Nahm's equation and the search for classical solutions in Yang-Mills theory

Wainwright, P.R.

NAHM'S EQUATION

AND THE SEARCH FOR CLASSICAL SOLUTIONS

IN YANG-MILLS THEORY

By P.R. Wainwright

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Supervised by Dr. E.F. Corrigan

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ABSTRACT

The history of the theory of magnetic monopoles in classical electrodynamics and unified gauge theories is reviewed, and the Atiyah-Ward and Atiyah-Drinfeld-Hitchin-Manin constructions of exact classical solutions to the self-dual Yang-Mills equations are described.

It is shown that the one-dimensional self-dual equation introduced by Nahm can be reformulated as a Riemann-Hilbert problem through the twistor transform previously used by Ward for monopole and instanton fields, and a general formula for the patching matrix is derived. This is evaluated in some special cases, and a few simple examples are given where Nahm's equation can be solved by this method.

An attempt is made to generalize the ADHM construction to treat nonselfdual Yang-Mills fields, with only partial success. The one-dimensional analogue of the second-order Yang-Mills equation, the so-called nonselfdual Nahm equation, is investigated, paying particular attention to a simple ansatz in which translation of the fields is equivalent to a mere scale transformation of the matrices $T_i(z)$. For these 'separable solutions' the matrices satisfy certain cubic equations, whose solution space depends critically on the nature of the Lie algebra under consideration. It is shown that corresponding to certain Riemannian symmetric pairs there are one-parameter families of 'interpolating solutions' to the cubic equations, which join oppositely oriented bases of a Lie subalgebra. The associated matrix-valued functions $T_i(z)$ therefore interpolate between solutions of 'selfdual' and 'antiselfdual' Nahm equations.

Statement of Originality and Publication History

No originality is claimed for the material reviewed in Chapter 0, Chapter 2, or the Appendices, nor for that in sections 1.1 to 1.3, which are included in preparation for the work which follows. With the exception of some brief comments at the start of sec. 1.4 (pp. 46, 47), the remainder of Chapter 1 is original except where references are cited, although the proof given of the fundamental result (1.56) was suggested by S. Rouhani, and replaces a lengthier derivation by the author.

In the brief sec. 3.1, pp. 92-94 contain a review of the work of Witten, Isenberg et.al.; but the study summarized in pp. 95-97 is entirely due to the author. The remaining work (sections 3.2 to 3.4) was done in collaboration with Dr. E. Corrigan. The general comments on the nonselfdual Nahm equations (sec. 3.3), and the original discovery of an 'interpolating solution' in SU(3) (sec. 3.4 as far as equ. (3.41)) are due to E.C., whereas the investigation of the tangent space (most of sec. 3.4) was done by the author. The study of interpolating solutions in algebras other than SU(3) was carried out by the author following a suggestion by E.C.

The speculations and suggestions in Ch. 4 are entirely the responsibility of the author.

Much of the original material in Ch. 1 has been published as "A Twistor approach to Nahm's Equation" by P. Wainwright, in J. Math. Phys. 26, 202 (1985). That in sections 3.2 and 3.3 was published as "Some Comments on the Non-Self-Dual Nahm Equations" by E. Corrigan, P. Wainwright and S. Wilson, in Comm. Math. Phys. 98, 259 (1985).

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Chapter 0. Introduction

The idea of magnetic monopoles was introduced to physics as long ago as 1931 by Dirac [1] in order to explain the observed quantization of electric charge. However, these did not become an essential feature of particle physics until t'Hooft [2] and Polyakov [3] pointed out that isolated magnetic charges appear quite naturally in many modern unified field theories. Since then the growth of the subject has been dramatic, and has involved many techniques drawn from the mathematical disciplines of vector bundle theory, homotopy theory, and algebraic geometry. In particular the complete integrability of the so-called Bogomolny equations which describe static magnetic monopoles has allowed the construction of explicit solutions by a variety of methods, which are distinct but interrelated. It is this aspect of the theory with which this work is concerned.

Before looking at the question of exact solutions, however, we shall briefly review the history of the monopole concept, starting with Dirac's original formulation.

1. Dirac Monopoles

Long before the advent of modern unified field theories involving the strong and weak interactions, Dirac considered the consequences of point monopoles in pure electromagnetism coupled to charged particles. Such magnetic charges produce singular magnetic fields of the form $\underline{B} = \underline{r}/4\pi r^3$. This field cannot be derived from a smooth vector potential $\underline{A}$. However, if the monopole is joined to the point at infinity by a curve known as the Dirac string, then a potential can be constructed which is regular except at points on the string and the wave equation for electrons or other charged particles may be solved in the background field $\underline{A}$. The string, however, is purely an artefact used in calculation, and its position is chosen arbitrarily; accordingly we must demand that no physical measurement can locate it. The path integral of the vector potential around a small curve encircling the string is

$$\oint \underline{A}.d\underline{x} = \mu$$

Thus the string appears as a tube of flux μ equal to the magnetic charge of the monopole. Magnetic flux enters the monopole from infinity through this tube, and then emerges as a normal inverse square law field.

To see how such a string may be detected, consider a flux μ concentrated on the x_3-axis; this may be derived from the potential $\underline{A} = \underline{\nabla}\omega$, where $\omega = \mu\chi/2\pi$, and χ is a polar co-ordinate around the x_3-axis. This field is obtained by applying the <u>singular</u> gauge transformation ω to the zero field. It follows that in order to solve the gauge-invariant Dirac or Schrödinger equation for an electron wave function we need only take a solution ψ_0 in the absence of magnetic fields, and apply a gauge transformation

$$\psi = e^{-iq\omega}\psi_0$$

where $q = -e$ is the charge on the electron. However, if ψ_0 is single-valued, ψ will in general be multivalued:

$$\psi(\chi = 2\pi) = e^{-iq\mu}\psi(\chi = 0)$$

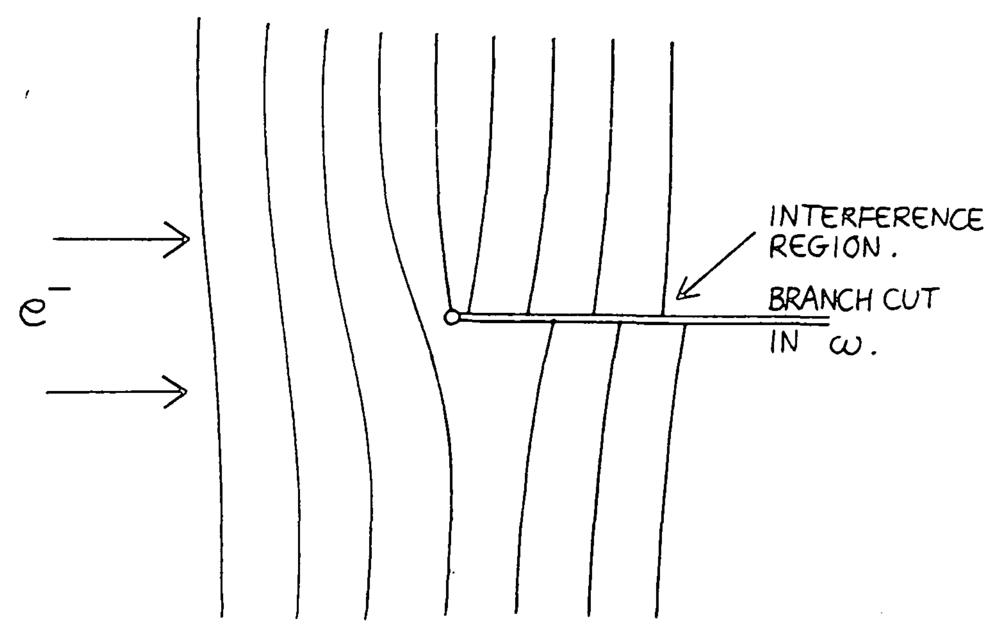


Fig. 0.1

Physically speaking, this means that plane waves incident on the string will interfere in the lee of the string: this is the so-called Bohm-Aharonov effect. The condition necessary for the string to be undetectable by this method is

$$q\mu = 2\pi l \quad , \qquad l \in Z \qquad (0.1)$$

This is the Dirac quantization condition, which tells us that if a monopole of charge μ exists anywhere, then electric charge is quantized in units $2\pi/\mu$; and conversely that magnetic charge is quantized in units $2\pi/q$, where q is the smallest electric charge.

The constraint (0.1) ensures that the gauge transformation $e^{-iq\omega}$ is single-valued, although it is singular on the string. Suppose we have a monopole with Dirac string C_1 and we wish to move this to the curve C_2. Let $C = -C_1 + C_2$, where $-C1$ denotes the curve C1 with opposite orientation.

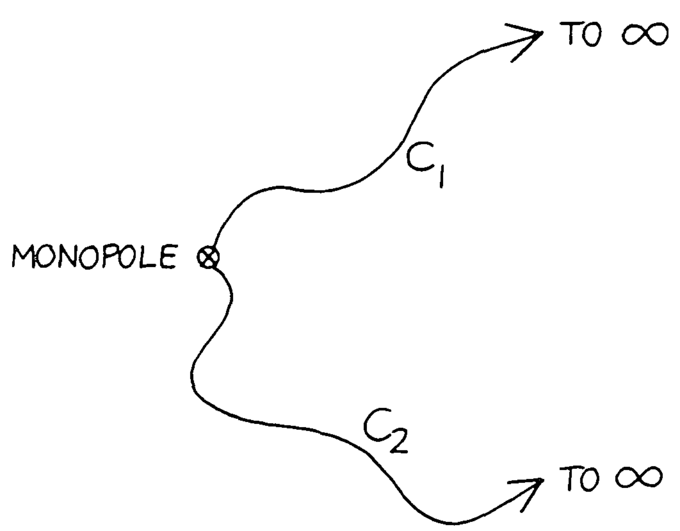


Fig. 0.2

Then we may apply a gauge transformation $e^{iq\omega}$, where ω is a multivalued function which increments by μ upon traversal of a curve encircling the line C. Such a gauge transformation will cancel the flux from the string C1 and create an equal flux in C2, as required. So we see that indeed all Dirac strings are equivalent under gauge transformations.

The following points should be noted concerning Dirac monopoles: Firstly, they are point singularities in the magnetic field; secondly, they have been introduced 'by hand', that is, while they are not inconsistent with standard electrodynamics they are certainly not neccessary features of it.

2. Monopoles in Unified Theories

The next major development in the theory of monopoles was the observation by t'Hooft [2] and Polyakov [3] that when the electromagnetic gauge group U(1) is obtained by spontaneous symmetry breaking from a simple group, there may exist classical solutions of the field equations which have magnetic charge. In contrast to Dirac monopoles, however, these are extended objects, and the surrounding magnetic fields are nonsingular.

The unbroken gauge theory contains a vector field $A_\mu(x)$ which takes values in the Lie algebra of the group G. We shall represent the elements of G by n x n unitary matrices, and as a basis of the Lie algebra we shall take the skew-hermitian matrices T_a normalized by

$$\operatorname{tr} T_a T_b = -\tfrac{1}{2}\delta_{ab}$$

In terms of this basis, we have, for example,

$$A_\mu(x) = A^a_\mu T_a$$

The Lagrangian density for the gauge fields alone is

$$L_G = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$$

where $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

It is invariant under the gauge transformations

$$A_\mu(x) \longrightarrow g(x)A_\mu(x)g(x)^{-1} + g(x)\partial_\mu g(x)^{-1}$$

To the gauge fields we couple a set of n complex scalar fields represented by the column vector $\phi(x)$ which transforms via the rule

$$\phi(x) \longrightarrow g(x)\phi(x)$$

The gauge-invariant kinetic term for this Higgs field is $(D_\mu\phi)^+ D^\mu\phi$, where the covariant derivative D_μ is defined by

$$D_\mu\phi = \partial_\mu\phi + A_\mu\phi$$

To break the gauge symmetry we also introduce a gauge-invariant potential $V(\phi)$ which attains its minimum when $|\phi| > 0$. The exact form of this function is not important, but for our purposes we must assume that the minimum $V = 0$ is unique up to gauge transformations, that is $V(\phi) = 0$ if and only if $\phi = g\phi_0$ for some $g \in G$, where ϕ_0 is a fixed vector. The complete action for the gauge and Higgs fields is now

$$S = \int d^4x \left(-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + (D_\mu\phi)^+ D^\mu\phi - V(\phi)\right) \tag{0.2}$$

We shall confine ourselves to seeking static, purely magnetic solutions $\partial_0 A_i = \partial_0 \phi = 0$, $A_0 = 0$. In this case we have to minimize the energy functional

$$E = \int d^3x \; (\tfrac{1}{2}B_i^a B_i^a + (D_i\phi)^+ D_i\phi + V(\phi)) \qquad (0.3)$$

where $\quad B_i^a = -\frac{1}{2}\varepsilon_{ijk}F_{jk}^a$

In order that the energy (0.3) be finite it is necessary that $V \longrightarrow 0$ as $r = |x| \longrightarrow \infty$. If $\underline{x} = r\underline{u}$, where $\underline{u}$ is a fixed unit vector, this implies that

$$\phi(\underline{x}) \longrightarrow g(\underline{u})\phi_0 \quad \text{as} \quad r \longrightarrow \infty \qquad (0.4)$$

where $g(\underline{u}) \in G$. Furthermore, the requirement of finite energy gives $D_i\phi \longrightarrow 0$ at infinity, leading to the asymptotic condition

$$A_i(\underline{x}) \sim g(\underline{u})\alpha_i(\underline{x})g(\underline{u})^{-1} + g(\underline{u})\partial_i g(\underline{u})^{-1} \qquad (0.5)$$

where $\alpha_i(\underline{x})\phi_0 = 0$. This means that A_i is asymptotically equivalent to the potential α_i which lies in the algebra of the <u>little group</u> H. This subgroup $H \subset G$ contains all the transformations which leave ϕ_0 invariant, $h\phi_0 = \phi_0$; these are the invariances which remain after symmetry breaking via the Higgs mechanism. Now the group element $g(\underline{u})$ is not uniquely determined by (0.4), since $g(\underline{u})\phi_0 = g(\underline{u})h\phi_0$ for any $h \in H$. What <u>is</u> determined is the coset $g(\underline{u})H$; this means that to each configuration of the Higgs field we may associate a mapping

$$\gamma : S^2 \dashrightarrow G/H$$
$$\underline{u} \dashrightarrow g(\underline{u})H \qquad (0.6)$$

of the two-sphere into the coset space G/H. If a configuration ϕ_1 may be deformed continuously into a configuration ϕ_2 then ϕ_1 and ϕ_2 are said to be homotopic. The mapping $\phi \dashrightarrow \gamma$ which we have described above is continuous, and it follows that if ϕ_1 and ϕ_2 are homotopic then the corresponding maps of spheres γ_1 and γ_2 are also homotopic. But conversely, if γ_1 and γ_2 are not homotopic, then ϕ_1 and ϕ_2 cannot be homotopic; in other words the configurations (A_1,ϕ_1) and (A_2,ϕ_2) lie in disjoint components of the space M. We therefore see that $M = \cup M_1$, where the disjoint components M_1 correspond to homotopy classes of maps $S^2 \dashrightarrow G/H$, that is, to elements of the so-called homotopy group $\pi_2(G/H)$.

It is possible to recast the above topological classification in a form which involves only the residual unbroken symmetry group H. This is done by noting that the two-sphere is topologically equivalent (homeomorphic) to a disc whose boundary is identified to a single point

$$S^2 = D/\partial D$$

More explicitly, if (θ,χ) are polar co-ordinates on S^2, the correspondence is established by using θ as the radial co-ordinate of the disc. The boundary of the disc, $\theta = \pi$, corresponds to the south pole of the sphere. It is therefore possible to regard γ as a mapping of the disc $D \longrightarrow G/H$, such that the boundary ∂D maps to a single point, which by a global gauge transformation may be taken to be the coset H. Now γ may be <u>lifted</u> to a continuous mapping $g : D \longrightarrow G$ such that $\gamma(\theta,\chi) = g(\theta,\chi)H$. This is equivalent to the statement that the coset representative $g(\underline{u})$ in (0.4) may be chosen smoothly except at the south pole of S^2. But since $\gamma(\partial D) = H$ we must have

$$g(\partial D) \subset H$$

Thus the restriction of g to the boundary of D, $g(\pi,\chi)$, is a mapping $S^1 \longrightarrow H$, which allows us to classify the field configuration by an element of the homotopy group $\pi_1(H)$. In particular, if the residual symmetry group contains a factor U(1), then this homotopy group will contain a factor Z. It is this integral topological invariant which is interpreted as magnetic charge.

In order to see that this interpretation is valid we turh to the specific class of models first considered by Bogomolny [4] and Sommerfield [5], in which exact solutions of arbitrary magnetic charge have been constructed. In these models the Higgs fields lie in the adjoint representation of G, and a limit is taken in which the Higgs potential vanishes. However, we retain the asymptotic condition on the Higgs field

$$\operatorname{tr} \phi^2 \longrightarrow \text{constant} \tag{0.7}$$

The energy functional to be minimized now takes the form

$$E = \int d^3\underline{x}\ \tfrac{1}{2}(B_i^a B_i^a + D_i\phi^a D_i\phi^a) \qquad (0.8)$$

which may be rewritten as the sum of a surface integral and a positive definite piece

$$E = \int dS_i\ B_i^a\phi^a + \int d^3\underline{x}\ \tfrac{1}{2}(B_i^a - D_i\phi^a)(B_i^a - D_i\phi^a)$$

The former integral is carried out over the sphere 'at infinity', and is therefore unaffected by variations of the fields in any bounded region. It follows that a local minimum of E with respect to such variations is given by the Bogomolny equations [4]

$$D_i\phi^a = B_i^a \qquad (0.9)$$

Now at each point of the sphere at infinity there is a certain subgroup $H_{\underline{u}} \subset G$ which leaves the Higgs field $\phi(\underline{x})$ invariant. When ϕ lies in the adjoint representation, the analogue of (0.4) is the asymptotic condition

$$\phi(\underline{x}) \dashrightarrow g(\underline{u})\phi_0 g(\underline{u})^{-1} \quad \text{as } r \dashrightarrow \infty \qquad (0.10)$$

It follows that all the subgroups $H_{\underline{u}}$ are in fact conjugate to one another

$$H_{\underline{u}} = g(\underline{u})Hg(\underline{u})^{-1}$$

In particular, $H_{\underline{u}}$ contains the U(1) factor $\exp t\phi(\underline{x})$ $(t \in R)$, which may be identified with the electromagnetic gauge group. This means that the asymptotic magnetic field is B_i^a, and the magnetic charge of the solution is the flux of this quantity through the sphere at infinity

$$\mu = \int dS_i \; B_i^a \phi^a \tag{0.11}$$

It follows that the energy of each field configuration satisfies the inequality $E > \mu$, and that equality is attained if and only if the Bogomolny equation (0.9) holds. The integral in (0.11) is gauge-invariant, and may therefore be evaluated in any convenient gauge. We shall use the gauge in which the Higgs field at infinity takes the constant value ϕ_0 and the asymptotic gauge field lies in the Lie algebra of H:

$$\alpha_i(\underline{x}) = g(\underline{u})^{-1}A_i(\underline{x})g(\underline{u}) + g(\underline{u})^{-1}\partial_i g(\underline{u}) \tag{0.12}$$

Recall, however, that the gauge transformation $g(\underline{u})$ is <u>singular</u> at the south pole of S^2. Using Stokes's theorem, we can express the magnetic charge in terms of a path integral around a small closed curve encircling this singularity

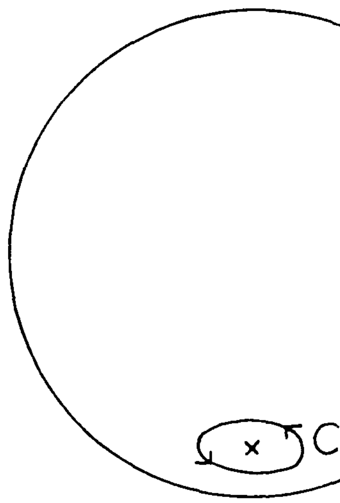


Fig. 0.3

$$\mu = \int dl_i \; \alpha_i^a \phi_0^a \tag{0.13}$$

where a limit is to be taken as the curve C shrinks to a point. Substituting (0.12) in (0.13) we find the expression

$$\mu = \int dl_i \; (g^{-1}\partial_i g)^a \phi_0^a \tag{0.14}$$

in terms of the left-invariant one-forms $(g^{-1}\partial_i g)^a$ on the group G. Now we can see the connection of magnetic charge with topology. Suppose that the residual symmetry group is $H = U(1) \times H'$, and the gauge transformation factorizes in the form $g = e^{-i\omega}g'$ on the curve C, where $g' \in H'$. Then the required one-form simplifies to

$$(g^{-1}\partial_i g)^a \phi_0^a = \partial_i \omega$$

and the magnetic charge is then $\mu = \Delta\omega$, the increment of ω upon traversal of the curve C. This is just the winding number or homotopy class of the gauge transformation g on the curve C.

Note that at the south pole $\theta = \pi$ of the sphere, the gauge transformation $g(\underline{u})$ is singular, and accordingly so is the H-gauge field $\alpha_i(\underline{x})$. At large distances the U(1) part of this field looks just like that of Dirac's point monopole, and this singularity occurs where the Dirac string meets the sphere at infinity.

3. Monopoles and Instantons

We have seen that the gauge and Higgs fields of a magnetic monopole are described by the Bogomolny equations (0.9). The construction of solutions to these equations will be described in chapter 1; however, it is convenient to start by rewriting them in a different form, which involves only a pure Yang-Mills field without explicit mention of the Higgs field ϕ.

To accomplish this we observe that scalar fields such as ϕ may be generated quite naturally by the process of dimensional reduction. In this way we are led to consider a four-dimensional gauge theory with vector potential $A_\mu(x)$ ($\mu = 0,1,2,3$) such that none of the fields is dependent on the extra co-ordinate x^0:

$$\partial_0 A_\mu = 0 \tag{0.15}$$

Under these circumstances we may identify the additional component A_0 with the Higgs field, so that $D_i\phi = F_{i0}$. The Bogomolny equation (0.9) is then simply the equation of self-duality for the field strength $F_{\mu\nu}$:

$$F_{\mu\nu} = {}^*F_{\mu\nu} \tag{0.16}$$

where the dual of F, $^*F$, is defined by

$$^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{0.17}$$

With a Euclidean metric $^{**}F = F$, whereas with a Minkowski metric $^{**}F = -F$. In order that the equations (0.16) be consistent it is therefore necessary to choose the Euclidean metric on R^4. The extra Euclidean dimension x^0 is unphysical and entirely unrelated to time.

Written in terms of the gauge field $F_{\mu\nu}$, the energy functional (0.8) becomes

$$E = \int d^3x \; \tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

It is natural at this point to consider a related problem, which is to determine the stationary points of the action functional

$$S = \int d^4x \; \tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (0.18)$$

when the fields are no longer constrained to be independent of x^0. By an argument similar to that which led to (0.9) we find that S has a local minimum at each field configuration which satisfies the self-duality equation (0.16). However, the boundary conditions are very different from those employed in the case of monopoles. For the action (0.18) to be finite it is necessary that $F = o(1/r^2)$ as $r = |x| \longrightarrow \infty$; this implies that the connection A_μ tends to a pure gauge at infinity

$$A_\mu(x) \sim g(u)^{-1}\partial_\mu g(u) \qquad \text{as } r \longrightarrow \infty \qquad (0.19)$$

where $x = ru$, $|u| = 1$.

When the self-duality equation (0.16) is satisfied, the action reaches a local minimum

$$Q = \int d^4x \ \frac{1}{4} F^a_{\mu\nu} \, {}^*F^{\mu\nu a}$$

Just as in the monopole case it is possible to express this as a surface integral over the sphere (in this case a three-sphere) at infinity.

$$\begin{aligned} Q &= \int dS_\mu \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} A^a_\nu (\partial_\rho A_\sigma + \frac{1}{3} [A_\rho, A_\sigma])^a \\ &= \frac{1}{12} \int dS_\mu \varepsilon_{\mu\nu\rho\sigma} C_{abc} (g^{-1}\partial_\nu g)^a (g^{-1}\partial_\rho g)^b (g^{-1}\partial_\sigma g)^c \end{aligned} \tag{0.21}$$

where the structure constants C_{abc} of the group G are defined by

$$[T_a, T_c] = C_a{}^b{}_c T_b \tag{0.22}$$

The expression (0.21) involves the canonical left-invariant three-form of G given by

$$C_{abc} (g^{-1}dg)^a (g^{-1}dg)^b (g^{-1}dg)^c$$

It may be shown that Q is a homotopy invariant of the mapping $g : S^3 \longrightarrow G$, that is, Q remains unchanged by any continuous deformation of the function $g(u)$. Furthermore, Q is additive in the sense that

$$Q(gh) = Q(g) + Q(h) \tag{0.23}$$

The interpretation of Q for the case $G = SU(2)$ is straightforward, since the three-form which appears in (0.21) is simply the standard volume measure on G, and therefore

$$Q = 8\pi^2 l \tag{0.24}$$

where l is the multiplicity with which the mapping g covers the group $SU(2)$.

The integral topological invariant l is known as the <u>instanton number</u> of the configuration; a selfdual gauge field for which $Q = 8\pi^2 l$ is said to be an l-instanton solution.

4. Nonselfdual Solutions

So far we have considered only the minima of the energy functional (0.8). It is possible that this functional also possesses saddle points which are solutions to the full second-order field equations

$$\varepsilon_{ijk} D_j B_k = [\phi, D_i \phi]$$
$$D_i D_i \phi = 0 \tag{0.26}$$

The Bogomolny equations (0.9) imply the equations (0.25) but are not implied by them.

The existence of saddle points may again be inferred from the topology of the configuration space M. To see how this is done, we shall consider a very simple model in which M is the two-dimensional torus embedded in R^3 with radii 1 and $a < 1$.

$$(r^2 - a^2 - 1)^2 = 4(a^2 - y^2)$$

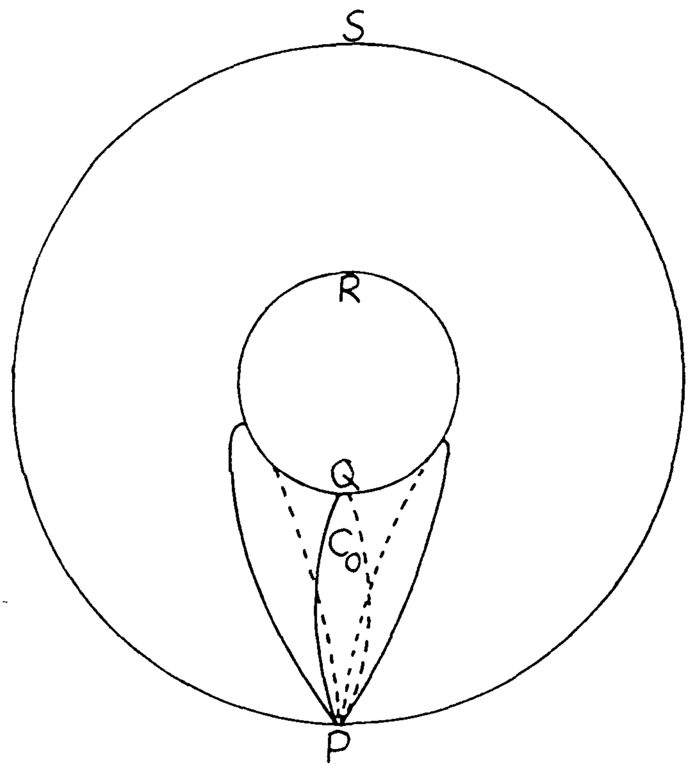


Fig. 0.4

Suppose we wish to find the stationary points of the function $f(x,y,z) = z$ on the surface M. The first step is to find the absolute minimum of f, which lies at the point $P = (0,0,-1-a)$. The next step is to consider a homotopy class of closed curves in M, passing through P, which cannot be shrunk to a point. On each curve C, f attains a maximum value F(C), say. In the given homotopy class there is a curve C_0 for which F(C) is minimized, and the maximum of f on C_0 is then a saddle point of f on M. In our simple model this is the point $Q = (0,0,-1+a)$. It is a stationary point with respect to variations along C_0 because it is the maximum of f on C_0. On the other hand it is a stationary point with respect to variations orthogonal to C_0, because otherwise there would be a neighbouring curve with a smaller value of

F(C).

The foregoing argument depends on the existence of curves in M which cannot be shrunk to a point, or in other words on a nontrivial homotopy group $\pi_1(M)$. In the case of the torus, $\pi_1(M)$ has two generators; the one illustrated leads us to the stationary point Q, while the other leads to the point $R = (0,0,1-a)$.

Now we can apply the idea developed above to the problem of stationary points for the energy functional (0.8). In this case we need to calculate the homotopy group $\pi_1(M)$ for the space of all finite energy Yang-Mills-Higgs fields. In section 2 we saw how to associate to each configuration a continuous mapping of the two-sphere into the coset space G/H; the space of all such mappings will be denoted $C^0(S^2,G/H)$. It is important to recall that all these mappings are understood to be <u>based</u>; that is, the south pole of the sphere always maps to the coset H. The problem is therefore reduced to the calculation of $\pi_1(C^0(S^2,G/H))$. An element of this group may be regarded as a mapping $S^1 \times S^2 \longrightarrow G/H$ with the property

$$\begin{aligned} 0 \times S^2 &\longrightarrow H \\ S^1 \times \underline{u}_0 &\longrightarrow H \end{aligned} \qquad (0.26)$$

In homotopy theory the subset $(0 \times S^2) \cup (S^2 \times 0)$ is called the <u>reduced sum</u> of S^1 and S^2, denoted $S^1 \vee S^2$, and the <u>reduced product</u> $S^1 \wedge S^2$ is constructed from $S^1 \times S^2$ by identifying the subset $S^1 \vee S^2$ to a single point:

$$S^1 \wedge S^2 = S^1 \times S^2 \,/\, S^1 \vee S^2$$

The importance of this construction lies in the fact that it may be used to produce higher-dimensional spheres [6], since $S^n \wedge S^m = S^{n+m}$. By virtue of (0.26) we therefore have a mapping

$$S^1 \wedge S^2 = S^3 \longrightarrow G/H$$

and so there is a relation

$$\pi_1(C^0(S^2, G/H)) = \pi_3(G/H) \tag{0.27}$$

An important example of this application of topology is an SU(2) gauge theory broken by a triplet Higgs field. In this case $G/H = S^2$, and any textbook will tell us that $\pi_3(S^2) = Z$. Since this homotopy group is nontrivial we expect a saddle point in the sector M_0 of zero magnetic charge. Of course, since the configuration space is infinite-dimensional, a rigorous proof of existence is more difficult than for our toy model, but nevertheless one has been given by Taubes [7]. Very little is known of the exact nature of this solution, but it is usually thought of as a <u>bound state</u> of a monopole and an antimonopole.

Saddle points may exist in theories where the configuration space M is connected and monopoles are absent. For example if SU(2) is broken by a <u>doublet</u> then there is no residual symmetry and $G/H = S^3$. There are no monopoles since $\pi_2(S^3) = 0$, but on the other hand $\pi_3(S^3) = Z$ so that saddle points are expected [8]. The same phenomenon occurs in the Weinberg-Salam theory [9].

Chapter 1. Nahm's Equation and the Transition Matrix

In this chapter we turn our attention to the solution of the self-duality equation (0.16), which describes either instantons or monopoles depending on the choice of boundary conditions. Two methods of solution will be described, the first due to Atiyah and Ward (AW) [10,11], and the second due to Atiyah, Drinfeld, Hitchin and Manin (ADHM) [12]. The main common feature of these two constructions is that both succeed in reducing the original nonlinear problem to an associated linear one, involving the solution of a linear or linear differential equation. Because of this underlying linear structure it is very convenient to use the language of vector bundles, which will be briefly summarized in section 1. Next the AW and ADHM constructions will be reviewed. The AW method associates to each selfdual field a holomorphic vector bundle over the twistor space CP^3. On the other hand the ADHM method sets up a 'reciprocity' between selfdual fields in d and 4-d dimensions which have been produced by dimensional reduction from d = 4 . In particular, the three-dimensional Bogomolny equation (0.9) is associated to a certain one-dimensional equation first considered in this context by Nahm [13]

$$\frac{d}{dz}T_i(z) = \frac{1}{2}\varepsilon_{ijk}[T_j(z),T_k(z)] \qquad (1.1)$$

In section 4 we shall address the question: What is the form of the holomorphic vector bundle which corresponds via the AW construction to a given solution of Nahm's equation (1.1)? We shall find that the transition matrix for this bundle takes a relatively simple form as a function of the complex 'spectral parameter' ζ.

1. Vector Bundles

The algebra of nonabelian gauge fields may be given an elegant geometrical interpretation within the theory of vector bundles, which we shall now describe. The formal definition of these objects runs as follows:

A vector bundle consists of a pair of manifolds E, M together with a smooth mapping $p : E \longrightarrow M$. Each point $x \in M$ has a neighbourhood U whose inverse image $p^{-1}(U)$ is diffeomorphic to the product $U \times V$, where V is a fixed vector space.

The manifold M is known as the base space of the bundle, and E is called the total space. By an abuse of terminology we usually speak of the vector bundle E. The inverse image $p^{-1}(x)$ is the fibre at x, which is denoted E_x. Each fibre is isomorphic to the vector space V, and the total space is the union of all the fibres

$$E = \bigcup_{x \in M} E_x$$

This means that the vector bundle E is simply a collection of vector spaces E_x, one for each point of the manifold M. To set up a co-ordinate system on the bundle we use a diffeomorphism

$$f : p^{-1}(U) \longrightarrow U \times V$$

$$f(v) = (p(v), q(v))$$

and then choose co-ordinates on the open set U and a basis for the vector space V.

The first factor of f(v) is fixed by the definition of the bundle; if v lies in the fibre E_x then $p(v) = x$. However, there is a good deal of freedom in the choice of the function $q : p^{-1}(U) \dashrightarrow V$. For each $x \in U$ let q_x be the restriction of q to the fibre at x

$$q_x : E_x \dashrightarrow V$$

The each q_x is an isomorphism. The crucial point is that because the E_x are distinct vector spaces, these isomorphisms may be chosen independently. In other words each fibre has its own basis which may be chosen independently of the others. It follows that an equally acceptable co-ordinate system related to f is provided by the diffeomorphism

$$f'(v) = (p(v), g(p(v))q(v))$$

where $g(x) \in GL(v)$ is a smooth function on the open set U.

If the fibre V has more structure than a vector space, then the automorphisms of V must preserve this. For example, if V has a hermitian metric and a measure, then g(x) lies in the special unitary group SU(V).

A <u>section</u> of the vector bundle $p : E \dashrightarrow M$ is a smooth mapping $\phi : M \dashrightarrow E$ such that the composition of p with ϕ is the identity mapping on M:

$$p \circ \phi = 1_M$$

This means that for each $x \in M$ the value $\phi(x)$ lies in the fibre at x. Taking co-ordinates on the bundle E, the section can be represented by a function on the chart U

$$\tilde{\phi} = q \circ \phi \quad : U \longrightarrow V$$

The function $\tilde{\phi}$ transforms under a change of co-ordinates through the familiar relation

$$\tilde{\phi}'(x) = g(x)\tilde{\phi}(x) \qquad (1.2)$$

In this way we see that the matter fields in a nonabelian gauge theory are described by sections of a vector bundle. The gauge transformations of these fields are simply co-ordinate changes on the bundle.

If the topology of the base space M is more complex than that of R^n, several charts may be needed to cover the whole bundle. In the intersection $U_1 \cap U_2$ of two charts, two alternative co-ordinate systems will exist. These are related by a transformation of the form

$$q_2(v) = g(p(v))q_1(v)$$

In this case the V-automorphism $g(x)$ is referred to as the <u>transition matrix</u> between the two co-ordinate systems.

The gauge field $A_\mu(x)$ may now be interpreted in the language of vector bundles. Suppose that $\phi(x)$ is a section of the bundle E, whose co-ordinate representative transforms via (1.2). Then the covariant derivative

$$D_\mu \tilde{\phi} = \partial_\mu \tilde{\phi} + A_\mu \tilde{\phi} \tag{1.3}$$

transforms in exactly the same way. This means that $D_\mu \tilde{\phi}$ is itself the co-ordinate representative of a section $D_\mu \phi$, the covariant derivative of ϕ. A differential operator D which maps sections to sections is called a <u>connection</u> on the bundle E. When E is the <u>tangent bundle</u> of spacetime M, the connection ∇_μ is identified with the gravitational field of general relativity.

It is important to note that the connection D_μ and the section ϕ provide co-ordinate-free descriptions of the gauge and matter fields, whereas A_μ and $\tilde{\phi}$ are co-ordinate dependent i.e. subject to gauge transformations. In future, however, we shall always write $\phi(x)$ for the co-ordinate representative $\tilde{\phi}(x) \in V$.

2. The Atiyah-Ward Method

It was shown by Ward [10] that to each selfdual connection on the gauge bundle over R^4 there corresponds a holomorphic vector bundle over the twistor space CP^3. The elegance of this construction lies in the fact that there is no connection on this latter bundle; the information concerning the gauge field has been coded into the holomorphic structure of the bundle.

The first step in the construction is to extend the gauge potential $A_\mu(x)$ to C^4 by analytic continuation. The points of C^4 may be represented as quaternions

$$x = x^0 - i\underline{x}.\underline{\sigma}$$

$$= \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix} \tag{1.4}$$

so defining the complex co-ordinates

$$\begin{aligned} y &= x^0 - ix^3 & \bar{y} &= x^0 + ix^3 \\ z &= x^2 - ix^1 & \bar{z} &= x^2 + ix^1 \end{aligned} \tag{1.5}$$

If $x \in R^4$ then y is complex conjugate to $\bar{y}$ and z is complex conjugate to $\bar{z}$. There are two classes of null planes in C^4, which may be distinguished by means of the skew-symmetric tensor $G_{\mu\nu} = V_\mu W_\nu - V_\nu W_\mu$ (unique up to a factor) formed from any two independent vectors in the plane. For one class of null planes $G_{\mu\nu}$ is selfdual; for the other class it is antiselfdual. An antiselfdual null plane can be described by an equation of the form

$$x\pi = \omega \tag{1.6}$$

where x is the quaternion introduced in (1.4) and π,ω are two-dimensional complex column vectors such that $\pi \neq 0$. It is clear that the pairs (π,ω) and (π',ω') define the same plane if and only if $\pi' = \lambda\pi$, $\omega' = \lambda\omega$ for some nonzero complex scale factor λ. It follows that the set of antiselfdual null planes, or <u>twistor space</u>, is isomorphic to $CP^3\backslash CP^1$, where the deleted CP^1 contains all those pairs (π,ω) for which $\pi = 0$.

We shall now introduce the following complex co-ordinates on CP^3:

$$\begin{aligned} \mu &= \omega_2/\pi_2 \\ \nu &= \omega_1/\pi_1 \\ \xi &= \pi_1/\pi_2 \end{aligned} \tag{1.7}$$

These functions, however, are singular as $\pi_1 \longrightarrow 0$ or $\pi_2 \longrightarrow 0$. To define a holomorphic structure on the twistor space we must use co-ordinates which are smooth at these points. Since $\pi \neq 0$ for each twistor, the manifold $CP^3\backslash CP^1$ can be covered by two charts

$$U_1 = \{(\pi,\omega) : \pi_1 \neq 0\} \quad \text{and} \quad U_2 = \{(\pi,\omega) : \pi_2 \neq 0\}$$

The standard co-ordinates in these regions are as follows:

$$\begin{aligned} &\text{in} \quad U_1: \quad \mu/\zeta \ , \ \nu \ , \ 1/\zeta \\ &\text{in} \quad U_2: \quad \mu \ , \ \nu\zeta \ , \ \zeta \end{aligned} \tag{1.8}$$

In terms of the parameters (1.7) the antiselfdual null plane (μ,ν,ζ) is given by

$$\begin{aligned} \bar{y} + z\zeta &= \mu \\ y - \bar{z}/\zeta &= \nu \end{aligned} \tag{1.9}$$

We shall now set up the correspondence between selfdual gauge fields and holomorphic vector bundles over the twistor space $T = CP^3 \backslash CP^1$. Let $\phi(x)$ be a field lying in the fundamental representation of the gauge group G. The restriction of the selfdual connection $F_{\mu\nu}$ to the any antiselfdual null plane θ vanishes identically. This condition is necessqry and sufficient for the consistency of the equations which state that $\phi(x)$ is covariantly constant on θ, namely

$$\begin{aligned} V_\mu D_\mu \phi &= 0 \\ W_\mu D_\mu \phi &= 0 \end{aligned} \tag{1.10}$$

where V and W are any two independent vectors in the plane θ. it is therefore possible to define a vector bundle over T whose fibre at θ contains all the covariant constants on θ; the fibre dimension is n, the dimension of the gauge group representation.

To study the structure of this bundle it is necessary to introduce co-ordinates, which may be done as follows. On each $\theta \in U_1$ choose a point $x_1(\theta)$, and on each $\theta \in U_2$ choose a point $x_2(\theta)$, such that the co-ordinates $x_i(\theta)$ are smooth functions on the respective charts U_i. To any covariant constant ϕ on the plane we may ascribe co-ordinates $\phi(x_1)$ if $\theta \in U_1$, or $\phi(x_2)$ if $\theta \in U_2$. If $\theta \in U_1 \cap U_2$ then there are two sets of co-ordinates for ϕ, related by a transition matrix $g(\theta)$:

$$\phi(x_1) = g(\theta)\phi(x_2) \tag{1.11}$$

where

$$g(\theta) = P \exp \int_{x_1}^{x_2} A_\mu(x)\, dx^\mu \tag{1.12}$$

and the integration is performed along any path from x_1 to x_2 in the plane θ. In principle, knowledge of $g(\theta)$ is sufficient to determine $A_\mu(x)$ up to a gauge transformation. This follows since

$$g(\theta) = P \exp \int_{x_1}^{x} A_\mu(x)\, dx^\mu \, P \exp \int_{x}^{x_2} A_\mu(x)\, dx^\mu$$

$$= h(x,\zeta)k(x,\zeta)^{-1}$$

where h is analytic for $\zeta \neq 0$ and k is analytic for $\zeta \neq \infty$. Liouville's theorem implies that h and k are uniquely determined up to transformations of the form

$h(x,\zeta) \longrightarrow h(x,\zeta)\gamma(x)$, $k(x,\zeta) \longrightarrow k(x,\zeta)\gamma(x)$. The gauge potential $A_\mu(x)$ may be calculated from either of the equivalent expressions

$$\begin{aligned} V^\mu A_\mu(x) &= V^\mu h(x,\zeta)^{-1}\partial_\mu h(x,\zeta) \\ &= V^\mu k(x,\zeta)^{-1}\partial_\mu k(x,\zeta) \end{aligned} \tag{1.14}$$

where V is an arbitrary vector lying in the plane $\theta(x,\zeta)$. From (1.14) it is evident that $A_\mu(x)$ is defined only up to gauge transformations $\gamma(x)$.

Now the vector V lies within the plane (μ,ν,ζ) if and only if $V^\mu\partial_\mu\mu = V^\mu\partial_\mu\nu = 0$. It follows using the expressions (1.9) that this plane is spanned by the vectors $\partial_y + \zeta\partial_{\bar{z}}$ and $\partial_z - \zeta\partial_{\bar{y}}$, and so the equations (1.14) lead to

$$\begin{aligned} A_y + \zeta A_{\bar{z}} &= h^{-1}(\partial_y + \zeta\partial_{\bar{z}})h \\ A_z - \zeta A_{\bar{y}} &= h^{-1}(\partial_z - \zeta\partial_{\bar{y}})h \end{aligned} \tag{1.15}$$

Thus it is straightforward to determine the gauge field $A_\mu(x)$ provided that we can perform the factorization of the transition matrix (1.12). This is the crucial step; it is termed the Riemann-Hilbert Problem for $g(x,\zeta)$.

The co-ordinates chosen on the holomorphic vector bundle E are by no means unique. In particular we may perform an independent change of basis in each of the charts U_1 and U_2; this would induce a transformation of the transition matrix $g(\theta)$

$$g(\theta) \longrightarrow A(\theta)g(\theta)\,B(\theta)^{-1} \tag{1.16}$$

where the unimodular matrices $A(\theta)$ and $B(\theta)$ are respectively analytic in U_1 and U_2. The corresponding transformation of $h(x,\zeta)$ is given by

$$h(x,\zeta) \longrightarrow A(\theta)h(x,\zeta) \tag{1.17}$$

The gauge potential calculated from (1.14) is unchanged because the matrix A depends only upon the null plane $\theta(x,\zeta)$ and the directional derivative $V^\mu\partial_\mu$ is tangential to this plane. Accordingly two transition matrices $g(\theta)$, $g'(\theta)$ are said to be equivalent if they are related by a transformation of the form (1.16); they describe the same vector bundle E.

The Riemann-Hilbert problem (1.13) for a general matrix-valued function $g(\theta)$ is too difficult for us. However, using certain theorems of algebraic geometry Atiyah and Ward have shown [11] that to construct all selfdual SU(2) instantons it is sufficient to use matrices with the special form

$$g(\theta) = \begin{pmatrix} \zeta^l & \rho(\theta) \\ 0 & \zeta^{-l} \end{pmatrix} \tag{1.18}$$

where l is a positive integer. A similar result was obtained more recently by Hitchin [14] for SU(2) monopoles; in this case l is the monopole number or topological charge. When the transition matrix has the triangular form (1.18) the splitting $g = hk^{-1}$ may be performed explicitly; this is done in [15,16], where a sequence of ansätze for the gauge potential is derived. We shall not describe these results here, since we are interested in the application of the twistor technique to a different situation in which the potential $A_\mu(x)$ depends upon one co-ordinate alone. The relevance of such a one-dimensional system to monopoles is the subject of the next section.

3. The ADHM Method

It has been shown in section 1 that a nonabelian gauge field may be described geometrically as a connection on a vector bundle $F = R^4 \times W$ over R^4. The fibre W of this bundle is a real, complex or quaternionic vector space of dimension n upon which acts the gauge group G. In what follows we shall restrict attention to O(n), U(n) and Sp(n); since all these groups arise as subgroups of unitary groups we shall only make explicit reference to U(n).

A submanifold $N \subset M$ may be curved even though the manifold M is itself flat; the curvature of N depends on the embedding. For example, the sphere S^n is most easily realized as a submanifold of R^{n+1}. The idea of the ADHM construction is to construct the curved gauge bundle F by embedding in a higher-dimensional flat bundle $E = R^4 \times V$. The vector space V, like W, is assumed to carry a Hermitian metric. The embedding $v : F \longrightarrow E$ can be represented by a linear mapping $v(x) : W \longrightarrow V$, depending on the base point x, which is normalized by

$$v(x)^+ v(x) = 1 \qquad (1.19)$$

The range of v(x) is the fibre of the subbundle vF at x. Suppose that $\phi(x)$ is a section of the gauge bundle F, that is, $\phi(x) \in W$. Using the embedding v it is possible to lift this to a section of the flat bundle E

$$\tilde{\phi}(x) = v(x)\phi(x) \in V \qquad (1.20)$$

The standard partial derivative $\partial_\mu\tilde{\phi}(x)$ does not generally lie in the subbundle vF, but it may be projected onto it using the operator

$$P(x) = v(x)v(x)^+ \tag{1.21}$$

A connection is now defined on F by

$$\widetilde{D_\mu\phi(x)} = P(x)\partial_\mu\tilde{\phi}(x) \tag{1.22}$$

or equivalently

$$D_\mu\phi(x) = v(x)^+\partial_\mu(v(x)\phi(x)) \tag{1.23}$$

From the latter definition it is easy to read off the gauge potential

$$A_\mu(x) = v(x)^+\partial_\mu v(x) \tag{1.24}$$

Under a change of basis in the gauge bundle F, the matrix v(x) transforms via

$$v(x) \dashrightarrow v(x)g(x)^{-1} \tag{1.25}$$

where g(x) is a unitary transformation of W. It is easy to see from (1.24) that this induces a gauge transformation of $A_\mu(x)$, as we have remarked earlier. Using the definition (1.22) it is possible to express the curvature of the connection in terms of the projection operator $P(x)$:

$$F_{\mu\nu}(x) = v(x)^{+}[P_{\mu}(x), P_{\nu}(x)]v(x) \qquad (1.26)$$

where $P_{\mu} = \partial_{\mu} P$.

In the work which follows we shall use the standard representation of quaternions by 2 x 2 complex matrices $e_0 = 1$, $e_i = -i\sigma_i$; points of R^4 correspond to quaternions through the relation (1.4). ADHM found [12] that in order to construct any selfdual SU(n) instanton solution with topological charge k it is sufficient to use a complex vector space V with dimension n+2k. The required embedding $v(x) : W \longrightarrow V$ is determined by the linear equation

$$\Delta(x)^{+} v(x) = 0 \qquad (1.27)$$

where $\Delta(x)$ is a linear mapping $W' \longrightarrow V$, and W' is a 2k-dimensional complex vector space. The space W' is also endowed with a quaternionic structure, that is, it may be viewed as a vector space over the quaternions. The mapping $\Delta(x)$ then has a particularly simple form: it is <u>linear</u> in the quaternion x

$$\Delta(x) = a + bx \qquad (1.28)$$

It is easily seen that the projection operator P(x) is given in terms of this mapping by

$$P = 1 - \Delta F \Delta^{+} \qquad (1.29)$$

where $F = (\Delta^+\Delta)^{-1}$. The curvature of the resulting gauge field then follows from the expression (1.26); it is

$$F_{\mu\nu} = v^+ b e_\mu F e_\nu^+ b^+ v - (\mu \longleftrightarrow \nu) \tag{1.30}$$

We readily see [17] that this tensor is selfdual provided that $\Delta^+\Delta$ commutes with the quaternions e_μ. However, the proof that <u>all</u> selfdual instanton solutions can be constructed in this way is more difficult; we shall approach this problem in the next chapter. It is clear that the gauge field (1.30) is nonsingular provided that the matrix $\Delta(x)^+\Delta(x)$ is invertible for all x. Furthermore, since $F(x) = O(r^{-2})$ and $b^+v(x) = O(r^{-1})$ as $r = |x| \longrightarrow \infty$, the asymptotic behaviour of the field strength is $F_{\mu\nu} = O(r^{-4})$; it follows that the action (0.18) is finite.

Recall that the spaces W' and V' have dimensions 2k and n+2k respectively. The components of $w \in W'$ will therefore be denoted $w^r_{B'}$, where $r = 1,2,\ldots,k$ and $B' = 1',2'$ is a dotted spinor index. Similarly the components of $v \in V$ will be denoted v^{rA} and v^α, where $A = 1,2$ is a spinor index and $\alpha = 1,2,\ldots,n$. It is possible to choose bases such that $b^{rA}{}_{sC} = \delta^r{}_s\delta^A{}_C$ and $b^\alpha{}_{sC} = 0$. The hermitian conjugate of (1.27) may now be written in component form

$$(v^+)_{rA}(\delta^r{}_s x^{AB'} + a_\mu{}^r{}_s e^{\mu AB'}) + (v^+)_\alpha a^\alpha{}_s{}^{B'} = 0 \tag{1.31}$$

In summary, then, the construction of a selfdual instanton field proceeds in two stages: Firstly, a matrix $\Delta(x)$ is chosen which is linear in x (1.28) and satisfies the quadratic constraint

$$[\Delta(x)^{+}\Delta(x) \, , \, e_{\mu}] = 0 \qquad (1.32)$$

Secondly, the <u>linear</u> equation (1.27) or (1.31) is solved for the n linearly independent vectors v(x) which span the curved subbundle representing the gauge field. The potential $A_{\mu}(x)$ is then easily found from (1.24).

The selfdual instanton solutions given by the above construction are localized in the co-ordinate x^0 as well as in x^i (hence the term 'instanton'); but a multimonopole is a selfdual gauge field <u>independent</u> of x^0. Such a solution may be regarded as an infinite string of instantons laid along the x^0-axis, that is, a limiting case of a multi-instanton as $k \longrightarrow \infty$ [18]; in this limit the spaces W' and V become infinite-dimensional. These considerations led Nahm to modify the ADHM construction to treat monopoles [13]. W' and V are now <u>Hilbert spaces</u> of functions, and $\Delta(x)^{+}$ is a differential operator whose kernel has finite dimension n.

After a gauge transformation the Higgs field $A_0(x)$ has the following asymptotic form as $r = |\underline{x}| \longrightarrow \infty$:

$$A_0^{\alpha\beta} = -i\delta^{\alpha\beta}(z_{\alpha} - \frac{k_{\alpha}}{2r}) + O(\frac{1}{r^2}) \qquad (1.33)$$

where $z_{\alpha} \leq z_{\alpha+1}$ and $k_{\alpha} \in Z$. Let $I = \{\alpha : k_{\alpha} = 0\}$ and define the integer-valued function

$$k(z) = \sum_\alpha k_\alpha \theta(z_\alpha - z) \tag{1.34}$$

Then the Hilbert space V of Nahm's construction consists of pairs $(v^{rA}(z), v^\alpha)$ where $r = 1,2,...,k(z)$ and $\alpha \in I$. The inner product is naturally defined by

$$\langle v_1, v_2\rangle = \int dz \; (\sum_{r=1}^{k(z)} (v_1^+)_{rA} v_2^{rA} + \sum_{\alpha \in I} (v_1^+)_\alpha v_2^\alpha) \tag{1.35}$$

and the gauge potential is calculated using the analogue of (1.24)

$$A_\mu^{\alpha\beta}(x) = \langle v^{(\alpha)}(x), \partial_\mu v^{(\beta)}(x)\rangle \tag{1.36}$$

The vectors used in this formula are the n orthonormal solutions to the differential equation

$$\begin{aligned}(v(z)^+)_{rA}(\delta^r{}_s x^{AB'} + iT_i{}^r{}_s(z) e^{iAB'} - i\delta^r{}_s \delta^{AB'}\frac{d}{dz}) \\ + \sum_{\alpha \in I} (v^+)_\alpha a^\alpha{}_s{}^{B'} \delta(z - z_\alpha) = 0 \end{aligned} \tag{1.37}$$

or suppressing indices

$$v(z)^+(x + iT - i\frac{d}{dz}) + \sum_{\alpha \in I} (v^+)_\alpha a^\alpha \delta(z - z_\alpha) = 0$$

which is entirely analogous to (1.31). The origin of the space V will be explained in the next chapter; it is in fact the solution space of a certain covariant linear equation in the background field $A_\mu(x)$. At the present time we shall not prove the completeness of Nahm's construction.

The quadratic constraint (1.32), which ensures selfduality of the monopole, implies firstly that the matrices $T_i(z)$ are antihermitian

$$T_i = -T_i^+ \tag{1.38}$$

and secondly that the following differential equation must be satisfied:

$$\frac{dT_i}{dz} = \varepsilon_{ijk} T_j T_k + \sum_{\alpha \in I} d_i^\alpha \delta(z - z_\alpha) \tag{1.39}$$

The discontinuity d_i^α of $T_i(z)$ at the 'jumping point' z_α is given by

$$d_{\mu}^{\alpha r}{}_s e^{\mu A B'} = (a^+)_\alpha{}^{rA} a^\alpha{}_s{}^{B'} \tag{1.40}$$

where no summation is intended on the index α. The equations (1.39) are known as <u>Nahm's equations</u>; between the points of discontinuity they take the simpler form

$$\frac{dT_i}{dz} = \varepsilon_{ijk} T_j T_k \tag{1.41}$$

It is a striking feature of Nahm's construction that the equation (1.41) expresses the selfduality of a gauge field $T_\mu(x)$ which depends only upon a single co-ordinate $z = x^0$.

In this case it is always possible to transform $T_0(z)$ to zero by a gauge transformation $g(z)$; then Nahm's equation is equivalent to the selfduality equation

$$F_{0i} = \frac{1}{2}\varepsilon_{ijk}F_{jk} \qquad (1.42)$$

It has also been remarked [19] that the relationship between the three-dimensional monopole potential $A_\mu(x)$ and the one-dimensional gauge field T (x) is a reciprocal one. In order to derive T_μ from A_μ or vice versa it is necessary to solve a covariant Weyl equation such as (1.37) in the background gauge field, and then take matrix elements of a suitable operator between the resulting solutions, as in (1.36). More details of the inverse construction (of T_μ in terms of A_μ) will be given later. For the present, however, we shall be content with the observation that the selfduality equation (0.16) can be written in a form which is highly reminiscent of the quadratic constraint (1.32). To see this we write the covariant derivative operator as a quaternion

$$\begin{aligned} D &= D_\mu e_\mu \\ D^+ &= D_\mu e^+_\mu \end{aligned} \qquad (1.43)$$

and use a well-known identity for the product of two quaternions

$$e^+_\mu e_\nu = \delta_{\mu\nu} + i\eta^-_{\mu\nu} \qquad (1.44)$$

where $\eta^-_{\mu\nu}$ is an antihermitian matrix and an antiselfdual tensor

$$\begin{aligned} \eta^-_{0i} &= -\eta^-_{i0} = -\sigma_i \\ \eta^-_{ij} &= \varepsilon_{ijk}\sigma_k \end{aligned} \tag{1.45}$$

It follows from (1.42) and (1.43) that

$$D^+D = D_\mu D_\mu + \tfrac{1}{2} i F_{\mu\nu}\eta^-_{\mu\nu} \tag{1.46}$$

and it is clear from this that the curvature $F_{\mu\nu}$ is self-dual if and only if the operator D^+D commutes with the quaternions

$$[D^+D \ , \ e_\mu] = 0 \tag{1.47}$$

In other words the covariant derivative D plays an analogous role to the matrix $\Delta(x)$ which appears in (1.31).

4. The Transition Matrix for Nahm's T_{μ}

We have seen in section 2 that there is a one-to-one correspondence between selfdual gauge fields and holomorphic vector bundles over the twistor space T. The argument presented there applies both to four-dimensional instanton fields and to three-dimensional monopoles; in the latter case a further constraint on the transition matrix is required to ensure the invariance of the gauge field with respect to translations in the x^0-direction. Let $\theta = (\mu, \nu, \zeta)$ be a general antiselfdual null plane, and let $\theta' = (\mu', \nu', \zeta')$ be the plane obtained by translating through a distance a along the x^0-axis. It is evident from (1.5) and (1.9) that

$$\begin{aligned} \mu' &= \mu + a \\ \nu' &= \nu + a \\ \zeta' &= \zeta \end{aligned} \tag{1.48}$$

In order that the gauge field $A_{\mu}(x)$ should be translation-invariant it is necessary that the translated patching matrix be equivalent to the original in the sense of (1.16)

$$g(\theta') = A(a,\theta) g(\theta) B(a,\theta)^{-1} \tag{1.49}$$

If this condition is satisfied then $g(\theta)$ is equivalent to a transition matrix $\bar{g}(\theta)$ which is <u>invariant</u> under x^0-translation, in other words $\bar{g}(\theta)$ is independent of x^0:

$$g(\theta) = H(\theta)\bar{g}(\theta)K(\theta)^{-1} \tag{1.50}$$

It is sufficient to choose the matrices $H(\theta)$ and $K(\theta)$ such that

$$\begin{aligned} H(\theta') &= A(a,\theta)H(\theta) \\ K(\theta') &= B(a,\theta)K(\theta) \end{aligned} \tag{1.51}$$

$H(\mu,0,\xi)$ and $K(0,\nu,\xi)$ are assigned arbitrary values, and the complete functions are then determined from (1.51). The new transition matrix $\bar{g}(\theta)$ is a function of $(\mu-\nu)/2$ and ξ only, since the third co-ordinate $(\mu+\nu)/2$ is not translation invariant.

A rather more direct way of ensuring translational invariance of the patching matrix is to choose the reference points $x^i(\theta)$ to satisfy

$$x_i(\theta') = x_i(\theta) + ae_0 \tag{1.52}$$

For example, a good choice is

$$\begin{aligned} x_1 &= \begin{pmatrix} \nu & 0 \\ (\mu-\nu)/\xi & \nu \end{pmatrix} \\ x_2 &= \begin{pmatrix} \mu & (\nu-\mu)\xi \\ 0 & \mu \end{pmatrix} \end{aligned} \tag{1.53}$$

If the gauge field $A_\mu(x)$ is independent of x^0, then the matrix defined by (1.12) is automatically invariant under translations, since the path of integration may simply be translated from θ to θ'.

The following question now arises: What is the form of the patching matrix for Nahm's reciprocal one-dimensional gauge field $T_\mu(x)$? In other words, what constraints must be placed on $g(\theta)$ in order that it should represent a gauge field which depends on one co-ordinate alone? To answer this question we shall perform explicitly the integral in (1.12).

We observe firstly that it is possible to choose two reference points $x_i(\theta)$ both of which lie on a given slice $x^0 = \lambda$. For example we might take

$$x_1 = \begin{pmatrix} \nu & 0 \\ (\mu+\nu-2\lambda)/\zeta & -\nu+2\lambda \end{pmatrix}$$
$$x_2 = \begin{pmatrix} -\mu+2\lambda & (\mu+\nu-2\lambda)\zeta \\ 0 & \mu \end{pmatrix} \tag{1.54}$$

Indeed, it is easy to see that x_i is an analytic function of the co-ordinates (1.8) in the region U_i; and in both cases $x^0 = (y+\bar{y})/2 = \lambda$. This choice of reference points greatly simplifies the integration in (1.12), since it can be performed along a straight line lying entirely within the slice $x^0 = \lambda$, on which T_μ is constant. The result is therefore

$$g(\lambda,\theta) = \exp T_\mu(\lambda)(x_2 - x_1)^\mu \tag{1.55}$$

On substitution of the expressions (1.54) for the endpoints, we find

$$g(\lambda,\theta) = \exp -\tau(\lambda,\theta)(\mu + \nu - 2\lambda) \qquad (1.56)$$

where

$$\tau(\lambda,\zeta) = T_y(\lambda) - T_{\bar{y}}(\lambda) + \frac{1}{\zeta}T_z(\lambda) + \zeta T_{\bar{z}}(\lambda)$$
$$= \frac{1}{2\zeta}\{i(1-\zeta^2)T_1(\lambda) + (1+\zeta^2)T_2(\lambda) + 2i\zeta T_3(\lambda)\}$$
$$(1.57)$$

Nahm's equation (1.41) is a first-order differential equation; it may therefore be regarded as an initial value problem. Given the initial value $T_i(z)$ at some point $z = \lambda$, the functions $T_i(z)$ are uniquely determined. This property is reflected in the fact that the transition matrix $g(\theta)$ depends only upon these initial values. It has already been remarked after (1.13) that the patching matrix determines the gauge potential only up to a gauge transformation; however, in the present case the condition $T_0 = 0$ is sufficient to fix the gauge.

The equivalence class of the transition matrix should not, of course, depend on the particular point where the initial conditions are used. We may verify this by the following argument: Let $\underline{v}(\zeta)$ be the null vector

$$\underline{v}(\zeta) = \frac{1}{2\zeta}(i(1-\zeta^2)\ ,\ 1+\zeta^2\ ,\ 2i\zeta) \qquad (1.58)$$

It is easily verified that for any nonzero vector $\underline{v}$ the mapping $\underline{u} \longrightarrow \underline{u} \wedge \underline{v}$ has rank 2, and it follows that the image of this mapping is the space of all vectors orthogonal to $\underline{y}$. Since $\underline{y}$ is a null vector, there exists some $\underline{u}$ such that

$$\underline{y} = \underline{u} \wedge \underline{y} \tag{1.59}$$

By taking the scalar product of $\underline{y}$ with Nahm's equation we obtain the following result for the $\underline{y}$-component of $\underline{T}$:

$$\frac{d}{dz}\underline{y}.\underline{T} = [\underline{u}.\underline{T}\ ,\ \underline{y}.\underline{T}] \tag{1.60}$$

This equation has solutions of the form

$$\underline{y}.\underline{T}(z) = h(z)\underline{y}.\underline{T}(\lambda)h(z)^{-1} \tag{1.61}$$

where $h(\lambda) = 1$. In other words, $\underline{y}.\underline{T}(z)$ and $\underline{y}.\underline{T}(\lambda)$ are equivalent; they differ only by a change of basis. The spectrum of $\underline{y}.\underline{T}$ is independent of z [20]. Now, $\underline{y}.\underline{T}$ is precisely the matrix which appears in the exponent of equation (1.56)

$$\tau(\lambda,\zeta) = \underline{y}(\zeta).\underline{T}(\lambda) \tag{1.62}$$

therefore $g(\lambda_1,\theta)$ and $g(\lambda_2,\theta)$ differ only by a gauge transformation, as asserted.

A relationship between the transition matrix (1.56) and the transition matrix for the monopole itself has been obtained by Rouhani [21], at least for the gauge group SU(2). For the case of SU(2) the parameters which characterize the asymptotic behaviour of the Higgs field (1.33) are

$$z_1 = -1/2 , \qquad z_2 = 1/2$$
$$k_1 = k , \qquad k_2 = -k \tag{1.63}$$

From (1.34) we may deduce that $T_i(z)$ is a $k \times k$ matrix defined on the interval $(-1/2, 1/2)$ and that $v(x,z)$ is a 2k-dimensional column vector defined on the same interval. If x_1 and x_2 are two reference points lying in a common antselfdual null plane, then the monopole patching matrix can be expressed in terms of $v(x,z)$ by [17]

$$g(x_1, x_2) = \int_{-1/2}^{1/2} dz\ \tilde{v}(x_1, z) v(x_2, z) \tag{1.64}$$

where $\tilde{v}(x,z)$ is the analytic continuation of $v(x,z)^+$ to complex x. We shall suppose that the reference points are chosen as in (1.54) so that $x_1^0 = x_2^0$. It may then be shown using the ideas of Panagopoulos [22] that the integrand in (1.64) is in fact a total derivative

$$\tilde{v}(x_1) v(x_2) = \frac{\partial}{\partial z}(v(x_1) H(x_1, x_2) v(x_2)) \tag{1.65}$$

To do this we need only find a kernel $H(x_1, x_2, z)$ such that

$$\frac{\partial H}{\partial z} + (-ix_1 + T)H + H(ix_2^+ + T) = 1 \tag{1.66}$$

Rouhani's solution to this equation is [21]

$$H = -\frac{1}{2}\frac{\underline{y}\cdot\sigma}{\underline{y}\cdot(\underline{x} + i\underline{T})} \qquad (1.67)$$

where $\underline{y}(\zeta)$ is the null vector defined previously in (1.58). It is now possible to express the patching matrix for the monopole in terms of the boundary values of v and H

$$g(x_1, x_2) = v(x_1)H(x_1, x_2)v(x_2)\Big|_{z=-1/2}^{z=1/2} \qquad (1.68)$$

From (1.5), (1.9) and (1.58) we see immediately that the scalar product $\underline{y}.\underline{x}$ depends only upon the complex parameters of the null plane containing x:

$$\underline{y}\cdot\underline{x} = \gamma = \frac{1}{2}(\mu - \nu) \qquad (1.69)$$

The inverse of $\gamma + i\underline{y}.\underline{T}$, which appears in (1.67), can be re-expressed as the Laplace transform of the transition matrix for the reciprocal selfdual field, that is,

$$(\gamma + i\underline{y}.\underline{T})^{-1} = \int_0^\infty ds\ e^{-s\gamma}e^{-is\underline{y}.\underline{T}} \qquad (1.70)$$

Indeed, the Laplace transform $\exp(-is\underline{y}.\underline{T})$ is precisely the transition matrix (1.56) with the argument $(\mu+\nu-2\lambda)$ replaced by is.

A further relationship with the Atiyah-Ward method is indicated by the appearance of the polynomial

$$P(x,\zeta) = \det(\gamma + i\underline{y}.\underline{T}) \qquad (1.71)$$

when calculating the inverse (1.67). This is the same polynomial which appears in the denominator of the monopole generating function $\rho(x,\zeta)$ (see (1.18)) according to Corrigan and Goddard [16]. What is more, the parametrization (1.58) shows that the space of null three-vectors has a complex structure; it is isomorphic to CP^1. The tangent space to this manifold at any point $\underline{y}$ is therefore isomorphic to the complex plane C. This means that the multivalued function $\gamma(\zeta)$ obtained by solving the characteristic equation $P = 0$ can be thaought of as a curve in the tangent bundle $T(CP^1)$; it is this which Hitchin refers to as the <u>spectral curve</u> of the monopole [14]. It can be shown that if two monopcle gauge fields have the same spectral curve then they are gauge-equivalent; the correspondence between gauge field and spectral curve is one-to-one.

5. Two-monopole Solutions for SU(2)

The simplest nontrivial example of Nahm's equation is that which describes a pair of monopoles in an SU(2) gauge field. Recall (1.63) that in this case $T_i(z)$ is a 2 x 2 matrix defined on the interval (-1/2, 1/2), and v(x,z) is a 4-dimensional column vector defined on the same interval. It is a condition of Nahm's construction that $T_i(z)$ has simple poles at the endpoints z = 1/2 , that is,

$$T_i(z) = -(z \pm 1/2)^{-1} \alpha_i^{\pm} + O(1) \qquad (1.72)$$

Nahm's equation (1.41) implies that the resides $\alpha_i^{\pm}$ must constitute a representation of the SO(3) Lie algebra,

$$[\alpha_i, \alpha_j] = \varepsilon_{ijk} \alpha_k \qquad (1.73)$$

For the two-monopole solution therefore a global gauge transformation may be found which reduces the residues at z = -1/2 to the form $\alpha_i^- = -(i/2)\sigma_i$. It is now possible to demonstrate that (1.37) has only two normalizable solutions v(z); to do this we consider the asymptotic form of this equation near z = -1/2 ,

$$\begin{aligned} \frac{dv}{dz} &= \frac{1}{2(z + 1/2)} \sigma_i \otimes \sigma_i v(z) \\ &= \frac{2 J_i K_i}{z + 1/2} v(z) \qquad (1.74) \end{aligned}$$

where $J_i = (\sigma_i/2) \otimes 1$ and $K_i = 1 \otimes (\sigma_i/2)$. The solution of

(1.74) takes the form $v(z) = V(z+1/2)^{\lambda}$, where λ is an eigenvalue of $2\underline{J}.\underline{K}$ and V is a corresponding eigenvector; so the asymptotic behaviour of $v(z)$ depends only upon the spectrum of this operator. Using the identity

$$2\underline{J}.\underline{K} = (\underline{J}+\underline{K})^2 - \underline{J}^2 - \underline{K}^2$$

we easily see that the eigenvalue 1/2 occurs with multiplicity 3, and the eigenvalue −3/2 occurs with multiplicity 1. This means that of the four independent solutions for $v(z)$ one blows up at $z = -1/2$; in the same way another blows up at $z = 1/2$. Thus we are indeed left with just two normalizable solutions from which to construct an SU(2) gauge field.

The fundamental solution of Nahm's equation when $k = 2$ is just

$$T_i(z) = \frac{i}{2z}\sigma_i \qquad (1.75)$$

This solution is not relevant to the physical two-monopole problem since it has only one pole; nevertheless we shall calculate the corresponding patching matrix as an example of (1.56). It is convenient to use the initial values at the point $z = \lambda = 1$; by substitution in (1.57) we obtain

$$\tau(1,\zeta) = \frac{i}{4\zeta}\{i(1-\zeta^2)\sigma_1 + (1+\zeta^2)\sigma_2 + 2i\sigma_3\} \qquad (1.76)$$

To evaluate the exponential in (1.56) we use the well-known identity

$$\exp i\alpha \underline{n}.\underline{\sigma} = 1 + i\alpha \underline{n}.\underline{\sigma} \tag{1.77}$$

valid when $\underline{n}$ is a null vector. In the present case we therefore obtain the patching matrix

$$g(\theta) = 1 + \frac{\mu+\nu-2}{4\zeta}\{(1-\zeta^2)\sigma_1 - i(1+\zeta^2)\sigma_2 + 2\zeta\sigma_3\}$$

$$= \begin{pmatrix} \frac{\mu+\nu}{2} & -\frac{\zeta}{2}(\mu+\nu-2) \\ \frac{\mu+\nu-2}{2\zeta} & 2-\frac{\mu+\nu}{2} \end{pmatrix} \tag{1.78}$$

Conversely, however, it is possible to start from the transition matrix (1.78) and to reclaim the original gauge field (1.75). The first step in this reverse procedure is to perform an equivalence transformation ot the type (1.16) in order to render $g(\theta)$ triangular. Once this has been done the calculation of $T_\mu(x)$ (up to a gauge transformation) proceeds by a straightforward application of the ansätze [15, 16]. As an illustration of the kind of manipulations which are necessary to find a triangular patching matrix we list here in full the sequence of transformations which must be carried out on (1.78)

$$g(\theta) \sim \begin{pmatrix} 1 & 0 \\ -\zeta^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\mu+\nu}{2} & -\frac{\zeta}{2}(\mu+\nu-2) \\ \frac{\mu+\nu-2}{2\zeta} & 2-\frac{\mu+\nu}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{\mu+\nu}{2} & -\frac{\zeta}{2}(\mu+\nu-2) \\ -\frac{1}{\zeta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} (\mu+\nu)/2 & \zeta \\ -\zeta^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \zeta & -(\mu+\nu)/2 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} \zeta & (\mu+\nu)/2 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad (1.79)$$

This matrix has the standard form (1.18) with $l = 1$; so to recover the gauge field we must apply the first ansatz, namely

$$T_\mu(x) = \tfrac{i}{2}\bar{\eta}_{\mu\nu}\partial_\nu \ln\Delta_0(x) \qquad (1.80)$$

where $\bar{\eta}_{\mu\nu}$ is the antiselfdual tensor defined in (1.45) and $\Delta_i(x)$ is the coefficient of ζ^{-i} in the Laurent expansion of $\rho(x,\zeta)$. In the present case $\Delta_0(x) = x_0$, and therefore the ansatz (1.80) leads directly to the original expression for the gauge field (1.75); we note that in this case the gauge conditions $T_0 = 0$, $\partial_i T_\mu = 0$ are already satisfied.

Our second example will be the most general two-monopole solution $T_i(z)$. It is known that the number of parameters needed to describe an l-monopole solution is 4l-1 [23]. Thus the two-monopole solution has 7 parameters of which 3# correspond to translations in $\underline{x}$-space and do not appear in T, and 3 correspond to rotations. There remains only one nontrivial parameter, which measures the distance between the two monopoles.

The gauge group SU(2) is isomorphic to Sp(1); in order to yield an Sp(1) potential the linear mapping $\Delta(x)$ of the ADHM construction must preserve a <u>symplectic structure</u>. In other words, there exist antilinear mappings $J : W' \longrightarrow W'$ and $J : V \longrightarrow V$ such that $J^2 = -1$, and the mapping commutes with J. In Nahm' construction for monopoles the symplectic structure is chosen to be

$$(Jw)(z) = C\otimes\mathcal{E}\, w(-z)^* \tag{1.81}$$

where C is a $k \times k$ 'charge conjugation' matrix such that $CC^* = 1$. We shall find it convenient to make a choice of basis in which $C = \sigma_1$. The operator $\Delta = x + iT + i\partial_z$ commutes with J provided that the three matrices $T_i(z)$ satisfy the reality conditions

$$T_i(z) = -\,CT_i(-z)^* C^{-1} \tag{1.82}$$

The most general SU(2) two-monopole solution of Nahm's equations is, up to rotations and gauge transformations

$$T_1(z) = -\frac{i}{2}\sigma_1\frac{q}{cn(pz)} \qquad T_2(z) = -\frac{i}{2}\sigma_2\frac{p\ dn(pz)}{cn(pz)}$$

$$T_3(z) = -\frac{i}{2}\sigma_3\frac{q\ sn(pz)}{cn(pz)} \qquad (1.83)$$

where $q = p(1-k^2)^{1/2}$ and k is the modulus of the elliptic functions. In order to place the poles at $z = \pm 1/2$ the parameters p and k must be related by $p = 2K(k)$, where K(k) is the smallest positive zero of cn(x). The remaining single parameter k is that which determines the separation of the two monopoles. It also measures the departure of the solution from axial symmetry, since when $k = 0$ there is an evident symmetry of the solution with respect to rotations in the (x^1, x^2)-plane.

The patching matrix for the two-monopole solution (1.83) is found using (1.56), (1.57) with initial values taken at $z = \lambda = 0$; it is

$$g(\theta) = \exp \frac{i}{4\zeta}\{i(1-\zeta^2)q\sigma_1 + (1+\zeta^2)p\sigma_2\}(\mu+\nu) \qquad (1.84)$$

To evaluate the exponential in this case we need to use the identity

$$\exp i\alpha\underline{n}.\underline{\sigma} = \cos\alpha + i\underline{n}.\underline{\sigma}\sin\alpha \qquad (1.85)$$

valid when $\underline{n}$ is a unit vector. From (1.84) we can read off the angle and axis of rotation, namely

$$\alpha(\mu,\nu,\zeta) = \sqrt{a(\zeta)b(\zeta)}(\mu+\nu)$$

$$\underline{n}(\zeta) = \frac{1}{2\sqrt{a(\zeta)b(\zeta)}}(i(a(\zeta)-b(\zeta)), \; a(\zeta)+b(\zeta), \; 0) \tag{1.86}$$

expressed in terms of the two rational functions

$$\begin{aligned} a(\zeta) &= \frac{1}{4\zeta}\{p(1+\zeta^2) + q(1-\zeta^2)\} \\ b(\zeta) &= \frac{1}{4\zeta}\{p(1+\zeta^2) - q(1-\zeta^2)\} \end{aligned} \tag{1.87}$$

Finally we obtain the following explicit representation of the patching matrix:

$$g(\theta) = \begin{pmatrix} \cos\sqrt{ab}(\mu+\nu) & \frac{b}{\sqrt{ab}}\sin\sqrt{ab}(\mu+\nu) \\ -\frac{a}{\sqrt{ab}}\sin\sqrt{ab}(\mu+\nu) & \cos\sqrt{ab}(\mu+\nu) \end{pmatrix} \tag{1.88}$$

The transition matrix (1.88) behaves in a very special way under translations in the three spatial co-ordinates x^i because the underlying gauge field admits these as invariances. In order that the translated gauge field $T_\mu(x')$ should be gauge-equivalent to $T_\mu(x)$ it is necessary that the translated patching matrix be equivalent to the original one (compare (1.49))

$$g(\theta') = A(\underline{a},\theta)g(\theta)B(\underline{a},\theta)^{-1} \tag{1.89}$$

where θ' is the plane obtained by subjecting θ to the displacement $\underline{a}$. The infinitesimal form of the translation invariance condition is

$$\delta g(\theta) = \delta x^i (X_i(\theta) g(\theta) - g(\theta) Y_i(\theta)) \qquad (1.90)$$

where the unimodular matrices $X_i(\theta)$ and $Y_i(\theta)$ are analytic in U_1 and U_2 respectively. It is possible to verify (1.90) directly for the general 2 x 2 patching matrix (1.88), whose variation is

$$\delta g(\theta) = (\delta z \zeta - \frac{\delta \bar{z}}{\zeta}) \begin{pmatrix} -\sqrt{ab}\,\sin\alpha & b\cos\alpha \\ -a\cos\alpha & -\sqrt{ab}\,\sin\alpha \end{pmatrix}$$

$$= \delta \bar{z} X(\zeta) g(\theta) - \delta z g(\theta) Y(\zeta) \qquad (1.91)$$

where the left and right infinitesimal gauge transformations are given by

$$\begin{aligned} X(\zeta) &= \zeta^{-1} Z(\zeta) \\ Y(\zeta) &= \zeta Z(\zeta) \end{aligned} \qquad (1.92)$$

$$Z(\zeta) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} \qquad (1.93)$$

It is readily checked using (1.92) and (1.93) that indeed $X(\zeta)$ is analytic when $\zeta \neq 0$ and $Y(\zeta)$ is analytic when $\zeta \neq \infty$; therefore the transition matrix (1.88) does indeed lead to a translation-invariant gauge field.

We should now like to reverse the procedure which led to $g(\theta)$ and recover from this the original field $T_\mu(x)$. However, the usual ansätze may only be used if the patching matrix can be reduced to the triangular form (1.18). Unfortunately, this is not possible for the general form of (1.88) - The function $a(\zeta)b(\zeta)$ possesses branch cuts which cannot be removed by our equivalence transformations (1.16). In order to recover the gauge field $T_{\mu}(x)$ in this case it would be necessary to solve the Riemann-Hilbert problem for matrix-valued functions with branches.

However, there is a special case for which the standard ansätze are sufficient, namely the case of axially symmetric monopoles. As we have already mentioned, the gauge field (1.83) becomes axially symmetric when $k = 0$, $p = q = \pi$, and the elliptic functions then reduce to the familiar circular ones

$$T_1(z) = -\frac{i}{2}\sigma_1\frac{\pi}{\cos\pi z} \qquad T_2(z) = -\frac{i}{2}\sigma_2\frac{\pi}{\cos\pi z}$$

$$T_3(z) = -\frac{i}{2}\sigma_3\frac{\pi\sin\pi z}{\cos\pi z} \tag{1.94}$$

The transition matrix (1.88) simplifies greatly since in this case ab is just the square of a rational function, and the result is just

$$g(\theta) = \begin{pmatrix} \cos\pi(\mu+\nu)/2 & \zeta\sin\pi(\mu+\nu)/2 \\ -\zeta^{-1}\sin\pi(\mu+\nu)/2 & \cos\pi(\mu+\nu)/2 \end{pmatrix} \tag{1.95}$$

The reduction of this matrix to upper-triangular form proceeds in a manner similar to (1.79); the first two stages consist in multiplying $g(\theta)$ by the two matrices

$$\begin{pmatrix} 1 & 0 \\ -i\zeta^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -i\zeta \\ 0 & 1 \end{pmatrix} \tag{1.96}$$

from the left and right respectively. The triangular form of the patching matrix is

$$g(\theta) = \begin{pmatrix} \zeta e^{i\pi(\mu+\nu)/2} & \cos \pi(\mu+\nu)/2 \\ 0 & \zeta^{-1} e^{-i\pi(\mu+\nu)/2} \end{pmatrix} \tag{1.97}$$

In order to reduce this to the form required for application of the first ansatz, we must remove the exponential factors which multiply the diagonal elements. This is possible because $e^{i\pi\nu/2}$ and $e^{i\pi\mu/2}$ are analytic in U_1 and U_2 respectively so that the further multiplication of (1.97) by

$$\begin{pmatrix} e^{-i\pi\nu/2} & 0 \\ 0 & e^{i\pi\nu/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{-i\pi\mu/2} & 0 \\ 0 & e^{i\pi\mu/2} \end{pmatrix}$$

from left and right is an equivalence transformation in the sense (1.16). After this transformation the patching matrix is left in the canonical form

$$g(\theta) = \begin{pmatrix} \zeta & \frac{1}{2}(e^{i\pi\mu} + e^{-i\pi\nu}) \\ 0 & \zeta^{-1} \end{pmatrix} \tag{1.99}$$

It is now possible to apply the first ansatz (1.80) with the substitution

$$\Delta_0(x) = e^{-\pi x^3}\cos \pi x^0 \tag{1.100}$$

This results in the selfdual gauge field

$$\begin{aligned}
T_0(x) &= \tfrac{i}{2}\pi\sigma_3 \\
T_1(x) &= \tfrac{i}{2}\pi(\sigma_2 - \sigma_1 \tan \pi x^0) \\
T_2(x) &= -\tfrac{i}{2}\pi(\sigma_1 + \sigma_2 \tan \pi x^0) \\
T_3(x) &= -\tfrac{i}{2}\pi\sigma_3 \tan \pi x^0
\end{aligned} \tag{1.101}$$

Unlike the previous example (1.75) this field does <u>not</u> satisfy the gauge condition $T_0 = 0$. In order to rectify this we must apply a gauge transformation

$$T_\mu \longrightarrow \gamma T_\mu \gamma^{-1} + \gamma \partial_\mu \gamma^{-1}$$

in which the group element $\gamma(x)$ obeys the differential equation $\partial_0 \gamma = \gamma T_0$. An appropriate choice might be

$$\gamma(x) = \exp -\tfrac{i}{2}\sigma_3\pi(\tfrac{1}{2} - x^0) \tag{1.102}$$

which represents a rotation through an angle $\pi(1/2 - x^0)$ in the (σ_1,σ_2)-plane. After this transformation we recover precisely the original field (1.94).

As our final example we shall consider the fundamental (single pole) solution of Nahm's equation for k x k matrices

$$T_i(z) = \frac{iJ_i}{z} \tag{1.103}$$

The three hermitian matrices J_i must constitute a representation of the angular momentum algebra so(3); we shall suppose that this is the irreducible representation of spin j, where $k = 2j+1$.

Taking boundary conditions on T at the point $z = \lambda = 1$ and using the equation (1.62) we obtain

$$\tau(1,\zeta) = i\underline{y}(\zeta).\underline{J} \tag{1.104}$$

Because $\underline{y}(\zeta)$ is a null vector, it is possible to choose co-ordinates in which $\underline{y}(\zeta) = c(1,i,0)$ for some scalar constant c; in this frame we can readily see that $\tau(1,\zeta)$ is proportional to the raising operator J_+, and is therefore nilpotent. The exponential series in (1.56) is therefore in fact a finite power series.

In order to evalate the expression (1.56) in this case we will use a neat little trick to eliminate the spectral parameter ζ. Define the k x k matrix M by

$$M = \exp(J_3 \ln \zeta) \tag{1.106}$$

which may be done in <u>any</u> representation of SU(2). This represents a rotation about the 3-axis through the complex angle $i \ln \zeta$. In the fundamental representation the generators of the Lie algebra are given by

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.107}$$

and hence, in the fundamental representation, our matrix M takes the particularly simple form

$$M = \begin{pmatrix} \zeta^{1/2} & 0 \\ 0 & \zeta^{-1/2} \end{pmatrix} \tag{1.108}$$

In the spin-1/2 representation it is trivial to verify the following relationships between M and the generators of the rotation group:

$$\zeta^{-1}J_- = MJ_-M^{-1}, \qquad \zeta J_+ = MJ_+M^{-1}$$
$$J_3 = MJ_3M^{-1} \tag{1.109}$$

However, these relationships tell us about the structure of the angular momentum group and its associated algebra; although we have chosen to prove them in a specially simple representation, in fact they hold in every representation of SU(2). They demonstrate that in order to multiply the raising and lowering operators by ζ and ζ^{-1} we need only conjugate by the matrix M.

Now we can use the above properties of M to simplify the exponential to be evaluated in (1.56). If we abbreviate our notation with the definition $s = (\mu+\nu-2)/2$, then the expression for the patching matrix is

$$
\begin{aligned}
g(1,\theta) &= \exp -\tau(1,\zeta).2s \\
&= \exp s(\zeta^{-1}J_- - \zeta J_+ + 2J_3) \\
&= M \exp s(J_- - J_+ + 2J_3)\, M^{-1} \qquad (1.110)
\end{aligned}
$$

To simplify the remaining exponential we again resort to the technique of proving an identity in the fundamental representation which can then be applied to an arbitrary representaion. In this way we can obtain the group multiplication law

$$
\begin{aligned}
\exp s(J_- - J_+ + 2J_3) &= \exp J_- \exp -(1+s)J_+ \\
&\quad . \exp i\pi J_2 \exp -J_+ \qquad (1.111)
\end{aligned}
$$

The matrix $C = \exp i\pi J_2$ is the so-called charge conjugation matrix for SU(2) which reverses the "charge" (or angular momentum) of any state, in other words $C^{-1}J_3C = -J_3$. Substituting the relation (1.111) into (1.110) and using (1.109), we obtain

$$
g(1,\theta) = \exp \zeta^{-1}J_- \exp -(1+s)\zeta J_+ \; MCM^{-1} \exp -\zeta J_+ \qquad (1.112)
$$

It is readily seen that the outer pair of exponentials in this expression are simply equivalence transformations on the patching matrix in the sense of (1.16); furthermore, it follows from the properties of the charge conjugation matrix that $C^{-1}MC = M^{-1}$. So an equivalent patching matrix for the fundamental solution of Nahm's equation is

$$g(1,\theta) = \exp -(1+s)\zeta J_+ \; M^2$$

$$= \exp - \frac{\mu+\nu}{2}\zeta J_+ \; \exp 2J_3 \ln\zeta \qquad (1.13)$$

This matrix is closely analogous to (1.79); it is upper-triangular, and the diagonal elements are simply the integral powers of ζ from ζ^{2j} down to ζ^{-2j}.

Chapter 2. The ADHM Construction and Reciprocity

In the previous chapter a description of the ADHM construction was given for instantons and for monopoles. The purpose of the present chapter is to show that this construction is complete, that is, that <u>all</u> instanton and monopole fields can be derived within it. We shall find that there is a reciprocal relationship between the monopole gauge field $A_\mu(x)$ and its associated one-dimensional partner $T_\mu(z)$, in that each is constructed from the other by solving a covariant Dirac (Weyl) equation using the other as background field, and then taking matrix elements of a suitable operator between the resulting solutions.

There are several versions of this completeness proof, but all start from the observation that in the background field of an instanton or monopole there are finitely many normalizable solutions to the covariant Weyl equation

$$D_{AB'}\psi^A = 0 \qquad , \qquad \text{or} \tag{2.1}$$

$$D^+\psi = 0$$

In fact, using the index theorem it can be shown that an instanton of topological charge k admits exactly k such solutions for which the normalization integral is finite:

$$\int \psi^+\psi \, d^4x < \infty$$

It is possible to arrange these solutions as the columns of a

$2n \times k$ matrix and to normalize this matrix so that

$$\int \psi^{+}\psi \, d^4x = 1_k \tag{2.2}$$

The most straightforward approach to the proof of completeness for the ADHM construction is that developed in [19] where the matrix $\Delta(x)$ is constructed directly using the matrix elements

$$a_{\mu} = -\int \psi^{+}(x) x_{\mu} \psi(x) \, d^4x \tag{2.3}$$

The ADHM equation $\Delta^{+}v = 0$ is then solved, and a new field $A'_{\mu}(x)$ derived using (1.24). By comparing the Green's functions of the covariant Laplacian operators D^2 and D'^2 it is then possible to show that this new field is gauge-equivalent to the original one.

The approach adopted here [25], although technically more difficult than that outlined above, has several advantages: firstly, instead of dealing with the matrix representatives of the linear maps a,b, we shall focus our attention on a co-ordinate-free description of the vector spaces V and W described in the preceding chapter, realizing these as solution spaces of partial differential equations. Secondly, we shall see how these spaces are related to the holomorphic bundle of covariant constants over the twistor space of antiselfdual null planes used in the Atiyah-Ward construction. We can thus uncover the connection between the two major approaches to the instanton/monopole problem.

1. Covariant Differential Equations

The proof of completeness for the ADHM method starts with a general selfdual connection D. By considering certain systems of covariant differential equations using this connection a sequence of three complex vector spaces U, V and W is constructed, and to each twistor θ is associated a linear mapping $A(\theta) : U \longrightarrow V$ and a similar mapping $B(\theta) : V \longrightarrow W$. Diagramattically,

$$U \xrightarrow{A(\theta)} V \xrightarrow{B(\theta)} W \qquad (2.4)$$

These mappings have the fundamental property

$$\begin{aligned} &B(\theta)A(\theta) = 0 \quad \text{for all } \theta \\ &\operatorname{im} A(\theta) \subset \ker B(\theta) \end{aligned} \qquad (2.5)$$

We shall show that the quotient space

$$E(\theta) = \ker B(\theta)/\operatorname{im} A(\theta) \qquad (2.6)$$

is naturally identified with the fibre of covariant constants on the null plane θ. If we denote by im A and ker B the vector bundles whose fibres at θ are im $A(\theta)$ and ker $B(\theta)$ respectively, then the holomorphic bundle of the Atiyah-Ward construction is isomorphic to

$$E = \ker B/\operatorname{im} A \qquad (2.7)$$

The covariant differential equations which give rise to the solution spaces U, V and W are most readily expressed in spinor form; therefore we shall briefly summarize the notation which will be employed.

The Lorentz group SL(2,C) has two inequivalent representations of lowest dimension which are complex conjugates of one another and act on two-component spinors v^A and $w^{B'}$ respectively. Because the representation matrices have unit determinant there are invariant skew-symmetric spinors ε_{AC} and $\varepsilon_{B'D'}$ which may be used to raise and lower indices according to the conventions

$$v_C = v^A \varepsilon_{AC} \quad , \quad v^A = \varepsilon^{AC} v_C$$
$$\varepsilon^{AE}\varepsilon_{CE} = \delta^A{}_C \tag{2.8}$$

The same conventions will hold for the corresponding dotted spinors. The transition between vector and spinor quantities is accomplished using the following matrices, which represent the quaternions used previously:

$$e_\mu^{AB'} = (1\ ,\ -i\sigma_i)$$
$$e_{\mu AB'} = (1\ ,\ -i\varepsilon^{-1}\sigma_i\varepsilon) = \overline{e^{AB'}} \tag{2.9}$$

These transition matrices satisfy the following orthogonality relations, used extensively in the sequel:

$$e_{\mu AB'} e_{\nu}^{AB'} = 2\delta_{\mu\nu}$$

$$e_{\mu AB'} e_{\mu}^{CD'} = 2\delta_{A}{}^{C}\delta_{B'}{}^{D'} \qquad (2.10)$$

if x_{μ} is a vector, we can define an associated spinor $x^{AB'}$ by

$$x^{AB'} = x_{\mu} e_{\mu}^{AB'} \qquad (2.11)$$

The connection D_{μ} may likewise be written in spinor form

$$D^{AB'} = D_{\mu} e_{\mu}^{AB'} \qquad (2.12)$$

The spaces U,V and W and the mappings $A(\theta)$ and $B(\theta)$ can <u>only</u> be constructed when the connection D_{μ} is selfdual. The condition of selfduality may be expressed in spinor form using the identity

$$e_{\mu AB'} e_{\nu}^{AD'} = (\delta_{\mu\nu} 1 + i\eta^{-}_{\mu\nu})_{B'}{}^{D'} \qquad (2.13)$$

where $\eta^{-}_{\mu\nu}$ is the antiselfdual tensor defined previously. If the curvature tensor $F_{\mu\nu}$ is selfdual, therefore,

$$\begin{aligned} D_{AB'} D^{AD'} &= D_{\mu} D_{\mu} \delta_{B'}{}^{D'} + \tfrac{1}{2} i F_{\mu\nu} (\eta^{-}_{\mu\nu})_{B'}{}^{D'} \\ &= D_{\mu} D_{\mu} \delta_{B'}{}^{D'} \end{aligned} \qquad (2.14)$$

This equation is used frequently in verifying the consistency of the sets of equations which define the spaces U,V,W.

The third space W in our sequence is simple to describe:

it is just the space of solutions of the covariant Weyl equation mentioned previously (2.1)

$$D_{AB'}\psi^A = 0 \tag{2.15}$$

We have already seen that this equation has only finitely many solutions in an instanton field if the normalization integral (2.2) is required to be finite. For monopoles the necessary modification is fairly obvious: we impose a condition of x^0-invariance

$$\psi(x^0,\underline{x}) = \exp(izx^0)\psi(0,\underline{x}) \tag{2.16}$$

and require finiteness of a 3-dimensional normalization integral, namely

$$\int \psi^{+}\psi \, d^3x = 1 \tag{2.17}$$

We have already noted that if instanton boundary conditions are imposed then the Weyl equation has just k solutions; if however we impose monopole boundary conditions as described above we find that the number of solutions depends on the 'frequency parameter' z. In fact, there are just k(z) solutions for any given value of z, where k(z) is the integer-valued function defined in (1.34). The number of solutions changes when z passes through one of the so-called 'jumping points' z_i (see discussion after (1.33)).

The second space in our sequence is rather more complex to describe: the fields upon which the differential equations are imposed are a scalar ϕ, a second rank spinor $\Omega^A{}_C$, and a mixed spinor $\Omega^{AB'}$. In order to abbreviate our notation somewhat we note that the direct sum of the dotted and undotted spinor representations is by definition the so-called Dirac spinor representation; the components of a Dirac spinor v are denoted v^α, where the index α runs over the standard spinor values A = 1,2 and the dotted values B' =1',2' . The inner product of two Dirac spinors is defined as

$$v_\alpha w^\alpha = v_A w^A + v^{B'} w_{B'},$$

The equations satisfied by the fields ϕ and Ω defining the space V may now be written

$$D^{AB'}\phi = \Omega^A{}_C x^{CB'} + \Omega^{AB'} = \Omega^A{}_\gamma x^{\gamma B'} \qquad (2.18)$$

$$D_{AB'}\Omega^A{}_\gamma = 0 \qquad (2.19)$$

where $x^{AB'}$ is the spinor representative of x (2.11) and $x_{D'}{}^{B'} = \delta_{D'}{}^{B'}$. The boundary condition on the spinor Ω is similar to that used on Ψ, namely that the integral of Ω^2 should be finite, the integral being taken over the entire 4-dimensional space in the case of instantons but over a 3-dimensional spacelike slice in the case of monopoles. In the latter case, this field should satisfy a condition of translational invariance analogous to (2.16). The boundary condition on the scalar ϕ is simply that it should be covariantly constant 'at infinity', and in the case of monopoles it should also be independent of x^0.

The equations (2.18) and (2.19) are consistent only if the background field A_μ is selfdual. This follows by differentiating (2.18) and using the fundamental relation (2.14) together with (2.19):

$$D_{AD'}D^{AB'}\phi = (D^2\delta_{D'}{}^{B'} + \tfrac{1}{2}iF_{\mu\nu}(\eta^-_{\mu\nu})_{D'}{}^{B'})\phi$$

$$= 2\Omega^A{}_A\delta_{D'}{}^{B'} \qquad (2.20)$$

It is clear that this equation is consistent only if the curvature term vanishes, i.e. if F is selfdual. In this case we have the simpler equation

$$D^2\phi = 2\Omega^A{}_A \qquad (2.21)$$

This may be used to deduce the dimension of the space V, that is, the number of linearly independent solutions of (2.18-2.19) which satisfy the appropriate boundary conditions. The space V may be represented by the fields ϕ and $\Omega^A{}_C$ alone, where ϕ satisfies (2.21) and $\Omega^A{}_C$ satisfies the covariant Weyl equation

$$D_{AB'}\Omega^A{}_C = 0 \qquad (2.22)$$

For given such fields we may define the remaining components $\Omega^{AB'}$ using (2.18), and then use (2.21-2.22) to verify that these components too satisfy the Weyl equation (2.19).

Now, since (2.22) represents a <u>pair</u> of solutions to the Weyl equation (2.1), namely $\Omega^A{}_1$ and $\Omega^A{}_2$, there are 2k solutions in the case of instantons, 2k(z) in the case of monopoles. On the other hand, the solution of (2.21) is defined only up to addition of a solution of the Laplace equation, covariantly constant at infinity

$$D^2\sigma = 0 \tag{2.23}$$

In an instanton field there are just n independent functions of this kind; Thus we have proved the assertion made earlier that in this case the dimension of the space V is n+2k.

However, in a monopole field which behaves like (1.33) there are fewer covariant constants at infinity. The essential difference between the two cases is that the curvature of a monopole field behaves like r^{-2} whereas that of an instanton field decreases more rapidly like r^{-3}. To estimate the change in a vector caused by parallel transport round a closed curve, the curvature is integrated over a 2-surface spanning the circuit. As this surface is magnified towards 'infinity', its area increases in proportion to r^2, and thus it may be seen that the effect of curvature remains finite for monopoles but tends to zero for instantons. However, there may remain vectors in the fundamental representation which are annihilated by the asymptotic curvature, in other words 'neutral' modes which do not interact with the asymptotic gauge field. These correspond to the indices $\alpha \in I$ mentioned after (1.33); they provide the covariantly constant boundary values for the solutions of the Laplace equation (2.23) as $r \longrightarrow \infty$. These solutions account for the extra components v^{α} needed to describe the ADHM vector space in Nahm's extension for monopoles (1.35),(1.37).

We see therefore that the differences between the original ADHM method (for instantons) and Nahm's construction (for monopoles) are largely due to the boundary conditions imposed on the fields. The actual equations which define the spaces U,V,W are the same!

The first space U may, like V, be described in two equivalent ways: The simplest representation is to consider it as the solution space of

$$D^{AB'} D^2 \lambda_A = 0 \tag{2.24}$$

where the Weyl spinor $D^2\lambda_A$ satisfies the same condition of normalization as ψ^A. The solutions of (2.24) may therefore be obtained by applying the inverse Laplacian operator

$$G = (D^2)^{-1}$$

to those of (2.15), and thus the dimensions of U and W are equal.

An alternative parametrization for U is constructed by introducing the third-rank spinors

$$P^A{}_{BC} = -\tfrac{1}{2} D^2 \lambda^A \varepsilon_{BC}$$

$$P^{A}{}_{C}{}^{B'} = D^{AB'}\lambda_{C} - P^{A}{}_{CE}x^{EB'}$$

$$P^{AB'}{}_{C} = - P^{A}{}_{C}{}^{B'} \qquad (2.25)$$

$$P^{AD'B'} = D^{AB'}\lambda^{D'} - P^{AD'}{}_{C}x^{CB'}$$

where the dotted spinor $\lambda^{B'}$ is related to its undotted namesake by

$$\lambda^{B'} = - \lambda_{A}x^{AB'} \qquad (2.26)$$

As usual, this complex set of equations may be much simplified by using four-component spinor indices; the purpose of the above definitions is to ensure that the fields λ_{γ} and $P^{A}{}_{\gamma\varepsilon}$ satisfy the system of equations

$$\lambda_{\gamma}x^{\gamma B'} = 0 \qquad (2.27)$$

$$D^{AB'}\lambda_{\gamma} = P^{A}{}_{\gamma\varepsilon}x^{\varepsilon B'} \qquad (2.28)$$

$$P^{A}{}_{\gamma\varepsilon} = - P^{A}{}_{\varepsilon\gamma} \qquad (2.29)$$

$$D_{AB'}P^{A}{}_{\gamma\varepsilon} = 0 \qquad (2.30)$$

which show a striking similarity to (2.18-2.19). In proving that these equations follow from (2.24-2.26) it is again necessary to use the identity (2.14) and the selfduality of the background field.

2. The Mappings and the Twistor Bundle

We are now in a position to describe the natural linear mappings $A(\theta)$ and $B(\theta)$ which exist between the solution spaces U,V and W described in the last section. We saw in section 1.2 that a twistor, or antiselfdual null plane, could be specified by a pair of two-component spinors ω^A and $\pi_{B'}$, such that the equation of the plane is

$$x^{AB'}\pi_{B'} = \omega^A$$

If we amalgamate these into a single four-component quantity θ^α, we can extend this equation to give

$$\theta^\alpha = x^{\alpha B'}\pi_{B'}, \tag{2.31}$$

using the additional components of x which were defined after (2.19).

Now we can see why the four-component representation of the equations (2.18-2.19) and (2.27-2.30) is so useful - it enables us to contract a field with the twistor θ^α giving a field with fewer indices. For example, if the pair (λ, P) is a solution of (2.27-2.30), i.e. an element of the space U, then we may form the contracted fields

$$\begin{aligned} \phi &= \lambda_\gamma \theta^\gamma \\ \Omega^A{}_\varepsilon &= P^A{}_{\gamma\varepsilon}\theta^\gamma \end{aligned} \tag{2.32}$$

It is now trivial to verify that these do indeed satisfy the the equations of V, namely (2.18-2.19). Given any other solution of the latter equations there exists a further contraction, namely

$$\psi^A = \Omega^A{}_\gamma \theta^\gamma \tag{2.33}$$

and the resulting field now satisfies the Weyl equation (2.15) which defines the space W.

Given any twistor θ, therefore, the transformations (2.32) and (2.33) may be used to define two linear mappings $A(\theta) : U \longrightarrow V$ and $B(\theta) : V \longrightarrow W$. It is at this point that the antisymmetry property (2.29) comes into its own; for if we apply both A and B successively using the <u>same</u> twistor θ as argument, then we find that the final result is

$$\begin{aligned}\psi^A &= \Omega^A{}_\varepsilon \theta^\varepsilon = P^A{}_{\gamma\varepsilon}\theta^\gamma\theta^\varepsilon \\ &= 0\end{aligned} \tag{2.34}$$

In other words, we have as promised the relation (2.5)

$$B(\theta)A(\theta) = 0$$

$$\operatorname{im} A(\theta) \subset \ker B(\theta)$$

The image of A and the kernel of B may be used to construct the holomorphic vector bundle of the Atiyah-Ward method using (2.6). In order to verify this we need several lemmas, full proofs of which are given in [25]. In the following outlines we use the notation

$$\rho^A = \omega^A - x^{AB'}\pi_{B'},$$
$$d_\pi^A = D^{AB'}\pi_{B'},\tag{2.35}$$

Therefore $\rho^A = 0$ is the equation of the null plane , and since $d_\pi^A \rho^C = 0$ it follows that the two directional derivatives d_π^A are the tangent vectors to this plane. As we have remarked previously, these operators commute with one another by virtue of the selfduality of the gauge field.

<u>Lemma 1:</u> The mapping $A(\theta)$ is surjective for each θ.

[Proof:] Let the spinor ψ^A be given, satisfying the Weyl equation (2.15), and consider the equation

$$d_\pi^A \phi = \psi^A \qquad \text{on the plane } \theta \tag{2.36}$$

This equation is integrable precisely because of the Weyl equation satisfied by ψ^A. If an arbitrary scalar field is chosen subject to this requirement, then

$$\psi^A = d_\pi^A \phi + \Omega^A{}_C \rho^C \tag{2.37}$$

for some smooth spinor field $\Omega^A{}_C$. The fields ϕ and Ω are not unique - ϕ is only determined on the plane θ, so is subject to transformations

$$\phi \longrightarrow \phi + \phi_C \rho^C \tag{2.38}$$

and similarly is subject to the transformations

$$\Omega^{A}{}_{C} \longrightarrow \Omega^{A}{}_{C} - d^{A}\phi_{C} + \Omega^{A}\rho_{C} \qquad (2.39)$$

It is possible to use these ambiguities to choose the two fields ϕ and Ω to satisfy the equations (2.21-2.22). When this has been done, the equation (2.37) leads to (2.33); in other words ψ^{A} is the image of the pair $(\phi, \Omega^{A}{}_{\gamma})$ under the mapping $B(\theta)$. Since a preimage may be found for any solution of (2.15), it follows that $B(\theta)$ is indeed surjective.

<u>Corollary.</u> If $\hat{\phi}$ is any covariant constant defined upon the null plane θ, then there exists a pair $(\phi, \Omega^{A}{}_{\gamma}) \in V$ such that $\hat{\phi}$ is the restriction of ϕ to the plane θ; Furthermore this pair lies in the kernel of $B(\theta)$.

[Proof:] Choose any extension $\hat{\phi}$ of $\hat{\phi}$ to R^4; Because this function is covariantly constant on θ we must have

$$0 = d^{A}_{\pi}\phi + \Omega^{A}{}_{C}\rho^{C} \qquad (2.40)$$

for some smooth spinor field $\Omega^{A}{}_{C}$. As in the main Lemma, we can use the remaining ambiguity in ϕ and Ω to choose these fields in such a way that they satisfy the equations (2.21-2.22). It is evident from (2.40) that $\psi^{A} = 0$, i.e. the pair $(\phi, \Omega^{A}{}_{\gamma})$ does indeed lie in the kernel of $B(\theta)$.

<u>Lemma 2.</u> Suppose the pair $(\phi, \Omega^{A}{}_{\gamma})$ lies in the kernel of $B(\theta)$; then it also lies in the image of $A(\theta)$ if and only if the scalar field ϕ vanishes on the null plane θ.

[Proof:] The 'only if' part of the proof is a matter of straightforward calculation. The 'if' part employs arguments similar to those used in Lemma 1. If ϕ vanishes on the plane θ then there is a smooth spinor field λ_A such that

$$\phi = \lambda_C \rho^C \tag{2.41}$$

By differentiating this equation and using the equations of the space V we can show that

$$\Omega^A{}_C = -d^A \lambda_C + T^A \rho_C \tag{2.42}$$

where T^A is another smooth spinor field. By using the remaining freedom in the choice of λ and T we may arrange that

$$\begin{aligned} D^2 \lambda_C &= -2T_C \\ D_{AB'} T^A &= 0 \end{aligned} \tag{2.43}$$

which of course are equivalent to the single equation (2.24) which defines the space U. Now a little further manipulation of (2.41-2.42) gives the relations (2.32), indicating that the pair $(\phi, \Omega^A{}_\gamma)$ is indeed the image of $(\lambda_A, P^A{}_{\gamma\epsilon})$ under the mapping $A(\theta)$.

Lemma 3. The kernel of $A(\theta)$ is isomorphic to the space of solutions of the covariant Laplace equation $D^2\sigma = 0$ subject to homogeneous boundary conditions at infinity.

[Proof:] Suppose the pair $(\lambda_A, P^A{}_{\gamma\epsilon})$ lies in the kernel of $A(\theta)$. It then follows that

$$\lambda_C \rho^C = 0 \tag{2.44}$$

and so there exists a scalar field, which we shall denote by σ, such that

$$\lambda_C = \sigma \rho_C \tag{2.45}$$

It is this scalar which may be shown [25] to satisfy the Laplace equation - it is plain that if the spinor λ^A is to satisfy the appropriate boundary conditions, the scalar must tend to zero at infinity. This proves homomorphism in one direction; to establish the other half of the isomorphism we just start with the field λ^A defined by (2.45), where σ is an arbitrary harmonic function, and verify by direct calculation that this field (i) satisfies the equations which define U, and (ii) lies in the kernel of $A(\theta)$.

Of course, in any physically reasonable gauge field, whether due to an instanton or to a monopole, the covariant Laplace equation has only one solution which satisfies homogeneous boundary conditions - the trivial zero solution. It follows that the sequence of spaces and mappings

$$U \xrightarrow[A(\theta)]{} V \xrightarrow[B(\theta)]{} W \tag{2.46}$$

has a very simple structure: A is injective, B is surjective, and the image of A is wholly contained by the kernel of B. If the pair $(\phi, \Omega^A{}_\gamma)$ lies in the image of $A(\theta)$, then the scalar field vanishes on the null plane θ. On the other hand, if (ϕ, Ω^A_γ) lies in the kernel of $B(\theta)$, then this scalar field is covariantly constant on θ. The corollary to Lemma 1 implies that any covariant constant on θ may be derived by restriction from the scalar field of a pair in the kernel of $B(\theta)$; of course, the pair is not uniquely determined - but if (ϕ, Ω) and (ϕ', Ω') are any two elements of ker $B(\theta)$ then according to Lemma 2,

$$\phi|\theta = \phi'|\theta'$$

if and only if

$$(\phi-\phi' , \Omega-\Omega') \in \text{im } A(\theta)$$

It follows that there is a natural isomorphism between the vector space of covariant constants on the plane θ, which we will call $E(\theta)$, and the quotient space which is constructed from ker $B(\theta)$ and its subspace im $A(\theta)$

$$E(\theta) = \ker B(\theta)/\text{im } A(\theta) \qquad (2.47)$$

As θ varies throughout the whole twistor space CP^3, this quotient space varies, describing the same holomorphic vector bundle which is used in the Atiyah-Ward approach to the solution of the selfduality equations.

To establish the explicit formula (1.24) or (1.36) for the gauge field a little more work must be done. Firstly, this formula involves an inner product on the space V, which has not yet been defined. Secondly, as one might expect from their equal dimensions, the spaces U and W are related - W is naturally anti-isomorphic with the dual space U^*, in other words there is a natural hermitian form on U x W.

In order to construct these hermitian forms, we view the fields $\psi^A(x)$, $\phi(x)$, $\Omega^A{}_\gamma(x)$, $\lambda_\gamma(x)$, $P^A{}_{\gamma\varepsilon}(x)$, not as particular solutions of the field equations, but as linear functions on the spaces U,V,and W. These may be represented numerically by using a linearly independent set of solutions as columns in a single matrix. Since the fields ψ^A satisfy the Weyl equation, they may be expressed as linear combinations of the ψ^A. Similarly, since the fields λ_γ and $P^A{}_{\gamma\varepsilon}$ satisfy the same equations as the fields of V (with an extra free index γ), they may be expressed as linear combinations of these fields

$$\Omega^A{}_\gamma = \psi^A B_\gamma \tag{2.48}$$

$$\lambda_\gamma = \phi A_\gamma \qquad\qquad P^A{}_{\gamma\varepsilon} = \Omega^A{}_\varepsilon A_\gamma \tag{2.49}$$

where A_γ : U ---> V and B_γ : V ---> W . It is not hard to verify that these mappings are the components of the $A(\Theta)$ and $B(\Theta)$, which are linear functions of the argument :

$$A(\Theta) = A_\gamma \Theta^\gamma \qquad\qquad B(\Theta) = B_\gamma \Theta^\gamma \tag{2.50}$$

Multiplying by the spinor co-ordinates $x^{\gamma B'}$, we then obtain two linear mappings, which also depend linearly upon position x:

$$\Delta^{B'}(x) = A_{\gamma} x^{\gamma B'} \qquad \tilde{\Delta}^{B'}(x) = B_{\gamma} x^{\gamma B'} \tag{2.51}$$
$$U \longrightarrow V \qquad\qquad V \longrightarrow W$$

By virtue of the fundamental relation (2.5), there exists a function $c(x) : U \longrightarrow W$ such that

$$\tilde{\Delta}^{B'}(x)\Delta^{D'}(x) = c(x)\varepsilon^{B'D'} \tag{2.52}$$

Let $F(x) : W \longrightarrow W^*$ be any inner product on W, and define the mappings

$$\begin{aligned} I &= \phi^+\phi + \tilde{\Delta}^{B'+}F\tilde{\Delta}^{B'} &&: V \longrightarrow V^* \\ L &= Fc &&: U \longrightarrow W^* \end{aligned} \tag{2.53}$$

The mapping $I(x)$ is positive definite hermitian, and represents an inner product on V; the mapping $L(x)$ is nonsingular, and is equivalent to a hermitian form on $U \times W$. Then it may be shown that the gauge connection must be given by the formula

$$A_{\mu} = \phi I^{-1}\partial_{\mu}\phi^+ \tag{2.54}$$

The inner products I and L will in general depend on the position x; however, if $F(x)$ is chosen to satisfy the Poisson equation

$$\partial^2 F(x) = -\psi^{A+}\psi^A \tag{2.55}$$

it is possible to show [25] that although the individual fields in (2.53) depend on x, the inner products I and L are actually constant. By choosing bases such that $I = L = 1$ we obtain from (2.54) a more familiar form of the ADHM formula (1.24) for the connection. The spaces U and W^* are now naturally identified, and the mappings $\Delta^{B'}(x)$ and $\tilde{\Delta}^{B'}(x)$ are adjoint to one another. The former mapping is that which appears in (1.27), and because the scalar field ϕ vanishes on the null plane θ whenever $(\phi,\Omega) \in \ker A(\theta)$ we have

$$\phi(x)\Delta^{B'}(x) = 0 \tag{2.56}$$

Thus this scalar field is recognized as identical with the adjoint of the mapping v(x) which appears in the ADHM construction: $\phi = v^+$.

In this chapter we have outlined a completeness proof, using partial differential equations, of the ADHM construction for selfdual gauge fields. In the conclusion to the paper [25] in which this method was first described, Osborn speculates that a similar approach may prove useful in the case of nonselfdual Yang-Mills equations. We shall investigate this possibility in the next chapter.

Chapter 3. The Nonselfdual Nahm Equation

The theorem of Taubes described in section 0.5 demonstrates that a simple theory which admits selfdual monopoles will also allow nonselfdual solutions of the field equations. So far we have concentrated our attention on selfdual gauge fields, since the selfduality equations are known to be completely integrable; as we have seen several methods of solution are available. The question which we would like to investigate is to what extent similar progress can be made towards solution of the full second-order field equations.

$$D_\mu F_{\mu\nu} = 0 \tag{3.1}$$

It must be said beforehand that there is no reason to suppose that the second-order Yang-Mills equations are completely integrable. A certain amount of success was achieved by Witten [26] and by Isenberg, Yasskin and Green [27] in generalizing the Atiyah-Ward construction to deal with nonselfdual gauge fields, but their method does not lead to explicit expressions for the gauge potential. It is therefore tempting to ask whether a nonselfdual parallel also exists for the ADHM method, and if so whether it may supply a direct way of evaluating A_μ.

The most sound way of approaching this question would be to search for systems of differential equations analogous to those investigated in the previous chapter - The condition for such systems to be integrable should be that the Yang-

Mills equation (3.1) is satisfied by the background gauge field, so that the very existence of these spaces would imply (3.1). We would then look for natural linear mappings between these spaces which might characterize the connection in the same way that the mappings $\Delta(x)$ and $\Delta(x,z)$ correspond to the selfdual instanton and monopole fields. We might speculate that as in the selfdual case the relationship between the connection D and these linear mappings should be a reciprocal one, and that the nonselfdual Δ's would satisfy essentially the same Yang-Mills equations.

This is an ambitious programme, and may prove impossible to implement. However, we have been able to obtain several interesting results concerning the so-called 'nonselfdual Nahm equation' which is satisfied by selfdual gauge fields in four dimensions which depend upon only one co-ordinate. This is just the type of field which we would expect to obtain if it were indeed possible to apply an ADHM type transformation to a static 3-dimensional solution in a model with one adjoint Higgs and zero Higgs potential (BPS limit).

1. Generalizing the Atiyah-Ward and ADHM Methods

By applying the Penrose transform to a general Yang-Mills gauge field A_μ Witten [26] and Isenberg, Yasskin and Green [27] obtain a holomorphic vector bundle E(A) over the space of null lines L; the fibre at L contains all covariant constants on L. Let the space of all selfdual null planes be denoted T_α, and that of all antiselfdual null planes be denoted T_β (both isomorphic to CP^3). Then since each null line lies in the intersection of a selfdual and an antiselfdual null plane, and since these planes are uniquely determined by the line ,the space L may be viewed as a 5-dimensional submanifold of $T_\alpha \times T_\beta$. The condition which must be imposed on the bundle E(A) to ensure that the associated connection A satisfies the vacuum Yang-Mills equations is rather subtle, and concerns the extension of the bundle to $T_\alpha \times T_\beta$.

If the bundle E(A) is represented by an open covering U_i and a set of transition matrices g_{ij} defined on the intersections $U_i \cap U_j$, then it is clear that the following 'cocycle conditions' must be satisfied:

$$\begin{aligned} g_{ij}g_{ji} &= 1 \\ g_{ij}g_{jk} &= g_{ik} \end{aligned} \tag{3.2}$$

That is, a transformation from the co-ordinate frame used in U_k to that used in U_i should have the same effect whether it is performed directly or through the intermediate frame in U_j. The bundle E(A) may be extended to the whole of $T_\alpha \times T_\beta$

provided that there exist

(i) extensions (supersets) $\tilde{U}_i$ of the U_i, which provide a covering of $T_\alpha \times T_\beta$, and
(ii) holomorphic extensions of g_{ij} to $\tilde{U}_i \cap \tilde{U}_j$, which satisfy the cocycle conditions (3.2).

Unfortunately, these conditions are not satisfied even for a connection which satisfies the Yang-Mills equation. However, if we only demand that the cocycle conditions (3.2) be satisfied to $O(t^3)$ in the neighbourhood of L, where $t = 0$ is a local equation of the submanifold L, then an extension can be found if and only if the field equation (3.1) holds.

$$\begin{aligned} g_{ij}g_{ji} &= 1 + O(t^4) \\ g_{ij}g_{jk} &= g_{ik} + O(t^4) \end{aligned} \tag{3.3}$$

As stated in the introduction to this chapter, our ultimate aim is the generalization of the ADHM construction to nonselfdual connections, since this would hopefully provide a much more explicit description of the gauge field than that just described. We found it helpful to use Witten's indirect proof of the equivalence of the Yang-Mills equation and the third-order extension property (3.3). He noticed [26] that if y_μ, z_μ are co-ordinates in a (real) 8-dimensional space, and if $D^{(y)}_\mu$, $D^{(z)}_\mu$ are the corresponing covariant derivatives of an 8-D connection, then it is possible to produce a 4-D Yang-Mills field by dimensional reduction as follows: Let the 8-D connection be selfdual in the 'y' co-ordinates but antiselfdual in the 'z' co-ordinates; furthermore let the 'y' and 'z' covariant derivatives commute

$$
\begin{aligned}
[D^{(y)}_{\mu}, D^{(y)}_{\nu}] &= \text{selfdual} \\
[D^{(z)}_{\mu}, D^{(z)}_{\nu}] &= \text{antiselfdual} \\
[D^{(y)}_{\mu}, D^{(z)}_{\nu}] &= 0
\end{aligned}
\tag{3.4}
$$

Now make a change of variables to x_{μ} and w_{μ} defined by

$$
y_{\mu} = x_{\mu} + w_{\mu} \qquad\qquad z_{\mu} = x_{\mu} - w_{\mu} \tag{3.5}
$$

The 4-dimensional subspace $w_{\mu} = 0$ corresponds to the 'diagonal' $y_{\mu} = z_{\mu}$ in terms of the original co-ordinates - this will be identified with 'physical' 4-D space. The restriction of the connection (3.4) to the diagonal subspace satisfies the (second-order) Yang-Mills equation.

By no means all solutions of (3.1) may be constructed by this method - but the remarkable finding of Witten was that the 4-dimensional solutions of (3.1) are precisely those which admit an extension

$$
D^{(x)}_{\mu} = \partial^{(x)}_{\mu} + A_{\mu}(x,w) \qquad\qquad D^{(w)}_{\mu} = \partial^{(w)}_{\mu} + B_{\mu}(x,w) \tag{3.6}
$$

to R^8, where A and B are power series which satisfy (3.4) as far as the second order in w. Note that $A_{\mu}(x,0)$ is the Yang-Mills gauge field on R^4, but $B_{\mu}(x,w)$ has no physical interpretation since it refers to covariant differentiation in a direction orthogonal to physical space.

Our attempt to extend the ADHM construction to cover nonselfdual gauge fields therefore began in 8 dimensions, with the simplest examples available; those for which the gauge group could be factored $G = H \times K$, and for which the 'y' components of the connection lay in H and the 'z' components lay in K. This ensured the commutation of the two sets of derivatives, as in (3.4), in a rather trivial way. Given an H-instanton $A_\mu(x)$ and a K-anti-instanton $B_\mu(x)$ it is easy to construct the 8-dimensional gauge field which when restricted to R^4 gives their tensor product; it is

$$D^{(y)}_\mu = \partial^{(y)}_\mu + A_\mu(y) \qquad D^{(z)}_\mu = \partial^{(z)}_\mu + B_\mu(z) \tag{3.7}$$

We can then construct a rather large set of spinor fields in 8 dimensions which satisfy covariant linear equations akin to those described in the previous chapter. These equations were modelled on those satisfied by the <u>tensor products</u> of the ϕ, Ψ, Ω, λ, etc. for the instanton and the anti-instanton. For example, the scalar fields would combine to form

$$\phi(y,z) = \phi(y) \otimes \phi(z) \tag{3.8}$$

where the first factor is a solution of (2.18) in the H-gauge field and the second satisfies a similar, but spatially reflected, equation in the K-gauge field (which is an <u>anti</u>-instanton). Differentiating (3.8) with respect to y and z we see that we require two further spinor fields

$$\Omega_1(y,z) = \Omega(y) \otimes \phi(z) \qquad \Omega_2(y,z) = \phi(y) \otimes \Omega(z) \tag{3.9}$$

and so on. If the spaces U, V and W used in Chapter 2 are renamed U_{-1}, U_0 and U_1, and the corresponding spaces for an anti-instanton are called W_{-1}, W_0 and W_1, then the tensor product of the instanton and anti-instanton complexes contains nine components $V_{ij} = U_i \otimes W_j$. For each antiselfdual null plane or twistor θ in 'y'-space there exists a mapping $B(\theta) : V_i \longrightarrow V_{i+1}$ (previously called $A(\theta)$ or $B(\theta)$) and for each selfdual plane η in 'z'-space there is a similar $C(\eta) : W_i \longrightarrow W_{i+1}$. Upon the tensor product of the two complexes there acts the mapping

$$\begin{aligned} A(\theta,\eta) &= B(\theta) \otimes 1 + (-1)^j 1 \otimes C(\eta) \\ &: V_{ij} \longrightarrow V_{i+1,j} + V_{i,j+1} \end{aligned} \tag{3.10}$$

which, like that used in the construction of instantons and monopoles, satisfies the fundamental relation

$$A(\theta,\eta)^2 = 0 \tag{3.11}$$

We note here that the index $k=i+j$ is incremented by the mapping A; we therefore define the direct sums

$$V_{k.} = \bigoplus_{i+j=k} V_{ij} \tag{3.12}$$

so that $A(\theta,\eta) : V_k \longrightarrow V_{k+1}$. It is clear that k can take the values $-2,-1,0,+1,+2$ - hence there are five spaces in the sequence

$$V_{-2} \xrightarrow[A]{} V_{-1} \xrightarrow[A]{} V_0 \xrightarrow[A]{} V_1 \xrightarrow[A]{} V_2 \qquad (3.13)$$

The equations defining these spaces and the mappings between them can be defined <u>without explicit reference</u> to the original derivation using tensor products. It is easy to see that the consistency of the differential equations is equivalent to Witten's conditions (3.4); the equations in the 'y'-variables and those in the 'z'-variables may be treated almost independently because the covariant derivatives commute. It is further possible to prove that the sequence (3.13) is <u>exact</u> at the points V_{-1} and V_1, that is, that the image of the incoming map is the kernel of the outgoing one. Furthermore, at the point V_0 we have $\mathrm{im}(A_{in}) \subset \ker(A_{out})$, and the quotient of these two spaces is naturally isomorphic to the space of covariant constants on the 4-D subspace $\theta \times \eta$.

These results are potentially very promising. However, in order to obtain 'nontrivial' nonselfdual fields (i.e. not merely tensor products) one really needs to construct complexes such as (3.13) using fields which are only defined by Taylor series in the neighborhood of the diagonal 'physical' space. The defining Dirac, Laplace equations should then be satisfied only to a low order in w (see (3.6)). This we have not been able to do.

2. The Nonselfdual Nahm Equations

By applying the process of dimensional reduction to the selfduality equation (0.16) we have obtained both the Bogomolny equations (with $\phi = A_0$) and Nahm's equation. Performing a similar reduction of the full second-order equation (3.1) produces the static Yang-Mills-Higgs system in 3 dimensions with the fourth component of the connection again playing the role of the Higgs field.

'RECIPROCITY'

	d = 3	<------->	d = 1
$F = *F$	Monopoles	<--->	Nahm's equation
$D F = 0$	Static Y-M-H	<--->	?

We know that the three and one-dimensional versions of the selfdual equation are related by the ADHM construction; it is at least possible that the one-dimensional Yang-Mills equation may play a similar role in the solution of the static Yang-Mills-Higgs equations. This was the original motivation for the study of a system which has since revealed many features of interest in their own right.

As before, we shall start our investigation by considering a Yang-Mills gauge field $T_\mu(x)$ in four Euclidean dimensions x_μ. The four components of this one-form take values in some matrix Lie algebra L, corresponding to a Lie group $G = \exp L$. The dimension of the vector space upon which this group acts, i.e the size of the matrices T_μ, is as yet undetermined. We recall that in the selfdual Nahm construction this dimension was determined by the topological charges k_i of the reciprocal gauge field A_μ since (by the index theorem) these determined the number of solutions of the Weyl equation (2.1). If indeed there exists some reciprocity in the nonselfdual case the size of the T_μ should be given by the dimension of another solution space, whose defining equation we have not been able to guess.

We proceed by supposing that there exists a gauge in which the connection matrices $T_\mu(x)$ depend upon one co-ordinate alone, namely x^0: Thus

$$\begin{aligned} F_{0j} &= D_0 T_j = \partial_0 T_j + [T_0, T_j] \\ F_{ij} &= [T_i, T_j] \end{aligned} \qquad (3.14)$$

The second-order Yang-Mills equations for the reduced gauge field take the form

$$\begin{aligned} D_\mu F_{\mu j} &= D_0 F_{0j} + D_i F_{ij} \\ &= D_0 D_0 T_j + [T_i, [T_i, T_j]] \end{aligned}$$

$$\begin{aligned} D_\mu F_{\mu 0} &= D_i F_{i0} \\ &= - [T_i, D_0 T_i] \end{aligned}$$

If in addition we let $z = x^0$ and choose a gauge in which $T_0 = 0$, these equations simplify further to

$$\frac{d^2T_i}{dz^2} = - [T_i,[T_i,T_j]] \tag{3.15}$$

$$[T_i,\frac{dT_i}{dz}] = 0 \tag{3.16}$$

As we might expect, any set of matrices which satisfies Nahm's equation (1.1) will also satisfy (3.15-3.16); but the converse is <u>not</u> true. We shall see that there are many solutions of our second-order system which do not solve the first-order equation. A particularly remarkable feature of this system is the existence of <u>continuous families</u> of solutions, distinguished by means of a single real parameter, whose extreme members satisfy the selfdual Nahm equation and its <u>antiselfdual</u> analogue

$$\frac{dT_i}{dz} = - \frac{1}{2}\varepsilon_{ijk}[T_j,T_k] \tag{3.17}$$

This equation differs from the standard Nahm equation (1.1) only by a single sign, and may in fact be derived from it by subjecting the fields $T_\mu(z)$ to the parity transformation $z \longrightarrow -z$.

To construct a conserved 'energy function' for equations (3.15-3.16) we are guided by analogy with 4-dimensional Yang-Mills theory, for which the following integral is conserved:

$$E = -\frac{1}{4}\int d^3x \; tr(F_{0i}F_{0i} + \frac{1}{2}F_{ij}F_{ij}) \qquad (3.18)$$

Of course, in the case we are currently considering there are two changes to be made - (i) the fields are independent of the three 'spatial' variables x^i, so that the integration cannot be performed, and (ii) the variable z represents 'Euclidean time', so that we expect the first term to change sign. Taking these points into account we can write down a candidate for the conserved 'energy', namely [28]

$$E = \frac{1}{4}\, tr(F_{0i}F_{0i} - \frac{1}{2}F_{ij}F_{ij})$$

$$= \frac{1}{4}\, tr(\sum_i (\frac{dT_i}{dz})^2 - \frac{1}{2}\sum_{ij} [T_i, T_j]^2) \qquad (3.19)$$

Indeed, a simple calculation shows that E is independent of z. However, unlike the true energy functional (3.18) this quantity is <u>not</u> positive definite. In fact, if we write it in the factorized form

$$E = -\, tr\{(\frac{dT_i}{dz} - \varepsilon_{ijk}T_jT_k)(\frac{dT_i}{dz} + \varepsilon_{ilm}T_lT_m)\} \qquad (3.20)$$

it becomes clear that selfdual (1.1) and antiselfdual (3.17) solutions are zeros of E.

The system of ordinary differential equations (3.15-3.16) has a very rich structure, and we have not been able to solve it completely, even when T_i lies in the simplest Lie algebra SU(2). However, there is a special class of solutions whose existence is easily verified, and whose structure depends in a complex and fascinating way on the nature of the underlying Lie algebra L. These are the 'separable solutions'

$$T_i(z) = iX_i f(z) \tag{3.21}$$

whose z-dependence is contained in a single scalar function f(z), which multiplies the constant matrices $X_i \in L$. It is clear that this function satisfies an equation of the form

$$f''(z) = cf(z)^3 \tag{3.22}$$

while the three elements of the Lie algebra must be subject to the following relations, which we have dubbed the 'cubic algebra'

$$[X_i,[X_i,X_j]] = cX_j \tag{3.23}$$

(as usual, a summation is understood on the repeated index i). The actual value of the real constant c is unimportant; it is clear from (3.22) that a change in c is equivalent to a rescaling of the variable z. It is most convenient to choose $c = 2$; then multiplying (3.22) by $2f'$ and integrating with respect to z, we obtain a first integral

$$f'(z)^2 = f(z)^4 \pm \sigma^4 \tag{3.24}$$

Note that the arbitrary constant σ may be adjusted using the scale transformations $z \longrightarrow \lambda^{-1}z$, $f \longrightarrow \lambda f$, under which

$\sigma \rightarrow \lambda\sigma$. When $\sigma \neq 0$ the general solution of this first-order equation may be constructed using the Jacobi elliptic functions: If the positive sign is taken in (3.24) then

$$f(z) = \sigma \frac{\operatorname{sn}\sigma(z-z_0)\operatorname{dn}\sigma(z-z_0)}{\operatorname{cn}\sigma(z-z_0)} \tag{3.25}$$

but if the constant in (3.24) is negative, then

$$f(z) = \frac{\sigma}{\operatorname{cn}\sqrt{2}\sigma(z-z_0)} \tag{3.26}$$

where the elliptic functions have modulus $k = 1/\sqrt{2}$; For this value their real and imaginary periods 4K, 4K' are equal. In the case $\sigma = 0$, the solution has only one pole, and no periodicity - it is just $f(z) = 1/(z-z_0)$. The arbitrary constant z_0 corresponds to an obvious translation-invariance of (3.24). The main feature which distinguishes the two kinds of solution shown above is that (3.25) possesses zeros (at $z = z_0+2nK$) as well as poles; it is an <u>odd</u> funcion of $(z-z_0)$. In contrast, the absolute value of (3.26) is never less than σ; it is an <u>even</u> function.

The conservation law (3.24) may be used to simplify the energy function (3.19) for the separable solutions (3.21), giving a gauge-invariant bilinear function of the three generators X_i:

$$E = \mp \tfrac{1}{4}\sigma^4 q \tag{3.27}$$

$$q = \mathrm{tr}(X_i X_i) \tag{3.28}$$

The sign in (3.27) opposes that which appears in (3.24). Furthermore, for a compact simple Lie group such as SU(n) the bilinear form (3.28) is positive definite. In fact, there is a theorem of Wigner [29] which states that any finite-dimensional representation of a compact group is equivalent to a unitary representation - it follows that without loss of generality the X_i may be assumed to be hermitian, and in that case $\mathrm{tr}\, XX = \mathrm{tr}\, X^+X$ is positive definite. Hence the two classes of solutions (3.25) and (3.26) are distinguished by the sign of the energy.

It is clear that for each SO(3) invariant tensor $T_{ij\ldots k}$ there exists a corresponding function of the X_i which is invariant under all the transformations $X_i \longrightarrow gX_i g^{-1}$, where $g \in G$, namely

$$I_T = T_{ij\ldots k}\mathrm{tr}(X_i X_j \ldots X_k) \tag{3.29}$$

The 'energy' q is only the first of the series; the next is the cubic invariant

$$t = -i\varepsilon_{ijk}\mathrm{tr}(X_i X_j X_k) \tag{3.30}$$

The two parameters (3.28) and (3.30) are most useful for characterizing the solutions of (3.23). By expanding the positive definite expression

$$\mathrm{tr}\{(X_i \pm i\varepsilon_{ijk}X_jX_k)(X_i \pm i\varepsilon_{ilm}X_lX_m)\} = 2(q \mp t) > 0 \tag{3.31}$$

we derive a bound on the cubic invariant, namely $|t| < q$. It is clear that for a compact simple Lie group the extreme values $t = q$ are attained if and only if the factors in (3.31) vanish, i.e.

$$X_i = \pm i\varepsilon_{ijk}X_jX_k \tag{3.32}$$

These are exactly the relations which must be satisfied by the residues at any pole of the T_i in Nahm's construction for selfdual (+) or antiselfdual (-) monopoles. For this reason solutions of (3.32) will be called selfdual or antiselfdual respectively. In either case the X_i span an SO(3) subalgebra of L - the difference lies in the orientation of the basis. The cubic invariant (3.30) changes sign under the 'parity' transformation $X_i \longrightarrow -X_i$; it is positive for selfdual solutions but negative for antiselfdual ones.

3. Interpolating Solutions

One of the most interesting features of the cubic algebra (3.23) is the existence, for certain choices of the Lie algebra L, of continuous families of solutions which interpolate between the 'selfdual' and 'antiselfdual' SO(3) embeddings (3.32). In this section we shall investigate the simplest Lie algebra for which a nontrivial 'interpolating solution' can be shown to exist, namely L = SU(3) . In 3.4 we shall show that this construction has an analogue in each Lie algebra, provided that we are prepared to consider sets of more than three matrices. In what follows L will always denote a compact semisimple Lie algebra, and G the Lie group obtained by exponentiating L. We shall find that each of our interpolating solutions is associated with an irreducible Riemannian symmetric space L/M.

In the following calculations it is usually convenient to use in place of the X_i the complex generators

$$X_+ = X_1 + iX_2 \qquad\qquad X_- = X_1 - iX_2$$
$$X_0 = X_3 \tag{3.33}$$

inspired of course by the raising and lowering operators J_+, J_- used in the theory of angular momentum. Written in terms of this basis the cubic algebra becomes

$$\tfrac{1}{2}[X_+,[X_-,X_0]] + \tfrac{1}{2}[X_-,[X_+,X_0]] = cX_0 \tag{3.34}$$

$$\tfrac{1}{2}[X_+,[X_-,X_+]] + [X_0,[X_0,X_+]] = cX_+ \tag{3.35}$$

The theorem of Wigner cited previously permits us to assume without loss of generality that the X_i are hermitian matrices, and therefore $X_- = (X_+)^+$. Hence the two terms in (3.34) are hermitian conjugates of one another; a third equation, which would have X_- on the right-hand side, is redundant, being the conjugate of (3.35).

We shall construct the 'interpolating solution' using a 'Chevalley basis' of SU(3), in which the commutation rules take a particularly simple form. A summary of the key facts about the canonical form of a Lie algebra, and particularly that of SU(3), will be found in Appendix A, where our conventions concerning normalization of root vectors and signs of structure constants are also collected.

The so-called <u>maximal embedding</u> of SU(2) in SU(3) is spanned by the three generators J_i, whose complex forms are

$$J_+ = 2(E_\alpha + E_\beta) \qquad J_- = 2(E_{-\alpha} + E_{-\beta}) \qquad J_0 = H_{\alpha+\beta} \tag{3.36}$$

which satisfy the familiar commutation rules

$$[J_0, J_+] = J_+ \qquad [J_0, J_-] = - J_- \qquad [J_+, J_-] = 2J_0 \tag{3.37}$$

In particular, therefore, the generators J_i provide a 'selfdual' solution to the cubic algebra (3.23). As we have noted before, the selfdual and antiselfdual solutions differ only by orientation; therefore by reversing one of these matrices, J_0 say, we obtain an 'antiselfdual' set

$$J_+ \quad , \quad -J_0 \quad , \quad J_-$$

These two extreme examples may be joined by a one-parameter family of 'interpolating' solutions [28]. This family is constructed by mixing the 0-component of the above sets with the root vector $E_{\alpha+\beta}$ and its conjugate, leaving the '+' and '−' components untouched:

$$\begin{aligned} X_+ &= J_+ \qquad\qquad X_- = J_- \\ X_0 &= \lambda J_0 + \mu(E_{\alpha+\beta} + E_{-\alpha-\beta}) \end{aligned} \tag{3.38}$$

where λ and μ are scalar variables which are linked by some relationship yet to be determined. Using the ansatz (3.38) in the cubic algebra (in its complex form (3.34-35)) we obtain after some calculation the conditions

$$\begin{aligned} 2X_0 &= \left(\lambda J_0 + 2\mu E_{\alpha+\beta}\right) + \text{herm. conj.} \\ 2X_+ &= (1 + \lambda^2 + \mu^2) J_+ \end{aligned} \tag{3.39}$$

The first of these equations is of course satisfied identically (i.e. for all λ and μ); the second holds provided only that

$$\lambda^2 + \mu^2 = 1 \tag{3.40}$$

It is clear that the pair $\lambda = 1$, $\mu = 0$ gives rise to the original selfdual embedding (3.36), whereas $\lambda = -1$, $\mu = 0$ gives the corresponding antiselfdual solution. These special solutions lie at opposite points on the circle (3.40) - the other points correspond to 'nonselfdual' solutions of the cubic algebra. As the parameter pair (λ,μ) moves round this circle, the gauge invariants q and t (cf. (3.28) and (3.30)) vary continuously; a simple calculation gives

$$q = 6 \qquad\qquad t = 6\lambda \tag{3.41}$$

The 'energy' invariant q remains constant, whereas the cubic invariant t changes sign as the solution is deformed from selfdual to antiselfdual.

The class of all sets $\{X_i\}$ which satisfy the cubic algebra has the structure of an <u>algebraic variety</u>, because the defining equation is polynomial in the elements of the X_i. Since the maximal embedding (3.36) is a member of the one-parameter family (3.38), the dimension of this variety is <u>at least</u> one; but we have not yet ruled out the possibility that it may be <u>more</u> than one. That is, there may be other families of solutions $X_i(\theta)$ which pass through the maximal embedding but are otherwise distinct from (3.38).

We can, however, place a bound upon the dimension of the solution space by using the fact that the dimension of an algebraic variety is no greater than the dimension of its tangent space at any point. Unlike a smooth differential manifold, however, there may be singular points where the dimension of the tangent space is strictly greater than that of the variety. Take for example case of the cone

$$f(x,y,z) = x^2 + y^2 - z^2 = 0 \tag{3.42}$$

The tangent space at the point (x,y,z) is by definition the vector space of all (u,v,w) such that

$$\frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w = 0 \qquad (3.43)$$

that is, the set of all variations for which the linear term in the Taylor series of $f(x+u,y+v,z+w)$ vanishes. However, in the case of the cone, all partial derivatives of f vanish at the origin – (3.43) is satisfied identically, and the tangent space at $\underline{0}$ is three-dimensional, although the dimension of the cone, and that of the tangent space at any other point, is only two.

With this reservation in mind, we shall calculate the dimension of the tangent space of the cubic algebra at the point (3.36), the maximal embedding. We accordingly consider a general variation in the configuration $\{X_i\}$, which we shall write

$$X_i = J_i + V_i \qquad (3.44)$$

where the J_i are the generators of the maximal embedding and the V_i are any three (Hermitian) generators of SU(3). The cubic algebra may be written in a form akin to (3.42), namely

$$f_j(X_1,X_2,X_3) = 0 \qquad (3.45)$$

where

$$f_j(X_1,X_2,X_3) = [X_i,[X_i,X_j]] - 2X_j \tag{3.46}$$

In order to determine the tangent space, we must calculate the Taylor series for f_j up to first order in V_i. The constant term vanishes, of course, because the maximal embedding is itself a particular solution of (3.45). The vector V_i lies in the tangent space if and only if the linear term is also zero:

$$\begin{aligned} MV_i &= [J_i,[J_i,V_j]] + [J_i,[V_i,J_j]] - [V_i,[J_i,J_j]] - 2V_j \\ &= 0 \end{aligned} \tag{3.47}$$

Now, the SU(2) subalgebra spanned by the generators J_i acts upon the entire Lie algebra SU(3) through the adjoint representation. Thus it is possible to classify the elements of the larger algebra into SU(2) or SO(3) multiplets. The J_i themselves form a triplet; the subspace orthogonal to these is <u>irreducible</u> under the rotation group, and therefore forms a quintuplet, or spin-2 representation.

It is worth noting that if M is a Lie subalgebra of L, and if P is a subalgebra which includes M and is included in L, then $[M,P] \subset [P,P] \subset P$; This means that P is invariant under the adjoint representation of M on L. However, if the quotient space L/M is irreducible under this representation, then no such intermediate P can be found, in other words, M is a <u>maximal</u> subalgebra of L. This is why the embedding (3.36) is called the 'maximal' embedding of SU(2) in SU(3) - it contrasts sharply with a more obvious embedding such as

$(E_\alpha, H_\alpha, E_{-\alpha})$ which admits an extension to $SU(2) \times U(1)$.

The cubic algebra (3.23) has a rotational symmetry

$$X_i \dashrightarrow R_{ij} X_j \,, \qquad R_{ij} \in SO(3) \tag{3.48}$$

as well as an invariance under the adjoint group

$$X_i \dashrightarrow g X_i g^{-1} \,, \qquad g \in SU(3) \tag{3.49}$$

Moreover, there is a certain 'diagonal' subgroup of the full symmetry group which leaves the maximal embedding as a fixed point; this is the group of all transformations of the form

$$X_i \dashrightarrow R_{ij} g X_j g^{-1} \tag{3.50}$$

where $g \in SU(2)$, and the orthogonal matrix R is chosen such that

$$g^{-1} J_i g = R_{ij} J_j \tag{3.51}$$

Under this group, the solutions of the cubic algebra are rotated about the fixed point J_i, thus inducing a group of transformations on the tangent space containing the V_i. This implies that the solutions of (3.47) may themselves be classified into SU(2) multiplets by their behaviour under the action of this group.

In order to solve the linear equation (3.47) we find it convenient to expand the Hermitian matrices V_i in terms of a standard basis of 'angular momentum eigenstates' T_{pm}, where

$$
\begin{array}{lll}
\text{(i)} & Ad(J_i)Ad(J_i)T_{pm} = p(p+1)T_{pm} & \\
\text{(ii)} & Ad(J_0)T_{pm} = mT_{pm} & \\
\text{(iii)} & Ad(J_{\pm})T_{pm} = N_{\pm}(p,m)T_{pm\pm 1} & (3.52) \\
\text{(iv)} & N_+(p,m) = N_-(p,m+1) = \sqrt{(p-m)(p+m+1)} & \\
\text{(v)} & T_{pm}{}^{+} = (-1)^m T_{p,-m} &
\end{array}
$$

(we use the letter p for the J^2 eigenvalue to avoid confusion with the three-vector index j). Thus

$$V_i = \sum_{pm} V_i{}^{pm} T_{pm} \tag{3.53}$$

In the case of SU(3), as we have seen, the only spins present in this expansion are $p = 1$ and $p = 2$; however, the argument which will be presented works for other Lie groups G and any SU(2) embedding - all we need to know is the spins which are present in the adjoint representation of this subgroup on L.

We can simplify the expansion (3.53) by using the observation that the linear operator M of (3.47) commutes with the square of the 'angular momentum', $Ad(\underline{J}^2)$. The solution space may be decomposed as a sum of spin eigenspaces, and we therefore need only consider one value of p at a time. A straightforward, though rather tedious, computation of the expression (3.47) using complex components of V_i and the expansion (3.53) gives finally

$$2\{p(p+1)-m^2-2\}V_0^{\;m} - (m-2)N_-V_-^{\;m-1} - (m+2)N_+V_+^{\;m+1} = 0 \tag{3.54}$$

$$\{p(p+1)+m(m+5)\}V_+^{\;m+1} - N_+N_-V_-^{\;m-1} - 2(m+2)N_+V_0 = 0 \tag{3.55}$$

There is in fact a third component in this set of equations, but this is somewhat redundant because of the Hermiticity of the V_i. It may be derived from (3.54) by taking the complex conjugate and using the reality condition

$$V_i^{\;m*} = (-1)^m V_i^{\;-m} \tag{3.56}$$

It is clear therefore that in any spin multiplet there is always at least one solution to these equations for each value of m; namely, if V_m are the components of a skew-Hermitian matrix V, then we set

$$V_+^{\;m+1} = N_+V_m \qquad V_0^{\;m} = mV_m \qquad V_-^{\;m-1} = N_-V_m$$

The significance of this solution is not hard to see - these are nothing but the components of the matrices

$$V_i = [J_i, V] \tag{3.57}$$

which are the variations of the J_i under an infinitesimal SU(3) gauge transformation of the form (3.49) generated by V.

In order that other, unexpected, dimensions should exist in the tangent space it is necessary that the equations (3.54-3.55) should be proportional, i.e. that the 2 x 2 determinants formed by their coefficients should vanish. These determinants have just one common factor; this is

$$p(p+1) - 6 = 0 \tag{3.58}$$

It follows that the linearised cubic algebra (3.47) has no other solutions but the gauge transformations (3.57) unless the adjoint representation of the SU(2) subalgebra on the including algebra L contains a spin-2 component. As we have seen, this condition holds for the maximal embedding in SU(3), but not for the three obvious embeddings of the form $(E_\alpha, H_\alpha, E_{-\alpha})$, where α is any root of SU(3). For these the adjoint representation is resolved in the form

$\underline{8} = \underline{3} + \underline{2} + \underline{2} + \underline{1}$

with no quintuplet available.

In the case $p = 2$, there is only one independent relation among the components of V for each value of m. Using (3.54), for reasons of symmetry, and solving for V_0, we obtain

$$V_0{}^m = \frac{(m+2)N_+V_+{}^{m+1} + (m-2)N_-V_-{}^{m-1}}{(2-m)(2+m)} \tag{3.59}$$

Since V_- is be the Hermitian conjugate of V_+, the $V_+{}^m$ are sufficient in general to determine the other components. Hovever, as seen from the vanishing numerator and denominator in (3.59), the $V_0^{\pm 2}$ remain free - apart from the constraint that they too shall be conjugates. Therefore there is a total of 12 real degrees of freedom in the cubic algebra for each spin-2 representation. Five of these are gauge transformations (3.57), one for each member of the quintuplet – but seven remain unaccounted for. Since the V_i carry a vector, or spin-1, index as well as a spin-2 internal symmetry, they belong to a representation of the form

$$\underline{3} \times \underline{5} = \underline{3} + \underline{5} + \underline{7} \qquad (3.60)$$

Since the linearised cubic algebra (3.47) is invariant under the 'diagonal' SO(3) group (3.50), its solution space can only contain <u>complete</u> irreducible components from this sum; therefore it must transform as $\underline{5} + \underline{7}$. Under the action of this group the original interpolating solution (3.38) sweeps out a four-dimensional submanifold; one co-ordinate being the parameter θ and the other three being the Euler angles of SO(3).

We have not been able to find any other interpolating solutions apart from those obtained by rotating (3.38). It seems likely, therefore, that the apparent 7 nontrivial dimensions of the tangent space are misleading, as in the case of the cone cited earlier. This is not surprising, since like the vertex of the cone, the maximal embedding is a point of high symmetry (cf. (3.51)); if there <u>is</u> any singular point on the variety, where else could it be!

4. Interpolating Solutions in Other Algebras

In the last section we considered in some detail the 'interpolating solution' (3.38); a set of three SU(3) matrices, depending upon a real parameter θ, which satisfies the cubic equations (3.23) for all values of the parameter and which reduces to the familiar (anti-) selfdual forms (3.32) for certain extreme values of θ. In this section we shall consider a generalization of this construction.

The cubic algebra as we have described it so far is a set of three elements of a Lie algebra L satisfying the cubic equations (3.23). In section 3.3 we considered only the simplest nontrivial case, L = SU(3) ; however, an obvious extension which cannot be neglected is the study of Lie algebras with higher dimension and/or rank.

The restriction to matrix triples which we have so far observed has been motivated by the hope that the nonselfdual Nahm equations may in some way be connected with three-dimensional Yang-Mills fields. If we now decide to study the cubic algebra for its own sake, or indeed if we are interested in gauge fields in higher-dimensional spaces, then we are free to consider larger sets of matrices.

In particular, if the X_i span a simple Lie subalgebra of L, M say, and if they are orthonormal with respect to the Killing form of M, then the cubic equations are certainly satisfied. For under the adjoint representation of exp M the basis X_i is subjected to orthogonal transformations:

$$g^{-1}X_i g = R_{ij}X_j \tag{3.61}$$

and it may therefore be shown that the mapping

$$Y \longrightarrow [X_i,[X_i,Y]] \tag{3.62}$$

of M into itself commutes with the adjoint group. Assuming that the subalgebra M is simple, it follows using Schur's Lemma that this mapping must be a constant multiple of the identity. The constant, c in (3.23), is called the quadratic Casimir invariant of the algebra M. Henceforth we shall denote by J_i the orthonormal basis of any subalgebra M (to maintain a similarity with the SO(3) embeddings), and use X_i for a general or 'nonselfdual' solution of the cubic algebra.

Now, not every embedding $M \subset L$ is connected to a one-parameter family of solutions such as (3.38). We have already found one constraint on the occurence of interpolating solutions in the case $M = SO(3)$, namely that the adjoint representation of M in L should contain a quintuplet. We saw in that case that we were forced to take a very special embedding of M in L, the so-called maximal embedding. If we look at the basis (3.36) for this embedding, we notice a very striking thing - it is constructed symmetrically with respect to the two simple roots α and β.

It is clear that there exists an isometry of the root space H^* which maps the set of roots onto itself and which exchanges the roots α and β; this is a reflection in the line of $\alpha+\beta$. There is a well-known theorem which asserts that for any such transformation mapping α to α' there exists a corresponding automorphism ϕ of the Lie algebra such that

$$\phi(E_\alpha) = E_\alpha, \qquad \phi(H_\alpha) = H_\alpha, \tag{3.63}$$

for each <u>simple</u> root α. In the present case, therefore,

$$\begin{aligned} \phi(H_\alpha) &= H_\beta & \phi(H_\beta) &= H_\alpha \\ \phi(E_\alpha) &= E_\beta & \phi(E_\beta) &= E_\alpha \\ \phi(E_{\alpha+\beta}) &= \phi([E_\alpha, E_\beta]) = - E_{\alpha+\beta} \end{aligned} \tag{3.64}$$

Furthermore, because it derives from a reflection of the root diagram, this mapping is an <u>involution</u>, that is, $\phi^2 = 1$. The significance of the maximal embedding (3.36) is clear; the subspace M spanned by the J_i is precisely the set of fixed points of this involution.

Another key result in the theory of Lie algebras is the fact that all simple sets of roots are equivalent under the adjoint group; that is, if $\{\alpha_i\}$ and $\{\alpha_i'\}$ are any two simple sets of roots then there exists an automorphism of the form (3.63) under which the two sets correspond, and furthermore there is some $g \in G$ such that

$$\phi(X) = gXg^{-1} \tag{3.65}$$

It follows that in order to find all involutions in a Lie algebra <u>up to equivalence</u> it is sufficient to consider those isometries of the root space diagram which permute a simple set of roots. However, the geometry of a simple set of roots is encoded in the so-called Dynkin diagram - each point of the diagram represents a root, and the number of lines joining each pair of points gives the angle between the corresponding roots according to the rule

$$(\text{number of lines}) = 4\cos^2(\text{angle}) \qquad (3.66)$$

Hence, up to equivalence under the adjoint group Inn(L), there is only one distinct involution of L for each possible symmetry of the Dynkin diagram.

In order to generalise the interpolating solution (3.38), it is convenient to use a somewhat different, though equivalent, embedding of SO(3). This is the set $M = L_+$ of fixed points of the involution

$$\begin{aligned} \phi(E_\alpha) &= -E_{-\alpha} \\ \phi(H_\alpha) &= -H_\alpha \end{aligned} \qquad (3.67)$$

which is spanned by the generators

$$J_\alpha = i(E_\alpha - E_{-\alpha}) \qquad (3.68)$$

corresponding one-to one with the positive roots of L. In order to see that this involution is indeed equivalent to the previous one $E_\alpha \dashrightarrow E_\beta$, we use the conventional basis of the complex algebra SL(n,C), in which the Cartan subalgebra contains all diagonal matrices and the simple roots $E_\alpha, E_\beta, \ldots$ are represented by matrices with a single unit element directly above the major diagonal. In SL(3,C)

$$E_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The negative simple roots $E_{-\alpha}, E_{-\beta}, \ldots$ are obtained by transposing the above roots, that is, by reflection about the major diagonal, so the involution (3.67) is that given by

$$\phi(X) = -\tilde{X} \tag{3.69}$$

However, the involution (3.64), which we used to construct the maximal embedding in the previous section, can clearly be expressed in the form

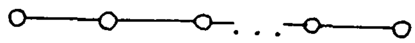
$$\phi(X) = -g\tilde{X}g^{-1} \tag{3.70}$$

where the matrix g for the case $n = 3$ is

$$g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.71}$$

The fixed point subalgebra of (3.70) is the Lie algebra of a group which preserves the bilinear metric g. It is clear from (3.71) that this form is <u>symmetric</u>, and therefore we may choose a (complex) basis in which it is represented by the identity matrix, as in (3.69).

The generalization of the maximal embedding for higher SL(n,C) algebras and their compact real forms SU(n) is clear. The Dynkin diagram for one of these algebras is

A_{n-1}: 

containing n-1 points and n-2 lines, which has an obvious P_2 symmetry. Each algebra therefore possesses an involution of the type (3.69) or (3.70), the equivalence matrix g being expanded accordingly - it always has an alternate sequence of 1's and -1's along the <u>minor</u> diagonal, as in (3.71). The fixed point algebra of (3.69) or (3.70) is SO(n,C) or its real form SO(n,R); The quotient space SU(n)/SO(n) is one example of a <u>Riemannian symmetric space</u>.

The Riemannian symmetric spaces are coset spaces of the form G/H, where H is the subgroup of fixed points of some involution of a Lie group G. This space clearly carries a representation of the subgroup H; if G is a <u>simple</u> group then this representation is irreducible, and G/H is called an <u>irreducible</u> symmetric space. The classification of irreducible symmetric spaces therefore amounts to the classification of all inequivalent involutions in a Lie group G or its associated Lie algebra L, and this is summarized in Appendix B.

We have not investigated the existence of interpolating solutions for all the symmetric spaces, but the involution (3.67) clearly exists in any Lie algebra, and the symmetric spaces so formed are the most promising candidates for further study. These are

(i) $SU(n)/SO(n)$ from A_n

(ii) $SO(2n+1)/SO(n)xSO(n+1)$ from B_n

(iii) $SO(2n)/SO(n)xSO(n)$ from D_n

(iv) $USp(2n)/U(n)$ from C_n

(v) G_2/A_1xA_1

(vi) F_4/C_3xA_1

(vii) E_6/C_4

(viii) E_7/A_7

(ix) E_8/D_8

For any of these algebras, the generators (3.68) constitute a selfdual solution of the cubic equations. Orthogonal to the fixed point subalgebra L_+ is the vector subspace L_- spanned by

$$K_\alpha = E_\alpha + E_{-\alpha} \qquad \text{and} \qquad H_\alpha \tag{3.72}$$

which carries an irreducible representation of L_+; the elements in this space are multiplied by -1 under the action of the involutive automorphism ϕ. We shall denote by ω the highest weight of this representation (with respect to some total ordering of the weight space). The most obvious generalization of the interpolating solution (3.38) is that given by

$$\begin{aligned} X_\alpha &= J_\alpha \qquad \text{for all } \alpha \neq \omega \\ X_\omega &= \lambda J_\omega + \mu H_\omega \end{aligned} \tag{3.73}$$
where $H_\omega = \omega . H$

The commutation relations of the Lie algebra in terms of the basis of the H, J and K are

$$
\begin{aligned}
[J_\alpha, J_\beta] &= N_{\alpha\beta} J_{\beta+\alpha} - N_{-\alpha\beta} J_{\beta-\alpha} \\
[J_\alpha, K_\beta] &= N_{\alpha\beta} K_{\beta+\alpha} - N_{-\alpha\beta} K_{\beta-\alpha} \qquad \text{if } \alpha \neq \beta \\
[J_\alpha, K_\alpha] &= 2H_\alpha \qquad\qquad (3.74) \\
[K_\alpha, K_\beta] &= N_{\alpha\beta} J_{\beta+\alpha} + N_{-\alpha\beta} J_{\beta-\alpha} \\
[H_\alpha, J_\beta] &= (\alpha,\beta) K_\beta \\
[H_\alpha, K_\beta] &= (\alpha,\beta) J_\beta
\end{aligned}
$$

Using these we can evaluate the various components of the cubic equation (3.23); in particular for any root β <u>other</u> than the highest weight ω we have

$$
\begin{aligned}
\sum_\alpha [X_\alpha, [X_\alpha, X_\beta]] = c_H J_\beta &- (\lambda^2 - 1)\{ N_{\omega\beta} N_{\omega,\beta+\omega} J_{\beta+2\omega} \\
&+ N_{-\omega,\beta} N_{-\omega,\beta-\omega} J_{\beta-2\omega} - (N_{\omega\beta}^2 + N_{-\omega\beta}^2) J_\beta \} \\
&- \lambda\mu\{ N_{\omega\beta}(\omega,\omega+2\beta) K_{\beta+\omega} - N_{-\omega\beta}(\omega,\omega-2\beta) K_{\beta-\omega} \} \\
&+ \mu^2 (\omega,\beta)^2 J_\beta \qquad (3.75)
\end{aligned}
$$

where c_H is the quadratic Casimir invariant for the subgroup $H = \exp L_+$. Now, the cubic algebra requires that (3.75) should be proportional to X_β for a continuous range of values of λ and μ; it is clear that this is impossible unless the root space of L and the choice of the particular root ω satisfy some very stringent conditions.

(1) The terms in $J_{\beta+2\omega}$ and in $J_{\beta-2\omega}$ must vanish, which means that $\beta \pm 2\omega$ can not be a root of L for any $\beta \neq \omega$.

This property may be expressed in terms of the geometry of the root space using the relation (A13), with ω in place of the generic root α. It implies that $p,q \leqslant 1$ - hence

$$-\frac{1}{2} \leqslant m \leqslant \frac{1}{2} \tag{3.76}$$

for all roots $\beta \neq \omega$, where m is the z-component of 'spin' with respect to the SU(2) subalgebra spanned by $(E_\omega, E_{-\omega}, H_\omega)$:

$$m = \frac{(\omega,\beta)}{\omega^2} \tag{3.77}$$

If this co-ordinate is used to order the root space then it is clear that ω must indeed be the highest weight, as claimed. This means that $\omega+\beta$ is not a root for any <u>positive</u> β, and the only possible values for the integers in (A13) are $q = 0$, $p = 0$ or 1.

(2) The terms in $K_{\beta+\omega}$ and in $K_{\beta-\omega}$ must vanish, which means that if $\omega-\beta$ is a root then

$$2(\omega,\beta) = \omega^2 \tag{3.78}$$

This relation holds by virtue of the constraints on p and q established in (1), provided only that ω is a highest weight.

(3) The terms in J_β itself must add up to $c_H J_\beta$; by equating coefficients we therefore obtain a relationship between the parameters λ and μ, analogous to (3.40):

$$(\lambda^2-1)(N_{\omega\beta}^2 + N_{-\omega\beta}^2) + \mu^2(\omega,\beta)^2 = 0 \tag{3.79}$$

if β is orthogonal to ω, this equation vanishes identically; otherwise $\omega-\beta$ must be a root of L. Using the result of (3.78) and the expression for the structure constants (A14), we can rewrite the condition in the form

$$\lambda^2 + \tfrac{1}{2}\omega^2\mu^2 = 1 \tag{3.80}$$

(4) Applying the cubic constraint (3.23) to the highest weight vector X_ω, we obtain the condition

$$\begin{aligned}\sum_\alpha [X_\alpha, [X_\alpha, X_\omega]] &= \lambda c_H J_\omega + \mu c_{G/H} H_\omega - 2\mu\omega^2 H_\omega \\ &= c_H(\lambda J_\omega + \mu H_\omega)\end{aligned} \tag{3.81}$$

where $c_{G/H}$ is the Casimir of the adjoint representation of H on G/H. It is clear that this invariant must be related to that of H itself by

$$c_H = c_{G/H} - 2\omega^2 \tag{3.82}$$

By direct calculation we can easily show that

$$c_H = \sum_\alpha (N_{\alpha\beta}^2 + N_{-\alpha\beta}^2) \tag{3.83}$$

where β is an arbitrary positive root of L and the sum is taken over all positive roots <u>except</u> β. By choosing β to be the highest weight ω and using the results (3.78), (A14) we may express this in terms of the geometry of the roots:

$$c_H = (\delta,\omega) - \omega^2 \tag{3.84}$$

where δ is the sum of all the positive roots of L. A similar calculation for the representation of H on G/H (spanned by K_α and H_α) leads to the Casimir

$$c_{G/H} = (\delta,\omega) + \omega^2 \tag{3.85}$$

Again, the necessary condition (3.82) is clearly satisfied.

We have therefore proved the following result:

<u>Theorem</u>: If L is a simple Lie algebra, with Chevalley basis (E_α, H_α), and if J_α are the generators of a subalgebra as defined in (3.68), then the cubic equation (3.23) has an 'interpolating solution' given by (3.73), in which the parameters λ and μ satisfy the quadratic relation (3.80).

Chapter 4. Conclusion

We have seen (Chapter 1) that in solving the Bogomolny equations which describe static magnetic monopoles, an associated one-dimensional selfdual field $T_\mu(z)$ introduced by Nahm plays a role of prime importance. We have further shown that since the ordinary differential equation (1.1) which determines this field may be realized by dimensional reduction of the four-dimensional selfduality equation (0.16), Ward's twistor transform may be applied to convert Nahm's system to an equivalent Riemann-Hilbert problem. The solution of Nahm's equation for any given initial conditions would follow straightforwardly if we could only perform the necessary factorization of the patching matrix $g(\mu,\nu,\zeta)$:

$$g(\mu,\nu,\zeta) = h(x,\zeta)k(x,\zeta)^{-1} \tag{4.1}$$

When performing a similar task in the construction of 3- or 4-dimensional selfdual fields (monopoles and instantons) the matrices h and k are analytic in the domains $\zeta \neq 0$ and $\zeta \neq \infty$ respectively; consequently the only singularities in g (as a function of x and ζ) are at $\zeta = 0$ and $\zeta = \infty$, the north and south 'poles' of the Riemann sphere. However, this condition is not necessary for the uniqueness of the solution to a Riemann-Hilbert problem. It is only necessary that g be analytic in the neighbourhood of a closed curve $\tilde{C}$ which bisects a compact Riemann surface S, and that h and k should be continuous on C and respectively analytic in the domains $\tilde{D}_-$ and $\tilde{D}_+$ bounded by $\tilde{C}$. This means that if C, D_- and D_+ are the projections of $\tilde{C}$, $\tilde{D}_-$ and $\tilde{D}_+$ onto the complex plane, we may allow h, k, and g to have branches provided that no branch cut crosses the curve C and that h and k are <u>finite</u> in D_- and D_+ respectively. The uniqueness of the solution to (4.1) (up to gauge transformation) now follows as before by Liouville's theorem, which holds on any <u>compact</u> Riemann surface.

The patching matrix which we have obtained for the general 2 x 2 Nahm equation has an unusual feature not found in those used for instantons and monopoles; this is the appearance of the double valued function $\sqrt{ab}$. Of course, this multivaluedness cancels in the particular form (1.88). As we have noted in section 1.5 it does not seem possible to reduce this matrix to upper-triangular form without introducing extra singularities; however, in view of the comments above, this need not present an insuperable obstacle to solving the RHP. By performing equivalence transformations analogous to (1.96), but using the double-valued functions $\sqrt{b/a}$ and $\sqrt{a/b}$ in place of ζ and ζ^{-1} as matrix elements, we can reduce the patching matrix to the triangular form

$$g(\theta) = \begin{pmatrix} \sqrt{\frac{b}{a}}e^{i\alpha} & \cos\alpha \\ 0 & \sqrt{\frac{a}{b}}e^{-i\alpha} \end{pmatrix} \tag{4.2}$$

where α is the angle defined in (1.86). It is clear from (1.87) that the zeros of $b(\zeta)$ lie within the curve $|\zeta| = 1$, whereas those of $a(\zeta)$ lie outside it. Therefore the points where $\sqrt{b/a}$ tends to infinity lie entirely in one domain D_-

whilst the infinities of $\sqrt{a/b}$ lie entirely in the other D_+; These regularity properties clearly generalize those of ζ and ζ^{-1}.

It seems then that it should be possible to solve the Riemann-Hilbert problem (4.1) even when the patching matrix is multivalued, as in (4.2); this would provide a systematic method for solving Nahm's equation. The solution will involve contour integrals of multivalued functions around branch cuts, which probably explains the appearance of the higher transcendental functions sn, cn and dn in the 2 x 2 case.

In the latter part of our work, which concerns the 'nonselfdual Nahm equations', we have uncovered a rich field for future study, which we have only barely begun to explore. We have shown that 'interpolating solutions' of the cubic algebra (3.23) are associated with those Riemannian symmetric pairs (L,M) for which the embedded subalgebra M is spanned by (3.68). It may well be that other symmetric pairs share this property, but so far this has not been fully investigated. It is possible to write down a set of conditions which would permit the existence of a solution analogous to (3.73)

$$\begin{aligned} c_{L/M} &= c_M + \omega^2 \\ (\omega,\beta)(2(\omega,\beta) - \omega^2) &= 0 \quad \text{for all } \beta \neq \omega \end{aligned} \tag{4.3}$$

where c_M and $c_{L/M}$ are the Casimir invariants for the adjoint representations of M in M and L/M respectively, ω is the highest weight of the latter representation, and β is an arbitrary positive root of M. In order to verify these conditions we therefore need considerable knowledge of the representation of M in L/M, and it may be that only by a case-by case study of each and every symmetric pair can results be established.

Appendix A. Lie Algebras and Root Spaces

If L is a semisimple Lie algebra, and L_0 a maximal abelian subalgebra of L, then L_0 is called a Cartan subalgebra (CSA); it is well-known that the group Inn(L) of all inner automorphisms

$$X \dashrightarrow gXg^{-1} \quad , \qquad g \in G \qquad \text{(A1)}$$

acts transitively on the set of Cartan subalgebras, that is, all CSA's are conjugate to one another. Each generator X of the algebra L gives rise to a linear mapping Ad(X) of L into itself, defined by

$$Ad(X)Y = [X,Y] \qquad \text{for each } Y \in L \qquad \text{(A2)}$$

It is a simple consequence of the Jacobi identity that the linear mapping X ---> Ad(X) constitutes a representation of L, called the adjoint representation (of L on L). Because the elements of the Cartan subalgebra L_0 commute with one another (L_0 is by definition an abelian algebra), the algebra L may be decomposed as the direct sum of a number of root spaces L_α, each of which is a simultaneous eigenspace of every member of the CSA under the adjoint representation. That is, if X lies in the root space L_α, then

$$[H,X] = \alpha(H)X \qquad \text{for each } H \in L_0 \qquad \text{(A3)}$$

the eigenvalue $\alpha(H)$ is clearly a linear function of the generator H; this linear function, or one-form, is called a root of the algebra L. The set of all nonzero roots, which is a subset of the dual space H^*, provides a unique characterization of the algebra. The root spaces have the following useful property under the Lie bracket operation:

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta} \tag{A4}$$

The symmetric bilinear form or metric

$$(X,Y) = tr(Ad(X)Ad(Y)) \tag{A5}$$

is invariant under the group of inner automorphisms, or adjoint group, Inn(L); it is called the Killing form (KF), and it can reveal much about the nature of the algebra L. In particular L is semisimple if and only if its Killing form is nonsingular; the group $G = \exp L$ is also compact if and only if the KF is negative definite. Any invariant metric on a semisimple Lie algebra may therefore be written in the form (X,AY) for some operator A, and if this algebra is simple (i.e the adjoint representation is irreducible) then Schur's Lemma may be used to show that $A = c$, a scalar. Thus there is only one invariant metric on L, up to a multiplicative constant, and so the calculation of the Killing form may be performed using any representation, not necessarily the adjoint as above.

It is shown in any standard text on Lie algebras [30] that if $\alpha \neq 0$ then the root space L_α is one-dimensional, and that if α is a root then $-\alpha$ is also a root. Furthermore, the root space corresponding to $\alpha = 0$ is precisely the Cartan subalgebra; this is why we use the notation L_0 for the CSA. A so-called Chevalley basis for the Lie algebra L may be constructed as follows: Firstly, we choose a basis for each root space L_α ($\alpha \neq 0$); each basis contains a single element E_α, and these are normalized with respect to the Killing metric by

$$(E_\alpha, E_{-\alpha}) = 1 \tag{A6}$$

Next, we define the commutators

$$H_\alpha = [E_\alpha, E_{-\alpha}] \tag{A7}$$

which belong to the Cartan subalgebra L_0 by virtue of (A4). It may be shown [30] that in fact the H_α span the Cartan subalgebra. They also satisfy

$$(H_\alpha, H) = \alpha(H) \tag{A8}$$

Therefore the commutator of an H with an E is

$$[H_\alpha, E_\beta] = (\alpha,\beta) E_\beta \tag{A9}$$

Since the H's lie in a Cartan subalgebra, and such a subalgebra is by definition abelian,

$$[H_\alpha, H_\beta] = 0 \tag{A10}$$

If $\alpha+\beta \neq 0$ then the subalgebra on the right-hand side of (A4) is one-dimensional; therefore the commutator of two E's is

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \tag{A11}$$

where $N_{\alpha\beta}$ is some real constant. The structure constants $N_{\alpha\beta}$ satisfy a number of symmetry properties, namely

$$\begin{aligned} N_{\alpha\beta} &= - N_{\beta\alpha} \\ N_{\alpha\beta} &= N_{\beta\gamma} = N_{\gamma\alpha} \qquad \text{if} \quad \alpha+\beta+\gamma = 0 \\ N_{\alpha\beta} &= - N_{-\alpha,-\beta} \end{aligned} \tag{A12}$$

Of these, the first follows from the antisymmetry of the Lie bracket, the second follows from the Jacobi identity, and the third is a further condition which we impose in order to fix the structure constants (almost) uniquely. In order to write down an expression for the $N_{\alpha\beta}$ we need some terminology: the α-chain containing β is the set of all nonzero root spaces of the form $L_{\beta+k\alpha}$, where k is an integer, positive, negative or zero. It follows from (A4) that any α-chain is invariant under the adjoint representation of the SU(2) subalgebra generated by E_α, $E_{-\alpha}$, and H_α. Suppose that

$$L_{\beta+k\alpha} \neq 0 \qquad \text{if and only if} \qquad -p \leq k \leq q$$

Then it may be shown that

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = p - q \tag{A13}$$

$$N^2_{\alpha,\beta+k\alpha} = \frac{1}{2}(\alpha,\alpha)(q-k)(p+k+1) \tag{A14}$$

$$N^2_{\alpha\beta} = \frac{1}{2}(\alpha,\alpha)q(p+1) \tag{A15}$$

Note here that $N_{\alpha,\beta+k\alpha} = 0$ when $k = -p-1$ or $k = q$; this follows since the root spaces with $k < -p$ and $k > q$ are empty, so the 'raising' and 'lowering' operators E_α, $E_{-\alpha}$ must eventually annihilate the $E_{\beta+k\alpha}$.

The Lie algebra SU(3) is said to be of <u>rank 2</u>, meaning that all its Cartan subalgebra H are two-dimensional. The root vectors of SU(3) therefore lie in a two-dimensional vector space H^*. There are in fact just six nonzero roots, which form a pleasingly symmetrical hexagonal pattern:

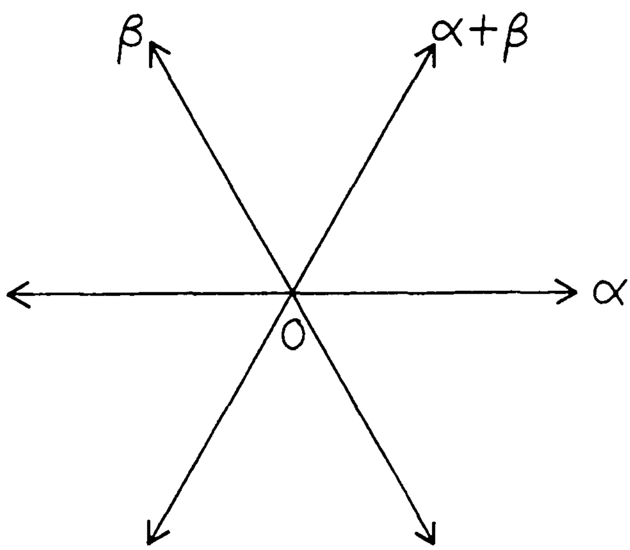


Fig. 5

The distances and angles in this <u>root space diagram</u> are calculated using the Killing form, which is transferred from

H to its dual space H^*. This metric is positive definite on the real space spanned by the H_α in any Lie algebra, so the roots can always be represented in a Euclidean space. The algebra SU(n) is said to be <u>simply laced</u> because all of its nonzero roots have the same length - we shall normalize them so that $|\alpha| = 2$. We mentioned before that the roots of a Lie algebra always occur in equal and opposite pairs; for the SU(n) groups H is usually taken to be the set of all diagonal $n \times n$ matrices, in which case a Chevalley basis can be chosen such that

$$E_{-\alpha} = E_\alpha^+ \tag{A16}$$

The equation (A15) is not quite sufficient to determine the structure constants; there are several possible sign conventions consistent with the symmetries (A12). The one we shall choose is

$$\begin{aligned} N_{\alpha\beta} &= N_{-\alpha,\beta+\alpha} = 1 \\ N_{\beta\alpha} &= N_{-\beta,\alpha+\beta} = -1 \end{aligned} \tag{A17}$$

Appendix B. Riemannian Symmetric Spaces

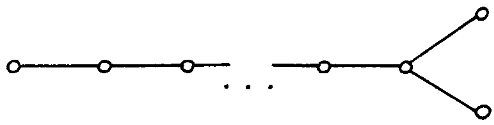
The space $SU(n)/SO(n)$ is by no means the only symmetric space based on the Lie group $SU(n)$. The involutions of the form

$$\phi(X) = gXg^{-1} \tag{B1}$$

where $g^2 = 1$, lead to fixed point subalgebras of the form $S[U(p) \times U(q)]$, where $p+q = n$, and symmetric spaces $U(p+q)/U(p)xU(q)$. It is also possible to take an involution of the form (3.70), but with a skew-symmetric bilinear form g; the dimension of the matrices must be even, $n = 2p$. Now the resulting subalgebra belongs to the unitary-symplectic group $USp(2p)$, that is, the group of unitary matrices which also preserve the symplectic form $g = \epsilon \otimes I_p$ - this is isomorphic to the group of unitary matrices whose elements are quaternions, $U(p,H)$. The corresponding symmetric space is $SU(2p)/USp(2p)$.

The classical Lie algebra B_n, or $SO(2n+1)$, has a Dynkin diagram with no symmetry; all involutions are therefore inner automorphisms of the form (B1), where $g \in SO(2n+1)$ and $g^2 = 1$. The equivalence classes of involutions are distinguished by the numbers p and q of positive and negative eigenvalues of g, and the only symmetric spaces based on B_n are $SO(p+q)/SO(p)xSO(q)$.

In the case of D_n, or $SO(2n)$, g is still an orthogonal matrix, but its determinant may be +1 or -1; this latter case arises because the Dynkin diagram

D_n: 

has a P_2 symmetry, so there is one coset in Aut(L)/Inn(L) which contains non-inner automorphisms. Since the automorphism (B1) is an involution, g^2 must be a constant multiple of the unit matrix, $g^2 = c1$. However, because the dimension of the matrices is even, taking the determinant of this equation no longer leads to $c = 1$. When c is positive we obtain the spaces SO(p+q)/SO(p)xSO(q) as before, but when $g^2 = -1$ we find instead SO(2n)/U(n) (note that the representation of complex numbers by 2 x 2 orthogonal matrices induces an embedding of U(n) in SO(2n)).

Finally, the symplectic groups C_n, or Sp(2n), allow only inner automorphisms (B1) with $g^2 = c1$. For positive c the possible symmetric spaces are USp(2p+2q)/USp(2p)xUSp(2q) , while for negative c there is only USp(2n)/U(n) .

In summary, then, the classical Lie groups give rise to the following symmetric spaces. We list here only the compact forms which carry positive definite metrics; to each of these there correspond many others which possess the same complex extension but have a metric of different signature.

(i) From A_n : U(p+q)/U(p)xU(q) ,
SU(n)/SO(n) , SU(2p)/USp(2p)

(ii) From B_n : SO(p+q)/SO(p)xSO(q)

(iii) From D_n : SO(p+q)/SO(p)xSO(q) , SO(2n)/U(n)

(iv) From C_n : USp(2p+2q)/USp(2p)xUSp(2q) ,
USp(2n)/U(n) .

REFERENCES

1. P. Dirac, "Quantised Singularities in the Electromagnetic Field", Proc. Roy. Soc. A133 (1931)

2. G. t'Hooft, "Magnetic Monopoles in Unified Gauge Theories", Nucl. Phys. B79, 276 (1974)

3. A. Polyakov, "Particle Spectrum in Quantum Field Theory", JETP Letters 20, 194 (1974)

4. E. Bogomolny, "The Stability of Classical Solutions", Sov. J. Nucl. Phys. 24, 449 (1976)

5. M. Prasad and C. Sommerfield, "Exact Classical Solution for the t'Hooft Monopole and the Julia-Zee Dyon", Phys. Rev. Lett. 35, 760 (1975)

6. C. Maunder, "Algebraic Topology" (C.U.P. 1980)

7. C. Taubes, "The Existence of a Non-Minimal Solution to the SU(2) Yang-Mills Higgs equations on R^3", Comm. Math. Phys. 86, 257 (1982)

8. J. Burzlaff, "A Classical Lump in SU(2) Gauge Theory with a Higgs Doublet", Nucl. Phys. B233, 262 (1984)

9. N. Manton, "Topology in the Weinberg-Salam Theory", Phys. Rev. D28, 2019 (1983)

   F. Klinkhamer and N. Manton, "A Saddle-Point Solution in the Weinberg-Salam Theory", Phys. Rev. D30, 2212 (1984)

10. R. Ward, "On Self-Dual Gauge Fields", Phys. Lett. 61A, 81 (1977)

11. M. Atiyah and R. Ward, "Instantons and Algebraic Geometry", Comm. Math. Phys. 55, 117 (1977)

12. M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, "Construction of Instantons", Phys. Lett. 65A, 185 (1978)

13. W. Nahm, "A Simple Formalism for the BPS Monopole", Phys. Lett. B90, 413 (1980)

"All Self-Dual Multimonopoles for Arbitrary Gauge Groups", in Structural Elements in Particle Physics and Statistical Mechanics, ed. J. Honerkamp et. al. (NATO, New York 1981)

"Multimonopoles in the ADHM Construction", in Proceedings of the Symposium on Particle Physics, ed. Z. Horvath et. al. (Visegrad 1981)

"The Construction of All Self-Dual Multimonopoles by the ADHM Method", in Monopoles in Quantum Field Theory, ed. N. Craigie et. al. (Trieste 1981)

14. N. Hitchin, "Monopoles and Geodesics", Comm. Math. Phys. 83, 579 (1982)

"On the Construction of Monopoles", Comm. Math. Phys. 89, 145 (1983)

15. E. Corrigan, D. Fairlie, P. Goddard and P. Yates, "The Construction of Self-Dual Solutions to SU(2) Gauge Theory", Comm. Math. Phys. 58, 223 (1978)

16. E. Corrigan and P. Goddard, "An N-Monopole Solution with 4N-1 Degrees of Freedom", Comm. Math. Phys. 80, 575 (1981)

17. E. Corrigan, D. Fairlie, P. Goddard and P. Templeton, "A Green Function for the General Self-Dual Gauge Field", Nucl. Phys. B140, 31 (1978)

N. Christ, N. Stanton and E. Weinberg, "General Self-Dual Yang-Mills Solutions", Phys. Rev. D18, 2013 (1978)

18. N. Manton, "Complex Structure of Monopoles", Nucl. Phys. B135, 319 (1978)

19. E. Corrigan and P. Goddard, "Construction of Instanton and Monopole Solutions and Reciprocity", Ann. Phys. 154, 253 (1984)

20. D. Olive, in Current Topics in Elementary Particle Physics, ed. K. Mutter et. al. (Plenum Press 1981), p. 199

21. S. Rouhani, "The Patching Matrix for SU(2) Monopoles of Arbitrary Charge", Durham University preprint DTP-83/25

22. H. Panagopoulos, "Multimonopoles in Arbitrary Gauge Groups and the Complete SU(2) 2-Monopole System", Phys. Rev. D28, 380 (1983)

23. E. Weinberg, "Parameter Counting for Multimonopole Solutions", Phys. Rev. D20, 936 (1979)

24. S. Brown, H. Panagopoulos and M. Prasad, "Two Separated SU(2) Yang-Mills-Higgs Monopoles in the Atiyah-Drinfeld-Hitchin-Manin Construction", Phys. Rev. D26, 854 (1982)

25. H. Osborn, "On the Atiyah-Drinfeld-Hitchin-Manin Construction for Self-Dual Gauge Fields", Comm. Math. Phys. 86, 195 (1982)

26. E. Witten, "An Interpretation of Classical Yang-Mills Theory", Phys. Lett. 77B, 394 (1978)

27. J. Isenberg, P. Yasskin and P. Green, "Non-Self-Dual Gauge Fields", Phys. Lett. 78B, 462 (1978)

28. E. Corrigan, P. Wainwright, and S. Wilson, "Some Comments on the Non-Self-Dual Nahm Equations", Comm. Math. Phys. 98, 259 (1985)

29. E. Wigner, "Group Theory" (Academic Press 1959) pp. 74, 101

30. R. Gilmore, "Lie Groups, Lie Algebras, and some of their Applications" (Wiley 1974)