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**SKYRMION AND OTHER EXTENDED SOLUTIONS OF  
NON-LINEAR  $\sigma$ -MODELS IN 2 AND (2+1) DIMENSIONS**

by

**Ian Stokoe**

Thesis submitted for the degree of  
Doctor of Philosophy  
at the  
University of Durham

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*September 1987*



23 JAN 1988

Thesis  
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**To Susy,  
with all my love.**

**x x x**

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## ABSTRACT

### “Skyrmion and other Extended Solutions of Non-Linear $\sigma$ -Models in 2 and (2+1) Dimensions”

*Ian Stokoe*

Low dimensional models are generally regarded to be a convenient theoretical laboratory for studying various aspects of elementary particle theory. In this thesis, the extended solutions of one particular class of such models, namely the  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models in 2 dimensions, are discussed. Special attention is paid to the shape of these extended structures and their dependence on the parameters of the solutions. Time dependence is introduced into the models, and properties of the moving objects in these (2 + 1)-dimensional theories are explored. In particular, the Hopf terms of the theories are investigated, and their relation to the spin of the extended solutions is discussed. Also the classical dynamics of these moving objects, and their explanation in terms of the geodesic motions on certain Hermitian and Kähler manifolds is considered. Finally the embedding of the  $\mathbb{C}P^{n-1}$  solutions into the 2-dimensional  $U(n)$  chiral models is studied, paying particular attention to the stability of these embedded solutions in the larger group space, and to the number of independent negative modes of the fluctuation operator around these solutions.

## DECLARATION

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1984 and September 1987. This material has not been submitted previously for any degree in this or any other university.

No claim of originality is made for chapter 2; the work in chapter 3 is claimed as original except where otherwise indicated. Apart from the introductory sections of chapters 4, 5 and 6, the remainder of these chapters are all claimed as original, except where authors have been specifically acknowledged in the text.

Much of chapter 5 has been published in a paper by the author in collaboration with W. J. Zakrzewski [1]; the work in chapter 6, undertaken by the author with W. J. Zakrzewski and B. Piette, is available as a Durham University preprint [2], and is to be published in *Zeitschrift für Physik C*.

The copyright of this thesis rests with the author. No quotation from it should be published without his prior written consent and information derived from it should be acknowledged.

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## 1. INTRODUCTION.

It is widely believed that non-Abelian gauge theories play an important role in any field theoretical description of the theory of elementary particles. For example, weak and electromagnetic interactions are described by such a theory, and it is generally felt that the same is true for strong interactions. Gauge theories are defined in terms of a Lagrangian density  $\mathcal{L}$ : for example, in the case of an  $SU(2)$  symmetry group,  $\mathcal{L}$  is given by

$$\mathcal{L} = \text{tr } F_{\mu\nu} F_{\mu\nu} \quad (1.1)$$

( $\mu, \nu = 1, 2, 3, 4$ ), where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (1.2)$$

and where  $A_\mu$  is a vector function of Euclidean four-dimensional space-time with values in  $SU(2)$ .

Many quantities in these theories are given in terms of functional integrations, and hence one of the major difficulties in making progress with any particular theory is the lack of understanding of how to perform many of these integrations. One approach is to calculate them numerically: the results of such attempts are encouraging, but unavoidably involve various approximations, making the results inconclusive.

If one attempts to calculate the functional integrations analytically, then the only viable approach available in many cases is based on an expansion around the stationary points of the action of the theory, followed by perturbation theory of the resultant effective theory. To proceed in this way, one therefore has to determine all the stationary points of the action, which as usual are given by the Euler-Lagrange equations of the theory:

$$D_\mu F_{\mu\nu} \equiv \partial_\mu F_{\mu\nu} - [A_\mu, F_{\mu\nu}] = 0. \quad (1.3)$$



Written in terms of the gauge potential  $A_\mu$ , these equations are second order, highly non-linear partial differential equations.

Now, due to the Bianchi identity

$$D_\mu {}^*F_{\mu\nu} = 0, \quad (1.4)$$

where

$${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (1.5)$$

and  $\epsilon_{\mu\nu\alpha\beta}$  is the totally anti-symmetric 4-tensor (with  $\epsilon_{1234} = +1$ ), it is well known that a subclass of solutions of the Euler-Lagrange equations is provided by the solutions of the first order equations

$$F_{\mu\nu} = \pm {}^*F_{\mu\nu}, \quad (1.6)$$

known as the self-duality equations. These equations can be thought of as resulting from requiring the Lagrangian density to be equal to the modulus of the topological charge density of the theory, that is, imposing the additional constraint

$$\mathcal{L} = \pm \mathcal{Q} \quad (1.7)$$

where

$$\mathcal{Q} = \text{tr } F_{\mu\nu} {}^*F_{\mu\nu} \quad (1.8)$$

is the topological charge density.

Now, the most interesting solutions of these equations are those for which the action is finite, since it is only for them that the perturbation theory of fluctuations around them can be set up. Indeed, *all* finite action solutions of (1.6) have been implicitly determined by Atiyah et al. [3]: in the case of the plus (minus) sign, these solutions are called instantons (anti-instantons), and they correspond to local minima of the action. Hence these solutions are stable under small fluctuations.

But what of finite action solutions of (1.3) which are *not* self-dual? It has turned out to be very difficult to find such solutions, although presumably they too will play a part in the calculation of the functional integrations.

Given the complexity of non-Abelian gauge theories in four dimensions, it is natural that people started looking at models in lower dimensions which exhibit some features of the four-dimensional theory, but where the relevant calculations are simpler to perform. In particular, for two (Euclidean) dimensions several classes of models have been proposed, namely the  $O(n)$  non-linear  $\sigma$ -models [4], the principal, or  $U(n)$ , chiral models [5], the  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models [6,7,8,9], and also the latter's non-Abelian generalizations, the complex Grassmannian models [10]. All of these models are interesting in their own right, and exhibit many properties similar to those of the four-dimensional theories; in this thesis, however, the focus of attention is the  $\mathbb{C}P^{n-1}$  models. This class of models additionally exhibits a first-order system of equations analogous to the four-dimensional self-duality equations (1.6), and has solutions which are also analogous to the instantons of gauge theories. Furthermore, the stability of these finite action solutions is guaranteed topologically for all  $n$  (in contrast to the  $O(n)$  models, which have stable instanton solutions only for  $n = 3$ , [11]). Of course, some of the properties exhibited by these models may not necessarily indicate what happens in non-Abelian gauge theories; on the other hand, some of these models may be directly relevant in other specific physical systems, for example, for the case  $n = 2$  the model provides a phenomenological description of Heisenberg ferromagnets in a two-dimensional system [12].

Chapter 2 therefore introduces in detail the basic quantities involved in the  $\mathbb{C}P^{n-1}$  models in two Euclidean dimensions: after the defining equations, the general instanton solution is presented and the stability of the solution is discussed. Other finite action solutions of the models do exist, however, and after a brief overview of the  $O(n)$  non-linear  $\sigma$ -models the chapter goes on to present

non-instanton solutions of the  $\mathbb{C}P^{n-1}$  models, constructed from instanton solutions of the  $O(n)$  models. The chapter ends by discussing a useful reformulation of the  $\mathbb{C}P^{n-1}$  models using projectors, a reformulation which allows the *general* finite action solution of the *full* Euler-Lagrange equations to be written down in an elementary way. This general solution contains both the instanton solutions and also non-instanton solutions, and an obvious application of this would be to attempt the evaluation of the functional integrals mentioned earlier in this introduction. However, the final pages of chapter 2 demonstrate that any solution of the  $\mathbb{C}P^{n-1}$  model which is neither instanton nor anti-instanton in nature is necessarily unstable under small perturbations, and at present there is no universal agreement on how to deal with instabilities in this type of calculation. Rather than trying to pursue this idea further, we therefore return our attention to the solutions of the models themselves.

Because of the ease with which explicit solutions can be constructed, the  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models offer a good theoretical laboratory for clarifying non-linear interactions in particle physics. To enable us to benefit from this, we clearly need a good understanding of the nature of the solutions we are constructing. To this end, the first part of chapter 3 presents a detailed examination of the action densities of various solutions in the simplest, ie.,  $\mathbb{C}P^1$  model, investigating the dependence these quantities have on the parameters of the solutions, and discussing the various interpretations which can be adopted. Similar techniques are then used to investigate possible analogous field configurations for a system containing both instantons and anti-instantons—no such exact solutions exist, but various field configurations approximate to solutions provided the instantons and anti-instantons are well separated in the complex plane. Even though they are approximate solutions, these configurations may still have an important part to play in the theory. For example, chapter 3 goes on to investigate an interaction which can be identified in these approximate

solutions, and which has an obvious physical analogue.

Up to this point in the thesis, all the structures so far considered have been solutions (or approximate solutions) of the two-dimensional  $\mathbb{C}P^{n-1}$  models, with no time dependence. If we wish to make further progress it is therefore of considerable interest to try to extend the models to  $(2+1)$ -dimensional (Minkowski) space, and to investigate time-dependent finite energy solutions of these models. The *static* solutions of these  $(2+1)$ -dimensional theories, known as skyrmions [13], have already been considered—they are simply the solutions of the two-dimensional theory discussed previously. To go beyond the static field configurations requires the solution of the full  $(2+1)$ -dimensional field equations, which is rather difficult. On the other hand, it has been suggested that one interprets the static field configurations as static extended objects, and then obtains *approximate* time-dependent solutions of the full theory by introducing a time dependence into these extended objects. This type of time evolution is discussed in the latter part of chapter 3, and by considering the various extended objects already discussed in this chapter as moving objects in  $(2+1)$  dimensions, various constraints are derived to ensure finiteness of the kinetic energy of each system.

If one wants to model non-linear interactions using these moving objects, as much information as possible needs to be known about the evolution of these structures. In general the movement may induce interaction forces to act, which will cause a distortion of the shape of the extended objects and may lead to some radiation effects, but even though the general problem may be intractable, observations can be made about the evolution of the system in the limit of small velocities. For example, in the theory of non-Abelian BPS monopoles, Manton [14] noted that the initial motion of a similar system of monopoles follows the geodesics in the space of parameters of the static extended solutions. This observation has led to a partial understanding of the evolution of a system of monopoles [15,16], and information about the way they scatter off each other. A

similar consideration for the  $(2 + 1)$ -dimensional  $\mathbb{C}P^{n-1}$  models would therefore be most welcome.

But before addressing this matter, we investigate a crucial observation made by Wilczek and Zee [17] on the  $(2 + 1)$ -dimensional models: namely, one is at liberty to introduce an additional topological term, known as the Hopf term, into the usual expression for the action of the models. This extra term, although it does not affect the classical equations of motion, does have a profound effect on the spin and statistics properties of the extended structures in the models, and in fact leads to the possibility of fractional spin for the skyrmions. This observation is of particular interest, not only because of the extra information it provides on the structures in the models, but also in view of its possible physical relevance: it has been suggested that an explanation of the fractional quantum Hall effect may be found in terms of such objects [18,19]. Chapter 4 therefore discusses in detail the construction of this Hopf term and its lack of effect on the equations of motion of the models. Explicit expressions for the Hopf terms are derived for the extended structures met in chapter 3, and the values of these terms are calculated for specific time evolutions of the objects, ending with a discussion on the spin properties these structures exhibit. The chapter ends with investigations into a more general method of calculating Hopf terms, for use with configurations which do not lend themselves easily to the methods discussed previously.

Chapter 5 returns to the consideration of the slowly moving skyrmions in the  $\mathbb{C}P^{n-1}$  models. For the case  $n = 2$ , Din and Zakrzewski [20] have shown that the classical dynamics of such structures can be described by the geodesic motion on a Kähler manifold of the structures' parameters. This result is reproduced in chapter 5, followed by detailed calculations aiming to extend the various observations that can be made to cases other than  $n = 2$ , particularly for structures of a non-instanton-like nature. These latter cases are far from

trivial, and chapter 5 is completed by a detailed investigation into the methods one can use to determine the nature of the metrics defined by these structures.

At the beginning of this introduction, several classes of two-dimensional models were mentioned, of which  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models was one. To round off this thesis, chapter 6 looks at the embedding of the  $\mathbb{C}P^{n-1}$  solutions into one of the other classes of models, namely the  $U(n)$  chiral models. Recently, much progress has been made in this class of models—this progress is discussed briefly at the beginning of the chapter, and the remainder of the chapter is devoted to studying the important question of stability of the  $\mathbb{C}P^{n-1}$  embedded solutions in this larger group space. Negative modes of the fluctuation operator around the solutions are considered, and the number of independent negative modes is investigated.

Finally, chapter 7 summarizes the major results in this thesis, and sets out to indicate some of the topics which could be studied in order to further expand our knowledge of this interesting and highly non-trivial area of elementary particle theory.

## 2. BASIC EQUATIONS.

The  $\mathbb{C}P^{n-1}$  model in two dimensions, as first discussed by Eichenherr [6], Cremmer and Scherk [7], Golo and Perelomov [8], and d'Adda et al. [9], can be defined in terms of an  $n$ -dimensional complex vector field  $Z_\alpha = Z_\alpha(x, y)$  where  $\alpha = 1, \dots, n$  on the Euclidean space  $E^2$ , subject to the constraint

$$|Z|^2 = 1. \quad (2.1)$$

Two such fields  $Z_\alpha$  and  $Z'_\alpha$  are taken to be equivalent if they are related by a regular gauge transformation

$$Z'_\alpha = Z_\alpha e^{i\Lambda(x,y)}, \quad (2.2)$$

that is, the theory is required to be  $U(1)$  gauge invariant. The covariant derivative is given by

$$D_\mu = \partial_\mu - \bar{Z} \cdot \partial_\mu Z \quad (2.3)$$

where  $\mu = x, y$  and the bar denotes complex conjugation, and the Lagrangian density for the theory is

$$\begin{aligned} \mathcal{L}(Z) &= \overline{D_\mu Z} \cdot D_\mu Z \\ &= \partial_\mu \bar{Z} \cdot \partial_\mu Z + (\bar{Z} \cdot \partial_\mu Z)^2. \end{aligned} \quad (2.4)$$

With the action  $S$  defined by

$$S = \int \mathcal{L} d^2x \quad (2.5)$$

then the Euler-Lagrange equations, corresponding to the stationary points of  $S$ , are

$$D_\mu D_\mu Z + \left( \overline{D_\mu Z} \cdot D_\mu Z \right) Z = 0. \quad (2.6)$$

The solutions of these equations which result in a finite action are the required classical solutions of the model. [To be more precise, imposing the condition of finiteness of the action means that the base space of the model is the compactified  $E^2$ , that is,  $S^2$ , since the points at infinity are identified.]



As demonstrated by Din and Zakrzewski [21], it is useful at this stage to introduce the complex variables

$$x_{\pm} = x \pm iy. \quad (2.7)$$

The Lagrangian density can then be rewritten as

$$\mathcal{L} = 2 \left[ |D_+ Z|^2 + |D_- Z|^2 \right] \quad (2.8)$$

where

$$D_{\pm} = \partial_{\pm} - \bar{Z} \cdot \partial_{\pm} Z, \quad (2.9)$$

and the Euler-Lagrange equations become

$$D_- D_+ Z + |D_+ Z|^2 = 0 \quad (2.10)$$

or

$$D_+ D_- Z + |D_- Z|^2 = 0. \quad (2.11)$$

It is also useful to introduce at this stage the quantity  $\mathcal{Q}$  defined by

$$\mathcal{Q} = 2 \left[ |D_+ Z|^2 - |D_- Z|^2 \right] \quad (2.12)$$

which, as shall be seen later, is the topological charge density.

Having rewritten the equations in this form, it is clear from (2.10) and (2.11) that there exists a subclass of equations called the self-duality equations,

$$D_{\pm} Z = 0 \quad (2.13)$$

which correspond to the situation

$$\mathcal{L} = \pm \mathcal{Q}. \quad (2.14)$$

The finite action solutions of  $D_- Z = 0$  are known as the instanton solutions of the  $\mathbb{C}P^{n-1}$  model (anti-instantons being the solutions of  $D_+ Z = 0$ ).

Explicitly, it has been shown by d'Adda et al. [9] that the general instanton solution is

$$Z_\alpha = \frac{f_\alpha(x_+)}{|f_\alpha(x_+)|} \quad (2.15)$$

where the  $f_\alpha$ 's are polynomials in  $x_+$  of the form

$$f_\alpha(x_+) = \lambda_\alpha \prod_{i=1}^k (x_+ - a_\alpha^i) \quad (2.16)$$

with no common roots, where  $\lambda_\alpha$  and  $a_\alpha^i$  are complex constants. [The general anti-instanton solution is obtained by simple complex conjugation.]

The degree  $k$  of the polynomials  $f_\alpha$  is called the instanton number, and the action for such solutions is

$$S = 2\pi k. \quad (2.17)$$

For example, in the  $\mathbb{C}P^1$  model, consider

$$f_\alpha(x_+) = (1, x_+). \quad (2.18)$$

This is an instanton solution of the model, and using (2.15) we find that

$$Z_\alpha = \frac{(1, x_+)}{\sqrt{1 + x_+x_-}}. \quad (2.19)$$

Hence from (2.8), after a few lines of algebra,

$$\mathcal{L} = \frac{2}{(1 + x_+x_-)^2} \quad (2.20)$$

and thus from (2.5) we obtain after simple integration

$$S = 2\pi, \quad (2.21)$$

which implies that (2.18) is a one-instanton solution, as we would expect from the form of  $f_\alpha$ .

The existence and stability of these instantons can be predicted topologically, and the observation that the base space is the compactified  $E^2$ , and not simply  $E^2$ , is essential for the existence of the non-trivial topological structure. This

can be seen by recalling that the complex projective space  $\mathbb{C}P^{n-1}$  is the space of all equivalence classes  $[Z_\alpha]$  of the complex  $n$ -dimensional vectors  $Z_\alpha \neq 0$ , with the equivalence relation defined by

$$Z'_\alpha \sim Z_\alpha \quad \text{if} \quad Z'_\alpha = \lambda Z_\alpha \quad \text{where} \quad \lambda \in \mathbb{C}. \quad (2.22)$$

D'Adda et al. [9] then noted that if the fields  $[Z_\alpha](\underline{x})$ , where  $\underline{x} = (x, y)$ , approach a constant value  $[Z^\infty]$  as  $|\underline{x}| \rightarrow \infty$ , then the fields  $Z_\alpha(\underline{x})$  in the equivalence class need not be continuously deformable into each other, because if

$$[Z_\alpha](\underline{x}) \rightarrow [Z^\infty] \quad \text{as} \quad |\underline{x}| \rightarrow \infty \quad (2.23)$$

then it follows that

$$Z_\alpha(\underline{x}) \rightarrow g\left(\frac{\underline{x}}{|\underline{x}|}\right) Z^\infty \quad \text{as} \quad |\underline{x}| \rightarrow \infty \quad (2.24)$$

where  $g(\underline{x}/|\underline{x}|)$  is a direction-dependent, singular phase factor. In other words the fields fall into different homotopy classes. The collection of these classes is known as the second homotopy group  $\Pi_2$ , and using well-known results from pure mathematics, d'Adda et al. concluded that

$$\Pi_2(\mathbb{C}P^{n-1}) = \mathbb{Z} \quad (2.25)$$

where  $\mathbb{Z}$  is the set of all integers. So each homotopy class can be labelled by an integer winding number  $Q$ , and therefore every field  $Z$  has a topological charge  $Q$  with values in  $\mathbb{Z}$ . Din and Zakrzewski [22], for example, exhibited a simple expression for  $Q$  as follows :-

$$Q = \int J^0 d^2x \quad (2.26)$$

where

$$J^0 = -\frac{i}{2\pi} \epsilon^{\mu\lambda} \overline{D_\mu Z} \cdot D_\lambda Z \quad (2.27)$$

$$\text{with} \quad \epsilon^{12} = +1. \quad (2.28)$$

Written in terms of the complex variables  $x_{\pm}$ , this becomes

$$Q = \frac{1}{2\pi} \int [2(|D_+Z|^2 - |D_-Z|^2)] d^2x, \quad (2.29)$$

that is, using equation (2.12),

$$Q = \frac{1}{2\pi} \int \mathcal{Q} d^2x, \quad (2.30)$$

demonstrating the previous claim that  $\mathcal{Q}$  is the topological charge density.

Defined in this way, if  $Z$  is an instanton (anti-instanton) field, then the topological charge will be positive (negative) and gives directly the instanton (anti-instanton) number by virtue of equations (2.14) and (2.17). For example, with  $f_{\alpha}$  as in equation (2.18), it is very easy to calculate that  $Q = 1$ , implying as it should that  $f_{\alpha}$  is a one-instanton solution.

Now  $Q$  is an invariant topological quantity: if a small complex fluctuation (the explicit form of which will be stated later) is introduced into  $Z$ , the field remains in the same homotopy class and  $Q$  is unchanged. Also equations (2.8) and (2.12) can be used to show that

$$\mathcal{L} = \mathcal{Q} + 4|D_-Z|^2 \quad (2.31)$$

or

$$\mathcal{L} = -\mathcal{Q} + 4|D_+Z|^2. \quad (2.32)$$

That is,

$$\mathcal{L} \geq |\mathcal{Q}| \quad (2.33)$$

or after integrating

$$S \geq |2\pi Q|. \quad (2.34)$$

The equality holds if and only if  $Z$  is either an instanton ( $\mathcal{L} = \mathcal{Q}$ ,  $D_-Z = 0$ ) or anti-instanton ( $\mathcal{L} = -\mathcal{Q}$ ,  $D_+Z = 0$ ) solution. Hence from this topological argument we can conclude the well-known result that the instanton (anti-instanton)

solutions are absolute minima of the action—and are therefore stable—and have definite positive (negative) integer topological charge.

The question now arises whether solutions of finite action other than the instantons and anti-instantons exist: the answer is yes, and Din and Zakrzewski [23] exhibited some of these non-instanton solutions by using certain results from two-dimensional Euclidean  $O(n)$  non-linear  $\sigma$ -models.

The  $O(n)$  models [4,11] are defined on the two-dimensional Euclidean space  $E^2$  in terms of real  $n$ -component fields  $q^i = q^i(x, y)$  where  $i = 1, \dots, n$ , subject to the constraint

$$q \cdot q = 1. \quad (2.35)$$

The Lagrangian density for this theory is defined to be

$$\mathcal{L}(q) = \partial_\mu q \cdot \partial_\mu q \quad \text{where} \quad \mu = x, y \quad (2.36)$$

and gives rise to the corresponding Euler-Lagrange equations

$$\partial_\mu \partial_\mu q + (\partial_\mu q \cdot \partial_\mu q) q = 0. \quad (2.37)$$

Again the base space  $E^2$  is compactified by requiring the solutions to be of finite action.

Belavin and Polyakov [12], and Woo [24] have shown that these models have stable instanton solutions only for the specific case of  $n = 3$ , given explicitly by

$$w = \lambda \frac{\prod_{i=1}^k (x_+ - a_i)}{\prod_{i=1}^k (x_+ - b_i)} \quad (2.38)$$

where  $w$  is a complex field related to  $q^i$  by the relation

$$w = \frac{q^1 + iq^2}{1 + q^3} \quad (2.39)$$

and where  $\lambda, a_i, b_i$  are complex constants such that  $a_i \neq b_j$  for all  $i$  and  $j$ . Anti-instanton solutions are given as before by complex conjugation of the field  $w$ , and the instanton number of the solution (2.38) is  $k$ . Finally the action  $S$  for this instanton solution can be calculated using (2.36) to be

$$S = 8\pi k. \quad (2.40)$$

The stability of the  $O(3)$  instantons can again be explained topologically: the solutions are characterized by different values of a conserved topological number because they fall into different homotopy classes. However, for  $n > 3$ , there is no corresponding non-trivial topological quantity for the  $O(n)$  solutions, and Din and Zakrzewski [11] have shown that its absence makes all non-trivial solutions unstable.

To conclude this brief overview of the  $O(n)$  non-linear  $\sigma$ -models, we state the observation of d'Adda et al. [9] that the  $\mathbb{C}P^1$  model is in fact equivalent to the  $O(3)$   $\sigma$ -model: if  $q^i$  and  $Z_\alpha$  are related by

$$q^i = \overline{Z}_\alpha \sigma_{\alpha\beta}^i Z_\beta \quad (\alpha, \beta = 1, 2) \quad (2.41)$$

$$(i = 1, 2, 3)$$

where  $\sigma^i$  are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.42)$$

then

$$q \cdot q = \overline{Z} \cdot Z = 1 \quad (2.43)$$

and simple algebraic manipulation shows that the Lagrangian densities of the two models are essentially the same:-

$$\partial_\mu q \cdot \partial_\mu q = 4 \overline{D_\mu Z} \cdot D_\mu Z. \quad (2.44)$$

Hence the two theories are equivalent.

Armed with these results, Din and Zakrzewski [23] have produced non-instanton solutions of the  $\mathbb{C}P^{n-1}$  model, after first hinting via the energy-momentum tensor that these solutions can exist: the energy-momentum tensor of the theory corresponding to the Lagrangian density (2.4) is

$$J_{\mu\nu} = -\delta_{\mu\nu} \overline{D_\lambda Z} \cdot D_\lambda Z + \overline{D_\mu Z} \cdot D_\nu Z + D_\mu Z \cdot \overline{D_\nu Z} \quad (2.45)$$

and this quantity must be conserved, ie.

$$\partial_\mu J_{\mu\nu} = 0. \quad (2.46)$$

Written in terms of the complex variables  $x_{\pm}$ , this conservation equation becomes

$$\partial_- [D_+ Z \overline{D_- Z}] = 0, \quad (2.47)$$

in other words,  $D_+ Z \overline{D_- Z}$  is a function of  $x_+$  only. Imposing finiteness of the action gives  $|D_{\mu} Z| \rightarrow 0$  as  $|\underline{x}| \rightarrow \infty$ , which essentially implies that

$$D_+ Z \overline{D_- Z} = 0. \quad (2.48)$$

As was noted, this does *not* further imply that

$$D_- Z = 0 \quad (2.49)$$

or

$$D_+ Z = 0 \quad (2.50)$$

in general; if true, then they could have concluded that the instantons and anti-instantons were the only existing solutions because (2.49) and (2.50) are the self-duality equations used to derive such solutions. In the  $n = 2$  case, since

$$\overline{Z} \cdot D_{\pm} Z = 0 \quad (2.51)$$

then (2.48) does in fact imply that at least one of equations (2.49) and (2.50) holds, but this is not so for general  $n$ . This demonstrates the known fact that only instanton and anti-instanton solutions exist in the  $\mathbb{C}P^1$  (or equivalently  $O(3)$ ) model.

For  $n \geq 3$  there are non-trivial solutions besides the instantons and anti-instantons: they obtained these by noting that as real  $(2p + 1)$ -dimensional vectors, solutions  $q^i$  of the  $O(2p + 1)$  models can easily be embedded in the  $(2p + 1)$ -dimensional space of the  $\mathbb{C}P^{2p}$  models by the simple identification

$$q^i \equiv Z_i \quad \text{where} \quad i = 1, \dots, 2p + 1. \quad (2.52)$$

These embedded solutions are still solutions of the  $\mathbb{C}P^{2p}$  models—because  $q$  is real and  $q \cdot q = 1$ , then

$$q \cdot \partial_{\mu} q = 0 \quad (2.53)$$

which means that

$$D_\mu q \equiv \partial_\mu q. \quad (2.54)$$

Hence, for  $q$  the Euler-Lagrange equations (2.6) and (2.37) are identical.

In fact, as they went on to explain, these embedded solutions can be interpreted as a mixture of  $\mathbb{C}P^{2p}$  instantons and anti-instantons. First notice that since these solutions are real, then their topological charge is zero: from (2.12)

$$\begin{aligned} \mathcal{Q} &= 2 \left[ |D_+ Z|^2 - |D_- Z|^2 \right] \\ &= 2 \left[ |\partial_+ Z|^2 - |\partial_- Z|^2 \right] \\ &= 0 \quad \text{because } Z \text{ is real,} \end{aligned} \quad (2.55)$$

and so

$$Q = \frac{1}{2\pi} \int \mathcal{Q} d^2x = 0. \quad (2.56)$$

Therefore if this interpretation is correct, these solutions should contain an equal number of instantons and anti-instantons. Now consider the  $O(3)$  model instanton solutions given by equation (2.38), which can be written as

$$w = \frac{A}{B} \quad (2.57)$$

where

$$\begin{aligned} A(x_+) &= \lambda \prod_{i=1}^k (x_+ - a_i), \\ B(x_+) &= \prod_{i=1}^k (x_+ - b_i). \end{aligned} \quad (2.58)$$

Using (2.39) this complex field can be related to the real field  $q$ , and it is found that

$$q = \frac{1}{(|A|^2 + |B|^2)} \left( A\bar{B} + \bar{A}B, -i(A\bar{B} - \bar{A}B), |B|^2 - |A|^2 \right). \quad (2.59)$$

Interpretation of this field as a solution of the  $\mathbb{C}P^2$  model is made simpler by introducing a translation defined by

$$A_s(x_+) = A(x_+ - s), \quad B_t(x_+) = B(x_+ - t) \quad (2.60)$$



and writing

$$q_{s,t} = \frac{1}{N} \left( A_s \overline{B_t} + \overline{A_t} B_s, -i(A_s \overline{B_t} - \overline{A_t} B_s), B_s \overline{B_t} - A_s \overline{A_t} \right) \quad (2.61)$$

where  $N$  is the normalization factor. Now, if  $s = t = 0$ , then  $q_{s,t} = q$ . Furthermore, if  $|s|, |t|$  and  $|s - t|$  are taken to be large compared to  $|a_i|$  and  $|b_i|$ , then if  $x_+$  is taken to be in the neighbourhood of  $s$ ,  $q_{s,t}$  looks like a  $k$ -instanton solution of the  $\mathbb{C}P^2$  model (see equations (2.15) and (2.16)); with  $x_+$  in the neighbourhood of  $t$ ,  $q_{s,t}$  resembles a  $k$ -anti-instanton solution. Hence (2.59) can be interpreted naturally as a mixture of  $k$  instantons and  $k$  anti-instantons.

Similarly, embeddings of  $O(2p + 1)$  solutions into  $\mathbb{C}P^{2p}$  can be thought of as mixtures of equal numbers of instantons and anti-instantons in  $\mathbb{C}P^{2p}$ —for more details see reference [23].

However, it turns out that *all* these embedded solutions are unstable: in fact, as shall be demonstrated later, *any* solution of the  $\mathbb{C}P^{n-1}$  model which is neither instanton nor anti-instanton in nature is necessarily unstable.

We end this chapter by discussing a reformulation of the  $\mathbb{C}P^{n-1}$  model in a gauge invariant way, looking at more general solutions of the model and considering their stability.

To reformulate the model, as discussed for example by Sasaki [25] and by Zakrzewski [26], an  $n \times n$  projection matrix  $\mathbb{P}$  is introduced, defined by

$$\mathbb{P} = Z.Z^\dagger \quad (2.62)$$

where the dagger denotes hermitian conjugation and where

$$\mathbb{P} = \mathbb{P}^\dagger = \mathbb{P}^2 \quad (\Leftrightarrow \overline{Z}.Z = 1). \quad (2.63)$$

The Lagrangian density (2.4) can be rewritten as

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial_\mu \mathbb{P} \cdot \partial_\mu \mathbb{P}) \quad (2.64)$$

and it can be shown by simple algebraic manipulation that the Euler-Lagrange equations (2.6) become

$$[\partial_\mu \partial_\mu \mathbb{P}, \mathbb{P}] = 0 \quad (2.65)$$

or, when written in terms of complex variables,

$$[\partial_+ \partial_- \mathbb{P}, \mathbb{P}] = 0. \quad (2.66)$$

The self-duality equations (2.13) in this formulation become

$$\partial_- \mathbb{P} \cdot \mathbb{P} = 0 \quad (\Leftrightarrow D_- Z = 0) \quad (2.67)$$

and

$$\mathbb{P} \cdot \partial_- \mathbb{P} = 0 \quad (\Leftrightarrow D_+ Z = 0) \quad (2.68)$$

or equivalently

$$\mathbb{P} \cdot \partial_+ \mathbb{P} = 0 \quad (\Leftrightarrow D_- Z = 0) \quad (2.69)$$

and

$$\partial_+ \mathbb{P} \cdot \mathbb{P} = 0 \quad (\Leftrightarrow D_+ Z = 0). \quad (2.70)$$

We now re-address ourselves to the question of non-instanton solutions of (2.6). Din and Zakrzewski [21] have shown that the general finite action solution of (2.6) can be expressed in terms of an arbitrary rational analytic vector  $f = f(x_+)$  by

$$Z = \frac{\hat{Z}^{(k)}}{|\hat{Z}^{(k)}|} \quad (2.71)$$

where  $k = 0, 1, \dots, n - 1$ , with

$$\hat{Z}^{(k)} = \partial_+^k f - \sum_{i,j=0}^{k-1} \partial_+^i f \left( M_{i,j}^{(k)} \right)^{-1} \partial_+ M_{j,k-1}^{(k)} \quad (2.72)$$

and the matrix  $M^{(k)}$  given by

$$M_{i,j}^{(k)} = \overline{\partial_+^i f} \cdot \partial_+^j f \quad \text{where} \quad i, j = 0, \dots, k - 1. \quad (2.73)$$

Taking  $k = 0$  gives the instanton solutions;  $k = n - 1$  results in the anti-instantons appearing. However, for any other choice of  $k$  within the specified range, new classes of solutions are obtained.

There is an alternative construction of this general solution using Gram-Schmidt orthonormalization which shall be more useful for our later discussions, and which uses the projector form of the model described above. This construction is described below using Zakrzewski's notation [26], and although by making certain identifications the two formulations of the solution can be seen to be equivalent, we shall nevertheless exhibit Sasaki's explicit proof [25] that the expressions obtained by this second method solve the  $\mathbb{C}P^{n-1}$  Euler-Lagrange equations in their projector form (2.66).

The construction starts by considering a vector field  $g \in \mathbb{C}^n - \{0\}$ . An operator  $P_+$  is defined to act on  $g$  by

$$P_+g = \partial_+g - \frac{g(\bar{g} \cdot \partial_+g)}{|g|^2} \quad (2.74)$$

and its repeated action is defined by

$$P_+^k g = P_+ (P_+^{k-1} g) \quad (2.75)$$

where

$$P_+^0 g \equiv g. \quad (2.76)$$

Analogous to the definition of  $P_+$ , an operator  $P_-$  can also be defined, involving differentiation with respect to  $x_-$  instead of  $x_+$ . It is not difficult to demonstrate that  $P_+P_-g \sim g$  and so  $P_\pm$  seems to behave like a kind of raising and lowering operator. With this definition of  $P_+^k$ , algebraic manipulation soon produces the fact that the  $\hat{Z}^{(k)}$  of (2.72) are given by

$$\hat{Z}^{(k)} = P_+^k f \quad (2.77)$$

where  $f$  again is an arbitrary rational analytic vector.

To proceed further, Zakrzewski [26] noted that

$$\begin{aligned} 1. \quad & \overline{P_+^k f} \cdot P_+^l f = 0 \quad \text{if} \quad l \neq k, \\ 2. \quad & \partial_- (P_+^k f) = -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2}, \end{aligned} \quad (2.78a)$$

$$3. \quad \partial_+ \left( \frac{P_+^{k-1} f}{|P_+^{k-1} f|^2} \right) = \frac{P_+^k f}{|P_+^{k-1} f|^2}, \quad (2.78b)$$

$$4. \quad P_+^n f = 0,$$

all of which either follow directly from the definitions, or are easy to prove from them. These orthogonality properties show that the  $P_+^k f$  vectors can be thought of as being obtained by Gramm-Schmidt orthogonalizing the sequence of vectors

$$f, \partial_+ f, \partial_+^2 f, \dots, \partial_+^l f, \dots \quad (2.79)$$

and if normalized, as shown by Sasaki [25], in fact provide solutions of the  $\mathbb{C}P^{n-1}$  model Euler-Lagrange equations: denote by

$$e_1, e_2, \dots, e_n \quad (2.80)$$

the vectors obtained by Gramm-Schmidt orthonormalizing the sequence (2.79).

Take the  $j^{\text{th}}$  element of the sequence and consider

$$\mathbb{P} = e_j e_j^\dagger. \quad (2.81)$$

Also consider another projector

$$\mathbb{Q} = \sum_{k=1}^{j-1} e_k e_k^\dagger. \quad (2.82)$$

Then, using the relations

$$\partial_- e_l = e_{l-1} \left( e_{l-1}^\dagger \partial_- e_l \right) + e_l \left( e_l^\dagger \partial_- e_l \right) \quad (2.83)$$

and

$$\partial_+ e_l = e_{l+1} \left( e_{l+1}^\dagger \partial_+ e_l \right) + e_l \left( e_l^\dagger \partial_+ e_l \right) \quad (2.84)$$

which follow easily from equations (2.78), various identities can be proved involving  $\mathbb{P}$  and  $\mathbb{Q}$ . First,  $\mathbb{Q}$  and  $(\mathbb{P} + \mathbb{Q})$  can be thought of as projectors describing instanton solutions in various complex Grassmannian  $\sigma$ -models—for more details see references [25] and [26]—and as such they satisfy the equations

$$\partial_- \mathbb{Q} \mathbb{Q} = 0 \quad (2.85)$$

and

$$\partial_-(\mathbb{P} + \mathbb{Q}) \cdot (\mathbb{P} + \mathbb{Q}) = 0 \quad (2.86)$$

(cf. equation (2.67)). However, due to (2.83) and (2.84), it can be shown that

$$\partial_- \mathbb{P} \cdot \mathbb{Q} = 0 \quad (2.87)$$

and so (2.85) and (2.86) give

$$\partial_- \mathbb{P} \cdot \mathbb{P} + \partial_- \mathbb{Q} \cdot \mathbb{P} = 0. \quad (2.88)$$

Also from (2.83) and (2.84) it is found that

$$\mathbb{P} \cdot \partial_+ \mathbb{Q} = \partial_+ \mathbb{Q} \quad (2.89)$$

and equivalently

$$\partial_- \mathbb{Q} \cdot \mathbb{P} = \partial_- \mathbb{Q}, \quad (2.90)$$

and using these two equations, (2.88) can be written as

$$\partial_- \mathbb{P} \cdot \mathbb{P} + \partial_- \mathbb{Q} = 0. \quad (2.91)$$

Taking the hermitian conjugate of this gives

$$\mathbb{P} \cdot \partial_+ \mathbb{P} + \partial_+ \mathbb{Q} = 0, \quad (2.92)$$

and finally if the combination  $\partial_+(2.91) - \partial_-(2.92)$  is considered, it is found that

$$[\partial_+ \partial_- \mathbb{P}, \mathbb{P}] = 0 \quad (2.93)$$

which is the required Euler-Lagrange equation (see (2.66)). This completes Sasaki's proof. So

$$Z = \frac{P_+^k f}{|P_+^k f|} \quad (2.94)$$

is a genuine solution of the  $\mathbb{C}P^{n-1}$  model:  $k = 0$  corresponds to instanton solutions;  $k = n - 1$  to anti-instantons. Any other choice of  $k$  gives new, non-instanton solutions since the  $P_+^k f$ ,  $k = 0, 1, \dots, n - 1$  are linearly independent.

We shall use these non-instanton solutions extensively in later chapters: let us conclude this chapter by considering their stability.

We have already noted that the instanton and anti-instanton solutions are relative minima of the action, that is, they are stable solutions. We now exhibit Din and Zakrzewski's proof [21] that any solution which is neither instanton nor anti-instanton in nature is necessarily unstable.

Assume for a certain solution  $Z$  that

$$D_{\pm}Z \neq 0 \quad (2.95)$$

and consider a small complex fluctuation  $\phi$  about  $Z$  of the form

$$Z' = \sqrt{1 - |\phi|^2} Z + \phi \quad (2.96)$$

with

$$\bar{Z} \cdot \phi = 0. \quad (2.97)$$

The action for this new field  $Z'$  is

$$S' = 2 \int (|D'_+ Z'|^2 + |D'_- Z'|^2) d^2x \quad (2.98)$$

where  $D'_{\pm}$  is the usual covariant derivative (2.9) written in terms of  $Z'$ . Since the integral of

$$Q' \equiv 2 (|D'_+ Z'|^2 - |D'_- Z'|^2) \quad (2.99)$$

is a topological invariant, it follows that

$$S' = \int Q' d^2x + 4 \int |D'_- Z'|^2 d^2x \quad (2.100)$$

where  $Q'$  is given by (2.12). To second order in  $\phi$  it can be shown that

$$\begin{aligned} |D'_- Z'|^2 &= |D_- Z|^2 + |D_- \phi|^2 - |\phi|^2 |D_- Z|^2 \\ &\quad - |\bar{Z} \cdot D_- \phi + \bar{\phi} \cdot D_- Z|^2 + \overline{D_- Z} \cdot D_- \phi + \overline{D_- \phi} \cdot D_- Z. \end{aligned} \quad (2.101)$$

Hence

$$S' = S + 4 \int V(\phi) d^2x \quad (2.102)$$

where  $S$  is the action of  $Z$  and

$$V(\phi) = |D_- \phi|^2 - |\phi|^2 |D_- Z|^2 - |\bar{Z} \cdot D_- \phi + \bar{\phi} \cdot D_- Z|^2. \quad (2.103)$$

They then chose  $\phi$  to satisfy

$$\phi = \epsilon D_+ Z \quad (2.104)$$

where  $\epsilon$  is a small complex number. This choice of  $\phi$  satisfies (2.97), but also leads to the additional results

$$D_- \phi = \epsilon D_- D_+ Z = -\epsilon |D_+ Z|^2 Z \quad (2.105)$$

and

$$\bar{\phi} \cdot D_- Z = 0 \quad (2.106)$$

and so  $V(\phi)$  becomes

$$\begin{aligned} V(\phi) &= |\epsilon|^2 |D_+ Z|^4 - |\epsilon|^2 |D_+ Z|^2 |D_- Z|^2 - |\epsilon|^2 |D_+ Z|^4 \\ &= -|\epsilon|^2 |D_- Z|^2 |D_+ Z|^2. \end{aligned} \quad (2.107)$$

Since  $D_{\pm} Z \neq 0$ ,  $V(\phi)$  is not identically zero, and so equation (2.102) implies that

$$S' < S. \quad (2.108)$$

So the solution  $Z$  does not correspond to a minima of the action: it is unstable under small fluctuations, and is in fact a saddle point of the action.

Hence, as stated earlier, *any* solution of the  $\mathbb{C}P^{n-1}$  model which is neither instanton nor anti-instanton in nature is necessarily unstable.

### 3. ACTION DENSITIES AND KINETIC ENERGIES.

In the previous chapter, we reviewed the basic quantities involved in the  $\mathbb{C}P^{n-1}$  model in two dimensions. In this chapter, we shall limit ourselves to the  $\mathbb{C}P^1$  model and concentrate particularly on the Lagrangian densities of various extended structures in the model. To avoid confusion at a later stage, we shall call the Lagrangian densities of these two-dimensional objects the “action densities”, and investigate in detail how these action densities depend on the various parameters of the structures. Towards the end of the chapter we shall go beyond the static limit of this model by introducing a time dependence into the system, and consider the resulting “kinetic energies” of our structures as moving objects in  $(2 + 1)$  dimensions, a concept to be used extensively in later chapters.

First we recall briefly the multi-instanton solution of the two-dimensional model defined in chapter 2—see equations (2.15) and (2.16). In  $\mathbb{C}P^1$ , we take

$$Z_\alpha = \frac{f_\alpha(x_+)}{|f_\alpha(x_+)|} \quad (3.1)$$

where  $\alpha = 1, 2$  and the  $f_\alpha$ 's are polynomials in  $x_+$  of the form

$$\begin{aligned} f_1 &= \lambda \prod_{i=1}^k (x_+ - a_i) \\ f_2 &= \prod_{i=1}^k (x_+ - b_i) \end{aligned} \quad (3.2)$$

where  $\lambda$ ,  $a_i$  and  $b_i$  are arbitrary complex constants (with  $a_i \neq b_j$ , and  $\lambda \neq 0$ ). This configuration solves the Euler-Lagrange equations (2.6) and its action is  $2\pi k$ , independent of the choice of  $\lambda$ ,  $a_i$  and  $b_i$ .

In particular, consider the simplest form of this field configuration, that is, the case  $k = 1$  which corresponds to a one-instanton solution, and take  $\lambda = 1$ , so that

$$f_1 = x_+ - a, \quad f_2 = x_+ - b. \quad (3.3)$$



Written another way, consider the configuration

$$w = \frac{x_+ - a}{x_+ - b} \quad (3.4)$$

where  $w$  is defined by

$$w \equiv \frac{f_1}{f_2}. \quad (3.5)$$

Simple algebraic manipulation to calculate the action density using equations (2.4), (3.1) and (3.3) yields the result

$$\overline{D_\mu Z} \cdot D_\mu Z = \frac{2|(b-a)/2|^2}{\left[|x_+ - (a+b)/2|^2 + |(b-a)/2|^2\right]^2} \quad (3.6)$$

(see also reference [27]). Figs. 3.1 and 3.2 show various plots of this action density for different values of  $a$  and  $b$ : in these and all following figures, the contour plots on the left are on the usual  $x-y$  complex plane, the horizontal axis being the real  $x$  direction and the vertical axis being the imaginary  $y$  direction, and the surface plots on the right plot the values of the action density as vertical distances above the same  $x-y$  complex plane, the point  $0+0i$  being the far left corner of the grid (as shown in Fig. 3.1a). [The values of the action density on all the plots are multiplied by a constant factor of 250 for ease of plotting, and the interval between each contour on the contour plots is 2 units.]

It is well known that this solution has finite total action, and the form of the action density, as illustrated by the figures, demonstrates that we can think of this solution (and other similar configurations which we shall meet later) as describing an extended structure in the model. Observe that the action density has an obvious maximum at  $x_+ = (a+b)/2$ —this can be seen both by looking at the form of (3.6), and by studying the pictorial evidence in Fig. 3.1—and so we identify the point

$$r = \frac{a+b}{2} \quad (3.7)$$

as the “position” of the instanton (that is, the position of the “centre” of the extended object). Note also from (3.6) and from Fig. 3.2 that the action density

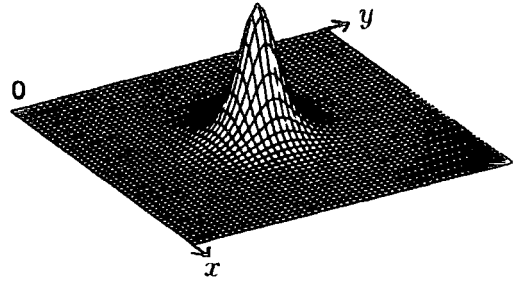
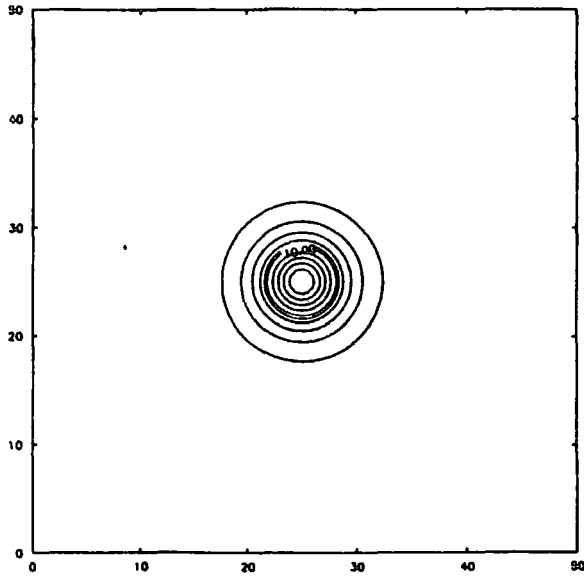


Fig. 3.1a: Action density (3.6) with  $a = 20 + 25i$ ,  $b = 30 + 25i$ .

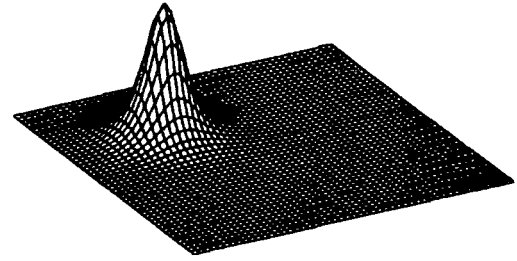
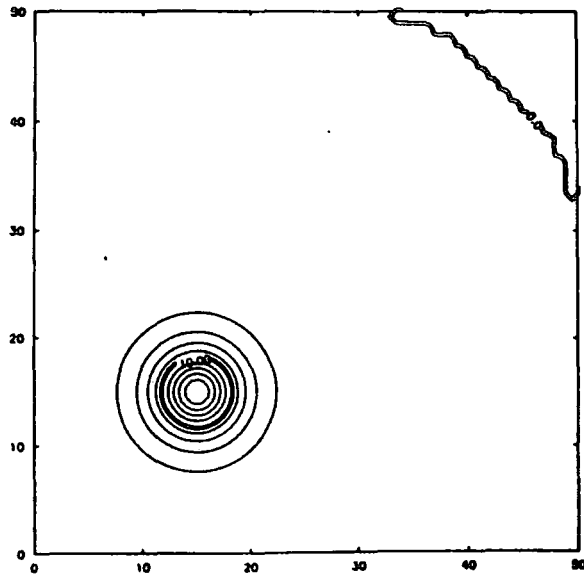


Fig. 3.1b: Action density (3.6) with  $a = 10 + 15i$ ,  $b = 20 + 15i$ .

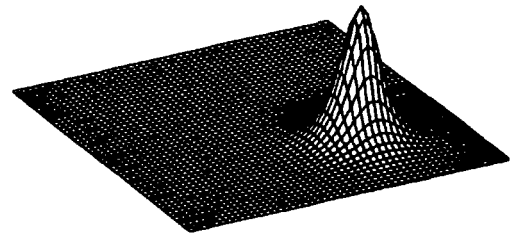
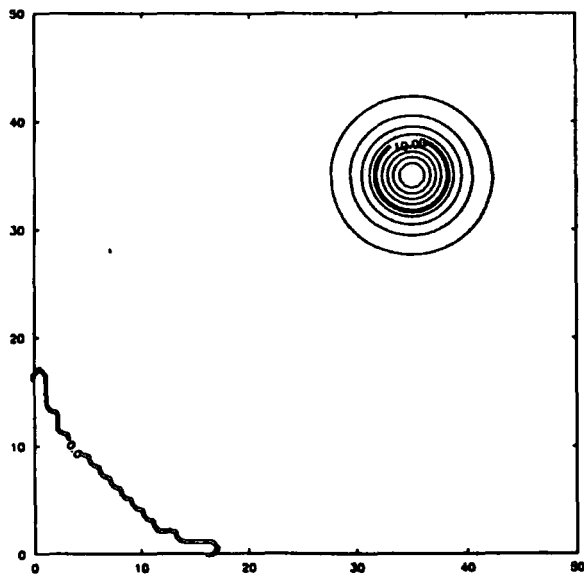


Fig. 3.1c: Action density (3.6) with  $a = 30 + 35i$ ,  $b = 40 + 35i$ .

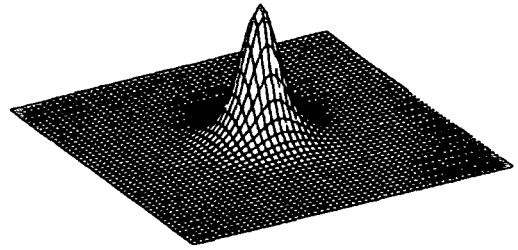
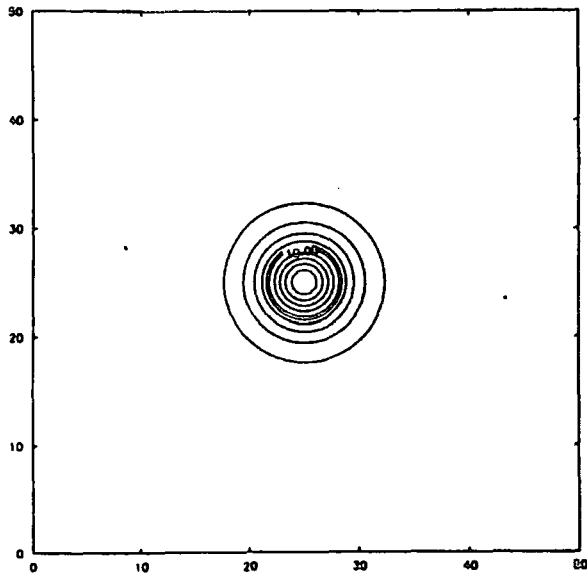


Fig. 3.2a: Action density (3.6) with  $a = 20 + 25i$ ,  $b = 30 + 25i$ .

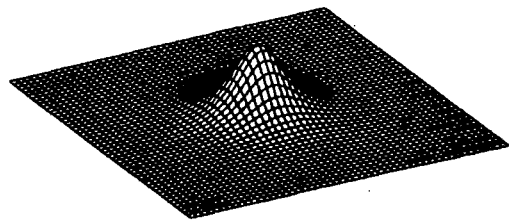
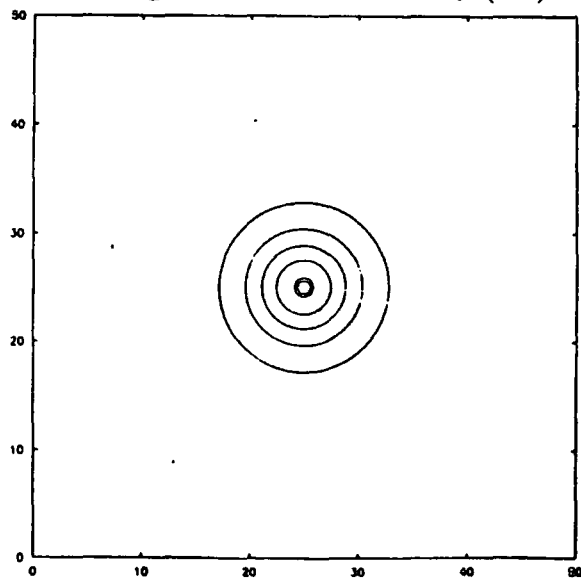


Fig. 3.2b: Action density (3.6) with  $a = 18 + 25i$ ,  $b = 32 + 25i$ .

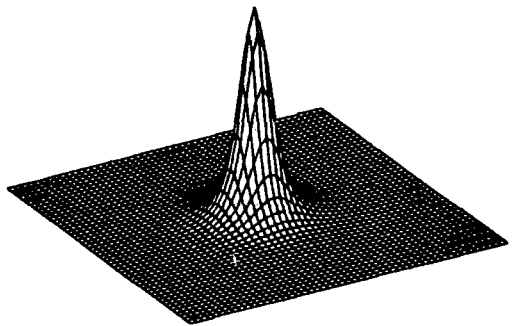
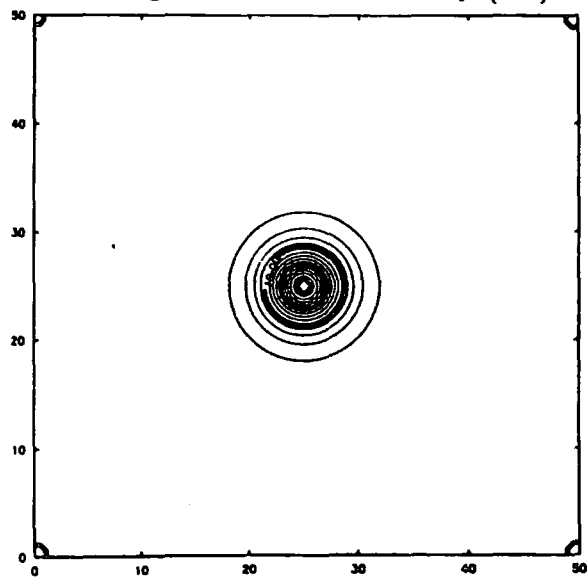


Fig. 3.2c: Action density (3.6) with  $a = 21 + 25i$ ,  $b = 29 + 25i$ .

is bell-shaped, the shape being controlled by the quantity  $|b - a|$ : as  $|b - a| \rightarrow 0$ , the action density becomes narrower and more peaked about the “position”  $r$ , the reverse being true when  $|b - a|$  increases, but *always* in such a way that the total action remains constant (we know this must occur since the action for a one-instanton solution is  $2\pi$ , independent of the choice of  $a$  and  $b$ ). We therefore identify the quantity

$$|s| = |b - a| \quad (3.8)$$

as describing the “size” of our extended structure—as the “size” decreases our extended object becomes more and more “point-like”.

Now we consider the slightly more general case of arbitrary  $\lambda$ , that is, we consider the configuration

$$w = \frac{\lambda(x_+ - a)}{(x_+ - b)}. \quad (3.9)$$

Calculation of the action density for this solution gives

$$\overline{D_\mu Z} \cdot D_\mu Z = \frac{2 \left| \frac{\lambda(b - a)}{|\lambda|^2 + 1} \right|^2}{\left[ \left| x_+ - \frac{(|\lambda|^2 a + b)}{|\lambda|^2 + 1} \right|^2 + \left| \frac{\lambda(b - a)}{|\lambda|^2 + 1} \right|^2 \right]^2} \quad (3.10)$$

so we have an extended structure similar to the previous configuration, with the “position” of the object given by

$$r = \frac{|\lambda|^2 a + b}{|\lambda|^2 + 1} \quad (3.11)$$

and its “size” governed by

$$|s| = \left| \frac{2\lambda(b - a)}{|\lambda|^2 + 1} \right| \quad (3.12)$$

if we work by analogy with the previous case.

Introducing the  $\lambda$  parameter in this way enables us to consider two limiting cases. First, take the limit

$$\lambda \rightarrow \infty \quad \text{and} \quad b \rightarrow \infty \quad \text{such that} \quad \lambda/b \rightarrow -c \quad (3.13)$$

where  $c$  is some complex number. In this limit, the configuration (3.9) becomes

$$w \rightarrow c(x_+ - a) \quad (3.14)$$

and also note that

$$\left| \frac{\lambda(b-a)}{|\lambda|^2 + 1} \right|^2 \rightarrow \left| \frac{1}{c} \right|^2 \quad (3.15)$$

and

$$\frac{|\lambda|^2 a + b}{|\lambda|^2 + 1} \rightarrow a. \quad (3.16)$$

Hence we can say, using equations (3.9)–(3.16), that the configuration

$$w = c(x_+ - a) \quad (3.17)$$

has an action density given by

$$\overline{D_\mu Z} \cdot D_\mu Z = \frac{2|1/c|^2}{[|x_+ - a|^2 + |1/c|^2]^2}, \quad (3.18)$$

which still exhibits the same characteristics as a true one-instanton configuration despite the simpler form for  $w$ . The action density this time indicates an extended structure with “position”

$$r = a \quad (3.19)$$

and “size”

$$|s| = |2/c|. \quad (3.20)$$

In a similar way, taking instead the limit

$$\lambda \rightarrow 0 \quad \text{and} \quad a \rightarrow \infty \quad \text{such that} \quad \lambda a \rightarrow -d \quad (3.21)$$

where  $d$  is another complex number, we can conclude that the configuration

$$w = d/(x_+ - b) \quad (3.22)$$

has an action density of the form

$$\overline{D_\mu Z} \cdot D_\mu Z = \frac{2|d|^2}{[|x_+ - b|^2 + |d|^2]^2}, \quad (3.23)$$

indicating an extended structure of one-instanton-like character, “position”

$$r = b, \tag{3.24}$$

and of “size”

$$|s| = |2d|. \tag{3.25}$$

We shall use these configurations later in this chapter—finally note at this stage, however, that the complex conjugates of all the configurations discussed above, which correspond to one anti-instanton models, give rise to *identical* action densities to those calculated above.

Returning to (3.2), we now consider the case  $k > 1$ , where the interpretation of the parameters is more complicated. If each  $a_i$  is assumed to be close to the corresponding  $b_i$  and far away from the other parameters in the complex plane, then an  $n$ -instanton solution looks like a classical system of  $n$  extended objects, whose “positions” are given by

$$r_i = \frac{a_i + b_i}{2} \tag{3.26}$$

and whose “sizes” are given by

$$|s_i| = |a_i - b_i|, \tag{3.27}$$

(assuming that  $\lambda = 1$ ). Various authors, for example Förster [33], d’Adda et al. [9], and Din and Zakrzewski [28], refer to this interpretation as the dilute instanton gas limit; Wilczek and Zee [29], and Wu [30] take this limit a little further and treat the widely separated instantons approximately as point particles. This form of interpretation may be deceptive however, since in general the  $a_i$ ’s and  $b_j$ ’s can take any value: Fateev et al. [31] prefer to interpret the system in terms of a gas of  $n$  “instanton quarks” and  $n$  “instanton antiquarks” positioned at  $a_i$  and  $b_j$  respectively.

We shall use various aspects of all these interpretations later: let us illustrate them by considering in more detail the  $k = 2$  case, that is, by considering the

configuration

$$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)}. \quad (3.28)$$

It is only a matter of algebra to produce the action density for this system:-

$$\begin{aligned} \overline{D_\mu Z} \cdot D_\mu Z = & \\ & \frac{2}{|f|^4} \left\{ |x_+ - a_1|^2 |x_+ - b_1|^2 |b_2 - a_2|^2 + |x_+ - a_2|^2 |x_+ - b_2|^2 |b_1 - a_1|^2 \right\} \\ & + \frac{4}{|f|^4} \operatorname{Re} \left\{ (x_+ - a_1)(x_- - \overline{a_2})(x_+ - b_1)(x_- - \overline{b_2})(\overline{b_1} - \overline{a_1})(b_2 - a_2) \right\} \end{aligned} \quad (3.29)$$

where

$$|f|^2 = |x_+ - a_1|^2 |x_+ - a_2|^2 + |x_+ - b_1|^2 |x_+ - b_2|^2, \quad (3.30)$$

and Figs. 3.3–3.5 illustrate this action density for various parameter values, using the same plotting specifications as Figs. 3.1 and 3.2. (Hatched contours imply a *depression* inside the contour.)

Fig. 3.3 demonstrates well the dilute instanton gas limit: the action density shows two distinct peaks, well separated in the complex plane and each with negligible distortion due to the other's presence. However, as depicted in Figs. 3.4 and 3.5, if the “positions” of the objects,  $r_1$  and  $r_2$ , are taken to be closer together, superposition of the extended structures occurs and when  $|r_1 - r_2|$  is small compared to the “sizes”  $|s_1|$  and  $|s_2|$ , the two distinct peaks are lost completely. At this stage, then, we must interpret the system as a gas of instanton quarks and antiquarks since the variables  $r_i$  and  $|s_i|$  have no well-defined interpretations as “positions” and “sizes”.

Despite this superposition, it is important to remember that the total action of the system is *constant* for *all* values of  $a_i$  and  $b_i$ : there is no interaction between the two instantons resulting in a change in the value of the action—the only change is the way in which we can interpret the system. Indeed, numerical calculations using the data for the surface plots confirm that the total action for the system is  $4\pi$  in every case.

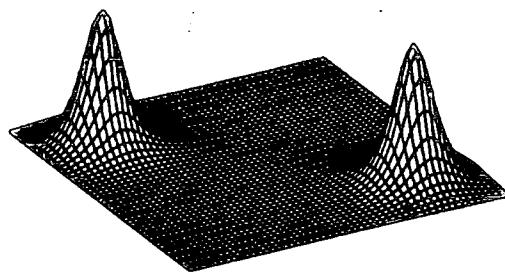
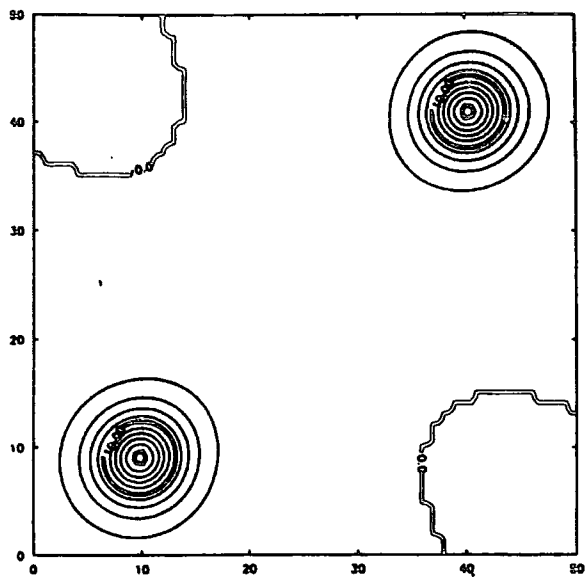


Fig. 3.3a: Action density (3.29) with  $\begin{cases} a_1 = 4 + 9i, & b_1 = 14 + 9i, \\ a_2 = 36 + 41i, & b_2 = 46 + 41i. \end{cases}$

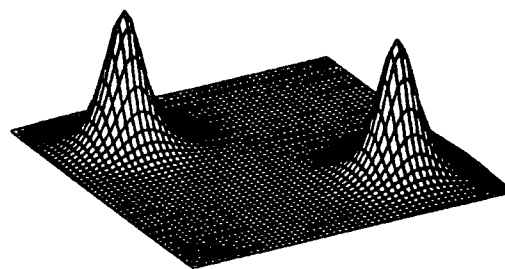
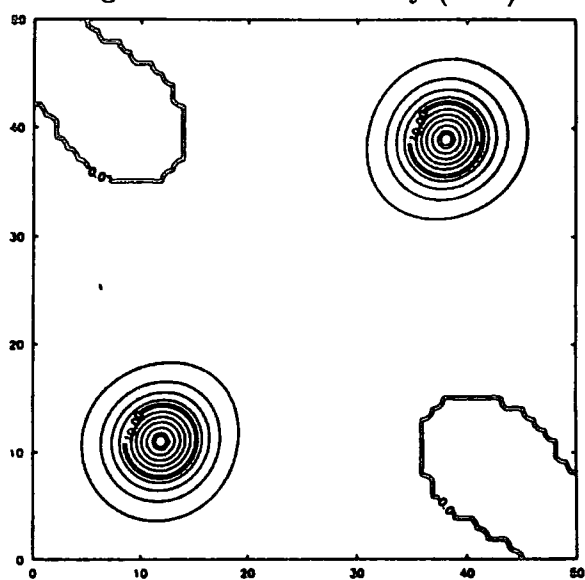


Fig. 3.3b: Action density (3.29) with  $\begin{cases} a_1 = 6 + 11i, & b_1 = 16 + 11i, \\ a_2 = 34 + 39i, & b_2 = 44 + 39i. \end{cases}$

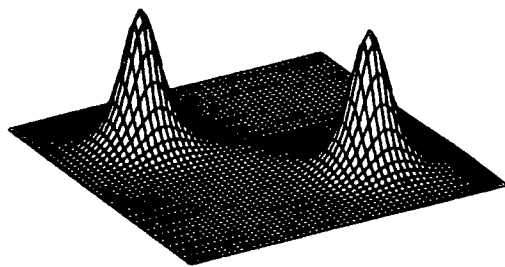
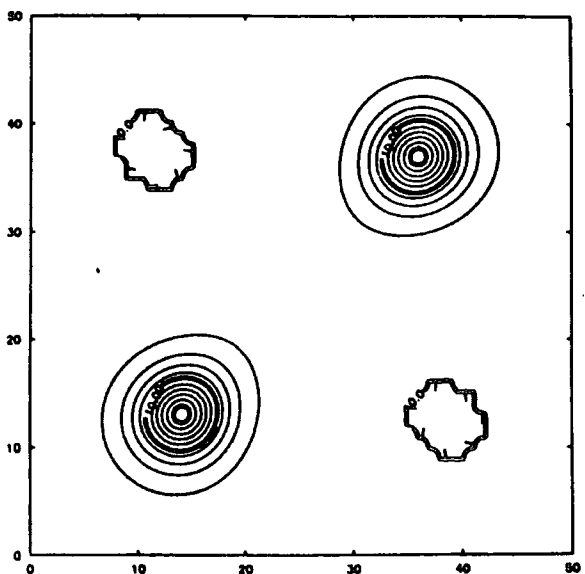


Fig. 3.3c: Action density (3.29) with  $\begin{cases} a_1 = 8 + 13i, & b_1 = 18 + 13i, \\ a_2 = 32 + 37i, & b_2 = 42 + 37i. \end{cases}$



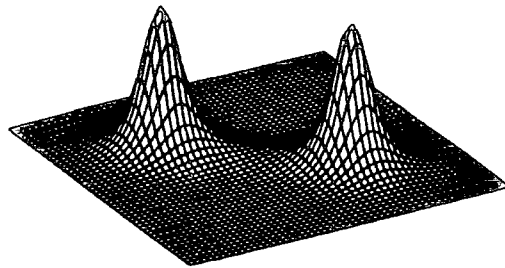
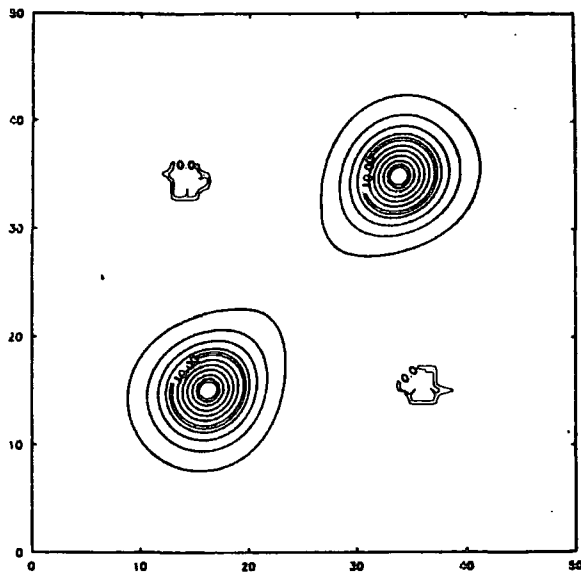


Fig. 3.4a: Action density (3.29) with  $\begin{cases} a_1 = 10 + 15i, & b_1 = 20 + 15i, \\ a_2 = 30 + 35i, & b_2 = 40 + 35i. \end{cases}$

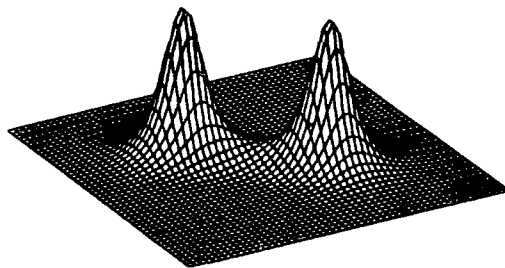
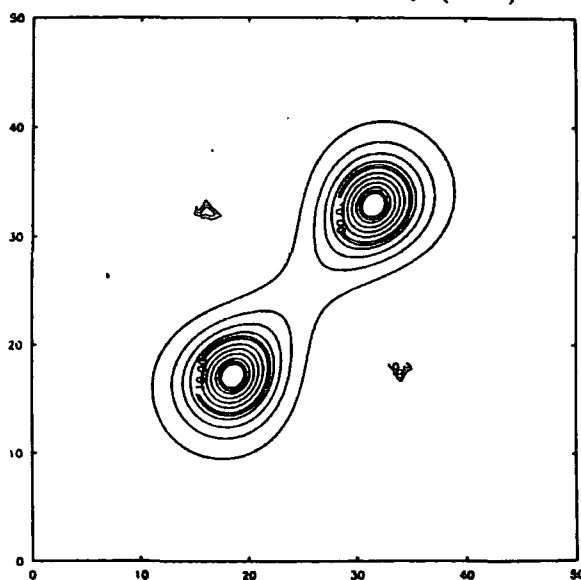


Fig. 3.4b: Action density (3.29) with  $\begin{cases} a_1 = 12 + 17i, & b_1 = 22 + 17i, \\ a_2 = 28 + 33i, & b_2 = 38 + 33i. \end{cases}$

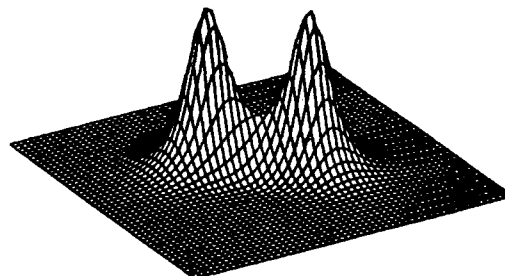
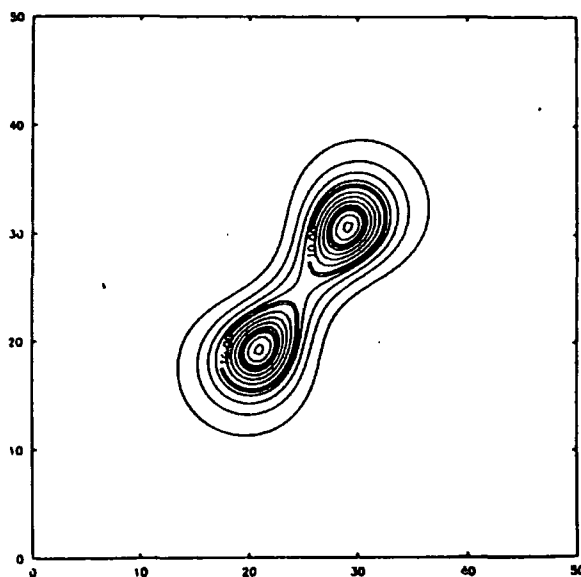


Fig. 3.4c: Action density (3.29) with  $\begin{cases} a_1 = 14 + 19i, & b_1 = 24 + 19i, \\ a_2 = 26 + 31i, & b_2 = 36 + 31i. \end{cases}$

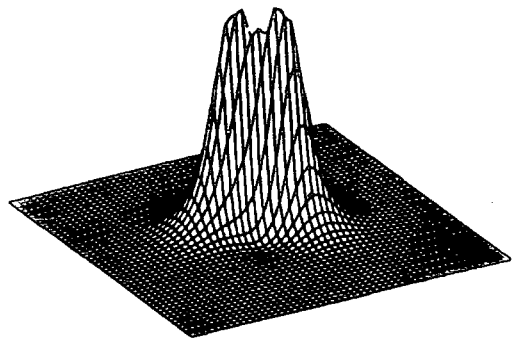
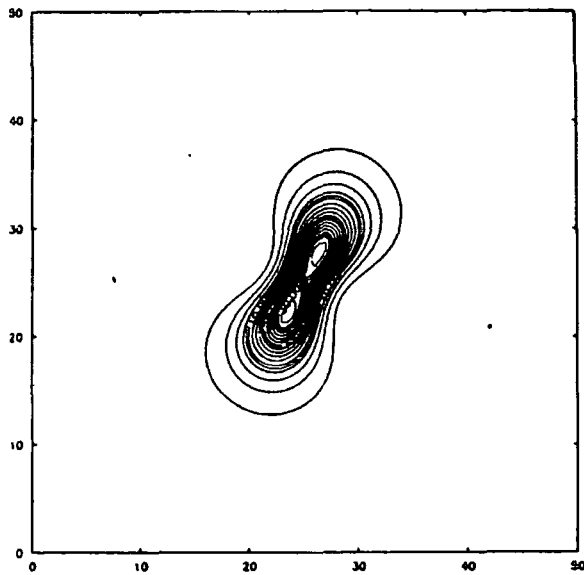


Fig. 3.5a: Action density (3.29) with  $\begin{cases} a_1 = 16 + 21i, & b_1 = 26 + 21i, \\ a_2 = 24 + 29i, & b_2 = 34 + 29i. \end{cases}$

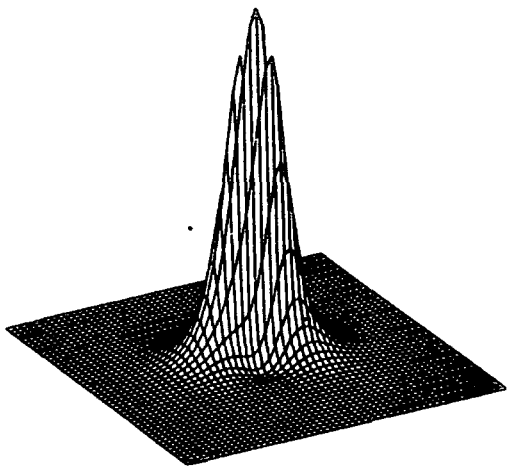
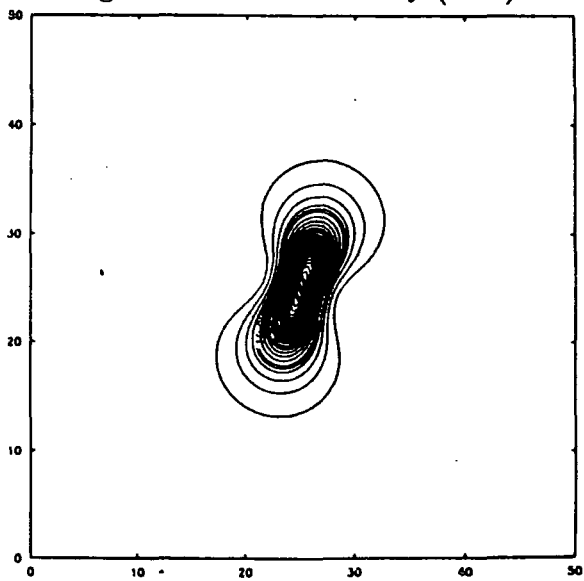


Fig. 3.5b: Action density (3.29) with  $\begin{cases} a_1 = 17 + 22i, & b_1 = 27 + 22i, \\ a_2 = 23 + 28i, & b_2 = 33 + 28i. \end{cases}$

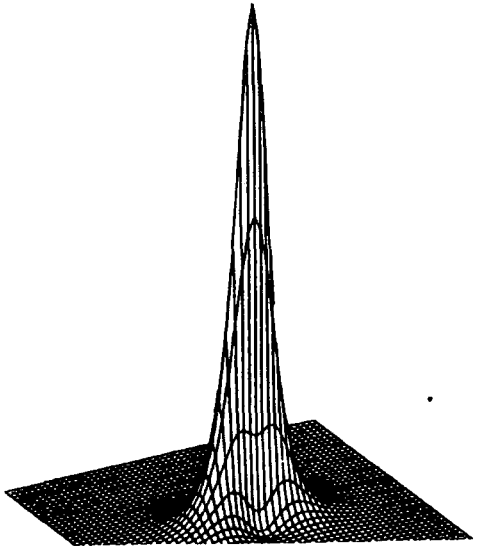
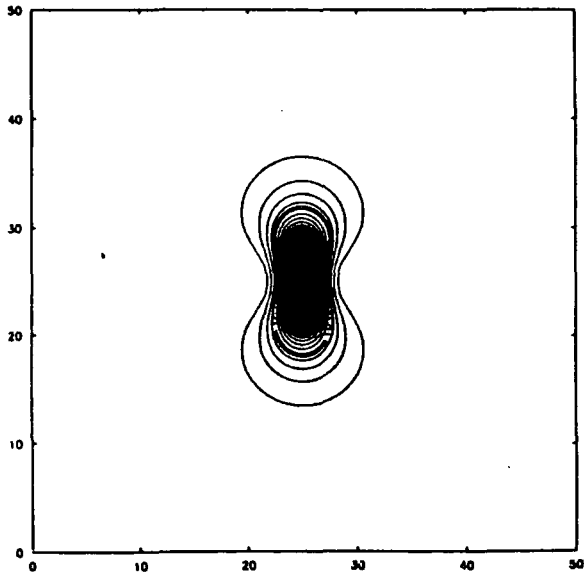


Fig. 3.5c: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 20 + 25i, & b_2 = 30 + 25i. \end{cases}$

On the other hand, the specific values of the parameters are crucial for determining the *nature* of the superposition—even for fixed values of  $r_i$  and  $|s_i|$ , the remaining degrees of freedom in the choice of  $a_i$  and  $b_i$  can greatly affect the form of the action density when superposition of the two extended objects is substantial. As an example, we fix

$$r_1 = r_2 = 25 + 25i \tag{3.31}$$

and

$$|s_1| = |s_2| = 10, \tag{3.32}$$

as in Fig. 3.5c, and illustrate in Figs. 3.6 and 3.7 the form of the action density (3.29) for various values of  $a_i$  and  $b_i$  subject to the constraints (3.31) and (3.32). [Remember that *all* these systems have the *same* total action, but note that a cut-off had to be introduced during the plotting of Figs. 3.7b and 3.7c as the peaks became too tall to plot accurately.]

In Fig. 3.6 it is clear that the only interpretation we can sensibly use is that of instanton quarks and antiquarks; as the specific values of  $a_2$  and  $b_2$  change, however, Fig. 3.7 clearly illustrates that the dilute instanton gas limit becomes meaningful, with  $a_1$  and  $b_2$  parametrizing one extended structure and  $a_2$  and  $b_1$  the other—indeed, in Fig. 3.7c where  $a_1 \approx b_2$  and  $b_1 \approx a_2$  (the action density is ill-defined for  $a_1 = b_2$  and  $b_1 = a_2$ ) there is strong evidence that the point-particle approximation becomes an acceptable interpretation for the system.

So far in this chapter, we have only considered the well-known instanton solutions of the  $\mathbb{C}P^1$  model, and we have looked in detail at these extended structures and their interpretations. We shall now go on to use these illustrative techniques to study possible analogous field configurations for a system containing instantons and anti-instantons.

As pointed out by Bukhlostov and Lipatov [32], although the classical Euler-Lagrange equations (2.6) have no exact solution containing both instantons and

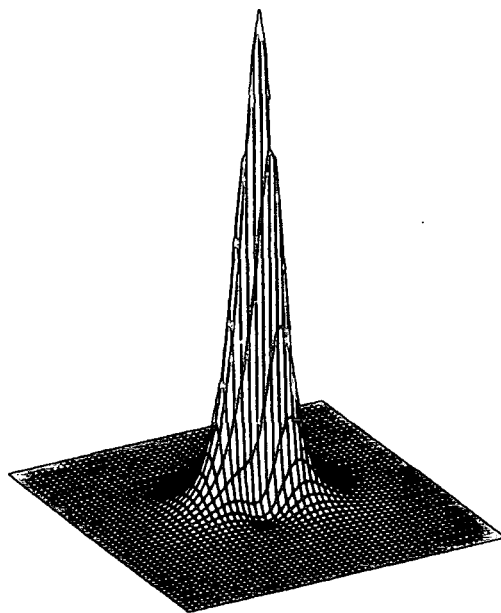
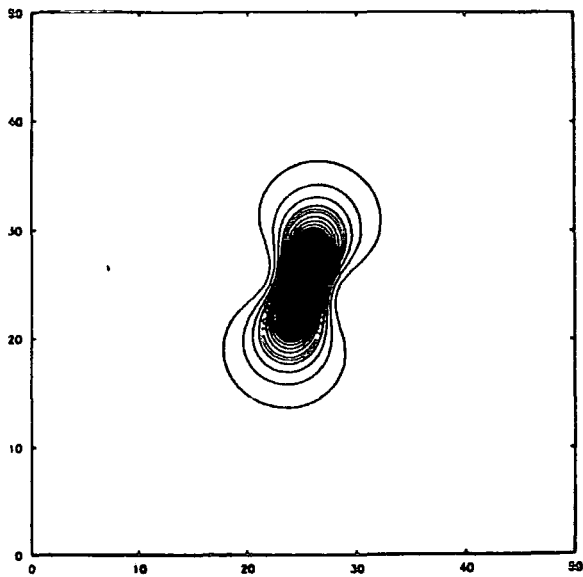


Fig. 3.6a: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 20.67 + 27.5i, & b_2 = 29.33 + 22.5i. \end{cases}$

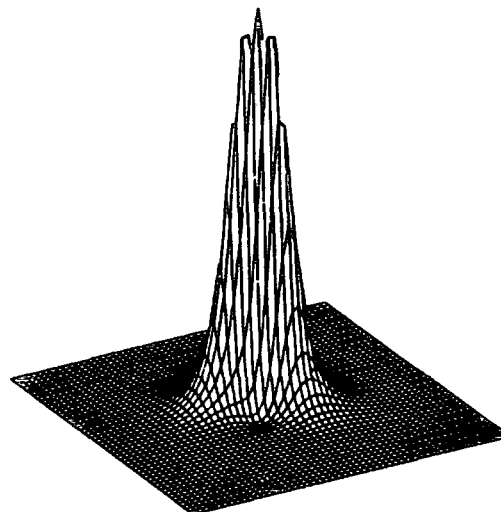
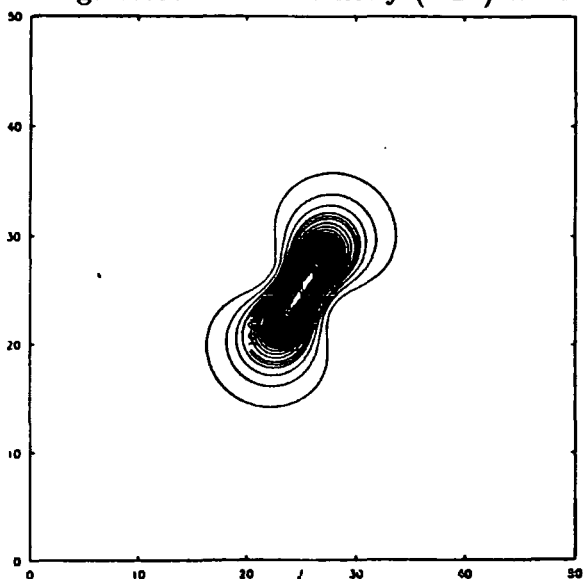


Fig. 3.6b: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 22.5 + 29.33i, & b_2 = 27.5 + 20.67i. \end{cases}$

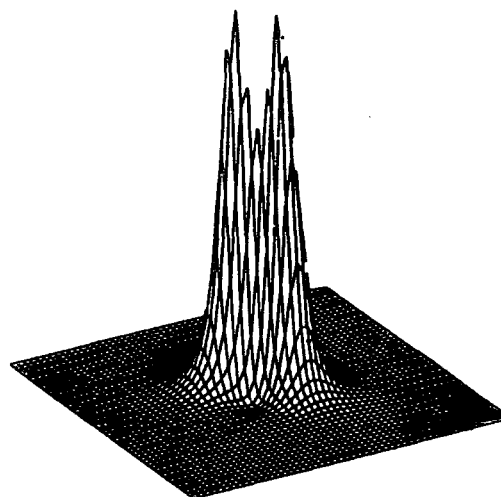
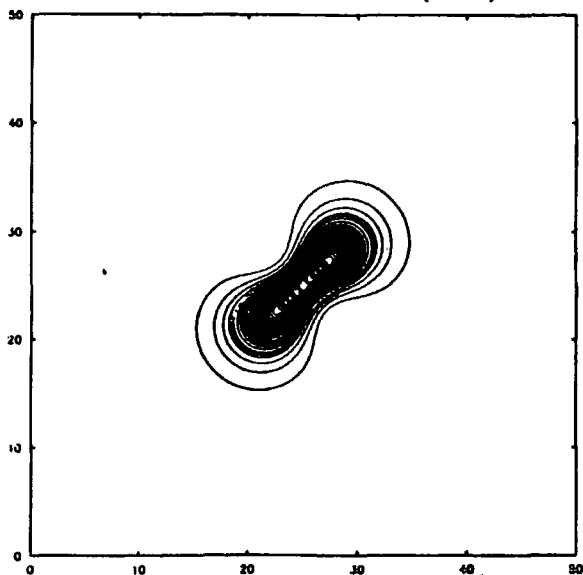


Fig. 3.6c: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 25 + 30i, & b_2 = 25 + 20i. \end{cases}$

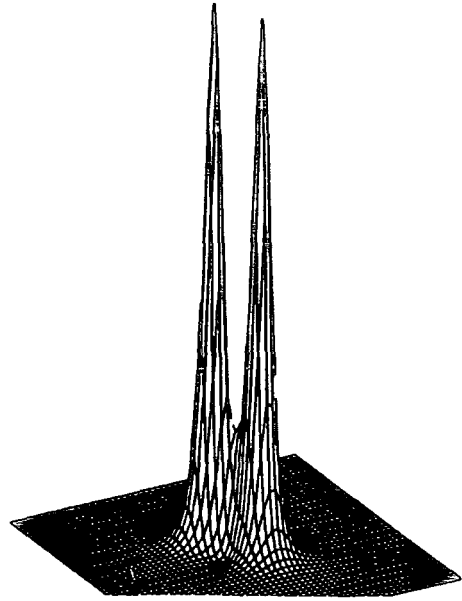
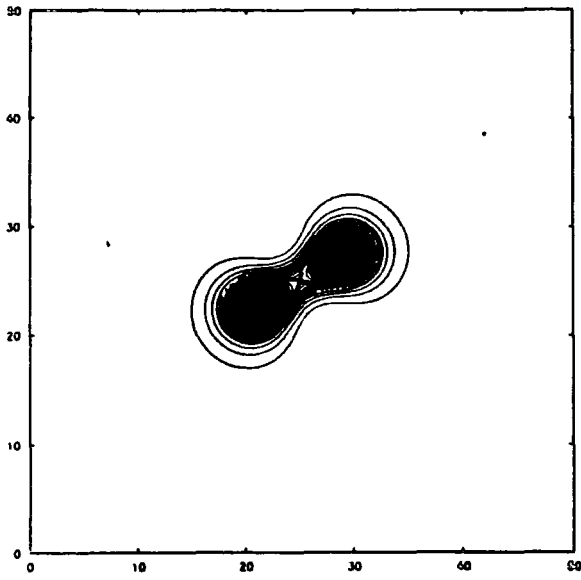


Fig. 3.7a: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 27.5 + 29.33i, & b_2 = 22.5 + 20.67i. \end{cases}$

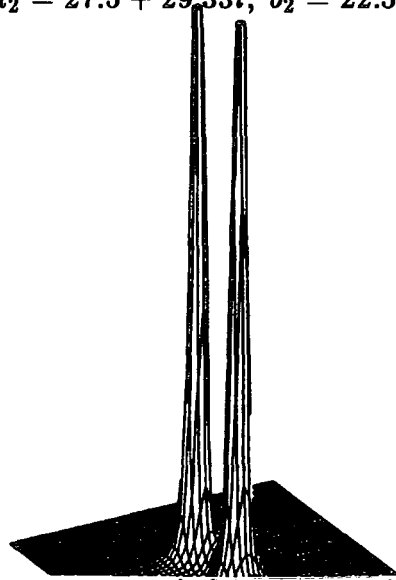
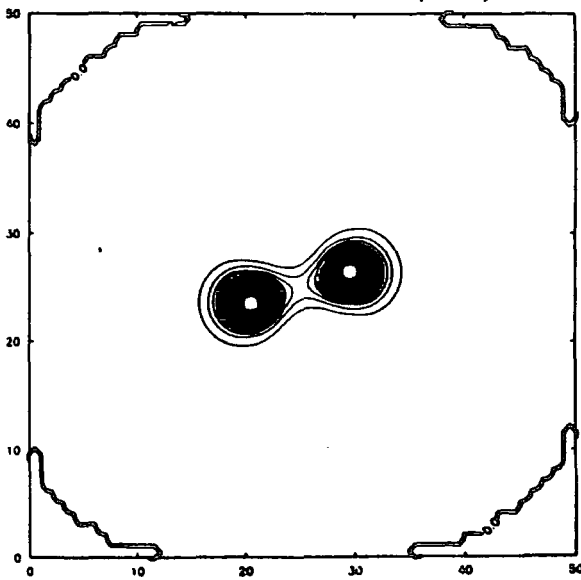


Fig. 3.7b: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 29.33 + 27.5i, & b_2 = 20.67 + 22.5i. \end{cases}$

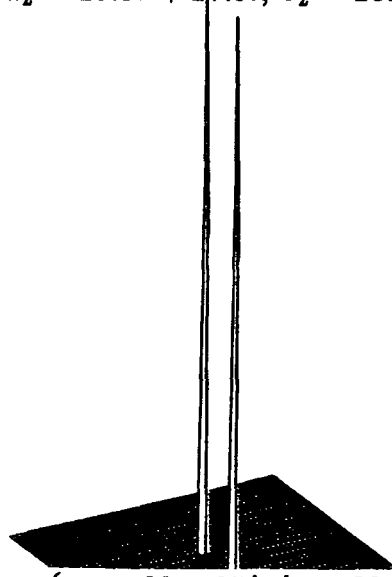
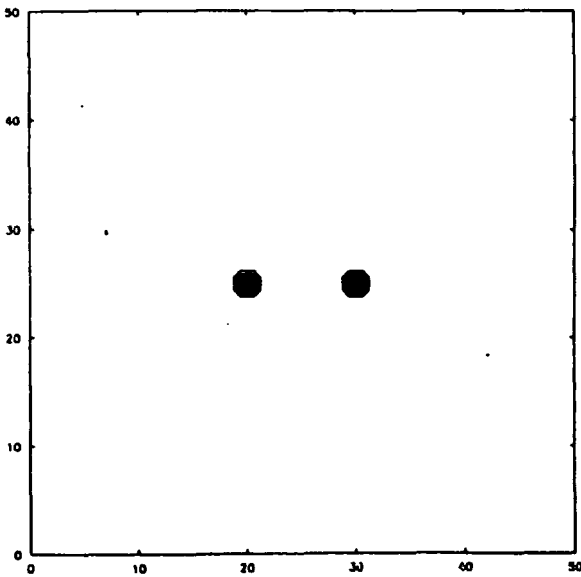


Fig. 3.7c: Action density (3.29) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 \approx 30 + 25i, & b_2 \approx 20 + 25i. \end{cases}$

anti-instantons, field configurations do exist which represent approximate solutions provided the instantons and anti-instantons are concentrated in different widely separated regions of the complex plane. Again we shall investigate in detail a simplified case, and consider a system containing only one instanton and one anti-instanton. Recalling from (3.4) that

$$w = \frac{x_+ - a}{x_+ - b} \quad (3.33)$$

describes one instanton and its complex conjugate one anti-instanton, we follow Bukhvostov and Lipatov's idea that this system can be described by the configuration

$$w = \frac{(x_+ - a_1)(x_- - \bar{a}_2)}{(x_+ - b_1)(x_- - \bar{b}_2)}. \quad (3.34)$$

This choice is of course not unique, and later in this chapter we will discuss other candidates. [For a more general system of  $n$  instantons and  $m$  anti-instantons, reference [32] suggests a configuration of the form

$$w = \lambda \prod_{i=1}^n \frac{x_+ - a_i}{x_+ - b_i} \prod_{j=1}^m \frac{x_- - \bar{c}_j}{x_- - \bar{d}_j}, \quad (3.35)$$

again a choice which is not unique, but which represents an approximate solution of the required form for large separations.]

Calculating the action density in the usual way for (3.34), algebraic manipulation produces the result

$$\begin{aligned} \overline{D_\mu Z} \cdot D_\mu Z = \\ \frac{2}{|f|^4} \left\{ |x_+ - a_1|^2 |x_+ - b_1|^2 |b_2 - a_2|^2 + |x_+ - a_2|^2 |x_+ - b_2|^2 |b_1 - a_1|^2 \right\} \end{aligned} \quad (3.36)$$

where

$$|f|^2 = |x_+ - a_1|^2 |x_+ - a_2|^2 + |x_+ - b_1|^2 |x_+ - b_2|^2. \quad (3.37)$$

Written in this form, the action density is not very illuminating; however, note that there are striking similarities between this and the action density (3.29) for two instantons. Therefore we shall illustrate this density for the same sets

of parameters as in Figs. 3.3–3.5 (using identical plotting parameters) and attempt to interpret the extended structures thus produced in Figs. 3.8–3.10 in an analogous way to the two instanton case.

The action densities depicted in Fig. 3.8 demonstrate well that the dilute instanton gas interpretation can easily be used to describe this system of extended objects too. With

$$r_1 = \frac{a_1 + b_1}{2} \quad (3.38)$$

as the “position” of the instanton and

$$r_2 = \frac{a_2 + b_2}{2} \quad (3.39)$$

for the anti-instanton, then for  $|r_1 - r_2|$  large compared to the “sizes”  $|s_1|$  and  $|s_2|$  (defined as before), two distinct peaks can be seen in the action density centred on  $r_1$  and  $r_2$ , as we would expect. Only when  $|r_1 - r_2|$  is small compared to the “sizes” of the instanton and anti-instanton does this interpretation break down, as shown by Fig. 3.10, when superposition of the extended object again becomes significant. Here we have to resort to the instanton quark and antiquark type of interpretation to obtain meaningful observations.

However, when we compare Figs. 3.9 and 3.10 to Figs. 3.4 and 3.5, there is one very noticeable and very important difference in the superpositions of the extended objects: despite starting with similar sized structures when the separation is fairly large, the resulting action density is dramatically smaller in Fig. 3.10c than that shown in Fig. 3.5c. Indeed, if we perform numerical calculations using the data for the plots in Figs. 3.8–3.10, we find that the total action for the system decreases from  $4\pi$  for Fig. 3.8a to  $2\pi$  for Fig. 3.10c, the action decreasing significantly once superposition is no longer negligible. In other words, when the instanton and anti-instanton are well separated in the complex plane, the total action of the system is approximately equal to the sum of the action for one instanton and the action for one anti-instanton; however, when the

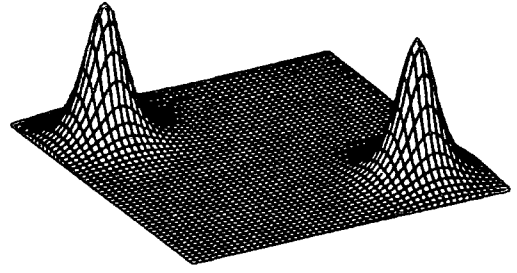
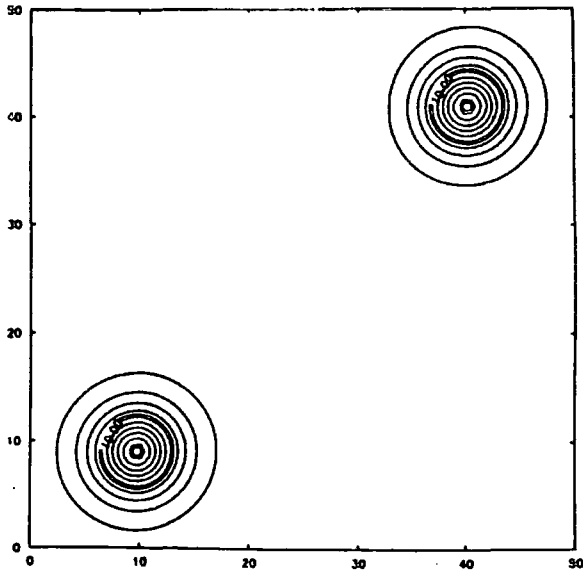


Fig. 3.8a: Action density (3.36) with  $\begin{cases} a_1 = 4 + 9i, & b_1 = 14 + 9i, \\ a_2 = 36 + 41i, & b_2 = 46 + 41i. \end{cases}$

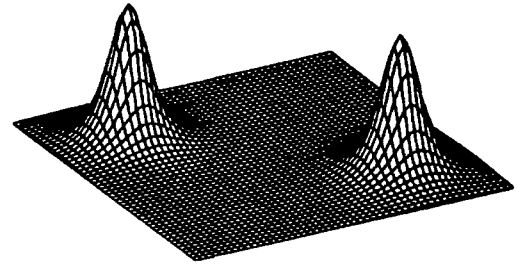
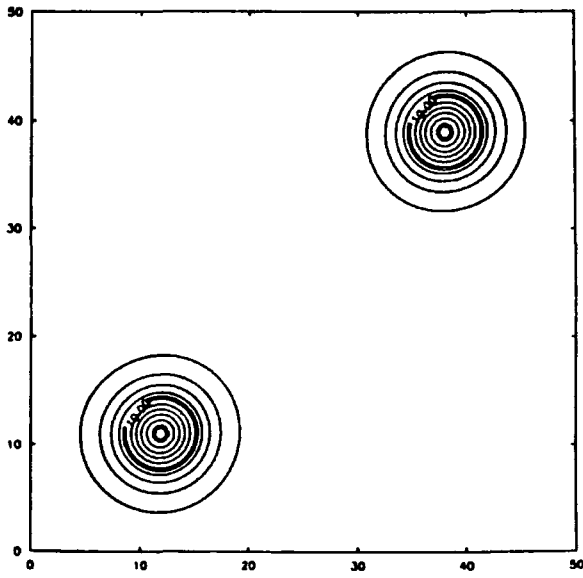


Fig. 3.8b: Action density (3.36) with  $\begin{cases} a_1 = 6 + 11i, & b_1 = 16 + 11i, \\ a_2 = 34 + 39i, & b_2 = 44 + 39i. \end{cases}$

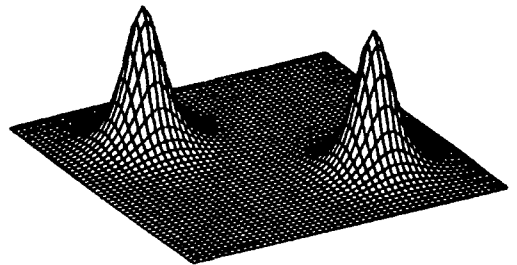
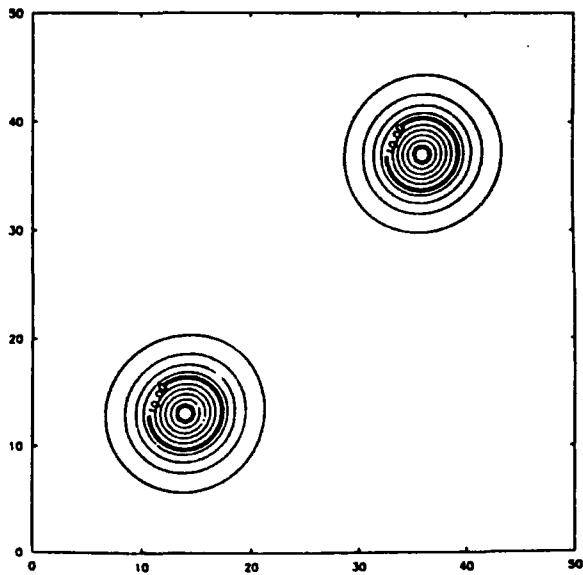


Fig. 3.8c: Action density (3.36) with  $\begin{cases} a_1 = 8 + 13i, & b_1 = 18 + 13i, \\ a_2 = 32 + 37i, & b_2 = 42 + 37i. \end{cases}$



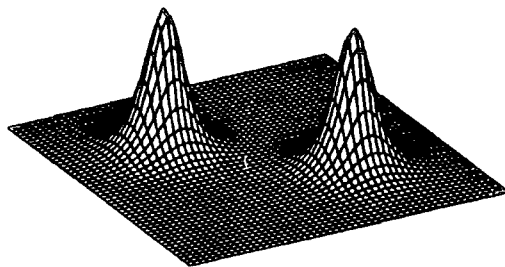
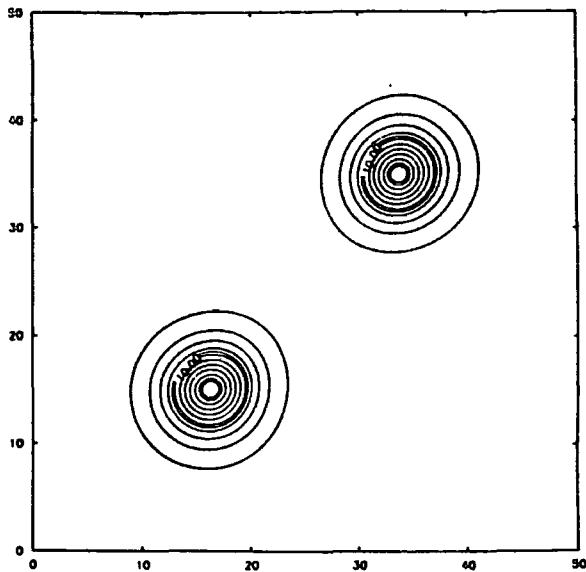


Fig. 3.9a: Action density (3.36) with  $\begin{cases} a_1 = 10 + 15i, & b_1 = 20 + 15i, \\ a_2 = 30 + 35i, & b_2 = 40 + 35i. \end{cases}$

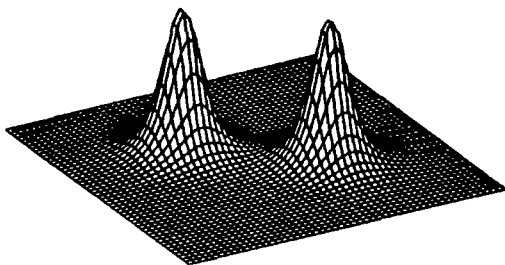
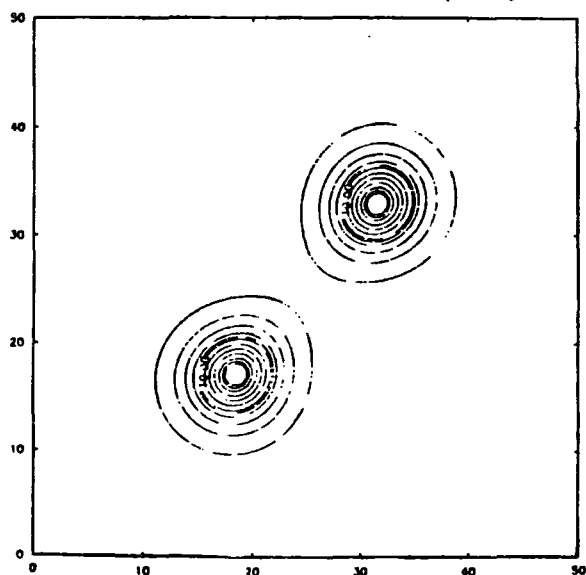


Fig. 3.9b: Action density (3.36) with  $\begin{cases} a_1 = 12 + 17i, & b_1 = 22 + 17i, \\ a_2 = 28 + 33i, & b_2 = 38 + 33i. \end{cases}$

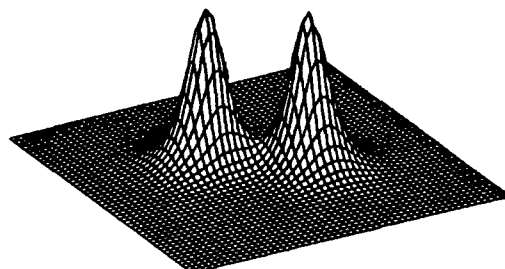
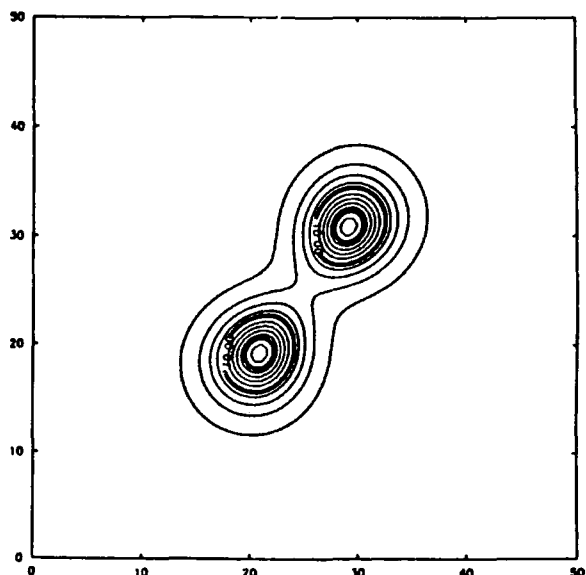


Fig. 3.9c: Action density (3.36) with  $\begin{cases} a_1 = 14 + 19i, & b_1 = 24 + 19i, \\ a_2 = 26 + 31i, & b_2 = 36 + 31i. \end{cases}$

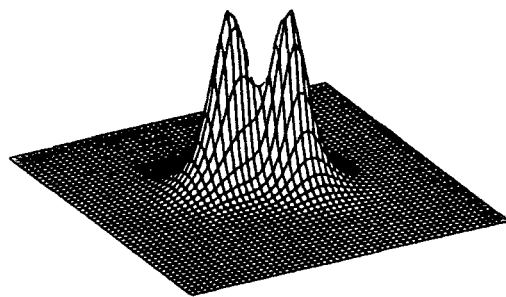
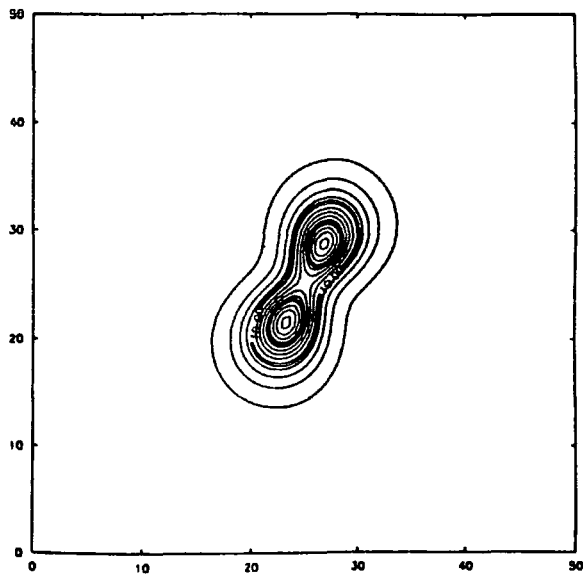


Fig. 3.10a: Action density (3.36) with  $\begin{cases} a_1 = 16 + 21i, & b_1 = 26 + 21i, \\ a_2 = 24 + 29i, & b_2 = 34 + 29i. \end{cases}$

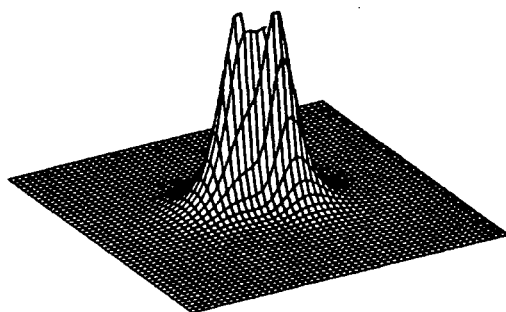
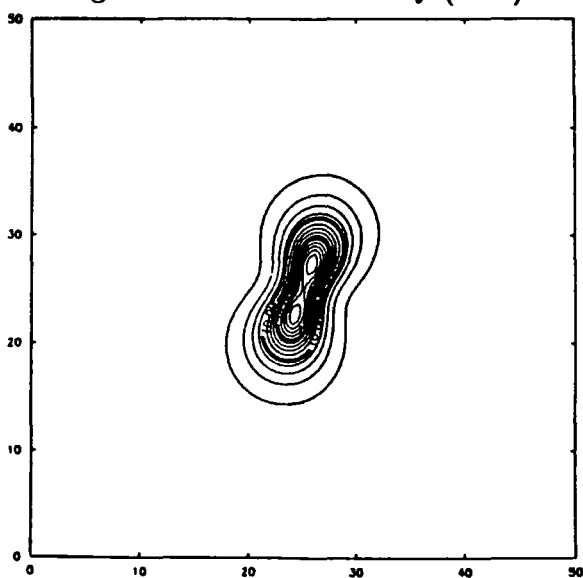


Fig. 3.10b: Action density (3.36) with  $\begin{cases} a_1 = 17 + 22i, & b_1 = 27 + 22i, \\ a_2 = 23 + 28i, & b_2 = 33 + 28i. \end{cases}$

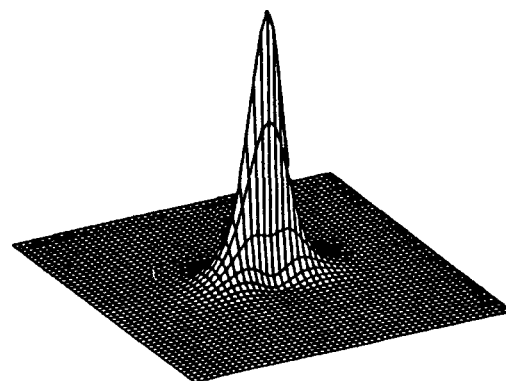
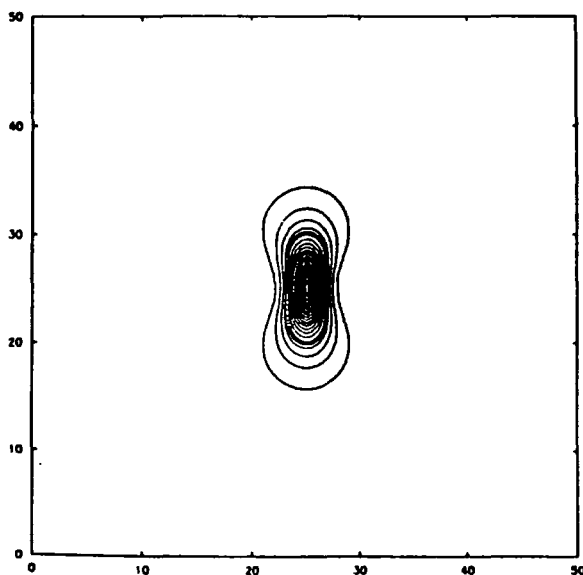


Fig. 3.10c: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 20 + 25i, & b_2 = 30 + 25i. \end{cases}$

separation is small, there is interaction between the objects, and associated with this interaction we can identify an “interaction energy”. Following Bukhvostov and Lipatov [32] this interaction energy  $S_{int}$  can be defined by

$$S_{int} = S_w - 2\pi(k_i + k_a) \quad (3.40)$$

where  $S_w$  is the total action for the system, and  $k_i$  and  $k_a$  are the numbers of instantons and anti-instantons respectively in the system. So in our case of one instanton and one anti-instanton, the interaction energy varies from 0 for large separation to a value of  $-2\pi$  when  $r_1 = r_2$ , which seems to indicate an attractive interaction. This is a very different situation to the two instanton case, where the total action for the system is always  $4\pi$ , and no interaction occurs.

Remembering that the form of the superposition in the two instanton case depended crucially on the specific values of the parameters  $a_i$  and  $b_j$ , it is natural to ask the same question for the instanton plus anti-instanton case, and to investigate their effect on the value of the interaction energy. To this end, we again fix

$$r_1 = r_2 = 25 + 25i \quad (3.41)$$

and

$$|s_1| = |s_2| = 10, \quad (3.42)$$

as we did in the two instanton case, and illustrate the action density (3.36) in Figs. 3.11 and 3.12 using the same sets of parameters as in Figs. 3.6 and 3.7 (once again using identical plotting parameters). [Again a cut-off had to be introduced during the plotting of Figs. 3.12b and 3.12c as the peaks of the action density became too tall to plot.]

Figs. 3.11 and 3.12 show that the form of the superposition does depend heavily on the specific values taken by  $a_i$  and  $b_j$ , in an analogous way to the two instanton case, and the various interpretations we can use for the system are once again clearly demonstrated (although the point particle limit is not as

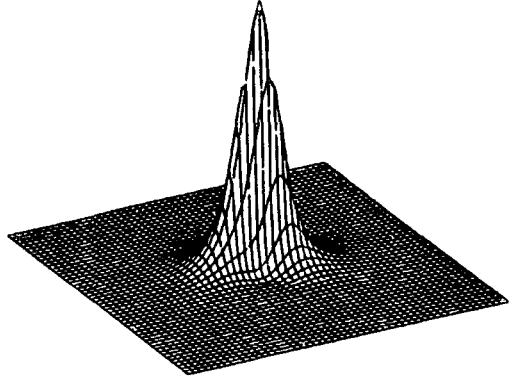
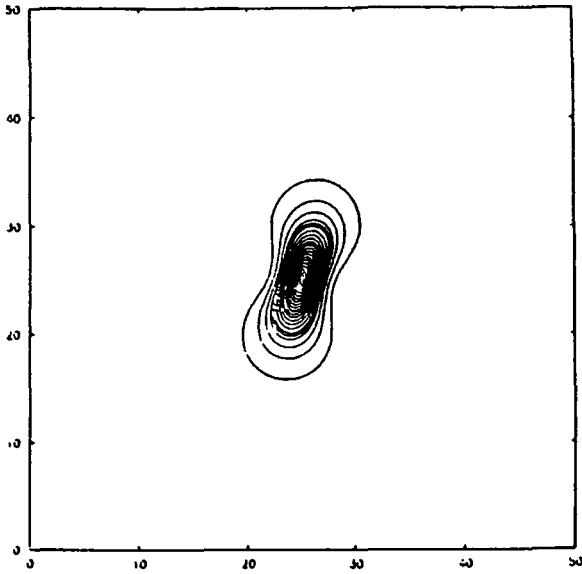


Fig. 3.11a: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 20.67 + 27.5i, & b_2 = 29.33 + 22.5i. \end{cases}$

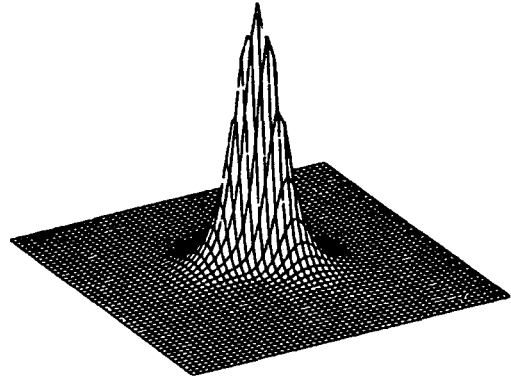
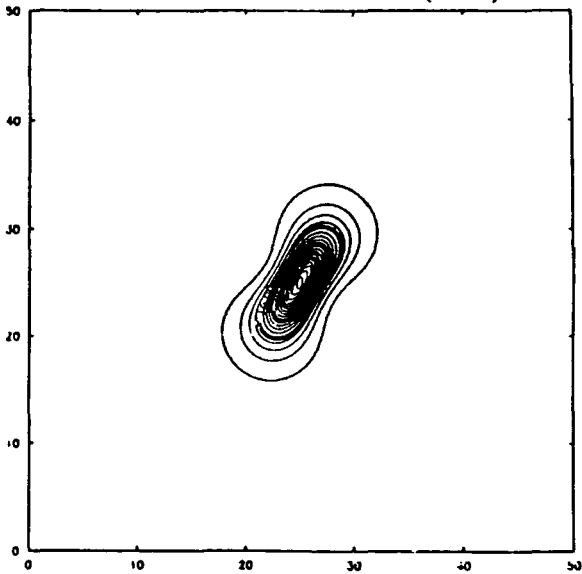


Fig. 3.11b: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 22.5 + 29.33i, & b_2 = 27.5 + 20.67i. \end{cases}$

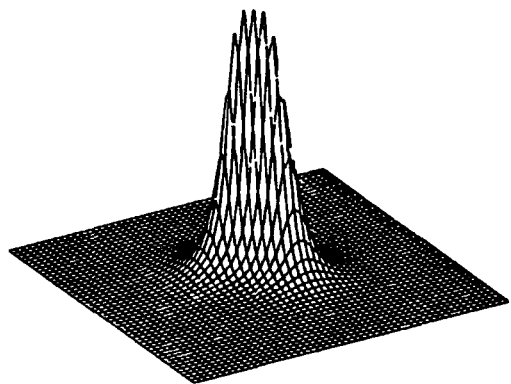
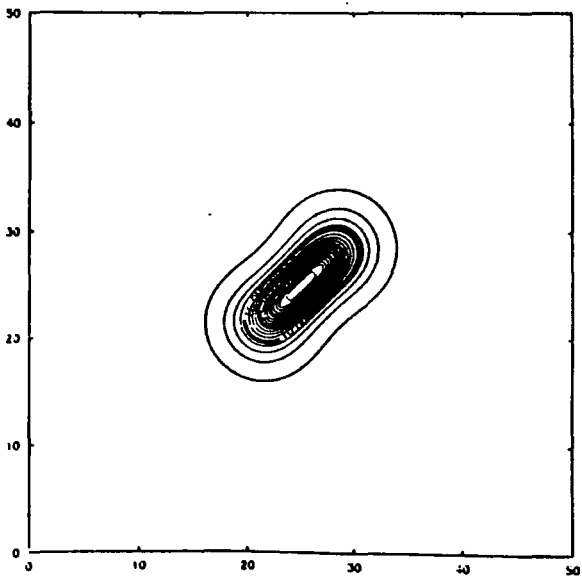


Fig. 3.11c: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 25 + 30i, & b_2 = 25 + 20i. \end{cases}$

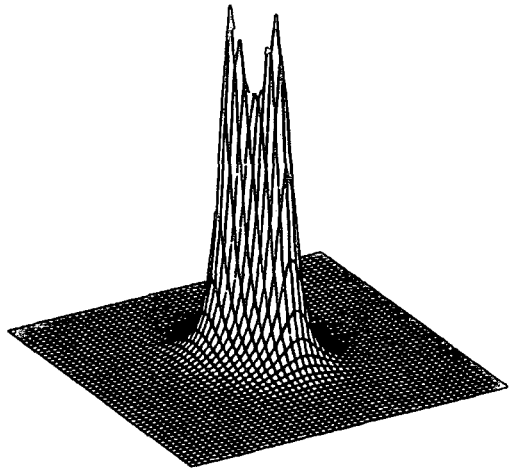
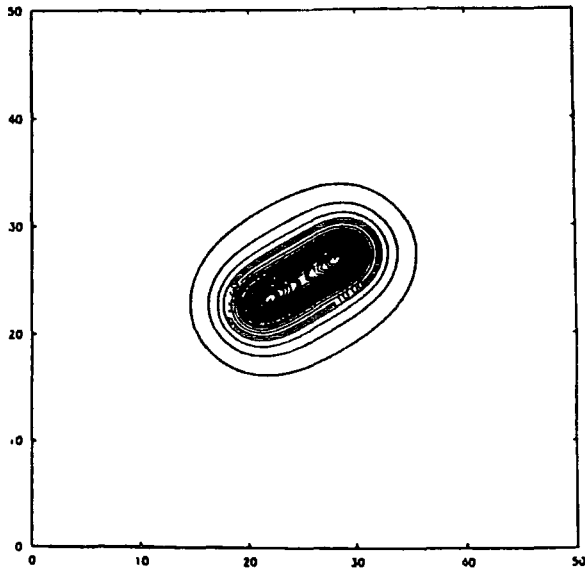


Fig. 3.12a: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 27.5 + 29.33i, & b_2 = 22.5 + 20.67i. \end{cases}$

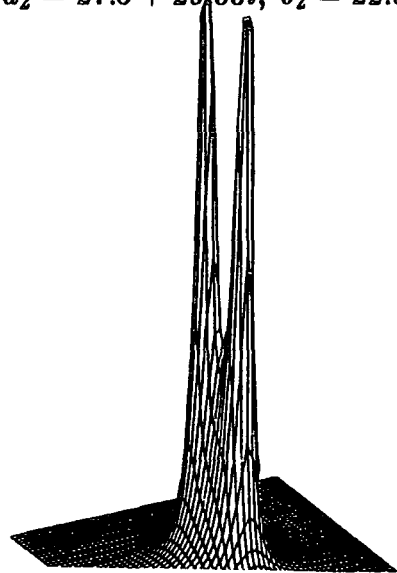
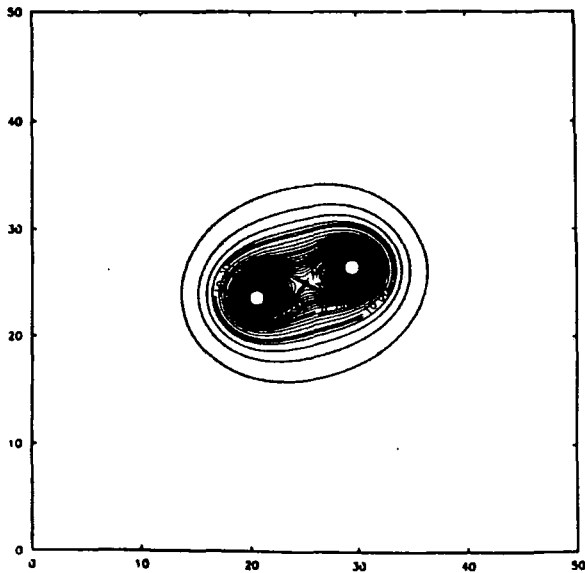


Fig. 3.12b: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 = 29.33 + 27.5i, & b_2 = 20.67 + 22.5i. \end{cases}$

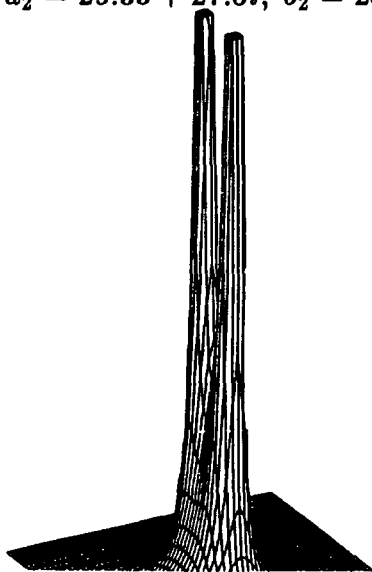
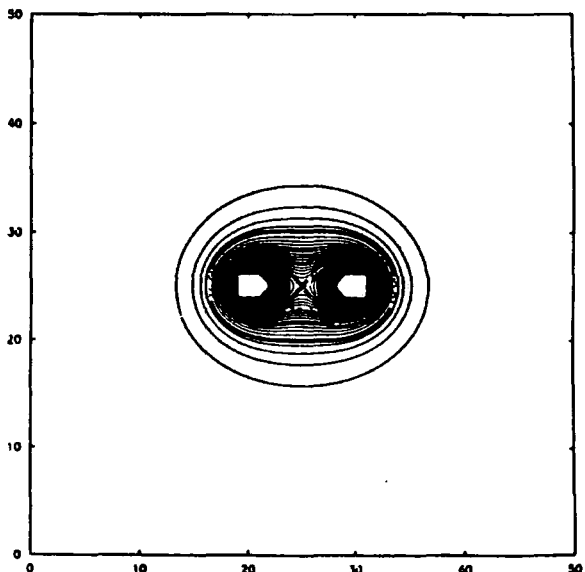


Fig. 3.12c: Action density (3.36) with  $\begin{cases} a_1 = 20 + 25i, & b_1 = 30 + 25i, \\ a_2 \approx 30 + 25i, & b_2 \approx 20 + 25i. \end{cases}$

appropriate in this case). The major difference for this system lies in the values of the total action for the various relative orientations of the instanton and anti-instanton. Numerical calculations using the data for Figs. 3.11 and 3.12 to estimate the total action and hence the interaction energy for the system indicate that the latter can vary from  $-2\pi$  as a lower limit to fairly large positive values—the action density is so sharply peaked in the later plots that the numerical approximations used in the calculations make the actual values inaccurate, but the magnitude of these results is clearly indicated (see later). In other words, the specific values for the parameters  $a_i$  and  $b_j$  can profoundly affect the interaction between the instanton and anti-instanton, making the interaction attractive or repulsive depending on their relative orientation.

This situation is in some ways reminiscent of the behaviour of two dipoles or magnets when brought together: two magnets will attract or repel depending on the relative positions of their north and south poles. Indeed, for large separations, Förster [33] has shown that the interaction energy for this system does have a dipole-dipole-type form. We need to take a closer look at how our numerically calculated interaction energies behave for small separations, to see if this physical analogy is still valid.

We do this by starting with a system of instanton and anti-instanton with a certain *fixed* relative orientation, and we plot the calculated interaction energies against “separation”, which we shall define as  $|r_1 - r_2|$ . (This definition makes obvious sense when the dilute instanton gas-type interpretation is valid: we simply extend this idea for *any* values of  $r_1$  and  $r_2$ .) We then change the relative orientation and repeat the procedure several times. The graphs thus produced are shown in Fig. 3.13a, and the relative orientations which produced these graphs are shown in Fig. 3.13b.

From the figure, we see that orientations 1–5 produce a negative interaction energy, that is, an attractive interaction, with orientation 1 being most attrac-

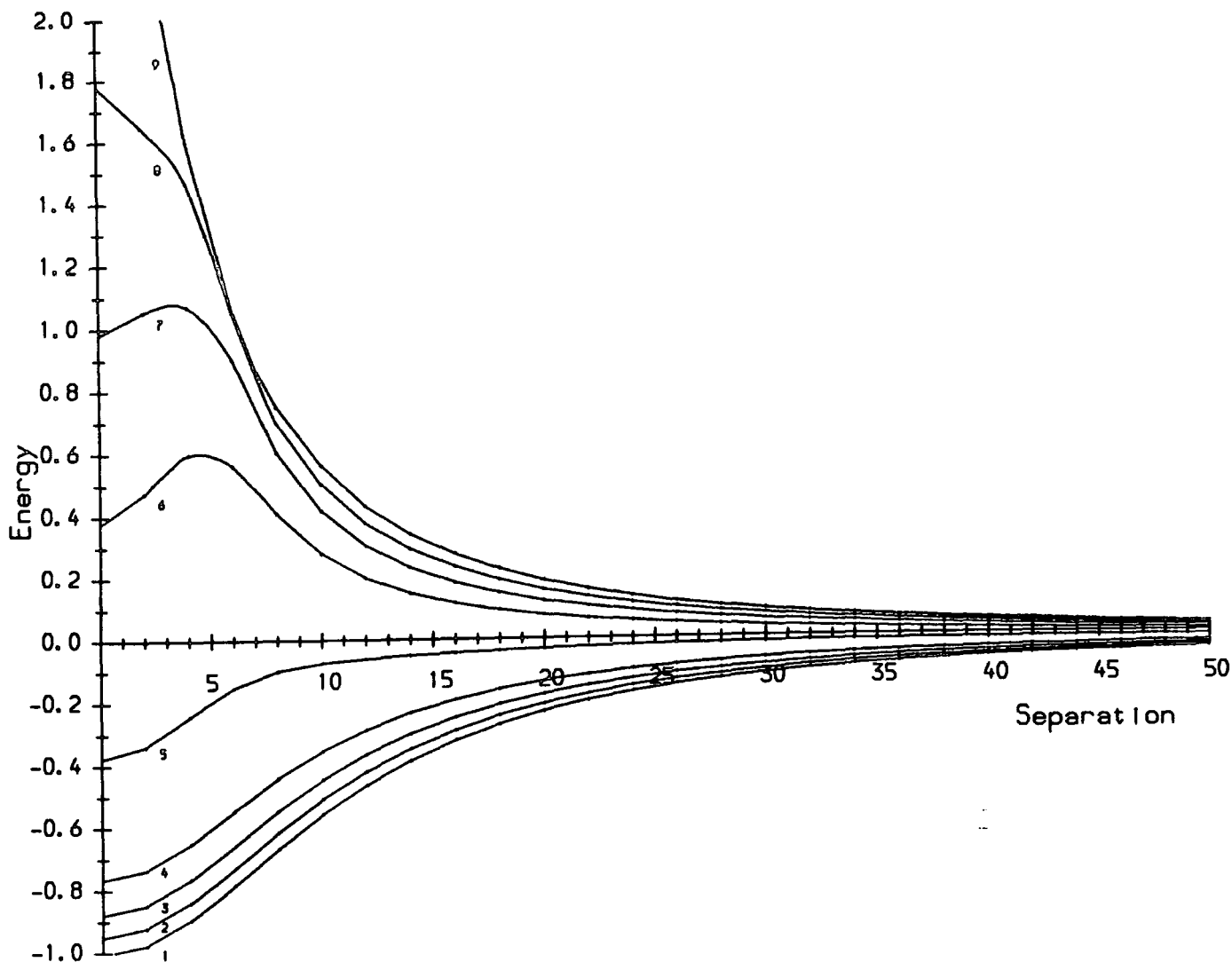


Fig. 3.13a: Graphs of interaction energies (in units of  $2\pi$ ) against separation.

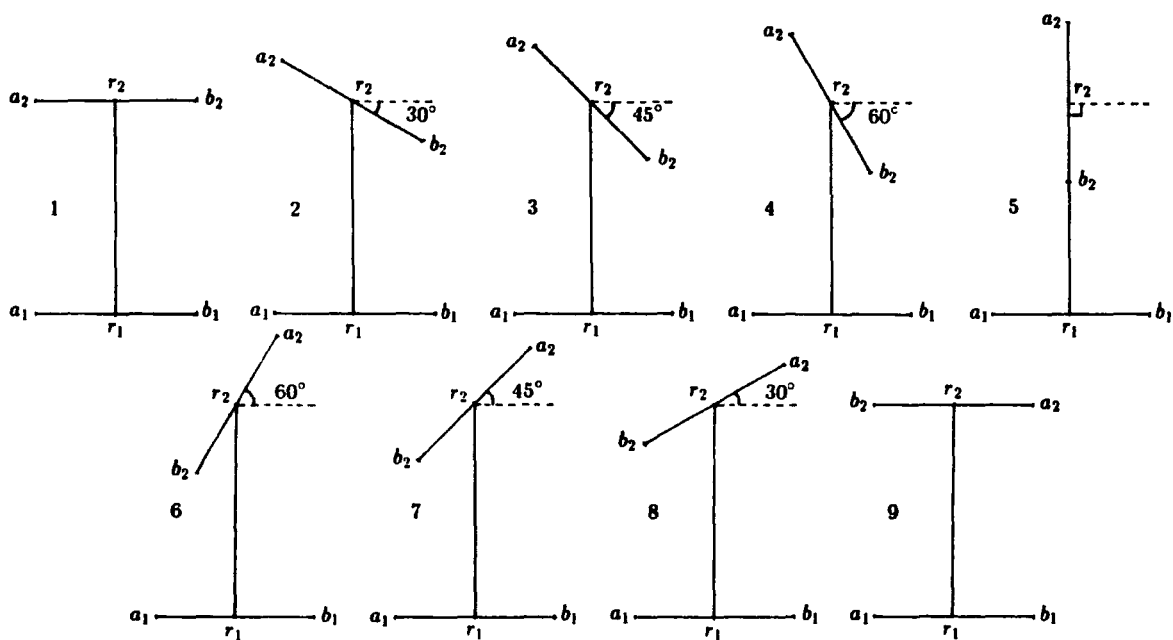


Fig. 3.13b: The nine orientations used to calculate the above curves.

( $|a_i - b_i| = 10$  units in all diagrams, and  $|r_2 - r_1|$  is the separation with  $0 \leq |r_2 - r_1| \leq 50$  units)

tive and 5 the least. Orientations 6–9, on the other hand, produce a repulsive interaction, the weakest being 6 and the strongest 9. This is exactly the type of behaviour we would expect to see exhibited by a pair of magnets positioned at  $r_1$  and  $r_2$ , with the north-south axes of the magnets perpendicular to the lines joining  $a_1$  and  $b_1$ , and  $a_2$  and  $b_2$  respectively. Then orientation 1 corresponds to the north pole of one magnet facing the south pole of the other, and orientations 2–8 represent rotation of one of the magnets about its centre point until two like poles (either north or south) face each other—orientation 9. In this scenario, we would certainly expect to see the interaction between the two magnets become less attractive as we rotate one of them, and become more repulsive until the two like poles are in line. (The graphs in Fig. 3.13a show some peculiarities for very small separation, particularly the repulsive ones. This is partly due to numerical inaccuracies because of the sharp peaking which occurs—see Fig. 3.12c—but for such small separation the magnet analogy becomes dubious anyway.)

So to conclude, the configuration (3.34) shows many promising features for it to represent a system containing an instanton and an anti-instanton: for large separations the instanton and anti-instanton are clearly seen as peaks in the action density, and for smaller separations the resulting superposition of the extended objects shows many similarities to the behaviour of the known two instanton configuration. The main difference is the appearance of the interaction between the instanton and anti-instanton: for large separation it has been shown [33] that the interaction is of a dipole-dipole form, and we have seen, at least qualitatively, that for smaller separations the comparison of the system with a system of two magnets is still feasible. (Obtaining an analytic expression for the form of the interaction for small separations is a highly non-trivial problem, and shall not be addressed in this thesis.)

We now return to an earlier comment made on the uniqueness of this instanton–anti-instanton configuration (3.34). This choice [32] is certainly *not*



unique, and we shall now look at some other possible configurations for this system.

Recall that the expression (3.34) was constructed from the known instanton configuration (3.4) and its complex conjugate. However, earlier in this chapter we derived other expressions which showed one-instanton-like character despite being of a simpler form—see equations (3.17) and (3.22)—namely

$$w = c(x_+ - a) \quad (3.43)$$

and

$$w = d/(x_+ - b). \quad (3.44)$$

In an analogous way, can we produce another possible configuration to describe a system containing one instanton and one anti-instanton? To this end, we investigate the configuration

$$w = \frac{x_+ - a}{x_- - b}. \quad (3.45)$$

This contains an instanton contribution of the form (3.43) and an anti-instanton contribution of the form (3.44): also the form of this expression is much simpler than (3.34). Does this represent the required system?

Calculating the action density for (3.45), we find

$$\overline{D_\mu Z} \cdot D_\mu Z = \frac{1}{\left| x_+ - \frac{(a+b)}{2} \right|^2 + \left| \frac{(b-a)}{2} \right|^2} \quad (3.46)$$

and Figs. 3.14 and 3.15 show various plots of this density for different values of  $a$  and  $b$ .

We see clearly from these plots that (3.45) is *not* a suitable candidate: the density shows behaviour more reminiscent of a one-instanton system than a system containing an instanton and an anti-instanton; however, this observation suggests we try one final configuration of a similar form, namely

$$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - b_1)(x_- - b_2)}. \quad (3.47)$$

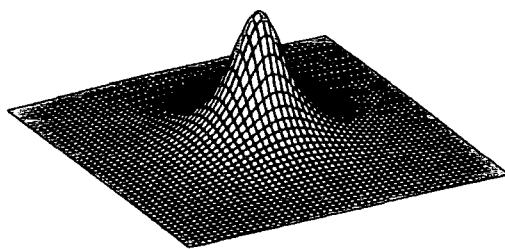
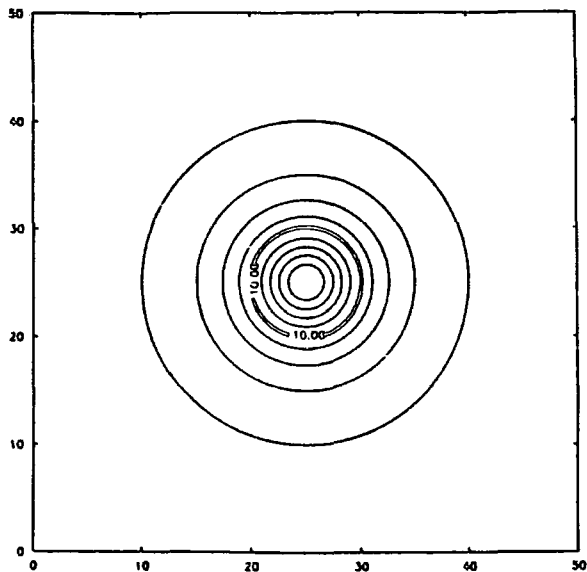


Fig. 3.14a: Action density (3.46) with  $a = 20 + 25i$ ,  $b = 30 + 25i$ .

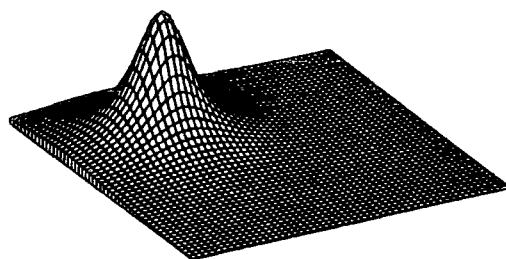
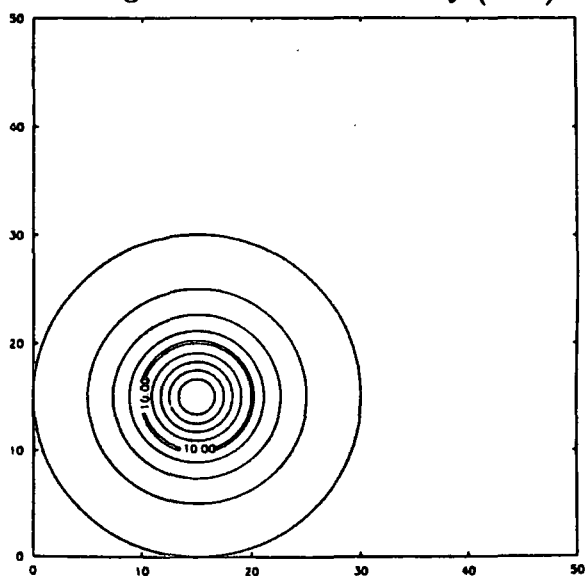


Fig. 3.14b: Action density (3.46) with  $a = 10 + 15i$ ,  $b = 20 + 15i$ .

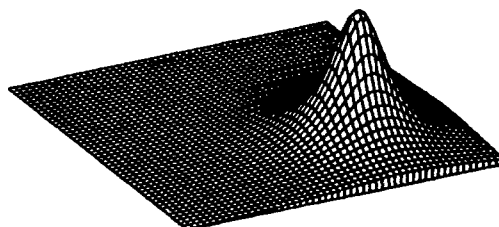
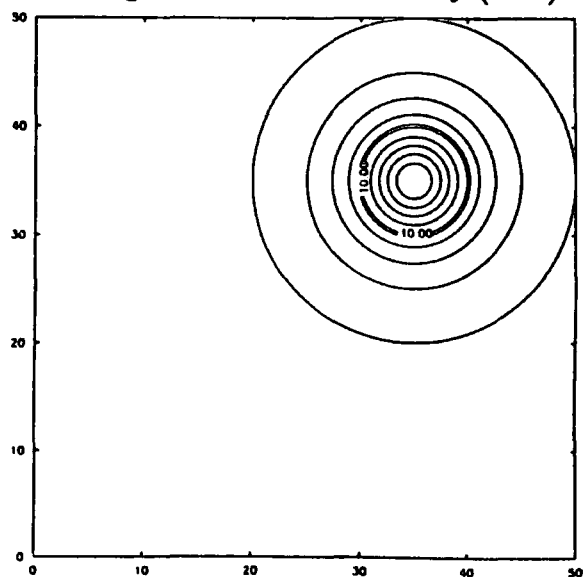


Fig. 3.14c: Action density (3.46) with  $a = 30 + 35i$ ,  $b = 40 + 35i$ .

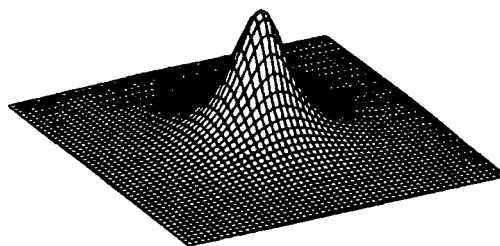
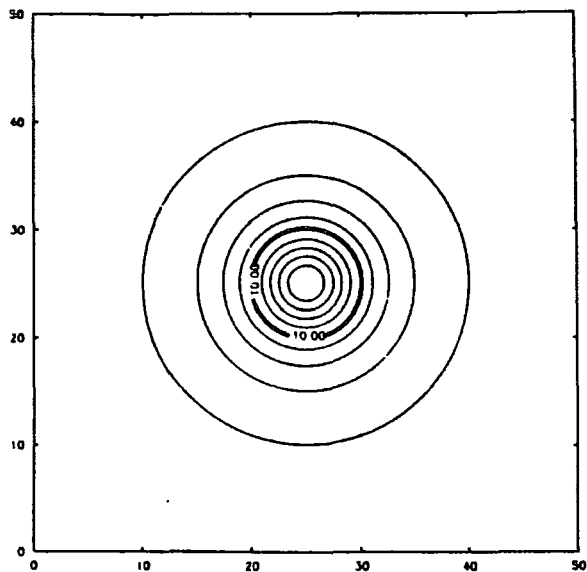


Fig. 3.15a: Action density (3.46) with  $a = 20 + 25i$ ,  $b = 30 + 25i$ .

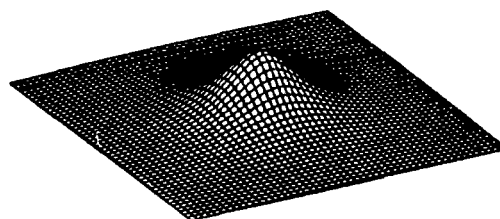
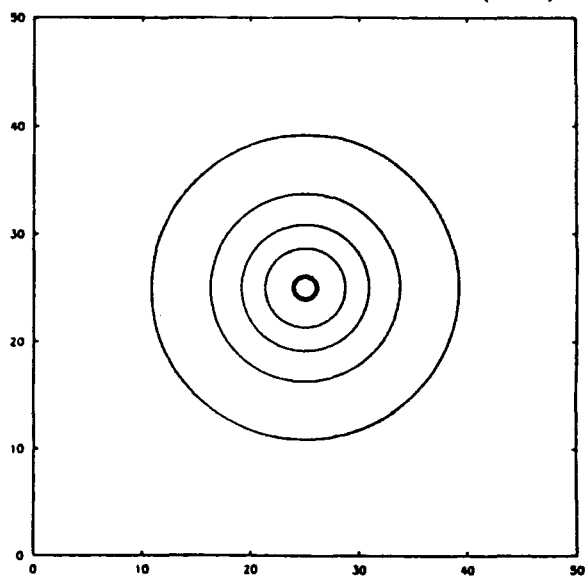


Fig. 3.15b: Action density (3.46) with  $a = 18 + 25i$ ,  $b = 32 + 25i$ .

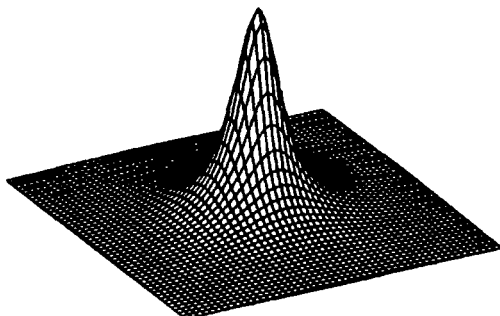
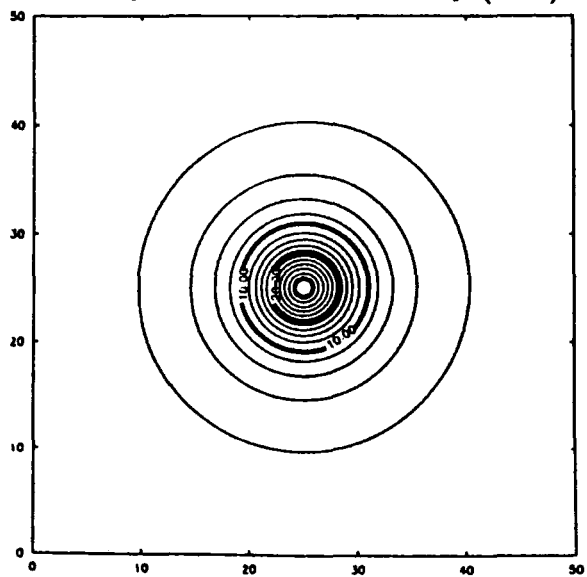


Fig. 3.15c: Action density (3.46) with  $a = 21 + 25i$ ,  $b = 29 + 25i$ .

The action density for this configuration is of the form

$$\begin{aligned} \overline{D_\mu Z} \cdot D_\mu Z = & \\ & \frac{2}{|f|^4} \left\{ |(x_+ - a_1)(x_+ - a_2)(2x_+ - (b_1 + b_2))|^2 \right\} \\ & + \frac{2}{|f|^4} \left\{ |(x_+ - b_1)(x_+ - b_2)(2x_+ - (a_1 + a_2))|^2 \right\} \end{aligned} \quad (3.48)$$

where

$$|f|^2 = |x_+ - a_1|^2 |x_+ - a_2|^2 + |x_+ - b_1|^2 |x_+ - b_2|^2, \quad (3.49)$$

and plots of this action density are shown in Figs. 3.16–3.18.

This time we do see two distinct peaks for large values of  $|r_1 - r_2|$  and superposition does occur in a similar way to previous configurations we have considered. A promising sign, although distortion of each extended structure due to the presence of the other appears to occur for much larger separations than was previously the case. There is one major failing of this model, however: if we consider the limit  $|x_+| \rightarrow \infty$ , we see from (3.48) that

$$\overline{D_\mu Z} \cdot D_\mu Z \sim \frac{4}{|x_+|^2} \quad \text{as} \quad |x_+| \rightarrow \infty. \quad (3.50)$$

Hence, the total action of the system, defined by

$$S = \int \overline{D_\mu Z} \cdot D_\mu Z \, d^2 x, \quad (3.51)$$

is logarithmically divergent. Since one of our first requirements when considering extended structures in this model was that the configurations should result in finite action (see chapter 2), the discovery that the configuration (3.47) gives a divergent action effectively precludes its use as a valid expression for an instanton–anti-instanton system.

So far in this chapter we have looked in detail at the action densities of various configurations in the  $\mathbb{C}P^1$  model in 2 dimensions: we have considered systems containing one or two instantons, and have looked at various candidate configurations for an instanton–anti-instanton system, of which one in particular showed many promising features. We shall now go beyond this static limit of

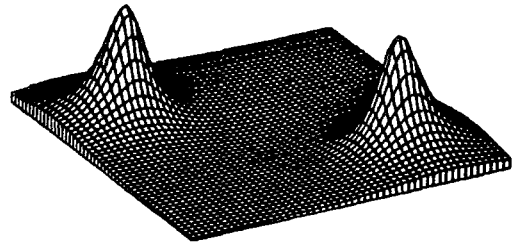
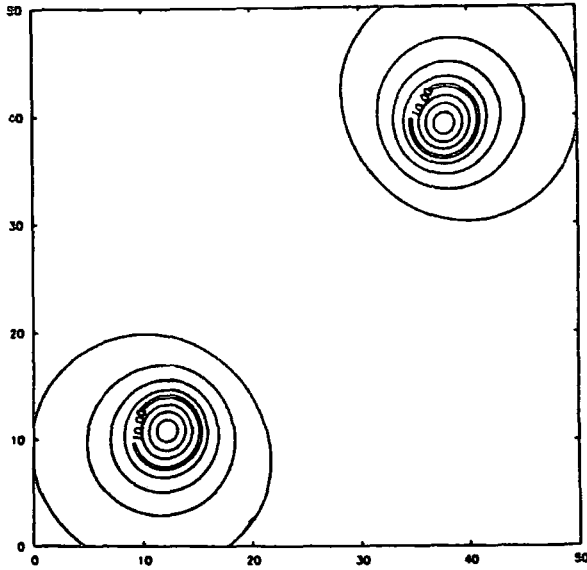


Fig. 3.16a: Action density (3.48) with  $\begin{cases} a_1 = 7 + 11i, & b_1 = 15 + 11i, \\ a_2 = 35 + 39i, & b_2 = 43 + 39i. \end{cases}$

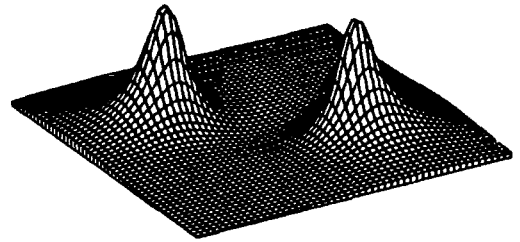
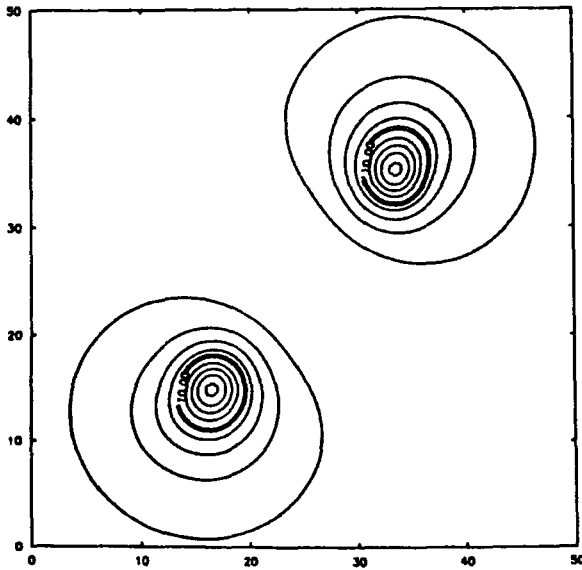


Fig. 3.16b: Action density (3.48) with  $\begin{cases} a_1 = 11 + 15i, & b_1 = 19 + 15i, \\ a_2 = 31 + 35i, & b_2 = 39 + 35i. \end{cases}$

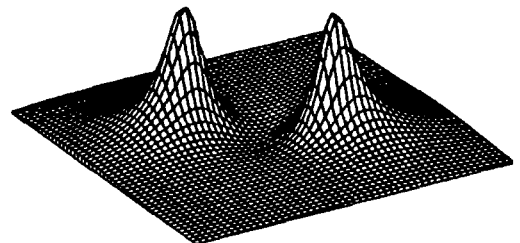
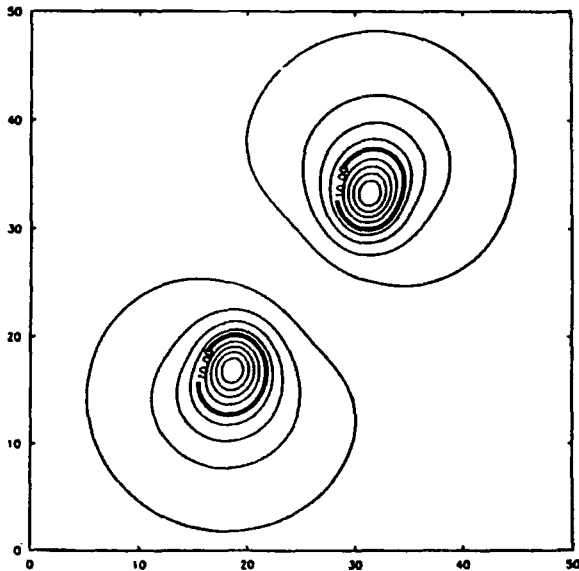


Fig. 3.16c: Action density (3.48) with  $\begin{cases} a_1 = 13 + 17i, & b_1 = 21 + 17i, \\ a_2 = 29 + 33i, & b_2 = 37 + 33i. \end{cases}$

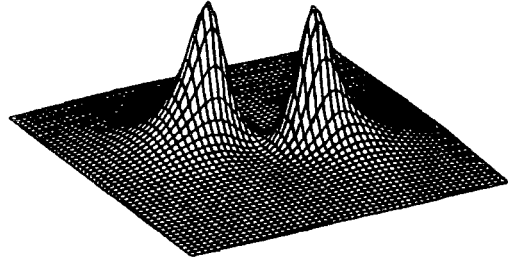
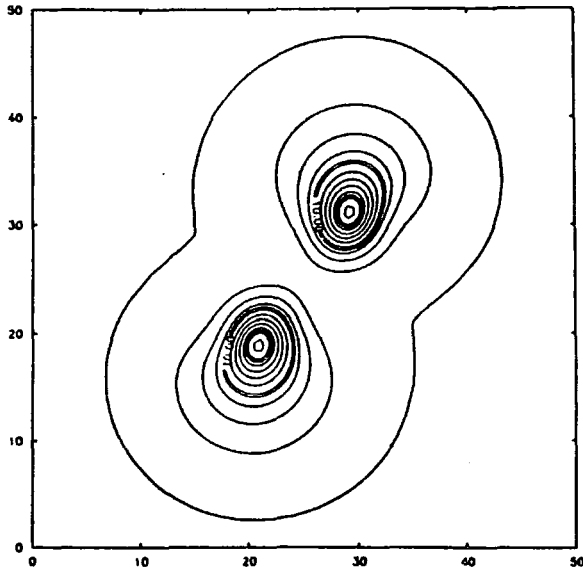


Fig. 3.17a: Action density (3.48) with  $\begin{cases} a_1 = 15 + 19i, & b_1 = 23 + 19i, \\ a_2 = 27 + 31i, & b_2 = 35 + 31i. \end{cases}$

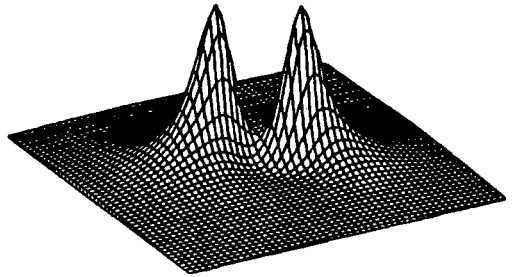
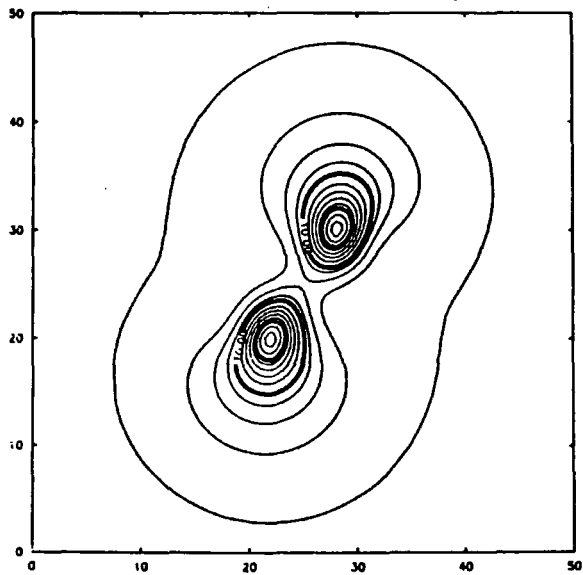


Fig. 3.17b: Action density (3.48) with  $\begin{cases} a_1 = 16 + 20i, & b_1 = 24 + 20i, \\ a_2 = 26 + 30i, & b_2 = 34 + 30i. \end{cases}$

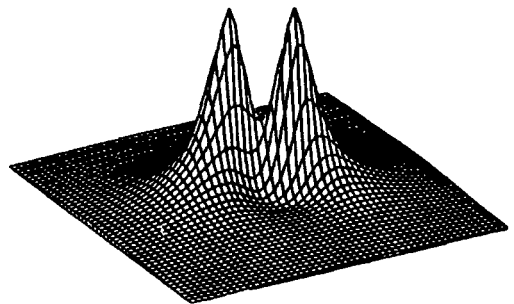
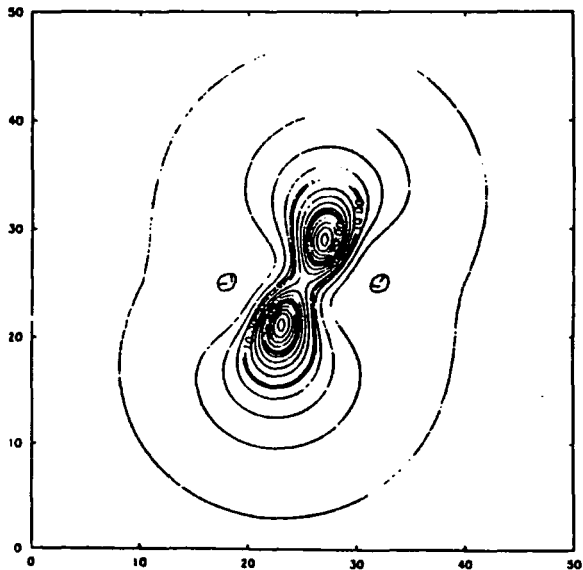


Fig. 3.17c: Action density (3.48) with  $\begin{cases} a_1 = 17 + 21i, & b_1 = 25 + 21i, \\ a_2 = 25 + 29i, & b_2 = 33 + 29i. \end{cases}$

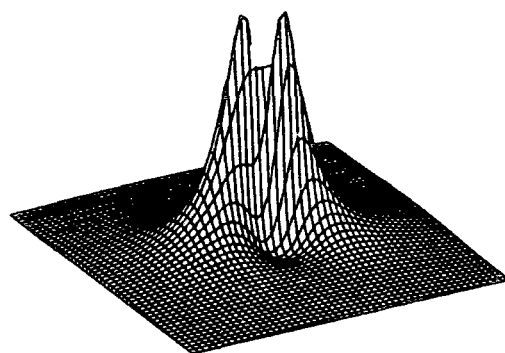
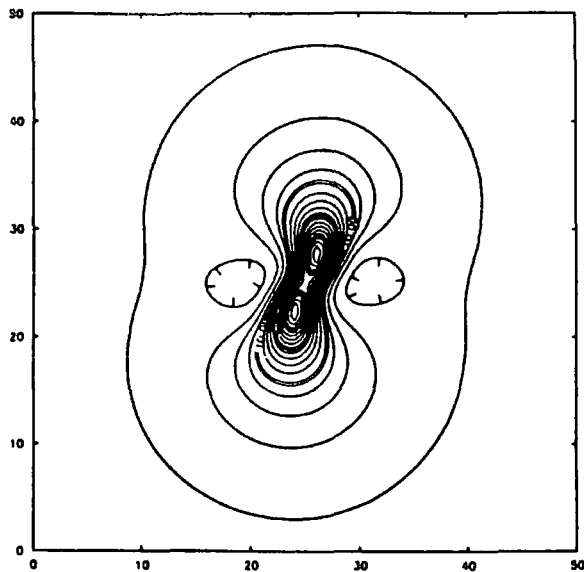


Fig. 3.18a: Action density (3.48) with  $\begin{cases} a_1 = 18 + 22i, & b_1 = 26 + 22i, \\ a_2 = 24 + 28i, & b_2 = 32 + 28i. \end{cases}$

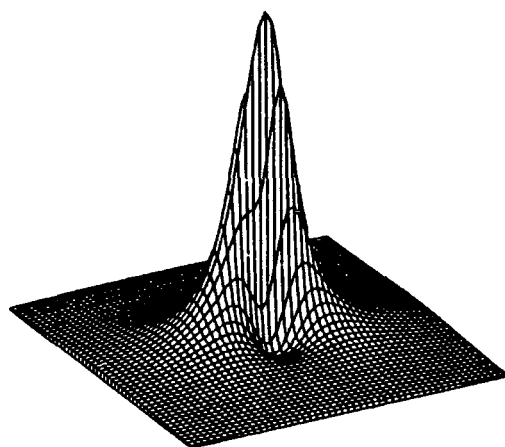
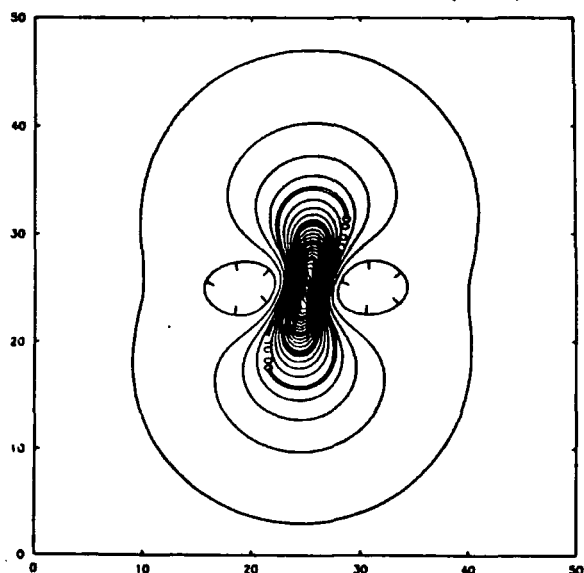


Fig. 3.18b: Action density (3.48) with  $\begin{cases} a_1 = 19 + 23i, & b_1 = 27 + 23i, \\ a_2 = 23 + 27i, & b_2 = 31 + 27i. \end{cases}$

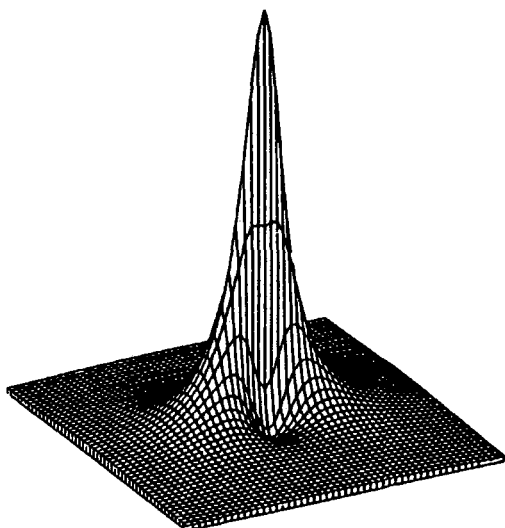
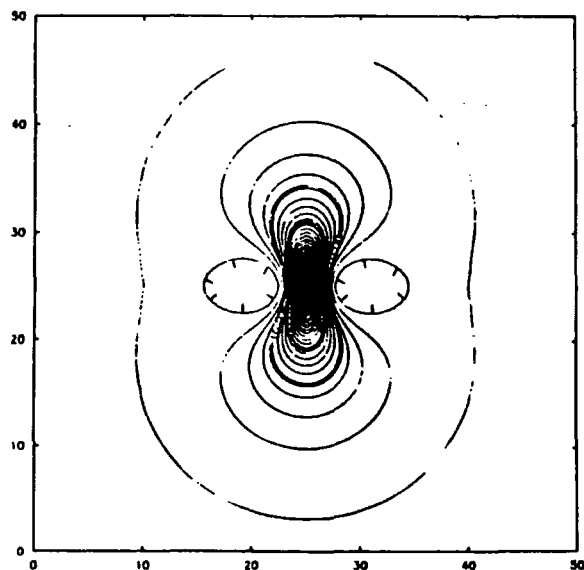


Fig. 3.18c: Action density (3.48) with  $\begin{cases} a_1 = 21 + 25i, & b_1 = 29 + 25i, \\ a_2 = 21 + 25i, & b_2 = 29 + 25i. \end{cases}$

the model by introducing a time dependence into the system, and consider our extended structures as moving objects in  $(2 + 1)$  dimensions.

The modified action for the  $\mathbb{C}P^1$  model in  $(2 + 1)$  dimensions is of the form

$$S = \int \left( -\overline{D_\mu Z} \cdot D_\mu Z + \overline{D_0 Z} \cdot D_0 Z \right) dt d^2x \quad (3.52)$$

where  $\mu = x, y$  and

$$D_0 = \partial_0 - \overline{Z} \cdot \partial_0 Z \quad (3.53)$$

where  $\partial_0$  denotes partial differentiation with respect to the time variable  $t$ . The first term in the integrand is simply the “action density” of the model in 2 dimensions which we studied extensively in the previous part of this chapter; the second term, for reasons which will become more obvious soon, we will call the “kinetic energy density”. [There is in fact another term which can be added to the integrand—this extra term will be discussed in detail in chapter 4.]

The equations of motion corresponding to  $S$  (ie. the Euler-Lagrange equations) have obvious time independent solutions, that is, soliton solutions—these static objects are simply the instantons (anti-instantons) of the  $\mathbb{C}P^1$  model in 2 dimensions discussed previously. Following Wilczek and Zee [17] we refer to these solitons as “skyrmions” [13] (or in the case of anti-instantons, “anti-skyrmions”).

To find time dependent solutions, we should try to solve the full equations of motion, bearing in mind of course that our interest lies in field configurations for which the total energy  $E$  is finite at any time  $t$ , that is,

$$E = \int \left( \overline{D_\mu Z} \cdot D_\mu Z + \overline{D_0 Z} \cdot D_0 Z \right) d^2x < \infty. \quad (3.54)$$

Forgács et al. [34] have studied this problem in detail and presented a general method of generating such solutions; however, complications were encountered in the construction of certain auxiliary functions in the solutions, and the existence of non-static solutions which satisfied all the necessary conditions was not fully resolved.



Therefore, rather than trying to solve the equations of motion exactly, we return to the skyrmion solutions described above, and *introduce* a time dependence into these fields (as described by Din and Zakrzewski [28]) by assuming that *all* time dependence resides in the parameters  $a_i$ ,  $b_i$  and  $\lambda$  which define the skyrmion, ie., we assume

$$a_i = a_i(t), \quad b_i = b_i(t), \quad \text{and} \quad \lambda = \lambda(t). \quad (3.55)$$

These proposed configurations do *not* solve the full equations of motion exactly, but if the parameters vary slowly with time it has been argued—see for example references [20] and [35]—that these fields approximate exact solutions in the limit of small velocities.

Take for example the  $n$ -skyrmion solution defined by (3.1) and (3.2), that is,

$$Z_\alpha = \frac{f_\alpha(x_+)}{|f_\alpha(x_+)|} \quad (3.56)$$

where  $\alpha = 1, 2$  and

$$\begin{aligned} f_1 &= \lambda \prod_{i=1}^n (x_+ - a_i), \\ f_2 &= \prod_{i=1}^n (x_+ - b_i). \end{aligned} \quad (3.57)$$

For the rest of this discussion we shall fix  $\lambda = 1$ , as this parameter only plays a minor role in the problem. Following Din and Zakrzewski [28] we now allow the parameters  $a_i$  and  $b_i$  to vary slowly as a function of the time variable  $t$  belonging to a finite interval  $[0, T]$ . If the configurations are *identical* at  $t = 0$  and  $t = T$ , then the skyrmion system defines a map  $S^3 \rightarrow \mathbb{C}P^1 \sim S^2$ , and since it is a well-known mathematical result that  $\Pi_3(S^2) = \mathbb{Z}$ , then for each topological charge sector the skyrmion manifold is multiply connected: this result is basically responsible for the spin and statistics properties of skyrmions—see in particular reference [36] and also discussions in chapter 4.

For the moment, however, we shall assume that  $a_i(t)$  and  $b_i(t)$  vary arbitrarily in the interval  $[0, T]$  (always subject to  $a_i(t_0) \neq b_j(t_0)$  for any particular

$t_0$ —see note at beginning of chapter). We now recall that we wish our configuration to have finite total energy at any time, as expressed by (3.54). Calculation of the first term in the integral (3.54) is straightforward—it is simply the “total action” of an  $n$ -instanton configuration in 2 dimensions, which we know takes the value  $2\pi n$ , independent of the values of  $a_i$  and  $b_j$ . So the contribution to the total energy of the  $n$ -skyrmion system due to the time dependence of the parameters  $a_i$  and  $b_j$  comes wholly from the second term in the integral (3.54)—hence the interpretation of  $\overline{D_0 Z} \cdot D_0 Z$  as the “kinetic energy density” of the system.

So we are left needing to calculate

$$\int \overline{D_0 Z} \cdot D_0 Z \, d^2 x. \quad (3.58)$$

Note that a small amount of simple algebraic manipulation yields the result

$$\overline{D_0 Z} \cdot D_0 Z = \overline{K_0} K_0 \quad (3.59)$$

where

$$K_0 = \frac{f_2 \partial_0 f_1 - f_1 \partial_0 f_2}{|f_1|^2 + |f_2|^2}. \quad (3.60)$$

There is much that can be said about this form of the kinetic energy density for various configurations and we shall discuss this in detail in chapter 5. Here we shall substitute explicitly for  $f_1$  and  $f_2$  from (3.57) (remembering that  $\lambda = 1$ ) and, as noted by Din and Zakrzewski [28], we obtain the result

$$K_0 = \frac{f_1 f_2}{|f_1|^2 + |f_2|^2} \sum_{i=1}^n \left[ \frac{\dot{b}_i}{x_+ - b_i} - \frac{\dot{a}_i}{x_+ - a_i} \right], \quad (3.61)$$

where

$$\dot{a}_i = \frac{da_i}{dt} \quad \text{and} \quad \dot{b}_i = \frac{db_i}{dt}. \quad (3.62)$$

Now, if the limit  $|x_+| \rightarrow \infty$  is considered, it is found that

$$\overline{K_0} K_0 \sim \frac{1}{|x_+|^2} \left[ \sum_i (\overline{\dot{b}_i} - \overline{\dot{a}_i}) \right] \left[ \sum_j (\dot{b}_j - \dot{a}_j) \right] + O\left(\frac{1}{|x_+|^5}\right) \quad (3.63)$$

and hence we see that in general  $\int \overline{D_0 Z} \cdot D_0 Z \, d^2 x$  diverges.

In order to ensure that our  $n$ -skyrmion system has *finite* kinetic energy, it is therefore necessary to impose a constraint on the parameters  $a_i$  and  $b_j$ , namely

$$\sum_{i=1}^n (\dot{b}_i - \dot{a}_i) = 0. \quad (3.64)$$

In the “instanton quark and antiquark” interpretation for the parameters  $a_i$  and  $b_i$ , this corresponds to requiring equality of the total quark and antiquark “momenta”; interpreted as a “dilute instanton gas” this constraint implies that the total internal “size” of the skyrmions must be conserved, if we make the obvious definition of “size”:-

$$|s_i(t)| = |b_i(t) - a_i(t)| \quad (3.65)$$

(cf. (3.8)). [If we interpret the system in the “point-particle” limit, then the kinetic energy of the system would *automatically* be finite: this interpretation contains the assumption that *each*  $s_i$  is small and constant, a constraint which is in fact stronger than the necessary minimum constraint of

$$\sum_{i=1}^n s_i = \text{constant}. \quad (3.66)$$

which we impose.]

Din and Zakrzewski then go on to calculate the leading terms for the kinetic energy—for more details, see reference [28]. It is sufficient for our purposes just to note that the kinetic energy can be made finite.

We are now in a position to return to the extended structures we considered in two dimensions in the earlier part of this chapter, and to introduce a similar time dependence into their parameters so that they can be thought of as moving objects in (2+1) dimensions. We can then look at their kinetic energy densities in a similar manner to the  $n$ -skyrmion case considered above, and determine what constraints it is necessary to impose on their parameters to ensure finiteness of the kinetic energy.

We have five configurations to consider: a one skyrmion configuration from

(3.4)

$$w = \frac{x_+ - a(t)}{x_+ - b(t)}, \quad (3.67)$$

a two skyrmion configuration from (3.28)

$$w = \frac{(x_+ - a_1(t))(x_+ - a_2(t))}{(x_+ - b_1(t))(x_+ - b_2(t))}, \quad (3.68)$$

and three candidate configurations for a skyrmion-antiskyrmion system from (3.34), (3.45) and (3.47), namely

$$w = \frac{(x_+ - a_1(t))(x_- - \overline{a_2(t)})}{(x_+ - b_1(t))(x_- - \overline{b_2(t)})}, \quad (3.69)$$

$$w = \frac{(x_+ - a(t))}{(x_- - b(t))}, \quad (3.70)$$

and

$$w = \frac{(x_+ - a_1(t))(x_+ - a_2(t))}{(x_- - b_1(t))(x_- - b_2(t))} \quad (3.71)$$

respectively. Configurations (3.67) and (3.68) are obviously just special cases of (3.57), and their kinetic energies can be made finite by imposing the constraints

$$\dot{b} - \dot{a} = 0 \quad (3.72)$$

and

$$(\dot{b}_1 - \dot{a}_1) + (\dot{b}_2 - \dot{a}_2) = 0 \quad (3.73)$$

respectively. In other words, in the one skyrmion system the skyrmion's "size" must remain constant, and in the two skyrmion system (when the two structures are well separated) the relative "sizes" of the two skyrmions can change, but the sum of the "sizes" must be constant.

Now we consider (3.69). Using (3.59) and (3.60), simple algebraic manipulation and then a consideration of the limit  $|x_+| \rightarrow \infty$  produces the result that

$$\overline{D_0 Z} \cdot D_0 Z \sim \frac{1}{|x_+|^2} \left[ \frac{|\dot{b}_1 - \dot{a}_1|^2 + |\dot{b}_2 - \dot{a}_2|^2}{4} \right]. \quad (3.74)$$

Hence, for the kinetic energy of this system to be finite, we need to impose

$$\dot{b}_1 - \dot{a}_1 = 0 \quad \text{and} \quad \dot{b}_2 - \dot{a}_2 = 0. \quad (3.75)$$

This constraint is noticeably stronger than the two skyrmion case, and seems to indicate that the skyrmion and antiskyrmion “sizes” must be conserved *independently*. Indeed, further consideration using the configuration

$$w = f_1/f_2, \quad (3.76)$$

where

$$\begin{aligned} f_1 &= \prod_{i=1}^k (x_+ - a_i)(x_- - \bar{c}_i) \\ f_2 &= \prod_{i=1}^k (x_+ - b_i)(x_- - \bar{d}_i), \end{aligned} \quad (3.77)$$

as a candidate for a system containing  $k$  skyrmions and  $k$  antiskyrmions soon produces the constraint

$$\sum_{i=1}^k \dot{b}_i - \dot{a}_i = 0 \quad \text{and} \quad \sum_{i=1}^k \dot{d}_i - \dot{c}_i = 0 \quad (3.78)$$

to ensure finiteness of the kinetic energy. This clearly indicates that the total “sizes” of the skyrmions and the antiskyrmions in the system must be conserved independently—certainly a stronger constraint than in the  $n$ -skyrmion case. Nevertheless, it is still possible to obtain a finite kinetic energy.

This turns out not to be the case for configuration (3.70). Calculating  $\overline{D_0 Z} \cdot D_0 Z$  for this field produces, in the limit  $|x_+| \rightarrow \infty$ ,

$$\overline{D_0 Z} \cdot D_0 Z \sim \frac{1}{|x_+|^2} \left[ \frac{|\dot{b}|^2 + |\dot{a}|^2}{4} \right] \quad (3.79)$$

and so the kinetic energy is divergent unless

$$|\dot{b}|^2 + |\dot{a}|^2 = 0, \quad (3.80)$$

which is only possible if  $a$  and  $b$  are both constants. This is not a valid situation, because the configuration would then be time-independent and its kinetic energy

density would be zero. Hence this configuration cannot result in a finite kinetic energy: if a large  $|x_+|$  cut-off  $\Lambda$  is introduced, it is found that the kinetic energy contains a  $\log \Lambda$  dependence, and is thus logarithmically divergent when we take the limit  $\Lambda \rightarrow \infty$ .

Finally, consideration of the kinetic energy density for (3.71) reveals in the limit  $|x_+| \rightarrow \infty$  that

$$\overline{D_0 Z} \cdot D_0 Z \sim \frac{1}{|x_+|^2} \left[ \frac{|\dot{a}_1 + \dot{a}_2|^2 + |\dot{b}_1 + \dot{b}_2|^2}{4} \right]. \quad (3.81)$$

Thus the kinetic energy of this system can be made finite by imposing the constraint

$$\dot{a}_1 + \dot{a}_2 = 0 \quad \text{and} \quad \dot{b}_1 + \dot{b}_2 = 0. \quad (3.82)$$

This constraint has no obvious interpretation in the “dilute instanton gas” limit; in the “instanton quark and antiquark” interpretation it corresponds to requiring the total quark and antiquark “momenta” both to be zero independently. Recall, however, that although the kinetic energy of this system can be finite, the “total action” of the system in the two-dimensional case was shown to be divergent, and hence the total energy of the system when thought of as a moving object in  $(2 + 1)$  dimensions is also divergent.

So we have now considered all our configurations as moving objects in  $(2 + 1)$  dimensions. We have looked at the constraints needed in each case to ensure finiteness of the energy, and interpreted these constraints as far as possible in terms of the characteristics of the skyrmions/antiskyrmions in the system. One configuration, namely

$$w = \frac{(x_+ - a_1)(x_- - \bar{a}_2)}{(x_+ - b_1)(x_- - \bar{b}_2)}, \quad (3.83)$$

is noticeable for exhibiting many promising features we would like to see in a system containing skyrmions and antiskyrmions.

We shall use all of these  $(2 + 1)$ -dimensional moving objects extensively in the following chapters.

#### 4. THE HOPF TERM.

It is well known that if a purely bosonic classical field theory admits topological solitons then it may have fermionic characteristics too—Finkelstein and Rubinstein [36] considered this possibility from the point of view of algebraic topology, and found that the multi-valuedness of the configuration space was crucial for the existence of spin and Fermi-Dirac statistics in the theory.

There are similar possibilities for the  $\mathbb{C}P^1$  model. In 2 dimensions we know that the field manifold of this model is  $\mathbb{C}P^1 \sim S^2$  and the configuration space is the space of continuous maps from the base space (which we know is the compactified Euclidean space  $E^2$ , that is,  $S^2$ ) to the field manifold. We have seen in chapters 2 and 3 that since  $\Pi_2(S^2) = \mathbb{Z}$  then this model admits topological solitons. In  $(2 + 1)$  dimensions however, with time evolution introduced as explained in the previous chapter, we have seen that approximate time-dependent solutions can be produced which evolve over the time interval  $[0, T]$ . If these configurations are *identical* at  $t = 0$  and  $t = T$ , then the base space of our model can be thought of as  $S^3$  and these skyrmions define maps  $S^3 \rightarrow \mathbb{C}P^1 \sim S^2$ . The additional well-known mathematical result  $\Pi_3(S^2) = \mathbb{Z}$  therefore means that the configuration space in this  $(2 + 1)$ -dimensional model is *infinitely* connected, which leads to the possibility of *fractional* spin and statistics. This possibility can in fact be realized, as was first shown by Wilczek and Zee [17] who noted that in  $(2 + 1)$  dimensions one was at liberty to introduce an additional topological term into the usual expression for the action. This extra term, known as the Hopf term of the theory, does not affect the classical equations of motion as we shall see later, but does alter the quantum properties of the model—it turns out that the Hopf term is related to the “spin” properties of the extended structures in the model, and the skyrmions can have *any* spin, determined by an arbitrary coefficient in the Hopf term, a fact reflected in their statistics properties.

Wilczek and Zee [17] produced an explicit expression for this term using the  $O(3)$  parametrization; Din and Zakrzewski [22] rewrote it using the equivalent  $\mathbb{C}P^1$  notation, and it is the latter that we shall present here.

The  $\mathbb{C}P^1$  model contains a topological current  $J^\mu$  (see reference [22]) given by

$$J^\mu = -\frac{i}{2\pi} \epsilon^{\mu\nu\lambda} \overline{D_\nu Z} \cdot D_\lambda Z \quad (4.1)$$

where all greek indices now run over 0, 1, 2 and  $\epsilon^{012} = +1$  (hence the notation  $J^0$  used in equations (2.26) and (2.27) whilst defining the topological charge). This current must be conserved, and its conservation enables us to define a “gauge potential”  $A_\mu$  through the curl equation:-

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \quad (4.2)$$

Explicitly,

$$A_\mu = i\overline{Z} \cdot \partial_\mu Z, \quad (4.3)$$

and using these quantities, the complete action for this model can now be presented, namely

$$S = \int \left[ \overline{D^\mu Z} \cdot D_\mu Z + \frac{\theta}{2\pi} A_\mu J^\mu \right] dt d^2x. \quad (4.4)$$

The  $\theta$ -term is the Hopf term of the theory, where  $\theta$  is an arbitrary coefficient and

$$H \equiv \int A_\mu J^\mu dt d^2x \quad (4.5)$$

is known as the Hopf invariant—a quantity formally analogous to the Chern-Simons term of gauge theories (see reference [37]) since from (4.2) and (4.3) it follows that

$$\begin{aligned} A_\mu J^\mu &= -\frac{1}{2\pi} A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \\ &= -\frac{1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \end{aligned} \quad (4.6)$$

where

$$F_{\nu\lambda} \equiv \partial_\nu A_\lambda - \partial_\lambda A_\nu. \quad (4.7)$$



The solitons of this model are the time independent solutions of the equations of motion corresponding to (4.4). However, since the  $\theta$ -term involves time derivatives it is identically zero for such solutions, and so the solutions are once again the instantons/anti-instantons of the  $\mathbb{C}P^1$  model in two dimensions discussed in chapter 2. In fact, as we shall see below, the addition of the Hopf term does not modify the classical equations of motion of the  $(2+1)$ -dimensional model at all since it can be shown to be a total divergence locally. [This last fact has been demonstrated independently for the  $\mathbb{C}P^1$  parametrization of the model by Din and Zakrzewski [22] and by Wu and Zee [38]; in the  $O(3)$  parametrization  $A_\mu J^\mu$  is not obviously a total divergence locally, but it can easily be shown that arbitrary variations of  $A_\mu J^\mu$  are total divergences—see for example reference [39]—from which it can also be deduced that the equations of motion remain unaltered.]

To see that the Hopf term is a total divergence in the  $n$ -skyrmion case, we reproduce here the arguments of Din and Zakrzewski [22], with the time parameter introduced into the skyrmions as described in the previous chapter, that is, the parameters  $a_i$ ,  $b_i$  and  $\lambda$  in the  $n$ -skyrmion configuration

$$Z = f/|f|, \quad (4.8)$$

where

$$\begin{aligned} f_1 &= \prod_{i=1}^n (x_+ - a_i) \\ f_2 &= \lambda \prod_{i=1}^n (x_+ - b_i), \end{aligned} \quad (4.9)$$

are allowed to vary arbitrarily as functions of  $t$  in the interval  $[0, T]$  (subject to  $a_i \neq b_j$  and  $\lambda \neq 0$  for all  $t$ ). Then

$$\begin{aligned} H &= \int_0^T dt \int d^2x A_\mu J^\mu \\ &= \frac{1}{2\pi} \int_0^T dt \int d^2x h \end{aligned} \quad (4.10)$$

where the integrand  $h$  is given by

$$h = 2\pi A_\mu J^\mu = \epsilon^{\mu\nu\lambda} (\bar{Z} \cdot \partial_\mu Z) (\partial_\nu \bar{Z} \cdot \partial_\lambda Z). \quad (4.11)$$

Writing in general

$$Z = f/|f|, \quad (4.12)$$

where  $f = (f_1(x_+), f_2(x_+))$ , after a few lines of algebra the quantity  $h$  in (4.11) becomes

$$h = \frac{i}{|f|^4} \left[ (\partial_0 f_1 \partial_+ f_2 - \partial_0 f_2 \partial_+ f_1) (f_1 \partial_+ f_2 - f_2 \partial_+ f_1)^\dagger - c.c. \right]. \quad (4.13)$$

Making use of the identity

$$\partial_+ \left( \frac{f_1}{\overline{f_2} |f|^2} \right) = \frac{1}{|f|^4} (f_2 \partial_+ f_1 - f_1 \partial_+ f_2), \quad (4.14)$$

we see that  $h$  can be rewritten as

$$h = i \left\{ \partial_- \left[ (\partial_0 f_2 \partial_+ f_1 - \partial_0 f_1 \partial_+ f_2) \frac{\overline{f_1}}{f_2 |f|^2} \right] - c.c. \right\}, \quad (4.15)$$

proving the claim that  $h$  is a total divergence. Hence the addition of the Hopf term does not affect the classical equations of motion, and the total derivative form of  $h$  enables us to write  $H$  as a line integral using Green's Theorem in the complex plane: we obtain

$$H = \frac{1}{2\pi} \int_0^T \frac{1}{2} \left[ \oint (\partial_0 f_2 \partial_+ f_1 - \partial_0 f_1 \partial_+ f_2) \frac{\overline{f_1}}{f_2 |f|^2} dx_+ + c.c. \right] dt \quad (4.16)$$

where the  $x_+$  contour in the line integral is the circle at infinity, which therefore encloses all the singularities of the integrand.

Inserting the explicit forms for  $f_1$  and  $f_2$  from (4.9), the integrand becomes

$$\left[ \sum_{i,j=1}^n \frac{\dot{a}_i - \dot{b}_j}{(x_+ - a_i)(x_+ - b_j)} + \sum_{i=1}^n \frac{1}{(x_+ - a_i)} \frac{\dot{\lambda}}{\lambda} \right] \frac{|f_1|^2}{|f|^2} \quad (4.17)$$

where as before the dots mean time derivatives. We can now use calculus of residues to evaluate the line integral in (4.16): from the first term of (4.17) there are contributions from the poles at  $x_+ = b_j$ , and from the second term the only contribution is at infinity. The final result for  $H$  is

$$H = \frac{i}{2} \int_0^T \left[ - \sum_{i,j=1}^n \frac{\dot{b}_j - \dot{a}_i}{b_j - a_i} + \frac{\dot{\lambda}}{\lambda} \frac{n}{(1 + |\lambda|^2)} - c.c. \right] dt. \quad (4.18)$$

Note that this expression involves line integrals of the relative vectors  $b_j - a_i$  and the scale  $\lambda$  in the complex plane. If the skyrmion configuration at  $t = 0$  coincides with the configuration at  $t = T$ , that is, if the set of parameters  $\{a_i(T)\}$  is some permutation of the set  $\{a_i(0)\}$ , and similarly for  $b_j(t)$  (and  $\lambda(0) = \lambda(T)$ ), then the  $t$  integral becomes a closed contour integral and the skyrmion defines a map  $S^3 \rightarrow \mathbb{C}P^1 \sim S^2$ : thus the value of  $H$  is always an integer multiple of  $2\pi$ , due to the mathematical result  $\Pi_3(S^2) = \mathbb{Z}$ .

As an example, Din and Zakrzewski considered the case of a one skyrmion configuration

$$f = (x_+ - a, x_+ - b) \quad (4.19)$$

and allowed the parameters  $a$  and  $b$  to perform a  $2\pi$  rotation, following the prescription

$$a(t) = ae^{it}, \quad b(t) = be^{it}, \quad t \in [0, 2\pi]. \quad (4.20)$$

With this time evolution, it is simple to calculate that

$$H = -\frac{i}{2} \int_0^{2\pi} (i + i) dt = 2\pi \quad (4.21)$$

and we recognize that the relative vector  $b(t) - a(t)$  makes one revolution about the origin during this evolution. In general, the value of  $H$  thus depends on how many times the relative position vector of the “instanton quark” and “antiquark” revolves around the origin.

The two skyrmion configuration can be considered as a further example; however, before discussing this case in more detail we shall derive expressions for the Hopf terms of other configurations we have studied in previous chapters. We can then carry out certain transformations on the parameters of these various extended structures, and compare the values of the Hopf terms thus produced in a systematic way.

First, for completeness, note that we exhibited above the Hopf term for the

configuration

$$\begin{aligned} f_1 &= \prod_{i=1}^m (x_+ - a_i) \\ f_2 &= \lambda \prod_{i=1}^n (x_+ - b_i) \end{aligned} \quad (4.22)$$

for the case  $m = n$ ; we shall now look briefly at the more general case  $m \neq n$ . (Such solutions do exist: we shall meet examples in later chapters.) Obviously,  $H$  can still be written in a line integral form as in equation (4.16):-

$$H = \frac{1}{2\pi} \int_0^T \frac{1}{2} \left[ \oint (\partial_0 f_2 \partial_+ f_1 - \partial_0 f_1 \partial_+ f_2) \frac{\bar{f}_1}{f_2 |f|^2} dx_+ + c.c. \right] dt, \quad (4.23)$$

and inserting expressions for  $f_1$  and  $f_2$  the integrand is found to be

$$\left[ \sum_{i=1}^m \sum_{j=1}^n \frac{\dot{a}_i - \dot{b}_j}{(x_+ - a_i)(x_+ - b_j)} + \sum_{i=1}^m \frac{1}{(x_+ - a_i) \lambda} \dot{\lambda} \right] \frac{|f_1|^2}{|f|^2}, \quad (4.24)$$

in complete analogy with (4.17). Again calculus of residues enables us to calculate this line integral around the circle at infinity: for  $m > n$  there are contributions from the poles at  $x_+ = b_j$  from the first term, and at infinity from the second term; for  $m < n$  the contribution at infinity disappears completely and the only contributions come from the simple poles. The final result is

$$H = \frac{i}{2} \int_0^T \left[ - \sum_{i=1}^m \sum_{j=1}^n \frac{\dot{b}_j - \dot{a}_i}{b_j - a_i} + \frac{\dot{\lambda}}{\lambda} m \vartheta(m - n) - c.c. \right] dt \quad (4.25)$$

where

$$\vartheta(m - n) = \begin{cases} 1 & \text{for } m > n \\ 0 & \text{for } m < n. \end{cases} \quad (4.26)$$

Now we shall attempt to calculate the Hopf terms in an analogous way for our candidate configurations for a skyrmion plus antiskyrmion system. To this end we consider first configurations of the form

$$\begin{aligned} f_1 &= f_1(x_+) = \prod_{i=1}^n (x_+ - a_i) \\ f_2 &= f_2(x_-) = \prod_{i=1}^n (x_- - \bar{b}_i), \end{aligned} \quad (4.27)$$

so that two of our candidates are considered, namely the cases  $n = 1$  and  $n = 2$  (cf. (3.45) and (3.47)). Can the Hopf terms for these structures also be written

in line integral form? Well, if we substitute (4.27) into our expression for  $h$  from (4.11), we find after a certain amount of algebra that

$$h = \frac{i}{|f|^4} \left[ (\partial_0 f_1 \partial_+ \bar{f}_2 - \partial_0 \bar{f}_2 \partial_+ f_1) (\bar{f}_2 \partial_+ f_1 - f_1 \partial_+ \bar{f}_2)^\dagger - c.c. \right], \quad (4.28)$$

and using the identity

$$\partial_+ \left( \frac{f_1}{f_2 |f|^2} \right) = \frac{1}{|f|^4} (\bar{f}_2 \partial_+ f_1 - f_1 \partial_+ \bar{f}_2) \quad (4.29)$$

we soon realize that

$$h = i \left\{ \partial_- \left[ (\partial_0 f_1 \partial_+ \bar{f}_2 - \partial_0 \bar{f}_2 \partial_+ f_1) \frac{\bar{f}_1}{f_2 |f|^2} \right] - c.c. \right\}, \quad (4.30)$$

again showing that  $h$  can be written in a total derivative form. So as before the addition of this Hopf term does not affect the classical equations of motion of the system, and  $H$  can be written as a line integral in the following manner:-

$$H = \frac{1}{2\pi} \int_0^T \frac{1}{2} \left[ \oint (\partial_0 f_1 \partial_+ \bar{f}_2 - \partial_0 \bar{f}_2 \partial_+ f_1) \frac{\bar{f}_1}{f_2 |f|^2} dx_+ + c.c. \right] dt \quad (4.31)$$

where the  $x_+$  contour is the circle at infinity.

This expression holds for  $f_1 = f_1(x_+)$ ,  $f_2 = f_2(x_-)$  in general; we shall now insert the explicit forms for  $f_1$  and  $f_2$  from (4.27), and we find that the integrand becomes

$$\sum_{i,j=1}^n \frac{\dot{b}_j - \dot{a}_i}{(x_+ - a_i)(x_+ - b_j)} \frac{|f_1|^2}{|f|^2} + c.c. \quad (4.32)$$

Applying calculus of residues once again to the  $x_+$  integral around the circle at infinity, the result obtained for the Hopf term is

$$H = \frac{i}{2} \int_0^T \left[ \sum_{i,j=1}^n \frac{\dot{b}_j - \dot{a}_i}{b_j - a_i} - c.c. \right] dt, \quad (4.33)$$

an expression which involves a line integral in the complex plane of the relative vectors  $b_j - a_i$ , and hence to which we can apply the same remarks as for the  $n$ -skyrmion case (indeed, (4.33) and (4.18) only differ by an overall minus sign if we take  $\lambda = 1$ ): we shall see in detail soon that the value of  $H$  from (4.33) is

again always an integer multiple of  $2\pi$  if the  $t$  integral becomes a closed contour integral as described previously.

But first there is one more configuration for which we would like to derive the Hopf term: our most likely candidate for a skyrmion/antiskyrmion system

$$w = \frac{(x_+ - a_1)(x_- - \bar{a}_2)}{(x_+ - b_1)(x_- - \bar{b}_2)}. \quad (4.34)$$

In other words, we would like to calculate the Hopf term for a configuration of the form

$$w = \frac{f_1(x_+, x_-)}{f_2(x_+, x_-)} \quad (4.35)$$

where  $f_1$  and  $f_2$  are functions of  $x_+$  and  $x_-$ . Inserting this into our expression for  $h$  in (4.11), we can eventually produce  $h$  in the form

$$h = \frac{i}{|f|^4} \times \left\{ \begin{array}{l} \partial_0 f_1 (-\partial_+ f_2 A + \partial_- f_2 B + f_2 C) - c.c. \\ + \partial_0 f_2 (\partial_+ f_1 A - \partial_- f_1 B - f_1 C) - c.c. \end{array} \right\} \quad (4.36)$$

where

$$\begin{aligned} A &= \partial_- \bar{f}_1 f_2 - \bar{f}_1 \partial_- \bar{f}_2 \\ B &= \partial_+ \bar{f}_1 f_2 - \bar{f}_1 \partial_+ \bar{f}_2 \\ C &= \partial_- \bar{f}_1 \partial_+ \bar{f}_2 - \partial_+ \bar{f}_1 \partial_- \bar{f}_2. \end{aligned} \quad (4.37)$$

Assuming that we can write

$$\begin{aligned} f_1(x_+, x_-) &= F_1(x_+) G_1(x_-) \\ f_2(x_+, x_-) &= F_2(x_+) G_2(x_-) \end{aligned} \quad (4.38)$$

—this is always the case for the expressions we use—then (4.36) becomes

$$\begin{aligned} h &= \frac{i(F_1 \partial_0 G_1 + G_1 \partial_0 F_1)}{|f|^4} \times \left\{ \begin{array}{l} -G_2 \partial_+ F_2 \bar{G}_1 \bar{G}_2 (\partial_- \bar{F}_1 \bar{F}_2 - \bar{F}_1 \partial_- \bar{F}_2) \\ + F_2 \partial_- G_2 \bar{F}_1 \bar{F}_2 (\partial_+ \bar{G}_1 \bar{G}_2 - \bar{G}_1 \partial_+ \bar{G}_2) \\ + F_2 G_2 (\bar{F}_2 \bar{G}_1 \partial_- \bar{F}_1 \partial_+ \bar{G}_2 - \bar{F}_1 \bar{G}_2 \partial_- \bar{F}_2 \partial_+ \bar{G}_1) \end{array} \right\} \\ &+ \frac{i(F_2 \partial_0 G_2 + G_2 \partial_0 F_2)}{|f|^4} \times \left\{ \begin{array}{l} + G_1 \partial_+ F_1 \bar{G}_1 \bar{G}_2 (\partial_- \bar{F}_1 \bar{F}_2 - \bar{F}_1 \partial_- \bar{F}_2) \\ - F_1 \partial_- G_1 \bar{F}_1 \bar{F}_2 (\partial_+ \bar{G}_1 \bar{G}_2 - \bar{G}_1 \partial_+ \bar{G}_2) \\ - F_1 G_1 (\bar{F}_2 \bar{G}_1 \partial_- \bar{F}_1 \partial_+ \bar{G}_2 - \bar{F}_1 \bar{G}_2 \partial_- \bar{F}_2 \partial_+ \bar{G}_1) \end{array} \right\} \\ &+ \text{complex conjugate.} \end{aligned} \quad (4.39)$$

It is not at all clear how to turn this expression into a total divergence, and hence how to express this Hopf term as a line integral. We can estimate what the expression may look like by writing down analogous terms to (4.15) and (4.30), for example,

$$\begin{aligned}
h = & -i\partial_- \left[ \frac{(\partial_0 F_1 \partial_+ F_2 - \partial_0 F_2 \partial_+ F_1) |f_1|^2}{F_1 F_2} \frac{|f_1|^2}{|f|^2} \right] + i\partial_+ [c.c.] \\
& -i\partial_- \left[ \frac{(\partial_0 \overline{G_1} \partial_+ \overline{G_2} - \partial_0 \overline{G_2} \partial_+ \overline{G_1}) |f_1|^2}{G_1 G_2} \frac{|f_1|^2}{|f|^2} \right] + i\partial_+ [c.c.] \\
& +i\partial_- \left[ \frac{(\partial_0 F_1 \partial_+ \overline{G_2} - \partial_0 \overline{G_2} \partial_+ F_1) |f_1|^2}{F_1 \overline{G_2}} \frac{|f_1|^2}{|f|^2} \right] - i\partial_+ [c.c.] \\
& +i\partial_- \left[ \frac{(\partial_0 \overline{G_1} \partial_+ F_2 - \partial_0 F_2 \partial_+ \overline{G_1}) |f_1|^2}{F_2 \overline{G_1}} \frac{|f_1|^2}{|f|^2} \right] - i\partial_+ [c.c.]
\end{aligned} \tag{4.40}$$

or more generally

$$\begin{aligned}
h = & -i\partial_- \left[ \frac{(\partial_0 f_1 \partial_+ f_2 - \partial_0 f_2 \partial_+ f_1) |f_1|^2}{f_1 f_2} \frac{|f_1|^2}{|f|^2} \right] + i\partial_+ [c.c.] \\
& -i\partial_- \left[ \frac{(\partial_0 \overline{f_1} \partial_+ \overline{f_2} - \partial_0 \overline{f_2} \partial_+ \overline{f_1}) |f_1|^2}{\overline{f_1} \overline{f_2}} \frac{|f_1|^2}{|f|^2} \right] + i\partial_+ [c.c.] \\
& +i\partial_- \left[ \frac{(\partial_0 f_1 \partial_+ \overline{f_2} - \partial_0 \overline{f_2} \partial_+ f_1) |f_1|^2}{f_1 \overline{f_2}} \frac{|f_1|^2}{|f|^2} \right] - i\partial_+ [c.c.] \\
& +i\partial_- \left[ \frac{(\partial_0 \overline{f_1} \partial_+ f_2 - \partial_0 f_2 \partial_+ \overline{f_1}) |f_1|^2}{\overline{f_2} \overline{f_1}} \frac{|f_1|^2}{|f|^2} \right] - i\partial_+ [c.c.]
\end{aligned} \tag{4.41}$$

but in each case, when we perform the copious amounts of algebra needed to compare these guesses with (4.36) and (4.39), we are always left with extra terms which cannot be eliminated. It is clear that we need a different approach to enable us to calculate the Hopf term for this type of configuration.

The approach we adopt is as follows: consider

$$w = \frac{(x_+ - a_1)(x_- - \overline{a_2})}{(x_+ - b_1)(x_- - \overline{b_2})} = \frac{f_1}{f_2} \tag{4.42}$$

where

$$\begin{aligned}
f_1 &= (x_+ - a_1)(x_- - \overline{a_2}), \\
f_2 &= (x_+ - b_1)(x_- - \overline{b_2}).
\end{aligned} \tag{4.43}$$

If we take

$$\begin{aligned}
f_1' &= \frac{x_+ - a_1}{x_+ - b_1}, \\
f_2' &= \frac{x_- - \overline{b_2}}{x_- - \overline{a_2}},
\end{aligned} \tag{4.44}$$

then we can say that

$$w = \frac{f'_1}{f'_2}, \quad (4.45)$$

but because we now have  $f'_1 = f'_1(x_+)$  and  $f'_2 = f'_2(x_-)$  we can use equations (4.27)–(4.31) to write the Hopf term, as a function of  $f'_1$  and  $f'_2$ , in the form of a line integral. Substituting the explicit forms for  $f'_1$  and  $f'_2$  from (4.44) into this line integral, the integrand becomes (cf. (4.32))

$$\begin{aligned} & \frac{|f_1|^2}{|f|^2} \left\{ \frac{(\dot{b}_1 - \dot{b}_2)}{(x_+ - b_1)(x_+ - b_2)} + \frac{(\dot{a}_1 - \dot{a}_2)}{(x_+ - a_1)(x_+ - a_2)} + c.c. \right\} \\ & - \frac{|f_1|^2}{|f|^2} \left\{ \frac{(\dot{b}_1 - \dot{a}_2)}{(x_+ - b_1)(x_+ - a_2)} + \frac{(\dot{a}_1 - \dot{b}_2)}{(x_+ - a_1)(x_+ - b_2)} + c.c. \right\} \end{aligned} \quad (4.46)$$

and using calculus of residues once more to evaluate the line integral we find that

$$H(f'_1, f'_2) = \frac{i}{2} \int_0^T \left[ \left( \frac{\dot{b}_2 - \dot{a}_1}{b_2 - a_1} - \frac{\dot{b}_1 - \dot{a}_2}{b_1 - a_2} \right) - c.c. \right] dt. \quad (4.47)$$

This is *not* the final result, however: we are seeking to evaluate  $H(f_1, f_2)$ , which will not in general be the same as  $H(f'_1, f'_2)$  even though  $f_1/f_2 = w = f'_1/f'_2$ . We shall now proceed to calculate the extra terms which may appear in the former due to this change of variables.

We start with the original definition of the Hopf term in terms of  $Z'$  from (4.10) and (4.11) where

$$Z' = \frac{(f'_1, f'_2)}{\sqrt{|f'_1|^2 + |f'_2|^2}}, \quad (4.48)$$

that is,

$$H(Z') = \frac{1}{2\pi} \int_0^T dt \int d^2x h(Z') \quad (4.49)$$

where

$$h(Z') = \epsilon^{\mu\nu\lambda} (\overline{Z'}^\mu \cdot \partial_\nu Z') (\partial_\nu \overline{Z'}^\lambda \cdot \partial_\lambda Z'), \quad (4.50)$$

and we consider a change of gauge given by

$$(f'_1, f'_2) \rightarrow (f_1, f_2) = (f'_1, f'_2)g \quad (4.51)$$



where

$$g = (x_+ - b_1)(x_- - \bar{a}_2). \quad (4.52)$$

Thus

$$Z' \rightarrow Z = \frac{Z'g}{|g|} = Z'e^{i\phi} \quad (4.53)$$

where

$$\phi = \tan^{-1} \left( \frac{\text{Im}(x_+ - b_1)(x_- - \bar{a}_2)}{\text{Re}(x_+ - b_1)(x_- - \bar{a}_2)} \right). \quad (4.54)$$

(Note that  $\bar{Z}'Z' = \bar{Z}Z = 1$ .) How does this change of gauge affect the Hopf term? Simple algebra yields

$$\begin{aligned} h(Z') &\rightarrow h(Z) = h(Z'e^{i\phi}) \\ &= h(Z') + i\epsilon^{\mu\nu\lambda} \partial_\mu \phi (\partial_\nu \bar{Z}' \cdot \partial_\lambda Z') \end{aligned} \quad (4.55)$$

and hence

$$H(f_1, f_2) = H(f'_1, f'_2) + \frac{i}{2\pi} \int_0^T dt \int d^2x \epsilon^{\mu\nu\lambda} \partial_\mu \phi (\partial_\nu \bar{Z}' \cdot \partial_\lambda Z'), \quad (4.56)$$

so our result for the required Hopf term is

$$\begin{aligned} H &= \frac{i}{2} \int_0^T dt \left[ \left( \frac{\dot{b}_2 - \dot{a}_1}{b_2 - a_1} - \frac{\dot{b}_1 - \dot{a}_2}{b_1 - a_2} \right) - c.c. \right] dt \\ &+ \frac{i}{2\pi} \int_0^T dt \int d^2x \left[ \epsilon^{\mu\nu\lambda} \partial_\mu \phi (\partial_\nu \bar{Z}' \cdot \partial_\lambda Z') \right]. \end{aligned} \quad (4.57)$$

Consider this second integral, say  $I$ :

$$I = \frac{i}{2\pi} \int_V \partial_\mu \left[ \epsilon^{\mu\nu\lambda} \phi (\partial_\nu \bar{Z}' \cdot \partial_\lambda Z') \right] dV \quad (4.58)$$

where  $V$  is the volume defined by  $-\infty < x < +\infty$ ,  $-\infty < y < +\infty$ ,  $0 < t < T$ .

Using the divergence theorem, we obtain

$$I = \frac{i}{2\pi} \int_S \left[ \epsilon^{\mu\nu\lambda} \phi (\partial_\nu \bar{Z}' \cdot \partial_\lambda Z') \right] dS_\mu \quad (4.59)$$

where  $S$  is the enclosing surface of  $V$ , and if we calculate this integral over the whole surface using the usual cylindrical polar coordinates we find that

$$I = \frac{i}{2\pi} \left[ \iint dx dy \phi (\partial_x \bar{Z}' \cdot \partial_y Z' - \partial_y \bar{Z}' \cdot \partial_x Z') \right]_{t=0}^{t=T} \quad (4.60)$$

since the contribution from the curved surface at infinity is zero. Written in terms of  $f'_1$  and  $f'_2$ , this becomes

$$I = \frac{1}{\pi} \left[ \iint dx dy \frac{\phi}{|f'|^4} (f'_1 \overline{f'_1} \partial_+ \overline{f'_2} \partial_- f'_2 - f'_2 \overline{f'_2} \partial_- \overline{f'_1} \partial_+ f'_1) \right]_{t=0}^{t=T} \quad (4.61)$$

and so, finally, the Hopf term for the configuration (4.42) is given by

$$H = \frac{i}{2} \int_0^T \left[ \left( \frac{b_2 - \dot{a}_1}{b_2 - a_1} - \frac{\dot{b}_1 - \dot{a}_2}{b_1 - a_2} \right) - c.c. \right] dt + \frac{1}{\pi} \left[ \iint dx dy \frac{\phi}{|f'|^4} (f'_1 \overline{f'_1} \partial_+ \overline{f'_2} \partial_- f'_2 - f'_2 \overline{f'_2} \partial_- \overline{f'_1} \partial_+ f'_1) \right]_{t=0}^{t=T} \quad (4.62)$$

where

$$f'_1 = \frac{x_+ - a_1}{x_+ - b_1}, \quad f'_2 = \frac{x_- - \overline{b_2}}{x_- - \overline{a_2}} \quad (4.63)$$

and

$$\phi = \tan^{-1} \left( \frac{\text{Im}(x_+ - b_1)(x_- - \overline{a_2})}{\text{Re}(x_+ - b_1)(x_- - \overline{a_2})} \right). \quad (4.64)$$

This expression may seem difficult to work with, and indeed it is, except in the special cases where the final and initial configurations of a time evolution are *identical*, in which case the second term vanishes and only the simpler first term remains.

More will be said about this Hopf term later; now we note that since we have derived expressions for our various Hopf terms we can return to the earlier comment about configurations which coincide at  $t = 0$  and  $t = T$ , namely, if the set  $\{a_i(T)\}$  is some permutation of the set  $\{a_i(0)\}$  and similarly for  $b$ , then the  $t$  integral reduces to a closed contour integral, and the homotopy  $\Pi_3(S^2) = \mathbb{Z}$  tells us that the value of  $H$  is always an integer multiple of  $2\pi$ . We illustrate this by considering the Hopf terms corresponding to the three configurations

$$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)} \quad (4.65)$$

$$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - \overline{b_1})(x_- - \overline{b_2})} \quad (4.66)$$

$$w = \frac{(x_+ - a_1)(x_- - \overline{a_2})}{(x_+ - b_1)(x_- - \overline{b_2})} \quad (4.67)$$

when the parameters are allowed to evolve in two ways:-

(a) Fixing  $b_1$  and  $b_2$  (so that  $\dot{b}_1 = \dot{b}_2 = 0$ ), and exchanging  $a_1$  and  $a_2$  according to the prescription

$$\begin{aligned} a_1(t) &= \frac{(a_1 + a_2)}{2} - \frac{(a_2 - a_1)}{2} e^{it} \\ a_2(t) &= \frac{(a_1 + a_2)}{2} + \frac{(a_2 - a_1)}{2} e^{it} \end{aligned} \quad (4.68)$$

where  $0 \leq t \leq \pi$  (or similarly exchanging  $b_1$  and  $b_2$  with  $a_1$  and  $a_2$  fixed).

(b) Fixing  $a_2$  and  $b_2$  (so that  $\dot{a}_2 = \dot{b}_2 = 0$ ), and rotating  $a_1$  through  $2\pi$  about  $b_1$  according to the prescription

$$a_1(t) = b_1 + (a_1 - b_1)e^{it} \quad (4.69)$$

where  $0 \leq t \leq 2\pi$  (or similarly rotating  $a_2$  about  $b_2$  with  $a_1$  and  $b_1$  fixed).

The Hopf terms of all these cases bar one (see below) are simple to calculate—a closed contour integral evaluated using calculus of residues again—and the various results are shown in Table 4.1. The problem case is the exchange prescription for the configuration (4.67): since the initial and final configurations are not *identical* (they are identical in the rotation prescription), the second expression in the Hopf term for this structure does not vanish, and hence the value of the Hopf term is not obviously calculable.

Notice from Table 4.1 that the values of  $H$  depend crucially on the positions of the various fixed parameters with respect to the closed contours traced out by the moving parameters, and also the values depend on how many times the contours are traced out by the moving parameters. For example, if we consider the configuration (4.65), if only  $b_1$  lies inside the closed contour then the value of  $H$  is  $2\pi$  (if the contour is traced out once) for both the exchange and the rotation prescriptions; for  $b_1$  and  $b_2$  both inside the contour,  $H = 4\pi$  for both prescriptions, and similarly for the configuration (4.66). This observation serves as an indication that the exchange and the  $2\pi$ -rotation belong to the same

Configuration	Motion performed	Hopf term (with conditions)
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed. Similarly for $1 \leftrightarrow 2$ .	$H = 4\pi$ if $b_2$ is inside the contour traced by $a_1$ $H = 2\pi$ if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	$H = 4\pi$ if $b_1$ & $b_2$ are both inside the contour traced by $a_1$ & $a_2$ $H = 2\pi$ if $b_1$ or $b_2$ is inside the contour traced by $a_1$ & $a_2$ $H = 0$ if $b_1$ & $b_2$ are both outside the contour traced by $a_1$ & $a_2$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - b_1)(x_- - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed. Similarly for $1 \leftrightarrow 2$ .	$H = -4\pi$ if $b_2$ is inside the contour traced by $a_1$ $H = -2\pi$ if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - b_1)(x_- - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	$H = -4\pi$ if $b_1$ & $b_2$ are both inside the contour traced by $a_1$ & $a_2$ $H = -2\pi$ if $b_1$ or $b_2$ is inside the contour traced by $a_1$ & $a_2$ $H = 0$ if $b_1$ & $b_2$ are both outside the contour traced by $a_1$ & $a_2$
$w = \frac{(x_+ - a_1)(x_- - \overline{a_2})}{(x_+ - b_1)(x_- - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed.	$H = -2\pi$ if $b_2$ is inside the contour traced by $a_1$ $H = 0$ if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_- - \overline{a_2})}{(x_+ - b_1)(x_- - b_2)}$	Rotation of $2\pi$ of $a_2$ about $b_2$ with $a_1$ and $b_1$ fixed.	$H = 2\pi$ if $b_1$ is inside the contour traced by $a_2$ $H = 0$ if $b_1$ is outside the contour traced by $a_2$
$w = \frac{(x_+ - a_1)(x_- - \overline{a_2})}{(x_+ - b_1)(x_- - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	The Hopf term for this motion is not easily determined.

Table 4.1

homotopy class, as discussed in reference [36], which is a manifestation of the connection between the spin and the statistics of the extended structures.

Now, according to Feynman (as quoted for example in reference [17]), to determine the spin of our skyrmions we simply rotate the skyrmion adiabatically through  $2\pi$  over a long time period, after which the wave function of the skyrmion acquires a phase factor  $\exp(iS)$  where  $S$  is the action corresponding to this rotation. Hence the effect of the Hopf term corresponding to the motions discussed above is simply to multiply the skyrmion wave function by a factor  $\exp(i\theta H/2\pi)$ , as can be seen from equation (4.4). For the choice of the parameter  $\theta = 0$ , this factor becomes  $+1$ , that is, the object is quantized as a boson; for the choice  $\theta = \pi$ , this factor becomes  $\exp(iH/2)$  which can be  $+1$ , that is, quantized as a boson, or  $-1$ , that is, quantized as a fermion, depending on the choice of the transformation path but independent of the orientation of the path. This is shown in Table 4.2 where this factor has been calculated for all the prescriptions considered in Table 4.1 (setting  $\theta = \pi$ ), and it is clearly demonstrated that although the Hopf term does not affect the classical equations of motion of the extended structures of this model as we saw previously, the term is intimately related to the spin properties of the objects and hence the quantum properties of the model.

If we choose the parameter  $\theta$  such that  $0 < \theta < \pi$ , to try to interpolate somehow between fermions and bosons, the transformation paths can be chosen so as to obtain *any* factor  $\exp(iq\theta)$  where  $q$  is an integer, thus realizing the possibility as claimed at the beginning of this chapter that this model can exhibit fractional spin and statistics.

We saw above that the form of the Hopf term in (4.62), corresponding to the configuration (4.34), was not very satisfactory when we tried to calculate its value for certain transformations. However, (4.34) turned out to be our best candidate for a system containing one skyrmion and one antiskyrmion in earlier

Configuration	Motion performed	Value of $\exp(iH/2)$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed. Similarly for $1 \leftrightarrow 2$ .	+1 if $b_2$ is inside the contour traced by $a_1$ -1 if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_+ - b_1)(x_+ - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	+1 if $b_1$ & $b_2$ are both either inside or outside the contour traced by $a_1$ & $a_2$ -1 if $b_1$ or $b_2$ is inside the contour traced by $a_1$ & $a_2$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - b_1)(x_- - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed. Similarly for $1 \leftrightarrow 2$ .	+1 if $b_2$ is inside the contour traced by $a_1$ -1 if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_+ - a_2)}{(x_- - b_1)(x_- - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	+1 if $b_1$ & $b_2$ are both either inside or outside the contour traced by $a_1$ & $a_2$ -1 if $b_1$ or $b_2$ is inside the contour traced by $a_1$ & $a_2$
$w = \frac{(x_+ - a_1)(x_- - \bar{a}_2)}{(x_+ - b_1)(x_- - b_2)}$	Rotation of $2\pi$ of $a_1$ about $b_1$ with $a_2$ and $b_2$ fixed. Similarly for $1 \leftrightarrow 2$ .	-1 if $b_2$ is inside the contour traced by $a_1$ +1 if $b_2$ is outside the contour traced by $a_1$
$w = \frac{(x_+ - a_1)(x_- - \bar{a}_2)}{(x_+ - b_1)(x_- - b_2)}$	Exchange of $a_1$ and $a_2$ , with $b_1$ and $b_2$ fixed. Similarly for $a_i \leftrightarrow b_i$ .	The value of $\exp(iH/2)$ for this motion is not easily determined.

Table 4.2

chapters, and so we end this chapter by trying to obtain a more suitable form for this Hopf term.

We start by recalling that for one skyrmion we know from (4.18) that

$$\int h d^2x = -\pi i \left[ \frac{\dot{b} - \dot{a}}{b - a} - \frac{\bar{\dot{b}} - \bar{\dot{a}}}{\bar{b} - \bar{a}} \right], \quad (4.70)$$

or if we put

$$a = p + iq, \quad b = r + is, \quad (4.71)$$

then

$$\int h d^2x = \frac{2\pi[(r - p)(\dot{s} - \dot{q}) + (s - q)(\dot{p} - \dot{r})]}{(r - p)^2 + (s - q)^2}. \quad (4.72)$$

We shall try to obtain this same result by a different method, which we hope to be able to generalize so that we can calculate the required Hopf term for (4.34).

First we note that in general we can always write

$$h = \partial_\mu J^\mu = \partial_t J^t + \partial_x J^x + \partial_y J^y \quad (4.73)$$

independently of the configuration we are considering. This result can be obtained simply by algebraic manipulation of

$$h = \epsilon^{\mu\nu\lambda} (\bar{Z} \cdot \partial_\mu Z) (\partial_\nu \bar{Z} \cdot \partial_\lambda Z) \quad (4.74)$$

after substituting

$$Z_1 = y_1 + iy_2 \quad (4.75)$$

$$Z_2 = y_3 + iy_4$$

where the  $y_i$ 's necessarily satisfy

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1, \quad (4.76)$$

and we find that

$$\begin{aligned} J^t &= 2f(y_i) [\partial_x y_1 \partial_y y_2 - \partial_y y_1 \partial_x y_2] \\ J^x &= 2f(y_i) [\partial_y y_1 \partial_t y_2 - \partial_t y_1 \partial_y y_2] \\ J^y &= 2f(y_i) [\partial_t y_1 \partial_x y_2 - \partial_x y_1 \partial_t y_2] \end{aligned} \quad (4.77)$$

where

$$f(y_i) = \sin^{-1} \left( \frac{y_3}{\sqrt{1 - y_1^2 - y_2^2}} \right). \quad (4.78)$$

For certain configurations (for example, the one skyrmion case),  $h$  can be further simplified to

$$h = \partial_x h^x + \partial_y h^y \quad (4.79)$$

and hence  $\int h d^2x$  is easily calculated using

$$\int h d^2x = \oint (n_x h^x + n_y h^y) ds. \quad (4.80)$$

This was the method we used to calculate all our previous Hopf terms. However, in some cases, simplification to the form (4.79) is not so straightforward, as we saw for the configuration (4.34) for example, so instead we use the following idea: consider instead the quantity

$$\int h d^3x. \quad (4.81)$$

Using cylindrical polar coordinates with the “ $z$  direction” being the  $t$  direction we can use (4.73) to write

$$\begin{aligned} \int h d^3x &= \int \partial_\mu J^\mu d^3x = \int J^\mu dS_\mu \\ &= \int_{t=T} J^t dx dy - \int_{t=0} J^t dx dy + \int_\sigma J^\mu dS_\mu \end{aligned} \quad (4.82)$$

where  $\sigma$  is the curved cylindrical surface at infinity. Now, by choosing a particular gauge the first two terms can be shown to *cancel* as we shall now demonstrate.

Recall that from (2.27) the topological charge density  $J^0$  is given by

$$\begin{aligned} J^0 &= -\frac{i}{2\pi} (\overline{D_x Z} \cdot D_y Z - \overline{D_y Z} \cdot D_x Z) \\ &= -\frac{i}{2\pi} (\partial_x \overline{Z} \cdot \partial_y Z - \partial_y \overline{Z} \cdot \partial_x Z) \end{aligned} \quad (4.83)$$

after a few lines of algebra. Substituting (4.75) into this expression yields

$$J^0 = \frac{1}{\pi} [\partial_x y_1 \partial_y y_2 - \partial_y y_1 \partial_x y_2] + \frac{1}{\pi} [\partial_x y_3 \partial_y y_4 - \partial_y y_3 \partial_x y_4]. \quad (4.84)$$



Also, from (4.77),

$$J^t = 2 \sin^{-1} \left( \frac{y_3}{\sqrt{y_3^2 + y_4^2}} \right) [\partial_x y_1 \partial_y y_2 - \partial_y y_1 \partial_x y_2]. \quad (4.85)$$

Now, if we choose a gauge such that

$$y_3 = \alpha y_4 \quad (4.86)$$

where  $\alpha$  is a constant, then (4.84) and (4.85) simplify to

$$J^0 = \frac{1}{\pi} [\partial_x y_1 \partial_y y_2 - \partial_y y_1 \partial_x y_2] \quad (4.87)$$

and

$$J^t = 2 \sin^{-1} \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) [\partial_x y_1 \partial_y y_2 - \partial_y y_1 \partial_x y_2]. \quad (4.88)$$

In other words, this choice of gauge produces the result

$$J^0 \propto J^t. \quad (4.89)$$

Hence,

$$\int_{t_0} J^t dx dy \propto \int_{t_0} J^0 dx dy \quad (4.90)$$

for any time  $t_0$ , but since  $J^0$  is the conserved topological charge density and is therefore independent of time, then the value of  $\int J^0 dx dy$  and hence  $\int J^t dx dy$  is independent of the time at which it is evaluated. Hence

$$\int_{t=T} J^t dx dy = \int_{t=0} J^t dx dy \quad (4.91)$$

and the first two terms in (4.82) cancel as claimed.

For the one skyrmion case, can such a gauge choice be made? In other words, if

$$Z \rightarrow Z' = Z e^{i\phi}, \quad (4.92)$$

can  $\phi$  be chosen so that  $y_3 = \alpha y_4$  and such that the Hopf term is *invariant*? (We saw earlier that for a general  $\phi$ ,  $H$  is not always invariant—see for example (4.56).)

Algebraic considerations soon show that  $\phi$  must be of the form

$$\phi = \tan^{-1} \left( \frac{y_3 - \alpha y_4}{\alpha y_3 + y_4} \right) \quad (4.93)$$

to ensure  $y_3 = \alpha y_4$  in the new gauge. We need to check the behaviour of  $H$  under this gauge transformation.

We saw earlier—see (4.56) and (4.60)—that under a gauge transformation  $Z \rightarrow Z e^{i\phi}$  the Hopf term transforms as

$$H \rightarrow H + \frac{i}{2\pi} \left[ \iint dx dy \phi \left( \partial_x \bar{Z} \cdot \partial_y Z - \partial_y \bar{Z} \cdot \partial_x Z \right) \right]_{t=0}^{t=T}. \quad (4.94)$$

Using (4.83), we realize this can be written in the form

$$H \rightarrow H - \left[ \iint dx dy \phi J^0 \right]_{t=0}^{t=T}, \quad (4.95)$$

and also in an  $n$ -skyrmion case  $J^0$  can be rewritten in the form

$$J^0 = \frac{1}{\pi} \partial_+ \partial_- \log |f|^2 \quad (4.96)$$

(this can be proved in only a couple of lines of algebra) and so we can say that

$$H \rightarrow H - \frac{1}{\pi} \left[ \iint dx dy \phi \partial_+ \partial_- \log |f|^2 \right]_{t=0}^{t=T}. \quad (4.97)$$

Integrating this second term by parts twice, it is fairly easy to show that the surface terms vanish and we are left with

$$H \rightarrow H - \frac{1}{\pi} \left[ \iint dx dy \partial_+ \partial_- \phi \log |f|^2 \right]_{t=0}^{t=T}. \quad (4.98)$$

Finally it can be shown that in our one skyrmion case

$$\partial_+ \partial_- \phi \equiv 0 \quad (4.99)$$

and so we are left with

$$H \rightarrow H, \quad (4.100)$$

that is, the Hopf term is *invariant* under this change of gauge.

Hence this choice of gauge can indeed be made, and equation (4.82) simplifies to

$$\int h d^3 x = \int_{\sigma} J^{\mu} dS_{\mu} \quad (4.101)$$

where the quantity on the right must be calculated in the new gauge, but the quantity on the left is invariant under this change of gauge. Now  $\sigma$  is the curved cylindrical surface at infinity and so on this surface

$$dS_{\mu} = \lim_{\rho \rightarrow \infty} n_{\mu} \rho d\psi dt \quad (4.102)$$

where  $n_{\mu}$  is the  $\mu^{\text{th}}$  component of the unit normal to the surface, and  $\rho$  and  $\psi$  are the usual radial and angular variables in cylindrical polar coordinates. Hence

$$\int h d^3 x = \lim_{\rho \rightarrow \infty} \int J^{\mu} n_{\mu} \rho d\psi dt, \quad (4.103)$$

that is,

$$\int h d^2 x dt = \lim_{\rho \rightarrow \infty} \int (J^x \cos \psi + J^y \sin \psi) \rho d\psi dt \quad (4.104)$$

and so we can write in this particular gauge

$$\int h d^2 x = \lim_{\rho \rightarrow \infty} \int_0^{2\pi} (J^x \cos \psi + J^y \sin \psi) \rho d\psi \quad (4.105)$$

where  $J^x$  and  $J^y$  are given by (4.77). Thus we have an expression for  $\int h d^2 x$  as required, which we need to calculate in the correct gauge for the one skyrmion case to compare with (4.72)

Recall that previously the one skyrmion configuration was given by

$$\begin{aligned} Z_1 &= \frac{(x + iy) - (p + iq)}{|f|} \\ Z_2 &= \frac{(x + iy) - (r + is)}{|f|} \end{aligned} \quad (4.106)$$

(using (4.71)). In the new gauge we know that  $y_3 = \alpha y_4$ , so the  $f(y_i)$  in (4.78) becomes a constant:

$$f(\alpha) = \sin^{-1} \left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) \quad (4.107)$$

and  $y_1$  and  $y_2$  can be shown to be given by

$$\begin{aligned} y_1 &= \frac{x-p}{|f|} \cos \phi - \frac{y-q}{|f|} \sin \phi \\ y_2 &= \frac{y-q}{|f|} \cos \phi + \frac{x-p}{|f|} \sin \phi \end{aligned} \quad (4.108)$$

where, using (4.93),

$$\phi = \tan^{-1} \left( \frac{x - \alpha y - (r - \alpha s)}{\alpha x + y - (\alpha r + s)} \right). \quad (4.109)$$

Several pages of algebra enable us to calculate the various derivatives of (4.108) needed to calculate  $J^x$  and  $J^y$ , and hence complete expressions for  $J^x$  and  $J^y$  can be written down. Here, however, we shall just consider the  $\dot{r}$  dependent terms for the sake of argument, and compare these terms with the  $\dot{r}$  dependent terms in (4.72).

It is found after much algebraic manipulation that the  $\dot{r}$  dependent terms in  $J^x$  and  $J^y$  are

$$\frac{2f(\alpha)\dot{r}[(x-p)^2 + (x-q)^2 - (x-r)(x-p) - (y-s)(y-q)]}{[(x-r)^2 + (y-s)^2 + (x-p)^2 + (y-q)^2]^2} \quad (4.110)$$

and

$$\frac{-2f(\alpha)\dot{r}[(x-r)(y-q) - (x-p)(y-s)]}{[(x-r)^2 + (y-s)^2 + (x-p)^2 + (y-q)^2]^2} \quad (4.111)$$

respectively, with  $f(\alpha)$  given by (4.107). Changing to polar coordinates

$$\begin{aligned} x &= \rho \cos \psi, \\ y &= \rho \sin \psi, \end{aligned} \quad (4.112)$$

the  $\dot{r}$  dependent term in  $\int h d^2x$  can therefore be shown to be given by

$$2f(\alpha)\dot{r} \lim_{\rho \rightarrow \infty} \int_0^{2\pi} \frac{[\rho^2(r-p) + \rho(p^2 + q^2 - rp - sq) \cos \psi - \rho(rq - sp) \sin \psi] d\psi}{[2\rho^2 - 2\rho(r+p) \cos \psi - 2\rho(s+q) \sin \psi + (r^2 + s^2 + p^2 + q^2)]^2}. \quad (4.113)$$

Now, as  $\rho \rightarrow \infty$ , the integrand appears to behave  $\sim 1/\rho^2$  and so if we take the limit  $\rho \rightarrow \infty$  this  $\dot{r}$  term would disappear. This contradicts our expected  $\dot{r}$  term from (4.72) of

$$\frac{2\pi(q-s)\dot{r}}{(r-p)^2 + (s-q)^2}. \quad (4.114)$$

Similar behaviour appears if we consider the terms dependent upon  $\dot{s}$ ,  $\dot{p}$  or  $\dot{q}$  and hence we are forced to conclude that our method contains a fundamental error: we believe that the gauge transformation interferes with the simple pole which gave us our contribution via the calculus of residues to the Hopf term (4.70) for the one skyrmion configuration, and so we cannot immediately produce the Hopf term we required. However, although we shall not pursue this idea further in this thesis, preliminary investigations indicate that if a way could be found to deal with this interference then this method provides a promising mode of attack to produce Hopf terms so far uncalculable by the method previously discussed in this chapter.

## 5. SKYRMION DYNAMICS AND KÄHLER METRICS.

We saw in chapter 3 that the motion of skyrmions in the  $\mathbb{C}P^{n-1}$  models can be studied by assuming that all time dependence of these skyrmions resides in the skyrmion parameters. This in fact amounts to studying the dynamics of slowly moving skyrmions which approximate to genuine time dependent classical solutions of the full  $(2 + 1)$ -dimensional theory. In that chapter we defined a quantity known as the “kinetic energy density” of the slowly moving skyrmions and studied the constraints it was necessary to impose on the skyrmion parameters to ensure finiteness of the kinetic energy. Here we shall look more closely at the form of the kinetic energy density so that we can make observations of a more general nature about the evolution of these systems in the limit of small velocities.

First we consider the  $\mathbb{C}P^1$  model, a case already studied in detail by Din and Zakrzewski [20], but one which is important enough in our later work to be presented again here in some detail. In chapter 3 we saw that the kinetic energy density of the  $\mathbb{C}P^1$  model is given by

$$\overline{D_0 Z} \cdot D_0 Z = \overline{K_0} K_0 \equiv \mathcal{K}, \quad \text{say,} \quad (5.1)$$

where

$$K_0 = \frac{f_2 \partial_0 f_1 - f_1 \partial_0 f_2}{|f_1|^2 + |f_2|^2} \quad (5.2)$$

(see equations (3.59) and (3.60)). Substituting the  $n$ -skyrmion configuration

$$\begin{aligned} f_1 &= \prod_{i=1}^n (x_+ - a_i) \\ f_2 &= \prod_{i=1}^n (x_+ - b_i), \end{aligned} \quad (5.3)$$

we saw that the constraint

$$\sum_{i=1}^n (b_i - a_i) = 0 \quad (5.4)$$

must be imposed to ensure finiteness of the kinetic energy of the system (see (3.64)). However, if we now look more closely at the form of (5.2), further observations can be made. Din and Zakrzewski [20] noted that the kinetic energy density  $\mathcal{K}$  for the  $n$ -skyrmion system can be written in a more convenient form as

$$\begin{aligned}\mathcal{K} &= \frac{1}{|f|^4} |f_2 \partial_0 f_1 - f_1 \partial_0 f_2|^2 \\ &= \frac{1}{|f|^4} \left| \dot{a}_i \frac{\partial}{\partial a_i} - \dot{b}_i \frac{\partial}{\partial b_i} \right|^2 |f_1|^2 |f_2|^2\end{aligned}\tag{5.5}$$

(where summation is implied over repeated indices). Introducing the compact notation

$$\{A^\alpha\} = \{a_1, \dots, a_n, b_1, \dots, b_n\}\tag{5.6}$$

the second equality in (5.5) implies that

$$\mathcal{K} = g_{\alpha\bar{\beta}} \dot{A}^\alpha \overline{\dot{A}^\beta}\tag{5.7}$$

where

$$\begin{aligned}g_{a_i \bar{a}_j} &= \frac{1}{|f|^4} \frac{\partial}{\partial a_i} \frac{\partial}{\partial \bar{a}_j} |f_1|^2 |f_2|^2 \\ g_{a_i \bar{b}_j} &= -\frac{1}{|f|^4} \frac{\partial}{\partial a_i} \frac{\partial}{\partial \bar{b}_j} |f_1|^2 |f_2|^2 \\ g_{b_i \bar{a}_j} &= -\frac{1}{|f|^4} \frac{\partial}{\partial b_i} \frac{\partial}{\partial \bar{a}_j} |f_1|^2 |f_2|^2 \\ g_{b_i \bar{b}_j} &= \frac{1}{|f|^4} \frac{\partial}{\partial b_i} \frac{\partial}{\partial \bar{b}_j} |f_1|^2 |f_2|^2\end{aligned}\tag{5.8}$$

and hence we see that the kinetic energy density simply defines an Hermitian metric on the manifold of skyrmion parameters. [Such a metric always exists on a complex analytic manifold: this means that for our complex local coordinates  $\{A^\alpha\}$ , the quadratic form  $ds^2$  which defines the metric on our manifold of skyrmion parameters can be written in the form

$$ds^2 = g_{\alpha\bar{\beta}} dA^\alpha d\overline{A^\beta}\tag{5.9}$$

where  $g_{\alpha\bar{\beta}}$  is an Hermitian matrix.]

In fact, a few lines of algebra easily show that the metric given by (5.8) can be written compactly as

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial \bar{A}^\beta} \log |f|^2, \quad (5.10)$$

which is of the form

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 V}{\partial A^\alpha \partial \bar{A}^\beta} \quad (5.11)$$

and hence we see that the metric defined by  $\mathcal{K}$  is Kähler-like, with Kähler potential

$$V = \log |f|^2. \quad (5.12)$$

[There are several other ways one can recognize a Kähler metric, some of which we shall meet later in this chapter—for a complete mathematical discussion of Hermitian and Kähler manifolds, see for example reference [40].]

The kinetic energy  $K$  is obtained by integrating the metric form over the space coordinates:

$$K = \int g_{\alpha\bar{\beta}} \dot{A}^\alpha \overline{\dot{A}^\beta} d^2x \quad (5.13)$$

and this integral is well defined on the submanifold given by the constraint (5.4) discussed previously. We can also write this as

$$K = G_{\alpha\bar{\beta}} \dot{A}^\alpha \overline{\dot{A}^\beta} \quad (5.14)$$

by introducing the integrated metric

$$G_{\alpha\bar{\beta}} = \int g_{\alpha\bar{\beta}} d^2x \quad (5.15)$$

but care is needed here to avoid divergent integrals: finiteness of (5.13) (due to the constraint) is not enough to ensure that individual terms such as (5.15) in the summation over  $\alpha$  and  $\beta$  are finite.

As Din and Zakrzewski have explained [20], it is possible to prevent divergences occurring, for example, by introducing a Lagrange multiplier term into the metric, or by introducing a space cut-off in all intermediate calculations and



then removing it in the final expressions. Suffice it to say that such a definition of the integrated metric can be made rigorous: we can then write down the corresponding equations of motion for the quark and antiquark coordinates, namely

$$G_{\alpha\bar{\beta}}\tilde{A}^\alpha + \frac{\partial G_{\epsilon\bar{\beta}}}{\partial A^\gamma}\dot{A}^\epsilon\dot{A}^\gamma = 0 \quad (5.16)$$

or

$$\tilde{A}^\alpha + G^{\alpha\bar{\beta}}\frac{\partial G_{\epsilon\bar{\beta}}}{\partial A^\gamma}\dot{A}^\epsilon\dot{A}^\gamma = 0. \quad (5.17)$$

Solutions of these equations, with the constraint taken into account, are precisely the geodesic motions of the skyrmion constituents which represent approximate solutions to the full (2+1)-dimensional theory if the velocity vectors are small.

So, in the  $\mathbb{C}P^1$  model we have seen that the classical dynamics of slowly moving skyrmions can be described by the geodesic motion on a Kähler manifold of the skyrmion quark and antiquark constituents. We shall now try to generalize these observations for the  $\mathbb{C}P^{n-1}$  models with  $n > 2$ , also in (2 + 1) dimensions, where similar time-dependent structures exist which approximate to exact solutions in the limit of small velocities.

In chapter 2 we studied in detail the  $\mathbb{C}P^{n-1}$  model in 2 dimensions, and apart from the  $k$ -instanton solutions

$$Z = f(x_+)/|f| \quad (5.18)$$

where

$$f_i = (x_+ - a_1^i)(x_+ - a_2^i) \dots (x_+ - a_k^i) \quad (5.19)$$

with  $i = 1, \dots, n$ , we saw that for  $n > 2$  there also exist non-instanton solutions of the full Euler-Lagrange equations, obtained from the instanton solutions, given by

$$Z = P_+f/|P_+f| \quad (5.20)$$

where

$$P_+ f = \partial_+ f - \frac{f(\bar{f} \cdot \partial_+ f)}{|f|^2} \quad (5.21)$$

(see equations (2.94) and (2.74)). We can think of all of these extended structures as moving objects in  $(2+1)$  dimensions simply by introducing time dependence as before: by supposing that the  $a_i^i$  depend on a time parameter  $t$ . As for the  $\mathbb{C}P^1$  case, these moving objects then approximate to exact solutions of the  $(2+1)$ -dimensional theory in the limit of small velocities. Can we extend the previous results in this chapter for the  $\mathbb{C}P^1$  model to both the instanton-like and the non-instanton-like skyrmions in the  $\mathbb{C}P^{n-1}$  model in  $(2+1)$  dimensions?

We consider first the  $k$ -instanton-like configuration (5.18) with

$$f_i = (x_+ - a_1^i(t))(x_+ - a_2^i(t)) \dots (x_+ - a_k^i(t)) \quad (5.22)$$

where  $i = 1, \dots, n$ . Starting from the definition of the kinetic energy density

$$\mathcal{K} \equiv \overline{D_0 Z} \cdot D_0 Z = |\partial_0 Z - (\bar{Z} \cdot \partial_0 Z) Z|^2 \quad (5.23)$$

we can rewrite  $\mathcal{K}$  as

$$\mathcal{K} = \partial_0 \bar{Z} \cdot \partial_0 Z - (\partial_0 \bar{Z} \cdot Z)(\bar{Z} \cdot \partial_0 Z). \quad (5.24)$$

Then if

$$Z_i = f_i / |f| \quad \text{where } i = 1, \dots, n \quad (5.25)$$

it is easy to show that

$$\begin{aligned} \mathcal{K} &= \frac{1}{|f|^4} [(\bar{f} \cdot f)(\partial_0 \bar{f} \cdot \partial_0 f) - (\partial_0 \bar{f} \cdot f)(\bar{f} \cdot \partial_0 f)] \\ &= \frac{1}{|f|^4} \sum_{i>j} |f_i \partial_0 f_j - f_j \partial_0 f_i|^2. \end{aligned} \quad (5.26)$$

Substituting

$$f_i = \prod_{l=1}^k (x_+ - a_l^i(t)), \quad (5.27)$$

we soon find that

$$\mathcal{K} = \frac{1}{|f|^4} \sum_{i>j} \left[ \left| \sum_{l=1}^k \left( \dot{a}_l^j \frac{\partial}{\partial a_l^j} - \dot{a}_l^i \frac{\partial}{\partial a_l^i} \right) \right|^2 |f_i|^2 |f_j|^2 \right]. \quad (5.28)$$

This result is a direct consequence of  $D_- Z = 0$ , simply stemming from the fact that all the parameters  $a_i^j$  in  $f_i$  are mutually independent.

Introducing the compact notation

$$\{A^\alpha\} = \{a_1^1, \dots, a_k^1, a_1^2, \dots, a_k^2, \dots, a_1^n, \dots, a_k^n\} \quad (5.29)$$

then equation (5.28) implies that

$$\mathcal{K} = g_{\alpha\bar{\beta}} \dot{A}^\alpha \overline{\dot{A}^\beta} \quad (5.30)$$

where the metric can again be written compactly, after a certain amount of manipulation, as

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial \overline{A^\beta}} \log |f|^2 \quad (5.31)$$

and hence we see again that the kinetic energy density defines a Kähler-like metric on the manifold of the skyrmion parameters, with Kähler potential

$$V = \log |f|^2 \quad (5.32)$$

in direct analogy with the  $\mathbb{C}P^1$  case. So once again the classical dynamics of slowly moving instanton-like skyrmions in this model can be described by the geodesic motion on a Kähler manifold of the skyrmion parameters, and to ensure finiteness of the kinetic energy of this system, we find that we need to impose the constraints

$$\sum_{l=1}^k (\dot{a}_l^i - \dot{a}_l^j) = 0 \quad \text{for all } i \neq j, \quad (5.33)$$

clearly a simple generalization of (5.4) for the  $\mathbb{C}P^1$  case.

We now turn our attention to the non-instanton-like skyrmions in the  $(2+1)$ -dimensional  $\mathbb{C}P^{n-1}$  model. Again the kinetic energy density  $\mathcal{K}$  for this system is given by

$$\mathcal{K} = \partial_0 \bar{Z} \cdot \partial_0 Z - (\partial_0 \bar{Z} \cdot Z)(\bar{Z} \cdot \partial_0 Z) \quad (5.34)$$

as in (5.24), but this time we make the substitution

$$Z_i = P_+ f_i / |P_+ f| \quad (5.35)$$

where  $i = 1, \dots, n$  and the  $f_i$  are given by (5.27), and a few lines of algebra soon produce the result

$$\mathcal{K} = \frac{1}{|P_+f|^4} [(\overline{P_+f} \cdot P_+f)(\partial_0 \overline{P_+f} \cdot \partial_0 P_+f) - (\partial_0 \overline{P_+f} \cdot P_+f)(\overline{P_+f} \cdot \partial_0 P_+f)]. \quad (5.36)$$

A significant amount of manipulation then follows, making use of the usual chain rule of differentiation, and again introducing the compact notation  $\{A^\alpha\}$  as in (5.29) for the skyrmion parameters. The manipulation involves various easily-proved identities containing  $P_+f$ ,  $\overline{P_+f}$  and their derivatives, for example,

$$\frac{\partial}{\partial A^\alpha} \overline{P_+f} \cdot P_+f = 0 \quad (5.37)$$

and

$$\frac{\partial}{\partial A^\alpha} \overline{P_+f} \cdot \frac{\partial}{\partial A^\beta} P_+f = 0, \quad (5.38)$$

and the final result of this algebra is that  $\mathcal{K}$  can be written as

$$\mathcal{K} = g_{\alpha\bar{\beta}} \dot{A}^\alpha \overline{\dot{A}^\beta} \quad (5.39)$$

where the metric  $g_{\alpha\bar{\beta}}$  is given by

$$\begin{aligned} g_{\alpha\bar{\beta}} = & \frac{1}{|P_+f|^2} \left( \frac{\partial}{\partial A^\alpha} \overline{P_+f} \cdot \frac{\partial}{\partial A^\beta} P_+f + \frac{\partial}{\partial A^\beta} \overline{P_+f} \cdot \frac{\partial}{\partial A^\alpha} P_+f \right) \\ & - \frac{1}{|P_+f|^4} \left( \frac{\partial}{\partial A^\beta} \overline{P_+f} \cdot P_+f \right) \left( \overline{P_+f} \cdot \frac{\partial}{\partial A^\alpha} P_+f \right). \end{aligned} \quad (5.40)$$

Just a few more lines of calculation, motivated by analogy with the instanton-like case, shows that this last expression can be rewritten in the form

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial A^\beta} \log |P_+f|^2 + \frac{2}{|P_+f|^2} \frac{\partial}{\partial A^\alpha} \overline{P_+f} \cdot \frac{\partial}{\partial A^\beta} P_+f. \quad (5.41)$$

So the metric defined by the kinetic energy density in this case is certainly Hermitian, and when written in the form (5.41) the first term is obviously Kähler-like, with Kähler potential

$$V = \log |P_+f|^2. \quad (5.42)$$

But what of the second term? Is it possible to write  $g_{\alpha\bar{\beta}}$  completely in an obviously Kähler form?

Naïvely for this metric to be Kähler, the second term

$$\frac{2}{|P_+f|^2} \frac{\partial}{\partial A^\alpha} \overline{P_+f} \cdot \frac{\partial}{\partial A^\beta} P_+f \equiv H_{\alpha\bar{\beta}}, \quad \text{say,} \quad (5.43)$$

must be Kähler-like itself, that is, a potential  $V$  must exist such that

$$H_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial A^\beta} V. \quad (5.44)$$

If such a potential  $V$  exists, then by symmetry of partial derivatives we have

$$\frac{\partial}{\partial A^\gamma} \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial A^\beta} V = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial A^\gamma} \frac{\partial}{\partial A^\beta} V, \quad (5.45)$$

that is,

$$\frac{\partial}{\partial A^\gamma} H_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} H_{\gamma\bar{\beta}}. \quad (5.46)$$

Hence, if this result does *not* hold, it seems that  $H_{\alpha\bar{\beta}}$  can not be Kähler-like.

As an example where the result should hold, we consider the specific case of a one-instanton-like skyrmion in  $\mathbb{C}P^2$ , that is, we take

$$\begin{aligned} f_1 &= (x_+ - a) \\ f_2 &= (x_+ - b) \\ f_3 &= (x_+ - c) \end{aligned} \quad (5.47)$$

where  $a$ ,  $b$  and  $c$  depend on the time parameter  $t$ . Applying the  $P_+$  operator to this solution we obtain

$$\begin{aligned} P_+f_1 &= [(a-b)(x_- - \bar{b}) + (a-c)(x_- - \bar{c})]/|f|^2 \\ P_+f_2 &= [(b-a)(x_- - \bar{a}) + (b-c)(x_- - \bar{c})]/|f|^2 \\ P_+f_3 &= [(c-a)(x_- - \bar{a}) + (c-b)(x_- - \bar{b})]/|f|^2 \end{aligned} \quad (5.48)$$

which in fact is essentially a one anti-instanton-like solution of the  $\mathbb{C}P^2$  model, that is, of the form

$$\begin{aligned} P_+f_1 &= \lambda(x_- - \bar{a}') \\ P_+f_2 &= \mu(x_- - \bar{b}') \\ P_+f_3 &= \nu(x_- - \bar{c}') \end{aligned} \quad (5.49)$$

and so we know that this solution should define a Kähler-like metric in the way discussed above for the instanton-like case.

However, when we try to compute (5.46) for the configuration (5.48), we find that the result does *not* hold in general, implying a non-Kähler metric! This contradiction leads us to reflect upon our “condition for Kählerity”, (5.46), and we realize that we are not using a covariant condition. This presents us with a very interesting problem: in one set of variables  $a', b', c'$  (that is, one complex structure) our metric appears to be Kähler-like whereas in another set of variables  $a, b, c$  (that is, a different complex structure) a non-Kähler-like metric is obtained. Hence, we clearly need a “condition for Kählerity” which will not produce contradictions due to changes of complex structure. We shall discuss this further later in this chapter; first we present configurations for which this prescription works perfectly reasonably, and the  $P_+$  operator does give us further solutions which clearly define Kähler-like metrics.

This situation occurs for example for very special configurations describing a one instanton-like skyrmion and one anti-instanton-like skyrmion located at the same point. Such solutions are real and are constructed as follows: we start with a one instanton solution of the 2-dimensional  $\mathbb{C}P^1$  model, that is,

$$\begin{aligned} f_1 &= \lambda(x_+ - a) \\ f_2 &= \mu(x_+ - b) \end{aligned} \tag{5.50}$$

and we translate this solution into its  $O(3)$  form using

$$q^i = \bar{Z}_\alpha \sigma_{\alpha\beta}^i Z_\beta \tag{5.51}$$

where  $\alpha, \beta = 1, 2$  and  $i = 1, 2, 3$ , and where as usual

$$Z_\alpha = f_\alpha / |f| \tag{5.52}$$

and  $\sigma^i$  are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.53}$$

This procedure yields the results

$$q^1 = \frac{\lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) + c.c.}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2},$$

$$q^2 = \frac{i\lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) + c.c.}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2}, \quad (5.54)$$

$$q^3 = \frac{|\lambda|^2|x_+ - a|^2 - |\mu|^2|x_+ - b|^2}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2}.$$

The next step is to identify  $q^i$  with the components  $Z_i^{(1)}$  of a configuration in the  $\mathbb{C}P^2$  model, that is, simply put  $q^i \equiv Z_i^{(1)}$  in  $\mathbb{C}P^2$ . These  $Z$ 's are real, and do indeed define a solution of the  $\mathbb{C}P^2$  model Euler-Lagrange equations—we met this type of embedded solution earlier, in chapter 2 (see equations (2.41)–(2.44) and (2.52)–(2.54)).

Now, defining

$$\begin{aligned} f_1^{(1)} &= \lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) + \mu\bar{\lambda}(x_- - \bar{a})(x_+ - b) \\ f_2^{(1)} &= i\lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) - i\mu\bar{\lambda}(x_- - \bar{a})(x_+ - b) \\ f_3^{(1)} &= |\lambda|^2|x_+ - a|^2 - |\mu|^2|x_+ - b|^2 \end{aligned} \quad (5.55)$$

we use the operator  $P_-$  defined by

$$P_- f = \partial_- f - \frac{f(\bar{f} \cdot \partial_- f)}{|f|^2} \quad (5.56)$$

to give us another solution of the  $\mathbb{C}P^2$  model (recall we discussed solutions produced by the operators  $P_{\pm}$  in chapter 2). Performing the algebra, we obtain

$$\begin{aligned} P_- f_1^{(1)} &= \frac{\bar{\lambda}\mu(\bar{b} - \bar{a})[\lambda^2(x_+ - a)^2 - \mu^2(x_+ - b)^2]}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2} \\ P_- f_2^{(1)} &= \frac{i\bar{\lambda}\mu(\bar{b} - \bar{a})[\lambda^2(x_+ - a)^2 + \mu^2(x_+ - b)^2]}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2} \\ P_- f_3^{(1)} &= \frac{-2\bar{\lambda}\mu(\bar{b} - \bar{a})\lambda\mu(x_+ - a)(x_+ - b)}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2} \end{aligned} \quad (5.57)$$

and ignoring overall factors, which we are at liberty to do since the model is invariant under a change of phase of  $Z = P_- f / |P_- f|$ , we can take this new solution to be defined by

$$Z_i = f_i^{(0)} / |f^{(0)}| \quad (5.58)$$

where

$$\begin{aligned}
f_1^{(0)} &= \lambda^2(x_+ - a)^2 - \mu^2(x_+ - b)^2 \\
f_2^{(0)} &= i\lambda^2(x_+ - a)^2 + i\mu^2(x_+ - b)^2 \\
f_3^{(0)} &= -2\lambda\mu(x_+ - a)(x_+ - b).
\end{aligned} \tag{5.59}$$

We see that this configuration describes a special two-instanton solution of our two-dimensional  $\mathbb{C}P^2$  model. We can now introduce time dependence into this system in the usual way, and if we consider the kinetic energy density of this  $(2 + 1)$ -dimensional moving object, we do indeed find that it gives rise to a Kähler-like metric of the form

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial \bar{A}^\beta} \log |f^{(0)}|^2 \tag{5.60}$$

as before, with the obvious definition of the compact notation  $\{A^\alpha\}$ .

Now, applying  $P_+$  to  $f_i^{(0)}$ , we get our  $(2 + 1)$ -dimensional non-instanton-like solution

$$Z_i = P_+ f_i^{(0)} / |P_+ f_i^{(0)}| \tag{5.61}$$

where explicitly

$$\begin{aligned}
P_+ f_1^{(0)} &= \frac{-2\lambda\mu(b-a)[\lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) + \mu\bar{\lambda}(x_+ - b)(x_- - \bar{a})]}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2} \\
P_+ f_2^{(0)} &= \frac{-2\lambda\mu(b-a)[i\lambda\bar{\mu}(x_+ - a)(x_- - \bar{b}) - i\mu\bar{\lambda}(x_+ - b)(x_- - \bar{a})]}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2} \\
P_+ f_3^{(0)} &= \frac{-2\lambda\mu(b-a)[|\lambda|^2|x_+ - a|^2 - |\mu|^2|x_+ - b|^2]}{|\lambda|^2|x_+ - a|^2 + |\mu|^2|x_+ - b|^2}.
\end{aligned} \tag{5.62}$$

Performing the copious amounts of algebra needed to calculate  $H_{\alpha\bar{\beta}}$  from (5.43) and then to compute (5.46), we find this time that the result (5.46) does hold, and thus this non-instanton-like skyrmion configuration can lead to a Kähler-like metric as claimed.

Note that, as expected, the  $P_+^2 f_i^{(0)}$ 's can be shown to describe essentially a two anti-instanton-like solution, which leads once more to a Kähler-like metric



from the kinetic energy density in a similar manner. Hence we have produced three related skyrmion systems in  $\mathbb{C}P^2$  which all lead to Kähler-like metrics, despite one of them being of non-instanton-like form.

In all our discussions of the  $\mathbb{C}P^{n-1}$  non-instanton-like cases, we have used the “unintegrated metric”  $g_{\alpha\bar{\beta}}$  given by (5.40). To discuss the dynamics of these systems this metric has to be integrated to the full metric, as in (5.15), that is,

$$G_{\alpha\bar{\beta}} = \int g_{\alpha\bar{\beta}} d^2x. \quad (5.63)$$

Then the dynamics are determined by the motion of the skyrmion parameters along the geodesics of this metric. Unfortunately it is very difficult to perform the integration in (5.63) analytically, even for the simplest configurations, and additionally the form of the constraints (analogous to (5.4) and (5.33)) needed to ensure finiteness of the kinetic energy of the system are more complicated now. Almost certainly any progress here would have to involve some numerical calculations, and so we shall not pursue this further in this thesis; however, it is difficult to conceive how the integrated metric, if it could be determined more explicitly, could avoid exhibiting similar properties to the ones we have discussed above for the unintegrated form.

So, in summary, we have seen that the classical dynamics of slowly moving  $\mathbb{C}P^{n-1}$  instanton-like skyrmions in  $(2+1)$  dimensions can be described by the geodesic motion on a Kähler-like manifold of the solution parameters. However, for the  $\mathbb{C}P^{n-1}$  non-instanton-like solutions, the Kähler or non-Kähler nature of the metric produced is unclear. For some special solutions the metric is obviously Kähler whereas for others the choice of complex structure affects the results dramatically. We end this chapter by taking some steps towards finding a “condition for Kählerity” which will not be so dependent upon the choice of complex structure.

Pope et al. [41] have produced expressions, involving the Riemann curvature tensor, Ricci tensor and covariant derivative constructed from a metric, which

may in principle be used to test whether the metrics we have been considering are Kähler or not. These expressions vanish for Kähler metrics, but not for general Riemannian metrics—the exact expression used depends on whether the metric in question is Einstein (ie. if the Ricci tensor is proportional to the metric tensor) or not.

Now, the metrics we have been considering are certainly Hermitian—are they Einstein? In other words, can we demonstrate that

$$\begin{aligned} R_{\alpha\beta} &= R_{\alpha\bar{\beta}} = 0 \\ R_{\alpha\bar{\beta}} &= \lambda g_{\alpha\bar{\beta}} \end{aligned} \tag{5.64}$$

where  $R_{\alpha\bar{\beta}}$  is the Ricci tensor and  $\lambda$  is a constant of proportionality? We start by considering the quantity  $R_{\alpha\beta}$ .

In terms of the Riemann curvature tensor  $R^\alpha_{\beta\gamma\delta}$ , we can define  $R_{\alpha\beta}$  by

$$R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta} + R^{\bar{\gamma}}_{\alpha\bar{\gamma}\beta}, \tag{5.65}$$

and the Riemann curvature tensor itself can be defined in terms of the Christoffel symbols  $\Gamma^\alpha_{\beta\gamma}$  by

$$\begin{aligned} R^\alpha_{\beta\gamma\delta} &= \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} \\ &\quad - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma} + \Gamma^\alpha_{\bar{\mu}\gamma} \Gamma^{\bar{\mu}}_{\beta\delta} - \Gamma^\alpha_{\bar{\mu}\delta} \Gamma^{\bar{\mu}}_{\beta\gamma} \end{aligned} \tag{5.66}$$

(these are simply the usual definitions for the Riemann curvature tensor and the Ricci tensor for a complex manifold). So we need to calculate the various Christoffel symbols for our Hermitian metrics. Working from first principles, the Christoffel symbols are determined by two conditions, namely requiring the metric to be covariantly constant, and requiring the manifold to be torsion-free. Coupled with the fact that our metrics are Hermitian, it takes only simple algebraic manipulation to produce the following results for the Christoffel symbols:

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\gamma\beta} = \frac{1}{2} g^{\alpha\bar{\mu}} (\partial_\beta g_{\gamma\bar{\mu}} + \partial_\gamma g_{\beta\bar{\mu}}) \\ \Gamma^{\bar{\alpha}}_{\beta\bar{\gamma}} &= \Gamma^{\bar{\alpha}}_{\bar{\gamma}\beta} = \frac{1}{2} g^{\mu\bar{\alpha}} (\partial_\beta g_{\mu\bar{\gamma}} - \partial_\mu g_{\beta\bar{\gamma}}) \\ \Gamma^{\bar{\alpha}}_{\beta\gamma} &= \Gamma^{\bar{\alpha}}_{\gamma\beta} = 0. \end{aligned} \tag{5.67}$$

Now that we have the Christoffel symbols, we can substitute (5.67) into (5.66) and hence evaluate (5.65)—performing the tensor analysis, it is straightforward to show that

$$R_{\alpha\beta} - R_{\beta\alpha} = 0, \quad (5.68)$$

that is,

$$R_{\alpha\beta} = R_{\beta\alpha} \quad (5.69)$$

so the Ricci tensor  $R_{\alpha\beta}$  is certainly symmetric; however, we find that

$$R_{\alpha\beta} + R_{\beta\alpha} = 0 \quad (5.70)$$

only if we impose an additional constraint on the metric, namely

$$\partial_\alpha g_{\beta\bar{\gamma}} - \partial_\beta g_{\alpha\bar{\gamma}} = 0. \quad (5.71)$$

If this condition is imposed, we see that

$$R_{\alpha\beta} \equiv 0 \quad (5.72)$$

as required; if we do not impose this condition then our metric cannot be Einstein.

But what have we assumed if we impose this condition? To understand this, we introduce a 2-form  $F$  which is known as the “Kähler form” associated with the Hermitian structure

$$ds^2 = g_{\alpha\bar{\beta}} dA^\alpha d\bar{A}^\beta, \quad (5.73)$$

given by

$$F = \frac{i}{2} g_{\alpha\bar{\beta}} dA^\alpha \wedge d\bar{A}^\beta. \quad (5.74)$$

Now, in order that an Hermitian metric is Kähler, it is necessary and sufficient that

$$(dF)_{\alpha\beta\bar{\gamma}} = 0, \quad (5.75)$$

that is, the Kähler form is closed (see for example reference [40] and also reference [42]). But a couple of lines of algebra shows us that this closure condition can be rewritten equivalently as

$$\partial_\alpha g_{\beta\bar{\gamma}} - \partial_\beta g_{\alpha\bar{\gamma}} = 0 \quad (5.76)$$

which is exactly the condition (5.71) we had to impose to ensure that  $R_{\alpha\beta} = 0$ .

So we find ourselves with a puzzle. To use the results of Pope et al. [41] to test the Kählerity of our Hermitian metrics, we need to know if the metrics are Einstein or non-Einstein. To this end we calculated  $R_{\alpha\beta}$  via the Riemann curvature tensor and the Christoffel symbols.  $R_{\alpha\beta}$  should be zero if the metric in question is Einstein, but we found that to achieve this we needed to impose a condition which, according to the above comments, is equivalent to already requiring our metric to be Kähler! Additionally, looking closely at the expression (5.76), we recognize that this condition is exactly the type of “condition for Kählerity” we studied earlier in the chapter (cf. (5.46)), which we found was subject to grave inconsistencies when different complex structures were considered. This is exactly the type of condition we were trying to avoid! One final problem is the method we used to calculate  $R_{\alpha\beta}$ —construction of the Christoffel symbols involved the use of the inverse metric  $g^{\alpha\bar{\beta}}$ , but consideration of  $g_{\alpha\bar{\beta}}$ , even for the simplest case of one instanton-like skyrmion in  $\mathbb{C}P^1$ , shows us that the metric  $g_{\alpha\bar{\beta}}$  is degenerate and hence non-invertible!

These inherent problems prevent us from being able to proceed further with this approach; however, we would still like to use the results of Pope et al. to test our metrics. This may still be possible if we calculate the Ricci tensor in a way which avoids the explicit use of the inverse metric. We may then be able to determine whether our metrics are Einstein or not without resorting to a condition of the form (5.76), and we would then be in a position to use the results of Pope et al. as required.

We shall therefore attempt to recalculate the Ricci tensor, this time using a

method described by Eguchi et al. in section 3 of reference [42] which appears to satisfy the needs outlined above. This method involves calculating for a given metric such quantities as the vierbeins, affine spin connection 1-forms, curvature 2-forms and hence the Riemann curvature tensor and the Ricci tensor—rigorous definitions of all these quantities are contained in [42]; we shall introduce the quantities as necessary as we work through the method.

For simplicity, we shall consider first the case of a one instanton-like skyrmion in the  $(2+1)$ -dimensional  $\mathbb{C}P^1$  model, and then try to extend the results to more general systems. For the one skyrmion case, we already know that the metric is given by

$$ds^2 = g_{\alpha\bar{\beta}} dA^\alpha d\bar{A}^\beta \quad (5.77)$$

where

$$\{A^\alpha\} = \{a, b\} \quad (5.78)$$

and

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial A^\alpha} \frac{\partial}{\partial \bar{A}^\beta} \log |f|^2, \quad (5.79)$$

with

$$f = (x_+ - a, x_+ - b) \quad (5.80)$$

(see equations (5.9), (5.6), (5.10) and (5.3) respectively) and so we know that this metric is Kähler-like. Nevertheless, we continue with this system in order to demonstrate how the Kähler nature of the metric may be concluded via the method of Eguchi et al. and Pope et al.

First, writing

$$a = p + iq, \quad b = u + iv \quad (5.81)$$



we see that our quadratic form  $ds^2$  in (5.77) can be rewritten as

$$\begin{aligned}
ds^2 &= \frac{\partial}{\partial a} \frac{\partial}{\partial \bar{a}} \log |f|^2 (dp^2 + dq^2) \\
&+ \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{b}} \log |f|^2 (du^2 + dv^2) \\
&+ \frac{\partial}{\partial a} \frac{\partial}{\partial \bar{b}} \log |f|^2 (dpdu + dqdv + idqdu - idp dv) \\
&+ \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{a}} \log |f|^2 (dudp + dvdq - idudq + idvdp),
\end{aligned} \tag{5.82}$$

that is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.83}$$

where  $\mu, \nu = 0, 1, 2, 3$  and

$$dx^0 \equiv dp, \quad dx^1 \equiv dq, \quad dx^2 \equiv du, \quad dx^3 \equiv dv. \tag{5.84}$$

We can now calculate the vierbeins  $e^a$  for this metric using the defining equations

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{5.85}$$

and

$$e^a = e^a{}_\mu dx^\mu \tag{5.86}$$

where  $a, b = 0, 1, 2, 3$  and  $\eta_{ab} = \text{diag}(1, 1, 1, 1)$ . In other words, calling

$$\begin{aligned}
\frac{\partial}{\partial a} \frac{\partial}{\partial \bar{a}} \log |f|^2 &\equiv A & \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{b}} \log |f|^2 &\equiv B \\
\frac{\partial}{\partial a} \frac{\partial}{\partial \bar{b}} \log |f|^2 &\equiv C & \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{a}} \log |f|^2 &\equiv D
\end{aligned} \tag{5.87}$$

we need to solve the set of simultaneous equations

$$\begin{aligned}
e^0_0 e^0_0 + e^1_0 e^1_0 + e^2_0 e^2_0 + e^3_0 e^3_0 &= A \\
e^0_1 e^0_1 + e^1_1 e^1_1 + e^2_1 e^2_1 + e^3_1 e^3_1 &= A \\
e^0_2 e^0_2 + e^1_2 e^1_2 + e^2_2 e^2_2 + e^3_2 e^3_2 &= B \\
e^0_3 e^0_3 + e^1_3 e^1_3 + e^2_3 e^2_3 + e^3_3 e^3_3 &= B \\
e^0_0 e^0_1 + e^1_0 e^1_1 + e^2_0 e^2_1 + e^3_0 e^3_1 &= 0 \\
e^0_2 e^0_3 + e^1_2 e^1_3 + e^2_2 e^2_3 + e^3_2 e^3_3 &= 0 \\
e^0_0 e^0_2 + e^1_0 e^1_2 + e^2_0 e^2_2 + e^3_0 e^3_2 &= \frac{1}{2}(C + D) \\
e^0_1 e^0_3 + e^1_1 e^1_3 + e^2_1 e^2_3 + e^3_1 e^3_3 &= \frac{1}{2}(C + D) \\
e^0_1 e^0_2 + e^1_1 e^1_2 + e^2_1 e^2_2 + e^3_1 e^3_2 &= \frac{i}{2}(C - D) \\
e^0_0 e^0_3 + e^1_0 e^1_3 + e^2_0 e^2_3 + e^3_0 e^3_3 &= -\frac{i}{2}(C - D),
\end{aligned} \tag{5.88}$$

a set of ten equations containing sixteen unknowns. With suitable choices for various of these unknowns, we find by the usual elimination/substitution technique for simultaneous equations that a consistent set of quantities can be obtained for all these unknowns. The details of this calculation are clearly tedious—we shall simply state here that during the calculation the result

$$AB = CD \tag{5.89}$$

was needed, a result which can be proved in a couple of lines for our one skyrmion system—and it was found that seven unknowns needed to be chosen. A suitable choice turned out to be

$$\begin{aligned}
e^1_1 = e^2_1 = e^3_1 = e^3_3 = e^2_0 &= 0 \\
e^2_3 = -\frac{i(C - D)}{(8A)^{1/2}}, \quad e^3_2 &= B^{1/2}
\end{aligned} \tag{5.90}$$

from which the other nine unknowns could be calculated. Using (5.86), the

vierbeins for this metric could then be written down: they are

$$\begin{aligned}
e^0 &= A^{1/2} dq + \frac{i(C-D)}{2A^{1/2}} du + \frac{(C+D)}{2A^{1/2}} dv \\
e^1 &= (2A)^{1/2} dp + \frac{[(C+D) + 2i(AB)^{1/2}]}{(8A)^{1/2}} du - \frac{i(C-D)}{(8A)^{1/2}} dv \\
e^2 &= \frac{[(C+D) - 2i(AB)^{1/2}]}{(8A)^{1/2}} du - \frac{i(C-D)}{(8A)^{1/2}} dv \\
e^3 &= -iA^{1/2} dp + B^{1/2} du
\end{aligned} \tag{5.91}$$

with  $A, B, C, D$  given by (5.87).

Having obtained the vierbeins, we are now in a position to calculate the affine spin connection 1-forms  $\omega^a_b$ , which are defined by

$$de^a + \omega^a_b \wedge e^b = 0 \tag{5.92}$$

for a torsion-free manifold. To calculate these quantities, Eguchi et al. advocate writing  $de^a$  in the form

$$\begin{aligned}
de^a &= c_{01}^a e^0 \wedge e^1 + c_{02}^a e^0 \wedge e^2 + c_{03}^a e^0 \wedge e^3 \\
&\quad + c_{23}^a e^2 \wedge e^3 + c_{31}^a e^3 \wedge e^1 + c_{12}^a e^1 \wedge e^2
\end{aligned} \tag{5.93}$$

where it should be easy to determine the  $c_{bd}^a$  simply by comparing independent terms on the left and right hand sides of (5.93). Having obtained the quantities  $c_{bd}^a$ , it is only a matter of simple substitution into a certain set of equations to obtain the affine spin connection 1-forms  $\omega^a_b$ —it is not very illuminating to present these equations here: they can be found in reference [42], in section 7 of Appendix A.

Carrying out this prescription for our vierbeins (5.91), however, surprisingly we find that a consistent set of expressions for the  $c_{bd}^a$  can *not* be found: the simultaneous equations we obtain by comparing independent terms on the left and right hand sides of (5.93) are not compatible, and hence we cannot calculate our affine spin connections in this manner.

What has caused this unfortunate breakdown? A couple of reasons present themselves: first, it is possible that the choices we made whilst calculating the



vierbeins were poor—perhaps more suitable choices would produce a consistent set of expressions for  $c_{bd}^a$ ? On the other hand, it is possible that the problem again lies in the non-invertibility of the metric  $g_{\alpha\bar{\beta}}$ , even though the inverse metric does not appear explicitly in this calculation.

One solution we suggest to this problem is to consider not  $g_{\alpha\bar{\beta}}$  but the integrated metric  $G_{\alpha\bar{\beta}}$  which, as we saw earlier in the chapter, can be suitably defined to avoid divergences. Preliminary investigations into this suggestion indicate that  $G_{\alpha\bar{\beta}}$  is *not* necessarily non-invertible, and hence we may be able to complete the above calculations using this integrated form of the metric. Unfortunately, calculations involving the integrated metric are very difficult to perform, and so far we have not succeeded in producing explicit expressions for the vierbeins associated with this metric. Nevertheless, it would be very interesting to proceed further with these calculations in an attempt to carry out our proposed verification of the Kähler nature of this metric. We could then proceed to more general skyrmion systems and comment on the nature of their associated metrics also.

The prospect of being able to make such comments, particularly in cases where the Kähler or non-Kähler nature of the metric in question was not previously clear, serves to highlight the importance of further research in this area.

### 6. LARGER SPACES AND NEGATIVE MODES.

In this thesis we have looked in detail at several important aspects of the  $\mathbb{C}P^{n-1}$  models and their solutions. Before concluding, we would like to discuss one more interesting item, namely the embedding of the  $\mathbb{C}P^{n-1}$  solutions in larger group spaces, and their stability in these new spaces.

It has been known for some time that all solutions of Grassmannian models are also solutions of the 2-dimensional  $U(n)$  chiral models [43]: in particular, as we shall see below,  $\mathbb{C}P^{n-1}$  solutions can be shown to be solutions of the  $SU(n)$  chiral model. To see this, we start with the Lagrangian density  $\mathcal{L}$  for the  $SU(n)$  chiral model

$$\mathcal{L} = \frac{1}{4} \text{tr}(\partial_\mu g^\dagger \partial_\mu g) \quad (6.1)$$

where  $g \in SU(n)$ , so that

$$g^\dagger g = g g^\dagger = 1 \quad \text{and} \quad \det g = 1. \quad (6.2)$$

Calculating the Euler-Lagrange equations in the usual way, we obtain

$$\partial_\mu (g^\dagger \partial_\mu g) = 0. \quad (6.3)$$

Now, if we write

$$g = \mathbb{I}(1 - 2\mathbb{P}) \quad (6.4)$$

where

$$\mathbb{I} = \text{diag}(\overbrace{1, 1, \dots, 1}^{n-1 \text{ times}}, -1) \quad (6.5)$$

then our Lagrangian density (6.1) becomes

$$\mathcal{L} = \text{tr}(\partial_\mu \mathbb{P} \partial_\mu \mathbb{P}) \quad (6.6)$$

and the Euler-Lagrange equations (6.3) become

$$[\partial_\mu \partial_\mu \mathbb{P}, \mathbb{P}] = 0. \quad (6.7)$$

But we saw in chapter 2 that if

$$\mathbb{P}_0 = Z.Z^\dagger \quad (6.8)$$

where  $Z$  is a solution of the  $\mathbb{C}P^{n-1}$  model, then  $\mathbb{P}_0$  satisfies the equation

$$[\partial_\mu \partial_\mu \mathbb{P}_0, \mathbb{P}_0] = 0 \quad (6.9)$$

(see equations (2.62) and (2.65)), and hence

$$g_0 = \mathbb{I}(1 - 2\mathbb{P}_0) \quad (6.10)$$

satisfies the Euler-Lagrange equations (6.3). Thus our  $\mathbb{C}P^{n-1}$  solution defines a solution of the  $SU(n)$  chiral model, as claimed.

Previously, not much was known about other solutions of the  $U(n)$  chiral models; recently, however, Uhlenbeck [44] has made great progress by proving that all classical solutions of the  $U(n)$  chiral models are of the form

$$g = A \prod_{i=1}^l (1 - 2\mathbb{P}_i) \quad (6.11)$$

where  $l$  is some number (which Uhlenbeck calls the “uniton” number),  $A$  is a constant matrix and the  $\mathbb{P}_i$  are projectors which satisfy certain first order differential equations. This theorem therefore provides a way of generating new solutions from old ones: namely one writes

$$g_1 = g_0(1 - 2\mathbb{R}) \quad (6.12)$$

and then, as Uhlenbeck has shown,  $g_1$  satisfies the Euler-Lagrange equations if  $g_0$  does and if  $\mathbb{R}$  satisfies the equations

$$(1 - \mathbb{R})A_+\mathbb{R} = 0 \quad (6.13)$$

and

$$(1 - \mathbb{R})[\partial_- \mathbb{R} + A_- \mathbb{R}] = 0, \quad (6.14)$$

where

$$A_{\pm} = \frac{1}{2} g_0^{\dagger} \partial_{\pm} g_0. \quad (6.15)$$

If  $g_0 = 1$ , equations (6.13) and (6.14) reduce to

$$\partial_- \mathbb{R} \mathbb{R} = 0, \quad (6.16)$$

that is, the self-duality equations for the instantons of the  $\mathbb{C}P^{n-1}$  model (see equation (2.67)). For  $g_0 \neq 1$  we obtain more general solutions, which include the  $\mathbb{C}P^{n-1}$  non-instanton solutions we met in previous chapters and also non-Grassmannian solutions. Piette and Zakrzewski have studied properties of these non-Grassmannian solutions and plan to report their findings soon [45]; in this thesis we shall be more concerned with the  $\mathbb{C}P^{n-1}$  solutions.

One of the most important properties of any solution is its stability when subjected to small fluctuations. We saw in chapter 2 that the  $\mathbb{C}P^{n-1}$  instanton solutions are stable whereas the non-instanton solutions are unstable with respect to small fluctuations: we also saw that the stability of the instantons was guaranteed by the topology of the field manifold. For the chiral model, however, there are no equivalent topological arguments to guarantee the stability of the non-trivial solutions, and so we would expect all these solutions to be unstable. For the  $\mathbb{C}P^{n-1}$  non-instanton embeddings this is obvious since they are already unstable in the  $\mathbb{C}P^{n-1}$  subspace: this leaves us with the instanton embeddings and the non-Grassmannian solutions.

Now, for solutions which are unstable it is important to know the number of directions of instability—the so-called “negative modes” of the fluctuation operator around the solution—and to check their independence. For example, in the functional integral approach to quantization one has to evaluate the determinant of the fluctuation operator which (at least formally) is given by a product of positive eigenvalues. All directions of instability have to be excluded from this calculation of the determinant and treated in a special way: hence the

importance of the *number* of negative modes, since this defines the index of the fluctuation operator.

It is in fact possible to show that *all* non-trivial solutions of the  $U(n)$  chiral model are unstable. To demonstrate this we take a general solution of the model and use Uhlenbeck's procedure (6.12) to construct from it a new solution. If we then define a third field which interpolates between the two solutions it is not difficult to show that the constructed solution is unstable. The details of this calculation have been presented in reference [2]; however, it is unclear using this method whether more than one mode of instability exists in general. In this chapter, therefore, we shall restrict ourselves to the instanton embeddings in the  $SU(n)$  chiral model. We shall introduce the fluctuation in a different way, and try to determine whether there is more than one negative mode.

We start with our instanton embedding from (6.10), that is,

$$g_0 = \mathbb{I}(1 - 2\mathbb{P}_0) \tag{6.17}$$

where  $\mathbb{P}_0$  satisfies

$$\begin{aligned} \mathbb{P}_0\mathbb{P}_0 &= \mathbb{P}_0 \\ \mathbb{P}_0\partial_+\mathbb{P}_0 &= 0 \end{aligned} \tag{6.18}$$

$$\partial_+\mathbb{P}_0\mathbb{P}_0 = \partial_+\mathbb{P}_0$$

(cf.(2.63) and (2.69)). We introduce a small fluctuation by assuming that the field  $g$  in the neighbourhood of  $g_0$  can be written in the form

$$g = g_0 \exp(i\epsilon X) \tag{6.19}$$

where  $\epsilon$  is small. For  $g \in SU(n)$ ,  $X$  must satisfy

$$X = X^\dagger \quad (\text{so that } g^\dagger g = 1) \tag{6.20}$$

and

$$\text{tr } X = 0 \quad (\text{so that } \det g = 1). \tag{6.21}$$

Substituting (6.19) into the Lagrangian density (6.1) and calculating to second order in  $\epsilon$ , it is straightforward to show that

$$\mathcal{L}(g) = \mathcal{L}(g_0) + \frac{\epsilon^2}{4} \text{tr}(\partial_\mu X \partial_\mu X + 2g_0^\dagger \partial_\mu g_0 X \partial_\mu X) \quad (6.22)$$

where we have used the Euler-Lagrange equations (6.3) for  $g_0$ , and assumed that  $X \rightarrow 0$  as  $|\underline{x}| \rightarrow \infty$  (this must be the case if we wish our fluctuation to be integrable). Finally, substituting for  $g_0$  from (6.17) and writing

$$\mathcal{L}(g) = \mathcal{L}(g_0) + \frac{\epsilon^2}{4} \delta \mathcal{L}, \quad (6.23)$$

we find that, after transferring to complex coordinates  $x_\pm$  and using the properties (6.18),  $\delta \mathcal{L}$  can be written in the form

$$\delta \mathcal{L} = 4 \text{tr}[\partial_+ X \partial_- X - 2\partial_+ \mathbb{P}_0 X \partial_- X + 2\partial_- \mathbb{P}_0 X \partial_+ X]. \quad (6.24)$$

Therefore, considering the action  $S = \int \mathcal{L} d^2x$  for the solutions  $g$  and  $g_0$ , we see that

$$S(g) = S(g_0) + \frac{\epsilon^2}{4} \delta S \quad (6.25)$$

where  $\delta S$ , the action associated with the fluctuation, is given by

$$\delta S = \int \delta \mathcal{L} d^2x. \quad (6.26)$$

So, to find negative modes of fluctuation, we simply have to find  $X$  such that  $\delta \mathcal{L}$  is negative.

It is relatively easy to find one negative mode. For example, consider

$$X = \mathbb{P}_0 K \mathbb{P}_0 - \frac{1}{n} \text{tr}(\mathbb{P}_0 K \mathbb{P}_0) \quad (6.27)$$

where  $K$  is a constant matrix which can be written in the form

$$K = V V^\dagger \quad (6.28)$$

where  $V$  is a column  $n$ -vector. This  $X$  certainly satisfies (6.20) and (6.21)—what value do we obtain for  $\delta \mathcal{L}$  for this trial fluctuation?

With  $X$  given by (6.27), then

$$\partial_+ X = \partial_+ \mathbb{P}_0 K \mathbb{P}_0 + \mathbb{P}_0 K \partial_+ \mathbb{P}_0 - \frac{1}{n} \text{tr}(\partial_+ \mathbb{P}_0 K) \quad (6.29)$$

(using the cyclic properties of trace), and similarly for  $\partial_- X$ . Performing the necessary algebra, using the properties (6.18) where appropriate and the cyclic properties of trace, substitution in (6.24) produces the result

$$\begin{aligned} \delta \mathcal{L} &= 4 \text{tr}[K \mathbb{P}_0 K (\partial_+ \mathbb{P}_0 \partial_- \mathbb{P}_0 - \partial_- \mathbb{P}_0 \partial_+ \mathbb{P}_0)] \\ &\quad - \frac{4}{n} \text{tr}(\partial_- \mathbb{P}_0 K) \text{tr}(\partial_+ \mathbb{P}_0 K) \end{aligned} \quad (6.30)$$

Now, since

$$\partial_- \mathbb{P}_0 \mathbb{P}_0 = 0 \quad (6.31)$$

we can write

$$\partial_+ \partial_- \mathbb{P}_0 \mathbb{P}_0 + \partial_- \mathbb{P}_0 \partial_+ \mathbb{P}_0 = 0 \quad (6.32)$$

and so

$$\begin{aligned} \partial_- \mathbb{P}_0 \partial_+ \mathbb{P}_0 &= -\partial_+ \partial_- \mathbb{P}_0 \mathbb{P}_0 \\ &= -\partial_- \partial_+ \mathbb{P}_0 \mathbb{P}_0 \\ &= -\partial_- (\partial_+ \mathbb{P}_0 \mathbb{P}_0) + \partial_+ \mathbb{P}_0 \partial_- \mathbb{P}_0 \\ &= -\partial_- \partial_+ \mathbb{P}_0 + \partial_+ \mathbb{P}_0 \partial_- \mathbb{P}_0, \end{aligned} \quad (6.33)$$

that is,

$$\partial_+ \mathbb{P}_0 \partial_- \mathbb{P}_0 - \partial_- \mathbb{P}_0 \partial_+ \mathbb{P}_0 = \partial_- \partial_+ \mathbb{P}_0 \quad (6.34)$$

so, substituting (6.34) into (6.30) we get

$$\delta \mathcal{L} = 4 \text{tr}[K \mathbb{P}_0 K \partial_- \partial_+ \mathbb{P}_0] - \frac{4}{n} \text{tr}(\partial_- \mathbb{P}_0 K) \text{tr}(\partial_+ \mathbb{P}_0 K). \quad (6.35)$$

The last term is clearly negative since it can be written as

$$-\frac{4}{n} |\text{tr}(\partial_+ \mathbb{P}_0 K)|^2 \quad (6.36)$$

but what of the first term?

Well, we can rewrite this term as

$$4 \text{tr}[\partial_- (K \mathbb{P}_0 K \partial_+ \mathbb{P}_0) - K \partial_- \mathbb{P}_0 K \partial_+ \mathbb{P}_0]. \quad (6.37)$$

If we now think in terms of the integrated quantity  $\delta S = \int \delta \mathcal{L} d^2 x$ , we realize that if  $K$  is chosen so that the divergence theorem can be used, then the first term in (6.37) becomes a surface term and, as we shall see below, can be dropped.

So the final expression for  $\delta \mathcal{L}$  is

$$\delta \mathcal{L} = -4 \operatorname{tr}[K \partial_- \mathbb{P}_0 K \partial_+ \mathbb{P}_0] - \frac{4}{n} |\operatorname{tr}(\partial_+ \mathbb{P}_0 K)|^2, \quad (6.38)$$

in other words, substituting  $K = V V^\dagger$  and using the cyclic properties of trace,

$$\delta \mathcal{L} = -4[V^\dagger \partial_- \mathbb{P}_0 V V^\dagger \partial_+ \mathbb{P}_0 V] - \frac{4}{n} |V^\dagger \partial_+ \mathbb{P}_0 V|^2, \quad (6.39)$$

that is,

$$\delta \mathcal{L} = -4\left(1 + \frac{1}{n}\right) |V^\dagger \partial_+ \mathbb{P}_0 V|^2. \quad (6.40)$$

This quantity is negative definite, and so we have found our required negative mode, provided we can justify the dropping of the surface term. Looking at this term, which we can essentially write as

$$\left[ V^\dagger \frac{f f^\dagger}{|f|^2} V \right] \left[ V^\dagger \partial_+ \left( \frac{f f^\dagger}{|f|^2} \right) V \right] \quad (6.41)$$

since  $K = V V^\dagger$  and

$$\mathbb{P}_0 = Z Z^\dagger = \frac{f f^\dagger}{|f|^2} \quad (6.42)$$

where  $f = f(x_+)$  for our instanton embedding, it is sufficient to require that

$$\frac{|V^\dagger f|^2}{|f|^2} \rightarrow 0 \quad \text{as} \quad |\underline{x}| \rightarrow \infty. \quad (6.43)$$

This condition is easy to impose: we shall look at one simple example for  $n = 3$ .

Consider the configuration

$$f_1 = x_+^2, \quad f_2 = \sqrt{2} x_+, \quad f_3 = 1, \quad (6.44)$$

so

$$|f| = (1 + x_+ x_-). \quad (6.45)$$



It is trivial to check that this is a valid  $\mathbb{C}P^2$  instanton solution, and if we consider the limit  $|x_+| \rightarrow \infty$  we see that

$$\frac{f}{|f|} \rightarrow (1, 0, 0) \quad \text{as} \quad |x_+| \rightarrow \infty. \quad (6.46)$$

Hence, if we choose  $V$  to be of the form

$$V = \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} \quad (6.47)$$

where  $v_2$  and  $v_3$  are constants, our condition (6.43) is satisfied, and we can rightly drop the surface term (6.41). Hence

$$X = \mathbb{P}_0 K \mathbb{P}_0 - \frac{1}{n} \text{tr}(\mathbb{P}_0 K \mathbb{P}_0) \quad (6.48)$$

does give us a genuine negative mode, which in our  $n = 3$  example depends on two parameters  $v_2$  and  $v_3$ . However, it is easy to check that no superposition principle exists for the independent parameters of  $V$ : the way the parameters appear in the expression for the fluctuation prevents us from taking the number of them as the number of negative modes. Do further independent negative modes of fluctuation exist?

First of all, observe that instead of (6.48) we could take

$$X = (1 - \mathbb{P}_0) M (1 - \mathbb{P}_0) - \frac{1}{n} \text{tr}[(1 - \mathbb{P}_0) M (1 - \mathbb{P}_0)] \quad (6.49)$$

where  $M$  is another constant matrix such that

$$M = W W^\dagger \quad (6.50)$$

where  $W$  is a column  $n$ -vector, that is, take

$$X = \mathbb{R} M \mathbb{R} - \frac{1}{n} \text{tr}(\mathbb{R} M \mathbb{R}) \quad (6.51)$$

where

$$\mathbb{R} = 1 - \mathbb{P}_0. \quad (6.52)$$

Since

$$\begin{aligned} g_0^\dagger \partial_\mu g_0 &= -2(1 - 2\mathbb{P}_0) \partial_\mu \mathbb{P}_0 \\ &= -2(1 - 2\mathbb{R}) \partial_\mu \mathbb{R}, \end{aligned} \quad (6.53)$$

then the above calculation can be repeated, with  $\mathbb{P}_0$  replaced by  $\mathbb{R}$  and  $K$  by  $M$ , to demonstrate that (6.49) is also a negative mode of the fluctuation, with  $\delta\mathcal{L}$  given by

$$\delta\mathcal{L} = -4\left(1 + \frac{1}{n}\right) |W^\dagger \partial_+ \mathbb{R} W|^2, \quad (6.54)$$

that is,

$$\delta\mathcal{L} = -4\left(1 + \frac{1}{n}\right) |W^\dagger \partial_+ \mathbb{P}_0 W|^2. \quad (6.55)$$

Again we must justify the dropping of a surface term, namely

$$\text{tr}[M(1 - \mathbb{P}_0)M\partial_+ \mathbb{P}_0], \quad (6.56)$$

which can be rewritten as

$$\left[ W^\dagger \frac{(|f|^2 - ff^\dagger)}{|f|^2} W \right] \left[ W^\dagger \partial_+ \left( \frac{ff^\dagger}{|f|^2} \right) W \right] \quad (6.57)$$

using (6.50) and (6.42). This time it is sufficient to require that

$$\frac{|W|^2 |f|^2 - |W^\dagger f|^2}{|f|^2} \rightarrow 0 \quad \text{as} \quad |\underline{x}| \rightarrow \infty, \quad (6.58)$$

which again is easy to impose: for our special case of  $n = 3$  and  $f$  given by (6.44), this corresponds to choosing

$$W = \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix}. \quad (6.59)$$

So (6.49) is another genuine negative mode. The important question we must now ask is whether these two modes are independent. To answer this question we consider the following superposition of the two modes:

$$X = \mathbb{P}_0 K \mathbb{P}_0 + \beta(1 - \mathbb{P}_0)M(1 - \mathbb{P}_0) - \frac{1}{n} \text{tr}[\mathbb{P}_0 K \mathbb{P}_0 + \beta(1 - \mathbb{P}_0)M(1 - \mathbb{P}_0)]. \quad (6.60)$$

If this superposition remains a negative mode of the fluctuation for *any* value of the constant  $\beta$ , then the two modes are truly independent.

If we perform the substitution of (6.60) into (6.24) and do the necessary manipulation, using the properties (6.18) as before, we find that  $\delta\mathcal{L}$  for this superposition is given by

$$\begin{aligned} \delta\mathcal{L} = & 4 \operatorname{tr}[K\mathbb{P}_0K\partial_-\partial_+\mathbb{P}_0 + 2\beta\partial_+\mathbb{P}_0K\partial_-\mathbb{P}_0M - \beta^2M(1 - \mathbb{P}_0)M\partial_-\partial_+\mathbb{P}_0] \\ & - \frac{4}{n} \operatorname{tr}[\partial_-\mathbb{P}_0K - \beta\partial_-\mathbb{P}_0M] \operatorname{tr}[\partial_+\mathbb{P}_0K - \beta\partial_+\mathbb{P}_0M]. \end{aligned} \quad (6.61)$$

Choosing  $K$  and  $M$  as before (so that the surface terms can again be dropped) this expression can be rewritten as

$$\begin{aligned} \delta\mathcal{L} = & -4[|V^\dagger\partial_+\mathbb{P}_0V|^2 + \beta^2|W^\dagger\partial_+\mathbb{P}_0W|^2 - 2\beta|W^\dagger\partial_+\mathbb{P}_0V|^2] \\ & - \frac{4}{n}|(V^\dagger\partial_+\mathbb{P}_0V - \beta W^\dagger\partial_+\mathbb{P}_0W)|^2. \end{aligned} \quad (6.62)$$

The last term here is clearly negative, but the appearance of the mixed term (proportional to  $\beta$ ) prevents us from drawing a conclusion in full generality. Let us therefore consider the first term for our special  $n = 3$  case, with  $\mathbb{P}_0 = \frac{ff^\dagger}{|f|^2}$  where

$$f = \begin{pmatrix} x_+^2 \\ \sqrt{2}x_+ \\ 1 \end{pmatrix} \quad (6.63)$$

as before, and with

$$V = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \quad W = \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} \quad (6.64)$$

for simplicity. It is not difficult to show that the first term in (6.62) for this special situation simplifies to

$$\delta\mathcal{L} = -\frac{16x_+x_-}{(1+x_+x_-)^6}[|v|^4 + \beta^2x_+x_-|w|^4 - 2\beta|v|^2|w|^2]. \quad (6.65)$$

Not being able to say much more at this stage, we therefore turn our attention to the integrated quantity  $\delta S = \int \delta\mathcal{L} d^2x$ . Demonstrating that  $\delta S$  is negative clearly indicates a negative mode in the same manner as showing  $\delta\mathcal{L}$  to be negative: we shall therefore calculate  $\delta S$  from (6.65). To do this, we change to polar coordinates in the normal way:

$$x_+ = re^{i\theta}, \quad x_- = re^{-i\theta} \quad (6.66)$$

so that

$$d^2x \rightarrow r dr d\theta \quad (6.67)$$

and the integration range is  $0 \leq r < \infty$ ,  $0 \leq \theta \leq 2\pi$ . Since  $x_+x_- = r^2$ , there is no  $\theta$  dependence in (6.65) and so the angular integration is trivial; to perform the radial integration we use the well-known results

$$\int \frac{r^m}{(r^2 + 1)^n} dr = \int \frac{r^{m-2}}{(r^2 + 1)^{n-1}} dr - \int \frac{r^{m-2}}{(r^2 + 1)^n} dr \quad (6.68)$$

and

$$\int \frac{r}{(r^2 + 1)^n} dr = -\frac{1}{2(n-1)(r^2 + 1)^{n-1}}, \quad (6.69)$$

which can be found in any table of indefinite integrals. The final result of this manipulation is

$$\delta S = -\frac{4}{5}\pi(|v|^2 - \beta|w|^2). \quad (6.70)$$

This quantity is clearly non-positive, only vanishing for  $|v|^2 = \beta|w|^2$ ; in the case  $|v|^2 = \beta|w|^2$ , however, the second term in (6.62) does *not* vanish and so the total expression for  $\delta S$ , that is,

$$\delta S = -\frac{4}{5}\pi(|v|^2 - \beta|w|^2) - \frac{4}{n} \int |(V^\dagger \partial_+ \mathbb{P}_0 V - \beta W^\dagger \partial_+ \mathbb{P}_0 W)|^2 d^2x \quad (6.71)$$

is negative definite for *all* values of  $\beta$ .

Thus, for our special  $n = 3$  solution we have found two genuinely independent negative modes of fluctuation. [If we had taken

$$V = \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} \quad (6.72)$$

as before, it can be shown easily that the two modes are independent only for  $v_2 \equiv 0$ —hence our stated choice for  $V$  in (6.64)]. It is very difficult to say much more in general about the number of negative modes—in this special  $n = 3$  case we have exhibited two such modes, but the question of more modes is unclear since we have been unable to find a general method of generating such objects: preliminary attempts using quantities of the form

$$\mathbb{P}_0 = \frac{f f^\dagger}{|f|^2}, \quad \mathbb{P}_1 = \frac{P_+ f (P_+ f)^\dagger}{|P_+ f|^2}, \quad \mathbb{P}_2 = \frac{P_+^2 f (P_+^2 f)^\dagger}{|P_+^2 f|^2}, \quad (6.73)$$

(where  $f$ ,  $P_+f$  and  $P_+^2f$  are all solutions we have met previously of the  $\mathbb{C}P^2$  model) have not produced results of the desired form. Similar considerations for  $n = 4$  have produced two negative modes analogous to our  $n = 3$  case, with

$$V = \begin{pmatrix} 0 \\ 0 \\ v_3 \\ v_4 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} w \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (6.74)$$

for the special solution

$$f = \begin{pmatrix} x_+^3 \\ \sqrt{3}x_+^2 \\ \sqrt{3}x_+ \\ 1 \end{pmatrix}, \quad (6.75)$$

but again we do not have a superposition principle for the independent parameters of  $V$ , so although our fluctuation contains three parameters this number cannot be taken as the number of independent negative modes, and so far we have been unable to find a third genuinely independent negative mode.

It would be very interesting to find a general method of generating such negative modes of fluctuation, and to find a procedure by which the number of negative modes for a particular fluctuation could be predicted. Although at present we have not succeeded in formulating these methods, this problem clearly merits further study.

## 7. CONCLUSION.

So, what have we learned from our consideration of the  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models?

Instanton solutions of the two-dimensional models have been known for some time; we have seen how these extended structures depend on the parameters of the solutions, and how various interpretations can be used to describe this behaviour. Using similar techniques, we have considered various field configurations which approximate to solutions of the theory, and which contain instanton *and* anti-instanton components, and have found that one configuration in particular exhibits many properties we would desire from such a system. Indeed, an interaction between the constituents was predicted, of a form very much reminiscent of the interaction between dipoles.

Introducing time dependence into the models, we saw that the full theory in  $(2 + 1)$  dimensions was rather difficult to solve, but that approximate solutions could be obtained by introducing time dependence into the static solutions of the theory. We have derived certain constraints which can be imposed on these systems to ensure finiteness of the kinetic energy, and have discussed their physical interpretations.

It is well known that the spin properties of such solutions are determined by the Hopf term of the theory; we have derived expressions for this term for both solutions and approximate solutions of the model, and have calculated the values of the Hopf term for certain explicit evolutions. This enabled us to make various remarks about the spin properties of the systems under consideration. One system did not lend itself easily to investigation in this way, however: to remedy this situation, we have initiated a more general method to derive Hopf terms for configurations of this form. Unfortunately, we have not been able to pursue this method to its conclusion, but the early indications are that this method

would provide a convenient way to produce Hopf terms as yet uncalculable, and is clearly worthy of further investigation.

What more have we learned about the motion of these extended objects? In the  $\mathbb{C}P^1$  case, it was already known that the dynamics of slowly moving skyrmions could be described by the geodesic motion on a Kähler manifold of the skyrmion parameters. We have been able to extend this result to  $\mathbb{C}P^{n-1}$  for instanton-like skyrmions, but we have seen that the situation for non-instanton-like skyrmions is far more complex. The metrics defined by the latter are certainly Hermitian, but we have found difficulty in producing a consistent way of testing the Kählerity of these metrics. Various approaches have been tried, and certainly further research in this area would be of great benefit.

Finally, we have looked at the stability of the  $\mathbb{C}P^{n-1}$  solutions when embedded in the 2-dimensional  $U(n)$  chiral models. We have seen that *all* non-trivial solutions of the  $U(n)$  chiral model are in fact unstable, and by using the  $\mathbb{C}P^{n-1}$  embedded solutions have exhibited negative modes of the fluctuation operators around these solutions. The number of *independent* negative modes, however, was not so easy to determine: for specific solutions we have been able to find two genuinely independent negative modes, but the problems of finding a general method of generating these modes, or of determining the number of such modes are still unsolved—clearly a third area of interest for future study.

The various topics considered in this thesis have certainly enhanced our knowledge of the  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$ -models, and have indicated certain areas where further research would definitely be of great benefit: what other topics could be studied to further improve our understanding?

In the approximation of considering only instanton and anti-instanton solutions, various authors [46,31] have shown that the quantum corrections to these classical solutions can be described in terms of a gas of instanton quarks. It would be interesting to consider the effect non-instanton solutions would have

on the properties of this gas. One may argue that classical solutions which are not stable give rise to unstable quantum states, and therefore it does not make sense to study quantum fluctuations around these states; however, although positive modes of fluctuation can be treated with standard methods, there is no universal agreement as yet on how to take into account negative modes with an appropriate mathematical apparatus.

Other authors have chosen to understand the physics of these models using a  $1/n$  expansion approach [9,47]. This approach has led to interesting features like dynamical mass generation and dynamically generated, confining long range fields. It has also been shown [48] that  $1/n$  expansion results can be simulated by classical solutions obeying certain boundary conditions. This line of approach would be particularly interesting if similar relations could be found in four-dimensional gauge theories.

Finally, one could consider a supersymmetric version of these models, and look at the various topics considered in this thesis again for the supersymmetrized models. A report on progress in this area can be found in reference [27].



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