The cosmological constant and dimensional reduction

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ABSTRACT;-

A brief history of the cosmological constant is given and its role in present day theories is discussed along with an indication of why it is a problem in physics today.

A discussion of the papers by Visser and Squires on the use of the cosmological constant for dimensional reduction is given and in the case of Squires' paper, an exact treatment of the bound states problem is given in 3-dimensions and some headway is made in the more general N-dimensional case.

Some preliminary work on the nonlocallity problem is also presented along with a brief discussion on the concept of nonlocallity.

In conjunction with this work a computer program, written in the computer language REDUCE, was developed to work out curvature components and associated variables.

DIMENSIONAL REDUCTION AND THE COSMOLOGICAL CONSTANT

BY

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CONTENTS.

CHAPTER 1

THE COSMOLOGICAL CONSTANT

§1. THE ORIGINS OF THE COSMOLOGICAL CONSTANT

Today, as seventy years ago, the Einstein field equations

$$R_{ab}-Rg_{ab}/2=kT_{ab} \qquad (1)$$

play an important role in Physics. Not only in the more classical aspects, such as Cosmology, but also in some of the more recent developments such as Supergravity. It is in these equations that the Cosmological constant has its origins.

It is well known that these equations must be divergence free, so without altering the physical significance of these equations one can add a further divergence free term so the equations then become

$$R_{ab}-Rg_{ab}+\lambda g_{ab}=kT_{ab} \qquad (2)$$

The constant, λ , introduced here has been named the cosmological constant. The reason for the name becomes apparent when one considers the reason why Einstein included this term in the field equations, which was to modify the law of gravitation at large distances to be one of the form;

$$\underline{r}''=\lambda\underline{r} \qquad (3)$$

No such effect has been observed either in our solar system or in the structure of our galaxy so this constant is very small and is only important at the cluster level or larger, in other words on the cosmological scale.

§2. Classical Cosmology and the Cosmological constant

If we assume that on a universal scale everything is isotopic and homogenous then the most general metric one can have is the Robertson-Walker metric;

$$ds^2 = c^2dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right\} \qquad (4)$$

where R(t) is a time dependent scale factor,

$K(t) = \frac{k}{\{R(t)\}^2}$ is the curvature.

If we substitute this into the field equations, (2), and further assume that all the matter and radiation contained in T_{ab} is like a perfect fluid, we then obtain the Freidman equations;

$$\ddot{R} = -4\pi G\rho_0 \frac{R_0^3}{2R^2} - \frac{\lambda R}{3} \qquad (5i)$$

$$\dot{R}^2 = 8\pi G\rho_0 \frac{R_0^3}{3R} - kc^2 + \frac{\lambda R^2}{3} = G(R) \qquad (5ii)$$

Where G is the Gravitational Constant

R_0 is the Curvature at t=0

ρ_0 is the Density at t=0

c is the speed of light.

From these equations we see that near R=0, the cosmological term has no effect, so the behavior of the big bang is unaltered.

Obviously for different values of λ , we may have various models for the universe and so the cases $\lambda > 0$ and $\lambda < 0$ will be considered separately.

$\underline{\lambda < 0}$

If we look at the Freidmann equations we can see that R must be finite for R' to remain real, also there is a value of R, R_c say, such that $G(R_c)=0$ i.e. $R'=0$ when $R=R_c$. Equation (4) then shows that $R''<0$ when $R=R_c$ so the universe starts to contract at this point. We therefore have an oscillating model, this is true whatever the value of k.

$\underline{\lambda > 0}$

If $K<0$, $R'^2>0$ for all R, so we have a monotonically expanding universe, the only difference from those with $\lambda = 0$ being at large R. $R'^2 \sim R^2/3$ so

$$R \propto \exp\left\{\left(\frac{\lambda}{3}\right)^{1/2} t\right\} \qquad (6)$$

If $K=0, \rho_0 = 0, \lambda > 0$ we have the De Sitter model for which (7) holds for all t.If we let K=1 then there is a critical value of λ , λ_c, such that R=0 and R"=0 can be both satisfied simutaneously. From (4) R"=0 implies

$$R = R_0\left(\frac{4\pi G\rho_0}{\lambda}\right)^{1/3} = R_c \qquad (7)$$

and then equation (6) gives

$$0 = (4\pi G\rho_0)^{2/3} \lambda^{1/3} R_0^2 - kc^2 \qquad (8)$$

So

$$\lambda_c = \frac{(kc^2)^3}{R_0^6 (4\pi G\rho_0)^2} \qquad (9)$$

This means that there is the possibility of a static model of the universe, with $R=R_C$, $\lambda = \lambda_c$, for all time , t, provided

$$\lambda = 4\pi G\rho_c = \frac{Kc^2}{R_c^2} \qquad (10)$$

and since $\rho_c > 0$, K must be positive for this to happen. This is of course the Einstein static model, which was the first solution of the general relativity to be found that satisfied the cosmological principle.

By studying the function G(R), (4), as a function of K, we can see what other possible $\lambda > 0$, K=-1 models there are. Clearly G(R) tends to infinity both for R and R" tending to infinity, and reaches a minimum at R_C with $G(R_C) >$ or < 0 according to $\lambda >$ or $< \lambda_c$.

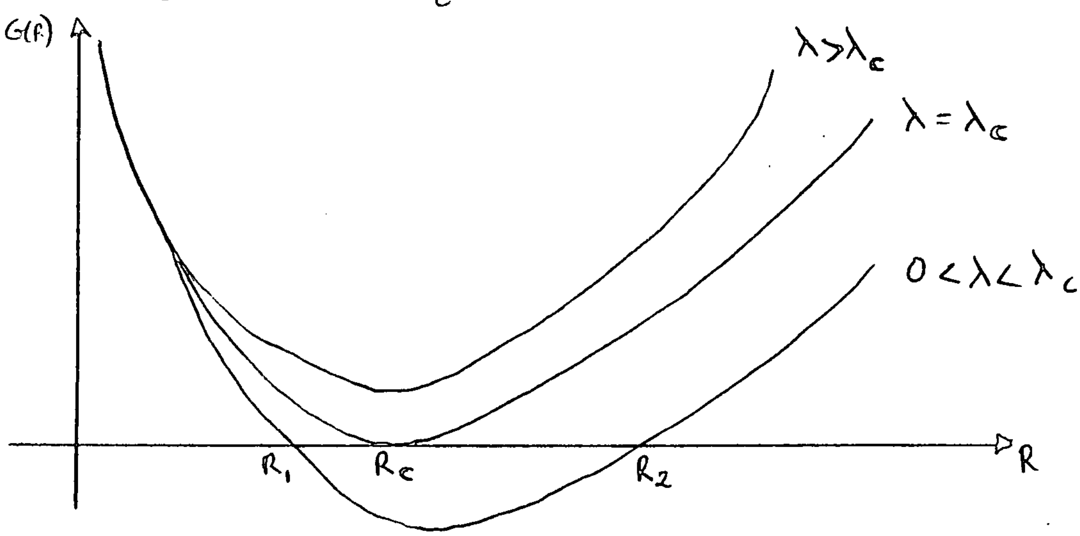

fig. 2 G(R) for >0, K=+1 Models.

$\underline{\lambda > \lambda_c}$

G(R)>0 for all R, so we have a monotonically expanding universe again.

$\lambda = \lambda_c$

Apart from the Einstein static model, there are two models that approach this asyptotically, corresponding to the two branches of G(R) (fig 2). One expands out gradualy from the Einstein state at t equals minus infinity and then turns into an exponential expansion. The other expands out from the usual big-bang and then tends asymptotically to the Einstein model as t tends to infinity. These are the Eddington-Lemaitre models shown below;-

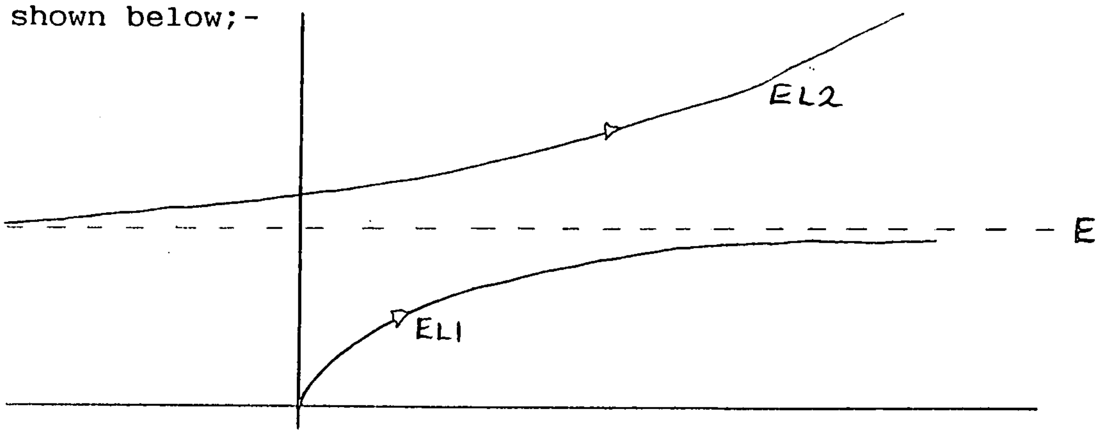

Fig 3. K=+1, $\lambda = \lambda_c$ models. E=Einstein static model;

EL1, EL2=Eddington-Lematre models.

If $\lambda = \lambda_c(1+e)$, $e \ll 1$, we have the Lemaitre models. For a long period of time R is close to R_c and the cosmological repulsion and gravitational attraction are almost in balence. Finally the repulsion wins and the expansion continues again.

$0 < \lambda < \lambda_c$

There are no solutions for $R_1 < R < R_2$ (fig 2). The solution with $R \leqslant R_1$ is an oscillating model. In the one with $R \geqslant R_2$ the universe bounces under the action of the cosmological repulsion as seen over;-

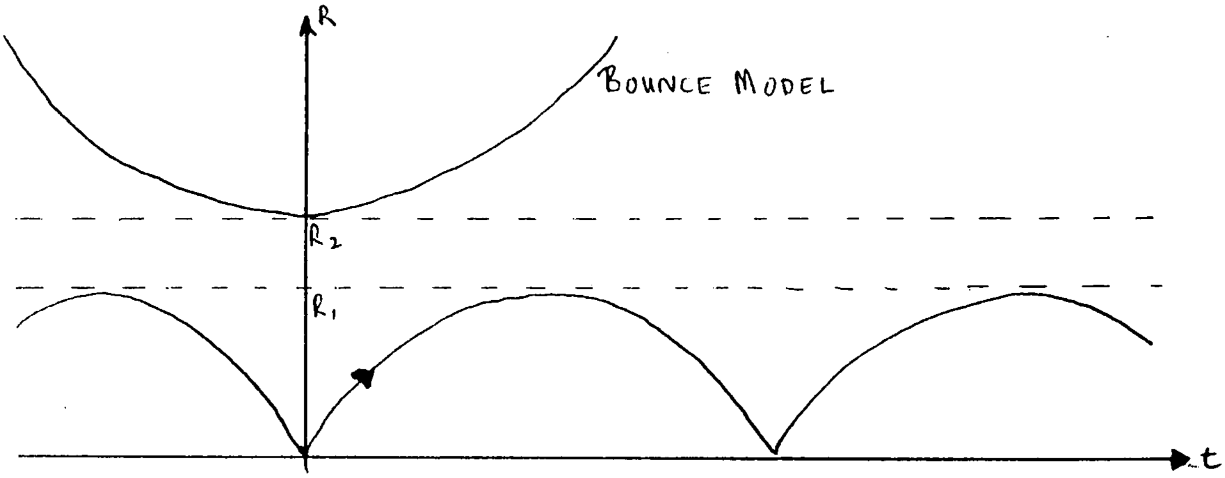

Fig. 4. $K=+1$, $0 < \lambda < \lambda_c$ models.

§3. The size of the cosmological constant.

If we define

$H(t) = \dfrac{R'(t)}{R(t)}$ is the Hubble parameter

$q(t) = \dfrac{-R(t)\,R''(t)}{\{R'(t)\}^2}$ is the deceleration parameter

$\sigma(t) = \dfrac{4\pi G\rho(t)}{3H^2(t)}$ is the mass density parameter.

then it only takes a little work to show

$$\frac{\lambda}{3} = H_0^2(\sigma_0 - q_0) \qquad (12)$$

The currently accepted values for these constants are

$$H_0 = 5 \times 10^{-9} \qquad (13i)$$

$$-1 < q < 2 \qquad (13ii)$$

$$0 \cdot 01 < \sigma < 0 \cdot 1 \qquad (13iii)$$

If we accept that the universe is expanding slower than what is was earlier in its history then $q>0$ and so, if we put the estimates in to (12) one obtains the upper limit for the cosmological constant of approximately $10^{-122}(t_{pl})^{-2}$, where t_{pl} is Plank time.

§4. The cosmological constant in present day theories.

The cosmological constant also plays an important role in present day theories,such as Supergravity or superstrings.

Both of these theories are built on a Quantum field theory approach in that all the states are found from variation on a lagrangian,which will include terms for the free particles, the mass associated with the particle and a term for any interactions of the particle i.e.

$$L_{tot}=L_{free}+L_{mass}+L_{int}$$

If we compare this lagrangian with the usual one used in general relativity then we see the cosmological constant comes into this lagrangian through the mass term. To illustrate this in a particular case let us look at the particular case of a supersymmetric unified theory, SU(5), for which the lagrangian is

$$L=-\sum_i |\partial^\mu \phi|^2 + i\sum_i \bar{\psi}_i \partial \psi^i - \sum_i \left|\frac{\partial W}{\partial \phi_i}\right| - \sum_{ij} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \bar{\Psi}_i \psi^j \qquad (14)$$

where in this case the mass term (cosmological term) is

$$\sum_i \left|\frac{\partial W}{\partial \phi_i}\right| \qquad (15)$$

and W is a super potential made up of SUSY multipets from the SU(5) group, and is described by

$$W = A\left[\frac{1}{3}\Sigma^x_y \Sigma^y_z \Sigma^z_x + \frac{m}{2}\Sigma^x_y \Sigma^y_x + \lambda H_x\left(\Sigma^x_y + 3m'\delta^x_y\right)\dot{H}^y \right. \qquad (16)$$
$$\left. + f_{ik}\,\varepsilon_{uvw} H^u_{xy} M^{uw}_i M^{xy}_k + g_{ij} H'_x M^{xy}_j M'_{ky}\right]$$

Under certain conditions, it is not hard to show that because of this cosmological term there is spontaneous breaking of this GUT into SU(4)*U(1) or SU(3)*SU(2) which corespond to the degenerate ground states of the vacuum in this theory. So the cosmological constant is connected to the expected energy density of the vacuum.

The cosmological constant comes into Superstrings in essentially the same way, as the states of the string are derived from a lagrangian in a similar fashion.

§5. The problem of the cosmological constant.

At first sight there may seem to be no problem with the fact the cosmological constant is very small, or zero. However if we look more closely we see that the cosmological constant is made up of contributions from the quantum fluctuations of the gravitational field and, also contributions from possible breaking of GUTS,SUSY and Salam-Weinberg symmetries all of which are large, in exponential order, but, all of which almost exactly cancel. This obviously is no accident and so it is the explination of this curious fact which is the problem. We should also consider that the constant comes into just about all Physics today as partly shown above, but also comes into particle Physics through the Higgs vacuum expectation values, fermion and gauge field condensates, and others, through the vacuum energy density, which is of course the cosmological constant.It is therefore a very serious problem indeed and one which at the present time is nowhere near coming to a satisfactory answer.

CHAPTER 2

Visser's model of dimensional reduction.

§1. Aims of the paper.

Kaluza-Klein type theories assume that the space-time is split into M_4*K where M_4 is a four dimensional manifold representing the observed universe and K is a compact manifold of internal coordinates. To illustrate this type of theory let us look at it in five dimensions. The metric, in this theory, breaks down into

$$g_{MN}=g_{ab} \quad M,N=1,2,3,4. \quad a,b=1,2,3,4. \tag{1.1}$$

$$g_{M5}=g_{5M} \quad M=1,2,3,4. \tag{1.2}$$

$$g_{55}=g_{55} \quad . \tag{1.3}$$

If we assume that the metric is periodic in x_5, so in this case K is a circle (this is sometimes called, assuming K to be compact), we can expand the metric

$$g_{MN}(\underline{x},y)=g_{MN}{}^{n}(\underline{x})e^{iny/\beta} \tag{2}$$

where the summation convention is used and ß is the radius of the circle in which g is periodic, and is considered to be small. The momentum in this case is given by n/β and the energy by n^2/β^2. It is assumed that the y direction is not observed and so ß is very small, consequently the energy is of the order of the Plank energy for all states with a nonzero n. It is not hard to see, therefore that all physics has n=0. This is sometimes called the cylinder condition and wrote

$$g_{ab',5}=0 \qquad (3)$$

If we take the metric to be of the form

$$g_{MN}=g_{ab}+K^2A_aA_b \quad M,N=1,2,3,4. \; a,b=1,2,3,4. \qquad (4.1)$$

$$g_{M5}=K\phi A_b \qquad M=1,2,3,4. \; b=1,2,3,4. \qquad (4.2)$$

$$g_{55}=\phi, \qquad (4.3)$$

and perturb this in the fields ϕ and A_a ,then it is not difficult to show that this automatically gives general relativity coupled with electromagnetism in 4 dimensions, as was shown by Kaluza and also independently by Klien in the earlier part of this century.

In Visser's paper (ref.1), he uses a Kaluza-Klein type of model in which he arranges for an absence of translational invariance in the internal coordinates. Then by using gravity, trap the particle near to the submanifold M_4.He then goes onto show that in the five dimensional case it is possible to obtain a zero cosmological constant in four dimensions from a nonzero one in five.

§2. The metric.

Visser assumes that the Kaluza-Klein type metric has the form

$$ds^2=-e^{2\phi(\beta)}dt^2+d\underline{x}.d\underline{x}+d\beta^2 \qquad (5.1)$$

where $x^a=(t,x,y,z,\beta) \qquad a=0,1,2,3,4. \qquad (5.2)$

This under certain conditions on $\phi(\beta)$ will produce a flat submanifold M_4.

Without looking at the field equations one can say something about any particle following geodesics:-integrating the geodesic equation one finds that

$$E=P^0e^{2\phi} \qquad (6.1)$$

and

$$p^i=P^i \qquad i=1,2,3,4. \qquad (6.2)$$

where E and p^i are constants of motion,and P^i are the momentum components of the mass in the five dimensional universe. The five momentum is defined in the same manner as the four momentum, namely

$$P^2=M^2-E^2.$$

Armed with this one can say any particle having a definite rest mass in the five dimensional universe will satisfy

$$\begin{aligned}(M_5)^2 &= P^a g_{ab} P^b \\ &= -e^{2\phi}(P^0)^2+\underline{p}.\underline{p}+(P^5)^2\end{aligned} \qquad (7)$$

Consequently the momentum component in the ß direction will satisfy

$$(p^5)^2=\{E^2e^{-2\phi}-(M_5)^2-p^2\}^{1/2} \qquad (8)$$

From this it is obvious that if

$$E<\{(M_5)^2+p^2\}^{1/2}\sup(e^{\phi}) \qquad (9)$$

any classical particle will be bounded in the ß direction by the potential ϕ.We do not see the extra dimensions so e^{ϕ} must rise very rapidly and as Visser says, it must be in a length scale of the order of the Plank mass, which is of course what we want for the extra dimensions not to be seen.

§3. Quantum Mechanical Trapping.

To see how this trapping works in quantum mechanical vein Visser assumes that there is a field,with the background meteric described above, which can be described by the Klein-Gordon equation;

$$\frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}\,g^{AB}\partial_B\right)\Phi = -M_5^2\Phi \qquad (10)$$

Substituting (1) into (10) yields the explicit form

$$\frac{\partial^2\Phi}{\partial\beta^2}+\frac{\partial\Phi}{\partial\beta}\frac{\partial\phi}{\partial\beta}+\left(e^{-2\phi}\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)=-M_5^2\Phi \qquad (11)$$

He now seeks a plane wave expansion of the form

$$\Phi = \exp\left(-i\left(\omega t-\underline{k}.\underline{x}\right)\right)e^{-\phi/2}\,\psi_1(\beta) \qquad (12)$$

which when substituted into (8) gives the Schrodinger like equation;

$$\frac{1}{2}\frac{\partial^2\psi_1}{\partial\beta^2} + \frac{1}{2}\left(\frac{\phi''}{2} + \frac{\phi'\phi'}{4} - \omega^2 e^{-2\phi}\right)\psi_1 = -\frac{1}{2}\left[M_S^2 + k^2\right]\psi_1 \tag{13}$$

or in terms of an eigenvalue problem

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\beta^2} + V(\omega,\beta)\right]\eta_n(\omega,\beta) = -\lambda_n(\omega,\beta)\eta_n(\omega,\beta) \tag{14}$$

where $V(w,\beta) = \frac{\phi''}{4} + \phi'\phi' - \frac{\omega^2 e^{-2\phi}}{2}$

The excitation spectrum is now found by solving

$$2\lambda_n(\omega) = M_S^2 + \underline{k}^2 \tag{15}$$

Visser notes that it is possible to put a bound on this equation.By a suitable normalization in t it is possible to get $\inf(\phi)=0$, and so $V(w,\beta)$ will be bounded by

$$V(w=0,\beta) \geq V(w,\beta) \geq V(w=0,\beta) - w^2/2 \tag{16}$$

and so the n^{th} exited state will be bounded by

$$\lambda_n(\omega) < \lambda_0(\omega=0) + \omega^2/2 \tag{17}$$

To get at the spectra in the lower dimensional world he defines

$$\omega = \lambda_n^{-1}(x) \tag{18}$$

to be the inverse of

$$x = \lambda_n(\omega) \tag{20}$$

and so

$$\omega(n,k) = \lambda_n^{-1}\left[\frac{M_S^2 + k^2}{2}\right] \tag{21}$$

Of course we would like a flat space to result and so we would like the spectrum to have O(1,3) invariance. This is not necessarily so but we would like it to show up in some approximation. Towards this end Visser examines the rest mass i.e. he sets k=0, and so

$$2\lambda_n(\Omega_n) = M_S^2 \tag{22}$$

He now uses first order perturbation techniques to evaluate $\lambda_n(\omega)$ near Ω_n and so

$$\lambda_n = \lambda(\Omega_n) + \frac{1}{2}C_n^{-2}(\omega^2 - \Omega_n^2)$$

(23)

with $c_n^{-2} = \langle \eta_n(\Omega_n,\beta) | e^{-2\phi(\beta)} | \eta_n(\Omega_n,\beta) \rangle$

Hence the spectrum may be obtained by solving

$$M_s^2 + c_n^{-2}(\omega^2 - \Omega_n^2) + \ldots\ldots = M_s^2 + k^2 \quad (24)$$

i.e. $\omega^2 = \Omega_n^2 + c_n^2 k^2$

Thus if we are in the region where first order perturbation is valid, which of course depends on the exact form of ϕ which will be discussed in the next section, the spectrum is Lorentz invariant and has a rest mass of Ω_n with c_n^{-2} being the expected speed of light for the n^{th} exited mode.

§4. The field equations.

To see the role of the cosmological constant in the trapping of particles, Visser considers the field equations where T at this time is unknown,

$$G_{ab} = \lambda g_{ab} + T_{ab} \quad (25)$$

By use of the reduce program (appendix 1) it was a simple matter to verify the nonzero components of the Einstein tensor are

$$G_{11} = G_{22} = G_{33} = e^{-\phi}(e^{\phi})'' \quad (26)$$

Putting (5.1) into (25) one easily sees that the nonzero components of the energy stress tensor are defined by

$$T^0{}_0 = T^4{}_4 \quad (27.1)$$

$$(e^{\phi})'' = (\lambda + T_{11})e^{\phi} \quad (27.2)$$

$$T_{11} = T_{22} = T_{33} \quad (27.3)$$

Therefore it is the pressure $\rho = T_{11} = T_{22} = T_{33}$, that produces the ß dependence of ϕ whereas the density must be independent of ß.

To construct the stress-energy tensor

$$T_{ab} = F_{ac}F_b{}^c - g_{ab}(F_{cd}F^{cd})/4 \quad (28)$$

Visser chooses the simple case of a constant electric field with components only in the ß direction. This of course is highly arbitrary and in the next chapter a more natural model will be considered.

With this type of field in mind Visser chooses

$$A_0=a(\beta) \tag{29.1}$$

$$A^1=A^2=A^3=A^4=0 \tag{29.2}$$

$$\Rightarrow \quad F_{04}=-F_{40}=a' \tag{29.3}$$

and

$$F_{ab}F^{ab}=-2e^{-2\phi}(a')^2 \tag{29.4}$$

By defining

$$E=e^{-\phi}a' \tag{30}$$

The components of the stress tensor turn out to be

$$T_0{}^0=a'(-e^{2\phi}a')+E^2/2 \tag{31.1}$$

$$=-E^2/2$$

$$T_4{}^4=-E^2/2 \tag{31.2}$$

$$T_1{}^1=T_2{}^2=T_3{}^3=E^2/2 \tag{31.3}$$

If we put (31.1-3) into (27.1-3) we obtain the equations

$$\lambda=E^2/2 \tag{32.1}$$

$$(e^{\phi})''=E^2(e^{\phi}) \tag{32.2}$$

Equations (30) and (31.2) are easily solved for ϕ and a to give

$$e^{\phi}=\cosh(E\beta) \tag{33.1}$$

$$a=\sinh(E\beta) \tag{33.2}$$

Vissers interpretation of this is a nonzero cosmological constant in five dimensions has in some way coupled with a constant electric field $E=(2\lambda)^{1/2}$ to produce a flat submanifold in which the effective cosmological constant is zero.The potential $\phi=\ln(\cosh(E\beta))$ traps the particles in the ß (x^4) direction and this trapping is governed by the Schrodinger like equation which on substitution of (33.1) into (13), is;

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\beta^2}+\frac{E^2}{8}-\left(\frac{w^2}{2}-\frac{E^2}{8}\right)\operatorname{sech}^2[E\phi]\right]\eta_n(w,\beta)=-\lambda_n\eta_n(w,\beta) \qquad (34)$$

The potential in (34) is the well known Rosen-Morse, for which the exact eigenvalues are

$$\lambda_n=\frac{1}{2}\left[\left(w-(n+1/2)E\right)^2-E^2/4\right] \qquad (35)$$

We may now solve for the exact spectrum, to give

$$w(n,\beta)=(n+1/2)E\pm\left[\left(\frac{E}{2}\right)^2+M_5^2+k^2\right]^{1/2} \qquad (36)$$

To go any further with the exact treatement Visser defines now the four dimensional mass by

$$m_4{}^2=M_5{}^2+(E/2)^2 \qquad (37)$$

One will note that this needs tachyonic masses in the higher dimension to give low ,and hence observed,masses in the lower dimensional world.

Putting (37) into (36) gives the spectrum

$$w(n,k)=(n+1/2)E-(m_4{}^2+K^2)^{1/2} \qquad (38)$$

There are two problems with equations (37) and (38) , in (37) it needs tachyonic masses to make the lower dimensional space low energy, but perhaps more importantly in (38) the momentum independent shift is not observed in nature where

$$w^2=m_4{}^2+k^2$$

and so Vissers model is unrealistic. Squires model (ref.2), discussed in the next chapter, has been constructed, hopefully at least, to overcome these problems.

§5. Conclusion.

If one takes Visser's paper at face value then indeed particles of the higher dimensional universe are gravitationally trapped on a four dimensional submanifold, whose energy spectrum can be made into low energy and which also possesses, generally at least, an approximate O(1,3) invarience.

Vissers model also gives a technique for cancelling a five dimensional cosmological constant in conjunction with an electric field.THe model is however unrealistic in three main ways;

a) the choice of the electro-magnetic field is contrived as mathematical convenience and no real justification was given.

b) it need a tachyonic mass in the higher dimensional universe to give low energy physics.

c)the momentum independent shift in (38) is not observed in nature.

CHAPTER 3.

Dimensional reduction and the Squires model.

§1. Basic aims.

As was stated in the previous chapter, Vissers model for dimensional reduction had several flaws in it which made it unrealistic.

In the paper by Squires, (ref.2), he aims for a similar process of dimensional reduction by a large cosmological constant, but which doesn't have the deficiencies of Vissers model. Namely there is no need for the introduction of arbitrary electric fields and the mass spectrum conforms to what we see in nature.

§2.The empty space field equations.

Towards this end Squires starts from the empty space field equations

$$R_{ab}=-\lambda g_{ab} \qquad (1)$$

Where the cosmological constant is positive so, classically at least, we have a monotonically expanding universe (see chapter 1, §2). This automatically avoids the introduction of arbitrary fields, as was done by Visser.

At the outset it was decided to work in three dimensions so as to gain an understanding of the gravitational trapping and then extend to a higher dimensional case.

The most general metric in three dimensions can be written as

$$ds^2=f(t,x,\beta)dt^2+g(t,x,\beta)dx^2+h(t,x,\beta)d\beta^2 \qquad (2)$$

since all other three dimensional metrics can be transformed into this metric (see Petrov).

Now, to solve the field equations for f,g and h without any simplifications would be very complex, so Squires makes the simplification that f,g,h all only depend on the variable ß. Also for a realistic solution he asserts that the space must be flat as $\lambda \to 0$. So the metric which Squires is seeking is of the form

$$ds^2=u^2(\beta)dt^2-v^2(\beta)dx^2-d\beta^2 \tag{3}$$

Substituting (3) into (1) yields the equations

$$\frac{u''}{u}+\frac{v'u'}{vu}=\lambda \tag{4.1}$$

$$\frac{v''}{v}+\frac{u'v'}{uv}=\lambda \tag{4.2}$$

$$\frac{u''}{u}+\frac{v''}{v}=\lambda \tag{4.3}$$

It is not hard to see that with a little manipulation one can obtain the solution

$$u=A\exp\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]+B\exp\left[-\left(\frac{\lambda}{2}\right)^{1/2}\beta\right] \tag{5.1}$$

$$v=C\exp\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]+D\exp\left[-\left(\frac{\lambda}{2}\right)^{1/2}\beta\right] \tag{5.2}$$

along with the condition

$$AD=-BC \tag{5.3}$$

Rescaling of t along with a shift of origin of ß provides the unique solution to the field equations

$$u=\cosh((\lambda/2)^{1/2}\beta) \tag{6.1}$$

$$v=\sinh((\lambda/2)^{1/2}\beta) \tag{6.2}$$

If however, one of A,B,C or D is zero then the above solution does not hold and so the following has to be used

$$u=A\exp\left[\pm\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]$$

$$v=B\exp\left[\mp\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]$$

Squires decides throughout his work to use the more general case as this is more unique.

§3. Classically bound particles.

Before continuing the discussion of Squires model it would be interesting to see if classically, particles moving in this space are bound. Of course here I will be assuming that there is negligible back effect on the metric through the field equations, of any masses moving in the space.

To examine this we must end up with some equation which is similar to the energy equation

$$E=T+V \tag{7}$$

If we adopt a similar approach to that of Visser we first note that all particles follow time-like geodesics (except light particles which for the moment we shall ignore) ,so the geodesic equation reads

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{8}$$

From (2) it is not hard to see that the equation in t and x are

$$\frac{d^2t}{ds^2} + \lambda\frac{\dot{u}}{u}\frac{dt}{ds}\frac{d\beta}{ds} = 0 \tag{9.1}$$

$$\frac{dx}{ds} + \lambda\frac{\dot{v}}{v}\frac{dx}{ds}\frac{d\beta}{ds} = 0 \tag{9.2}$$

These of course may be integrated to give

$$u^2\frac{dt}{ds} = E \quad , \quad E \text{ is a constant} \tag{10.1}$$

and

$$v^2\frac{dx}{ds} = L \quad , \quad L \text{ is a constant} \tag{10.2}$$

(3) now yields

$$1 = u^2\left(\frac{dt}{ds}\right)^2 - v^2\left(\frac{dx}{ds}\right)^2 - \left(\frac{d\beta}{ds}\right)^2 \tag{11}$$

We can now substitute equation (10.1) and (10.2) into (11) to give

$$1 = \frac{E^2}{\cosh^2[(\frac{\lambda}{2})^{1/2}\beta]} - \frac{L^2}{\sinh^2[(\frac{\lambda}{2})^{1/2}\beta]} - \left(\frac{d\beta}{ds}\right)^2 \qquad (12)$$

If we now let $z=\sinh((\lambda/2)^{1/2}\ )$ (13)

then it is a simple matter to show

$$\frac{1}{2\lambda}\left(\frac{d\beta}{dz}\right)^2 + \frac{1}{4}\left\{L^2\left(\frac{1}{z^2}+1\right) + (1+z^2)\right\} = E^2/4 \qquad (14)$$

which is obviously the energy equation (7) for a particle of mass λ^{-1} moving under the action of the repulsive potential

$$V(z) = \frac{1}{4}\left\{L^2\left(\frac{1}{z^2}+1\right) + (1+z^2)\right\} \qquad (15)$$

One should note that although the potential is repulsive it is strongly so at the origin and also at $+\infty$, so there is still the possibility of trapping so this will be looked at further.

Consider

$$V(z)= \frac{1}{4}\left\{L^2\left(\frac{1}{z^2}+1\right) + (1+z^2)\right\} \qquad (16)$$

so

$$V'(z)= \frac{1}{4}\left(-\frac{2L^2}{z^3} + 2z\right) \qquad (17)$$

for turning values

$$V'(z)=0$$

hence

$$z^4-L^2=0$$

i.e. $z=+L^{1/2}$ since $z\geqslant 0$

Also

$$\frac{d^2V}{dz^2} = \frac{3L^2}{2z^4} > 0 \quad \text{at turning value} \qquad (18)$$

Hence the potential has a minimum at $z=L^{1/2}$ (see fig 1.) and by Occums razor the particle will be trapped at this minimum

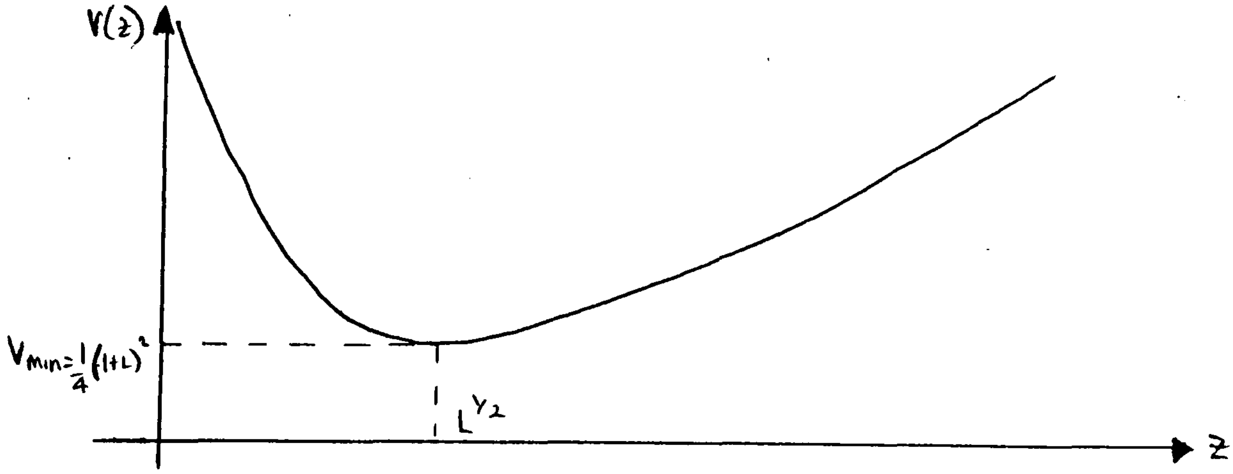

fig 1. A sketch of the potential (16).

Therefore classically at least particles of mass λ^{-1} are bound in the β direction and so the effective number of dimensions are reduced from 3 to 2.

For the case of light particles it turns out that they are not bounded, however we can neglect this case as we expect the particles, in the lower dimension to have some mass.

§4. Quantum field theory and bound states.

To return to Squires paper, he, to set up quantum mechanical states, assumes that there exists a field with the background metric defined by (2) and (5i-ii), which can be described by the Klien-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}\,g^{AB}\partial_B\right)\Phi = -M_H^2\,\Phi \qquad (19)$$

This of course suffers from the same problem as the classical approach in that the field Φ may have a back effect on the metric through the field equations, but this will be discussed later in the chapter.

If one puts the metric into (19) then it is easy to arrive at

$$\frac{1}{u^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial\beta^2} - \left(\frac{1}{u}\frac{du}{d\beta} + \frac{1}{v}\frac{dv}{d\beta}\right)\frac{\partial\Phi}{\partial\beta} = -M_H^2\,\Phi \qquad (20)$$

Squires now seeks a plane wave expansion in the t and x directions and so

$$\Phi(t, x, \beta) = \exp(-i\omega t + ikx)\,\chi(\beta) \tag{21}$$

where $\chi(\beta)$ satisfies the Schrodinger like equation

$$\left(\frac{-\omega^2}{u^2} + \frac{k^2}{v^2}\right)\chi - \frac{d^2\chi}{d\beta^2} - \left(\frac{1}{u}\frac{du}{d\beta} + \frac{1}{v}\frac{dv}{d\beta}\right)\frac{d\chi}{d\beta} = -M_H^2\chi \tag{22}$$

or on substitution (5i-ii)

$$\left(\frac{-\omega^2}{\cosh^2[(\frac{\lambda}{2})^{1/2}\beta]} + \frac{k^2}{\sinh^2[(\frac{\lambda}{2})^{1/2}\beta]}\right)\chi - \frac{d^2\chi}{d\beta^2} \tag{23}$$

$$-\left[\tanh[(\tfrac{\lambda}{2})^{1/2}\beta] + \coth[(\tfrac{\lambda}{2})^{1/2}\beta]\right]\frac{d\chi}{d\beta} = -M_H^2\chi$$

To show that particles, in the β direction are bounded, and to get the mass spectrum in the lower dimensional, observed universe, one not only has to find the mass spectrum but also one has to show that M_H^2 is discrete.

To show the mass spectrum, of M_H^2, is discrete Squires examines two things, the asymptotic behavior of (23) for large $|\beta|$ and the invariant measure.

For large $|\beta|$ the solution to (23) behaves like

$$\chi(\beta) \sim \exp\left[-(\tfrac{\lambda}{2})^{1/2} \pm (\tfrac{\lambda}{2} + M_H^2)^{1/2}\right]|\beta| \tag{24}$$

The invariant measure is

$$\sqrt{g}\, dt\, dx\, d\beta \equiv u v\, dt\, dx\, d\beta \tag{25}$$

so he states that for normalisable states only one of (24) is allowable and so the spectrum of $M^2{}_H$ is discrete, consequently we are dealing with bound states.

To get the mass spectrum in the lower dimension he makes the further assumption of u and v being essentially the same. This is true, as he points out, except for the region $|\beta| \leq \lambda^{-1}$.

He then substitutes

$$\psi = u\chi \tag{26}$$

to find the Schrodinger like equation

$$-\frac{d^2\psi}{d\beta^2} - \frac{(\omega^2-k^2)}{u^2}\psi = -(M_H^2+\lambda/2)\psi \tag{27}$$

So the existence of a solution for a given M^2_H and λ will provide a condition on the square of the observed particle mass, where m^2_{obs}, where

$$m^2_{obs}=w^2-k^2 \tag{28}$$

Since we would wish the observed particle mass to be small, it is obvious that

$$M^2_H \sim\lambda/2 \tag{29}$$

To now get a relation between M^2_H and m^2_{obs} he assumes that the kinetic and the potential energies in (27) are approximately equal in magnitude, and so obtains,

$$m^2_{obs} \; M^2_H+\lambda/2 \tag{30}$$

which according to (29) is indeed small.

He justifies the assertion of u and v being the same by saying, the wave function ψ will only spread a distance of the order m^{-1}_{obs} in ß because of (30). He does point out however, around ß=0 the solution needs more care, but the error is of the order m^2_{obs}/λ and so he believes the assertion justified.

To round off he says that the ß direction is not observed because all states have the same ψ(ß) factor and so the only way in which an exited state, ψ'(ß), can happen would be for w^2-k^2 to be large, say $O(\lambda)$, so that $-(M^2+\lambda/2)$ now becomes that eigenvalue of the deeper potential in (27)

§5. Conclusion.

Although this paper produces some good results there are a lot of assumptions in it, and so may be prone to error. In the next chapter I shall present an exact treatment of this problem in three dimensions and will make some headway in the general case.

Chapter 4.

An exact treatment of the Squires model.

§1. Introduction.

As stated in the last chapter, Squires solution to the model although good, from an approximation point of view does however contain some dubious steps.In this chapter the problem is attacked analytically by the use of special functions and indeed it comes to light that there are bound states, in the three dimensional case at least,but still more work will have to be done on this as the particles in the lower dimensional,observed, space have a mass which puts them squarely in the unobserved region.

§2. Topology of the three dimensional space.

Before we start to seek for the bound states of particles, it would be of benifit to see what sort of space these particles move through close to the origin, which is where we expect them to be bounded.

The line element in three dimensions is

$$ds^2=\cosh^2\{(\lambda/2)^{1/2}\beta\}dt^2-\sinh^2\{(\lambda/2)^{1/2}\beta\}dx^2-d\beta^2 \quad (1)$$

which if β is close to zero, reduces to the form

$$ds^2=dt^2-\beta^2 d\{(\lambda/2)^{1/2}x\}^2-d\beta^2 \quad (2)$$

If we now look at just the x and β part of this, namely

$$ds'^2=\beta^2dx'^2+d\beta^2 \quad (3)$$

we can see that this indicates the natural topology of the space is, x' and β form polar coordinates, where x' is equivalent to   and β to r, as shown on the top of the next page

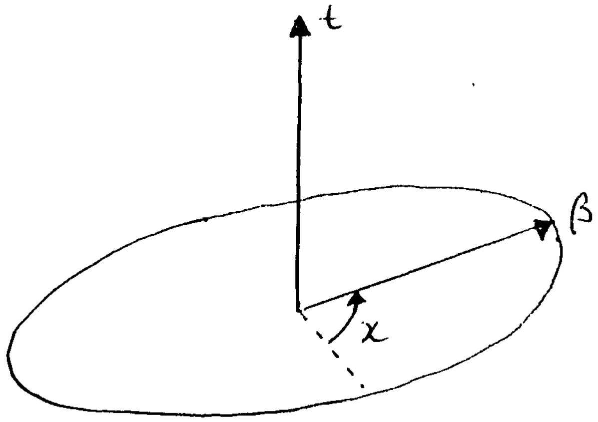

fig 1;-Topology of the space.

Of course the other possibility is that the topology forms a corkscrew type space but this seems unnatural and so only the more natural topology described on the previous page will be considered.

§3. Exact solutions in three dimensions

If we take equation (23) from the chapter 3 and make the substitution

$$z=\cosh\{(\lambda/2)^{1/2}\beta\} \qquad (4)$$

with

$$\frac{d\chi}{d\beta}=2\left(\frac{\lambda}{2}\right)^{1/2}\sinh\left\{\left(\frac{\lambda}{2}\right)^{1/2}\beta\right\}\cosh\left\{\left(\frac{\lambda}{2}\right)^{1/2}\beta\right\} \qquad (5)$$

and

$$\frac{d^2\chi}{d\beta^2}=-\lambda\left\{\cosh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]+\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]\right\}\frac{d\chi}{d\beta}+2\lambda\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]\cosh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\beta\right]\frac{d^2\chi}{d\beta^2}, \qquad (6)$$

we end up with the differential equation

$$z(1-z)\frac{d^2\chi}{dz}+(2z-1)\frac{d\chi}{dz}+\left(\frac{\omega^2}{2\lambda z}+\frac{k^2}{2\lambda(z-1)}\right)\chi=\frac{M_n^2}{2\lambda}\chi \qquad (7)$$

At first sight it does'nt seem that we have made much progress, but if we compare this with the equation

$$z(1-z)\chi''+\{(c-2\alpha)-(a+b+1)z\}\chi' \qquad (8)$$

$$+\left\{\frac{\alpha(\alpha-c+1)}{z}+\frac{\beta(\beta-a-b-c)}{1-z}-\left[(\alpha+\beta)(\alpha+\beta-a-b)-ab\right]\right\}\chi=0$$

which may be transformed, by the use of

$$\chi = z^{\alpha}(1-z)^{\beta} Y \tag{9}$$

into the well known Gauss's differential equation with hypergeometric solutions (ref.3)

$$z(1-z)Y'' + \{c - (a+b+1)z\}Y' - abY = 0 \tag{10}$$

Comparing equations (7) and (8) we end up with the set of relations

$$c - 2\alpha = 1 \tag{11i}$$
$$a + b + 1 - 2\alpha - 2\beta = 2 \tag{11ii}$$
$$\alpha(\alpha - c + 1) = -\omega^2/2\lambda \tag{11iii}$$
$$\beta(\beta - a - b - c) = -k^2/2\lambda \tag{11iv}$$
$$(\alpha+\beta)(\alpha+\beta-a-b) + ab = -M_H^2/2\lambda \tag{11v}$$

So setting aside the problem of bound states for the moment we should be able to find a solution to the differential eqation by just solving these five equations.

(11i) $\Rightarrow$ $c = 1 + 2\alpha$

$\therefore$(11iii) $\Rightarrow$ $\alpha(\alpha - 1 - 2\alpha + 1) = -\omega^2/2\lambda$

$$\Rightarrow \quad \alpha^2 = \omega^2/2\lambda \tag{12}$$

(11i)+(11ii) $\Rightarrow$ $a + b + c = 2 + 4\alpha + 2\beta$

$\therefore$(11iv) $\Rightarrow$ $\beta(\beta - 2 - 4\alpha - 2\beta) = -k^2/2\lambda$

$\Rightarrow$ $\beta^2 + (2+4\alpha)\beta - k^2/2\lambda = 0$

hence

$$\beta = \tfrac{1}{2}\left[-(2+4\alpha) \pm \left\{(2+4\alpha)^2 + 2k^2/\lambda\right\}^{1/2}\right] \tag{13}$$

Although it does'nt matter about the choice of sign for transforming (7) into (10) there may be other criteria and for this reason, the sign shall be left arbitary.

(11ii) $$a = 1 + 2\alpha + 2\beta - b \tag{14}$$

$$\therefore (\alpha+\beta)(\alpha+\beta-1-2\alpha-2\beta+b-b) + (1+2\alpha+2\beta-b)b = -M_H^2/2\lambda$$

for convenience set

$$\varepsilon = \alpha + \beta \tag{15}$$

so $$-\varepsilon(1+\varepsilon) + b(1+2\varepsilon) - b^2 = -m_H^2/2\lambda$$

$$\Rightarrow \quad b = \frac{1}{2}\left\{(1+2\varepsilon) \pm \left[(1+2\varepsilon)^2 - 4\varepsilon(\varepsilon+1) + \frac{2m_H^2}{\lambda}\right]^{1/2}\right\}$$

$$\Rightarrow \quad b = \frac{1}{2}\left\{(1+2\varepsilon) \pm \left[1 + 2m_H^2/\lambda\right]^{1/2}\right\} \tag{16}$$

Putting (16) into (14) yields

$$a = \frac{1}{2}\left\{(1+2\varepsilon) \mp \left[1 + 2m_H^2/\lambda\right]^{1/2}\right\} \tag{17}$$

The choice of sign for a and b does'nt matter since there are relationships between the two possibilities (ref.3). So without any loss of generality I shall assume that a has the possitive sign.

Hence the solution too (11i-11v) is

$$a = \frac{1}{2}\left\{(1+2\varepsilon) + \left[1 + \frac{2m_H^2}{\lambda}\right]^{1/2}\right\} \tag{18i}$$

$$b = \frac{1}{2}\left\{(1+2\varepsilon) - \left[1 + \frac{2m_H^2}{\lambda}\right]^{1/2}\right\} \tag{18ii}$$

$$c = 1 + 2\alpha \tag{18iii}$$

$$\varepsilon = \alpha + \beta \tag{18iv}$$

with

$$\alpha^2 = \omega^2/2\lambda \tag{18v}$$

and

$$\beta = -(1+2\alpha) \pm \left[(1+2\alpha)^2 - 2k^2/\lambda\right]^{1/2} \tag{18vi}$$

So the complete solution to (7) is

$$\chi(z) = z^{\alpha}(1-z)^{\beta}\left\{A F(a,b,c;z) + B F(a+1-c, b+1-c, 2-c; z)\right\} \tag{19}$$

We should also note that because of the topology, x close to the origin is essentialy an angle and, since we do not want a many valued wave function, the only values k can take are integer.We can therefore, without loosing validity, take k=0, and so look for a ground state to the masses.This simplifies equations (18i-18vi) too

$$a = \frac{1}{2}\left\{(1+2\varepsilon) + \left[1 + 2m_H^2/\lambda\right]^{1/2}\right\} \tag{20i}$$

$$b = \frac{1}{2}\left\{(1+2\varepsilon) - \left[1 + 2m_H^2/\lambda\right]^{1/2}\right\} \tag{20ii}$$

$$c = 1 \tag{20iii}$$

$$\Sigma = \alpha + \beta \tag{20iv}$$

with

$$\alpha = -\omega/\sqrt{2\lambda} \quad \text{(Chosen to be -ve for consistency)} \tag{20v}$$

and

$$\beta = -(1+2\alpha) \pm (1+2\alpha) \tag{20vi}$$

So, let us now examine in detail the bound states, if any, of the differential equation.We should note before continuing that the solution (19) is given as an expansion about the origin and if we examine in more detail the variable we have, we will note that for the boundary conditions we have, namely χ finite at the origin and at infinity, then in the variable z this would equate to χ finite at z=1 and z finite at infinity so the expansion we require is one around z=1 and not around z=0. We should also note that c=1 and so the second solution is a degenerate case, and for consistency when both k and w are zero we must take the plus sign in β.

If we look at Abramovitz and Stegun (ref.3) we see that the two unique solutions are

$$w_1 = F(a, b, 1; 1-z) \tag{21i}$$

and

$$w_2 = F(a,b,1;1-z)\ln(1-z) + \sum_n \frac{(a)_n (b)_n z^n}{(c)_n} * \left\{\psi(a+n) + \psi(b+n) - \psi(a) - \psi(b) - 2\psi(n+1) - \psi(1)\right\} \tag{21ii}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)} \quad (b)_n = \frac{\Gamma(b+n)}{\Gamma(n)} \quad (c)_n = \frac{\Gamma(c+n)}{\Gamma(n)} \tag{21iii}$$

and

$$\psi(z) = \frac{d}{dz}\Gamma(z) \tag{21iv}$$

which is usually called the diagamma function (ref.3).

Hence the complete solution is

$$\chi = z^{\alpha}(Aw_1 + Bw_2) \tag{22}$$

If we now impose the boundary condition at z=1, namely χ finite there, we can see that since the second solution w_2 contains logarithms and so B=0. We also want χ to be finite at infinity, so if we use (15.3.7) of Abramovitz & Stegun, it is obvious that for large z

$$F(a,b,c;z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} \tag{23}$$

If we now look at the powers of z more closely, and remembering (22) it is not hard to see

$$\chi \sim \frac{\Gamma(c)\Gamma(b-a)}{\{\Gamma(b)\}^2}(1+z)^{-a+\alpha} + \frac{\Gamma(c)\Gamma(a-b)}{\{\Gamma(a)\}^2}(1+z)^{-b+\alpha} \tag{24}$$

Since χ must be finite at infinity we require the second term of this to vanish, and to do so requires a to be a negative integer which is because of the properties of the Gamma function.

So we have our bound state solution, which is

$$\chi = Az^{\alpha}F(-n,-(1+n+2\alpha),1;1-z) \tag{25}$$

If we wish to see what the mass spectrum is like, all we have to do is rearrange equation (20i) to obtain

$$M_H^2 = \frac{\lambda}{2}\left\{(1+2\alpha+2n)^2-1\right\} \tag{26}$$

If we examine the ground state of the mass spectrum then it is easy to see the energy spectrum in the lower dimension is

$$\omega^2 = 2\lambda\left\{\frac{1}{2}+n+\left(1+\frac{2M_H^2}{\lambda}\right)^{1/2}\right\}^2$$

The only problem with this is that, for any M_H the particle mass w >2λ and so they are once again in the hidden sector of the mass spectrum .

To sum up, it is possible in three dimensions to obtain bound states in two of the dimensions and so get reduction from three to one dimension, but in this case it does not seem possible to get observed particles in the lower dimension but perhaps this problem will be resolved in N-dimensions.

§4. Extension to higher dimensions.

In order to extend this discussion to higher dimensions we first put the same two conditions on the metric, i.e. we use orthogonal coordinates in which g_{ab} is diagonal and we restrict the elementary g_{ab} to be functions of one variable only.

So the metric that Squires uses is

$$ds^2=u^2(\beta)dt^2-v_i^2(\beta)(dx^i)^2-d\beta^2 \tag{27}$$

with the solution (ref.4)

$$u = \sinh^{1/n+p}(\alpha\beta)\cosh^{1/n-p}(\alpha\beta) \tag{28i}$$

$$v_i = \sinh^{1/n+q_i}(\alpha\beta)\cosh^{1/n-q_i}(\alpha\beta) \tag{28ii}$$

where

$$\alpha = \left(\frac{\lambda N}{4}\right)^{1/2} \tag{28iii}$$

and

$$p+\sum_i q_i = 0 \tag{28iv}$$

$$p^2+\sum_i q_i^2 = 1-1/N \tag{28v}$$

Squires in his paper only states that these types of metric defined by (28i-v) produce confinement so I will now give a partial treatment of this problem.

We note that the metric in (27) is not unique since there are many solutions to (28i-28v), hence we impose now the additional condition that the space is flat when $\lambda=0$, i.e. $R_{abcd}\to 0$ when $\lambda\to 0$.

By use of the reduce program (appendix 1) it was found that a condition on the curvature tensor to be zero was

$$\ddot{V}_\alpha V_\alpha \to 0 \quad \text{as } \lambda\to 0 \;,\; \alpha = 1,2,3 \ldots N \tag{29}$$

where this is not summed over.

If we let $p=q_0$ then the functions of the metric may be written

$$V_i \doteq \cosh^{1/N+q_i}(\alpha\beta)\sinh^{1/N-q_i}(\alpha\beta \tag{30}$$

so

$$\dot{V}_i = \alpha\left(\frac{1}{N}+q_i\right)\cosh^{1/N+q_i-1}(\alpha\beta)\sinh^{1/N-q_i+1}(\alpha\beta) + \alpha\left(\frac{1}{N}-q_i\right)\cosh^{1/N+q_i+1}(\alpha\beta)\sinh^{1/N-q_i-1}(\alpha\beta \tag{31i}$$

$$\therefore \ddot{V}_i = \alpha^2\cosh^{1/N+q_i-1}(\alpha\beta)\sinh^{1/N-q_i-1}(\alpha\beta)\left[\left(\frac{1}{N}+q_i\right)\cosh(\alpha\beta) + \left(\frac{1}{N}-q_i\right)\sinh(\alpha\beta)\right]$$
$$+ \alpha^2\cosh^{1/N+q_i-2}(\alpha\beta)\sinh^{1/N-q_i-2}(\alpha\beta)\left[\left(\frac{1}{N}+q_i\right)\sinh(\alpha\beta) + \left(\frac{1}{N}-q_i\right)\cosh(\alpha\beta)\right] \tag{32ii}$$
$$* \left[\left(\frac{1}{N}+q_i-1\right)\sinh(\alpha\beta) + \left(\frac{1}{N}-q_i-1\right)\cosh(\alpha\beta)\right]$$

hence (32)

$$\ddot{V}_iV_i = \alpha^2\cosh^{2/N+2q_i+2}(\alpha\beta)\sinh^{2/N-2q_i-2}(\alpha\beta)\Big\{\left(\frac{1}{N}+q_i\right)\cosh(\alpha\beta)\sinh^2(\alpha\beta) + \left(\frac{1}{N}-q_i\right)\cosh^2(\alpha\beta)\sinh(\alpha\beta)$$
$$+\left(\frac{1}{N}+q_i\right)\left(\frac{1}{N}+q_i-1\right)\sinh^2(\alpha\beta) + \left(\frac{1}{N}-q_i\right)\left(\frac{1}{N}-q_i-1\right)\cosh^2(\alpha\beta)$$
$$+\cosh(\alpha\beta)\sinh(\alpha\beta)\left[\left(\frac{1}{N}+q_i\right)\left(\frac{1}{N}-q_i-1\right)+\left(\frac{1}{N}+q_i-1\right)\left(\frac{1}{N}-q_i\right)\right]\Big\}$$

Before going any further one should remember the behavior of sinh and cosh for small λ.

i.e. $\sinh(\alpha\beta) \sim \alpha$, $\cosh(\alpha\beta) \sim 1$ for small λ .

Hence

$$\ddot{V}_iV_i \sim \alpha^{2/N-2q_i}\Big\{\left(\frac{1}{N}+q_i\right)\alpha^2 + \left(\frac{1}{N}-q_i\right)\alpha + \left(\frac{1}{N}+q_i\right)\left(\frac{1}{N}+q_i-1\right)\alpha^2 + \left(\frac{1}{N}-q_i\right)\left(\frac{1}{N}-q_i-1\right) \tag{33}$$
$$+\alpha\left[\left(\frac{1}{N}+q_i\right)\left(\frac{1}{N}-q_i-1\right) + \left(\frac{1}{N}+q_i-1\right)\left(\frac{1}{N}-q_i\right)\right]\Big\}$$

Although at first sight this appears to go to zero no matter what values q_i take, a closer inspection shows that one has to take more care since there is a factor α which may go to infinity without a judicious choice of q_i.

It takes very little working to see that the only choice of q_i, to prevent this and also to have $R_{abcd} \to 0$ as $\lambda \to 0$ are

$$q_i = -1/N \qquad , i = 0, 1, \dots N-2. \tag{34i}$$
$$q_{N-1} = 1 - 1/N \tag{34ii}$$

Of course we should now check to whether these values are solutions of the field equations, however it takes very little work to see that they do.

Hence the only solution to the empty space field equations which automatically produces an asymptotically($\lambda \to 0$) flat space, in N dimensions is

$$U = \cosh^{2/N}(\alpha\beta) \tag{35i}$$

$$V_i = \cosh^{2/N}(\alpha\beta) \quad , \quad i = 1, 2, \cdots N-2 \tag{35ii}$$

$$V_{N-1} = \sinh(\alpha\beta)\cosh^{2/N-1}(\alpha\beta) \tag{35iii}$$

It is this unique solution that I shall now concentrate.

§5. Exact solutions in N+1 dimensions.

The metric we are dealing with is now,

$$ds^2 = \cosh^{4/N}(\alpha\beta)\left\{dx_0^2 - \sum_{i=1}^{N-2} dx_i^2\right\} - \sinh^2(\alpha\beta)\cosh^{4/N-2}(\alpha\beta)\,dx_{N-1}^2 - d\beta^2 \tag{36i}$$

where

$$\alpha^2 = \frac{\lambda N}{4} \tag{36ii}$$

On substitution into the Klien-Gordon equation we find

$$\cosh^{-4/N}(\alpha\beta)\left\{\frac{\partial^2\psi}{\partial x_0^2} - \sum_{i=1}^{N-2}\frac{\partial^2\psi}{\partial x_i^2}\right\} - \cosh^{2-4/N}(\alpha\beta)\sinh^{-2}\frac{\partial^2\psi}{\partial x_{N-1}^2} \tag{37}$$

$$- \alpha\left(\frac{\cosh(\alpha\beta)}{\sinh(\alpha\beta} + \frac{\sinh(\alpha\beta)}{\cosh(\alpha\beta}\right)\frac{\partial\psi}{\partial\beta} - \frac{\partial^2\psi}{\partial\beta^2} = -M_n^2\psi$$

So if we now search for plane wave solutions in N dimensions such that

$$\psi = \exp\left\{-i\left(\omega x_0 - \sum_{\mu=1}^{N-1} k_\mu x_\mu\right)\right\}\chi(\beta) \tag{38}$$

with the aim of bounding a particle of mass

$$m^2 = \omega^2 - \sum_{\mu=1}^{N-1} k_\mu^2 \tag{39}$$

in the ß direction, we obtain

$$\frac{d^2\chi}{d\beta^2} + \alpha\left(\frac{\cosh(\alpha\beta)}{\sinh(\alpha\beta} + \frac{\sinh(\alpha\beta)}{\cosh(\alpha\beta)}\right)\frac{d\chi}{d\beta} + \left(\frac{m^2}{\cosh^{4/N}(\alpha\beta)} + \frac{k_n^2\cosh^2(\alpha\beta)}{\cosh^{4/N}(\alpha\beta\sinh^2(\alpha\beta))}\right)\chi = -M_n^2\chi \tag{40}$$

If we now substitute

$$z = \cosh^2(\alpha\beta) \tag{41}$$

it is not hard to find the equation

$$z(1-z)\frac{d^2\chi}{dz^2} + (1-2z)\frac{d\chi}{dz} + \left(\frac{M_H^2}{2\lambda} - \frac{k_{N-1}^2 z}{2\lambda(1-z)z^{2/N}} - \frac{m^2}{z^{2/N}}\right)\chi = 0 \tag{42}$$

which is the generalization of the equation (7) in §3 of this chapter. We should now take note of of the topology of this space as was done in §2.If we do so, again it is not hard to see that we have a polar pair (x_{n-1},β) hence again without loosing validity we can take $k_{n-1}=0$ and so seek a ground state to the mass in the x_{n-1} direction.We therefore end up with the equation

$$z(1-z)\frac{d^2\chi}{dz^2} + (1-2z)\frac{d\chi}{dz} - \frac{m^2\chi}{z^{2/N}} = -\frac{M_H^2\chi}{2\lambda} \tag{43}$$

Notice that this equation determines the value of m^2 (39), and there are no other degeneracies on any of the k_μ so the model predicts exact Lorentz invariance. This follows from the nature of the solution given in 34i and of course is not true of the general solution to (28).

This however is as far as we can get with the analysis of the differential equation as it cannot be transformed into the hypergeometric equation or anything similar.

§6. Conclusion.

From this work is is therefore possible to get dimensional reduction from not only three to two dimensions, as Squires indicated, but from three to one, but the masses in the lower dimension are not in the observed region of the spectrum, so there will be considerable back effect on the metric hence this suggests that we should be considering the field equations with the energy-momentum tensor but then we come back to the problem of how to construct this so that it is the most natural choice.

In the more general case by imposing another restriction on the space, namely $R_{abcd} \to 0$ as $\lambda \to 0$, it is possible to have bound states, although as yet it is not possible to find the mass spectrum, that have exact Lorentz invariance.

It seems therefore that many of the aims of Squires and Visser have been achieved, however there still remains the problem of finding the mass spectrum in the more general case.

Chapter 5.

The nonlocallity problem.

§1. Introduction.

Quantum Mechanics today, can be based around one of two interpretations, the Copenhagen School of Thought or the Einstein Podolsky-Rosen (EPR), (ref.5).The Copenhagen interpretation asserts that quantum mechanical description is complete. This is based around the principle of non-separability of two physical systems A and B ,and, more generally, of the physical system, the instrument, and the observer.

On the other hand, E.P.R argued that quantum mechanics is not a complete theory, and they based this on two assumptions:-

(1) The assumption of the existence of an objective reality independent of any observation;

(2) Physical interaction are local in character, the velocity of which implies the actual separability of two physical systems, separated by a space time interval.

After formulating their idea about locality, Einstein put forward the argument that if the Copenhagen interpretation is true' that there must be action at a distance, or non-locallity as it is now called.

One possible way in which one can understand nonlocallity is too consider 'wave function reduction', which certainly happens in more orthodox theories, then the nonlocallity can be seen from the fact that a position observation automatically removes the whole wave function to one point. Previous to the measurement of the position, for example a particle that might have been reflected or transmitted by a potential barrier, the wave function consists of two pieces, as in fig. 1, whereas after a measurement it collapses to fig. 2.

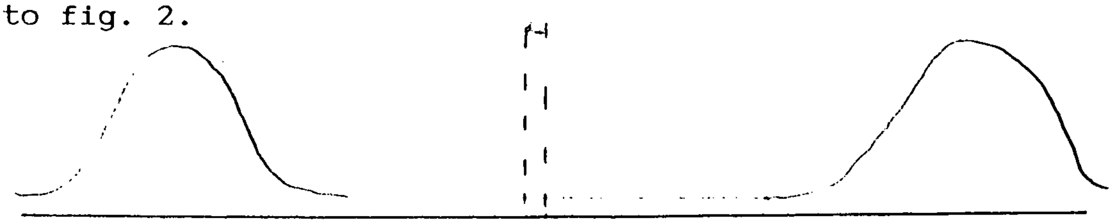

fig.1

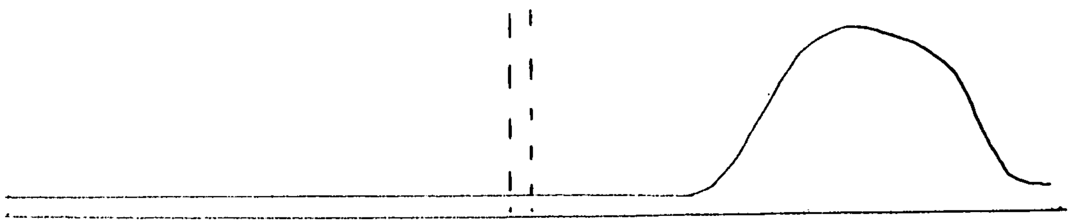

fig.2

A more precise understanding comes however from the E.P.R. type of experiment where Bell's theorem shows that locality and quantum mechanics are not compatible and in fact recent experiments have show that quantum mechanics and hence non-locality is correct (ref.5).

Since we are very attached to the concept of locality it has been suggested (ref.2 & 6) that some form of it might be maintained if we live in a "ball of string" universe. The idea here is that our 4-dimensional space-time is embedded in a flat N-dimensional space in such a way that distance measured "along the string" (i.e. in the physical 4-dimensions) might be large, although distances measured by an observer, not constrained to this physical space, might be small. In this analogy we can embed an infinitely long string (of zero thickness, of course) in an arbitrary small region of three space (or indeed two space, since it is of zero thickness). Since we want our 4-dimensional physical space to be flat (or at least approximately) we cannot achieve this in

5-dimensions (as is used in Kaluza-Klien's original idea), on the other hand we can clearly achieve it in 8 (or more), since we can embed each dimension in 2-space.

Whilst it is not completely clear what all this means, it is clearly of interest to see whether there are solutions to the Einstein field equations in which the "bottom of the valley" (i.e. =0 in chapter 4) follows a curve in some higher dimensional space, in which the intrinsic curvature is small but the extrinsic curvature (ref.7) is large.

§2. The metric.

As a preliminary to this work we develop a metric in 3-dimensions which, for realism, is a solution of the empty space field equations :

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{1}$$

The usual metric for flat three dimensional space is

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2}$$

if we make the change of variables

$$x = \xi \sin\theta \tag{3i}$$

$$y = \sin\theta\,(a + \xi\cos\phi) \tag{3ii}$$

$$z = \cos\theta\,(a + \xi\cos\phi) \tag{3iii}$$

then this is still flat space, but now ξ =0 is a circle in space

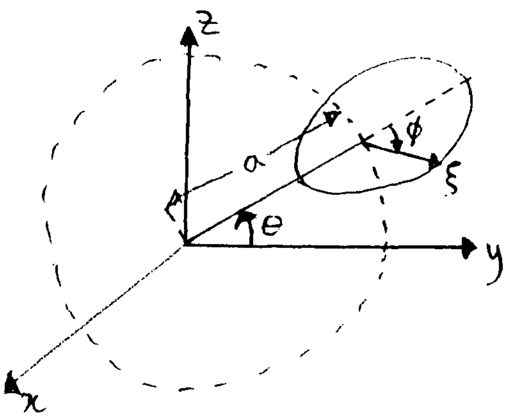

Diagram of coordinates used in 3i-iii

Note that this transformation is used so that in the generalisation we can break the symmetry of the space to achieve the ball of string model.

It this metric which I shall try and generalise so that it solves the empty space field equations but also is, as this one is, asyptotically flat as $\lambda \to 0$, (so this correlates with observation). The metric turns out to be, when (3i-iii) are substituted into (2)

$$ds^2 = (a + \xi \cos\phi)\,d\theta^2 + \xi\,d\phi^2 + d\xi^2 \tag{4}$$

It is obvious that a generalization of this would be

$$ds^2 = F(\phi,\xi)\,d\theta^2 + G(\xi)\,d\phi^2 + d\xi^2 \tag{5}$$

so the question now is does this satisfy the field equations. Hence if we substitute (5) into (1), and by use of the reduce program (Appendix 1) we end up with the following set of equations to solve

$$\lambda F = \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2} - \frac{1}{4F}\left(\frac{\partial F}{\partial \xi}\right)^2 + \frac{1}{4G}\frac{\partial F}{\partial \xi}\frac{dG}{d\xi} + \frac{1}{2G}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{4FG}\left(\frac{\partial F}{\partial \phi}\right)^2 \tag{6i}$$

$$\lambda G = \frac{1}{4F}\frac{\partial F}{\partial \xi}\frac{dG}{d\xi} + \frac{1}{2F}\frac{\partial^2 F}{\partial \phi^2} - \frac{1}{4F^2}\left(\frac{\partial F}{\partial \phi}\right)^2 + \frac{1}{2}\frac{d^2 G}{d\xi^2} \tag{6ii}$$

$$\lambda = \frac{1}{2F}\frac{\partial^2 F}{\partial \xi^2} - \frac{1}{4F^2}\left(\frac{\partial F}{\partial \xi}\right)^2 + \frac{1}{2G}\frac{d^2 G}{d\xi^2} - \frac{1}{4G^2}\left(\frac{dG}{d\xi}\right)^2 \tag{6iii}$$

$$0 = \frac{1}{2F}\frac{\partial^2 F}{\partial \phi\,\partial \xi} - \frac{1}{4F^2}\frac{\partial F}{\partial \xi}\frac{\partial F}{\partial \phi} - \frac{1}{4FG}\frac{\partial F}{\partial \phi}\frac{dG}{d\xi} \tag{6iv}$$

It should also be noted that the choice of sign of λ was made so we have an attractive potential

So G*(6i)-(6ii)-(6iii) gives the equation for consistency

$$2G''G - G'^2 - 2\lambda G^2 = 0 \tag{7}$$

$$\Rightarrow \quad \frac{G''}{G} - \frac{G'^2}{2G} = \lambda$$

$$\Rightarrow \quad \left(\frac{G'}{G}\right)' + \frac{1}{2}\left(\frac{G'}{G}\right) = \lambda$$

$$\Rightarrow \quad G = A\cosh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]$$

Also one notes that

$$G = A\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \tag{8}$$

is also a solution of (7). If we now use the boundary conditions which we are employing, namely the space is flat as $\lambda \to 0$, it doesn't take that much work to see that the only solution is

$$G = \frac{2}{\lambda}\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \tag{9}$$

Hence the equations (6i-iv) boil down to the set of equations

$$2F\frac{\partial^2 F}{\partial\phi^2} + \frac{4F}{\lambda}\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]\frac{\partial^2 F}{\partial\xi^2} - \left(\frac{\partial F}{\partial\xi}\right)^2 - \frac{2}{\lambda}\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]\left(\frac{\partial F}{\partial\xi}\right)^2 + F\left(\frac{2}{\lambda}\right)^{1/2}\sinh\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]\frac{\partial F}{\partial\xi} = 8F^2\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \tag{10i}$$

$$2F\frac{\partial^2 F}{\partial\phi^2} - \left(\frac{\partial F}{\partial\phi}\right)^2 + \left(\frac{2}{\lambda}\right)^{1/2}F\sinh\left[2\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]\frac{\partial F}{\partial\xi} = 4F^2\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \tag{10ii}$$

$$2F\frac{\partial^2 F}{\partial\xi^2} - \left(\frac{\partial F}{\partial\xi}\right)^2 = 2\lambda F^2 \tag{10iii}$$

Here though (10i) is just the sum of the equations (10ii) and (10iii), so the problem of finding F boils down to the solution of

$$2F\frac{\partial^2 F}{\partial\xi^2} - \left(\frac{\partial F}{\partial\xi}\right)^2 = 2\lambda F^2 \tag{11}$$

Equation (11) may be rearranged to

$$\frac{\ddot{F}}{F^{1/2}} - \frac{\dot{F}^2}{2F^{3/2}} = 2\lambda F^{1/2} \tag{12}$$

and so if we make the substitution

$$u = F^{1/2} \tag{13}$$

it is easy too see

$$\ddot{u} = \frac{\lambda}{2}u \tag{14}$$

which of course has the solution

$$u = A(\phi)\cosh\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] + B(\phi)\sinh\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \tag{15}$$

where A and B are functions of ϕ.

Upon substitution of F and G into (10i-ii) it is not hard to find that

$$A'' = 0 \tag{16i}$$

$$B'' = -B \tag{16ii}$$

So the complete metric is

$$g_{11} = \left[(a\phi + b)\cosh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right] + (c\cos\phi + d\sin\phi)\sinh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right] \right]^2 \tag{17i}$$

$$g_{22} = \frac{2}{\lambda}\sinh^2\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right] \tag{17ii}$$

$$g_{33} = 1 \tag{17iii}$$

$$g_{12} = g_{13} = g_{21} = g_{13} = g_{23} = g_{31} = 0 \tag{17iv}$$

Where $a, b; c$ and d are different constants than before.

If we now substitute the values of F and G into the off diagonal term we easily see that this vanishes and so the metric defined by equations (17i-iv) is indeed the most general one.

§3. Asyptotically flat spaces.

Although it is well known that all Einstienian spaces in three dimensions have constant curvature, we have introduced another degree of freedom into the equations with the cosmological constant, λ , hence the necessity of looking for asymtotically flat spaces.

There now comes the problem of, which values of a, b, c and d, will produce the asyptotically flat space that we require. To do this we shall have to look at the curvature tensor and if once again we use the REDUCE program of appendix 1 we see that the nonzero components are R_{1212}, R_{1313}, R_{2323} and R_{1312}, so if we use the values we have for F and G we obtain

$$R_{1212} = \frac{\sinh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right]\left\{-(c\cos\phi + d\sin\phi) + \cosh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right]\left[(a\phi+b)\sinh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right] + (c\cos\phi + d\sin\phi)\cosh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right]\right]\right\}}{\left[(a\phi+b)\cosh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right] + (c\cos\phi + d\sin\phi)\sinh\left[\left(\tfrac{\lambda}{2}\right)^{1/2}\xi\right]\right]} \tag{18i}$$

$$R_{1312} = 0 \tag{18ii}$$

$$R_{1313} = \lambda/2 \tag{18iii}$$

$$R_{2323} = \lambda/2 \qquad (18iv)$$

So if now recall the behavior of sinh and cosh for small values of the argument we find the equations

$$R_{1312} = 0 \qquad (19i)$$

$$R_{1313} = \lambda/2 \qquad (19ii)$$

$$R_{2323} = \lambda/2 \qquad (19iii)$$

$$R_{1212} = \frac{\left(\frac{\lambda}{2}\right)^{1/2}\xi\left\{-(c\cos\phi + d\sin\phi) + \left[(a\phi+b)\left(\frac{\lambda}{2}\right)^{1/2}\xi + (c\cos\phi + d\sin\phi)\right]\right\}}{\left[(a\phi+b) + (c\cos\phi + d\sin\phi)\left(\frac{\lambda}{2}\right)^{1/2}\xi\right]} \qquad (19iv)$$

Hence when we remember the boundary condition i.e. this metric must tend to the one in equations (3i-iii) it is not hard to see that the only values that a, b, c and d can take are

$$a = 0 \qquad (20i)$$

$$b = \text{radius of circle in } 3i-iii \qquad (20ii)$$

$$c = \left(\frac{2}{\lambda}\right)^{1/2} \qquad (20iii)$$

$$d = 0 \qquad (20iv)$$

Therefore the only metric of the type first defined in (5) that tends to a flat space as $\lambda \to 0$ is

$$g_{11} = \left[a\cosh\left(\left(\frac{\lambda}{2}\right)^{1/2}\xi\right) + \left(\frac{2}{\lambda}\right)^{1/2}\cos\phi\sinh\left(\left(\frac{\lambda}{2}\right)^{1/2}\xi\right)\right]^2 \qquad (21i)$$

$$g_{22} = \frac{2}{\lambda}\sinh^2\left[\left(\frac{\lambda}{2}\right)^{1/2}\xi\right] \qquad (21ii)$$

$$g_{33} = 1 \qquad (21iii)$$

$$g_{ij} = 0 \qquad i = 1,2,3;\ j = 1,2,3 \qquad i \neq j \qquad (21iv)$$

Actually this question of uniqueness need to be looked into a little closer as it is possible that this metric and the one used in chapter 3 are isomorphic however if one examines the killing vectors (appendix 2) of both these metrics then indeed the metric found here is unique. The structure of Squires metric is isomorphic to an Abelian group of order 2 and the one found here is generated by only one element hence these two metrics cannot be isomorphic.

§4. Conclusion.

Although we have found a unique metric which produces flat space as $\lambda \to 0$, and one which looks likely to be extended into the "ball of string", it is not clear which curves we should be examining for intrinsic and extrinsic curvature. Since particles follow geodesics it would seem reasonable to assume that we should be examining these for the afore mentioned properties, however this is something which shall have to be left for the future.

CHAPTER 6

Possibilities for future research.

Throughout this work, the metric has only been in orthogonal coordinates,or if we concider the group structure of the isometry group then (ref.4) it is the simple G(n),where n is the number of independent killing vectors of the metric. Hence one possibility would be to have a more complex metric or to put it more mathematically ,to have a non-simple group structure, perhaps one such as SU(3)*SU(2)*U(1) which of course would have the benifit of more realistic as well.

The meric as a consequence of the coordinate system is diagonal and so if we concider it a la Kaluza-Klien, there are no internal fields. We could therefore 'turn on' these fields and see what consequence they have in the Klien-Gordon equation.

These fields are of course Bosonic in nature, and so possess spin properties, it therefore would not be unreasonable to concider, the Dirac instead of the Klien-Gordon equation.This would nessesitate working in a Clifford algebra for the now non-constant spin matricies, but it should be feasable to attack the problem of bound states using this method.

Aside from Vissers paper all the work concidered here has been on the empty space field equations , it therefore seems reasonable to try to extend the work to include a non-zero energy-momentum tensor, which should be chosen realistically and not arbitarily to satisfy the equations.

There is of course no reason why we should have the cosmological constant as an overall constant, after all as long as the field equations are satisfied the cosmological constant can be whatever we like, and so in this vein we could take it to be now a matrix where each of the components are a function of the cosmological time(ref.7 & 8).Alternatively we could take it to be a function of the variables ,so long as we remember that this must satisfy the boundary conditions of inflation i.e. a large cosmological constant, and a small cosmological constant now.

There is also work to be done in the nonlocallity area since as was stated in the chapter on it that work was only a preliminary to working in N-dimensions. This should be of a geometrical nature and in essence be about the embedding of one space inside another with special attention being payed to the properties of curves in both the larger space and also the embedded space.

Acknowledgments.

I would like to take this opportunity to thank all members of the Theory group at Durham for many interesting and thought provoking lectures and discussions, in particular those of the MAPHIA. I would especially like to thank Prof.E.J.Squires for his support and guidance throughout this work, without which this wouldn't have been possible.

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Appendix 1.

Reduce program with some results.

§1. Introduction.

In this appendix I have used the algebraic manipulation language REDUCE to construct a program that will work out, at least in the case of orthogonal coordinates, such things as the Christoffel symbols, curvature tensor and other related tensors also some of the results which have been used throughout this work are presented here

As previously stated this program only works for the simple case of the metric tensor in orthogonal coordinates so anyone else wishing to use this program, not in orthogonal coordinates will have to alter the program accordingly.

§2. How the program works.

The only full program shown will be for the metric which Squires used, namely

$$ds^2 = v^2(\xi)\,dt^2 - v^2(\xi)\,dx^2 - d\xi^2 \quad ,$$

for the metrics

$$ds^2 = (a+\xi\cos\phi)^2\,d\theta^2 + \xi^2\,d\phi^2 + d\xi^2$$

and

$$ds^2 = F(\phi,\xi)\,d\theta^2 + G(\xi)\,d\phi^2 + d\xi^2$$

only the relevant changes to the code will be shown.

Line 1:-This tells the computer that all zeros to be printed out, are suppressed.

Line 2:-It is this line which defines all functions that the program will use and one will note that all the variables used in this case are stored in the indexed function X, which is the one the computer automatically links with variables.The variables in this case, have to be stored so as, there is no other way to the create loops required to calculate such things as Christoffel symbols, otherwise.

Lines 3-4;-It is in these lines that the functions U and V are defined to be functions of the variable X(3).

Lines 5-13;-In these lines, the covarient andcontravarient metric tensor are given the dimension 3 and the contravarient metric tensor is computed.Therefore, it is in this place the modification required for the more general inverse of the metric tensor should be made.Also, it should be noted that no other modifications are needed throughout the program as it has been set up for the most general case already.

lines 16-20;-Calculate and print the Christoffel symbols.

lines 21-33;-Calculation and print out of the curvature tensor.

lines 34-38;-Calculation and print out of the Ricci tensor.

lines 39-40;-These calculate and print out the curvature scalar.

lines 41-44;-These calculate and print out the Einstein tensor. It should be noted, in this case the program works in 3 dimensions (DIM=3) but could easily be made to work in 10 or some other fixed value, however the program cannot be made too work in an arbitrary number of dimensions, as is required sometimes in generalized theories.

When defining the metric for this case, general functions were used but if one has, instead, specific functions then these would be entered directly into the metric without having the need to use the DEPEND statement. However in all cases one must still define X to be an OPERATOR as it is the indexing of this, which is used throughout the program, in calculations.

§3. Conclusion.

This program, with some minor modifications, works for a metric defined in specific number of dimensions, whose components are either definite or general functions,or indeed a mixture. However this program only works in orthogonal coordinates and so this may be a topic for future research.

§3. Programs and results.

The program detailed below is for the metric used by Squires in chapter 3, namely

$$ds^2 = u^2(\xi)\,dt^2 - v^2(\xi)\,dx^2 - d\xi^2$$

```
ON NERO;
OPERATOR U,V,X;
DEPEND U,X(3);
DEPEND V,X(3);
DIM=3;
ARRAY GG(DIM,DIM),H(DIM,DIM);
WRITE "COVARIENT METRIC COMPONENTS ARE:-";
     GG(1,1):=U^2;
     GG(2,2):=-V^2;
     GG(3,3):=-1;
WRITE "";
WRITE "CONTRAVARIENT METRIC COMPONENTS ARE:-";
FOR I:=1:DIM DO WRITE H(I,I):=1/GG(I,I);
ARRAY CS1(DIM,DIM,DIM),CS2(DIM,DIM,DIM);
WRITE "NON-ZERO CHRISTOFFEL SYMBOLS ARE:-";
FOR I:=1:DIM DO FOR J:=1:DIM DO BEGIN
     FOR K:=1:DIM DO
          CS1(J,I,K):=CS1(I,J,K):=(DF(GG(I,K),X(J))
               +DF(GG(J,K),X(I))-DF(GG(I,J),X(K)))/2;
```

```
FOR K:=1:DIM DO
     WRITE CS2(J,I,K):=CS2(I,J,K):=FOR P:=1:DIM SUM
                                             H(K,P)*CS1(I,J,P) END;
ARRAY R(DIM,DIM,DIM,DIM),CUR(DIM,DIM,DIM,DIM);
WRITE "NON-ZERO CURVATURE COMPONENTS ARE:-";
  FOR I:=1:DIM DO FOR J:=I+1:DIM DO FOR K:=I:DIM DO
    FOR L:=K+1:IF K=I THEN J ELSE DIM DO BEGIN
      R(J,I,L,K):=R(I,J,K,L):=FOR Q:=1:DIM
        SUM GG(I,Q)*(DF(CS2(K,J,Q),X(L))-DF(CS2(J,L,Q),X(K))
          + FOR P:=1:DIM SUM (CS2(P,L,Q)*CS2(K,J,P)-CS2(P,K,Q)
                                             *CS2(L,J,P)));
    WRITE CUR(I,J,L,K):=R(I,J,K,L);
      R(I,J,L,K):=-R(I,J,K,L);R(J,I,K,L):=-R(I,J,K,L);
    IF I=K AND J<=L THEN GO TO A;
     R(k,L,I,J):=R(L,K,J,I):=R(I,J,K,L);
     R(L,K,I,J):=-R(I,J,K,L);R(K,L,J,I):=-R(I,J,K,L);
A: END;
ARRAY RICCI(DIM,DIM);
WRITE "NON-ZERO RICCI COMPONENTS ARE:-";
  FOR I:=1:DIM DO FOR J:=1:DIM DO
    WRITE RICCI(J,I):=RICCI(I,J):=FOR P:=1:DIM SUM FOR Q:=1:DIM
                                  SUM H(P,Q)*R(Q,I,P,J);
WRITE "RICCI SCALAR IS:-";
  RS:=FOR I:=1:DIM SUM FOR J:=1:DIM SUM H(I,J)*RICCI(I,J);
ARRAY EINSTEIN(DIM,DIM);
WRITE "NON-ZERO COMPONENTS OF THE EINSTEIN TENSOR ARE:-";
FOR I:=1:DIM DO FOR J:=1:DIM DO
  WRITE EINSTEIN(I,J):=RICCI(I,J)-RS*GG(I,J)/2;
```

The results for this program are presented on the next page.

COVARIENT METRIC COMPONENTS ARE:-

GG(1,1):=U^2

GG(2,2):=-V^2

GG(3,3):=-1

CONTRAVARIENT METRIC COMPONENTS ARE:-

H(1,1):=1/U^2

H(2,2):=(-1)/V^2

H(3,3):=(-1)

NON-ZERO CHRISTOFFEL SYMBOLS ARE:-

CS2(1,1,3):=CS2(1,1,3):=DF(U,X(3))*U

CS2(3,1,1):=CS2(1,3,1):=DF(U,X(3))/U

CS2(2,2,3):=CS2(2,2,3):=-DF(V,X(3))*V

CS2(3,2,2):=CS2(2,3,2):=DF(V,X(3))/V

NON-ZERO CURVATURE COMPONENTS ARE:-

CUR(1,2,1,2):=DF(U,X(3))*DF(V,X(3))*U*V

CUR(1,3,1,3):=DF(U,X(3),2)*U

CUR(2,3,2,3):=-DF(V,X(3),2)*V

NON-ZERO RICCI COMPONENTS ARE:-

RICCI(1,1):=RICCI(1,1):=(-U(DF(U,X(3),2)*V+DF(U,X(3))*DF(V,X(3))))/V

RICCI(2,2):=RICCI(2,2):=(V*(DF(U,X(3))*DF(V,X(3))+DF(V,X(3),2)*U))/U

RICCI(3,3):=RICCI(3,3):=(DF(U,X(3),2)*V+DF(V,X(3),2)*U)/(U*V)

RICCI SCALAR IS:-

RS:=(-2*(DF(U,X(3),2)*V+DF(U,X(3))*DF(V,X(3))+DF(V,X(3),2)*U))/(U*V)

NON-ZERO COMPONENTS OF THE EINSTEIN TENSOR ARE:-

EINSTEIN(1,1):=(DF(V,X(3),2)*U^2)/V

EINSTEIN(2,2):=(-DF(U,X(3),2)*V^2)/U

EINSTEIN(3,3):=(-DF(U,X(3))*DF(V,X(3)))/(U*V)

Where DF(A,X(N),K) means differentiate the function A with respect to the variable X(N),K times. Elsewhere one will see

DF(A,X(N),X(M)), which of course means differentiate the function A with respect to the variable X(N) then differentiate with respect to X(M).

For the case of exact functions in the metric, such as

$$ds^2 = (a+\xi\cos\phi)^2 d\theta^2 + \xi^2 d\phi^2 + d\xi^2$$

which was used in chapter 5, the relevant changes in the code are made between lines 2-11 and are;-

```
2   OPERATOR X;
3   DIM:=3;
4   ARRAY GG(DIM,DIM),H(DIM,DIM);
5   WRITE "COVARIENT METRIC COMPONENTS ARE:-";
6           GG(1,1):=(A+X(3)*COS(X(2)))**2;
7           GG(2,2):=X(3)**2;
8           GG(3,3):=1
```

So, as one can see the lines of code with DEPEND in, have been missed out since we are now dealing with exact functions, as was stated in the introduction.

The results for this are shown below and indeed show that the space defined by the metric, is flat.

COVARIENT METRIC COMPONENTS ARE:-

GG(1,1):=X(3)^2*COS(X(2))^2+2*X(3)*COS(X(2))*A+A^2

GG(2,2):=X(3)^2

GG(3,3):=1

CONTRAVARIENT METRIC COMPONENTS ARE:-

H(1,1):=1/(X(3)^2*COS(X(2))^2+2*COS(X(2))*A+A^2)

H(2,2):=1/(X(3)^2)

H(3,3):=1

NON-ZERO CHRISTOFFEL SYMBOLS ARE:-

CS2(1,1,2):=CS2(1,1,2):=(SIN(X(2))*(X(3)*COS(X(2))+A))/X(3)

```
CS2(1,1,3):=CS2(1,1,3):=-COS(X(2))*)X(3)*COS(X(2))+A)

CS2(2,1,1):=CS2(1,2,1):=(-(X(3)*SIN(X(2)))*(X(3)*COS(X(2))+A))/
                        (X(3)^2*COS(X(2))^2+2*X(3)*COS(X(2))*A+A^2)

CS2(3,1,1):=CS2(1,3,1):=(COS(X(2))*(X(3)*COS(X(2))+A))/
                        (X(3)^2*COS(X(2))^2+2*X(3)*COS(X(2))*A+A^2)

CS2(2,2,3):=CS2(2,2,3):=-X(3)

CS2(3,2,2):=CS2(2,3,2):=1/X(3)

NON-ZERO CURVATURE COMPONENTS ARE:-

NON-ZERO RICCI COMPONENTS ARE:-

RICCI SCALAR IS:-

NON-ZERO COMPONENTS OF THE EINSTEIN TENSOR ARE:-
```

For the more general metric of this type, namely

$$ds^2 = F(\phi,\zeta)\,d\theta^2 + G(\zeta)\,d\phi^2 + d\zeta^2$$

the relevant changes to the code would again be between lines 2-11 and are

```
2    OPERATOR F,G,X;
3    DEPEND F,X(2),X(3);
4    DEPEND G,X(3);
5    DIM:=3;
6    ARRAY GG(DIM,DIM),H(DIM,DIM);
7    WRITE "COVARIENT METRIC COMPONENTS ARE:-";
8         GG(1,1):=F;
9         GG(2,2):=G;
10        GG(3,3):=1;
```

The results produced by the program for this metric are;-

```
COVARIENT METRIC COMPONENTS ARE:-

GG(1,1):=F

GG(2,2):=G
```

```
GG(3,3):=1

CONTRAVARIENT METRIC COMPONENTS ARE:-

H(1,1):=1/F

H(2,2):=1/G

H(3,3):=1

NON-ZERO CHRISTOFFEL SYMBOLS ARE:-

CS2(1,1,2):=CS2(1,1,2):=(-DF(F,X(2)))/(2*G)

CS2(1,1,3):=CS2(1,1,3):=(-DF(F,X(3)))/2

CS2(2,1,1):=CS2(1,2,1):=DF(F,X(2))/(2*F)

CS2(3,1,1):=CS2(1,3,1):=DF(F,X(3))/(2*F)

CS2(2,2,3):=CS2(2,2,3):=(-DF(G,X(3)))/2

CS2(3,2,2):=CS2(2,3,2):=DF(G,X(3))/(2*G)

NON-ZERO CURVATURE COMPONENTS ARE:-

CUR(1,2,1,2):=(DF(F,X(3))*DF(G,X(3))*F+2*DF(F,X(2),2)*F
                        -DF(F,X(2))²)/(4*F)

CUR(1,3,1,2):=(2*DF(F,X(3),X(2))*F*G-DF(F,X(3))*DF(F,X(2))*G
                        -DF(F,X(2))*DF(G,X(3))*F)/(4*F*G)

CUR(1,3,1,3):=(2*SD(F,X(3),2)*F-DF(F,X(3))²)/(4*F)

CUR(2,3,2,3):=(2*SD(G,X(3),2)*G-DF(G,X(3))²)/(4*G)

NON-ZERO RICCI COMPONENTS ARE:-

RICCI(1,1):=RICCI(1,1):=(2*DF(F,X(3),2)*F*G-DF(F,X(3))²*G
                        +DF(F,X(3))*DF(G,X(3))*F+"*DF(F,X(2),2)*F
                        -DF(F,X(2))²)/(4*F*G)

RICCI(2,2):=RICCI(2,2):=(DF(F,X(3))*DF(G,X(3))*F*G+2*DF(F,X(2),2)*F*
                        -DF(F,X(2))²*G+2*DF(G,X(3),2)*F²*G
                        -DF(G,X(3))²*F²)/(4*F²*G)

RICCI(3,2):=RICCI(2,3):=(2*DF(F,X(3),X(2))*F*G
                        -DF(F,X(3))*DF(F,X(2))*G
                        -DF(F,X(2))*DF(G,X(3))*F)/(4*F²*G)
```

```
RICCI(3,3):=RICCI(3,3):=(2*DF(F,X(3),2)*F*G²-DF(F,X(3))²*G²
                              +2*DF(G,X(3),2)*F²*G-F(G,X(3))²*F²)/(4*F²*G)
RICCI SCALAR IS:-
RS:=(2*DF(F,X(3),2)*F*G²-DF(F,X(3))²*G²+DF(F,X(3))*DF(G,X(3))*F*G
    +2*DF(F,X(2),2)*F*G-DF(F,X(2))²*G+2*DF(G,X(3),2)*F²*G
    -DF(G,X(3))²*F²)/(2*F²*G²)
NON-ZERO COMPONENTS OF THE EINSTEIN TENSOR ARE:-
EINSTEIN(1,1):=((F*(-2*DF(G,X(3),2)*G+DF(G,X(3))²))/(4*G²
EINSTEIN(2,2):=((G*(-2*DF(F,X(3),2)*F+DF(F,X(3))²))/(4*F²)
EINSTEIN(2,3):=(2*DF(F,X(3),X(2))*F*G-DF(F,X(3))*DF(F,X(2))*G
               -DF(F,X(2))*DF(G,X(3))*F)/(4*F²*G)
EINSTEIN(3,2):=((2*DF(F,X(3),X(2))*F*G-DF(F,X(3))*DF(F,X(2))*G
               -DF(F,X(2))*DF(G,X(3))*F)/(4*F²*G)
EINSTEIN(3,3):=(-DF(F,X(3))*DF(G,X(3))*F-2*DF(F,X(2),2)*F
               +DF(F,X(2))²)/(4*F²*G)
```

Of course throughout the chapters I have simplified the results into a more manageable form although this could also have been done by the REDUCE language.

You might say that the results here are questionable, however the results used throughout the chapters was only verified by this program, which in turn has been checked against many metrics for which the coresponding curvature,Ricci etc, are known.

Appendix 2.

Killing vectors and isometry groups.

§1. Introduction.

If one wants to get onto a more structured approach towards unification or dimensional reduction, some sort of classification of the metric tensor is required. Towards this end, there are several ways in which the metric tensor may be classified, and for a detailed treatise see Petrov. One of the ways in which the metric is classified is by killing vectors which are used to form an isometry group, and it is this isometry group that is used to classify the metric. It is this approach that I have used, but moreover it is used in Kaluza-Klien type theories to classify the internal space (denoted by K in the case of Visser) whereas the other approaches, for example, classify the metric tensor by eigenvalues of the Ricci tensor in a null-'n'rad of coordinates, and so not only are more difficult to use but also do not relate to current theories so well.

§2. killing vectors.

Killing vectors are best described as being the direction in which, when the metric is transformed into another coordinate system it retains its form.

The metric in primed and unprimed coordinates are related by

$$g_{ab} = \frac{\partial y'^c}{\partial y^a} \frac{\partial y^d}{\partial y^b} \, g'_{cd}(y') \qquad (1)$$

Now form invariance under such a transform requires

$$g'_{ab}(y')=g_{ab}(y') \qquad (2)$$

If we now consider an infinitesimal transformation

$$y'^{\alpha} = y^{\alpha} + \varepsilon k^{\alpha}(y) \qquad (3)$$

where $k^{\alpha}(y)$ are the components of a vector field (these are in fact the killing vectors on the metric), and equate first orders in ε we obtain

$$g_{\alpha\beta}(y) = \left(\delta_{\gamma\alpha}\delta_{\delta\beta} + \varepsilon\frac{\partial k^{\gamma}}{\partial y^{\alpha}}\delta_{\delta\beta} + \varepsilon\delta_{\gamma\alpha}\frac{\partial k^{\delta}}{\partial y^{\beta}}\right)\left(g_{\gamma\delta}(y) + \varepsilon\frac{\partial g_{\gamma\delta}}{\partial y^{\gamma}}k^{\beta}\right) \qquad (4)$$

or

$$\frac{\partial k^{\gamma}}{\partial y^{\alpha}} g_{\gamma\beta} + \frac{\partial k^{\gamma}}{\partial y^{\beta}} g_{\gamma\alpha} + k^{\gamma}\frac{\partial g_{\alpha\beta}}{\partial y^{\gamma}} = 0 \qquad (5)$$

Which is called the Killing equation after the man who first derived it, and it is from this that with a given metric the killing vectors are derived, as will be shown later in §3 and §4.

If we now form elements

$$X_a = K_a^{\ n}\partial_n \qquad (6)$$

where $K^n_{\ a}$ is the a^{th} component of the n^{th} killing vector, then they form a group which is of course a lie group, since they are continuous. It is this, that is called an isometry group, which is used to classify the metric tensor.

This method of classification is particularly good as the purely geometric qualities are eliminated and so what we end up with is the description of the gravitational features.

§3. Killing vectors and Squires model.

The metric from chapter 3 on Squires model for dimensional reduction is

$$g_{11} = \cosh^2(\alpha x_3) \quad , \quad \alpha = \left(\frac{\lambda}{2}\right)^{1/2} \qquad (7i)$$

$$g_{22} = -\sinh^2(\alpha x_3) \qquad (7ii)$$

$$g_{33} = -1 \qquad (7iii)$$

From this we can see that the variables x_1 and x_2 are missing and so, by one of the many theories associated with killing vectors, we automatically have two independent killing vectors;

$$k^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad k^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (8)$$

If we now substitute the metric into the killing equation (5), one ends up with six equations to solve for the remaining killing vectors,

$$\alpha k^3 \cosh(\alpha x_3) + \sinh(\alpha x_3)\,\partial_1 k^3 = 0 \qquad (9i)$$

$$\cosh^2(\alpha x_3)\,\partial_2 k^1 - \sinh^2(\alpha x_3)\,\partial_1 k^2 = 0 \qquad (9ii)$$

$$\cosh^2(\alpha x_3)\,\partial_3 k^1 - \partial_1 k^3 = 0 \qquad (9iii)$$

$$\alpha k^3 \cosh(\alpha x_3) + \sinh(\alpha x_3)\,\partial_2 k^2 = 0 \qquad (9iv)$$

$$\sinh^2(\alpha x_3)\,\partial_3 k^2 + \partial_2 k^3 = 0 \qquad (9v)$$

$$\partial_3 k^3 = 0 \qquad (9vi)$$

We can now make use of the two killing vectors we have already found to give four equations

$$\alpha k^3 \cosh(\alpha x_3) = 0 \qquad (10i)$$

$$\partial_1 k^3 = 0 \qquad (10ii)$$

$$\partial_2 k^3 = 0 \qquad (10iii)$$

$$\partial_3 k^3 = 0 \qquad (10iv)$$

which obviously have only one solution

$$k^3 = 0 \qquad (11)$$

If we now form elements of the Lie group

$$X_a = k_a^n \partial_n \qquad (12)$$

we can see that this is Abelian with

$$[X_1, X_2] = 0 \qquad (13)$$

So the metric Squires finds in his paper has its structure defined by an Abelian Lie group of order 2.

§4. Killing vectors and the nonlocallity problem.

The metric which was found in chapter 5 on the nonlocallity problem was

$$g_{11} = F(x_2, x_3) \quad (14i)$$

$$g_{22} = G(x_3) \quad (14ii)$$

$$g_{33} = 1 \quad (14iii)$$

where

$$F = \left[a\cosh(\alpha x_3) + \frac{1}{\alpha}\cos(x_2)\sinh(\alpha x_3)\right]^2 , \quad \alpha = \left(\lambda/_2\right)^{1/2} \quad (14iv)$$

$$G = \frac{1}{\alpha^2}\sinh^2(\alpha x_3) \quad (14v)$$

If we now use a similar method to what was used in §3 then we end up with six equations to solve (the reason why we end up with six once again is that these equations are formed from the metric which of course is symmetric with six independent terms, or potentials);

$$k^2F_2 + k^3F_3 + 2F\partial_1 k^1 = 0 \quad (15i)$$

$$F\partial_2 k^1 + G\partial_1 k^2 = 0 \quad (15ii)$$

$$F\partial_3 k^1 - \partial_1 k^3 = 0 \quad (15iii)$$

$$G' k^3 + 2G\partial_2 k^2 = 0 \quad (15iv)$$

$$G\partial_3 k^2 - \partial_2 k^3 = 0 \quad (15v)$$

$$\partial_3 k^3 = 0 \quad (15vi)$$

However if we take note of the metric then we can see that it is independent of x_1 and so immediately one killing vector is

$$k^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (16)$$

If we now make use of this then the above equations (15i-vi) boil down too

$$k_2 F_2 + k^3 F_3 = 0 \tag{17i}$$

$$\partial_1 k^2 = 0 \tag{17ii}$$

$$\partial_1 k^3 = 0 \tag{17iii}$$

$$G' k^3 + 2G\partial_2 k^2 = 0 \tag{17iv}$$

$$G\partial_3 k^2 - \partial_2 k^3 = 0 \tag{17v}$$

$$\partial_3 k^3 = 0 \tag{17vi}$$

Equations (17iii) and (17vi) give

$$k^3 = f(x_2) \tag{18}$$

If we now look at equation (17ii) then we see

$$k^2 = g(x_2, x_3) \tag{19}$$

With some work it is not hard to show that the only functions that f and g can be are the zero function

$$k^2 = 0 \tag{20i}$$

$$k^3 = 0 \tag{20ii}$$

This leads us to a simple translation group of order 1 for this metric.

§5. Conclusion.

Since the group of isometries formed by the metric from Squires model is an Abelian group of order 2 and the metric from the non-locality problem has a simple translational group of order 1 ~~and~~ they are both independent solutions of the empty space field equations. This I must say though is surprising as one might expect from the geometry of the space at least one rotational isometry, which may bare closer investigation at some later date.

The above brief discussion, I hope, shows the power and simplicity of isometry groups which in a view shared by many, is the rigorous way to attack the problems confronting Physics today.