Elliptic operators, connections and gauge transformations

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ELLIPTIC OPERATORS, CONNECTIONS AND GAUGE TRANSFORMATIONS

A study is made of the action of various Banach Lie groups of principal bundle automorphisms (gauge transformations) on corresponding spaces of connections on some principal bundle, using standard theorems of global analysis together with elliptic regularity theorems. A proof of elliptic regularity theorems in Sobolev and Hölder norms for linear elliptic partial differential operators with smooth coefficients acting on sections of smooth vector bundles is presented. This proof assumes acquaintance with the theory of tempered distributions and their Fourier transforms and with the theory of compact and Fredholm operators, and also uses results from the papers of Calderon and Zygmund and from the early papers of Hörmander on pseudo-differential operators, but is otherwise intended to be self-contained. Elliptic regularity theorems are proved for elliptic operators with non-smooth coefficients, using only the regularity theorems for elliptic operators with smooth coefficients, together with the Sobolev embedding theorems, the Rellich-Kondrakov theorem and the Sobolev multiplication theorems. For later convenience these elliptic regularity results are presented as a generalization of the analytical aspects of Hodge theory. Various theorems concerning the action of automorphisms on connections are proved, culminating in the slice theorems obtained in chapter VIII. Regularity theorems for Yang-Mills connections and for Yang-Mills-Higgs systems are obtained. In chapter IX analytical properties of the covariant derivative operators associated with a connection are related to the holonomy group of the connection via a theorem which shows the existence of an upper bound on the length of loop required to generate the holonomy group of a connection with compact holonomy group.

ELLIPTIC OPERATORS, CONNECTIONS AND GAUGE TRANSFORMATIONS

by

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DECLARATION

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A statement as to which parts of the thesis are claimed as original and the sources from which the rest has been derived has been included in Chapter I.

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CONTENTS

Chapter I

A DESCRIPTION OF THE MAIN RESULTS

In this chapter we give a brief outline of the results obtained and of their relationship to results occurring in the literature.

We begin with a discussion of the slice theorems proved in chapter VIII for the action of principal bundle automorphisms (gauge transformations) on connections and of the elliptic regularity results for Yang-Mills connections and Yang-Mills-Higgs systems. These results are consequences of general elliptic regularity results for elliptic partial differential operators with smooth coefficients, which imply results generalizing the analytical aspects of Hodge theory to the study of Hodge-de Rham Laplacians with respect to connections which need not be smooth.

Slice theorems are used when studying the properties of suitably differentiable functionals defined on Banach manifolds with the property that the functional is constant along the orbits of some infinite dimensional symmetry group. We give a brief survey of occasions in geometry where this situation arises. An account is given of the methods of Morse theory and Lyusternik-Schnirelmann theory for relating the topology of a Banach manifold to the critical sets of functionals defined on that manifold. The role of slice theorems in circumstances where the functional is invariant under the action of an infinite dimensional symmetry group is described.

We also give a survey of the physical origins of gauge theories and of recent work on the topological, geometric and analytical aspects of gauge theories.

We give an account of the plan of the thesis. A statement is given which specifies those parts of the thesis believed to be original research and the sources on which other results in the thesis are based.

The Slice Theorem and Yang-Mills Connections

We describe the results of chapter VIII. These results are the main objective of the thesis, for which the earlier chapters prepare the necessary foundation.

We study the action of various Banach Lie groups of principal bundle automorphisms on corresponding affine spaces of connections on a given principal bundle over a compact manifold with compact structural group. These Banach Lie groups of automorphisms are modelled on Sobolev, C^k and Hölder spaces, and the affine spaces of connections correspond to Sobolev, C^k and Hölder spaces of sections of the appropriate vector bundle.

Theorem 2.3 of chapter VIII is a slice theorem giving sufficient conditions for the existence of a differentiable structure on the quotient space obtained by quotienting a Sobolev or Hölder space of connections by the action of the corresponding group of principal bundle automorphisms in such a way that the quotient map is a smooth map between Banach manifolds which admits smooth local sections. This theorem generalizes corresponding slice theorems in [Singer, I.M., 1978], [Narasimhan, M.S. and Ramadas, T.R., 1979], [Mitter, P.K. and Viallet, C.M., 1981] and [Parker, T., 1982] and is closely related to corresponding results in [Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978] and [Donaldson, S.K., 1983b].

The proof of the slice theorem uses the results of chapter VI in which a fairly detailed study of the action of Banach Lie groups of principal bundle automorphisms on the corresponding spaces of connections is undertaken. It is shown that the actions of the Banach Lie groups of L^p_{k+1}, C^{k+1} and $C^{k+1,\alpha}$ principal bundle automorphisms on the spaces of L^p_k, C^k and $C^{k,\alpha}$ connections respectively are smooth, privided that $1 \leq p < \infty$ and $p(k + 1) > n$, where n is the dimension of the base manifold of the bundle, and provided that $0 < \alpha < 1$. In all these cases the quotient of the space of connections by the corresponding group of automorphisms is Hausdorff, and the stabilizer of a connection in the appropriate group of automorphisms is compact (see theorems VI.4.1 and VI.4.2). Note that this result holds even for L^p connections which are not continuous, provided that $p > n$. It is also shown that if (ω_i) is a sequence in any of the above spaces of connections, if (Ψ_i) is a sequence in the corresponding group of principal bundle automorphisms and if both (ω_i) and $(\Psi_i^* \omega_i)$ converge in the space of connections, then some subsequence of (Ψ_i) converges in the group of automorphisms (see corollary VI.3.3). Indeed if (Ψ_i) converges on some given fibre of the principal bundle, then (Ψ_i) converges in the group of automorphisms (see theorem VI.3.2).

The proof of the slice theorem (theorem VIII.2.3) uses both the results of chapter VI described above and also a generalization of the analytical aspects of Hodge theory, presented in chapter VII, which describes the properties of the covariant Hodge-De Rham Laplacian with respect to a connection that need not be smooth.

This generalization of Hodge theory is also used to prove regularity theorems for Yang-Mills connections (theorems 3.1, 3.2 and 3.3 of chapter VIII) which place sufficient conditions on p and k in order that, for every L^p_k connection ω satisfying the Yang-Mills equation, there should exist an L^p_{k+1} principal bundle automorphism Ψ such that $\Psi^*\omega$ is smooth. An informal discussion of the regularity of Yang-Mills-Higgs systems is given. Regularity theorems for Yang-Mills fields and Yang-Mills-Higgs systems are given in [Uhlenbeck, K.K., 1982b] and [Parker, T., 1982].

Elliptic Regularity and Hodge Theory

In chapter III we shall prove a general regularity theorem for elliptic differential operators with smooth coefficients. This may be stated as follows. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be smooth vector bundles over a compact smooth manifold M and let $L : C^\infty(E_1) \to C^\infty(E_2)$ be a linear elliptic differential operator of order m with smooth coefficients. If k is an integer and if p satisfies $1 < p < \infty$ then L extends to a bounded Fredholm map

$$L : L^p_{k+m}(E_1) \to L^p_k(E_2).$$

Moreover if u is an E_1-valued distribution with the property that $Lu \in L^p_k(E_2)$ then $u \in L^p_{k+m}(E_1)$. Similarly if k is a non-negative integer and if α satisfies $0 < \alpha < 1$ then L extends to a bounded Fredholm map

$$L : C^{k+m,\alpha}(E_1) \to C^{k,\alpha}(E_2)$$

and if u is an E_1-valued distribution with the property that $Lu \in C^{k,\alpha}(E_2)$ then $u \in C^{k+m,\alpha}(E_1)$. Rather surprisingly, I have not found this theorem stated in the above form in the literature. The nearest approach to this theorem that I have yet discovered in the literature is theorem 3.54 of the book 'Nonlinear analysis on manifolds. Monge-Ampere equations' by Aubin.

We prove this regularity theorem using the theory of singular integrals, due to Calderon and Zygmund, and the theory of pseudo-differential operators. A parametrix for L is defined to be a linear operator $P : C^\infty(E_2) \to C^\infty(E_1)$ with the property that the operators $I - LP$ and $I - PL$ are smoothing operators (a smoothing operator is a continuous linear operator whose distribution kernel is smooth). Hörmander has shown that every linear elliptic differential operator of order m with smooth coefficients has a parametrix which is a pseudodifferential operator of order -m in the class of such operators introduced by Kohn and Nirenberg and by Hörmander. Using the pseudolocal property of pseudodifferential operators and a partition of unity argument it suffices to show that a pseudodifferential operator $Q : C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ of order -m in the class of pseudodifferential operators introduced by Kohn and Nirenberg extends to continuous linear maps

$$Q : L^p_{k,loc}(\mathbb{R}^n) \longrightarrow L^p_{k+m,loc}(\mathbb{R}^n),$$

$$Q : C^{k,\alpha}_{loc}(\mathbb{R}^n) \longrightarrow C^{k+m,\alpha}_{loc}(\mathbb{R}^n).$$

Now pseudodifferential operators on $\mathbb{R}^n$ in the class introduced by Kohn and Nirenberg have the form

$$Q\varphi(x) = (2\pi)^{-n} \int e^{ix.\xi} q(x,\xi)\, \hat{\varphi}(\xi)\, d\xi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, where the symbol $q(x,\xi)$ has an asymptotic expansion in ξ for large ξ in which each term is a positively homogeneous function of ξ. Now if sufficiently many terms of this asymptotic expansion are taken, then the distribution kernel of the pseudodifferential operator corresponding to the remainder term is C^r for r as large as required. Thus it suffices to consider the boundedness in Sobolev and Hölder norms of the pseudodifferential operator corresponding to each individual term in the asymptotic expansion. But one can express such an operator as a sum of compositions of operators which are either the singular integral operators with variable kernels sutdied by Calderon and Zygmund, or are convolution operators with summable kernels, or are other well-behaved operators. Thus the boundedness of the pseudodifferential operator will follow using the results of Calderon and Zygmund, together with Young's theorem on convolutions. This enables us to prove the elliptic regularity results of chapter III.

We use the regularity theorems proved in chapter III, together with the Sobolev embedding theorems, the Rellich-Kondrakov theorem and the Sobolev multiplication theorems in order to derive a generalization of the analytical aspects of Hodge theory, which we present in chapter VII. This applies to the covariant Hodge-de Rham Laplacian with respect to connections which need not be smooth. More specifically, let k be a non-negative integer, let p satisfy the conditions $1 < p < \infty$ and $p(k + 1) > n$, where n is the dimension of the compact smooth manifold under consideration. If $k = 0$, let p also satisfy the condition $p \geq 2$. Let p' be the exponent conjugate to p, defined by

$$\frac{1}{p'} = 1 - \frac{1}{p}.$$

If $\pi : E \longrightarrow M$ is a smooth vector bundle and if Δ^ω is the covariant Hodge-de Rham Laplacian with respect to an L^p_k connection ω on an associated principal bundle, and if $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy the conditions

$$-k \leq l \leq k,$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

then

$$\Delta^\omega : L^q_{l+1}(E \otimes \Lambda^j T^*M) \longrightarrow L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

is a Fredholm operator. Moreover if $u \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^\omega u \in L^q_{l-1}(E \otimes \Lambda^j T^*M)$ then $u \in L^q_{l+1}(E \otimes \Lambda^j T^*M)$. Using these results one can prove an analoque of the Hodge decomposition theorem and define the Green's operator

$$G^{\omega} : L^{p'}_{-k-1}(E \otimes \Lambda^{j} T^*M) \to L^{p'}_{-k+1}(E \otimes \Lambda^{j} T^*M)$$

of Δ^{ω} in the usual manner, and this restricts to a bounded linear map

$$G^{\omega} : L^{q}_{l-1}(E \otimes \Lambda^{j} T^*M) \to L^{q}_{l+1}(E \otimes \Lambda^{j} T^*M).$$

Similar results can be proved if ω is a $C^{k,\alpha}$ connection.

Why are Slice Theorems important?

In the mathematical literature one may find various instances where a study has been made of the following type of problem. Suppose that one is given a smooth manifold M and that on this manifold is defined some class X of geometric structures, where X may be identified with some Banach (or Frechet) space of sections of some fibre bundle over M. Suppose that there is a naturally defined infinite dimensional symmetry group H which permutes the elements of X. We shall suppose that H is a Banach (or Frechet) Lie group acting (smoothly) on X and that H acts freely on some open set X_0 of generic elements of X. The problem is then to show that the topological quotient X_0/H of X by the action of H is a Hausdorff topological space which admits a canonical differentiable structure with the property that the natural projection from X_0 to X_0/H is smooth and admits smooth local sections.

A classic example is provided by Teichmüller theory. We let M be a smooth surface and define Con(M) to be the space of conformal structures on M. Given a conformal structure on M and a diffeomorphism of M we may form a new conformal structure which is the pullback of the given conformal structure by the diffeomorphism. Thus the group Diff(M) of orientation-preserving diffeomorphisms of M acts on the space Con(M) of conformal structures on M, and thus we may form the quotient space Con(M)/Diff(M). This quotient space is referred to as the moduli space of Riemann surfaces whose topological type is that of M. Similarly the Teichmüller space $Con(M)/Diff_0(M)$ of marked Riemann surfaces whose topological type is that of M is defined to be the quotient of the space Con(M) of conformal structures on M by the identity component $Diff_0(M)$ of the group Diff(M) of orientation-preserving diffeomorphisms of M (see

[Earle, C.J. and Eells, J., 1969]. When M is a torus then the Teichmüller space of marked Riemann surfaces of genus 1 is identified with the upper half plane $\mathbb{C}_+$ and the moduli space of Riemann surfaces of genus 1 is identified with the quotient $\mathbb{C}_+/SL(2,\mathbb{Z})$. In general we see that the Teichmüller space is a covering space of the moduli space.

A second example is provided by the action of the group of diffeomorphisms of a smooth manifold on the space of Riemannian metrics on this manifold. This action has been studied in [Ebin, D.G., 1970], in [Fischer, A.E. and Marsden, J.E., 1977] and in [Bourguignon, J.-P., 1975].

Let M be a compact smooth manifold of dimension n, let N be a compact Riemannian manifold of dimension k, where $k \geq n$, and let B be a smooth submanifold of N of dimension n − 1 which is diffeomorphic to ∂M. The n-dimensional Plateau problem is to find a map $f : M \to N$ which sends ∂M diffeomorphically onto B with the property that f(M) has minimal volume among such maps from M to N. To study this problem one might take X to be the space of maps $f : M \to N$ which send ∂M diffeomorphically onto B and define $vol : X \to \mathbb{R}$ to be the map sending $f \in X$ to the volume of f(M). If $\varphi : M \to M$ is a diffeomorphism of M and if $f \in X$ then so does $f \circ \varphi$, and $vol(f \circ \varphi) = vol(f)$. Thus $vol : X \to \mathbb{R}$ induces a map $vol : X/Diff(M) \to \mathbb{R}$.

We shall be studying the action of groups $\mathcal{G}$ of principal bundle automorphisms on spaces $\mathcal{A}$ of connections on a smooth principal bundle over a compact smooth manifold M. The Yang-Mills functional $YM : \mathcal{A} \to \mathbb{R}$ is invariant under $\mathcal{G}$ and thus induces a functional $YM : \mathcal{A}/\mathcal{G} \to \mathbb{R}$. If the dimension of M is 4 then the minimum of the Yang-Mills functional is attained by the set $\mathcal{A}_{min}$

of instantons (or anti-instantons) on the principal bundle. The moduli space of instantons (or anti-instantons) is defined to be the quotient $\mathcal{A}_{min}/\mathcal{G}$, and has been studied in /Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978/ and in /Donaldson, S.K., 1983b/.

Given a suitably well-behaved functional $f : X \longrightarrow \mathbb{R}$ defined on a Banach manifold X one may relate the critical sets of f to the topology of X by means of either Morse theory or Lyusternik-Schnirelmann theory.

First we discuss Morse theory on Hilbert manifolds. Let $f : X \longrightarrow \mathbb{R}$ be a non-trivial C^3 function defined on a connected Hilbert manifold X and let $df : X \longrightarrow T^*X$ be the differential of f. A critical point of f is an element of X at which df vanishes. Suppose that f satisfies the following condition: given any subset S of X on which $|f|$ is bounded and $\|df\|$ is not bounded away from zero, there exists a critical point of f adherent to S. Then Palais and Smale have shown that the conclusions of Morse theory apply to the function f, relating the critical sets of f to the topology of the manifold X (see /Palais, R.S., 1963/). The above condition on f is referred to as the Palais-Smale condition. It ensures that if $\gamma : (a, b) \longrightarrow X$ is a maximal integral curve of the gradient vector field ∇f of f, where $-\infty \leq a < b \leq +\infty$, then either

$$\lim_{t \to b-} f(t) = +\infty$$

or there exists a sequence $(t_i \in (a, b) : i \in \mathbb{N})$ converging to b such that the sequence $(\gamma(t_i) : i \in \mathbb{N})$ converges to a critical point of f, and similarly when t converges to a from above. In particular the critical values of f are isolated and if c is a critical value of f then the set of critical points x satisfying $f(x) = c$ is compact (a critical value of f is the image under f of a critical point of f).

$f : X \longrightarrow \mathbb{R}$ is said to be a Morse function if and only if the critical set of f consists of isolated points and the Hessian of f at those critical points is nondegenerate. For all $c \in \mathbb{R}$ let

$$X_c = \{ x \in X : f(x) \leq c \} .$$

If $f : X \longrightarrow \mathbb{R}$ is a Morse function and if c is a critical value of f then for all sufficiently small $\varepsilon > 0$ the pair $(X_{c+\varepsilon}, X_{c-\varepsilon})$ is homotopy equivalent to a relative CW complex, where $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by attaching a cell of dimension k for each critical point in $f^{-1}(c)$ at which the index of the Hessian of f is k (see [Milnor, J.W., 1963] or [Palais, R.S., 1963]).

Suppose that H is a group acting on the Hilbert manifold X and that f is H-invariant. Then $f : X \longrightarrow \mathbb{R}$ will not in general be a Morse function, unless the critical points of f were fixed points for the action of H on X. However one may apply the equivariant Morse theory described in [Atiyah, M.F. and Bott, R., 1982].

One may also study the relationship between the topology of a topological space X and the critical sets of a continuous function $f : X \longrightarrow \mathbb{R}$ on this space by means of Lyusternik-Schnirelmann theory (see [Lyusternik, L.A., 1966]). Let (X, f, K) be a triple, where X is a topological space, $f : X \rightarrow \mathbb{R}$ is a continuous function and K is a closed subset of X. For all $c \in \mathbb{R}$ let

$$X_c = f^{-1} ((-\infty , c]).$$

We may apply the techniques of Lyusternik-Schnirelmann theory to (X, f, K) provided that the following three conditions are satisfied:

(i) f(K) is discrete,

(ii) if $c \in \mathbb{R} \setminus f(K)$ then $X_{c+\varepsilon}$ may be deformed into $X_{c-\varepsilon}$ for all sufficiently small $\varepsilon > 0$,

(iii) if $c \in f(K)$ then for every open neighbourhood U of $K \cap f^{-1}(c)$ there exists $\varepsilon > 0$ such that $X_{c+\varepsilon}$ may be deformed into $U \cup X_{c-\varepsilon}$.

We refer to K as the critical set of f and to $f(K)$ as the set of critical values of f.

In Lyusternik-Schnirelmann theory one proves the existence of one or more distinct critical values of $f : X \to \mathbb{R}$ from a knowledge of the topology of X. One method of doing this may be described as follows. Let (Y, B) be a topological pair, let $a \in \mathbb{R}$ and let $\Gamma \in [(Y, B), (X, X_a)]$ be a homotopy class of continuous maps $q : (Y, B) \to (X, X_a)$ with the property that $q(Y) \not\subset X_a$ for all $q \in \Gamma$. Define

$$c_\Gamma = \inf_{q \in \Gamma} \sup_{y \in Y} f(q(y)) .$$

Then c_Γ is a critical value of $f : X \to \mathbb{R}$. Indeed if c_Γ were not a critical value of f then there would exist a map $h : X_{c+\varepsilon} \to X_{c-\varepsilon}$ which was homotopic in $X_{c+\varepsilon}$ to the identity map of $X_{c+\varepsilon}$ for all sufficiently small $\varepsilon > 0$, by condition (ii) above. But by definition of c_Γ there would exist $q \in \Gamma$ such that $q(Y) \subset X_{c+\varepsilon}$. But then $h \circ q \in \Gamma$ and $h \circ q(Y) \subset X_{c-\varepsilon}$, contradicting the definition of c_Γ. Hence c_Γ is a critical value of f. Using this method Lyusternik and Fet proved the existence of at least one closed geodesic on a compact Riemannian manifold (see /Klingenberg, W., 1978/ of /Klingenberg, W., 1982/).

One may prove the existence of more than one critical point of $f : X \to \mathbb{R}$ using the concept of Lyusternik-Schnirelmann category. If A is a subset of X then the Lyusternik-Schnirelmann category of A in X, $\mathrm{cat}(A; X)$, is the least integer n such that A may be covered by n closed subsets of X, each of which is contractible in X. If no such integer exists then $\mathrm{cat}(A; X)$ is defined to be ∞. We denote $\mathrm{cat}(X; X)$ by $\mathrm{cat}(X)$.

For all $m \leq \mathrm{cat}(X)$, define

$$c_m(f) = \inf \{ a \in \mathbb{R} : \mathrm{cat}(X_a; X) \geq m \} .$$

It can be shown that

$$c_m(f) \leq c_{m+1}(f)$$

and that if $c_m(f) \in (-\infty, \infty)$ then $c_m(f)$ is a critical value of f. Also $f : X \to \mathbb{R}$ has at least $\mathrm{cat}(X)$ critical points. Indeed if the number of critical points of $f : X \to \mathbb{R}$ is finite then it may be shown that

$$c_m(f) < c_n(f)$$

for all m and n satisfying

$$1 \leq m < n \leq \mathrm{cat}(X)$$

(see [Palais, R.S., 1966]).

The Lyusternik-Schnirelmann category of a topological space may be related to the homology of that space. However one may deduce information about the critical point structure of $f : X \to \mathbb{R}$ directly from the homology of X without the need to introduce the concept of Lyusternik-Schnirelmann category.

Given $a \in \mathbb{R}$ define i_a and j_a to be the inclusions $i_a : X_a \hookrightarrow X$ and $j_a : X \hookrightarrow (X, X_a)$. i_a and j_a induce homomorphisms

$$i_{a*} : H_* (X_a) \to H_* (X),$$

$$j_{a*} : H_* (X) \to H_* (X, X_a),$$

and the kernel of j_{a*} is the image of i_{a*} by the homology exact sequence of the pair (X, X_a). Given $z \in H_*(X)$, define

$$c(z) = \inf \{ a \in \mathbb{R} : j_{a*} z = 0 \}$$

$$= \inf \{ a \in \mathbb{R} : z \in i_{a*} H_*(X_a) \} .$$

$c(z)$ is a critical value of $f : X \to \mathbb{R}$. For suppose $c(z)$ were not a critical value. Then there would exist a continuous map $h : X_{c+\varepsilon} \to X_{c-\varepsilon}$ homotopic to the identity map of $X_{c+\varepsilon}$ for all sufficiently small $\varepsilon > 0$, where $c = c(z)$. But then $z = i_{c+\varepsilon\,*} w$ for some $w \in H_*(X_{c+\varepsilon})$ and hence $z = i_{c-\varepsilon\,*} h_* w$, contradicting the definition of $c(z)$. Thus $c(z)$ is a critical value of f.

Using the homology exact sequences and the naturality of the cap product one may easily show that if $z \in H_*(X)$ and $\varphi \in H^*(X)$ then

$$c(\varphi \cap z) \leq c(z)$$

and that if equality holds then $k^*_U \varphi \neq 0$ for all open neighbourhoods U of $k \cap f^{-1}(c(z))$, where k_U denotes the inclusion $k_U : U \hookrightarrow X$.

One may also use variants of the methods described above. For example the proof of the Lyusternik-Schnirelmann theorem on the existence of at least three simple closed geodesics on a Riemannian manifold diffeomorphic to a 2-sphere given in [Klingenberg, W., 1982] does not fall strictly within the purview of the above methods though it is closely related to the homology method described above (see also [Ballmann, W., Thorbergsson, G. and Ziller, W., 1983]).

Given a C^{2-} function $f : X \to \mathbb{R}$ on a complete C^2 Finsler manifold X satisfying the Palais-Smale condition described above one may verify that (X, f, K) satisfies conditions (i), (ii) and (iii) above, where

$$K = \{ x \in X : df_x = 0 \}$$

(see /Palais, R.S., 1966/), and thus any of the above methods of Lyusternik-Schirelmann theory are applicable. However Lyusternik-Schnirelmann theory is more robust than Morse theory in that by verifying that conditions (i), (ii), and (iii) above are satisfied using methods other than by studying the flow of an approximate gradient field, one may apply the techniques of Lyusternik-Schnirelmann theory in situations where the Palais-Smale condition is not applicable (see /Klingenberg, W., 1982/).

Having summarized the basic methods of Morse theory and Lyusternik-Schnirelmann theory for relating the topology of a Banach manifold to the critical point structure of continuous functions defined on it we now indicate how one might apply these methods in situations where the function in question is constant along the orbits of the action of some infinite dimensional symmetry group. Let $f : X \to \mathbb{R}$ be such a function defined on the Banach manifold X and let f be constant on the orbits of the action of the Banach Lie group H acting on X. One would not in general expect to be able to verify the Palais-Smale condition for $f : X \to \mathbb{R}$. Indeed given a sequence $(x_i \in X : i \in \mathbb{N})$ for which $|f(x_i)|$ is bounded and the norm of df at x_i converges to zero then this sequence would not in general contain a convergent subsequence, for even if $(x_i \in X : i \in \mathbb{N})$ were to converge, one would expect to find a sequence $(h_i \in H : i \in \mathbb{N})$ such that $(x_i.h_i : i \in \mathbb{N})$ contains no convergent subsequence, yet $f(x_i.h_i)$ is bounded and the norm of df at $x_i.h_i$ converges to zero.

In order to overcome this problem it is necessary to factor out the action of the symmetry group H from the Banach manifold X. Let us suppose that H has a Banach Lie subgroup H_0 of finite codimension in H which acts freely on X and such that $\bar{H}$ is compact, where $\bar{H} = H/H_0$.

Define $\bar{X} = X/H_o$. In order to apply critical point theory to the function $\bar{f} : \bar{X} \rightarrow \mathbb{R}$ induced by $f : X \rightarrow \mathbb{R}$ on $\bar{X}$, one would aim to prove a slice theorem which would state that $\bar{X}$ admits a unique differentiable structure with the property that the natural projection from X onto $\bar{X}$ is smooth and admits smooth local sections. Then one has to reduce the problem to one of studying the behaviour of $\bar{f} : \bar{X} \rightarrow \mathbb{R}$, where $\bar{f}$ is constant along the orbits of the action of the compact Lie group $\bar{H}$. In these circumstances one has more hope of being able to verify the Palais-Smale condition for the function $f : \bar{X} \rightarrow \mathbb{R}$. Moreover if X satisfies the second axiom of countability then X and $\bar{X}$ will be paracompact. Then the natural projection $X \rightarrow \bar{X}$ will be a principal bundle with fibre H_o over a paracompact Hausdorff base space $\bar{X}$ and thus will be a Hurewicz fibration (see [Spanier, E.H., 1966; pp.92-96]). Thus the homotopy groups of H_o, X and $\bar{X}$ are related by the homotopy exact sequence

$$\ldots \rightarrow \pi_i(H_o) \rightarrow \pi_i(X) \rightarrow \pi_i(\bar{X}) \rightarrow \pi_{i-1}(H_o) \rightarrow \ldots$$

of the fibration, and the relationship between the homology of H_o, X and $\bar{X}$ may be studied using the Serre spectral sequence.

Applying these remarks in the context of Yang-Mills theory we see that it is sensible to consider the Yang-Mills functional as a smooth map

$$YM : L_1^2\mathcal{A} / L_2^2\mathcal{G}^m \rightarrow \mathbb{R},$$

where $L_1^2\mathcal{A}$ is the space of L_1^2 connections on a principal bundle $\pi : P \rightarrow M$ over a compact Riemannian manifold M with structural group G, and where $L_2^2\mathcal{G}^m$ is the group of L_2^2 principal bundle automorphisms of $\pi : P \rightarrow M$ which fix the fibre of $\pi : P \rightarrow M$ over m, for some $m \in M$. $L_2^2\mathcal{G}^m$ is a well-defined Banach Lie group acting smoothly on $L_1^2\mathcal{A}$ when the dimension of M does not exceed 3, and in /Uhlenbeck, K.K., 1982c/ it is conjectured that the Palais-Smale condition is satisfied in these circumstances.

However interesting problems in geometry occur in circumstances where the Palais-Smale condition just fails to apply (see /Uhlenbeck, K.K., 1982c/), notably in the study of harmonic maps whose domain is a compact 2-dimensional surface and also in the study of the Yang-Mills functional for principal bundles over a compact 4-dimensional manifold. In the theory of harmonic maps whose domain is a 2-dimensional surface interesting results may be obtained by perturbing the functional in question to nearby functionals which satisfy the Palais-Smale condition on the appropriate Banach manifold (see /Sacks, J. and Uhlenbeck, K.K., 1981/). By analogy this suggests that, to obtain results for the Yang-Mills functional for connections on a principal bundle over a 4-manifold, it might be fruitful to study the functional

$$YM_p([\omega]) = \int (1 + |F^{\omega}|^2)^{p/2} \, d(vol)$$

defined on the Banach manifold $L_1^p \mathcal{A} / L_2^p \mathcal{G}^m$ for $p > 2$, where F^{ω} is the curvature of the connection ω .

Gauge Theories

Non-Abelian gauge theories were introduced by Yang and Mills in [Mills, R.L. and Yang, C.N., 1954] as a generalization of Maxwell's theory of electromagnetism. We recall that the electric and magnetic fields on spacetime are described by a 2-form F satisfying Maxwell's equations

$$dF = 0$$

$$\delta F = J$$

where J is some constant multiple of the current density, considered as a 1-form on spacetime, where d is the exterior derivative operator and where δ is the codifferential determined by the metric on spacetime. Since $dF = 0$ there exists a 1-form Λ such that

$$F = d\Lambda$$

by the Poincaré lemma. The 1-form Λ is often referred to as a 4-potential of F. This 1-form Λ is not unique. Indeed if ψ is a smooth function on spacetime, then $\Lambda + d\psi$ is also a 4-potential of F. The correspondence sending Λ to $\Lambda + d\psi$ is referred to as a gauge transformation (this terminology arose from Weyl's attempt to unify gravitation and electromagnetism in a single theory in which the length of a measuring rod in spacetime would change under parallel transport around closed loops in spacetime). It became customary to 'fix a gauge' by demanding that Λ also satisfy the condition

$$\delta \Lambda = 0$$

since if $\Lambda + d\psi$ also satisfied this condition then ψ would have to be harmonic, and thus if ψ satisfied appropriate boundary conditions at infinity then ψ would have to be constant. The condition that the divergence of Λ vanish is often referred to as the Lorentz gauge condition. If A satisfies the Lorentz gauge condition then Maxwell's equations become

$$F = dA,$$

$$\Delta A = J$$

where Δ is the Hodge-de Rham Laplacian acting on 1-forms, defined by

$$\Delta = \delta \circ d + d \circ \delta .$$

Thus

$$-\nabla^2 A + \mathrm{Ric} \bullet A = J$$

using the Bochner-Weizenböck formula, where $-\nabla^2$ is the rough Laplacian acting on 1-forms and where Ric is the symmetric endomorphism determined by the Ricci curvature of spacetime.

The vacumm Maxwell equations are the Euler-Lagrange equations for the action

$$I(A) = \int |dA|^2 \, d(\mathrm{vol}).$$

Yang and Mills introduced a non-Abelian gauge theory with many similarities to the theory of electromagnetism just described. In this theory the gauge potentials are 1-forms A on $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of some compact Lie group G. Yang and Mills consider the case when G is SU (2). The group G is referred to by physicists as the gauge group. Corresponding to the gauge potential A we have a covariant derivative operator D. If V is a representation space for G and if $f : \mathbb{R}^4 \longrightarrow V$ is differentiable then

$$Df = df + A.f.$$

The appropriate analogue of the electromagnetic field tensor is the field strength F. F is a $\mathfrak{g}$-valued 2-form whose components F are given by

$$F_{\mu\nu} = [D_\mu, D_\nu]$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Yang-Mills equation is

$$\delta^A F = 0$$

where δ^A, the covariant codifferential, is the formal adjoint of the covariant exterior derivative. The Yang-Mills equation is the Euler-Lagrange equation of the Yang-Mills functional

$$YM(A) = \int |F|^2 \, d(vol).$$

Given a map $g : \mathbb{R}^4 \to G$, g determines a gauge transformation sending the covariant derivative operator $f \mapsto Df$ to the operator $f \mapsto D^g f$, where

$$D^g f = g^{-1} D(gf).$$

If $D^g = d + A^g$, then

$$A^g = g^{-1} Ag + g^{-1} dg.$$

The field strength F transforms to $g^{-1} Fg$ under the gauge transformation.

For an account of non-Abelian gauge theories from the physicist's point of view, see [Taylor, J.C., 1976] or chapter 12 of [Itzykson, C. and Zuber, J.-B., 1980].

Physicists study gauge theories both on Minkowski spacetime and on four-dimensional Euclidean spacetime. We shall here be concerned exclusively with the Euclidean case and its generalization to gauge theories on Riemannian manifolds, since we wish to apply the theory of elliptic partial differential equations.

Yang and Mills originally proposed their theory as a possible model for the isospin symmetry between protons and neutrons in elementary particle physics. In the standard theory an isospin 'rotation', determined by an element of SU (2), would 'rotate' all protons in the universe to the appropriate linear combination of proton and neutron eigenstates, and the relative proportion of the proton component and the neutron component of the dynamical state of the 'rotated' particle would be the same for all protons in the universe. Yang and Mills wished to construct a theory of isospin which permitted symmetries which might 'rotate' a proton into a neutron at one event in spacetime yet which fixed a proton at some other event. Such a symmetry would be determined by a map from spacetime to the isospin group SU (2). However gauge theories found their application not in this context but in the context of unified field theories of the forces of nature, once it was shown that gauge theories were renormalizable and once spontaneous symmetry breaking had been introduced into the theory via the Higgs mechanism. Those theories currently regarded as standard include the Salam-Weinberg unification of the electromagnetic and weak interactions, and also quantum chromodynamics, which is the theory of the strong interaction in which quarks interact via the exchange of gluons.

Physicists imposed the appropriate analogue of the Lorentz gauge condition, namely the condition

$$\sum_{\mu} \partial^{\mu} A_{\mu} = 0$$

or their gauge potentials on the assumption that this would determine a unique gauge potential from each orbit of the group of gauge transformations. That this was not the case was pointed out in /Gribov, V.N., 1978/ in the case where the gauge potentials satisfied appropriate boundary conditions at infinity. An explanation of why this had to be so was given in /Singer, I.M., 1978/.

Singer observes that the boundary conditions at infinity imposed by Gribov are such as to enable one to extend the gauge transformations to the compactification S^4 of $\mathbb{R}^4$. The gauge potentials studied by physicists correspond to connections defined on a principal bundle $\pi : P \to M$ over the manifold M being considered (in this case $M = S^4$). Similarly the gauge transformations introduced by physicists correspond to principal bundle automorphisms of $\pi : P \to M$. Let $C^\infty\mathcal{A}$ denote the Frechet space of smooth connections on $\pi : P \to M$ (strictly speaking this is an affine space modelled on a Frechet space) and let $C^\infty\mathcal{G}$ be the group of smooth principal bundle automorphisms of $\pi : P \to M$. $C^\infty\mathcal{G}$ acts on $C^\infty\mathcal{A}$ on the right where each principal bundle automorphism in $C^\infty\mathcal{G}$ acts on $C^\infty\mathcal{A}$ by sending each connection on $\pi : P \to M$ to its pullback under the given automorphism. What to the physicist is a choice of gauge condition corresponds to the construction of a section of the natural projection

$$C^\infty\mathcal{A} \to C^\infty\mathcal{A}/C^\infty\mathcal{G} .$$

Let $C^\infty\mathcal{A}_{irr}$ denote the open dense subset of $C^\infty\mathcal{A}$ consisting of all smooth irreducible connections on $\pi : P \to M$. Let $C^\infty\mathcal{G}_o$ be the quotient $C^\infty\mathcal{G}/Z(G)$ of $C^\infty\mathcal{G}$ by the subgroup naturally isomorphic to the centre of the structural group G. Then $C^\infty\mathcal{G}_o$ acts continuously on $C^\infty\mathcal{A}_{irr}$. Singer states that the map

$$\nu : C^\infty\mathcal{A}_{irr} \to C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o$$

is a principal bundle with structural group $C^\infty\mathcal{G}_o$. Singer also shows that

$$\pi_j(C^\infty\mathcal{A}_{irr}) = 0$$

for all non-negative integers j. Thus

$$\pi_{j+1}(C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o) \cong \pi_j(C^\infty\mathcal{G}_o).$$

Now if the map ν above has a section, then the identity automorphism of $\pi_{j+1}(C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o)$ factors through the zero homomorphism and thus

$$\pi_{j+1}(C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o) = 0.$$

Singer shows by standard methods of homotopy theory that if $M = S^4$ or S^3 and if $G = SU(N)$ for some $N > 1$ then $\pi_j(C^\infty\mathcal{G}_o) \neq 0$ for some j and hence no continuous choice of gauge exists in the sense that there is no continuous section

$$s : C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o \rightarrow C^\infty\mathcal{A}_{irr}$$

of the natural projection

$$\nu : C^\infty\mathcal{A}_{irr} \rightarrow C^\infty\mathcal{A}_{irr}/C^\infty\mathcal{G}_o .$$

The slice theorem stated in /Singer, I.M., 1978/ was proved in /Narasimham, M.S. and Ramadas, T.R., 1979/ and in /Mitter, P.K. and Viallet, C.M., 1981/. Narasimhan and Ramadas restrict their attention to SU(2) gauge fields over S^3 and prove theorems for the actions of L^2_{k+1} principal bundle automorphisms on L^2_k connections for $k \geqslant 3$. Mitter and Viallet prove slice theorems for the action of L^2_{k+1} principal bundle automorphisms on L^2_k connections where

$$k > \frac{n}{2} + 1 ,$$

n being the dimension of the base manifold of the principal bundle.

A connection on a principal bundle over a four-dimensional manifold is an instanton (or anti-instanton) if and only if the curvature of the connection is self-dual (or anti self-dual). Instantons or anti-instantons attain the minimum of the Yang-Mills functional, provided that they exist. The moduli space of instantons is defined to be the quotient $C^\infty\mathcal{A}_{min}/C^\infty\mathcal{G}$ of the Banach manifold $C^\infty\mathcal{A}_{min}$ of instantons by the group $C^\infty\mathcal{G}$ of principal bundle automorphisms. In /Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978/

it is shown that the moduli space of irreducible instantons over a compact (self-dual) half conformally flat 4-manifold with positive scalar curvature is either empty or is a manifold of dimension

$$p_1(\mathfrak{g}_P) - \tfrac{1}{2}(\dim G)(\chi - \tau)$$

where $p_1(\mathfrak{g}_P)$ is the first Pontryagin number of the adjoint bundle $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$, χ is the Euler characteristic of the base manifold and τ is its signature.

In general it is known that the moduli space of all instantons over a 4-manifold (not necessarily half conformally flat) will have singularities, though the regular set will have the dimension given above. This dimension is calculated using the Atiyah-Singer index theorem.

All instantons on S^4 have been classified using methods of algebraic geometry applied via twistor theory (see /Atiyah, M.F., Hitchin, N.J., Drinfeld, U. and Manin, Yu., 1977/ and /Atiyah, M.F., 1979/).

Bourguignon, Lawson and Simons have proved stability, isolation and non-existence theorems for Yang-Mills fields on compact homogeneous Riemannian manifolds (see /Bourguignon, J.-P. and Lawson, H.B., 1980/ or /Bourguignon, J.-P. and Lawson, H.B., 1982/).

Taubes has proved an existence theorem for instantons on compact Riemannian 4-manifolds whose intersection form is positive definite (see /Taubes, C.H., 1982/).

Uhlenbeck has provided various analytical tools that are useful in the study of connections whose curvature is bounded in some appropriate norm. In /Uhlenbeck, K.K., 1982b/ it is shown that there exist constants κ and c, depending only on n, such that if $\tfrac{1}{2}n \leq p < n$ and if $d + A$ is an L_1^p connection on a trivial bundle over the unit ball B^n in $\mathbb{R}^n$ whose curvature F(A) satisfies

$$\| F(A) \|_{L^p} < \varkappa$$

then d + A is gauge equivalent to a connection d + A satisfying the conditions

$$\delta A = 0 ,$$

$$\| A \|_{L_1^p} \leq c \, \| F(A) \|_{L^p} .$$

Using this result Uhlenbeck shows that if 2p > n and if $(\omega_i \,;\, i \in \mathbb{N})$ is a sequence of connections on a principal bundle $\pi : P \to M$ over a compact Riemannian manifold M of dimension n with compact structural group and if the L^p norms of the curvatures of (ω_i) are uniformly bounded then there exists a sequence $(\Psi_i : i \in \mathbb{N})$ of L_2^p gauge transformations such that the sequence $(\Psi_i{}^* \omega_i : i \in \mathbb{N})$ is weakly convergent in the space of L_1^p connections to some connection ω. Moreover the curvature F^ω of ω satisfies

$$\| F^\omega \|_{L^p} \leq \lim \sup \| F^{\omega_i} \|_{L^p} .$$

A similar theorem may be proved in the limiting case when 2p = n, though here one finds that the sequence of connections will converge weakly only over the complement of some finite set of points in the base manifold of the principal bundle (see [Sedlacek, S., 1982] and [Donaldson, S.K., 1983b]). For Yang-Mills connections on 4-manifolds one may then extend this limiting connection to a connection on some principal bundle defined over the whole of M using Uhlenbeck's removal of signularities theorem (discussed below), though the topological type of this new bundle may differ from that of the principal bundle on which the original sequence of connections was defined.

Uhlenbeck's removal of signularities theorem states that if a connection on a principal bundle over $B^4 \setminus \{0\}$ satisfies the Yang-Mills equation and if the curvature of the connection is bounded on $B^4 \setminus \{0\}$, in the L^2 norm, then the principal bundle and the connection may be extended over the whole of B^4 (see [Uhlenbeck, K.K., 1982a]).

Uhlenbeck's results have been extended to Yang-Mills-Higgs systems by Parker (see [Parker, T., 1982]). Parker also proves slice theorems in Sobolev L_k^p norms for $2 < p < 4$.

Donaldson has made a study of the topology of the moduli space of instantons introduced by Atiyah, Hitchin and Singer using the analytical tools developed by Taubes and Uhlenbeck. In consequence he was able to prove his celebrated theorem that if the intersection form of a smooth 4-manifold is positive definite then it is a sum of squares.

Atiyah and Bott have made a study of the Morse theory for the Yang-Mills functional for connections on a bundle over a Riemann surface (see [Atiyah, M.F. and Bott, R., 1982]. Donaldson has used the weak compactness theorem of Uhlenbeck in giving a differential geometric characterization of stable bundles over projective algebraic varieties (see [Donaldson, S.K., 1983a] and [Donaldson, S.K., 1985].

Plan of the Thesis

In chapter II we review the definitions and basic properties of Sobolev and Hölder spaces. We also discuss slice theorems in the general context of a Banach Lie group acting smoothly and freely on a Banach manifold.

In chapter III we prove general elliptic regularity theorems in Sobolev and Hölder norms for linear elliptic differential operators with smooth coefficients defined over a compact smooth manifold. The proof uses the theory of singular integrals, developed by Calderon and Zygmund, and the theory of pseudodifferential operators.

In chapter IV we shall prove an inequality satisfied by continuous functions on a compact manifold which is closely related to the Sobolev embedding theorem for embeddings of Sobolev spaces in Hölder spaces.

In chapter V we give an account of the theory of Ehresmann connections on principal bundles and of principal bundle automorphisms in preparation for subsequent chapters.

In chapter VI we study the action of Banach Lie groups of principal bundle automorphisms of connections and prove various results that will be used in chapter VIII, where we prove slice theorems for this action.

In chapter VII we produce a generalization of the analytical aspects of Hodge theory which is applicable to covariant Hodge-de Rham Laplacians with respect to connections that need not be smooth. This chapter uses the general elliptic regularity theorems of chapter III, together with the Sobolev embedding theorems, the Rellich-Kondrakov theorem and the Sobolev multiplication theorems.

In chapter VIII we prove slice theorems in Sobolev and Hölder norms for the action of principal bundle automorphisms on connections, using the results of chapters VI and VII. We shall also prove regularity theorems for Yang-Mills connections and discuss the regularity of Yang-Mills-Higgs systems.

In chapter IX we shall show the existence of an upper bound on the length of loops required to generate the holonomy group of a principal bundle over a compact Riemannian manifold. We shall show how this result can be used to derive inequalities satisfied by sections of a fibre bundle associated to the given principal bundle.

The Relationship of the Results to Published Material

I give here a discussion of the sources from which the research contained in this work has been derived.

Chapter II contains no original research, being a summary of the basic theorems of global analysis that we shall be using. However I have not come across the general slice theorem (theorem II.3.1) in the literature in the form in which I have stated it, though it is implicit in the proofs of slice theorems occurring in the literature and it is stated in the more abstract formulation given here mainly for reasons of economy (for not only do we need theorem II.3.1 in proving theorem VIII.2.3 but also in section 4 of chapter VI in forming the quotients of the groups of principal bundle automorphisms by the centre of the structural group).

Sections 2 and 3 of chapter III do not contain any original research, being summaries of the results of Calderon and Zygmund and of Hörmander on which the proofs of the elliptic regularity theorems are based. A partial exception to this is the proof that smoothing operators are pseudodifferential operators in the sense of Hörmander, which we prove using the methods of Hörmander. Section 4 of chapter III is original research, at least I have not yet come across a proof of L_k^p and Hölder estimates for pseudo-differential operators in the literature which employs this method. The elliptic regularity results of section 5 do not appear to be stated explicitly in the literature; their proofs are immediate generalizations to the L_k^p and Hölder cases of standard results in the L_k^2 case obtained by merely replacing the standard L_k^2 estimates for pseudodifferential operators by the results of section 4 at the appropriate steps in the proofs.

Chapter IV consists of original research. The proof of theorem IV.3.3 was suggested by the ideas underlying the proof of the Sobolev embedding theorem for embeddings of Sobolev spaces in Hölder spaces.

Chapter V is basically an expanded and freely adapted account of the theory of Ehresmann connections and principal bundle automorphisms based on the papers by Bourguignon and Lawson and by Atiyah, Hitchin and Singer listed in the references at the end of chapter V. Any result not found in these papers may be taken to be 'original research', though many of these results are either 'obvious' or 'well-known'. Note however that theorem V.4.2 is stated as lemma 2.2 of [Narasimhan, M.S. and Ramadas, T.R., 1979].

Chapter VI is original research, apart from theorems VI.2.1 and VI.2.2 which are stated in the Sobolev case as lemma 1.2 of [Uhlenbeck, K.K., 1982b] and there proved when $k = 0$ or 1. The differences between the proofs given in chapter VI and the proofs given by Uhlenbeck are essentially cosmetic in nature.

Chapter VII consists of original research.

Chapter VIII contains original research. The slice theorem (theorem VIII.2.3) generalizes theorems stated or proved in [Singer, I.M., 1978], [Narasimhan, M.S. and Ramadas, T.R., 1979], [Mitter, P.K. and Viallet, C.M., 1981] and [Parker, T., 1982]. Of these authors, only Parker proves his results in Sobolev spaces other than L^2_k spaces. The regularity theorems for Yang-Mills connections in section 3 and the corresponding results for Yang-Mills-Higgs systems discussed in section 4 generalize results of [Uhlenbeck, K.K., 1982b] and [Parker, T., 1982].

Chapter IX consists of original research.

Appendix A contains no original research.

References for Chapter I

Atiyah, M.F., "Geometry of Yang-Mills fields", Fermi Lectures, Scuola Normale Pisa, 1979.

Atiyah, M.F. and Bott, R., "The Yang-Mills equations over Riemann surfaces", Phil. Trans. Roy. Soc. Lond. A308 (1982), pp.524-615.

Atiyah, M.F., Drinfeld, U., Hitchin, N.J. and Manin, Yu., "Construction of instantons", Proc. Nat. Acad. Sci. U.S.A. 74 (1977), pp.2662-2663.

Atiyah, M.F., Hitchin, N.J. and Singer, I.M., "Self-duality in four-dimensional Riemannian geometry", Proc. R. Soc. Lond. A362 (1978), pp.425-461.

Ballmann, W., Thorbergsson, G., Ziller, W., "On the existence of short closed geodesics and their stability properties", in "Seminar on minimal submanifolds", Ann. of Math. Studies, 103, Prirceton Univ. Press, Princeton N.J., 1982.

Bourguignon, J.-P., "Une stratification de l'espace des structures Riemannians", Comp. Math., 30 (1975), pp.193-209.

Bourguignon, J.-P. and Lawson, H.B., "Stability and isolation phenomena for Yang-Mills fields", Comm. Math. Phys. 79 (1980), pp.189-230.

Bourguignon, J.-P. and Lawson, H.B., "Yang-Mills theory: its physical origins and differential geometric aspects", in "Seminar on differential geometry", Ann. of Math. Studies, 102, Princeton Univ. Press, Princeton N.J., 1982.

Donaldson, S.K., "A new proof of a theorem of Narasimhan and Seshadri", J. Diff. Geom. 18 (1983), pp.269-277.

Donaldson, S.K., "An application of gauge theory to four-dimensional topology", J. Diff. Geom. 18 (1983), pp.279-315.

Donaldson, S.K., "Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles", Proc. Lond. Math. Soc. 50 (1985), pp.1-26.

Earle, C.J. and Eells, J., "A fibre bundle description of Teichmüller theory", J. Diff. Geom., 3 (1969), pp.19-43.

Ebin, D.G., "The manifold of Riemannian metrics", in "Global Analysis", Proc. Sympos. Pure Math., Vol.15, Amer. Math. Soc., Providence R.I., 1970.

Fischer, A.E. amd Marsden, J.E., "The manifold of conformally equivalent metrics", Can. J. Math., 29 (1977), pp.193-209.

Freed, D.S. and Uhlenbeck, K.K., Instantons and four-manifolds, Springer-Verlag, New York, 1984.

Gribov, V.N., "Quantization of non-abelian gauge theories", Nucl. Phys., B139 (1978), pp.1-19.

Itzykson, C. and Zuber, J.-B., Quantum field theory, McGraw-Hill, 1980.

Klingenberg, W., Lectures on closed geodesics, Springer-Verlag, Berlin, 1978.

Klingenberg, W., Riemannian geometry, de Gruyter, Berlin, 1982.

Lyusternik, L.A., "The topology of the calculus of variations in the large", Translations of Russian monographs Vol.16. Amer. Math. Soc., Providence R.I., 1966.

Mills, R.L. and Yang, C.N., "Conservation of isotopic spin and isotopic gauge invariance", Phys. Rev., 96 (1954), pp.191.195.

Milnor, J.W., "Morse theory", Ann. of Math. Studies, No.51, Princeton Univ. Press, Princeton N.J., 1963.

Mitter, P.K. and Viallet, C.M., "On the bundle of connections and the gauge orbit manifold in Yang-Mills theory", Comm. Math. Phys. , 79 (1981), pp.457-472.

Narasimhan, M.S. and Ramadas, T.R., "Geometry of SU(2) gauge fields", Comm. Math. Phys., 67 (1979), pp.121-136.

Palais, R.S., "Morse theory on Hilbert manifolds", Topology, 2 (1963), pp.299-340.

Palais, R.S., "Lusternik-Schnirelman theory of Banach manifolds", Topology, 5 (1966), pp.115-132.

Parker, T., "Gauge theories on 4-dimensional Riemannian manifolds", Comm. Math. Phys., 85 (1982), pp.563-602.

Sacks, J. and Uhlenbeck, K.K., "The existence of minimal immersions of 2-spheres", Ann. of Math., 113 (1981), pp.1-24.

Sedlacek, S., "A direct method for minimizing the Yang-Mills functional", Comm. Math. Phys., 86 (1982), pp.515-527.

Singer, I.M., "Some remarks on the Gribov ambiguity", Comm. Math. Phys., 60 (1978), pp.7-12.

Spanier, E.H., Algebraic topology, McGraw-Hill, 1966.

Taubes, C.H., "The existence of self-dual connections on non self-dual 4-manifolds", J. Diff. Geom., 17 (1982), pp.139-170.

Taylor, J.C., Gauge theories of weak interactions, Cambridge Univ. Press, Cambridge, 1976.

Uhlenbeck, K.K., "Removeable singularities in Yang-Mills fields", Comm. Math. Phys., 83 (1982), pp.11-30.

Uhlenbeck, K.K., "Connections with L^p bounds on curvature", Comm. Math. Phys., 83 (1982), pp.31-42.

Uhlenbeck, K.K., "Variational problems for gauge fields" in "Seminar on differential geometry", Ann. of Math. Studies No.102, Princeton Univ. Press, Princeton N.J., 1982.

Chapter II

BASIC RESULTS OF GLOBAL ANALYSIS

§1. Introduction

In this chapter, we give an account of the basic results of global analysis which we shall be using.

In §2 we define the Sobolev spaces and Hölder spaces of sections of a smooth vector bundle over a compact smooth manifold. We list some of their important properties. In particular we state the Sobolev embedding theorem (theorem 2.1), the Rellich-Kondrakov theorem (theorem 2.3) and the Sobolev multiplication theorems (theorem 2.4). Some sources in the literature give only restricted versions of the Sobolev multiplication theorems, such as the result that $L^p_k(\Omega)$ is a Banach algebra when Ω is a bounded domain in $\mathbb{R}^n$ and $pk > n$. Other sources (for example /Palais, R.S., 1968; chapter 9/) give very general statements of these theorems. The statement of theorem 2.4 is an attempt to strike a balance by stating a theorem which is sufficiently general for the applications which we shall make of it, yet which is not so general as to be difficult to remember and apply. We conclude §2 with a statement of the results proved in /Palais, R.S., 1968/ which give sufficient conditions for one to be able to define Banach manifolds of sections of a smooth fibre bundle over a compact manifold modelled on Sobolev and Hölder spaces (theorems 2.5 and 2.6). We present also a simple corollary (corollary 2.7) of theorem 2.6.

Palais proves these results in a more general setting. Let $\mathfrak{M}$ be a functor which associates to every smooth vector bundle $E \rightarrow M$ over a compact n-dimensional manifold M a complete normable topological vector space $\mathfrak{M}(E)$ of continuous sections of $E \rightarrow M$ satisfying the following two axioms:

(i) if M and N are compact smooth n-dimensional manifolds, if $\varphi : M \longrightarrow N$ is a diffeomorphism of M into N and if $E \longrightarrow N$ is a vector bundle over N then the map sending s to $s \circ \varphi$ defines a continuous linear map from $\mathfrak{M}(E)$ into $\mathfrak{M}(\varphi^* E)$,

(ii) if $E_1 \longrightarrow M$ and $E_2 \longrightarrow M$ are smooth vector bundles over a compact smooth n-dimensional manifold M and if $f : E_1 \longrightarrow E_2$ is a smooth fibre preserving map then the induced map from $\mathfrak{M}(E_1)$ to $\mathfrak{M}(E_2)$ is continuous.

Palais shows that any functor $\mathfrak{M}$ satisfying these two axioms extends to a unique functor which associates to any smooth fibre bundle $B \longrightarrow M$ a Banach manifold $\mathfrak{M}(B)$ of sections of $B \longrightarrow M$ and which associates to any smooth fibre preserving map between fibre bundles a smooth map between Banach manifolds. Palais shows that the functors C^k, $C^{k,\alpha}$ and L^p_k satisfy axioms (i) and (ii) for all non-negative integers k and for all α and p satisfying the conditions $0 < \alpha < 1$, $1 \leq p < \infty$, $pk > n$.

In §3 we consider Banach Lie groups H acting smoothly and freely on Banach manifolds X. In theorem 3.1 we give necessary and sufficient conditions for the existence of a unique differentiable structure on X/H with the property that the projection map $X \longrightarrow X/H$ is smooth and admits smooth local sections. We observe that these conditions are automatically satisfied when H is compact (corollary 3.2). In particular if H is a compact normal subgroup of a Banach Lie group then G/H is a Banach Lie group (corollary 3.3).

§2. Sobolev and Hölder Spaces

In this section we shall define Sobolev and Hölder spaces and review some of their basic properties.

Given a domain $U \subset \mathbb{R}^n$, given a non-negative integer k and given $p \in [1, \infty)$, the Sobolev space $L^p_k(U)$ is defined to be the Banach space consisting of all functions $f : U \to \mathbb{R}$ with the property that, for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying $|\alpha| \leq k$, $\partial^\alpha f$ belongs to $L^p(U)$, where

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} .$$

The norm $\| . \|_{p,k}$ on $L^p_k(U)$ is given by

$$\| f \|_{p,k} = \left(\sum_{|\alpha| < k} \int_U | \partial^\alpha f |^p \, d\mu \right)^{\frac{1}{p}}$$

where μ is Lebesque measure on $\mathbb{R}^n$.

If $\alpha \in (0,1)$, we define the space $C^k(U)$ to be the Banach space of all functions $f : U \to \mathbb{R}$ whose partial derivatives of order not exceeding k are continuous, and we define the Hölder space $C^{k,\alpha}(U)$ to be the Banach space of all functions $f : U \to \mathbb{R}$ all of whose partial derivatives of order not exceeding k are continuous and satisfy a Holder condition of order α. Norms $\| . \|_k$ and $\| . \|_{k,\alpha}$ on $C^k(U)$ and $C^{k,\alpha}(U)$ may be taken to be

$$\| f \|_k = \sum_{|\beta| \leq k} \sup_{x \in U} | \partial^\beta f |$$

and

$$\| f \|_{k,\alpha} = \| f \|_k + \sum_{|\beta| \leq k} \sup_{x,y \in U} \frac{| \partial^\beta f(x) - \partial^\beta f(y) |}{| x - y |^\alpha} .$$

Let M be a compact smooth manifold and let $\pi : E \to M$ be a smooth vector bundle over M. The Sobolev spaces $L^p_k(E)$, the spaces

$C^k(E)$ and the Hölder spaces $C^{k,\alpha}(E)$ are the Banach spaces of sections $s : M \longrightarrow E$ of $\pi : E \longrightarrow M$ with the following property: for all smooth charts $\varphi : U \rightarrow \mathbb{R}^n$, for all smooth functions $f : M \longrightarrow \mathbb{R}$ with compact support contained in U, and for all smooth sections $\sigma : M \rightarrow E^*$ of the dual bundle $E^* \longrightarrow M$ of E, the composition

$$(f \langle \sigma , s \rangle) \circ \varphi^{-1}$$

belongs to $L^p_k(\varphi(U))$, $C^k(\varphi(U))$ or $C^{k,\alpha}(\varphi(U))$ respectively. Using the fact that M is compact, it can be shown that $L^p_k(E)$, $C^k(E)$ and $C^{k,\alpha}(E)$ are complete normable topological vector spaces together with norms that are well-defined up to equivalence of norms (see [Palais, R.S., 1968]).

If k is a non-negative integer and if $p \in (1, \infty)$ (i.e. we exclude the cases $p = 1$ and $p = \infty$), then $L^p_k(E)$ is a reflexive Banach space (see [Adams, R.A., 1975; p.47]), and if p' is the exponent conjugate to p, defined by the condition

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

then $L^{p'}(E^*)$ is the dual space of $L^p(E)$. If $k \in \mathbb{Z}$ and $k < 0$ we define $L^p_k(E)$ to be the dual space of $L^{p'}_{-k}(E^*)$.

The space $C^\infty(E)$ of smooth sections of $\pi : E \rightarrow M$ is dense in $L^p_k(E)$ for all $p \in (1, \infty)$ and $k \in \mathbb{Z}$, and in $L^1_k(E)$, $C^k(E)$ and $C^{k,\alpha}(E)$ for all $k \in \mathbb{Z}$ satisfying $k \geqslant 0$ and for all $\alpha \in (0, 1)$ (see [Palais, R.S., 1968; pp.24-25]).

There are various embeddings amongst the Sobolev spaces, C^k spaces and Hölder spaces. These are given by the Sobolev embedding theorem.

<u>Theorem 2.1</u> (Sobolev Embedding Theorem)

Let $\pi : E \longrightarrow M$ be a smooth vector bundle over a compact smooth manifold M of dimension n. Let $p, q \in [1, \infty)$, let $k, l \in \mathbb{Z}$ and let $\alpha \in (0, 1)$. Then

(i) if $l \leq k$ and if

$$\frac{1}{q} - \frac{1}{n} \geq \frac{1}{p} - \frac{k}{n}$$

then we have a continuous embedding

$$L^p_k(E) \hookrightarrow L^q_l(E)$$

(where $k \geq 0$ if $p = 1$ and $l \geq 0$ if $q = 1$),

(ii) if $k, l \geq 0$ and if

$$l < k - \frac{n}{p}$$

then we have a continuous embedding

$$L^p_k(E) \hookrightarrow C^l(E)$$

(iii) if $k, l \geq 0$, if $\alpha \in (0,1)$ and if

$$l + \alpha \leq k - \frac{n}{p}$$

then we have a continuous embedding

$$L^p_k(E) \hookrightarrow C^{l,\alpha}(E).$$

Proof

See [Aubin, T., 1982; chapter 2] or [Adams, R.A., 1975; chapter 5].

□

A map between Banach spaces is said to be compact if it maps bounded sets to sets with compact closure. The following theorem is a corollary of the Ascoli-Arzela theorem.

Theorem 2.2

Let $\pi : E \longrightarrow M$ be a smooth vector bundle over a compact smooth manifold M. Let k and l be non-negative integers and let $\alpha, \beta \in (0,1)$. Suppose that $l + \beta < k + \alpha$. Then the embeddings

$$C^{k,\alpha}(E) \hookrightarrow C^{l,\beta}(E)$$

$$C^{k,\alpha}(E) \hookrightarrow C^{l}(E)$$

$$C^{k}(E) \hookrightarrow C^{l,\beta}(E)$$

$$C^{k}(E) \hookrightarrow C^{l}(E)$$

are compact.

Proof

See [Adams, R.A., 1975; p.117].

□

Theorem 2.3 (Rellich-Kondrakov)

Let $\pi : E \longrightarrow M$ be a smooth vector bundle over a compact smooth manifold M of dimension n. Let $p,q \in [1,\infty)$, let $k,l \in \mathbb{Z}$ and let $\alpha \in (0,1)$. Then

(i) if $l < k$ and if

$$\frac{1}{q} - \frac{l}{n} > \frac{1}{p} - \frac{k}{n}$$

then we have a compact embedding

$$L^p_k(E) \hookrightarrow L^q_l(E)$$

(where $k \geq 0$ if $p = 1$ and $l \geq 0$ if $q = 1$),

(ii) if $k,l \geq 0$ and if

$$l < k - \frac{n}{p}$$

then we have a compact embedding

$$L^p_k(E) \hookrightarrow C^l(E),$$

(iii) if $k,l \geq 0$, if $\alpha \in (0,1)$ and if

$$l + \alpha < k - \frac{n}{p}$$

then we have a compact embedding

$$L^p_k(E) \hookrightarrow C^{l,\alpha}(E).$$

Proof

See [Aubin, T., 1982; chapter 2] or [Adams, R.A., 1975; chapter 6].

□

The following theorem is the basic multiplication theorem for Sobolev spaces which generalizes Hölder's inequality. Other multiplication theorems for multilinear maps between vector bundles may be deduced from the given theorem by induction on the degree of the multilinear map, by using the Sobolev embedding theorem, and by using the duality between the Sobolev spaces $L^p_k(E)$ and $L^{p'}_{-k}(E^*)$ where $E^* \longrightarrow M$ is the vector bundle dual to $E \longrightarrow M$ and where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

(for more details, see [Palais, R.S., 1968; chapter 9]).

<u>Theorem 2.4</u>

Let M be a compact smooth manifold of dimension n, let $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ and $\pi_3 : E_3 \to M$ be smooth vector bundles over M, and let $B : E_1 \otimes E_2 \longrightarrow E_3$ be a smooth morphism of vector bundles. Let

$$\bar{B} : C^\infty(E_1) \times C^\infty(E_2) \to C^\infty(E_3)$$

be the map sending (s_1, s_2) to $B(s_1 \otimes s_2)$, for all $s_1 \in C^\infty(E_1)$ and $s_2 \in C^\infty(E_2)$. Let k be a non-negative integer and let $p, q, r \in [1, \infty)$. Then

(i) if $r < p$, $r < q$ and

$$\frac{1}{r} \geqslant \frac{1}{p} + \frac{1}{q} - \frac{k}{n}$$

then $\bar{B}$ extends to a continuous bilinear map

$$\bar{B} : L^p_k(E_1) \times L^q_k(E_2) \to L^r_k(E_3),$$

(ii) if $q > p$ and $qk > n$, then $\bar{B}$ extends to a continuous bilinear map

$$\bar{B} : L^p_k(E_1) \times L^q_k(E_2) \to L^p_k(E_3),$$

(iii) if $pk > n$, then $\bar{B}$ extends to a continuous bilinear map

$$\bar{B} : L^p_k(E_1) \times L^p_k(E_2) \to L^p_k(E_3),$$

(iv) $\bar{B}$ extends to a continuous bilinear map

$$\bar{B} : L^p_k(E_1) \times C^k(E_2) \rightarrow L^p_k(E_3).$$

Proof

It suffices to prove the result for trivial bundles over the unit ball in $\mathbb{R}^n$ and for the map B sending $s_1 \otimes s_2$ to the product $s_1 s_2$. Expand the partial derivatives of $s_1 s_2$ of order not exceeding k by Leibnitz' rule. Then use the Sobolev embedding theorems and Hölder's inequality.

□

Next we consider the continuity on Sobolev and Hölder norms of maps on sections induced by smooth fibre preserving maps (not necessarily linear) between vector bundles over a compact manifold.

Theorem 2.5

Let $\pi_1 : E_1 \rightarrow M$ and $\pi_2 : E_2 \rightarrow M$ be smooth vector bundles over a compact smooth manifold M of dimension n. Let $f : E_1 \rightarrow E_2$ be a smooth fibre preserving map. Then, for all non-negative integers k, for all $p \in [1, \infty)$ satisfying $pk > n$ and for all $\alpha \in (0,1)$, the map f induces smooth maps

$$L^p_k(E_1) \rightarrow L^p_k(E_2)$$

$$C^k(E_1) \rightarrow C^k(E_2)$$

$$C^{k,\alpha}(E_1) \rightarrow C^{k,\alpha}(E_2)$$

of Banach spaces, mapping a section s of $\pi_1 : E_1 \rightarrow M$ to the section $f \circ s$ of $\pi_2 : E_2 \rightarrow M$.

Proof

This follows from theorem 9.10, the remarks at the beginning of section 11 and theorem 11.3 of [Palais, R.S., 1968].

□

Let $\pi : B \to M$ be a smooth fibre bundle over a compact smooth manifold M of dimension n. Then for all non-negative integers k, for all $p \in [1,\infty)$ satisfying $pk > n$ and for all $\alpha \in (0,1)$ there are well-defined smooth Banach manifolds $L^p_k(B)$, $C^k(B)$ and $C^{k,\alpha}(B)$ with the property that if $\pi_1 : E \to M$ is a smooth vector bundle, if U is an open set in B and if $f : U \to E$ is a fibre preserving diffeomorphism onto an open subset of E, then f induces a diffeomorphism from

$$\{ s \in L^p_k(B) : s(M) \subset U \}$$

onto an open subset of $L^p_k(E)$, and similarly for $C^k(B)$ and $C^{k,\alpha}(B)$.

Theorem 2.6

Let $\pi_1 : B_1 \to M$ and $\pi_2 : B_2 \to M$ be smooth fibre bundles over a compact smooth manifold M of dimension n. Let $f : B_1 \to B_2$ be a smooth morphism of fibre bundles. Then, for all non-negative integers k, for all $p \in [1,\infty)$ satisfying $pk > n$ and for all $\alpha \in (0,1)$, the map f induces smooth maps

$$L^p_k(B_1) \to L^p_k(B_2)$$
$$C^k(B_1) \to C^k(B_2)$$
$$C^{k,\alpha}(B_1) \to C^{k,\alpha}(B_2)$$

of Banach manifolds, mapping a section s of $\pi_1 : B_1 \to M$ to the section $f \circ s$ of $\pi_2 : B_2 \to M$.

Proof

See [Palais, R.S., 1968; theorem 13.5].

□

Corollary 2.7

Let $\pi : B \to M$ be a smooth fibre bundle and let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be smooth vector bundles over a compact smooth manifold M of dimension n. Let $f : B \times_M E_1 \to E_2$ be a smooth morphism of fibre bundles with the property that for all $s \in C^{\infty}(B)$ the map from $C^{\infty}(E_1)$ to $C^{\infty}(E_2)$ sending $s_1 \in C^{\infty}(E_1)$ to $f(s, s_1)$ is linear. Then, for all non-negative integers k, for all $p \in [1, \infty)$, for all $q \in [1, \infty)$ satisfying $qk > n$ and for all $\alpha \in (0,1)$, the map f induces smooth maps

$$L_k^q(B) \times L_k^p(E_1) \to L_k^p(E_2)$$

$$C^k(B) \times L_k^p(E_1) \to L_k^p(E_2)$$

$$C^k(B) \times C^k(E_1) \to C^k(E_2)$$

$$C^{k,\alpha}(B) \times C^{k,\alpha}(E_1) \to C^{k,\alpha}(E_2)$$

of Banach manifolds, mapping sections s of $\pi : B \to M$ and s_1 of $\pi_1 : E_1 \to M$ to the section $m \mapsto f(s(m), s_1(m))$ of $\pi_2 : E_2 \to M$.

Proof

Let $\bar{f} : B \to \mathrm{Hom}\,(E_1, E_2)$ be the smooth map defined by

$$\bar{f}(s)s_1 = f(s, s_1)$$

for all $s \in C^{\infty}(B)$ and $s_1 \in C^{\infty}(E_1)$. $\bar{f}$ defines smooth maps

$$L_k^q(B) \to L_k^q(\mathrm{Hom}(E_1, E_2)),$$

$$C^k(B) \to C^k(\mathrm{Hom}(E_1, E_2)).$$

$$C^{k,\alpha}(B) \to C^{k,\alpha}(\mathrm{Hom}(E_1, E_2)).$$

Let $e : \mathrm{Hom}(E_1, E_2) \otimes E_1 \to E_2$ be the evaluation map. e defines continuous bilinear maps

$$L_k^q(\mathrm{Hom}(E_1, E_2)) \times L_k^p(E_1) \to L_k^p(E_2),$$

$$C^k(\mathrm{Hom}(E_1, E_2)) \times L_k^p(E_1) \to L_k^p(E_2),$$

$$C^k(\mathrm{Hom}(E_1, E_2)) \times C^k(E_1) \to C^k(E_2),$$

$$C^{k,\alpha}(\mathrm{Hom}(E_1, E_2)) \times C^{k,\alpha}(E_1) \to C^{k,\alpha}(E_2).$$

The result then follows from the identity

$$f(s, s_1) = e(\bar{f}(s), s_1).$$

□

§3. Quotients of Banach Manifolds by Banach Lie Groups

In this section, we prove a theorem giving necessary and sufficient conditions for the existence of smooth local slices for a smooth free action of a Banach Lie group H on a Banach manifold X. Proving the existence of such slices is equivalent to proving the existence of a smooth structure on the quotient X/H of X by H with the property that the projection $X \rightarrow X/H$ is smooth and admits smooth local sections around each element of X/H. These necessary and sufficient conditions are satisfied when H is a compact Lie group. We deduce that the quotient of a Banach Lie group G by a compact Lie subgroup H normal in G is a Banach Lie group G/H, and any smooth action of G on a Banach manifold which restricts to a trivial action of H induces a smooth action of G/H on this manifold.

Theorem 3.1

Let X be a connected Banach manifold and let H be a Banach Lie group (i.e. a Banach manifold with a group structure such that the group operations are smooth). Let H act smoothly and freely on X (on the right). Suppose that the action of H on X satisfies the following three conditions:

(i) for all $x \in X$, the derivative at the identity element e of the map from H to X sending $h \in H$ to x.h defines an isomorphism of $T_e H$ onto a closed subspace of $T_x X$ tangent to the orbit of H containing x,

(ii) for all $x \in X$, this tangent space to the orbit of H containing x has a closed complement in $T_x X$,

(iii) if $(x_i : i \in \mathbb{N})$ is a sequence converging in X and if $(h_i : i \in \mathbb{N})$ is a sequence in H such that the sequence $(x_i . h_i : i \in \mathbb{N})$ also converges in X, then the sequence $(h_i : i \in \mathbb{N})$ has a subsequence converging in H.

Then the quotient space X/H may be given the structure of a Banach manifold in such a way that the projection map $p : X \to X/H$ is smooth and such that every point of X/H has an open neighbourhood which is the domain of a smooth local section of $p : X \to X/H$. Moreover, this smooth structure on X/H is the unique smooth structure satisfying these conditions, and if X/H has such a smooth structure, then the action of H on X satisfies (i), (ii) and (iii).

Proof

Suppose that the action of H on X satisfies (i), (ii) and (iii). First we show that X/H is a Hausdorff topological space. Let

$$R = \{ (x,\bar{x}) \quad X \times X : \exists\, h \in H \text{ such that } x_1.h = x_2 \} .$$

Let $(x,\bar{x})$ belong to the closure of R in $X \times X$. X satisfies the first axiom of countability, hence there exist sequences $(x_i : i \in \mathbb{N})$ and $(h_i : i \in \mathbb{N})$ in X and H respectively such that the sequences $(x_i : i \in \mathbb{N})$ and $(x_i.h_i : i \in \mathbb{N})$ converge to x and $\bar{x}$ respectively. By condition (iii), some subsequence of $(h_i : i \in \mathbb{N})$ converges to h, for some $h \in H$, and $x.h = \bar{x}$ by the continuity of the action of H on X. Hence $(x,\bar{x}) \in R$. Thus R is closed. Hence X/H is Hausdorff.

Let $x \in X$. Then there exists a smooth chart $\varphi : U \to X$, where U is an open neighbourhood of zero in T_xX, such that φ maps sero to x and such that the derivative of φ at zero is the identity map of T_XX. Let Z be the subspace of T_xX tangent to the orbit of H through x. Z is closed by (i). By (ii), there exists a closed complement Z' of Z in T_XX. Let $U_1 = U \cap Z'$. By (i), the derivative of the smooth map $\theta : U_1 \times H \to X$ sending (u,h) to $\varphi(u).h$ is an isomorphism at (0,e), where e is the identity of element of H. By the inverse function theorem for Banach manifolds, there exist an open neighbourhood U_2 of zero in U_1 and an open neighbourhood V_2 of e in H such that $\theta/U_2 \times V_2 : U_2 \times V_2 \to X$ is a diffeomorphism onto an open set in X. Using the fact that

$$\theta(u,h) = \theta(u,hh_1^{-1}).h_1$$

for all $h_1 \in H$, we see that $\theta/U_2 \times H$ is a local diffeomorphism from $U_2 \times H$ onto an open set in X. We claim that there exists a neighbourhood U_3 of zero in U_2 such that $\theta/U_3 \times H$ is a diffeomorphism from $U_3 \times H$ onto an open set in X. Suppose this were not so. Then, for each neighbourhood N of zero in U_2, there would exist $u,u' \in N$ and $h,h' \in H$ such that $\varphi(u).h = \varphi(u').h'$ though $h \neq h'$, and then we would have $hh'^{-1} \neq e$, $u \in N$ and $\varphi(u).hh'^{-1} \in \varphi(N)$. Since this would be true for all such neighbourhoods N of zero, we would be able to construct sequences $(u_i : i \in \mathbb{N})$ and $(u_i' : i \in \mathbb{N})$ in H such that $\varphi(u_i).h_i = \varphi(u_i')$, such that $h_i \neq e$ for all i, and such that the sequences $(\varphi(u_i) : i \in \mathbb{N})$ and $(\varphi(u_i') : i \in \mathbb{N})$ would converge in X to x. By (iii), a subsequence of $(h_i : i \in \mathbb{N})$ would converge to some element of H and, by the continuity of the action of H on X, this element would stabilize x and so would be the identity element e of H. Thus there would exist positive integers i such that $h_i \in V_2$. But then for these values of i, we would have $h_i \neq e$ and

$$\theta(u_i,h_i) = \theta(u_i',e),$$

contradicting the fact that $\theta|U_2 \times V_2$ is injective. It follows that there exists a neighbourhood U_3 of zero in U_2 such that $\theta|U_3 \times H$ is a diffeomorphism from $U_3 \times H$ onto an open subset of X. Thus for all $x \in X$ there exist an open neighbourhood U_x of zero in some Banach space and a smooth map $\varphi_x : U_x \to X$ mapping zero to x with the property that the map from $U_x \times H$ to X sending $(u,h) \in U_x \times H$ to $\varphi_x(u).h$ is a diffeomorphism onto an open subset of X.

Define $V_x = p\varphi_x(U_x)$. Then V_x is an open neighbourhood of p(x) in X/H. The map $p\varphi_x : U_x \to X/H$ is continuous, injective and open, and is thus a homeomorphism onto V_x (where X/H is given the quotient topology). There is then a unique smooth structure on V_x such that

$p\varphi_x$ is a diffeomorphism from U_x onto V_x. Let $s_x : V_x \to X$ be the composition $s_x = \varphi_x \circ (p\varphi_x)^{-1}$. Then s_x is smooth and $p \circ s_x$ is the identity map on V_x. If $x, y \in X$ and V_x and V_y intersect, then $p \circ s_x$ is a smooth map from the open set $V_x \cap V_y$ in V_x to the open set $V_x \cap V_y$ in V_y, where V_x and V_y are given the smooth structures defined above, $p \circ s_x$ has inverse $p \circ s_y$, and the map defined by $p \circ s_x$ between the underlying topological spaces is the identity map. Thus the smooth structures on V_x and V_y are compatible. It follows that there is a unique smooth structure on X/H such that the open sets V_x are open Banach submanifolds of X/H for all $x \in X$. This smooth structure on X/H has the property that $p : X \to X/H$ is smooth and has smooth local sections around every element of X/H. This smooth structure is the unique smooth structure with this property, since if X/H is given two such smooth structures, then the identity map between the underlying topological spaces factors locally as the composition of a smooth local section and the smooth projection, and is thus smooth and has a smooth inverse. Conditions (i), (ii), and (iii) for the action of H on X follow immediately from the existence of a smooth structure on X/H with the above property.

□

Corollary 3.2

Let X be a connected Banach manifold and let H be a compact Lie group acting smoothly and freely on X (on the right). Then the quotient space X/H may be given the structure of a smooth Banach manifold with the property that the projection map $p : X \to X/H$ is smooth and has smooth local sections around every element of X/H.

Proof

We must verify that the action of H on X satisfies conditions (i), (ii) and (iii) of the theorem. But, for all $x \in X$, the derivative at the identity element e of the map from H to X sending $h \in H$ to x.h defines a continuous linear injection from $T_e H$ onto a finite dimensional subspace of $T_x H$, and this injection is necessarily an isomorphism onto a closed subspace of $T_x X$ which splits in $T_x X$. Thus (i) and (ii) are satisfied. (iii) is satisfied since H is compact.

□

Corollary 3.3

Let G be a Banach Lie group and let H be a compact Lie subgroup of G. Then G/H may be given the structure of a smooth Banach manifold in such a way that the projection map $p : X \longrightarrow X/H$ is smooth and has smooth local sections around every element of G/H. If H is normal in G, then the group operations on G induce smooth group operations on G/H, giving G/H the structure of a Banach Lie group, and if G acts smoothly on a Banach manifold X and if the subgroup H acts trivially on X via the action of G, then the action of G on X defines a smooth action of G/H on X.

Proof

The existence of the required smooth structure on G/H follows from the previous corollary. The smoothness of the group multiplication $\bar{\mu} : G/H \times G/H \longrightarrow G/H$ follows from the fact that $\bar{\mu}$ factors locally as $\bar{\mu} = p \circ \mu \circ (s_1 \times s_2)$, where $p : G \rightarrow G/H$ is the smooth projection, where $\mu : G \times G \longrightarrow G$ is the group multiplication on G and where s_1 and s_2 are smooth local sections of p. The smoothness of the map sending an element of G/H to its inverse follows from a similar local factorization, as does the smoothness of the action of G/H on X.

□

Let $i : Y \longrightarrow X$ be an injection of Banach manifolds. We say that Y is a locally closed submanifold of X if and only if for all $y \in Y$ there exist a Banach space S, a closed subspace S_1 of S, an open neighbourhood N of zero in S and charts $\theta : N \to X$ and $\varphi : N \cap S_1 \to Y$ mapping zero to y with the property that $\theta|N \cap S_1 = i \circ \varphi$.

Corollary 3.4

Let X be a connected Banach manifold and let H be a Banach Lie group acting smoothly and freely on X. Let Y be an H- invariant locally closed submanifold of X. If X/H admits a smooth structure with the property that the projection $p : X \to X/H$ is smooth and has smooth local sections around every element of X/H, then Y/H admits a smooth structure with the property that the projection $p\,Y : Y \to Y/H$ is smooth and has smooth local sections around every element of Y/H. Then Y/H is a locally closed submanifold of X/H.

Proof

We must show that if the action of H on X satisfies conditions (i), (ii) and (iii) of theorem 3.1, then so does the action of H on Y. For all $y \in Y$, T_yY is a closed subspace of T_yX containing the closed subspace Z_y of T_yX tangent to the orbit of H containing y. Since Z_y splits in T_yX, there exists a continuous projection $\pi : T_yX \to Z_y$. Then $\pi|T_yY$ is a continuous projection of T_yY onto Z_y hence Z_y splits in T_yY. Z_y is also closed in T_yY since it is closed in T_yX. Thus the action of H on Y satisfies conditions (i) and (ii). Condition (iii) is also satisfied. Hence Y/H has a unique smooth structure such that the natural projection Y Y/H is smooth and has smooth local sections around every element of Y/H.

Since Y is a locally closed submanifold of X, there exists a chart $\theta : N \to X$, where N is a neighbourhood of zero in T_yX, such that θ maps zero to y and $\theta|N \cap T_yY : N \cap T_yY \to Y$ is a chart for Y. Let S_y be a closed complement of Z_y in T_yX. Then $S_y \cap T_yY$ is a closed complement of Z_y in T_yY. By definition of the smooth structures on X/H and Y/H it follows that if N is chosen sufficiently small, then

$p \circ \theta \mid N \cap S_y$ is a smooth chart for X/H and $p \circ \theta \mid N \cap T_y Y \cap S_y$ is a smooth chart for Y/H. Thus Y/H is a locally closed submanifold of X/H.

$\square$

References for Chapter II

Adams, R., Sobolev Spaces. Academic Press, New York, (1975).

Aubin, T., Nonlinear analysis on manifolds, Monge-Ampere equations. Springer-Verlag, New York, (1982).

Palais, R.S., Foundations of global non-linear analysis. Benjamin, New York, (1968).

Chapter III

ELLIPTIC REGULARITY THEOREMS

§1. Introduction

In this chapter, we give a proof of general elliptic regularity theorems for linear elliptic differential operators with smooth coefficients.

Let $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ be smooth vector bundles over a compact manifold M. A linear map $L : C^\infty(E_1) \to C^\infty(E_2)$ is said to be a differential operator of order not exceeding m with smooth coefficients if and only if there is a smooth vector bundle morphism $T : J^m(E_1) \to E_2$, where $J^m(E_1) \to M$ is the bundle of m-jets of sections of $E_1 \to M$, such that

$$Ls = T \circ j_m(s)$$

where $j_m : C^\infty(E_1) \to C^\infty(J^m(E_1))$ is the m-jet extension map (see [Palais, R.S., 1968; chapter 3]). Let $\pi : T^*M \to M$ denote the co-tangent bundle of M. L determines a map $\sigma_m(L) : \pi^*E_1 \to \pi^*E_2$ as follows: let $\omega \in T^*M$ and $e \in E_1$ satisfy

$$\pi(\omega) = \pi_1(e) = x$$

so that (ω, e) represents an element of the pullback π^*E_1 of E_1 by $\pi : T^*M \to M$, choose $f \in C^\infty(M)$ and $s \in C^\infty(E_1)$ such that $df(x) = \omega$ and $s(x) = e$, then define

$$\sigma_m(L)\ (\omega, e) = \frac{1}{m!}\ L\ (f^m s)\ (x).$$

It may be verified that $\sigma_m(L)$ is well-defined, homogeneous of degree m in ω, and that $L \to \sigma_m(L)$ is linear in L (see [Palais, R.S., 1968; chapter 3]). The map $\sigma_m(L) : \pi^*E_1 \to \pi^*E_2$ is referred to as the leading symbol of L. L is said to be elliptic if and only if the map

$$e \mapsto \sigma_m(L)\ (\omega, e)$$

is an isomorphism from the fibre of π^*E_1 over ω to that of π^*E_2

over ω , for all non-zero $\omega \in T^*M$.

It is well-known that if $L : C^\infty(E_1) \to C^\infty(E_2)$ is a linear elliptic operator of order not exceeding m with smooth coefficients, if $s : M \to E_1$ is a section of $E_1 \to M$, and if $Ls \in L^2_{k-m}(E_2)$, then $s \in L^2_k(E_1)$, and that L extends to a Fredholm operator

$$L : L^2_k(E_1) \to L^2_{k-m}(E_2)$$

for all integers k. We shall show that the analogous results hold for the operator

$$L \; : \; L^p_k(E_1) \to L^p_{k-m}(E_2)$$

in the case where $p \in (1,\infty)$, and for the operator

$$L : C^{k,\alpha}(E_1) \to C^{k-m,\alpha}(E_2)$$

in the case where $k \geqslant m$ and $\alpha \in (0,1)$. In the special case where $L : C^\infty(M) \to C^\infty(M)$ is a linear elliptic differential operator of even order with smooth coefficients acting on the ring $C^\infty(M)$ of smooth functions on M, the above results are stated in [Aubin, T., 1982; p.85] where a proof is indicated, using results contained in [Morrey, C.B., 1966] and in [Bers, L., John, F. and Schechter, M., 1964]. Here we give an alternative proof, valid for linear elliptic differential operators with smooth coefficients of arbitrary order acting on sections of vector bundles over a compact manifold.

The proof given here uses the theory of pseudodifferential operators. The class of pseudodifferential operators used is that defined in [Kohn, J.J. and Nirenberg, L., 1965] and [Hörmander, L., 1965]. This class was historically the first class of pseudo-differential operators to be considered, and is the most suitable for our purposes.

A smoothing operator $k : C^\infty(E_1) \to C^\infty(E_2)$ on $C^\infty(E_1)$ is defined to be a linear operator which extends to a linear operator

$$k : \mathcal{D}'(E_1) \to C^\infty(E_2)$$

mapping the space $\mathcal{D}'(E_1)$ of distribution-valued sections of $E_1 \to M$ to the space $C^\infty(E_2)$ of smooth sections of the vector bundle $E_2 \to M$. The distribution kernel of k is then smooth. A parametrix $P : C^\infty(E_2) \to C^\infty(E_1)$ of a linear elliptic differential operator $L : C^\infty(E_1) \to C^\infty(E_2)$ is a linear operator with the property that

$$PL - I : C^\infty(E_1) \longrightarrow C^\infty(E_1)$$
$$LP - I : C^\infty(E_2) \longrightarrow C^\infty(E_2)$$

are smoothing operators. If P_1 and P_2 are parametrices of L, then

$$P_1 = P_1LP_2 - P_1(LP_2 - I)$$
$$= P_2 + (P_1L - I)\, P_2 - P_1(LP_2 - I)$$

and the linear operator

$$(P_1L - I)\, P_2 - P_1(LP_2 - I)$$

is a smoothing operator. Thus any two parametrices of L differ by a smoothing operator. It is a well-known result that if $L : C^\infty(E_1) \longrightarrow C^\infty(E_2)$ is a linear elliptic differential operator of order m with smooth coefficients, then L has a parametrix $P : C^\infty(E_2) \longrightarrow C^\infty(E_1)$ which is a pseudodifferential operator of order $-m$ in the class of pseudodifferential operators that we are considering. The required results in the L^2_k case follow from the fact that P extends to a continuous linear operator

$$P : L^2_{k-m}(E_2) \longrightarrow L^2_k(E_1).$$

This result is proved using Fourier transform methods stemming from the Plancherel theorem, which states that the Fourier transform, acting on functions from $\mathbb{R}^n$ to $\mathbb{R}$, defines an automorphism of $L^2(\mathbb{R}^n)$. In order to obtain elliptic regularity results in the L^p_k case for $p \in (1,\infty)$ and in the $C^{k,\alpha}$ case for $k \geqslant m$ and $\alpha \in (0,1)$, it is sufficient to show that P extends to continuous operators

$$P : L^p_{k-m}(E_2) \longrightarrow L^p_k(E_1),$$

$$P : C^{k-m,\alpha}(E_2) \longrightarrow C^{k,\alpha}(E_1).$$

There is a class of linear operators acting on functions from $\mathbb{R}^n$ to $\mathbb{R}$ which is closely related to the class of pseudodifferential operators. This is the class of singular integral operators defined by Calderon and Zygmund. In a series of papers, these authors prove that a singular integral operator $H : C^\infty_0(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ extends to continuous linear operators

$$H : L^p_k(\mathbb{R}^n) \longrightarrow L^p_k(\mathbb{R}^n) \quad (K \geq 0,\ p \in (1,\infty)),$$

$$H : C^{k,\alpha}_{loc}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}_{loc}(\mathbb{R}^n) \quad (k \geq 0, \alpha \in (0,1)).$$

In the case when H is translation-invariant, the result for the L^p norm is the well-known Calderon-Zygmund inequality. The result for the $C^{0,\alpha}$ norm is also well-known. The results for more general singular integral operators, not necessarily translation-invariant, can be proved from the translation-invariant case using expansions in spherical harmonics.

The proof of the required continuity results for pseudo-differential operators is obtained by showing that, locally, such an operator is a sum of products of singular integral operators, in local coordinates, and other well-behaved translation-invariant operators. Then the local results are pieced together using a partition of unity argument.

In §2, we present a summary, without proofs, of the theory of singular integral operators due to Calderon and Zygmund. In §3 we summarize the invariant definition and properties of the class of pseudodifferential operators studied in [Hörmander, L., 1965]. None of the material in these two sections is new. In §4, we develop the local theory of the continuity, in Sobolev and Hölder norms, of pseudodifferential operators defined on Euclidean space. In §5, the continuity, in Sobolev and Hölder norms, of pseudodifferential perators on sections of vector bundles over compact manifolds is deduced from the local theory presented in §4, and the required elliptic regularity results (theorems 5.2 and 5.3) are deduced.

An alternative proof of the boundedness of classical pseudodifferential operators in Sobolev L_k^p norms for $k \in \mathbb{Z}$ and for p satisfying $1 < p < \infty$ is to be found in chapter IV of [Coifman, R. and Meyer, Y., 1978] employing methods pioneered in [Calderon, A.P. and Zygmund, 1952]. This proof uses the Marcinkiewicz interpolation theorem (see chapter V of [Stein, E.M. and Weiss, G., 1972]. The principles of this proof have been employed by Muramatu and Illner to derive Sobolev L_k^p estimates for more general (non-classical) classes of pseudodifferential operators (see [Illner, R., 1975]).

§2. Singular Integrals on Euclidean Space

We give an account of the main results of [Calderon, A.P. and Zygmund, A., 1956] and [Calderon, A.P. and Zygmund, A., 1957]. Calderon and Zygmund study singular integrals $P : C_o^{\infty}(\mathbb{R}^n) \to C^{\infty}(\mathbb{R}^n)$ of the form

$$P\varphi(x) = \lim_{\varepsilon \to o+} \int_{|x-y| > \varepsilon} \Omega\left(x, \frac{x-y}{|x-y|} \right) \frac{\varphi(y)}{|x-y|^n} \, dy$$

where $\Omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ is a smooth function satisfying

$$\int_{S^{n-1}} \Omega(x, z')\, dz' = 0$$

for all $x \in \mathbb{R}^n$, where dz' is the volume measure on S^{n-1}. Every such singular integral operator may be expressed in the form

$$P\varphi(x) = (2\pi)^{-n} \int e^{ix.\xi} \omega\left(x, \frac{\xi}{|\xi|}\right) \hat{\varphi}(\xi)\, d\xi$$

where $\hat{\varphi}$ is the Fourier transform of φ and where $\omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ is a smooth function with the property that

$$\int_{S^{n-1}} \omega(x, \xi')\, d\xi' = 0.$$

Conversely every smooth function $\omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ with this property arises from a singular integral operator in this way. ω is referred to as the symbol of P.

If Ω and its derivatives of all orders are bounded on $\mathbb{R}^n \times S^{n-1}$ then the singular integral operator P determined by Ω as above defines bounded linear maps

$$P : L_k^p(\mathbb{R}^n) \to L_k^p(\mathbb{R}^n),$$

$$P : C_o^{k,\alpha}(B_R) \to C^{k,\alpha}(B_R)$$

for all non-negative integers k and for all p and α satisfying $1 < p < \infty$ and $0 < \alpha < 1$, where B_R denotes the ball of radius R about the origin in $\mathbb{R}^n$.

We now discuss the above results in more detail.

Theorem 2.1

Let $\Omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ be a smooth function with the property that

$$\int_{S^{n-1}} \Omega(x,z')dz' = 0$$

for all $x \in \mathbb{R}^n$, and let $P : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be the singular integral operator determined by Ω, defined by

$$P\varphi(x) = \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \Omega\left(x, \frac{x-y}{|x-y|}\right) \frac{\varphi(y)}{|x-y|^n}\, dy.$$

Then there exists a smooth function $\omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$, the symbol of P, such that

$$P\varphi(x) = (2\pi)^{-n} \int e^{ix.\xi} \omega\left(x, \frac{\xi}{|\xi|}\right) \hat{\varphi}(\xi)\, d\xi,$$

where $\hat{\varphi}$ is the Fourier transform of φ, defined by

$$\hat{\varphi}(\xi) = \int e^{-ix.\xi} \varphi(x)\, dx.$$

$\omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ has the property that

$$\int_{S^{n-1}} \omega(x, \xi')\, d\xi' = 0$$

for all x $\mathbb{R}^n$. Conversely, given a smooth function $\omega : \mathbb{R}^n \times S^{n-1}$ with this property, there exists a smooth function $\Omega : \mathbb{R}^n \times S^{n-1}$ satisfying

$$\int_{S^{n-1}} \Omega(x,z')\, dz' = 0$$

for all $x \in \mathbb{R}^n$, such that ω is the symbol of the singular integral operator P determined by Ω as above.

$\Omega(x,z')$ and its derivatives of all orders are bounded on $\mathbb{R}^n \times S^{n-1}$ if and only if ω and its derivatives of all orders are bounded on $\mathbb{R}^n \times S^{n-1}$, where $\omega(x, \xi')$ is the symbol of the singular integral operator P determined by ω as above.

Proof

We sketch the proof. For more details see /Calderon, A.P. and Zygmund, A., 1957/ and /Stein, E.M. and Weiss, G., 1972; chapter IV/. In particular the latter has a well-written account of the definition and properties of spherical harmonics.

We expand $\Omega : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$ in spherical harmonics.

Let

$$\Omega(x,z') = \sum_{m=0}^{\infty} \sum_{j} a_{mj}(x) Y_{mj}(z')$$

where Y_{mj} is a spherical harmonic of degree m and where

$$(Y_{mj} : m,j \in \mathbb{Z},\ m \geq 0,\ 1 \leq j \leq d_m)$$

is an orthonormal basis of the Hilbert space $L^2(S^{n-1})$. One can show that the partial derivatives of $\Omega(x,z')$ with respect to z' of all orders are bounded on $\mathbb{R}^n \times S^{n-1}$ if and only if for all non-negative integers $k \geq 0$ there exist constants A_k independent of x such that

$$\sum_{m=0}^{\infty} \sum_{j} (1+m^2)^k\, a_{mj}(x)^2 \leq A_K.$$

Thus if Ω is smooth then the expansion of Ω in spherical harmonics converges rapidly. Let

$$P_{mj}\varphi = \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} Y_{mj}\left(\frac{x-y}{x-y}\right) \frac{\varphi(y)}{|x-y|^n}\, dy.$$

We claim that there exists a constant γ_m such that

$$P_{mj}\varphi = (2\pi)^{-n} \gamma_m \int e^{ix.\xi}\, Y_{mj}\left(\frac{\xi}{|\xi|}\right) \hat{\varphi}(\xi)\, d\xi.$$

To show this, define tempered distributions $K_{mj} : S(\mathbb{R}^n) \to \mathbb{C}$ and $K_{mj\delta} : S(\mathbb{R}^n) \to \mathbb{C}$ for all δ satisfying $0 < \delta < \frac{1}{2}n$ by

$$K_{mj}(\varphi) = \lim_{\varepsilon \to 0+} \int_{|x|>\varepsilon} Y_{mj}\left(\frac{x}{|x|}\right) \frac{\varphi(x)}{|x|^n}\, dx$$

$$K_{mj\delta}(\varphi) = \int Y_{mj}\left(\frac{x}{|x|}\right) \frac{\varphi(x)}{|x|^{n-\delta}}\, dx$$

(where $S(\mathbb{R}^n)$ denotes the class of smooth rapidly decreasing test functions on $\mathbb{R}^n$). Then

$$P_{mj}\varphi = K_{mj} * \varphi$$

where $K_{mj} * \varphi$ is the convolution of the distribution K_{mj} and the test function φ. Hence

$$(P_{mj}\varphi)^{\wedge} = \hat{K}_{mj}\hat{\varphi}.$$

By the Fourier inversion formula we see that it suffices to show that

$$K_{mj}(\xi) = \gamma_m Y_{mj}\left(\frac{\xi}{|\xi|}\right).$$

Now $K_{mj\delta} \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ for all δ satisfying $0 < \delta < \frac{1}{2}n$, hence $\hat{K}_{mj\delta} \in C^0(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Moreover it can be shown that the Fourier transform of the function

$$Y_{mj}\left(\frac{x}{|x|}\right) |x|^{-n+\delta}$$

is of the form

$$\psi(|\xi|)\, Y_{mj}\left(\frac{\xi}{|\xi|}\right)$$

for some function ψ. But $K_{mj\delta}$ is a distribution which is homogeneous of degree $-n+\delta$ in $|x|$, hence $\hat{K}_{mj\delta}$ is homogeneous of degree $-\delta$ in $|\xi|$. Thus

$$K_{mj\delta} = \gamma_{m,\delta} |\xi|^{-\delta} Y_{mj}\left(\frac{\xi}{|\xi|}\right)$$

for some constant $\gamma_{m,\delta}$. One can evaluate $\gamma_{m,\delta}$ by applying $\hat{K}_{mj\delta}$ to the test function to show that

$$\gamma_{m,\delta} = 2^{\delta} i^{-m} \pi^{n/2} \frac{\Gamma(\tfrac{1}{2}(m+\delta))}{\Gamma(\tfrac{1}{2}(n+m-\delta))} .$$

But $K_{mj\delta} \rightarrow K_{mj}$ in $S'(\mathbb{R}^n)$ as $\delta \rightarrow 0$, hence $\hat{K}_{mj\delta} \rightarrow \hat{K}_{mj}$ in $S'(\mathbb{R}^n)$ as $\delta \rightarrow 0$, by the continuity of the Fourier transform on the space of tempered distributions. It follows that

$$\hat{K}_{mj}(\xi) = \gamma_m Y_{mj}\left(\frac{\xi}{|\xi|}\right)$$

and hence that

$$P_{mj}\varphi = (2\pi)^{-n} \gamma_m \int e^{ix.\xi} Y_{mj}\left(\frac{\xi}{|\xi|}\right)\hat{\varphi}(\xi)\, d\xi ,$$

where

$$\gamma_m = i^{-m} \pi^{n/2} \frac{\Gamma(\tfrac{1}{2}m)}{\Gamma(\tfrac{1}{2}(n+m))} .$$

Define

$$\omega(x, \xi') = \sum_{m=o}^{\infty} \sum_j \gamma_m a_{mj}(x) Y_{mj}(\xi').$$

Then

$$P\varphi = (2\pi)^{-n}\int e^{ix.\xi}\, \omega\left(x, \frac{\xi}{|\xi|}\right)\hat{\varphi}(\xi)\, d\xi .$$

Since $\gamma_m = O(m^{-n/2})$ and $\gamma_m^{-1} = O(m^{n/2})$ we see that Ω and all its derivatives are bounded on $\mathbb{R}^n \times S^{n-1}$ if and only if ω and all its derivatives are bounded on $\mathbb{R}^n \times S^{n-1}$. The theorem follows directly from this.

□

An important example is provided by the Riesz operators $R_j : C_o^{\infty}(\mathbb{R}^n) \rightarrow C^{\infty}(\mathbb{R}^n)$, where j takes integer values from 1 to n. The Riesz operators are defined by

$$R_j \varphi(x) = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{\frac{1}{2}(n+1)}} \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} \varphi(y)\, dy.$$

The Riesz operators have the property that

$$(R_j \varphi)^\wedge(\xi) = i \frac{\xi_j}{|\xi|} \hat{\varphi}(\xi)$$

for all $\varphi \in C_o^\infty(\mathbb{R}^n)$. From this it follows that

$$\sum_{j=1}^{n} R_j(R_j \varphi) = -\varphi,$$

$$R_j R_k \varphi = R_k R_j \varphi,$$

$$\frac{\partial}{\partial x_k}(R_j \varphi) = R_j \frac{\partial f}{\partial x_k} = R_k \frac{\partial f}{\partial x_j}.$$

When $n = 1$ the Riesz operator R_1 is the Hilbert transform $H : C_o^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$H\varphi(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \frac{\varphi(y)}{x-y}\, dy.$$

A classical theorem, due to M. Riesz, states that the Hilbert transform extends to a bounded linear map

$$H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$$

(for a proof, see appendix A). This theorem is the basis of the proof of the following theorem, due to Calderon and Zygmund.

<u>Theorem 2.2</u> (Calderon-Zygmund)

Let $\Omega \in C^\infty(\mathbb{R}^n \times S^{n-1})$ and suppose that

$$\int_{S^{n-1}} \Omega(x,y')\, dy' = 0$$

for all $x \in \mathbb{R}^n$. Further suppose that Ω and its derivatives of all orders are bounded on $\mathbb{R}^n \times S^{n-1}$. Then the singular integral operator $P : C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ defined by

$$P\varphi(x) = \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \Omega\left(x, \frac{x-y}{|x-y|}\right) \frac{(y)}{|x-y|^n}\, dy$$

extends to bounded linear maps

$$P : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n)$$

for all non-negative integers k and for all p satisfying $1 < p < \infty$.

Proof

First we show that P is bounded on $L^p(\mathbb{R}^n)$ whenever p satisfies $1 < p < \infty$ and Ω is odd, that is

$$\Omega(x, -z') = -\Omega(x, z').$$

In this case

$$P\varphi = \frac{\pi}{2} \int_{S^{n-1}} \Omega(x, z')\, H_{z'}\varphi\, dz'$$

where

$$H_{z'}\varphi = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|s|>\varepsilon} \frac{(x - sz')}{s}\, ds$$

for all $z' \in S^{n-1}$. It follows from M. Riesz' theorem on the boundedness of the Hilbert transform on $L^p(\mathbb{R}^n)$ and from Fubini's theorem that there exists a constant C_p such that

$$\| H_{z'}\varphi \|_p \leq C_p \| \varphi \|_p$$

where $\|\varphi\|_p$ denotes the L^p norm of φ. Hence

$$\| P\varphi \|_p \leq \tfrac{1}{2}\pi \, \| \Omega \|_0 \, C_p \, \mathrm{vol}(S^{n-1}) \, \| f \|_p$$

by the integral form of Minkowski's inequality, where

$$\| \Omega \|_0 = \sup \left\{ |\Omega(x,z')| : x \in \mathbb{R}^n, \ z' \in S^{n-1} \right\}.$$

This proves that P is bounded on $L^p(\mathbb{R}^n)$ when Ω is odd.

To prove the result when Ω is even, that is

$$\Omega(x, -z') = \Omega(x, z'),$$

we use the Riesz operators $R_j : C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ defined above.

Since

$$\sum_{j=1}^{n} R_j(R_j \varphi) = -\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ it follows that

$$P = -\sum_{j=1}^{n} (R \circ R_j) \circ R_j .$$

One can show that $P \circ R_j$ is a singular integral operator with odd kernel, either directly or by observing that if

$$P\varphi(x) = (2\pi)^{-n} \int e^{ix.\xi}\, \omega\left(x, \frac{\xi}{|\xi|}\right) \hat{\varphi}(\xi)\, d\xi$$

where $\omega : \mathbb{R}^n \times S^{n-1} \rightarrow \mathbb{C}$ is the symbol of P, then ω is smooth and

$$PR_j \varphi = (2\pi)^{-n} \int e^{ix.\xi}\, \omega_j\left(x, \frac{\xi}{|\xi|}\right) \hat{\varphi}(\xi)\, d\xi$$

where $\omega_j : \mathbb{R}^n \times S^{n-1}$ is defined by

$$\omega_j(x, \xi') = -i\,\omega(x, \xi')\, \xi'_j$$

for $j = 1, \ldots, n$. Note that

$$\omega_j(x, -\xi') = -\omega_j(x, \xi'),$$

since

$$\omega(x, -\xi') = \omega(x, \xi'),$$

and thus

$$\int_{S^{n-1}} \omega_j(x, \xi')\, d\xi' = 0.$$

By theroem 2.1 it follows that ω_j is the symbol of a singular integral operator with odd kernel. Thus $P \circ R_j$ is a singular integral operator with odd kernel. The boundedness of P on $L^p(\mathbb{R}^n)$ then follows from the boundedness of $P \circ R_j$ and R_j on $L^p(\mathbb{R}^n)$.

The boundedness on $L^p(\mathbb{R}^n)$ of a singular integral operator whose kernel is neither even nor odd follows by expressing the kernel as a sum of an even kernel and an odd kernel and applying the above results.

Define $P_j : C_0^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^n)$ by

$$P_j \varphi(x) = \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \Omega_j\left(x, \frac{x-y}{|x-y|}\right) \frac{\varphi(y)}{|x-y|^n} \, dy$$

where

$$\Omega_j(x, z') = \frac{\partial}{\partial x_j} \Omega(x, z').$$

Then

$$P \frac{\partial \varphi}{\partial x_j} = P_j \varphi + P \frac{\partial \varphi}{\partial x_j}.$$

Since P and P_j are bounded on $L^p(\mathbb{R}^n)$ it follows that P is bounded on $L^p_1(\mathbb{R}^n)$. By induction P is bounded on $L^p_k(\mathbb{R}^n)$ for all non-negative integers k.

□

When $k = 0$ and $\Omega(x, z')$ is independent of x this result is known as the Calderon-Zygmund inequality (see [Calderon, A.P. and Zygmund, A., 1956], [Stein, E.M. and Weiss, G., 1972; chapter VI], [Bers, L., John, F. and Schechter, M., 1964; pp.224, 245-250] of [Morrey, C.B., 1966; pp.55-61]).

The corresponding theorem for Hölder spaces is the following classical result.

Theorem 2.3

Let $\Omega \in C^{\infty}(\mathbb{R}^n \times S^{n-1})$ and suppose that

$$\int_{S^{n-1}} \Omega(x, y') \, dy' = 0$$

for all $x \in \mathbb{R}^n$. Then the singular integral operator $P : C_0^{\infty}(\mathbb{R}^n) \to C^{\infty}(\mathbb{R}^n)$ defined by

$$P\varphi(x) = \lim_{\varepsilon \to 0+} \int_{|x-y|>\varepsilon} \Omega\left(x, \frac{x-y}{|x-y|}\right) \frac{\varphi(y)}{|x-y|^n}\, dy$$

extends to bounded linear maps

$$P : C_o^{k,\alpha}(B_R) \longrightarrow C^{k,\alpha}(B_R)$$

for all non-negative integers k, for all α satisfying $0 < \alpha < 1$ and for all $R > 0$, where B_R denotes the ball of radius R about the origin in $\mathbb{R}^n$.

Proof

When $k = 0$ and $\Omega(x, z')$ is independent of X, the result is classical and proofs may be found in /Bers, L., John, F. and Schechter, M., 1964; pp.223, 244-245)/, (/Morrey, C.B., 1966; pp.50-53/ and /Calderon, A.P. and Zygmund, A., 1957/. The result in the general case follows by a straightforward adaptation of the proof in the case when $\Omega(x, z')$ is independent of x or by expanding Ω in spherical harmonics as in theorem 2.1 and proving that each term in this expansion is bounded on $C^{k,\alpha}(B_R)$ and then using the rapid convergence of the expansion.

$\square$

Calderon and Zygmund have considered versions of the above theorems when the assumption that the kernel of the singular integral operator is smooth is relaxed (see /Calderon, A.P. and Zygmund, A., 1956/ and /Calderon, A.P. and Zygmund, A., 1957/).

§3. Pseudo-Differential Operators on Manifolds

In this section, we study the properties of pseudo-differential operators on smooth manifolds. There are several definitions of pseudo-differential operators in the literature (see, for example [Kohn, T.J. and Nirenberg, L., 1965], [Hörmander, L., 1965], [Palais, R.S. et al, 1965; chapter XVI], where they are referred to as Calderon-Zygmund operators, [Hörmander, L., 1967], [Atiyah, M.F. and Singer, I.M., 1968], [Nirenberg, L., 1970], [Wells, R.O., 1973]). We adopt here the definition due to Hörmander in his paper "Pseudo-differential operators", [Hörmander, L., 1965]. This definition has the advantage of defining pseudo-differential operators invariantly on smooth manifolds, without reference to local coordinates. If a pseudo-differential operator is defined in this way on open sets in $\mathbb{R}^n$ then it can be shown that it is the sum of a pseudo-differential operator in the sense of Kohn and Nirenberg [Kohn, J.J. and Nirenberg, L., 1965] and a smoothing operator. Hörmander's paper can thus be regarded as giving a proof of the invariance of pseudo-differential operators defined in the sense of Kohn and Nirenberg under change of coordinates, modulo the smoothing operators. The Calderon-Zygmund operators of Palais and Seeley as defined in chapter XVI of "Seminar on the Atiyah-Singer index theorem" [Palais, R.S. et al, 1965], defined on smooth manifolds and vector bundles, are the pseudo-differential operators of Hörmander [Hörmander, L., 1965].

In proving the continuity properties of pseudo-differential operators when extended to Sobolev and Hölder spaces, we shall relate pseudo-differential operators to the singular integral operators of Calderon and Zygmund. For this purpose, some of the later definitions (such as in [Hörmander, L., 1967], [Atiyah, M.F. and Singer, I.M., 1968], [Nirenberg, L., 1970] or [Wells, R.O., 1973]) of pseudo-differential operators are less suitable.

We shall state in this section the definition of a pseudo-differential operator and its symbol, discuss pseudo-differential operators on open sets in $\mathbb{R}^n$, smoothing operators, the composition of pseudo-differential operators, the adjoint of a pseudo-differential operator, pseudo-differential operators acting on sections of vector bundles, elliptic pseudo-differential operators and their parametrices.

Let M be a smooth manifold. We recall the definition of a bounded subset of the Frechet space $C^\infty(M)$. A subset B of $C^\infty(M)$ is bounded if for every compact set $K \subset M$ and for every differential operator L with smooth coefficients, there is a uniform bound for $|Lf|$ on K whenever $f \in B$. We can now give Hörmander's invariant definition of a pseudo-differential operator on a smooth manifold (see [Hörmander, L., 1965]).

Definition 3.1

A pseudo-differential operator P on a smooth manifold M is a continuous linear operator.

$$P : C_0^\infty(M) \longrightarrow C^\infty(M)$$

such that there exists a strictly decreasing sequence $(s_j : j = 0,1,2,\ldots)$ of real numbers converging to $-\infty$ as $j \to \infty$ such that for all $f \in C_0^\infty(M)$, for all $g \in C^\infty(M)$ with g real-valued and $dg \neq 0$ in the support of f, and for all $\lambda > 0$, there is an asymptotic expansion

$$e^{-i\lambda g} P(fe^{i\lambda g}) \sim \sum_{j=0}^{\infty} P_j(f,g)\lambda^{s_j}$$

with the property that for every integer $N > 0$ and for every compact set G of real-valued functions $g \in C^\infty(M)$ with $dg \neq 0$ in the support of f, the error

$$\lambda^{-s_N}\left(e^{-i\lambda g}\,P(fe^{i\lambda g}) - \sum_{j=0}^{N-1} P_j\,(f,g)\,\lambda^{s_j}\right)$$

belongs to a bounded set in $C^{\infty}(M)$ whenever $g \in G$ and $\lambda \geq 1$. If $P_0 \not\equiv 0$, we say that P is of order s_0, and if all P_j vanish identically, the order is defined as $-\infty$.

It follows from this definition that $P_j(f,g)$ is a positively homogeneous function of g of degree s_j. Thus

$$e^{-i\lambda g}\,P(fe^{i\lambda g}) \sim \sum_{j=0}^{\infty} P_j\,(f,\lambda g).$$

We define the symbol $\sigma_P(f,g)$ of P to be the formal sum

$$\sigma_P(f,g) = \sum_{j=0}^{\infty} P_j(f,\ g).$$

In his paper [Hörmander, L., 1965], Hörmander studies the action of pseudo-differential operators on smooth functions whose support is contained in the domain of a coordinate chart on the manifold, obtaining an expression for the symbol in local coordinates, and uses it to study the properties of pseudo-differential operators. The following theorems characterize the local behaviour of pseudo-differential operators (see [Hörmander, L., 1965; lemma 2.3 and theorems 3.3 and 3.7 and proposition 3.1].

Theorem 3.2

Let M be a smooth manifold of dimension n, and let $P : C_0^{\infty}(M) \rightarrow C^{\infty}(M)$ be a pseudo-differential operator on M. Let Ω be an open subset of M and let $x : \Omega \rightarrow \mathbb{R}^n$ be a chart giving local coordinates x on Ω. For every $f \in C_0^{\infty}(M)$ with

$$\operatorname{supp} f \subset \Omega$$

define $p_f : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$p_f\,(x, \xi) = e^{-ix.\xi}\,P(fe^{ix.\xi})$$

and let

$$p_f(x, \lambda\xi) \sim \sum_{j=0}^{\infty} p_{f,j}(x, \lambda\xi)$$

be the asymptotic expansion of p_f as $\lambda \to +\infty$, where $p_{f,j}(x,\xi)$ is homogeneous of degree s_j in ξ. Then p_f is smooth, and the asymptotic expansion of $p_f(x, \xi)$, and all its derivatives, in the variable ξ is uniformly asymptotic in x for all x belonging to some given compact subset of Ω. Thus for all multi indices α and β and for all compact subsets K of Ω, there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(p_f(x,\xi) - \sum_{j=0}^{N-1} p_{f,j}(x,\xi) \right) \right| \leq C_{\alpha,\beta,K} \, |\xi|^{s_N - |\beta|}$$

whenever $x \in K$ and $|\xi| \geq 1$, and also

$$P(fu) = (2\pi)^{-n} \int e^{ix.\xi} \, p_f(x,\xi) \, \hat{u}(\xi) \, d\xi .$$

Theorem 3.3

Let Ω be an open set in $\mathbb{R}^n$ and let $q : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function with an asymptotic expansion

$$q(x, \lambda\xi) \sim \sum_{j=0}^{\infty} q_j(x, \lambda\xi)$$

in λ, for $\lambda > 0$, where q_j is positively homogeneous of degree s_j and smooth in $\Omega \times (\mathbb{R}^n \setminus \{0\})$, such that, for all multi-indices α and β and for all compact subsets K of Ω, there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(q(x,\xi) - \sum_{j=0}^{N-1} q_j(x,\xi) \right) \right| \leq C_{\alpha,\beta,K} \, |\xi|^{s_N - |\beta|}$$

whenever $x \in K$ and $|\xi| \geq 1$. Then we can define an operator

$$Q : C_0^\infty(\Omega) \to C^\infty(\Omega)$$

by the identity

$$Qu = (2\pi)^{-n} \int e^{ix.\xi} q(x, \xi) \hat{u}(\xi) d\xi$$

and $Q : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is a pseudo-differential operator with symbol

$$\sigma_Q(f,g) = \sum_{\alpha, j} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha q_j(x, \xi_x) \partial_x^\alpha (fe^{ihx})$$

where

$$\xi_x = \text{grad } g(x)$$

and

$$h_x(y) = g(y) - g(x) - \langle y - x, \xi_x \rangle.$$

If $q_j(x, \xi) = 0$ for all j, then Q is a smoothing operator

$$Qu(x) = \int_\Omega K(x,y) u(y) dy,$$

where $K \in C^\infty(\Omega \times \Omega)$ is given by the identity

$$K(x,x-y) = (2\pi)^{-n} \int e^{iy.\xi} q(x, \xi) d\xi.$$

Also given a strictly decreasing sequence s_j which converges to $-\infty$ as $j \to \infty$ and smooth functions $q_j : \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ such that $q_j(x, \xi)$ is positively homogeneous of degree s_j in ξ, then there exists a pseudo-differential operator

$$Q : C_0^\infty(\Omega) \to C^\infty(\Omega)$$

where

$$Qu = (2\pi)^{-n} \int e^{ix.\xi} q(x, \xi) \hat{u}(\xi) d\xi$$

and where $q : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is smooth and has an asymptotic expansion

$$q(x, \lambda\xi) \sim \sum_{j=0}^{\infty} q_j(x, \lambda\xi)$$

for $\lambda > 0$, which satisfies the conditions stated at the beginning of the statement of this theorem. This pseudo-differential operator Q is unique up to a smoothing operator on Ω.

Let M be a smooth manifold and let μ be a smooth measure on M. We define a pairing

$$\langle \cdot , \cdot \rangle : C^{\infty}(M) \otimes C_0^{\infty}(M) \to \mathbb{R}$$

by the identity

$$\langle g, f \rangle = \int gf \, d\mu.$$

A continuous linear map $P : C_0^{\infty}(M) \to C^{\infty}(M)$ is referred to as a smoothing operator if there exists a smooth function $K : C^{\infty}(M \times M) \to \mathbb{R}$ such that

$$(Pf)(x) = \langle K_x, f \rangle$$

where

$$K_x : M \to \mathbb{R} : y \mapsto K(x,y).$$

Theorem 3.4

Let M be a smooth manifold of dimension n and let $P : C_0^{\infty}(M) \to C^{\infty}(M)$ be a smoothing operator. Then P is a pseudo-differential operator whose symbol vanishes everywhere.

Proof

It is sufficient to prove that for all functions $\varphi, \psi \in C_0^{\infty}(M)$, the smoothing operator $\varphi P \psi$ is a pseudo-differential operator. But, by employing a partition of unity subordinate to a locally finite covering of M by domains of coordinate charts, it suffices to prove the result when supp $\varphi \in U$ and supp $\psi \in U'$ where $x : U \to \mathbb{R}^N$ AND $y : U' \to \mathbb{R}^n$ are coordinate charts. Then if $Q = \varphi P \psi$, we have

$$Qu(x) = \int_{U'} K(x,y)\, u(y)\, dy$$

where $K : U \times U' \to \mathbb{R}$ is smooth and has compact support in both variables. Thus if $f \in C_0^\infty(M)$ and $g \in C^\infty(M)$ and if $dg \neq 0$ on supp f, we have

$$e^{-i\lambda g}\, Q(fe^{i\lambda g}) = \int_{U'} K(x,y)\, f(y)\, e^{i\lambda(g(y)-g(x))} dy.$$

Let

$$G_K = \frac{\partial g}{\partial y_K}$$

and

$$\|G\|^2 = \sum_{K=1}^{n} G_K^2 .$$

Then $\|G\|^2 \neq 0$ on supp f, by assumption, and

$$e^{-i\lambda g}\, Q(fe^{i\lambda g}) = -i\lambda^{-1}\int_{U'} \frac{K(x,y)f(y)}{\|G\|^2} \sum_{K=1}^{n} G_K \frac{\partial}{\partial y_K} e^{i\lambda(g(y)-g(x))} dy$$

$$= i\lambda^{-1} \sum_{K=1}^{n} \int_{U'} \frac{\partial}{\partial u_K}\left(\frac{K(x,y)f(y)}{\|G\|^2} G_K\right) e^{i\lambda(g(y)-g(x))} dy$$

on integrating once by parts. If we continue integrating by parts in this way, we obtain

$$e^{-i\lambda g}\, Q(fe^{i\lambda g}) = \lambda^{-N}\int_{U'} \frac{L(x,y)}{\|G\|^{2m}}\, e^{i\lambda(g(y)-g(x))} dy$$

where m is a positive integer and L is a polynomial in a finite number of derivatives of the functions K, f and g. L has compact support contained in the support of K. It follows that

$$e^{-i\lambda g}\, Q(fe^{i\lambda g}) = O(\lambda^{-N})$$

as $\lambda \to +\infty$, for all non-negative integers N, and moreover, if B is a compact subset of $C^\infty(M)$ and $dg \neq 0$ on supp f for all $g \in B$, then there is a constant C such that for all $g \in B$

$$e^{-i\lambda g}\, Q(fe^{i\lambda g}) \qquad c\lambda^{-N}$$

for all $\lambda \geq 1$. Thus $Q = \varphi P \psi$ is a pseudo-differential operator. Hence P is a pseudo-differential operator.

□

The next theorem, expressing the pseudolocal character of P, is immediate from theorem 4.5 of [Hörmander, L., 1965].

Theorem 3.5

Let M be a smooth manifold of dimension n, and let $P : C_0^\infty(M) \to C^\infty(M)$ be a pseudo-differential operator. If $f \in C_0^\infty(M)$ and $g \in C^\infty(M)$ and if

$$\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$$

then $gPf : C_0^\infty(M) \to C^\infty(M)$ is a smoothing operator.

The next two theorems are theorems 4.3 and 4.4 of [Hörmander, L., 1965]. The asymptotic expansions are due to [Kohn, J.J. and Nirenberg, L., 1965] (see also [Palais, R.S. et al, 1965; chapter XVI], [Hörmander, L., 1967] and [Nirenberg, L., 1970]).

Theorem 3.6

Let M be a smooth manifold, let $P : C_0^\infty(M) \to C^\infty(M)$ and $Q : C_0^\infty(M) \to C^\infty(M)$ be pseudo-differential operators of order s and t respectively and let $f \in C_0^\infty(M)$. Then

$$QfP : C_0^\infty(M) \to C^\infty(M)$$

is a pseudo-differential operator of order not exceeding s + t. In particular if $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$ and $Q : C_0^\infty(\Omega) \to C^\infty(\Omega)$ are pseudo-differential operators on an open set Ω in $\mathbb{R}^n$, and if $f \in C_0^\infty(\Omega)$ and if

$$Pu = (2\pi)^{-n} \int e^{ix.\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi ,$$

$$p(x,\xi) \sim \sum_j p_j(x,\xi),$$

$$Qu = (2\pi)^{-n} \int e^{ix.\xi} \; q(x, \xi) \, \hat{u}(\xi) \, d\xi ,$$

$$q(x, \xi) \sim \sum_{k} q_k(x, \xi),$$

$$Ru = QfPu = (2\pi)^{-n} \int e^{ix.\xi} \; r(x, \xi) \, \hat{u}(\xi) \, d\xi ,$$

$$r(x, \xi) \sim \sum_{i} r_i(x, \xi),$$

where $p_j(x, \xi)$, $q_k(x, \xi)$ and $r_i(x, \xi)$ are positively homogeneous in ξ, then we have an equality of formal sums

$$r_i(x, \xi) = \sum_{\alpha, j, k} \frac{(-i)^{|\alpha|}}{\alpha !} \partial_\xi^\alpha q_k(x, \xi) \, \partial_x^\alpha (fp_j(x, \xi))$$

Theorem 3.7

Let M be a smooth manifold. To every pseudo-differential operator $P : C_o^\infty(M) \to C^\infty(M)$ of order s, there is one and only one pseudo-differential operator ${}^tP : C_o^\infty(M) \to C^\infty(M)$ of order s, called its adjoint, such that

$$\langle Pu, v \rangle = \langle u, {}^tPv \rangle$$

if $u, v \in C_o^\infty(M)$. In particular, if $P : C_o^\infty(\Omega) \to C^\infty(\Omega)$ is a pseudo-differential operator on an open set Ω in $\mathbb{R}^n$, and is given by the identity

$$Pu = (2\pi)^{-n} \int e^{ix.\xi} \; p(x, \xi) \, \hat{u}(\xi) \, d\xi ,$$

$$p(x, \xi) \sim \sum_{j} p_j(x, \xi),$$

$${}^tPu = (2\pi)^{-n} \int e^{ix.\xi} \; {}^tp(x, \xi) \, \hat{u}(\xi) \, d\xi ,$$

$${}^tp(x, \xi) \quad \sum_{k} {}^tp_k(x. \xi),$$

where $p_j(x, \xi)$ and ${}^tp_k(x, \xi)$ are positively homogeneous in ξ, then we have an equality of formal sums

$${}^tp_k(x, \xi) = \sum_{\alpha, j} \frac{i^{|\alpha|}}{\alpha !} \partial_x^\alpha \, \partial_\xi^\alpha \, p_j(x, -\xi).$$

We can define pseudo-differential operators acting on sections of vector bundles over a smooth manifold. Let M be a smooth manifold and $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M. A continuous linear operator $P : C_0^\infty(E) \to C^\infty(F)$ is a pseudo-differential operator if for all smooth sections $f \in C_0^\infty(E)$ and smooth functions $g \in C^\infty(M)$ with $dg \neq 0$ on supp f, there is an asymptotic expansion

$$e^{-i\lambda g} P(fe^{i\lambda g}) \sim \sum_{j=0}^{\infty} P_j (f,g) \lambda^{s_j}$$

which is uniformly asymptotic for all g belonging to any given compact subset of $C^\infty(M)$, exactly as in the definition of pseudo-differential operators acting on smooth functions. All the results stated so far go over without change, when applied to pseudo-differential operators acting on sections of vector bundles.

Let M be a smooth manifold and let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M. Let $\pi : T^*M \setminus M \to M$ be the co-tangent bundle over M with the zero section removed, and let $\pi^*E \to T^*M \setminus M$ and $\pi^*F \to T^*M \setminus M$ be the pullbacks of E and F. Then there is a correspondence σ which assigns to a pseudo-differential operator $P : C_0^\infty(E) \to C^\infty(F)$ or order s_0 a homomorphism $\sigma(P) : \pi^*E \to \pi^*F$ of vector bundles over $T^*M \setminus M$ such that if $\omega \in T^*M \setminus M$ and $\lambda > 0$, then

$$\sigma(P)(\lambda\omega) = \lambda^{s_0} \sigma(P)(\omega),$$

$\sigma(P)$ is referred to as the leading symbol of P (or is often simply referred to as the symbol of P). $\sigma(P)$ is defined as follows. Let $m \in M$, let $e \in E_m$, the fibre of E over m, and let $\omega \in T_m^*M \setminus \{0\}$. Choose $f \in C_0^\infty(E)$ and $g \in C^\infty(M)$ such that $dg \neq 0$ on supp f, and such that $f(m) = e$ and $dg(m) = \omega$. We then have an asymptotic expansion

$$e^{-i\lambda g} P(fe^{i\lambda g}) \sim \sum_{j=0} P_j(f,g) \lambda^{s_j}.$$

Define

$$\sigma(P)_\omega (e) = P_o(f,g)(m).$$

We claim that $\sigma(P)$ is well-defined, independent of the choice of f and g. To verify this, it is sufficient to consider the case when the support of f is contained in the domain of a coordinate chart $x : \Omega \to \mathbb{R}^n$ of M. But then there are uniquely defined functions $p_k : \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$, where $p_k(x,\xi)$ is positively homogeneous of degree s_k in ξ such that we have an equality of formal sums

$$\sum_j P_j(f,g) = \sum_{\alpha,k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p_k(x,\xi_x) \partial_x^\alpha (fe^{ihx}),$$

where ξ_x = grad g (x) and

$$h_x(y) = g(y) - g(x) - \langle y - x, \xi_x \rangle$$

(see /Hörmander, L., 1965/, theorem 4.2), but then

$$P_o(f,g)(m) = fp_o(x, \text{grad } g(x)).$$

This shows that $\sigma(P)$ is well-defined.

Definition 3.8

Let M be a smooth manifold, let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M, and let $P : C_o^\infty(E) \to C^\infty(F)$ be a pseudo-differential operator. P is an elliptic pseudo-differential operator if and only if, for all $m \in M$ and $\omega \in T_m^*M \setminus M$, the homomorphism

$$\sigma(P)(\omega) : E_m \to F_m$$

of vector spaces is an isomorphism (i.e. $\sigma(P)$ is an isomorphism of vector bundles over $T^*M \setminus M$).

A very important property of elliptic pseudo-differential operators on smooth manifolds is the existence of a parametrix, guaranteed by the next theorem (see /Hörmander, L., 1965/, theorem 4.8

or /Nirenberg, L., 1970/, p.157).

Theorem 3.9

Let M be a smooth manifold, let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M, and let $P : C_0^\infty(E) \to C^\infty(F)$ be an elliptic pseudo-differential operator of order s. Then for every $f \in C_0^\infty(M)$, there exists a pseudo-differential operator $Q : C_0^\infty(F) \to C^\infty(E)$ of order -s such that for any open set U in M on which f is identically equal to 1, the operators

$$(QfP - I)|U : C_0^\infty(E|U) \to C^\infty(E|U)$$

$$(PfQ - I)|U : C_0^\infty(F|U) \to C^\infty(F|U)$$

are smoothing operators. In particular, if M is compact then there exists a parametrix $Q : C_0^\infty(F) \to C^\infty(E)$ of P such that the operators QP - I and PQ - I are smoothing operators.

§4. Pseudo-Differential Operators on Euclidean Space

In this section, we study some of the properties of pseudo-differential operators on open sets in $\mathbb{R}^n$. We degin by establishing a convenient notation and using it to reformulate some of the standard properties of pseudo-differential operators on open sets in $\mathbb{R}^n$. The rest of the section is devoted to showing that if $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is a pseudo-differential operator of order M, defined on an open set Ω of $\mathbb{R}^n$, and if $f,g \in C_0^\infty(\Omega)$, then gPf extends to continuous linear operators

$$gPf : L_k^p(\mathbb{R}^n) \to L_{k-m}^p(\mathbb{R}^n),\ 1 < p < \infty\ ,\ k \in \mathbb{Z}\ ,$$

$$gPf : C^{k,\alpha}(\mathbb{R}^n) \to C^{k-m,\alpha}(\mathbb{R}^n),\ 0 < \alpha < 1,\ k \geq \max(m,0).$$

First we make a number of definitions. Except for the definition of $\Sigma^m(\Omega)$, all the following definitions are taken from /Nirenberg, L., 1970/ or /Hörmander, L., 1967/.

Definitions 4.1

Let Ω be an open set in $\mathbb{R}^n$ and let m be a real number. We denote by $S^m(\Omega)$ the set of all $p \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for every compact set $K \subset \Omega$ and for all multi indices α and β, there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta p(x,\xi) \right| \leq C_{\alpha,\beta,K}\,(1 + |\xi|)^{m-|\beta|}$$

We denote by $\Sigma^m(\Omega)$ the subset of $S^m(\Omega)$ consisting of all $p \in S^m(\Omega)$ which possess an asymptotic expansion

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_j(x,\xi)$$

as $|\xi| \to \infty$, where p_j is homogeneous of degree s_j in ξ, with the property that if $\eta : \mathbb{R}^n \to [0,1]$ is any smooth function that vanishes in a neighbourhood of 0 and is identically equal to 1 outside some compact set in $\mathbb{R}^n$, we have that

$$p(x,\xi) - \sum_{j=1}^{N-1} \eta(\xi)\, p_j(x,\xi) \in S^{s_N}(\Omega).$$

Given $p \in S^m(\Omega)$ we define a linear operator $p(x,D) : C_0^\infty(\Omega) \to C^\infty(\Omega)$ by the identity

$$p(x,D)\, u = (2\pi)^{-n} \int e^{ix.\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi .$$

Given a strictly decreasing sequence m_j converging to $-\infty$ as $j \to \infty$ and given $p \in S^m(\Omega)$ and $q_j \in S^{m_j}(\Omega)$, we write

$$p \sim \sum_j q_j$$

if for all $N \geq 0$

$$p - \sum_{j<N} q_j \in S^{m_N}.$$

Also if $p \in \Sigma^m(\Omega)$, we write

$$p \sim \sum_j p_j$$

to denote the asymptotic expansion of p in functions positively homogeneous in ξ .

Let Ω be an open set in $\mathbb{R}^n$ and let $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$ be a continuous linear operator. Then P is a pseudo-differential operator in the sense of Hörmander ([Hörmander, L., 1965]) if and only if for every $f \in C_0(\Omega)$, there exists $p_f \in \Sigma^m(\Omega)$ such that

$$P(fu) = p_f(x,D)\, u.$$

Also P is a pseudo-differential operator in the sense of Kohn and Nirenberg ([Kohn, J.J. and Nirenberg, L., 1965]) if and only if there exists $p \in \Sigma^m(\Omega)$ such that

$$Pu = p(x,D)\, u.$$

We may express the asymptotic expansions of compositions and adjoints of pseudo-differential operators, due to Kohn and Nirenberg, as follows. Let $p \in \Sigma^{m_1}(\Omega)$, $q \in \Sigma^{m_2}(\Omega)$ and let $P = p(x, D)$, $Q = q(x, D)$ be the corresponding pseudo-differential operators, and let $f \in C_0^{\infty}(\Omega)$. Then there exist symbols $r \in \Sigma^{m_1 + m_2}$ and ${}^t p \in \Sigma^{m_1}$ such that

$$QfPu = r(x, D)\, u,$$
$${}^tPu = {}^tp(x, D)\, u,$$

and r and ${}^tp$ have asymptotic expansions

$$r(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} q(x, \xi)\, \partial_x^{\alpha} (f(x)p(x, \xi)),$$
$${}^tp(x, \xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \partial_x^{\alpha} \partial_{\xi}^{\alpha} p(x, -\xi).$$

Indeed, if $p \in S^{m_1}(\Omega)$ and $q \in S^{m_2}(\Omega)$ then $QfP = r(x, D)$ and ${}^tP = {}^tp(x, D)$ for some $r \in S^{m_1 + m_2}(\Omega)$ and ${}^tp \in S^{m_1}(\Omega)$ and r and ${}^tp$ have asymptotic expansions as above (see [Hörmander, L., 1967] or [Nirenberg, L., 1970]).

Let (m_j) be a strictly decreasing sequence of real numbers converging to $-\infty$ and let $q_j \in S^{m_j}(\Omega)$. Then there exists $p \in S^{m_0}(\Omega)$ such that

$$p \sim \sum_j q_j.$$

If $q_j \in \Sigma^{m_j}(\Omega)$, then $p \in \Sigma^{m_0}(\Omega)$. Also, given $p \in S^m(\Omega)$, $p(x, D)$ is a smoothing operator if and only if $p \sim 0$ (see [Hörmander, L., 1967] of [Nirenberg, L., 1970] for proofs).

Note also that if $p \in S^m(\mathbb{R}^n)$, then $p(x, D)$ is translation-invariant if and only if $p(x, \xi)$ is a function of ξ alone. For if we define $\tau_h : C^{\infty}(\mathbb{R}^n) \to C^{\infty}(\mathbb{R}^n)$ by the identity

$$(\tau_h u)(x) = u(x - h)$$

then

$$(\tau_h u)^\wedge(\xi) = d^{-ih.\xi}\hat{u}(\xi)$$

hence

$$\tau_h^{-1} p(x, D)\tau_h = p(x + h, D)$$

so that

$$p(x + h, D) - p(x, D) = 0$$

and thus

$$p(x + h, \xi) - p(x, \xi) = 0.$$

We write $p(D) = p(x, D)$ whenever p is translation-invariant, and we then have that

$$(p(D)u)^\wedge = p(\xi)\hat{u}.$$

Given a pseudo-differential operator $P : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ and functions $f, g \in C_0^\infty(\mathbb{R}^n)$, we wish to know when the pseudo-differential operator $gPf : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ extends to a continuous linear operator between Sobolev or Hölder spaces. First we will prove a number of lemmas in preparation for the study of this question.

Lemma 4.2

Let $\varphi \in L^1_{loc}(\mathbb{R}^n)$ and let $P : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be the linear operator defined by the identity

$$Pu = \varphi * u$$

where $\varphi * u$ denotes the convolution of φ and u. Then, for all $f,g \in C_0^\infty(\mathbb{R}^n)$, for all non-negative integers k, for all $p \in [1, \infty)$ and for all $\alpha \in [0, 1)$, gPf extends to continuous linear operators

$$gPf : L^p_k(\mathbb{R}^n) \to L^p_k(\mathbb{R}^n),$$

$$gPf : C^{k,\alpha}(\mathbb{R}^n) \to C^{k,\alpha}(\mathbb{R}^n).$$

Proof

Given $f,g \in C_0^\infty(\mathbb{R}^n)$, let

$$R = \sup\{|x - y| : x \in \text{supp } g, y \in \text{supp } f\}$$

and let $\eta \in C_0^{\infty}(\mathbb{R}^n)$ be a smooth function with compact support with the property that $\eta(x) = 1$ for all x satisfying $|x| \leq R$. Then, for all $u \in C_0^{\infty}(\mathbb{R}^n)$

$$gP(fu) = g(\varphi * (fu)) = g((\eta\varphi) * (fu)).$$

Thus it suffices to prove that $P_0 : C_0^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^n)$ extends to continuous linear operators

$$P_0 : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n),$$

$$P_0 : C^{k,\alpha}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}(\mathbb{R}^n),$$

where $\psi = \eta\varphi$ and $P_0 u = \psi * u$. But $\psi \in L^1(\mathbb{R}^n)$ and thus by Young's theorem on convolutions

$$\|\psi * u\|_{L^p} \leq \|\psi\|_{L^1} \|u\|_{L^p}.$$

Also

$$\partial^{\beta}(\psi * u) = \psi * \partial^{\beta} u$$

for all multi indices β, hence

$$\|\psi * u\|_{L_k^p} \leq \|\psi\|_{L^1} \|u\|_{L_k^p}.$$

Thus P_0 extends to a continuous linear operator

$$P_0 : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n).$$

Also

$$\sup |\psi * u| \leq \|\psi\|_{L^1} \sup |u|$$

hence P_0 extends to a continuous linear operator

$$P_0 : C^k(\mathbb{R}^n) \longrightarrow C^k(\mathbb{R}^n).$$

For $u \in C^{\alpha}(\mathbb{R}^n)$ let

$$|u|_{\alpha} = \sup_{x,y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

Then if $v = \psi * u$

$$\begin{aligned}|v(x+h) - v(x)| &\leq \int |\psi(y)|\, |u(x+h-y) - u(x-y)|\, dy \\ &\leq h^{\alpha} |u|_{\alpha} \|\psi\|_{L^1}\end{aligned}$$

hence

$$|v|_{\alpha} \leq |u|_{\alpha} \, \|\Psi\|_{L^1}$$

and thus

$$\|\Psi \circ u\|_{C^{\alpha}} \leq \|\Psi\|_{L^1} \, \|u\|_{C^{\alpha}}$$

$$\|\Psi \circ u\|_{C^{k,\alpha}} \leq \|\Psi\|_{L^1} \, \|u\|_{C^{k,\alpha}}$$

hence P_0 extends to a continuous linear operator

$$P_0 : C^{k,\alpha}(\mathbb{R}^n) \to C^{k,\alpha}(\mathbb{R}^n).$$

$\square$

<u>Lemma 4.3</u>

Let Ω be an open set in $\mathbb{R}^n$. Let $P : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega)$ be the translation-invariant linear operator

$$Pu = p(D)\,u$$

where $p \in S^{m_1}(\Omega)$, and let $Q : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega)$ be the linear operator

$$Qu = q(x,D)\,u$$

where $q \in S^{m_2}(\Omega)$. Then

$$QP = R,$$

$$R = r(x,D)u,$$

where $r \; S^{m_1+m_2}(\Omega)$ and

$$r(x,\xi) = q(x,\xi)\,p(\xi).$$

If $p \in \Sigma^{m_1}(\Omega)$ and $q \in \Sigma^{m_2}(\Omega)$, then $r \in \Sigma^{m_1+m_2}(\Omega)$. Moreover, if gPf and gQf extend to continuous linear operators

$$gPf : L_k^p(\Omega) \to L_{k-l_1}^p(\Omega), \quad (\forall k \geq l_1),$$

$$gQf : L_k^p(\Omega) \to L_{k-l_2}^p(\Omega), \quad (\forall k \geq l_2),$$

for all $f,g \in C_0^{\infty}(\Omega)$, then gRf extends to a continuous linear operator

$$gRf : L_k^p(\Omega) \to L_{k-l}^p(\Omega) \quad (\forall k \geq l)$$

for all $f,g \in C_0^{\infty}(\Omega)$, where $l = l_1 + l_2$. Similarly if gPf and gQf extend to continuous linear operators

$$gPf : C^{k,\alpha}(\Omega) \to C^{k-l_1,\alpha}(\Omega), \quad (\forall k \geq l_1),$$

$$gQf : C^{k,\alpha}(\Omega) \to C^{k-l_2,\alpha}(\Omega), \quad (\forall k \geq l_2),$$

for all $f, g \in C_0^\infty(\Omega)$, then gRf extends to a continuous linear operator

$$gRf : C^{k,\alpha}(\Omega) \to C^{k-l,\alpha}(\Omega), \quad (\forall k \geq l)$$

for all $f, g \in C_0^\infty(\Omega)$, where $l = l_1 + l_2$.

<u>Proof</u>

It is immediate that

$$QPu = r(x,D)\, u$$

where

$$r(x,\xi) = q(x,\xi)\, p(\xi)$$

and that if $p \in S^{m_1}(\Omega)$ and $q \in S^{m_2}(\Omega)$ then $r \in S^{m_1+m_2}(\Omega)$ and that if $p \in \Sigma^{m_1}(\Omega)$ and $q \in \Sigma^{m_2}(\Omega)$, then $r \in \Sigma^{m_1+m_2}(\Omega)$. Thus it only remains to check that gRf extends to the given linear operators between Sobolev and Hölder spaces, for all $f, g \in C_0^\infty(\Omega)$. Choose $h \in C_0^\infty(\Omega)$ such that $h \equiv 1$ on supp g and define $T : C_0^\infty(\Omega) \to C^\infty(\Omega)$ by the identity

$$T = gQh^2P.$$

Then $T = t(x,D)$ for some $t \in S^{m_1+m_2}(\Omega)$ and

$$t(x,\xi) \sim \sum_\alpha \frac{(-i)}{\alpha!} \partial_\xi^\alpha \Big(g(x)\, q(x,\xi)\Big)\, \partial_x^\alpha \Big(h(x)^2\, p(\xi)\Big)$$

$$\sim g(x)\, q(x,\xi)\, p(\xi)$$

by the asymptotic expansion of Kohn and Nirenberg (cf. the remarks after definitions 4.1).

$$gR - T : C_0^\infty(\Omega) \to C^\infty(\Omega)$$

is a smoothing operator, and hence gRf - Tf extends to continuous linear operators

$$gRf - Tf : L_k^p(\Omega) \to C_0^\infty(\Omega),$$

$$gRf - Tf : C^{k,\alpha}(\Omega) \to C_0^\infty(\Omega).$$

But Tf = (gQh) (hPf), hence Tf extends to the required linear operators between Sobolev and Hölder spaces, and hence so does gRf.

□

The next result is taken from [Stein, E.M. and Weiss, G., 1972; theorem IV.4.1] (but note that the authors adopt a different definition of the Fourier transform from that adopted here).

Lemma 4.4

Let $s \in (0, \frac{n}{2})$ and let $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the function

$$u(x) = |x|^{s-n}.$$

Then the Fourier transform $\hat{u}$ of u is given by

$$\hat{u}(\xi) = \gamma_s |\xi|^{-s}$$

where

$$\gamma_s = \pi^{1/2} 2^s \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{n-s}{2})}.$$

Proof

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be identically equal to 1 on a neighbourhood of 0. Then $u = u_1 + u_2$, where $u_1 = \varphi u$ and $u_2 = (1 - \varphi)u$, $u_1 \in L^1(\mathbb{R}^n)$ and $u_2 \in L^2(\mathbb{R}^n)$. Then $\hat{u}_1 \in C^0(\mathbb{R}^n)$ and $\hat{u}_2 \in L^2(\mathbb{R}^n)$, using the Plancherel theorem. Hence $\hat{u} \in L^2_{loc}(\mathbb{R}^n)$. But u is homogeneous of degree $s - n$, hence $\hat{u}$ is a tempered distribution, homogeneous of degree $-s$. Hence

$$\hat{u}(\xi) = \gamma_s |\xi|^{-s}$$

for some constant γ_s. Now if $e \in S(\mathbb{R}^n)$ is a rapidly decreasing test function, then

$$\langle u, \hat{e} \rangle = \langle \hat{u}, e \rangle .$$

Let

$$e(\xi) = e^{-\frac{1}{2}|\xi|^2}.$$

Then

$$e(x) = (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}|x|^2}$$

hence

$$(2\pi)^{n/2} \int |x|^{s-n} \; e^{-\frac{1}{2}|x|^2} \, dx = \gamma_s \int |\xi|^{-s} \; e^{-\frac{1}{2}|\xi|^2} \, d\xi$$

and thus

$$(2\pi)^{n/2} \omega_{n-1} \int_0^\infty r^{s-n} \; e^{-\frac{1}{2}r^2} \; r^{n-1} \, dr = \gamma_s \, \omega_{n-1} \int_0^\infty r^{-s} \; e^{\frac{1}{2}r^2} \; r^{n-1} \, dr$$

where ω_{n-1} is the volume of the unit (n - 1) - sphere. Hence

$$(2\pi)^{\frac{n}{2}} \; 2^{\frac{s}{2}-1} \, \Gamma\left(\frac{s}{2}\right) = \gamma_s \; 2^{\frac{n-s}{2}-1} \, \Gamma\left(\frac{n-s}{2}\right)$$

hence

$$\gamma_s = \pi^{\frac{n}{2}} \; 2^s \; \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} .$$

□

Lemma 4.5

Let s be a positive real number, let $\eta : \mathbb{R}^n \to [0,1]$ be a smooth function which is identically equal to 0 in a neighbourhood of 0 and is identically equal to 1 outside a compact set, and let $Z_{-s} : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be the translation-invariant pseudo-differential operator defined by the identity

$$(Z_{-s}u)^\wedge(\xi) = \eta(\xi) \, |\xi|^{-s} \, \hat{u}(\xi).$$

Then for all functions $f, g \in C_0^\infty(\mathbb{R}^n)$, $gZ_{-s}f$ extends to continuous linear operators

$$gZ_{-s}f : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n),$$

$$gZ_{-s}f : C^{k,\alpha}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}(\mathbb{R}^n)$$

for all $p \in [1,\infty)$ and $\alpha \in [0,1)$, and for all non-negative integers k.

Proof

It suffices to consider the case when $s \in (0, \frac{n}{2})$, since for all $s \in (0,\infty)$ there exists an integer m such that

$$0 < \frac{s}{m} < \frac{n}{2}$$

and then

$$Z_{-s}u = Q^m u$$

where

$$(Qu)^{\wedge}(\xi) = (\eta(\xi))^{\frac{1}{m}} |\xi|^{-\frac{s}{m}} \hat{u}(\xi)$$

and if gQf extends to the required continuous linear operators between Sobolev and Hölder spaces for all $f, g \in C_0^\infty(\mathbb{R}^n)$, then so does $gZ_{-s}f$ by Lemma 4.3. Thus we now restrict ourselves to the case where $s \in (0, \frac{n}{2})$.

Let $c_1 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ and $c_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the functions defined by

$$c_1(\xi) = |\xi|^{-s},$$

$$c_2(\xi) = (1 - \eta(\xi)) |\xi|^{-s}.$$

Then

$$\eta(\xi) |\xi|^{-s} = c_1 - c_2,$$

since $s < \frac{n}{2}$, $c_2 \in L^2_{loc}(\mathbb{R}^n)$ and has compact support. Hence

$$(1 + |\xi|^2)^{K/2} c_2 \in L^2(\mathbb{R}^n)$$

for all k, and hence $c_2 = \hat{\varphi}_2$ where

$$\varphi_2 \in L^2_k(\mathbb{R}^n)$$

for all k, by the Plancherel theorem. Thus by the Sobolev embedding theorem

$$\varphi_2 \in C^\infty(\mathbb{R}^n).$$

Also $c_1 = \hat{\varphi}_1$, where

$$\varphi_1(x) = \gamma_s^{-1} |x|^{s-n},$$

$$\gamma_s = \pi^{\frac{n}{2}} 2^s \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{n-s}{2})}$$

by Lemma 4.4. Since $s > 0$

$$\varphi \in L^1_{loc}(\mathbb{R}^n).$$

Hence

$$\varphi \in L^1_{loc}(\mathbb{R}^n)$$

where $\varphi = \varphi_1 - \varphi_2$. But then

$$Z_{-s}u = \varphi * u$$

where $\varphi * u$ is the convolution of φ and u. Hence $gZ_{-s}f$ extends to the required continuous linear operators between Sobolev and Hölder spaces, for all $f, g \in C_0^\infty(\mathbb{R}^n)$ by Lemma 4.2.

$\square$

Lemma 4.6

Let m be an integer, let $\eta : \mathbb{R}^n \to [0, 1]$ be a smooth function which is identically equal to 0 in a neighbourhood of 0 and is identically equal to 1 outside a compact set, and let $Z_m : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be the translation-invariant pseudo-differential operator defined by the identity

$$(Z_m u)^\wedge(\xi) = \eta(\xi)\,|\xi|^m\,\hat{u}(\xi).$$

Then for all functions $f, g \in C_0^\infty(\mathbb{R}^n)$, $gZ_m f$ extends to continuous linear operators

$$gZ_m f : L^p_k(\mathbb{R}^n) \to L^p_{k-m}(\mathbb{R}^n), \quad (\forall\, k \geq m),$$

$$gS_m f : C^{k,\alpha}(\mathbb{R}^n) \to C^{k-m,\alpha}(\mathbb{R}^n), \quad (\forall\, k \geq m)$$

for all $p \in (1, \infty)$ and $\alpha \in (0,1)$.

Proof

It suffices to consider the cases $m \in \{-1, 0, 1\}$, since if $m \neq 0$ then

$$Z_m u = Q_\pm^{|m|} u$$

where

$$(Q_\pm u)^\wedge = (\eta(\xi))^{\frac{1}{|m|}}\,|\xi|^{\pm 1}\,\hat{u}(\xi)$$

and if $gQ_{\pm} f$ extends to the required continuous operators between Sobolev and Hölder spaces, for all $f, g \in C_0^\infty(\mathbb{R}^n)$, then so does $gZ_m f$ for all $m \in \mathbb{Z}$, by Lemma 4.3.

If $m = 0$

$$(Z_0 u)^\wedge(\xi) = \hat{u}(\xi) - (1 - \eta(\xi))\hat{u}(\xi).$$

But since $1 - \eta(\xi)$ is a smooth function with compact support, the last term is a smoothing operator, hence

$$Z_0 u = u - \varphi * u$$

for some $\varphi \in C^\infty(\mathbb{R}^n)$, hence $gZ_0 f$ extends to the required continuous linear operators, for all $f, g \in C_0(\mathbb{R}^n)$, by Lemma 4.2.

Next we consider the case $m = -1$. Note that

$$\left(\frac{\partial}{\partial x_j}(Z_{-1} u)\right)^\wedge(\xi) = i\xi_j (Z_{-1} u)^\wedge = -\eta(\xi)\left(-i\frac{\xi_j}{|\xi|} u\right).$$

But

$$-i\frac{\xi_j}{|\xi|}\hat{u} = (R_j u)^\wedge$$

where $R_j : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is the Riesz transform

$$R_j u = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \text{P.V.} \int \frac{x_j - y_j}{|x - y|^{n+1}} u(y)\, dy.$$

Let $c_j : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the function defined by

$$c_j(\xi) = -i(1 - \eta(\xi))\frac{\xi_j}{|\xi|}.$$

Then $c_j \in L^2_{loc}(\mathbb{R}^n)$ and has compact support. Hence

$$(1 + |\xi|^2)^{\frac{k}{2}} c_j \in L^2(\mathbb{R}^n),$$

and hence $c_j = \hat{\varphi}_j$, where

$$\varphi_j \in L^2_k(\mathbb{R}^n)$$

for all k, by the Plancherel theorem. Thus by the Sobolev embedding theorem

$$\varphi_j \in C^\infty(\mathbb{R}^n).$$

Thus

$$\frac{\partial}{\partial x_j}(Z_{-1}u) = -R_j u + \varphi_j * u.$$

Let f, $g \in C_o^{\infty}(\mathbb{R}^n)$. Then

$$\frac{\partial}{\partial x_j}(gZ_{-1}(fu)) = \frac{\partial g}{\partial x_j} Z_{-1}(fu) - gR_j(fu) + g(\varphi_j * (fu)).$$

Now $(\partial_j g) Z_{-1} f$ extends to continuous linear operators

$$(\partial_j g) Z_{-1} f : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n),$$

$$(\partial_j g) Z_{-1} f : C^{k,\alpha}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}(\mathbb{R}^n)$$

by Lemma 4.5. Also $\varphi_j \in L^1_{loc}(\mathbb{R}^n)$ hence if $T : C_o^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^n)$ is defined by

$$Tu = \varphi_j * u$$

then gTf extends to continuous linear operators

$$gTf : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n)$$

$$gTf : C^{k,\alpha}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}(\mathbb{R}^n)$$

by Lemma 4.2. By the theorems of Calderon and Zygmund (theorems 2.2 and 2.3) the Riesz operators extend to continuous linear operators

$$R_j : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n),$$

$$R_j : C^{k,\alpha}(\mathbb{R}^n) \longrightarrow C^{k,\alpha}(\mathbb{R}^n).$$

Hence

$$\left\| \frac{\partial}{\partial x_j}(gZ_{-1}(fu)) \right\|_{L_k^p} \leq A_{p,k} \|u\|_{L_k^p}$$

$$\left\| \frac{\partial}{\partial x_j}(gZ_{-1}(fu)) \right\|_{C^{k,\alpha}} \leq A_{k,\alpha} \|u\|_{C^{k,\alpha}}$$

for some constants $A_{p,k}$ and $A_{k,\alpha}$. Hence $gZ_{-1}f$ extends to continuous linear operators

$$gZ_{-1}f : L^p_k(\mathbb{R}^n) \rightarrow L^p_{k+1}(\mathbb{R}^n),$$

$$gZ_{-1}f : C^{k,\alpha}(\mathbb{R}^n) \rightarrow C^{k+1,\alpha}(\mathbb{R}^n),$$

for all $p \in (1,\infty)$, all $\alpha \in (0,1)$ and for all non-negative integers k.

Finally we consider the case $m = 1$.

$$(Z_1 u)^\wedge = \eta(\xi)|\xi| \hat{u}(\xi)$$

$$= \sum_{j=1}^{n} \eta(\xi)(i\xi_j)\left(-i\frac{\xi_j}{|\xi|}\right)\hat{u}$$

$$= \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(R_j u)\right)^\wedge - \left(1 - \eta(\xi)\right)|\xi|\hat{u}(\xi).$$

Now $(1 - \eta(\xi))|\xi| = \hat{\varphi}(\xi)$ for some $\varphi \in C^\infty(\mathbb{R}^n)$, as before.

Hence

$$Z_1 u = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(R_j u) - \varphi * u.$$

If $T : C_0^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ is defined by

$$Tu = \varphi * u$$

then for all $f,g \in C_0^\infty(\mathbb{R}^n)$, gTf extends to continuous linear operators

$$gTf : L^p_k(\mathbb{R}^n) \rightarrow L^p_k(\mathbb{R}^n),$$

$$gTf : C^{k,\alpha}(\mathbb{R}^n) \rightarrow C^{k,\alpha}(\mathbb{R}^n)$$

and $\partial_j \circ R_j$ extends to continuous linear operators

$$\partial_j \circ R_j : L^p_k(\mathbb{R}^n) \rightarrow L^p_{k-1}(\mathbb{R}^n),$$

$$\partial_j \circ R_j : C^{k,\alpha}(\mathbb{R}^n) \rightarrow C^{k-1,\alpha}(\mathbb{R}^n).$$

Hence gZ_1f extends to continuous linear operators

$$gZ_1f : L^p_k(\mathbb{R}^n) \rightarrow L^p_{k-1}(\mathbb{R}^n),$$

$$gZ_1f : C^{k,\alpha}(\mathbb{R}^n) \rightarrow C^{k-1,\alpha}(\mathbb{R}^n)$$

for all $p \in (1,\infty)$, all $\alpha \in (0,1)$ and for all positive integers k.

□

Lemma 4.7

Let Ω be an open set in $\mathbb{R}^n$, let $m \in \mathbb{R}$ and let $p : \Omega \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$ be a smooth function with the property that for all multi indices α and compact subsets K of Ω, there exists a constant $c_{\alpha, K}$ $(1 + |\xi|)^m$

for all $x \in K$ and $\xi \neq 0$. Then if $m + j < -n$, the function $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the identity

$$A(x,z) = (2\pi)^{-n} \int e^{iz.\xi} p(x, \xi) \, d\xi$$

is continuous and given multi indices α and β with $|\beta| \leq j$ and a compact subset K of Ω, then $\partial^{\alpha}_x \partial^{\beta}_z A(x,z)$ is continuous and bounded for all $(x,z) \in K \times \mathbb{R}^n$. Let $P : C^{\infty}_0(\Omega) \rightarrow C^{\infty}(\Omega)$ be the continuous linear operator defined by the identity

$$Pu(x) = (2\pi)^{-n} \int e^{ix.\xi} p(x, \xi) \, \hat{u}(\xi) \, d\xi .$$

Then

$$Pu(x) = \int A(x,z) \, u(x - z) \, dz$$

if $m + j < -n$, and hence gPf extends to continuous linear operators

$$gPf : L^p_k(\Omega) \rightarrow L^p_{k+j}(\Omega),$$

$$gPf : C^{k,\alpha}(\Omega) \rightarrow C^{k+j,\alpha}(\Omega)$$

for all $f,g \in C^{\infty}_0(\Omega)$, all $p \in [1,\infty)$, all real numbers $\alpha \in [0,1)$ and all non-negative integers k.

Proof

Let $m + j < -n$, and let K be a compact subset of Ω. The integrals defining A and all its derivatives $\partial^{\alpha}_x \partial^{\beta}_z A$ with $|\beta| < j$ are absolutely and uniformly convergent for $(x,z) \in K \times \mathbb{R}^n$ and the integrals are continuous functions of x and z. Also

$$Pu(x) = (2\pi)^{-n} \int e^{ix.\xi} \, p(x,\xi) \left(\int e^{-i(x-z).\xi} u(x-z) dz \right) d\xi$$

$$= \int A(x,z) \, u\,(x-z) \, dz.$$

Given $f,g \in C_0^\infty(\Omega)$, we see that

$$gP(fu)\,(x) = \int B(x,z) \, u\,(x-z) \, dz$$

where

$$B(x,z) = f(x) \, A\,(x,z) \, g\,(x-z).$$

$B(x,z)$ has compact support in x and z and all partial derivatives $\partial_x^\alpha \partial_z^\beta B$ with $|\beta| \leq j$ are continuous and uniformly bounded. It follows easily, using integration by parts, that gPf extends to the required continuous linear operators.

□

Corollary 4.8

Let Ω be an open set in $\mathbb{R}^n$, let $p \in S^m(\Omega)$ and define $P = p(x,D)$. If $m + j < -n$ then gPf extends to continuous linear operators

$$gPf : L_k^p(\Omega) \rightarrow L_{k+j}^p(\Omega),$$

$$gPf : C^{k,\alpha}(\Omega) \rightarrow C^{k+j,\alpha}(\Omega)$$

for all f,g $C_0^\infty(\Omega)$, all p $[1,\infty)$ all $\alpha \in [0,1)$ and all non-negative integers k.

Lemma 4.9

Let Ω be an open set in $\mathbb{R}^n$ and let $q : \Omega \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$ be a smooth function such that $q(x,\xi)$ is positively homogeneous of degree 0 in ξ, and let $\eta : \mathbb{R}^n \rightarrow [0,1]$ be a smooth function which is identically equal to 0 on a neighbourhood of 0 and is identically equal to 1 outside a compact set, and let $Q : C_0^\infty(\Omega) \rightarrow C^\infty(\Omega)$ be the pseudo-differential operator defined by

$$Qu = (2\pi)^{-n} \int e^{ix.\xi} q(x,\xi)\,\eta(\xi)\,\hat{u}(\xi)\,d\xi .$$

Then for all functions $f, g \in C_o(\Omega)$, gQf extends to continuous linear operators

$$gQf : L^p(\Omega) \to L^p_k(\Omega),$$

$$gQf : C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\Omega)$$

for all $p \in (1,\infty)$, for all $\alpha \in (0,1)$ and for all non-negative integers k.

Proof

Let

$$p(x,\xi) = (1-\eta(\xi))q(x,\xi)$$

and

$$Pu = (2\pi)^{-n} \int e^{ix.\xi} p(x,\xi)\,\hat{u}(\xi)\,d\xi .$$

Then for all multi indices α, all compact sets K of Ω and all $m \in \mathbb{R}$ there exists a constant $C_{\alpha,K,m}$ such that

$$\partial_x^\alpha p(x,\xi) \leq C_{\alpha,K,m}(1+|\xi|)^m.$$

Thus gPf extends to continuous linear maps

$$gPf : L^p_k(\Omega) \to C^\infty_o(\Omega),$$

$$gPf : C^{k,\alpha}(\Omega) \to C^\infty_o(\Omega)$$

for all $p \in (1,\infty)$, all $\alpha \in (0,1)$ and all non-negative integers k, by Lemma 4.7. But

$$Q = Q_o - P$$

where Q_o is the singular integral operator

$$Q_o u = (2\pi)^{-n} \int e^{ix.\xi} q(x,\xi)\,\hat{u}(\xi)\,d\xi .$$

Hence Q_o extends to continuous linear operators

$$Q_o : L^p_k(\mathbb{R}^n) \to L^p_k(\mathbb{R}^n),$$

$$Q_o : C^{k,}_{loc}(\mathbb{R}^n) \to C^{k,}_{loc}(\mathbb{R}^n)$$

for all $p \in (1,\infty)$, all $\alpha \in (0,1)$ and all non-negative integers k, by the theorems of Calderon and Zygmund (theorems 2.1, 2.2 and 2.3).

Hence gQf extends to the required continuous linear operators between Sobolev and Hölder spaces.

□

We are now ready to prove the main theorem of this section.

Theorem 4.10

Let Ω be an open set in $\mathbb{R}^n$, let $m \in \mathbb{Z}$, let $p \in \Sigma^m(\Omega)$ and let $P : C_o^\infty(\Omega) \rightarrow C^\infty(\Omega)$ be the pseudo-differential operator defined by

$$Pu(x) = p(x,D)\,u = (2\pi)^{-n} \int e^{ix.\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi .$$

Let $f,g \in C_o(\Omega)$. Then gPf extends to continuous linear operators

$$gPf : L_k^p(\Omega) \longrightarrow L_{k-m}^p(\Omega)\ , \quad 1 < p < \infty\ ,\ k \geq \max(m,0)$$

$$gPf : C^{k,\alpha}(\Omega) \rightarrow C^{k-m,\alpha}(\Omega), \quad 0 < \alpha < 1,\ k \geq \max(m,0)$$

Proof

Let the asymptotic expansion of $p(x,\xi)$ be

$$p(x,\xi) \sim \sum_j p_j(x,\xi)$$

where $p_j(x,\xi)$ is positively homogeneous of degree s_j in ξ, $s_j \leq m$. Let $\eta : \mathbb{R}^n \rightarrow [0,1]$ be a smooth function which is identically equal to 0 in a neighbourhood of 0 and is identically equal to 1 outside a compact set. Define smooth functions $q_j : \Omega \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$ by the identity

$$q_j(x,\xi) = p_j(x, \frac{\xi}{|\xi|})$$

and define $r_N \in S^{s_N}(\Omega)$ for all positive integers N by the identity

$$r_N(x,\xi) = p(x,\xi) - \eta(\xi)^3 \sum_{j=0}^{N-1} p_j(x,\xi)$$

and let $R_N = r_N(x,D)$. Also let $Z_s : C_o^\infty(\Omega) \rightarrow C^\infty(\Omega)$ be the translation-invariant pseudo-differential operator defined by the identity

$$(Z_s u) \hat{} = \eta(\xi)\,|\xi|^{-s}\,\hat{u}$$

and let $Q_j : C_o^{\infty}(\Omega) \to C^{\infty}(\Omega)$ be the pseudo-differential operator defined by the identity

$$Q_j u\,(x) = (2\pi)^{-n} \int e^{ix.\xi}\, q_j\,(x,\xi)\;\; \eta(\xi)\,\hat{u}(\xi)\,d\xi$$

for each non-negative integer j. We have shown that if $f,g \in C_o^{\infty}(\Omega)$, then $gZ_s f$, $gQ_j f$ and $gR_N f$ extend to continuous linear operators

$$gZ_m f : L_k^p(\Omega) \to L_{k-m}^p(\Omega),\; m \in \mathbb{Z},\; 1 < p < \infty,\; k \geq \max(m,0),$$

$$gZ_m f : C^{k,\alpha}(\Omega) \to C^{k-m,\alpha}(\Omega),\; m \in \mathbb{Z},\; 0 < \alpha < 1,\; k \geq \max(m,0),$$

$$gZ_s f : L_k^p(\Omega) \to L_k^p(\Omega),\; s < 0,\; 1 < p < \infty,\; k \geq 0,$$

$$gZ_s f : C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\Omega),\; s < 0,\; 0 < \alpha < 1,\; k \geq 0,$$

$$gQ_j f : L_k^p(\Omega) \to L_k^p(\Omega),\; 1 < p < \infty,\; k \geq 0,$$

$$gQ_j f : C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\Omega),\; 0 < \alpha < 1,\; k \geq 0,$$

$$gR_N f : L_k^p(\Omega) \to L_{k+l}^p(\Omega),\; s_N + l < -n,\; 1 < p < \infty,\; k \geq 0,$$

$$gR_N f : C^{k,\alpha}(\Omega) \to C^{k+l,\alpha}(\Omega),\; s_N + l < -n,\; 0 < \alpha < 1,\; k \geq 0,$$

by Lemmas 4.6, 4.9 and corollary 4.8. Moreover

$$P = \sum_{j=0}^{N-1} Q_j\, Z_{s_j - m}\, Z_m + R_N$$

where $s_j - m < 0$ for all j. By Lemma 4.3, $gZ_{s_j-m}\, Z_m\, f$ extends to continuous linear operators

$$gZ_{s_j-m}\, Z_m\, f : L_k^p(\Omega) \to L_{k-m}^p(\Omega),\; 1 < p < \infty,\; k \geq \max(m,0).$$

$$gZ_{s_j-m}\, S_m\, f : C^{k,\alpha}(\Omega) \to C^{k-m,\alpha}(\Omega),\; 0 < \alpha < 1,\; k \geq \max(m,0)$$

for all $f,g \in C_o(\Omega)$ and since $Z_{s_j-m}\, Z_m$ is a translation-invariant pseudo-differential operator, $gQ_j\, Z_{s_j-m}\, Z_m\, f$ extends to continuous linear operators

$$gQ_j Z_{s_j - m} Z_m f \ : \ L^p_k(\Omega) \to L^p_{k-m}(\Omega), \ 1 < p < \infty \ , \ k \geq \max(m,0),$$

$$gQ_j Z_{s_j - m} Z_m f \ : \ C^{k,\alpha}(\Omega) \to C^{k-m,\alpha}(\Omega), \ 0 < \alpha < 1, \ k \geq \max(m,0)$$

The result follows on choosing N sufficiently large.

□

We can extend the action of pseudo-differential operators to the dual spaces of the Sobolev spaces. We recall that if k is a non-positive integer and $1 < p < \infty$, then we define

$$L^p_k(\Omega) = L^{p'}_{-k}(\Omega)^*$$

for all open sets Ω in $\mathbb{R}^n$, where $L^{p'}_{-k}(\Omega)^*$ is the dual space of $L^p_{-k}(\Omega)$, and where $p' \in (1, \infty)$ satisfies the identity

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Note that by Hölder's inequality and by the Riesz representation theorem,

$$L^p_0(\Omega) = L^{p'}_0(\Omega)^*$$

so that the definition is consistent when $k = 0$.

<u>Theorem 4.11</u>

Let Ω be an open set in $\mathbb{R}^n$, let $m \in \mathbb{Z}$, let $p \in \Sigma^m(\Omega)$ and let $P : C^\infty_0(\Omega) \to C^\infty(\Omega)$ be the pseudo-differential operator defined by

$$Pu(x) = p(x,D)\,u = (2\pi)^{-n} \int e^{ix.\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi .$$

Let $f,g \in C^\infty_0(\Omega)$. Then gPf extends to a continuous linear operator

$$gPf \ : \ L^p_k(\Omega) \to L^p_{k-m}(\Omega), \ 1 < p < \infty \ , \ k \in \mathbb{Z}$$

<u>Proof</u>

If $k \geq \max(m,0)$, then the result follows from the previous theorem. If $k \geq \min(m,0)$ then $f\,{}^tPg$ extends to a continuous linear operator

$$f\,{}^tPg : L^{p'}_{k+m}(\Omega) \rightarrow L^{p'}_{-k}(\Omega)$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

But then

$$(f\,{}^tPg)^* = g\,{}^tP^*f$$

and hence the dual of $f\,{}^tPg$ is a continuous linear operator

$$g\,{}^tP^*f : L^p_k(\Omega) \rightarrow L^p_{k-m}(\Omega)$$

and if $u \in C^\infty_0(\Omega)$ then ${}^tP^*(j(u)) = Pu$ where $j : C^\infty_0(\Omega) \rightarrow L^p_k(\Omega)$ is the natural embedding. Since the image of j is dense in $L^p_k(\Omega)$ it follows that $g\,{}^tP^*f$ is the unique continuous extension of gPf to $L^p_k(\Omega)$. It remains to consider the cases $0 \leq k \leq m$ and $m \leq k \leq 0$. In these cases, let $r \in \Sigma^k(\Omega)$ be defined by

$$r(\xi) = (1 - \eta(\xi)) + \eta(\xi)|\xi|^k$$

where $\eta : \mathbb{R}^n \rightarrow [0, 1]$ is a smooth function which is identically equal to 0 in a neighbourhood of 0 and is identically equal to 1 outside a compact subset. Note that $r(\xi) > 0$ for all $\xi \in \mathbb{R}^n$. Define also $q \in \Sigma^{m-k}(\Omega)$ by the identity

$$q(x, \xi) = \frac{p(x, \xi)}{r(\xi)}.$$

Then

$$p(x, D) = q(x, D)\, r(D)$$

and if $h \in C^\infty_0(\Omega)$ and $h \equiv 1$ on supp g then the linear operator

$$g(x)\, p(x,D)\, f(x) - g(x)\, q(x,D)\, h(x)^2\, r(D)\, f(x)$$

is a smoothing operator, by the asymptotic expansion of Kohn and Nirenberg. Thus since $h(x)\, r(D)\, f(x)$ and $g(x)\, q(x,D)\, h(x)$ extend to continuous linear operators

$$h(x)\ r(D)\ f(X)\ :\ L^p_k(\Omega) \rightarrow L^p_0(\Omega),$$

$$g(x)\ q(x,D)\ h(x)\ :\ L^p_0(\Omega) \longrightarrow L^p_{k-m}(\Omega),$$

it follows that gPf extends to a continuous linear operator

$$gPf\ :\ L^p_k(\Omega) \longrightarrow L^p_{k-m}(\Omega)$$

when $0 \leq k \leq m$ or $m \leq k \leq 0$.

$\square$

§5. Some Elliptic Regularity Results

In this section, we prove some elliptic regularity results concerning linear elliptic differential operators with smooth coefficients. The theorems follow immediately from the following theorem on the continuity of pseudo-differential operators on compact manifolds.

Theorem 5.1

Let M be a compact smooth manifold, let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M and let $P : C^\infty(E) \to C^\infty(F)$ be a pseudo-differential operator of order not exceeding m, for some $m \in \mathbb{Z}$. Then P extends to continuous linear operators

$$P : L_k^p(E) \to L_{k-m}^p(F), \quad 1 < p < \infty, \quad k \in \mathbb{Z},$$

$$P : C^{k,\alpha}(E) \to C^{k-m,\alpha}(F), \quad 0 < \alpha < 1, \quad k \geq \max(m,0).$$

Proof

By using a partition of unity subordinate to a finite cover of M by coordinate neighbourhoods in M over which the vector bundles E and F are trivial, it suffices to show that $P\varphi$ extends to the required continuous linear operators between Sobolev and Hölder spaces whenever $\varphi : M \to \mathbb{R}$ is a smooth function with its support in the domain of some coordinate chart $x : \Omega \to \mathbb{R}^n$ over which the bundles E and F are trivial. Let ψ be a smooth function whose support is contained in Ω and which is identically equal to 1 on the support of φ. Then the operator

$$(1 - \psi)P\varphi : C^\infty(E) \to C^\infty(F)$$

is a smoothing operator, hence it suffices to show that the pseudo-differential operator

$$\psi P \varphi : C_0^\infty(E|\Omega) \to C^\infty(F|\Omega)$$

extends to continuous linear operators

$$\Psi P \varphi : L^p_k(E|\Omega) \to L^p_{k-m}(F|\Omega), \quad 1 < p < \infty, \; k \in \mathbb{Z}$$

$$\Psi P \varphi : C^{k,\alpha}(E|\Omega) \to C^{k-m,\alpha}(F|\Omega), \quad 0 < \alpha < 1, \; k \geq \max(m,0).$$

But this follows immediately from the corresponding results for pseudo-differential operators defined on open sets in $\mathbb{R}^n$ (theorems 4.10 and 4.11).

□

Theorem 5.2

Let M be a compact smooth manifold, let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M and let $L : C^\infty(E) \to C^\infty(F)$ be a linear elliptic differential operator of order m with smooth coefficients. Let f be a section of $\pi_2 : F \to M$ and let $u \in \mathcal{D}'(E)$ be a weak solution of the equation

$$Lu = f.$$

If p is a real number satisfying $1 < p < \infty$, K is an integer and $f \in L^p_k(F)$, then $u \in L^p_{k+m}(E)$. If α is a real number satisfying $0 < \alpha < 1$, k is a non-negative integer and $f \in C^{k,\alpha}(F)$, then $u \in C^{k+m,\alpha}(E)$.

Proof

Let $P : C^\infty(F) \to C^\infty(E)$ be a parametrix for L. Then

$$u = PLu + Ku = Pf + Ku$$

where $K : \mathcal{D}'(E) \to C^\infty(E)$ is a smoothing operator. But P extends to continuous linear operators

$$P : L^p_k(F) \to L^p_{k+m}(E),$$

$$0 : C^{k,\alpha}(F) \to C^{k+m,\alpha}(E),$$

from which the result follows immediately.

□

Let M be a compact smooth manifold, let μ be a smooth measure on M, let $\pi : E \to M$ be a smooth vector bundle over M, and let

$\beta \in C^{\infty}(E^* \otimes E^*)$ be a smooth section of $E^* \otimes E^*$ which restricts to a positive definite symmetric bilinear form on each fibre of E. We define an inner product on sections of E by

$$(e_1, e_2) = \int_M \beta(e_1, e_2)\, d\mu .$$

A linear differential operator $L : C^{\infty}(E) \to C^{\infty}(E)$ is self-adjoint if and only if

$$(Le_1, e_2) = (e_1, Le_2).$$

The results of Hodge theory for self-adjoint elliptic differential operators apply to the Sobolev spaces $L^p_k(E)$ for p satisfying $1 < p < \infty$, and are given in the following theorem.

Theorem 5.3

Let M be a compact smooth manifold, let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be smooth vector bundles over M and let $L : C^{\infty}(E) \to C^{\infty}(F)$ be a linear elliptic differential operator with smooth coefficients of order m. If k is an integer and if p satisfies $1 < p < \infty$, then the extension

$$L : L^p_k(E) \longrightarrow L^p_{k-m}(F)$$

of L to $L^p_k(E)$ is a Fredholm operator. If $k \geq m$ and $0 < \alpha < 1$, then the extension

$$L : C^{k,\alpha}(E) \longrightarrow C^{k-m,\alpha}(F)$$

of L to $C^{k,\alpha}(E)$ is a Fredholm operator.

Moreover if

$$(\,\cdot\,,\,\cdot\,) : C^{\infty}(E) \times C^{\infty}(E) \to \mathbb{R}$$

is a smooth inner product structure of E and $L : C^{\infty}(E) \to C^{\infty}(E)$ is a self-adjoint elliptic differential operator with smooth coefficients of order m, then the index of the Fredholm operators

$$L : L^p_k(E) \to L^p_{k-m}(E),$$

$$L : C^{k,\alpha}(E) \to C^{k-m,\alpha}(E)$$

is equal to zero and there exists a pseudodifferential operator

$$G : C^{\infty}(E) \to C^{\infty}(E)$$

of order $-m$ such that if $H : C^{\infty}(E) \to C^{\infty}(E)$ is the projection whose image is $H(E)$, where

$$H(E) = \{ e \in C^{\infty}(E) : Le = 0 \} ,$$

and whose kernel is the immage of L, then

$$I - LG = I - GL = H.$$

Proof

Let $P : C^{\infty}(E) \to C^{\infty}(E)$ be a pseudodifferential operator of order $-m$ which is parametrix of L. Then $LP - I$ and $PL - I$ are smoothing operators, hence

$$LP - I : L^p_{k-m}(F) \to L^p_{k-m}(E)$$

$$PL - I : L^p_k(E) \to L^p_k(E)$$

$$LP - I : C^{k-m,\alpha}(F) \to C^{k-m,\alpha}(F)$$

$$PL - I : C^{k,\alpha}(E) \to C^{k,\alpha}(E)$$

are compact operators. Hence

$$L : L^p_k(E) \to L^p_{k-m}(F)$$

$$L : C^{k,\alpha}(E) \to C^{k-m,\alpha}(F)$$

are Fredholm operators, thus proving the first part of the theorem.

Let $L : C^{\infty}(E) \to C^{\infty}(E)$ be self-adjoint with respect to the given inner product structure. Let $V_{p,k}(E)$ be the orthogonal complement of $H(E)$ with respect to the inner product structure. Then

$$L^p_k(E) = H(E) \oplus V_{p,k}(E)$$

and

$$V_{p,k}(E) = \text{image} \{ L : L^p_{k+m}(E) \to L^p_k(E) \}$$

since $H(E)$ is the orthogonal complement of the image of L, using the fact that L is self-adjoint. Then

$$L \mid V_{p,k}(E) : V_{p,k}(E) \rightarrow V_{p,k-m}(E)$$

is continuous and bijective and hence has a bounded inverse, by the Banach isomorphism theorem. Define $G : L^p_{k-m}(E) \rightarrow L^p_k(E)$ by

$$G \mid H(E) = 0$$

$$G \mid V_{p,k-m}(E) = (L \mid V_{p,k}(E))^{-1}$$

Then

$$I - LG = I - GL = H.$$

Since H is a smoothing operator, G is a parametrix of L. But any two parametrices of L differ by a smoothing operator, hence G is a pseudodifferential operator of order $-m$.

References for Chapter III

Atiyah, M.F. and Singer, I.M., "The index of elliptic operators, I." Ann. of Math. (1968), pp.484-530.

Aubin, T., Nonlinear analysis on manifolds, Monge-Ampere equations, Springer-Verlag, New York, (1982).

Bers, L., John, F. and Schechter, M., Partial differential equations, Wiley, New York, (1964).

Calderon, A.P. and Zygmund, A., "On the existence of certain singular integrals", Acta Math., 88 (1952), pp.85-139.

Calderon, A.P. and Zygmund, A., "On singular integrals", Amer. Journ. Math., 78 (1956), pp.289-309.

Calderon, A.P. and Zygmund, A. "Singular integral operators and differential equations", Amer. Journ. Math., 79 (1957), pp.901-921.

Coifman, R. amd Meyer, Y., "Au dela des operateurs pseudo-differentiels", Asterisque, no.57, 1978.

Hörmander, L. "Pseudodifferential operators", Comm. Pure Appl. Math., 18 (1965), pp.501-517.

Hörmander, L. "Pseudodifferential operators and hypoelliptic equations", in "Singular integrals". Proc. Sympos. Pure Math. Vol.10, Amer. Math. Soc., Providence R.I. (1967).

Illner, R., "A class of L^p bounded pseudo-differential operators", Proc. Amer. Math. Soc., 51 (1975), pp.347-355.

Kohn, J.J. and Nirenberg, L. "On the algebra of pseudodifferential operators", Comm. Pure Appl. Math., 18 (1965), pp.269-305.

Morrey, C.B., Multiple integrals in the calculus of variations, Springer-Verlag, New York, (1966).

Nirenberg, L., "Pseudo-differential operators" in "Global Analysis", Proc. Sympos. Pure Math., Vol.16, Amer. Math. Soc., Providence R.I. (1970).

Palais, R.S. et al, Seminar on the Atiyah-Singer index theorem, Princeton Univ. Press, Princeton, New Jersey, (1965).

Palais, R.S., Foundations of global non-linear analysis, Benjamin, New York, (1968).

Stein, E.M. and Weiss, G., Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, New Jersey, (1972).

Wells, R.O., Differential analysis on complex manifolds, Prentice Hall, Englewood Cliffs, New Jersey, (1973).

Chapter IV

AN INEQUALITY FOR FUNCTIONS ON RIEMANNIAN MANIFOLDS

§1. Introduction

In this chapter we prove an inequality (theorem 3.3) satisfied by continuous functions $f : M \to \mathbb{R}$ on a compact Riemannian manifold M. Given $f : M \to \mathbb{R}$ and given $m_1, m_2 \in M$, let $\mu_f(m_1, m_2)$ denote the infimum of the integrals of f with respect to arclength taken over all piecewise smooth paths from m_1 to m_2. Also let $d(m_1, m_2)$ be the distance from m_1 to m_2 defined using the Riemannian metric on M. Theorem 3.3 states that if $\dim M < p < \infty$ and if $\alpha \in (0, 1)$ is defined by

$$\alpha = 1 - \frac{\dim M}{p}$$

then

$$\mu_f(m_1, m_2) \leq K_p(d(m_1, m_2))^{\alpha} \| f \|_p$$

where K_p is a constant depending only on p and the Riemannian geometry of M, and where $\|f\|_p$ is the L^p norm of f with respect to the Riemannian volume measure on M.

In section §2, we shall study tubes about length minimizing geodesics in a compact Riemannian manifold, in preparation for section §3. In section §3 we shall prove the main result (theorem 3.3) and deduce from it a result (corollary 3.4) which applies when the manifold M is noncompact. We shall show how the Sobolev embedding theorem for the embedding of a Sobolev space in a Hölder space may be deduced from theorem 3.3.

§2. Geodesic Tubes about Length Minimizing Geodesics

In this section, we use compactness arguments to show that, given a compact Riemannian manifold M, there exists a positive constant R such that, for any length minimizing geodesic $\gamma : [a, b] \to M$, the geodesic tube of radius R about γ is embedded in M. R is independent of the choice of minimizing geodesic. Furthermore, we may choose R such that, on considering the exponential map as a diffeomorphism from the tube of radius R about the zero section of the normal bundle of γ to the geodesic tube of radius R about , the derivatives of the exponential map and its inverse both increase the lengths of tangent vectors by a factor of at most 2.

First we prove a topological lemma. A continuous map $f : X \to Y$ between topological spaces X and Y is said to be locally injective if, for all $x \in X$, there exists a neighbourhood U_x of x such that $f | U_x$ is injective.

Lemma 2.1

Let $f : X \to Y$ be a continuous map from a topological space X to a Hausdorff topological space Y, and let K be a compact subset of X. Suppose that $f : X \to Y$ is locally injective and that $f | K : K \to Y$ is injective. Then there exists an open neighbourhood U of K such that $f | U : U \to Y$ is injective.

Proof

For all $x \in K$, there exists an open neighbourhood U_x of x such that $f | U_x$ is injective. For all $y \in K \setminus U_x$, it follows that $f(y) \neq f(x)$ (since $f | K$ is injective) and hence that there exist open neighbourhoods $V_{x,y}$ of x and $W_{x,y}$ of y such that

$$f(V_{x,y}) \cap f(W_{x,y}) = \emptyset$$

(since Y is Hausdorff). Since K is compact, there exist $y_1, \ldots, y_n \in K$ such that $K \subset U_x \cup W_x$, where

$$W_x = \bigcup_{i=1}^{n} W_{x,y_i}.$$

Let

$$V_x = U_x \cap \bigcap_{i=1}^{n} V_{x,y_i}.$$

Then $x \in V_x$, $V_x \subset U_x$ and

$$f(V_x) \cap f(W_x) = \emptyset.$$

Let $N_x = U_x \cup W_x$. Then N_x is an open neighbourhood of K. If $z \in V_x$, $w \in N_x$ and $f(z) = f(w)$, then $w \notin W_x$, hence $w \in U_x$, and thus $z = w$, since $f | U_x$ is injective. Since K is compact, there exist $x_1, \ldots, x_m$ such that $K \subset V$, where

$$V = \bigcup_{i=1}^{m} V_{x_i}.$$

Let

$$N = \bigcap_{i=1}^{m} N_{x_i}.$$

Then N is an open neighbourhood of K. If $z \in V$, $w \in N$ and $f(z) = f(w)$, then $z = w$. Thus if $U = V \cap N$, then U is an open neighbourhood of K and $f | U$ is injective.

□

Theorem 2.2

Let M be a compact smooth Riemannian manifold. Given any geodesic $\gamma : [a, b] \to M$, let $N\gamma \to [a, b]$ denote the normal bundle of γ with its canonical flat Riemannian metric, let $B_R N\gamma$ denote the tube

$$B_R N_\gamma = \{ X \in N_\gamma : \| X \| \leqslant R \}$$

of radius R about the zero section of $N\gamma$, let $\exp_\gamma : N\gamma \to M$ denote the exponential map of γ, and let $\exp_{\gamma *} : TN\gamma \to TM$ denote its derivative. Then there exists a constant R, independent of the choice of geodesic, with the following property: if $\gamma : [a, b] \to M$ is a length minimizing geodesic in M, then

$$\exp_\gamma \mid B_R N\gamma : B_R N\gamma \to M$$

is a diffeomorphism onto its image, and if $Z \in T_V N\gamma$ for some $V \in B_R N\gamma$, then

$$\tfrac{1}{2} \| Z \| \leq \| \exp_{\gamma *} Z \| \leq 2 \| Z \| .$$

<u>Proof</u>

Let

$$STM = \{ X \in TM : \| X \| = 1 \}$$

and define a map $E : STM \times_M TM \to M$ in the following manner. Let $X \in ST_m M$ and $Y \in T_m M$. Then $Y = Y_1 + Y_2$ where Y_1 is a scalar multiple of X and Y_2 is perpendicular to X. Let $\gamma : [0, 1] \to M$ be the geodesic

$$\gamma(t) = \exp_m t Y_1$$

(recall that M is compact, and hence geodesically complete), let $q = \gamma(1)$, and let $V \in T_q M$ be the vector obtained from Y_2 by parallel transport along γ. Then define

$$E(X,Y) = \exp_q V.$$

Also, for all $X \in ST_m M$, define $E_X : T_m M \to M$ to be the map sending $Y \in T_m M$ to $E(X,Y) \in M$. Note that if Y is a scalar multiple of X, then the derivative of E_X at Y is an isometry.

Let K be the subset of $STM \times_M TM$ consisting of all elements (X,Y) such that $Y = \lambda X$ for some real number $\lambda \geq 0$, and also such that

$$\gamma : [0, 1] \to M : t \mapsto \exp t Y$$

is a length minimizing geodesic from $\gamma(0)$ to $\gamma(1)$. K is a closed subset of $STM \times_M M$. If $(X,Y) \in K$, then $\| Y \| \leq \operatorname{diam}(M)$. Since M is compact, K is also compact. Also if $(X,Y) \in K$, then the derivative of E_X at Y is an isometry. Hence there exists a neighbourhood U_1 of K such that if $X,Y \in T_m M$, if $\| X \| = 1$, if $(X,Y) \in U_1$, and if $Z \in T_Y T_m M$, then

$$\tfrac{1}{2} \| Z \| \leq \| E_{X*} Z \| \leq 2 \| Z \|$$

where $E_{X*} : TT_m M \rightarrow TM$ is the derivative of E_X. In particular, if $(X,Y) \in U_1$, then the derivative of E_X at Y is an isomorphism.

Now define $f : STM \times_M TM \rightarrow STM \times M$ by

$f(X,Y) = (X, E(X,Y))$.

The derivative of f at $(X, Y) \in STM \times_M TM$ is an isomorphism if and only if the derivative of E_X at Y is an isomorphism. In particular, if $(X, Y) \in U_1$, then the derivative of f at (X, Y) is an isomorphism, hence $f \mid U_1$ is a local diffeomorphism, by the inverse function theorem.

By definition of E, if $(X, Y) \in K$ then

$f(X, Y) = (X, \exp Y)$.

Hence if $(X_1, Y_1) \in K$ and $(X_2, Y_2) \in K$ and if

$f(X_1, Y_1) = f(X_2, Y_2)$

then there exists $m \in M$ such that $X_1, X_2, Y_1, Y_2 \in T_m M$. Also $X_1 = X_2 = X$ for some $X \in T_m M$. Then $Y_1 = \lambda X$ and $Y_2 = \mu X$ for some $\lambda, \mu \geq 0$, and the geodesics

$$\gamma_1 = [0, 1] \rightarrow M : t \mapsto \exp_m tY_1$$

$$\gamma_2 = [0, 1] \rightarrow M : t \mapsto \exp_m tY_2$$

are length minimizing geodesics satisfying $\gamma_1(1) = \gamma_2(1)$. It follows that $Y_1 = Y_2$. Thus we have shown that $f \mid K$ is injective.

Now $f \mid U_1$ is a local diffeomorphism and $f \mid K$ is injective, hence there exists an open neighbourhood U of K, contained in U_1, such that $f \mid U$ is a diffeomorphism, by the previous lemma. Then if $X, Y \in T_m M$, $\| X \| = 1$, $(X, Y) \in U$ and if $Z \in T_Y T_m M$ then

$$\tfrac{1}{2} \| Z \| \leq \| E_{X*} Z \| \leq 2 \| Z \|.$$

Using the compactness of K, it follows that there exists $R > 0$ such that if $(X, Y) \in STM \times_M TM$, if $Y = Y_1 + Y_2$ where Y_1 is parallel to X and Y_2 is perpendicular to X, if $(X, Y_1) \in K$, and if $\| Y_2 \| \leq R$, then $(X, Y) \in U$. We claim that R is the required constant.

Let $\gamma : [0, b) \to M$ be a length minimizing geodesic parameterized by arclength, and let $\pi : N\gamma \to [0, b]$ be the normal bundle of γ. Let $m = \gamma(0)$ and let $X = \gamma'(0)$. Then $X \in STM$. Define $\nu : N\gamma \to T_mM$ by

$$\nu(V) = \pi(V)X + \tau(V)$$

where $\tau : N\gamma \to T_mM$ is the map sending $V \in N\gamma$ to the vector $\tau(V)$ in T_mM obtained from V by parallel transport along γ from $\gamma(\pi(V))$ to m. Then ν is an isometry from N onto its image in T_mM, and if $V \in B_RN\gamma$, then $(X, \nu(V)) \in U$, and also

$$\exp_\gamma V = E(X, \nu(V)).$$

Hence $\exp_\gamma | B_RN\gamma : B_RN\gamma \to M$ is a diffeomorphism onto its image in M with the required properties.

$\square$

§3. An Inequality concerning Functions on Riemannian Manifolds

In this section, we consider the following situation. Let M be a compact smooth Riemannian manifold of dimension n, and let p be a real number satisfying $n < p < \infty$. We show that there exists a constant K_p, depending only on p and the Riemannian geometry of M such that for all continuous functions $f : M \to \mathbb{R}$, and for all $m_1, m_2 \in M$, the infimum $\mu_f(m_1, m_2)$ of the integrals of f with respect to arclength along all piecewise smooth paths from m_1 to m_2 satisfies

$$\mu_f(m_1, m_2) \leq K_p(d(m_1, m_2))^{\alpha} \| f \|_p$$

where $\|f\|_p$ is the L^p norm of f, $d(m_1, m_2)$ is the Riemannian distance from m_1 to m_2, and

$$\alpha = 1 - \frac{\dim M}{p} .$$

Those cases of the Sobolev embedding theorems dealing with embeddings of Sobolev spaces into C^k spaces or Hölder spaces follow easily from this inequality.

In what follows, we regard $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R}^{n-1}$. We denote the unit ball $\{ u \in \mathbb{R}^{n-1} : \|u\| \leq 1 \}$ in $\mathbb{R}^n$ by B^{n-1}. The volume of B^{n-1} with respect to the Euclidean metric is given by

$$\text{vol}\,(B^{n-1}) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} .$$

The main theorem of the section will follow from a Lemma concerning the behaviour of functions defined on cones in Euclidean space.

Lemma 3.1

Given $\varepsilon > 0$, $l > 0$, let $\Gamma \subset \mathbb{R}^n$ be the cone defined by

$$\Gamma = \{ (w, y) \in [0, l] \times \mathbb{R}^{n-1} : \| y \| \leq \varepsilon w \}$$

and, for all $u \in B^{n-1}$, let $c_u : [0, 1] \to \Gamma$ be the ray defined by

$$c_u(t) = (Lt, \varepsilon Ltu)$$

(on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R}^{n-1}$). Let $p \in (n, \infty)$, let $f \in L^p(\Gamma)$, and let

$$I = \frac{1}{\mathrm{vol}(B^{n-1})} \int_{B^{n-1}} \left(\int_0^1 |f(c_u(t))| \, \| c_u'(t) \| \, dt \right) du$$

be the mean value, taken over B^{n-1}, of the integrals of $|f|$ with respect to arclength along the rays c_u. Then

$$I \leq K_{p,n,\varepsilon} \, L^{\alpha} \, \| f \|_{\Gamma, p}$$

where

$$\alpha = 1 - \frac{n}{p},$$

where

$$\| f \|_{\Gamma, p} = \left(\int_{\Gamma} |f|^p \, dx \right)^{\frac{1}{p}}$$

and where

$$K_{p,n,\varepsilon} = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}} \left(\frac{p-1}{p-n} \right)^{\frac{p-1}{p}} \varepsilon^{\frac{1-n}{p}} \left(\int_{B^{n-1}} (1 + \varepsilon^2 u^2)^{\frac{p}{2(p-1)}} du \right)^{\frac{p-1}{p}}$$

is a constant depending on p, n and ε.

Proof

Since $C^0(\Gamma)$ is dense in $L^p(\Gamma)$, it suffices to prove the inequality for all $f \in C^0(\Gamma)$. Let $\varphi : [0, 1] \times B^{n-1} \to \Gamma$ be the map defined by

$$\varphi(t, u) = (Lt, \varepsilon Ltu).$$

Let dx be the volume form on $\mathbb{R}^n$ and let du be the volume form on $B^{n-1} \subset \mathbb{R}^{n-1}$. Then

$$\varphi^* dx = L(\varepsilon Lt)^{n-1} \, dt \wedge du.$$

Also

$$\| c_u'(t) \| = L(1 + \varepsilon^2 u^2)^{\frac{1}{2}},$$

hence

$$I = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}} \int_{[0,1]\times B^{n-1}} L(1+\varepsilon^2u^2)^{1/2}(\varphi^* f)\, dt\wedge du$$

$$= \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}} \int_{[0,1]\times B^{n-1}} (\varepsilon L t)^{1-n}(1+\varepsilon^2u^2)^{1/2}(\varphi^* f)\varphi^* dx$$

$$\leq \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}\, C_{p,n,\varepsilon,L}^{1-\frac{1}{p}} \left(\int_{[0,1]\times B^{n-1}} |f\circ\varphi|^p \varphi^* dx \right)^{1/p}$$

$$= \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}\, C_{p,n,\varepsilon,L}^{1-\frac{1}{p}} \left(\int_{\Gamma} |f|^p dx \right)^{1/p}$$

by Hölder's inequality, where

$$C_{p,n,\varepsilon,L} = \int_{[0,1]\times B^{n-1}} (\varepsilon L t)^{(1-n)q}(1+\varepsilon^2u^2)^{\frac{q}{2}}\, \varphi^* dx$$

and where q satisfies

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$C_{p,n,\varepsilon,L} = \int_{[0,1]\times B^{n-1}} L(\varepsilon L t)^{(n-1)(1-q)}(1+\varepsilon^2u^2)^{\frac{q}{2}}\, dt\wedge du$$

$$= L(\varepsilon L)^{(n-1)(1-q)} \int_0^1 t^{(n-1)(1-q)} dt \int_{B^{n-1}} (1+\varepsilon^2u^2)^{\frac{q}{2}}\, du$$

$$= L\frac{(L\varepsilon)^{(n-1)(1-q)}}{(n-1)(1-q)+1} \int_{B^{n-1}} (1+\varepsilon^2u^2)^{\frac{q}{2}}\, du,$$

provided that

$$(n-1)(1-q)+1 > 0.$$

But

$$q - 1 = \frac{1}{p-1}$$

hence

$$(n - 1)(1 - q) + 1 = 1 - \frac{n - 1}{p - 1} = \frac{p - n}{p - 1}.$$

Since $p > n$,

$$(n - 1)(1 - q) + 1 > 0$$

as required. Hence

$$C_{p,n,\varepsilon,L} = L^{\frac{p-n}{p-1}} \varepsilon^{\frac{1-n}{p-1}} \left(\frac{p-1}{p-n}\right) \int_{B^{n-1}} (1 + \varepsilon^2 u^2)^{\frac{p}{2(p-1)}} du.$$

Thus

$$C_{p,n,\varepsilon,L}^{1-\frac{1}{p}} = L^{1-\frac{n}{p}} \varepsilon^{\frac{1-n}{p}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}} \left(\int_{B^{n-1}} (1+\varepsilon^2 u^2)^{\frac{p}{2(p-1)}} du\right)^{\frac{p-1}{p}}.$$

Hence

$$I \leq K_{p,n,\varepsilon} L^{1-\frac{n}{p}} \| f \|_{\Gamma,p}.$$

$\square$

<u>Corollary 3.2</u>

Given $\varepsilon > 0$, $L > 0$, let $T \subset \mathbb{R}^n$ be the tube

$$T = \{ (w,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : w \in [0, 2L], \| y \| \leq \varepsilon L \}$$

about $[0, 2L]$ of radius εL, and let P be the set of piecewise smooth paths $c : [0, 1] \to T$ from $(0, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to $(2L, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ which are contained in T. For any continuous function $f : T \to \mathbb{R}$, define

$$m(f) = \inf \left\{ \int_0^1 |f(c(t))| \; \| c'(t) \| \, dt : c \in P \right\}.$$

m(f) is the infimum of the integrals of $|f|$ with respect to arc-length along all paths belonging to P. Let $p \in (n,\infty)$. Then there exists a constant $K_{p,n,\varepsilon}$ depending only on p,n and ε such that if

$f : T \to \mathbb{R}$ is a continuous function on T, then

$$m(f) \leq K_{p,n,\varepsilon}\, L^{\alpha} \left(\int_T |f|^p \, dx \right)^{1/p}$$

where $\alpha \in (0, 1)$ is defined by

$$\alpha = 1 - \frac{n}{p}$$

<u>Proof</u>

Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ denote the reflection defined by

$$\tau(w, y) = (2L - w, y)$$

for all $(w, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and let $\Gamma \subset \mathbb{R}^n$ be the cone defined by

$$\Gamma = \{ (w, y) \in [0, L] \times \mathbb{R}^{n-1} : \| y \| \leq \varepsilon w \}.$$

Then $\Gamma \subset T$ and $\tau(\Gamma) \subset T$. For $u \in B^{n-1}$, let $c_u : [0, 1] \to \Gamma$ be the ray defined by

$$c_u(t) = (Lt, \varepsilon L t u)$$

(on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R}^{n-1}$). Then the product path $v_u = c_u * (\tau c_u)^{-1}$ consisting of c_u followed by τc_u reversed is a piecewise smooth path from $(0, 0)$ to $(2L, 0)$ (via $(L, \varepsilon L u)$), and if $f : T \to \mathbb{R}$ is continuous and

$$J_f = \frac{1}{\mathrm{vol}(B^{n-1})} \int_{B^{n-1}} \left(\int_0^1 |f(v_u(t))| \, \| v_u'(t) \| \, dt \right) du,$$

then

$$\begin{aligned} \mu(f) &\leq J_f \\ &\leq K_{p,n,\varepsilon}\, L^{1-\frac{n}{p}} \left(\left(\int_\Gamma |f|^p dx \right)^{\frac{1}{p}} + \left(\int_{\tau(\Gamma)} |f|^p dx \right)^{1/p} \right) \\ &\leq K_{p,n,\varepsilon}\, L^{\alpha} \left(\int_{\Gamma \cup \tau(\Gamma)} |f|^p dx \right)^{1/p} \\ &\leq K_{p,n,\varepsilon}\, L^{\alpha} \left(\int_T |f|^p dx \right)^{1/p} \end{aligned}$$

by the previous lemma.

□

Using this corollary, and using properties of geodesic tubes about length minizing geodesics, we may prove the main theorem of this section.

Theorem 3.3

Let M be a compact Riemannian manifold of dimension n (possibly with boundary). For all continuous functions $f : M \to \mathbb{R}$, and for all $m_1, m_2 \in M$, let $\mu_f(m_1, m_2)$ denote the infimum, over all piecewise smooth paths $c : [a, b] \to M$ from m_1 to m_2, of the integrals

$$\int_a^b |f(c(t))| \; \| c'(t) \| \; dt$$

of $|f|$, with respect to arclength, along c. Then, for all $p \in (n, \infty)$, there exists a constant K_p, depending only on p and on the Riemannian geometry of M, such that if $f : M \to \mathbb{R}$ is a continuous function on M, then

$$\mu_f(m_1, m_2) \leq K_p \, (d(m_1, m_2))^{\alpha} \, \| f \|_{M,p}$$

for all $m_1, m_2 \in M$, where $\alpha \in (0, 1)$ is defined by

$$\alpha = 1 - \frac{n}{p} ,$$

where $d(m_1, m_2)$ is the distance from m_1 to m_2 with respect to the Riemannian metric, and where

$$\| f \|_{M,p} = \left(\int_M |f|^p \, d(\mathrm{vol}) \right)^{\frac{1}{p}} .$$

Proof

First, we restrict our attention to the case when $\partial M = \emptyset$. Then there exists a constant R such that, for all length minimizing geodesics $\gamma : [a, b] \to M$, the exponential map

$$\exp_\gamma \big| B_R N\gamma : B_R N\gamma \to M$$

is a diffeomorphism onto its image, and such that if $Z \in T_V N\gamma$ for some $V \in B_R N\gamma$, then

$$\tfrac{1}{2} \| Z \| \leq \| \exp_* Z \| \leq 2 \| Z \|$$

(here $B_R N\gamma$ is the tube of radius R about the sero section of $N\gamma$, consisting of all vectors of length not exceeding R in the normal bundle $N\gamma \to [a, b]$ of γ). This follows from theorem 2.2.

Let diam (M) be the diameter of M, and let

$$\varepsilon = \frac{2R}{\text{diam }(M)} .$$

Let m_1, $m_2 \in M$ and let $\gamma : [0,\sigma] \to M$ be a length minimizing geodesic from m_1 to m_2, parameterized by arclength (such a geodesic always exists, since M is compact, and since we are restricting our attention to the case when M is without boundary). Let $\pi : N\gamma \to [0, \sigma]$ be the normal bundle of γ. Then the tube T of radius $\tfrac{1}{2}\sigma\varepsilon$ about the zero section of $N\gamma$ is contained in $B_R N\gamma$. Let P be the set of piecewise smooth paths $v : [0, 1] \to T$ in the tube T from $\exp_\gamma^{-1}(m_1)$ to $\exp_\gamma^{-1}(m_2)$. T has a natural flat Riemannian metric and is isometric to the corresponding tube in $\mathbb{R}^n$. By the corollary to the previous Lemma, if $g : T \to \mathbb{R}$ is a continuous function on T and if

$$m(g) = \inf \left\{ \int_0^1 \left| f(v(t)) \right| \| v'(t) \| \, dt : v \in P \right\}$$

is the infimum, taken over piecewise smooth paths in T from $\exp_\gamma^{-1}(m_1)$ to $\exp_\gamma^{-1}(m_2)$, of the integrals of $|g|$ with respect to arclength along the curves, then

$$m(g) \leq K'_{p,n,\varepsilon} \left(\frac{\sigma}{2} \right)^{\alpha} \left(\int_T |g|^p \, dx \right)^{\frac{1}{p}}$$

where $K'_{p,n,\varepsilon}$ is a constant depending only on p, n and ε. But $\exp_\gamma | T : T \to M$ increases the length of tangent vectors to T by a factor of at most 2, hence

$$\mu_f(m_1, m_2) \leq 2\, m\, (f \circ \exp)$$

$$2\, K'_{p,n,\varepsilon} \left(\frac{\sigma}{2}\right)^{\alpha} \left(\int_T |f \circ \exp|^p \, dx \right)^{1/p}$$

$$2^{n+1}\, K'_{p,n,\varepsilon} \left(\frac{d(m_1, m_2)}{2}\right)^{\alpha} \left(\int_M |f|^p \, d(\mathrm{vol}) \right)^{1/p}$$

$$K_p \left(d\,(m_1, m_2) \right)^{\alpha} \left(\int_M |f|^p \, d\,(\mathrm{vol}) \right)^{1/p}$$

where

$$K_p = 2^{n+1-\alpha}\, K_{p,n,\varepsilon} \,.$$

Here we have used the fact that

$$|dx| \leq 2^n \, | \exp_{\gamma}{}^* \, d\,(\mathrm{vol}) |$$

where d (vol) is the volume measure on m and dx is the volume measure on T. This fact follows from the fact that if $Z \in T_V N\gamma$ for some $V \in B_R N\gamma$, then

$$2 \, \| Z \| \geq \| \exp_{\gamma *} Z \| \,.$$

The constant K_p depends only on p, n, R and diam (M), and thus only on p and on the Riemannian geometry of M. Thus we have proved the theorem in the case when $\partial M = \emptyset$.

Now suppose that $\partial M \neq \emptyset$. First note that if we prove the theorem for one particular Riemannian metric on M, then the result holds for any Riemannian metric on M.

Let $j_1 : M \to M_1$ and $j_2 : M \to M_2$ be diffeomorphisms of M onto disjoint copies M_1 and M_2 of M. Let 2M be the smooth manifold obtained from the disjoint union of M_1 and M_2 by identifying $j_1(m) \in M_1$ and with $j_2(m) \in M_2$ for all $m \in \partial M$. Let $p : M_1 \vee M_2 \to 2M$ be the identification map, and let $i_1 : M \to 2M$ and $i_2 : M \to 2M$ be the maps $i_1 = p \circ j_1$ and $i_2 = p \circ j_2$. Then 2M is a compact smooth manifold without boundary, $i_1 : M \to 2M$ and $i_2 : M \to 2M$ are embeddings of M in 2M, $2M = i_1(M) \cup i_2(M)$ and $i_1(M) \cap i_2(M) \cong \partial M$. We have a smooth involution $\tau : 2M \to 2M$ defined by the property that $\tau \circ i_1 = i_2$ and $\tau \circ i_2 = i_1$. Given a smooth Riemannian metric g_0 on 2M, we obtain a smooth τ-invariant Riemannian metric g on 2M by defining

$$g = \tfrac{1}{2}(g_0 + \tau^* g_0).$$

Then there exists a unique smooth Riemannian metric on M such that i_1 and i_2 are isometric embeddings.

We have a piecewise smooth map $\nu : 2M \to M$ sending $i_1(m)$ and $i_2(m)$ to m, for all $m \in M$. Then $i_1 \circ \nu : 2M \to 2M$ is a piecewise smooth map whose image is contained in $i_1(M)$ and which preserves the lengths of piecewise smooth curves. Let $m_1, m_2 \in i_1(M)$. Suppose that γ is a length minimizing geodesic from m_1 to m_2. Then $i_1 \circ \nu \circ \gamma$ is a path from m_1 to m_2 with the same length as γ, and is thus also a length minimizing geodesic from m_1 to m_2. Thus $i_1 \circ \nu \circ \gamma = \gamma$, unless both $m_1 \in i_1(\partial M)$ and $m_2 \in i_1(\partial M)$, in which case either $i_1 \circ \nu \circ \gamma = \gamma$ or $i_1 \circ \nu \circ \gamma = \tau \circ \gamma$. Thus any two points in $i_1(M)$ may be joined by a length minimizing geodesic lying wholly within $i_1(M)$. If $f : M \to \mathbb{R}$ is continuous, then $f \circ \nu : 2M \to \mathbb{R}$ is continuous, and we have already shown that

$$\mu_{f\circ\nu}(m_1, m_2) \leq K_p \left(d(m_1, m_2)\right)^{\alpha} \left(\int_{2M} |f \circ \nu|^p \, d(\mathrm{vol}) \right)^{1/p}$$

for some constant K_p depending only on the Riemannian geometry of 2M. Thus

$$\mu_f(m_1, m_2) \leq 2^{1/p} K_p \left(d(m_1, m_2)\right)^{\alpha} \left(\int_{M} |f|^p \, d(\mathrm{vol}) \right)^{1/p}.$$

Hence the theorem is true when $\partial M \neq \emptyset$ also.

□

One may easily deduce part of the Sobolev embedding theorems from this theorem. For let M be a compact n-dimensional manifold, let $p \in (n,\infty)$, and let f be a C^1 function on M. If we define

$$\| f \|_p = \left(\int_M |f|^p \, d(\mathrm{vol}) \right)^{1/p}$$

then there exists a point $m \in M$ such that

$$|f(m)| \leq \mathrm{vol}(M)^{-\frac{1}{p}} \| f \|_p .$$

Let $g = |df|$. Then g is continuous, and if $m_1, m_2 \in M$, then

$$|f(m_1) - f(m_2)| \leq \mu_g(m_1,m_2)$$
$$K_p(d(m_1,m_2))^{\alpha} \| df \|_p$$

for some constant K_p depending only on p and on the Riemannian geometry of M, where

$$\alpha = 1 - \frac{\dim M}{p}$$

and where $\mu_g : M \times M \to \mathbb{R}$ is the function defined in the previous theorem. Applying this result with $m_1 = m$, we obtain

$$\sup_M |f| \leq \mathrm{vol}(M)^{-\frac{1}{p}} \| f \|_p + K_p(\mathrm{diam}(M))^{\alpha} \| df \|_p$$
$$\leq C_p \| f \|_{p,1}$$

where C_p is a constant depending only on p and on the Riemannian geometry of M, and where $\| . \|_{p,1}$ is the L_1^p-norm. Since $C^1(M)$ is dense in $L_1^p(M)$, it follows that we have a continuous embedding

$$L_1^p(M) \hookrightarrow C^0(M).$$

Also

$$\sup_{m_1 \neq m_2} \frac{|f(m_1) - f(m_2)|}{(d(m_1,m_2))^{\alpha}} \leq K_p \| df \|_p$$
$$K_p \| f \|_{p,1}$$

and thus we have a continuous embedding

$$L_1^p(M) \hookrightarrow C^{0,\alpha}(M)$$

where $C^{0,\alpha}(M)$ is the Hölder space with exponent $\alpha \in (0,1)$ given by

$$\alpha = 1 - \frac{\dim M}{p}.$$

There is an analogue of theorem 3.3 when M is not compact.

Corollary 3.4

Let M be a Riemannian manifold of dimension n. For all continuous functions $f : M \to \mathbb{R}$, for all bounded domains D in M and for all $m_1, m_2 \in D$, let $\mu_{f,D}(m_1,m_2)$ denote the infimum, over all piecewise smooth paths $c : [a, b] \to D$ from m_1 to m_2 of the integrals

$$\int_a^b |f(c(t))| \, \|c'(t)\| \, dt$$

of $|f|$, with respect to arclength, along c. Then, for all $p \in (n, \infty)$ and for all bounded domains D and D_1 satisfying

$$\bar{D} \subset \operatorname{int} D_1$$

there exists a constant K_{p,D,D_1} depending only on p, on the Riemannian geometry of M and on the domains D and D_1 such that if $f : M \to \mathbb{R}$ is a continuous function on M, then

$$\mu_{f,D}(m_1,m_2) \leq K_{p,D,D_1} (d(m_1,m_2))^{\alpha} \, \|f\|_{D_1,p}$$

for all $m_1, m_2 \in D$, where $\alpha \in (0, 1)$ is defined by

$$\alpha = 1 - \frac{n}{p},$$

where $d(m_1,m_2)$ is the distance from m_1 to m_2 with respect to the Riemannian metric on M, and where

$$\|f\|_{D_1,p} = \left(\int_{D_1} |f|^p \, d(\mathrm{vol}) \right)^{\frac{1}{p}}$$

Proof

As in the proof of theorem 3.3 we may assume that $\partial M = \emptyset$. Then there exists a smooth function $\varphi : M \to [0, 1]$ such that φ has compact support contained in D_1 and $\varphi \equiv 1$ on a neighbourhood of D. By Sard's theorem, there exists a regular value t of φ in the open interval (0, 1). Then $\varphi^{-1}([t, 1])$ is a compact manifold with boundary. The result then follows from theorem 3.3.

□

An alternative proof of corollary 3.4 not involving Sard's theorem could be constructed as follows. If M is a smooth Riemannian manifold (possibly noncompact) we could apply theorem 3.3 to compact geodesically convex sets in M with smooth boundary. By a well-known theorem of J.H.C. Whitehead, the interiors of such sets form a base for the topology of M. The domain D in the statement of corollary 3.4 may be covered by a finite number of compact geodesically convex balls with smooth boundary which are contained in the interior of the domain D_1. By the Lebesque covering theorem, there exists $\delta > 0$ such that if $m_1, m_2 \in D$ then either m_1 and m_2 both belong to one of these geodesically convex balls or else $d(m_1, m_2) \geq \delta$. If $d(m_1, m_2) < \delta$ then the required inequality follows from theorem 3.3 applied to the geodesically convex ball containing m_1 and m_2. If $d(m_1, m_2) \geq \delta$ then we can find a finite sequence of points of D whose first member is m_1 and whose last member is m_2 with the property that any pair of successive members of the sequence is contained in one of the geodesically convex balls. One can then bound $\mu_{f,D}(m_1, m_2)$ in terms of $\| f \|_{D_1, p}$ as required.

An examination of the proof of theorem 6.3 shows that if M is a sufficiently well-behaved Riemannian manifold, such as a symmetric space, then it is in principle possible to find an upper bound on the constant K_p by studying the properties of geodesic tubes about length minimizing geodesics using Jacobi fields.

Chapter V

PRINCIPAL BUNDLES AND CONNECTIONS

§1. Introduction

In this chapter, we give an account of Ehresmann connections on a principal bundle and of the action of principal bundle automorphisms on such connections, in preparation for subsequent chapters. This account is based to some extent on [Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978] and [Bourguignon, J.-P. and Lawson, H.B., 1981].

In §2 we review the construction of fibre bundles associated to a given principal bundle $\pi : P \to M$ as described in, for example, chapter 9 of [Auslander, L. amd MacKenzie, R.E., 1963]. Two such associated bundles of particular importance are the adjoint bundles $P \times_{ad} G \to M$ and $P \times_{Ad} \mathfrak{g} \to M$, where G is the structural group of $\pi : P \to M$ and $\mathfrak{g}$ is the Lie algebra of G. We shall denote $P \times_{ad} G$ by G_P and $P \times_{Ad} \mathfrak{g}$ by $\mathfrak{g}_P$. It is easily seen that each fibre of $G_P \to M$ acts on the corresponding fibre of any fibre bundle associated to $\pi : P \to M$. An important general principle is the following: given some structure on the fibre F of some fibre bundle $\pi_F : F_P \to M$ associated to $\pi : P \to M$, where this structure is invariant under the action of G on F, then we may define a corresponding structure on every fibre of the map $\pi_F : F_P \to M$ in a canonical way, and moreover this structure is invariant under the action of each fibre of $G_P \to M$ on the corresponding fibre of $\pi_F : F_P \to M$. For example, this structure on F may be a group structure on F, a Riemannian metric on F, a vector space structure on F, a Lie algebra structure on F or a vector space norm on F. Using this principle, we shall show that if G and M are compact, then any biinvariant

Riemannian metric on M determine a canonical biinvariant distance function on $C^0(G_P)$ and canonical norms on the Banach spaces $C^0(\mathfrak{g}_P)$, $L^q(\mathfrak{g}_P)$, $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$, where q satisfies $1 \leq q < \infty$. Moreover these canonical norms are invariant under the action of $C^0(G_P)$ on these Banach spaces. We shall make use of this property in chapter VI. There appears to be no obvious analogue of this result which applies to the Sobolev spaces $L^q_k(\mathfrak{g}_P)$ and $L^q_k(\mathfrak{g}_P \otimes T^*M)$ when $k \neq 0$.

In §3 we define Ehresmann connections and holonomy groups and review their basic properties. This material is standard.

In §4 we define principal bundle automorphisms and study their action on connections. We show that the stabilizer of a smooth Ehresmann connection in the group of smooth principal bundle automorphisms is naturally isomorphic to the centralizer of the holonomy group of the connection (theorem 4.2). This result has been used in studying the singularities in the moduli space of instantons over a 4-manifold that play an important role in the proof of Donaldson's theorem on the intersection form of a smooth 4-manifold (see [Donaldson, S.K., 1983]).

In §5 we show that given two Ehresmann connections ω_1 and ω_2 on the principal bundle $\pi : P \longrightarrow M$, then their difference $\omega_1 - \omega_2$ may be identified with a section of the vector bundle $\mathfrak{g}_P \otimes T^*M \longrightarrow M$. We shall then show that if $\Psi : P \longrightarrow P$ is a principal bundle automorphism and if $\|.\|$ is the canonical norm on $C^0(\mathfrak{g}_P \otimes T^*M)$ or $L^q(\mathfrak{g}_P \otimes T^*M)$ then

$$\|\Psi^*\omega_1 - \Psi^*\omega_2\| = \|\omega_1 - \omega_2\|$$

(Lemma 5.1). We also prove a theorem (theorem 5.2) which relates the distance between two principal bundle automorphisms Ψ_1 and Ψ_2 evaluated at the endpoints of a piecewise smooth curve in M to the integral of $|\Psi_1{}^*\omega - \Psi_2{}^*\omega|$ along the curve, for any Ehresmann connection ω on the principal bundle. We shall make use of this result in chapter VI.

In §6 we review the well-known construction whereby, given an Ehresmann connection ω on a principal bundle $\pi : P \longrightarrow M$ we may split the tangent bundle $TE \longrightarrow E$ of an associated fibre bundle $E \longrightarrow M$ into the Whitney sum of the vertical bundle $VE \longrightarrow E$ and a horizontal bundle $HE \longrightarrow E$, where VE consists of all vectors tangent to the fibres of $E \longrightarrow M$. This enables us to define the covariant $D^{\omega}s : TM \longrightarrow VE$ of a section $s : M \longrightarrow E$ of $E \longrightarrow M$ with respect to the given connection in the obvious way. If $E \longrightarrow M$ is a vector bundle then we may use this construction to define the covariant differential $d^{\omega}s : M \longrightarrow E \otimes T^*M$ of a section $s : M \longrightarrow E$ of $E \longrightarrow M$. Given a smooth connection ω we shall define a first order differential operator

$$\mathcal{X}^{\omega} : C^{\infty}(G_P) \longrightarrow C^{\infty}(\mathfrak{g}_P \otimes T^*M)$$

and a fibre bundle morphism

$$B : C^{\infty}(\mathfrak{g}_P) \longrightarrow C^{\infty}(\mathrm{End}\ (\mathfrak{g}_P \otimes T^*M)$$

with the properties that

$$\mathcal{X}^{\omega}(\Psi) = \Psi^*\omega - \omega$$
$$\mathcal{X}^{\omega}(\exp \xi) = B(\xi)\, d^{\omega}\xi$$

for all principal bundle automorphisms Ψ and for all $\xi \in C^{\infty}(\mathfrak{g}_P)$, where

$$\exp : C^{\infty}(\mathfrak{g}_P) \longrightarrow C^{\infty}(G_P)$$

is the exponential map. The reason for introducing these operators is that in chapter VI we shall express the equations governing the action of principal bundle automorphisms on connections in terms of and B and by examining the form of these operators we may use the results of /Palais, R.S., 1968/, which we have summarized in chapter II, in order to deduce smoothness results for the action of

Banach Lie groups of principal bundle automorphisms on the appropriate Banach spaces of connections.

In section §7 we review the basic formalism of the covariant exterior derivative, covariant codifferential and covariant Hodge-de Rham Laplacian as developed in [Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978] and in [Bourguignon, J.-P. and Lawson, H.B., 1981].

§2. The Adjoint Bundles

In this section, we summarize first the construction of fibre bundles associated to a given principal bundle. We apply this construction to define the adjoint bundles $\pi_{ad} : G\mathbf{p} \to M$ and $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$ of a principal bundle $\pi : P \to M$ with structural group G whose Lie algebra is $\mathfrak{g}$. It is shown that, given a biinvariant metric on G, the biinvariant distance function on G resulting from this Riemannian metric determines a distance function on each fibre of $\pi_{ad} : G\mathbf{p} \to M$, and hence determines a biinvariant distance function, the canonical distance function, on the group $C^o(G\mathbf{p})$ of continuous sections of $\pi_{ad} : G\mathbf{p} \to M$. Similarly, the G-invariant norm on $\mathfrak{g}$ resulting from the Riemannian metric on G determines a norm on each fibre of $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$, and hence determines norms, the canonical norms, on the vector spaces $C^o(\mathfrak{g}\mathbf{p})$ and $L^q(\mathfrak{g}\mathbf{p})$ of continuous sections and L^q sections respectively of $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$ for all $q \in [1, \infty)$; also, given a Riemannian metric on M, the invariant norm on $\mathfrak{g}$ determines norms on each fibre of $\mathfrak{g}\mathbf{p} \otimes T^*M \to M$, and hence determines norms, the canonical norms, on $C^o(\mathfrak{g}\mathbf{p} \otimes T^*M)$ and $L^q(\mathfrak{g}\mathbf{p} \otimes T^*M)$. The norms on $C^o(\mathfrak{g}\mathbf{p})$, $L^q(\mathfrak{g}\mathbf{p})$, $C^o(\mathfrak{g}\mathbf{p} \otimes T^*M)$ and $L^q(\mathfrak{g}\mathbf{p} \otimes T^*M)$ are shown to be invariant under the adjoint action of $C^o(G\mathbf{p})$.

Let $\pi : P \to M$ be a smooth principal bundle with structural group G, acting on P on the right. Let F be a smooth manifold on which G acts smoothly, with action $\theta : G \to \mathrm{Diff}(F)$. Then we can construct a fibre bundle $\pi_\theta : P \times_\theta F \to M$ with fibre F associated to the principal bundle $\pi : P \to M$. The total space $P \times_\theta F$ of this fibre bundle is defined to be the quotient space of $P \times F$ by the equivalence relation $\sim$, where

$$(p.\gamma^{-1}, f) \sim (p, \theta(\gamma) f)$$

for all $p \in P$, $\gamma \in G$, $f \in F$. Let $[p, f] \in P \times_\theta F$ denote the equivalence class of $(p, f) \in P \times F$. The projection $\pi_\theta : P \times_\theta F \to M$ is defined by

$$\pi_\theta([p, f]) = \pi(p)$$

and each element p of P determines a diffeomorphism $\nu_p : \pi^{-1}(m) \to F$, where $m = \pi(p)$ and

$$\nu_p([p, f]) = f$$

for all $f \in F$. If $\gamma \in G$ then

$$\nu_{p.\gamma^{-1}} = \theta \circ \nu_p,$$

since

$$\nu_{p.\gamma^{-1}}([p, f]) = \nu_{p.\gamma^{-1}}([p.\gamma^{-1}, \theta(\gamma) f]).$$

The structural group G acts on P on the right and on F on the left. A map $\beta : P \to F$ is said to be <u>G-equivariant</u> if and only if

$$\beta(p.\gamma^{-1}) = \theta(\gamma)\beta(p)$$

for all $p \in P$ and $\gamma \in G$. There is a bijective correspondence between sections of $\pi_\theta : P \times_\theta F \to M$ and G-equivariant maps $\beta : P \to F$. This correspondence sends a section $s : M \to P \times_\theta F$ to the G-equivariant map

$$p \mapsto \nu_p(s(\pi(p))).$$

This correspondence sends C^k sections of $P \times_\theta F$ to G-equivariant C^k maps from P to F.

Suppose that (F, ρ) is a metric space with distance function $\rho : F \times F \to \mathbb{R}$ (we refer to such a function as a 'distance function' rather than as a 'metric' in order to distinguish between 'distance functions' and 'Riemannian metrics'; note however that a Riemannian metric on F determines a distance function on F, the distance between two points of F being the infimum of the lengths of all piecewise smooth paths joining these points, assuming that F is connected). The distance function $\rho : F \times F \to \mathbb{R}$ is G-invariant if and only if

$$\rho\,(\theta\,(\gamma)\,f_1,\ \theta\,(\gamma)\,f_2) = \rho\,(f_1, f_2)$$

for all $f_1, f_2 \in F$ and $\gamma \in G$. If $\rho : F \times F \longrightarrow \mathbb{R}$ is a G-invariant distance function, then for all $m \in M$ there is a unique distance function ρ_m on the fibre $\pi_\theta^{-1}(m)$ of $\pi_\theta : P \times_\theta F \longrightarrow M$ over m with the property that

$$\rho_m\,(e_1, e_2) = \quad (\nu_p\,(e_1),\ \nu_p\,(e_2))$$

for all $e_1, e_2 \in \pi_\theta^{-1}(m)$ and for all $p \in \pi^{-1}(m)$, where $\nu_p : \pi_\theta^{-1}(m) \longrightarrow F$ is the diffeomorphism determined by p.

If M is compact, then we obtain a distance function $\hat{\rho}$ on $C^0(P \times_\theta F)$, the space of continuous sections of $\pi_\theta : P \times_\theta F \longrightarrow M$ by defining

$$\hat{\rho}\,(s_1, s_2) = \sup_{m \in M} \rho_m\,(s_1(m),\ s_2(m)\,).$$

If $s_1, s_2 \in C^0(P \times_\theta F)$ and if $\sigma_1 : P \longrightarrow F$ and $\sigma_2 : P \longrightarrow F$ are the corresponding G-equivariant maps, then

$$\hat{\rho}\,(s_1,\ s_2) = \sup_{p \in P}\,(\sigma_1\,(p), \sigma_2\,(p)\,).$$

A special case occurs when the fibre F of the fibre bundle is a normed vector space on which G acts as a group of vector space automorphisms preserving the norm $|\,.\,|$. Then there is a unique norm $|\,.\,|_m$ on the fibre $\pi_\theta^{-1}(m)$ of $\pi_\theta : P \times_\theta F \longrightarrow M$ over $m \in M$ such that

$$|\,e\,|_m = |\,\nu_p\,(e)\,|$$

for all $e \in \pi_\theta^{-1}(m)$ and $p \in \pi^{-1}(m)$, where $\nu_p : \pi_\theta^{-1}(m) \longrightarrow F$ is the isomorphism determined by p. If M is compact, then we have a norm $\|\,.\,\|$ on $C^0(P \times_\theta F)$ defined by

$$\|\,s\,\| = \sup_{m \in M} |\,s(m)\,|_m\,.$$

If $\sigma : P \longrightarrow F$ is the G-equivariant map determined by the section $s : M \longrightarrow P \times_\theta F$, then

$$\|\,s\,\| = \sup_{p \in P} |\,\sigma\,(p)\,|\,.$$

If we are given a smooth measure μ on M, then, for all $q \in [1, \infty)$ we may define the norm $\| \cdot \|_q$ on $L^q(P \times_\theta F)$ by

$$\| s \|_q = \left(\int_M | s(m) |_m^q \, d\mu \right)^{1/q} .$$

Now consider the case when F_1 and F_2 are smooth manifolds with smooth actions $\theta_1 : G \longrightarrow \mathrm{Diff}(F_1)$ and $\theta_2 : G \longrightarrow \mathrm{Diff}(F_2)$, and let $\varphi : F_1 \longrightarrow F_2$ be a G-equivariant smooth map (φ is G-equivariant if and only if $\theta_2(\gamma) \circ \varphi = \varphi \circ \theta_1(\gamma)$ for all $\gamma \in G$). Then φ induces a smooth fibre bundle morphism $\varphi_P : P \times_{\theta_1} F_1 \longrightarrow P \times_{\theta_2} F_2$.

We apply these general results to the bundles $P \times_{ad} G$ and $P \times_{Ad} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a compact Lie group G with a biinvariant Riemannian metric, G being the structural group of a smooth principal bundle $\pi : P \longrightarrow M$ over a compact Riemannian manifold M. The adjoint action

$$ad : G \longrightarrow \mathrm{Diff}(G)$$

of G on G is the map sending $\gamma \in G$ to the inner automorphism $\beta \mapsto \gamma \beta \gamma^{-1}$ of G. We denote the manifold $P \times_{ad} G$ by G_P. G_P is the total space of a smooth fibre bundle $\pi_{ad} : G_P \longrightarrow M$. The adjoint representation

$$Ad : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

of G on $\mathfrak{g}$ is the map sending $\gamma \in G$ to the derivative $Ad(\gamma) : \mathfrak{g} \to \mathfrak{g}$ of $ad(\gamma) : G \longrightarrow G$ at the identity element of G. We denote the manifold $P \times_{Ad} \mathfrak{g}$ by $\mathfrak{g}_P$. $\mathfrak{g}_P$ is the total space of a smooth vector bundle $\pi_{Ad} : \mathfrak{g}_P \longrightarrow M$. We denote the fibre of $\pi_{ad} : G_P \longrightarrow M$ over $m \in M$ by $G_P[m]$.

The biinvariant Riemannian metric on G determines a biinvariant distance function $\rho : G \times G \longrightarrow \mathbb{R}$. It also determines a norm $| \cdot |$ on $\mathfrak{g}$ invariant under the adjoint representation of G. Also the Riemannian metric on M determines a smooth measure μ on M, the volume measure on M.

Proposition 2.1

Let the principal bundle $\pi : P \to M$, the structural group G, the adjoint bundle $\pi_{ad} : G_P \to M$, and the biinvariant distance function $\rho : G \times G \to \mathbb{R}$ on G be as above.

Then, for all $m \in M$ and for all $p \in \pi^{-1}(m)$, $G_P[m]$ is a Lie group and the diffeomorphism from $G_P[m]$ to G determined by p is an isomorphism of Lie groups. These group operations on each fibre of $\pi_{ad} : G_P \to M$ induce a corresponding group structure on the space $C^0(G_P)$ of continuous sections of $\pi_{ad} : G_P \to M$.

For all $m \in M$, there is a unique distance function $\rho_m : G_P[m] \times G_P[m] \to \mathbb{R}$ with the property that, for all $p \in G_P[m]$, the isomorphisms from $G_P[m]$ to G determined by p is an isometry of metric spaces. ρ_m is then a biinvariant distance function on the Lie group $G_P[m]$. The biinvariant metric on G thus determines a distance function $\hat{\rho}$ on $C^0(G_P)$ defined by

$$\hat{\rho}(s_1, s_2) = \sup_{m \in M} \rho_m(s_1(m), s_2(m))$$

and if $\sigma_1 : P \to G$, $\sigma_2 : P \to G$ are the G-equivariant maps corresponding to $s_1 : M \to G_P$ and $s_2 : M \to G_P$, then

$$\sigma_1(p.\gamma^{-1}) = \gamma\,\sigma_1(p)\,\gamma^{-1}$$

and similarly for σ_2, and

$$(s_1, s_2) = \sup_{p \in P} (\sigma_1(p), \sigma_2(p)).$$

The distance function $\hat{\rho}$ on $C^0(G_P)$ is biinvariant, and thus the group operations on the metric space $(C^0(G_P), \hat{\rho})$ are continuous.

Proof

Let $m \in M$, let $p \in \pi^{-1}(m)$ and let $\nu_p : G_P[m] \to G$ be the diffeomorphism determined by p. Then

$$\nu_{p.\gamma^{-1}} = ad(\gamma) \circ \nu_p$$

for all $\gamma \in G$, hence $\nu_{p.\gamma^{-1}} \circ \nu_p^{-1} : G \to G$ is an inner automorphism of G. Thus there is a unique group structure on $G_p[m]$ such that $\nu_p : G_p[m] \rightarrow G$ is an isomorphism of Lie groups for all $p \in \pi^{-1}(m)$. We have already seen that there is a well-defined distance function $\rho_m : G_p[m] \times G_p[m] \rightarrow \mathbb{R}$ such that $\nu_p : G_p[m] \rightarrow G$ is an isometry of metric spaces for all $p \in \pi^{-1}(m)$. Since $\rho : G \times G \rightarrow \mathbb{R}$ is biinvariant and since $\nu_p : G_p[m] \rightarrow G$ is a Lie group isomorphism, it follows that $\rho_m : G_p[m] \times G_p[m] \rightarrow \mathbb{R}$ is also biinvariant.

The map $\sigma : P \rightarrow G$ is G-equivariant with respect to the adjoint action of G on G if and only if

$$\sigma(p.\gamma^{-1}) = \gamma\, \sigma(p)\gamma^{-1}$$

and we have already seen that

$$\hat{\rho}(s_1,s_2) = \sup_{p \in P} (\sigma_1(p), \sigma_2(p))$$

where σ_1 and σ_2 are the G-equivariant maps from P to G corresponding to s_1 and s_2. It remains to show that $\hat{\rho}$ is biinvariant and that the group operations on the metric space $(C^0(G_p), \hat{\rho})$ are continuous. But the biinvariance of $\hat{\rho}$ is an immediate corollary of the biinvariance of ρ_m for all $m \in M$. But then

$$\hat{\rho}(s_1,s_2) = \hat{\rho}(s_1^{-1}\, s_1\, s_2^{-1},\ s_1^{-1}\, s_2\, s_2^{-1})$$
$$= \hat{\rho}(s_2^{-1},\ s_1^{-1})$$

for all sections $s_1, s_2 \in C^0(G_p)$, hence the map sending s to s^{-1} is an isometry of $(C^0(G_p), \hat{\rho})$. Also if $s_1, s_2, s_1', s_2' \in C^0(G_p)$, then

$$\hat{\rho}(s_1 s_1', s_2 s_2') \leq \hat{\rho}(s_1 s_1', s_2 s_1') + \hat{\rho}(s_2 s_1', s_2 s_2')$$
$$\leq \hat{\rho}(s_1, s_2) + \hat{\rho}(s_1', s_2')$$

hence the map $(s,s') \mapsto ss'$ is a continuous map from $C^0(G_p) \times C^0(G_p)$ to $C^0(G_p)$. Hence the group operations on $(C^0(G_p), \hat{\rho})$ are continuous.

□

We refer to the distance function $\hat{\rho} : C^0(G\mathbf{p}) \times C^0(G\mathbf{p}) \to \mathbb{R}$, defined in the above proposition, as the <u>canonical distance function</u> (or <u>canonical metric</u>) on $C^0(G\mathbf{p})$ determined by the biinvariant Riemannian metric on G.

<u>Proposition 2.2</u>

Let the principal bundle $\pi : P \to M$, the Lie algebra $\mathfrak{g}$ of the structural group, the adjoint bundle $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$, the invariant norm $|\cdot|$ on $\mathfrak{g}$, and the Riemannian volume measure μ on M be as above.

Then, for all $m \in M$ and for all $p \in \pi^{-1}(m)$, $\mathfrak{g}\mathbf{p}[m]$ is a Lie algebra and the vector space isomorphism from $\mathfrak{g}\mathbf{p}[m]$ to $\mathfrak{g}$ defined by the element p of P is an isomorphism of Lie algebras. The Lie bracket on each fibre of $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$ induces a corresponding Lie bracket on the space $C^0[\mathfrak{g}\mathbf{p}]$ of continuous sections of $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$.

For all m M there is a unique norm $|\cdot|_m$ on $\mathfrak{g}\mathbf{p}[m]$ with the property that, for all $p \in \mathfrak{g}\mathbf{p}[m]$, the isomorphism from $\mathfrak{g}\mathbf{p}[m]$ to $\mathfrak{g}$ is an isometry of normed vector spaces. $|\cdot|_m$ determines a norm $\|\cdot\|$ on $C^0(\mathfrak{g}\mathbf{p})$ defined by

$$\|a\| = \sup_{m \in M} |a(m)|_m$$

and if $\alpha : P \to \mathfrak{g}$ is the G-equivariant map corresponding to $a : M \to \mathfrak{g}\mathbf{p}$, then

$$\alpha(p.\gamma^{-1}) = Ad(\gamma)\alpha(p)$$

and

$$\|a\| = \sup_{p \in P} |\alpha(p)| .$$

The Lie bracket is a continuous map $C^0(\mathfrak{g}\mathbf{p}) \times C^0(\mathfrak{g}\mathbf{p}) \to C^0(\mathfrak{g}\mathbf{p})$.

For all $q \in [1, \infty)$ we have a norm $\|\cdot\|_q$ on $L^q(\mathfrak{g}\mathbf{p})$ defined by

$$\|a\|_q = \left(\int_M |a(m)|_m^q \, d\mu\right)^{1/q}$$

and if $q,r,s \in [1, \infty)$ and

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{s}$$

then the Lie bracket on each fibre of $\pi_{Ad} : \mathfrak{g}_P \to M$ induces a continuous bilinear map $L^q(\mathfrak{g}_P) \times L^r(\mathfrak{g}_P) \to L^s(\mathfrak{g}_P)$.

Proof

The proof is analogous to that of the previous proposition, with the exception of the last part, which follows from Hölder's inequality.

$\square$

We refer to the norm $\|\cdot\|$ on $C^0(\mathfrak{g}_P)$, defined in the above proposition, as the canonical norm on $C^0(\mathfrak{g}_P)$ determined by the biinvariant Riemannian metric on G. We refer to the norms $\|\cdot\|_q$ on $L^q(\mathfrak{g}_P)$, for any $q \in [1, \infty)$, as the canonical norms on $L^q(\mathfrak{g}_P)$ determined by the biinvariant Riemannian metric on G and the smooth measure on M. If this measure is the volume measure of a given Riemannian metric on M, we refer to $\|\cdot\|_q$ as the canonical norm on $L^q(\mathfrak{g}_P)$ determined by the biinvariant Riemannian metric on G and the Riemannian metric on M.

For all $\gamma_1, \gamma_2 \in G$, the exponential map $\exp : \mathfrak{g} \to G$ satisfies

$$Ad(\gamma_1)(\exp \gamma_2) = \exp(ad(\gamma_1)\gamma_2)$$

and is thus G-equivariant and so induces a smooth fibre bundle morphism $\exp_P : \mathfrak{g}_P \to G_P$. Also we have a smooth fibre bundle morphism $Ad_P : G_P \to End(\mathfrak{g}_P)$, induced by $Ad : G \to End(\mathfrak{g})$.

Proposition 2.3

Let the principal bundle $\pi : P \to M$, the structural group G with Lie algebra $\mathfrak{g}$, the adjoint bundles $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$, the biinvariant distance function $\rho : C^0(G_P) \times C^0(G_P) \to \mathbb{R}$ on $C^0(G_P)$, and the canonical norms $\|\cdot\|$ and $\|\cdot\|_q$ on $C^0(\mathfrak{g}_P)$ and $L^q(\mathfrak{g}_P)$ respectively be as above.

Then the map exp : $\mathfrak{g} \rightarrow G$ induces a smooth fibre bundle morphism $\exp_P : \mathfrak{g}_P \rightarrow G_P$. This in turn induces a continuous map $\exp : C^0(\mathfrak{g}_P) \rightarrow C^0(G_P)$ between metric spaces, and if $a \in C^0(\mathfrak{g}_P)$ and if e is the identity section of $C^0(G_P)$, then

$$\hat{\rho}(\exp a, e) = \| a \| .$$

The adjoint representation Ad : $G \rightarrow \mathrm{End}(\mathfrak{g})$ induces a smooth fibre bundle morphism $\mathrm{Ad}_P : G_P \rightarrow \mathrm{End}(\mathfrak{g}_P)$. This in turn induces adjoint representations

$$C^0(G_P) \times C^0(\mathfrak{g}_P) \rightarrow C^0(\mathfrak{g}_P) : (s,a) \mapsto \mathrm{Ad}(s)a,$$

$$C^0(G_P) \times L^q(\mathfrak{g}_P) \rightarrow L^q(\mathfrak{g}_P) : (s,a) \mapsto \mathrm{Ad}(s)a.$$

If $s : M \rightarrow G_P$ is a section of $G_P \rightarrow M$ and if $a : M \rightarrow \mathfrak{g}_P$ is a section of $\mathfrak{g}_P \rightarrow M$, then

$$\| \mathrm{Ad}(s)\, a \| = \| a \| ,$$

$$\| \mathrm{Ad}(s)\, a \|_q = \| a \|_q$$

for all $q \in [1, \infty)$. Thus the canonical norms of $C^0(\mathfrak{g}_P)$ and $L^q(\mathfrak{g}_P)$ are invariant under the action of the group $C^0(G_P)$.

Proof

The continuity of $\exp : C^0(\mathfrak{g}_P) \rightarrow C^0(G_P)$ and $\mathrm{Ad} : C^0(G_P) \rightarrow \mathrm{End}(C^0(\mathfrak{g}_P))$ follow by elementary compactness arguments. If $|\cdot|_m$ is the canonical norm on the fibre $\mathfrak{g}_P[m]$ of $\mathfrak{g}_P \rightarrow M$ over $m \in M$ and if ρ_m is the canonical distance function on the fibre $G_P[m]$ of $G_P \rightarrow M$ over m, and if $a \in C^0(\mathfrak{g}_P)$, then

$$\rho_m(\exp_P(a(m)), e(m)) = |a(m)|_m,$$

hence

$$\hat{\rho}(\exp a, e) = \| a \| .$$

Also if $s \in C^0(G_P)$, then

$$| \mathrm{Ad}_P(s(m))\, a(m) |_m = | a(m) |_m$$

hence

$$\| \mathrm{Ad}(s)\, a \| = \| a \|$$

and $\| \mathrm{Ad}(s)\, a \|_q = \| a \|_q$

for all $q \in [1, \infty)$.

$\square$

Finally we consider the bundle $\mathfrak{g}_P \otimes T^*M \to M$.

<u>Proposition 2.4</u>

Let the principal bundle $\pi : P \to M$ over the Riemannian amnifold M, the structural group G with Lie algebra $\mathfrak{g}$, the adjoint bundles $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$, the invariant norm $|\cdot|$ on $\mathfrak{g}$, and the norms $|\cdot|_m$ on the fibres $\mathfrak{g}_P[m]$ of $\mathfrak{g}_P \to M$ be as above.

Consider the vector bundle $\mathfrak{g}_P \otimes T^*M \to M$. The fibre of this bundle over $m \in M$ is isomorphic to the vector space $L(T_mM, \mathfrak{g}_P[m])$ of linear transformations from the tangent space T_mM of M at m to the fibre $\mathfrak{g}_P[m]$ of $\mathfrak{g}_P \to M$ over m. This vector space has a norm $|\cdot|_m$ defined by

$$|S|_m = \sup \{ |SX|_m : X \in T_mM,\ |X| = 1 \}$$

for any $S \in L(T_mM, \mathfrak{g}_P[m])$. If μ is the volume measure of the Riemannian manifold M, we may define norms $\|\cdot\|$ and $\|\cdot\|_q$ on $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$ respectively, for all $q \in [1, \infty)$, by

$$\|\tau\| = \sup_{m \in M} |\tau(m)|_m$$

$$\|\tau\|_q = \left(\int_M |\tau(m)|_m^q \, d\mu \right)^{1/q}$$

The smooth bundle morphism $\mathrm{Ad}_P : G_P \to \mathrm{End}(\mathfrak{g}_P)$ induces a smooth bundle morphism $\mathrm{Ad}_P : G_P \to \mathrm{End}(\mathfrak{g}_P \otimes T^*M)$, and hence induces continuous adjoint representations

$$C^0(G_P) \times C^0(\mathfrak{g}_P \otimes T^*M) \to C^0(\mathfrak{g}_P \otimes T^*M) : (s, \tau) \mapsto \mathrm{Ad}(s)\tau,$$

$$C^0(G_P) \times L^q(\mathfrak{g}_P \otimes T^*M) \to L^q(\mathfrak{g}_P \otimes T^*M) : (s, \tau) \mapsto \mathrm{Ad}(s)\tau.$$

Then

$$\| \mathrm{Ad}(s)\, \tau \| = \| \tau \|$$

$$\| \mathrm{Ad}(s)\, \tau \|_q = \| \tau \|_q$$

for all $q \in [1, \infty)$. Thus the canonical norms of $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$ are invariant under the action of the group $C^0(G_P)$.

Proof

The proof is exactly analogous to that of the previous proposition.

□

We refer to the norms $\| \cdot \|$ and $\| \cdot \|_q$ on $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$ as the canonical norms determined by the biinvariant Riemannian metric on G and the Riemannian metric on M.

§3. Connections and Holonomy

This section is a review of basic facts about Ehresmann connections on principal bundles and their holonomy groups. See [Ambrose, W. and Singer, I.M., 1952] or [Kobayashi, S. and Nomizu, K., 1963] for more details.

Let $\pi : P \to M$ be a smooth principal bundle with structural group G whose Lie algebra is $\mathfrak{g}$. A tangent vector to P is vertical if and only if it is annihilated by the derivative $\pi_* : TP \to TM$ of π. There is a canonical mapping

$\mathfrak{g} \to$ vertical vector fields on P

sending $a \in \mathfrak{g}$ to the vector field $\sigma(a)$ on P whose value $\sigma_p(a)$ at $p \in P$ is tangent to the curve $t \mapsto p.\exp ta$ at $t = 0$. The map $a \mapsto \sigma(a)$ is a Lie algebra homomorphism. $\sigma(a)$ is referred to as the fundamental vector field on P determined by a.

An Ehresmann connection $\omega : TP \to \mathfrak{g}$ on $\pi : P \to M$ is a 1-form on P with the following two properties:

(i) $\omega(\sigma_p(a)) = a$ for all $p \in P$ and $a \in \mathfrak{g}$,

(ii) $R_\gamma^* \omega = \mathrm{Ad}(\gamma^{-1}) \circ \omega$ for all $\gamma \in G$,

where $R_\gamma^* \omega$ is the pullback of ω under the map $R_\gamma : P \to P$ mapping p to $p.\gamma$.

A tangent vector to P is horizontal if and only if it is annihilated by $\omega : TP \to \mathfrak{g}$. If VP and HP are the subbundles of TP consisting of all vertical and all horizontal vectors respectively, then TP decomposes as a Whitney sum

$$TP = VP \oplus HP.$$

A piecewise differentiable curve in P is horizontal if and only if all tangent vectors to the curve are horizontal. If $c : [t_0, t_1] \to M$ is a piecewise smooth curve, and if $p \in P$ satisfies $\pi(p) = c(t_0)$, then there is a unique piecewise smooth horizontal curve $\tilde{c}_p = [t_0, t_1] \to P$

such that $\tilde{c}_p(t_0) = p$. If $\gamma \in G$, then

$$\tilde{c}_{p.\gamma}(t) = (\tilde{c}_p(t)).\gamma .$$

Let $h : TP \to HP$ be the projection onto the horizontal bundle HP over P whose kernel is the vertical bundle VP. The curvature $F^{\omega} : \Lambda^2 TP \to \mathfrak{g}$ of ω is the $\mathfrak{g}$-valued 2-form on P defined by

$$F^{\omega}(X,Y) = d\omega(hX, hY)$$

for all vector fields X and Y on P. F^{ω} has the following two properties:

(i) $F^{\omega}(X, Y) = 0$ if either X or Y is vertical,

(ii) $R^*_{\gamma} F = Ad(\gamma^{-1}) \circ F$ for all $\gamma \in G$.

Given a point $p \in P$, the holonomy bundle attached to p, B(p), is the set of all points of P which may be joined to p by a piecewise smooth horizontal curve in P. The holonomy group attached to p, H(p), is a group of all $\gamma \in G$ such that $p.\gamma \in B(p)$. The null holonomy group attached to p, $H_0(p)$, is the subgroup of H(p) consisting of all $\gamma \in H(p)$ with the property that p may be joined to $p.\gamma$ by a piecewise smooth horizontal curve in P whose image under π is a null-homotopic loop in M. If $c : [t_0, t_1] \to M$ is a loop in M beginning and ending at $\pi(p)$, then $\gamma \in H(p)$ is said to be generated by c if and only if the horizontal lift $\tilde{c}_p : [t_0, t_1] \to P$ of c beginning at p ends at $p.\gamma$. If c_1 and c_2 are loops in M beginning and ending at $\pi(p)$ and generating the elements γ_1 and γ_2 respectively of the holonomy group H(p) attached to p, then the product loop $c_1 * c_2$ (c_1 followed by c_2) generates $\gamma_2 . \gamma_1 \in H(p)$. If c is a loop based at $\pi(p)$ generating $\gamma \in H(p)$ and if $\eta \in G$, then c generates $\eta^{-1}\gamma\eta \in H(p.\eta)$. If $c_0 : [t_0, t_1] \to M$ is a piecewise smooth curve which lifts to a horizontal curve in P from q to p for some $p, q \in P$ and if c is a loop in M based at $\pi(p)$ generating $\gamma \in H(p)$, then the loop $c_0 * c * c_0^{-1}$ (c_0 followed by c followed by c_0 reversed) generates the same element $\gamma \in H(q)$.

<u>Theorem 3.1</u>

Given a smooth principal bundle $\pi : P \to M$ with structural group G and given a smooth Ehresmann connection on $\pi : P \to M$, the holonomy bundle B(p) is a smooth immersed submanifold of P, the holonomy group H(p) is an immersed Lie subgroup of G with identity component $H_0(p)$, and $\pi | B(p) : B(p) \to M$ is a smooth principal bundle with structure group H(p) ($H_0(p)$ is the null holonomy group).

<u>Theorem 3.2</u> (Ambrose-Singer holonomy theorem)

Let $\pi : P \to M$ be a smooth principal bundle with structural group G, and let $\mathfrak{g}$ be the Lie algebra of G. Let $\omega : TP \to \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \to M$. For all $p \in P$, let $\mathfrak{h}(p)$ be the subalgebra of $\mathfrak{g}$ generated by all $F^{\omega}(X, Y)$, where F^{ω} is the curvature of ω and where X and Y run through all pairs of tangent vectors to P at all points of B(p), the holonomy bundle attached to p. Then the subgroup of G generated by $\mathfrak{h}(p)$ is the null holonomy group $H_0(p)$ of ω attached to p.

An Ehresmann connection on a principal bundle is said to be <u>irreducible</u> if and only if the holonomy group attached to any point of P is the whole structural group of the principal bundle $\pi : P \to M$.

§4. Principal Bundle Automorphisms

In this section, we define principal bundle automorphisms and present some of their properties. In particular, we examine the action of principal bundle automorphisms on connections.

Let $\pi : P \to M$ be a smooth principal bundle with structure group G whose Lie algebra is $\mathfrak{g}$. A principal bundle automorphism $\Psi : P \to P$ is a fibre-preserving G-equivariant diffeomorphism of P (i.e.

$$\pi(\Psi(p)) = \pi(p)$$

and

$$\Psi(p.\gamma) = \Psi(p).\gamma$$

for all $p \in P$ and $\gamma \in G$). The set of all principal bundle automorphisms of $\pi : P \to M$ is in bijective correspondence with the set of all G-equivariant maps from P to G, where the group G acts on G by the adjoint action (recall that $\psi : P \to G$ is G-equivariant if and only if

$$\psi(p.\gamma) = \gamma^{-1}\psi(p)\gamma$$

for all $p \in P$ and $\gamma \in G$). This correspondence maps the principal bundle automorphism $\Psi : P \to P$ to the map $\psi : P \to G$ with the property that

$$\Psi(p) = p.\psi(p)$$

for all $p \in P$. If $\Psi : P \to P$ is C^k for some non-negative integer k, then so is $\psi : P \to G$.

Let $\psi_1 : P \to G$ and $\psi_2 : P \to G$ be G-equivariant maps corresponding to principal bundle automorphisms $\Psi_1 : P \to P$ and $\Psi_2 : P \to P$. Then

$$\begin{aligned}\Psi_1 \circ \Psi_2(p) &= \Psi_1(p.\psi_2(p)) \\ &= \Psi_1(p).\psi_2(p) \\ &= p.\psi_1(p)\psi_2(p).\end{aligned}$$

Thus if the space of G-equivariant maps from P to G is considered as a group under the operations of pointwise multiplication and inversion of maps, then the above correspondence is a group isomorphism from the group of C^k principal bundle automorphisms on $\pi : P \to M$ to the group of C^k G-equivariant maps from P to G. But this latter group is isomorphic to the group of C^k sections of the adjoint bundle $\pi_{ad} : G_P \to M$ whose total space G_P is $P \times_{ad} H$. Thus the group of C^k principal bundle automorphisms of $\pi : P \to M$ is isomorphic to the group $C^k(G_P)$.

Given $\gamma \in G$, let $R_\gamma : P \to P$ denote the map sending $p \in P$ to $p.\gamma$. Given $a \in \mathfrak{g}$, let $\sigma(a)$ be the fundamental vector field on P determined by a, and let $\sigma_p(a)$ denote the value of $\sigma(a)$ at $p \in P$. If $\Psi : P \to P$ is a principal bundle automorphism, then

$$\Psi \circ R_\gamma = R_\gamma \circ \Psi \quad .$$

Also the flow of $\sigma(a)$ is given by

$$(p, t) \mapsto p. \exp ta.$$

It follows that the flow of $\sigma(a)$ commutes with $\Psi : P \to P$, and hence

$$\Psi_* \sigma(a) = \sigma(a).$$

Thus if $\omega : TP \to \mathfrak{g}$ is an Ehresmann connection on $\pi : P \to M$, then so is $\Psi^*\omega$: if $a \in \mathfrak{g}$, then

$$\begin{aligned} (\Psi^*\omega)(\sigma(a)) &= \omega(\Psi_* \sigma(a)) \\ &= \omega(\sigma(a)) \\ &= a \end{aligned}$$

and if $\gamma \in G$ then

$$\begin{aligned} R_\gamma^* \Psi^*\omega &= \Psi^* R_\gamma^* \omega \\ &= \mathrm{Ad}(\gamma^{-1}) \circ \Psi^*\omega . \end{aligned}$$

Given $\gamma \in G$, let $L_{\gamma^{-1}*} : T_\gamma G \to \mathfrak{g}$ be the derivative at γ of the map $L_{\gamma^{-1}} : G \to G$ multiplying elements of G on the left by γ^{-1}, and define

$$\Phi(X) = L_{\gamma^{-1}*} X$$

for all $X \in T_\gamma G$. The map $\Phi : TG \to \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on G, the Maurer-Cartan form on G.

Lemma 4.1

Let $\pi : P \to M$ be a smooth principal bundle with structure group G whose Lie algebra is $\mathfrak{g}$. Let $\Phi : TG \to \mathfrak{g}$ denote the Maurer-Cartan form on G. Let $\Psi : P \to P$ be a differentiable principal bundle automorphism and let $\psi : P \to G$ be the corresponding G-equivariant map with the property that $\Psi(p) = p.\psi(p)$ for all $p \in P$. Let $\psi_* : TP \to TG$ be the derivative of ψ. Then, for all $X \in T_pP$,

$$\Psi^*\omega(X) = Ad(\psi(p)^{-1})\,\omega(X) + \Phi(\psi_*(X)).$$

Proof

Let $c : (-\varepsilon, \varepsilon) \to P$ be a short curve with tangent vector $X \in T_pP$ at $t = 0$, where $p = c(0)$. Let $q = \Psi(p)$. Then, by Leibnitz' rule,

$$\begin{aligned}
\Psi_* X &= \frac{d}{dt}\Big(c(t)\,.\,\psi(c(t))\Big)\Big|_{t=0} \\
&= R_{\psi(p)*} X + \frac{d}{dt}\Big(p.\psi(p)\big(\psi(p)^{-1}\psi(c(t))\big)\Big)\Big|_{t=0} \\
&= R_{\psi(p)*} X + \sigma_q\Big(\frac{d}{dt}\big(\psi(p)^{-1}\psi(c(t))\big)\Big|_{t=0}\Big) \\
&= R_{\psi(p)*} X + \sigma_q\big(\Phi(\psi_*(X))\big),
\end{aligned}$$

hence

$$\Psi^*\omega(X) = Ad(\psi(p)^{-1})\,\omega(X) + \Phi(\psi_*(X)).$$

$\square$

We now describe the stabilizer of a smooth Ehresmann connection under the action of the group of smooth principal bundle automorphisms.

Theorem 4.2

Let $\pi : P \to M$ be a smooth principal bundle, with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\omega : TP \to \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \to M$ and let Stab(ω) be the stabilizer of ω in the group of smooth principal bundle automorphisms. Then Stab(ω) is isomorphic to the centralizer in G of the holonomy group of ω attached to any point of P.

Proof

Let $\Psi : P \to P$ be a smooth principal bundle automorphism and let $\psi : P \to G$ be the G-equivariant map corresponding to Ψ, where $\Psi(p) = p.\psi(p)$ for all $p \in P$. Now if $X \in TP$ is vertical, then

$$\Psi^* \omega(X) = \omega(X) .$$

Since TP decomposes as the Whitney sum of its vertical and horizontal subbundles with respect to ω, a necessary and sufficient condition for Ψ to belong to Stab(ω) is that $\Psi^*\omega(X) = 0$ for all $X \in TP$ satisfying $\omega(X) = 0$. But if $\omega(X) = 0$ then

$$\Psi^*\omega(X) = \Phi(\psi_*(X))$$

by lemma 4.1. Thus $\Psi \in$ Stab(ω) iff

$$\psi_*(X) = 0$$

for all $X \in TP$ satisfying $\omega(X) = 0$. This condition is satisfied if and only if $\psi : P \to G$ is constant along all piecewise smooth curves in P which are horizontal with respect to the connection ω. Thus Ψ belongs to Stab(ω) if and only if $\psi : P \to G$ is constant along all the holonomy bundles attached to points of P.

But if $\gamma \in G$ and if $R_\gamma : P \to P$ is the map sending p to $p.\gamma$, then the smooth maps $R_\gamma : P \to P$ for all $\gamma \in G$ permute the holonomy bundles of the connection ω, this action of G on the set of holonomy bundles of the connection ω is transitive, and

$$\psi \circ R_\gamma = \mathrm{ad}(\gamma^{-1}) \circ \psi .$$

Hence if ψ is constant on one of the holonomy bundles in P of the connection ω, then ψ is constant on all of these holonomy bundles. Let $p \in P$. Then $\Psi \in \mathrm{Stab}(\omega)$ if and only if ψ is constant on the holonomy bundle attached to p. It follows that ψ is determined uniquely by its value $\psi(p)$ at p. Thus we have a monomorphism from $\mathrm{Stab}(\omega)$ to the structural group G mapping $\Psi : P \to P$ to $\psi(p)$, where $\psi : P \to G$ is the G-equivariant map corresponding to Ψ.

We recall that if $\gamma \in G$ then γ belongs to H(p), the holonomy group of ω attached to p, if and only if $p.\gamma$ belongs to the holonomy bundle of ω attached to p. Thus if $\Psi \in \mathrm{Stab}(\omega)$ and if $\gamma \in H(p)$, then

$$\psi(p.\gamma) = \psi(p)$$

and thus

$$\gamma^{-1}\psi(p)\gamma = \psi(p).$$

Hence $\psi(p)$ belongs to the centralizer of the holonomy group of ω attached to p. Thus the image of the monomorphism from $\mathrm{Stab}(\omega)$ to G mapping $\Psi : P \to P$ to $\psi(p)$ is contained in H(p). It remains to show that if γ belongs to the centralizer of H(p) in G, then there exists $\Psi \in \mathrm{Stab}(\omega)$ such that $\psi(p) = \gamma$ where $\psi : P \to G$ is the G-equivariant map corresponding to Ψ.

Let γ belong to the centralizer of H(p) in G. For all $p_1 \in P$, there exists $\gamma_1 \in G$ such that $p.\gamma_1$ belongs to the holonomy bundle of ω attached to p_1. Define

$$\psi(p_1) = \gamma_1^{-1}\gamma\gamma_1.$$

$\psi : P \to G$ is well-defined since γ centralizes H(p): if $p.\gamma_1$ and $p.\gamma_2$ belong to the same holonomy bundle of ω then so do p and $p.\gamma_2\gamma_1^{-1}$, hence $\gamma_2\gamma_1^{-1} \in H(p)$, and hence

$$\begin{aligned}\gamma_1^{-1}\gamma\gamma_1 &= \gamma_1^{-1}(\gamma_1\gamma_2^{-1}\gamma\gamma_2\gamma_1^{-1})\gamma_1 \\ &= \gamma_2^{-1}\gamma\gamma_2.\end{aligned}$$

Using the fact that we have a smooth foliation of P by the holonomy bundles of the connection ω , we may easily show that $\psi : P \rightarrow G$ is smooth. Thus we have an isomorphism from Stab(ω) onto the centralizer of the holonomy group of the connection ω attached to p.

$\square$

§5. Connections and Canonical Norms

In this section, we review facts about the adjoint bundles $G_P \to M$ and $\mathfrak{g}_P \to M$ associated to a smooth principal bundle $\pi : P \to M$ over a compact manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. We then show that the difference of two Ehresmann connections determines a section of $\mathfrak{g}_P \otimes T^*M \to M$, where $\mathfrak{g}_P \to M$ is the adjoint bundle with total space $P \times_{Ad} \mathfrak{g}$, where $Ad : G \to Aut(\mathfrak{g})$ is the adjoint representation of G. Denote by $\| . \|$ and $\| . \|_q$ the canonical norms on $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$, for $q \in [1, \infty)$, determined by a given biinvariant metric on G and a given Riemannian metric on M. We shall show that if ω_1 and ω_2 are continuous connections and if $\Psi : P \to P$ is a C^1 principal bundle automorphism, then

$$\omega_1 - \omega_2 = \| \Psi^* \omega_1 - \Psi^* \omega_2 \| .$$

Similarly,

$$\| \omega_1 - \omega_2 \|_q = \| \Psi^* \omega_1 - \Psi^* \omega_2 \|_q$$

if $\omega_1 - \omega_2 \in L^q(\mathfrak{g}_P \otimes T^*M)$. We shall then obtain results comparing principal bundle automorphisms along curves in the total space P of the principal bundle P which are horizontal with respect to some Ehresmann connection on P.

We recall that the group of C^k principal bundle automorphisms is isomorphic to the group of C^k sections of the bundle $\pi_{ad} : G_P \to M$ whose total space G_P is $P \times_{ad} G$. Also, given a biinvariant Riemannian metric on the structural group G, determining a distance function $\rho : G \times G \to \mathbb{R}$ on G and a G-invariant norm $| . |$ on $\mathfrak{g}$, there is for all $m \in M$, a unique distance function $\rho_m : G \times G \to \mathbb{R}$ on the fibre $G_P[m]$ of G_P over m, and a unique norm $| . |_m$ on the fibre $\mathfrak{g}_P[m] \otimes T^*_m M$ of $\mathfrak{g}_P \otimes T^*M \to M$ over m, with the property that every $p \in \pi^{-1}(m)$ determines an isometry from $G_P[m]$ to G, and from

$\mathfrak{g}_P[m] \otimes T^*_mM$ to $\mathfrak{g} \otimes T^*_mM$ (where the norm on $\mathfrak{g} \otimes T^*_mM$ is the usual norm

$$|S| = \sup \{ |SX| : X \in T_mM \text{ and } |X| = 1 \}$$

obtained when we regard $\mathfrak{g} \otimes T^*_mM$ as the space of linear transformations from T_mM to $\mathfrak{g}$). We recall that the canonical norms $\| . \|$ and $\| . \|_q$ on $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$ are then defined by

$$\| \tau \| = \sup_{m \in M} |\tau(m)|_m$$

$$\| \tau \|_q = \left(\int_M |\tau(m)|_m^q \, d(\text{volume}) \right)^{1/q}.$$

The adjoint representation Ad : $G_P[m] \to \operatorname{Aut} \mathfrak{g}_P[m]$ induces continuous maps

$$C^0(G_P) \times C^0(\mathfrak{g}_P \otimes T^*M) \to C^0(\mathfrak{g}_P \otimes T^*M),$$

$$C^0(G_P) \times L^q(\mathfrak{g}_P \otimes T^*M) \to L^q(\mathfrak{g}_P \otimes T^*M),$$

for $q \in [1, \infty)$, which are linear over sections of $\mathfrak{g}_P \otimes T^*M \to M$. The norms $\| . \|$ and $\| . \|_q$ on $C^0(\mathfrak{g}_P \otimes T^*M)$ and $L^q(\mathfrak{g}_P \otimes T^*M)$ are invariant under this adjoint action (see proposition 2.4).

Let $\tau : TP \to \mathfrak{g}$ be a $\mathfrak{g}$-valued 1-form on P. τ is said to be <u>horizontal</u> if and only if $\tau(X) = 0$ for all vertical tangent vectors $X \in TP$ (a tangent vector to P is said to be vertical if and only if it is tangent to the fibres of $\pi : P \to M$ and is thus mapped to zero under the derivative of $\pi : P \to M$). τ is said to be <u>G-equivariant</u> if and only if

$$R_\gamma^* \tau = \operatorname{Ad}(\gamma^{-1}) \tau$$

where $R_\gamma^* \tau$ is the pullback of τ under the map $R_\gamma : P \to P$ sending p to $p.\gamma$ for all $p \in P$ and $\gamma \in G$.

Let $\tau : TP \to \mathfrak{g}$ be a horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P. Then, for all $p \in P$, there is a unique well-defined linear map $\tilde{\tau}_p : T_mM \to \mathfrak{g}$, where $m = \pi(p)$, such that

$$\tilde{\tau}_p(\pi_* X) = \tau(X)$$

for all $X \in T_pP$, where $\pi_* : TP \to TM$ is the derivative of $\pi : P \to M$. If $R_{\gamma *} : TP \to TP$ is the derivative of $R_\gamma : P \to P$, then

$$\begin{aligned} \tilde{\tau}_{p.\gamma}(\pi_* X) &= \tilde{\tau}_{p.\gamma}(\pi_* R_{\gamma *} X) \\ &= \tau(R_{\gamma *} X) \\ &= R^*_\gamma \tau(X) \\ &= \mathrm{Ad}(\gamma^{-1}) \tau(X) \\ &= \mathrm{Ad}(\gamma^{-1}) \tilde{\tau}_p(\pi_* X). \end{aligned}$$

Thus, for all vector fields Y on M, the map sending $p \in P$ to $\hat{\tau}_p(Y_{\pi(p)})$ is a G-equivariant map from P to $\mathfrak{g}$ and thus determines a unique section of $\pi_{Ad} : \mathfrak{g}_P \to M$. Thus every horizontal, G-equivariant $\mathfrak{g}$-valued 1-form $\tau : TP \to \mathfrak{g}$ determines a unique section of $\mathfrak{g}_P \otimes T^*M \to M$, which we also denote by τ on identifying sections of this bundle with the corresponding 1-forms on P.

If $|.|_m$ is the norm on the fibre of $\mathfrak{g}_P \otimes T^*M \to M$ over $m \in M$, then, for all $p \in \pi^{-1}(m)$ and for all $X \in T_pP$,

$$|\tau(X)| \leq |\tau(m)|_m |\pi_* X|$$

for all horizontal G-equivariant $\mathfrak{g}$-valued 1-forms on P.

<u>Lemma 5.1</u>

Let $\pi : P \to M$ be a smooth principal bundle over a compact manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $G_P = P \times_{ad} G$ and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$ be the total spaces of the adjoint bundles $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$. For all $m \in M$, let $|.|_m$ be the norm on the fibre of $\mathfrak{g}_P \otimes T^*M \to M$ over $m \in M$ determined by some biinvariant Riemannian metric on G and some Riemannian metric on M, and let $\|.\|$ and $\|.\|_q$ be the corresponding canonical norms on $C^0(\mathfrak{g}_{\mathbf{P}} \otimes T^*M)$ and $L^q(\mathfrak{g}_{\mathbf{P}} \otimes T^*M)$ for $q \in [1, \infty)$.

Let $\omega_1 : TP \to \mathfrak{g}$ and $\omega_2 : TP \to \mathfrak{g}$ be continuous Ehresmann connections on $\pi : P \to M$ and let $\Psi : P \to P$ be a C^1 principal bundle automorphism of $\pi : P \to M$.

Then $\omega_1 - \omega_2 : TP \to \mathfrak{g}$ is a continuous horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P, and thus determines a unique continuous section of the vector bundle $\mathfrak{g}_{\mathbf{P}} \otimes T^*M \to M$, which we also denote by $\omega_1 - \omega_2$. Ψ determines a C^1 section of $G_{\mathbf{P}} \to M$. Then

$$\| \Psi^*\omega_1 - \Psi^*\omega_2 \| = \| \omega_1 - \omega_2 \|$$

and

$$\| \Psi^*\omega_1 - \Psi^*\omega_2 \|_q = \| \omega_1 - \omega_2 \|_q.$$

<u>Proof</u>

The fact that $\omega_1 - \omega_2$ is horizontal and G-equivariant follows from the definition of an Ehresmann connection. The final statement of the lemma follows from the invariance of $\| \cdot \|$ and $\| \cdot \|_q$ under the adjoint action of $C^0(G_{\mathbf{P}})$, provided that we can show that

$$\Psi^*\omega_1 - \Psi^*\omega_2 = \mathrm{Ad}\,(\Psi^{-1})(\omega_1 - \omega_2)$$

where the right hand side of this identity should be interpreted as the image of the section Ψ^{-1} of $G_{\mathbf{P}} \to M$ and the section $\omega_1 - \omega_2$ of $\mathfrak{g}_{\mathbf{P}} \otimes T^*M \to M$ under the adjoint action. But if

$$\Psi(p) = p.\psi(p)$$

for some G-equivariant map $\psi : P \to G$, then

$$\Psi^*\omega_1(p) = \mathrm{Ad}\,(\psi(p)^{-1})\,\omega_1(p) + L_{\psi(p)^{-1}*}\,\psi_*(p)$$

where $L_{\gamma*} : TG \to TG$ is the derivative of $L_\gamma : G \to G$ sending $\beta \in G$ to $\gamma\beta$, by Lemma 4.1. Thus

$$\Psi^*\omega_1 - \Psi^*\omega_2 = \mathrm{Ad}\,(\psi(p)^{-1})\,(\omega_1 - \omega_2).$$

The result follows using the correspondences between G-equivariant maps $\psi : P \to G$ and sections Ψ of G_P and between horizontal G-equivariant $\mathfrak{g}$-valued 1-forms on P and sections of $\mathfrak{g}_P \otimes T^*M$.

$\square$

Let $p \in P$ and let $c : [a, b] \to M$ be a loop beginning and ending at m, where $m = \pi(p)$. Let $\tilde{c} : [a, b] \to P$ be a lift of c beginning at p which is horizontal with respect to a C^1 Ehresmann connection ω. Then

$$\tilde{c}(b) = p.\gamma$$

for some $\gamma \in G$. The image of (p, γ) under the natural projection $P \times G \to P \times_{ad} G$ is an element hol(c) of $G_P[m]$. hol(c) is independent of the choice of $p \in \pi^{-1}(m)$. The elements of $G_P[m]$ of the form hol(c) form a subgroup $Hol_m(\omega)$ of $G_P[m]$ identified with the holonomy group of the connection ω.

Theorem 5.2

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $\rho : G \times G \to \mathbb{R}$ be the distance function of a given biinvariant Riemannian metric on G, and let $\rho_m : G_P[m] \times G_P[m] \to \mathbb{R}$ be the corresponding distance function on the fibre $G_P[m]$ over $m \in M$ of the adjoint bundle $G_P \to M$, where $G_P = P \times_{ad} G$. Let $|\cdot|_m$ be the norm on the fibre $\mathfrak{g}_P[m] \otimes T^*_m M$ over $m \in M$ of the bundle $\mathfrak{g}_P \otimes T^*M \to M$ where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$.

Let $\omega : TP \to \mathfrak{g}$ be a C^1 Ehresmann connection on $\pi : P \to M$, and let $Hol_m(\omega)$ denote the holonomy group of ω generated by loops based at $m \in M$. Let $\Psi_1 : P \to P$ and $\Psi_2 : P \to P$ be C^2 principal bundle automorphisms. Let $c : [a, b] \to M$ be a piecewise smooth curve in M parameterized by arclength s, and define $\Delta : M \to \mathbb{R}$ by

$$\Delta(m) = \rho_m(\Psi_1(m), \Psi_2(m)).$$

Then

$$|\Delta(c(b)) - \Delta(c(a))| \leq \int_a^b |\Psi_1^*\omega - \Psi_2^*\omega|_{c(s)}\, ds.$$

Further, if $c : [a, b] \longrightarrow M$ is a piecewise smooth loop beginning and ending at m and generating the element h of the holonomy group $Hol_m(\omega)$ of ω, then

$$\rho_m(h^{-1}\Psi_1(m)\Psi_2(m)^{-1}h, \Psi_1(m)\Psi_2(m)^{-1}) \leq \int_a^b |\Psi_1^*\omega - \Psi_2^*\omega|_{c(s)}\, ds.$$

Proof

Since ρ_m is biinvariant and $|\cdot|_m$ is G-invariant

$$\Delta(m) = \rho_m(\Psi_1(m)\Psi_2^{-1}(m), e(m))$$

where e is the identity section of $G_P \longrightarrow M$, and

$$|\Psi_1{}^*\omega - \Psi_2{}^*\omega|_{c(s)} = |\Psi_2^{-1*}\Psi_1^*\omega - \omega|_{c(s)}$$

$$= |(\Psi_1\Psi_2^{-1})^*\omega - \omega|_{c(s)}$$

it suffices to prove the theorem when $\Psi_1 = \Psi$ and $\Psi_2 = e$.

Let $\tilde{c} : [a, b] \longrightarrow P$ be a lift of $c : [a, b] \longrightarrow M$ which is horizontal with respect to ω. Let $\psi : P \longrightarrow G$ be the G-equivariant function defined by

$$\Psi(p) = p.\psi(p)$$

for all $p \in P$. Let $\eta : [a, b] \longrightarrow G$ be defined by

$$\eta(s) = \psi(\tilde{c}(s))$$

and let $\hat{c} : [a, b] \longrightarrow P$ be defined by

$$\hat{c}(s) = \tilde{c}(s).\eta(s).$$

Then

$$\hat{c}(s) = \Psi \circ \tilde{c}(s).$$

Thus the tangent vectors $\tilde{c}'$ and $\hat{c}'$ to $\tilde{c}$ and $\hat{c}$ are related by

$$\hat{c}'(s) = \Psi_*\tilde{c}'(s)$$

where $\Psi_* : TP \to TP$ is the derivative of Ψ. Thus

$$\omega(\hat{c}'(s)) = (\Psi^*\omega)(\tilde{c}'(s))$$
$$= \tau(\tilde{c}'(s))$$

where

$$\tau = \Psi^*\omega - \omega,$$

since

$$\omega(\tilde{c}'(s)) = 0.$$

But by Leibnitz' rule

$$\hat{c}'(s) = R_{\eta(s)*}\tilde{c}'(s) + \sigma_{\hat{c}(s)}(\Phi(\eta(s)))$$

where $\Phi : TG \to \mathfrak{g}$ is the Maurer-Cartan form on G and where $\sigma_{\hat{c}(s)}$ is the map sending an element of $\mathfrak{g}$ to the value at $\hat{c}(s)$ of the corresponding fundamental vertical vector field. Hence

$$\omega(\hat{c}'(s)) = \Phi(\eta(s)).$$

Thus

$$\Phi(\eta(s)) = \tau(\tilde{c}'(s)).$$

But

$$|\tau(c'(s))| \leq |\tau|_m |c'(s)| = |\tau|_m$$

hence

$$\rho(\eta(b), \eta(a)) \leq \int_a^b |\Phi(\eta(s))| \, ds$$
$$\leq \int_a^b |\tau|_{c(s)} \, ds.$$

But

$$\rho(\eta(b), \eta(a)) = \rho(\Psi(\tilde{c}(b)), \Psi(\tilde{c}(a)))$$

hence

$$|\Delta(c(b)) - \Delta(c(a))| = |\rho(\Psi(c(b)), e) - \rho(\Psi(c(a)), e)|$$
$$\leq \rho(\Psi(c(b)), \Psi(c(a)))$$
$$\leq \int_a^b |\Psi^*\omega - \omega|_{c(s)} \, ds.$$

If in addition $c : [a, b] \to M$ is a loop generating $h \in Hol_m(\omega)$ where $h = hol(c)$, and if h is the image of $(c(a), \gamma) \in P \times G$ under the natural projection $P \times G \to G_P$, then

$$\begin{aligned} \eta(b) &= \Psi(\tilde{c}(b)) \\ &= \Psi(\tilde{c}(a).\gamma) \\ &= \gamma^{-1}\Psi(\tilde{c}(a))\gamma \end{aligned}$$

and

$$\eta(a) = \Psi(\tilde{c}(a)),$$

hence

$$\begin{aligned} \rho_m(h^{-1}\bar{\Psi}(m)h, \bar{\Psi}(m)) &= \rho(\gamma^{-1}\Psi(\tilde{c}(a))\gamma, \Psi(\tilde{c}(a))) \\ &= (\eta(b), \eta(a)) \\ &\leq \int_a^b |\Psi^*\omega - \omega|_{c(s)}\, ds \end{aligned}$$

as required.

$\square$

§6. Covariant Derivatives of Sections of Fibre Bundles

In this section, we show that, given a smooth fibre bundle $F_P \to M$ associated to a smooth principal bundle $\pi : P \to M$, and given a smooth connection ω on $\pi : P \to M$, we may define, for any C^1 section $s : M \to F_P$ of $F_P \to M$, the covariant derivative $D^\omega s : TM \to TF_P$ of the section s. The image of $D^\omega s$ is contained in the vector bundle VF_P over F_P consisting of all vectors in TF_P which are tangent to the fibres of $F_P \to M$. We first consider the covariant derivative of sections of the adjoint bundle $\pi_{ad} : G_P \to M$, where G is the structural group of $\pi : P \to M$ and $G_P = P \times_{ad} G$. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. The Maurer-Cartan form $\Phi : TG \to \mathfrak{g}$ induces a fibre bundle morphism $\Phi_P : VG_P \to \mathfrak{g}_P$, and the composition $\Phi_P \circ D^\omega \Psi$ of the covariant derivative $D^\omega \Psi : TM \to VG_P$ of some C^1 section Ψ of $G_P \to M$ with Φ_P defines a $\mathfrak{g}_P$-valued 1-form $\mathcal{X}^\omega(\Psi)$ on M. We show that if $\Psi \in C^1(G_P)$ corresponds to a principal bundle automorphism $\Psi: P \to P$, then $\mathcal{X}^\omega(\Psi)$ corresponds to the horizontal G-equivariant $\mathfrak{g}$-valued 1-form $\Psi^*\omega - \omega$. We define the covariant differential $d^\omega s$ of a section s of a vector bundle associated to $\pi : P \to M$. We then show the existence of a fibre bundle morphism $B : \mathfrak{g}_P \to \mathrm{End}\,(\mathfrak{g}_P \to T^*M)$ such that, for all $\xi \in C^1(\mathfrak{g}_P)$,

$$\mathcal{X}^\omega(\exp \xi) = B(\xi)\, d^\omega \xi$$

where $\exp : C^1(\mathfrak{g}_P) \to C^1(G_P)$ is induced by the exponential map $\exp : \mathfrak{g} \to G$, and where $d^\omega \xi$ is the covariant differential of . Also we show that there exists a neighbourhood of the zero section of $\mathfrak{g}_P$ such that if ξ belongs to this neighbourhood, then $B(\xi)$ is a section of $\mathrm{Aut}\,(\mathfrak{g}_P \otimes T^*M)$. Then we compare the covariant derivative operators with respect to different connections on the principal bundle $\pi : P \to M$.

Definition

Let $p : E \longrightarrow M$ be a smooth fibre bundle and let $p_* : TE \longrightarrow TM$ be the derivative of p. The vertical bundle $VE \longrightarrow M$ of $E \longrightarrow M$ is the fibre bundle over M defined by

$$VE = \{ X \in TE : p_* (X) = 0 \}$$

(we shall also regard VE as the total space of a vector bundle over E whenever appropriate).

If $p_1 : E_1 \longrightarrow M$ and $p_2 : E_2 \longrightarrow M$ are smooth fibre bundles over M, and if $\varphi : E_1 \longrightarrow E_2$ is a smooth fibre bundle morphism, we define $V\varphi : VE_1 \longrightarrow VE_2$ to be the restriction of the derivative $\varphi_* : TE_1 \longrightarrow TE_2$ of φ to VE_1.

Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structural group G. Let F be a smooth manifold on which G acts smoothly on the left with action $\theta : G \longrightarrow \mathrm{Diff}(F)$. Let $\pi_\theta : F\mathbf{p} \longrightarrow M$ be the fibre bundle with total space $F\mathbf{p} = P \times_\theta F$ associated to $\pi : P \longrightarrow M$ by the action θ. Let TF be the tangent bundle of F and let $T\theta : G \longrightarrow \mathrm{Diff}(TF)$ be the smooth left action sending $\gamma \in G$ to the derivative $T\theta(\gamma) : TF \longrightarrow TF$ of $\theta(\gamma) : F \rightarrow F$. Then $VF\mathbf{p} \longrightarrow M$ is a fibre bundle with fibre TF and total space $VF\mathbf{p} = P \times_{T\theta} TF$ associated to $\pi : P \longrightarrow M$ by the action $T\theta$.

Now let $\omega : TP \longrightarrow \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \longrightarrow M$, where $\mathfrak{g}$ is the Lie algebra of the structural group G. ω determines a splitting of TP as a direct sum

$$TP = VP \oplus HP$$

of vector bundles over P, where

$$P = \{ X \in TP : \omega(X) = 0 \}.$$

Let V_pP and H_pP denote the fibres of VP and HP respectively over $p \in P$. For all $\gamma \in G$, let $R_{\gamma *} : TP \longrightarrow TP$ denote the derivative of the smooth map $R_\gamma : P \longrightarrow P$ sending p to $p.\gamma$.

Then

$$V_{p.\gamma} P = R_{\gamma *} V_p P$$

$$H_{p.\gamma} P = R_{\gamma *} H_p P$$

The derivative $\pi_* : TP \longrightarrow TM$ of $\pi : P \longrightarrow M$ restricts to an isomorphism

$$\pi_* \mid H_p P : H_p P \longrightarrow T_{\pi(p)} M.$$

If $F\mathbf{p} \longrightarrow M$ is the fibre bundle with total space $P \times_\theta F$ associated to $\pi: P \longrightarrow M$ by the smooth left action $\theta : G \longrightarrow \mathrm{Diff}(F)$, then the natural projection

$$(p, f) \mapsto [p, f]$$

from $P \times F$ to $F\mathbf{p}$ determines, for each $f \in F$, a map $e_f : P \longrightarrow F\mathbf{p}$ sending p to $[p, f]$. The derivative $e_{f*} : TP \longrightarrow TF\mathbf{p}$ of e_f satisfies

$$\pi_{\theta *} e_{f*} = \pi_*$$

and thus e_{f*} maps VP into $VF\mathbf{p}$. Then

$$e_f \circ R_\gamma = e_{\theta(\gamma^{-1})f} ,$$

hence

$$e_{f*} \circ R_{\gamma *} = e_{\theta(\gamma^{-1})f*}.$$

Then

$$e_{\theta(\gamma^{-1})f*} (H_p P) = e_{f*} R_{\gamma *} (H_p P)$$

$$= e_{f*} (H_{p.\gamma} P).$$

Hence there is a well-defined subbundle $HF\mathbf{p}$ of $TF\mathbf{p}$ with the property that if $x = [p, f]$ for some $p \in P$ and $f \in F$, and if $H_x F_\mathbf{p}$ is the fibre of $HF\mathbf{p}$ over x, then

$$H_x F_p = e_{f*} (H_\mathbf{p} P).$$

Since $\pi_* \mid H_p P : H_p P \longrightarrow T_m M$ is an isomorphism, where $m = \pi(p)$, and since

$$\pi_{\theta *} e_{f*} = \pi_*$$

it follows that

$$e_{f*} \mid H_pP \; : \; H_pP \longrightarrow H_xF\mathbf{p}$$

and

$$\pi_{\theta *} \mid H_xF\mathbf{p} \; : \; H_xF\mathbf{p} \longrightarrow T_mM$$

are both isomorphisms, and we have a splitting of $TF\mathbf{p}$ as a direct sum

$$TF\mathbf{p} = VF\mathbf{p} \oplus HF\mathbf{p} .$$

Definition

Let $\pi_\theta : F\mathbf{p} \longrightarrow M$ be a smooth fibre bundle associated to a smooth principal bundle $\pi : P \longrightarrow M$ with structural group G by a smooth left action $\theta : G \longrightarrow \mathrm{Diff}(F)$ of G on F. Let ω be a smooth Ehresmann connection on P with horizontal bundle $HP \longrightarrow P$.

Let $e_{f*} : TP \longrightarrow TF\mathbf{p}$ be the derivative of the map $e_f : P \longrightarrow F\mathbf{p}$ sending $p \in P$ to the image of (p, f) under the natural projection $P \times F \longrightarrow F\mathbf{p}$, for all $f \in F$. Then the horizontal bundle $HF\mathbf{p} \longrightarrow F\mathbf{p}$ is the subbundle of the tangent bundle $TF\mathbf{p} \longrightarrow F\mathbf{p}$ with the property that if $x = e_f(p)$ then the fibre $H_xF\mathbf{p}$ of $HF\mathbf{p}$ over $x \in F\mathbf{p}$ is given by

$$H_xF\mathbf{p} = e_{f*}(H_\mathbf{p}P).$$

The vertical projection $pr_V^\omega : TFp \longrightarrow VFp$ is the projection mapping the tangent space $T_xF\mathbf{p}$ of $F\mathbf{p}$ at x onto the vertical subspace $V_xF\mathbf{p}$, the kernel of pr_V^ω at x being $H_xF\mathbf{p}$. Given a C^1 section $s : M \longrightarrow F\mathbf{p}$ of $F\mathbf{p} \longrightarrow M$, the covariant derivative $D^\omega s : TM \longrightarrow VF\mathbf{p}$ of s is the map

$$D^\omega s = pr_V^\omega \circ s_*$$

where $s_* : TM \longrightarrow TF\mathbf{p}$ is the derivative of s.

Now suppose that $\pi_1 : E_1 \longrightarrow M$ and $\pi_2 : E_2 \longrightarrow M$ are smooth fibre bundles associated to the principal bundle $\pi : P \longrightarrow M$, and that ω is a smooth Ehresmann connection on $\pi : P \longrightarrow M$. Suppose also that $\varphi_P : E_1 \to E_2$ is a smooth morphism of fibre bundles over M induced by a smooth equivariant map $\varphi : F_1 \to F_2$ between the fibres F_1 and F_2 of $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$. If $pr_V^\omega : TE_1 \to VE_1$ and $pr_V^\omega : TE_2 \to VE_2$ are the vertical projections, and if $V\varphi_P : VE_1 \to VE_2$ is the restriction of the derivative $\varphi_{P*} : TE_1 \to TE_2$ of φ_P to the vertical bundle, then

$$pr_V^\omega \circ \varphi_{P*} = V\varphi_P \circ pr_V^\omega$$

since φ_{P*} maps the horizontal bundle of E_1 to that of E_2. Hence the covariant derivatives of a given section $s : M \to E_1$ and of $\varphi_P \circ s$ satisfy

$$\begin{aligned} D^\omega(\varphi_P \circ s) &= pr_V^\omega \circ \varphi_{P*} \circ s_* \\ &= V\varphi_P \circ D^\omega s. \end{aligned}$$

Thus the correspondence sending a section of a fibre bundle associated to a given principal bundle to its covariant derivative with respect to some Ehresmann connection on the principal bundle is functorial with respect to morphisms of fibre bundles induced by equivariant maps between their respective fibres.

We recall that, given a fibre bundle $\pi_\theta : F_P \to M$ associated to a principal bundle $\pi : P \to M$ with structural group G, there is a natural bijective correspondence between sections of $\pi_\theta : F_P \to M$ and G-equivariant maps from P to the fibre F of $\pi_\theta : F_P \to M$.

<u>Lemma 6.1</u>

Let $\pi_\theta : F_P \to M$ be a smooth fibre bundle associated to a smooth principal bundle $\pi : P \to M$ with structural group G. Let ω be a smooth Ehresmann connection on $\pi : P \to M$. Let $\mu_{p*} : TF \to TF_P$ be the derivative of the map $\mu_p : F \to F_P$ from the fibre F of $\pi_\theta : F_P \to M$ into F_P mapping $f \in F$ to the image $\lambda(p, f)$ of $(p, f) \in P \times F$ under the natural projection $\lambda : P \times F \to F_P$, for all $p \in P$.

Let $s : M \longrightarrow F_P$ be a C^1 section of $F_P \longrightarrow M$, and let $\hat{s} : P \longrightarrow F$ be the corresponding G-equivariant map from P to F. If $p \in P$, if $m = \pi(p)$, if $X \in T_m M$ and if $\tilde{X}$ is the horizontal lift of X to $T_p P$, then

$$D^{\omega} s\,(X) = \mu_{p*}\, \hat{s}_*\,(\tilde{X}).$$

<u>Proof</u>

s and $\hat{s}$ are related by the identity

$$s(\pi(p)) = \mu_p\, \hat{s}\,(p) = \lambda(p, \hat{s}\,(p)).$$

We recall that $e_f : P \longrightarrow F_P$ is the map sending $p \in P$ to $\lambda(p, f)$, for all $f \in F$. Let $\lambda_* : T_p P \oplus T_f F \longrightarrow T_x F_P$ be the derivative of λ at (p,f), where $x = \lambda(p,f)$, and let $e_{f*} : T_p P \longrightarrow T_x F_P$ be the derivative of e_f at p. Then, for all $(Y_1, Y_2) \in T_p P \oplus T_f F$,

$$\lambda_*\,(Y_1, Y_2) = e_{f*}\,(Y_1) + \mu_{p*}\,(Y_2).$$

Thus

$$\begin{aligned} s_*\,(X) &= (s \circ \pi)_*\,(\tilde{X}) \\ &= \lambda_*\,(\tilde{X}, \hat{s}_*\,(\tilde{X})) \\ &= e_{f*}\,(\tilde{X}) + \mu_{p*}\,\hat{s}_*\,(\tilde{X}). \end{aligned}$$

But $e_{f*}\,\tilde{X} \in HF_P$ and $\mu_{p*}\,\hat{s}_*\,(\tilde{X}) \in VF_P$, hence

$$\begin{aligned} D^{\omega} s\,(X) &= pr_V^{\omega} s_*\,(X) \\ &= \mu_{p*}\,\hat{s}_*\,(\tilde{X}). \end{aligned}$$

$\square$

We recall that if $\pi_\theta : F_P \longrightarrow M$ is a smooth fibre bundle associated to the principal bundle $\pi : P \longrightarrow M$ with structural group G, where $F_P = P \times_\theta F$, and where $\theta : G \longrightarrow \mathrm{Diff}\,(F)$ is a smooth left action on the fibre F of $\pi_\theta : F_P \longrightarrow M$, then the vertical bundle $VF_P \longrightarrow M$ of $F_P \longrightarrow M$ is a fibre bundle with fibre TF associated to $\pi : P \longrightarrow M$ by the action $T\theta : G \longrightarrow \mathrm{Diff}\,(TF)$, where $T\theta(\gamma) : TF \longrightarrow TF$

is the derivative of $\theta(\gamma) : F \longrightarrow F$. In particular, if $\pi_{ad} : G_P \longrightarrow M$ is the adjoint bundle, G_P being given by $G_P = P \times_{ad} G$, then the vertical bundle $VG_P \longrightarrow M$ of $G_P \longrightarrow M$ is the bundle with fibre TG associated to $\pi : P \longrightarrow M$ by the action $T(ad) : G \longrightarrow Diff(TG)$. The adjoint bundle $\pi_{Ad} : \mathfrak{g}_P \longrightarrow M$ is the vector bundle whose fibre $\mathfrak{g}$ is the Lie algebra of G, where $\pi_{Ad} : \mathfrak{g}_P \longrightarrow M$ is associated to $\pi : P \longrightarrow M$ by the adjoint representation $Ad : G \longrightarrow Aut(\mathfrak{g})$ of G. The Maurer-Cartan form $\Phi : TG \longrightarrow \mathfrak{g}$ on G is the $\mathfrak{g}$-valued 1-form mapping $X \in T_\gamma G$ to $L_{\gamma^{-1}*} X \in \mathfrak{g}$, for all $\gamma \in G$, where $L_{\gamma^{-1}*} : TG \rightarrow TG$ is the derivative of the map $L_{\gamma^{-1}} : G \rightarrow G$ sending $\eta \in G$ to $\gamma^{-1}\eta$. One may easily verify that $\Phi : TG \rightarrow \mathfrak{g}$ is G-equivariant, where G acts on TG by the left action $T(ad) : G \rightarrow Diff(TG)$ and on $\mathfrak{g}$ by the adjoint representation $G \rightarrow Aut(\mathfrak{g})$. It follows that Φ induces a smooth fibre bundle morphism $\Phi_P : VG_P \longrightarrow \mathfrak{g}_P$.

Let $\omega : TP \rightarrow \mathfrak{g}$ by a smooth connection on $\pi : P \longrightarrow M$. Let $\Psi : M \longrightarrow G_P$ be a C^1 section of $\pi_{ad} : G_P \longrightarrow M$, and let $D^\omega\Psi : TM \longrightarrow VG_P$ be the covariant derivative of Ψ. We may compose $D^\omega\Psi$ with $\Phi_P : VG_P \rightarrow \mathfrak{g}_P$ to obtain a map

$$\Phi_P \circ D^\omega \Psi : TM \rightarrow \mathfrak{g}_P.$$

It may easily be verified that this map is a morphism of vector bundles over M, and thus determines a c^0 section $\chi^\omega(\Psi) : M \rightarrow \mathfrak{g}_P \otimes T^*M$ of $\mathfrak{g}_P \otimes T^*M \longrightarrow M$. We shall show that if $\Psi \in C^1(G_P)$ corresponds to a principal bundle automorphism $\Psi : P \longrightarrow P$, then $\chi^\omega(\Psi) \in C^0(\mathfrak{g}_P \otimes T^*M)$ corresponds to the horizontal G-equivariant $\mathfrak{g}$-valued 1-form $\Psi^*\omega - \omega$ on P.

<u>Theorem 6.2</u>

Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G_P \longrightarrow M$ and $\pi_{Ad} : \mathfrak{g}_P \rightarrow M$ be the adjoint bundles with total spaces $G_P = P \times_{ad} G$ and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. Let $\omega : TP \rightarrow \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \rightarrow M$ and let $\Psi : P \rightarrow P$ be a C^1 principal bundle automorphism of $\pi : P \rightarrow M$, identified with the section $\Psi \in C^1(G_P)$ of $G_P \rightarrow M$. Let $D^{\omega}\Psi : TM \rightarrow VG_P$ be the covariant derivative of Ψ, and let $\chi^{\omega}(\Psi) \in C^0(\mathfrak{g}_P \otimes T^*M)$ be defined by

$$\chi^{\omega}(\Psi) = \Phi_P D^{\omega}\Psi$$

where $\Phi_P : VG_P \rightarrow \mathfrak{g}_P$ is the map induced by the Maurer-Cartan form $\Phi : TG \rightarrow \mathfrak{g}$ on G. Then $\chi^{\omega}(\Psi)$ is the section of $C^0(\mathfrak{g}_P \otimes T^*M)$ determined by the horizontal G-equivariant $\mathfrak{g}$-valued 1-form $\Psi^*\omega - \omega$ on P.

Proof

Ψ determines a G-equivariant C^1 map $\psi : P \rightarrow G$ such that

$$\Psi(p) = p.\psi(p)$$

for all $p \in P$. Given $p \in P$, let $\mu_p : G \rightarrow G_P$ be the map sending $\gamma \in G$ to $\lambda(p, \gamma)$, where $\lambda : P \times G \rightarrow G_P$ is the natural projection. Let $m \in \pi(p)$, let $X \in T_mM$, and let $\tilde{X} \in T_pP$ be the horizontal lift of X. Then

$$(D^{\omega}\Psi)(X) = \mu_{p*}\psi_*(\tilde{X})$$

by the preceding lemma.

Let $\bar{\mu}_p : \mathfrak{g} \rightarrow \mathfrak{g}_P$ be the map sending $a \in \mathfrak{g}$ to the image of (p, a) under the natural projection from $P \times \mathfrak{g}$ onto $\mathfrak{g}_P$. Then

$$\Phi_P \circ \mu_{p*} = \bar{\mu}_p \circ \Phi$$

by definition of Φ_P. Hence

$$\chi^{\omega}(\Psi)(X) = \Phi_P(\mu_{p*}\psi_*(\tilde{X}))$$
$$= \bar{\mu}_p \Phi(\psi_*\tilde{X}).$$

Now consider the horizontal G-equivariant G-valued 1-form τ, where

$$\tau = \Psi^*\omega - \omega .$$

Now

$$\tau(\tilde{X}) = \omega(\Psi_* \tilde{X}) - \omega(\tilde{X})$$

$$= \omega(\Psi_* \tilde{X}),$$

and by lemma 4.1

$$\omega(\Psi_* \tilde{X}) = \mathrm{Ad}(\psi(p)^{-1})\,\omega(\tilde{X}) + L_{\psi(p)^{-1}*}\,\psi_* \tilde{X}$$

$$= \Phi(\psi_* \tilde{X}).$$

Thus

$$\Phi(\psi_* \tilde{X}) = \tau(\tilde{X})$$

and hence

$$\chi^{\omega}(\Psi)(X) = \tilde{\mu}_p(\Psi^*\omega - \omega)(\tilde{X})$$

showing that $\chi^{\omega}(\Psi)$ is the section of $\mathfrak{g}_P \otimes T^*M \to M$ determined by the horizontal G-equivariant $\mathfrak{g}$-valued 1-form $\Psi^*\omega - \omega$.

□

We see that, given a smooth vector bundle $p : E \to M$ associated to the principal bundle $\pi : P \to M$, and given a smooth Ehresmann connection ω on $\pi : P \to M$, each section $s : M \to E$ has a covariant derivative $D^{\omega}s : TM \to VE$. We now show how to define the covariant differential of $s : M \to E$. There is a natural smooth isomorphism $\nu : E \oplus E \to VE$ of vector bundles over M such that if $(X,Y) \in E \oplus E$ then $\nu(X,Y)$ is tangent to the curve

$$t \mapsto X + tY$$

at $t = 0$. It follows that for all C^1 sections $s : M \to E$ of E there is a C^1 map $d^{\omega}s : TM \to E$ such that

$$D^{\omega}s = \nu(s, d^{\omega}s).$$

$d^{\omega}s$ is linear on each fibre of $TM \to M$, and may thus be identified with a smooth section of the vector bundle $E \otimes T^*M \to M$. One may easily verify that

$$d^{\omega}(s_1 + s_2) = d^{\omega}s_1 + d^{\omega}s_2$$

$$d^{\omega}(fs) = fd^{\omega}s + s \otimes df$$

for all sections s, s_1, s_2 of E and for all C^1 functions f on M. $d^{\omega}s$ is the covariant differential of s.

We now consider the composition of the map $\exp : C^1(\mathfrak{g}_P) \to C^1(G_P)$, induced by the fibre bundle morphism $\exp_P : \mathfrak{g}_P \to G_P$ determined by $\exp : \mathfrak{g} \to G$, and the differential operator $\mathcal{X}^{\omega}$, mapping sections of $G_P \to M$ to sections of $\mathfrak{g}_P \otimes T^*M \to M$, defined above.

Theorem 6.3

Let $\pi : P \to M$ be a smooth principal bundle with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$ be the adjoint bundles with total spaces $G_P = P \times_{ad} G$ and $\mathfrak{g}_P : P \times_{Ad} \mathfrak{g}$. Let $\omega : TP \to \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \to M$. Let $\exp : C^1(\mathfrak{g}_P) \to C^1(G_P)$ and $\mathcal{X}^{\omega} : C^1(G_P) \to C^0(\mathfrak{g}_P \otimes T^*M)$ be defined as above.

Then there exists a smooth morphism $B : \mathfrak{g}_P \to \mathrm{End}(\mathfrak{g}_P \otimes T^*M)$ of fibre bundles such that

$$\mathcal{X}^{\omega}(\exp \xi) = B(\xi)(d^{\omega}\xi)$$

for all $\xi \in C^1(\mathfrak{g}_P)$, where $d^{\omega}\xi \in C^0(\mathfrak{g}_P \otimes T^*M)$ is the covariant differential of ξ.

Let $|\cdot|_m$ be the norm on the fibre $\mathfrak{g}_P[m]$ of $\mathfrak{g}_P \to M$ over $m \in M$ determined by a given biinvariant metric on G, and let $i(G)$ be the injectivity radius of G. If the section $\xi \in C^0(\mathfrak{g}_P)$ satisfies

$$|\xi(m)|_m < i(G)$$

for all $m \in M$, then $B(\xi)$ is a section of the bundle $\mathrm{Aut}(\mathfrak{g}_P \otimes T^*M) \to M$ of vector bundle automorphisms of $\mathfrak{g}_P \otimes T^*M \to M$. If $0 \in C^0(\mathfrak{g}_P)$ is the zero section, then $B(0)$ is the identity automorphism of $\mathfrak{g}_P \otimes T^*M \to M$.

<u>Proof</u>

We recall that if $\exp_P : \mathfrak{g}_P \to G_P$ is the fibre bundle morphism induced by $\exp : \mathfrak{g} \to G$, then

$$D^{\omega}(\exp_P \circ \xi) = V(\exp_P)\, D^{\omega}\xi$$

by the functorality property of the covariant derivative. Thus

$$\begin{aligned} \mathcal{X}^{\omega}(\exp \xi) &= \Phi_P\left(D^{\omega}(\exp_P \xi)\right) \\ &= \Phi_P\left(V(\exp_P)\, D^{\omega}\xi\right) \\ &= \Phi_P\left(V(\exp_p)\, \nu(\xi, d^{\omega}\xi)\right) \end{aligned}$$

where $\nu : \mathfrak{g}_P \oplus \mathfrak{g}_P \to V\mathfrak{g}_P$ is the natural vector bundle isomorphism defined above. Thus we may define

$$B(\xi)\eta = \Phi_P\left(V(\exp_p)\, \nu(\xi, \eta)\right).$$

Let $a \in \mathfrak{g}_P[m]$ for some $m \in M$ and let $\gamma = \exp_p a$. Then the maps

$$\mathfrak{g}_P[m] \to V_a\mathfrak{g}_P : b \mapsto \nu(a, b)$$

$$V_a\mathfrak{g}_p \to V_{\gamma} G_P \quad : \quad A \mapsto V(\exp_P)\, A$$

$$V_{\gamma} G_P \to \mathfrak{g}_P \quad : \quad X \to \Phi_P(X)$$

are linear. Hence $B(\xi)$ is linear. If $|a|_m < i(G)$, then $V(\exp_p) \mid V_a\mathfrak{g}_p$ is an isomorphism from $V_a\mathfrak{g}_P$ onto $V_{\gamma} G_P$, and so

$$B(a) \; : \; \mathfrak{g}_P[m] \otimes T_mM \longrightarrow \mathfrak{g}_P[m] \otimes T_mM$$

is an automorphism of vector spaces whenever $|a|_m < i(G)$.

$\square$

Let $\theta : G \longrightarrow \mathrm{Diff}(F)$ be a smooth left action of the structural group G of $\pi : P \longrightarrow M$ on a smooth manifold F and let $\pi_\theta : F_P \longrightarrow M$ be a fibre bundle with fibre F and total space $P \times_\theta F$ associated to $\pi : P \longrightarrow M$ by the action θ. Then there is a natural fibre bundle morphism

$$\theta_P : \; G_P \times_M F_P \longrightarrow F_P$$

which is induced by the map from $G \times F$ to F sending (γ, f) to $\theta(\gamma)f$. θ_P in turn induces a left action of $C^\infty(G_P)$ on F_P mapping $\Psi \in C^\infty(G_P)$ to the diffeomorphism $x \mapsto \Psi.x$, where

$$\Psi.x = \theta_P(\Psi(\pi_\theta(x)), x)$$

for all $x \in F_P$. A smooth section ξ of $\mathfrak{g}_P \rightarrow M$ then determines a vertical vector field $\alpha(\xi)$ on F_P whose flow is given by

$$(x, t) \mapsto (\exp t\xi).x.$$

The map α, sending smooth sections of $\mathfrak{g}_P \rightarrow M$ to vertical vector fields on fibre bundles associated to some given principal bundle is natural with respect to those fibre bundle morphisms that are induced by smooth G-equivariant maps between the fibres of the bundles.

<u>Theorem 6.4</u>

Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G\mathbf{p} \rightarrow M$ and $\pi_{Ad} : \mathfrak{g}\mathbf{p} \rightarrow M$ be the adjoint bundles, with total spaces $P \times_{ad} G$ and $P \times_{Ad} \mathfrak{g}$. Let $\theta : G \rightarrow \mathrm{Diff}(F)$ be a smooth left action of G on F and let $\pi_\theta : F\mathbf{p} \rightarrow M$ be the fibre bundle with total space $F\mathbf{p} = P \times_\theta F$. Given $\xi \in C^\infty(\mathfrak{g}\mathbf{p})$, let $\alpha_x(\xi)$ be the value at $x \in F\mathbf{p}$ of the vertical vector field $\alpha(\xi)$ on $F\mathbf{p}$ whose flow is given by

$$(x, t) \mapsto (\exp t\xi) \,.\, x.$$

Let $\pi : TP \rightarrow \mathfrak{g}$ be a smooth Ehresmann connection on $\pi : P \longrightarrow M$ and let $\tau : TP \rightarrow \mathfrak{g}$ be a smooth horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P. Then, for all sections $s : M \longrightarrow F\mathbf{p}$ of $\pi_\theta : F\mathbf{p} \longrightarrow M$ and for all vectors $X \in T_m M$,

$$D^{\omega + \tau} s(X) = D^{\omega} s(X) + \alpha_{s(m)} (\hat{\tau}(X))$$

where $\hat{\tau}(X) \in C^\infty(\mathfrak{g}\mathbf{p})$ is the image of X under the vector bundle morphism $\hat{\tau} : TM \rightarrow \mathfrak{g}\mathbf{p}$ corresponding to τ.

<u>Proof</u>

First we compare the vertical projections $pr_V^{\omega} : TP \rightarrow VP$ and $pr_V^{\omega+\tau} : TP \rightarrow VP$. For all $a \in \mathfrak{g}$, let $\sigma_p(a)$ be the value at $p \in P$ of the fundamental vertical vector field $\sigma(a)$ on P determined by a. Then, for all $Y \in T_p P$,

$$pr_V^{\omega}(Y) = \sigma_p(\omega(Y))$$

and hence

$$pr_V^{\omega+\tau}(Y) = pr_V^{\omega}(Y) + \sigma_p(\tau(Y)).$$

But the principal bundle $\pi : P \rightarrow M$ may itself be regarded as a fibre bundle with fibre G associated to $\pi : P \rightarrow M$ by the left action of G on G by left multiplication. Thus $C^{\infty}(G\mathbf{p})$ acts on P on the left, and any section ξ of $\mathfrak{g}\mathbf{p} \to M$ determines a vertical vector field $\alpha(\xi)$ on P, and it is easily seen that

$$\sigma_p(\tau(Y)) = \alpha_p(\hat{\tau}(\pi_* Y))$$

where $\pi_* : TP \longrightarrow TM$ is the derivative of $\pi : P \longrightarrow M$. Hence

$$pr_V^{\omega+\tau}(Y) = pr_V^{\omega}(Y) + \alpha_p(\hat{\tau}(\pi_* Y))$$

and thus

$$pr_H^{\omega+\tau}(Y) = pr_H^{\omega}(Y) - \alpha_p(\hat{\tau}(\pi_* Y))$$

where $pr_H^{\omega} : TP \longrightarrow TP$ is the horizontal projection on TP determined by ω. But the derivative of a fibre bundle morphism induced by a G-equivariant map between the fibres of the bundles has the property that it maps the horizontal bundle of one bundle onto the horizontal bundle of the other, and it also maps the vertical vector field $\alpha(\xi)$ on one bundle to that on the other. Hence $pr_H^{\omega} : TF\mathbf{p} \longrightarrow TF\mathbf{p}$ and $pr_H^{\omega+\tau} : TF\mathbf{p} \longrightarrow TF\mathbf{p}$ satisfy

$$pr_H^{\omega+\tau}(Z) = pr_H^{\omega}(Z) - \alpha_x(\hat{\tau}(\pi_{\theta *} Z))$$

for all $Z \in T_xF\mathbf{p}$, and hence

$$pr_V^{\omega+\tau}(Z) = pr_V^{\omega}(Z) + \alpha_x(\hat{\tau}(\pi_{\theta *} Z)).$$

Then, for any C^1 section $s : M \longrightarrow F\mathbf{p}$ of $\pi_\theta : F\mathbf{p} \longrightarrow M$, and for any vector $X \in T_mM$, we have

$$\begin{aligned} D^{\omega+\tau}s(X) &= pr_V^{\omega+\tau} s_*(X) \\ &= pr_V^{\omega} s_*(X) + \alpha_{s(m)}(\hat{\tau}(\pi_{\theta *} s_*(X))) \\ &= D^{\omega}s(X) + \alpha_{s(m)}(\hat{\tau}(X)). \end{aligned}$$

<u>Corollary 6.5</u>

Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G\mathbf{p} \longrightarrow M$ and

$\pi_{Ad} : \mathfrak{g}_P \to M$ be the adjoint bundles, with total spaces $G_P = P \times_{ad} G$ and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. For any smooth connection on $\pi : P \to M$ and for any differentiable section s of $\pi_{ad} : G_P \to M$, let $\mathcal{X}^{\omega}(s)$ denote the section $\Phi_P \circ D^{\omega} s$ of $\mathfrak{g}_P \otimes T^*M \to M$, where $D^{\omega} s : M \to VG_P$ is the covariant derivative of s and $\Phi_P : VG_P \to \mathfrak{g}_P$ is the fibre bundle morphism induced by the Maurer-Cartan form $\Phi : TG \to \mathfrak{g}$. Let τ be a horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P, corresponding to a section $\hat{\tau}$ of $\mathfrak{g}_P \otimes T^*M \to M$. Then

$$\mathcal{X}^{\omega+\tau}(s) = \mathcal{X}^{\omega}(s) + \mathrm{Ad}\,(s^{-1})\hat{\tau} - \hat{\tau}.$$

<u>Proof</u>

By the preceding theorem, we must show that

$$\Phi_P\left(\alpha_{s(m)}\,(\hat{\tau}(X))\right) = \mathrm{Ad}\,(s(m)^{-1})\,\hat{\tau}(X) - \hat{\tau}(X)$$

for all $X \in T_m M$. It is thus sufficient to show that

$$\Phi_P\,(\alpha_{s(m)}\,(\xi)) = \mathrm{Ad}\,(s\,(m)^{-1})\,\xi - \xi$$

for all $\xi \in \mathfrak{g}_P[m]$, the fibre of $\mathfrak{g}_P \to M$ over m. But

$$\alpha_{s(m)}(\xi) = \frac{d}{dt}\left((\exp t\xi)\, s(m)\, \exp(-t\xi)\right)\Big|_{t=0}$$

hence

$$\Phi_P(\alpha_{s(m)}(\xi)) = \frac{d}{dt}\left(s(m)^{-1}(\exp t\xi)\, s(m)\, \exp(-t\xi)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(s(m)^{-1}(\exp t\xi)\, s(m)\right)\Big|_{t=0} + \frac{d}{dt}\left(\exp(-t\xi)\right)\Big|_{t=0}$$

$$= \mathrm{Ad}\,(s(m)^{-1})\,\xi - \xi$$

by Leibnitz' rule, as required.

Now let us consider the case when $\pi_{\theta} : F_P \to M$ is a vector bundle with fibre F associated to the principal bundle $\pi : P \to M$ by the representation $\theta : G \to \mathrm{Aut}(F)$. The representation θ determines a representation $\bar{\theta} : \mathfrak{g} \longrightarrow \mathrm{End}(F)$ of $\mathfrak{g}$ which induces a smooth morphism $\bar{\theta}_P : \mathfrak{g}_P \longrightarrow \mathrm{End}(F_P)$. If $\xi : M \rightarrow \mathfrak{g}_P$ is a section of $\mathfrak{g}_P$ and $s : M \longrightarrow F_P$ is a section of F_P, we shall denote $\bar{\theta}_P(\xi)s$ by $\xi.s$. If ν: $F_P \oplus F_P \longrightarrow VF_P$ is the natural isomorphism, then

$$\alpha_{s(x)}(\xi) = \nu(s(x), \xi.s(x)).$$

If follows that if $\omega : TP \rightarrow \mathfrak{g}$ is a smooth connection on $\pi : P \longrightarrow M$ and $\tau : TP \rightarrow \mathfrak{g}$ is a horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P, then

$$d^{\omega+\tau}s = d^{\omega}s + \hat{\tau}.s$$

for all differentiable sections $s : M \longrightarrow F_P$ of $F_P \longrightarrow M$, where $\hat{\tau} : M \longrightarrow \mathfrak{g}_P \otimes T^*M$ is the section of $\mathfrak{g}_P \otimes T^*M \longrightarrow M$ determined by τ, and where $(\hat{\tau}.s)(X) \equiv \hat{\tau}(X).s$ for all vector fields X on M.

Theorem 6.6

Let $\pi : P \longrightarrow M$ be a smooth principal bundle over a compact Riemannian manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $\theta : G \longrightarrow \mathrm{Aut}(F)$ be a representation of G as a group of isometries of a normed vector space F. Let $\pi_\theta : E \longrightarrow M$ be the vector bundle with total space $E = P \times_\theta F$. For all $m \in M$, let $|\cdot|_m$ be the norm on the fibre $E[m]$ of $\pi_\theta : E \rightarrow M$ over m determined by the norm $|\cdot|$ on F.

Let $\omega : TP \rightarrow \mathfrak{g}$ be a C^1 Ehresmann connection on $\pi : P \longrightarrow M$, and let $\mathrm{Hol}_m(\omega)$ denote the holonomy group of ω generated by loops based at $m \in M$. Let $\xi : M \longrightarrow E$ be a C^1 section of $\pi_\theta : E \longrightarrow M$ and let $c : [a, b] \longrightarrow M$ be a piecewise smooth curve in M parameterized by arclength s. Then

$$\left| \, |\xi(c(b))|_{c(b)} - |\xi(c(a))|_{c(a)} \right| \leq \int_a^b \left| d^{\omega}\xi(c(s)) \right|_{c(s)} ds.$$

Furthermore if $c : [a, b] \to M$ is a piecewise smooth loop beginning and ending at m and generating the element h of the holonomy group $Hol_m(\omega)$ of ω, then

$$\left| (h^{-1}.\xi)(m) - \xi(m) \right| \leq \int_a^b \left| d^\omega \xi (c(s)) \right|_{c(s)} ds.$$

Proof

Let $\tilde{c} : [a, b] \to P$ be a lift of $c : [a, b] \to M$ which is horizontal with respect to ω. Let $\hat{\xi} : P \to F$ be the G-equivariant map corresponding to $\xi : M \to E$. Then

$$\left| \frac{d}{ds} \hat{\xi}(c(s)) \right| = \left| (d^\omega \xi) \left(\frac{dc(s)}{dt} \right) \right|_{c(s)}$$

$$\leq \left| d^\omega \xi \right|_{c(s)}$$

since $c : [a, b] \to M$ is parameterized by arclength. Hence

$$\left| |\xi|_{c(b)} - |\xi|_{c(a)} \right| = \left| |\hat{\xi}(\tilde{c}(b))| - |\hat{\xi}(\tilde{c}(a))| \right|$$

$$\leq \left| \hat{\xi}(\tilde{c}(b)) - \hat{\xi}(\tilde{c}(a)) \right|$$

$$\leq \int_a^b \left| d^\omega \xi \right|_{c(s)} ds$$

as required. If $c : [a, b] \to M$ is a loop based at m, generating $h \in Hol_m(\omega)$, then

$$\left| h^{-1}.\xi - \xi \right|_m = \left| \hat{\xi}(\tilde{c}(b)) - \hat{\xi}(\tilde{c}(a)) \right|$$

$$\leq \int_a^b \left| d^\omega \xi \right|_{c(s)} ds$$

as required.

□

§7. The Covariant Exterior Derivative and Codifferential

In this section, we review the definition and properties of the covariant exterior derivative, covariant codifferential and covariant Hodge-de Rham Laplacian. The material is all standard, and is to be found in [Atiyah, M.F., Hitchin, N.J. and Singer, I.M., 1978], [Bourguignon, J.-P. and Lawson, H.B., 1981], [Bourguignon, J.-P. and Lawson, H.B., 1982].

Let $\pi_E : E \longrightarrow M$ be a vector bundle associated to a principal bundle $\pi : P \longrightarrow M$ over a compact Riemannian manifold M with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\omega : TP \longrightarrow \mathfrak{g}$ be a sufficiently differentiable Ehresmann connection on $\pi : P \longrightarrow M$. We have seen that ω determines a differential operator $d^{\omega} : C^1(E) \longrightarrow C^0(E \otimes T^*M)$, where $d^{\omega}s : M \longrightarrow E \otimes T^*M$ is the covariant differential of $s : M \longrightarrow E$ with respect to ω, for all $s \in C^1(E)$. Let $\langle .,. \rangle : E \otimes E \longrightarrow \mathbb{R}$ be a smooth inner product structure (i.e. an inner product defined on each fibre of $\pi_E : E \longrightarrow M$ by a suitable smooth section of $E^* \otimes E^*$). We say that the connection preserves the inner product structure $\langle .,. \rangle$ on $\pi_E : E \longrightarrow M$ if and only if

$$d \langle s_1, s_2 \rangle = \langle d^{\omega} s_1, s_2 \rangle + \langle s_1, d^{\omega} s_2 \rangle ,$$

where $\langle e_1 \otimes \eta , e_2 \rangle \equiv \langle e_1, e_2 \rangle \eta$ for all $e_1 \otimes \eta \in C^0(E \otimes T^*M)$ and $e_2 \in C^0(E)$.

For all non-negative integers p, let the <u>covariant exterior derivative</u>

$$d^{\omega} : C^1(E \otimes \Lambda^p T^*M) \rightarrow C^0(E \otimes \Lambda^{p+1} T^*M)$$

be the differential operator defined by

$$d^{\omega}(s \otimes \eta) = d^{\omega}s \wedge \eta + s \otimes d\eta$$

for all $s \in C^1(E)$ and $\eta \in C^1(\Lambda^p T^*M)$. If θ is an E-valued differential form on M and φ is a differential form on M, then

$$d^{\omega}(\theta \wedge \varphi) = d^{\omega}\theta \wedge \varphi + (-1)^{\deg\theta}\, \theta \wedge d\varphi .$$

However $(d^{\omega})^2 \neq 0$ in general. In fact, if ω is a C^1 Ehresmann connection on the principal bundle $\pi : P \rightarrow M$ associated to $\pi_E : E \rightarrow M$, then the curvative $F : \Lambda^2 TP \rightarrow \mathfrak{g}$ of ω determines a $\mathfrak{g}_P$-valued 2-form F^{ω} on M, where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. But for all $m \in M$, the fibre $\mathfrak{g}_P[m]$ of $\mathfrak{g}_P \rightarrow M$ over m is a Lie algebra acting naturally on the fibre $E[m]$ of $E \rightarrow M$ over m. This action defines a bilinear map

$$C^0(\mathfrak{g}_P) \times C^0(E) \rightarrow C^0(E)$$

and thus determines bilinear maps

$$C^0(\mathfrak{g}_P \otimes \Lambda^p T^*M) \times C^0(E \otimes \Lambda^q T^*M) \rightarrow C^0(E \otimes \Lambda^{p+q} T^*M)$$

for all non-negative integers p and q, mapping $(\xi \otimes \eta_1, s \otimes \eta_2)$ to $(\xi . s) \otimes (\eta_1 \wedge \eta_2)$ for all $\xi \in C^0(\mathfrak{g}_P)$, $s \in C^0(E)$, $\eta_1 \in C^0(\Lambda^p T^*M)$ and $\eta_2 \in C^0(\Lambda^q T^*M)$. If $\theta \in C^0(\mathfrak{g}_P \otimes \Lambda^p T^*M)$ and $\varphi \in C^0(E \otimes \Lambda^q T^*M)$, we denote the image of (θ, φ) under this bilinear map by $\theta \wedge \varphi$. It is well-known that

$$d^{\omega} d^{\omega}\theta = F^{\omega} \wedge \theta$$

for all E-valued differential forms θ on M, where $F^{\omega} \in C^0(\mathfrak{g}_P \otimes \Lambda^2 T^*M)$ is determined by the curvature of ω.

We now suppose that $\langle \, , \, \rangle : E \otimes E \rightarrow \mathbb{R}$ is a smooth inner product structure on the vector bundle $\pi_E : E \rightarrow M$ over the compact Riemannian manifold M and that this inner product structure is preserved by the connection ω on the associated principal bundle $\pi : P \rightarrow M$. The Riemannian metric on M determines an inner product structure on $\Lambda^p T^*M$: if $(\sigma_1, \ldots, \sigma_n)$ is an orthonormal coframe on M, then

$$\sigma_{j_1} \wedge \ldots \wedge \sigma_{j_p} : 1 \leq j_1 \quad \ldots \quad j_p \leq \dim M$$

is an orthonormal basis of sections of $\Lambda^p T^*M \rightarrow M$ over the domain of definition of the coframe. If $e_1 \otimes \eta_1$ and $e_2 \otimes \eta_2$ are sections of $E \otimes \Lambda^p T^*M \to M$, we define

$$e_1 \otimes \eta_1, e_2 \otimes \eta_2 = \langle e_1, e_2 \rangle \langle \eta_1, \eta_2 \rangle .$$

This defines a smooth inner product structure on $E \otimes \Lambda^p T^*M \to M$. Given E-valued p-forms θ, and θ_2 on M, we define

$$(\theta_1, \theta_2) = \int_M \langle \theta_1, \theta_2 \rangle \, d(vol)$$

where the integral is taken with respect to the Riemannian volume measure on the compact manifold M.

Let M be oriented, let n be the dimension of M and, for all integers p satisfying $0 \leqslant p \leqslant n$, let

$$* : \Lambda^p T^*M \to \Lambda^{n-p} T^*M$$

be the Hodge star operator. If η_1 and η_2 are p-forms on M then

$$(\eta_1, \eta_2) = \int_M \eta_1 \wedge * \eta_2 = \int_M \eta_2 \wedge * \eta_1 .$$

The Hodge star operator from $\Lambda^p T^*M$ to $\Lambda^{n-p} T^*M$ satisfies $** = (-1)^{p(n-p)}$. If $e \otimes \eta \in E \otimes \Lambda^p T^*M$, we define $*(e \otimes \eta)$ to be $e \otimes * \eta$.

The codifferential $\delta : C^1(\Lambda^p T^*M) \to C^0(\Lambda^{p-1} T^*M)$ is defined by

$$\delta \eta = (-1)^{n(p+1)+1} * d * \eta$$

for all C^1 p-forms η on M.

We may define the covariant codifferential

$$\delta^\omega : C^1(E \otimes \Lambda^p T^*M) \longrightarrow C^0(E \otimes \Lambda^{p-1} T^*M)$$

with respect to the connection ω by

$$\delta^\omega \theta = (-1)^{n(p+1)+1} * d^\omega * \theta$$

for all E-valued p-forms θ. It may be verified that

$$(d^\omega \theta, \varphi) = (\theta, \delta^\omega \varphi)$$

for all E-valued p-forms θ and E-valued (p + 1)-forms φ on M. The covariant Hodge-de Rham Laplacian

$$\Delta^{\omega} : C^2(E \otimes \Lambda^p T^*M) \to C^0(E \otimes \Lambda^p T^*M)$$

is defined by

$$\Delta^{\omega} = d^{\omega} \delta^{\omega} + \delta^{\omega} d^{\omega} .$$

Δ^{ω} is an elliptic operator which is self-adjoint with respect to the inner product (,) on E-valued differential forms. If θ is an E-valued differential form on the compact manifold M then $\Delta^{\omega}\theta = 0$ if and only if $d^{\omega}\theta = 0$ and $\delta^{\omega}\theta = 0$.

Let ω be a smooth connection on $\pi : P \to M$ and let τ be a $\mathfrak{g}_P$-valued 1-form on M determining a horizontal G-equivariant $\mathfrak{g}$-valued 1-form on P which we also denote by τ. If $s : M \to E$ is a section of $E \to M$, then

$$d^{\omega+\tau}s = d^{\omega}s + \tau . s$$

where $\tau . s$ is the image of $\tau \otimes s$ under the natural action $\mathfrak{g}_P \otimes E \to E$ of the bundle $\mathfrak{g}_P$ of Lie algebras on the vector bundle E. It follows that

$$d^{\omega+\tau}\theta = d^{\omega}\theta + \tau \wedge \theta$$

and

$$\delta^{\omega+\tau}\theta = \delta^{\omega}\theta + (-1)^{n(p+1)+1} *(\tau \wedge * \theta)$$

for all E-valued p-forms θ.

The Lie bracket on $C^0(\mathfrak{g}_P)$ determines a bilinear map

$$C^0(\mathfrak{g}_P \otimes \Lambda^p T^*M) \times C^0(\mathfrak{g}_P \otimes \Lambda^q T^*M) \to C^0(\mathfrak{g}_P \otimes \Lambda^{p+q} T^*M)$$

mapping $(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2)$ to $[\xi_1, \xi_2] \otimes (\eta_1 \wedge \eta_2)$ for all $\xi_1, \xi_2 \in C^0(\mathfrak{g}_P)$, $\eta_1 \in C^0(\Lambda^p T^*M)$ and $\eta_2 \in C^0 9 \Lambda^q T^*M)$. If $\theta_1 \in C^0(\mathfrak{g}_P \otimes \Lambda^p T^*M)$ and $\theta_2 \in C^0(\mathfrak{g}_P \otimes \Lambda^q T^*M)$, we denote the image of (θ_1, θ_2) under this bilinear map by $[\theta_1, \theta_2]$. One may verify that

$$[\theta_1, \theta_2] = (-1)^{pq+1} [\theta_2, \theta_1]$$

and that if in addition $q = p$ then

$$[\theta_1, * \theta_2] = - [\theta_2, * \theta_1] .$$

Thus if θ is a $\mathfrak{g}P$-valued p-form on M, then

$$[\theta, * \theta] = 0 .$$

In particular if τ is a $\mathfrak{g}P$-valued 1-form on M, then

$$\delta^{\omega+\tau}\tau = \delta^{\omega}\tau - *([\tau, *\tau])$$

$$= \delta^{\omega}\tau$$

The curvatures F^{ω} and $F^{\omega+\tau}$ of the connections ω and $\omega + \tau$ are related by the identity

$$F^{\omega+\tau} = F^{\omega} + d^{\omega}\tau + \tfrac{1}{2}[\tau, \tau] .$$

The curvature F^{ω} of ω satisfies the Bianchi identity

$$d^{\omega} F^{\omega} = 0$$

The connection ω is said to be a Yang-Mills connection if its curvature satisfies the Yang-Mills equation

$$\delta^{\omega} F^{\omega} = 0 .$$

Using the Bianchi identity we see that this condition is equivalent to the condition

$$\Delta^{\omega} F^{\omega} = 0$$

(i.e. the curvature of ω is harmonic). Yang-Mills connections are critical points of the Yang-Mills functional

$$YM(\omega) = (F, F) = \int_M \|F\|^2 \, d(vol).$$

If ξ is a differentiable section of $\mathfrak{g}P \to M$, we have seen that

$$(\exp \xi)^* \omega - \omega = B(\xi)(d^{\omega}\xi)$$

where $B : \mathfrak{g}P \to \mathrm{End}(\mathfrak{g}P \otimes T^*M)$ is a smooth fibre bundle morphism mapping the zero section of $\mathfrak{g}P \to M$ to the identity section of

$\operatorname{End}(\mathfrak{g}_P \otimes T^*M) \to M$, and where $(\exp \xi)^*\omega$ is the pullback of ω by the principal bundle automorphism determined by $\exp \xi$ (see theorems 6.2 and 6.3). Thus if $\omega_t = (\exp t\xi)^*\omega$, then

$$\left.\frac{d\omega_t}{dt}\right|_{t=0} = d^{\omega}\xi .$$

We collect together some of the above facts in the following proposition.

Proposition 7.1

Let $\pi_E : E \to M$ be a vector bundle associated to a principal bundle $\pi : P \to M$ over a Riemannian manifold M with structural group G whose Lie algebra is $\mathfrak{g}$. Let $G_P = P \times_{ad} G$ and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. Let ω be a smooth connection on $\pi : P \to M$, let F^{ω} be the curvature of ω, let τ be a differentiable $\mathfrak{g}_P$-valued 1-form on M, let ξ be a differentiable section of $\mathfrak{g}_P \to M$, and let θ be an E-valued p-form on M. Then

(i) $d^{\omega}d^{\omega}\theta = F^{\omega} \wedge \theta,$

(ii) $d^{\omega}F^{\omega} = 0,$

(iii) $d^{\omega+\tau}\theta = d^{\omega}\theta + \tau \wedge \theta,$

(iv) $\delta^{\omega+\tau}\theta = \delta^{\omega}\theta + (-1)^{n(p+1)+1} * (\tau \wedge * \theta),$

(v) $F^{\omega+\tau} = F^{\omega} + d^{\omega}\tau + \tfrac{1}{2}[\tau,\tau],$

(vi) $\delta^{\omega+\tau}\tau = \delta^{\omega}\tau$

(vii) $\left.\frac{d}{dt}(\exp t\xi)^*\omega\right|_{t=0} = d^{\omega}\xi .$

If M is compact and E has an inner product structure invariant with respect to the connection ω, then

(viii) $(d^{\omega}\varphi, \theta) = (\varphi, \delta^{\omega}\theta)$

for all E-valued p-forms θ and E-valued (p − 1)-forms φ,

(ix) Δ^{ω} is elliptic, where

$$\Delta^{\omega} = \delta^{\omega}d^{\omega} + d^{\omega}\delta^{\omega}$$

(x) $(\varphi, \Delta^{\omega}\theta) = (\Delta^{\omega}\varphi, \theta)$

for all E-valued p-forms θ and φ on M,

(xi) $\Delta^{\omega}\theta = 0$ if and only if $d^{\omega}\theta = \delta^{\omega}\theta = 0$.

References for Chapter V

Ambrose, W. and Singer, I.M., "A theorem on holonomy", Trans. Amer. Math. Soc., 75 (1953), pp.428-433.

Atiyah, M.F., Hitchin, N.J. and Singer, I.M., "Self-duality in four-dimensional Riemannian geometry", Proc. R. Soc. Lond. A. 362 (1978), pp.425-461.

Auslander, L. and MacKenzie, R.E., Introduction to differentiable manifolds, McGraw-Hill, New York, 1963.

Bourguignon, J.-P. and Lawson, H.B., "Stability and isolation phenomena for Yang-Mills fields", Comm. Math. Phys., 79 (1981), pp.189-230.

Donaldson, S.K.,"An application of gauge theory to four dimensional topology," J. Diff. Geom., 18 (1983), pp.237-315.

Kobayashi, S. and Nomizu, K., Foundations of differential geometry, Interscience, New York, 1963.

Palais, R.S., Foundations of global non-linear analysis, Benjamin, New York, 1968.

Chapter VI

BANACH MANIFOLDS OF AUTOMORPHISMS AND CONNECTIONS

§1. Introduction

In this chapter, we study the action of various Banach manifolds of automorphisms of a principal bundle $\pi : P \to M$ on the corresponding spaces of connections on that bundle. We define $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ and $C^{k,\alpha} \mathcal{A}$ to be the spaces of L^p_K connections, C^k connections and $C^{k,\alpha}$ connections respectively on $\pi : P \to M$. These are affine spaces modelled on the corresponding Banach spaces of sections of the vector bundle $\mathfrak{g}_P \otimes T^*M$, where $\mathfrak{g}_P$ is the adjoint bundle of $\pi : P \to M$. When $p(k + 1) > n$, where n is the dimension of M, we define $L^p_{k+1} \mathcal{G}$, $C^{k+1} \mathcal{G}$ and $C^{k+1,\alpha} \mathcal{G}$ to be the corresponding Banach Lie groups of principal bundle automorphisms.

We show that $L^p_{k+1} \mathcal{G}$ acts smoothly on $L^p_k \mathcal{A}$ when $p(k + 1) > n$ (theorem 2.1) and that if Ψ is a principal bundle automorphism which maps an L^p_k connection ω to another L^p_k connection $\Psi^* \omega$ and if Ψ corresponds to a continuous section of the adjoint bundle G_P which is differentiable almost everywhere on M then $\Psi \in L^p_{k+1} \mathcal{G}$ (theorem 2.2). These results are stated in [Uhlenbeck, K.K., 1982], where they are proved in the cases $k = 0$ and $k = 1$. Similar results are proved for the action of $C^{k+1} \mathcal{G}$ on $C^k \mathcal{A}$ and for the action of $C^{k+1,\alpha} \mathcal{G}$ on $C^{k,\alpha} \mathcal{A}$.

Our main result of this chapter is theorem 3.2 where it is shown that if the base manifold M and the structural group G are compact, if $pk > n$, if $(\omega_i \in L^p_k \mathcal{A} : i \in \mathbb{N})$ and $(\Psi_i \in L^p_{k+1} \mathcal{G} : i \in \mathbb{N})$ are sequences of connections and automorphisms respectively, if the sequences (ω_i) and $(\Psi_i^* \omega_i)$ converge in $L^p_k \mathcal{A}$ and if the sequence of automorphisms converges on some fibre of the map $\pi : P \to M$, then the sequence (Ψ_i) of automorphisms converges in $L^p_{k+1} \mathcal{G}$. Similar results are proved for C^k and Hölder spaces. From this result we shall deduce that the topological spaces $L^p_k \mathcal{A} / L^p_{k+1} \mathcal{G}$, $C^k \mathcal{A} / C^{k+1} \mathcal{G}$ and $C^{k,\alpha} \mathcal{A} / C^{k+1,} \mathcal{G}$ are Hausdorff (theorem 4.1). Also the stabilizer of any connection in $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ or $C^{k,\alpha} \mathcal{A}$ is a compact subgroup of $L^p_{k+1} \mathcal{G}$, $C^{k+1} \mathcal{G}$ or $C^{k+1,\alpha} \mathcal{G}$ respectively (theorem 4.2).

The above results will be used in chapter VIII, where we shall prove various slice theorems for the action of automorphisms on connections.

In §5 we consider various properties of the covariant differential withr espect to a given connection, considered as a map between Banach spaces of sections of the appropriate vector bundles.

§2. Basic Properties of the Action of Automorphisms on Connections

In this section, we shall study the action of principal bundle automorphisms on Ehresmann connections on a smooth principal bundle over a compact manifold with compact structural group. The group $C^\infty \mathcal{G}$ of smooth principal bundle automorphisms acts on the space $C^\infty \mathcal{A}$ of smooth Ehresmann connections on the right, sending (ω, Ψ) to $\Psi^* \omega$, the pullback of ω by Ψ, for all $\omega \in C^\infty \mathcal{A}$ and $\Psi \in C^\infty \mathcal{G}$. Let k be an integer. We shall define, for all $p \in [1, \infty)$, the space $L^p_k \mathcal{A}$ of L^p_k connections and if also $p(k + 1) > \dim M$ we shall define the group $L^p_{k+1} \mathcal{G}$ of L^p_{k+1} principal bundle automorphisms and show that the action of $C^\infty \mathcal{G}$ on $C^\infty \mathcal{A}$ on the right extends to a smooth right action of $L^p_{k+1} \mathcal{G}$ on $L^p_k \mathcal{A}$ (provided that $p(k + 1) > \dim M$). If k is non-negative and if $\alpha \in (0, 1)$, we shall define $C^k \mathcal{A}$, and $C^{k+1} \mathcal{G}$, $C^{k,\alpha} \mathcal{A}$ and $C^{k+1,\alpha} \mathcal{G}$ similarly and show that $C^{k+1} \mathcal{G}$ and $C^{k+1,\alpha} \mathcal{G}$ act smoothly on $C^k \mathcal{A}$ and $C^{k,\alpha} \mathcal{A}$ respectively on the right. We shall then show that if $\Psi : P \rightarrow P$ is a continuous principal bundle automorphism satisfying certain mild differentiability conditions, then $\Psi \in L^p_{k+1} \mathcal{G}$ provided that Ψ maps some element of $L^p_k \mathcal{A}$ into $L^p_k \mathcal{A}$ (where $p(k + 1) > \dim M$), $\Psi \in C^{k+1} \mathcal{G}$ provided that Ψ maps some element of $C^k \mathcal{A}$ into $C^k \mathcal{A}$, and that $\Psi \in C^{k+1,\alpha} \mathcal{G}$ provided that Ψ maps some element of $C^{k,\alpha} \mathcal{A}$ into $C^{k,\alpha} \mathcal{A}$. Finally, we shall obtain a theorem which will enable us to prove results concerning the action of $L^p_{k+1} \mathcal{G}$, $C^{k+1} \mathcal{G}$ or $C^{k+1,\alpha} \mathcal{G}$ on $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ or $C^{k,\alpha} \mathcal{A}$ respectively for large **k** from similar results for small k by a 'bootstrap' procedure using induction on k.

Throughout this section, $\pi : P \longrightarrow M$ will be a smooth principal bundle over a compact smooth manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$, and $\pi_{ad} : G\mathbf{p} \rightarrow M$ and $\pi_{Ad} : \mathfrak{g}\mathbf{p} \rightarrow M$ will be the adjoint bundles, with total spaces $G\mathbf{p} = P \times_{ad} G$ and

$\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$.

We have seen that G-equivariant horizontal $\mathfrak{g}$-valued 1-forms on $\pi : P \to M$ are in natural bijective correspondence with sections of the vector bundle $\mathfrak{g}_P \otimes T^*M \to M$ over M. Thus if $\omega_1 : TP \to \mathfrak{g}$ and $\omega_2 : TP \to \mathfrak{g}$ are Ehresmann connections on $\pi : P \to M$, then their difference $\omega_1 - \omega_2$ may be identified with a section of $\mathfrak{g}_P \otimes T^*M \to M$, and conversely if $\omega : TP \to \mathfrak{g}$ is an Ehresmann connection on $\pi : P \to M$ and if $\tau : M \to \mathfrak{g}_P \otimes T^*M$ is a section of $\mathfrak{g} \otimes T^*M \to M$, then we may construct an Ehresmann connection on $\pi : P \to M$, denoted by $\omega + \tau$, such that the 1-form $(\omega + \tau) - \omega$ on P corresponds to the section τ of $\mathfrak{g}_P \otimes T^*M \to M$. Thus the space $C^\infty\mathcal{A}$ of all smooth connections on $\pi : P \to M$ may be regarded as an affine space modelled on the Frechet space $C^\infty(\mathfrak{g}_P \otimes T^*M)$. We have seen also that the group $C^\infty\mathcal{G}$ of smooth principal bundle automorphisms of $\pi : P \to M$ may be identified with the group $C^\infty(G_P)$ of smooth sections of $\pi_{ad} : G_P \to M$.

Definition

For all integers k and for all $p \in [1, \infty)$, define $L^p_k\mathcal{A}$, the space of L^p_k connections on $\pi : P \to M$, to be the completion of $C^\infty\mathcal{A}$ with respect to the metric on $C^\infty\mathcal{A}$ defined by a norm on $C^\infty(\mathfrak{g}_P \otimes T^*M)$ generating the L^p_k topology. If in addition $p(k + 1) > \dim M$, define $L^p_{k+1}\mathcal{G}$, the group of L^p_{k+1} principal bundle automorphisms on $\pi : P \to M$ to be the subset of the group of continuous principal bundle automorphisms identified with the Banach manifold $L^p_{k+1}(G_P)$. Similarly, for all non-negative integers k and for all $\alpha \in (0, 1)$, define $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ to be the completions of $C^\infty\mathcal{A}$ with respect to the metrics defined by C^k and $C^{k,\alpha}$ norms on $C^\infty(\mathfrak{g}_P \otimes T^*M)$. Define also $C^k\mathcal{G}$ and $C^{k,\alpha}\mathcal{G}$ to be the subgroups of the group of continuous principal bundle automorphisms identified with the Banach manifolds $C^k(G_P)$ and $C^{k,\alpha}(G_P)$ respectively.

The group operations on the Banach manifolds $L^p_{k+1}\mathcal{G}$ (for $p(k+1) > \dim M$), $C^k\mathcal{G}$ and $C^{k,\alpha}\mathcal{G}$ (for $\alpha \in (0, 1)$) are smooth by the results proved in [Palais, R.S., 1968] (see theorem II.2.6). The Lie algebras of these groups are identified with $L^p_{k+1}(\mathfrak{g}_P)$, $C^k(\mathfrak{g}_P)$ and $C^{k,\alpha}(\mathfrak{g}_P)$ respectively and the exponential maps $L^p_{k+1}(\mathfrak{g}_P) \to L^p_{k+1}\mathcal{G}$, $C^k(\mathfrak{g}_P) \to C^k\mathcal{G}$ and $C^{k,\alpha}(\mathfrak{g}_P) \to C^{k,\alpha}\mathcal{G}$ are smooth.

Let ω_0 be a smooth connection on $\pi : P \to M$. For all integers k and for all $p \in [1, \infty)$, every element of $L^p_k\mathcal{A}$ may be expressed uniquely as $\omega_0 + \tau$ for some $\tau \in L^p_k(\mathfrak{g}_P \otimes T^*M)$, and similarly for $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$.

We have seen that if ω_0 is a smooth connection, then

$$\Psi^*\omega_0 - \omega_0 = \mathcal{X}^{\omega_0}(\Psi)$$

for all differentiable principal bundle automorphisms $\Psi : P \to P$, where $\mathcal{X}^{\omega_0}$ is the first order non-linear differential operator defined in section V.6 (see theorem V.6.2). We have also seen that there exists a smooth fibre bundle morphism $B : \mathfrak{g}_P \to \mathrm{End}(\mathfrak{g}_P \otimes T^*M)$ such that

$$\mathcal{X}^{\omega_0}(\exp \xi) = B(\xi)\, d^{\omega_0}\xi$$

for all differentiable sections ξ of $\mathfrak{g}_P \to M$. B maps the zero section of $\mathfrak{g}_P \to M$ to the identity automorphism of $\mathfrak{g}_P \otimes T^*M$. Also if G is given a biinvariant Riemannian metric determining a canonical C^0-norm $\|\,.\,\|$ on $C^0(\mathfrak{g}_P)$ and if the norm $\|\xi\|$ of $\xi \in C^0(\mathfrak{g}_P)$ does not exceed the injectivity radius of G, then $B(\xi)$ is a vector bundle automorphism of $\mathfrak{g}_P \otimes T^*M$ (see theorem V.6.3).

Let $\omega_0 : TP \to \mathfrak{g}$ be a smooth connection on $\pi : P \to M$, let τ be a section of $\mathfrak{g}_P \otimes T^*M \to M$ and let $\Psi : P \to P$ be a differentiable principal bundle automorphism of $\pi : P \to M$. We have seen that

$$\Psi^*(\omega_0 + \tau) - \Psi^*\omega_0 = \mathrm{Ad}(\Psi^{-1})\tau$$

on regarding Ψ as a section of $\pi_{ad} : G_P \rightarrow M$ (see the proof of V.5.1). Thus

$$\Psi^*(\omega_0 + \tau) = \mathrm{Ad}(\Psi^{-1})\tau + \chi^{\omega_0}(\Psi).$$

Theorem 2.1

Let $\pi : P \rightarrow M$ be a smooth principal bundle over a compact smooth manifold with compact structural group. For all non-negative integers k, for all $p \in [1, \infty)$ satisfying $p(k+1) > \dim M$, and for all $\alpha \in (0, 1)$, the right action of the group $C^\infty \mathcal{G}$ of smooth principal bundle automorphisms on the space $C^\infty \mathcal{A}$ of smooth connections on $\pi : P \rightarrow M$ extends to smooth right actions

$$L^p_k \mathcal{A} \times L^p_{k+1} \mathcal{G} \rightarrow L^p_k \mathcal{A},$$

$$C^k \mathcal{A} \times C^{k+1} \mathcal{G} \rightarrow C^k \mathcal{A},$$

$$C^{k,\alpha} \mathcal{A} \times C^{k+1,\alpha} \mathcal{G} \rightarrow C^{k,\alpha} \mathcal{A}.$$

Proof

Consider the action of $L^p_{k+1}\mathcal{G}$ on $L^p_k \mathcal{A}$. Given an open neighbourhood of the zero section in $L^p_{k+1}(\mathfrak{g}_P)$, any element Ψ of $L^p_{k+1}\mathcal{G}$ may be expressed as $\Psi = \Psi_0 \exp \xi$, where $\Psi_0 \in C^\infty \mathcal{G}$, $\xi \in L^p_{k+1}(\mathfrak{g}_P)$ and ξ is contained in the given neighbourhood of the zero section. Also the map $\exp : L^p_{k+1}(\mathfrak{g}_P) \rightarrow L^p_{k+1}\mathcal{G}$ is a chart for $L^p_{k+1}\mathcal{G}$ when restricted to some neighbourhood of the zero section. The map from $L^p_k \mathcal{A}$ to itself sending ω to $\Psi_0{}^*\omega$ is smooth, hence it suffices to verify that the map from $L^p_k \mathcal{A} \times L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k \mathcal{A}$ sending (ω, ξ) to $(\exp \xi)^*\omega$ is smooth. By the remarks above, it suffices to verify that the map

$$(\tau, \xi) \mapsto \mathrm{Ad}(\exp(-\xi))\tau + \chi^{\omega_0}(\exp \xi)$$

from $L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ is smooth.

If $k > 0$, there exists $q \in [1, \infty)$ such that

$$\frac{1}{p} - \frac{1}{\dim M} < \frac{1}{q} < \frac{k}{\dim M}.$$

Then we have a smooth Sobolev embedding $L^p_{k+1}(\mathfrak{g}_P) \hookrightarrow L^q_k(\mathfrak{g}_P)$, the map from $L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ sending ξ to $d^{\omega_0}\xi$ is smooth, and the map from $L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^q_k(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ sending (η, ξ) to $B(\xi)\eta$ is smooth by corollary II.2.7, where $B : \mathfrak{g}_P \to \mathrm{End}(\mathfrak{g}_P \otimes T^*M)$ is the fibre bundle morphism with the property that

$$\mathcal{X}^{\omega_0}(\exp \xi) = B(\xi)\, d^{\omega_0}\xi .$$

Composing these smooth maps, we see that the map from $L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ sending ξ to $\mathcal{X}^{\omega_0}(\exp \xi)$ is smooth. Similarly the map from $L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ sending (τ, ξ) to $\mathrm{Ad}(\exp(-\xi))\tau$ is smooth, again using the Sobolev embedding theorem and corollary II.2.7. Thus the map

$$(\tau, \xi) \mapsto \mathrm{Ad}(\exp(-\xi))\tau + \mathcal{X}^{\omega_0}(\exp \xi)$$

from $L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ is smooth. Thus the action of $L^p_{k+1}\mathcal{G}$ on $L^p_k\mathcal{A}$ is smooth whenever $p(k+1) > \dim M$ and $k > 0$. If $p(k+1) > \dim M$ and $k = 0$, then the same argument applies on replacing $L^q_k(\mathfrak{g}_P)$ by $C^0(\mathfrak{g}_P)$ and using corollary II.2.7 again.

An analogous argument again using corollary II.2.7 shows that the actions of $C^{k+1}\mathcal{G}$ and $C^{k+1,\alpha}\mathcal{G}$ on $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ respectively are smooth.

□

<u>Theorem 2.2</u>

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$, and let $\pi_{ad} : G_P \to M$ be the adjoint bundle with total space $G_P = P \times_{ad} G$. Let $\Psi : P \to P$ be the continuous principal bundle automorphism corresponding to a continuous section of $\pi_{ad} : G_P \to M$ that is differentiable almost everywhere on M. Let k be a non-negative integer. Then

(i) if $p \in [1, \infty)$ satisfies $p(k+1) > \dim M$, if $\omega \in L^p_k \mathcal{A}$ and if $\Psi^* \omega \in L^p_k \mathcal{A}$, then $\Psi \in L^p_{k+1} \mathcal{G}$,

(ii) if $\omega \in C^k \mathcal{A}$ and if $\Psi^* \omega \in C^k \mathcal{A}$, then $\Psi \in C^{k+1} \mathcal{G}$,

(iii) if $\alpha \in (0, 1)$, if $\omega \in C^{k,\alpha} \mathcal{A}$ and if $\Psi^* \omega \in C^{k,\alpha} \mathcal{A}$, then $\Psi \in C^{k+1,\alpha} \mathcal{G}$.

Proof

Choose a biinvariant Riemannian metric on G. Then $\Psi = (\exp \xi)\Psi_0$, where $\Psi_0 : P \to P$ is a smooth principal bundle automorphism, and where ξ is a continuous section of the adjoint bundle $\pi_{Ad} : \mathfrak{g}_P \to M$ whose canonical C^0 norm does not exceed the injectivity radius of G. It is sufficient to prove the theorem when Ψ_0 is the identity automorphism of P, since $\Psi^* \omega \in L^p_k \mathcal{A}$, $C^k \mathcal{A}$ or $C^{k,\alpha} \mathcal{A}$ if and only if $(\Psi \Psi_0^{-1})^* \omega \in L^p_k \mathcal{A}$, $C^k \mathcal{A}$ or $C^{k,\alpha} \mathcal{A}$ respectively. Let $\omega = \omega_0 + \tau$ where ω_0 is a smooth connection. Then

$$\Psi^*(\omega_0 + \tau) = Ad(\Psi^{-1})\tau + \chi^{\omega_0}(\Psi)$$
$$= Ad(\exp(-\xi))\tau + B(\xi)\, d^{\omega_0} \xi$$

where χ^{ω_0} and B are defined above. Since the canonical C^0 norm of ξ is strictly less than the injectivity radius of G, $B(\xi)$ is a vector bundle automorphism of $\mathfrak{g}_P \otimes T^*M$ (see theorem V.6.3). Thus

$$d^{\omega_0} \xi = B(\xi)^{-1} (\Psi^* \omega - Ad(\exp(-\xi))(\omega - \omega_0)).$$

We prove the theorem by induction on k. Suppose $k = 0$. $\xi \in C^0(\mathfrak{g}_P)$, hence both $B(\xi)^{-1}$ and $Ad(\exp(-\xi))$ belong to $C^0(\mathrm{End}(\mathfrak{g}_P \otimes T^*M))$. Thus if ω and $\Psi^* \omega$ belong to $L^p \mathcal{A}$ for some $p \in [1, \infty)$ satisfying $p > \dim M$, then $d^{\omega_0} \xi \in L^p(\mathfrak{g}_P \otimes T^*M)$, and hence $\xi \in L^p_1(\mathfrak{g}_P)$ and $\Psi \in L^p_1 \mathcal{G}$; if ω and $\Psi^* \omega$ belong to $C^0 \mathcal{A}$, then $d^{\omega_0} \xi \in C^0(\mathfrak{g}_P \otimes T^*M)$ and hence $\xi \in C^1(\mathfrak{g}_P)$ and $\Psi \in C^1 \mathcal{G}$. This proves (i) and (ii) when $k = 0$. If ω and $\Psi^* \omega$ belong to $C^{0,\alpha} \mathcal{A}$, then ω and $\Psi^* \omega$ belong to $C^0 \mathcal{A}$ and $\xi \in C^1(\mathfrak{g}_P)$, and thus $\xi \in C^{0,\alpha}(\mathfrak{g}_P)$. Then $B(\xi)^{-1}$ and $Ad(\exp(-\xi))$ belong to $C^{0,\alpha}(End(\mathfrak{g}_P \otimes T^*M))$. Thus $d^{\omega_0}\xi$ belongs to $C^{0,\alpha}(\mathfrak{g}_P \otimes T^*M)$, and hence $\xi \in C^{1,\alpha}(\mathfrak{g}_P)$. This proves (iii) when $k = 0$. We now use the induction hypothesis to prove the theorem when $k > 0$.

Consider case (i) when $k > 0$. We have Sobolev embeddings $L^p_k\mathcal{A} \hookrightarrow L^q_{k-1}\mathcal{A}$ and $L^p_{k+1}\mathcal{G} \hookrightarrow L^q_k\mathcal{G}$, where $q \in [1, \infty)$ may be chosen to satisfy

$$\frac{1}{p} - \frac{1}{\dim M} < \frac{1}{q} < \frac{k}{\dim M}.$$

Then ω and $\Psi^*\omega$ belong to $L^q_{k-1}\mathcal{A}$, and hence $\Psi \in L^q_k\mathcal{G}$, by the induction hypothesis. Thus $\xi \in L^q_k(\mathfrak{g}_P)$. It follows that $B(\xi)^{-1}$ and $Ad(\exp(-\xi))$ belong to $L^q_k(End(\mathfrak{g}_P \otimes T^*M))$. Hence $d^{\omega_0}\xi \in L^p_k(\mathfrak{g}_P \otimes T^*M)$, by corollary II.2.7 and thus $\xi \in L^p_{k+1}(\mathfrak{g}_P)$. Hence $\Psi \in L^p_{k+1}\mathcal{G}$. This proves (i). The proof of (ii) and (iii) is analogous.

□

<u>Theorem 2.3</u>

Let $\pi : P \longrightarrow M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group. Let $(\omega_i : i \in \mathbb{N})$ and $(\Psi_i : i \in \mathbb{N})$ be sequences of connections on $\pi : P \rightarrow M$ and continuous principal bundle automorphisms of $\pi : P \rightarrow M$ respectively. Let k be a non-negative integer. Then

(i) if $p \in [1, \infty)$ satisfies $p(k + 1) > \dim M$, if $\omega_i \in L^p_k\mathcal{A}$ and $\Psi_i \in L^p_{k+1}\mathcal{G}$, if the sequences (ω_i) and $(\Psi_i^*\omega_i)$ converge in $L^p_k\mathcal{A}$ to ω and $\bar{\omega}$ respectively, and if the sequence (Ψ_i) converges in $C^k\mathcal{G}$ to Ψ, where $\Psi \in C^k\mathcal{G}$, then $\Psi \in L^p_{k+1}\mathcal{G}$, (Ψ_i) converges to Ψ in $L^p_{k+1}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$,

(ii) if $k > 0$, if $p,q \in [1, \infty)$ satisfy $p(k+1) > \dim M$, $qk > \dim M$, $q \geq p$, if $\omega_i \in L^p_k \mathcal{A}$ and $\Psi_i \in L^p_{k+1} \mathcal{G}$, if the sequences (ω_i) and $(\Psi^* \omega_i)$ converge in $L^p_k \mathcal{A}$ to ω and $\bar{\omega}$ respectively and if the sequence (Ψ_i) converges in $L^q_k \mathcal{G}$ to Ψ where $\Psi \in L^q_k \mathcal{G}$, then $\Psi \in L^p_{k+1} \mathcal{G}$, (Ψ_i) converges to Ψ in $L^p_{k+1} \mathcal{G}$, and $\Psi^* \omega = \bar{\omega}$,

(iii) if $\omega_i \in C^k \mathcal{A}$ and $\Psi_i \in C^{k+1} \mathcal{G}$, if the sequences (ω_i) and $(\Psi_i^* \omega_i)$ converge in $C^k \mathcal{A}$ to ω and $\bar{\omega}$ respectively and if the sequence (Ψ_i) converges in $C^k \mathcal{G}$ to Ψ, where $\Psi \in C^k \mathcal{G}$, then $\Psi \in C^{k+1} \mathcal{G}$, (Ψ_i) converges to Ψ in $C^{k+1} \mathcal{G}$, and $\Psi^* \omega = \bar{\omega}$,

(iv) if $\alpha \in (0, 1)$, if $\omega_i \in C^{k,\alpha} \mathcal{A}$ and $\Psi_i \in C^{k+1,\alpha} \mathcal{G}$, if the sequences (ω_i) and $(\Psi_i^* \omega_i)$ converge in $C^{k,\alpha} \mathcal{A}$ to ω and $\bar{\omega}$ respectively and if the sequence (Ψ_i) converges in $C^{k,\alpha} \mathcal{G}$ to Ψ where $\Psi \in C^{k,\alpha} \mathcal{G}$, then $\Psi \in C^{k+1,\alpha} \mathcal{G}$, (Ψ_i) converges to Ψ in $C^{k+1,\alpha} \mathcal{G}$, and $\Psi^* \omega = \bar{\omega}$.

Proof

We claim that, without loss of generality we may assume that $\Psi_i = \exp \xi_i$ and $\Psi = \exp \xi$, where ξ_i and ξ are differentiable sections of $\pi_{Ad} : \mathfrak{g}_P \to M$ for all $i \in \mathbb{N}$ and where the canonical C^0 norms of ξ and ξ_i determined by a given biinvariant metric on G do not exceed some constant which is strictly less than the injectivity radius of G. For since (Ψ_i) converges to Ψ in the C^0 topology in all cases (i), (ii), (iii), and (iv), it follows that if a smooth principal bundle automorphism Ψ_0 is sufficiently close to Ψ in the C^0 topology, then there exist differentiable sections ξ_i of $\pi_{Ad} : \mathfrak{g}_P \to M$ for sufficiently large i, that are bounded in the canonical C^0 norm by some constant strictly less than the injectivity radius of G, such that $\Psi_i \Psi_0^{-1} = \exp \xi_i$, and the conclusions of the theorem hold for the sequence (Ψ_i) if they hold when the sequence (Ψ_i) is replaced by the sequence $(\Psi_i \Psi_o^{-1})$ in the statement of the theorem. Thus we may assume that $\Psi_i = \exp \xi_i$, $\Psi = \exp \xi$ and that the canonical C^o-norms of ξ_i and ξ are bounded by a constant strictly less than the injectivity radius of G.

If ω_o is a smooth connection, then

$$d^{\omega_o}\xi_i = B(\xi_i)^{-1} (\Psi_i{}^*\omega_i - Ad(\exp(-\xi_i))(\omega_i - \omega_o)).$$

Also let η be the section of $\mathfrak{g}_P \otimes T^*M \to M$ defined by

$$\eta = B(\xi)^{-1} (\bar{\omega} - Ad(\exp(-\xi))(\omega - \omega_o)).$$

By corollary II.2.7 it follows that $\eta \in L_k^p(\mathfrak{g}_P \otimes T^*M)$ in cases (i) and (ii), $\eta \in C^k(\mathfrak{g}_P \otimes T^*M)$ in case (iii) and $\eta \in C^{k,\alpha}(\mathfrak{g}_P \otimes T^*M)$ in case (iv). It follows also that $(d^{\omega_o}\xi_i)$ converges to η in the L_k^p norm in cases (i) and (ii), $(d^{\omega_o}\xi_i)$ converves to η in the C^k norm in case (iii) and $(d^{\omega_o}\xi_i)$ converges to η in the $C^{k,\alpha}$ norm in case (iv). But $(d^{\omega_o}\xi_i)$ converges to $d^{\omega_o}\xi$ in $L_{k-1}^p(\mathfrak{g}_P \otimes T^*M)$ in cases (i) and (ii), $(d^{\omega_o}\xi_i)$ converges to $d^{\omega_o}\xi$ in $C^{k-1}(\mathfrak{g}_P \otimes T^*M)$ if $k > 0$ in cases (iii) and (iv), and $(d^{\omega_o}\xi_i)$ converges to $d^{\omega}\xi$ in $L_{-1}^r(\mathfrak{g}_P \otimes T^*M)$ for all $r \in (1,\infty)$ if $k = 0$ in cases (iii) and (iv). Hence $\eta = d^{\omega_o}\xi$ in all cases. Thus $\Psi \in L_{k+1}^p \mathcal{G}$ and (Ψ_i) converges to Ψ in the L_{k+1}^p topology in cases (i) and (ii), $\Psi \in C^{k+1}\mathcal{G}$ and (Ψ_i) converges to Ψ in the C^{k+1} topology in case (iii), and $\Psi \in C^{k+1,\alpha}\mathcal{G}$ and (Ψ_i) converges to Ψ in the $C^{k+1,\alpha}$ topology in case (iv). Then $\bar{\omega} = \Psi^*\omega$ by the continuity of the actions of the appropriate groups of principal bundle automorphisms on the corresponding spaces of connections.

□

§3. A Convergence Criterion for Principal Bundle Automorphisms

In this section, we give a condition for a sequence of principal bundle automorphisms to converge in the groups of L^p_{k+1}, C^k or $C^{k,\alpha}$ principal bundle automorphisms of a smooth principal bundle over a compact manifold with compact structural group, in terms of the action of the automorphisms on connections.

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$, let G be given a biinvariant Riemannian metric, and let M be given a Riemannian metric. Let $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$ be the adjoint bundles associated to $\pi : P \to M$ with total spaces $G_P = P \times_{ad} G$ and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$.

We recall that the biinvariant Riemannian metric on G determines a biinvariant Riemannian metric on each fibre of $\pi_{ad} : G_P \to M$ which in turn determines a distance function on this fibre. We let $\rho_m : G_P[m] \times G_P[m] \to \mathbb{R}$ denote this distance function on the fibre $G_P[m]$ of $\pi_{ad} : G_P \to M$ over $m \in M$. We recall also that the biinvariant metric on G determines a G-invariant norm on $\mathfrak{g}$ which in turn induces norms on the fibres of $\pi_{Ad} : \mathfrak{g}_P \to M$ and $\mathfrak{g}_P \otimes T^*M \to M$. We let $|\cdot|_m$ denote both the norm on the fibre $\mathfrak{g}_P[m]$ of $\pi_{Ad} : \mathfrak{g}_P \to M$ over $m \in M$ and also the norm on $\mathfrak{g}_P[m] \otimes T^*_mM$. The canonical distance function $\bar{\rho} : C^0(G_P) \times C^0(G_P) \to \mathbb{R}$ on $C^0(G_P)$, and the canonical C^0 norms $\|\cdot\|$ on $C^0(\mathfrak{g}_P)$ and $C^0(\mathfrak{g}_P \otimes T^*M)$, and the canonical L^p norms $\|\cdot\|_p$ on $L^p(\mathfrak{g}_P)$ and $L^p(\mathfrak{g}_P \otimes T^*M)$ for $p \in [1, \infty)$ are defined by

$$\bar{\rho}(\Psi_1, \Psi_2) = \sup_{m \in M} \rho_m(\Psi_1(m), \Psi_2(m)),$$

$$\|\eta\| = \sup_{m \in M} |\eta(m)|_m,$$

$$= \left(\int_M |\eta(m)|^p \, d(vol) \right)^{1/p}$$

$\bar{\rho}$ may be regarded as a biinvariant distance function on $C^0\mathcal{G}$, on identifying $C^0\mathcal{G}$ and $C^0(G\mathbf{p})$, and the canonical norms on $C^0(\mathfrak{g}\mathbf{p})$, $C^0(\mathfrak{g}\mathbf{p} \otimes T^*M)$, $L^p(\mathfrak{g}\mathbf{p})$ and $L^p(\mathfrak{g}\mathbf{p} \otimes T^*M)$ are invariant under the action of $C^0\mathcal{G}$ (see propositions V.2.3 and V.2.4). Thus if ω_1 and ω_2 belong to $C^0\mathcal{A}$, the space of continuous connections on $\pi : P \to M$ and if $\Psi \in C^1\mathcal{G}$, then

$$\| \Psi^*\omega_1 - \Psi^*\omega_2 \| = \| \omega_1 - \omega_2 \|$$

and if ω_1 and ω_2 belong to $L^p\mathcal{A}$ for some $p \in [1, \infty)$ satisfying $p > \dim M$ and if $\Psi \in L^p_1\mathcal{G}$, then

$$\| \Psi^*\omega_1 - \Psi^*\omega_2 \|_p = \| \omega_1 - \omega_2 \|_p.$$

Lemma 3.1

Let $\pi : P \to M$ be a smooth principal bundle over a compact manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G\mathbf{p} \to M$ and $\pi_{Ad} : \mathfrak{g}\mathbf{p} \to M$ be the adjoint bundles with total spaces $G\mathbf{p} = P \times_{ad} G$ and $\mathfrak{g}\mathbf{p} = P \times_{Ad} \mathfrak{g}$. Let M be given a Riemannian metric and let G be given a biinvariant Riemannian metric, determining a distance function ρ_m on the fibre $G\mathbf{p}[m]$ of $\pi_{ad} : G\mathbf{p} \to M$ over $m \in M$ and determining canonical norms $\| \cdot \|$ on $C^0(\mathfrak{g}\mathbf{p} \otimes T^*M)$ and $\| \cdot \|_p$ on $L^p(\mathfrak{g}\mathbf{p} \otimes T^*M)$, where $\dim M < p < \infty$. Given a compact subset K of M, let

$$\bar{\rho}_K(\Psi_1, \Psi_2) = \sup_{m \in K} (\rho_m(\Psi_1(m), \Psi_2(m)))$$

for all $\Psi_1, \Psi_2 \in C^0\mathcal{G}$. Then there exists a constant A_p, depending only on p and on the Riemannian geometry of M, such that

(i) if $\omega \in C^0\mathcal{A}$, $\Psi_1, \Psi_2 \in C^1\mathcal{G}$ and $m \in K$, then

$$\bar{\rho}_K(\Psi_1, \Psi_2) \leq \rho_m(\Psi_1(m), \Psi_2(m)) + \| \Psi_1^*\omega - \Psi_2^*\omega \| (\text{diam } K),$$

(ii) if $\omega \in L^p\mathcal{A}$, $\Psi_1, \Psi_2 \in L^p_1\mathcal{G}$ and $m \in K$, then

$$\bar{\rho}_K(\Psi_1, \Psi_2) \leq \rho_m(\Psi_1(m), \Psi_2(m)) + A_p \| \Psi_1^*\omega - \Psi_2^*\omega \|_p (\text{diam } K)^{\alpha},$$

where

$$\alpha = 1 - \frac{\dim M}{p}.$$

Proof

Since the smooth connections are dense in $C^0\mathcal{A}$ and $L^p\mathcal{A}$, the smooth principal bundle automorphisms are dense in $C^1\mathcal{G}$ and $L^p_1\mathcal{G}$, and since $C^1\mathcal{G}$ and $L^p_1\mathcal{G}$ act continuously on $C^0\mathcal{A}$ and $L^p\mathcal{A}$ respectively, one may assume that ω, Ψ_1 and Ψ_2 are smooth. By theorem V.5.2, if $c : [a, b] \to M$ is a piecewise smooth curve parameterized by arclength s, $c(a) = m$ and $c(b) = m'$, then

$$\rho_{m'}(\Psi_1(m'),\Psi_2(m')) - \rho_m(\Psi_1(m),\Psi_2(m)) \leq \int_c f(c(s))\, ds$$

where

$$f(x) = |(\Psi_1^*\omega - \Psi_2^*\omega)(x)|_x,$$

and hence

$$\rho_{m'}(\Psi_1(m'),\Psi_2(m')) \leq \rho_m(\Psi_1(m),\Psi_2(m)) + \mu_f(m,m')$$

and

$$\bar\rho_K(\Psi_1,\Psi_2) \leq \rho_m(\Psi_1(m),\Psi_2(m)) + \sup_{m'\in K} \mu_f(m,m'),$$

where $\mu_f(m,m')$ is the infimum of the integrals of f with respect to arclength along all piecewise smooth curves from m to m'.

If $\omega \in C^0\mathcal{A}$ and $\Psi_1, \Psi_2 \in C^1\mathcal{G}$, then $\Psi_1^*\omega - \Psi_2^*\omega$ $C^0(\mathfrak{g}_P \otimes T^*M)$, and

$$f(x) \leq \| \Psi_1^*\omega - \Psi_2^*\omega \|,$$

and thus

$$\mu_f(m,m') \leq \| \Psi_1^*\omega - \Psi_2^*\omega \| \operatorname{dist}(m,m')$$

and hence

$$\bar\rho_K(\Psi_1,\Psi_2) \leq \rho_m(\Psi_1(m),\Psi_2(m)) + \| \Psi_1^*\omega - \Psi_2^*\omega \| (\operatorname{diam} K).$$

Also it follows from theorem IV.3.3 that there exists a constant A_p depending only on p and the Riemannian geometry of M, such that

$$\mu_f(m_1, m_2) \leq A_p \left(\int_M f(x)^p \, d(\text{vol}) \right)^{1/p} (\text{dist}\,(m, m'))^{\alpha}$$

$$\leq A_p \| \Psi_1^* \omega - \Psi_2^* \omega \|_p \, (\text{dist}\,(m,m'))^{\alpha}$$

and hence

$$\bar{\rho}_K(\Psi_1, \Psi_2) \leq \rho_m(\Psi_1(m), \Psi_2(m)) + A_p \| \Psi_1^* \omega - \Psi_2^* \omega \| (\text{diam}\, K)^{\alpha}.$$

□

Theorem 3.2

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G. Let $G_P[m]$ be the fibre over some given $m \in M$ of the adjoint bundle $\pi_{ad} : G_P \to M$ with total space $G_P = P \times_{ad} G$. Let $(\omega_i : i \in \mathbb{N})$ be a sequence of connections on $\pi : P \to M$ and let $(\Psi_i : i \in \mathbb{N})$ be a sequence of continuous principal bundle automorphisms of $\pi : P \to M$ with the property that the sequence $(\Psi_i(m) : i \ \mathbb{N})$ converges in $G_P[m]$. Let k be a non-negative integer. Then

(i) if $p \in [1, \infty)$, if $p(k+1) > \dim M$, if $\omega_i \in L^p_k$ and $\Psi_i \in L^p_{k+1}\mathcal{G}$, and if (ω_i) and $(\Psi_i^* \omega_i)$ converge in $L^p_k\mathcal{A}$ to ω and $\bar{\omega}$ respectively then (Ψ_i) converges in $L^p_{k+1}\mathcal{G}$ to Ψ, for some $\Psi \in L^p_{k+1}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$,

(ii) if $\omega_i \in C^k\mathcal{A}$ and $\Psi_i \in C^{k+1}\mathcal{G}$, and if (ω_i) and $(\Psi_i^*\omega_i)$ converge in $C^k\mathcal{A}$ to ω and $\bar{\omega}$ respectively, then (Ψ_i) converges in $C^{k+1}\mathcal{G}$ to Ψ for some $\Psi \in C^{k+1}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$,

(iii) if $\alpha \in (0, 1)$, if $\omega_i \in C^{k,\alpha}\mathcal{A}$ and $\Psi_i \in C^{k+1,\alpha}\mathcal{G}$, and if (ω_i) and $(\Psi_i^*\omega_i)$ converge in $C^{k,\alpha}\mathcal{A}$ to ω and $\bar{\omega}$ respectively, then (Ψ_i) converges in $C^{k+1,\alpha}\mathcal{G}$ to Ψ for some $\Psi \in C^{k+1,\alpha}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$.

Proof

The proof is by induction on k. First consider the case $k = 0$. If $\omega_i \in L^p\mathcal{A}$ and $\Psi_i \in L_1^p\mathcal{A}$, where $p > \dim M$, then for all positive integers i and j

$$\|\Psi_j^*\omega_i - \Psi_i^*\omega_i\|_p = \|\omega_i - (\Psi_i\Psi_j^{-1})^*\omega_i\|_p$$

$$\leq \|\omega_i - \omega_j\|_p + \|\omega_j - (\Psi_i\Psi_j^{-1})^*\omega_i\|_p$$

$$= \|\omega_i - \omega_j\|_p + \|\Psi_j^*\omega_j - \Psi_i^*\omega_i\|_p.$$

Thus if the sequences (ω_i) and $(\Psi_i^*\omega_i)$ converge in $L^p\mathcal{A}$, then

$$\lim_{i,j\to+\infty}\|\omega_i - \omega_j\|_p = \lim_{i,j\to+\infty}\|\Psi_j^*\omega_j - \Psi_i^*\omega_i\|_p = 0$$

and hence

$$\lim_{i,j\to+\infty}\|\Psi_j^*\omega_i - \Psi_i^*\omega_i\|_p = 0.$$

But by the previous lemma there exists a constant A_p depending only on p and the Riemannian geometry of M such that

$$\bar{\rho}(\Psi_j,\Psi_i) \leq \rho_m(\Psi_j(m),\Psi_i(m)) + A_p\|\Psi_j^*\omega_i - \Psi_i^*\omega_i\|(\operatorname{diam} M)^\alpha$$

where

$$\alpha = 1 - \frac{\dim M}{p}.$$

But the sequence $(\Psi_i(m))$ converges, hence

$$\lim_{i,j\to+\infty}\rho_m(\Psi_i(m),\Psi_j(m)) = 0$$

and thus

$$\lim_{i,j\to+\infty}\bar{\rho}(\Psi_i,\Psi_j) = 0.$$

But $C^0\mathcal{G}$ is complete, hence there exists $\Psi \in C^0\mathcal{G}$ such that the sequence (Ψ_i) converges to Ψ. Then $\Psi \in L_1^p\mathcal{G}$ and (Ψ_i) converges to Ψ in $L^p_1 \mathcal{G}$ by theorem 2.3. This proves (i) when $k = 0$. The proof of (ii) when $k = 0$ is completely analogous to that of (i) when $k = 0$. To prove (iii) when $k = 0$, note that if (ω_i) and $(\Psi_i{}^*\omega_i)$ converge in $C^{0,\alpha}\mathcal{A}$ then (Ψ_i) converges to Ψ in $C^1\mathcal{G}$ for some $\Psi \in C^1\mathcal{G}$ (by (ii) with $k = 0$). Thus (Ψ_i) converges to Ψ in $C^{0,\alpha}\mathcal{G}$, and hence $\Psi \in C^{1,\alpha}\mathcal{G}$ and (Ψ_i) converges to Ψ in $C^{1,\alpha}\mathcal{G}$ by theorem 2.3. This proves (iii) when $k = 0$.

We now prove (i) for $k > 0$ using induction on k. Suppose the result is true for $k - 1$. Let $p \in [1, \infty)$ satisfy $p(k + 1) > \dim M$. Then there exists $q \in [1, \infty)$ such that

$$\frac{1}{p} - \frac{1}{\dim M} \leq \frac{1}{q} < \frac{k}{\dim M} .$$

Then there exists a Sobolev embedding $L^p_k\mathcal{A} \hookrightarrow L^q_{k-1}\mathcal{A}$. If the sequences (ω_i) and $(\Psi_i{}^*\omega_i)$ converge in $L^p_k\mathcal{A}$, then they converge in $L^q_{k-1}\mathcal{A}$, hence the sequence (Ψ_i) converges to Ψ in $L^q_k\mathcal{G}$ for some $\Psi \in L^q_k\mathcal{G}$, by induction. Then, by theorem 2.3, $\Psi \in L^p_{k+1}\mathcal{G}$, the sequence (Ψ_i) converges to Ψ in $L^p_{k+1}\mathcal{G}$ and $\Psi^*\omega = \bar{\omega}$. This proves (i). (ii) and (iii) are proved similarly using induction, again by theorem 2.3.

$\square$

Corollary 3.3

Let $\pi : P \longrightarrow M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G. Let $(\omega_i : i \in \mathbb{N})$ be a sequence of connections on $\pi : P \longrightarrow M$ and let $(\Psi_i : i \in \mathbb{N})$ be a sequence of principal bundle automorphisms of $\pi : P \longrightarrow M$. Let k be a non-negative integer. Then

(i) if $p \in [1, \infty)$, if $p(k + 1) > \dim M$, if $\omega_i \in L^p_k\mathcal{A}$, if $\Psi_i \in L^p_{k+1}\mathcal{G}$, and if the sequences (ω_i) and $(\Psi_i{}^*\omega_i)$ converge in $L^p_k\mathcal{A}$ to ω and $\bar{\omega}$ respectively, then a subsequence of $(\Psi_i : i \in \mathbb{N})$ converges in $L^p_{k+1}\mathcal{G}$ to Ψ, for some

$\Psi \in L^p_{k+1}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$,

(ii) if $\omega_i \in C^k\mathcal{A}$, if $\Psi_i \in C^{k+1}\mathcal{G}$, and if the sequences (ω_i) and $(\Psi_i{}^*\omega_i)$ converge in $C^k\mathcal{A}$ to ω and $\bar{\omega}$ respectively, then a subsequence of $(\Psi_i : i \in \mathbb{N})$ converges in $C^{k+1}\mathcal{G}$ to Ψ , for some $\Psi \in C^{k+1}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$,

(iii) if $\alpha \in (0, 1)$, if $\omega_i \in C^{k,\alpha}\mathcal{A}$, if $\Psi_i \in C^{k+1,\alpha}\mathcal{G}$, and if the sequences (ω_i) and $(\Psi_i{}^*\omega_i)$ converge in $C^{k,\alpha}\mathcal{A}$ to ω and $\bar{\omega}$ respectively, then a subsequence of $(\Psi_i : i \in \mathbb{N})$ converges in $C^{k+1,\alpha}\mathcal{G}$ to Ψ , for some $\Psi \in C^{k+1,\alpha}\mathcal{G}$, and $\Psi^*\omega = \bar{\omega}$.

Proof

In all cases, we may suppose that Ψ_i is continuous for all i. Since the structural group of $\pi : P \to M$ is compact, for any given fibre of $\pi : P \to M$ there exists a subsequence of $(\Psi_i : i \in \mathbb{N})$ converging on that fibre. By theorem 3.2, this subsequence has the required properties.

□

Lemma 3.1 has an analogue for sections of vector bundles associated to a given principal bundle.

Lemma 3.4

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $\theta : G \to \mathrm{Aut}(F)$ be a representation of G as a group of isometries of a normal vector space F. Let $\pi_\theta : E \to M$ be the vector bundle associated to $\pi : P \to M$ with total space $E = P \times_\theta F$. For all $m \in M$, let $|\cdot|_m$ be the norm on the fibre $E[m]$ of $\pi_\theta : E \to M$ over m determined by the norm $|\cdot|$ on F. For all compact subsets K of M let the C^0 norms and L^p norms of C^0 sections and L^p sections respectively of $\pi : P \to M$ over K be defined by

$$\|\xi\|_{K,C^0} = \sup_{m \in k} |\xi(m)|_m$$

$$\|\xi\|_{K,L^p} + \left(\int_K |\xi(m)|_m^p \, d(\mathrm{vol}) \right)^{1/p}$$

for all p satisfying $\dim M < p < \infty$.

Then there exists a constant A_p, depending only on p and on the Riemannian geometry of M such that

(i) if $\omega \; C^0\mathcal{A}$, $\xi \in C^1(E)$ and $m \in K$, then

$$\|\xi\|_{K,C^0} \leq |\xi(m)|_m + \|d^\omega \xi\|_{M,C^0} \,(\mathrm{diam}\ K)$$

(ii) if $\omega \in L^p\mathcal{A}$, $\xi \in L_1^p(E)$ and $m \in K$, then

$$\|\xi\|_{K,C^0} \leq |\xi(m)|_m + A_p \|d^\omega \xi\|_{M,L^p} \,(\mathrm{diam}\ K)^\alpha$$

where

$$\alpha = 1 - \frac{\dim M}{p} .$$

<u>Proof</u>

It suffices to verify that the inequalities are satisfied when ω and ξ are smooth. By theorem V.6.6, if $c : [a, b] \to M$ is a piecewise smooth curve parameterized by arclength s, c(a) = m and c(b) = m, then

$$\Big| |\xi(m')|_{m'} - |\xi(m)|_m \Big| \leq \int_c |d^\omega \xi|_{c(s)} \, ds.$$

Let $f : M \to \mathbb{R}$ be defined by

$$f(m) = |d^\omega \xi(m)|_m .$$

Then

$$\|\xi\|_{K,C^0} \leq |\xi(m)|_m + \sup_{m' \in K} \mu_f(m,m')$$

where $\mu_f(m,m')$ is the infimum of the integrals of f with respect to arclength along all piecewise smooth curves from m to m'. But

$$\mathcal{N}_f(m,m') \leq \| d^{\omega}\xi \|_{M,C^0} \ \text{dist}\ (m,m')$$

and

$$\mathcal{N}_f(m,m') \leq A_p \| d^{\omega}\xi \|_{M,L^p} \ (\text{dist}\ (m,m')\ )^{\alpha}$$

by theorem IV, 3.3, and hence

$$\|\xi\|_{K,C^0} \qquad |\xi(m)|_m + \| d^{\omega}\xi \|_{M,C^0} \ (\text{diam}\ K) ,$$

$$\|\xi\|_{K,C^0} \qquad |\xi(m)|_m + A_p \| d^{\omega}\xi \|_{M,L^p} \ (\text{diam}\ K)^{\alpha} .$$

$\square$

§4. Further Properties of the Action of Automorphisms on Connections

In this section we investigate the consequences of corollary 3.3 for the action of the various groups of principal bundle automorphisms on the corresponding spaces of connections on a principal bundle over a compact manifold with compact structural group. It is shown that the quotients of the various spaces of connections by the action of the corresponding groups of principal bundle automorphisms are Hausdorff, and that the stabilizer of any connection in these spaces is a compact subgroup of the appropriate group of automorphisms and it contains a subgroup naturally isomorphic to the centre of the structural group of the bundle. It is shown that the subset of each space of connections consisting of those connections whose stabilizer is the centre of the structural group form an open subset of the space of connections. We shall also consider the action on the spaces of connections of the subgroups of the corresponding groups of principal bundle automorphisms consisting of those automorphisms which fix the fibre of the bundle over some given element of the base space.

Theorem 4.1

Let $\pi : P \rightarrow M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group. Let k be a non-negative integer. Then, for all $p \in [1, \infty)$ satisfying $p(k + 1) > \dim M$ and for all $\alpha \in (0, 1)$, the quotients $L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}$, $C^k\mathcal{A}/C^{k+1}\mathcal{G}$ and $C^{k,\alpha}\mathcal{A}/C^{k+1,\alpha}\mathcal{G}$ of the spaces $L^p_k\mathcal{A}$, $C\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ of connections on $\pi : P \rightarrow M$ by the corresponding groups of principal bundle automorphisms are Hausdorff.

Proof

If $\sim$ is an equivalence relation on a topological space X, then $X/\sim$ is Hausdorff if and only if R is closed in X x X, where

$$R = \{ (x_1, x_2) \in X \times X : x_1 \sim x_2 \} .$$

Thus $L^p_k \mathcal{A}/L^p_{k+1}\mathcal{G}$ is Hausdorff if and only if R is closed in $L^p_k \mathcal{A} \times L^p_k \mathcal{A}$, where

$$R = \{ (\omega, \bar{\omega}) \quad L^p_k \mathcal{A} \times L^p_k \mathcal{A} : \exists \Psi \in L^p_{k+1}\mathcal{G} \text{ such that } \Psi^* \omega = \omega \} .$$

Let $(\omega, \bar{\omega})$ belong to the closure of R. Then there exists a sequence $(\omega_i : i \in \mathbb{N})$ in $L^p_k \mathcal{A}$ and a sequence $(\Psi_i : i \in \mathbb{N})$ in $L^p_{k+1}\mathcal{G}$ such that (ω_i) converges in $L^p_k \mathcal{A}$ to ω and $(\Psi_i{}^* \omega_i)$ converges in $L^p_k \mathcal{A}$ to $\bar{\omega}$. By corollary 3.3, a subsequence of $(\Psi_i : i \in \mathbb{N})$ converges in $L^p_{k+1}\mathcal{G}$ to $\Psi \in L^p_{k+1}\mathcal{G}$, and $\Psi^* \omega = \bar{\omega}$. .lence $(\omega, \bar{\omega}) \in R$. Hence R is closed in $L^p_k \mathcal{A} \times L^p_k \mathcal{A}$, and thus $L^p_k \mathcal{A}/L^p_{k+1}\mathcal{G}$ is Hausdorff. Similarly $C^k \mathcal{A}/C^{k+1}\mathcal{G}$ and $C^{k,\alpha}\mathcal{A}/C^{k+1,\alpha}\mathcal{G}$ are Hausdorff.

□

<u>Theorem 4.2</u>

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k + 1) > \dim M$, and let $\alpha \in (0, 1)$. Let $L^p_{k+1}\mathcal{G}$, $C^{k+1}\mathcal{G}$ and $C^{k+1,\alpha}\mathcal{G}$ be the groups of principal bundle automorphisms acting on the corresponding spaces $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ of connections on $\pi : P \to M$. Then,

(i) if $\omega \in L^p_k \mathcal{A}$, then the stabilizer of ω in $L^p_{k+1}\mathcal{G}$ is compact,

(ii) if $\omega \in C^k \mathcal{A}$, then the stabilizer of ω in $C^{k+1}\mathcal{G}$ is compact,

(iii) if $\omega \in C^{k,\alpha}\mathcal{A}$, then the stabilizer of ω in $C^{k+1,\alpha}\mathcal{G}$ is compact.

<u>Proof</u>

Let $\omega \in L^p_k \mathcal{A}$ and let $(\Psi_i : i \in \mathbb{N})$ be a sequence of principal bundle automorphisms in $L^p_{k+1}\mathcal{G}$ such that $\Psi_i{}^* \omega = \omega$. By corollary 3.3, there exists a subsequence of $(\Psi_i : i \in \mathbb{N})$ converging in $L^p_{k+1}\mathcal{G}$ to $\Psi \in L^p_{k+1}\mathcal{G}$, and $\Psi^*\omega = \omega$. Thus Ψ belongs to the stabilizer of ω. Thus the stabilizer of ω is a compact subgroup of $L^p_{k+1}\mathcal{G}$. This proves (i). (ii) and (iii) are proved similarly.

□

Let G be the structural group of $\pi : P \to M$ and let Z(G) be the centre of G. If $\gamma \in Z(G)$, then γ defines a smooth principal bundle automorphism of $\pi : P \to M$ mapping p to $p.\gamma$. Thus we have natural smooth embeddings $Z(G) \hookrightarrow L^p_{k+1}\mathcal{G}$ (where $p(k + 1) > \dim M$), $Z(G) \hookrightarrow C^k\mathcal{G}$ and $Z(G) \in C^{k,\alpha}\mathcal{G}$ (where $\alpha \in (0, 1)$) for all non-negative integers k. Moreover if $\gamma \in Z(G)$ and if $\Psi : P \to P$ is the principal bundle automorphism sending $p \in P$ to $p.\gamma$, then $\Psi^*\omega = \omega$ for all $\omega \in C^\infty\mathcal{A}$, and hence for all $\omega \in L^p_k\mathcal{A}$, $\omega \in C^k\mathcal{A}$ and $\omega \in C^{k,\alpha}\mathcal{A}$. Define

$$L^p_{k+1}\mathcal{G}_0 = L^p_{k+1}\mathcal{G}/Z(G),$$

$$C^k\mathcal{G}_0 = C^k\mathcal{G}/Z(G),$$

$$C^{k,\alpha}\mathcal{G}_0 = C^{k,\alpha}\mathcal{G}/Z(G).$$

$L^p_{k+1}\mathcal{G}_0$, $C^{k+1}\mathcal{G}_0$ and $C^{k+1,\alpha}\mathcal{G}_0$ are smooth Banach Lie groups acting smoothly on the spaces $L^p_k\mathcal{A}$, $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ respectively, by corollary II.3.3. Define $L^p_k\mathcal{A}_0$, $C^k\mathcal{A}_0$ and $C^{k,\alpha}\mathcal{A}_0$ to be the subsets of $L^p_k\mathcal{A}$, $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ respectively consisting of connections on $\pi : P \to M$ whose stabilizers in $L^p_{k+1}\mathcal{G}$, $C^{k+1}\mathcal{G}$ and $C^{k+1,\alpha}\mathcal{G}$ respectively are the subgroups of these groups corresponding to the centre Z(G) of G. Thus $L^p_{k+1}\mathcal{G}_0$, $C^{k+1}\mathcal{G}_0$ and $C^{k+1,\alpha}\mathcal{G}_0$ act freely on $L^p_k\mathcal{A}_0$, $C^k\mathcal{A}_0$ and $C^{k,\alpha}\mathcal{A}_0$ respectively.

Lemma 4.3

Let G be a compact Lie group and let N be a closed normal subgroup of G. Then there exists a neighbourhood U of N such that if H is a subgroup of G and $H \subset U$ then $H \subset N$.

Proof

Without loss of generality, we may assume that N is the trivial group consisting of the identity element of G, for otherwise we may apply the theorem to the subgroup HN/N of G/N. Choose a biinvariant Riemannian metric on G and let U be the ball of radius $\frac{1}{3}i(G)$ about the identity element e, where i(G) is the injectivity radius of G. If $\gamma \in U$ and $\gamma \neq e$, then there exist $a \in T_e G$ and $t \in \mathbb{R}$, where $|a| = 1$ and $0 < t < \frac{1}{3}i(G)$, such that $\gamma = \exp(ta)$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{i(G)}{n+1} \leqslant t < \frac{i(G)}{n} .$$

It follows that $n \geqslant 3$ and

$$\tfrac{3}{4}i(G) \leqslant \frac{n}{n+1}\, i(G) \leqslant nt < i(G),$$

and thus γ^n does not belong to U. It follows that if H is a subgroup of G satisfying $H \subset U$ then $H = \{e\}$, as required.

□

Theorem 4.4

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > \dim M$, and let $\alpha \in (0, 1)$. Let $L^p_k \mathcal{A}_0$, $C^k \mathcal{A}_0$ and $C^{k,\alpha} \mathcal{A}_0$ be the subsets of $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ and $C^{k,\alpha} \mathcal{A}$ respectively consisting of all connections whose stabilizer in $L^p_{k+1} \mathcal{G}$, $C^{k+1} \mathcal{G}$ and $C^{k+1,\alpha} \mathcal{G}$ respectively is the subgroup corresponding to the centre Z(G) of G. Then $L^p_k \mathcal{A}_0$, $C^k \mathcal{A}_0$ and $C^{k,\alpha} \mathcal{A}_0$ are open sets in $L^p_k \mathcal{A}$, $C^k \mathcal{A}$ and $C^{k,\alpha} \mathcal{A}$ respectively containing all smooth irreducible connections on $\pi : P \to M$.

Proof

Let $Z(G)_M$ denote the trivial fibre bundle $M \times Z(G) \to M$. The inclusion $Z(G) \hookrightarrow G$ induces an inclusion $Z(G)_M \hookrightarrow G_P$ of fibre bundles over M, where $G_P = P \times_{ad} G$. Suppose that $\omega \in L^p_k \mathcal{A}$, $C^k \mathcal{A}$ or $C^{k,\alpha} \mathcal{A}$ and that $\Psi : P \to P$ is a continuous principal bundle automorphism stabilizing ω, identified with a section $\Psi \in C^0(G_P)$ of $G_P \to M$. Suppose that $\Psi(m) \in Z(G)_M$ for some $m \in M$. Let Ψ_0 be the principal bundle automorphism corresponding to the element of $Z(G)$ defined by $\Psi(m)$. Then Ψ_0 also stabilizes ω and $\Psi_0(m) = \Psi(m)$. But then $\Psi_0 = \Psi$, by lemma 3.1, and thus Ψ belongs to the subgroup of the group of principal bundle automorphisms corresponding to $Z(G)$. We deduce that if $\Psi : M \to G_P$ defines a principal bundle automorphism of $\pi : P \to M$ stabilizing some connection on $\pi : P \to M$, and if Ψ is not a member of the subgroup of $C^0(G_P)$ corresponding to $Z(G)$, then $\Psi(M)$ and $Z(G)_M$ are disjoint subsets of G_P.

Let U be an open neighbourhood of $Z(G)$ in G with the property that if H is a subgroup of G satisfying $H \subset U$ then $H \subset Z(G)$ (such a neighbourhood U exists by the previous lemma). We may choose U such that U is invariant under all inner automorphisms of G. Then U determines an open neighbourhood V of $Z(G)_M$ in G_P such that if H is a subgroup of $C^0(G_P)$ consisting of sections of $V \to M$ then $H \in C^0(Z(G)_M)$.

Let $\omega \in L^p_k \mathcal{A}$. If the stabilizer of ω in $L^p_{k+1} \mathcal{G}$ corresponded to a subgroup of $L^p_{k+1}(G_P)$ consisting of sections of $V \to M$, then it would correspond to a subgroup of $L^p_{k+1}(Z(G)_M)$ and hence the stabilizer of ω would be the subgroup of $L^p_{k+1}(G_P)$ corresponding to $Z(G)$. Thus $\omega \in L^p_k \mathcal{A} \setminus L^p_k \mathcal{A}_0$ if and only if there exists $\Psi \in L^p_{k+1} \mathcal{G}$ and $m \in M$ such that $\Psi(m) \in G_P \setminus V$.

Let $\omega \in L^p_k \mathcal{A}$ belong to the closure of $L^p_k \mathcal{A} \setminus L^p_k \mathcal{A}_0$. Then there exists a sequence $(\omega_i : i \in \mathbb{N})$ of elements of $L^p_k \mathcal{A} \setminus L^p_k \mathcal{A}_0$ converging to ω. Then there exist a sequence $(\Psi_i : i \in \mathbb{N})$ of elements of $L^p_{k+1} \mathcal{G}$ and a sequence $(m_i : i \in \mathbb{N})$ of elements of M such that

$\Psi_i^*\omega_i = \omega_i$ and $\Psi_i(m_i) \in Gp \setminus V$. By corollary 3.3, a subsequence of $(\Psi_i : i \in \mathbb{N})$ converges to $\Psi \in L^p_{k+1}\mathcal{G}$ and $\Psi^*\omega = \omega$, and this subsequence may be chosen such that $(m_i : i \in \mathbb{N})$ converges to $m \in M$, since M is compact. But then $\Psi(m) \in Gp \setminus V$, since the chosen subsequence converges uniformly to Ψ. Thus $\omega \in L^p_k\mathcal{A} \setminus L^p_k\mathcal{A}_o$. Thus $L^p_k\mathcal{A} \setminus L^p_k\mathcal{A}_o$ is closed, and hence $L^p_k\mathcal{A}_o$ is open in $L^p_k\mathcal{A}$. Similarly $C^k\mathcal{A}_o$ and $C^{k,\alpha}\mathcal{A}$ are open sets in $C^k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ respectively.

By theorem V.4.2, the stabilizer of a smooth connection is isomorphic to the centralizer of the holonomy group of the connection. It follows that the stabilizer of a smooth irreducible connection is isomorphic to Z(G). Thus $L^p_k\mathcal{A}_o$, $C^k\mathcal{A}_o$ and $C^{k,\alpha}\mathcal{A}_o$ contain all smooth irreducible connections on $\pi : P \to M$.

□

Theorem 4.5

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold with compact structural group. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > \dim M$ and let $\alpha \in (0, 1)$. Let $L^p_k\mathcal{A}_o$, $C^k\mathcal{A}_o$, $C^{k,\alpha}\mathcal{A}_o$, $L^p_{k+1}\mathcal{G}_o$, $C^{k+1}\mathcal{G}_o$ and $C^{k+1,\alpha}\mathcal{G}_o$ be defined as above. Then

(i) $L^p_{k+1}\mathcal{G}_o$ acts smoothly and freely on $L^p_k\mathcal{A}_o$ on the right, $L^p_k\mathcal{A}_o / L^p_{k+1}\mathcal{G}_o$ is Hausdorff, and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $L^p_k\mathcal{A}_o$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $L^p_{k+1}\mathcal{G}_o$, and if the sequences (ω_i) and $(\omega_i.\Psi_i)$ converge in $L^p_k\mathcal{A}_o$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to Ψ, for some $\Psi \in L^p_{k+1}\mathcal{G}_o$, and $\omega.\Psi = \bar{\omega}$,

(ii) $C^{k+1}\mathcal{G}_o$ acts smoothly and freely on $C^k\mathcal{A}_o$ on the right, $C^k\mathcal{A}_o / C^{k+1}\mathcal{G}_o$ is Hausdorff and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $C^k\mathcal{A}_o$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $C^{k+1}\mathcal{G}_o$

and if the sequences (ω_i) and $(\omega_i.\Psi_i)$ converge in $C^k\mathcal{A}_o$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to Ψ, for some $\Psi \in C^{k+1}\mathcal{G}_o$, and $\omega.\Psi = \bar{\omega}$,

(iii) $C^{k+1,\alpha}\mathcal{G}_o$ acts smoothly and freely on $C^{k,\alpha}\mathcal{A}_o$ on the right, $C^{k,\alpha}\mathcal{A}_o / C^{k+1,\alpha}\mathcal{G}_o$ is Hausdorff, and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $C^{k,\alpha}\mathcal{A}_o$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $C^{k+1,\alpha}\mathcal{G}_o$ and if the sequences (ω_i) and $(\omega_i.\Psi_i)$ converge in $C^{k,\alpha}\mathcal{A}_o$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to Ψ for some $\Psi \in C^{k+1,\alpha}\mathcal{G}_o$, and $\omega.\Psi = \bar{\omega}$.

Proof

The action of $L^p_{k+1}\mathcal{G}_o$ on $L^p_k\mathcal{A}_o$ is well-defined and smooth by corollary II.3.3, it is free by the definition of $L^p_k\mathcal{A}_o$. Suppose that (ω_i) and $(\omega_i.\Psi_i)$ converged in $L^p_k\mathcal{A}_o$ to ω and $\bar{\omega}$ but (Ψ_i) did not converge to the unique $\Psi \in L^p_{k+1}\mathcal{G}_o$ with the property that $\omega.\Psi = \bar{\omega}$ (such a Ψ exists by corollary 3.3). Then there would exist a neighbourhood N of Ψ in $L^p_{k+1}\mathcal{G}_o$ and a subsequence of $(\Psi_i : i \in \mathbb{N})$ with the property that $\Psi_i \notin N$. But then by corollary 3.3, some subsequence of this subsequence converges to some $\Psi_o \in L^p_{k+1}\mathcal{G}_o$ and $\omega.\Psi_o = \bar{\omega}$, which would imply that $\Psi_o = \Psi$. But this is a contradiction. Thus if (ω_i) and $(\omega_i.\Psi_i)$ converge, then so does (Ψ_i). It follows immediately that $L^p_k\mathcal{A}_o / L^p_{k+1}\mathcal{G}_o$ is Hausdorff. This proves (i). The proofs of (ii) and (iii) are similar.

□

Choose $m \in M$ and let $L^p_{k+1}\mathcal{G}^m$, $C^{k+1}\mathcal{G}^m$ and $C^{k+1,\alpha}\mathcal{G}^m$ denote the subgroups of $L^p_{k+1}\mathcal{G}$, $C^{k+1}\mathcal{G}$ and $C^{k+1,\alpha}\mathcal{G}$ consisting of those principal bundle automorphisms of $\pi : P \to M$ which restrict to the identity automorphism on the fibre of $\pi : P \to M$ over m.

Theorem 4.6

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold with compact structural group. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > \dim M$ and let $\alpha \in (0, 1)$. Let $m \in M$ and let $L^p_k \mathcal{A}$, $C^k \mathcal{A}$, $C^{k,\alpha} \mathcal{A}$, $L^p_{k+1} \mathcal{G}^m$, $C^{k+1} \mathcal{G}^m$ and $C^{k+1,\alpha} \mathcal{G}^m$ be defined as above. Then

(i) $L^p_{k+1} \mathcal{G}^m$ acts smoothly and freely on $L^p_k \mathcal{A}$ on the right, $L^p_k \mathcal{A} / L^p_{k+1} \mathcal{G}^m$ is Hausdorff, and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $L^p_k \mathcal{A}$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $L^p_{k+1} \mathcal{G}^m$ and if the sequences (ω_i) and $(\omega_i . \Psi_i)$ converge in $L^p_k \mathcal{A}$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to the unique $\Psi \in L^p_{k+1} \mathcal{G}^m$ such that $\omega . \Psi = \bar{\omega}$,

(ii) $C^{k+1} \mathcal{G}^m$ acts smoothly and freely on $C^k \mathcal{A}$ on the right, $C^k \mathcal{A} / C^{k+1} \mathcal{G}^m$ is Hausdorff and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $C^k \mathcal{A}$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $C^{k+1} \mathcal{G}^m$ and if the sequences (ω_i) and $(\omega_i . \Psi_i)$ converge in $C^k \mathcal{A}$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to the unique $\Psi \in C^{k+1} \mathcal{G}^m$ such that $\omega . \Psi = \bar{\omega}$,

(iii) $C^{k+1,\alpha} \mathcal{G}^m$ acts smoothly and freely on $C^{k,\alpha} \mathcal{A}$ on the right $C^{k,\alpha} \mathcal{A} / C^{k+1,\alpha} \mathcal{G}^m$ is Hausdorff, and if $(\omega_i : i \in \mathbb{N})$ is a sequence in $C^{k,\alpha} \mathcal{A}$, $(\Psi_i : i \in \mathbb{N})$ is a sequence in $C^{k+1,\alpha} \mathcal{G}^m$ and if the sequences in (ω_i) and $(\omega_i . \Psi_i)$ converge in $C^{k,\alpha} \mathcal{A}$ to ω and $\bar{\omega}$ respectively, then the sequence (Ψ_i) converges to the unique $\Psi \in C^{k+1,\alpha} \mathcal{G}^m$ such that $\omega . \Psi = \bar{\omega}$.

Proof

$L^p_{k+1} \mathcal{G}^m$, $C^{k+1} \mathcal{G}^m$ and $C^{k+1,\alpha} \mathcal{G}^m$ act freely on the appropriate spaces of connections, by lemma 3.1. The convergence of the sequences (Ψ_i) of principal bundle automorphisms follows immediately from theorem 3.2.

□

§5. Analytical Properties of the Covariant Differential

In this section, we shall study some properties of the covariant differential d^{ω} mapping sections of a vector bundle $E \to M$ to sections of $E \otimes T^*M \to M$, where ω is a smooth connection on a principal bundle $\pi : P \to M$ to which $E \to M$ is associated. We shall prove a priori inequalities for the map d^{ω} and deduce that d^{ω} maps $L^p_{k+1}(E)$, $C^{k+1}(E)$ and $C^{k+1,\alpha}(E)$ onto closed subspaces of $L^p_k(E \otimes T^*M)$, $C^k(E \otimes T^*M)$ and $C^{k,\alpha}(E \otimes T^*M)$ respectively, where $p(k+1) > \dim M$.

We have seen that the group $L^p_{k+1}\mathcal{G}$ of L^p_{k+1} principal bundle automorphisms acts smoothly on the space $L^p_k\mathcal{A}$ of L^p_k connections on $\pi : P \to M$ whenever $p(k+1) > \dim M$. The Lie algebra of $L^p_{k+1}\mathcal{G}$ may be identified with $L^p_{k+1}(\mathfrak{g}_P)$. For any $\omega \in L^p_k\mathcal{A}$, the map from $L^p_{k+1}\mathcal{G}$ to $L^p_k\mathcal{A}$ sending Ψ to $\Psi^*\omega$ is smooth and its derivative at the identity may be identified with the map from $L^p_{k+1}(\mathfrak{g}_P)$ to $L^p_k(\mathfrak{g}_P \otimes T^*M)$ sending $\xi \in L^p_{k+1}(\mathfrak{g}_P)$ to $d^{\omega}\xi$ (see proposition V.7.1(vii)). Similar considerations apply to the actions of $C^{k+1}\mathcal{G}$ on $C^k\mathcal{A}$ and of $C^{k+1,\alpha}\mathcal{G}$ on $C^{k,\alpha}\mathcal{A}$. The theorems proved in this section will thus be applicable to the study of these actions.

Let $\omega_0 : TP \to \mathfrak{g}$ be a smooth Ehresmann connection on a smooth principal bundle $\pi : P \to M$ over a compact smooth manifold M. Then for all vector bundles $E \to M$ associated to $\pi : P \to M$, for all differentiable sections ξ of $E \to M$ and for all connections ω on $\pi : P \to M$

$$d^{\omega}\xi = d^{\omega_0}\xi + \tau \wedge \xi$$

where $\tau = \omega - \omega_0$. The map

$$d^{\omega_0} : L^p_{k+1}(E) \to L^p_k(E \otimes T^*M)$$

is a continuous linear map. Also if $p(k+1) > \dim M$ and $k > 0$

there exists $q \in [1, \infty)$ such that

$$\frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{\dim M}$$

and $qk > \dim M$. Then there is a continuous Sobolev embedding

$$L^p_{k+1}(E) \hookrightarrow L^q_k(E)$$

and a continuous bilinear map

$$L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^q_k(E) \rightarrow L^p_k(E \otimes T^*M)$$

sending (τ, ξ) to $\tau \wedge \xi$, where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. Hence the map

$$d^{\omega} : L^p_{k+1}(E) \rightarrow L^p_k(E \otimes T^*M)$$

is a continuous whenever $p(k + 1) > \dim M$ and $k = 0$. The continuity of this map when $p > \dim M$ and $k = 0$ is proved similarly, as is the continuity of the maps

$$d^{\omega} : C^{k+1}(E) \rightarrow C^k(E \otimes T^*M)$$

$$d^{\omega} : C^{k+1,\alpha}(E) \rightarrow C^{k,\alpha}(E \otimes T^*M).$$

Theorem 5.1

Let $\pi : P \rightarrow M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let $E \rightarrow M$ be a vector bundle associated to $\pi : P \rightarrow M$.

Let $\omega : TP \rightarrow \mathfrak{g}$ be an Ehresmann connection on $\pi : P \rightarrow M$ and let $\xi : M \rightarrow E$ be a continuous section of $E \rightarrow M$ which is differentiable almost everywhere. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k + 1) > \dim M$ and let $\alpha \in (0, 1)$. Then

(i) if $\omega \in L^p_k \mathcal{A}$ and if $d^{\omega}\xi \in L^p_k(E \otimes T^*M)$, then $\xi \in L^p_{k+1}(E)$ and there exists a constant $K_{\omega} > 0$, independent of ξ, such that

$$\| \xi \|_{L^p_{k+1}} \leqslant K_{\omega} \left(\| d^{\omega} \xi \|_{L^p_k} + \| \xi \|_{C^0} \right),$$

(ii) If $\omega \in C^k \mathcal{A}$ and if $d^\omega \xi \in C^k(E \otimes T^*M)$, then $\xi \in C^{k+1}(E)$ and there exists a constant $K_\omega > 0$ independent of ξ, such that

$$\|\xi\|_{C^{k+1}} \leq K_\omega \left(\|d^\omega \xi\|_{C^k} + \|\xi\|_{C^0} \right),$$

(iii) if $\omega \in C^{k,\alpha}\mathcal{A}$ and if $d^\omega \xi \in C^{k,\alpha}(E \otimes T^*M)$ then $\xi \in C^{k+1,\alpha}(E)$ and there exists a constant $K_\omega > 0$, independent of ξ, such that

$$\|\xi\|_{C^{k+1,\alpha}} \leq K_\omega \left(\|d^\omega \xi\|_{C^{k,\alpha}} + \|\xi\|_{C^0} \right).$$

Proof

Let $\omega_0 : TP \to \mathfrak{g}$ be a smooth connection and let

Then

$$d^{\omega_0} \xi = d^\omega \xi - \tau \wedge \xi$$

There are continuous bilinear maps

$$L^p_k(\mathfrak{g}_P \otimes T^*M) \times L^q_k(E) \to L^p_k(E \otimes T^*M)$$

$$L^p_k(\mathfrak{g}_P \otimes T^*M) \times C^k(E) \to L^p_k(E \otimes T^*M)$$

$$C^k(\mathfrak{g}_P \otimes T^*M) \times C^k(E) \to C^k(E \otimes T^*M)$$

$$C^{k,\alpha}(\mathfrak{g}_P \otimes T^*M) \times C^{k,\alpha}(E) \to C^{k,\alpha}(E \otimes T^*M)$$

where $qk > \dim M$, by corollary II.2.7. Thus if $qk > \dim M$, if $\xi \in L^q_k(E)$, if $\omega \in L^p_k\mathcal{A}$ and if $d^\omega \xi \in L^p_k(E \otimes T^*M)$, then $d^{\omega_0}\xi \in L^p_k(E \otimes T^*M)$, and hence $\xi \in L^p_{k+1}(E)$. Furthermore there exist $K_1 > 0$ and $K_2 > 0$ such that

$$\|\xi\|_{L^p_{k+1}} \leq K_1 \left(\|d^{\omega_0}\xi\|_{L^p_k} + \|\xi\|_{L^q_k} \right)$$

$$\leq K_1 \left(\|d^\omega \xi\|_{L^p_k} + K_2 \|\omega - \omega_0\|_{L^q_k} + \|\xi\|_{L^q_k} \right).$$

Hence there exists a constant K_3, depending on ω but independent of ξ, such that

$$\|\xi\|_{L^p_{k+1}} \leq K_3\left(\|d^\omega \xi\|_{L^p_k} + \|\xi\|_{L^q_k}\right).$$

Similarly, if $\xi \in C^k(E)$, if $\omega \in L^p_k\mathcal{A}$ and if $d^\omega\xi \in L^p_k(E \otimes T^*M)$, then $\xi \in L^p_{k+1}(E)$ and

$$\|\xi\|_{L^p_{k+1}} \leq K_3\left(\|d^\omega \xi\|_{L^p_k} + \|\xi\|_{C^k}\right)$$

for some constant K_3 depending on ω but independent of ξ ; if $\xi \in C^k(E)$ if $\omega \in C^k\mathcal{A}$ and if $d^\omega\xi \in C^k(E \otimes T^*M)$, then $\xi \in C^{k+1}(E)$ and

$$\|\xi\|_{C^{k+1}} \leq K_3\left(\|d^\omega \xi\|_{C^k} + \|\xi\|_{C^k}\right)$$

for some constant K_3; if $\xi \in C^{k,\alpha}(E)$, if $\omega \in C^{k,\alpha}\mathcal{A}$ and if $d^\omega\xi \in C^{k,\alpha}(E \otimes T^*M)$, then $\xi \in C^{k+1,\alpha}(E)$ and

$$\|\xi\|_{C^{k+1,\alpha}} \leq K_3\left(\|d^\omega \xi\|_{C^{k,\alpha}} + \|\xi\|_{C^{k,\alpha}}\right)$$

for some constant K_3. It follows that (i) and (ii) are satisfied when $k = 0$. To prove (iii) when $k = 0$, we observe that $\xi \in C^1(E)$ and a fortiori $\xi \in C^{0,\alpha}(E)$. It then follows that $\xi \in C^{1,\alpha}(E)$ and the required inequality is satisfied, proving (iii) when $k = 0$. (ii) and (iii) for $k > 0$ follow from the case $k = 0$ by induction on k, using the a priori estimates derived above.

We now proceed to prove (i) by induction on k. If $p(k + 1) > \dim M$ and $k > 0$ then there exists $q \in [1, \infty)$ satisfying

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\dim M}$$

and $qk > \dim M$. If $\omega \in L^p_k\mathcal{A}$ and $d^\omega\xi \in L^p_k(E \otimes T^*M)$, then $\omega \in L^q_{k-1}\mathcal{A}$ and $d^\omega\xi \in L^q_{k-1}(E \otimes T^*M)$, and furthermore there exists $K_4 > 0$ such that

$$\|d^\omega \xi\|_{L^q_{k-1}} \leq K_4 \|d^\omega \xi\|_{L^p_k}$$

By the induction hypothesis, $\xi \in L^q_k(E)$ and there exists $K_5 > 0$, depending on ω but independent of ξ, such that

$$\|\xi\|_{L^q_k} \leq K_5 \left(\|d^\omega \xi\|_{L^q_{k-1}} + \|\xi\|_{C^0} \right).$$

Hence $\xi \in L^p_{k+1}(E)$ and

$$\|\xi\|_{L^p_{k+1}} \leq K_3 \left(\|d^\omega \xi\|_{L^p_k} + \|\xi\|_{L^q_k} \right)$$

$$\leq (K_3 + K_3 K_4 K_5) \|d^\omega \xi\|_{L^p_k} + K_3 K_5 \|\xi\|_{C^0}$$

as required.

□

Combining this theorem with lemma 3.4 we obtain the following analogue of theorem 3.2.

Theorem 5.2

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group. Let $E \to M$ be a smooth vector bundle associated to $\pi : P \to M$.

Let $\omega : TP \to \mathfrak{g}$ be an Ehresmann connection on $\pi : P \to M$ and let $\xi : M \to E$ be a section of $E \to M$. Let $m \in M$. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k + 1) > \dim M$ and let $\alpha \in (0, 1)$. Then

(i) if $\omega \in L^p_k \mathcal{A}$ and $\xi \in L^p_{k+1}(E)$ then there exists a constant $K_\omega > 0$, independent of ξ, such that

$$\|\xi\|_{L^p_{k+1}} \leq K_\omega \left(\|d^\omega \xi\|_{L^p_k} + |\xi(m)|_m \right),$$

(ii) if $\omega \in C^k \mathcal{A}$ and $\xi \in C^{k+1}(E)$ then there exists a constant $K_\omega > 0$ independent of ξ, such that

$$\|\xi\|_{C^{k+1}} \leq K_\omega \left(\|d^\omega \xi\|_{C^k} + |\xi(m)|_m \right),$$

(iii) if $\omega \in C^{k,\alpha} \mathcal{A}$ and $\xi \in C^{k+1,\alpha}(E)$ then there exists a constant $K_\omega > 0$ independent of ξ, such that

$$\|\xi\|_{C^{k+1,\alpha}} \leq K_\omega \left(\|d^\omega \xi\|_{C^{k,\alpha}} + |\xi(m)|_m \right).$$

Proof

Since $p(k+1) > \dim M$, there exists $q > \dim M$ satisfying

$$\frac{1}{q} \geqslant \frac{1}{p} - \frac{k}{\dim M}$$

By the Sobolev embedding theorem, there exists $K_1 > 0$, independent of ξ and ω, such that

$$\|d^\omega \xi\|_{L^q} \leq K_1 \|d^\omega \xi\|_{L^p_k}.$$

By theorem 5.1 and lemma 3.4 there exist constants K_2 and K_3, independent of ξ, such that

$$\|\xi\|_{L^p_{k+1}} \leq K_2 \left(\|d^\omega \xi\|_{L^p_k} + \|\xi\|_{C^0} \right),$$

$$\|\xi\|_{C^0} \leq K_3 \left(\|d^\omega \xi\|_{L^q} + |\xi(m)|_m \right).$$

Combining these inequalities, we see that

$$\|\xi\|_{L^p_{k+1}} \leq (K_1 + K_1 K_2 K_3) \|d^\omega \xi\|_{L^p_k} + K_2 K_3 |\xi(m)|_m$$

thus proving (i). The proofs of (ii) and (iii) are similar.

□

Corollary 5.3

Let $\pi' : P \to M$ be a smooth principal bundle over a compact smooth manifold M with compact structural group. Let $E \to M$ be a vector bundle associated to $\pi : P \to M$. Let $\omega : TP \to \mathfrak{g}$ be an Ehresmann connection on $\pi : P \to M$.

Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > \dim M$ and let $\alpha \in (0, 1)$. Then

(i) if $\omega \in L^p_k \mathcal{A}$ then the continuous linear map

$$d^{\omega} : L^p_{k+1}(E) \rightarrow L^p_k(E \otimes T^*M)$$

has finite dimensional kernel and maps $L^p_{k+1}(E)$ onto a closed subspace of $L^p_k(E \otimes T^*M)$,

(ii) if $\omega \in C^k\mathcal{A}$, then the continuous linear map

$$d^{\omega} : C^{k+1}(E) \rightarrow C^k(E \otimes T^*M)$$

has finite dimensional kernel and maps $C^{k+1}(E)$ onto a closed subspace of $C^k(E \otimes T^*M)$,

(iii) if $\omega \in C^{k,\alpha}\mathcal{A}$, then the continuous linear map

$$d^{\omega} : C^{k+1,\alpha}(E) \rightarrow C^{k,\alpha}(E \otimes T^*M)$$

has finite dimensional kernel and maps $C^{k+1,\alpha}(E)$ onto a closed subspace of $C^{k,\alpha}(E \otimes T^*M)$.

Proof

Let $m \in M$ and let X be the subspace of $L^p_{k+1}(E)$ consisting of all $\xi \in L^p_{k+1}(E)$ satisfying the condition $\xi(m) = 0$. X has finite codimension in $L^p_{k+1}(E)$, thus it suffices to show that $d^{\omega}(X)$ is a closed subspace of $L^p_k(E \otimes T^*M)$ and that $d^{\omega}| X$ is a monomorphism. X is a Banach space, hence in order to show that $d^{\omega}(X)$ is closed it is sufficient to show that the map

$$d^{\omega}| X : X \rightarrow d^{\omega}(X)$$

is an isomorphism of normed vector spaces. Thus it is sufficient to verify that

$$(d^{\omega}| X)^{-1} : d^{\omega}(X) \rightarrow X$$

is bounded. But by the previous theorem

$$\|\xi\|_{L^p_{k+1}} \leq K_{\omega} \|d^{\omega}\xi\|_{L^p_k} ,$$

for all $\xi \in X$ (since $\xi(m) = 0$). Thus $(d^{\omega}| X)^{-1}$ is bounded, and hence $d^{\omega}(X)$ is closed. Thus $d^{\omega}(L^p_{k+1}(E))$ is closed in $L^p_k(E \otimes T^*M)$. This proves (i). The proofs of (ii) and (iii) are similar.

□

References for Chapter VI

Palais, R.S., *Foundations of global non-linear analysis*, Benjamin, New York, 1968.

Uhlenbeck, K.K., "Connections with L^p bounds on curvature", *Comm. Math. Phys.*, 83 (1982), pp.31-42.

Chapter VII

A GENERALIZATION OF HODGE THEORY

§1. Introduction

Given a smooth vector bundle over a compact Riemannian manifold and given a connection on this bundle we obtain results for the covariant Hodge-de Rham Laplacian, acting on differential forms with values in the given vector bundle. These results generalize the results obtained by Hodge in his theory of harmonic differential forms on a compact Riemannian manifold.

We first outline the main results of Hodge theory. Let M be a compact Riemannian manifold of dimension n and let d and δ be the exterior derivative and codifferential respectively, acting on differential forms on M. The Hodge-de Rham Laplacian Δ is defined by

$$\Delta = \delta d + d\delta .$$

The vector bundle $\Lambda^j T^*M \to M$ has a natural inner product structure

$$\langle . , . \rangle : \Lambda^j T^*M \otimes \Lambda^j T^*M \to \mathbb{R},$$

determined by the Riemannian metric on M, for all integers j satisfying $0 \leq j \leq n$. Then Δ is a self-adjoint elliptic differential operator of order 2. Define an inner product $(. , .)$ on $L^2(\Lambda^j T^*M)$ by

$$(\eta , \xi) = \int_M \langle \eta , \xi \rangle \, d(\mathrm{vol})$$

for all j-forms η and ξ on M. Then

$$(\Delta\eta , \xi) = (\eta , \Delta\xi).$$

Also if η is a j-form and ζ is a (j + 1)-form, then

$$(d\eta , \zeta) = (\eta , \delta\zeta).$$

Thus

$$(\Delta\eta , \xi) = (d\eta , d\xi) + (\delta\eta , \delta\xi)$$

for all j-forms η and ξ on M. Thus if $\eta \in L^2_1(\Lambda^j T^*M)$ then $\Delta\eta = 0$ if and only if $d\eta = 0$ and $\delta\eta = 0$.

The Laplacian defines Fredholm operators

$$\Delta : L^2_{k+2}(\Lambda^j T^*M) \rightarrow L^2_k(\Lambda^j T^*M)$$

of index zero, and if $u \in L^2_{-\infty}(\Lambda^j T^*M)$ is a current with the property that $\Delta u \in L^2_k(\Lambda^j T^*M)$ for some $k \in \mathbb{Z}$ then $u \in L^2_{k+2}(\Lambda^j T^*M)$ (see /Warner, F.W.; 1971, chapter 6/ or /Wells, R.O., 1973; chapter 4/). Let $H^j(M)$ be the space of harmonic j-forms, defined by

$$H^j(M) = \{\eta \in C^\infty(\Lambda^j T^*M) : \Delta\eta = 0\}.$$

Using the above results, one may show that

$$L^2(\Lambda^j T^*M) = H^j(M) \oplus \Delta(L^2_{k+2}(\Lambda^j T^*M))$$

$$(L^2_{k+2}(\Lambda^j T^*M)) = d(L^2_{k+1}(\Lambda^{j-1}T^*M)) + \delta(L^2_{k+1}(\Lambda^{j+1}T^*M))$$

for all $k \in \mathbb{Z}$. We deduce that every smooth j-form η on M is uniquely expressible in the form

$$\eta = \zeta + d\alpha + \delta\beta$$

for some harmonic j-form ζ and for some smooth (j - 1)-form α and (j + 1)-form β. Let

$$G : C^\infty(\Lambda^j T^*M) \rightarrow C^\infty(\Lambda^j T^*M)$$

be the unique linear map with the properties that $G\eta = 0$ if $\eta \in H^j(M)$, and if $\eta \in \Delta(C^\infty(\Lambda^j T^*M))$ then $G\eta$ is the unique element of $\Delta(C^\infty(\Lambda^j T^*M))$ satisfying $\Delta(G\eta) = \eta$. Let

$$H : C^\infty(\Lambda^j T^*M) \rightarrow H^j(M)$$

be the orthogonal projection with kernel $\Delta(C^\infty(\Lambda^j T^*M))$ and image $H^j(M)$. Then

$$I - \Delta G = I - G\Delta = H.$$

Using the regularity results described above together with the Banach isomorphism theorem, it follows easily that G and H extend to bounded linear maps

$$G : L^2_k(\Lambda^j T^*M) \longrightarrow L^2_{k+2}(\Lambda^j T^*M),$$

$$H : L^2_k(\Lambda^j T^*M) \longrightarrow H^j(M).$$

The results of Hodge theory may be extended to differential forms on M with values in some smooth vector bundle $E \longrightarrow M$ over M. A smooth connection ω on $E \longrightarrow M$ and an inner product structure on $E \longrightarrow M$ preserved by this connection determine a covariant exterior derivative d^ω, a covariant codifferential δ^ω and a covariant Hodge-de Rham Laplacian Δ^ω, all acting on E-valued differential forms on M. All the results described above have obvious analogues with two exceptions. While it is true that

$$\Delta^\omega (L^2_{k+2}(E \otimes \Lambda^j T^*M)) = d^\omega (L^2_{k+1}(E \otimes \Lambda^{j-1} T^*M))$$
$$+ \quad \delta^\omega (L^2_{k+1}(E \otimes \Lambda^{j+1} T^*M))$$

it is in general no longer true that this sum is direct. This is a consequence of the fact that $(d^\omega)^2 \neq 0$ in general. Thus though every smooth E-valued j-form η on M is expressible in the form

$$\eta = \zeta + d^\omega \alpha + \delta^\omega \beta$$

for some E-valued j-form ζ satisfying $\Delta^\omega \zeta = 0$ and for some smooth E-valued $(j - 1)$-form α and $(j + 1)$-form β, it is no longer true that this decomposition of η is unique.

One may extend these results to Sobolev spaces and Hölder spaces using the regularity results of chapter III. Let $k \in \mathbb{Z}$ and let

$$1 < p < \infty .$$

Then Δ^ω defines a Fredholm operator

$$\Delta^{\omega} : L^p_{k+2}(E \otimes \Lambda^j T^*M) \rightarrow L^p_k(E \otimes \Lambda^j T^*M)$$

of index zero. Moreover if u is an E-valued current with the property that $\Delta^{\omega} u \in L^p_k(E \otimes \Lambda^j T^*M)$ then $u \in L^p_{k+2}(E \otimes \Lambda^j T^*M)$, where $k \in \mathbb{Z}$ and $p \in (1, \infty)$. Similarly if $k \in \mathbb{Z}$ satisfies $k \geq 0$ and if $\alpha \in (0, 1)$, then Δ^{ω} defines a Fredholm operator

$$\Delta^{\omega} : C^{k+2,\alpha}(E \otimes \Lambda^j T^*M) \rightarrow C^{k,\alpha}(E \otimes \Lambda^j T^*M)$$

of index zero. Moreover if u is an E-valued current with the property that $\Delta^{\omega} u \in C^{k,\alpha}(E \otimes \Lambda^j T^*M)$ then $u \in C^{k+2,\alpha}(E \otimes \Lambda^j T^*M)$.

In this chapter we shall relax the condition that ω be smooth. Instead we shall demand that ω be an L^p_k connection on $E \rightarrow M$ where $k \in \mathbb{Z}$ and $p \in (1, \infty)$ satisfy the condition $p(k + 1) > n$, where n is the dimension of M, and where also $p \geq 2$ in the case where $k = 0$ (note that this last condition follows immediately from the condition $p(k + 1) > n$ when $n \geq 2$). Let $p' \in (1, \infty)$ be the exponent conjugate to p, defined by the condition that

$$\frac{1}{p} + \frac{1}{p'} = 1 .$$

Then we shall show that Δ^{ω} defines Fredholm operators

$$\Delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \rightarrow L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

for all $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfying the conditions

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

(theorem 3.4). Moreover if $u \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^{\omega} u \in L^q_{l-1}(E \otimes \Lambda^j T^*M)$ then $u \in L^q_{l+1}(E \otimes \Lambda^j T^*M)$ (theorem 3.5). From these results we shall deduce results corresponding to results in the theory of harmonic forms on a Riemannian manifold described above (theorem 4.1). We shall also prove analogous results when ω is a $C^{k,\alpha}$ connection for some integer $k \geq 1$ and for some $\alpha \in (0, 1)$ (theorem 4.2).

§2. Lemmas concerning Maps between Sobolev Spaces

We study the linear maps between Sobolev spaces of sections of vector bundles $E_1 \to M$ and $E_2 \to M$ over a compact manifold M induced by a vector bundle morphism $\theta \in L^p_k(\mathrm{Hom}(E_1,E_2))$, where $k \geq 0$ and $p(k+1) > \dim M$.

Lemma 2.1

Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be smooth vector bundles over a compact smooth manifold M of dimension n. Let the non-negative integer k and the real numbers p and ε satisfy

$$1 \leq p < \infty ,$$

$$0 \leq \varepsilon < \frac{1}{n} ,$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon ,$$

and let $\theta \in L^p_k(\mathrm{Hom}(E_1, E_2))$. Let $l \in \mathbb{Z}$ and $q,r \in (1,\infty)$ satisfy

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{n} - \varepsilon .$$

Then θ defines a bounded linear map from $L^q_l(E_1)$ to $L^r_l(E_2)$ sending $f \in L^q_l(E_1)$ to $\theta \circ f$ provided that

$$-k \leq l \leq k,$$

$$\frac{1}{r} - \frac{l}{n} \geq \frac{1}{p} - \frac{k}{n} ,$$

$$\frac{1}{q} - \frac{l}{n} \leq 1 - \left(\frac{1}{p} - \frac{k}{n} \right) .$$

Proof

First note that

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{n} + \varepsilon$$

$$< 1 - \frac{1}{n} + \varepsilon$$

$$< 1 - \left(\frac{1}{p} - \frac{k}{n} \right)$$

and thus the condition

$$\frac{1}{q} - \frac{\ell}{n} < 1 - \left(\frac{1}{p} - \frac{k}{n} \right)$$

is automatically satisfied when $\ell \geq 0$. Similarly

$$\frac{1}{r} > \frac{1}{q} + \frac{1}{p} - \frac{k}{n}$$

$$> \frac{1}{p} - \frac{k}{n}$$

and thus the condition

$$\frac{1}{r} - \frac{\ell}{n} > \frac{1}{p} - \frac{k}{n}$$

is automatically satisfied when $\ell \leq 0$. Note that if

$$\frac{1}{r} - \frac{\ell}{n} = \frac{1}{p} - \frac{k}{n}$$

then $\ell > 0$.

First consider the case when $\ell \geq 0$ and

$$\frac{1}{r} - \frac{\ell}{n} > \frac{1}{p} - \frac{k}{n} .$$

Choose $s \in (1, \infty)$ such that

$$\frac{1}{p} - \frac{k - \ell}{n} < \frac{1}{s} < \frac{\ell + 1}{n} - \varepsilon$$

and $s > r$. This is possible since

$$\frac{1}{p} - \frac{k - \ell}{n} < \frac{\ell + 1}{n} - \varepsilon ,$$

$$\frac{\ell + 1}{n} - \varepsilon > 0 ,$$

$$\frac{1}{p} - \frac{k - \ell}{n} < 1 ,$$

$$\frac{1}{p} - \frac{k - \ell}{n} < \frac{1}{r} .$$

Then $\theta \in L^s_\ell(\mathrm{Hom}(E_1, E_2))$ by the Sobolev embedding theorem. Moreover

$r < q$, $r < s$ and

$$\frac{1}{q} + \frac{1}{s} - \frac{l}{n} < \left(\frac{1}{r} - \frac{1}{n} + \varepsilon\right) + \left(\frac{l+1}{n} - \varepsilon\right) - \frac{l}{n} < \frac{1}{r}$$

and thus the evaluation map defines a continuous bilinear map

$$L^q_l(E_1) \times L^s_l(\mathrm{Hom}(E_1, E_2)) \to L^r_l(E_2)$$

by theorem II.2.4, part (i). This proves the theorem when $l \geq 0$ and

$$\frac{1}{r} - \frac{l}{n} > \frac{1}{p} - \frac{k}{n}.$$

Next we prove the theorem when $l > 0$ and

$$\frac{1}{r} - \frac{l}{n} = \frac{1}{p} - \frac{k}{n}$$

(we have already seen that this equality implies that $l > 0$ given that the hypotheses of the lemma are satisfied). Then

$$\frac{1}{q} - \frac{l}{n} = \frac{1}{p} - \frac{k+1}{n} + \varepsilon < 0$$

and hence $ql > n$. Now $\theta \in L^r_l(\mathrm{Hom}(E_1, E_2))$ by the Sobolev embedding theorem, and the evaluation map defines a continuous biliear map

$$L^q_l(E_1) \times L^r_l(\mathrm{Hom}(E_1, E_2)) \to L^r_l(E_2)$$

by theorem II.2.4 part (ii). This completes the proof of the lemma when $l \geq 0$.

We prove the lemma when $l < 0$ by duality. Let q' and r' be the exponents conjugate to q and r respectively, defined by

$$\frac{1}{q} + \frac{1}{q'} = 1,$$

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Then

$$\frac{1}{q'} = \frac{1}{r'} + \frac{1}{n} - \varepsilon,$$

$$\frac{1}{q'} - \frac{(-l)}{n} \geq \frac{1}{p} - \frac{k}{n}.$$

Let $\theta' \in L^p_k(\mathrm{Hom}(E_2{}^*, E_1{}^*))$ be the section of $\mathrm{Hom}(E_2{}^*, E_1{}^*) \to M$ which is dual to the section θ of $\mathrm{Hom}(E_1, E_2) \to M$ on each fibre of these vector bundles. From what we have already proved we see that θ' defines a bounded linear map from $L^{r'}_{-l}(E_2{}^*)$ to $L^{q'}_{-l}(E_1{}^*)$. The Banach space dual θ'^* of θ' thus defines a bounded linear map from $L^q_l(E_1)$ to $L^r_l(E_2)$, by duality. But θ'^* and θ coincide on $C^\infty(E_1)$. Thus $\theta = \theta'^*$ by definition of θ. Thus proves the lemma when $l < 0$.

□

Lemma 2.2

Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be smooth vector bundles over a compact smooth manifold M of dimension n. Let the non-negative integer k and the real numbers p and ε satisfy

$$1 \leq p < \infty$$

$$0 \leq \varepsilon < \frac{1}{n}$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon$$

and let $\theta \in L^p_k(\mathrm{Hom}(E_1, E_2))$. Let $l \in \mathbb{Z}$ and $q, r \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{q} - \varepsilon.$$

Then θ defines a compact linear map from $L^q_{l+1}(E_1)$ to $L^r_l(E_2)$ sending $f \in L^q_{l+1}(E_1)$ to $\theta \circ f$ provided that

$$-k-1 \leq l \leq k,$$

$$\frac{1}{r} - \frac{l}{n} \geq \frac{1}{p} - \frac{k}{n}$$

$$\frac{1}{q} - \frac{l+1}{n} \leq 1 - \left(\frac{1}{p} - \frac{k}{n}\right).$$

Proof

First we prove the result when $l \geq 0$ and

$$\frac{1}{r} - \frac{l}{n} > \frac{1}{p} - \frac{k}{n}.$$

As in the proof of the previous lemma we may choose $s \in (1, \infty)$ such that

$$\frac{1}{p} - \frac{k - \ell}{n} < \frac{1}{s} < \frac{\ell + 1}{n} - \varepsilon$$

and $s > r$. Then $\theta \in L^s_\ell(\mathrm{Hom}(E_1, E_2))$ by the Sobolev embedding theorem. Now

$$\frac{\ell + 1}{n} - \frac{1}{s} - \varepsilon > 0$$

hence

$$\frac{1}{q} - \frac{1}{n} < \frac{1}{q} + \frac{\ell}{n} - \frac{1}{s} - \varepsilon .$$

Also

$$\frac{1}{q} + \frac{\ell}{n} - \frac{1}{s} - \varepsilon = \frac{1}{r} - \frac{1}{s} + \frac{\ell}{n}$$

$$> 0$$

since $s > r$ and $\ell \geq 0$. Clearly

$$\frac{1}{q} - \frac{1}{n} < \frac{1}{r} < 1$$

hence there exists $t \in (1, \infty)$ such that $t > r$ and

$$\frac{1}{q} - \frac{1}{n} < \frac{1}{t} < \frac{1}{q} + \frac{\ell}{n} - \frac{1}{s} - \varepsilon .$$

Since

$$\frac{1}{t} > \frac{1}{q} - \frac{1}{n}$$

we have a compact embedding

$$L^q_{\ell+1}(E_1) \hookrightarrow L^t_\ell(E_1)$$

by the Rellich-Kondrakov theorem. Also the evaluation map defines a continuous bilinear map

$$L^t_\ell(E_1) \times L^s_\ell(\mathrm{Hom}(E_1, E_2)) \to L^r_\ell(E_2)$$

by theorem II.2.4 since $s > r$, $t > r$ and

$$\frac{1}{s} + \frac{1}{t} - \frac{\ell}{n} < \frac{1}{r} .$$

It follows that θ defines a compact linear map from $L^q_{\ell+1}(E_1)$ to $L^r_\ell(E_2)$.

Next we prove the theorem when $\ell = 0$ and

$$\frac{1}{r} - \frac{\ell}{n} = \frac{1}{p} - \frac{k}{n} .$$

Then $\theta \in L^r(\text{Hom}(E_1, E_2))$ by the Sobolev embedding theorem. Now

$$\frac{1}{q} - \frac{1}{n} = \frac{1}{p} - \frac{k+1}{n} + \varepsilon$$

$$< 0$$

and hence we have a compact embedding

$$L^q(E_1) \hookrightarrow C^0(E_1)$$

by the Rellich-Kondrakov theorem. Also the evaluation map defines a continuous bilinear map

$$C^0(E_1) \times L^r(\text{Hom}(E_1, E_2)) \to L^r(E_2).$$

Thus θ defines a compact linear map from $L^q_1(E_1)$ to $L^r(E_2)$.

Next we prove the theorem when $\ell > 0$ and

$$\frac{1}{r} - \frac{\ell}{n} = \frac{1}{p} - \frac{k}{n} .$$

$\theta \in L^r_\ell(\text{Hom}(E_1, E_2))$ by the Sobolev embedding theorem. Now

$$\frac{1}{q} - \frac{\ell+1}{n} = \frac{1}{r} - \frac{\ell}{n} - \frac{1}{n} + \varepsilon$$

$$= \frac{1}{p} - \frac{k+1}{n} + \varepsilon$$

$$< 0$$

and thus there exists $t \in (1, \infty)$ such that

$$\frac{1}{q} - \frac{1}{n} < \frac{1}{t} < \frac{\ell}{n}$$

and $t > r$. Then we have a compact embedding

$$L^q_{\ell+1}(E_1) \hookrightarrow L^t_\ell(E_1)$$

by the Rellich-Kondrakov theorem. Also the evaluation map defines a continuous bilinear map

$$L^t_\ell(E_1) \times L^r_\ell(\mathrm{Hom}(E_1, E_2)) \longrightarrow L^r_\ell(E_2)$$

by theorem II.2.4, since $t > r$ and $t\ell > n$. Hence θ defines a compact linear map from $L^q_{\ell+1}(E_1)$ to $L^r_\ell(E_2)$. This completes the proof when $\ell \geq 0$.

We prove the lemma when $\ell < 0$ by duality. Let q' and r' be the exponents conjugate to q and r respectively, defined by

$$\frac{1}{q} + \frac{1}{q'} = 1 ,$$

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Then

$$\frac{1}{q'} = \frac{1}{r'} - \varepsilon ,$$

$$\frac{1}{q'} + \frac{\ell + 1}{n} \geq \frac{1}{p} - \frac{k}{n}.$$

Let $\theta' \in L^p_k(\mathrm{Hom}(E_2{}^*, E_1{}^*))$ be the section of $\mathrm{Hom}(E_2{}^*, E_1{}^*) \longrightarrow M$ which is dual to the section θ of $\mathrm{Hom}(E_1, E_2) \longrightarrow M$ on each fibre of these vector bundles. θ' defines a compact linear map from $L^{r'}_{-\ell}(E_2{}^*)$ to $L^{q'}_{-\ell-1}(E_1{}^*)$. Thus θ defines a compact linear map from $L^q_{\ell+1}(E_1)$ to $L^r_\ell(E_2)$, by duality.

□

<u>Corollary 2.3</u>

Let $\pi_1 : E_1 \longrightarrow M$ and $\pi_2 : E_2 \longrightarrow M$ be smooth vector bundles over a compact smooth manifold M of dimension n. Let k be a non-negative integer and let $p \in [1, \infty)$ satisfy $pk > n$. Let $\theta \in L^p_k(\mathrm{Hom}(E_1,E_2))$. Then θ defines a compact linear map from $L^q_{\ell+1}(E_1)$ to $L^q_\ell(E_2)$ sending $f \in L^q_\ell(E_1)$ to $\theta \circ f$ provided that $\ell \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy

$$- k - 1 \leq \ell \leq k,$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k+1}{n},$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

<u>Proof</u>

This follows immediately from the previous theorem on taking $\varepsilon = 0$.

□

§3. Continuity of some Differential Operators between Sobolev Spaces

In this section we study the covariant exterior derivative, codifferential and Hodge-de Rham Laplacian, with respect to a not necessarily smooth connection, of differential forms with values in some vector bundle.

Let $\tilde{\pi} : E \to M$ be a smooth vector bundle associated to a smooth principal bundle $\pi : P \to M$ over a compact Riemannian manifold M with structural group G whose Lie algebra is $\mathfrak{g}$. Let $\pi_{ad} : G_P \to M$ and $\pi_{Ad} : \mathfrak{g}_P \to M$ be the adjoint bundles, with total spaces $G_P = P \times_{ad} G$, $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. Let $\tilde{\pi} : E \to M$ be given a smooth inner product structure $\langle . , . \rangle : E \otimes E \to \mathbb{R}$ which is preserved by every connection on $\tilde{\pi} : E \to M$ arising from an Ehresmann connection on $\pi : P \to M$.

Let $\omega_1 : TP \to \mathfrak{g}$ and $\omega_2 : TP \to \mathfrak{g}$ be Ehresmann connections on $\pi : P \to M$. We have seen that the covariant exterior derivatives $d^{\omega_1}\eta$ and $d^{\omega_2}\eta$ and the covariant codifferentials $\delta^{\omega_1}\eta$ and $\delta^{\omega_2}\eta$ of an E-valued differential form η on M satisfy

$$d^{\omega_2}\eta = d^{\omega_1}\eta + \tau \wedge \eta$$
$$\delta^{\omega_2}\eta = \delta^{\omega_1}\eta + (-1)^{n(\deg \eta + 1)} * (\tau \wedge * \eta)$$

where $\tau : M \to \mathfrak{g}_P \otimes T^*M$ is the $\mathfrak{g}_P$-valued 1-form on M corresponding to $\omega_2 - \omega_1$ (see proposition V.7).

Proposition 3.1

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the covariant exterior derivative and covariant codifferential of E-valued differential forms with respect to connections on $\pi : P \to M$ be as above. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k + 1) > n$ and let $L^p_k\mathcal{A}$ be the space of L^p_k connections on $\pi : P \to M$.

Let $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy the conditions

$$-k-1 \leq l \leq k,$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k+1}{n},$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1 .$$

Then the covariant exterior derivative d^{ω} and the covariant codifferential δ^{ω} define bounded linear operators

$$d^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_l(E \otimes \Lambda^{j+1} T^*M)$$

$$\delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^{j+1} T^*M) \to L^q_l(E \otimes \Lambda^j T^*M)$$

for all $\omega \in L^p_k \mathcal{A}$. If $\omega_1, \omega_2 \in L^p_k \mathcal{A}$ then the linear operators

$$d^{\omega_2} - d^{\omega_1} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q(E \otimes \Lambda^{j+1} T^*M)$$

$$\delta^{\omega_2} - \delta^{\omega_1} : L^q_{l+1}(E \otimes \Lambda^{j+1} T^*M) \to L^q(E \otimes \Lambda^j T^*M)$$

are compact.

<u>Proof</u>

The second part of the proposition follows from corollary 2.3 and the fact that

$$d^{\omega_2}\eta - d^{\omega_1}\eta = \tau \wedge \eta$$

$$\delta^{\omega_2}\eta - \delta^{\omega_1}\eta = (-1)^{n(\deg \eta + 1) + 1} * (\tau \wedge * \eta).$$

On applying this result when $\omega_2 = \omega$ and when ω_1 is a smooth connection we obtain the first part of the proposition.

□

Lemma 3.2

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the covariant exterior derivative and covariant co-differential of E-valued differential forms with respect to connections on $\pi : P \to M$ be as above. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > n$ and let $L^p_k \mathcal{A}$ be the space of L^p_k connections on $\pi : P \to M$. Let $\omega_1, \omega_2 \in L^p_k \mathcal{A}$.

Let $\varepsilon \in [0, \infty)$ satisfy the conditions

$$0 \leq \varepsilon < \frac{1}{n},$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon$$

and let $l \in \mathbb{Z}$ and $q, r \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{q} - \varepsilon$$

$$-k-1 \leq l \leq k,$$

$$\frac{1}{r} - \frac{l}{n} \geq \frac{1}{p} - \frac{k}{n},$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k+1}{n},$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then $d^{\omega_2} - d^{\omega_1}$ and $\delta^{\omega_2} - \delta^{\omega_1}$ define bounded linear operators

$$d^{\omega_2} - d^{\omega_1} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^r_l(E \otimes \Lambda^{j+1} T^*M)$$

$$\delta^{\omega_2} - \delta^{\omega_1} : L^q_{l+1}(E \otimes \Lambda^{j+1} T^*M) \to L^r_l(E \otimes \Lambda^j T^*M).$$

Proof

This follows immediately from lemma 2.2.

□

Lemma 3.3

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the covariant exterior derivative and covariant codifferential of E-valued differential forms with respect to connections on $\pi : P \to M$ be as above. Let k be a non-negative integer, let $p \in [1, \infty)$ satisfy $p(k+1) > n$ and let $L^p_k \mathcal{A}$ be the space of L^p_k connections on $\pi : P \to M$. Let $\omega_1, \omega_2 \in L^p_k \mathcal{A}$.

Let $\varepsilon \in [0, \infty)$ satisfy the conditions

$$0 \leq \varepsilon < \frac{1}{n},$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon ,$$

and let $l \in \mathbb{Z}$ and $q, r \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{n} - \varepsilon ,$$

$$-k \leq l \leq k$$

$$\frac{1}{r} - \frac{l}{n} \geq \frac{1}{p} - \frac{k}{n},$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then $d^{\omega_2} - d^{\omega_1}$ and $\delta^{\omega_2} - \delta^{\omega_1}$ define bounded linear operators

$$d^{\omega_2} - d^{\omega_1} : L^q_l(E \otimes \Lambda^j T^*M) \to L^r_l(E \otimes \Lambda^{j+1} T^*M),$$

$$\delta^{\omega_2} - \delta^{\omega_1} : L^q_l(E \otimes \Lambda^{j+1} T^*M) \to L^r_l(E \otimes \Lambda^j T^*M).$$

Proof

This follows immediately from lemma 2.1.

□

We recall that if ω is a connection on $\pi : P \to M$ then the covariant Hodge-de Rham Laplacian Δ^{ω} with respect to ω is the elliptic differential operator acting on E-valued differential forms defined by

$$\Delta^{\omega} = \delta^{\omega} d^{\omega} + d^{\omega} \delta^{\omega} .$$

Theorem 3.4

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the covariant Hodge-de Rham Laplacian of E-valued differential forms with respect to connections on $\pi : P \to M$ be as above. Let k be a non-negative integer, let $p \in (1, \infty)$ satisfy $p(k + 1) > n$ and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let $L^p_k \mathcal{A}$ be the space of L^p_k connections on $\pi : P \to M$ and let $\omega \in L^p_k \mathcal{A}$.

Let $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy the conditions

$$-k \leq l \leq k$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then the covariant Hodge-de Rham Laplacian defines a Fredholm linear operator

$$\Delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

of index 0.

Proof

First suppose that ω is smooth. Then Δ^{ω} is a self-adjoint elliptic differential operator and defines a Fredholm operator

$$\Delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

of index 0, using theorem III.5.3.

Now consider the case when $\omega \in L^p_k \mathcal{A}$ but ω is not necessarily smooth. The operators

$$d^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_l(E \otimes \Lambda^{j+1} T^*M)$$

$$\delta^{\omega} : L^q_l(E \otimes \Lambda^{j+1} T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

$$\delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_l (E \otimes \Lambda^{j-1} T^*M)$$

$$d^{\omega} : L^q_l(E \otimes \Lambda^{j-1} T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

are bounded by proposition 3.1, and moreover, in each of these four cases, the operator $d^{\omega} - d^{\omega_0}$ or $\delta^{\omega} - \delta^{\omega_0}$ is compact, where ω_0 is any smooth connection. Thus

$$\Delta^{\omega} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

is bounded, provided that l and q satisfy the hypotheses of the theorem, and also

$$\Delta^{\omega} - \Delta^{\omega_0} : L^q_{l+1}(E \otimes \Lambda^j T^*M) \to L^q_{l-1}(E \otimes \Lambda^j T^*M)$$

is compact, using the fact that

$$\Delta^{\omega} - \Delta^{\omega_0} = (\delta^{\omega} - \delta^{\omega_0}) d^{\omega} + \delta^{\omega_0}(d^{\omega} - d^{\omega_0}) + (d^{\omega} - d^{\omega_0})\delta^{\omega} + d^{\omega_0}(\delta^{\omega} - \delta^{\omega_0}).$$

It follows that Δ^{ω} is Fredholm, being the sum of a Fredholm operator and a compact operator. Also

$$\begin{aligned} \text{index } \Delta^{\omega} &= \text{index } \Delta^{\omega_0} \\ &= 0. \end{aligned}$$

□

<u>Theorem 3.5</u>

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the covariant Hodge-de Rham Laplacian of E-valued differential forms with respect to connections on $\pi : P \to M$ be as above. Let k be a non-negative integer, let $p \in (1, \infty)$ satisfy $p(k+1) > n$ and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let $L^p_k \mathcal{A}$ be the space of L^p_k connections on $\pi : P \to M$ and let $\omega \in L^p_k \mathcal{A}$.

Let $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy the conditions

$$-k \leq l \leq k,$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n},$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If $\eta \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^\omega \eta \in L^q_{l-1}(E \otimes \Lambda^j T^*M)$ then $\eta \in L^q_{l+1}(E \otimes \Lambda^j T^*M)$.

<u>Proof</u>

Let ω be a smooth connection on $\pi : P \to M$. There exists a real number ε satisfying the conditions

$$0 < \varepsilon < \frac{1}{n},$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon .$$

First we show that if $m \in \mathbb{Z}$ and $r, s \in (1, \infty)$ satisfy the conditions

$$-k \leq m \leq l$$

$$\frac{1}{r} - \varepsilon \leq \frac{1}{s} \leq \frac{1}{r},$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{s} - \frac{m}{n} \leq \frac{1}{r} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k}{m}$$

and if $\eta \in L^r_{m+1}(E \otimes \Lambda^j T^*M)$ then $\eta \in L^s_{m+1}(E \otimes \Lambda^j T^*M)$. By proposition 3.1 and lemma 3.2 it follows that

$$\Delta^\omega \eta - \Delta^{\omega_0} \eta \in L^s_{m-1}(E \otimes \Lambda^j T^*M)$$

using the fact that

$$\Delta^{\omega}-\Delta^{\omega_0} = (\delta^{\omega}-\delta^{\omega_0})d^{\omega}+\delta^{\omega_0}(d^{\omega}-d^{\omega_0})+(d^{\omega}-d^{\omega_0})\delta^{\omega}+d^{\omega_0}(\delta^{\omega}-\delta^{\omega_0})$$

But $\Delta^{\omega}\eta \in L^{s}_{m-1}(E \otimes \Lambda^{j}T^{*}M)$ by the Sobolev embedding theorem, since

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{s} - \frac{m}{n}\,.$$

Hence $\Delta^{\omega_0}\eta \in L^{s}_{m-1}(E \otimes \Lambda^{j}T^{*}M)$. But Δ^{ω_0} is an elliptic differential operator with smooth coefficients, hence $\eta \in L^{s}_{m+1}(E \otimes \Lambda^{j}T^{*}M)$ by the elliptic regularity theorem III.5.2.

By iteration it follows that if $\eta \in L^{r}_{m+1}(E \otimes \Lambda^{j}T^{*}M)$ for some $m \in \mathbb{Z}$ and $r \in (1, \infty)$ satisfying

$$-k \leq m \leq l$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{r} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

then $\eta \in L^{s}_{m+1}(E \otimes \Lambda^{j}T^{*}M)$ for all $s \in (1, \infty)$ satisfying

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{s} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

(note that if $s < r$ we have an embedding

$$L^{r}_{m+1}(E \otimes \Lambda^{j}T^{*}M) \hookrightarrow L^{s}_{m+1}(E \otimes \Lambda^{j}T^{*}M)$$

and thus the result follows trivially in this case). The theorem is the case when $l = -k$ follows directly from this result.

Now let $m \in \mathbb{Z}$ and $r,s \in (1, \infty)$ satisfy the conditions

$$-k + 1 \leq m \leq l$$

$$\frac{1}{r} - \frac{1}{n} \leq \frac{1}{s} \leq \frac{1}{r} - \frac{1}{n} + \varepsilon$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{r} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{s} - \frac{m-1}{n} \leq \frac{1}{p'} + \frac{k}{n}.$$

We shall show that if $\eta \in L^{s}_{m}(E \otimes \Lambda^{j}T^{*}M)$ then $\eta \in L^{r}_{m+1}(E \otimes \Lambda^{j}T^{*}M)$. Using proposition 3.1 and lemma 3.3 it follows that if $\eta \in L^{s}_{m}(E \otimes \Lambda^{j}T^{*}M)$

then

$$\Delta^{\omega}\eta - \Delta^{\omega_0}\eta \in L^r_{m-1}(E \otimes \Lambda^j T^*M).$$

But $\Delta^{\omega}\eta \in L^r_{m-1}(E \otimes \Lambda^j T^*M)$ by the Sobolev embedding theorem. Hence $\Delta^{\omega_0}\eta \in L^r_{m-1}(E \otimes \Lambda^j T^*M)$ and thus $\eta \in L^r_{m+1}(E \otimes \Lambda^j T^*M)$ by the elliptic regularity theorem III.5.2.

Now let us suppose that $n > 1$ and that $l > -k$. Let $m \in \mathbb{Z}$ satisfy

$$-k + 1 \leq m \leq l$$

and suppose that $\eta \in L^t_m(E \otimes \Lambda^j T^*M)$ for some $t \in (1, \infty)$ satisfying the condition

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{t} - \frac{m-1}{n} \leq \frac{1}{p'} + \frac{k}{n}.$$

Then there exists $s \in (1, \infty)$ satisfying the conditions

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{s} - \frac{m-1}{n} \leq \frac{1}{p'} + \frac{k}{n},$$

$$\frac{1}{s} < 1 - \frac{1}{n},$$

since $n > 1$ and

$$\frac{1}{q} - \frac{l-(m-1)}{n} \leq \frac{1}{q} - \frac{1}{n}$$

$$< 1 - \frac{1}{n}.$$

But we have seen that if $\eta \in L^t_m(E \otimes \Lambda^j T^*M)$ then $\eta \in L^s_m(E \otimes \Lambda^j T^*M)$ and hence $\eta \in L^r_{m+1}(E \otimes \Lambda^j T^*M)$, where $r \in (1, \infty)$ is defined by

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{n}.$$

Iterating this procedure, we see that if $\eta \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ then there exists $r \in (1, \infty)$ such that

$$\frac{1}{q} - \frac{l}{n} \leq \frac{1}{r} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n}$$

and $\eta \in L^r_{l+1}(E \otimes \Lambda^j T^*M)$. But we have seen that this implies that

$\eta \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M)$ as required. This completes the proof of the theorem when $n > 1$.

It only remains to prove the theorem when $n = 1$ and $\ell > -k$. But then

$$\frac{1}{q} - \ell < \frac{1}{p'} + k$$

since $p', q \in (1, \infty)$, $\ell, k \in \mathbb{Z}$, $\ell \neq k$ and

$$\frac{1}{q} - \ell \leq \frac{1}{p'} + k .$$

Without loss of generality, ε may be chosen such that ε also satisfies the condition

$$\varepsilon \leq \frac{1}{p'} \quad , \quad \frac{1}{q} - \ell < \frac{1}{p'} + k - \varepsilon$$

as well as the conditions

$$0 < \varepsilon < \frac{1}{n} \quad ,$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon .$$

Suppose that $m \in \mathbb{Z}$ satisfies

$$-k + 1 \leq m \leq \ell$$

and that $\eta \in L^t_m(E \otimes \Lambda^j T^*M)$ for some $t \in (1, \infty)$ satisfying the condition

$$\frac{1}{q} - \ell \leq \frac{1}{t} - (m - 1) \leq \frac{1}{p'} + k.$$

Then there exists $s \in (1, \infty)$ satisfying the conditions

$$\frac{1}{q} - \ell \leq \frac{1}{s} - (m - 1) \leq \frac{1}{p'} + k - \varepsilon$$

$$\frac{1}{s} < \varepsilon .$$

Then $\eta \in L^s_m(E \otimes \Lambda^j T^*M)$ and hence $\eta \in L^r_{m+1}(E \otimes \Lambda^j T^*M)$ where $r \in (1, \infty)$ satisfies

$$\frac{1}{r} > \frac{1}{s} + 1 - \varepsilon$$

$$\frac{1}{q} - l \leq \frac{1}{r} - m \leq \frac{1}{p'} + k.$$

As before, if $\eta \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ then $\eta \in L^{r}_{l+1}(E \otimes \Lambda^j T^*M)$ for some r satisfying the condition

$$\frac{1}{q} - l \leq \frac{1}{r} - l \leq \frac{1}{p'} + k,$$

and hence $\eta \in L^{q}_{l+1}(E \otimes \Lambda^j T^*M)$. This proves the theorem when n = 1.

□

<u>Corollary 3.6</u>

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \rightarrow M$, the principal bundle $\pi : P \rightarrow M$ and the covariant Hodge-de Rham Laplacian of E-valued differential forms with respect to connections on $\pi : P \rightarrow M$ be as above. Let k be a non-negative integer, let $p \in (1, \infty)$ satisfy $p(k + 1) > n$ and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let $L^p_k \mathcal{A}$ be the space of L^p_k connections on $\pi : P \rightarrow M$ and let $\omega \in L^p_k \mathcal{A}$.

Let $p' \in (1, \infty)$ be the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If $\eta \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^{\omega}\eta = 0$ then $\eta \in L^{p}_{k+1}(E \otimes \Lambda^j T^*M)$.

<u>Proof</u>

Take $q = p$ and $l = k$ in the above theorem.

□

§4. Covariant Hodge Theory with respect to Non-Smooth Connections

In this section we derive properties of the covariant Hodge-de Rham Laplacian with respect to a connection that is not necessarily smooth, considered as a mapping between Sobolev spaces of differential forms with values in a given vector bundle over a smooth manifold. These properties generalize properties of the Hodge-de Rham Laplacian acting on differential forms defined on a compact manifold which form the basis of Hodge's theory of harmonic differential forms.

Theorem 4.1

Let M be a compact Riemannian manifold of dimension n and let $\tilde{\pi} : E \to M$ be a smooth vector bundle associated to the smooth principal bundle $\pi : P \to M$. Let $\tilde{\pi} : E \to M$ have a smooth inner product structure which is preserved by every connection on $\tilde{\pi} : E \to M$ arising from an Ehresmann connection on $\pi : P \to M$. Let k be a non-negative integer, let $p \in (1, \infty)$ satisfy $p(k + 1) > n$ and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let ω be an L^p_k connection on $\pi : P \to M$. Let d^{ω}, δ^{ω} and Δ^{ω} denote the covariant exterior derivative operator, the covariant codifferential operator and the covariant Hodge-de Rham Laplacian respectively with respect to the connection ω.

Let $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy the conditions

$$-k \leq l \leq k,$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n},$$

where p' is the exponent conjugate to p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Define

$$H^j(E) = \{ \eta \in L^p_{k+1}(E \otimes \Lambda^j T^*M) : \Delta^\omega \eta = 0 \},$$

$$(\ker \Delta^\omega)_{j,q,\ell+1} = \{ \eta \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M) : \Delta^\omega \eta = 0 \},$$

$$(\ker d^\omega)_{j,q,\ell+1} = \{ \eta \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M) : d^\omega \eta = 0 \},$$

$$(\ker \delta^\omega)_{j,q,\ell+1} = \{ \eta \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M) : \delta^\omega \eta = 0 \},$$

$$(\operatorname{im} \Delta^\omega)_{j,q,\ell-1} = \{ \Delta^\omega \eta : \eta \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M) \},$$

$$(\operatorname{im} d^\omega)_{j,q,\ell-1} = \{ d^\omega \eta : \eta \in L^q_{\ell}(E \otimes \Lambda^{j-1} T^*M) \},$$

$$(\operatorname{im} \delta^\omega)_{j,q,\ell-1} = \{ \delta^\omega \eta : \eta \in L^q_{\ell}(E \otimes \Lambda^{j+1} T^*M) \}.$$

Then

(i) $H^j(E)$ is finite dimensional,

(ii) $(\ker \Delta^\omega)_{j,q,\ell+1} = H^j(E)$,

(iii) $L^q_{\ell-1}(E \otimes \Lambda^j T^*M) = H^j(E) \oplus (\operatorname{im} \Delta^\omega)_{j,q,\ell-1}$,

(iv) $(\ker \Delta^\omega)_{j,q,\ell+1} = (\ker d^\omega)_{j,q,\ell+1} \cap (\ker \delta^\omega)_{j,q,\ell+1}$,

(v) $(\operatorname{im} \Delta^\omega)_{j,q,\ell-1} = (\operatorname{im} d^\omega)_{j,q,\ell-1} + (\operatorname{im} \delta^\omega)_{j,q,\ell-1}$.

<u>Proof</u>

(i) follows immediately from the fact that

$$\Delta^\omega : L^p_{k+1}(E \otimes \Lambda^j T^*M) \to L^p_{k+1}(E \otimes \Lambda^j T^*M)$$

is Fredholm (see theorem 3.4), and (ii) follows immediately from corollary 3.6.

Let q' be the exponent conjugate to q and let

$$(.,.) : L^q_{\ell-1}(E \otimes \Lambda^j T^*M) \times L^{q'}_{-\ell+1}(E \otimes \Lambda^j T^*M) \to \mathbb{R}$$

be the pairing induced by the inner product structure on $\tilde{\pi} : E \to M$ and the Riemannian metric on M. Note that

$H^j(E) \subset C^0(E \otimes \Lambda^j T^*M)$ by the Sobolev embedding theorem, since $p(k + 1) > n$. Thus

$$(\eta, \eta) = \int_M |\eta|^2 \; d(\text{vol})$$

for all $\eta \in H^j(E)$. If $\eta \in H^j(E) \cap (\text{im } \Delta^\omega)_{j,q,\ell-1}$ then there exists $\xi \in L^q_{\ell+1}(E \otimes \Lambda^j T^*M)$ such that $\eta = \Delta^\omega \xi$. Then

$$\begin{aligned}(\eta, \eta) &= (\eta, \Delta^\omega \xi) \\ &= (\Delta^\omega \eta, \xi) \\ &= 0\end{aligned}$$

since Δ^ω is self-adjoint. Thus $\eta = 0$. Hence

$$H^j(E) \cap (\text{im } \Delta^\omega)_{j,q,\ell-1} = \{0\}.$$

Also $(\text{im } \Delta^\omega)_{j,q,\ell-1}$ is closed and has finite codimension, since

$$\Delta^\omega : L^q_{\ell+1}(E \otimes \Lambda^j T^*M) \to L^q_{\ell-1}(E \otimes \Lambda^j T^*M)$$

is Fredholm. Also $(\ker \Delta^\omega)_{j,q',-\ell+1}$ is the annihilator of $(\text{im } \Delta^\omega)_{j,q,\ell-1}$, since Δ^ω is self-adjoint. Thus

$$\begin{aligned}\text{codim}(\text{im } \Delta^\omega)_{j,q,\ell-1} &= \dim(\ker \Delta^\omega)_{j,q',-\ell+1} \\ &= \dim H^j(E)\end{aligned}$$

by (ii). Thus

$$L^q_{\ell-1}(E \otimes \Lambda^j T^*M) = H^j(E) \oplus (\text{im } \Delta^\omega)_{j,q,\ell-1}.$$

This proves (iii).

We observe that

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{2} \leq \frac{1}{p'} + \frac{k}{n}$$

if p and k satisfy the hypotheses of the theorem. If $n \geq 2$ this is a consequence of the condition $p(k + 1) > n$. If $n = 1$ and $k > 0$ then the above inequalities follow immediately. If $n = 1$ and $k = 0$ then the result is true by hypothesis. By the Sobolev embedding theorem there exist embeddings

$$L^p_{k+1}(E \otimes \Lambda^j T^*M) \hookrightarrow L^2_1(E \otimes \Lambda^j T^*M)$$

$$L^2_1(E \otimes \Lambda^j T^*M) \hookrightarrow L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M).$$

If $\eta \in (\ker \Delta^\omega)_{j,q,\ell+1}$ then $\eta \in H^j(E)$ and hence $\eta \in L^2_1(E \otimes \Lambda^j T^*M)$. Then

$$0 = (\Delta^\omega \eta, \eta) = (d^\omega \eta, d^\omega \eta) + (\delta^\omega \eta, \delta^\omega \eta)$$

and hence

$$d^\omega \eta = \delta^\omega \eta = 0.$$

Thus

$$(\ker \Delta^\omega)_{j,q,\ell+1} \subset (\ker d^\omega)_{j,q,\ell+1} \cap (\ker \delta^\omega)_{j,q,\ell+1}.$$

The reverse inclusion is trivial. This proves (iv).

Clearly

$$(\mathrm{im}\, \Delta^\omega)_{j,q,\ell-1} \subset (\mathrm{im}\, d^\omega)_{j,q,\ell-1} + (\mathrm{im}\, \delta^\omega)_{j,q,\ell-1}.$$

Thus in order to prove (v) it suffices to show that

$$(\mathrm{im}\, d^\omega)_{j,q,\ell-1} \subset (\mathrm{im}\, \Delta^\omega)_{j,q,\ell-1},$$

$$(\mathrm{im}\, \delta^\omega)_{j,q,\ell-1} \subset (\mathrm{im}\, \Delta^\omega)_{j,q,\ell-1}.$$

But $(\mathrm{im}\, \Delta^\omega)_{j,q,\ell-1}$ is the annihilator of $(\ker \Delta^\omega)_{j,q',-\ell+1}$ since Δ^ω is self-adjoint. Thus it suffices to show that $(\mathrm{im}\, d^\omega)_{j,q,\ell-1}$ and $(\mathrm{im}\, \delta^\omega)_{j,q,\ell-1}$ annihilate $H^j(E)$. But if $\xi \in L^q_\ell(E \otimes \Lambda^{j-1} T^*M)$ and $\eta \in H^j(E)$, then

$$\begin{aligned}(d^\omega \xi, \eta) &= (\xi, \delta^\omega \eta) \\ &= 0\end{aligned}$$

by (iv). Similarly if $\xi \in L^q_\ell(E \otimes \Lambda^{j+1} T^*M)$ and $\eta \in H^j(E)$ then

$$\begin{aligned}(\delta^\omega \xi, \eta) &= (\xi, d^\omega \eta) \\ &= 0.\end{aligned}$$

This proves (v). $\square$

Let the compact Riemannian manifold M, the vector bundle $\tilde{\pi} : E \to M$ and the principal bundle $\pi : P \to M$ be as in the above theorem. Let k be a non-negative integer let $p \in (1, \infty)$ satisfy $p(k+1) > n$ and let ω be an L^p_k connection on $\pi : P \to M$. Define

$$(H^j(E)^{\perp})_{r,m} = \{ \eta \in L^r_m(E \otimes \Lambda^j T^*M) : (\eta, \zeta) = 0 \text{ for all } \zeta \in H^j(E) \}$$

for all $m \in \mathbb{Z}$ and $r \in (1, \infty)$ satisfying

$$-k-1 \leq m \leq k+1,$$

$$\frac{1}{p} - \frac{k+1}{n} \leq \frac{1}{r} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k+1}{n} .$$

Since

$$H^j(E) \cap (H^j(E)^{\perp})_{r,m} = 0$$

and since

$$\operatorname{codim} (H^j(E)^{\perp})_{r,m} = \dim H^j(E)$$

it follows that

$$L^r_m(E \otimes \Lambda^j T^*M) = H^j(E) \oplus (H^j(E)^{\perp})_{r,m}.$$

Let $l \in \mathbb{Z}$ and $q \in (1, \infty)$ satisfy

$$-k \leq l \leq k$$

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{l}{n} \leq \frac{1}{p'} + \frac{k}{n} .$$

Since

$$(\operatorname{im} \Delta^{\omega})_{j,q,l-1} \subset (H^j(E)^{\perp})_{q,l-1}$$

and

$$\operatorname{codim} (\operatorname{im} \Delta^{\omega})_{j,q,l-1} = \dim H^j(E),$$

it follows that

$$(\operatorname{im} \Delta^{\omega})_{j,q,l-1} = (H^j(E)^{\perp})_{q,l-1}.$$

Since

$$(\ker \Delta^{\omega})_{q,l+1} = H^j(E)$$

if follows that

$$\Delta^{\omega} | (H^j(E)^{\perp})_{q,l+1} : (H^j(E)^{\perp})_{q,l+1} \to (H^j(E)^{\perp})_{q,l-1}$$

is a bijection, and is thus an isomorphism of Banach spaces, by the Banach isomorphism theorem. Define

$$G^{\omega} : L^{p'}_{-k-1}(E \otimes \Lambda^j T^*M) \to L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$$

by the properties

$$G^{\omega} | H^j(E) = 0,$$

$$G^{\omega} | (H^j(E)^{\perp})_{p',-k-1} = (\Delta^{\omega} | (H^j(E)^{\perp})_{p',-k+1})^{-1} .$$

Then

$$G^{\omega} | (H^j(E)^{\perp})_{q,l-1} = (\Delta | (H^j(E)^{\perp})_{q,l+1})^{-1}$$

and hence G^{ω} restricts to a bounded linear map

$$G^{\omega} : L^q_{l-1}(E \otimes \Lambda^j T^*M) \to L^q_{l+1}(E \otimes \Lambda^j T^*M).$$

We refer to G^{ω} as the Green's operator of the covariant Hodge-de Rham Laplacian Δ^{ω} . Define

$$H^{\omega} : H^{p'}_{-k-1}(E \otimes \Lambda^j T^*M) \to L^{p'}_{-k-1}(E \otimes \Lambda^j T^*M)$$

to be the projection mapping with kernel $(H^j(E)^{\perp})_{p',-k-1}$ and image $H^j(E)$. Then H^{ω} restricts to a compact linear operator

$$H^{\omega} : L^r_m(E \otimes \Lambda^j T^*M) \to L^r_m(E \otimes \Lambda^j T^*M)$$

for all $m \in \mathbb{Z}$ and $r \in (1, \infty)$ satisfying the conditions

$$-k-1 \leq m \leq k+1,$$

$$\frac{1}{p} - \frac{k+1}{n} \leq \frac{1}{r} - \frac{m}{n} \leq \frac{1}{p'} + \frac{k+1}{n} .$$

We see that

$$(I - \Delta^{\omega} G^{\omega})\eta = H^{\omega}\eta$$

for all $\eta \in L^{p'}_{-k-1}(E \otimes \Lambda^j T^*M)$ and

$$(I - G^\omega \Delta^\omega)\eta = H^\omega \eta$$

for all $\eta \in L^{p'}_{-k+1}(E \otimes \Lambda^j T^*M)$.

We can obtain results similar to those above when ω is a $C^{k,\alpha}$ connection on $\pi : P \to M$.

Theorem 4.2

Let M be a compact Riemannian manifold of dimension n and let the vector bundle $\tilde{\pi} : E \to M$, the principal bundle $\pi : P \to M$ and the inner product structure on $\tilde{\pi} : E \to M$ be as in the previous theorem. Let k be a strictly positive integer, let $\alpha \in (0, 1)$ and let ω be a $C^{k,\alpha}$ connection on $\pi : P \to M$.

The covariant Hodge-de Rham Laplacian Δ^ω defines Fredholm operators

$$\Delta^\omega : L^p_{l+1}(E \otimes \Lambda^j T^*M) \to L^p_{l-1}(E \otimes \Lambda^j T^*M)$$

$$\Delta^\omega : C^{m+1,\beta}(E \otimes \Lambda^j T^*M) \to C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$$

of index zero for all $p \in (1, \infty)$ and for all $l, m \in \mathbb{Z}$ and $\beta \in (0, 1)$ satisfying

$$-k \leq l \leq k$$

$$1 < m + \beta \leq k + \alpha .$$

If $\eta \in L^p_{-k+1}(E \otimes \Lambda^j T^*M)$ for some $q \in (1, \infty)$ and if $\Delta^\omega \eta \in L^p_{l-1}(E \otimes \Lambda^j T^*M)$ then $\eta \in L^p_{l+1}(E \otimes \Lambda^j T^*M)$, where l satisfies the condition above. If $\eta \in L^q_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^\omega \eta \in C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$ then $\eta \in C^{m+1,\beta}$, where m and β satisfy the conditions given above. The Green's operator

$$G^\omega : L^q_{-k-1}(E \otimes \Lambda^j T^*M) \to L^q_{-k+1}(E \otimes \Lambda^j T^*M)$$

restricts to bounded linear operators

$$G^\omega : L^p_{l-1}(E \otimes \Lambda^j T^*M) \to L^p_{l+1}(E \otimes \Lambda^j T^*M),$$

$$G^\omega : C^{m-1,\beta}(E \otimes \Lambda^j T^*M) \to C^{m+1,\beta}(E \otimes \Lambda^j T^*M),$$

where l , m and β satisfy the conditions given above. Also

$$C^{m-1,\beta}(E \otimes \Lambda^j T^*M) = H^j(E) \oplus \Delta^\omega (C^{m+1,\beta}(E \otimes \Lambda^j T^*M))$$

where

$$H^j(E) = \{ \eta \in C^{k+1,\alpha}(E \otimes \Lambda^j T^*M) : \Delta^\omega \eta = 0 \} .$$

Moreover

$$\Delta^\omega (C^{m+1,\beta}(E \otimes \Lambda^j T^*M)) = d^\omega (C^{m,\beta}(E \otimes \Lambda^{j-1} T^*M)) + \delta^\omega (C^{m,\beta}(E \otimes \Lambda^{j+1} T^*M))$$

where these spaces are considered as subspaces of $C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$.

Proof

Let ω_0 be a smooth connection on $\pi : P \to M$. Δ^{ω_0} defines Fredholm operators between Sobolev and Hölder spaces by theorem III.5.3. Also $\Delta^\omega - \Delta^{\omega_0}$ defines compact operators between the Sobolev and Hölder spaces under consideration. Thus Δ^ω defines Fredholm operators between these spaces.

If $\eta \in L^q_{-k+1}(E \otimes \Lambda^j T^*M)$ and $\Delta^\omega \eta \in L^p_{l-1}(E \otimes \Lambda^j T^*M)$ then $\eta \in L^p_{l+1}(E \otimes \Lambda^j T^*M)$ by theorem 3.5. If $\Delta^\omega \eta \in C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$ then $\Delta^\omega \eta \in L^r_{m-1}(E \otimes \Lambda^j T^*M)$ for all $r \in (1, \infty)$ and hence $\eta \in L^r_{m+1}(E \otimes \Lambda^j T^*M)$. On choosing r sufficiently large we see that $\eta \in C^{m,\beta}(E \otimes \Lambda^j T^*M)$ by the Sobolev embedding theorem. Then

$$(\Delta^\omega - \Delta^{\omega_0})\eta \in C^{m-1,\beta}(E \otimes \Lambda^j T^*M).$$

Since $\Delta^\omega \eta \in C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$ it follows that $\Delta^{\omega_0}\eta \in C^{m-1,\beta}(E \otimes \Lambda^j T^*M)$ and hence that $\eta \in C^{m+1,\beta}(E \otimes \Lambda^j T^*M)$.

By the previous theorem

$$L^2_{-1}(E \otimes \Lambda^j T^*M) = H^j(E) \oplus \Delta^\omega (L^2_1(E \otimes \Lambda^j T^*M)).$$

Also

$$\Delta^\omega (L^2_1(E \otimes \Lambda^j T^*M)) \cap C^{m-1,\beta}(E \otimes \Lambda^j T^*M) = \Delta^\omega (C^{m+1,\beta}(E \otimes \Lambda^j T^*M))$$

by the result proved above, hence

$$\begin{aligned} C^{m-1,\beta}(E \otimes \Lambda^j T^*M) &= L^2_{-1}(E \otimes \Lambda^j T^*M) \cap C^{m-1,\beta}(E \otimes \Lambda^j T^*M) \\ &= H^j(E) \oplus \Delta^\omega (C^{m+1,\beta}(E \otimes \Lambda^j T^*M)). \end{aligned}$$

We deduce that

$$G^{\omega} : C^{m-1,\beta}(E \otimes \Lambda^{j} T^*M) \rightarrow C^{m+1,\beta}(E \otimes \Lambda^{j} T^*M)$$

is bounded. The final statement of the theorem follows from the fact that $d^{\omega}(C^{m,\alpha}(E \otimes \Lambda^{j-1} T^*M)$ and $\delta^{\omega}(C^{m,\alpha}(E \otimes \Lambda^{j+1} T^*M))$ annihilate $H^{j}(E)$, as in the proof of the previous theorem.

□

References for Chapter VII

Warner, F.W., Foundations of Differentiable manifolds and Lie groups, Scott, Foresman and company, Glenview, Illinois, 1971.

Wells, R.O., Differential analysis on complex manifolds, Prentice Hall, Englewood Cliffs, New Jersey, 1973.

Chapter VIII

SLICE THEOREMS AND REGULARITY THEOREMS

§1. Introduction

This chapter contains the slice theorems and the regularity theorems towards which we have been working.

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold of dimension n with compact structural group G. Let k be a non-negative integer and let p satisfy the conditions $1 < p < \infty$ and $p(k + 1) > n$. Let p also satisfy the condition $p \geq 2$ in the case when $k = 0$.

In chapter VI we define $L^p_{k+1}\mathcal{G}$ to be the group of all L^p_{k+1} principal bundle automorphisms of $\pi : P \to M$, we also defined $L^p_{k+1}\mathcal{G}_0$ to be the quotient $L^p_{k+1}\mathcal{G}/Z(G)$ where $Z(G)$ is the centre of G, and given $m \in M$ we defined $L^p_{k+1}\mathcal{G}^m$ to be the subgroup of $L^p_{k+1}\mathcal{G}$ consisting of those automorphisms which fix the fibre of $\pi : P \to M$ over m. We defined $L^p_k\mathcal{A}$ to be the space of all L^p_k connections on $\pi : P \to M$ and we defined $L^p_k\mathcal{A}_0$ to be the subset of $L^p_k\mathcal{A}$ consisting of those connections whose stabilizer in $L^p_{k+1}\mathcal{G}$ is $Z(G)$. We prove that $L^p_{k+1}\mathcal{G}_0$ acts smoothly and freely on $L^p_k\mathcal{A}_0$ and that $L^p_{k+1}\mathcal{G}^m$ acts smoothly and freely on $L^p_k\mathcal{A}$.

We use theorem II.3.1 to prove a slice theorem (theorem 2.3) which states that $L^p_k\mathcal{A}_0/L^p_{k+1}\mathcal{G}_0$ and $L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}^m$ admit unique differentiable structures such that the projections

$$L^p_k\mathcal{A}_0 \to L^p_k\mathcal{A}_0/L^p_{k+1}\mathcal{G}_0$$

$$L^p_k\mathcal{A} \to L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}^m$$

are smooth and admit smooth local sections. Analogous results are proved for the action of $C^{k+1,\alpha}$ principal bundle automorphisms on $C^{k,\alpha}$ connections, where k is an integer satisfying $k \geq 1$ and where α satisfies $0 < \alpha < 1$.

A weakened form of the slice theorem (theorem 2.2) is proved which states that if ω is an L^p_k connection then there exists a smooth connection ω_o and an L^p_{k+1} principal bundle automorphism Ψ such that

$$\delta^{\omega_o}(\Psi^*\omega - \omega_o) = 0$$

where δ^{ω_o} is the covariant codifferential operator with respect to the connection ω_o. This result has applications in proving the regularity results of §3.

In §3 we prove various regularity theorems for Yang-Mills connections (theorems 3.1, 3.2 and 3.3). In §4 we give an informal discussion on how these results may be extended to Yang-Mills-Higgs systems.

§2. Slice Theorems for Connections

We use the results of chapter VII to prove a theorem (theorem 2.2) which will be useful in proving elliptic regularity results. Then we shall prove a slice theorem (theorem 2.3) for the action of principal bundle automorphisms on connections.

Lemma 2.1

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n, let ω be a connection on $\pi : P \to M$ and let $E \to M$ be a smooth vector bundle associated to $\pi : P \to M$ with a smooth inner product structure preserved by the connection ω. Let k be a non-negative integer and let p satisfy the conditions $1 < p < \infty$ and $p(k+1) > n$, and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. If ω is an L^p_k connection then $L^p_k(E \otimes T^*M)$ decomposes as the direct sum

$$L^p_k(E \otimes T^*M) = \operatorname{im} d^\omega \oplus \ker \delta^\omega$$

of the image of

$$d^\omega : L^p_{k+1}(E) \to L^p_k(E \otimes T^*M)$$

and the kernel of

$$\delta^\omega : L^p_k(E \otimes T^*M) \to L^p_{k-1}(E).$$

Moreover the image of d^ω is closed and the kernel of d^ω is finite dimensional.

Similarly if α satisfies $0 < \alpha < 1$ and if k is strictly positive then $C^{k,\alpha}(E \otimes T^*M)$ decomposes as the direct sum

$$C^{k,\alpha}(E \otimes T^*M) = \operatorname{im} d^\omega \oplus \ker \delta^\omega$$

of the image of

$$d^\omega : C^{k+1,\alpha}(E) \to C^{k,\alpha}(E \otimes T^*M)$$

and the kernel of

$$\delta^\omega : C^{k,\alpha}(E \otimes T^*M) \to C^{k-1,\alpha}(E).$$

Moreover the image of d^{ω} is closed and the kernel of d^{ω} is finite dimensional.

Proof

Let ω be an L^p_k connection. By theorem VII.4.1 it follows that

$$L^p_{k-1}(E) = H^o(E) \oplus \text{im } \Delta^{\omega}$$
$$= H^o(E) \oplus \text{im } \delta^{\omega}$$

where

$$H^o(E) = \{ s \in L^p_{k+1}(E) : \Delta^{\omega} s = 0 \} .$$

$H^o(E)$ is finite dimensional by theorem VII.4.1. Let

$$G^{\omega} : L^p_{k-1}(E) \rightarrow L^p_{k+1}(E)$$

be the Green's operator of Δ^{ω} (see §4 of chapter VII). Then, since

$$\text{im } \delta^{\omega} = \text{im } \Delta^{\omega}$$

it follows that

$$\Delta^{\omega} G^{\omega} \delta^{\omega} = \delta^{\omega}.$$

If $\eta \in L^p_k(E \otimes T^*M)$ and if

$$d^{\omega} G^{\omega} \delta^{\omega} \eta = 0$$

then $\delta^{\omega}\eta = 0$, since G^{ω} is injective on the image of δ^{ω} and d^{ω} is injective on the image of G^{ω}. Also if $\eta = d^{\omega} s$ then

$$d^{\omega} G^{\omega} \delta^{\omega} \eta = d^{\omega} G^{\omega} \Delta^{\omega} s$$
$$= \eta ,$$

since

$$d^{\omega} G^{\omega} \Delta^{\omega} = d^{\omega}.$$

Now

$$(d^{\omega} G^{\omega} \delta^{\omega})^2 = d^{\omega} G^{\omega} \Delta^{\omega} G^{\omega} \delta^{\omega}$$
$$= d^{\omega} G^{\omega} \delta^{\omega}$$

hence

$$d^{\omega} G^{\omega} \delta^{\omega} : L^p_k(E \otimes T^*M) \longrightarrow L^p_k(E \otimes T^*M)$$

is a bounded idempotent linear map whose kernel is ker δ^{ω} and whose image is im d^{ω}. It follows immediately that

$$L^p_k(E \otimes T^*M) = \text{im } d^{\omega} \oplus \ker \delta^{\omega}$$

and that im d^{ω} is closed. The kernel of d^{ω} is $H^0(E)$ by theorem VII.4.1 and is thus finite dimensional.

The proof of the lemma when $\omega \in C^{k,\alpha}\mathcal{A}$ is exactly analogous, using theorem VII.4.2.

□

Let G be the structural group of the principal bundle $\pi : P \to M$, let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. A given biinvariant metric on G determines an inner product structure on $\mathfrak{g}_P$ and thus determines the codifferential δ^{ω} acting on $\mathfrak{g}_P$-valued differential forms.

Theorem 2.2

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group. Let k be a non-negative integer and let p satisfy the conditions $1 < p < \infty$ and $p(k+1) > n$, and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let ω be an L^p_k connection on $\pi : P \to M$. Then for all $\varepsilon > 0$ and for all neighbourhoods of the identity in the group $L^p_{k+1}\mathcal{G}$ of L^p_{k+1} principal bundle automorphisms there exist a smooth connection ω_0 on $\pi : P \to M$ and an L^p_{k+1} princiapl bundle automorphism $\Psi : P \to P$, contained in the given neighbourhood of the identity in $L^p_{k+1}\mathcal{G}$, such that

$$\| \omega - \omega_0 \|_{L^p_k} < \varepsilon$$

and

$$\delta^{\omega_0}(\Psi^*\omega - \omega_0) = 0$$

Similarly if α satisfies $0 < \alpha < 1$ and if $k \geq 1$ and if ω is a $C^{k,\alpha}$ connection, then, for all $\varepsilon > 0$ and for all neighbourhoods of the identity in $C^{k+1,\alpha}\mathcal{G}$ there exist a smooth connection ω_0 and a $C^{k+1,\alpha}$ principal bundle automorphism contained in the given neighbourhood such that

$$\| \omega - \omega_0 \|_{C^{k,\alpha}} < \varepsilon$$

and

$$\delta^{\omega_0} (\Psi^* \omega - \omega_0) = 0 .$$

Proof

Let ω be an L^p_k connection and let $H^0(\mathfrak{g}\mathbf{P})^{\perp}$ be the orthogonal complement of the kernel $H^0(\mathfrak{g}\mathbf{P})$ of d^{ω} in $L^p_{k+1}(\mathfrak{g}\mathbf{P})$. Consider the map

$$\varphi : \ker \delta^{\omega} \oplus H^0(\mathfrak{g}\mathbf{P})^{\perp} \to L^p_k(\mathfrak{g}\mathbf{P} \otimes T^*M)$$

defined by

$$\varphi(\tau, \xi) = (\exp \xi)^* (\omega + \tau) - \omega$$

where $\ker \delta^{\omega}$ is the kernel of

$$\delta^{\omega} : L^p_k(\mathfrak{g}\mathbf{P} \otimes T^*M) \to L^p_{k-1}(\mathfrak{g}\mathbf{P}).$$

By theorem V.7.1, part (vii), the derivative $D\varphi$ of φ at the origin is given by

$$D\varphi(\tau, \xi) = \tau + d^{\omega}\xi$$

and is thus an isomorphism. Thus φ is a diffeomorphism from a neighbourhood $U \times V$ of the origin in $\ker \delta^{\omega} \oplus H^0(\mathfrak{g}\mathbf{P})$ to a neighbourhood of the origin in $L^p_k(\mathfrak{g}\mathbf{P} \otimes T^*M)$, by the inverse function theorem for Banach spaces. Given any neighbourhood of the identity in $L^p_{k+1}\mathcal{G}$ we may choose V sufficiently small such that exp(V) is contained in the given neighbourhood of the identity. Since $C^{\infty}\mathcal{A}$ is dense in $L^p_k\mathcal{A}$, there exists a smooth connection ω_0 such that

$$\| \omega - \omega_o \| < \varepsilon$$

and such that

$$\omega - \omega_o \in \varphi(U \times V).$$

Let

$$\omega - \omega_o = \varphi(\tau, \xi).$$

Then

$$\partial^{\omega}(\omega - \exp(-\xi)^* \omega_o) = 0.$$

Let $\omega_1 = (\exp \xi)^* \omega$. Then

$$(\exp \xi)^* \delta^{\omega} \tau = \delta^{\omega_1}((\exp \xi)^* \tau)$$

for all $\tau \in L^p_k(\mathfrak{g}_P \otimes T^*M)$, and hence

$$\delta^{\omega_1}(\omega_1 - \omega_o) = 0.$$

Thus

$$\delta^{\omega_o}(\Psi^* \omega - \omega_o) = \delta^{\omega_1}(\omega_1 - \omega_o)$$
$$= 0$$

by theorem V.7.1, part (vi), where $\Psi = \exp \xi$.

The proof when ω is a $C^{k,\alpha}$ connection is exactly analogous.

□

In chapter VI, section §4, we identified the centre Z(G) of the structural group G with a subgroup of $L^p_{k+1}\mathcal{G}$ and of $C^{k+1,\alpha}\mathcal{G}$, and we defined

$$L^p_{k+1}\mathcal{G}_o = L^p_{k+1}\mathcal{G}/Z(G),$$

and

$$C^{k+1,\alpha}\mathcal{G}_o = C^{k+1,\alpha}\mathcal{G}/Z(G).$$

We saw that $L^p_{k+1}\mathcal{G}_o$ and $C^{k+1,\alpha}\mathcal{G}_o$ act smoothly on $L^p_k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ respectively and we defined $L^p_k\mathcal{A}_o$ and $C^{k,\alpha}\mathcal{A}_o$ to be the subsets of $L^p_k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ consisting of all connections on which $L^p_{k+1}\mathcal{G}_o$ and $C^{k+1,\alpha}\mathcal{G}_o$ act freely. We showed that $L^p_k\mathcal{A}_o$ and $C^{k,\alpha}\mathcal{A}_o$ are open subsets of $L^p_k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ containing all smooth irreducible connections on $\pi : P \rightarrow M$ (theorem VI.4.4).

Also given $m \in M$ we defined $L^p_{k+1}\mathcal{G}^m$ and $C^{k+1,\alpha}\mathcal{G}^m$ to be the subgroups of $L^p_{k+1}\mathcal{G}$ and $C^{k+1,\alpha}\mathcal{G}$ consisting of all automorphisms of $\pi : P \rightarrow M$ which fix the fibre of $\pi : P \rightarrow M$ over m. We showed that $L^p_{k+1}\mathcal{G}^m$ and $C^{k+1,\alpha}\mathcal{G}^m$ act freely on $L^p_k\mathcal{A}$ and $C^{k,\alpha}\mathcal{A}$ (see theorem VI.4.6).

Theorem 2.3

Let $\pi : P \rightarrow M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group. Let $m \in M$. Let k be a non-negative integer and let p satisfy the conditions $1 < p < \infty$ and $p(k + 1) > n$, and in the case when $k = 0$ let p also satisfy the condition $p \geq 2$. Let $L^p_{k+1}\mathcal{G}_o$, $L^p_k\mathcal{A}_o$ and $L^p_{k+1}\mathcal{G}^m$ be defined as above. Then $L^p_k\mathcal{A}_o/L^p_{k+1}\mathcal{G}_o$ and $L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}^m$ admit unique differentiable structures such that the natural projections

$$L^p_k\mathcal{A}_o \rightarrow L^p_k\mathcal{A}_o/L^p_{k+1}\mathcal{G}_o,$$

$$L^p_k\mathcal{A} \rightarrow L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}^m$$

are smooth maps between Banach manifolds and admit smooth local sections.

Similarly if $k \geq 1$ and if α satisfies $0 < \alpha < 1$ then $C^{k,\alpha}\mathcal{A}_o/C^{k+1,\alpha}\mathcal{G}_o$ and $C^{k,\alpha}\mathcal{A}/C^{k+1,\alpha}\mathcal{G}^m$ admit unique differentiable structures such that the natural projections

$$C^{k,\alpha}\mathcal{A}_o \rightarrow C^{k,\alpha}\mathcal{A}_o/C^{k+1,\alpha}\mathcal{G}_o$$

$$C^{k,\alpha}\mathcal{A} \rightarrow C^{k,\alpha}\mathcal{A}/C^{k+1,\alpha}\mathcal{G}^m$$

are smooth maps between Banach manifolds and admit smooth local sections.

Proof

We must check that the conditions of theorem II.3.1 are satisfied. First consider $L^p_k\mathcal{A}_o/L^p_{k+1}\mathcal{G}_o$. Let $\omega \in L^p_k\mathcal{A}_o$. The centre $z(\mathfrak{g})$ of the Lie algebra of the structural group may be identified with a subalgebra of the Lie algebra $L^p_{k+1}(\mathfrak{g}_P)$ of $L^p_{k+1}\mathcal{G}$, corresponding to the identification of $Z(G)$ with a subgroup of $L^p_{k+1}\mathcal{G}$. The kernel of

$$d^{\omega} : L^p_{k+1}(\mathfrak{g}_P) \to L^p_k(\mathfrak{g}_P \otimes T^*M)$$

is $z(\mathfrak{g})$, since $\omega \in L^p_k\mathcal{A}_o$. Let

$$\theta : L^p_{k+1}\mathcal{G}_o \to L^p_k\mathcal{A}_o$$

be the smooth map sending the coset $Z(G).\Psi$ to $\Psi^*\omega$. The derivative of θ at the identity sends $\xi + z(\mathfrak{g})$ to $d^{\omega}\xi$ for all $\xi \in L^p_{k+1}(\mathfrak{g}_P)$, by theorem V.7.1, part (vii). By lemma 2.1 we see that the derivative of θ at the identity maps $L^p_{k+1}(\mathfrak{g}_P)/z(\mathfrak{g})$ isomorphically onto a closed complemented subspace of the tangent space to $L^p_k\mathcal{A}_o$ at ω. Thus the first two conditions of theorem II.3.1 are satisfied.

If $(\omega_i : i \in \mathbb{N})$ is a sequence in $L^p_k\mathcal{A}_o$, if $(\Psi_i : i \in \mathbb{N})$ is a sequence in $L^p_{k+1}\mathcal{G}_o$ and if the sequences (ω_i) and $(\omega_i.\Psi_i)$ converge in $L^p_k\mathcal{A}_o$ then the sequence (Ψ_i) converges in $L^p_{k+1}\mathcal{G}_o$ by theorem VI.4.5. This verifies the remaining condition of theorem II.3.1. From theorem II.3.1 we deduce that $L^p_k\mathcal{A}_o/L^p_{k+1}\mathcal{G}_o$ admits the required differentiable structure.

The proof for $L^p_k\mathcal{A}/L^p_{k+1}\mathcal{G}^m$ is similar, using theorem VI.4.6, the fact that $d^{\omega}(L^p_{k+1}(\mathfrak{g}_P))$ is a closed complemented subspace of $L^p_k(\mathfrak{g}_P \otimes T^*M)$ (by lemma 2.1) and the fact that

$$\{\xi \in L^p_{k+1}(\mathfrak{g}_P) : \xi(m) = 0\}$$

is a closed subspace of $L^p_{k+1}(\mathfrak{g}_P)$ of finite codimension (since $p(k+1) > n$).

The proof when $\omega \in C^{k,\alpha}\mathcal{A}_0$ or $\omega \in C^{k,\alpha}\mathcal{A}$ is exactly analogous.

□

We have used the elliptic regularity results of chapter VII to prove that conditions (i) and (ii) of theorem II.3.1 are satisfied. The proof that (iii) is satisfied stems ultimately from theorem VI.3.2 (or corollary VI.3.3). In fact condition (iii) of theorem II.3.1 is satisfied by the actions of groups of L^p_{k+1}, C^{k+1} or $C^{k+1,\alpha}$ automorphisms on manifolds of L^p_k, C^k or $C^{k,\alpha}$ connections respectively, provided that k is a non-negative integer and p and α satisfy $1 \leq p < \infty$, $0 < \alpha < 1$ and $p(k+1) > n$. Moreover if the relevant group acts freely then condition(i) of theorem II.3.1 is satisfied in these cases by corollary VI.5.3. It follows that in order to prove a slice theorem for the action of a group of automorphisms on some Banach manifold of connections in any of the above cases it suffices to verify that the action of the group of automorphisms is free and that the image under d^{ω} of the Lie algebra of the group of automorphisms is complemented in the tangent space to the manifold of connections. In some cases it may be possible to prove this by methods other than by using elliptic regularity. In particular if p = 2 then the above theorem follows from corollary VI.3.3 and theorem VI.5.3, without the need to use any elliptic regularity results at all, since every closed subspace of a Hilbert space has a closed complement.

§3. Regularity Theorems for Yang-Mills Connections

We prove regularity theorems for Yang-Mills connections on principal bundles over compact Riemannian manifolds.

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group G whose Lie algebra is $\mathfrak{g}$. Let G be given a biinvariant Riemannian metric, determining a smooth inner product structure on the adjoint bundle $\mathfrak{g}_P \to M$, where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. This inner product structure is preserved by all connections on $\pi : P \to M$, and it determines the covariant codifferential acting on $\mathfrak{g}_P$-valued differential forms.

Let ω be an L^p_k connection, where k is a non-negative integer, where p satisfies $1 < p < \infty$ and where p satisfies $p \geq 2$ in the case when k = 0. We recall that by theorem 2.2 there exists a smooth connection ω_0 and an L^p_{k+1} principal bundle automorphism $\Psi : P \to P$ such that

$$\delta^{\omega_0} \tau = \delta^{\omega_0 + \tau} \tau = 0$$

where

$$\tau = \Psi^* \omega - \omega_0 .$$

We recall that a Yang-Mills connection ω on $\pi : P \to M$ is a connection whose curvature F^{ω} satisfies the Yang-Mills equation

$$\delta^{\omega} F^{\omega} = 0.$$

If ω_0 is a smooth connection with curvature F_0 then

$$\begin{aligned} F^{\omega} &= F_0 + d^{\omega_0} \tau + \tfrac{1}{2}[\tau, \tau] \\ &= F_0 + d^{\omega} \tau - \tfrac{1}{2}[\tau, \tau] \end{aligned}$$

where $\tau = \omega - \omega_0$, by proposition V.7.1, using the fact that

$$d^{\omega} \tau = d^{\omega_0} \tau + [\tau, \tau] .$$

Thus if ω is a Yang-Mills connection and if τ satisfies the condition

$$\delta^{\omega}\tau = \delta^{\omega_o}\tau = 0$$

where $\tau = \omega - \omega_o$ for some smooth connection ω_o, then

$$\Delta^{\omega}\tau = \frac{1}{2}\delta^{\omega}[\tau, \tau] - \delta^{\omega}F_o .$$

Theorem 3.1

Let $\pi : P \longrightarrow M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group. Let the structural group of the principal bundle be given a biinvariant Riemannian metric. Let k be an integer satisfying $k \geq 2$ and let p satisfy the conditions $1 < p < \infty$ and $p(k + 1) > n$. If ω is an L^p_k connection on $\pi : P \rightarrow M$ satisfying the Yang-Mills equation, then there exists an L^p_{k+1} principal bundle automorphism $\Psi : P \longrightarrow P$ such that $\Psi^*\omega$ is smooth.

Proof

In view of theorem 2.2, it suffices to prove that if ω is an L^p_k connection satisfying the Yang-Mills equation and if ω_o is a smooth connection such that

$$\delta^{\omega_o}(\omega - \omega_o) = 0$$

then ω is smooth. Let $\tau = \omega - \omega_o$. From the remarks above we see that $\tau \in L^p_k(\mathfrak{g}_P \otimes T^*M)$ satisfies the equation

$$\Delta^{\omega}\tau = \frac{1}{2}\delta^{\omega}[\tau, \tau] - \delta^{\omega}F_o$$

where F_o is the curvature of ω_o. Let ε satisfy the conditions

$$0 < \varepsilon < \frac{1}{n} ,$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon ,$$

$$\varepsilon < \frac{1}{p} ,$$

and define $q \in (1, \infty)$ by

$$\frac{1}{q} = \frac{1}{p} - \varepsilon .$$

Then

$$\left(\frac{1}{p} - \frac{1}{n}\right) + \left(\frac{1}{p} - \frac{1}{n}\right) - \frac{k-1}{n} = \frac{1}{p} + \left(\frac{1}{p} - \frac{k+1}{n}\right) < \frac{1}{q} .$$

Using the Sobolev embedding theorem, theorem II.2.4 (the Sobolev multiplication theorems) and the condition $p(k+1) > n$, we deduce that

$$\tfrac{1}{2}[\tau, \tau] - F_0 \in L^p_{k-1}(\mathfrak{g}_P \otimes \Lambda^2 T^*M)$$

(see lemma VII.2.2). Hence

$$\delta^\omega(\tfrac{1}{2}[\tau, \tau] - F_0) \in L^q_{k-2}(\mathfrak{g}_P \otimes T^*M)$$

by proposition VII.3.1. Thus $\Delta^\omega \tau \in L^q_{k-2}(\mathfrak{g}_P \otimes T^*M)$ and hence $\tau \in L^q_k(\mathfrak{g}_P \otimes T^*M)$ by theorem VII.3.5.

If we iterate this procedure a finite number of times we see that ω is an L^q_k connection for some q satisfying the conditions $1 < q < \infty$ and $qk > n$. But then

$$\tfrac{1}{2}[\tau, \tau] - F_0 \in L^q_k(\mathfrak{g}_P \otimes T^*M),$$

hence $\Delta^\omega \tau \in L^q_{k-1}(\mathfrak{g}_P \otimes T^*M)$, and hence $\tau \in L^q_{k+1}(\mathfrak{g}_P \otimes T^*M)$ by theorem VII.3.5. By induction on k it follows that τ is smooth.

□

<u>Theorem 3.2</u>

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group. Let the structural group of the principal bundle be given a biinvariant Riemannian metric. Let p satisfy the conditions $1 < p < \infty$ and

$2p > n$, and in the case $n = 2$ let p also satisfy the condition $p > \frac{4}{3}$.
If ω is an L_1^p connection on $\pi : P \to M$ satisfying the Yang-Mills equation (weakly), then there exists an L_2^p principal bundle automorphism $\Psi : P \to P$ such that $\Psi^*\omega$ is smooth.

Proof

In view of theorem 2.2 it suffices to prove that if ω is an L_1^p connection satisfying the Yang-Mills equation and if ω_0 is a smooth connection such that

$$\delta^{\omega_0}(\omega - \omega_0) = 0$$

then ω is smooth. Let $\tau = \omega - \omega_0$.

Note that

$$\frac{2}{p} - \frac{1}{n} < 1.$$

When $n \geq 3$ this is a consequence of the condition $2p > n$. When $n = 2$ this is a consequence of the condition $p > \frac{4}{3}$. When $n = 1$ this is a consequence of the condition $p > 1$. Also since $2p > n$ it follows that

$$\frac{2}{p} - \frac{1}{n} < \frac{1}{p} + \frac{1}{n}.$$

Hence there exists q satisfying the conditions $1 < q < p$ and

$$\frac{2}{p} - \frac{1}{n} \leq \frac{1}{q} \leq \frac{1}{p} + \frac{1}{n}.$$

By theorem II.2.4 (the Sobolev Multiplication theorems) it follows that

$$\tfrac{1}{2}[\tau, \tau] - F_0 \in L_1^q(\mathfrak{g}_P \otimes \Lambda^2 T^*M),$$

$\Delta^\omega \tau \in L^q(\mathfrak{g}_P \otimes T^*M)$ by proposition VII.3.1, since ω is a Yang-Mills connection. Hence $\tau \in L_2^q(\mathfrak{g}_P \otimes T^*M)$ by theorem VII.3.5. But $3q > n$, hence τ is smooth by the proof of the previous theorem.

□

Theorem 3.3

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M of dimension n with compact structural group. Let the structural group of the principal bundle be given a biinvariant Riemannian metric. Let ω be a connection on $\pi : P \to M$ with the following properties:

(i) ω is an L^p connection for some p satisfying the condition

$$n < p < \infty ,$$

(ii) ω is an L_1^q connection for some q satisfying the condition

$$\frac{1}{q} \leq 1 - \frac{1}{p} ,$$

(iii) ω satisfies the Yang-Mills equation (weakly),

(iv) there exists a smooth connection ω_o such that

$$\delta^{\omega_o} (\omega - \omega_o) = 0$$

Then ω is smooth.

Proof

First we show that ω is an L_1^p connection. If $q \geq p$ this is trivial. Otherwise $p > 2$. Let ε satisfy the conditions

$$0 < \varepsilon < \frac{1}{n} ,$$

$$\frac{1}{p} < \frac{1}{n} - \varepsilon ,$$

$$\frac{1}{q} - \varepsilon \geq \frac{1}{p}$$

and let

$$\frac{1}{r} = \frac{1}{q} - \varepsilon .$$

If $\tau = \omega - \omega_o$ then

$$\tfrac{1}{2}[\tau,\tau] - F_o \in L^r(\mathfrak{g}_P \otimes \Lambda^2 T^*M)$$

by lemma VII.2.2, where F_o is the curvature of ω_o. Hence

$$\delta^{\omega}(\tfrac{1}{2}[\tau,\tau] - F_o) \in L^r_{-1}(\mathfrak{g}_P \otimes \Lambda^2 T^*M)$$

by proposition VII.3.1. Hence $\tau \in L^r_1(\mathfrak{g}_P \otimes T^*M)$ by theorem VII.3.5. By a finite number of iterations of this procedure we see that $\tau \in L^p_1(\mathfrak{g}_P \otimes T^*M)$. Since $p > n$ it follows that $\tau \in L^p_k(\mathfrak{g}_P \otimes T^*M)$ for all k, by induction on k as in the conclusion of the proof of theorem 3.1.

□

§4. Regularity of Yang-Mills-Higgs Systems

We give an informal discussion of regularity theorems for Yang-Mills-Higgs systems corresponding to the results proved in §3.

Let $\pi : P \to M$ be a smooth principal bundle over a compact smooth Riemannian manifold M of dimension n with compact structural group G, and let G be given a biinvariant Riemannian metric. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_P \to M$ be the adjoint bundle of $\pi : P \to M$, where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$. Let $\pi_1 : E \to M$ be a smooth vector bundle associated to $\pi : P \to M$ and let $\langle . , . \rangle : E \otimes E \to \mathbb{R}$ be a smooth inner product structure on E which is preserved by all connections on $\pi : P \to M$.

Let ω be a connection on $\pi : P \to M$ and let $\Phi : M \to E$ be a section of $\pi : E \to M$. The Yang-Mills-Higgs equations are the Euler-Lagrange equations for the functional

$$I(\omega, \Phi) = \int_M \left(\langle F^\omega, F^\omega \rangle + \langle d^\omega \Phi, d^\omega \Phi \rangle - V(|\Phi|) \right) d\mu$$

where μ is the Riemannian volume measure on M, where F^ω is the curvature of ω and where $V(|\Phi|)$ is an even polynomial in $|\Phi|$. In the standard Higgs model, as used in the Salam-Weinberg unification of the electromagnetic and the weak forces, the potential V is given by

$$V(|\Phi|) = \frac{\lambda}{4!} \left(|\Phi|^2 - c^2 \right)^2$$

for some constants λ and c in order to induce spontaneous symmetry breaking in the quantum field theory with the above Lagrangian via the Higgs mechanism (of course quantum field theories occurring in nature are formulated in the first instance in Minkowski space-time rather than on a Riemannian manifold). Here we shall allow V to be arbitrary, subject to constraints on the degree d of the polynomial $V(|\Phi|)$.

The Yang-Mills-Higgs equations have the form

$$\partial^\omega F^\omega = j(\Phi \otimes d^\omega \Phi),$$

$$\Delta^\omega \Phi + U(\Phi) = 0$$

where

$$j : E \otimes (E \otimes T^*M) \to \mathfrak{g}_P \otimes T^*M$$

is a smooth vector bundle morphism and where $U(\Phi)$, the derivative of $V(|\Phi|)$ with respect to Φ, is a polynomial of degree $d - 1$.

We recall that if $p(k + 1) > n$ then the group $L^p_{k+1}\mathcal{G}$ of L^p_{k+1} principal bundle automorphisms of $\pi : P \to M$ acts on $L^p_l(E)$ on the left for all l satisfying $0 \leq l \leq k + 1$. If ω is an L^p_k connection, Φ is an L^p_k section of $E \to M$ and (ω, Φ) satisfies the Yang-Mills-Higgs equations then so does $(\Psi^*\omega, \Psi^{-1}.\Phi)$ for all L^p_{k+1} principal bundle automorphisms $\Psi : P \to P$.

If the degree d of the potential $V(|\Phi|)$ does not exceed 4, then one can prove regularity theorems for Yang-Mills-Higgs systems exactly analogous to theorems 3.1, 3.2 and 3.3 for Yang-Mills connections. For instance suppose that ω is an L^p_k connection and that Φ is an L^p_k section of $E \to M$, where k is an integer satisfying $k \geq 2$ and where p satisfies $1 < p < \infty$ and $p(k + 1) > n$, where n is the dimension of M. Suppose also that there exists a smooth connection ω_0 such that

$$\delta^{\omega_0}(\omega - \omega_0) = 0.$$

Let $\tau = \omega - \omega_0$. The Yang-Mills-Higgs equations have the form

$$\Delta^\omega \tau = j(\Phi \otimes d^\omega \Phi) + \tfrac{1}{2}\delta^\omega[\tau, \tau] - F_0,$$

$$\Delta^\omega \Phi = U(\Phi)$$

where F_0 is the curvature of ω_0. Let ε satisfy the conditions

$$0 < \varepsilon < \frac{1}{n}$$

$$\frac{1}{p} < \frac{k+1}{n} - \varepsilon$$

$$\varepsilon < \frac{1}{p}$$

and define $q \in (1, \infty)$ by

$$\frac{1}{q} = \frac{1}{p} - \varepsilon .$$

Then

$$\delta^{\omega}(\tfrac{1}{2}[\tau, \tau] - F_0) \in L^q_{k-2}(\mathfrak{g}_P \otimes T^*M)$$

as in the proof of theorem 3.1. Now $d^{\omega}\Phi \in L^p_{k-1}(E \otimes T^*M)$ by proposition VII.3.1, hence

$$j(\Phi \otimes d^{\omega}\Phi) \in L^q_{k-2}(\mathfrak{g}_P \otimes T^*M)$$

by lemma VII.2.2, since $\Phi \in L^p_k(E)$ and $p(k + 1) > n$. Hence $\Delta^{\omega}\tau \in L^q_{k-2}(\mathfrak{g}_P \otimes T^*M)$. Also the degree of $U(\Phi)$ does not exceed 3 and since $p(k + 1) > n$ it follows that $U(\Phi) \in L^p_{k-2}(E)$, by two applications of lemma VII.2.2. Thus $\Delta^{\omega}\Phi \in L^q_{k-2}(E)$. Hence $\tau \in L^q_k(\mathfrak{g}_P \otimes T^*M)$ and $\Phi \in L^q_k(E)$ by theorem VII.3.5.

By iterating this procedure a finite number of times we see that $\tau \in L^q_k(\mathfrak{g}_P \otimes T^*M)$ and $\Phi \in L^q_k(E)$ for some $q \in (1, \infty)$ satisfying $qk > n$. As in theorem 3.1, one may easily show that $\tau \in L^q_{k+1}(\mathfrak{g}_P \otimes T^*M)$ and $\Phi \in L^q_{k+1}(E)$ and hence show by induction that τ and Φ are smooth. Thus we can prove the analogue of theorem 3.1 for Yang-Mills-Higgs systems the degree of whose potential does not exceed 4. Similar analogues of 3.2 and 3.3 may be proved.

When the degree d of the potential exceeds 4 the hypotheses of these regularity theorems must be strengthened. In addition to the condition $p(k + 1) > n$ in the above proof we must also require that p be sufficiently large in order that U may map $L^p_k(E)$ to $L^q_{k-2}(E)$ for some q satisfying

$$\frac{1}{p} > \frac{1}{q} > \frac{1}{p} - \frac{1}{n} .$$

It is sufficient to require that

$$\frac{1}{p} - \frac{k-2}{n} > (d-1)\left(\frac{1}{p} - \frac{k}{n}\right)$$

by the Sobolev embedding and multiplication theorems. Thus we require that

$$p\left(k + \frac{2}{d-2}\right) > n$$

in order to show that an L_k^p Yang-Mills-Higgs system with $d > 4$ and $k \geq 1$ may be transformed to a smooth Yang-Mills-Higgs system by an L_{k+1}^p principal bundle automorphism, where p must also satisfy the condition $p > \frac{4}{3}$ in the case when $n = 2$ and $k = 1$.

Chapter IX

COVARIANT DERIVATIVES AND HOLONOMY

§1. Introduction

Let $\pi : P \to M$ be a smooth principal bundle over a Riemannian manifold M and let ω be an Ehresmann connection on $\pi : P \to M$ whose holonomy group is compact. In theorem 2.1 we shall show that, given $m \in M$, there exists a constant $L_{\omega,m}$ such that every element of the holonomy group of ω may be generated by a loop based at m of length not exceeding $L_{\omega,m}$. In essence, the proof is by showing that every element of the holonomy group of ω is generated by a loop based at m which is a concatenation of lassos and their reversals, where the lassos are taken from a finite set of one-parameter families of lassos which generate the Lie algebra of the holonomy group of ω . The existence of such a set of one-parameter families of lassos is guaranteed by the Ambrose-Singer holonomy theorem.

We give two applications of this theorem in §3. We show that, given any continuously differentiable section of a vector bundle associated to $\pi : P \to M$ there exists a covariantly constant section whose distance from the given section is bounded by some constant multiple of the supremum of the magnitude of the covariant derivative of the given section (theorem 3.1). A similar result (theorem 3.3) is proved for principal bundle automorphisms.

The main problem is, of course, to estimate $L_{\omega,m}$. The proof of theorem 2.1 shows the existence of $L_{\omega,m}$ but provides no effective means of calculating it in general. Indeed one can easily visualise pathological examples of connections where the stalks of the lassos, whose existence is guaranteed by the Ambrose-Singer holonomy theorem, wander around the base manifold of the bundle in a complicated manner.

However if one imposes suitable restrictions on the curvature tensor of the connection and its covariant derivatives it may be possible to estimate $L_{\omega,m}$. For example one could place restrictions on the curvature so that the holonomy group was generated by curves in any arbitrarily small neighbourhood of some given point. An interesting problem might be to study the Levi-Civita connection on a Riemannian manifold in this way.

§2. The Length of Loops generating the Holonomy Group

In this section we prove a theorem which provides an upper bound on the length of loops required to generate any element of the holonomy group of a smooth Ehresmann connection on a principal bundle $\pi : P \longrightarrow M$ over a compact Riemannian manifold M, provided that the holonomy group is a compact subgroup of the structural group.

Theorem 2.1

Let ω be a smooth Ehresmann connection on a smooth principal bundle $\pi : P \rightarrow M$ over a Riemannian manifold M with the property that the holonomy group of ω is compact. Let $m \in M$. Then there exists a constant $L_{\omega,m}$, depending on ω and m, with the following property: given any element of the holonomy group attached to some element of the fibre of $\pi : P \rightarrow M$ over m, there exists a piecewise smooth loop, of length not exceeding $L_{\omega,m}$, beginning and ending at m which generates the required element of the holonomy group.

Proof

First we introduce some terminology. A piecewise smooth curve in M is a piecewise smooth map $c : [a, b] \rightarrow M$. A piecewise smooth path in M is an equivalence class of piecewise smooth curves in M where two curves are equivalent if and only if each is a reparameterization of the other. A loop based at $m \in M$ is a path beginning and ending at $m \in M$. Given paths in M, represented by curves

$c_1 : [a, x] \to M$ and $c_2 : [x, b] \to M$, the concatenation $c_1 * c_2$ of c_1 and c_2 is the path represented by the curve $c_1 * c_2 : [a, b] \to M$, where

$$(c_1 * c_2) \mid [a, x] = c_1 ,$$

$$(c_1 * c_2) \mid [x, b] = c_2 .$$

Given a path, represented by the curve $c : [0, 1] \to M$ the reverse $c^{\leftarrow}$ of c is represented by the curve $c^{\leftarrow} : [0, 1] \to M$ defined by

$$c^{\leftarrow}(t) = c(1 - t)$$

for all $t \in [0, 1]$.

Now consider curves in the structural group G of $\pi : P \to M$. If $\gamma_1 : (-\varepsilon, \varepsilon) \to G$ and $\gamma_2 : (-\varepsilon, \varepsilon) \to G$ are continuously differentiable curves in G then the product curve $\gamma_1 \cdot \gamma_2 : (-\varepsilon, \varepsilon) \to G$ and the inverse curve $\gamma_1^{-1} : (-\varepsilon, \varepsilon) \to G$ are defined by

$$(\gamma_1 \cdot \gamma_2)(t) = \gamma_1(t) . \gamma_2(t)$$

$$\gamma_1^{-1}(t) = \gamma_1(t)^{-1}$$

for all $t \in (-\varepsilon, \varepsilon)$. Let $\mathfrak{g}$ be the Lie algebra of G. If $\gamma_1(0) = \gamma_2(0) = e$, where e is the identity element of G and if X_1 and X_2 are tangent to γ_1 and γ_2 at $t = 0$, then $X_1 + X_2$ is tangent to $\gamma_1 \cdot \gamma_2$ at $t = 0$ and $-X_1$ is tangent to γ_1^{-1} at $t = 0$. Let us also define a curve

$$\operatorname{comm}(\gamma_1, \gamma_2) : (-\varepsilon^2, \varepsilon^2) \to G$$

by the conditions that

$$\operatorname{comm}(\gamma_1, \gamma_2)(t) = \gamma_1(\sqrt{t})^{-1} \gamma_2(\sqrt{t})^{-1} \gamma_1(\sqrt{t}) \gamma_2(\sqrt{t})$$

when $0 \leq t < \varepsilon^2$ and

$$\operatorname{comm}(\gamma_1, \gamma_2)(t) = \gamma_1(\sqrt{-t})^{-1} \gamma_2(\sqrt{-t}) \gamma_1(\sqrt{-t}) \gamma_2(\sqrt{-t})^{-1}$$

when $-\varepsilon^2 < t \leq 0$. If γ_1 and γ_2 are continuously differentiable, if

$$\gamma_1(0) = \gamma_2(0) = e,$$

and if X_1 and X_2 are the vectors in $\mathfrak{g}$ tangent to γ_1 and γ_2 at $t = 0$ then comm(γ_1 , γ_2) is continuously differentiable and $[X_1, X_2]$ is tangent to comm(γ_1 , γ_2) at $t = 0$.

Let $m, m' \in M$ and let c be a piecewise smooth path from m to m'. We define a <u>one-parameter family of lassos</u> based at m with <u>stalk</u> c and <u>vertex</u> m' to be a family $\{\lambda^t : t \in (-\varepsilon, \varepsilon)\}$ of loops based at m with the property that there exists a one-parameter family $\{c^t : t \in (-\varepsilon, \varepsilon)\}$ of loops based at m' such that

$$\lambda^t = c * c^t * c^{\leftarrow},$$

where the family of $\{c^t\}$ satisfies the following conditions: c^0 is the constant path at m' and the paths c^t are represented by a family of curves $c^t : [0, 1] \to M$ with the property that the map from $(-\varepsilon, \varepsilon) \times [0, 1]$ to M sending (t, u) to $c^t(u)$ is piecewise smooth and the map from $(-\varepsilon, \varepsilon)$ to G sending t to the element of the holonomy group of ω generated by c^t is continuously differentiable.

Let $\{c^t : t \in (-\varepsilon, \varepsilon)\}$ be a one-parameter family of loops based at m with the property that c_0 generates the identity element of the holonomy group H_p of ω attached to p, where p is an element of the fibre of $\pi : P \to M$ over m. We say that $\{c^t : t \in (-\varepsilon, \varepsilon)\}$ <u>generates</u> the short curve $\gamma : (-\varepsilon, \varepsilon) \to H_p$ if and only if $\gamma(t)$ is the element of H_p generated by c^t for all $t \in (-\varepsilon, \varepsilon)$. Under these circumstances we say that $\{c^t\}$ <u>generates</u> $X \in \mathfrak{h}_p$ where X is the element of the Lie algebra $\mathfrak{h}_p$ of the holonomy group H_p which is tangent to $\gamma : (-\varepsilon, \varepsilon) \to H_p$ at $t = 0$.

Let $\{c_1^t : t \in (-\varepsilon, \varepsilon)\}$ and $\{c_2^t : t \in (-\varepsilon, \varepsilon)$ be one-parameter families of loops based at m generating curves $\gamma_1 : (-\varepsilon, \varepsilon) \to H_p$ and $\gamma_2 : (-\varepsilon, \varepsilon) \to H_p$ respectively whose tangent vectors at $t = 0$ are $X_1 \in \mathfrak{h}_p$ and $X_2 \in \mathfrak{h}_p$. Then the one-parameter families $\{c_1^t * c_2^t : t \in (-\varepsilon, \varepsilon)\}$ and

$\{ c_1^{t\leftarrow} : t \in (-\varepsilon, \varepsilon)$ generate curves $\gamma_2 \cdot \gamma_1 : (-\varepsilon, \varepsilon) \to H_p$ and $\gamma_1^{-1} : (-\varepsilon, \varepsilon) \to H_p$, and hence generate $X_1 + X_2 \in \mathfrak{h}_p$ and $-X_1 \in \mathfrak{h}_p$ respectively. We see that if $t \geq 0$ then $\mathrm{comm}(\gamma_1, \gamma_2)\,(t^2)$ is generated by

$$c_2^{t} * c_1^{t} * c_2^{t\leftarrow} * c_1^{t\leftarrow}$$

and $\mathrm{comm}\,(\gamma_1, \gamma_2)\,(-t^2)$ is generated by

$$c_2^{t\leftarrow} * c_1^{t} * c_2^{t} * c_1^{t\leftarrow} .$$

We now proceed with the proof of the theorem. Let p be an element of the fibre of $\pi : P \to M$ over m, and let B_p be the holonomy bundle of the connection ω attached to p. By the Ambrose-Singer holonomy theorem there exist $\{ p_i \in B_p : i = 1, \ldots, d \}$ and vectors $U_i \in T_{p_i}P$, $V_i \in T_{p_i}P$ such that the Lie algebra $\mathfrak{h}\,p$ of the holonomy group H_p of ω attached to p is generated by $\{ F^{\omega}(U_i \wedge V_i) : i = 1, \ldots, d \}$, where $F^{\omega} : \Lambda^2 TP \to \mathfrak{g}$ is the curvature of ω. Expressed geometrically, this implies that there exist one-parameter families $\{ \lambda_i^t : t \in (-\varepsilon, \varepsilon) \}$ of lassos with vertices $\pi(p_i)$ such that the elements $\{ X_i \in \mathfrak{h}_p : i = 1, \ldots, d \}$ generated by the $\{ \lambda_i^t \}$ form a set of generators for $\mathfrak{h}_p$. Let $\gamma_i : (-\varepsilon, \varepsilon) \to H_p$ be the curve generated by the one-parameter family $\{ \lambda^t_i : t \in (-\varepsilon, \varepsilon) \}$ of lassos. Let L_1 be an upper bound on the length of the lassos in

$$\{ \lambda^t_i : t \in (-\varepsilon, \varepsilon),\ i = 1, \ldots, d \} .$$

We may construct a basis $(Y_j : j = 1, \ldots, r)$ of the vector space $\mathfrak{h}_p$ whose elements are (iterated) Lie brackets of elements of the set $\{ X_i : i = 1, \ldots, d \}$. The vectors Y_j are tangent to continuously differentiable curves $\beta_j : (-\delta, \delta) \to H_p$ at $t = 0$, where the curves $\{ \beta_j : j = 1, \ldots, r \}$ are constructed out of the curves $\{ \gamma_i : i = 1, \ldots, d \}$ using the operation comm defined above.

Since every element of the image of β_j for $j = 1, \ldots, r$ may be generated by a concatenation of lassos in

$$\{ \lambda^t_i : t \in (-\varepsilon, \varepsilon), i = 1, \ldots, d \}$$

and their reverses, and since the length of these lassos does not exceed L_1, it follows that there exists a constant L_2 such that, given $t \in (-\delta, \delta)$ and an integer j in the range $1 \leq j \leq r$, we can find a loop c^t_j based at m of length not exceeding L_2 which generates $\beta_j(t)$.

Define a continuously differentiable map

$$\varphi : (-\delta, \delta)^r \to H_p$$

sending $(t_1, \ldots, t_r)$ to $\beta_1(t_1) \ldots \beta_r(t_r)$. The derivative of φ at the origin is an isomorphism since $\{ Y_j : j = 1, \ldots, r \}$ is a basis of $\mathfrak{h}_p$. Thus the image of φ contains an open neighbourhood of the identity, by the inverse function theorem. But every element of the image of φ is generated by a concatenation of r loops based at m, each of length not exceeding L_2. Thus there exists a neighbourhood N of the identity in H_p such that every element of N is generated by a loop based at m of length not exceeding rL_2.

Since H_p is compact, there exists a positive integer k such that every element of the identity component of H_p is of the form γ^k for some $\gamma \in N$, and thus may be generated by a loop of length not exceeding krL_2. Also H_p has finitely many components. Thus we may find representatives $h_1, \ldots, h_m$ in each coset of the identity component. Then there exists L_3 such that $h_1, \ldots, h_m$ may all be generated by a loop of length not exceeding L_3. Since every element of H_p is of the form $h_i . \gamma$ for some h_i and for some element γ of the identity component, every element of H_p may be generated by a loop of length not exceeding $L_{\omega,m}$, where

$$L_{\omega,m} = krL_2 + L_3 .$$

□

Note that if the holonomy group of ω is not compact then for all $m \in M$ and for all compact subsets K of the holonomy group attached to some element of the fibre of $\pi : P \rightarrow M$ over m there exists a constant $L_{\omega,m,K}$ such that every element of k may be generated by a loop of length not exceeding $L_{\omega,m,K}$. For as in the above proof we see that there exists a neighbourhood N of the identity in the holonomy group and a constant L' such that every element of N may be generated by a loop of length not exceeding L'. The required result follows easily on noting that K is covered by a finite number of translates of the neighbourhood N.

§3. Further Inequalities for Sections of Fibre Bundles

We present two theorems in which theorem 2.1 of the previous section is applied to prove inequalities satisfied by sections of some fibre bundle associated to a given principal bundle.

Let H be a compact Lie group and let $H \rightarrow \mathrm{End}(V)$ be a representation of H. Let $\langle \,.\, , \,.\, \rangle$ be an H-invariant inner product on V. Let V_0 be the subspace on which H acts trivially and let $V_0^{\perp}$ be its orthogonal complement. Then there exists a constant λ such that

$$|v| \leq \lambda \sup_{h \in H} |h.v - v|$$

for all $v \in V_0^{\perp}$. For let S be the unit sphere in $V_0^{\perp}$. Then $f : S \rightarrow \mathbb{R}$ is continuous, where

$$f(v) = \sup_{h \in H} |h.v - v|$$

since H is compact. Moreover $f(v) > 0$ for all $v \in S$ hence there exists a constant λ such that $f(v) \geqslant \lambda^{-1}$, since S is compact. This is the required constant.

We recall that if ω is a smooth connection on a principal bundle $\pi : P \rightarrow M$ over a Riemannian manifold M and if the holonomy group of ω is compact then for all $m \in M$ there exists a constant $L_{\omega,m}$ such that every element of the holonomy group of ω attached to an element of the fibre of $\pi : P \rightarrow M$ over m may be generated by a loop based at m of length not exceeding $L_{\omega,m}$, by theorem 2.1.

Theorem 3.1

Let $\pi : P \rightarrow M$ be a principal bundle over a Riemannian manifold M whose diameter $\mathrm{diam}(M)$ is finite. Let ω be a smooth Ehresmann connection on $\pi : P \rightarrow M$ whose holonomy group is compact. Let $m \in M$ and let $L_{\omega,m}$ be an upper bound on the length of loops based at m required to generate the holonomy group of ω .

Let $\pi_1 : E \to M$ be a vector bundle associated to $\pi : P \to M$ with fibre V, and let V be given an inner product invariant under the action of the structural group of $\pi : P \to M$. Let $\| \cdot \|$ denote the canonical C^0 norm on $C^0(E)$ and on $C^0(E \otimes T^*M)$ determined by the inner product on V and the Riemannian metric on M. Let V_0 be the subspace of V on which the holonomy group H of ω acts trivially and let λ be a constant such that

$$|v| \leq \lambda \sup_{h \in H} |h.v - v|$$

for all $v \in V_0^{\perp}$. Then for all C^1 sections $\sigma : M \to E$ of $\pi_1 : E \to M$ there exists a section $\sigma_0 : M \to E$ such that

$$d^\omega \sigma_0 = 0$$

and

$$\| \sigma - \sigma_0 \| \leq (\lambda L_{\omega,m} + \operatorname{diam}(M)) \| d^\omega \sigma \| .$$

Proof

Let $| \cdot |_m$ be the norm on the fibre of $\pi_1 : E \to M$ over m determined by the inner product on V. Let p be an element of the fibre of $\pi : P \to M$ over m and let γ be an element of the holonomy group H of ω attached to p. Then there exists a loop $c : [0, L] \to M$ based at m of length not exceeding $L_{\omega,m}$ and parameterized by arclength s which generates γ. Then

$$|\gamma . \sigma(m) - \sigma(m)|_m \leq \int_c |d^\omega \sigma| \, ds$$
$$\leq L_{\omega,m} \| d^\omega \sigma \|$$

by theorem V.6.6. But there exists $e_0 \in \pi_1^{-1}(m)$ such that $\gamma . e_0 = e_0$ for all $\gamma \in H$ and such that $\sigma(m) - e_0$ is orthogonal to the subspace of $\pi_1^{-1}(m)$ on which H acts trivially. Then

$$|\sigma(m) - e_0| \leq \lambda \sup_{\gamma} |\gamma . \sigma(m) - \sigma(m)|$$
$$\leq \lambda L_{\omega,m} \| d^\omega \sigma \| .$$

But since $\gamma . e_0 = e_0$ for all $\gamma \in H$ there exists a unique section $\sigma_0 : M \to E$ such that $d^{\omega}\sigma_0 = 0$ and $\sigma_0(m) = e_0$. Then

$$\| \sigma - \sigma_0 \| \leq | \sigma(m) - \sigma_0(m) |_m + \text{diam}(M) \| d^{\omega}\sigma - d^{\omega}\sigma_0 \|$$
$$\leq (\lambda L_{\omega,m} + \text{diam}(M)) \| d^{\omega}\sigma \|$$

by theorem VI.3.4.

□

One can also combine the inequality

$$| \sigma(m) - \sigma_0(m) |_m \leq \lambda L_{\omega,m} \| d^{\omega}\sigma \|$$

with the inequalities stated in theorem VI.5.2, provided that in case (i) of VI.5.2, p and k satisfy the stronger condition $pk > \dim M$.

A result for principal bundle automorphisms corresponding to theorem 3.1 will be proved using the following lemma.

Lemma 3.2

Let G be a compact Lie group with a biinvariant Riemannian metric whose distance function is $\rho : G \times G \to \mathbb{R}$. Let H be a closed subgroup of G and let C(H) be the centralizer of H. For all $\gamma \in G$ define

$$\rho(\gamma, C(H)) = \inf_{\eta \in C(H)} \rho(\gamma, \eta).$$

Then there exists a positive constant A such that

$$\rho(\gamma, C(H)) \leq A \sup_{h \in H} \rho(h^{-1}\gamma h, \gamma)$$

Proof

Let $\mathfrak{g}$ be the Lie algebra of G, let

$V_0 = \{ X \in \mathfrak{g} : \text{Ad}(h^{-1}) X = X \text{ for all } h \in H \}$

and let $V_0^{\perp}$ be the orthogonal complement of V_0. We have seen that there exists a constant λ such that

$$| X | \leq \lambda \sup_{h \in H} | \text{Ad}(h^{-1}) X - X |$$

for all $X \in V_0^{\perp}$. Also given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|X - Y| \leq (1 + \varepsilon)\, \rho(\exp X, \exp Y)$$

whenever $|X| < \delta$ and $|Y| < \delta$. Thus if $X \in V_0^{\perp}$ and $|X| < \delta$ then

$$|X| \leq (1 + \varepsilon)\, \lambda \sup_{h \in H} (h^{-1} (\exp X)\, h, \exp X).$$

Now suppose that $\gamma \in G$ and that

$$0 < \rho(\gamma, C(H)) < \delta.$$

Since H is compact, there exists an element $\eta_0 \in C(H)$ such that

$$\rho(\gamma, \eta_0) = \rho(\gamma, C(H)).$$

Then

$$\rho(\gamma \eta_0^{-1}, e) = \rho(\gamma, C(H))$$

since ρ is biinvariant. But then $\gamma \eta_0^{-1}$ is joined to e by a length minimizing geodesic of length strictly less than δ whose tangent vector at e is orthogonal to the tangent space V_0 to C(H) at e. Thus

$$\gamma \eta_0^{-1} = \exp X$$

for some $X \in V_0^{\perp}$ satisfying $|X| < \delta$. Then

$$\begin{aligned} \rho(\gamma, C(H)) &= \rho(\exp X, e). \\ &= |X| \\ &\leq (1 + \varepsilon)\, \lambda \sup_{h \in H} (h^{-1} \gamma \eta_0^{-1} h, \gamma \eta_0^{-1}) \\ &\leq (1 + \varepsilon)\, \lambda \sup_{h \in H} (h^{-1} \gamma h \eta_0^{-1}, \gamma \eta_0^{-1}) \\ &\leq (1 + \varepsilon)\, \lambda \sup_{h \in H} (h^{-1} \gamma h, \gamma). \end{aligned}$$

Define $f : G \setminus C(H) \to \mathbb{R}$ by

$$f(\gamma) = \sup_{h \in H} \frac{\rho(h^{-1} \gamma h, \gamma)}{\rho(\gamma, C(H))}.$$

f is continuous since H is compact. Also we have just shown that

$$f(\gamma) \geqslant \frac{1}{(1+\varepsilon)\lambda}$$

whenever $\rho(\gamma, C(H)) < \delta$. Since G is compact there exists a constant A such that

$$A \geqslant (1+\varepsilon)\lambda$$

and such that $f(\gamma) > A^{-1}$ for all γ satisfying

$$\rho(\gamma, C(H)) \geqslant \tfrac{1}{2}\delta .$$

Then

$$\rho(\gamma, C(H)) \leqslant A \sup_{h \in H} \rho(h^{-1}\gamma h, \gamma)$$

for all $\gamma \in G$.

$\square$

Let $\pi : P \to M$ be a smooth principal bundle with compact structural group G. Let G be given a biinvariant Riemannian metric with distance function $\rho : G \times G \to \mathbb{R}$. We recall that this determines a biinvariant distance function $\rho_m : \pi_{ad}^{-1}(m) \times \pi_{ad}^{-1}(m) \to \mathbb{R}$ on the fibre $\pi_{ad}^{-1}(m)$ of the adjoint bundle $\pi_{ad} : G\mathbf{p} \to M$ over m for all $m \in M$, where $G\mathbf{p} = P \times_{ad} G$. This distance function has the property that the group isomorphism from G to $\pi_{ad}^{-1}(m)$ determined by any element of $\pi : P \to M$ is an isometry. If M is compact then the canonical distance function $\bar{\rho} : C^0(G\mathbf{p}) \times C^0(G\mathbf{p}) \to \mathbb{R}$ is defined by

$$\bar{\rho}(\Psi_1, \Psi_2) = \sup_{m \in M} \rho_m(\Psi_1(m), \Psi_2(m)).$$

<u>Theorem 3.3</u>

Let $\pi : P \to M$ be a smooth principal bundle over a compact Riemannian manifold M with compact structural group G. Let G be given a biinvariant Riemannian metric with distance function $\rho : G \times G \to \mathbb{R}$, determining the canonical distance function $\bar{\rho} : C^0(G\mathbf{p}) \times C^0(G\mathbf{p}) \to \mathbb{R}$ on $C^0(G\mathbf{p})$, where $\pi_{ad} : G\mathbf{p} \to M$ is the adjoint bundle with total space

$G_P = P \times_{ad} G$. Let $\| . \|$ be the canonical norm on $C^0(\mathfrak{g}_P \otimes T^*M)$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$.

Let ω be a smooth connection on $\pi : P \to M$ with compact holonomy group H attached to some element p of the fibre of $\pi : P \to M$ over m, for some $m \in M$. Let A be a constant such that

$$\rho(\gamma, C(H)) \leq A \sup_{h \in H} \rho(h^{-1}\gamma h, \gamma)$$

for all $\gamma \in G$. Let $L_{\omega,m}$ be an upper bound on the length of loops based at m required to generate the holonomy group of ω.

If Ψ is a C^1 principal bundle automorphism then there exists a principal bundle automorphism Ψ_0 stabilizing ω such that

$$\bar{\rho}(\Psi, \Psi_0) \leq (AL_{\omega,m} + \mathrm{diam}(M)).$$

<u>Proof</u>

Let $h \in H$ and let $c : [0, l] \to M$ be a loop based at m generating h of length not exceeding $L_{\omega,m}$ which is parameterized by arclength s. Let $\psi : P \to G$ be the unique C^1 function with the property that

$$\Psi(p) = p.\psi(p)$$

for all $p \in P$. Then

$$\rho(h^{-1}\psi(p)h, \psi(p)) \leq \int_c |\Psi^*\omega - \omega| \, ds$$

$$\leq L_{\omega,m} \|\Psi^*\omega - \omega\|$$

by theorem V.5.2. Then

$$\rho(\psi(p), C(H)) \leq A \sup_{h \in H} \rho(h^{-1}\psi(p)h, \psi(p))$$

$$AL_{\omega,m} \|\Psi^*\omega - \omega\|$$

hence there exists $\eta \in C(H)$ such that

$$\rho(\psi(p), \eta) \leq AL_{\omega,m} \|\Psi^*\omega - \omega\|.$$

By theorem V.4.2 there exists a principal bundle automorphism $\Psi_0 : P \to P$

which stabilizes ω such that

$$\Psi_o(p) = p.\eta .$$

Then

$$\rho_m(\Psi(m), \Psi_o(m)) \leq AL_{\omega,m} \| \Psi^*\omega - \omega \|$$

and hence

$$\bar{\rho}(\Psi, \Psi_o) \leq (AL_{\omega,m} + \mathrm{diam}(M)) \| \Psi^*\omega - \omega \|$$

by lemma VI.3.1.

□

Appendix A

THE HILBERT TRANSFORM

The Hilbert transform $Hf : \mathbb{R} \longrightarrow \mathbb{R}$ of a measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \, dt$$

whenever this principal value exists. We give a proof, derived from [Calderon, A.P., 1966], of a theorem due to M. Riesz, which states that the Hilbert transform is bounded on $L^p(\mathbb{R}^n)$ for all p satisfying $1 < p < \infty$.

Theorem (M. Riesz)

The Hilbert transform defines a bounded linear map

$H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$

for all p satisfying $1 < p < \infty$.

Proof

First we show that if $1 < p < 3$ then there exist positive constants c_1 and c_2, depending on p, such that if $w = u + iv$ is a complex number satisfying $u \geq 0$, then

$$|v|^p \leq c_1 u^p - c_2 \mathrm{Re}(w^p)$$

(where we define $(re^{i\theta})^p = r^p e^{ip\theta}$ for all $p \in \mathbb{R}$ and for all θ satisfying $-\pi < \theta < \pi$).

It suffices to verify this inequality when $|w| = 1$, by homogeneity. Since $1 < p < 3$ there exists δ satisfying $0 < \delta < \pi/4$ such that

$$p \cdot \frac{\pi}{2} < \frac{3\pi}{2} - \delta ,$$

$$p\left(\frac{\pi}{2} - \delta\right) > \frac{\pi}{2} + \delta .$$

If $|w| = 1$ and

$$\tfrac{1}{2}\pi - \delta < |\arg w| < \tfrac{1}{2}$$

then

$$\tfrac{1}{2}\pi + \delta < \arg(w^p) < \tfrac{3}{2}\pi - \delta$$

and hence

$$|v|^p \leq -c_2 (\mathrm{Re}\ w^p)$$

where

$$c_2 = \frac{1}{\sin \delta}$$

since $|v|^p \leq 1$ and

$$-\mathrm{Re}\ w^p > \sin \delta \ .$$

If $|w| = 1$ and

$$-\tfrac{1}{2}\pi + \delta < \arg w < \tfrac{1}{2}\pi - \delta$$

then

$$\left| |v|^p + c_2 \mathrm{Re}\ (w^p) \right| \leq 1 + c_2$$

and

$$u^p \geq (\sin \delta)^p .$$

Thus

$$|v|^p + c_2 \mathrm{Re}\ (w^p) \leq c_1 u^p$$

where

$$c_1 = \frac{1 + c_2}{(\sin \delta)^p} .$$

This completes the proof that

$$|v|^p \leq c_1 u^p - c_2 \mathrm{Re}\ (w^p)$$

for all complex numbers $w = u + iv$ satisfying $u \geq 0$, where $1 < p < 3$.

Let $f \in C_0^\infty(\mathbb{R})$ be a non-negative function. Define an analytic function F on the upper half complex plane by

$$F(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{z - t}\, dt$$

and let

$$F(x + iy) = u(w, y) + iv(x, y)$$

for some harmonic functions $u : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ and $v : \mathbb{R}^2_+ \rightarrow \mathbb{R}$. If $z = x + iy$ then

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(t)}{(x - t)^2 + y^2},$$

$$v(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - t)\, f(t)}{(x - t)^2 + y^2}.$$

Note that $u(x, y) \geq 0$ for all x and for all $y > 0$.

Applying the inequality derived above we see that

$$\int_{-\infty}^{\infty} |v(x, y)|^p \, dx \leq c_1 \int_{-\infty}^{\infty} u(x, y)^p \, dx - c_2 \mathrm{Re}\left[\int_{-\infty}^{\infty} F(x + iy)^p \, dx\right]$$

when $y > 0$ and $1 < p < 3$. Using the fact that f has compact support, we see that $F(z) = O(|z|)$ and hence $F(z)^p = O(|z|^p)$ as $z \rightarrow \infty$ in the upper half plane. Thus the integrals of $F(z)^p$ around the semicircles $z = Re^{i\theta} + iy$, where $0 \leq \theta \leq \pi$, converge to zero as $R \rightarrow +\infty$. Thus

$$\int_{-\infty}^{\infty} F(x + iy)^p \, dx = 0$$

for all $y > 0$ by Cauchy's theorem, since $F(z)^p$ has no poles in the upper half plane. It follows that

$$\int_{-\infty}^{\infty} |v(x, y)|^p \, dx \leq c_1 \int_{-\infty}^{\infty} u(x, y)^p \, dx.$$

Now $u(x, y)$ is equal to the value at x of the convolution $k_y * f$ of k_y and f, where

$$k_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$

But

$$\int_{-\infty}^{\infty} |k_y(x)| \; dx = \int_{-\infty}^{\infty} k_y(x) \; dx = 1$$

hence

$$\int_{-\infty}^{\infty} |u(x, y)|^p \, dx \leq \int_{-\infty}^{\infty} |f(x)|^p \, dx$$

for all $y > 0$, by Young's theorem on convolutions. Hence

$$\int_{-\infty}^{\infty} | v (x, y) |^p \, dx \leq c_1 \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

Now

$$Hf(x) = \frac{1}{\pi}\left(\int_{|x-t|<1} \frac{f(t) - f(x)}{x - t} \, dt + \int_{|x-t| \geq 1} \frac{f(t)}{x - t} \, dt \right)$$

$$= \frac{1}{\pi} \lim_{y \to 0+} \int_{-\infty}^{\infty} \frac{(x - t) \, f(t)}{(x - t)^2 + y^2} \, dt$$

$$= \lim_{y \to 0+} v(x, y).$$

By Fatou's lemma

$$\int_{-\infty}^{\infty} | Hf(x) |^p \, dx = \int_{-\infty}^{\infty} \lim_{y \to 0+} |v(x, y)|^p \, dx$$

$$\leq \liminf_{y \to 0+} \int_{-\infty}^{\infty} |v(x, y)|^p \, dx$$

$$\leq c_1 \int_{-\infty}^{\infty} |f(x)|^p \, dx$$

for all p satisfying $1 < p < 3$ and for all non-negative $f \in C_o^{\infty}(\mathbb{R})$. To extend this result to general $f \in C_o^{\infty}(\mathbb{R})$ we observe that for all $\varepsilon > 0$ there exist non-negative functions $f_1, f_2 \in C_o^{\infty}(\mathbb{R})$ such that $f = f_1 - f_2$ and

$$\| f_1 \|_p + \| f_2 \|_p \leq \| f \|_p + \varepsilon$$

where $\| f \|_p$ is the L^p norm of f. On applying the above result to f_1 and f_2 we see that

$$\| Hf \|_p \leq \| Hf_1 \|_p + \| Hf_2 \|_p$$
$$\leq C(\| f_1 \|_p + \| f_2 \|_p)$$
$$\leq C(\| f \|_p + \varepsilon)$$

where $C^p = c_1$. But $\varepsilon > 0$ is arbitrary, hence

$$\| Hf \|_p \leq C \| f \|_p$$

for all $f \in C_0^\infty(\mathbb{R})$ and p satisfying $1 < p < 3$. Hence H is bounded on $L^p(\mathbb{R})$ when $1 < p < 3$. Since H is self-adjoint, H is bounded on $L^p(\mathbb{R})$ when $\frac{3}{2} < p \leq \infty$, by duality.

□

In /Calderon, A.P., 1966/ it is asserted that v(x, y) - Hf in the above proof is the convolution of f with an integrable function. However

$$v(x, y) - Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|t-x| > \varepsilon} \frac{y^2 f(t)\, dt}{(x - t)\,((x - t)^2 + y^2)}$$

and the function

$$\frac{y^2}{x^2(x^2 + y^2)}$$

is not an integrable function of x in a neighbourhood of zero. Fatou's lemma has been used in the above proof to overcome this difficulty.

The theorem may also be deduced as a corollary of the Marcinkiewicz interpolation theorem (see /Stein, E.M. and Weiss, G., 1972; pp.183-188/).

References to Appendix A

Calderon, A.P., "Singular integrals", Bull. Amer. Math. Soc., 72 (1966), pp.427-465.

Riesz, M., "Sur les fonctions conjugees", Math. Z., 27 (1927), pp.218-244.

Stein, E.M. and Weiss, G., Introduction to Fourier analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, New Jersey, 1972.