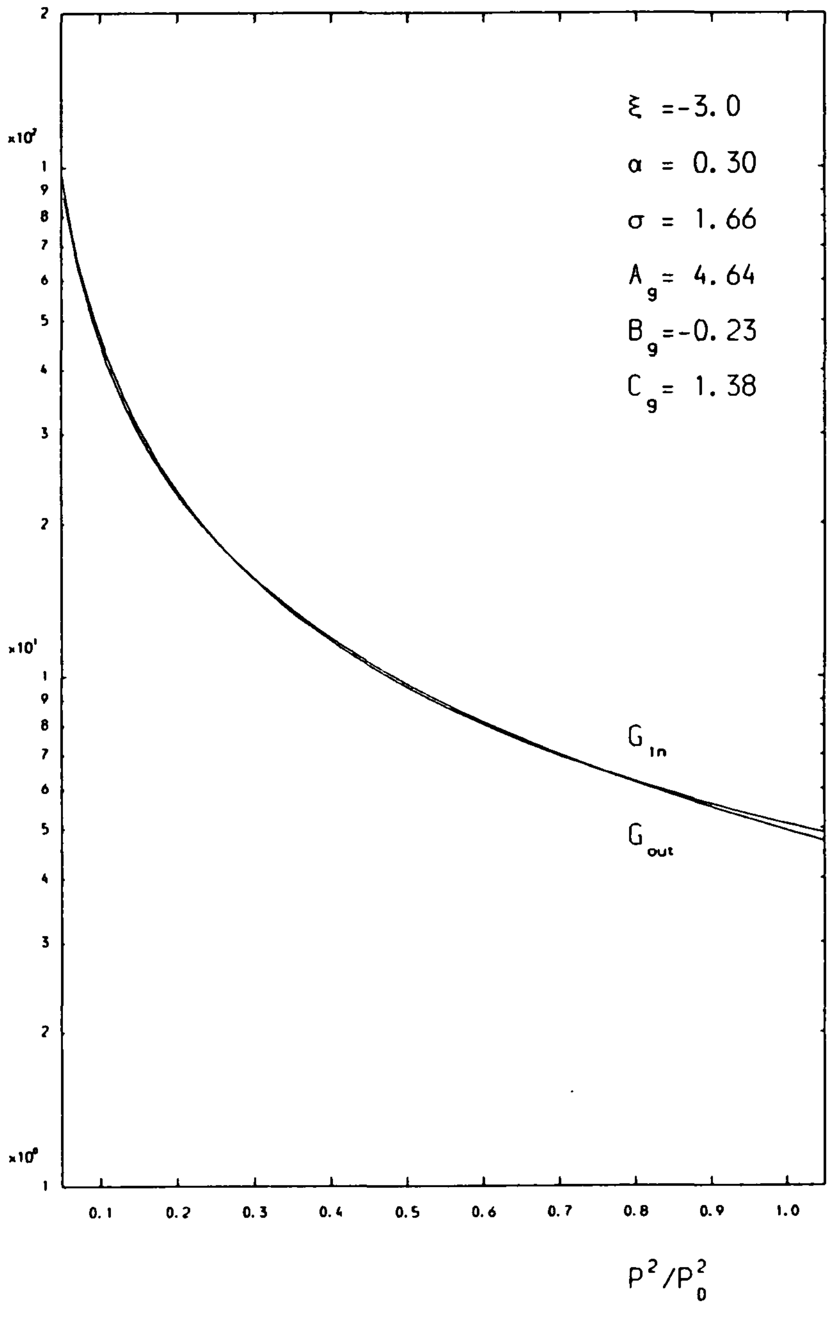

Figure 4.2b

The gluon function determined by the least squares fit method in the gauge $\xi = -3$.

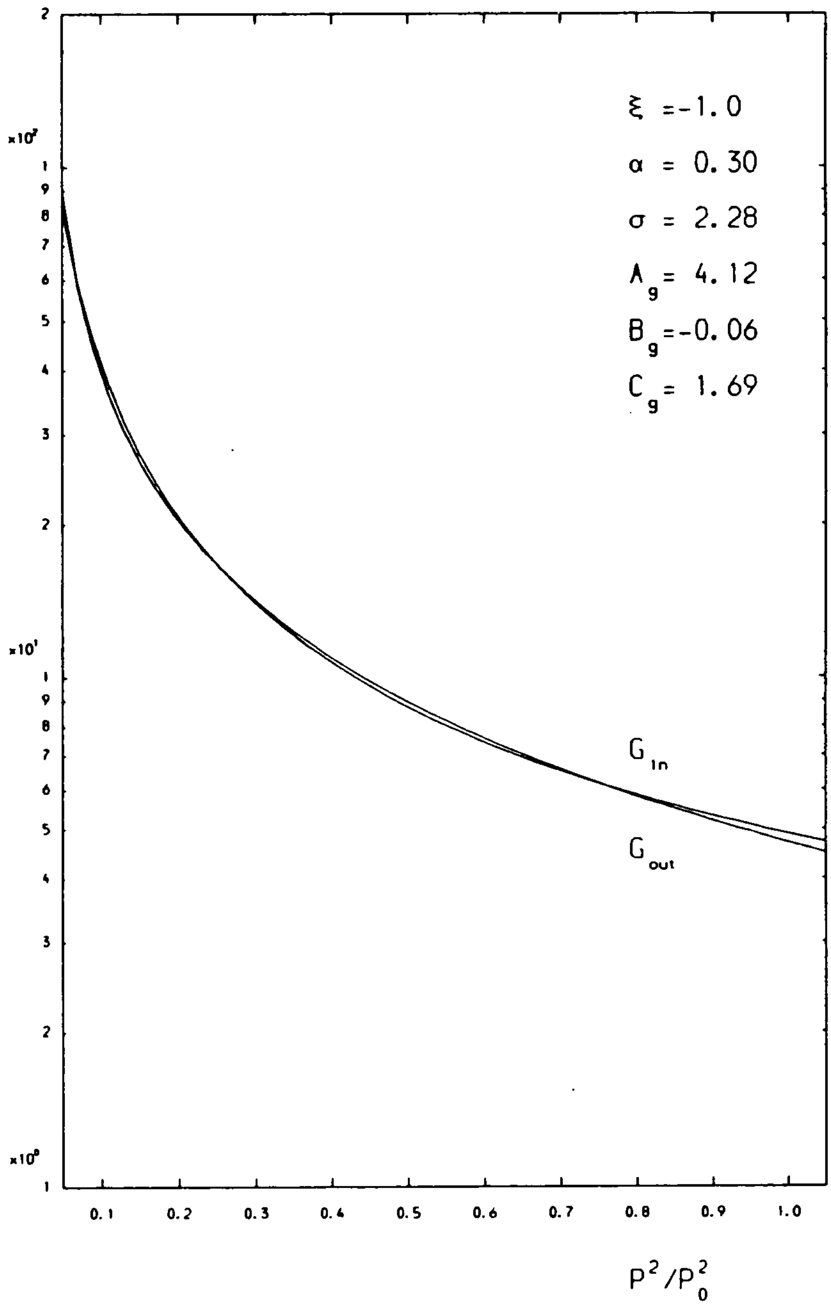

Figure 4.2c

The gluon function determined by the least squares fit method in the gauge $\xi = -1$.

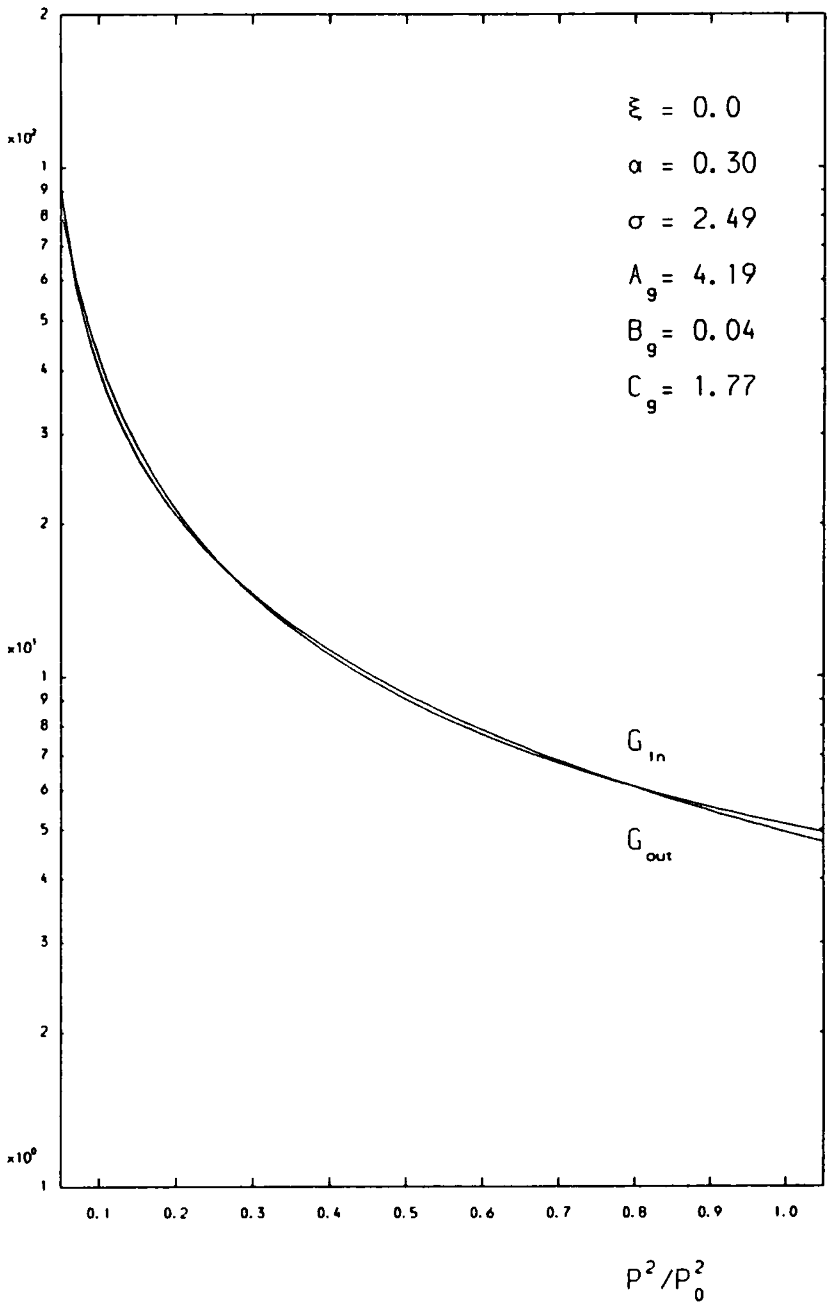

Figure 4.2d

The gluon function determined by the least squares fit method in the gauge $\xi = 0$.

Self-Consistent Gluon

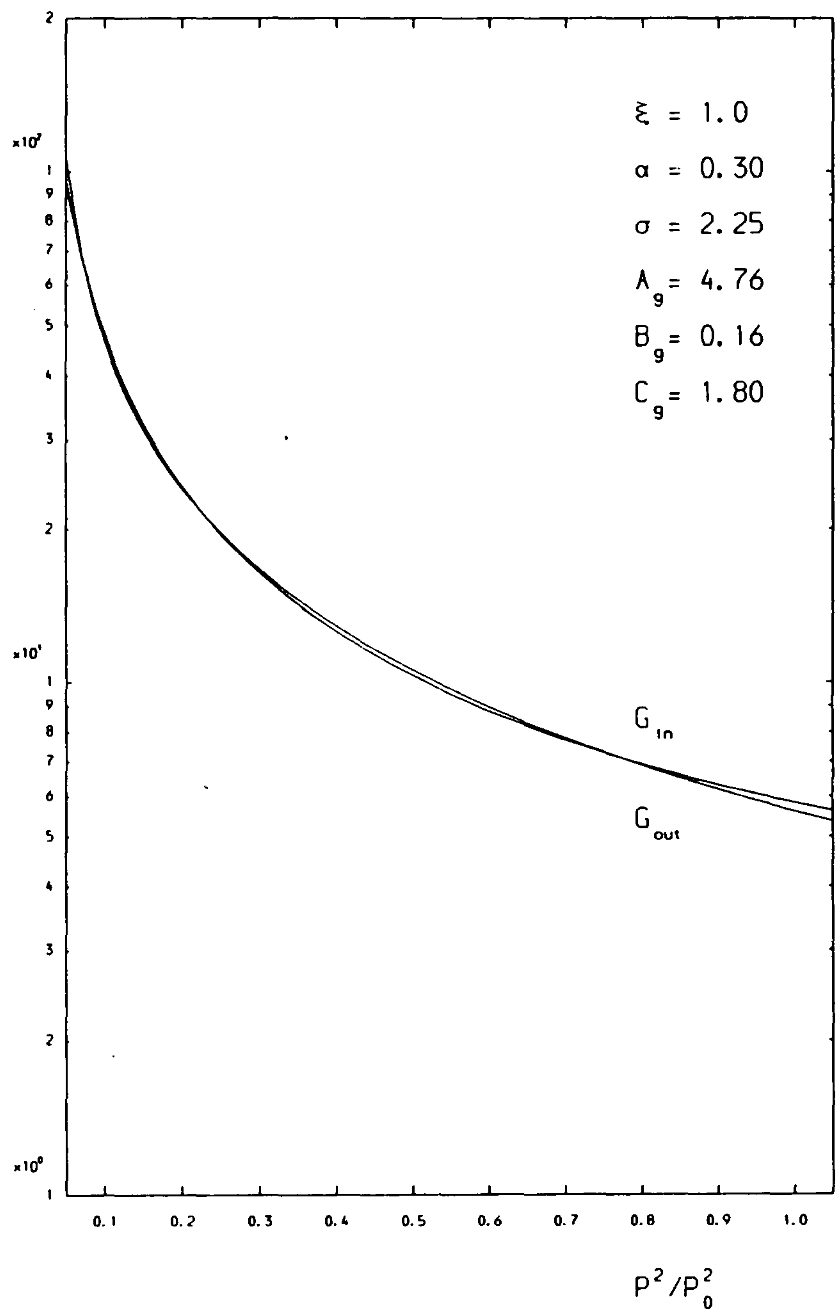

Figure 4.2e

The gluon function determined by the least squares fit method in the gauge $\xi = 1$.

Self-Consistent Gluon

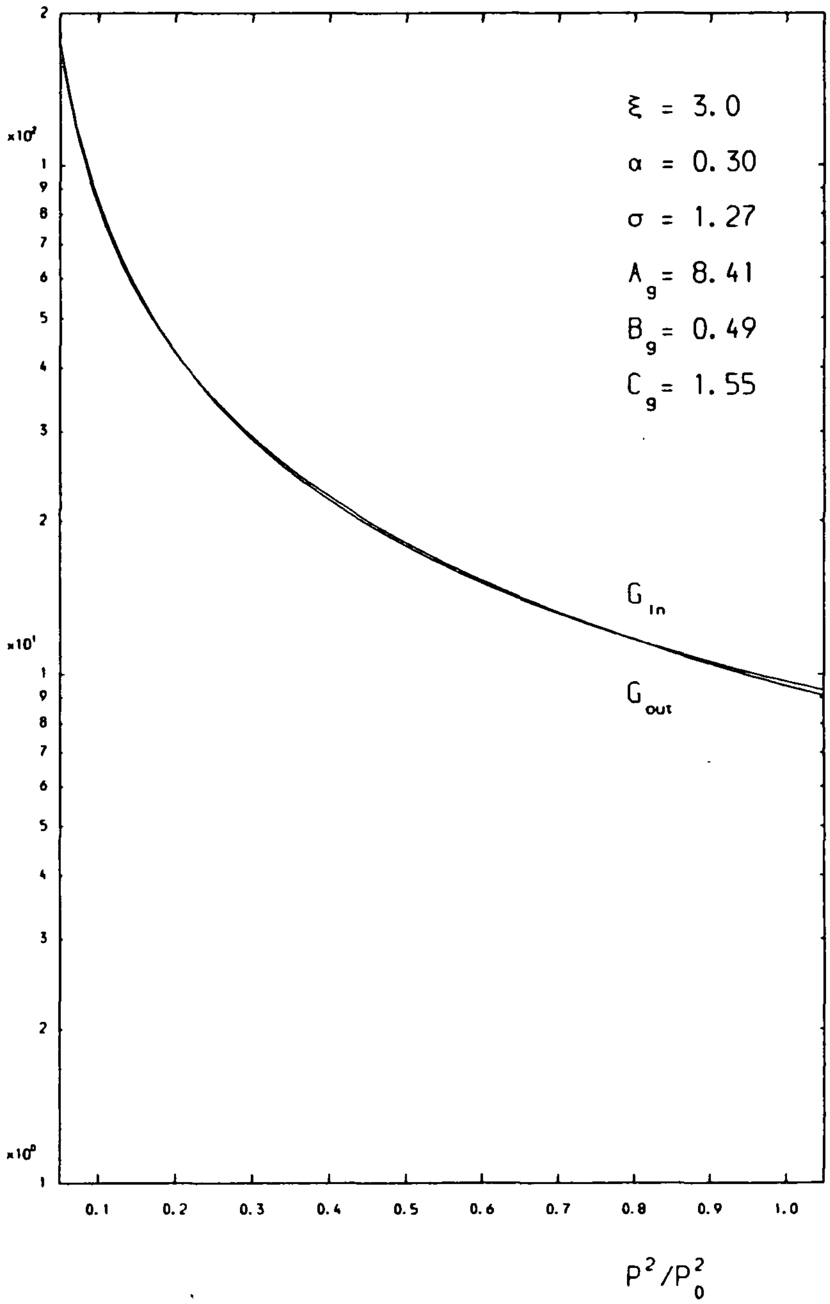

Figure 4.2f

The gluon function determined by the least squares fit method in the gauge $\xi = 3$.

Self-Consistent Gluon

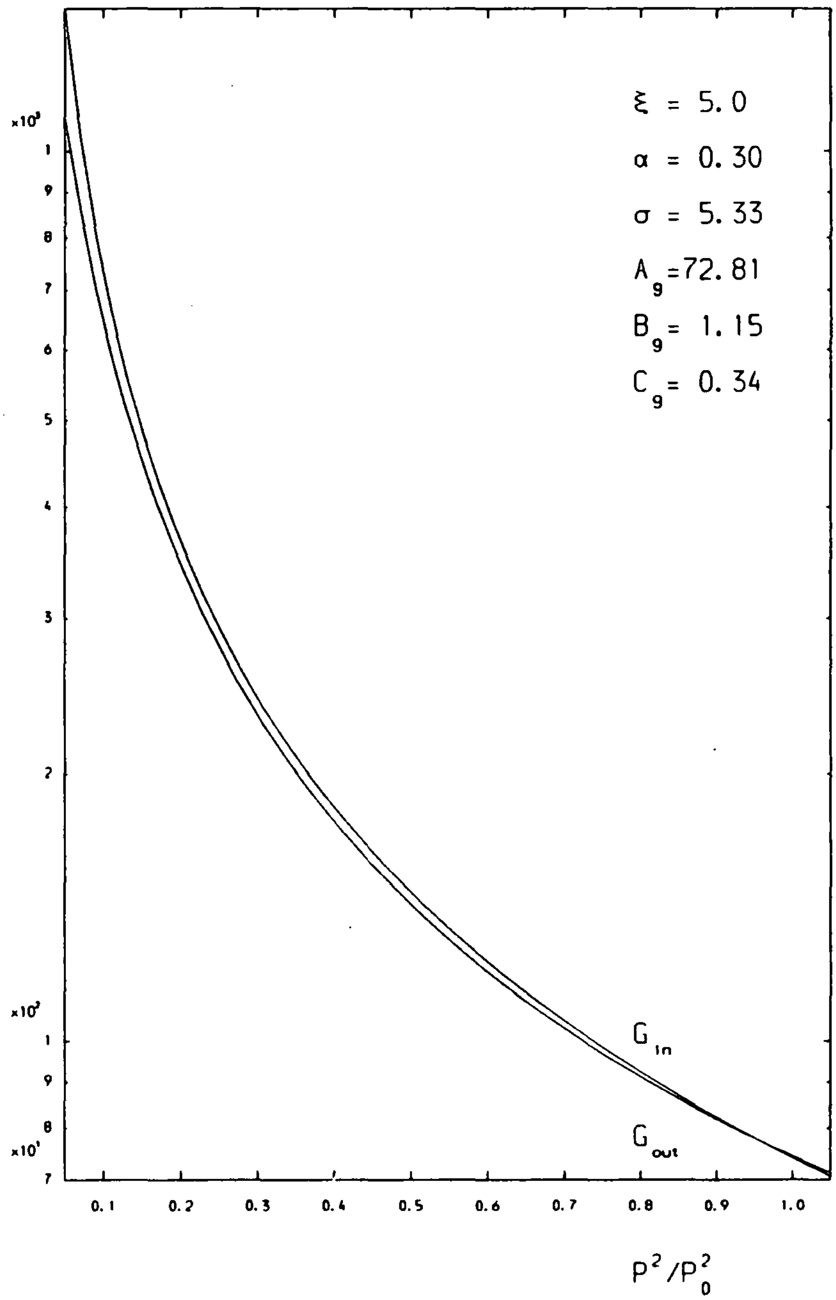

Figure 4.2g

The gluon function determined by the least squares fit method in the gauge $\xi = 5$.

Self-Consistent Gluon

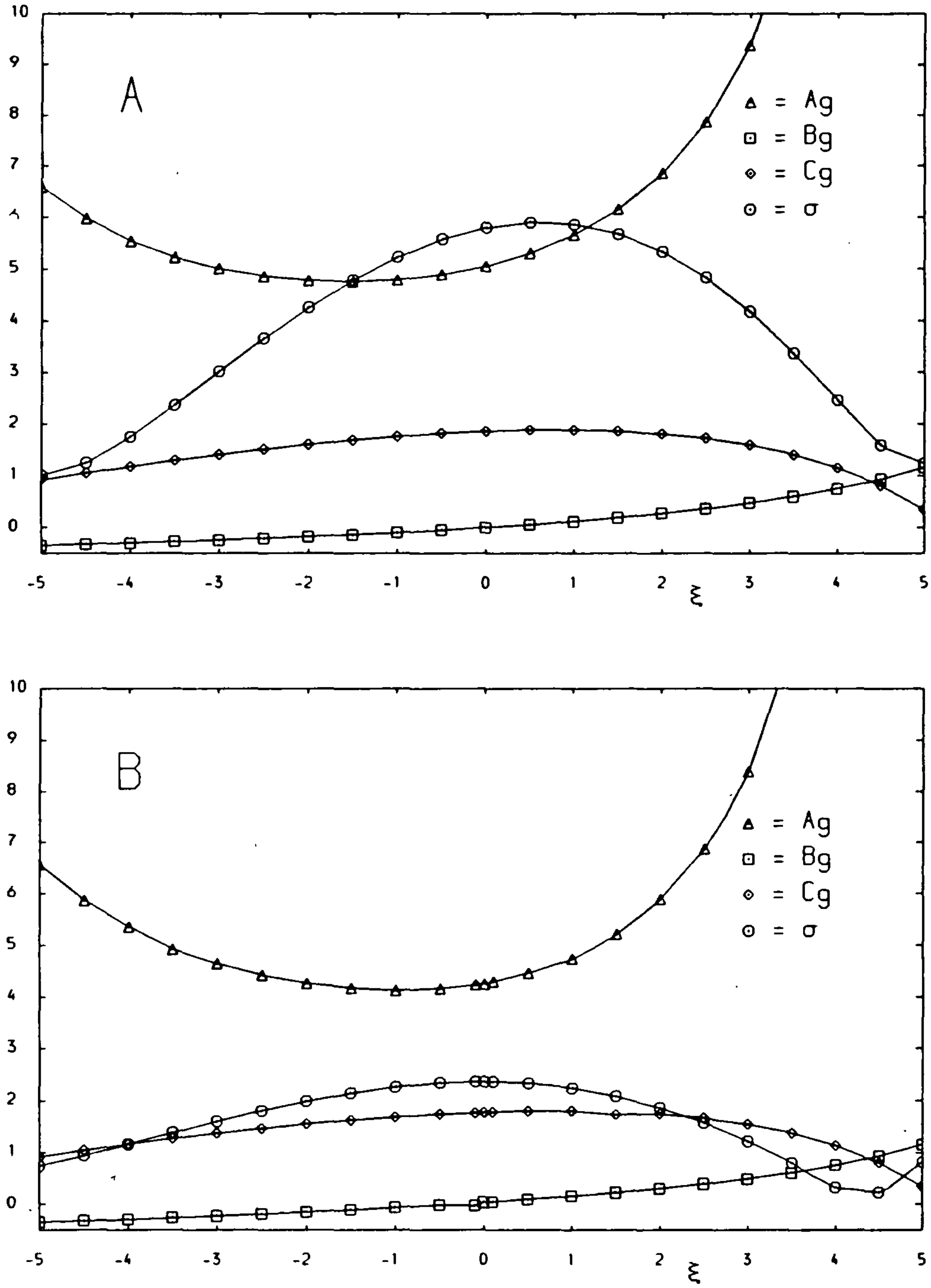

Figure 4.3

The parameters of the gluon function and the relative error σ determined by A) the Fourier method and B) the least squares fit method in the gauges ξ = - 5, 5.

renormalization point is determined by the gauge. Thus the value of the gluon function at lower momentum than the renormalization point is higher for higher gauges. This also explains why the fit is much better, as the effect of increasing the gauge can be countered by increasing the momentum scale. Thus, with a fixed scale, higher gauges effectively are only fitted over a smaller region of momentum.

The effect of changing the top limit of the momentum range can be seen in the next set of graphs figures (4.4a-d). These results are for the Feynman gauge $\xi = 1$, using the least squared fit method. As we expect the fit gets worse as the top limit is increased but is still reasonable at less than 6 %. Improvements on this will be discussed with refernce to later work in chapter 8.

Lastly, we look at the effect of varying the coupling strength for the Feynman gauge (4.5a-c). Perhaps surprisingly the fit is better for the higher value of the coupling constant. We can see the reason for this in the fact that the coupling constant enters inversely in the determination of C_g (3.8), this means that for higher values of the coupling parameter the enhanced term A_g will be more dominant. We might be tempted then to work with a much higher value of the coupling than we might naively expect, but this would be a mistake as it would then call into doubt our assumption that the one loop terms dominate over the two loop terms.

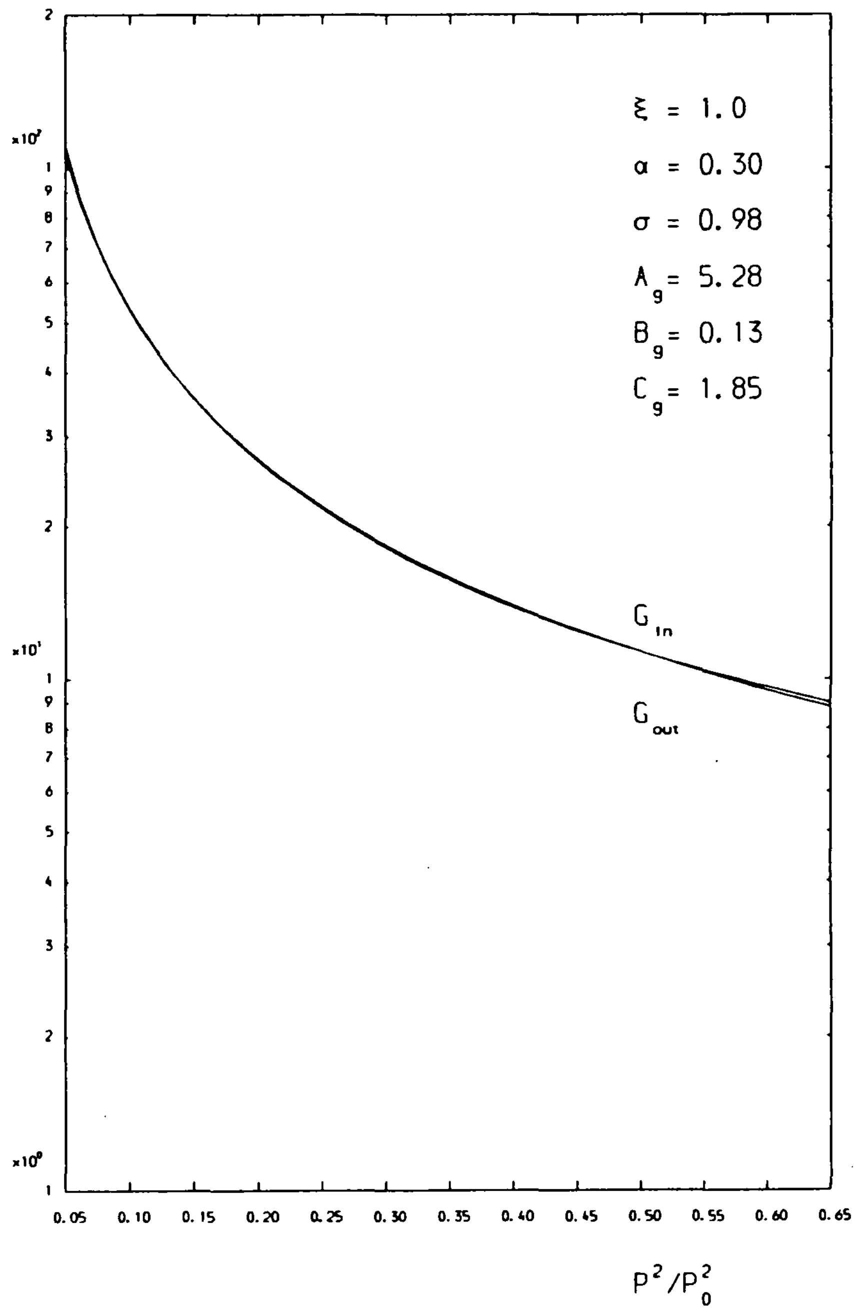

Figure 4.4a

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, fitting upto $0.65p_0^2$.

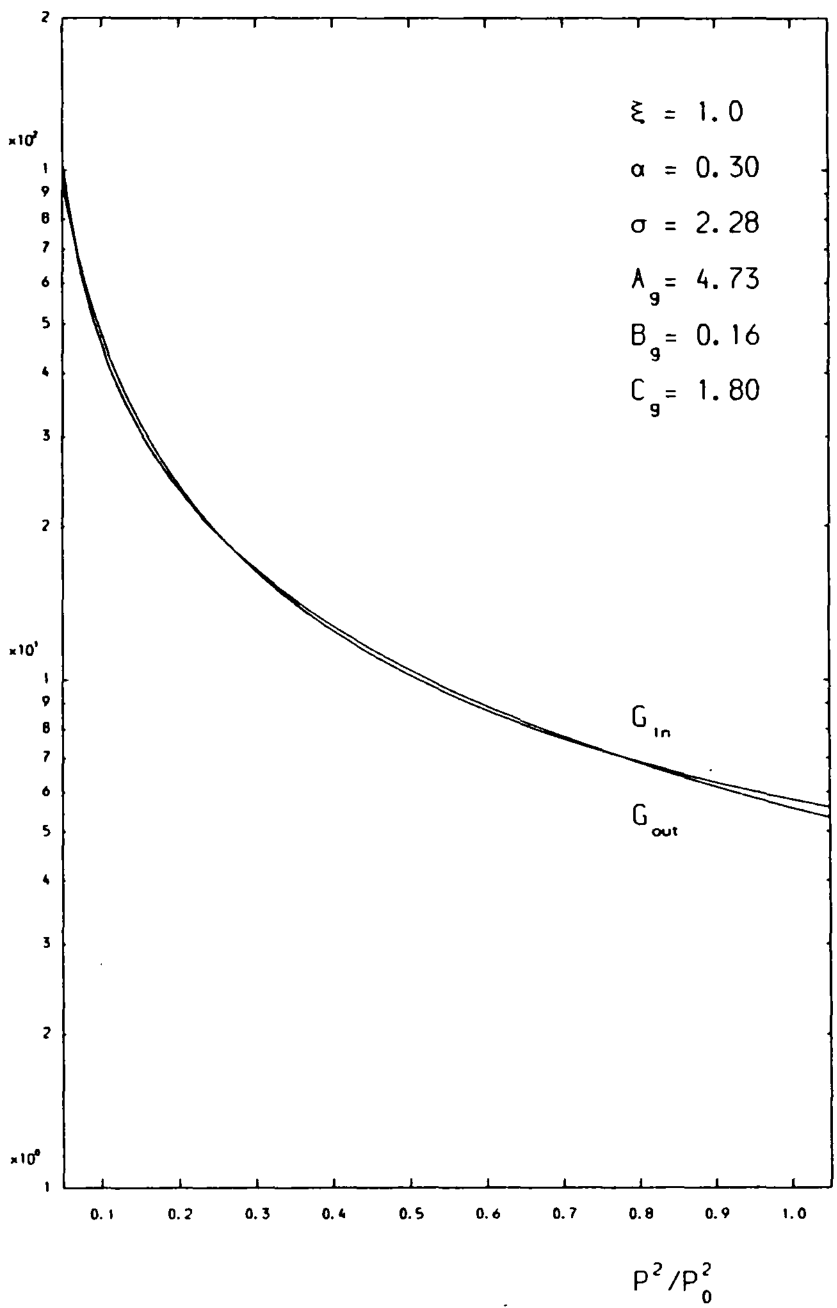

Figure 4.4b

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, fitting upto $1.05p_0^2$.

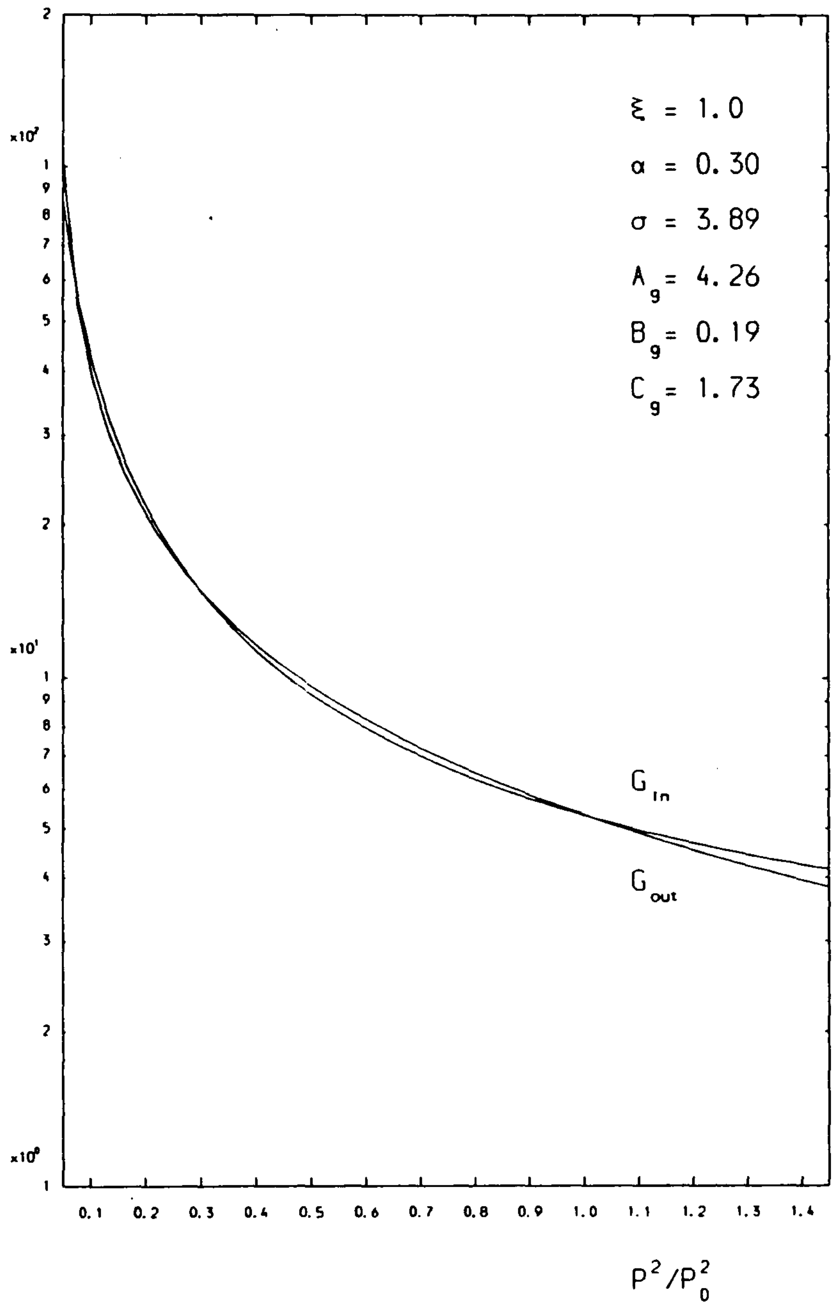

Figure 4.4c

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, fitting upto $1.45p_0^2$.

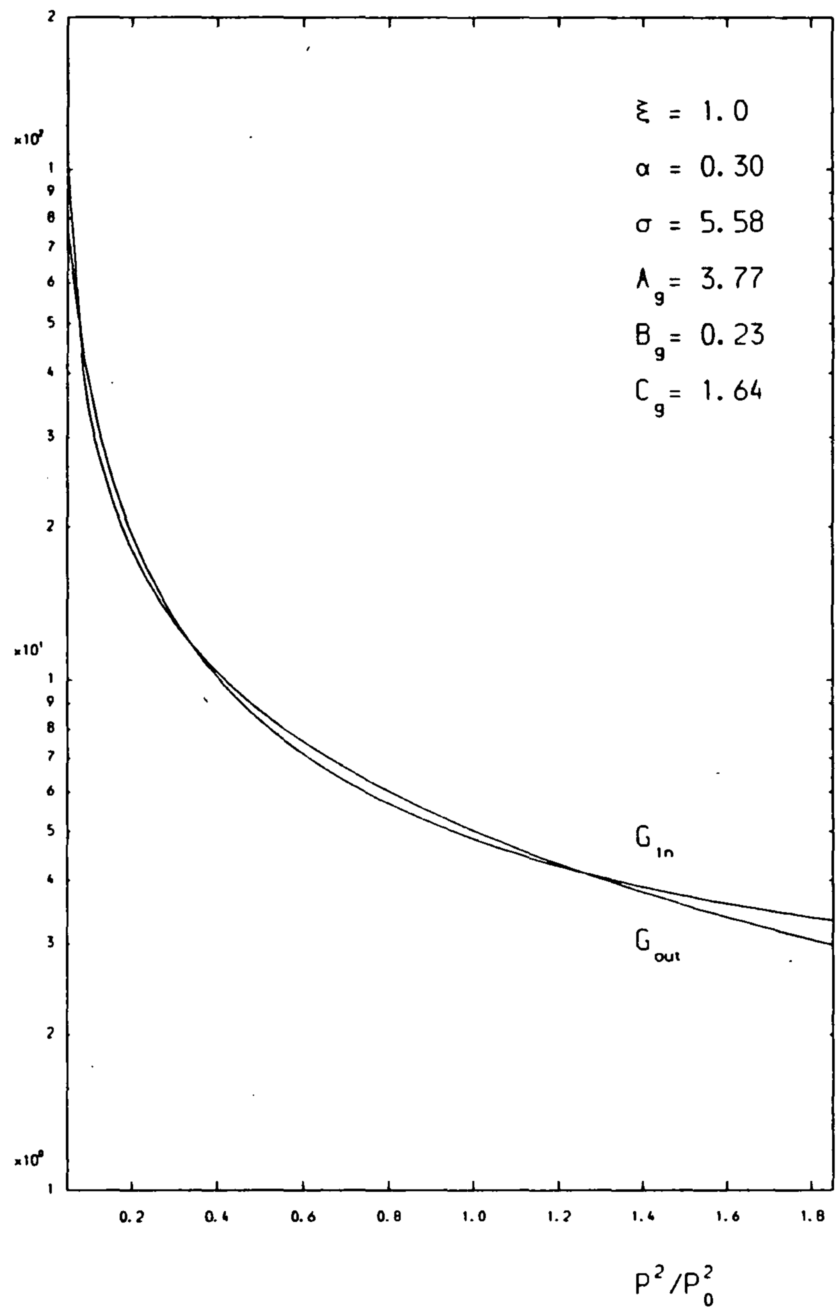

Figure 4.4d

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, fitting upto $1.85p_0^2$.

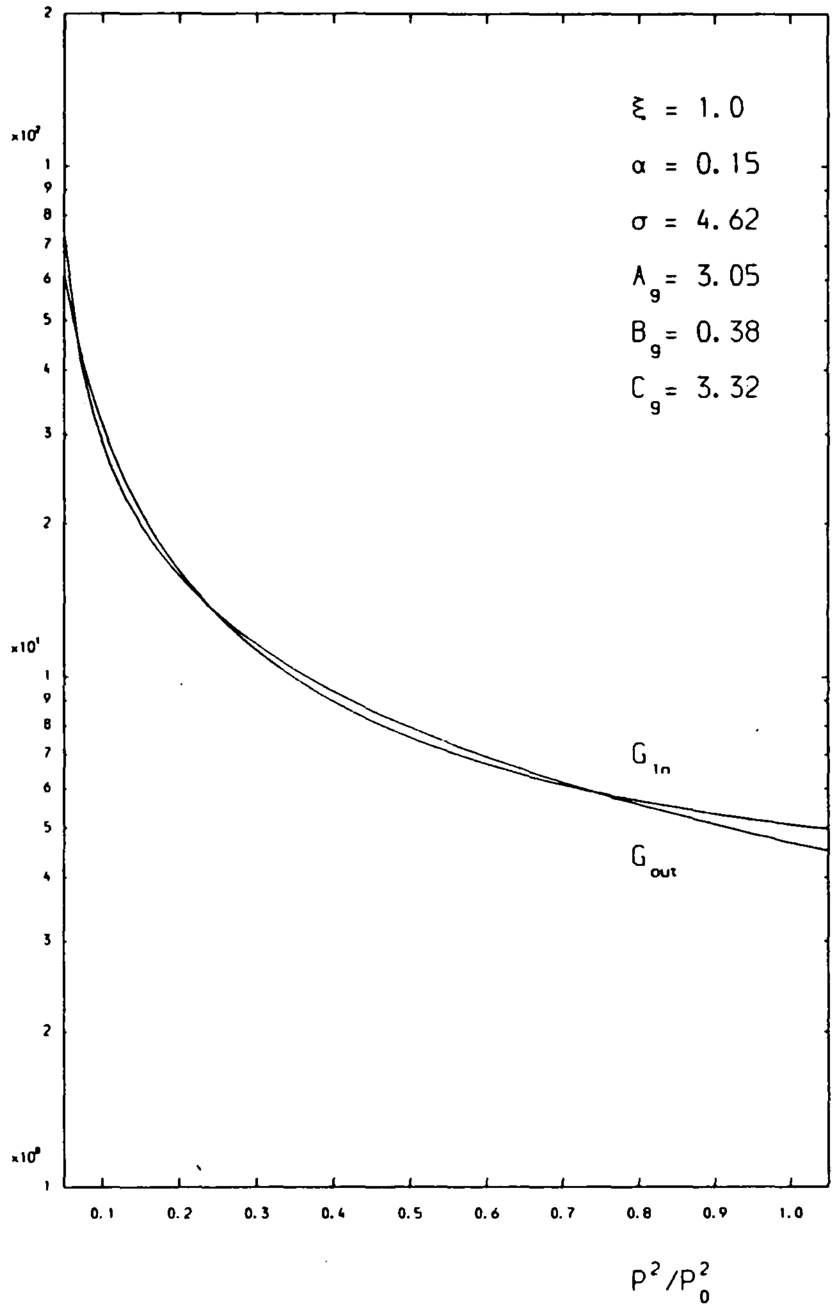

Figure 4.5a

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, with the coupling constant $\alpha_s = 0.15$.

Self-Consistent Gluon

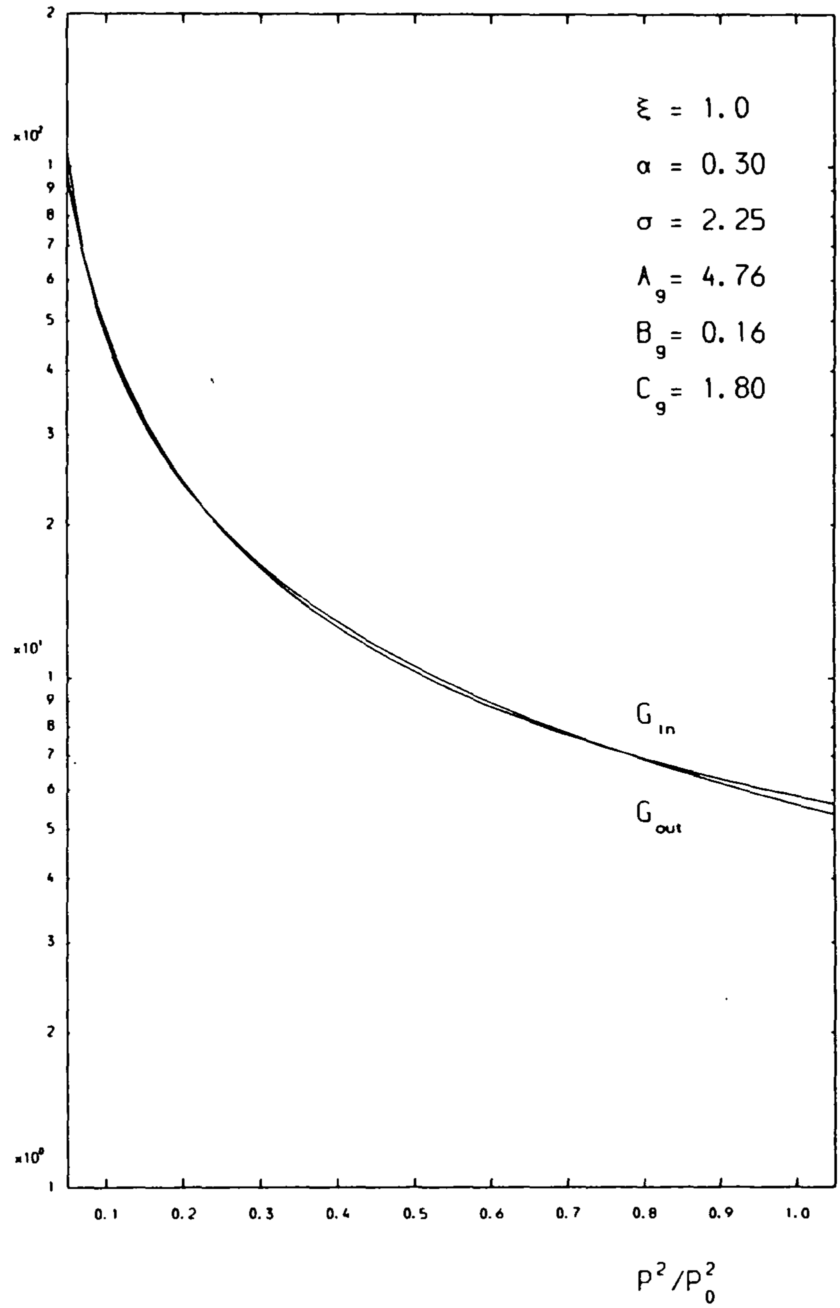

Figure 4.5b

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, with the coupling constant $\alpha_s = 0.3$.

Self-Consistent Gluon

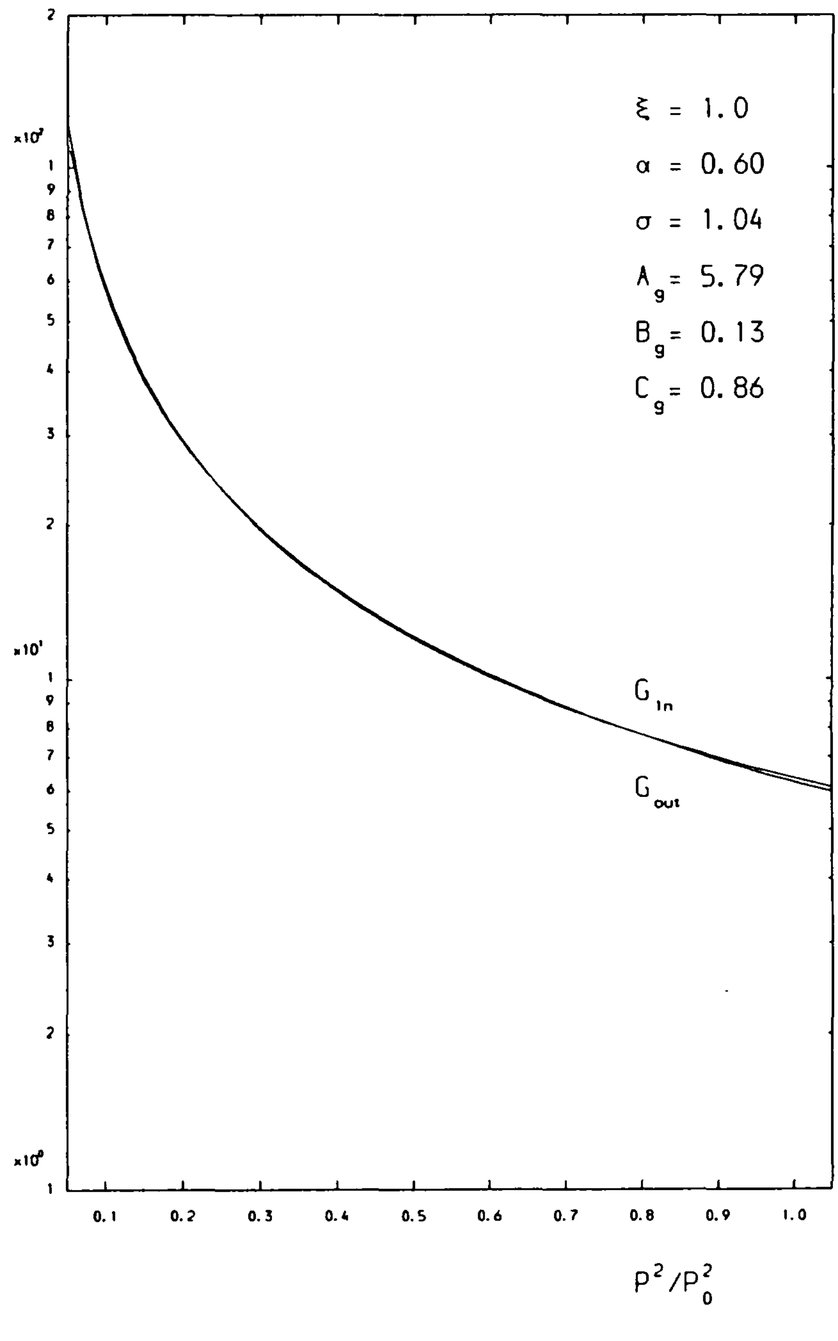

Figure 4.5c

The gluon function determined by the least squares fit method in the Feynman gauge, $\xi = 1$, with the coupling constant $\alpha_0 = 0.6$.

4:7 The Static Potential

We have now obtained a form of the gluon renormalization function in the absence of dynamical quarks. Although this was obtained at small momenta and so is not necessarily correct at large momenta, the form of the function (approximately constant) is not unreasonable even if its value is not.

From our result we can calculate the potential between two static quarks (see section 1:3). The energy associated with the field at large times is approximately,

$$E_{ijkl}(r) = T^a_{ij}\ T^a_{kl}\ g^2 \int_{-\infty}^{\infty} \Delta_{00}(r,\ t)\ dt \tag{7.1}$$

where r is the distance separating the quarks. For a colour singlet state the colour charges, i,j and k,l must each sum to zero. By introducing the Fourier transform of the propagator,

$$V_m(r) = -\ C_2(F)\ g^2 \int_{-\infty}^{\infty} dt \int \frac{d^4k}{(2\pi)^4}\ \Delta_{00}(k^2)\ \exp(\ -i\ k^\mu\ r_\mu\ )$$

The integral over time gives rise to a delta function of k_0. This allows the k_0 integral to be done trivially leading to the result.

$$V_m(r) = -\ C_2(F)\ g^2 \int \frac{d^3k}{(2\pi)^3}\ \Delta_{00}(k^2)\ \exp(\ -i\ \underline{k}.\underline{r}\ ) \tag{7.2}$$

where $\underline{k}$ and $\underline{r}$ are three vectors. Transforming to spherical polar coordinates in three dimensions,

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$$V_m(r) = -\frac{C_2(F)\alpha_s}{2\pi^2}\int_0^\infty k^2\,dk\int_{-1}^{1}dz\int_0^{2\pi}d\varphi\,\Delta_{00}(k^2)\exp(-ikrz)$$

where the axes have been choosen so that $\underline{k}.\underline{r} = k\,r\,z$, $z = \cos\theta$, and $\alpha_s = g^2/4\pi$. It is then a simple matter to perform the angular integrations so that

$$V_m(r) = -\frac{4C_2(F)\alpha_s}{\pi\,r^3}\int_0^\infty K\,dK\,\sin K\,\Delta_{00}(K/r) \tag{7.3}$$

where $K = kr$.

Now the general form of the gluon propagator is (3.2)

$$\Delta^{\mu\nu}(k^2) = \frac{G(k^2)}{k^2}\left\{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right\} + \xi\frac{k^\mu k^\nu}{k^2}$$

Hence the time time component of the propagator evaluated at $k_0 = 0$

$$\Delta_{00}(k^2)\Big|_{k_0=0} = \frac{G(k^2)}{k^2} \qquad \text{where} \quad k^2 = k^2\Big|_{k_0=0}$$

If we were talking about QED then the renormalization function is approximately one. Thus the potential is

$$V_{QED}(r) = \frac{q_1\,q_2}{2\pi^2\,r}\int_0^\infty\frac{dK}{K}\sin K$$

$$= \frac{q_1\,q_2}{4\,\pi\,r}$$

where q_1 and q_2 are the charges on the two particles. However, from our work we know that the renormalization function for gluons is very different from that for leptons, e.g.

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$$G(k^2) = A_g \frac{p_0^2}{k^2} + B_g + C_g \frac{k^2}{k^2 + p_0^2} \tag{7.4}$$

Substituting this result into the integral for the potential (7.3) gives

$$V_m(r) = - \frac{4C_2(F)\alpha_s}{\pi r} \int_0^\infty \frac{dK}{K} \sin K \left\{ A_g \frac{p_0^2 r^2}{K^2} + B_g + C_g + \frac{C_g r^2}{K^2 + p_0^2 r^2} \right\}$$

The integral over the last term can be done by consulting a table of standard definite integrals [3.1].

$$\int_0^\infty \frac{dK}{K} \frac{\sin K}{K^2 + p_0^2 r^2} = \frac{\pi}{2 p_0^2 r^2} [1 - \exp (- r p_0)]$$

The second integral we have already evaluated in the case of QED. The first integral is infra-red divergent. However, by regularising it (ie. an infra-red cutoff or by changing $K^2 \to K^2 + \varepsilon$ in the denominator) we get the result

$$\int_0^\infty \frac{dK}{K} \sin K = - \frac{\pi}{4} + \operatorname*{Limit}_{\varepsilon \to 0} \frac{\pi}{2 \varepsilon}$$

Subtracting the pole away from this term we get a static potential of

$$V_m(r) = - C_2(F) \alpha_s \left\{ - \frac{A_g p_0^2}{2} r + \frac{B_g}{r} + \frac{C_g}{r} \exp(- r p_0) \right\} - D \tag{7.5}$$

where D is a renormalization constant such that $V_m(r_0) = 0$ for some physical point r_0. This means that the string tension defined by,

$$V_m(r) = \kappa r$$

is,

$$\kappa' = 4/3\ \alpha_s\ A_g\ p_0^2$$

By comparing this potential to one obtained from fitting the spectrum of heavy mesons, for example the potential of Quigg and Rosner [4.1],

$$V(r) = -\frac{4\alpha_s}{3\ r} + \frac{r}{a^2} \qquad (7.6)$$

where $\alpha_s = 0.38$, $a = 2.43$ and r is measured in GeV^{-1}, we can determine the renormalization constant D and the gluon scale p_0^2. Notice that we cannot just compare the string tensions, as the tension is only an effective one for the region in which data is available. From the graphs (4.6a-c) we can see that the linear part of the potential in the phenomenological region is in good agreement.

It is not surprising that in the Coulomb part of the potential we get a different result as we do not have any constraint in the perturbative regime. If in our parameterization of the gluon function we were to add another parameter then we could constrain the function to go to the perturbative form. This would mean that p_0 would be determined and so the potential would be defined except for the constant D. This will be discussed in our summary of latter work beyond this thesis in chapter 8.

From this comparison with the experimentally determined function we can see how p_0 varies with gauge, see figure (4.7). Using these results we can then replot the gluon function in units of GeV. This is done in figure (4.8) for three different values of the gauge parameter and we can see, rather remarkably, that the function is now largely gauge independent.

Self-Consistent Gluon

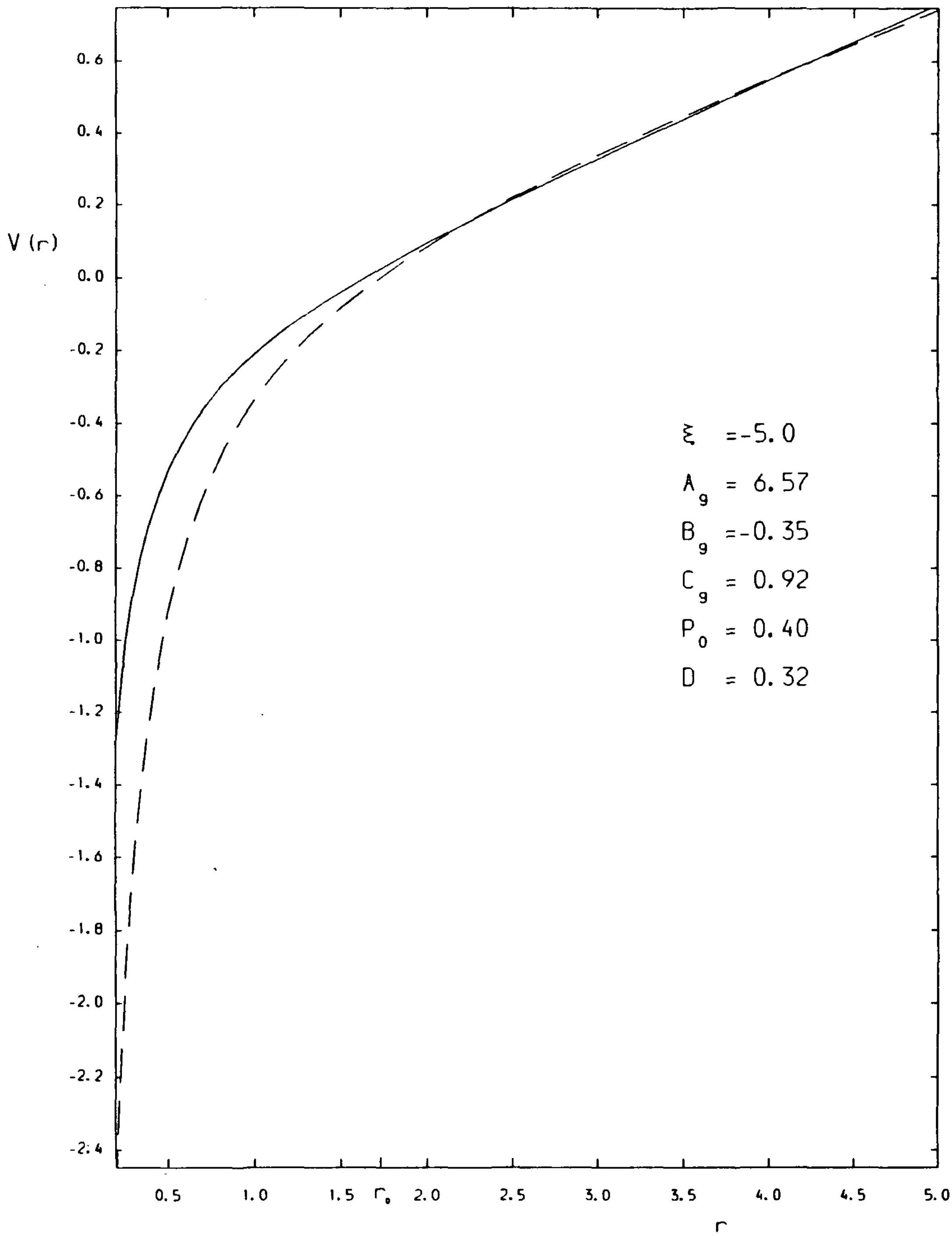

Figure 4.6a

The static potential (solid line) fitted to the potential of Quigg and Rosner (broken line) from r_0 to 5 GeV^{-1} in the gauge $\xi = -5$.

Self-Consistent Gluon

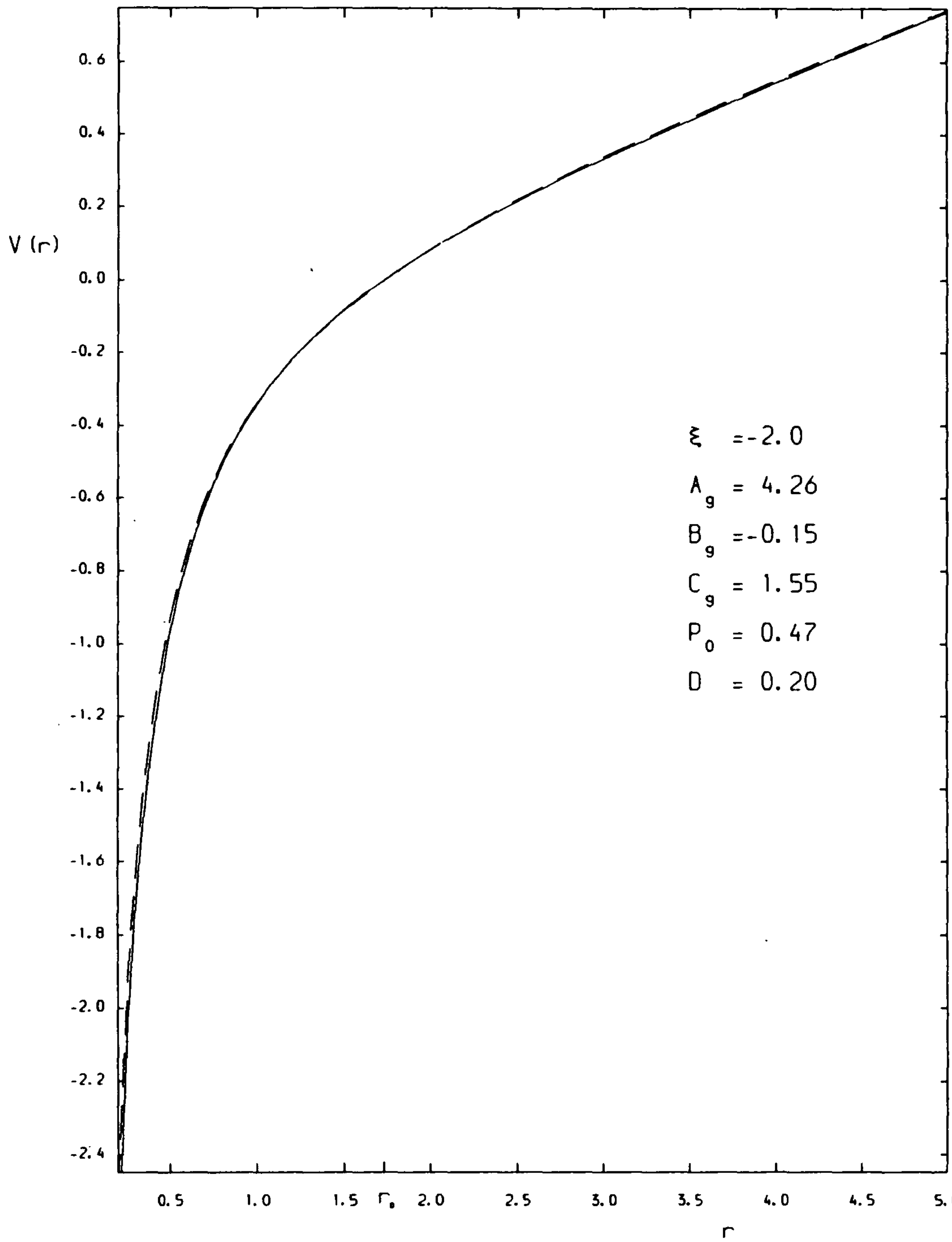

Figure 4.6b

The static potential (solid line) fitted to the potential of Quigg and Rosner (broken line) from r_0 to 5 GeV^{-1} in the gauge $\xi = -2$.

Self-Consistent Gluon

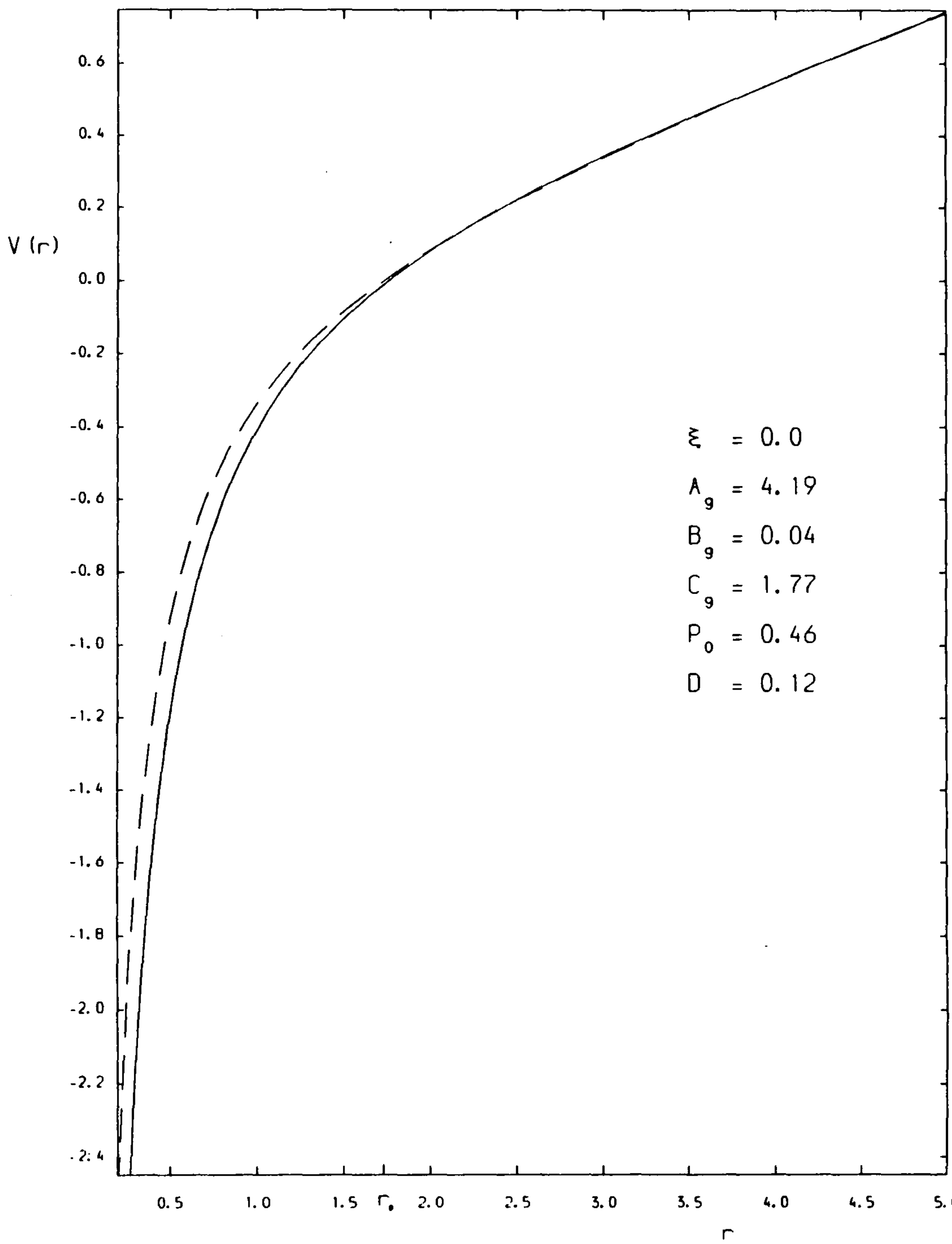

Figure 4.6c

The static potential (solid line) fitted to the potential of Quigg and Rosner (broken line) from r_0 to 5 GeV^{-1} in the gauge $\xi = 0$.

Self-Consistent Gluon

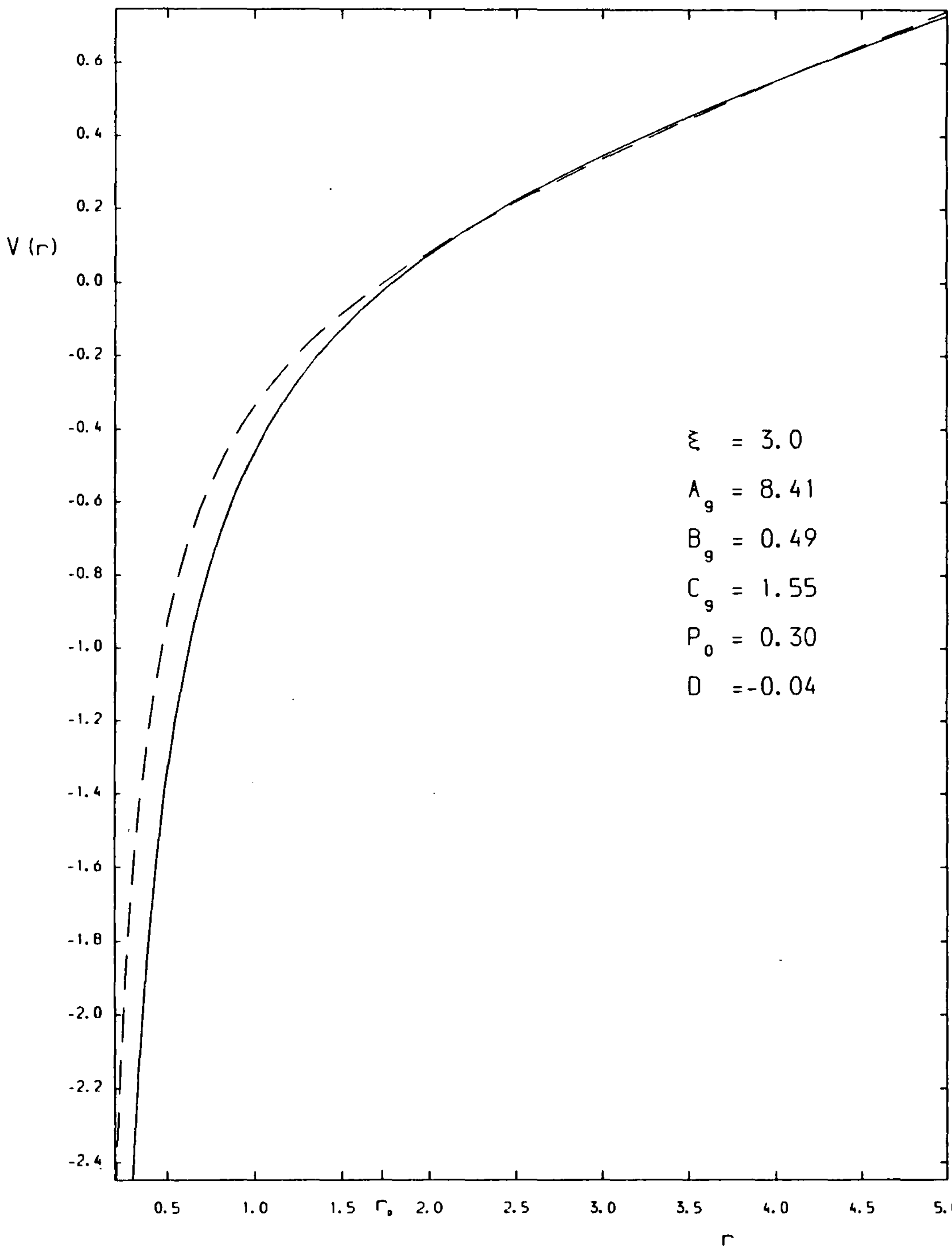

Figure 4.6d

The static potential (solid line) fitted to the potential of Quigg and Rosner (broken line) from r_0 to 5 GeV^{-1} in the gauge $\xi = 3$.

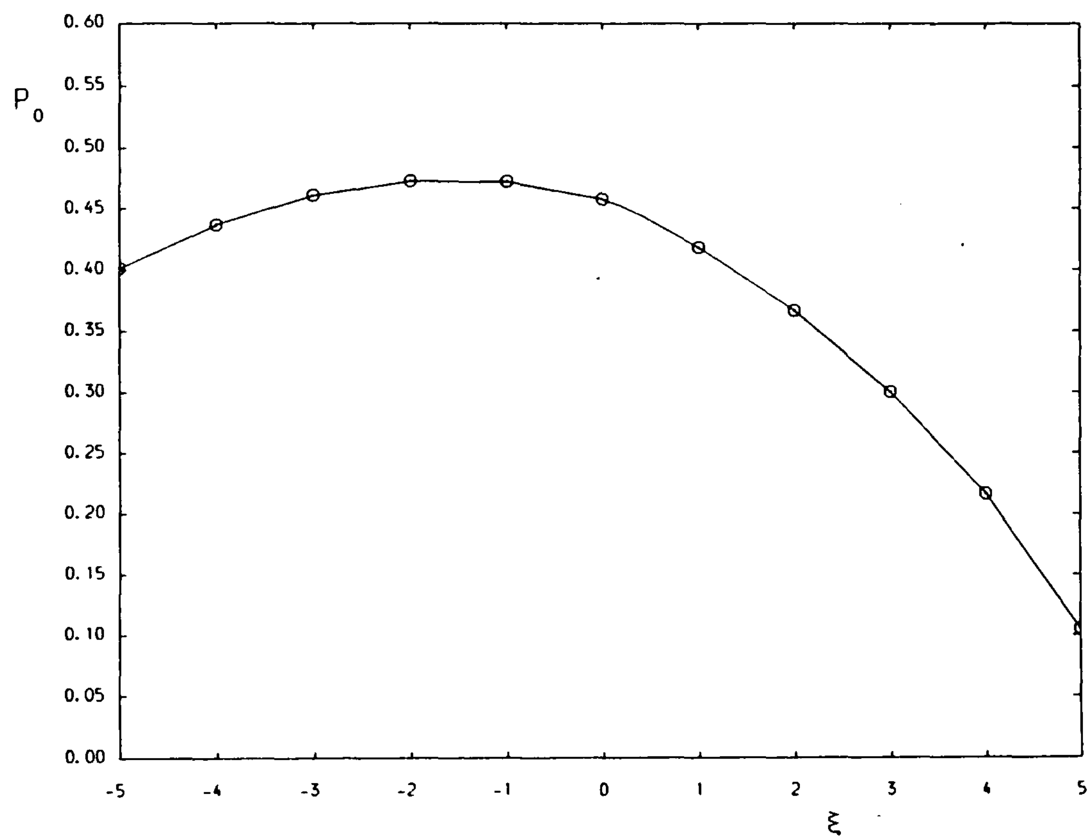

Figure 4.7

The gluon scale p_0^2 determined by fitting the potentials in the gauges $\xi = -5, 5$.

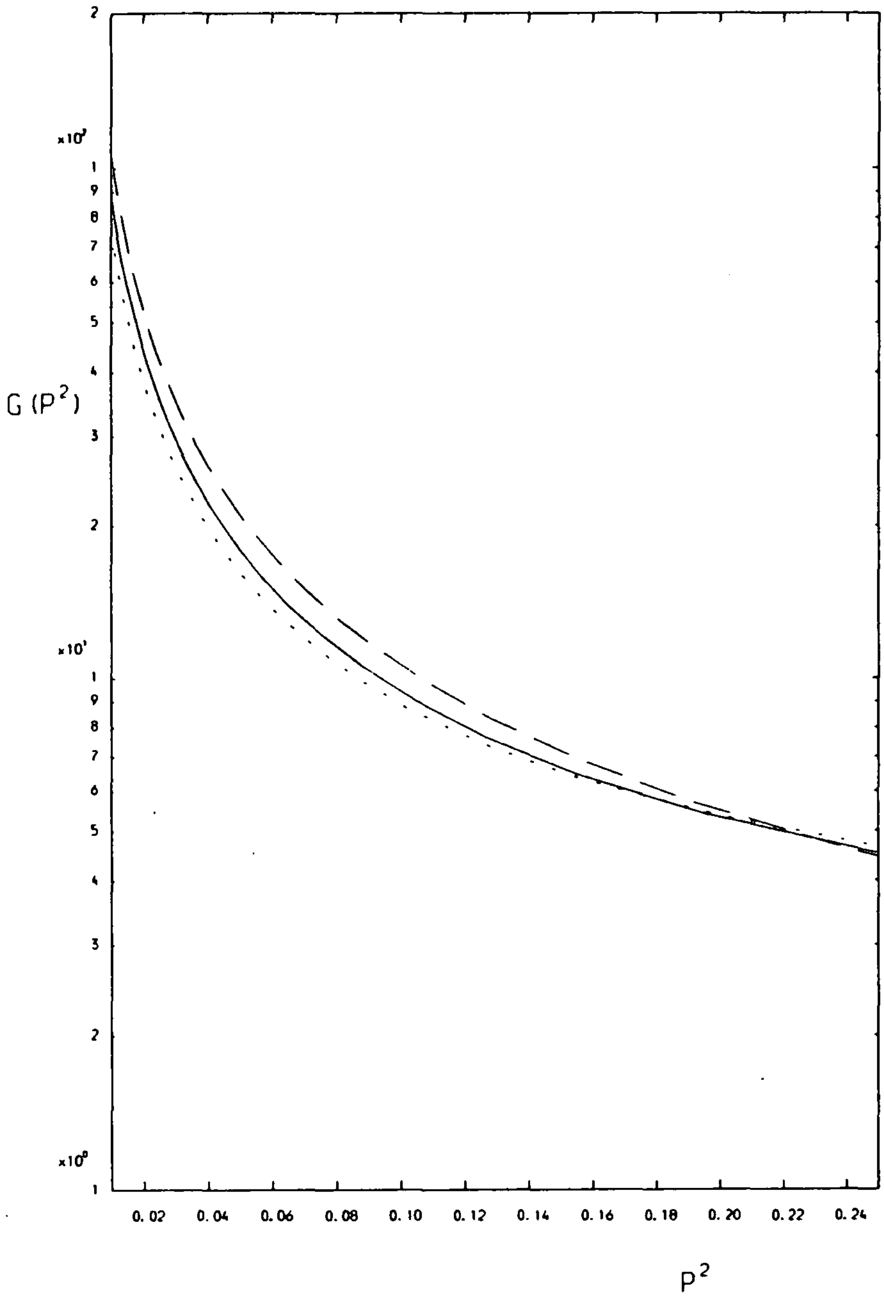

Figure 4.8

The static potential with the gluon scale p_0^2 determined by fitting the potential in the gauges $\xi = 0$ (solid line), $\xi = -5$ (broken line) and $\xi = 3$ (dotted line).

4:8 The Axial Gauge Gluon Propagator

Now that we have completed our calculation of the gluon propagator in the covariant gauge, let us discuss in some detail the work that has already been done for the axial gauge. In a series of papers Baker, Ball, Zachariasen [2.4] and others have been studying the infra-red behaviour of the gluon propagator in the axial gauge. In this gauge the Schwinger-Dyon equation for the gluon propagator in terms of the full 3 point and 4 point vertices can be written as (cf figure 2.1 without the ghost loop)

$$\Pi_{\mu\nu}(p) = - p^2 \left[\delta_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right] +$$

$$+ \frac{1}{2} g^2 \int \frac{d^4k}{(2\pi)^4}\; i\Gamma^{(0)}_{\mu\alpha\gamma}(p,-k,-q)\; \Delta_{\alpha\beta}(k)\; \Delta_{\gamma\delta}(q)\; i\Gamma_{\beta\delta\nu}(k,q,-p) +$$

$$+ \frac{1}{2} g^2 \int \frac{d^4k}{(2\pi)^4}\; i\Gamma^{(0)}_{4\,\mu\nu\alpha\beta}(p,-k,-q)\; \Delta_{\alpha\beta}(k) + \text{two loop terms} \tag{8.1}$$

where Γ is the full triple gluon vertex $\Gamma^{(0)}$ and $\Gamma^{(0)}_4$ are the bare three point and four point vertices respectively. In the axial gauge the propagator and its inverse satisfy the relation

$$\Pi_{\mu\alpha}(p)\; \Delta^{\alpha}{}_{\nu}(p) = \delta_{\mu\nu} - \frac{n_\mu p_\nu}{n.p} \tag{8.2}$$

where n is the direction of the gauge choice, and hence, n.A = 0 which implies

$$n^\mu\, \Delta_{\mu\nu} = 0. \tag{8.3}$$

In general the propagator, and its inverse, can be split into two pieces multiplied by scalar functions of the variables, p^2/M^2, M some renormalization point, and a gauge variable $\gamma = p^2 n^2 / (n.p)^2$. Thus the Schwinger-Dyson equation (8.1) can be split into two scalar equations by contracting it with $\delta^{\mu\nu}$ or $n^\mu n^\nu / n^2$. (We could have done a similar projection in the covariant gauge into equations for the transverse part and the longitudinal part, which would be an identity because of the Ward-Takahashi identity).

At this point, Baker, Ball and Zachariasen make the simplifying ansatz that only one of the scalar functions in the propagator contains any infra-red singularities. Specifically that as $p^2 \to 0$, γ fixed

$$\Delta_{\mu\nu}(p) \to -\frac{Z(p,\gamma)}{p^2}\Delta^{\circ}_{\mu\nu}(p)$$

and hence, from (8.2)

$$\Pi_{\mu\nu}(p) \to \frac{p^2}{Z(p,\gamma)}\left\{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right\} \tag{8.4}$$

They later verify that this ansatz is self-consistent with the result. This simplification also means that it is sufficient to consider only one of the two scalar equations coming from the Schwinger-Dyson equation (8.1). In particular they look at the equation obtained by contracting with $n^\mu n^\nu / n^2$

$$-\frac{p^2}{Z(p)}\left[1-\frac{1}{\gamma}\right] = -p^2\left[1-\frac{1}{\gamma}\right] +$$

$$+\frac{1}{2}g^2\int\frac{d^4k}{(2\pi)^4}\frac{n.(k-q)}{n^2}\Delta_{\alpha\beta}(k)\,\Delta_{\gamma\delta}(q)\,i\Gamma_{\beta\delta\nu}(k,q,-p)\,n_\nu +$$

$$+\frac{1}{2}g^2\int\frac{d^4k}{(2\pi)^4}\Delta_{\alpha\alpha}(k) \tag{8.5}$$

This result is considerably simplified by the condition (8.3), which not only reduces the spin structure coming from the bare vertices but also guarantees that the two loop contributions vanish.

The only unknown in this equation (8.5) is the full triple gluon vertex. Baker, Ball and Zachriasen then follow the procedure outlined in section 2:4 to find a closed form for the gluon propagator equation. The Slavnov-Taylor identity in the axial gauge is

$$i\Gamma_{\alpha\beta\mu}(k,q,-p)(-p)^\mu = \Pi_{\alpha\beta}(k) - \Pi_{\alpha\beta}(q) \tag{8.6}$$

which leads to the longitudinal form of the triple gluon vertex

$$\Gamma^{(L)}_{\mu\nu\sigma}(p,q,r) = \delta_{\mu\nu}(\,Z^{-1}(p)\,p_\sigma - Z^{-1}(q)\,q_\sigma\,) -$$

$$-\frac{Z^{-1}(p)-Z^{-1}(q)}{p^2-q^2}(\,p.q\,\delta_{\mu\nu} - q_\mu\,p_\nu\,)\,(p-q)_\sigma +$$

$$+\text{ cyclic permutations} \tag{8.7}$$

Which is the same form as we found for the covariant gauge with bare ghost (section 3:3). We see that the vertex is then totally determined in terms of the function Z(p). It is in the demand that the vertex is free of kinematic singularities that this work differs from that of others in the axial gauge [4.2], and that guarantees not only that the longitudinal part is unique, but also that it dominates over the transverse part [2.7].

Using the central assumption that the replacement of the full triple gluon vertex by its longitudinal component does not affect the infra-red singularity in the gluon propagator, the Schwinger-Dyson equation (8.5) becomes a closed integral equation (see 8.7). The value of this result outside the infra-red region, particularly at intermediate momenta, is not clear although it is expected to give the correct form in the ultra-violet limit.

Thus by replacing the full vertex by its longitudinal component (8.7) in the Schwinger-Dyson equation (8.5) they find

$$\frac{n_\mu \, \Pi^{\mu\nu} \, n_\nu}{n^2} = - \frac{p^2}{Z(p)} \left[1 - \frac{1}{\gamma} \right]$$

$$= - p^2 \left[1 - \frac{1}{\gamma} \right] + \lambda \int d^4k \, \frac{n.(k-q) \; n.q}{n^2} \, \Delta^{(0)}_{\alpha\beta}(k) \, \Delta^{(0)}_{\gamma\delta}(q) \; \times$$

$$\times \left\{ - \frac{Z(k)}{Z(p)} \frac{Z(q) - Z(p)}{q^2 - p^2} (p+q)^\beta p^\delta + \frac{Z(k) - Z(q)}{k^2 - q^2} \left(k.q \, \delta^{\beta\delta} - q^\beta k^\delta \right) + \right.$$

$$\left. + Z(k) \, \delta^{\beta\delta} \right\} + \lambda \int \frac{d^4k}{k^2} \, Z(k) \left[2 + \frac{k^2 \, n^2}{(k.n)^2} \right] \qquad (8.8)$$

where

$$\lambda = \frac{C_2(A) g^2}{(2\pi)^4}$$

This result, complex as it is, is again much simplified by the condition (8.3). (This simplification does not occur in the covariant gauge which is why we were forced to make a further assumption, ie. the Mandelstam approximation).

Unfortunately this equation is divergent and would not lead to a propagator that satisfies the Ward identity

$$\Pi_{\mu\nu}(p) \to 0 \quad \text{as} \quad p_\alpha \to 0 \tag{8.9}$$

which guarantees that the propagator is massless. In order to rectify this the equation (8.8) must be renormalised in such a way as to preserve the Ward identity (8.9). The standard way to do this is to use dimensional regularization and subtract away the poles, but in this case this cannot be done since the function Z(q) is not known as an analytic function. The alternative is to simply subtract $\Pi_{\mu\nu}(0)$ from the right hand side of the Schwinger-Dyson equation (8.8), which can then be written in the form

$$\frac{1}{Z(p)} = 1 + \lambda \int d^4k\; K(k,p,n)\; Z(k) + \frac{\lambda}{Z(p)} \int d^4k\; L(k,p,n)\; Z(k)\; Z(p-k) \tag{8.10}$$

This contains logarithmic divergences which can be handled by renormalising the charge. Define $Z(p) = Z(m)\, Z_R(p)$ for some fixed space-time vector M^μ where $Z_R(M) = 1$. Then the renormalised coupling constant is defined by (modifying the kernels slightly to avoid the pole at $q^2 = p^2$)

$$g^2(M) = \frac{g^2 \quad Z(M)}{1 + \lambda\; Z(M) \int d^4\; K(k,M,n)\; Z_R(k)}$$

This is not the standard definition of the axial running coupling constant, nor does it coincide with it even in the infra-red limit. From these two equations they get the final form of the renormalised equation for Z

$$\frac{1}{Z(p)} = 1 + \lambda(M) \int d^4k \, [\, K(k,p,n) - K(k,M,n) \,] \, Z_R(k)$$

$$- \frac{\lambda(M)}{Z_R(p)} \int d^4k \, [\, L(k,p,n) - L(k,M,n) \,] \, Z_R(k) \, Z_R(M-k) \tag{8.11}$$

Now they try and find the possible form for Z. By consideration of the dimensionality and convergence of the integrals together with the general structure of the kernels, Baker, Ball and Zachariasen find that the only consistent form is

$$Z_{in} = A \, M^2 / p^2 + B \, (p^2/M^2)^{\nu} \tag{8.12}$$

The reason why Baker, Ball and Zachariasen cannot have a constant in their parameterization is that it would lead to a $\log(p^2)$ term which not would give a self-consistent result. In our case such a logarithm is essential in order to cancel the logarithm coming from the gauge dependant term.

From numerical studies of equation (8.11), Baker, Ball and Zachariasen found that for a range of γ between 2 and 10, there exists an input function of the form (8.12) with a finite value of ν. This has also been verified by a later analytic study of the equation with the gauge choice $n.p = 0$.

So far only the scalar equation coming from the contraction of the Schwinger-Dyson equation with $n_\mu \, n_\nu \, / \, n^2$ has been considered. It can be shown that this behaviour of the propagator is consistent with the second scalar equation, coming from contracting the Schwinger-Dyson equation with $\delta_{\mu\nu}$, neglecting the two loop contributions. This is because the basic structure of the equation is the same, although the kernels K and L are different. However, the value of the coefficient A coming from this equation is different from the one obtained in the first case. This is to be expected and confirms the fact that although $\Gamma^{(T)}$ does not affect the the infra-red singularity it does affect the value of the coefficient in front of the singularity. (This case is closely resembles the structure of the equation in the covariant gauge.)

Thus we have seen how in both the covariant and axial gauges the infra-red behaviour of the gluon propagator is approximately of the form

$$\Delta_{R\,\mu\nu}(p) \;=\; - A \frac{M^2}{p^2} \Delta^{\circ}{}_{\mu\nu}(p)$$

where $\Delta^{\circ}{}_{\mu\nu}(p)$ is the bare propagator. This result is a self-consistant solution of the Schwinger-Dyson equations because the gauge invariance of the theory requires a zero mass for the gluon. The fact that the longitudinal part of the triple gluon vertex dominates over the transverse part (which has been neglected) means that the transverse part of the vertex will not affect the infra-red singularity. However, since the coefficient of the singular term is determined by the next to leading order term in the gluon function, the transverse part will affect its value.

Chapter 5

The Quark Propagator

5:1 Introduction

In this chapter we investigate the full quark propagator using the relevant Schwinger-Dyson equation. In the second section we follow through the usual one loop perturbative calculation of the quark propagator in the general covariant gauge. To do this we use the technique of dimensional regularization in $4-2\varepsilon$ dimensions in Euclidean space. Then by applying the renormalization group equation we get the leading log result for the propagator in terms of the running coupling constant raised to a power, which is the anomalous dimension of the quark propagator.

In the third section we consider the Schwinger-Dyson equation itself. By using the Ward-Takahashi identity we determine a form for the longitudinal part of the quark-gluon vertex involving only the quark renormalization function. It can be seen from this result that the longitudinal part of the vertex dominates in the infra-red limit over the transverse part, Thus by substituting the longitudinal part for the full vertex we have a closed integral equation for the quark function. Notice that this is different from the gluon case where we are forced to make further assumptions because of the complexity of the general solution obtained by this method.

In this case we choose not to use dimensional regularization to calculate the integrals for two reasons. Firstly the presence of the gluon function makes the denominators more complicated. Secondly, studies of simple models indicate that the quark function is more sensitive to the treatment of the ultra-violet limit than was the case for the gluon equation. For these reasons we therefore choose to use a method that allows us to use the perturbative result in the ultra-violet limit. Since the quark function is dependent only upon the magnitude of the momentum flowing along the quark line, the obvious thing to do is to integrate over the angular variables first to leave a scalar equation for the quark function. This is done in the fourth section by a suitable choice of the spherical polar coordinate frame. The angular integrals are then, if not trivial, considerably simplified.

The resulting scalar integral equation is unfortunately not finite as the integrals diverge at the ultra-violet end, where we use the leading log approximation for the quark function, and also diverge as the loop momentum goes to the external momentum. With the present truncation of the Schwinger-Dyson equations, the ultra-violet subtractions are essentially renormalizations at the one loop level, and so only involve α_s to this order. As remarked in the general discussion (chapter 2), successively higher order definitions of the coupling α_s only arise if the coupled Schwinger-Dyson equations include Green's functions with an increasing number of external legs. The enhancement of the gluon propagator at low momentum introduces an infra-red divergence beyond that found in perturbation theory. By introducing cutoffs δ and Y, in the fifth section, these divergent pieces are explicitly removed from under the integrals leaving them finite. The problem then is to subtract away the divergent terms from the scalar integral equation. This is done by introducing three subtraction points in the perturbative region, where the quark function is approximated by the leading logarithm perturbative result, and subtracting from the equation its form at these points multiplied by appropriate functions.

5:2 Perturbative Result

First we take a look at the perturbative calculation of the quark renormalization function. We choose to calculate the quark propagator in Euclidean space with a general gauge parameter. To ensure that the calculation is finite, we shall use the technique of dimensional regularization. The full propagator S_F is given by the perturbative expansion,

$$S_F = S_F^{\circ} + S_F^{\circ}\,\Sigma\,S_F^{\circ} + \ldots$$

which can be resummed as

$$S_F = S_F^{\circ} + S_F^{\circ}\,\Sigma\,S_F + \ldots .$$

where S_F° is the bare propagator and Σ is the quark self energy. If we multiply from the right by S_F^{-1} and on the left by $S_F^{\circ -1}$ then, truncating the series we get,

$$S_F^{\circ -1} = S_F^{-1} + \Sigma$$

ie.

$$S_F^{-1} = S_F^{\circ -1} - \Sigma . \qquad (2.1)$$

For a massless quark, the propagator can only depend upon a scalar function of it s momentum squared times it s momentum contracted with the gamma matrix, so it can be written in the form,

Quark Propagator

$$S_F(p) = -i \frac{F(p^2)}{\not p} \qquad \text{ie} \qquad S_F^{-1}(p) = i \frac{\not p}{F(p^2)} \tag{2.2}$$

Thus, substituting in equation (2.1) with $F(p^2) = 1$ for the bare propagator,

$$\frac{i\not p}{F(p^2)} = i\not p - \Sigma$$

multiplying by $- i \not p / p^2$,

$$\frac{1}{F(p^2)} = 1 + \frac{i\not p}{p^2} \Sigma . \tag{2.3}$$

The self-energy is given by,

$$\Sigma = \int \frac{d^n k}{(2\pi)^n} \Gamma^\circ_\mu S^\circ_F \Gamma^\circ_\mu \Delta^{\mu\nu}_\circ(k-p) . \tag{2.4}$$

where the gluon propagator is, suppressing colour indices,

$$\Delta^{\mu\nu}_\circ(q) = \frac{1}{q^2} \left\{ \delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right\} + \xi \frac{q^\mu q^\nu}{q^4} . \tag{2.5}$$

If we let the dimension of space-time be $n = 4 - 2\varepsilon$, then the bare quark gluon vertex is,

$$\Gamma^\circ_\mu = -i\, g\, \mu^\varepsilon \gamma_\mu T^a{}_{ij} \tag{2.6}$$

By substituting equations (2.2), (2.5) and (2.6) into equation (2.4), letting $\theta = 1 - \xi$, $q = k - p$ the quark self energy becomes,

Quark Propagator

$$\Sigma = i\, g^2\, C_2(F)\, \mu^{2\varepsilon} \int \frac{d^n k}{(2\pi)^n}\, \gamma_\mu\, \not{k}\, \gamma_\nu \left\{ \delta^{\mu\nu} - \theta\, \frac{q^\mu q^\nu}{q^2} \right\}. \tag{2.7}$$

Using the anti-commutator relation for the gamma matrices

$$\begin{aligned} \gamma_\mu \not{k} \gamma_\mu &= (2k_\mu - \not{k} \gamma_\mu)\, \gamma_\mu \\ &= (2 - n)\, \not{k} \\ &= -2\,(1 - \varepsilon)\, \not{k} \end{aligned}$$

and

$$\begin{aligned} \gamma_\mu \not{k}\, q^\mu \not{q} &= (2\, k_\mu - \not{k}\, \gamma_\mu)\, q_\mu \not{q} \\ &= 2\, k.q\, \not{q} - \not{k}\, q^2 \,. \end{aligned}$$

Therefore the integral in the self energy (2.7) is,

$$\int \frac{d^n k}{k^2 (k-p)^2} \left\{ -2(1-\varepsilon)\not{k} - \theta\, \frac{2k.q\not{q} - \not{k}q^2}{q^2} \right\} .$$

Collecting terms and using the transformation k —> k + p on the second term,

$$= -(2-2\varepsilon-\theta) \int \frac{d^n k\, \not{k}}{k^2 (k-p)^2} - 2\theta \int d^n k\, \frac{(k^2 + k.p)\, \not{k}}{k^4\, (k + p)^2} .$$

Using the techniques of dimensional regularization (see Appendex A) the self-energy is up to zeroth order in ε,

$$\Sigma = \frac{i g^2 C_2(F) \not{p}}{16\pi^2} \left\{ - \frac{(2-2\varepsilon-\theta)}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi^2 \mu^2}{p^2} + 2 \right] + \frac{\theta}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right] \right\}$$

Quark Propagator

$$= -\frac{ig^2 C_2(F)\not{p}\xi}{16\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{p^2} + 2 - \frac{1}{\xi}\right\}.$$

Using the $\overline{\text{MS}}$ scheme to remove the pole in ε, as well as some unwanted constants.

$$\Sigma_{reg} = -\frac{ig^2 C_2(F)\not{p}\xi}{16\pi^2}\left\{\ln\frac{\mu^2}{p^2} + 2 - \frac{1}{\xi}\right\}.$$

Therefore substituting this result into equation (2.3), the quark function is

$$\frac{1}{F(p^2)} = 1 - \frac{g^2 C_2(F)\xi}{16\pi^2}\left\{\ln\frac{\mu^2}{p^2} + 2 - \frac{1}{\xi}\right\}. \tag{2.8}$$

We can remove the dependence on the unphysical scale μ by subtracting at some momentum p_1 say,

$$\frac{1}{F(p^2)} = \frac{1}{F(p_1^2)} - \frac{g^2 C_2(F)\xi}{16\pi^2}\ln\frac{p^2}{p_1^2}.$$

By applying the renormalization group equation to this result, the quark function can be related to the running coupling constant to give the result,

$$F(p^2) = F(p_1^2)\left[\frac{\alpha_s(p^2)}{\alpha_s(p_1^2)}\right]^{\gamma}. \tag{2.9}$$

where

$$\gamma = \xi\, C_2(F) / \beta_0$$

Now α_s is given by the perturbative series,

$$\alpha_s(p^2) = \alpha_s(p_1^2) - \beta_0/(4\pi)\,\alpha_s(p_1^2)^2 \ln(p^2/p_1^2) + \ldots$$

$$= \frac{\alpha_s(p_1^2)}{1 + \beta_0/(4\pi)\ \alpha_s(p_1^2)\ \ln(p^2/p_1^2)}$$

where $\beta_0 = 11/3\ C_2(A) - 4/3\ N_f\ T_2(F)$. If we introduce the QCD scale parameter Λ where,

$$\Lambda = p_1 \exp\left[\frac{2\pi}{\alpha_s(p_1^2)\ \beta_0}\right].$$

Then α_s can be expressed as,

$$\alpha_s(p^2) = \frac{4\pi}{\beta_0 \ln(p^2/\Lambda^2)}. \qquad (2.10)$$

Thus substituting this in equation (2.9) the quark function can be written as,

$$F(p^2) = e\left[\ln(p^2/\Lambda^2)\right]^{\gamma} \qquad (2.11)$$

where

$$e = \frac{F(p_1^2)}{\left[\ln(p_1^2/\Lambda^2)\right]^{\gamma}}. \qquad (2.12)$$

5:3 Schwinger-Dyson Equation

The Schwinger-Dyson equation for the quark propagator is illustrated graphically in fig (5.1) which in terms of the bare and full propagators and the self-energy can be written as,

$$S_F = S_F^{\circ} + S_F^{\circ}\ \Sigma\ S_F .$$

As before by substituting the form of the quark propagator (2.2) this can be reduced to

Quark Propagator

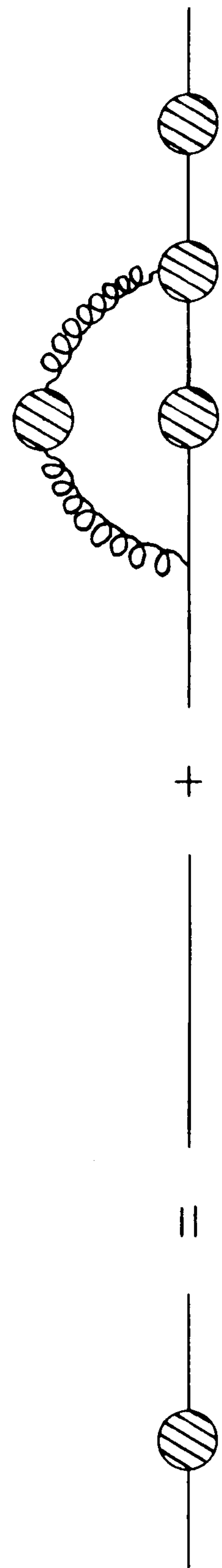

Figure 5.1

The Schwinger-Dyson equation for the quark propagator.

Quark Propagator

$$\frac{1}{F(p^2)} = 1 + \frac{i\not{p}}{p^2}\,\Sigma\ . \tag{3.1}$$

Where now the self-energy is,

$$\Sigma = \int \frac{d^4k}{(2\pi)^4}\, \Gamma^{\bullet}_{\mu}\, S^{\bullet}_{F}\, \Gamma_{\mu}(k,p)\, \Delta^{\mu\nu}(k-p)\ . \tag{3.2}$$

The full gluon propagator being given by,

$$\Delta^{\mu\nu}(q) = \frac{G(q^2)}{q^2}\left\{ \delta^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} \right\} + \xi\,\frac{q^{\mu}q^{\nu}}{q^4} \tag{3.3}$$

where the function $G(q^2)$ is determined by the Schwinger-Dyson equation for the gluon propagator as discussed earlier. The problem now is to determine the full quark-gluon vertex in terms of the quark function. The Ward-Takahashi identities provide a method of determining the vertex. The identity for the quark gluon vertex is,

$$(\,k - p\,)^{\mu}\, \Gamma_{\mu}(k,p) = S_F^{-1}(k) - S_F^{-1}(p) \tag{3.4}$$

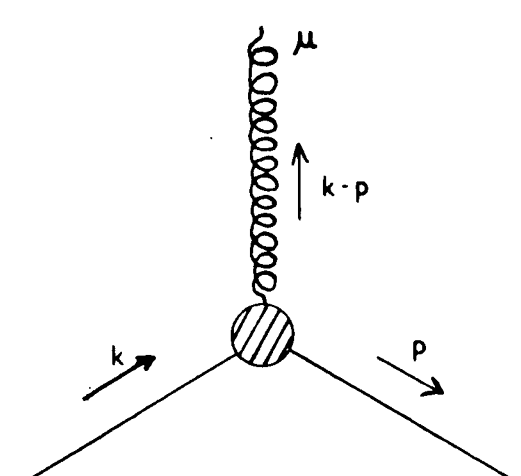

Figure 5.2

The full quark-gluon vertex

where the quark momenta k is flowing into the vertex and p is flowing out of the vertex (see fig 5.2).The gluon momentum is then, by momentum conservation, just $k-p$ flowing out of the vertex. The vertex can be separated into two parts, a transverse part and a longitudinal part,

$$\Gamma_\mu(k,p) = \Gamma^L_\mu(k,p) + \Gamma^T_\mu(k,p)$$

where the transverse part is defined to vanish when contracted with the gluon momentum ie.

$$(k - p)^\mu \; \Gamma^T_\mu(k,p) = 0 \; . \tag{3.5}$$

The longitudinal part of the quark-gluon vertex can be parameterised by the form,

$$\Gamma^L_\mu(k,p) = A(k,p)\, \gamma_\mu + B(k,p)\, (\not{p} + c \not{k})\, (p + d\, k)_\mu$$

where A and B must be free of kinematic singularities. By substituting this into the Ward-Takahashi identity (3.4), we see [5.1] that

$$\frac{\not{k}}{F(k^2)} - \frac{\not{p}}{F(p^2)} = A(k,p)\, (\not{k} - \not{p})$$

$$+ B(k,p)\, (\not{p} + c \not{k})\, (d\, k^2 - p^2 - d\, k.p + k.p) \; .$$

Equating the coeficients of $\not{p}$ and $\not{k}$ we get,

$$\frac{1}{F(p^2)} = A(k,p) - B(k,p)\, (d\, k^2 - p^2 - d\, k.p + k.p) \tag{3.6}$$

$$\frac{1}{F(k^2)} = A(k,p) + c\, B(k,p)\, (d\, k^2 - p^2 - d\, k.p + k.p) \tag{3.7}$$

respectively. Subtracting equation (3.7) from equation (3.6) we get that,

$$B(k,p) = \left[\frac{1}{F(p^2)} - \frac{1}{F(k^2)} \right] / \left[(1+c)\ (p^2-dk^2+dpk-pk) \right] .$$

Because of Bose symmetry $B(k,p) = B(p,k)$, and hence $d = 1$. Adding c times equation (3.6) to equation (3.7) we get,

$$A(k,p) = \frac{c}{1+c} \left[\frac{c}{F(p^2)} + \frac{1}{F(k^2)} \right] .$$

Again because of Bose symmetry $A(k,p) = A(p,k)$, and hence $c = 1$. Thus the vertex is given by,

$$\Gamma^L_\mu(k,p) = \frac{1}{2} \left[\frac{1}{F(p^2)} + \frac{1}{F(k^2)} \right] \gamma_\mu$$

$$+ \frac{1}{2} \left[\frac{1}{F(p^2)} - \frac{1}{F(k^2)} \right] \frac{(\not{p}+\not{k})}{p^2-k^2} (p+k)_\mu . \tag{3.8}$$

Thus the longitudinal part of the vertex is completely determined by the quark renormalization function.

The transverse part of the vertex which trivially satisfies the Ward-Takahashi identity (3.5) must be proportional to terms like,

$$p^\mu(k.p) \quad - k^\mu(p.q) \quad \text{and} \quad \sigma^{\mu\nu}q_\nu$$

where

$$q^\mu = (k-p)^\mu \text{ and } \sigma^{\mu\nu} = i/2[\gamma^\mu,\gamma^\nu] .$$

It can be seen that because the vertex must be free of kinematic singularities that the transverse part is of at least one order higher in momentum than the longitudinal part. Therefore the longitudinal part will dominate in the infra-red region. Then substituting the longitudinal part of the vertex (3.8) for the full vertex and using the form of the full propagators (3.3) the equation for the quark self-energy (3.2) becomes,

$$\Sigma = \alpha_s \frac{i\, C_2(F)}{(2\pi)^3} \int d^4k \, \frac{\gamma_\mu \, \not{k} \, F(k^2)}{k^2 q^2}$$

$$\times \left\{ \left[\frac{1}{F(p^2)} + \frac{1}{F(k^2)} \right] \gamma_\nu + \left[\frac{1}{F(p^2)} - \frac{1}{F(k^2)} \right] \frac{(\not{p}+\not{k})}{p^2-k^2} (p+k)_\nu \right\}$$

$$\times \left\{ G(q^2)\, \delta^{\mu\nu} + [\xi - G(q^2)\,] \frac{q^\mu q^\nu}{q^2} \right\} \tag{3.9}$$

with the standard definition of $\alpha_s = g^2/(4\pi)$. Using the anti-commutator relation for the gamma matrices and doing a little algebra, and splitting off the gauge dependent piece, we obtain

$$\Sigma = \frac{i\,\lambda}{\pi^2} \int d^4k \, \frac{G(q^2)}{k^2 q^2} \left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not{q}}{q^2} - \not{k} \right] + \right.$$

$$\left. + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\frac{2(p.k+k^2)}{p^2 - k^2} \not{q} + \frac{2(p.k)}{q^2} \not{q} \right] \right\}$$

$$- \frac{i\,\lambda\,\xi}{\pi^2} \int \frac{d^4k}{k^2 q^2} \left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not{q}}{q^2} + \not{k} \right] + \right.$$

$$\left. + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\not{k} + \frac{2(p.k)}{q^2} \not{q} \right] \right\} . \tag{3.10}$$

where $\lambda = \alpha_s C_2(F)/8\pi$.

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If we let the gluon function have the form:

$$G(q^2) = A_g \frac{p_0^2}{q^2} + B_g + C_g \frac{q^2}{q^2+p_0^2} \qquad (3.11)$$

where A_g, B_g and C_g have the values previously determined, then we can write the quark self-energy as,

$$\Sigma = i \lambda \not{p} (A_g \Sigma_A + B_g \Sigma_B + C_g \Sigma_C + \xi \Sigma_\xi) \qquad (3.12)$$

With this definition the Scwinger-Dyson equation (3.1) becomes,

$$\frac{1}{F(p^2)} = 1 - \lambda (A_g \Sigma_A + B_g \Sigma_B + C_g \Sigma_C + \xi \Sigma_\xi) \qquad (3.13)$$

5:4 Angular Integration

The problem now is to determine these self-energy parts. Since the quark function only depends upon the momenta squared, the angular integrals are completely determined. We now consider how to actually perform the angular integration. A fuller treatment, together with all the calculations of the integrals we use in determining the self-energy, is given in appendix B.

Let us transform to spherical polar coordinates in four dimensions when the measure of integration becomes,

$$d^4k \rightarrow k^3 dk \sin^2\psi d\psi \sin\theta d\theta \, d\varphi$$

where the new integration variables are constrained such that,

$$0 \leqslant k \leqslant \infty, \quad 0 \leqslant \psi, \ \theta \leqslant \pi \quad \text{and} \quad 0 \leqslant \varphi \leqslant 2\pi.$$

For convenience, let us choose the time axis to be along the p momentum direction. This does not affect the Lorentz invariance of the equation as any choice of the axis will lead to the same results. However, the choice of p along the time axis simplifies the integrals considerably. Then the components of the momenta p and k are,

$$p^{\mu} = (\ p,\ 0,\ 0,\ 0\ ),$$
$$k^{\mu} = k(\ \cos\psi,\ \sin\psi\ \sin\theta\ \cos\varphi,\ \sin\psi\ \sin\theta\ \sin\varphi,\ \sin\psi\ \cos\theta\ )\ .$$

Notice that $p.k = p\ k\ \cos\psi$.

There are two types of integral over a function of k,p and z, where $z=\cos\psi$, one just over the function, and the other over the function multiplied by $\not{k}$. The former is reasonably straightforward. In the latter, we extract the γ matrix and consider the vector integral component by component.

For the first and second component the integral is zero, since

$$\int_0^{2\pi} \sin\varphi\ d\varphi = \int_0^{2\pi} \cos\varphi\ d\varphi = 0\ .$$

Also the third component is zero, as

$$\int_0^{\pi} \sin\theta\ d\theta\ \ \cos\theta = 0\ .$$

This leaves only the zeroth component and since $k^{(0)} = k\cos\varphi$, it has the same form as the integral not involving $\not{k}$. A typical integral that is left to be done is,

$$4\pi \int_0^{\infty} k^3 dk\ f(k,p) \int_{-1}^{1} \sqrt{1-z^2}\ \ dz\ \frac{(bz/2)}{(a - bz)} \tag{4.1}$$

where $b = 2pk$ and $a = p^2 + k^2$ or $p^2 + k^2 + p_0^2$ depending on the form of the gluon function. Substitution of $y = a - bz$ and the integral becomes,

$$\frac{1}{2b^2} \int_{a-b}^{a+b} dy \, \frac{(a-y)}{y} \sqrt{b^2 - a^2 + 2ay - y^2} \; .$$

This can then be reduced to the standard integrals

$$\frac{1}{2b^2} \left\{ a\,(b^2 - a^2) \int_{a-b}^{a+b} \frac{dy}{\sqrt{b^2 - a^2 + 2ay - y^2}} + \right.$$

$$\left. + \,(a^2 - b^2/2) \int_{a-b}^{a+b} \frac{dy}{y\sqrt{b^2 - a^2 + 2ay - y^2}} \right\} . \qquad (4.2)$$

Now since $a \geqslant p^2 + k^2$, $b = 2pk$ then $b^2 - a^2 \leqslant 0$ and the discriminant of the square root is $\Delta = -4b \leqslant 0$. Thus the angular integral (4.2) is,

$$\pi \left[(a^2 - b^2/2) - a\sqrt{a^2 - b^2} \right] / (2b^2)$$

$$= \pi/2 \left[a\,(a - \sqrt{a^2 - b^2}\,)/b^2 - 1/2 \right] .$$

Therefore the total integral (4.1) is,

$$2\pi^2 \int_0^\infty k^3 dk\, f(k,p) \; \left[a\,(a - \sqrt{a^2 - b^2}\,)/b^2 - 1/2 \right] .$$

In particular, for $a = p^2 + k^2$ and $b = 2pk$

$$\sqrt{a^2 - b^2} = \sqrt{(p^2 + k^2)^2 - 4p^2k^2} = | p^2 - k^2 |$$

hence

$$a - \sqrt{a^2 - b^2} = p^2 - k^2 - | p^2 - k^2 |$$

$$= \begin{cases} 2k^2 & \text{for } p^2 > k^2 \\ 2p^2 & \text{for } p^2 < k^2 \end{cases}$$

then (4.1) is,

$$\int \frac{f(k^2)\not{k}}{k^2 (k-p)^2} d^4k = \pi^2 \not{p} \int_0^\infty f(k^2)\ h(k^2/p^2)\ p^2/k^2\ dk$$

$$\text{where } h(x) = \begin{cases} x \text{ for } x < 1 \\ 1 \text{ otherwise.} \end{cases} \tag{4.3}$$

This technique can be used to evaluate all of the angular integrals of interest to leave an integral equation in p^2 over the unknown function $F(p^2)$. We can now proceed to evaluate each of the self-energy parts in turn.

5:5 The Angular Integrals

5:5a The Gauge Dependent Term

Consider first the part of the integral in equation for the self-energy (3.10) that is multiplied by the gauge parameter ξ.

$$\Sigma_\xi = \frac{\not{p}}{\pi^2 p^2} \int \frac{d^4k}{k^2 q^2} \left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not{q}}{q^2} + \not{k} \right] + \right.$$

$$\left. + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\not{p} + \frac{2(p.k)}{q^2} \not{q} \right] \right\}.$$

Doing the angular integrals by the technique outlined above, this becomes, after a little manipulation,

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$$\Sigma_\xi = 2\left\{ \frac{1}{p^2}\int_0^p k^3 dk \left[\frac{F(k^2)}{F(p^2)} + 1 \right] + \int_p^\infty dk/k \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \right.$$

$$\left. - \frac{1}{p^2}\int_0^p k^3 dk \left[\frac{F(k^2)}{F(p^2)} - 1 \right] + \int_p^\infty dk/k \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \right\}$$

$$= 4\left\{ \frac{1}{p^4}\int_0^p k^3 dk + \int_p^\infty dk/k \frac{F(k^2)}{F(p^2)} \right\}$$

$$= 4\left\{ \frac{1}{4} + \int_p^\infty dk/k \frac{F(k^2)}{F(p^2)} \right\} . \tag{5.1}$$

5:5b The Constant Term

Now let us consider the part of the integral in the equation for the quark self-energy (3.10) involving just the constant B_g in the gluon function. Then,

$$\Sigma_B = \frac{\not p}{\pi^2 p^2}\int d^4k \frac{1}{k^2 q^2}\left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not q}{q^2} - \not k \right] + \right.$$

$$\left. + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\frac{2(p.k+k^2)}{p^2 - k^2}\not p + \frac{2(p.k)}{q^2}\not q \right] \right\}$$

$$= 3\int_0^\infty dk/k \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \frac{p^2 + k^2}{p^2 - k^2} h(k^4/p^4). \tag{5.2}$$

The angular integrals over $2\,(k.p)\,\not q/q^2 + \not k$ gives zero, so that the result only depends upon the difference of the quark function evaluated at the two momenta.

5:5c The Enhanced Term

Now it is necessary to calculate the angular intergrals for the parts of the parameterization of the gluon function that differ from a constant. Consider the part of the self-energy Σ_A arising from the term $A_g\ p_0^2/k^2$ in the gluon renormalization function

$$\Sigma_A = \frac{\not p}{\pi^2 p^2} \int d^4k \, \frac{p_0^2}{k^2 q^4} \left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not q}{q^2} - \not k \right] + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\frac{2(p.k+k^2)}{p^2 - k^2} \not p + \frac{2(p.k)}{q^2} \not q \right] \right\}$$

$$= 3\, p_0^2 \int_0^\infty dk/k \, \frac{h(k^4/p^4)}{|p^2 - k^2|} \left\{ - \left[\frac{F(k^2)}{F(p^2)} + 1 \right] + \frac{k^2 + p^2}{p^2 - k^2} \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \right\}.$$

$$= 6\, p_0^2 \int_0^\infty dk/k \, \frac{h(k^4/p^4)}{|p^2 - k^2|} \left\{ \frac{k^2}{p^2 - k^2} \left[\frac{F(k^2)}{F(p^2)} - 1 \right] - 1 \right\}. \qquad (5.3)$$

5:5d The Intermediate Term

Finally consider the part of the quark self-energy Σ_C coming from the term multiplying the C_g parameter in the gluon function.

$$\Sigma_C = \frac{\not p}{\pi^2 p^2} \int d^4k \, \frac{q^2}{k^2 q^4 (q^2 + p_0^2)} \left\{ \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left[\frac{-2(k.q)\not q}{q^2} - \not k \right] + \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left[\frac{2(p.k+k^2)}{p^2 - k^2} \not p + \frac{2(p.k)}{q^2} \not q \right] \right\} \qquad (5.4)$$

Because of the difference in the denominator, these integrals are a little more complicated. However, we introduce the functions,

$$2\, h_1(x,y) = 1 + x + y - \sqrt{(1-x)^2 + 2\,x\,(1+x) + y^2} \tag{5.5}$$

and

$$h_2(x,y) = (1 + x + y)\, h_1(x,y) - x\ . \tag{5.6}$$

Note that as y goes to zero ,

$$h_1(x,y) \rightarrow h(x)$$

and

$$h_2(x,y) \rightarrow h(x^2).$$

Then using these functions, the angular integrals in (5.4) can be done and the result written in the form,

$$\Sigma_C = \frac{1}{p^2 p_0^2} \int_0^\infty dk/k \left[\frac{F(k^2)}{F(p^2)} + 1 \right] \left\{ 2k^2 p_0^2 - (k^2 - p^2)^2\, h(k^2/p^2) \right.$$

$$\left. + \left[(k^2 - p^2)^2 - p_0^2\, (k^2 + p^2) - p_0^4 \right] h_1(k^2/p^2, p_0^2/p^2) \right\}$$

$$+ \frac{1}{p_0^2} \int_0^\infty dk/k \left[\frac{F(k^2)}{F(p^2)} - 1 \right] \left\{ \frac{k^4 - p^4}{p^2}\, h(k^2/p^2) + \frac{k^2 p_0^2}{p^2} - \frac{2k^2 p_0^2}{p^2 - k^2} \right.$$

$$\left. + \left[2p_0^2\, \frac{3k^2 + p^2 + p_0^2}{p^2 - k^2} - \frac{k^4 - p^4 + 2k^2 p_0^2 + p_0^4}{p^2} \right] h_1(k^2/p^2, p_0^2/p^2) \right\} . \tag{5.7}$$

5:6 Consistency with Perturbation Theory

Before proceeding any further let us check that this method does reproduce the perturbative result. If the gluon takes on its bare value the Schwinger-Dyson equation (3.13) reduces to,

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$$\frac{1}{F(p^2)} = 1 - \lambda \left(\Sigma_B + \xi \Sigma_\xi \right) \tag{6.1}$$

The part of the self energy coming from the constant Σ_B is given by equation (5.2). This term only depends upon the difference of the quark function evaluated at the two momenta. So if the quark function is constant, this contribution to the self-energy is zero. Substituting in the form of the self-energy from the gauge dependant part (5.1) with the quark function set to a constant, we get,

$$\frac{1}{F(p^2)} = 1 + \lambda \xi \left\{ 1 + 4 \int_p^\infty dk/k \right\} .$$

Introducing an ultra-violet cutoff R to regularise the integral,

$$1/F(p^2) = 1 + \lambda \xi \{ 1 + 2 \ln (R^2/p^2) \} .$$

Finally renormalising to remove the dependence upon the cutoff R by subtracting at some momentum p_1^2 say,

$$\frac{1}{F(p^2)} = \frac{1}{F(p_1^2)} + 2 \lambda \xi \ln \frac{p^2}{p_1^2} .$$

$$= \frac{1}{F(p_1^2)} - \frac{g^2 C_2(F)\xi}{16\pi^2} \ln \frac{p^2}{p_1^2} .$$

since $\lambda = \alpha_s C_2(F)/8\pi$. This is the perturbative result (2.8) obtained in section2. For $F(p_1^2) = 1$ we get,

$$F(p^2) = 1 + \alpha_s(p_1^2)\, C_2(F)/4\pi \ln (p^2/p_1^2)$$

Notice that this result is consistent with the Schwinger-Dyson equation upto first order in the coupling constant as the integral over the logarithm gives rise to terms of order α_s^2.

5:7 Regularising the Integrals

Now collecting the terms (5.1), (5.2), (5.3) and (5.7) together and substituting into (3.13) with x for p^2, x_0 for p_0^2 and y for k^2 the equation for the quark self-energy becomes,

$$\frac{1}{F(p^2)} = 1 - \lambda \left(A_g \Sigma_A + B_g \Sigma_B + C_g \Sigma_C + \xi \Sigma_\xi \right) \qquad (7.1)$$

where

$$\Sigma_A = 3 x_0 \int_0^\infty dy/y \, \frac{h(y^2/x^2)}{| x - y |} \left\{ \frac{y}{x - y} \left[\frac{F(y)}{F(x)} - 1 \right] - 1 \right\}. \qquad (7.2a)$$

$$\Sigma_B = \frac{3}{2} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - \right] \frac{x + y}{x - y} h(y^2/x^2) . \qquad (7.2b)$$

$$\begin{aligned}
\Sigma_C = {} & \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} + 1 \right] \left\{ 2yx_0 - (y - x)^2 \, h(y/x) \right. \\
& \left. + \left[(y - x)^2 - x_0 (y + x) - x_0^2 \right] h_1(y/x, x_0/x) \right\} \\
& + \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ (y^2 - x^2) \, h(y/x) + x_0 y \frac{y + x}{x - y} \right. \\
& \left. + \left[2xx_0 \frac{3y + x + x_0}{x - y} - \frac{(y - x)^2}{x} + x \right] h_1(y/x, x_0/x) \right\}
\end{aligned}$$

(7.2c)

$$\Sigma_\xi \;=\; -\left\{ 1 + 2 \int_x^\infty dy/y \,\frac{F(y)}{F(x)} \right\} . \tag{7.2d}$$

5:8a Infra-red Regularization

Notice that the part of the quark self-energy Σ_A (5.3) diverges as $y \rightarrow x$. Such a divergence only arises from the enhanced gluon function when its momentum goes to zero, of course. Consequently, its cancellation goes beyond the usual perturbative treatment of infra-red divergences. This divergence can be extracted from under the integral by adding and subtracting the second term in the Taylor series for $F(y)$, $y \approx x$. Then the integral can be written in the form,

$$\Sigma_A \;=\; 3\, x_0 \int_0^\infty dy/y \,\frac{h(y^2/x^2)}{|\,x - y\,|} \left\{ \left[\frac{y}{y-x} \left[\frac{F(y)}{F(x)} - 1 \right] + x\,\frac{F'(x)}{F(x)} \right] - \left[x\,\frac{F'(x)}{F(x)} + 1 \right] \right\} . \tag{8.1}$$

Now by using,

$$\int_0^{x-\delta_-} dy/y \,\frac{y^2/x^2}{x-y} \;+\; \int_{x+\delta^+}^\infty dy/y \,\frac{1}{y-x} \;=\; \left[\ln(x^2/\delta^2) - 1 \right] / \,x$$

where $\delta^2 = \delta^+\delta_-$, (8.1) becomes,

$$\Sigma_A \;=\; \Sigma_A^{reg} \;+\; 3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \ln(\delta^2)$$

where Σ_A^{reg} is defined by,

$$\Sigma_A^{reg} = 3\, x_0 \int_0^\infty dy/y \, \frac{h(y^2/x^2)}{|\, x - y \,|} \left\{ \frac{y}{y - x} \left[\frac{F(y)}{F(x)} - 1 \right] + x \frac{F'(x)}{F(x)} \right\}$$

$$- 3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \left[\ln(x^2) - 1 \right] . \qquad (8.2)$$

Consider now the infra-red behaviour of the equation paying particular attention to the region where $y \approx x \ll 1$. Then the integrand of Σ_B (5.2) is proportional to $F'(x)/F(x)$ and so the integral is infra-red safe. Similarly for Σ_C (5.7). The part of self-energy multiplied by the gauge parameter, Σ_ξ (5.1), is infra-red safe if, for small y, $F(y) \propto y^c$, c some positive constant, and logarithmically divergent if F(y) goes to a constant as y goes to zero.

Therefore if A_g is zero, the quark function is approximately constant for small x (modulo logarithms in the Landau gauge). This is presumably the case in QED where the leptons are not confined and the associated gauge bosons (photons) propagate out to infinite distances with vanishingly small momenta.

However, if A_g is non-zero, then there is an infra-red divergence in the part of the self energy multiplied by A_g (8.2) arising from the integrand of the form,

$$[\, F'(x) + F''(x) \,] \,/\, [\, y\, F(x) \,]$$

which leads to a logarithmic divergence. Moreover, there is the term that has already been extracted from under the integral, which has an explicit pole term multiplying the logarithmic divergence as $y \to x$. ie,

$$3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \left[\ln(x^2) \;-\; 1 \right] . \qquad (8.3)$$

Since the right hand side of the equation goes to infinity as x goes to zero this forces F(x) on the left hand side to go to zero. The possibility that F(x) = 1/x is excluded by the existence of the Σ_B and Σ_C terms in the total self-energy.

5:8b Ultra-violet Regularization

Let us now turn our attention to the ultra-violet behaviour of the equation. For large y, F(y) is given by the result obtained from renormalization group equation for the quark propagator (2.9) in section 2 with $x_1 = p_1^2$ in the perturbative regime,

$$F(x) \;=\; F(x_1) \left[\frac{\alpha_s(x)}{\alpha_s(x_1)} \right]^{\gamma} .$$

where α_s is the running coupling constant and γ is the anomalous dimension for the quark propagator, ie.

$$\gamma \;=\; \xi\, C_2(F) \,/\, \beta_0 \;.$$

Which we write in the form

$$F(x) \;=\; e \left[\ln\, (x/\Lambda^2) \right]^{\gamma}$$

where

$$e \;=\; \frac{F(x_1)}{\left[\ln\, (x_1/\Lambda^2) \right]^{\gamma}} .$$

The self-energy part Σ_A is ultra-violet safe so the regularization we have already applied is sufficient to render it finite.

The Constant Term

The Σ_B part of quark self-energy is, on the other hand, ultraviolet divergent. The term in the integral that leads to the ultraviolet divergence is,

$$\frac{3}{2\,y}\left[\frac{F(y)}{F(x)} - 1\right] .$$

In order to remove this divergence we add and subtract the integral ,

$$\frac{3}{2}\int_R^{\infty} dy/y \left[\frac{F(y)}{F(x)} - 1\right] \tag{8.4}$$

where R is some momentum squared less than x_1 in the perturbative region. By introducing an ultraviolet cutoff these integrals in (8.4) can be calculated. By substituting the perturbative form for F(y) derived above the part of the integral containing F(y) may be calculated as follows,

$$\int_R^{Y} dy/y\; F(y) \;=\; e\int_R^{Y} dy/y\; [\; \ln(y/\Lambda^2)\;]^{\gamma}$$

Using the substitution $z = \ln(y/\Lambda^2)$,

$$= e\int_a^b dz\; z^{\gamma} \qquad \text{where } a=\ln(R/\Lambda^2),\ b=\ln(Y/\Lambda^2)$$

$$= e\;\{\;[\;\ln(Y/\Lambda^2)\;]^{\gamma+1} \;-\; [\;\ln(R/\Lambda^2)\;]^{\gamma+1}\;\}/(\gamma + 1)\;.$$

combining this result with the integral over the constant part we get the result,

Quark Propagator

$$3/2\ e\ \{\ [\ \ln(Y/\Lambda^2)\ ]^{\gamma+1}\ -\ [\ \ln(R/\Lambda^2)\ ]^{\gamma+1}\ \}/[\ (\gamma+1)F(x)\ ]$$

$$-\ 3/2\ \{\ \ln(Y/\Lambda^2)\ -\ \ln(R/\Lambda^2)\ \} \qquad (8.5)$$

Then the self-energy part from the constant in the gluon propagator can be written as,

$$\Sigma_B = \Sigma_B^{reg} - 3/2\ \{\ e\ [\ \ln(Y/\Lambda^2)\ ]^{\gamma+1}\ /\ [\ (\gamma+1)F(x)\ ]\ -\ \ln(Y/\Lambda^2)\ \}$$

where,

$$\Sigma_B^{reg} = 3/2 \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ \frac{x+y}{x-y}\, h(y^2/x^2) + \theta_+(y-R) \right\}.$$

$$+\ 3/2\ \{\ \ e\ [\ \ln(R/\Lambda^2)\ ]^{\gamma+1}\ /\ [\ (\gamma+1)F(x)\ ]\ -\ \ln(R/\Lambda^2)\ \}$$

and

$$\theta_+(y-R) = \begin{cases} 0 & \text{for } y < R \\ 1 & \text{for } y > R \end{cases}$$

Notice that although at first sight Σ_B^{reg} seems to depend upon the mass scale R it is in fact independent of R because the second line is cancelled by the lower limit from the integral over the step function.

The Intermediate term

Let us consider the limit as y goes to infinity of the function $h_1(y/x,x_0/x)$ in the self-energy part Σ_C (5.5). We can be expand the function in terms of 1/y such that,

$$h_1(y/x,x_0/x) \approx 1 - x_0/y - x_0(x_0-x)/y^2 + O(1/y^3)\ . \qquad (8.6)$$

for $y \gg x, x_0$.

Using this result the ultra-violet form of the integrand of the Σ_C

term (5.7) becomes,

$$- \frac{3}{2\,y} \left[\frac{F(y)}{F(x)} - 1 \right]$$

So in the same way as for the Σ_B term the divergence can be extracted and the self-energy term Σ_C can be written as,

$$\Sigma_C = \Sigma_C^{reg} - 3/2 \{ e [\ln(Y/\Lambda^2)]^{\gamma+1} / [(\gamma+1)F(x)] - \ln(Y/\Lambda^2) \}$$

where,

$$\Sigma_C^{reg} = \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} + 1 \right] \left\{ 2yx_0 - (y - x)^2\, h(y/x) \right.$$
$$\left. + \left[(y - x)^2 - x_0 (y + x) - x_0^2 \right] h_1(y/x, x_0/x) \right\}$$
$$+ \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ (y^2 - x^2)\, h(y/x) + x_0 y \frac{y + x}{x - y} + 3xx_0 \theta_+(y-R) \right.$$
$$\left. + \left[2xx_0 \frac{3y+x+x_0}{x - y} - \frac{(y - x)^2}{x} + x \right] h_1(y/x, x_0/x) \right\}$$
$$+ 3/2 \{ e [\ln(R/\Lambda^2)]^{\gamma+1} / [(\gamma+1)F(x)] - \ln(R/\Lambda^2) \}$$

Σ_C^{reg} is also independant of the mass scale R for the same reason as the self-energy term Σ_B^{reg} is independent of R.

The Gauge dependant Term

Since the integrand in this term does not change for y > R, all we have to do is to perform the integral from R to infinity and then we can write the contribution from the gauge term written as,

$$\Sigma_\xi = \Sigma_\xi^{reg} - 2\, e [\ln(Y/\Lambda^2)]^{\gamma+1} / [(\gamma+1)F(x)]$$

where,

$$\Sigma_\xi^{reg} = - \left\{ 1 + 2 \int_x^R dy/y \, \frac{F(y)}{F(x)} \right\} .$$

$$+ 2 \, e \, [\, \ln(R/\Lambda^2) \,]^{\gamma+1} \, / \, [\, (\gamma+1) F(x) \,]$$

5:9 The Renormalization

Let us introduce the function J(x) which contains the regularised self-energy parts and the constant on the right hand side of the Schwinger-Dyson equation (7.1)

$$J(x) = 1 - \lambda \, (\, \Sigma_A^{reg} + \Sigma_B^{reg} + \Sigma_C^{reg} + \Sigma_\xi^{reg} \,)$$

Then the equation can be written in the more managable form of,

$$1/F(x) = J(x) + c \ln(Y/\Lambda^2) + g/F(x) \, [\ln(Y/\Lambda^2)]^{\gamma+1}$$

$$+ 3 \, A_g \, d(x) \ln(\delta^2) \, . \tag{9.1}$$

where

$$c = 3/2 \, \lambda \, (B_g + C_g) \, , \quad g = -e \, \lambda \, \{3/2(B_g + C_g) + 2\xi\} \, / \, (\gamma+1)$$

and $$d(x) = x_0 \, \{ \, F'(x)/F(x) + 1/x \, \} \tag{9.2}$$

Obviously F(x) is now dependant upon the cutoffs, δ and Y, that have been introduced. To remove these dependences we do three subtractions on the equation at the points x_1, x_2 and x_3. We might expect that since there are only two cutoffs, it is, in fact, only necessary to do two subtractions. However, our use of the renormalization group improved form of the perturbative quark function leads to complications, such that it is easier just to do three subtractions rather than try to find a method of only doing two subtractions. The use of three subtractions is in no way incorrect, the extra subtraction is just a finite renormalization instead of an infinite renormalization.

Now let us denote the F(x) on the left hand side of equation (9.1) (which at the moment is also a function of the cutoffs) by $F^{\circ}(x)$ and multiply through by F(x). Then subtract from the equation (9.1) its value at the points x_1, x_2 and x_3 multiplied by the arbitary function A(x), B(x) and C(x) respectivly. Thus,

$$F(x)/F^{\circ}(x) - A(x)F(x_1)/F^{\circ}(x_1) - B(x)F(x_2)/F^{\circ}(x_2) - C(x)F(x_3)/F^{\circ}(x_3)$$

$$= J(x)F(x) - A(x)J(x_1)F(x_1) - B(x)J(x_2)F(x_2) - C(x)J(x_3)F(x_3)$$

$$+ c\,[\,F(x) - A(x)F(x_1) - B(x)F(x_2) - C(x)F(x_3)\,]\ln(Y/\Lambda^2)$$

$$+ g\,[\,1 - A(x) - B(x) - C(x)\,]\,\{\ln(Y/\Lambda^2)\}^{\gamma+1}$$

$$+ 3A_g\,[F(x)d(x)-A(x)F(x_1)d(x_1)-B(x)F(x_2)d(x_2)-C(x)F(x_3)d(x_3)]\ln(\delta^2)\,. \tag{9.3}$$

For $F^{\circ}(x)$ to be independent of the cutoffs, the square brackets must vanish, which leads to the equations,

$$1 - A(x) - B(x) - C(x) = 0 \tag{9.4a}$$

$$F(x) - A(x)F(x_1) - B(x)F(x_2) - C(x)F(x_3) = 0 \tag{9.4b}$$

$$F(x)d(x)-A(x)F(x_1)d(x_1)-B(x)F(x_2)d(x_2)-C(x)F(x_3)d(x_3) = 0 \tag{9.4c}$$

Notice that, since d(x) only appears in equation (9.4c), it is not uniquely defined and we are free to multiply it by an arbitrary constant. It was for this reason that the factor 3 A_g was extracted in the definition (5,j). These equations are then sufficient to define the functions A(x), B(x) and C(x). Solving these three equations simultaneously leads to the result,

$$A(x) = [\ F(x)(F_3 d_3 - F_2 d_2) + F(x)d(x)(F_2 - F_3) + F_2 F_3 (d_2 - d_3)\ ]/D$$

$$B(x) = [\ F(x)(F_1 d_1 - F_3 d_3) + F(x)d(x)(F_3 - F_1) + F_3 F_1 (d_3 - d_1)\ ]/D$$

$$C(x) = [\ F(x)(F_2 d_2 - F_1 d_1) + F(x)d(x)(F_1 - F_2) + F_1 F_2 (d_1 - d_2)\ ]/D$$

where

$$D = F_1 F_2 (d_1 - d_2) + F_2 F_3 (d_2 - d_3) + F_3 F_1 (d_3 - d_1) \tag{9.5}$$

and $F_1 = F(x_1)$, $d_1 = d(x_1)$, etc.

Since the output function F° is now independent of the cutoffs, the left hand side of the equation vanishes. So rearranging the equation (9.3) becomes,

$$1/F(x) = J(x)/\{\ A(x)J_1 F_1 + B(x)J_2 F_2 + C(x)J_3 F_3\ \} \tag{9.6}$$

where,

$$J(x) = 1 - \lambda\ (\ A_g\ \Sigma_A^{reg} + B_g\ \Sigma_B^{reg} + C_g\ \Sigma_C^{reg} + \xi\ \Sigma_\xi^{reg}\ )$$

and

$$\Sigma_A^{reg} = 3\ x_0 \int_0^\infty dy/y\ \frac{h(y^2/x^2)}{|\ x - y\ |} \left\{ \frac{y}{x - y} \left[\frac{F(y)}{F(x)} - 1 \right] + x\ \frac{F'(x)}{F(x)} \right\}$$

$$- \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] [\ \ln(x^2) - 1\ ]\ .$$

Quark Propagator

$$\Sigma_B^{reg} = 3/2 \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ \frac{x+y}{x-y} h(y^2/x^2) + \theta_+(y-R) \right\}.$$

$$+ \; 3/2 \; \{ \; e \; [\; \ln(R/\Lambda^2) \;]^{\gamma+1} \; / \; [\; (\gamma+1)F(x) \;] - \ln(R/\Lambda^2) \; \}$$

$$\Sigma_C^{reg} = \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} + 1 \right] \left\{ 2yx_0 - (y-x)^2 \; h(y/x) \right.$$

$$\left. + \; [\; (y-x)^2 - x_0 \; (y+x) - x_0^2 \;] \; h_1(y/x, x_0/x) \right\}$$

$$+ \; \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ (y^2-x^2) \; h(y/x) + x_0 y \frac{y+x}{x-y} + 3xx_0 \theta_+(y-R) \right.$$

$$\left. + \left[2xx_0 \frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x \right] h_1(y/x, x_0/x) \right\}$$

$$+ \; 3/2 \; \{ \; e \; [\; \ln(R/\Lambda^2) \;]^{\gamma+1} \; / \; [\; (\gamma+1)F(x) \;] - \ln(R/\Lambda^2) \; \}$$

$$\Sigma_\xi^{reg} = - \left\{ 1 + 2 \int_x^R dy/y \frac{F(y)}{F(x)} \right\} .$$

$$+ \; 2 \; e \; [\; \ln(R/\Lambda^2) \;]^{\gamma+1} \; / \; [\; (\gamma+1)F(x) \;]$$

There is a good reason for leaving the Schwinger-Dyson equation in this form and not recasting it in a form more closely resembling the original unrenormalised equation. If we were to try to recast the equation in the original form, equation (9.4a) would lead to the removal of the explicit dependence upon the parameter λ which contains the strong coupling constant α_s. Analytically, this is not of importance, since the coupling is implicit in the function F(x) and the perturbative result. However, numerically, it can make quite a difference to the convergence of the minimization and stability of the equation about the minimum. We might be tempted to write the equation in a form in which F(x) and not it's inverse appears on the left hand side of the equation. In such a form the right hand side would be the difference of a number of terms. These terms would have to conspire to give zero as the momentum goes to zero. Analytically this is all right, but, numerically, it is much easier to get zero by dividing by a large number than by subtracting numbers.

Chapter 6

Evaluation of the Scalar Integrals

6:1 Introduction

We now turn to the evaluation of the scalar integrals contained in the definition of the regularized quark self-energy. In regularizing the self-energy, we introduced the mass scale R and claimed that for momenta squared above this scale the perturbative approximation for the quark function was valid. So we now need to have some approximation for the quark renormalization function which is valid for momenta less than R in the non-perturbative regime and continuous with the perturbative result.

We choose to parameterise the quark function as a power series in the momenta squared upto the value R in momentum squared. As we have seen in the beginning of section 5 in the previous chapter, we expect the quark function F(x) to vanish as x goes to zero ie, that the leading term in an expansion about zero would be x. But to be on the safe side, let us start the series at unity and have a sufficient number of terms so that we can join the series on to the perturbative form at R such that the derivative is continuous at R. ie let

$$F(x) = \begin{cases} A_q + B_q\, z + C_q\, z^2 + D_q\, z^3 + E_q\, z^4 & \text{for } x < R \qquad (1.1) \\ e\, [\, \ln (x/\Lambda^2)\,]^{\gamma} & \text{for } x > R \qquad (1.2) \end{cases}$$

where

$$z = x / R$$

and

Scalar Integral

$$e = \frac{F(p_1^2)}{[\ \ln\ (p_1^2/\Lambda^2)\ ]^{\gamma}}.$$

The requirement that our parameterization matches on to the perturbative result and that its derivative is continuous at the point R gives us two constraints on the parameters,

$$A_q + B_q + C_q + D_q + E_q = F_p(R)$$

$$B_q + 2\ C_q + 3\ D_q + 4\ E_q = F_p'(R)$$

where

$$F_p(R) = e\ [\ \ln\ (R/\Lambda^2)\ ]^{\gamma} \tag{1.3}$$

and

$$F_p'(R) = e\ \gamma\ [\ \ln\ (R/\Lambda^2)\ ]^{\gamma-1}\ /R$$

If we solve these equations for D_q and E_q, say, then we find that,

$$D_q = 4\ F_p(R) - F_p'(R) - 4\ A_q - 3\ B_q - 2\ C_q \tag{1.4}$$

$$E_q = 3\ A_q + 2\ B_q + C_q - 3\ F_p(R) + F_p'(R) \tag{1.5}$$

We now have a smooth approximation for the quark renormalization function which depends upon the parameters outlined above and the perturbative approximation. Notice that although the quark parameters may be dependent upon the value of the mass scale R, the numerical value will be largely independent of R (totally independent only if our approximation is in fact the exact result).

We now turn to the evaluation of the scalar integrals in the regularized quark self-energy. Dropping the superscript reg, the self-energy can be written as,

Scalar Integral

$$\Sigma = \lambda \left\{ A_g \Sigma_A + B_g \Sigma_B + C_g (\Sigma_C^+ + \Sigma_C^-) + \xi \Sigma_\xi \right\}$$

where we have split the self-energy contribution from the intermediate term in the gluon in to two pieces. The term Σ_C^+ contains the integral over F(y) + F(x) and Σ_C^- contains the integral over F(y) - F(x) together with the logarthmic contributions from the regularization.

Because the quark renormalization function F(y) is given by two different forms (1.1), in the non-perturbative regime ($y < R$), and in the perturbative regime ($y > R$), it is natural to split the integrals into two parts corresponding to the internal variable in the two different regimes. In the integrals involving F(y) - F(x) the denominator goes like y - x , this means that it is necessary to explictly substitute the form of the approximation for F(x) when the external and internal momenta are in the same regime. Since this occurs in most of the integrals, we split all of the integrals into four parts.

1) Both the internal and external momenta in the non-perturbative regime ($y, x < R$).

2) The internal momentum in the perturbative regime and the external momentum in the non-perturbative regime ($x < R < y$).

3) The internal momentum in the non-perturbative regime and the external momentum in the perturbative regime ($y < R < x$).

4) Finally both the internal and external momena in the perturbative regime ($R < x, y$).

Scalar Integral

Now we evaluate the integrals in the parts of the self energy coming from each of the parameters in the gluon renormalization function and the gauge term in turn. In each case we evaluate the integrals for the four conditions outlined above.

6:2 The Enhanced Term

First let us consider the contribution to the quark self-energy coming from the enhanced term in the gluon propagator. From equation (5:9.4a) we see that the self-energy part Σ_A coming from the enhanced term in the gluon propagator is given by,

$$\Sigma_A = 3\, x_0 \int_0^\infty dy/y \, \frac{h(y^2/x^2)}{|\, x - y \,|} \left\{ \frac{y}{x - y} \left[\frac{F(y)}{F(x)} - 1 \right] + x \, \frac{F'(x)}{F(x)} \right\}$$

$$3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \left[\ln(x^2) - 1 \right] \qquad (2.1)$$

where,

$$h(y^2/x^2) = \begin{cases} y^2/x^2 & \text{for } y^2 < x^2 \\ 1 & \text{otherwise.} \end{cases} \qquad (2.2)$$

Let us consider the first two cases when the external momentum is in the non-perturbative regime ($x < R$), so that the quark function $F(x)$ is given by the power series parametrization (1.1). Then the integral in (2.1) is,

$$\frac{3x_0}{F(x)} \int_0^\infty dy/y \, \frac{h(y^2/x^2)}{|\, x - y \,|} \left\{ \frac{y}{x - y} \left[F(y) - F(x) \right] + x\, F'(x) \right\} \qquad (2.3)$$

Now in the first case we take the part of the integral (2.3) from 0 upto the mass scale R. Then both of the momenta are in the perturbative regime (x, y < R) and we can use the power expansion (1.1) for F(y) as well as F(x). In doing this integral we have to be careful of the point y = x. We know that the integrand as a whole does does not diverge as y goes to x, since we have extracted the divergent piece in section 5 of the last chapter. However, the individual terms are less well behaved. First let us just consider the difference between the quark function evaluated at the two momenta x and y pulling out an explicit y - x.

$$\begin{aligned} F(y) - F(x) = (y - x) \, [\, & B_q + C_q \, z + D_q \, z^2 + E_q \, z^3 + \\ & + (C_q + D_q \, z + E_q \, z^2) \, y/R + (D_q + E_q \, z) \, (y/R)^2 + \\ & + E_q \, (y/R)^3 \,] \, / \, R \end{aligned} \qquad (2.4)$$

We can write this in a more compact form by introducing the primed parameters

$$\begin{aligned} B'_g &= B_q + C_q \, z + D_q \, z^2 + E_q \, z^3 \\ C'_g &= C_q + D_q \, z + E_q \, z^2 \\ D'_g &= D_q + E_q \, z. \end{aligned} \qquad (2.5)$$

so that,

$$F(y) - F(x) = (y - x) \, [\, B'_q + C'_q \, y/R + D'_q \, (y/R)^2 + E_q \, (y/R)^3 \,] \, / \, R \qquad (2.6)$$

Using the result (2.4), the part of the integrand of (2.1) in the curly brackets can be written as,

Scalar Integral

$$\frac{y}{y - x} \left[F(y) - F(x) \right] + x F'(x)$$

$$= (x - y) [B_q + 2 C_q z + 3 D_q z^2 + 4 E_q z^3 +$$

$$+ (C_q + 2 D_q z + 3 E_q z^2) y/R + (D_q + 2 E_q z) (y/R)^2 +$$

$$+ E_q (y/R)^3] / R$$

$$= (x - y) [B''_q + C''_q y/R + D''_q (y/R)^2 + E_q (y/R)^3] / R \tag{2.7}$$

where we have introduced the double primed parameters,

$$\begin{aligned} B''_g &= B_q + 2 C_q z + 3 D_q z^2 + 4 E_q z^3 \\ C''_g &= C_q + 2 D_q z + 3 E_q z^2 \\ D''_g &= D_q + 2 E_q z. \end{aligned} \tag{2.8}$$

Therefore, by substituting this result into the integral (2.3) we get,

$$\frac{3x_0}{RF(x)} \int_0^R dy/y \, h(y^2/x^2) \frac{y - x}{| x - y |} \left[B''_q + C''_q y/R + D''_q (y/R)^2 + \right.$$

$$\left. + E_q (y/R)^3 \right]$$

$$= \frac{3 z_0}{F(x)} \left\{ B''_q \left[\frac{1}{2} - \ln z \right] + C''_q \left[\frac{4}{3} z - 1 \right] + \right.$$

$$\left. + \frac{D''_q}{2} \left[\frac{3}{2} z^2 - 1 \right] + \frac{E_q}{3} \left[\frac{8}{5} z^3 - 1 \right] \right\} \qquad x < R \tag{A1}$$

where $z_0 = x_0 / R$

In the second case we consider the upper half of the integral (2.3) when the internal momentum is in the perturbative regime ($R < y$), which we can write as,

Scalar Integral

$$\frac{3x_0}{F(x)} \int_R^\infty dy/y \, \frac{1}{y - x} \left\{ \frac{y}{x - y} \left[F(y) - F(x) \right] + x\, F'(x) \right\} \qquad (2.9)$$

as the function $h(y^2/x^2) = 1$. Because F(y) is now given by the perturbative quark renormalization function (1.2), it is not possible to do the integrals over the quark function analytically. But we can still do the integral over the part independent of F(y), since $x < y$ and so the denominator does not vanish in this region. Thus the integral (2.9) becomes,

$$= \frac{3\, x_0}{F(x)} \left\{ - \int_R^\infty dy \, \frac{F(y)}{(y - x)^2} + \frac{F(x)}{R(1 - z)} - F'(x) \ln (1 - z) \right\}$$

$$x < R \qquad (A2)$$

The remaining integral we have to do numerically. Since the top limit of the integral is infinity, we have to make a change of variable to map the integral on to a finite interval. A convenient choice of change of variables is,

$$z = 1 / \ln(y/\Lambda^2) \quad \Rightarrow \quad dy/y = - dz/z^2$$

The limits of the integral are then $1 / \ln(R/\Lambda^2)$ and 0 (in fact due to machine accuracy, the lower limit was taken to be 0.0015). The quark function can be written with this change of variables as,

$$\begin{aligned} F(y) &= e \left[\ln (x/\Lambda^2) \right]^{\gamma} \\ &= e\, z^{-\gamma}. \end{aligned}$$

All the subsequent numerical integrals in the quark self-energy are done by using this same substitution.

Now let us consider the second two cases, where the external momentum is in the perturbative regime ($R < x$). For the third case when the internal momentum is in the non-perturbative regime ($y < R$), the lower half of the integral in (2.1) can be written as,

$$\frac{3\, x_0}{x^2\, F(x)} \int_0^R y\, dy\, \frac{1}{x - y} \left\{ \frac{y}{x - y} \left[F(y) - F(x) \right] + x\, F'(x) \right\}$$

as now the function $h(y^2/x^2) = y^2/x^2$. Since the external momentum is greater than the internal momentum ($y < R < x$) we do not need to worry about the form of F(x) as the denominator does not vanish. Thus by substituting our power series parameterization (1.1) for F(y) and performing the integration we get,

$$= \frac{3\, x_0}{x\, F(x)} \left\{ \left(A_q - F(x) \right) \left[\frac{1}{z} + 2 \ln \frac{z - 1}{z} + \frac{1}{z - 1} \right] - \right.$$

$$- R\, F'(x) \left[1 + z \ln \frac{z - 1}{z} \right] +$$

$$+ B_q \left[\frac{1}{2\, z} + 2 + 3\, z \ln \frac{z - 1}{z} + \frac{z}{z - 1} \right] +$$

$$+ C_q \left[\frac{1}{3\, z} + 1 + 3\, z + 4\, z^2 \ln \frac{z - 1}{z} + \frac{z^2}{z - 1} \right] +$$

$$+ D_q \left[\frac{1}{4\, z} + \frac{2}{3} + \frac{3}{2} z + 4\, z^2 + 5\, z^3 \ln \frac{z - 1}{z} + \frac{z^3}{z - 1} \right] +$$

$$\left. + E_q \left[\frac{1}{5\, z} + \frac{1}{2} + z + 2\, z^2 + 5\, z^3 + 6\, z^4 \ln \frac{z - 1}{z} + \frac{z^4}{z - 1} \right] \right\}$$

$$x > R \qquad \text{(A3)}$$

Scalar Integral

Finally the fourth case is when the internal momentum is in the perturbative regime together with the external momentum. In this case, we have to evaluate the integral totally numerically, because of the logarithm to a power in the perturbative quark renormalization function. The integrand is finite in the limit y goes to x and provided our quadrature routine does not attempt to evaluate the integrand at that point the integral can be calculated numerically. Nevertheless the integral can be simply written as,

$$\frac{3x_0}{F(x)} \int_R^\infty dy/y \, \frac{h(y^2/x^2)}{|\, x - y \,|} \left\{ \frac{y}{x - y} \left[F(y) - F(x) \right] + x\, F'(x) \right\}$$

$$x > R \qquad \text{(A4)}$$

6:3 The Constant Term

We now turn to the evaluation of the self-energy part coming from the constant term in the gluon propagator. From equation (5:9.4b) we see that this contribution to the self-energy is given by,

$$\Sigma_B = 3/2 \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left\{ \frac{x + y}{x - y} h(y^2/x^2) + \theta_+(y-R) \right\} +$$

$$+ \; 3/2 \; \{ \; e \; [\; \ln(R/\Lambda^2) \;]^{\gamma+1} \; / \; [\; (\gamma+1)F(x) \;] - \ln(R/\Lambda^2) \; \} \qquad (3.1)$$

First we evaluate the integral in the regime in which the external momentum is non-perturbatve (x < R). Then the quark renormalization function F(x) is given by our power series parameterization (1.1). Consider the first case when the internal momentum is also in the non-perturbative regime (y, x < R). In this case the step function gives zero contribution and we can write the integral in (3.1) as,

Scalar Integral

$$\frac{3}{2}\int_0^R dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \frac{x+y}{x-y}\, h(y^2/x^2). \tag{3.2}$$

From the previous section we recall that for y and x < R

$$F(y) - F(x) = (y - x) \left[B'_q + C'_q\, y/R + D'_q\, (y/R)^2 + E_q\, (y/R)^3 \right] / R$$

where the primed parameters are defined in equations (2.5). Substituting this result into the integral (3.2), we get that,

$$\frac{-3}{2F(x)}\int_0^R dy/y\ (x + y)\, h(y^2/x^2) \left[B'_q + C'_q\, y/R + D'_q\, (y/R)^2 + \right.$$

$$\left. + E_q\, (y/R)^3 \right] / R$$

$$= \frac{3}{2F(x)} \left\{ B'_q \left[-\frac{1}{6} z + z \ln z - 1 \right] + C'_q \left[\frac{11}{12} z^2 - z - 1/2 \right] + \right.$$

$$\left. + D'_q \left[\frac{23}{60} z^3 - z/2 - 1/3 \right] + E_q \left[\frac{13}{60} z^4 - z/3 - 1/4 \right] \right\}$$

$$x < R \tag{B1}$$

If we now consider the second case, where the internal momentum is in the perturbative regime (R < y), then the function $h(y^2/x^2)$ and the step function $\theta_+(y-R)$ both give unit contribution. This means, doing a little algebra, that upper half of the integral in (3.1) can be written as,

$$-\frac{3}{2}\int_R^\infty dy/y\, \frac{2x}{y-x} \left[\frac{F(y)}{F(x)} - 1 \right]. \tag{3.3}$$

Since F(y) is given by the perturbative quark renormalization function (1.2) we cannot do the integrals over F(y) analytically.

However, we can do the integral over the constant part as $y \neq x$ leading to the result for the integral (3.3) of,

$$= - \frac{1}{F(x)} \int_R^\infty dy/y \, \frac{3\,x}{y - x} F(y) - 3 \ln (1 - z) \qquad x < R \qquad \text{(B2)}$$

We now go on to consider the second two cases when the external momentum is in the perturbative regime ($R < x$). For the third case when the internal momentum is in the non-perturbative regime the function $h(y^2/x^2) = y^2/x^2$ and the step function is zero. This means that the lower half of the integral in (3.1) can be written as,

$$\frac{1}{x^2\ F(x)} \int_0^R y\ dy \frac{y + x}{x - y} [\ F(y) - F(x)\ ]\ . \qquad (3.4)$$

Now by substituting in the power series parameterization (1.1) for F(y) we can do the integral (3.4) leading to the result,

$$\begin{aligned}
= \frac{-\,3}{2F(x)} \Big\{ & [\ A_g - F(x)\ ] \left[\frac{1}{2z^2} + \frac{2}{z} + 2 \ln \frac{z - 1}{z} \right] + \\
& + B_q \left[\frac{1}{3z^2} + \frac{2}{z} + 2 + 2\ z \ln \frac{z - 1}{z} \right] + \\
& + C_q \left[\frac{1}{4z^2} + \frac{2}{3\ z} + 1 + 2\ z + 2\ z^2 \ln \frac{z - 1}{z} \right] + \\
& + D_q \left[\frac{1}{5z^2} + \frac{1}{2\ z} + \frac{2}{3} + z + 2\ z^2 + 2\ z^3 \ln \frac{z - 1}{z} \right] + \\
& + E_q \left[\frac{1}{6z^2} + \frac{2}{5\ z} + \frac{1}{2} + \frac{2\ z}{3} + z^2 + 2\ z^3 + 2\ z^4 \ln \frac{z - 1}{z} \right] \Big\}
\end{aligned}$$

$$x > R \qquad \text{(B3)}$$

Scalar Integral

Finally, we consider the fourth case when both of the momenta are in the perturbative regime ($R < y, x$). In this case the step function gives unit contribution. Thus the upper half of the integral in (3.1) can be written as,

$$\int_R^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \left[\frac{x + y}{x - y} h(y^2/x^2) + 1 \right] \qquad x > R. \qquad \text{(B4)}$$

Now as the quark function is given by the perturbative result (1.2) and the individual pieces diverge we cannot do the integral analytically. However, the integral can be done numerically.

6:4 The Intermediate term

We now move on to consider the contribution to the quark self-energy coming from the intermediate term in the gluon propagator. As can be seen from equation (5:9.4c), this part of the self-energy involves integrals over the function $h_1(y/x, x_0/x)$, which we recall from equation (5:5.6) is defined to be,

$$h_1(y/x , x_0/x) = y + x + x_0 + \sqrt{A(y)} \qquad (4.1)$$

where

$$A(y) = (x + x_0)^2 - 2 (x - x_0) y + y^2 \qquad (4.2)$$

It is the integrals over this square root that will cause some complication of the results. Since this square root appears in the integrals multiplied by various powers of y, particularly those coming from the power series parameterization of the quark function, it is advantageous to make the following definitions.

Scalar Integral

Let $$J_a(y) = \int dy\, y^a \sqrt{A(y)} \tag{4.3}$$

Then

$$J_0(y) = \int dy \sqrt{A(y)}$$

$$= 1/2\ (\ y + x_0 - x\ ) \sqrt{A(y)} + 2\ x\ x_0\ I_0(y) \tag{4.4a}$$

where

$$I_0(y) = \int \frac{dy}{\sqrt{A(y)}}$$

$$= \ln\ (\ \sqrt{A(y)} + y + x_0 + x\ ) \tag{4.5}$$

also

$$J_1(y) = \int dy\ y \sqrt{A(y)}$$

$$= 1/3\ A(y)^{3/2} - 1/2\ (\ y + x_0 - x\ )\ (\ x_0 - x\ ) \sqrt{A(y)} -$$

$$- 2\ x\ x_0\ (\ x_0 - x\ )\ I_0(y) \tag{4.4b}$$

In general, we can use the recurrence relation,

$$J_a(y) = y^{a-1}\ A(y)^{3/2} - (\ 2\ a + 1\ )\ (\ x_0 - x\ )\ J_{a-1}(y) -$$

$$- (\ a - 1\ )\ (\ x_0 + x\ )^2\ J_{a-2}(y) \tag{4.6a}$$

to generate the result of the integrals involving powers of y greater than one.

Let us now consider the cases when the square root is divided by powers of y. For just one power of y, we have,

Scalar Integral

$$J_{-1}(y) = \int dy/y \sqrt{A(y)}$$

$$= \sqrt{A(y)} + (x_0 - x) I_0(y) + (x_0 + x) I_1(y) \tag{4.4c}$$

where

$$I_1(y) = - (x + x_0) \int \frac{dy}{y \sqrt{A(y)}}$$

$$= \ln \frac{(x + x_0)^2 + (x_0 - x) y + (x + x_0) \sqrt{A(y)}}{x_0 \; y} \tag{4.7}$$

From this result for $J_{-1}(y)$ and the result for $J_0(y)$ (4.4a) we can use the recurrence relation,

$$J_{-a}(y) = \Big[- y^{-a+1} A(y)^{3/2} + (5 - 2 a) (x_0 - x) J_{-a+1}(y) + + (4 - a) J_{-a+2}(y) \Big] / \Big[(a - 1) (x_0 + x)^2 \Big] \tag{4.6b}$$

to generate all the results for the integrals where the square root is divided by a power of y greater than one.

Having made these defintions, it is a simple matter for the algebraic manipulation package MACSYMA to calculate the indefinite part of the integrals involving the the power series parameterization of the quark renormalization function. The reason that only the indefinite integrals were calculated by MACSYMA is that the individual terms in the integral are not finite in the limits y goes to infinity and y goes to zero. These divergent limits are then handled by making an appropriate expansion of the square root. We can see that the integrals must be finite by expanding the function $h_1(y/x, x_0/x)$ in the two limits. We find that, for y small,

$$h_1(y/x , x_0/x) \approx \frac{y}{x_0 + x} \qquad (4.8a)$$

so the integral is indeed infra-red safe. For y large

$$h_1(y/x , x_0/x) \approx 1 + \frac{1}{y} + \frac{x - x_0}{y^2} \qquad (4.8b)$$

and so the integrals is also ultra-violet safe.

Since the contribution from the intermediate term to the self-energy is somewhat complicated, we split it into two parts depending upon whether the integral contains the difference or the sum of the quark function evaluated at the two different momenta:

$$\Sigma_C = \Sigma_C^+ + \Sigma_C^-$$

The part containing the difference (Σ_C^-) also contains the logarithmic contribution from the regularization which will cancel the dependence of the integral on the mass scale R coming from the step function.

6:5 The Self-Energy Part Σ^+

Let us first consider the part of the self-enegy contribution, from the intermediate term in the gluon, containing the sum of the quark functions. This can be written as,

Scalar Integral

$$\Sigma_C^+ = \frac{1}{2xx_0} \int_0^\infty dy/y \left[\frac{F(y)}{F(x)} + 1 \right] \left\{ 2yx_0 - (y - x)^2 \, h(y/x) + \right.$$

$$\left. + \left[(y - x)^2 - x_0 \, (y + x) - x_0^2 \right] h_1(y/x, x_0/x) \right\} \tag{5.1}$$

We now proceed to evaluate this contribution in each of the four cases that we outlined in the introduction.

Let us consider the first two cases when external momentum is in the non-perturbative regime ($x < R$). In the first case, when the internal momentum is also in the non-perturbative regime ($y < R$), we can use the power series parametrization for F(y). There is no need to substitute for F(x) as the denominator in the integral does not vanish for y = x. Thus, using our definitions in the previous section, we can calculate the lower half of the integral in (5.1), splitting it into five pieces ,one for each of the quark parameters, for simplicity,

$$\frac{(A_q + F(x))}{24 \, x_0 \, x^2 \, F(X)} \left\{ 2 R^3 - 9 x R^2 - 18 (x_0^2 - x^2) R - 22 x^3 + \right.$$

$$+ 12 x^3 \ln R/x - (2 R^2 - (7 x + 2 x_0) R) \sqrt{A(R)} -$$

$$- (11 x^2 + x x_0 - 16 x_0^3) (\sqrt{A(R)} - (x_0 + x)) +$$

$$+ 6 (x^3 + 3 x_0^2 x + 2 x_0^3) I_0 +$$

$$\left. + 6 (x - 2 x_0) (x_0 + x)^2 I_1 \right\}$$

$$x < R \qquad (C^+A1)$$

where, from equation (4.5), we have that,

$$I_0 = I_0(R) - I_0(0)$$

$$= \ln \frac{(\sqrt{A(R)} + R + x_0 + x)}{2 x_0}. \tag{5.2}$$

Also, from equation (4.7), we have that,

$$I_1 = I_1(R) - \lim_{\varepsilon \to 0} (I_1(\varepsilon) + \ln \varepsilon/R)$$

$$= \ln \frac{(x + x_0)^2 + (x_0 - x) R + (x + x_0) \sqrt{A(R)}}{2 (x + x_0)^2} \tag{5.3}$$

Let me reiterate that the integral is not infra-red divergent. The divergence coming from the lower limit in the integral I_1 is cancelled exactly by a divergence coming from the other part of the integral over the function $h_1(y/x , x_0/x)$. The remaining contributions to the integral from the power series parametrization can be written as,

$$\frac{B_q}{16x^2 x_0 F(x)} \left\{ \frac{2x^4}{R} + R^3 - 4 x R^2 + 6 (x^2 - x_0^2) R - \right.$$

$$- 8 x_0^3 - 12 x x_0^2 - 4 x^3 + 24 x^2 x_0^2 I_0 / R -$$

$$- (R^2 - (3 x + x_0) R + 3 x^2 - 5 x_0^2) \sqrt{A(R)} +$$

$$\left. + (x^3 - x^2 x_0 + 13 x x_0^2 + 3 x_0^3) (\sqrt{A(R)} - (x_0 + x)) \right\}$$

$$x < R \tag{C$^+$B1}$$

Scalar Integral

$$\frac{C_q}{80x^2 x_0 F(x)} \left\{ \frac{2x^5}{R^2} + 4\,R^3 - 15\,x\,R^2 - 20\,(\,x_0^2 - x^2\,)\,R - \right.$$

$$- 10\,(\,x^3 + 3\,x\,x_0^2 + 2\,x_0^3\,) + 120\,\frac{x^2}{R^2}\,x_0^2\,(\,x - 2\,x_0\,)\,I_0 -$$

$$- \Big[\,4\,R^2 - (\,11\,x + 4\,x_0\,)\,R + 9\,x^2 - x\,x_0 - 16\,x_0^2 -$$

$$- (\,x^3 - 4\,x_0\,x^2 + 29\,x\,x_0^2 + 4\,x_0^3\,)\,/\,R\,\Big]\,\sqrt{A(R)} +$$

$$\left. + \Big[\,x^4 - x_0\,x^3 + 141\,x^2\,x_0^2 - 41\,x\,x_0^3 - 4\,x_0^4\,\Big]\,\frac{\sqrt{A(R)} - (x_0 + x)}{R^2} \right\}$$

$$x < R \qquad (C^{+}C1)$$

Scalar Integral

$$\frac{D_q}{120x^2 x_0 F(x)} \left\{ \frac{2x^6}{R^3} + 10\,R^3 - 36\,x\,R^2 - 45\,(\,x_0^2 - x^2\,)\,R - \right.$$

$$- 20\,(\,x^3 + 3\,x\,x_0^2 + 2\,x_0^3\,) +$$

$$+ 120\,\frac{x^2}{R^3}\,x_0^2\,(\,3\,x^2 - 16\,x\,x_0 + 9\,x_0^2\,)\,I_0 -$$

$$- \Big[\,10\,R^2 - (\,26\,x + 10\,x_0\,)\,R + 19\,x^2 - 4\,x\,x_0 - 35\,x_0^2 -$$

$$- (\,x^3 - 9\,x_0\,x^2 + 51\,x\,x_0^2 + 5\,x_0^3\,)\,/\,R -$$

$$- (\,x^4 - 4\,x_0\,x^3 + 204\,x\,x_0^2 - 76\,x\,x_0^3 - 5\,x_0^4\,)\,/\,R^2\,\Big]\,\sqrt{A(R)} +$$

$$+ \Big[\,x^5 - x_0\,x^4 + 556\,x^3\,x_0^2 - 1036\,x^2\,x_0^3 + 91\,x\,x_0^4 +$$

$$\left. + 5\,x_0^5\,\Big]\,\frac{\sqrt{A(R)} - (x_0 + x)}{R^3} \right\}$$

$$x < R \qquad (C^{+}D1)$$

<u>Scalar Integral</u>

$$\frac{E_q}{560x^2 x_0 F(x)} \Bigg\{ \frac{2x^7}{R^4} + 20\,R^3 - 70\,x\,R^2 - 84\,(\,x_0^2 - x^2\,)\,R -$$

$$- 35\,(\,x^3 + 3\,x\,x_0^2 + 2\,x_0^3\,) +$$

$$+ 840\,\frac{x^2}{R^4}\,x_0^2\,(\,x^3 - 10\,x^2\,x_0 + 15\,x\,x_0^2 - 4\,x_0^3\,)\,I_0 -$$

$$- \Big[\,20\,R^2 - 10\,(\,5\,x + 2\,x_0\,)\,R + 34\,x^2 - 10\,x\,x_0 - 64\,x_0^2 -$$

$$- (\,x^3 - 16\,x_0\,x^2 + 79\,x\,x_0^2 + 6\,x_0^3\,)\,/\,R -$$

$$- (\,x^4 - 9\,x_0\,x^3 + 295\,x^2\,x_0^2 - 121\,x\,x_0^3 - 6\,x_0^4\,)\,/\,R^2 -$$

$$- (\,x^5 - 4\,x_0\,x^4 + 688\,x^2\,x_0^2 - 1262\,x^2\,x_0^3 + 151\,x\,x_0^4 +$$

$$+ 6\,x_0^4\,)\,/\,R^3\,\Big]\,\sqrt{A(R)} +$$

$$+ \Big[\,x^6 - x_0\,x^5 + 1520\,x^4\,x_0^2 - 6770\,x^3\,x_0^3 + 4145\,x^2\,x_0^4 -$$

$$- 169\,x\,x_0^5 - 6\,x_0^6\,\Big]\,\frac{\sqrt{A(R)} - (x_0 + x)}{R^3}\,\Bigg\}$$

$$x < R \qquad (C^{+}E1)$$

Scalar Integral

Now we consider the upper half of the integral from R to infinity with the external momentum in the perturbative regime. Then the function h(y/x) gives unit contribution and so the integral is,

$$\frac{1}{2xx_0}\int_R^\infty dy/y\left[\frac{F(y)}{F(x)}+1\right]\left\{2yx_0-(y-x)^2\right.$$
$$\left.+\left[(y-x)^2-x_0(y+x)-x_0^2\right]h_1(y/x,x_0/x)\right\}$$

The quark function F(y) is now given by the renormalised perturbative result (1.1) and so, as before, the integrals containing F(y) have to be done numerically. On the other hand, the integral over the unit term can be performed analytically leading to the result,

$$\frac{1}{2xx_0}\int_R^\infty dy/y\,\frac{F(y)}{F(x)}\left\{2yx_0-(y-x)^2+\right.$$
$$\left.+\left[(y-x)^2-x_0(y+x)-x_0^2\right]h_1(y/x,x_0/x)\right\}$$

$$\frac{1}{24x^2x_0}\left\{(2R^2-(7x+2x_0)R+11x^2+xx_0-16x_0^2)\sqrt{A(R)}+\right.$$
$$+11x^3-9xx_0^2+16x_0^3-2R^3+9xR^2+18(x_0^2-x^2)R-$$
$$\left.-6(x^3+3xx_0^2+2x_0^3)\tilde{I}_0-6(x-2x_0)(x_0+x)^2\tilde{I}_1\right\}$$
$$x < R \qquad (C^+2)$$

where from equation (4.c) we have that,

$$\tilde{I}_0 = \lim_{Y\to\infty}(I_0(Y)-\ln 2Y) - I_0(R)$$

$$= \ln \frac{\sqrt{A(R)} + R + x_0 - x}{2 R}.$$

The logarithmic divergence in $I_0(y)$ as y goes to infinity is exactly cancelled by a divergence coming from the integral over the rest of the function $h_1(y/x,x_0/x)$ which agrees with the large y expansion of the function $h_1(y/x,x_0/x)$ given in equation (4.8b).

Notice that from equation (C^+A1) and (C^+2), we can calculate the integral over the unit term from 0 to infinity,

$$\frac{1}{2xx_0} \int_0^\infty dy/y \Big\{ 2yx_0 - (y - x)^2 h(y/x) +$$

$$+ \Big[(y - x)^2 - x_0 (y + x) - x_0^2 \Big] h_1(y/x,x_0/x) \Big\}$$

$$= \frac{1}{4x_0 x^2} \Big\{ (x - 2 x_0) \Big[2 x x_0 + (x_0 + x)^2 \ln \frac{x x_0}{(x_0 + x)^2} \Big] -$$

$$- (x^3 + 3 x x_0^2 + 2 x_0^3) \ln \frac{x}{x_0} \Big\} .$$

Now let us consider the self-energy part Σ_C^+ for the last two cases when the external momentum is in the perturbative regime ($R < x$). Then for the third case when the internal momentum is in the non-perturbative regime the function $h(y/x) = y/x$ and so the lower half of the integral is,

$$\frac{1}{2xx_0} \int_0^R dy/y \Big[\frac{F(y)}{F(x)} + 1 \Big] \Big\{ 2yx_0 - (y - x)^2 y/x +$$

$$+ \Big[(y - x)^2 - x_0 (y + x) - x_0^2 \Big] h_1(y/x,x_0/x) \Big\}$$

Scalar Integral

By substituting the power series parametrization (1.1) for F(y) the integrals can be done using the definitions in section 4. To make things manageable we again write down the result in five pieces.

$$\frac{(A_q + F(x))}{24\, x_0\, x^2\, F(X)} \Big\{ - 2 R^3 + 9 x R^2 - 18 (x_0^2 + x^2) R -$$

$$- (2 R^2 - (7 x + 2 x_0) R) \sqrt{A(R)} -$$

$$- (11 x^2 + x x_0 - 16 x_0^3) (\sqrt{A(R)} - (x_0 + x)) +$$

$$+ 6 (x^3 + 3 x_0^2 x + 2 x_0^3) I_0 +$$

$$+ 6 (x - 2 x_0) (x_0 + x)^2 I_1 \Big\}$$

$$x < R \qquad (C^+A3)$$

$$\frac{B_q}{16 x^2 x_0 F(x)} \Big\{ - R^3 + 4 x R^2 - 6 (x^2 + x_0^2) R -$$

$$- 8 x_0^3 - 12 x x_0^2 + 4 x^3 + 24 x_0^3 \frac{x^2}{R} I_0$$

$$- (R^2 - (3 x + x_0) R + 3 x^2 - 5 x_0^2) \sqrt{A(R)} +$$

$$+ (x^3 - x^2 x_0 + 13 x x_0^2 + 3 x_0^3) \frac{\sqrt{A(R)} - (x_0 + x)}{R} \Big\}$$

$$x > R \qquad (C^+B3)$$

Scalar Integral

$$\frac{C_q}{80x^2 x_0 F(x)} \Big\{ - 4 R^3 + 15 x R^2 - 20 (x_0^2 + x^2) R +$$

$$+ 10 (x^3 - 3 x x_0^2 - 2 x_0^3) +$$

$$+ 120 \frac{x^2}{R^2} x_0^2 (x - 2 x_0) I_0 -$$

$$- \Big[4 R^2 - (11 x + 4 x_0) R + 9 x^2 - x x_0 - 16 x_0^2 -$$

$$- (x^3 - 4 x_0 x^2 + 29 x x_0^2 + 4 x_0^3) / R \Big] \sqrt{A(R)} +$$

$$+ \Big[x^4 - x_0 x^3 + 141 x^2 x_0^2 - 41 x x_0^3 + 4 x_0^4 \Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^2} \Big\}$$

$$x > R \qquad (C^+C3)$$

Scalar Integral

$$\frac{D_q}{240x^2 x_0 F(x)} \Bigg\{ - 10\, R^3 + 36\, x\, R^2 - 45\, (\, x_0^2 + x^2\,)\, R +$$

$$+ 20\, (\, x^3 - 3\, x\, x_0^2 - 2\, x_0^3\,) +$$

$$+ 120\, \frac{x^2}{R^3}\, x_0^2\, (\, 3\, x^2 - 16\, x\, x_0 + 9\, x_0^2\,)\, I_0 -$$

$$- \Big[\, 10\, R^2 - (\, 26\, x + 10\, x_0\,)\, R + 19\, x^2 - 4\, x\, x_0 - 35\, x_0^2 -$$

$$- (\, x^3 - 9\, x_0\, x^2 + 51\, x\, x_0^2 + 5\, x_0^3\,) / R -$$

$$- (\, x^4 - 4\, x_0\, x^3 + 204\, x\, x_0^2 - 76\, x\, x_0^3 - 5\, x_0^4\,) / R^2 \,\Big] \sqrt{A(R)} +$$

$$+ \Big[\, x^5 - x_0\, x^4 + 556\, x^3\, x_0^2 - 1036\, x^2\, x_0^3 + 91\, x\, x_0^4 +$$

$$+ 5\, x_0^5 \,\Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^3} \Bigg\}$$

$$x > R \qquad (C^+D3)$$

Scalar Integral

$$
\frac{E_q}{560x^2 x_0 F(x)} \Bigg\{ - 20\ R^3 + 70\ x\ R^2 - 84\ (\ x_0^2 + x^2\ )\ R +
$$

$$
+ 35\ (\ x^3 - 3\ x\ x_0^2 - 2\ x_0^3\ ) +
$$

$$
+ 840\ \frac{x^2}{R^4}\ x_0^2\ (\ x^3 - 10\ x^2\ x_0 + 15\ x\ x_0^2\ - 4\ x_0^3\ )\ I_0 -
$$

$$
- \Big[\ 20\ R^2 - 10\ (\ 5\ x + 2\ x_0\ )\ R + 34\ x^2 - 10\ x\ x_0 - 64\ x_0^2 -
$$

$$
- (\ x^3 - 16\ x_0\ x^2 + 79\ x\ x_0^2 + 6\ x_0^3\ )\ /\ R -
$$

$$
- (\ x^4 - 9\ x_0\ x^3 + 295\ x^2\ x_0^2 - 121\ x\ x_0^3 - 6\ x_0^4\ )\ /\ R^2 -
$$

$$
- (\ x^5 - 4\ x_0\ x^4 + 688\ x^2\ x_0^2 - 1262\ x^2\ x_0^3 + 151\ x\ x_0^4 +
$$

$$
+ 6\ x_0^4\ )\ /\ R^3\ \Big]\ \sqrt{A(R)} +
$$

$$
+ \Big[\ x^6 - x_0\ x^5 + 1520\ x^4\ x_0^2 - 6770\ x^3\ x_0^3 + 4145\ x^2\ x_0^4 -
$$

$$
- 169\ x\ x_0^5 - 6\ x_0^6\ \Big]\ \frac{\sqrt{A(R)} - (x_0 + x)}{R^3}\ \Bigg\}
$$

$$
x > R \qquad (C^{+}E3)
$$

Scalar Integral

For the fourth case, when both of the momenta are in the perturbative regime ($R < x, y$), the integral can be written as,

$$\frac{1}{2xx_0}\int_R^\infty dy/y \left[\frac{F(y)}{F(x)} + 1 \right] \Big\{ 2yx_0 - (y-x)^2\ h(y/x) +$$

$$+ \left[(y-x)^2 - x_0\ (y+x) - x_0^2 \right] h_1(y/x, x_0/x) \Big\}$$

$$x > R \qquad (C^+4)$$

Since the quark renormalization function is in this case given by the perturbative result (1.2) we cannot perform the integration analytically and so it has to be done numerically.

6.1 The Self-Energy Part Σ^-

We now move on to consider the contribution to the self-energy coming from the intermediate term in the gluon containing the intergrals over the difference of the quark functions at the two momenta. If we include the logarithmic terms coming from the regularization this part of the self-energy will be independent of the mass scale R. Thus we have,

$$\Sigma_C^- = \frac{1}{2xx_0}\int_0^\infty dy/y \left[\frac{F(y)}{F(x)} - 1 \right] \Big\{ (y^2 - x^2)\ h(y/x) + x_0 y\frac{y+x}{x-y} +$$

$$+ 3xx_0\theta_+(y-R) + \left[2xx_0\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x \right] h_1(y/x, x_0/x) \Big\}$$

$$+ 3/2\ \{ e\ [\ \ln(R/\Lambda^2)\ ]^{\gamma+1} / [\ (\gamma+1)F(x)\ ] - \ln(R/\Lambda^2)\ \} \qquad (6.1)$$

We now have to consider the integral in this self-energy part in the four different cases for the internal and external momenta.

Let us consider first the two cases when the external momentum is in the non-perturbative regime. Then the lower half of the integral in (6.1) can be written as,

$$\frac{1}{2xx_0}\int_0^R dy/y\left[\frac{F(y)}{F(x)}-1\right]\left\{ (Y^2-x^2)\; h(y/x) + x_0 y\,\frac{y+x}{x-y} + \right.$$
$$\left. + \left[2xx_0\,\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x \right] h_1(y/x, x_0/x) \right\}$$

(6.2)

since the step function is zero for this part of the integral. We can see that the denominator of this integral vanishes as y goes to x. However we recall from section 2 that for both x and y in the non-perturbative regime the difference between the quark functions can be written as,

$$F(y) - F(x) = (\; y - x\;)\;[\; B_q' + C_q'\; y/R + D_q'\; (y/R)^2 + E_q\; (y/R)^3\;]\;/\;R$$

where the primed parameters are defined in equations (2.c) Substituting this result into the integral (6.2) we can perform the integration. For simplcity, we split the result into four pieces, one for each of the primed parameters.

Scalar Integral

$$\frac{B_q'}{48x^2 x_0 F(x)} \left\{ \frac{2x^4}{R} \left[19 + 12 \ln \frac{R}{x} \right] - 3 R^3 - \right.$$

$$- 4 (3 x_0 - 2 x) R^2 - 18 x_0 (x_0 + x) R -$$

$$- 12 (2 x_0^3 + 3 x_0 x^2 + 6 x x_0^2 + x_0^3) +$$

$$+ \left[3 R^2 + (9 x_0 - 5 x) R - 5 x^2 \right.$$

$$\left. + 26 x x_0 + 9 x_0^2 \right] \sqrt{A(R)} +$$

$$+ (19 x^3 - x^2 x_0 + 43 x x_0^2 + 3 x_0^3) \frac{\sqrt{A(R)} - (x_0 + x)}{R}$$

$$- 12 x_0/R (x^3 + 3 x^2 x_0 - 9 x_0^2 x - x_0^3) I_0 -$$

$$\left. - 12 x_0/R (x_0 + x)^3 I_1 \right\}$$

$$x < R \qquad (C^- B' 1)$$

where I_0 and I_1 are defined by equations (5.2) and (5.3).

Scalar Integral

$$
\frac{C'_q}{80x^2 x_0 F(x)} \Bigg\{ - 4 R^3 - 5 (3 x_0 - 2 x) R^2 - 20 x_0 (x_0 + x) R -
$$

$$
- 10 (2 x_0^3 + 3 x_0 x^2 + 6 x x_0^2 + x_0^3) -
$$

$$
- 20 \frac{x}{R} (x_0^3 + 3 x x_0^2 + 3 x^2 x_0 - x^3) -
$$

$$
- 12 \frac{x^5}{R^2} + 120 \frac{x^2}{R^2} x_0^2 (2 x - x_0) I_0 +
$$

$$
+ \Big[4 R^2 + (11 x_0 - 6 x) R - 6 x^2 + 29 x x_0 + 9 x_0^2 +
$$

$$
+ (14 x^3 + 9 x_0 x^2 + 26 x x_0^2 + x_0^3) / R \Big] \sqrt{A(R)} -
$$

$$
- \Big[6 x^4 + 49 x_0 x^3 - 129 x_0^2 x^2 + 9 x x_0^3 + x_0^4 \Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^2} \Bigg\}
$$

$$
x < R \qquad (C^- C'1)
$$

Scalar Integral

$$\frac{D'_q}{240x^2 x_0 F(x)} \Big\{ - 10 R^3 - 12 (3 x_0 - 2 x) R^2 - 45 x_0 (x_0 + x) R -$$

$$- 20 (2 x^3 + 3 x_0 x^2 + 6 x x_0^2 + x_0^3) -$$

$$- 30 \frac{x}{R} (x_0^3 + 3 x x_0^2 + 3 x^2 x_0 - x^3) - 8 \frac{x^6}{R^3} +$$

$$+ 120 \frac{x^2}{R^3} x_0^2 (2 x - 3 x_0)(3 x - x_0) I_0$$

$$+ \Big[10 R^2 - (14 x - 26 x_0) R - 14 x^2 + 65 x x_0 + 19 x_0^2 +$$

$$+ (26 x^3 + 27 x_0 x^2 + 42 x_0^2 x + x_0^3) / R -$$

$$- (4 x^4 + 53 x_0 x^3 - 135 x_0^2 x^2 + 17 x x_0^3 +$$

$$+ x_0^4) / R^2 \Big] \sqrt{A(R)} -$$

$$- \Big[4 x^5 + 65 x_0 x^4 - 680 x_0^2 x^3 + 440 x_0^3 x^2 -$$

$$- 20 x x_0^4 - x_0^5 \Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^2} \Big\}$$

$$x < R \qquad (C^- D' 1)$$

Scalar Integral

$$\frac{E_q}{1680 x^2 x_0 F(x)} \Bigg\{ - 60 R^3 - 70 (3 x_0 - 2 x) R^2 - 252 x_0 (x_0 + x) R -$$

$$- 105 (2 x^3 + 3 x_0 x^2 + 6 x x_0^2 + x_0^3) -$$

$$- 140 \frac{x}{R} (x^3 + 3 x x_0^2 + 3 x^2 x_0 - x^3) - 20 \frac{x^6}{R^3} +$$

$$+ 840 \frac{x^2}{R^4} x_0^2 (6 x^3 - 23 x_0 x^2 + 18 x x_0^2 - 3 x_0^3) I_0 +$$

$$+ \Big[60 R^2 + 10 (15 x_0 - 8 x) R - 80 x^2 + 362 x x_0 + 102 x_0^2 +$$

$$+ (130 x^3 + 167 x_0 x^2 + 190 x_0^2 x + 3 x_0^3) / R -$$

$$- (10 x^4 + 183 x_0 x^3 - 459 x_0^2 x^2 + 71 x x_0^3 + 3 x_0^4) -$$

$$- (10 x^5 + 233 x_0 x^4 - 1884 x_0^2 x^3 + 1310 x_0^3 x^2 -$$

$$- 86 x x_0^4 - 3 x_0^4) / R^3 \Big] \sqrt{A(R)} -$$

$$- \Big[10 x^6 + 263 x_0 x^5 - 6185 x_0^2 x^4 + 10910 x_0^3 x^3 -$$

$$- 3560 x_0^4 x^2 + 95 x x_0^5 + 3 x_0^5 \Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^4} \Bigg\}$$

$$x > R \qquad (C^{-}E1)$$

Scalar Integral

For the second case, where the internal momentum is in the perturbative regime, the upper half of the integral in (6.1) can be written as,

$$\frac{1}{2xx_0}\int_A^\infty dy/y \left[\frac{F(y)}{F(x)} - 1\right]\left\{ (y^2-x^2) + x_0\, y\,\frac{y+x}{x-y} + \right.$$

$$\left. + 3xx_0 + \left[2xx_0\,\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x\right] h_1(y/x, x_0/x) \right\}$$

since the function h(y/x) and the step function are both unity. Because the quark function F(y) is given by the renormalization perturbative result (1.2) we cannot do the integrals over F(y) analytically. However, we can do the integral over the unit term as $y \neq x$, leading to the result,

<u>Scalar Integral</u>

$$\frac{1}{2xx_0}\int_0^\infty dy/y\,\frac{F(y)}{F(x)}\left\{ (y^2-x^2)\,h(y/x) + x_0 y\,\frac{y+x}{x-y} + \right.$$

$$\left. + 3xx_0 + \left[2xx_0\,\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x \right] h_1(y/x, x_0/x) \right\} +$$

$$- \frac{1}{24x^2x_0}\left\{ 7\,x^3 - 45\,x^2\,x_0 + 39\,x\,x_0^2 + 11\,x_0^3 + 2\,R^3 + \right.$$

$$+ 3\,(\,3\,x_0 - x\,)\;R^2 - 6\,(\,x^2 - 6\,x\,x_0 - 3\,x^2\,)\,R +$$

$$+ 12\,(\,x_0^2 + 6\,x\,x_0 + 6\,x^2\,)\,\ln\frac{R-x}{x} -$$

$$- (\,2\,R^2 - (\,x - 7\,x_0\,)\,R - 7\,x^2 + 40\,x\,x_0 + 11\,x_0^2\,)\,\sqrt{A(R)} +$$

$$- 6\,(\,x^3 + 3\,x^2\,x_0 + 9\,x\,x_0^2 + x_0^3\,)\,\tilde{I}_0 -$$

$$\left. - 6\,(\,x_0 + x\,)^3\,\tilde{I}_1 + 12\,(\,x_0^2 + 4\,x\,x_0\,)^{3/2}\,\tilde{I}_x \right\}$$

$$x < R \qquad (C^-2)$$

where

$$I_x(y) = -(\,x_0 + 4\,x_0 x\,)^{1/2}\int\frac{\sqrt{A(y)}}{y-x}\,dy$$

$$= \ln\frac{x_0 + 3\,x + y + [\,(1+4\,x)\,A(y)\,]^{1/2}/x_0}{y-x} \qquad (6.3)$$

$$\tilde{I}_x = I_x(\infty) - I_x(R)$$

Now let us consider the self-energy part Σ_C^- for the second two cases when the external momentum is in the perturbative regime ($R < x$). Then for the third case when the internal momentum is in the non-perturbative regime, the step function gives zero contribution and the function $h(y/x) = y/x$. Thus the lower half of the integral in (6.1) can be written as,

$$\frac{1}{2xx_0}\int_0^R dy/y\left[\frac{F(y)}{F(x)} - 1\right]\left\{(y^2-x^2)\,y/x + x_0 y\,\frac{y+x}{x-y} + \right.$$

$$\left. + \left[2xx_0\,\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x\right] h_1(y/x, x_0/x)\right\}$$

Since the internal momentum is in the non-perturbative regime we can substitute the power series parametrization (1.1) for $F(y)$. Then we can perform the integration as $y \neq x$ using the definitions in section 4 and (6.3). To make things manageable we again write down the result in five pieces, one for each of the quark parameters.

$$\frac{(A_q - F(x))}{24\,x_0\,x^2\,F(X)}\left\{6\,(x^2 + 6\,xx_0 + 3\,x_0^2)\,R + 3\,(x + 3\,x_0)\,R^2 - \right.$$

$$- 2\,R^3 + 12\,x_0\,(x_0^2 + 6\,x\,x_0 + 6\,x^2)\,\ln\frac{R-x}{x} -$$

$$- (2\,R^2 - (x - 7\,x_0)\,R)\,\sqrt{A(R)} -$$

$$+ (7\,x^2 + 40\,x\,x_0 - 11\,x_0^2)\,(\sqrt{A(R)} - (x_0 + x)) +$$

$$+ 6\,(x^3 + 3\,x^2\,x_0 + 9\,x_0^2\,x + x_0^3)\,I_0 +$$

$$\left. + 6\,(x_0 + x)^3\,I_1 \quad + 12\,(x_0^2 + 4\,x\,x_0)^{3/2}\,I_x\right\}$$

$$x > R \qquad (C^- A3)$$

Scalar Integral

$$\frac{B_q}{48x^2 x_0 F(x)} \Big\{ 3 R^3 - 4 (x + 3 x_0) R^2 - 6 (x^2 + 6 x x_0 + 3 x_0^2) R +$$

$$+ 12 (x^3 - 9 x^2 x_0 - 9 x x_0^2 - x_0^3)$$

$$+ 24 \frac{x}{R} x_0 (x_0^2 + 6 x x_0 + 6 x^2) \ln \frac{R - x}{x} +$$

$$+ (3 R^2 - (x - 9 x_0) R - 7 x^2 + 40 x x_0 + 9 x_0^2) \sqrt{A(R)} +$$

$$+ (5 x^3 + 79 x^2 x_0 + 65 x x_0^2 + 3 x_0^3) \frac{\sqrt{A(R)} - (x_0 + x)}{R} +$$

$$+ 24 \frac{x}{R} x_0^3 (x_0 + 9 x) I_0 - 24 \frac{x}{R} (x_0^2 + 4 x x_0)^{3/2} I_x \Big\}$$

$$x > R \qquad (C^{-}B3)$$

Scalar Integral

$$\frac{C_q}{240x^2 x_0 F(x)} \Bigg\{ 12\,R^3 - 15\,(\,x + 3\,x_0\,)\,R^2 - 20\,(\,x^2 + 6\,xx_0 + 3\,x_0^2\,)\,R +$$

$$+ 30\,(\,x^3 - 9\,x^2\,x_0 - 9\,x\,x_0^2 - x_0^3\,) -$$

$$- 120\,\frac{x}{R}\,x_0\,(\,x_0^2 + 6\,x\,x_0 + 6\,x^2\,)\left[\,1 + \frac{x}{R}\ln\frac{R - x}{x}\,\right] -$$

$$+ \Big[\,12\,R^2 - 3\,(\,x - 11\,x_0\,)\,R - 23\,x^2 + 132\,x\,x_0 + 27\,x_0^2 +$$

$$+ (\,7\,x^3 + 227\,x^2\,x_0 + 123\,x\,x_0^2 + 3\,x_0^3\,)\,/\,R\,\Big]\,\sqrt{A(R)} +$$

$$+ \Big[\,7\,x^4 + 248\,x_0\,x^3 + 712\,x_0^2\,x^2 - 12\,x\,x_0^3 -$$

$$- 3\,x_0^4\,\Big]\,\frac{\sqrt{A(R)} - (x_0 + x)}{R^2} -$$

$$- 120\,\frac{x^2}{R^2}\,x_0^2\,(\,2\,x_0 - 15\,x\,)\,I_0 - 120\,\frac{x^2}{R^2}\,(\,x_0^2 + 4\,x\,x_0\,)^{3/2}\,I_x \Bigg\}$$

$$x > R \qquad (C^-C3)$$

Scalar Integral

$$\frac{D_q}{240x^2 x_0 F(x)} \Bigg\{ 10\,R^3 - 12\,(x + 3\,x_0)\,R^2 - 15\,(x^2 + 6\,xx_0 + 3\,x_0^2)\,R +$$

$$+ 20\,(x^3 - 9\,x^2\,x_0 - 9\,x\,x_0^2 - x_0^3) -$$

$$- 120\,\frac{x}{R}\,x_0\,(x_0^2 + 6\,x\,x_0 + 6\,x^2)\left[\frac{1}{2} + \frac{x}{R} + \frac{x^2}{R^2}\ln\frac{R - x}{x}\right] +$$

$$+ \Big[10\,R^2 - 2\,(x - 13\,x_0)\,R - 17\,x^2 + 98\,x\,x_0 + 19\,x_0^2 +$$

$$+ (3\,x^3 + 159\,x^2\,x_0 + 69\,x\,x_0^2 + x_0^3)\,/\,R +$$

$$+ (3\,x^4 + 174\,x^3\,x_0 + 258\,x^2\,x_0^2 -$$

$$- 14\,x\,x_0^2 - x_0^3)\,/\,R^2 \Big]\,\sqrt{A(R)} +$$

$$+ \Big[3\,x^5 + 183\,x_0\,x^4 + 1392\,x_0^2\,x^3 - 452\,x_0^3\,x^2 +$$

$$+ 17\,x\,x_0^4 + x_0^5 \Big]\,\frac{\sqrt{A(R)} - (x_0 + x)}{R^2} -$$

$$+ 120\,\frac{x^2}{R^3}\,x_0^2\,(21\,x^2 - 13\,x\,x_0 + 3\,x_0^2)\,I_0 -$$

$$- 120\,\frac{x^3}{R^3}\,(x_0^2 + 4\,x\,x_0)^{3/2}\,I_x \Bigg\}$$

$$x > R \qquad (C^{-}D3)$$

Scalar Integral

$$\frac{E_q}{1680 x^2 x_0 F(x)} \Big\{ 60 R^3 - 70 (x + 3 x_0) R^2 - 84 (x^2 + 6 x x_0 + 3 x_0^2) R +$$

$$+ 105 (x^3 - 9 x^2 x_0 - 9 x x_0^2 - x_0^3) -$$

$$- 840 \frac{x}{R} x_0 (x_0^2 + 6 x x_0 + 6 x^2) \Big[\frac{1}{3} + \frac{x}{2 R} + \frac{x^2}{R^2} + \frac{x^3}{R^3} \ln \frac{R - x}{x} \Big] -$$

$$+ \Big[60 R^2 - 10 (x - 15 x_0) R - 94 x^2 + 544 x x_0 + 102 x_0^2 +$$

$$+ (11 x^3 + 853 x^2 x_0 + 323 x x_0^2 + 3 x_0^3) / R +$$

$$+ (11 x^4 + 930 x^3 x_0 + 942 x^2 x_0^2 - 64 x x_0^2 - 3 x_0^3) / R^2$$

$$+ (11 x^5 + 985 x^4 x_0 + 3690 x^3 x_0^2 - 1408 x^2 x_0^3 +$$

$$+ 79 x x_0^4 + 3 x_0^5) / R^3 \Big] \sqrt{A(R)} +$$

$$+ \Big[11 x^6 + 1018 x_0 x^5 + 15929 x_0^2 x^4 - 14074 x_0^3 x^3 +$$

$$+ 3679 x_0^4 x^2 - 88 x x_0^5 - 3 x_0^6 \Big] \frac{\sqrt{A(R)} - (x_0 + x)}{R^2} -$$

$$+ 2520 \frac{x^2}{R^4} x_0^2 (9 x^3 - 12 x^2 x_0 + 7 x x_0^2 - x_0^3) I_0 -$$

$$+ 840 \frac{x^4}{R^4} (x_0^2 + 4 x x_0)^{3/2} I_x \Big\}$$

$$x > R \qquad (C^- E3)$$

Scalar Integral

In the fourth case, when both of the momenta are in the perturbative regime, the step function gives a unit contribution. Hence the upper half of the integral in (6.1) can be written as,

$$\frac{1}{2xx_0}\int_R^\infty dy/y \left[\frac{F(y)}{F(x)} - 1\right]\left\{ (y^2-x^2)\ h(y/x) + x_0 y\,\frac{y+x}{x-y} + \right.$$

$$\left. + 3xx_0 + \left[2xx_0\,\frac{3y+x+x_0}{x-y} - \frac{(y-x)^2}{x} + x \right] h_1(y/x,x_0/x) \right\}.$$

$$x > R \qquad (C^{-}4)$$

Since the quark function is given by the perturbative result (1.2) and the part of the integral independent of the quark function diverges (although of course the total integral does not) we have to perform this integration numerically.

6.7 The Gauge Dependant Term

Finally, but not least, we come to the contribution form the gauge dependent part of the self-energy. From equation (5:9.4d) we see that this contribution to the self energy is,

$$\Sigma_\xi = - \left\{ 1 + 2\int_x^R dy/y\,\frac{F(y)}{F(x)} \right\} .$$

$$+ 2\ e\ [\ \ln(R/\Lambda^2)\ ]^{\gamma+1}\ /\ [\ (\gamma+1)F(x)\ ] \qquad (7.1)$$

In this case, because of the limits on the integral, we only have to consider the cases when the two momenta are in the same regime. For the external momentum in the non-perturbative regime, we can use the power series parameterization (1.1) for F(y). Substituting this into the integral (7.1), we get the result,

Scalar Integral

$$\frac{-2}{F(x)}\left\{ A_q \ln\frac{R}{x} + B_q\,(1-z) + \frac{C_q}{2}(1-z^2) + \frac{D_q}{3}(1-z^3) + \frac{E_q}{4}(1-z^4) \right\}. \qquad x < R \qquad (\xi.1)$$

The other case is when the two momenta are in the perturbative regime. Then the quark function is given by the perturbative result (1.2). In this case the integral in (7.1) can be done analytically and leads to the result,

$$-\,2\,e\,[\,\ln(R/x)\,]^{\gamma+1}\,/\,[\,(\gamma+1)F(x)\,] \qquad x > R \qquad (\xi.4)$$

6:8 Summary

Let us review what has been achieved in this rather technical chapter. We started off with a scalar equation (5:9.6), which was the result of doing the angular integrals in the Schwinger-Dyson equation. This scalar equation involved integrals over the largely unknown quark renormalization function. At large momenta, the quark function is known from perturbation theory (1.2) and this fact was used in the last chapter to regularise the Schwinger-Dyson equation.

So what was required was an approximation for the quark function that was valid in the non-perturbative regime. For this, we choose to use a power series expansion starting with a constant (although we argued in section 5 of the last chapter that we expect the constant to be zero) upto fourth power in momenta. Since we were using this approximation only for momenta upto the mass scale R, we choose to do the expansion in the dimensionless variable $z = x/R$, so that $z < 1$.

Scalar Integral

We required that the quark function is continuous and differentiably continuous at the point R. This gave us two constraints between the parameters (1.a), which we then solved for the last two parameters.

We then separated the self-energy into parts multiplied by the gluon parameters, two parts for the intermediate term, and the gauge parameter. Since the quark function has different forms for the two regimes, perturbative (> R) and non-perturbative (< R), it was natural to split the integral into two pieces, a lower half with the internal momentum in the non-perturbative regime and an upper half with the internal momentum in the perturbative regime. It is less obvious why it was necessary to consider the case when the external momentum (x) is in the two different regimes. This was because the denominator in the integrals vanishes as y goes to x. The numerator contained terms like F(y) - F(x), which we needed to know in the limit y goes to x, so we could explicitly cancel the divergence from the denominator. We therefore had to consider the integrals in the four possible cases for the external and internal momentum in the perturbative and non-perturbative regimes seperately.

This we did for each of the parts multiplying the gluon parameters and the gauge parameter in turn. The results of this integrals were somewhat complicated, particularly for the intermediate term in the gluon propagator and when the internal momentum was in the non-perturbative regime where we had to expand the quark function. We can summarise the results in the folowing symbolic manner. For the external momentum in the non-perturbative regime the self-energy parts can be written as,

<u>Scalar Integral</u>

$$\Sigma_A = A_g \left\{ (A1) + (A2) - 3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \right\}$$

$$\Sigma_B = B_g \left\{ (B1) + (B2) + \frac{3}{2} \left[e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} - \ln \frac{R}{\Lambda^2} \right] \right\}$$

$$\Sigma_C = C_g \left\{ (C^+1) + (C^+2) + (C^-1) + (C^-2) + \right.$$

$$\left. + \frac{3}{2} \left[e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} - \ln \frac{R}{\Lambda^2} \right] \right\}$$

$$\Sigma_\xi = \xi \left\{ (\xi.1) - 1 + 2\, e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} \right\}$$

The self-energy parts with the external momentum in the perturbative regime can be written in a similar manner as,

$$\Sigma_A = A_g \left\{ (A3) + (A4) - 3\, x_0 \left[\frac{F'(x)}{F(x)} + \frac{1}{x} \right] \right\}$$

$$\Sigma_B = B_g \left\{ (B3) + (B4) + \frac{3}{2} \left[e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} - \ln \frac{R}{\Lambda^2} \right] \right\}$$

$$\Sigma_C = C_g \left\{ (C^+3) + (C^+4) + (C^-3) + (C^-4) + \right.$$

$$\left. + \frac{3}{2} \left[e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} - \ln \frac{R}{\Lambda^2} \right] \right\} +$$

$$\Sigma_\xi = \xi \left\{ (\xi.4) - 1 + 2\, e \frac{[\ \ln(R/\Lambda^2)\ ]^{\gamma+1}}{(\ \gamma + 1\ )\ F(x)} \right\}$$

Chapter 7

The Self-Consistent Quark

7:1 Introduction

Having calculated the quark self-energy (6:8.1), we can now evaluate the Schwinger-Dyson equation using (5:9.6).

$$1/F(x) \;=\; J(x)/\{\; A(x)J_1F_1 \;+\; B(x)J_2F_2 \;+\; C(x)J_3F_3 \;\} \tag{1.1}$$

where,

$$J(x) \;=\; 1 \;-\; \lambda\,(\; A_g\,\Sigma_A \;+\; B_g\,\Sigma_B \;+\; C_g\,\Sigma_C \;+\; \xi\,\Sigma_\xi\;)$$

To do this we need to know the value of the quark function F(x) at the three points x_1, x_2 and x_3. We choose these to be in the perturbative region starting with x_1 where we choose the value of the function to one and then the remaining two points equally spaced with the value of the function determined by perturbation theory. Then the function J(x) can be determined for these three points by using the self energies (6:8.1b). Equation (1.1) can be seen to be true at these three points since the coefficient functions, A(x), B(x) and C(x) defined by (5:9.5) go to one and zero in the appropriate manner.

$$A(x) \;=\; [\; F(x)(F_3d_3 \;-\; F_2d_2) + F(x)d(x)(F_2-F_3) + F_2F_3(d_2-d_3)\;]/D$$

$$B(x) \;=\; [\; F(x)(F_1d_1 \;-\; F_3d_3) + F(x)d(x)(F_3-F_1) + F_3F_1(d_3-d_1)\;]/D$$

$$C(x) \;=\; [\; F(x)(F_2d_2 \;-\; F_1d_1) + F(x)d(x)(F_1-F_2) + F_1F_2(d_1-d_2)\;]/D \tag{1.2}$$

where

$$D \;=\; F_1F_2(d_1-d_2) + F_2F_3(d_2-d_3) + F_3F_1(d_3-d_1)$$

and

$$d(x) = x_0 \{ F'(x)/F(x) + 1/x \} \tag{1.3}$$

with

$$F_1 = F(x_1),\ d_1 = d(x_1),\ \text{etc.}$$

The value of the quark function in the non-perturbative regime can then be calculated from the Schwinger-Dyson equation using equation (6:8.1a). The output function, $F_{out}(x)$, from the Schwinger-Dyson equation is not necessarily the same as the function we put in,

$$F_{in}(x) = A_q + B_q\, x/R + C_q\, (x/R)^2 + D_q\, (x/R)^3 + E_q\, (x/R)^4 \tag{1.4}$$

To make them the same, or at least as close as possible, we use the least squares fit method developed for the gluon function. Thus we will seek to minimise the relative error squared

$$\sigma = \frac{1}{a-b} \int_b^a dx \left\{ \frac{F_{out}(x)}{F_{in}(x)} - 1 \right\}^2 \tag{1.5}$$

where a and b are the limit over which we choose to fit the functions. We cannot use the Fourier method since the output function is dependent upon the quark parameters.

We now have to confront the problem of over what range should we fit the quark function. As we have just seen, we need to evaluate the integrals in the perturbative region in order to perform the renormalization and we have constrained two of the quark parameters so that the parameterization of the quark function is continuous and differentiably continuous at the point R where it joins onto the perturbative result.

$$D_q = 4\,F_p(R) - F_p'(R) - 4\,A_q - 3\,B_q - 2\,C_q$$
$$E_q = 3\,A_q + 2\,B_q + C_q - 3\,F_p(R) + F_p'(R) \tag{1.6}$$

However, the problem arises that our parameterization of the gluon function has not been determined in the perturbative regime. This problem will be discussed in the next chapter with reference to some later work. Our parameterization of the gluon function although not agreeing with the perturbative result numerically is approximately constant in the perturbative regime and this should not adversely affect our results for the quark propagator. Indeed as we shall see the value of the gluon parameters is not as important as the existence of the enhanced term in the gluon propagator. This is signalled by the fact that d(x), equation (1.3), which comes from the infra-red regularization of the enhanced term is independent of the value of A_g. Also we notice that because we have parameterised the quark function in terms of x/R the quark parameters will not be strongly dependent upon the gluon scale p_0^2.

Therefore, in general then we will choose the lower limit of the fit to be 0.1 p_0^2 and the upper limit to be 20.1 p_0^2 with x_1 = 21 p_0^2 and x_2 and x_3 separated by p_0^2. The point R will be chosen to be midway between the upper limit and the first subtraction point, x_1, that is R = 20.5 p_0^2, since our method of splitting the self-energy means that we cannot evaluate it at the point x = R.

7:2 The Results

We now turn to the results obtained by minimising the relative error squared (1.5). Firstly we investigate the effects of changing the gauge parameter (figure 7.1a-e). The first thing we notice is that the constant in the quark function is zero, in general the result for A_q is less than 10^{-4}. These fits were obtained by starting from a zero value for A_q. Starting from a non-zero value, for example $A_q = 1$, did not produce a consistent result. The minimum is very narrow and the minimiser has to work very hard to move A_q from a non-zero value towards a zero value.

We also can see that the results depend upon the value of the gauge parameter, although the quark parameters do not seem to vary smoothly. This would tend to indicate that the number of free parameters, effectively only two, is not enough to get an absolute fit. The fit for the gauge parameter approximately zero (figure 7.1c) is particularly good. The reason why the quark function was not evaluated in the Landau gauge, $\xi = 0$, is that the renormalization has been done on the assumption that $\xi \neq 0$, and the special case of the Landau gauge, where the perturbative result is constant, has to be treated separately. As the gauge parameter increases the fit becomes worse (figure 7.1e) this is because the gluon function in these gauges is not accurately determined. The fits have been repeated using the value of the gluon scale p_0^2 determined from the static potential and it was found that the results were not significantly effected.

The next thing we look at is the effect of changing the subtraction points, keeping their separation constant, in the Feynman gauge (figure 7.2a,b). We can see that the result is remarkably independant of the value of the subtraction points.

In figure (7.3a,b) we can see that the quark function is also independent of changes in the range over which the gluon function was

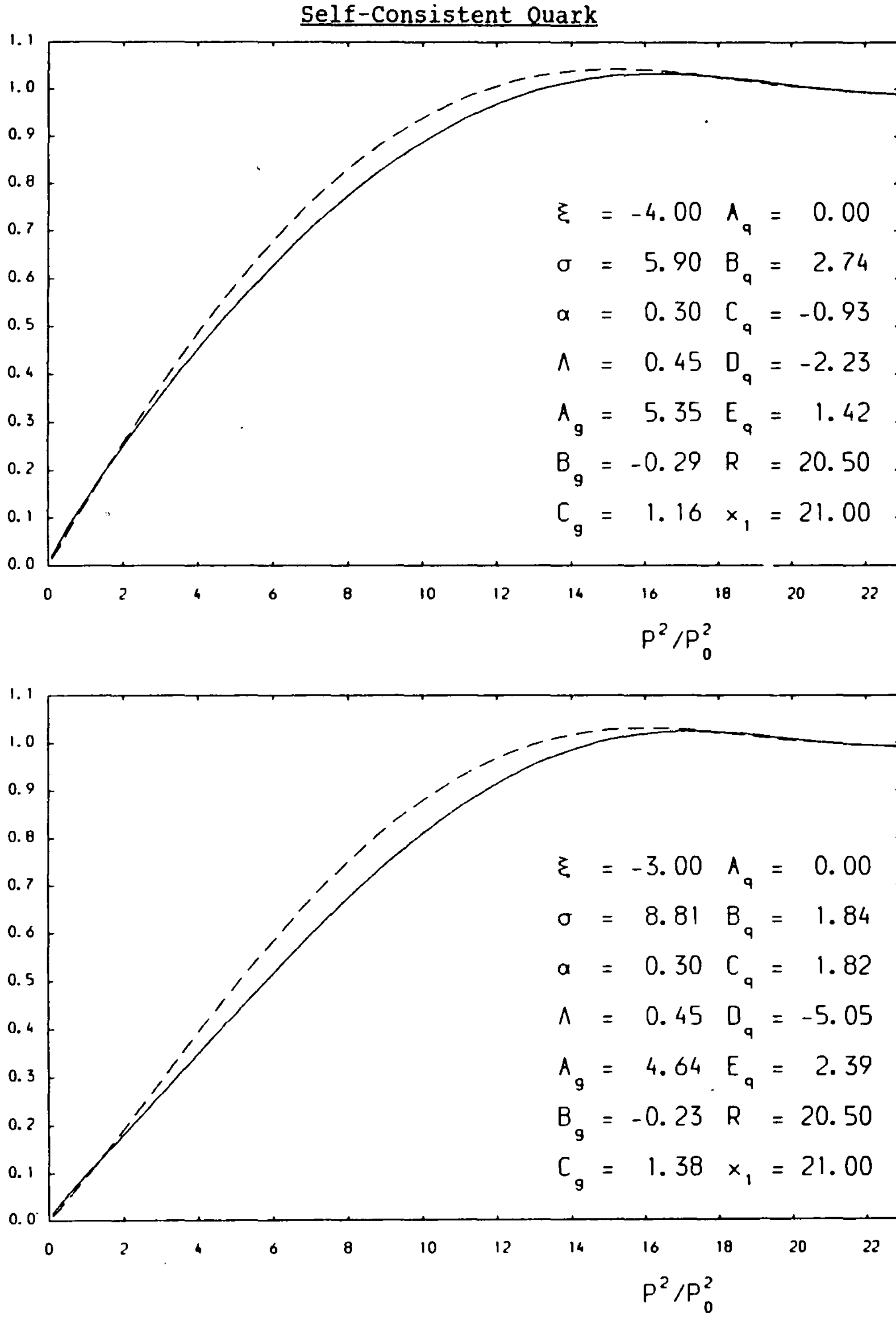


Figure 7.1a

The quark function, input broken line and output solid line for the gauges $\xi = -4$ and -3.

Self-Consistent Quark

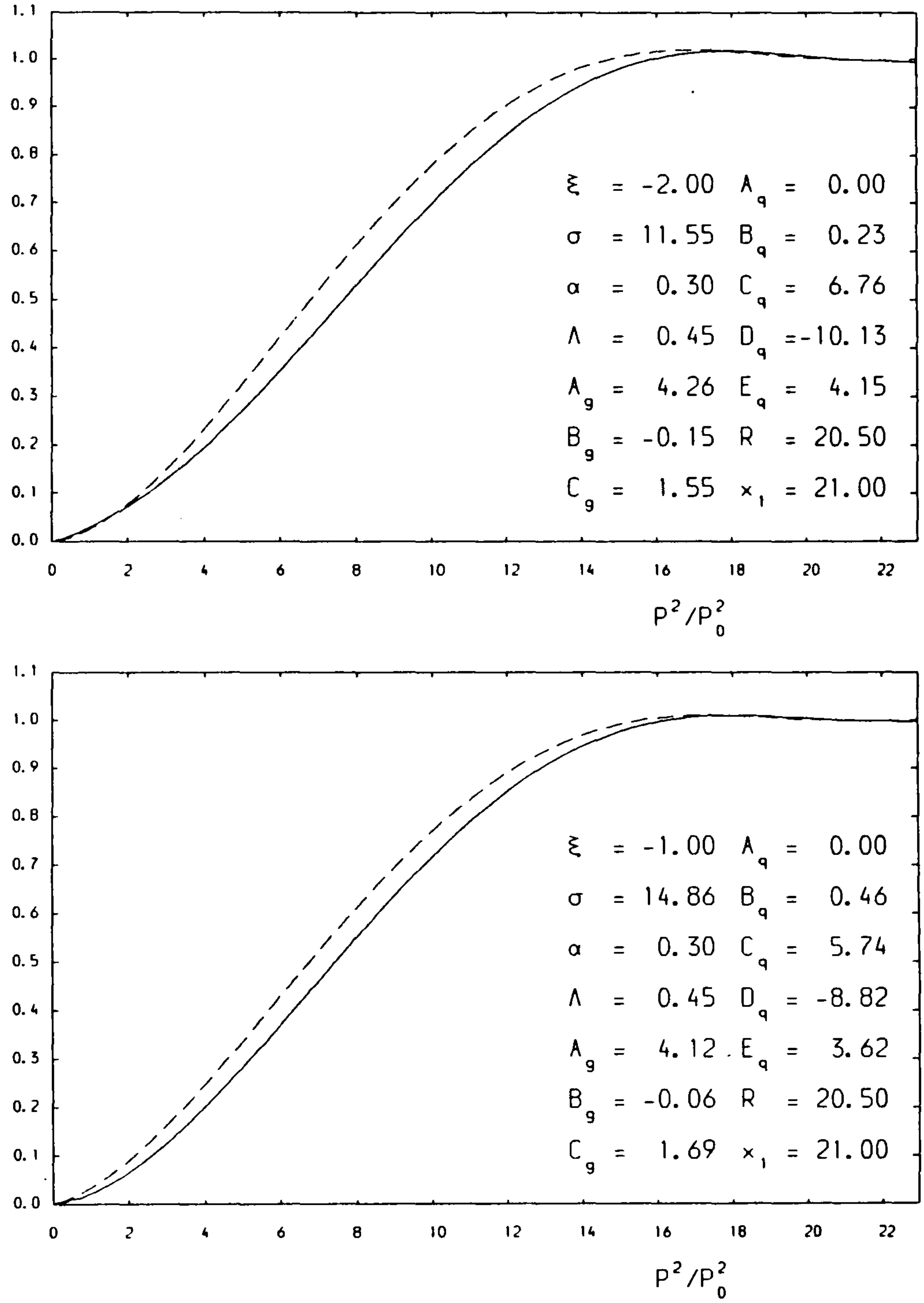


Figure 7.1b

The quark function, input broken line and output solid line for the gauges $\xi = -2$ and -1.

Self-Consistent Quark

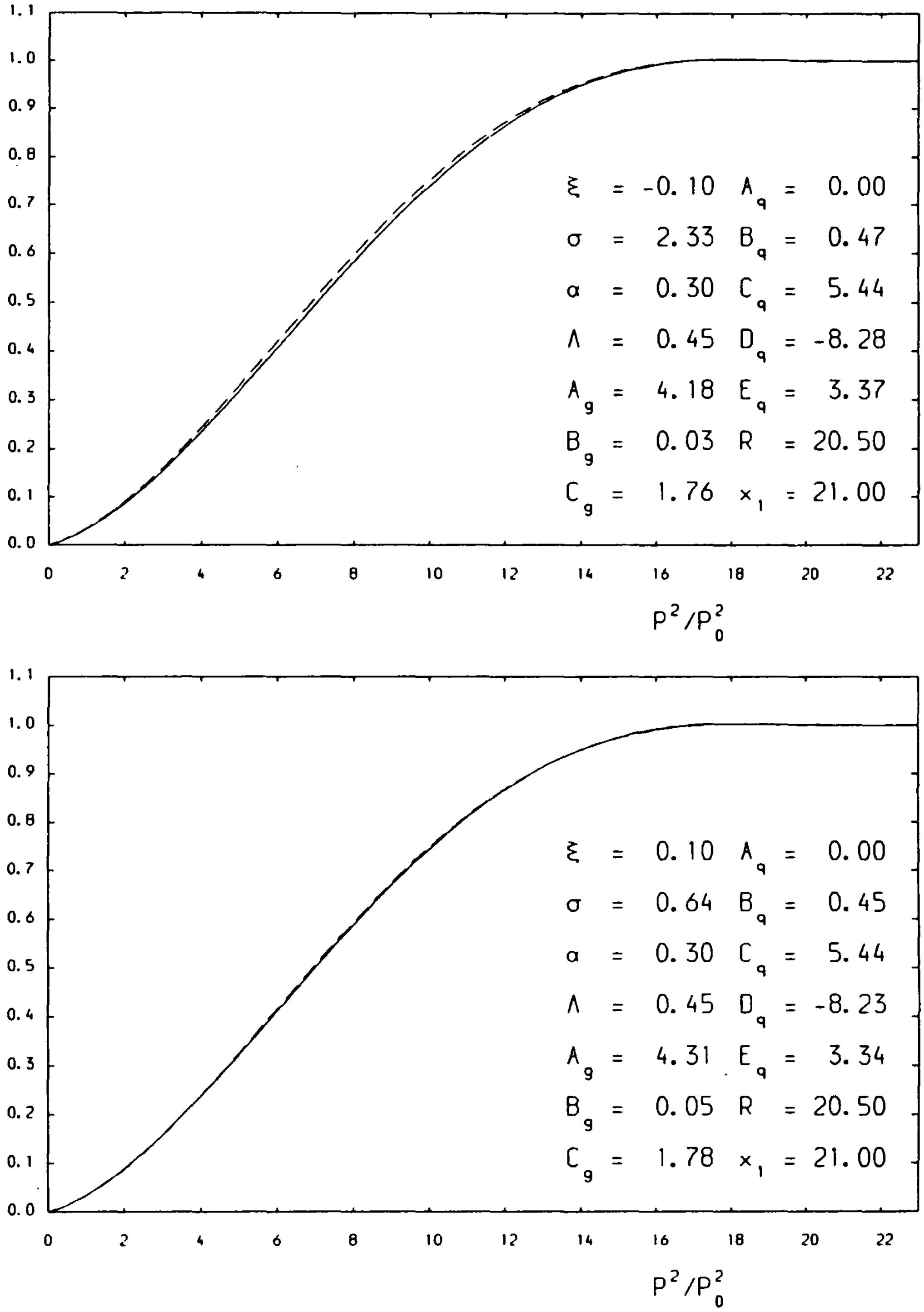


Figure 7.1c

The quark function, input broken line and output solid line for the gauges $\xi = -0.1$ and 0.1.

Self-Consistent Quark

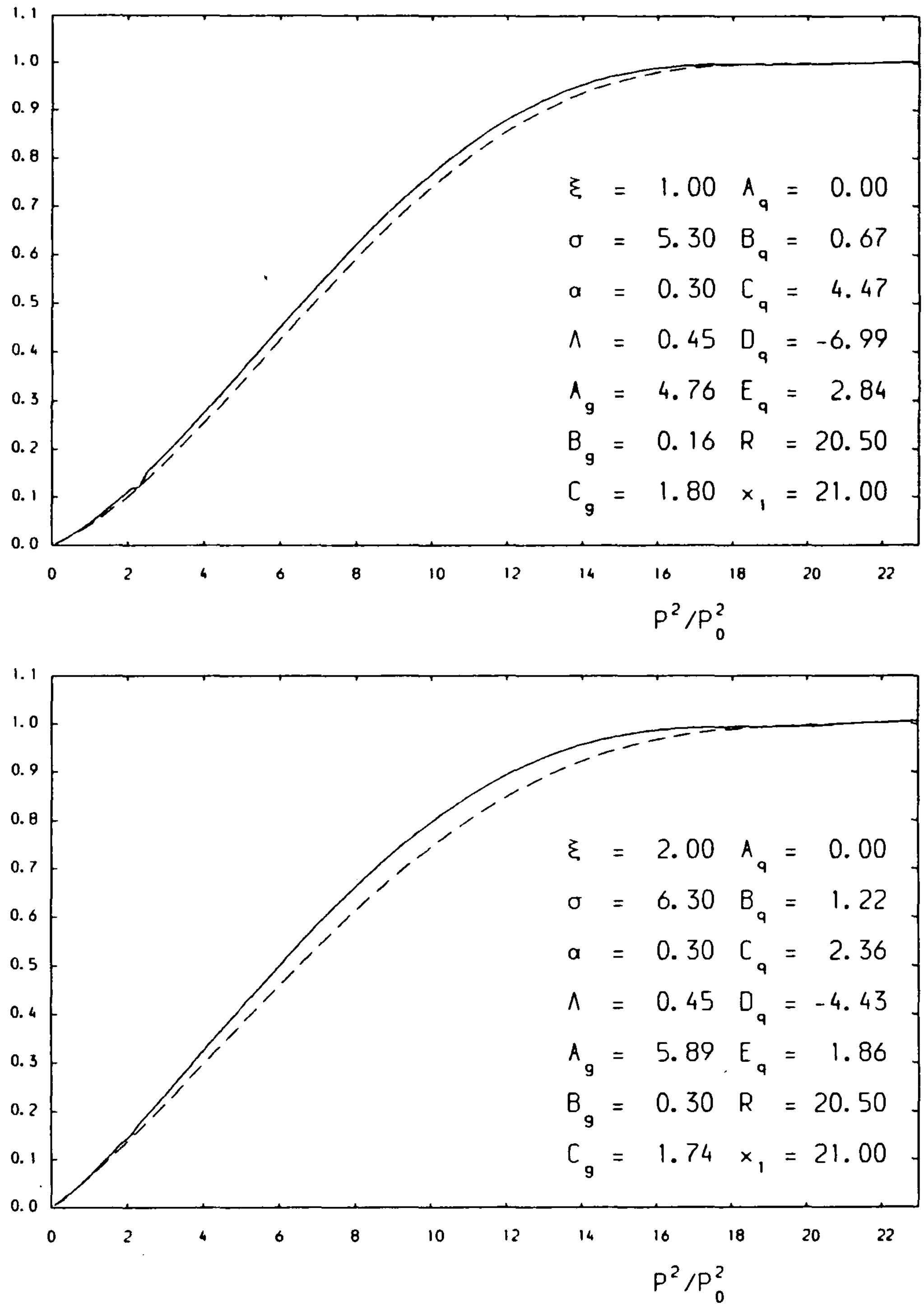
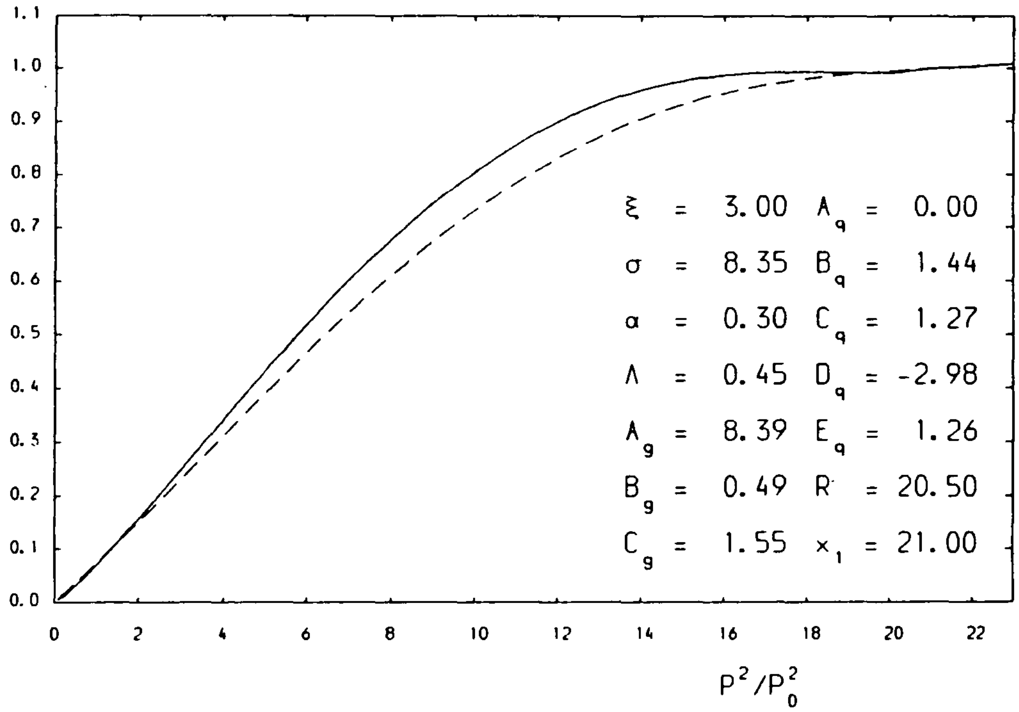


Figure 7.1d

The quark function, input broken line and output solid line for the gauges $\xi = 1$ and 2.



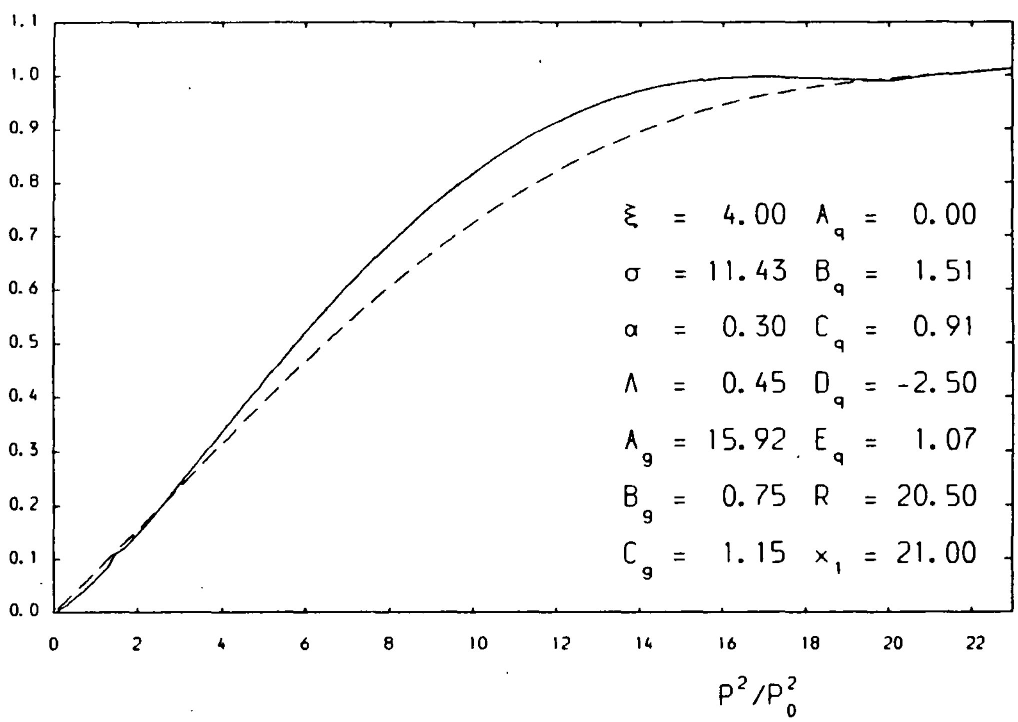


Figure 7.1e

The quark function, input broken line and output solid line for the gauges ξ = 3 and 4.

Self-Consistent Quark

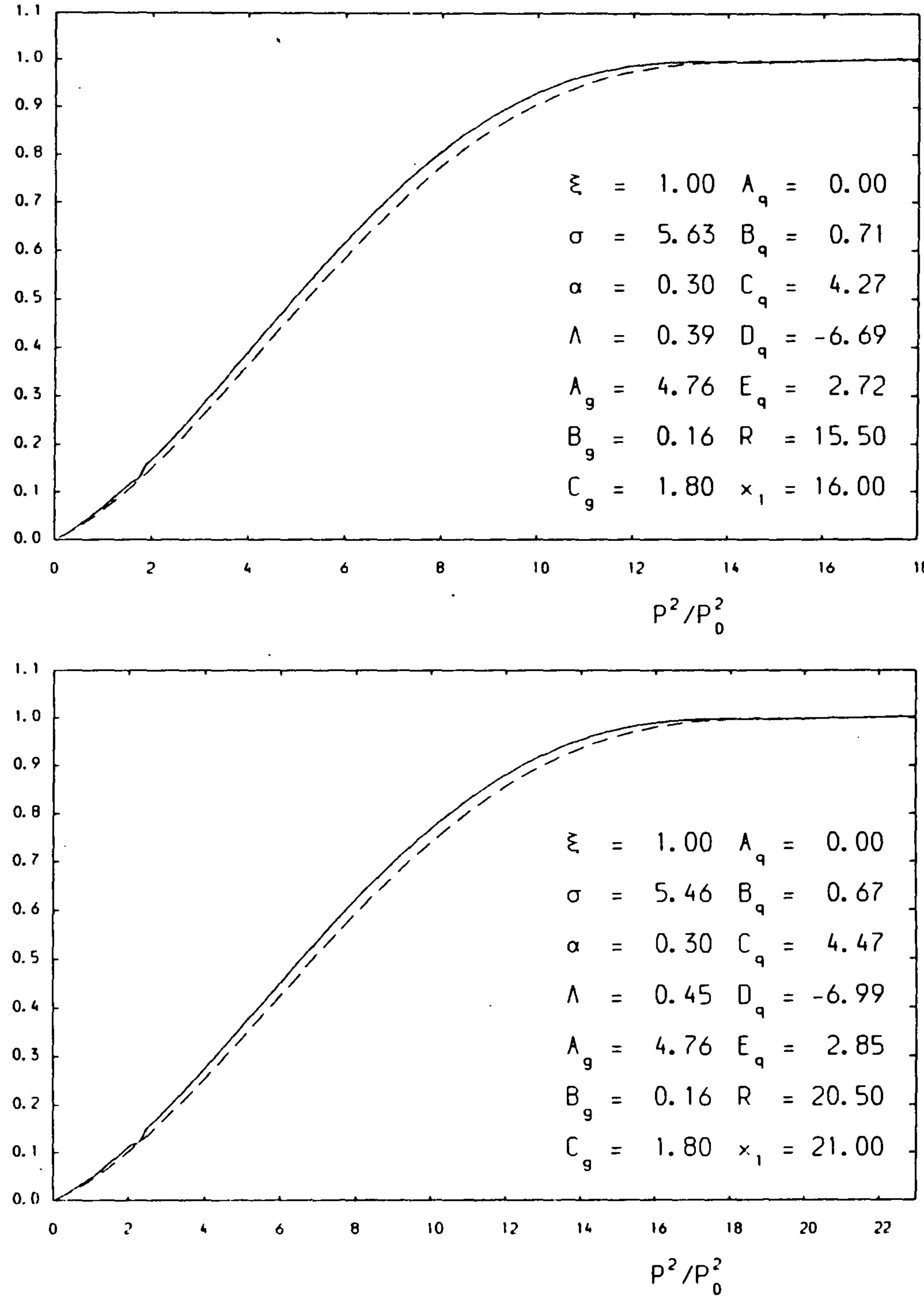
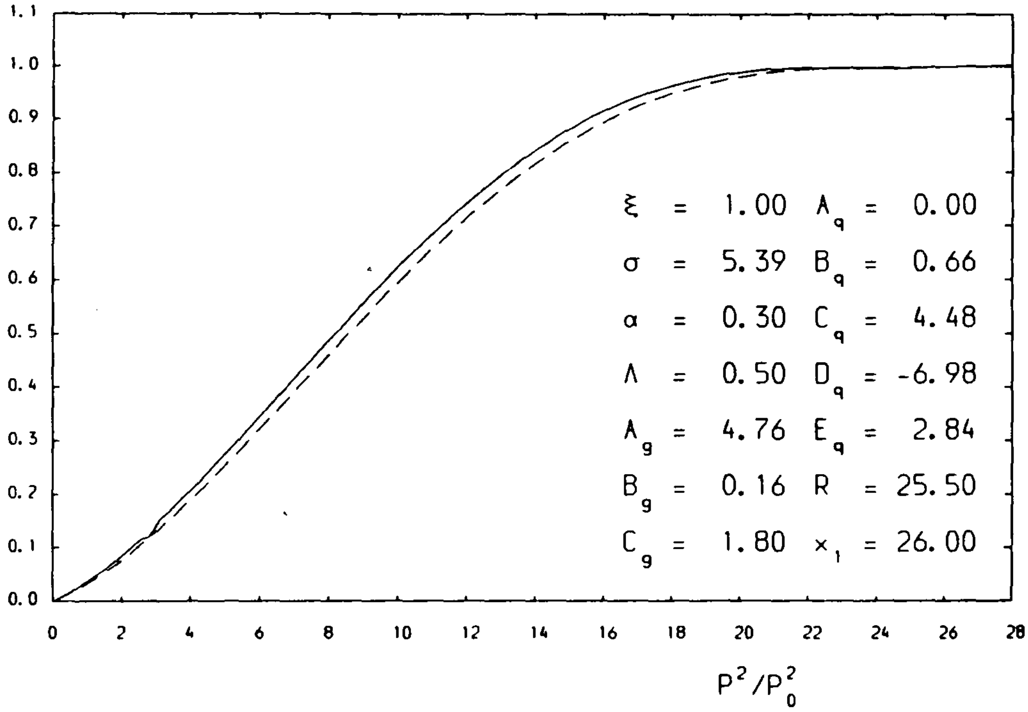


Figure 7.2a

The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ with $x_1 = 16$ and 21, $x_2 - x_1 = x_3 - x_2 = 1$.

Self-Consistent Quark



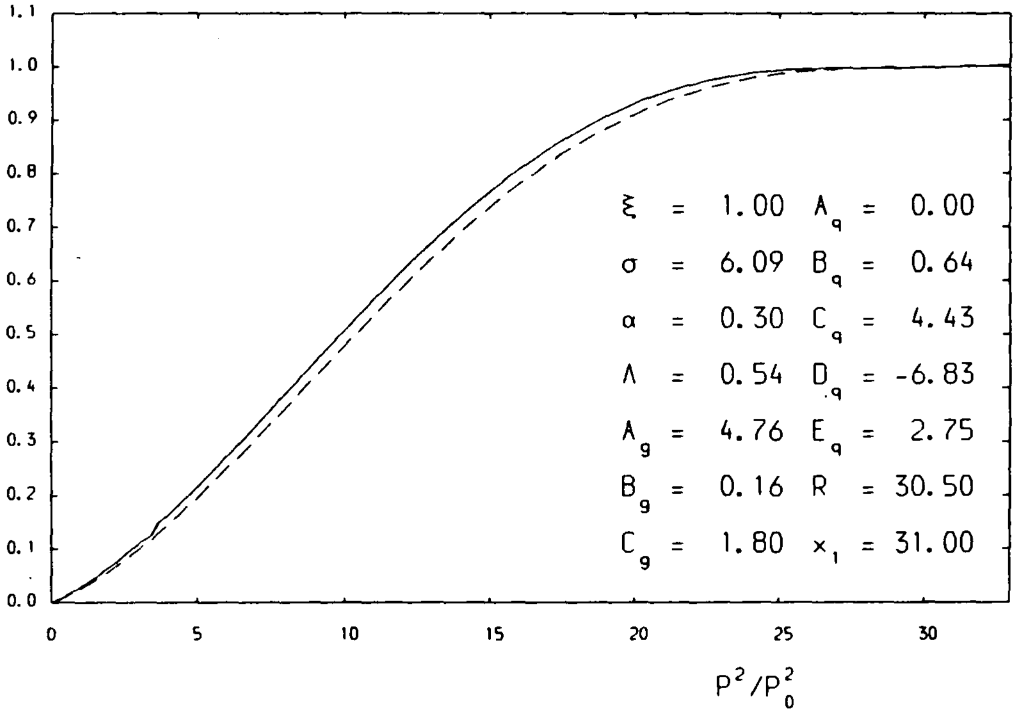
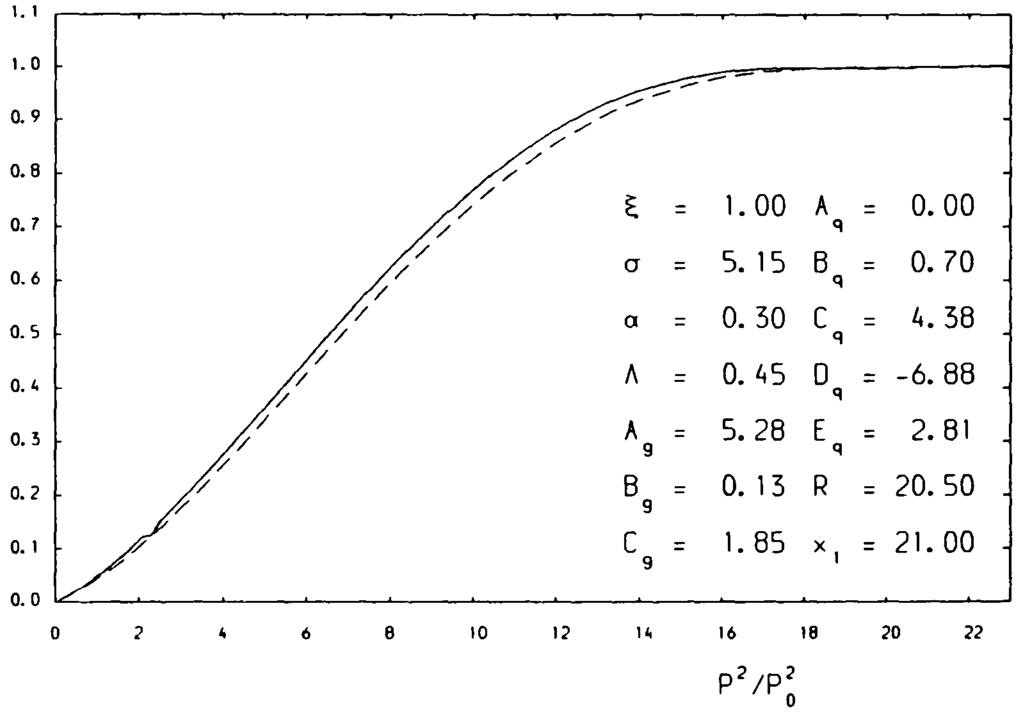


Figure 7.2b

The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ with $x_1 = 26$ and 31, $x_2 - x_1 = x_3 - x_2 = 1$.

Self-Consistent Quark



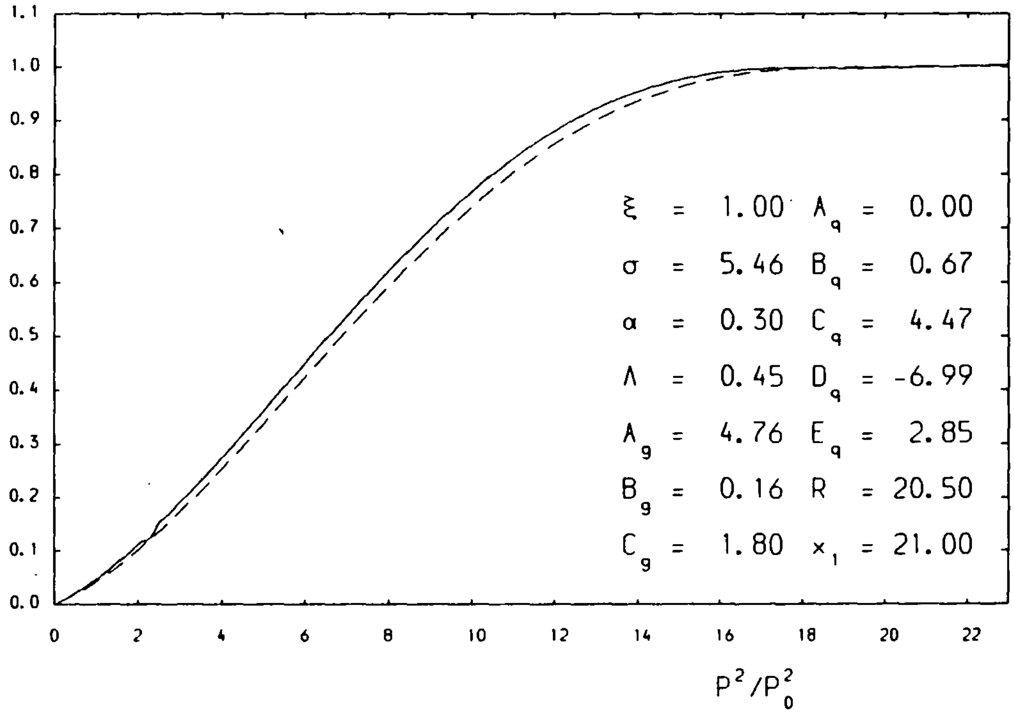
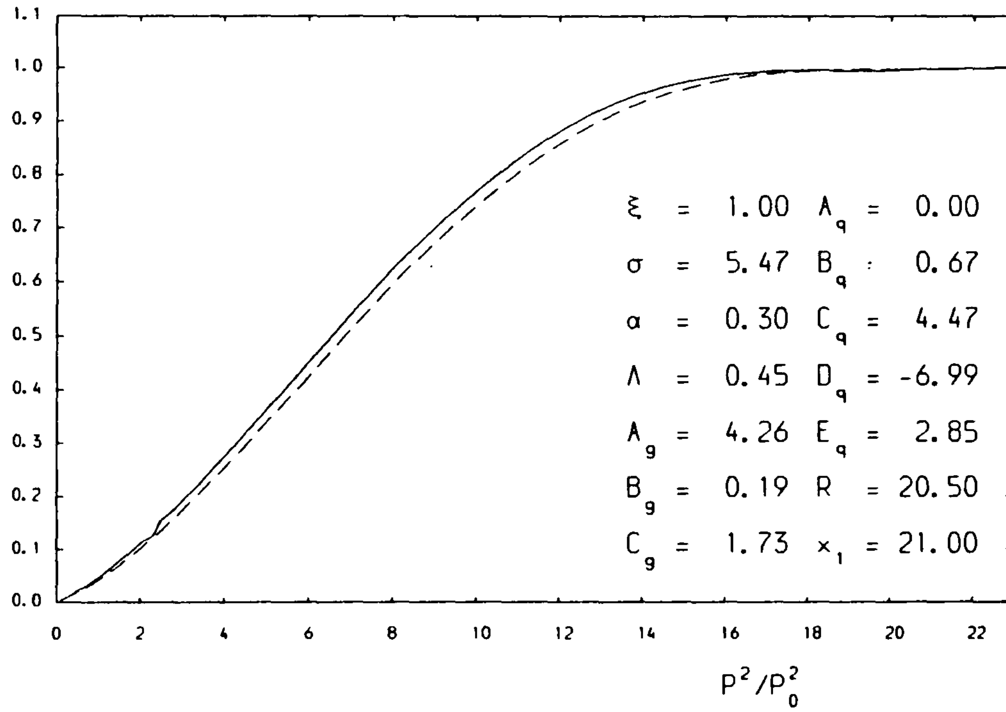


Figure 7.3a

The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ using the gluon function fitted upto 0.65 and 1.05 p_0^2.

Self-Consistent Quark



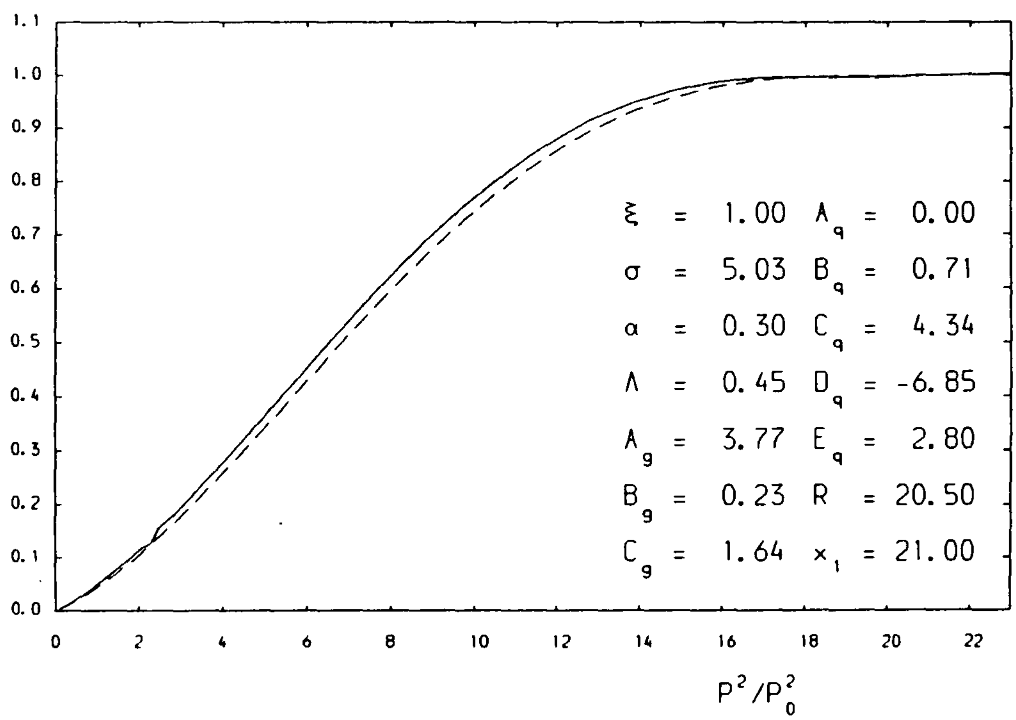


Figure 7.3b

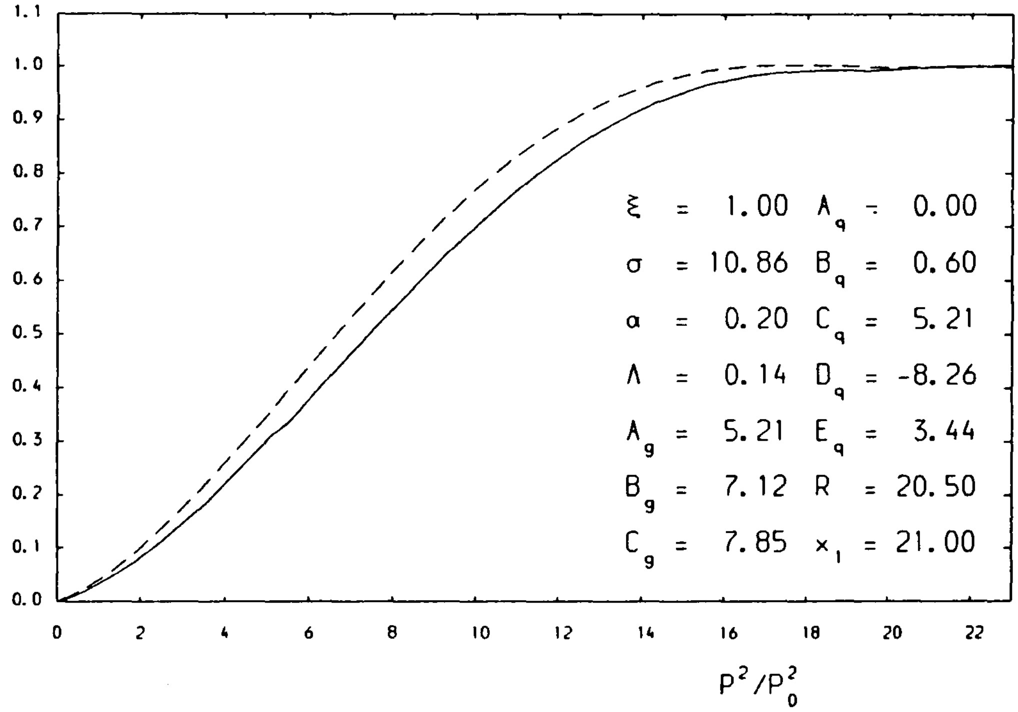
The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ using the gluon function fitted upto 1.45 and 1.85 p_0^2.

fitted. This result was remarked on earlier with reference to the fact that the function d(x) (1.3) coming from the regularization of the enhanced term is independent of A_g.

Lastly, we consider the effects of changing the coupling constant in the Feynman gauge (figures 7.4a,b). We can see that changing the coupling does have some effect, although the explanation of these effects is far from transparent involving, as it does, the gluon function.

The main conclusion that we would wish to draw from these results is that a constant in the quark renormalization function is not consistent in the infra-red, unlike the case of the electron. The details of the behaviour in the infra-red are less clear cut and further work (see later) is needed. We can also see that the change over from a perturbative behaviour to a non-perturbative one is relatively sudden. This would then support the observation (further borne out by work done beyond this thesis) that perturbation theory seems to work well right upto the confining region.

Self-Consistent Quark



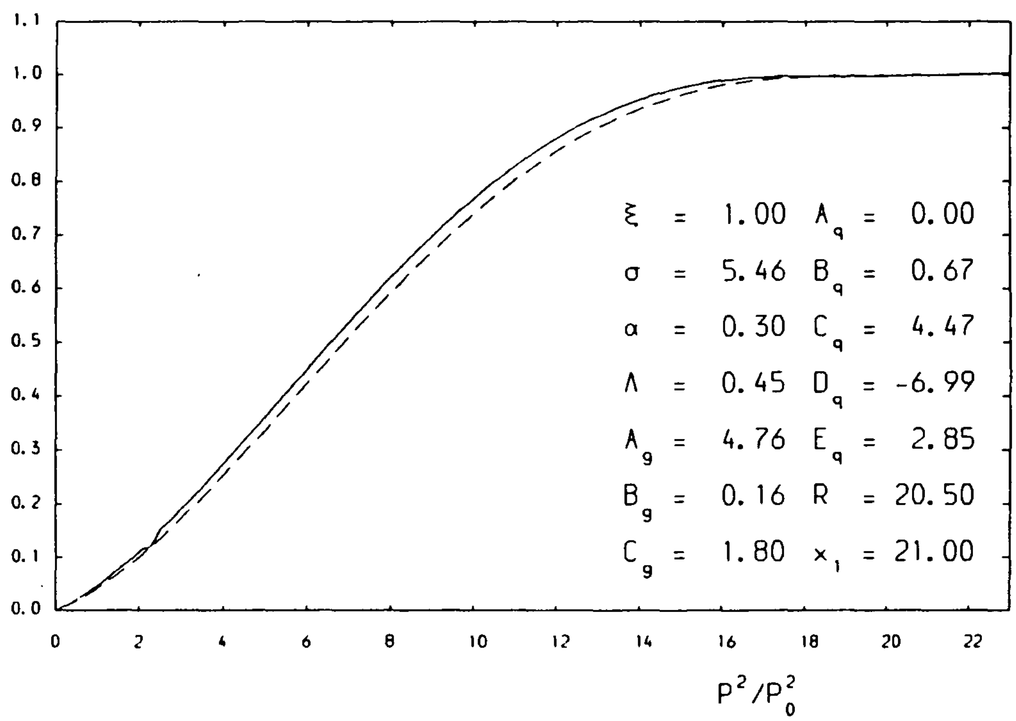
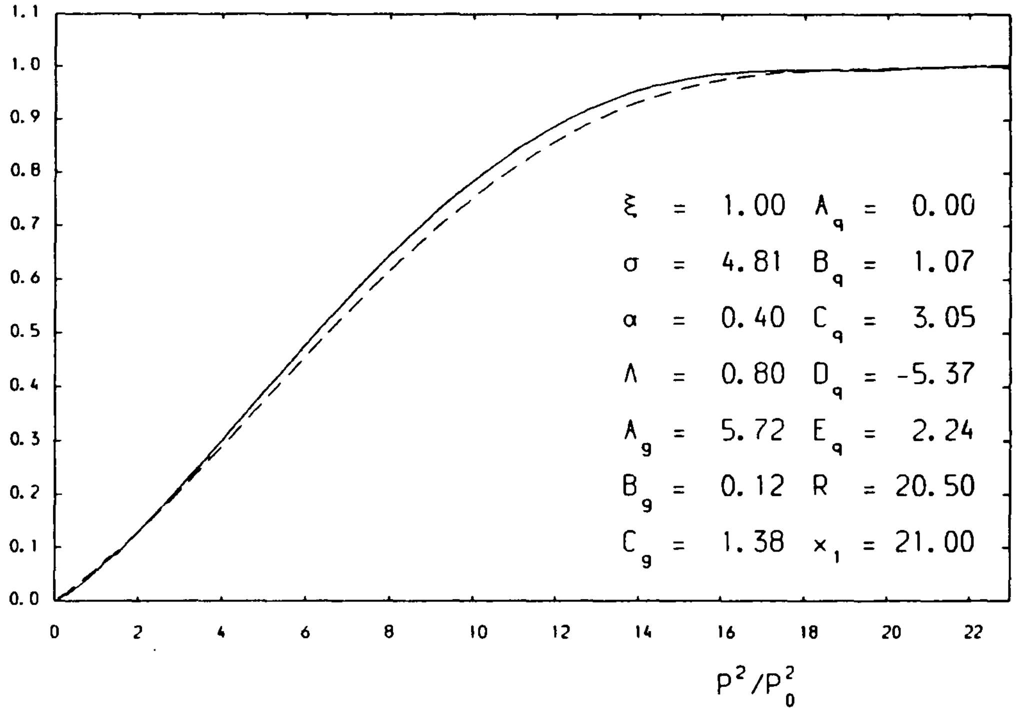


Figure 7.4a

The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ with a coupling constant of $\alpha_s = 0.2$ and 0.3.

Self-Consistent Quark



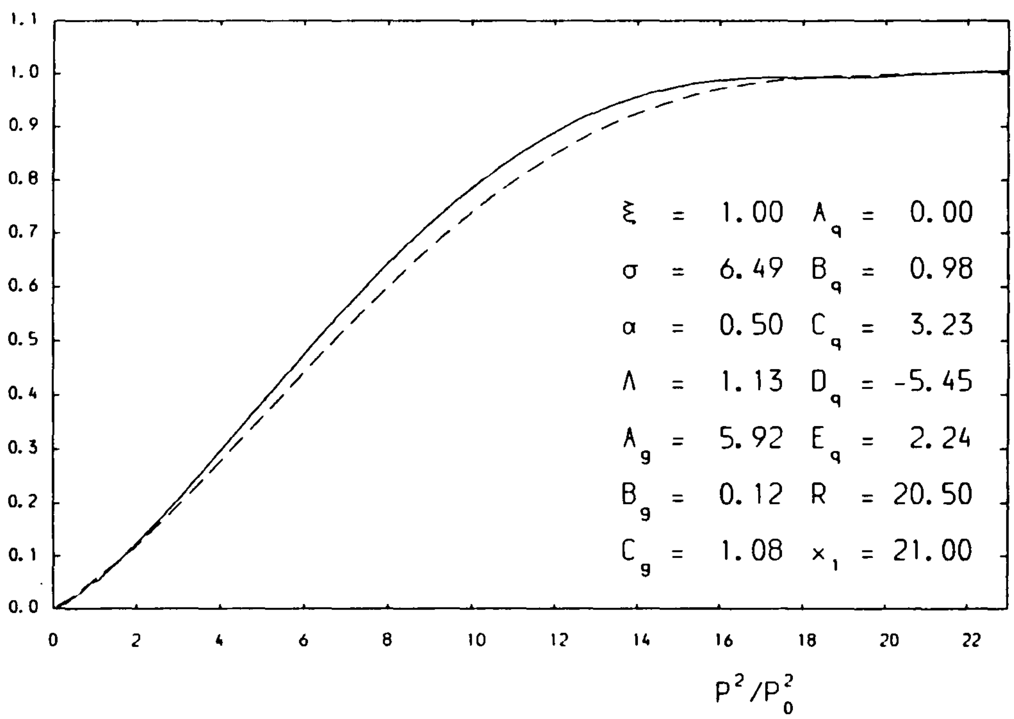


Figure 7.4b

The quark function, input broken line and output solid line in the Feynman gauge $\xi = 1$ with a coupling constant of $\alpha_s \doteq 0.4$ and 0.5.

Chapter 8

Summary and Conclusion

8:1 The Gluon

The Schwinger-Dyson equation for the gluon propagator involves integrals over the full three and four point Green's functions, which in turn depend in priciple on a heirarchy of multi-point functions. A consistent gauge invariant truncation of the Schwinger-Dyson equation for the two-point function (viz. the propagator) is to neglect all but the three-point vertex, the longitudinal part of which is determined by the Slavnov-Taylor identity. Moreover, if the coupling constant is small, the four-point vertices in the two loop graphs will make very little numerical difference to the answer, just as in perturbation theory, and so this is a sensible approximation. This then left the problem of finding the full three point gluon vertex. From the Slavnov-Taylor identity we found a form for the longitudinal part of the vertex in the limit of a bare ghost propagator. Unfortunately this form was found to be too complicated to be amenable to calculation. However, its general structure motivated our adoption of the Mandelstam approximation, which is its infra-red limit. This then gave us a closed equation for the gluon renormalization function $G(p^2)$.

This equation is non-linear and cannot be solved analytically and so we must resort to numerical methods. Attempts to solve a simplified version iteratively were found to be unstable. We therefore chose to approximate the gluon function by a parameterization and then solve the equation for the parameters over some range of momenta. The parameterization, obviously, must have the correct ultra-violet and infra-red limits, so we chose to approximate the gluon function by

$$G(p^2) = A_g \frac{p_0^2}{p^2} + B_g + C_g \frac{p^2}{p^2 + p_0^2}$$

for some mass scale p_0^2 where consistancy may have required the infra-red enhanced term, A_g, to be zero. Such forms allowed us to calculate the integrals in the vacuum polarization by dimensional regularization, and so obtain an equation for the gluon function by projecting out the transverse part. We then have to remove a mass term proportional to the coefficient of the enhanced term, A_g. This means that the output from the Schwinger-Dyson equation does not depend upon A_g. The equation for the gluon function can then be written as

$$\frac{1}{G(p^2)} = 1 + B_g (b_0 + b_1 \ln p^2/p_0^2) + \xi (\xi_0 + \xi_1 \ln p^2/p_0^2) + C_g (c_0 + g(p^2))$$

where $g(p^2)$ is a complicated function with the property that $g(0) = 0$. Thus if A_g is non-zero the right hand side must vanish as $p^2 \to 0$ which can only happen if

$$B_g = \xi \frac{\xi_1}{b_1} = \frac{3\xi}{28 - 3\xi}$$

This is sufficient to guarantee that the right hand side is finite in the limit $p^2 \to 0$. To ensure that it is zero requires that

$$C_g = 1 + \frac{B_g b_0 + \xi \xi_0}{c_0}$$

$$= \frac{36/\lambda - (214 + 18 \xi) B_g - 18 \xi (\xi + 1)}{(168 - 18 \xi) \ln \mu^2/p_0^2 + 155}$$

thus the equation for the gluon function becomes

$$\frac{1}{G(p^2)} = C_g \, g(p^2)$$

This is not true in the analytic sense, since there is no value of A_g, the only free parameter, for which the equality holds for all values of p^2. Fortunately we only require that this equation is approximately true over a finite region of momentum. At very small momenta the creation of real pions takes place and for this reason we have argued that we do not require the equation to be true for very small momenta. Conversely at large momenta the Mandelstam approximation for the full triple gluon vertex breaks down and so we do not require that the equation be true in this region either. The meaning of large and small in this case depends upon the value of p_0^2 which is unknown. However we expect that is of the order of 1 Gev^2, an expectation that was later verified by consideration of the static potential. We therefore took the limits of our fit to be 0.05 p_0^2 to 1.05 p_0^2. The reason for not taking a higher value for the upper limit is the intimate relation between the parameters A_g and C_g.

Since, with the constraints upon B_g and C_g, the output from the Schwinger-Dyson equation is independent of the parameters, we found that we could use a Fourier method to determine A_g. However, the constraints were found by taking the limit $p^2 \to 0$ and we have argued that we need not expect consistency in that limit. This led us to adopt the more general method of least squares fit to determine the parameters. The constraints were then just used to provide the initial values of B_g and C_g. Using this method we could show that a zero value for A_g is not consistent.

From the time-time component of the propagator the renormalised static potential was calculated, giving a linear potential with a string tension dependent upon A_g and p_0^2. By comparing this potential with a phenomenological one, p_0 was determined to be slightly less than 1 GeV. The string tensions could not be compared directly as the potential is not necessarly exactly linear in the phenomenological region. The gluon function was then replotted with the value of p_0 determined by the potential and was found to be remarkably gauge independent.

8:2 The Quark

The Schwinger-Dyson equation for the quark propagator is much simpler than the one for the gluon, and the only unknown is the quark-gluon vertex. The dominant longitudinal part of the vertex can be determined in terms of the quark function by solving the Ward-Takahashi identity. This then gave us a closed integral equation for the quark renormalization function.

The angular integrations are independent of the quark function and can be performed by assuming the form of the gluon function previously determined. This then left us with a scalar integral equation. Attempts to solve a simplified version of this equation by an iterative method had been found to be unstable, particularly in the ultra-violet limit. For this reason it was decided to parameterise the quark function in the non-perturbative regime and to use the perturbative result in the ultra-violet limit. The integrals could then be seen to contain divergences in the ultra-violet limit and an infra-red divergence because of the presence of the enhanced term in the gluon propagator. These divergences were handled by extracting the divergent parts and introducing cutoffs. The dependance upon the cutoffs was then removed by renormalising. The infra-red divergence is particularly important as this is peculiar to the enhanced term and is not analogous to any perturbative divergences. It is this divergent part that gives rise to the function d(x) that is responsible for the suppression of the quark function at low momenta.

Having removed the divergences, the radial integrals could then be calculated, although they were far from trivial, using a power series parameterization of the quark function that starts with a constant and is smoothly connected to the perturbative result at the point R.

$$F(x) = A_q + B_q\, x/R + C_q\, (x/R)^2 + D_q\, (x/R)^3 + E_q\, (x/R)^4$$

where

$$D_q = 4\, F_p(R) - F_p'(R) - 4\, A_q - 3\, B_q - 2\, C_q$$

$$E_q = 3\, A_q + 2\, B_q + C_q - 3\, F_p(R) + F_p'(R)$$

By using the least squares fit method developed for the gluon case the values of the parameters could then be determined. It was found that the result was only consistent if the constant A_q was zero, $< 10^{-4}$, compared to the other parameters, which were found to be of order one. It was also found that this result was largely independent of the values of the gluon parameters and the gluon scale. This is because the function d(x) coming from the enhanced term in the gluon propagator is independent of the coefficient of the enhanced term A_g.

8:3 Conclusion

Our main conclusions from this work are then: that the gluon propagator has an enhanced singularity in the infra-red, which is consistant with a $1/p^4$ behaviour in the confining region, and, as a direct consequence of this, the propagation of quarks is supressed at low momenta. Before going on to look at how this work can be, and is being, extended, let us look at some of the different approaches that have been taken by other people.

A radically different approach in spirit is that of J.M.Cornwall, [4.2] who has studied the behaviour of the gluon over large distances in the absence of quarks. In his work, in the light-cone gauge, he has been investigating, in an approximation to the Schwinger-Dyson equation, the dynamics of the creation of hadrons (glueballs) in the purely gluonic sector, at large distances. This leads him to propose a mass gap in the gluon spectrum of about 500 MeV and an 0^+ glueball mass of twice this value. We might expect that the inclusion of quarks will dramatically alter this result.

R.Delbourgo has also been studying the Schwinger-Dyson equations in the axial gauge using a spectral representation [8.1]. In this work the triple gluon vertex is given by the so called gauge approximation, based on an earlier suggestion of A.Salam in connection with QED. This approximation for the gluon vertex contains arbitrary transverse pieces with kinematic singularities, which makes the solution of the Slavnov-

Taylor identity somewhat ambiguous. The approximations used by Delbourgo allow him to find a linear form of the Schwinger-Dyson equations which the give rise to a gluon propagator that does not have an enhanced singularity in the infra-red limit. However, as has been pointed out by J.E.King [8.2] in the case of QED, one must be very careful in the inclusion of transverse parts of the vertices in order to handle the overlapping divergences correctly. This gauge approximation has also been studied by E.J.Gardner [8.3] who finds two alternative solutions to her equation, one with an enhanced singularity in the gluon propagator and the other where the gluon develops an effective mass, which illustrates the ambiguities that arise when arbitrary amounts of the transverse part of the gluon vertex are included [2.7]. In a study of the gluon propagator using a Lehmann representation, G.B.West [8.4] has concluded that, because of the positivity of the spectral function and analytic properties, it is impossible for the $g^{\mu\nu}$ term to be more singular than $1/q^2$,[†] though other parts of the propagator maybe. West has also pointed out that [8.5] if the propagator is as singular as $1/q^4$ in any one gauge then the Willson loop will decay exponentially with an area law.

Some other groups have been more concerned with the analytic properties of the solution in the infra-red limit, which is unphysical since it neglects the effect of pion creation but is, nevertheless, an interesting mathematical problem. A.I.Alekseev [8.6], studying the Schwinger-Dyson equation in the axial gauge finds that the infra-red limit of the gluon propagator is $1/(k^2)^{2.5374}$ although a later paper by B.A.Arbuzov, et. al. [8.7] find a different behaviour. In a series of papers D.Atkinson et. al. [8.8] have studied the infra-red limit of

† In the axial gauge

the Mandelstam approximation in the Landau gauge. They find that the solution does have the expected double pole at the origin of the propagator momentum, but that there are also find an accumulation of unphysical branch points in the complex plane. In their last paper, they suggest an ansatz for the gluon vertex which produces a solution, which is not plagued by such branch points.

We now turn to improvements upon the method we have presented. As we have mentioned in the previous chapter the problem with our determination of the gluon function when applied to the quark propagator is that it has been required to be consistent only over a small range. One way of overcoming this is to adopt an approach akin to that used for the quark function. That is the gluon function is parameterised by a power series starting at p^{-2} for values of momenta in the non-perturbative regime and use the perturbative result for higher values of momenta. One of the advantages of this scheme is that the integrals are much simpler. Because this is to be used in the quark equation where the argument of the gluon function is $(k-p)^2$ the parameterization has to be valid well into the perturbative regime. This is not as troublesome as it might at first sight appear to be, and satisfactory results for the gluon propagator have already been found with only five parameters [8.9]. The calculation of the quark propagator using this new gluon function is at present under study.

The next stage is to calculate the massless quark loop in the gluon propagator and determine its affect on the gluon renormalization function. Preliminary results indicate that the quark loop does indeed have an effect on the gluon parameters, reducing the value of A_g, but not affecting the general form of the solution, at least for small numbers of quark flavours.

The quark loop also provides a method of testing the Mandelstam approximation, as applied to the quark-gluon vertex, since we can calculate the quark loop in the Mandelstam approximation and in the case where the vertex is given by the solution of the Slavnov-Taylor identity.

We expect that, though these improvements will clear up some of the details, particularly the gluon scale and the exact behaviour of the quark at intermediate momenta, they will not affect our fundamental conclusions. The gluon propagator exhibits a self-consistant form in the confining region with a dominant behaviour of $1/q^4$, and, as a direct consequence of this, the propensity of massless quarks to propagate is suppresed at low momenta. The fact that the gluon propagator exhibits this enhanced behaviour causes the Willson loop to decay exponentially with an area law, which is indicative of a confining potential. We also see that massless quarks are inhibited from propagating out to large distances, which is another sign of confinement. A solution of the gluon equation which is valid in the perturbative regime fixes the point at which these confining effects become dominant to be of order Λ [8.9].

Appendix A

Dimensional Integral

In a general number of dimensions n, say, the integral of a function can be written in generalized spherical polar coordinates as,

$$\int d^n x \; f(x) \;=\; \int f(x)\; r^{n-1}\, dr \,\sin^{n-2}\theta_{n-1}\, d\theta_{n-1} \;\times$$

$$\times \sin^{n-3}\theta_{n-2}\, d\theta_{n-2} \;.\;.\;.\; d\theta_1 \qquad \text{(A.1)}$$

where the angular integration variables are constrained to be in the range 0 to π for θ_i, $i \neq 1$ and 0 to 2π for θ_1 with the radial variable given by,

$$r \;=\; \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

If the function f(x) depends upon only the radius r and not the angular variables, then the angular integrals θ_2 to θ_{n-1} can be done using the result,

$$\int_0^{\pi} \sin^m\theta \; d\theta \;=\; \sqrt{\pi}\; \frac{\Gamma[(m+1)/2]}{\Gamma[(m+2)/2]} \;. \qquad \text{(A.2)}$$

This means that the integral (A.1) becomes,

$$\int d^n k \; f(r) \;=\; \frac{2\;\pi^{n/2}}{\Gamma[n/2]} \int_0^{\infty} f(r)\; r^{n-1}\; dr \;. \qquad \text{(A.3)}$$

Now consider the integral,

Appendix A

$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2}\int d^n k \,\frac{1}{k^{2a}\,(\,k - p\,)^{2b}} \tag{A.4}$$

where $2\,\varepsilon = 4 - n$. By making a Feynman parameterization, which has the general form,

$$\frac{1}{D_1^{a_1}\,D_2^{a_2}\ldots D_k^{a_k}} = \frac{\Gamma(a_1+a_2+\ldots+a_k)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_k)}\int_0^1\ldots\int_0^1 dx_1\;\ldots\;dx_k$$

$$\times\;\frac{\delta(1-x_1-\ldots-x_k)\,x_1^{a_1-1}\ldots x_k^{a_k-1}}{(D_1x_1+\ldots+D_kx_k)^{a_1+\ldots+a_k}} \tag{A.5}$$

we get,

$$\frac{(2\pi\mu)^{2\varepsilon}}{\pi^2}\int_0^1 dx\,\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int d^n k\,\frac{x^{b-1}\,(\,1-x\,)^{a-1}}{(\,k^2 - 2\,x\,p.k + xp^2\,)^{(a+b)}} \tag{A.6}$$

In order to make the angular integrals trivial let us make a shift in the integration variable such that the denominator does not depend upon the angles ie.

$$k'^{\mu} = k^{\mu} - x\,p^{\mu}\;.$$

Then the denominator becomes,

$$k'^{2} + x\,(\,1 - x\,)p^2$$

This enables us to do the angular integrals using the result (A.2), thus (A.6) becomes,

$$\frac{(4\pi\mu^2)^{\varepsilon}}{\Gamma(n/2)}\int_0^1 dx\,\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{\infty} k'^{\,n-1}dk'\;\frac{x^{b-1}\,(\,1-x\,)^{a-1}}{(\,k'^{2} + x\,(\,1-x\,)p^2\,)^{(a+b)}} \tag{A.7}$$

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To perform this integral we use the standard integral,

$$\int_0^\infty dy \frac{y^b}{(y^2 + M^2)^a} = \frac{\Gamma[(b+1)/2)]\Gamma[a-(b+1)/2)]}{2\,\Gamma(a)\, M^{2[a-(b+1)/2]}}$$

Thus the integral (A.7) becomes,

$$(4\pi\mu^2)^\varepsilon \frac{\Gamma(a + b - 2 + \varepsilon)}{\Gamma(a)\Gamma(b)p^{2(a+b-2+\varepsilon)}} \int_0^1 dx\, x^{1-a-\varepsilon} (1 - x)^{1-b-\varepsilon}$$

$$= (4\pi\mu^2)^\varepsilon \frac{\Gamma(a + b - 2 + \varepsilon)}{\Gamma(a)\Gamma(b)p^{2(a+b-2+\varepsilon)}} \beta(2 - a - \varepsilon , 2 - b - \varepsilon) \qquad (A.8)$$

where,

$$\beta(a , b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a + b)}$$

The gamma function can be expanded in powers of ε to give ,

$$\Gamma(m + n\,\varepsilon) = \Gamma(m) [1 + n\,\varepsilon\,\psi(m) + O(\varepsilon^2)] \qquad (A.9)$$

with $\psi(m) = \frac{d}{dz} \ln \Gamma(z) \Big|_{z=m} = \sum_{k=1}^{m-1} \frac{1}{k} - \gamma_E$

and $\psi(1) = - \gamma_E$.

For different values of indices a and b, it is necessary to manipulate the gamma functions differently using the shift relation

$$\Gamma(n+1) = n\,\Gamma(n). \qquad (A.10)$$

This is to make sure that the gamma function $\Gamma(m)$ in the expansion (A.9) is finite. So let us specialise to the case where, $a = b = 1$, then the integral (A.8) is equal to,

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$$(4\pi\mu^2)^\varepsilon \frac{\Gamma(\varepsilon)}{p^{2\varepsilon}} \frac{\Gamma(1-\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(\,2 - 2\,\varepsilon\,)}.$$

Using the shift property of the gamma function, (A.10), this becomes

$$= \quad (4\pi\mu^2)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon\, p^{2\varepsilon}} \frac{\Gamma(1-\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(\,2 - 2\,\varepsilon\,)}$$

Expanding the gamma function using the result (A.9),

$$= \quad (4\pi\mu^2)^\varepsilon \frac{1}{\varepsilon} \left\{ 1 - \varepsilon\,\psi(1) + 2\,\varepsilon\,\psi(2) + \varepsilon \ln \frac{4\pi\mu^2}{p^2} \right\}$$

$$= \quad \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2\ .$$

Where we have used the result,

$$a^b \;=\; e^{b\,\ln(a)} \;\approx\; 1 + b\ln(a) \quad \text{for} \quad b \ll 1\ . \qquad \text{(A.11)}$$

This method can easily be applied to integrals with arbitrary powers a and b and with numerators involving the integration variable with an external index. This leads to the results quoted below.

The "B" and "C" integrals are calculated in a similar manner using the expantion of the hypergeometric functions as explained in section 3:5.

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$$A_{11} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^2 \, (k-p)^2}$$

$$= \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2 \, .$$

$$A_{11}^{\mu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu}{k^2 \, (k-p)^2}$$

$$= \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2 \right\} p^\mu .$$

$$A_{11}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^2 \, (k-p)^2}$$

$$= \frac{1}{3} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{13}{6} \right\} p^\mu \, p^\nu$$

$$- \frac{1}{12} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{8}{3} \right\} p^2 \delta^{\mu\nu} .$$

$$A_{11}^{\mu\nu\sigma} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu \, k^\sigma}{k^2 \, (k-p)^2}$$

$$= \frac{1}{4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{7}{3} \right\} p^\mu \, p^\nu \, p^\sigma$$

$$- \frac{1}{24} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{8}{3} \right\} p^2 (\delta^{\mu\nu} p^\sigma + \delta^{\nu\sigma} p^\mu + \delta^{\sigma\mu} p^\nu) \, .$$

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$$A_{12} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^2 \, (k - p)^4}$$

$$= - \frac{1}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right\} .$$

$$A^{\mu}_{12} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu}{k^2 \, (k - p)^4}$$

$$= - \frac{p^\mu}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\} .$$

$$A^{\mu\nu}_{12} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^2 \, (k - p)^4}$$

$$= - \frac{p^\mu \, p^\nu}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{3}{2} \right\}$$

$$+ \frac{\delta^{\mu\nu}}{4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2 \right\} .$$

$$A^{\mu\nu\sigma}_{12} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu \, k^\sigma}{k^2 \, (k - p)^4}$$

$$= - \frac{1}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{11}{6} \right\} p^\mu \, p^\nu \, p^\sigma$$

$$+ \frac{1}{6} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{13}{6} \right\} (\delta^{\mu\nu} p^\sigma + \delta^{\nu\sigma} p^\mu + \delta^{\sigma\mu} p^\nu) .$$

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$$A_{13} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^2 \, (k-p)^6}$$

$$= -\frac{1}{p^4}$$

$$A^{\mu}_{13} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu}{k^2 \, (k-p)^6}$$

$$= \frac{p^\mu}{2\,p^4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} - 1 \right\} .$$

$$A^{\mu\nu}_{13} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^2 \, (k-p)^6}$$

$$= \frac{p^\mu \, p^\nu}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{1}{2} \right\}$$

$$- \frac{\delta^{\mu\nu}}{4p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\}$$

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$$A_{21} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^4 \; (k - p)^2}$$

$$= \frac{1}{p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} \right\} .$$

$$A_{21}^{\mu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu}{k^4 \; (k - p)^2}$$

$$= \frac{p^\mu}{p^2} .$$

$$A_{21}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^4 \; (k - p)^2}$$

$$= \frac{p^\mu \, p^\nu}{2 \, p^2} + \frac{\delta^{\mu\nu}}{4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2 \right\} .$$

$$A_{21}^{\mu\nu\sigma} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu \, k^\sigma}{k^4 \; (k - p)^2}$$

$$= \frac{p^\mu \, p^\nu \, p^\sigma}{3 \, p^2}$$

$$+ \frac{1}{12} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{5}{3} \right\} (\delta^{\mu\nu} p^\sigma + \delta^{\nu\sigma} p^\mu + \delta^{\sigma\mu} p^\nu) .$$

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$$A_{22} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{k^4 \, (k - p)^4}$$

$$= - \frac{1}{2p^4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\} .$$

$$A^{\mu}_{22} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^{\mu}}{k^4 \, (k - p)^4}$$

$$= - \frac{p^{\mu}}{p^4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\} .$$

$$A^{\mu\nu}_{22} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^{\mu} \, k^{\nu}}{k^4 \, (k - p)^4}$$

$$= - \frac{p^{\mu} \, p^{\nu}}{p^4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 2 \right\} + \frac{\delta^{\mu\nu}}{2p^2} .$$

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$$A_{31}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^6 \, (\, k - p \,)^2}$$

$$= \frac{p^\mu \, p^\nu}{2 \, p^4} - \frac{\delta^{\mu\nu}}{4p^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\} \; .$$

$$A_{31}^{\mu\nu\sigma} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu \, k^\sigma}{k^6 \, (\, k - p \,)^2}$$

$$= \frac{p^\mu \, p^\nu \, p^\sigma}{4 \, p^4} + \frac{1}{8p^2} \, (\delta^{\mu\nu} p^\sigma + \delta^{\nu\sigma} p^\mu + \delta^{\sigma\mu} p^\nu) \; .$$

$$A_{32}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^6 \, (\, k - p \,)^4}$$

$$= - \frac{p^\mu \, p^\nu}{p^6} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{3}{2} \right\}$$

$$- \frac{\delta^{\mu\nu}}{4p^4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + 1 \right\} \; .$$

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$$B_{11} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{1}{(k^2 + p_0^2)(k-p)^2}$$

$$= \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + 1 + \varphi_{11}(q) \ .$$

$$B_{11}^{\mu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{k^{\mu}}{(k^2 + p_0^2)(k-p)^2}$$

$$= \frac{1}{2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{3}{2} + \varphi_{12}(q) \right\} p^{\mu}.$$

$$B_{11}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{k^{\mu} k^{\nu}}{(k^2 + p_0^2)(k-p)^2}$$

$$= \frac{1}{3} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} + \varphi_{13}(q) \right\} p^{\mu} p^{\nu}$$

$$- \frac{1}{12} \left\{ \left[3 + \frac{p^2}{p_0^2} \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{3}{2} \right] + 3\, \varphi_{21}(q) \right\} p_0^2 \delta^{\mu\nu}.$$

$$B_{11}^{\mu\nu\sigma} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{k^{\mu} k^{\nu} k^{\sigma}}{(k^2 + p_0^2)(k-p)^2}$$

$$= \frac{1}{4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{25}{12} + \varphi_{14}(q) \right\} p^{\mu} p^{\nu} p^{\sigma}$$

$$- \frac{1}{24} \left\{ \left[2 + \frac{p^2}{p_0^2} \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} \right] + 2\, \varphi_{22}(q) \right\} \times$$

$$\times\ p_0^2 (\delta^{\mu\nu} p^{\sigma} + \delta^{\nu\sigma} p^{\mu} + \delta^{\sigma\mu} p^{\nu}) \ .$$

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$$B_{12} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{1}{(k^2 + p_0^2)(k-p)^4}$$

$$= -\frac{q}{p_0^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + 1 + \frac{\varphi_{02}(q)}{q} \right\} .$$

$$B_{12}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{(k^2 + p_0^2)(k-p)^4}$$

$$= -\frac{q}{p_0^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} + \frac{\varphi_{04}(q)}{q} \right\} p^\mu \, p^\nu$$

$$+ \frac{1}{4} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{3}{2} + \varphi_{21}(q) \right\} \delta^{\mu\nu} .$$

$$B_{12}^{\mu\nu\sigma\tau} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu \, k^\sigma \, k^\tau}{(k^2 + p_0^2)(k-p)^4}$$

$$= -\frac{q}{p_0^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{137}{60} + \frac{\varphi_{06}(q)}{q} \right\} p^\mu \, p^\nu \, p^\sigma \, p^\tau$$

$$+ \frac{1}{8} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{25}{12} + \varphi_{14}(q) \right\} \times$$

$$\times (\delta^{\mu\nu} p^\sigma p^\tau + \delta^{\nu\sigma} p^\tau p^\mu + \delta^{\sigma\tau} p^\mu p^\nu + \delta^{\tau\mu} p^\nu p^\sigma + \delta^{\mu\sigma} p^\nu p^\tau + \delta^{\tau\nu} p^\sigma p^\mu) .$$

$$- \frac{p_0^2}{48} \left\{ \left[2 + \frac{p^2}{p_0^2} \right] \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} \right] + 2 \, \varphi_{22}(q) \right\} \times$$

$$\times \, (\delta^{\mu\nu} \delta^{\sigma\tau} + \delta^{\nu\sigma} \delta^{\tau\mu} + \delta^{\mu\sigma} \delta^{\nu\tau}) .$$

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$$C_{111}^{\mu\nu} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{k^\mu\, k^\nu}{k^2\ (\ k^2 + p_0^2\ )\ (\ k - p\ )^2}$$

$$= \frac{1}{p_0^2}\ (\ A_{11}^{\mu\nu} - B_{11}^{\mu\nu}\ )$$

$$= \frac{1}{3}\left\{ \ln \frac{p^2}{p_0^2} - \frac{1}{3} + \varphi_{13}(q) \right\} p^\mu\, p^\nu \;+\; \frac{p^2}{12}\left\{ \ln \frac{p^2}{p_0^2} - \frac{7}{6} \right\} \frac{\delta^{\mu\nu}}{p_0^2}$$

$$+ \frac{1}{4}\left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{3}{2} + \varphi_{21}(q) \right\} p_0^2 \delta^{\mu\nu} .$$

$$C_{111}^{\mu\nu\sigma} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \frac{k^\mu\, k^\nu\, k^\sigma}{k^2\ (\ k^2 + p_0^2\ )\ (\ k - p\ )^2}$$

$$= \frac{1}{p_0^2}\ (\ A_{11}^{\mu\nu\sigma} - B_{11}^{\mu\nu\sigma}\ )$$

$$= -\frac{1}{4}\left\{ \ln \frac{p^2}{p_0^2} - \frac{1}{4} + \varphi_{14}(q) \right\} \frac{p^\mu\, p^\nu\, p^\sigma}{p_0^2}$$

$$+ \frac{1}{24}\left\{ \ln \frac{p^2}{p_0^2} - \frac{5}{6} \right\} \frac{p^2}{p_0^2}\ (\delta^{\mu\nu}p^\sigma + \delta^{\nu\sigma}p^\mu + \delta^{\sigma\mu}p^\nu)$$

$$+ \frac{1}{12}\left\{ \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} + 2\ \varphi_{22}(q) \right\} \times$$

$$\times\ (\delta^{\mu\nu}p^\sigma + \delta^{\nu\sigma}p^\mu + \delta^{\sigma\mu}p^\nu)\ .$$

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$$C^{\mu\nu}_{112} = \frac{(2\pi\mu)^{2\varepsilon}}{\pi^2} \int d^n k \, \frac{k^\mu \, k^\nu}{k^2 \, (k^2 + p_0^2) \, (k - p)^4}$$

$$= \frac{1}{p_0^2} \left(A^{\mu\nu}_{12} - B^{\mu\nu}_{12} \right)$$

$$= \left\{ \frac{q}{p_0^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p_0^2} + \frac{11}{6} + \frac{\varphi_{04}(q)}{q} \right] \right.$$

$$\left. - \frac{1}{p^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{p^2} + \frac{3}{2} \right] \right\} \frac{p^\mu \, p^\nu}{p_0^2}$$

$$- \frac{1}{4} \left\{ \ln \frac{p^2}{p_0^2} + \frac{3}{2} + \varphi_{21}(q) \right\} \frac{\delta^{\mu\nu}}{p_0^2} .$$

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$$\varphi_{02}(q) = q \left\{ \left[1 - \frac{q}{1-q} \ln q \right] - 1 \right\} .$$

$$\varphi_{04}(q) = q \left\{ \left[1 - \frac{q^3}{(1-q)^3} \ln q \right] - \frac{q^2}{2} + \frac{q}{2} - \frac{1}{3} - q^3 \left[\frac{1}{(1-q)^2} + \frac{1}{2(1-q)} \right] \right\}$$

$$\varphi_{06}(q) = q \left\{ \left[1 - \frac{q^5}{(1-q)^5} \ln q \right] - \frac{q^4}{4} + \frac{q^3}{12} - \frac{q^2}{12} + \frac{q}{4} - \frac{1}{5} - q^3 \left[\frac{1}{(1-q)^4} + \frac{1}{2(1-q)^3} + \frac{1}{3(1-q)^2} + \frac{1}{4(1-q)} \right] \right\}$$

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$$\varphi_{11}(q) = 1 + \frac{\ln q}{1-q}$$

$$\varphi_{12}(q) = \frac{3}{2} - \frac{1}{1-q} + \left[1 - \frac{q^2}{(1-q)^2} \right] \ln q$$

$$\varphi_{13}(q) = \frac{11}{6} + \frac{1}{(1-q)^2} - \frac{5}{2(1-q)} + \left[1 - \frac{q^3}{(1-q)^3} \right] \ln q$$

$$\varphi_{14}(q) = \frac{25}{12} - \frac{1}{(1-q)^3} + \frac{7}{2(1-q)^2} - \frac{5}{2(1-q)} + \\ + \left[1 - \frac{q^4}{(1-q)^4} \right] \ln q$$

$$\varphi_{21}(q) = \frac{1}{3q} \left\{ \frac{4}{3} q + \frac{7}{6} + \frac{q^2}{1-q} + \frac{1}{(1-q)^2} \ln q \right\}$$

$$\varphi_{22}(q) = \frac{1}{2q} \left\{ \frac{5}{6} q + \frac{4}{3} - \frac{q^2}{2(1-q)^2} + \frac{1}{2q(1-q)^2} \\ + \frac{2q-1}{q(1-q)^2} \ln q \right\}$$

Appendix B

Angular Integrals for the Quark Propagator

If we let the gluon function have the form,

$$G(q^2) = \frac{A_g}{q^2} + B_g + C_g \frac{q^2}{q^2+p_0^2} .$$

Then transforming to spherical polar coordinates in four dimension

$$d^4k \longrightarrow k^3 \, dk \, \sin^2\psi \, d\psi \, \sin\theta \, d\theta \, d\varphi$$

where the integration variables are constrained such that

$$0 \leqslant k \leqslant \infty, \qquad 0 \leqslant \psi, \theta \leqslant \pi, \qquad 0 \leqslant \varphi \leqslant 2\pi .$$

For convenience let us choose the time axis to be along the p momentum direction. This does not affect the Lorentz invariance of the equation as any choice of the axis will lead to the same results. However, the choice of p along the time axis simplifies the integrals considerably. Then the conponents of the momenta p and k are:

$$p^\mu = (p, 0, 0, 0),$$
$$k^\mu = k(\cos\psi, \sin\psi \sin\theta \cos\varphi, \sin\psi \sin\theta \sin\varphi, \sin\psi \cos\theta) .$$

notice that $p.k = p \, k \cos\psi$.

There are two types of integral over a function of k,p and z, where $z=\cos\psi$, one just over the function, the over multiplied by $\not{k}$. The first one is reasonably straight forward, for the second, extract the γ matrix and consider the vector integral component by component.

For the first and second component the intregral is zero since,

$$\int_0^{2\pi} \sin\varphi \, d\varphi = \int_0^{2\pi} \cos\varphi \, d\varphi = 0 .$$

The third component is zero as,

$$\int_0^{\pi} \sin\theta \, d\theta \quad \cos\theta \;\; = 0 \;.$$

This leaves only the zeroth component and since $k^{(0)} = k \cos\varphi$ it has the same form as the integral not involving $\not{k}$. By doing a partial fraction decomposition the integrals can be written in the form,

$$4\pi \int_0^{\infty} k^3 dk \; f(k,p) \int_{-1}^{1} \sqrt{1-z^2} \quad dz \; \frac{1}{(a - bz)^j}$$

where $b = 2 \; p \; k$ and $a = p^2 + k^2$ or $p^2 + k^2 + p_0^2$ depending on the form of the gluon function and the same integral without a denominator.

Thus, the basic angular integrals involved in the equation for the quark propagator are of the form,

$$\int d\Omega \; \frac{1}{(\; a - b \cos\psi \;)^j} \tag{B.1}$$

where $d\Omega \;=\; \sin^2\psi \; d\psi \; \sin\theta \; d\theta \; d\varphi$ and $j = \{ \; 1, \; 2, \; 3 \; \}$.

Consider the integral with $j = 1$. Then (B.1) becomes,

$$\int d\Omega \; \frac{1}{a - b \cos\psi}$$

$$= \int_0^{\pi} \sin^2\psi \; d\psi \int_0^{\pi} \sin\theta \; d\theta \int_0^{2\pi} d\varphi \; \frac{1}{a - b \cos\psi}$$

$$= 4\pi \int_{-1}^{1} dz \; \frac{\sqrt{1 - z^2}}{a - b \; z}$$

where $z = \cos\psi$. If we let $y = a - b \; z$, then

$$\frac{4\pi}{b^2}\int_{a-b}^{a+b}\sqrt{R}\;dy/y$$

where $R = b^2 - a^2 + 2\,a\,y - y^2$

$$= \frac{4\pi}{b^2}\left\{ (b^2 - a^2)\int_{a-b}^{a+b}\frac{dy}{y\,\sqrt{R}} + a\int_{a-b}^{a+b}\frac{dy}{\sqrt{R}}\right\}.$$

Since $a \geqslant p^2 + k^2$, $b = 2\,p\,k$ and the discrimenant of R is $\Delta = -\,4\,b$ then the integral is

$$\frac{4\pi^2}{b^2}\left\{ a - \sqrt{a^2 - b^2}\right\}$$

$$= 4\pi^2\,\frac{a - \sqrt{a^2 - b^2}}{b^2}. \qquad \text{(B.2)}$$

Now let us consider the integral (B.1) with $j = 2$

$$\int d\Omega\,\frac{1}{(a - b\cos\psi)^2}$$

$$= \int_0^{\pi}\sin^2\psi\;d\psi\int_0^{\pi}\sin\theta\;d\theta\int_0^{2\pi}d\varphi\,\frac{1}{(a - b\cos\psi)^2}$$

$$= 4\pi\int_{-1}^{1}dz\,\frac{\sqrt{1 - z^2}}{(a - b\,z)^2}$$

where $z = \cos\psi$. If we again let $y = a - b\,z$ then,

$$\frac{4\pi}{b^2}\int_{a-b}^{a+b}\sqrt{R}\;dy/y^2$$

where $R = b^2 - a^2 + 2ay - y^2$

$$= \frac{4\pi}{b^2}\left\{ a\int_{a-b}^{a+b} \frac{dy}{y\sqrt{R}} - \int_{a-b}^{a+b} \frac{dy}{\sqrt{R}} \right\}.$$

Since $a \geqslant p^2 + k^2$, $b = 2pk$ and the discrimenant of R is $\Delta = -4b$

then the integral is

$$= \frac{4\pi^2}{b^2}\left\{ \frac{a}{\sqrt{a^2 - b^2}} - 1 \right\}. \tag{B.3}$$

Finally, consider the integral (B.1) with $j = 3$

$$\int d\Omega \frac{1}{(a - b\cos\psi)^3}$$

$$= \int_0^{\pi} \sin^2\psi\, d\psi \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi \frac{1}{(a - b\cos\psi)^2}$$

$$= 4\pi \int_{-1}^{1} dz \frac{\sqrt{1 - z^2}}{(a - bz)^3}$$

where $z = \cos\psi$.

Making the same transformation $y = a - bz$

$$\frac{4\pi}{b^2}\int_{a-b}^{a+b} \sqrt{R}\, dy/y^3$$

where $R = b^2 - a^2 + 2ay - y^2$

$$= -\frac{4\pi}{b^2}\left\{ \frac{4a^2}{8(b^2-a^2)} + \frac{1}{2} \right\} \int_{a-b}^{a+b} \frac{dy}{y\sqrt{R}}.$$

Appendix B

Since $a \geqslant p^2 + k^2$, $b = 2\,p\,k$ and the discrimenant of R is $\Delta = -4\,b$ then the intergal is

$$\frac{2\pi^2}{b^2}\left\{\frac{a^2}{a^2 - b^2} - 1\right\}\frac{1}{\sqrt{a^2 - b^2}}$$

$$= \frac{2\,\pi^2}{(a^2 - b^2)^{3/2}}. \tag{B.4}$$

For the integrals of interest $a = p^2 + k^2$ or $p^2 + k^2 + p_0^2$ and $b = 2\,p\,k$. Thus, for $a = p^2 + k^2$

$$\sqrt{a^2 - b^2} = \sqrt{(p^2 + k^2)^2 - 4\,p^2\,k^2} = |\,p^2 - k^2\,|$$

thus introducing the function h(x) where

$$h(x) = \begin{cases} x & \text{for } x < 1 \\ 1 & \text{otherwise} \end{cases}$$

Thus the integrals can be written as,

$$\int \frac{d\Omega}{(\,k - p\,)^2} = \frac{2\,\pi^2}{k^2}\,h(k^2/p^2) \tag{B.5}$$

$$\int \frac{d\Omega}{(\,k - p\,)^4} = \frac{2\,\pi^2\,h(k^2/p^2)}{k^2\,|\,p^2 - k^2\,|} \tag{B.6}$$

$$\int \frac{d\Omega}{(\,k - p\,)^6} = \frac{2\,\pi^2}{|\,p^2 - k^2\,|^3} \tag{B.7}$$

The numerators of the integrals of interest have only three forms unity , $z\,k$ where $z = \cos\psi$ which is either $k.p/p$ or the third component of k^μ contracted with γ^μ which is the only non-zero contribution to the integral, or $z\ k\ p.k = z^2\ k^2\ p$. These integrals can be calculated from the unit numerator integrals by partial fractions and some simple integrals over z. thus the nine integrals are,

$$\int \frac{d\Omega}{(\ k - p\ )^2} = \frac{2\ \pi^2}{k^2}\ h(k^2/p^2)$$

$$\int \frac{d\Omega\ k^\mu}{(\ k - p\ )^2} = \frac{\pi^2 p^\mu}{k^2}\ h(k^4/p^4)$$

$$\int \frac{d\Omega\ k^\mu p.k}{(\ k - p\ )^2} = \frac{\pi^2 p^\mu}{2\ k^2}\ (\ k^2 + p^2\ )\ h(k^4/p^4)$$

$$\int \frac{d\Omega}{(\ k - p\ )^4} = \frac{2\ \pi^2\ h(k^2/p^2)}{k^2\ |\ p^2 - k^2\ |}$$

$$\int \frac{d\Omega\ k^\mu}{(\ k - p\ )^4} = \frac{2\pi^2 p^\mu\ h(k^4/p^4)}{k^2\ |\ p^2 - k^2\ |}$$

$$\int \frac{d\Omega\ k^\mu p.k}{(\ k - p\ )^4} = \frac{2\pi^2 p^\mu\ [\ 3\ p^2\ h(k^6/p^6) + k^2\ h(k^2/p^2)\ ]}{k^2\ |\ p^2 - k^2\ |}$$

$$\int \frac{d\Omega}{(\ k - p\ )^6} = \frac{2\ \pi^2}{|\ p^2 - k^2\ |^3}$$

$$\int \frac{d\Omega\ k^\mu}{(\ k - p\ )^6} = \frac{\pi^2 p^\mu\ [\ 3\ k^2\ h(k^2/p^2) + p^2\ h(k^6/p^6)\ ]}{k^2\ |\ p^2 - k^2\ |^3}$$

$$\int \frac{d\Omega\ k^\mu p.k}{(\ k - p\ )^6} = \frac{\pi^2 p^\mu\ [\ k^4 + 6\ k^2\ p^2\ h(k^4/p^4) - 3\ p^4\ h(k^8/p^8)\ ]}{2\ k^2\ |\ p^2 - k^2\ |^3}$$

Appendix B

For the integrals with $a = p^2 + k^2 + p_0^2$ we introduce the functions $h_1(x,y)$ and $h_2(x,y)$ where,

$$2\, h_1(x,y) = 1 + x + y - \sqrt{(1 - x)^2 + 2\, x\, (1 + x) + y^2}$$

$$h_2(x,y) = (1 + x + y)\, h_1(x,y) - x$$

Notice that as y goes to zero,

$$h_1 \rightarrow h(x)$$

and

$$h_2 \rightarrow h(x^2)$$

Then the three angular integrals needed in the evaluation of the part of vacuum polarization coming from the intermediate term can be written as,

$$\int \frac{d\Omega}{(k - p)^2 + p_0^2} = \frac{2\,\pi^2}{k^2}\, h_1(k^2/p^2, k_0^2/p^2)$$

$$\int \frac{d\Omega\, k^\mu}{(k - p)^2 + p_0^2} = \frac{\pi^2 p^\mu}{k^2}\, h_2(k^2/p^2, k_0^2/p^2)$$

$$\int \frac{d\Omega\, k^\mu\, p.k}{(k - p)^2 + p_0^2} = \frac{\pi^2 p^\mu}{2\, k^2}\, (k^2 + p^2 + p_0^2)\, h_2(k^2/p^2, k_0^2/p^2)$$

These twelve integrals are all that are needed to do the angular integrals in the equation for the quark propagator.

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