Superstructures on graded phase space

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SUPERSTRUCTURES
ON
GRADED PHASE SPACE

by
William Speares

First submitted for the degree of Doctor of Philosophy, Durham University, in April 1988. Resubmitted after amendments in March 1989.

Abstract

In this thesis we study problems associated with the generalisation, to include Grassmann type variables, of the 'group theoretical' approach to quantisation of C.Isham [37]. Although a full generalisation of this quantisation scheme is not achieved, consideration of this problem leads us to make studies in four principle sectors:

(A) Graded Poisson brackets and graded 'vector field like' constructs. A graded version of the Hamiltonian vector field is defined and it is found that both left acting and right acting vector fields are necessary. Properties of these vector fields are investigated.

(B) Local graded canonical transformations and graded function groups. Simple examples of these structures are studied.

(C) The realisation of a general superalgebra by the use of graded 'functions' and the graded Poisson bracket. The graded generalisation of a standard classical result is presented. Also the question of central extensions to these algebras is studied and a partial generalisation of a classical result on this is given.

(D) Investigations into a model of quantum mechanics on a 2-sphere which incorporates fermions. This model is similar to that derived by Spiegelglas [56] and Barcelos-Neto et al.[6,7] from the O(3) supersymmetric sigma model first studied by Witten in [62,63], except that an additional primary constraint has been included. The graded Dirac brackets of this model are calculated.

Statement of research content.

The thesis is organised in two parts. The sections which make up Part I are of an introductory nature and therefore unoriginal. The original work of this thesis is contained in Part II. Section 2.1 is claimed as original in approach and formulation, although later it was discovered that there is some overlap between this section and work by De Witt in [22]. Sections 2.2 and 2.3 are claimed as original. Parts of sections 2.4 and 2.5 are claimed as original - work not claimed as original in these sections is indicated in the text. Finally, the inclusion of an additional constraint in the model presented in section 2.6 is claimed as a new feature, and which results in a substantially different model from that of [6,56].

Dedication

This work is dedicated to my mother and father, for their help and for their support through some difficult years.

Acknowledgements

First I wish to thank the Science and Engineering Research Council for giving me the opportunity to do post graduate research in particle physics at Durham University, between the years of 1983 to 1986, and which by so doing made this work possible. Also I wish to thank Durham University Departement of Mathematical Sciences, the academic staff and the secretaries, for a variety of assistance at different times and for endeavouring to make my stay in Durham a happy one.

I would like to thank the following individuals for their assistance:

Dr. D.B. Fairlie and Dr. E. Corrigan, my supervisors, without whose help, patience and encouragement this thesis would not have come into being.

Professor C. Isham for his understanding and tolerance during the viva voce examination and his numerous suggestions for improvements, and also for drawing my attention to various useful references.

Finally I would like to thank Mellisa Girling and my father for their efforts at typing the various draft copies, and also my sister for proof reading the manuscript and Michael Brookes for the use of his personal computer.

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ERRATA

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(107) $$P = p + C\alpha\beta.\theta\alpha.\pi\beta + \tfrac{1}{2}E(\theta\alpha.\pi\alpha)^2$$

(108) $$Q = q + c\alpha\beta.\theta\alpha.\pi\beta + \tfrac{1}{2}e(\theta\alpha.\pi\alpha)^2$$

(111) $$P = p - \frac{1}{X}\frac{\partial X}{\partial q}(\theta\alpha.\pi\alpha) - \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial q}, X \right\}(\theta\alpha.\pi\alpha)^2$$

(112) $$Q = q + \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha.\pi\alpha) + \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial p}, X \right\}(\theta\alpha.\pi\alpha)^2$$

(116) $$..+ \left(\left\{ \frac{1}{X^2}\left\{ \frac{\partial X}{\partial p}, X \right\}(\theta\alpha.\pi\alpha)^2, p \right\} - \left\{ q, \frac{1}{X^2}\left\{ \frac{\partial X}{\partial q}, X \right\}(\theta\alpha.\pi\alpha)^2 \right\} \right) - ..$$

(118) $$..- \left\{ \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha.\pi\alpha), \frac{1}{X^2}\left\{ \frac{\partial X}{\partial q}, X \right\}(\theta\alpha.\pi\alpha)^2 \right\} - ..$$

(119) $$..- \left\{ \frac{1}{X^2}\left\{ \frac{\partial X}{\partial p}, X \right\}(\theta\alpha.\pi\alpha)^2, \frac{1}{X}\frac{\partial X}{\partial q}(\theta\alpha.\pi\alpha) \right\} - ..$$

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(120) $$..- \left(\left\{ \frac{1}{X^2}\left\{ \frac{\partial X}{\partial p}, X \right\}(\theta\alpha.\pi\alpha)^2, \frac{1}{X^2}\left\{ \frac{\partial X}{\partial q}, X \right\}(\theta\alpha.\pi\alpha)^2 \right\} \right)$$

$$\left(-\frac{1}{X^3}\frac{\partial X}{\partial q}\left\{ \frac{\partial X}{\partial p}, X \right\} - \frac{1}{2X^2}\left\{ X, \frac{\partial^2 X}{\partial q \partial p} \right\} - \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial q}, \frac{\partial X}{\partial p} \right\} \right)(\theta\alpha.\pi\alpha)^2$$

$$- \left\{ \frac{1}{X}\frac{\partial X}{\partial p}, \frac{1}{X}\frac{\partial X}{\partial q} \right\}(\theta\alpha.\pi\alpha)^2$$

$$Pi = pi - \frac{1}{X}\frac{\partial X}{\partial qi}(\theta\alpha.\pi\alpha) - \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial qi}, X \right\}(\theta\alpha.\pi\alpha)^2$$

$$Qi = qi + \frac{1}{X}\frac{\partial X}{\partial pi}(\theta\alpha.\pi\alpha) + \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial pi}, X \right\}(\theta\alpha.\pi\alpha)^2$$

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(121)

$$Pi = pi - \frac{1}{X}\frac{\partial X}{\partial qi}(\theta\alpha.\pi\alpha) - \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial qi}, X \right\}(\theta\alpha.\pi\alpha)^2 - \frac{1}{2.3X^3}\left\{ \left\{ \frac{\partial X}{\partial qi}, X \right\}, X \right\}(\theta\alpha.\pi\alpha)^3$$

(122)

$$Qi = qi + \frac{1}{X}\frac{\partial X}{\partial pi}(\theta\alpha.\pi\alpha) + \frac{1}{2X^2}\left\{ \frac{\partial X}{\partial pi}, X \right\}(\theta\alpha.\pi\alpha)^2 + \frac{1}{2.3X^3}\left\{ \left\{ \frac{\partial X}{\partial pi}, X \right\}, X \right\}(\theta\alpha.\pi\alpha)^3$$

(125)

$$Pi = pi - \frac{1}{X}\frac{\partial X}{\partial qi}(\theta\alpha.\pi\alpha) - \ldots - \frac{1}{N!X^N}\left\{ \ldots\left\{ \frac{\partial X}{\partial qi}, X \right\}, X \right\}, \ldots, X \right\}(\theta\alpha.\pi\alpha)^N$$

(126)

$$Qi = qi + \frac{1}{X}\frac{\partial X}{\partial pi}(\theta\alpha.\pi\alpha) + \ldots + \frac{1}{N!X^N}\left\{ \ldots\left\{ \frac{\partial X}{\partial pi}, X \right\}, X \right\}, \ldots, X \right\}(\theta\alpha.\pi\alpha)^N$$

CONTENTS

INTRODUCTION

The relationship between a quantum theory and the classical theory associated with it is often not straightforward. Broadly speaking there are two 'directions' which this relationship may take. Either a quantum theory arises as a result of taking the constituents of a classical theory and applying a "quantisation map" to them, which relates the classical observables and phase space to linear quantum operators in some Hilbert space. Or the classical theory arises as the result of taking some low energy limit of an a priori quantum theory. An example of this is the Green and Schwarz Superstring theory giving rise to N=1,D=10 Supergravity [54].

In this thesis we are concerned with the first of these two routes. The motivation for the work coming from a desire to generalise to include fermionic variables one particular approach to the construction of such a 'quantistion map': that due to C.Isham and described in his Les Houche lectures [37] and also [36,38]. In this work Isham outlines a general procedure for moving from a classical theory to a quantum theory, the methodology of which centres round the use of the symmetry group of the classical phase space in the construction of the quantum operators of the new theory. There are several features of this group theoretic quantisation scheme which are attractive, for instance the use of a globally well defined framework in which to discuss topological and cohomological aspects of the quantisation process. However, as it stands [37] only deals with theories which are bosonic in nature. The work of this thesis is to study problems associated with generalising this quantisation scheme to include theories which incorporate Grassmannian, which is to say anticommuting, variables. While ultimately a full generalisation of Isham's work in [37] is not achieved, the path to such a fermi/bose unified framework is clarified and some new features of 'classical' theories which include graded variables are found.

The canonical quantisation of classical theories - that is theories in which all the variables commute - through the use of the group theoretical programme described in [37], is in essence taking the principle "replace Poisson brackets by commutator brackets, observables by operators", proposed by Dirac, Heisenberg and others to its logical conclusion. Rather than have this rule for theories without constraints and a different rule for when constraints are present (for example the underlying phase space of the theory might be some general non-flat space. Expressed as an embedding this would result in relationships between the local phase space variables), in [37] it is proposed that quantisation of a classical theory should always revolve round the natural symplectic 2-form of the space under study. The objective of the procedure is to construct the "quantum group" associated with the classical theory under quantisation, and then find irreducible operator representations of the algebra of this group on some Hilbert space. These operators then form the basis of the quantum theory. So how is the quantum group constructed and which set of operators are chosen to represent it ? To construct the quantum group Isham appeals to the group of symplectic translational symmetries of the classical phase space. This symplectic symmetry group, assuming such a group exists, is required to meet various conditions before it can be a suitable candidate for this quantisation process. The action of the group on the underlying phase space must be transitive and almost effective [37], where these conditions are obtained as a result of certain technical considerations. Once this candidate quantum group has been chosen, the next step in the procedure is to realise the algebra of this group using classical observables on the phase space and the natural symplectic 2-form (Poisson bracket) of the theory as the Lie combination principle. If this is possible to do, and sometimes it is not due to the existence of obstructions to this process, then it is this set of classical observables that will determine the structure of the quantum operators of the theory. The complete quantum theory is then uncovered by studying the irreducible representations of these quantum operators, which act in some (to be chosen) Hilbert space and obey the same (or almost the same, the algebra might have to be adjusted to take into account the requirement that the quantum operators are hermitian) commutation relations as their classical counterparts.

The programme described above really involves knowledge of three areas:

(A) the differential geometry of symplectic spaces,

(B) the realisation of Lie transformation groups which act symplectically on those spaces by observables, and

(C) the representation theory of semi-direct products of Lie transformation groups (Mackey theory [43]).

The investigations of this thesis seek to generalise (to include Grassmannian variables) results and constructions on the first two of these subjects, with the thought being ultimately to produce a graded version of the quantisation programme developed by Isham in [37]. The thesis is arranged in two parts, the first consisting of an introduction to some key concepts in classical mechanics and differential geometry and concluded by a brief overview of the group theoretical approach to quantisation found in [37]. And the second, where the original work of this thesis is presented and where graded generalisations of structures and results aired in Part I are produced. The remainder of this introduction will be taken up by discussion of those sections of Part II that are claimed as original.

The original work of this thesis falls under four broad headings as follows:

(A) Investigations into various orthosymplectic structures - graded Poisson brackets, vector fields and so on - found in section 2.1.

(B) The presentation of a class of local graded canonical transformations and the introduction to the concept of the graded function group. This is found in sections 2.2 and 2.3.

(C) Studies concerning problems encountered when attempting to realise a general superalgebra by graded functions on a super phase space: section 2.4.

(D) And finally the last two section 2.5 and 2.6. In the first of these a graded version of the group theoretical quantisation scheme is studied, for a space which is the direct product of Rn with an N-dimensional Grassmann parameter space GN. And in the second, a quantum mechanical model of fermions on a 2-sphere is studied, very similar in construction to the standard O(3) supersymmetric sigma model [6,56,63,64], and the graded Dirac brackets of this model are calculated.

The original contributions presented under each of these general headings will now be examined, starting with a brief description of the work on graded orthosymplectic structures.

Graded orthosymplectic structures.

In order to make a graded generalisation to a quantisation prescription of the type suggested in [37], one must first have a well understood theory of 'classical' dynamics that incorporates Grassmannian variables - preferably stated in a way which makes the role of any supersymmetries in the theory explicit. Such a theory has been developed in a series of papers by R.Casalbuoni and others [8,17,18,19], and also briefly and in a very different setting by B.Kostant [40]. However neither of these approaches seemed suitable to base a generalisation of [37] around: the language of the first was not precise enough, where as conversely the algebraic geometry used in the second seemed over abstract. The investigations of this thesis begin by taking the definition of the graded Poisson bracket derrived by R.Casalbuoni in [17] and trying to couch it in a more 'geometrical' setting (in a sense we will describe shortly). It is well known that the Poisson bracket between two functions on phase space can be obtained from the symplectic 2-form product of the Hamiltonian vector fields associated with those two functions. An immediate question to ask was whether the definition of the graded Poisson bracket in [17] could also be thought of arising in this manner. In order to discuss the notion of the graded vector field one must first have some idea of what a graded manifold-like structure is. Unfortunately the definitions of these objects in the literature tend to be highly technical, but broadly speaking there are two approaches to dealing with these structures. Work has been done by F.A. Berezin et al. [13,15], B. Kostant [40] and others, using techniques taken from algebraic geometry, on what are called 'graded manifolds'. Where as B. De Witt originated an alternative 'supermanifold' approach [22]. Work has also been done on variations of supermanifolds by M. Batchelor [10, 11] and A.Rogers [49,50,51] (we will return to this in a moment). The approach we take here is close to that of [22], which the author only descovered after the completion of section 2.1 (the text cites where any duplication has occurred). So as to be able to proceed further with the graded vector field question without becoming overly technical, we assume the existence of some underlying graded manifold-like structure whose local co-ordinate expansion is of the form: $(x_1, \ldots, x_n; \theta_1, \ldots, \theta_N)$, with the x_i variables commuting and the θ_α variables belonging to some anticommuting Grassmann algebra. The precise nature of the underlying space is not defined, but we assume such a definition is possible. Under this assumption we produce a graded generalisation of the Hamiltonian vector field which is structured from the definition of the graded Poisson bracket made by Casalbuoni in [17]. From these investigations the following structural feature of the graded Poisson bracket becomes clear. For the graded bracket to be thought of as arising from a graded Hamiltonian vector field, two sorts of graded vector field are required: a field associated with even generators which acts to the left, and a field associated with odd generators which acts to the right. These left acting and right acting fields are built up from left and right acting derivatives respectively, as defined in F.A. Berezin's book [12]. This result was arrived at independently, although the idea of a dual vector field structure is closely linked to Berezin's two types of graded form [13]. Throughout the remainder of this section the properties of the left and right acting graded Hamiltonian vector field are explored, in particular how these graded vector fields combine under graded commutation - the classical result on this type of combination is generalised. Also the 'function of a function' Poisson bracket identity is extended to the graded case. The 'symmetrisation with respect to handedness' first observed in the graded Hamiltonian vector field is found to carry through as a general feature of graded vector calculations, and it is postulated at the end of this section that the natural structural object of this type of graded manifold is the 'left/right' tensor.

In fact De Witt's book [22] also explores the properties of objects very similar to the left/right tensor suggested in section 2.1.7. For this reason we did not persue further structural development of the graded vector fields introduce in section 2.1. The similarities which exist between the dual 'left/right' structure we uncover here and which is further developed in [22], and the work of Berezin et al [13,15] on the Kostant type graded manifold, might seem to point to a more subtle correspondance. This is actually the case. The Kostant approach studies the geometry of the space through the algebraic structure of its sheaf of functions [40], where as the De Witt supermanifold incorporates the Grassmann variables into a manifold-like structure from the outset. After the cessation of work on this subject, the work of M.Batchelor [10,11] was brought to the author's attention *. In [11] it is demonstrated that the category of Kostant graded manifolds are equivalent to the category of De Witt supermanifolds. Hence, the type of structural similarity between these two approaches that we descover in this work is not incredible, but is merely an expression of the fundamental equivalence of these theories.

Graded canonical transformations and graded function groups.

We now discuss the work on graded canonical transformations and graded function groups. The work presented in sections 2.2 and 2.3 is claimed as being wholley original in content. In 2.2 a class of local graded canonical transformations is introduced and explored, using at first a graded phase space consisting of one even and one odd conjugate pair, and then extending this to higher orders in even and odd conjugate pairs of variables. In the simplest case a local'co-ordinate net' $(q,p;\theta,\pi)$ with variables which satisfy the graded fundamental Poisson brackets [17], is transformed into a second co-ordinate net $(Q,P;\Sigma,\Gamma)$ which we demand also satisfy the fundamental brackets. It is found that the form of the 'twisted' variables $(Q,P;\Sigma,\Gamma)$ may be determined up to an arbitary function of the bosonic sector of the phase space. This result is generalised for arbitary dimension in both odd and even sectors of the phase space. Interestingly, it is found not possible to generalise the result by making the arbitary function graded. However, it <u>is</u> possible to raise the set of transformations to form a group, by demanding that the underlying bosonic functions satisfy two partial differential equations as conditions (by this we mean that the super Jacobi identities are satisfied by the associated graded Hamiltonian vector fields, but only for special sets of generator functions). A seperable variable solution of these conditions is given.

* The author wishes to thank Professor Isham for bringing these references to his attention.

The highly constrained requirement that the graded fundamental Poisson brackets be satisfied, makes extension of these results difficult. The transformations we describe above all depend on a conformal-like scaling in the fermionic sector, which seems highly resistant to generalisation of any sort. There is some cause to believe that more interesting structures in the fermionic sector are possible if one imposes conditions on the bosonic functions which generate the symmetries (for instance demanding that they form a commuting function group), but this would have to be an objective of further research. It is clear that these structures represent some interplay between the odd and even sectors of the graded Poisson bracket, but just how much more is waiting to be uncovered here is hard to estimate.

In section 2.3 the notion of the graded function group is introduced. Before discussing this work we look briefly at why the classical version of this concept is important.

The standard reference quoted on function groups is surprisingly still Eisenhart's book [27], which was written at a time before the modern index free differential geometric language was widely in use. In [27] a function group is defined as a collection of functions fi, which is closed under Poisson bracket combination. Thus: $\{ f_i, f_j \} = \Phi_{ij}(f)$, where Φ_{ij} is some function of the fi's antisymmetric in i and j. Eisenhart proves a number of theorems about function groups (two of which are quoted in Part I) and which indicate to what extent one is 'free' to reduce a given set of canonical variables to a smaller set, which also obeying the canonical commutation relations. It is this process which underpins the construction of the Dirac bracket, first introduced in Dirac's paper [25], and which under certain conditions is actually just the Poisson bracket of a special reduced set of variables (for a good contemporary account of this see N. Mukunda and E.C.G. Sudarshan's book [44]). When such a reduction is possible, the reduced set of variables can be thought as representing the 'true' variables of the theory, in the sense that each one corresponds to some actual physical degree of freedom of the system (up to some possible rotations of basis). This process of 'phase space reduction' - that is stating a constrained theory in a manner which manifestly preserves the constraints - is one which potentially has great interest for theories involving Grassmannian variables. Graded theories generically have large amounts of constraints associated with them (for instance see P.Fayet and S.Ferrara's review [29]), which by their nature are hard to grasp intuitively. Discovering whether it is possible to 'lock in' graded constraints by the local redefinition of graded variables, in a fashion analogous to the classical case, could be interesting from both a mathematical and physical perspective, and it was for this reason that the investigations of section 2.3 were undertaken.

In section 2.3 a simple example of the formation of a graded function group is studied, corresponding to the reduction of a canonical set of two even and two odd variables $(q,p;\theta,\pi)$ to a pair of graded variables, where these variables form a simple graded function group. There are three possibilities for the graded character of this pair of variables, and each of these is studied in turn. The graded Jacobi identities impose strong conditions on the form of the function group in each case, ruling out in this simple example any truely non trivial 'mixed' odd-even and odd-odd function group. However, even though the form of the group in these cases is heavily constrained, the functional form of the two graded variables in the group show what we term a 'compression' of the graded canonical transformation found in the previous section. It is intriguing to speculate that the apparent difficulty in generalising the canonical transformation in section 2.2.1 to include anything more than a real scaling function in the fermionic sector, together with the more general form of this variable in the 'compressed' example of section 2.3.2, might hint of a more fundamental result. In the case of the even-even function group, the structure of the standard bosonic sector of the group dominates the overall form of the two even functions. Some speculation as to the underlying nature of the graded function group concludes this section.

Graded Poisson bracket realisations.

Section 2.4 contains work on the realisation of a general superalgebra by odd and even functions over a graded phase space. The Lie combination rule is provided by the graded Poisson bracket introduced earlier. In the first subsection the question of graded central extensions is studied. The objective of this work is to try and reproduce Whitehead's theorem [37] on cocycles in a graded setting. Broadly speaking cocycles are numbers that can appear on the R.H.S. of some (super) Lie algebras, when one attempts to realise those algebras by, for example, functions on (super) phase space. They arise in a variety of settings and are endemic to particular classes of Lie algebra [37,39,59,65]. Our interests lie in 'removing them', as they represent an obstruction to the quantisation process. Provided the superalgebra satisfies various conditions all, with the exception of one caveat, similar to the classical case, it is found that graded central extensions associated with these realisations of superalgebras may be removed. In the second subsection a graded generalisation of a well known classical realisation is presented. This result appears in a less general form in [8], and deals with the case when the dimensions of the graded phase space exactly match the numbers of odd and even generators of the superalgebra. The graded Hamiltonian vector fields associated with this realisation are calculated.

Graded examples.

The last two sections of the thesis contain graded examples designed to illustrate some of the concepts introduced in the text. In the first of of these, section 2.5, we examine how a graded generalisation of the group theoretical approach to quantisation might look when applied to a 'flat' space of real and Grassmann variables, mimicing the example of the quantisation of Rn in Part I. Although this work has not been done elsewhere to the author's knowledge, the lack of a fully developed theory means that it is only of limited use at this stage. However it does serve to illustrate the previous work in section 2.4.1 on cocycles occurring in a 'non removable' manner. This section also outlines the form a graded generalisation of the group theoretical approach to quantisation might be expected to take.

Perhaps of greater interest is the example presented in section 2.6. Here we study a model of quantum mechanics on a 2-sphere with fermions. The model is derived in a simple way using an argument first presented by E. Witten in [63,64] for the supersymmetric sigma model, and also used by J. Barcelos-Neto et al in [6,7] and by M. Spiegelglas in [56]. The model we use differs from that used in the afore mentioned papers by the inclusion of an extra constraint, which arises in a simple way from the natural superfield constraint for this model [56]. The presence of this extra constraint results in some substantial structural differences occuring to the model. While the manifestly supersymmetric construction of the model ensures that all the primary constraints are preserved under supersymmetry transformations, it is found that the secondary constraints required to maintain the closure of the system under time developement are not all supersymmetric invariants. This casts some doubt on whether this model is in fact supersymmetric or not, and represents a substantial departure from the work in [6,56]. These differences become even more apparent when the graded Dirac brackets of the model are calculated.

Here the presence of the extra primary constraint causes non-trivial additions to the Dirac brackets between the fundamental variables of the theory, when compaired with similar bracket calculations performed in [6]. This work is claimed as original, though similar in formulation to work in [6] and [56]. Further work is required to understand fully the relationship between the model we use here, incorporating the extra primary constraint, and the model studied in [6,56]. The model might also be useful as a theoretical laboratory to study the graded group theoretical quantisation process in a non trivial setting. The topology of the sphere might make it possible to insert a ´twist´ into the coupling together of the quantum theory on the local graded co-ordinate patches covering the sphere, with potentially interesting consequences* (the bosonic version of this idea is explored in [37]). It is regretable that sufficient time is not availible to investigate this possibility.

* The author wishes to thank Professor Isham for suggesting this avenue for future investigation.

PART I

1.0.0 Classical Mechanics Overview.

There exist many excellent accounts of classical mechanics in mathematical physics literature, varying widely in sophistication, abstration and style. Two texts which illustrate this well are Abrahams and Marsden [1] and Mukunda & Sudarshan [44], and the purpose of the introductory first half of this thesis is not to attempt to give another, more of less comprehensive, account of classical mechanics - that would not be useful in aiding the illumination of the second part of the work. What the first part is intended to do is merely to introduce some of the relationships and ideas, which in the second part of the thesis are looked at from a graded point of view. In this sense Part I should be viewed as being a detailed guide to Part II.

The classical mechanics described and introduced below is draw chiefly from [33],[44] and [58].

1.0.1 Lagrange's Equations.

It turns out that Newton's Laws of Motion can be stated in a very elegant fashion using the notion of generalised co-ordinates and velocities to describe completely some mechanical system, and a function of them know as the Lagrangian. A particularly fast way of reaching the equation's of motion is by the use of Hamilton's Stationary Action Principal. The system is described by K generalised co-ordinates qs, s = 1,...,K and by K generalised velocities $\dot{q}s$, s = 1,...,K, where qs and $\dot{q}s$ are all smooth functions of a real time parameter denoted t. And also by the Lagrangian function $L = L(qs,\dot{q}s,t)$. We define an action integral, thus:

$$A(qs(t)) := \int L(qs,\dot{q}s,t)dt \qquad (1)$$

Where the integral is carried out over some evolutionary path $C(t_0,t_1)$ of the system, determined by the functions qs(t), for $t = t_0$ to $t = t_1$ and where $t \in (t_0,t_1) \subset R$.

We now demand that this action is stationary with respect to variations round the classical path $C(t_0, t_1)$. That is:

(2) $$\delta A := A(C') - A(C) = 0$$

With:

(3) $$C' = C + \delta C \quad , \quad \delta C = 0 \mid t_0, t_1$$

(4a) $$q's(t) = qs(t) + \delta qs(t)$$

(4b) $$\dot{q}'s(t) = \dot{q}s(t) + \delta\dot{q}s(t)$$

(4c) $$\delta qs(t)|t_0, t_1 = 0$$

Writing:

$$L' := L(q's, \dot{q}'s, t) = L(qs+\delta qs, \dot{q}s+\delta\dot{q}s, t)$$

and Fourier expanding in powers of δqs we are led to the following expression for δA (up to 1st order in δqs):

(5) $$\delta A = \int \left(\frac{\partial}{\partial qs} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}s}\right)\right) L dt$$

Demanding stationary action $\delta A = 0$ yields the following K partial differential equations for $qs(t)$:

(6) $$\frac{\partial L}{\partial qs} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}s}\right) = 0 \quad , \quad \text{where } s = 1, \ldots, K$$

These are Lagrange's equations of motion, equivalent to Newton's equations, and which must be solved to determine the $qs(t)$ and find the time evolution of the system. In the above equations, the functions qs and qs are regarded as being functionally independant, although, clearly, $\dot{q}s = \frac{dqs}{dt}$.

Actually the Lagrangian formulation of mechanics is not the most useful to employ when moving between classical mechanics and quantum mechanics using the route first taken by Heisenberg and Dirac - historically the formulation due to Hamilton has played a stronger role. For this reason we now continue by describing in a formal manner Hamilton's approach to classical mechanics, which results in 2K first order equations of motion, rather than the K second order equations above.

1.0.2 Hamilton's Equations.

Proceeding formally at this stage following Arthurs [3], we assume it is possible to introduce a new variable $p_s(t)$, independent of the generalised co-ordinates $q_s(t)$ and defined:

$$(7) \qquad p_s := \frac{\partial L(q_s,q_s,t)}{\partial \dot{q}_s} \quad , \qquad s = 1,\ldots,K$$

Where $L(q_s,q_s,t)$ is the previously introduced Lagragian function. The variables $p_s(t)$ are said to be conjugate to the generalised co-ordinates $q_s(t)$ and are called 'generalised momenta'. Furthermore, we assume that (7) may be solved to give <u>all</u> the $\dot{q}_s$ in terms of (q_s,p_s,t), that is the system is non-singular (we will return to this condition later). We now define a new function $H(q_s,p_s,t)$ thus:

$$(8) \qquad H(q_s,p_s,t) := \sum_s p_s\dot{q}_s - L(q_r,q_r,t)$$

This function H is known as Hamiltonian. We now rewrite the action integral (1) in terms of the Hamiltonian:

$$(9) \qquad A(q_s,p_s) = \left(\sum_s p_s\dot{q}_s - H(q_s,p_s,t)\right)dt$$

where $q_s(t)$, $p_s(t)$ are independant functions of t. Applying the stationary action principle as before, equation (2) leads us to the following Lagrange's equations:

$$(10a) \qquad \left(\frac{\partial}{\partial q_s} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_s}\right)\left(\sum_r p_r\dot{q}_r - H\right) = 0 \quad , \qquad r,s = 1,\ldots,K$$

$$(10b) \qquad \left(\frac{\partial}{\partial p_s} - \frac{d}{dt}\frac{\partial}{\partial \dot{p}_s}\right)\left(\sum_r p_r\dot{q}_r - H\right) = 0 \quad , \qquad r,s = 1,\ldots,K$$

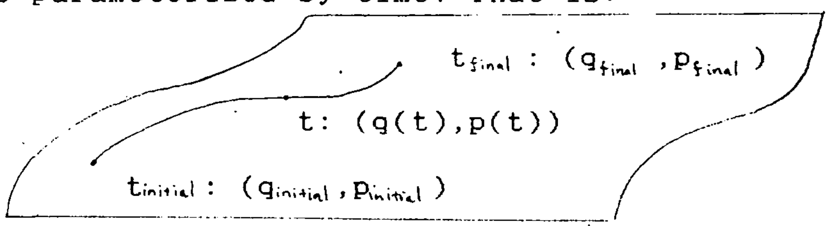
(10a) and (10b) give us the following:

$$(11a) \qquad -\frac{\partial H}{\partial q_s} - \dot{p}_s = 0 \quad , \qquad s = 1,\ldots,K$$

$$(11b) \qquad -\frac{\partial H}{\partial p_s} + \dot{q}_s = 0 \quad , \qquad s = 1,\ldots,K$$

These 2K first order P.D.E's are known as Hamilton's equations and are equivalent to the K second order Lagrange equations (6).

In the Lagrangian description of dynamics the fundamental variables are $(q_s,\dot{q}_s,t)$. With the Hamiltonian approach they are (qs,ps,t). In the language of differential geometry, Lagrangian mechanics takes place on the 'tangent bundle', with local co-ordinates (qs,$\dot{q}$s), of the configuration space. Where as Hamiltonian mechanics taken place on the 'cotangent bundle', or phase space with (qs,ps) as local co-ordinates, of the configuration space. One can picture the time evolution of a physical system as a path through phase space parameterised by time. That is:



The path, or phase space trajectory, is determined by the initial values of the generalised variables ($q_{initial}$, $p_{initial}$) and by equations (11a) and (11b). The Hamiltonian H(q,p,t) is the energy function of the system. We now introduce the key notion of the 'Poisson bracket'.

1.0.3 The Poisson bracket.

Consider the time derivative of a function f defined on phase space (qs,ps), f = f(qs,ps,t), where f: {Phase space} ---> R, thus:

(12) $$\frac{df(q,p,t)}{dt} = \sum_s \left(\frac{dqs}{dt}\frac{\partial f}{\partial qs} + \frac{dps}{dt}\frac{\partial f}{\partial ps} \right) + \frac{\partial f}{\partial t} , \quad s = 1,\ldots,K$$

Now let us employ equations (11a) and (11b):

(13) $$\frac{df}{dt} = \sum_s \left(\frac{\partial f}{\partial qs}\frac{\partial H}{\partial ps} - \frac{\partial f}{\partial ps}\frac{\partial H}{\partial qs} \right) + \frac{\partial f}{\partial t} , \quad s = 1,\ldots,K$$

We now define the Poisson bracket of two functions A and B defined on phase space, denoted { A,B } to be:

(14) $$\{ A,B \} := \sum_s \left(\frac{\partial A}{\partial qs}\frac{\partial B}{\partial ps} - \frac{\partial A}{\partial ps}\frac{\partial B}{\partial qs} \right) , \quad s = 1,\ldots,K$$

In this notation (12) becomes:

(15) $$\frac{df}{dt} = \{ f,H \} + \frac{\partial f}{\partial t}$$

The property of skew-symmetry of the P.B. implies immediately:

(16) $$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Thus change with time of the Hamiltonian can only come about via an explicit time dependence, thus energy is conserved. More special cases of equation (13) and definition (14) are:

(17)

$$\dot{q}s = \{ qs,H \} \quad , \quad \dot{p}s = \{ ps,H \}$$

$$\{ qs,qt \} = 0 = \{ ps,pt \} \quad , \quad \{ qs,pt \} = \delta st$$

These last three relationships are called the <u>fundamental Poisson brackets</u> of the system. The following properties of the Poisson bracket may be verified by direct calculation:

(P1) Skew-Symmetry:

$$\{ A,B \} = - \{ B,A \} \quad , \quad \text{for } A,B,C \in F(M)$$

(P2) Linearity:

$$\{ aA+bB,C \} = a\{ A,C \} + b\{ B,C \} \quad , \quad \text{for } a,b \in R$$

(P3) Product Rule:

$$\{ AB,C \} = A\{ B,C \} + \{ A,C \}B \quad , \quad \text{for } A,B,C \in F(M)$$

(P4) Jacobi Identity:

$$\{ A,\{ B,C \} \} + \{ C,\{ A,B \} \} + \{ B,\{ C,A \} \} = 0$$

Where F(M) is the set of suitably differentiable function over the phase space M.

At this stage we make the observation that (P1),(P2) and (P4), together define an infinite dimensional lie algebra over the function space F(M), with the Poission bracket acting as the combination principle. The Poisson bracket (P.B.) is the fundamental structural object of classical mechanics and forms the starting point for the so called ´canonical quantisation´ of dynamical systems. In more geometrical language, in defining the P.B. on phase space, one is requiring that there exist a closed, non-degenerate, 2-form on the cotangent bundle of the configuration space. This co-ordinate free approach is the natural one in which to address global questions, and will be briefly described later in ∮ 1.2.

1.0.4 <u>Canonical Transformations</u>.

A canonical transformation is one which does not alter the functional form of Hamilton´s equations of motion. More precisely, given a transformation of phase space co-ordinates thus:

(18) $Qs = Qs(q,p,t) \quad , \quad Ps = Ps(q,p,t) \quad , \quad \text{for } s = 1,...,K$

where Qs,Ps are 2K independent functions of (qs,ps,t). We demand that there exist a new Hamiltonian H′(Q,P,t) such that:

$$(19) \qquad \dot{Q}s = \frac{\partial H'}{\partial Ps} \quad , \quad \dot{P}s = -\frac{\partial H'}{\partial Qs} \quad , \quad \text{for } s = 1,\ldots,K$$

This requirement implies that the two Lagrangians L(q,p,t) and L′(Q,P,t) differ only by a total derivative. That is:

$$(20) \qquad \sum_s Ps\dot{Q}s - H'(Q,P,t) = \sum_s ps\dot{q}s - H(q,p,t) + \frac{dF}{dt} \quad , \quad s = 1,\ldots,K$$

(Note that we exclude two transformations from our definition:

qs ---> ps and (qs,ps) ---> (fqs,fps) , f ∈ F(M).

That is, interchange of qs & ps and dilations.)

Following [44], if we expand the R.H.S. of (20) in terms of Qs and Ps , s = 1,...,K , and compare coefficients, we obtain 2K+1 conditions:

$$(21) \qquad Ps - \sum_t pt\frac{\partial qt}{\partial Qs} = \frac{\partial F}{\partial Qs}(Q,P,t) \quad , \quad s,t = 1,\ldots,K$$

$$(22) \qquad - \sum_t pt\frac{\partial qt}{\partial Ps} = \frac{\partial F}{\partial Ps}(Q,P,t) \quad , \quad s,t = 1,\ldots,K$$

$$(23) \qquad H'(Q,P,t) = H(q,p,t) - \sum_s ps\, qs - \frac{\partial F}{\partial t}(Q,P,t)$$

For F(Q,P,t) to exist the following integrability conditions, which are obtained directly from (21),(22) and (23) must be satisfied

$$(24) \qquad \sum_s \left(\frac{\partial qs}{\partial Qt}\frac{ps}{Qr} - \frac{\partial qs}{\partial Qr}\frac{\partial ps}{\partial Qt} \right) = 0 \quad , \quad r,s,t = 1,\ldots,K$$

$$(25) \qquad \sum_s \left(\frac{\partial qs}{\partial Pt}\frac{\partial ps}{\partial Pr} - \frac{\partial qs}{\partial Pr}\frac{\partial ps}{\partial Pt} \right) = 0 \quad , \quad r,s,t = 1,\ldots,K$$

$$(26) \qquad \sum_s \left(\frac{\partial qs}{\partial Qt}\frac{\partial ps}{\partial Pr} - \frac{\partial qs}{\partial Pr}\frac{\partial ps}{\partial Qt} \right) = \delta rt \quad , \quad r,s,t = 1,\ldots,K$$

The form of these conditions leads us to define the ′Lagrange bracket′ of 2K independent functions of qs,ps denoted Ri, as the following:

$$(27) \qquad (Ri,Rj) := \sum_s \left(\frac{\partial qs}{\partial Ri}\frac{\partial ps}{\partial Rj} - \frac{\partial qs}{\partial Rj}\frac{\partial ps}{\partial Ri} \right) \quad , \quad \text{for } i,j = 1,\ldots,2K$$

Where it is required that the qs,ps may be written as functions of the Ri′s , for i = 1,..., 2K. In this notation the integrability conditions (24) to (26) become:

(24a) $(Q_t, Q_r) = 0$, $t,r = 1,\dots,K$

(25a) $(P_t, P_r) = 0$, $t,r = 1,\dots,K$

(26a) $(Q_t, P_r) = \delta_{tr}$, $t,r = 1,\dots,K$

A simple calculation reveals the true nature of the Lagrange bracket:

$$\sum_j (R_i,R_j).\{ R_j,R_k \} = \sum_{s,j} \left(\frac{\partial q_s}{\partial R_i}\frac{\partial p_s}{\partial R_j} - \frac{\partial q_s}{\partial R_j}\frac{\partial p_s}{\partial R_i} \right)\left(\frac{\partial R_j}{\partial q_t}\frac{\partial R_k}{\partial p_t} - \frac{\partial R_k}{\partial q_t}\frac{\partial R_j}{\partial p_t} \right)$$

$$= - \sum_{s,t} \left(\frac{\partial q_s}{\partial R_i}\frac{\partial R_k}{\partial q_t}\delta_{st} - \frac{\partial p_s}{\partial R_i}\frac{\partial R_k}{\partial p_t}\delta_{st} \right) = -\delta_{ik} \;, \quad \text{with } i,j,k = 1,\dots,2K$$

That is the Lagrange bracket is the inverse of the Poisson bracket. This means we may write (24a) to (26a) in the following form :

(28) $\{ Q_t,Q_s \} = 0$, $s,t = 1,\dots,K$

(29) $\{ P_s,P_t \} = 0$, $s,t = 1,\dots,K$

(30) $\{ Q_s,P_t \} = \delta_{st}$, $s,t = 1,\dots,K$

Thus we see that the fundamental P.B.'s (17) have been preserved by the canonical transformations (18). To clarify what is going on here we introduce a new notation (for further details of this see for example [44] or [58]) and look at the canonical transformation from a more group theoretical stand point. However, before proceeding with the group theory of the Poisson bracket, we state without proof the following easily verifiable identity:

(P5) $$\{ F,G \} \equiv \sum_{i,j} \frac{\partial F}{\partial h_i} \{ h_i,h_j \} \frac{\partial G}{\partial h_j}$$

Where $h_i = h_i(q,p) \in F(M)$, $i,j = 1,\dots,2K$, and $F = F(h)$, $G = G(h)$.

1.0.5 <u>Group theoretical aspects of canonical transformations</u>.

So far we have treated the canonical variables q_s and p_s separately. However, to get at the invariance aspects of the P.B. it is easier to unify them and to define a general phase space co-ordinate:

(31) $(x_\alpha) = (q_1,\dots,q_i,\dots,q_n;p_1,\dots,p_i,\dots,p_n)$, with $\alpha = 1,\dots,2n$

In this notation the P.B of two observables $F(q,p)$, $G(p,q)$ becomes:

(32) $$\{ F,G \} = \frac{\partial F}{\partial x_\alpha} \{ x_\alpha,x_\beta \} \frac{\partial G}{\partial x_\beta} = \frac{\partial F}{\partial x_\alpha} \Gamma_{\alpha\beta} \frac{\partial G}{\partial x_\beta}$$

Where from now on we use the repeated index summation convention.

Direct calculation shows that the matrix $\Gamma\alpha\beta$ has the following form:

(33)
$$(\Gamma\alpha\beta) = \begin{pmatrix} 0n & 1n \\ -1n & 0n \end{pmatrix}$$

where 0n and 1n are the n x n block matrices:

(34)
$$0n = \begin{pmatrix} 0 & \dots & 0 \\ & 0 & \\ 0 & \dots & 0 \end{pmatrix} \quad , \quad 1n = \begin{pmatrix} 1 & \dots & 0 \\ & 1 & \\ 0 & \dots & 1 \end{pmatrix}$$

This is the fundamental P.B. in the unified notation. The matrix $(\Gamma\alpha\beta)$ has the following properties:

(35)
$$\Gamma^T = -\Gamma$$

(36)
$$\Gamma^2 = -1$$

We can now state a canonical transformation in a far simpler way. For the fundamental P.B. to be preserved in a transformation:

(37)
$$x \dashrightarrow x' \quad , \quad \text{where} \quad x'\alpha = x'\alpha(\dots x\beta \dots)$$

we require:

(38)
$$\{ x'\alpha, x'\beta \} = \Gamma\alpha\beta$$

However,

$$\{ x\alpha, x\beta \} = \frac{\partial x\alpha}{\partial x\mu} \{ x\mu, x\tau \} \frac{\partial x\beta}{\partial x\tau} .$$

but since $\{ x\alpha, x\beta \} = \Gamma\alpha\beta$ for a canonical transformation, we have:

(39)
$$\frac{\partial x\alpha}{\partial x\mu} \Gamma\mu\tau \frac{\partial x\beta}{\partial x\tau} = \Gamma\alpha\beta$$

Or, if we call

(40)
$$(\Sigma\alpha\beta) = \left(\frac{\partial x\alpha}{\partial x\beta}\right)$$

Then we may write the condition (39) as:

(41)
$$\Sigma \, \Gamma \, \Sigma^T = \Gamma$$

It is easy to show two canonical transformations carried out in succession is also a canonical transformation, and clearly identity and inverse transformation exist. It follows that the set of matrices Σ satisfying (41) form a group. This group is called the symplectic group, denoted Sp(2n,R), of 2n $\otimes$ 2n dimensional matrices which preserve the sympletic form Γ. Also it is easy to see that the P.B. of two functions w.r.t. two different, but canonically related co-ordinate sets are equal. That is:

(42) $$\{ A(x'),B(x') \}_{x'} = \{ A(x'(x)),B(x'(x)) \}_x$$

The canonical group of those transformations (37) whose Jacobians (40) satisfy (41) is infinite dimensional as a set. Various sub-groups of the canonical group exist, for example the subgroup of canonical transformations which satisfy the following relationship:

(43) $$\sum_s P_s dQ_s = \sum_s p_s dq_s$$

These are know as "contact transformations", and have an elegant geometrical interpretation as the preservation of the Louville form [37]. More important for the canonical quantisation path between the classical and the quantum, is the notation of a 1-parameter sub-group of canonical transformations sometimes known as a 'canonical flow'. As will be shown, the time development of a mechanical system is one such canonical flow. Associated with these 1-parameter sub-groups are certain functions, called observables, which are said to 'generate' the transformations. To discuss this firstly we need the canonical transformation in the infinitesimal.

1.0.6 Infinitesimal canonical transformations.

Following the analysis in [44] and [58], let us now make an infinitesimal change to the variables (x_α) and demand that (39) is still satisfied:

(44) $$x_\alpha \longrightarrow x_\alpha + \delta x_\alpha \quad , \quad \text{for } \alpha = 1,\ldots,2K$$

substituting in (39):

(45) $$\frac{\partial(x_\alpha + \delta x_\alpha)}{\partial x_\mu} \Gamma_{\mu\tau} \frac{\partial(x_\beta + \delta x_\beta)}{\partial x_\tau} = \Gamma_{\alpha\beta}$$

Expanding (45), and neglecting $O(\delta)$ terms we obtain:

(46) $$\frac{\partial(\delta x_\alpha)}{\partial x_\mu} \Gamma_{\mu\tau} \delta_{\tau\beta} + \delta_{\mu\alpha} \Gamma_{\mu\tau} \frac{\partial(\delta x_\beta)}{\partial x_\tau} = 0$$

or $$\frac{\partial(\delta x_\alpha)}{\partial x_\mu} \Gamma_{\mu\beta} - \frac{\partial(\delta x_\beta)}{\partial x_\mu} \Gamma_{\mu\alpha} = 0$$

Multiplying through by $\Gamma_{\tau\alpha}\Gamma_{\phi\beta}$ and using $\Gamma^2 = -1$ gives:

(47) $$\frac{\partial(\delta x_\beta \Gamma_{\phi\beta})}{\partial x_\tau} - \frac{\partial(\delta x_\alpha \Gamma_{\tau\alpha})}{\partial x_\phi} = 0$$

Equation (47) implies that $\delta x\alpha\ \Gamma\tau\alpha$ is in fact a gradient of some function $\theta f(x)$ say, where $\theta \ll 1$. That is:

(48)
$$\delta x\alpha\ \Gamma\tau\alpha = -\frac{\partial(\theta f)}{\partial x\tau}$$

Thus (44) becomes:

(49)
$$x'\alpha = x\alpha + \theta\Gamma\alpha\tau\frac{\partial f}{\partial x\tau} = x\alpha + \theta\{ x\alpha, f \}$$

The function f(q,p) associated in this way with the infintesimal canonical transformation (44) is called the 'generator' of the infinitesimal transformation.

Further insight into the P.B. is obtained by taking the commutator of two infinitesimal transformations, each associated with different functions f and g, say. Denoting these by:

(50)
$$(\delta 1f)x\alpha := \theta 1\{ x\alpha, f \}$$

(51)
$$(\delta 2g)x\alpha := \theta 2\{ x\alpha, g \}$$

and taking the commutator of these transformations:

$$\begin{aligned}[\delta 1f,\delta 2g]x\alpha &:= (\delta 1f\delta 2g - \delta 2g\delta 1f)x\alpha \\ &= \delta 1f(\mu 2\{x\alpha,g\}) - \delta 2g(\mu 1\{x\alpha,f\}) \\ &= \mu 1\mu 2\{\{x\alpha,g\},f\} - \mu 1\mu 2\{\{x\alpha,f\},g\}\end{aligned}$$

Now employing the Jacobi identity gives:

(52)
$$\begin{aligned}[\delta 1f,\delta 2g]x\alpha &= \mu 1\mu 2\{x\alpha,\{g,f\}\} \\ &:= -\ \mu 1\mu 2\ \delta\{f,g\}x\alpha\end{aligned}$$

So we see that the commutator of two infinitesimal canonical transformations is itself an infinitesimal canonical transformation, whose generator is the (minus) P.B. of the generators of those two transformations. Thus the set of infinitesimal canonical transformations form a Lie algebra whose combination bracket is the (negative) Poisson bracket.

By employing (13) we see that Hamilton's equations in this notation become simply:

(53)
$$\frac{dx\alpha}{dt} = \{x\alpha,H\} + \frac{\partial H}{\partial t}$$

Thus we see that the time development of the system for a non-explicitely time-dependent Hamiltonian is none other than the unfolding of a 1-parameter canonical transformation in which the Hamiltonian is the generator function.

At this stage we raise one other point concerning infinitesimal canonical transformations. Suppose a collection of functions of qi and pj denoted fi, for i = 1,...,N , satisfy:

(54) $$\{ f_i, H \} = 0 \quad , \quad \text{for all } N$$

Clearly (15) implies that the functions fi are conserved quantities, sometimes known as conserved charges, which do not change with the time evolution of the system. Since the Poisson bracket satisfies the Jacobi identity, it follows that any combination { fi,fj } also commutes with the Hamiltonian. In fact we are able to give the set of functions {fi} the structure of a Lie algebra under Poisson bracket combinations. Hamiltonians of this sort are said to exhibit an internal symmetry. A good example of this behavior is given by the first class constraints in § 1.3, which produce an effect rather similar to gauge invarience in Yang-Mills theory [4,58]. This will be discussed in more detail later on.

1.1.0 An Introduction to some Differential Geometry.

Before proceeding further with the structure of classical mechanics, we will briefly look at the some of the concepts and language of differential geometry. This will enable us to reformulate some of the previous results in a far more compact and illuminating fashion. The notation we use here is drawn principly from Hawking and Ellis [35] and O'Neill [47].

1.1.1 Definitions of fundamental objects.

A topological space S is a pair of objects (X,U), where X is a non empty set and U is a topology on X. That is, a class of subsets of X which is closed under the formation of arbitary unions and finite intersections.

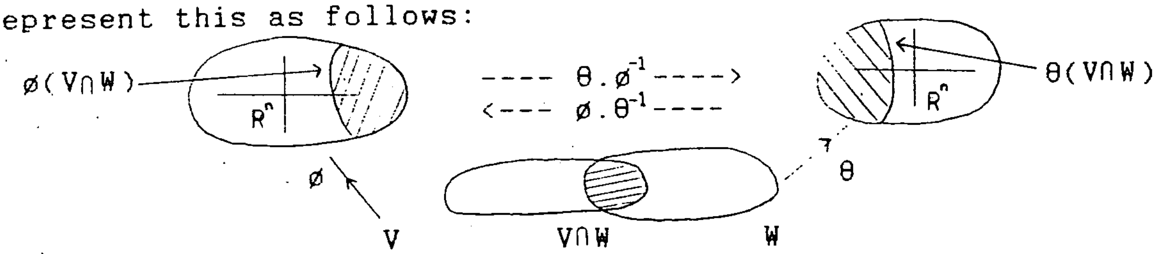
A topological space is called a <u>Hausdorff space</u> if for every pair of distinct points x and y in X there exist neighborhoods V and W such that $x \in V$, $y \in W$ and $V \cap W$ = Empty Set.

A <u>co-ordinate system</u> or <u>chart</u>, of dimension n in a topolgical space S, is a homeomorphism ø of an open set V of S onto an open set ø(V) of Rn. That is :

$$ø(p) = (ø_1(p), \ldots, øn(p)) \quad \text{for each } p \in V ,$$

where the functions $ø_1, \ldots, øn$ are called the co-ordinate functions of ø, which has <u>dimension</u> n .

Two n-dimensional charts (ø,V) and (θ,W) in a Hausdorff space S are said to <u>overlap smoothly</u> provided the functions ø.θ and $θ.ø^{-1}$ are infinitely differentiable. Diagramatically we can represent this as follows:



An infinitely differentiable n-dimensional <u>atlas</u> A on a Hausdorff space S, is a collection of charts (øi,Ui), $i \in P$, where P is a labelling set, such that:

a) $S = \bigcup_{i \in P} Ui$ (that is, the domain of charts cover S),

b) For each $i,j \in P$, the map øi.øj is infinitely differentiable.

We now come to a key definition. A smooth n-dimensional <u>manifold</u> M is a Hausdorff space S furnished with an infinitely differentiable n-dimensional atlas A. Thus locally the manifold may be made to resemble Rn in a smooth, well behaved, manner. An example of a manifold is the two dimensional sphere S(2).

A <u>differentiable function</u> on M, is a function f: M ---> R such that $f.ø^{-1}$: Rn ---> R is differentiable w.r.t. the relevant local co-ordinates. A similar definition may be given for a differentiable map between two manifolds of the same dimension.

A <u>curve</u> Γ on M is a smooth map from some open interval of the real line to M:

$$\Gamma: (-\delta,\delta) \subset R \longrightarrow M$$

Using our local co-ordinate functions we can express Γ as a curve in Rn parameterised by some real parameter $t \in (-\delta,\delta)$:

$$\phi.\Gamma: R \longrightarrow Rn \quad , \quad \phi.\Gamma(t) = (x_1(t),...,x_n(t))$$

where $xi(t) = \phi i(\Gamma(t))$.

Suppose $\Gamma(t) = p$ at $t = 0$,then the <u>contravariant vector</u> $(\frac{\partial}{\partial t})\Gamma(0)$ tangent to the curve Γ in M at $p = \Gamma(0)$ is the operator which maps a smooth, differentiable function $f \in F(M)$, say, to the number:

$$(\frac{\partial f}{\partial t})\Gamma|_{t=0}$$

That is, maps the function f to its derivative in the direction of the curve $\Gamma(t)$ at the point $p = \Gamma(0) \in M$. Thus we have:

$$(1) \quad (\frac{\partial f}{\partial t})\Gamma|_{t=0} = \lim_{s \to 0} \frac{1}{s}\{f(\Gamma(s)) - f(\Gamma(0))\}$$

$$= \sum_i \frac{dxi}{dt}\frac{\partial f}{\partial xi}|_{\Gamma(0)=p} \quad , \quad \text{where } i = 1,...,n$$

Clearly a <u>tangent vector</u> to any curve $\Gamma(t)$ passing through p can be expressed as a linear combination of the coordinate derivatives:

$$(2) \quad \frac{\partial}{\partial xi}|_{\Gamma(0)=p} \quad , \quad i = 1,...,n$$

Conversely, it is easy to see every linear sum of derivatives comes from a curve in M. In this way the collection of all tangent vectors to all curves through point $p \in M$ form a vector space called the <u>tangent space</u> to M at p, denoted Tp(M). The operators (2) are the basis vectors to this space. In general, a vector can be thought of as a linear operator which maps the space of functions over M into R. That is:

$$(3) \quad V: F(M) \longrightarrow R \quad , \quad V \in Tp(M)$$

for $f \in F(M)$, $V(f) \in R$ with value given by (1).

Note : There are alternative ways of defining the tangent space to a manifold. In particular one may define a tangent vector as an equivalence class of curves which are tangent to one another at some point p. The tangent space at p is then the complete set of tangent vectors at p.

The tangent bundle of M, denoted T(M), is defined as the union of all tangent spaces over M: $T(M) = \bigcup_p T_p(M)$.

A vector field is a smooth assignment of tangent vectors $V_p \in T_p(M)$ to each point $p \in M$, in the sense that the function defined on the manifold M, whose value at p is given by the number $V_p(f)$ for some arbitrary $f \in F(M)$, is smooth. Thus:

$$V = \bigcup_p V_p : F(M) \longrightarrow F(M) .$$

In components:

$$Vf(p) := V_p(f) \in R .$$

and

$$V_i(p) = V_p(x_i) \quad , \quad \text{for } i = 1,\ldots,n$$

$$V_p(f) = \sum_i V_i(p) \frac{\partial f}{\partial x_i}(p) \quad , \quad \text{for all } p \in M$$

This is true for all $f \in F(M)$ so:

$$(4) \qquad V_p = \sum_i V_p(x_i) \left(\frac{\partial}{\partial x_i}\right)_p$$

The cotangent space at $p \in M$ is the dual space to $T_p(M)$, and is denoted by $T_p^*(M)$. A covariant vector W_p at point $p \in M$ is a linear map of $T_p(M)$ into R:

$$W_p : T_p(M) \longrightarrow R$$

We denote the product between contravariant and covariant vectors $W_p \in T^*p(M)$ and $V_p \in T_p(M)$ by $\langle W_p, V_p \rangle_p \in R$.

A 1-Form W is a smooth assignment of covariant vectors to each point in M. We can naturally transform a smooth real function $f \in F(M)$ into a 1-form, called the differential of f and denoted df, in the following manner. At $p \in M$, $df \in T^*p(M)$ and is defined by $\langle df, V_p \rangle_p := V_p(f)$ for any $V_p \in T_p(M)$.

Clearly $\{ dx_i \}$ $i = 1,\ldots,n$ form a basis for the cotangent space at $p \in M$, since from the definition $\langle dx_i, \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$. Also we have:

$$W_p = \sum_i (W_i dx_i)_p \quad \text{and} \quad V_p = \sum_i \left(V_i \frac{\partial}{\partial x_i}\right)_p \quad , \text{ and therefore:}$$

$$(5) \qquad \langle W_p, V_p \rangle_p = \sum_i (W_i V_i)_p$$

which is denoted Wp(V). Clearly, if V is a vector field over M and W is a 1-form, then < W,V > ∈ F(M), that is, a function over M whose value at p is defined by equation (5).

A <u>smooth 1-form</u> is one such that for all vector fields V on M, < W,V > is a smooth function on M.

Note: A smooth function f ∈ F(M) is regarded as being a 0-form on M.

A <u>K-Form</u> is a totally skew-symmetric, multi-linear map:

(6) $$Wp: Tp(M) \otimes .. \otimes Tp(M) \text{ (k-times) } \longrightarrow R$$

such that the function: p ---> Wp(V1,...,Vk) is smooth for all p ∈ M, for any set of K vector fields V1,...,VK on M.

The <u>exterior derivative d</u> is a map which takes a K-form to a (K+1)-form, as follows. For the K-form:

(7) W = Aij...k dxi*dxj*...*dxk , with Aij...k ∈ F(M)

the (K+1)-form dW is defined by:

(8) dW := dAij...k*dxi*dxj*...*dxk.

Where the '*' denotes the <u>wedge product</u> and summation occurs over the i,j,...,k indices.

The <u>cotangent bundle</u> T*M is defined as the union of all the cotangent spaces over M. That is:

(9) $$T^*M := \bigcup_p T^*p(M).$$

We define the <u>commutator</u> of two vector fields V1 and V2, denoted by [V1,V2], as the following vector field :

(10) [V1,V2] := V1(V2f) - V2(V1f) , for all f ∈ F(M).

Note that vector fields combined in this manner satisfy the Jacobi identity:

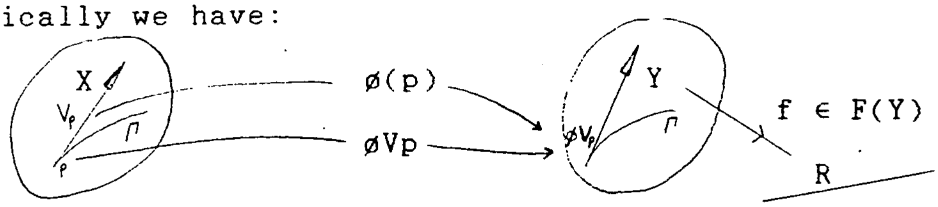
(11) [V1,[V2,V3]] + [V3,[V1,V2]] + [V2,[V3,V1]] = 0

To conclude this brief introduction to differential geometry we describe one last useful concept - this is the notion of 'pushing forward' and 'pulling back' tangent and cotangent vectors between manifolds.

Given a smooth differentiable map ø between two manifolds X and Y, say, it is possible for a tangent vector on X to define a tangent vector on Y, and a cotangent vector on Y to define a cotangent vector on X. These correspondences are defined as follows. For a tangent vector $V_p \in T_p(X)$ we define the corresponding push forward vector of V_p, $\phi V_p \in T_{\phi(p)}(Y)$, by:

(12) $$(\phi V_p)_{\phi(p)}(f) := V_p(f.\phi)$$

where $f \in F(Y)$ and ϕ is known as the push forward operator.
Diagramatically we have:



Similarly, to pull back a cotangent vector $W_{\phi(p)} \in T^*_{\phi(p)}(Y)$, we define $\phi^* W_{\phi(p)} \in T^*_p(X)$ by:

(13) $$< \phi^* W_{\phi(p)}, V_p >_p := < \phi V_p, W_{\phi(p)} > \quad , \quad \text{for any } V_p \in T_p(X).$$

These definitions are well behaved for individual tangent and cotangent vectors, the question arises whether it is possible to apply the same map to vector fields on X and 1-forms on Y. In the case of the 1-form the definition of the pull back operator generalises as expected:

(14) $$< (\phi^* W)_p, V_p >_p := < W_{\phi(p)}, \phi V_p >_{\phi(p)}, \quad \text{for any } V_p \in T_p(X).$$

Note: In the case of a 0-form (i.e a function on Y), we define the pull back simply as:

$$(\phi^* f)(p) := f(\phi(p)) \text{ , for } f \in F(Y).$$

However, with the case of the vector field there can be a problem. If the map ϕ is many-to-one then there will be points where the push forward vector field is not well defined. Similarly, if ϕ is not surjective then there will be points in Y where there isn't a vector field defined at all. To extricate ourselves from these two possible pathologies we demand that ϕ is a diffeomorphism. Under these circumstances we may define a ϕ - related vector field on Y, ϕV, from a vector field V on X by:

(15) $$(\phi V)_{\phi(p)}(f) := (\phi^* V_p)(f) := V_p(f.\phi)$$

for any $f \in F(Y)$ and where $V_p := (V)_p \in T_p(X)$, $(\phi^* V)_{\phi(p)} \in T_{\phi(p)}(Y)$.
ϕ-related vector fields have the property of preserving the

commutator. That is:

(16) $$\phi^*[V1,V2] = [\phi^*V1,\phi^*V2] .$$

We can proceed to make use of this new formalism to reformulate some of the previous results on classical mechanicas in a new and more illuminating manner in the next subsection.

1.2.0 Differential Geometry in Classical Mechanics.

In the last section the reader was introduced to some central concepts and definitions used in differential geometry. Employing this compact and elegant language it is possible to reformulate the previous work on the classical mechanics of non-singular systems in a far more revealing way. This reworking also serves to introduce the reader to some constructions, graded analogues of which will be studied in Part II. A good reference for this approach to classical mechanics is found in Abraham and Marsden [1] or Sundermeyer [58], for example.

1.2.1 The symplectic manifold.

A manifold S is a symplectic manifold if there exists a closed, non-degenerate 2-form W, where closure means $dW = 0$ and non-degenerate means that if $W(V1,V2) = 0$ for all $V2 \in Tp(S)$ and some $p \in S$, then it follows that $V1 = 0$.

The cotangent bundle T*M of some configuration space M has a natural symplectic structure associated with it. We lable the local co-ordinates of T*M : $(q_1,\ldots,qn;p_1,\ldots,pn)$, where the {pi} are the 'fibre co-ordiantes' associated with each cotangent space at some particular point $p \in M$. We now consider 1-forms defined on the 2n-dimensional cotangent bundle T*M. We define the Liouville 1-form on T*M by:

(1) $$L := \sum_i pi(\mu)(dqi)\mu \quad , \quad \text{where } \mu \in T^*M .$$

The natural symplectic structure on T*M is given by:

(2) $$W := - dL$$

or, in words, the exterior derivative of the Liouville form. In local co-ordinates on T^*M this symplectic 2-form has the following expression:

(3) $$W = \sum_i dqi * dpi \quad , \quad i = 1,\ldots,n$$

This is the natural symplectic structure on a cotangent bundle, and we shall see that it is very closely related to the previously defined Poisson bracket between two functions on phase space. Before that though, we define a <u>symplectic transformation</u> on phase space.

1.2.2 <u>The symplectic transformation</u>.

A <u>symplectic transformation</u> (or 'canonical transformation') is a smooth differentiable map taking the phase space M to itself, in such a way as to preserve the symplectic 2-form of the space. That is:

(4) $$ø: M \text{ ---> } M \quad \text{and} \quad ø^*W = W$$

where $ø^*$ is the pull-back operator. Basically this statement is the equivalent of the "canonical transformations are those which preserve the Poission bracket" statement of ∮ 1.0.5. As before, a symplectic transformation can be thought of as arising from an infinitesimal generator. This comes about in the following way. A 1-paramenter family of smooth differentiable maps on M gives rise to a field of tangent vectors on M, where the direction of these vectors at each point is determined by the tangent to the flow line of the 1-parameter family of diffeomorphisms $øt$ through that point. That is:

$$øo(x) = x \text{ for all } x \in M$$

where

$$øt(øs(x)) = øs+t(x) \text{ for all } x \in M, \text{ with } s,t \in R$$

Conversely, a vector field on M gives rise to an unique integral curve through each point $x \in M$, which can be used to construct a 1-parameter family of diffeomorphisms of M for some range of the real paramenter t. If the range of t is R - the whole real line - then the vector field is said to be <u>complete</u>. In this way vector fields generate 1-parameter groups of diffeomorphisms, and a group of 1-parameter diffeomorphisms determines a vector field. This relationship becomes pertinent to the symplectic transformation when we consider one vector field in particular, known as the Hamiltonian vector field.

1.2.3 The Hamiltonian vector field.

The nature of the canonical transformation is further illuminated by the introduction of the following vector field on M, associated with some function $f \in F(M)$. We define the Hamiltonian vector field (H.V.F.) on M associated with f, denoted Hf by:

$$Hf := \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) \tag{5}$$

Why is this vector field important ? The H.V.F is important because the transformations assocated with flows along its integral curves are symplectic. Thus, if ϕ_t^H , $t \in R$

is the family of diffeomorphisms associated with a flow of parameter 'distance' t down an integral curve of Hf from some point $x \in M$, then:

$$\phi_t^H * W = W \tag{6}$$

In this way, every function on M can be thought of as determining a canonical transformation via the intergral curves of its associated Hamiltonian vector field. If Hf is complete, then the 1-parameter group of canonical tranformations is defined on the whole phase space.

How about the converse process of determining if a given vector field is Hamiltonian (that is, if it arises from some function $f \in F(M)$ and the definition (5))? This question is more subtle and depends on the cohomology group of the phase space [37]. If the first cohomology group is trivial - resulting in every closed 1-form being exact - then for the vector fields which cause the Lie derivative of the symplectic 2-form to vanish, it is possible to find a function which will give the fields the structure of the definition

(5). Thus to answer even such a straight forward question, immediately involves one in considerations about the global nature of the phase space. If the cohomology of the space is non-trivial the Lie derivative condition only guarantees that the vector field is 'locally Hamiltonian' - a far weaker statement [37]. We now ask the question: does the commutator of two Hamiltonian vector fields yield a third H.V.F. and, if so, what is the function which corresponds to it in definition (5)? That is if Hf, Hg ∈ { H.V.F. on M }, with f, g ∈ F(M), then does there exist a function h ∈ F(M) which satisfies [Hf,Hg] = Hh ? After some calculation one finds [37]:

(7) $$[Hf,Hg] = H_{H_f(g)}$$

What is this function Hf(g)? Substitution in the explicit co-ordinate form of (5) quickly reveals that it is none other than the negative Poisson bracket between f and g. That is we have:

(8) $$[Hf,Hg] = - H\{f,g\}$$

This relationship is the rigourous form of (51) in § 1.0.6. We now can understand the time development of a dynamical system as being a flow along the Hamiltonian vector field generated by the Hamiltonian function H, and with a structure given locally by definition (5). This gives us back Hamilton's equations (52) section 1.0.6 simply by reading off the components of the H.V.F.:

(9) $$\frac{dqi(t)}{dt} = \frac{\partial H(q,p)}{\partial pi} \quad , \quad \frac{dpi(t)}{dt} = - \frac{\partial H}{\partial qi}$$

Similarly the time development of a function f can be thought of as being its evolution along a line of flow of the system, giving the previous result (52):

(10) $$\frac{df}{dt}(\Gamma(t)) = \{ f,H \}(\Gamma(t))$$

for some integral curve of HH in M, Γ(t). The Poisson bracket between two functions f,g ∈ M may now be understood in terms of their H.V.F.s. In fact calculation shows:

(11) $$W(Hf,Hg) = \{ f,g \}$$

How is the underlying group theory expressed in this new approach? The set of all (complete) vector field on phase space together with the commutator bracket operation (7) form a Lie algebra - actually it is the Lie algebra of the group of all smooth 1-parameter differentiable maps of M onto M, or in other words the diffeomorphism group Diff M [37]. Clearly the group of all symplectic transformations of M is a subgroup of Diff M, with its Lie algebra being the set of all complete locally Hamiltonian vector fields. The set of all functions defined on M together with the Poisson bracket operation form another Lie algebara, which, through the correspondence: f ---> - Hf, is homomorphic to the Lie algebra of Hamiltonian vector fields. This map is not 1-1, as clearly the constant functions are the kernel and lie in the centre of the Poisson bracket algebra (because the Poisson bracket between a constant and a function on M is zero). This is a feature which has important repercussions later on regarding the occurence of cocycles in the quantisation process [37].

This concludes this brief geometric interlude - it is intended to show the earlier sections on classical mechanics from a different perspective. Also it introduces some important objects and language which will be of use in the later section on the group theoretical approach to quantisation, and in the work of Part II on graded analogue structures. However for the next section we return to the more pedestrian language of the earlier sections to deal with the problem of dynamical systems which incorporate a singular Lagrangian - that is, when constraints are present in the theory.

1.3.0 Constrained Hamiltonian Systems.

Introduction

Early in the introduction to Hamilton mechanics it was assumed that equation (7) in section 1.0.2 defining the canonical momenta, led to equations in which it was possible to explicitly solve the system for the generalised velocities. These would then be expressed as functions of the generalised co-ordiantes and their conjugate momenta. In the presence of constraints, that is, functional relationships between phase space variables valid throughout the system's time evolution, this process is made complicated by the inherent dependence of some of the co-ordinates. In the following section we describe the canonical approach to dealing with these problems, which was pioneered by Dirac [23,25], Bergmann [2,16] and others, culminating in the introduction of the 'Dirac bracket'. And, by way of an example, we demonstrate this construction for particles moving on an N-Sphere. The following review relies on [44] and [58] , and also with contributions from [23] and [34].

1.3.1 Singular systems.

A singular lagrangian L , that is one in which the generalised velocities may not be uniquely expressed in terms of phase space variables, is characterised by the vanishing of the determinant $|(W)_{rs}|$, where:

(1) $$(W)_{rs} = \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s} \quad \text{where} \quad |(W)_{rs}| = 0 \ , \text{ with } r,s = 1,\ldots,K$$

This property is invariant under canonical transformations, unlike the corresponding one for the Hamiltonian [44]. At this point one may now proceed in the Lagrangian formalism if so desired. However here we decide to follow the Hamiltonian approach because traditionally this formulation has played a stronger role in relation to quantum mechanics*. Following this formalism equation (1) leads

* Note: there are alternative approaches to quantum mechanics, for instance see Feynmann's path intergral method [30].

to the following phase space constraints:

(2) $$\phi_\mu(q,p) = 0 \quad , \quad \text{with } \mu = R+1,\ldots,K$$

where, as before, ps is defined:

(3) $$p_s := \frac{\partial L}{\partial \dot{q}_s} \quad , \quad \text{with } s = 1,\ldots,K$$

and where R is the rank of the matrix (W). Equation (2) embodies the so-called 'primary constraints' of the system and, in principle, it allows one to express momenta $p_{R+1},\ldots,p_K$ in terms of the $p_1,\ldots,p_R$. That is [44]:

(4) $$p_\mu = \theta_\mu(q_s,p_j) \quad , \quad \text{for } \mu = R+1,\ldots,K \text{ and } j = 1,\ldots,R$$

This means that K-R generalised velocities are arbitrary functions, so that the first R generalised velocities may be solved as functions of (q_s,p_j), and K-R arbitrary generalised velocities. The canonical Hamiltonian is defined as before, with:

(5) $$H_c := \sum_s p_s.\dot{q}_s - L(q,\dot{q})$$

At first sight it appears that this should depend on the arbitrary unsolved generalised velocities, however, analysis shows that this is not the case. Thus:

(6) $$H_c := H_c(q_s.p_j) .$$

Differentiating (5) w.r.t. q_s and p_j, and using Lagrange's equation (6) in ∮ 1.0.1 leads to the following equations of motion [44]:

(7) $$\dot{q}_j = \frac{\partial H_c}{\partial p_j} - \sum_\mu \dot{q}_\mu \frac{\partial \theta_\mu}{\partial p_j} \quad , \quad \text{with } j = 1,\ldots,R$$

(8) $$\dot{p}_s = -\frac{\partial H_c}{\partial q_s} + \sum_\mu \dot{q}_\mu \frac{\partial \theta_\mu}{\partial q_s} \quad , \quad \text{with } s = 1,\ldots,K.$$

and with $\mu = R+1,\ldots,K$.

(Note : It is possible to identify the K-R arbitrary generalised velocities with the K-R $\dot{q}_\mu$). The above (7) and (8) are Hamilton's equations for a constrained system.)

1.3.2 Weak and Strong Equality

By using an idea first introduced by Dirac [23], it is possible to rearrange the constrained Hamilton's equations (7,8) in an illuminating manner. Before we can do this though, two definitions are needed. When the two functions f and g, say, defined over the whole phase space, are equal when valued on the hypersurface satisfying the constraints of the system, they are said to be 'weakly equal'. This is denoted:

(9) $$f(q_s,p_s) \approx g(q_s,p_s) <=> f = g \mid p_\mu = \theta_\mu(q_s,p_j)$$

Two functions f(q,p) and g(q,p) are 'strongly equal' when they are weakly equal and their gradients are also weakly equal. This is denoted:

$$f(q_s,p_s) \equiv g(q_s,p_s)$$

It is easy to prove the following lemma (Dirac) [23]:

$$f(q_s,p_s) \approx g(q_s,p_s) \quad <=> \quad f - \sum_\mu \phi_\mu \frac{\partial f}{\partial p_\mu} \equiv g - \sum_\mu \phi_\mu \frac{\partial g}{\partial p_\mu}$$

where:

$$\phi_\mu = \phi_\mu(q_s,p_s) := p_\mu - \theta_\mu(q_s,p_j)$$

Employing this theorem one is quickly led to a more elegant form of equations (7) and (8):

(10) $$q_s \approx \frac{\partial H}{\partial p_s} + \sum_\mu \dot{q}_\mu \frac{\partial \phi_\mu}{\partial p_s} \qquad s = 1,\ldots,K$$

(11) $$p_s \approx \frac{\partial H}{\partial q_s} - \sum_\mu \dot{q}_\mu \frac{\partial \phi_\mu}{\partial q_s} \qquad \mu = R+1,\ldots,K$$

where:

(12) $$H = H' - \phi_\mu . \frac{\partial H'}{\partial p_\mu}$$

(13) $$H' \approx H_c$$

Equations (10) and (11) may be written thus [44]:

(14) $$q_s \approx \{ q_s,H \} + \sum_\mu \{ q_s,\phi_\mu \}\dot{q}_\mu \text{ , for } s = 1,\ldots,K$$

(15) $$p_s \approx \{ p_s,H \} + \sum_\mu \{ p_s,\phi_\mu \}\dot{q}_\mu \text{ , and } \mu = R+1,\ldots,K$$

Where we have the primary constraints $\phi_\mu(q_s,p_s) \approx 0$ and where also $H \equiv H_c$. If $g(q_s,p_t)$ is some function on phase space, then (14) and (15) give:

(16) $$\frac{dg(q_s,p_t)}{dt} \approx \{ g,H \} + \sum_\mu \{ g,\phi_\mu \}\dot{q}_\mu$$

We now demand that the constraint functions ϕ_μ remain weakly zero throughout the time development of the system - this gives us the further equation:

$$\{ \phi_\mu, H \} + \{ \phi_\mu, \phi_\alpha \} \dot{q}_\alpha \approx 0 , \quad \alpha = R+1, \ldots, K \tag{17}$$

Assuming $\det|\{\phi_\mu, \phi_\alpha\}|$ is non-zero then it is possible to solve for the $\dot{q}_\mu$ in (16) to give:

$$\frac{dg(q_s,p_t)}{dt} \approx \{ g, H \} - \sum_{\mu,\alpha} \{ g, \phi_\mu \} C_{\mu\alpha} \{ \phi_\alpha, H \} \tag{18}$$

Where $C_{\mu\alpha} = (\{\phi_\mu, \phi_\alpha\})^{-1}$, the inverse of the matrix of constraints. So enters the 'Dirac bracket' on to the scene, and which is defined:

$$\{ A, B \}^* := \{ A, B \} - \sum_{\mu,\alpha} \{ A, \phi_\mu \} C_{\mu\alpha} \{ \phi_\alpha, B \} \tag{19}$$

Clearly if the matrix of constraints is non-singular we now have:

$$\frac{dg(q_s,p_t)}{dt} \approx \{ g, H \}^* \tag{20}$$

Thus we see that by employing Dirac's notion of weak and strong equality, we are able to rewrite Hamilton's equations of motion for a constrained system in a form that is reminiscent of the free equations. That one is able to do this is, in fact, no accident and is a topic we will return to later.

1.3.3 First and second class constraints

How do we proceed if the matrix $(\{\phi_\mu, \phi_\alpha\})$ is singular? In this event (17) will give rise to further constraint equations between the (q_s, p_t), called secondary constraints, which must be added to the primary constraints already present. Once again the time development of the full set of constraints must be checked. If their time development produces further independent conditions which must be set weakly to zero, then these should be added to the existing collection and the process repeated until all the constraints have been determined. (It must be said that the general case of this procedure is quite complex as the rank of the constraints matrix must also be checked). The upshot of all this is that to the R primary constraints are now added a number, say S, of secondary constraints, the total number R + S representing the complete set of constraints associated with the system.

The next step, following Dirac and others [23,34,44] is to partition the primary and secondary constraints into so called first class and second class constraints. A first class constraint has a weakly vanishing Poisson bracket with <u>all</u> other contraints; second class otherwise (note: first and second class mixes primary and secondary). Suppose we call the first class constraints:

(21) $$\epsilon_i(q,p) \approx 0 \quad , \quad \text{for} \quad i = 1,\ldots,I$$

and the second class constraints:

(22) $$\Gamma_\alpha(q,p) \approx 0 \quad , \quad \text{for} \quad \alpha = 1,\ldots,N \quad \text{and } I+N = S+R$$

Dirac has a neat proof that the constraints matrix associated with the second class constraints $D_{\alpha\beta} = (\{\Gamma_\alpha,\Gamma_\beta\})$ is non-singular and therefore invertable [23]. It can be shown, that the first class constraints are responsible for contributing various linear combinations, which we denote by v_i for $i = 1,\ldots,I$, of the unsolved generalised velocities $\dot{q}_\alpha$ to the time derivative of a function on phase space [44]. In fact if $g(q_s,p_t)$ is some function on phase space, then:

(23) $$\frac{dg(q_s,p_t)}{dt} \approx \{ g,H \} + \sum_i \{ g,\epsilon_i \}v_i - \sum_{\alpha,\beta} \{ g,\Gamma_\alpha \}(D)_{\alpha\beta}\{ \Gamma_\beta,H \}$$

These arbitrary functions may be included within the Hamiltonian by defining a 'total Hamiltonian' H_t ,thus:

(24) $$H_t := H + \sum_i \epsilon_i v_i$$

Then the time derivative once again becomes:

(25) $$\frac{dg(q_s,p_t)}{dt} \approx \{ g,H_t \}^*$$

The fact that $\{ A,\Gamma_\alpha \}^* = 0$ identically for any $A = A(q_s,p_t)$ means that, provided Dirac brackets are used throughout, weak equations may now be replaced by strong ones - it no longer matters that the constraints are imposed <u>after</u> calculating a bracket expression.

To discover why the Dirac bracket is structurally the way it is, or what transformations preserve it, detailed analysis must be carried out. This was first done in [16], and we briefly review some of the results of this work in the next section. We shall see that the Dirac bracket has many properties in common with the standard Poisson bracket - in fact structurally it is a Poisson bracket, only constructed out of a reduced set of variables. These variables are independant from and complementary to the functions which embed the 'constraint hypersuface' - the surface obtained by resolving the phase space constraints - in the higher dimensional phase space which has natural co-ordinates (qs,pt). First class constraints are in fact generators of internal symmetries rather similar to gauge degrees of freedom in Yang-Mills theory [4]. That is, you can show that the functions $\epsilon\alpha$ generate infinitesimal contact transformations which do not alter the state of the system physically. In this sense the presence of first class constraints obscures the relationship between the phase space variables and the states of the theory [58].

1.3.4 The generalised Poisson bracket

In their elegant book Mukunda and Sudarshan [44] define the notion of the 'generalised Poisson bracket' of which they demonstrate the Dirac bracket to be an example. They consider a general construction of the following nature:

Let Xi, for i = 1,...,N , be a set of real variables defined on certain open intervals in R, and let there exist a set of real functions of {Xi}, Bij(X), which are antisymmetric in i and j. If f(x) and g(x) are two functions of the {Xi} we define a third function h(x) by:

(26) $$h(X) = \{ f,g \}^{**} := Bij(X)\frac{\partial f}{\partial xi}\frac{\partial g}{\partial xj}$$

Furthermore, we demand that the { , }** operation obeys the Jacobi identity:

(27) $\{ f,\{ g,h \}^{**}\}^{**}$ + Cyclic permutations = 0

A combination operation defined in this way is called a generalised Poisson bracket (G.P.B.) after [44].

We may now define a Generalised Canonical Transformation (G.C.T.) as being a map $\{X_i\} \longrightarrow \{X'_i = X'_i(X)\}$ which preserves the G.P.B. The condition that the variables $\{X_i\}$ must satisfy in order to achieve this is [44]:

(28) $$\{ X'_i, X'_j \}^{**} = B_{ij}(X')$$

Notice that this condition is just like the usual Poisson bracket condition with B_{ij} replacing the Γ_{ij} from ∮ 1.0.6. Imposing the Jacobi identity (27) on this bracket gives us further conditions that the B_{ij} must satisfy:

(29) $$\frac{\partial B_{jk}}{\partial x_i} B_{mi} + \frac{\partial B_{mj}}{\partial x_i} B_{ki} + \frac{\partial B_{km}}{\partial x_i} B_{ji} = 0 \quad , \quad i = 1,\ldots,N$$

As in the case of the straight Poisson bracket, we can use functions to generate Infinitesimal Generalised Canonical Transformations (I.G.C.T.) by demanding that the following differential equation is satisfied:

(30) $$\frac{dX_i(t)}{dt} = \{ X_i(t), \phi(X_i,t) \}^{**}$$

where $X_i(t) = f_i(X_{oi},t)$, and $f_i(X_{oi},0) = X_{oi} = X_i(0)$, $t \in R$.
The function ϕ is known as the 'generator' of the I.G.C.T., and it is simple to verify using the Jacobi identity that the $\{f_i\}$ which satisfy (30) do also satisfy (28). The G.P.B. is said to be singular if $\det|B_{ij}| = 0$. This implies that there exists a function or functions W_α such that:

(31) $$\{ f, W_\alpha \}^{**} = 0 \quad \text{for all } f \in F(X)$$

where $F(X)$ = {The set of all smooth functions over $\{X_i\}$}. These functions W_α are called neutral functions [44]. If the rank of the matrix $(B)_{ij}$ is $N-M$ then you can show that there are M neutral functions, which form a basis for the space of functions satisfying (31), each one being responsible for generating a null eigenvector of matrix $(B)_{ij}$. Suppose there are M neutral functions W_α of the above G.P.B. Let us adjoin to them N-M independent functions ϕa so that together the set $(\phi a, W\alpha)$ form N independent functions over phase space. Then it is possible to show that the G.P.B. (26) reduces to the following:

$$(32) \qquad \{ f,g \}^{**} = \sum_{a,b} Bab \frac{\partial f}{\partial \phi a}\frac{\partial g}{\partial \phi b}$$

where $Bab(X) = \{\phi a, \phi b\}^{**} |x$ and is non-singular. Thus a singular G.P.B. may be made to look like a non-singular G.P.B. in a lower number of variables, that number of variables being determined by the rank of the original matrix $(B)ij$.

The importance of the above discussion lies in its use in understanding the underlying nature of the Dirac bracket. In the next section we will see that the Dirac bracket is none other than a special case of a generalised Poisson bracket.

1.3.5 The Dirac Bracket as an example of a generalised Poisson bracket

Reverting to the notation used in § 1.0.6, we now consider the case when the variables $X\alpha$, $\alpha = 1,\ldots,2K$, are the natural co-ordinates on phase space. Suppose that there exist an even number of secondary constraint functions ϕa, $a = 1,\ldots,2N < 2K$, with a non-singular matrix of Poisson brackets, whose inverse is denoted $Cab = (\{\phi a, \phi b\})^{-1}$. To this set of 2N independent functions $\{\phi a\}$ we add the further 2K-2N independent functions Ws, where $s = 2K-2N+1,\ldots,2K$, such that the 2K functions $(\phi a, Ws)$ form an independent co-ordinate set. Then Mukunda and Sudarshan show that the G.P.B. defined by:

$$(33) \qquad \{ f,g \}^{**}(\phi,W) := \sum_{s,t} Bst \frac{\partial f}{\partial Ws}\frac{\partial g}{\partial Wt}$$

and where $(Bst) = (-(Ws,Wt))^{-1}$ is the inverse of negative Lagrange bracket of Ws and Wt w.r.t. the variables $X\alpha$, are <u>precisely</u> that of the Dirac bracket. That is:

$$(34) \qquad \{ f,g \}^{**} \equiv \{ f,g \}^{*}$$

This is straightforward to demonstrate, relying mainly on the properties of the Lagrange bracket, that it is the inverse of the Poisson bracket. This shows neatly why the Jacobi identity holds for the Dirac bracket. Rather than being a miracle it is merely a result of the matrix (Bst) satisfying condition (29) through the identity properties of the Lagrange bracket.

Before going on to discuss the significance of the transformations which preserve the Dirac bracket, we introduce an idea which is returned to in Part II, and which really lies at the heart of manipulations involving Poisson brackets: this is the notion of the 'function group' [27,44].

1.3.6 Function groups.

Definition

A set of R independent functions $Fa(X\alpha)$, $a = 1,\ldots,R$, over phase space $\{X\alpha\} = (q_1,\ldots qn;p_1,\ldots,pn)$ and such that:

(35) $$\{ Fa,Fb \} = Gab(Fc) \qquad \text{with } a,b,c = 1,\ldots,R$$

where Gab is some function antisymmetric in a and b, is said to form a function group G of rank R. A function group G is commutative if $\{ Fa,Fb \} = 0$ for all values of a and b. If a subset of functions of the function group G forms a function group then they constitute a subgroup of the group G. There are two theorems on functions groups which are of special interest to us and we state them here without proof (for a proof see [27]).

Theorem 1

A non-commutative function group G of rank R is a subgroup of a function group of rank 2n whose basis $(\phi_1,\ldots,\phi n;\pi_1,\ldots,\pi n)$ can be chosen so that:

$$\{ \phi i,\phi j \} = \{ \pi i,\pi j \} = 0 \;, \quad \text{and} \quad \{ \phi i,\pi j \} = \delta ij$$

where $i,j = 1,\ldots,n$

Theorem 2

A system of 2m+q independent equations which define a surface of dimension $D = 2n-2m-q$, denoted $\Gamma\mu = 0$, for $\mu = 1,\ldots,2m+q$, and such that the rank of $(\{\Gamma\mu,\Gamma\delta\}) = 2m$, for $\mu,\delta = 1,\ldots,2m+q$, can be substituted for a locally equivalent system:

$$\phi a = 0 \quad , \quad \text{for } a = 1,\ldots,m+q \quad \text{and}$$

$$\pi\alpha = 0 \quad , \quad \text{for } \alpha = 1,\ldots,m$$

and for which the relations:

$$\{ \phi a, \phi b \} = \{ \pi\alpha, \pi\beta \} = 0 \text{ , and } \{ \phi a, \pi\alpha \} = \delta a\alpha$$

hold <u>locally</u> in phase space.

These theorems make it possible to, locally at least, set the matrix (Bab) from § 1.3.5 equal to the natural symplectic 2-form in the reduced number of variables. This makes the Dirac bracket look like a Poisson bracket in that locality of the embedded constraint hyper-space. That is locally we can choose N independent functions on phase space $(\phi r;\pi s;Wi)$, such that:

(36) $$\{ f,g \}^* = \sum_r \left(\frac{\partial f}{\partial \phi r}\frac{\partial g}{\partial \pi r} - \frac{\partial f}{\partial \pi r}\frac{\partial g}{\partial \phi r} \right)$$

where $f = f(\phi,\pi,W)$ and $g = g(\phi,\pi,W)$, and with Rank(Bab) = 2R.

<u>Note</u>: In general the transformation: $(X\alpha) \longrightarrow (\phi r,\pi s,Wi)$ is <u>not</u> a generalised canonical transformation. For this reason often it is not convenient to state the Dirac bracket in this manner.

1.3.7 <u>Generalised canonical transformations and the Dirac bracket.</u>

Consider the I.G.C.T. generated by some function $\epsilon(X\alpha,t)$, where $\{X\alpha\}$ are the phase space coordinates with $\alpha = 1,\ldots,2K$, and the functions:

(37) $$X\alpha(t) = f\alpha(X\alpha(0),t) \quad , \text{ with } \epsilon(X\alpha(0),0) = X\alpha(0)$$

are determined from the differential equation:

(38) $$\frac{dX\alpha(t)}{dt} = \{ X\alpha(t),\epsilon(X\alpha,t) \}^*$$

It is easy to that for all constraints functions ϕm used in deriving the Dirac bracket :

(39) $$\frac{d\phi m(t)}{dt} = 0 \quad , \text{ for all } m .$$

This comes about directly as a result of how the bracket is constructed and, as a consequence, means that the transformation

$X\alpha(0) \to X\alpha(t)$, for $t > 0$, preserves the constraint hypersurface. For a generating function ϵ such that:

$$\{ \phi_m(X\alpha), \epsilon \} = 0 \quad \text{, for all } m \tag{40}$$

then the generalised canonical transformation becomes an ordinary canonical transformation. If equation (40) is weakly zero then it is possible to show that there exists a function $\epsilon'(X\alpha,t)$, which generates a normal canonical transformation the same as that generated by using $\epsilon(X\alpha,t)$ in (38) on the constraint hypersurface, but which differs when away from it in areas where $\phi_m = 0$. The function $\epsilon(X\alpha,t)$ is arbitrary and $\epsilon'(X\alpha,t)$ is constructed from it [44]. For the case of transformations generated by the Hamiltonian this works as follows. The time development of a system which incorporates constraints can either be regarded as a generalised canonical transformation generated by H, thus:

$$\frac{dg(X\alpha)}{dt} = \{ g,H \}^* \tag{41}$$

(note: we assume only second class contraints are present), or as an ordinary canonical transformation confined to the constraint hypersurface and generated by the associated Hamiltonian H', where:

$$H' = H - \sum_{m,n} \phi_m C_{mn} \{ \phi_n,H \} \tag{42}$$

In a phase space where we could globally make our Dirac bracket look like a Poisson bracket, the group of generalised canonical transformations is merely a group of ordinary canonical transformations in a phase space of reduced dimension - the fact that we can only usually do this procedure locally means that normally these groups only coincide near their origins. A necessary and sufficient condition that this reduction process is possible other than locally is that the constraint equations themselves form a function group [44].

1.3.8 An example: the Dirac Bracket on S(N).

An N-sphere is defined as: $\{ \underline{X} \in S(N)\ ;\ \underline{X} \in Rn,\ \underline{X}.\underline{X} = 1 \}$ where $\underline{X}.\underline{X}$ denotes the inner product $\sum_i XiXi$, for i = 1,...,N+1 , and it is this which defines the configuration space of our example. For the Hamiltonian of the system we choose the standard:

(43) $$H = \sum_i PiPi = \underline{P}.\underline{P}$$

The distinction of primary and secondary constraints is not nearly as important as knowing whether a constraint is first or second class. But before determining that, we must find all the constraints of the system. Following ∮ 1.3 we take the P.B. between any and every two constraints in the theory, and between those and the Hamiltonian, until we have a closed set. Thus:

$$H = \underline{P}.\underline{P} \quad , \quad \emptyset 1 = \underline{X}.\underline{X} - 1$$

and

$$\{ H,\emptyset 1 \} = 2\underline{P}.\underline{X} := \emptyset 2 \quad , \{ H,\emptyset 2 \} = 2H \ , \text{ and } \ \{ \emptyset 1,\emptyset 2 \} = 2H$$

so we see that the set (H,ø1,ø2) is closed w.r.t. Poisson bracket combination. Furthermore the constraints ø1 and ø2 are second class. We now form the constraints matrix ({øa,øb}), for a,b = 1,2. We have:

(44) $$(\{\emptyset a,\emptyset b\}) = 2\begin{pmatrix} 0 & \underline{X}.\underline{X} \\ -\underline{X}.\underline{X} & 0 \end{pmatrix}$$

so the inverse of ({øa,øb}), denoted (C)ab is:

(45) $$(C)ab = \tfrac{1}{2}\begin{pmatrix} 0 & \frac{-1}{\underline{X}.\underline{X}} \\ \frac{1}{\underline{X}.\underline{X}} & 0 \end{pmatrix}$$

From ∮ 1.3, the Dirac bracket between two function is given by:

$$\{ F,G \}^* := \{ F,G \} - \sum_{a,b} \{ F,\emptyset a \}Cab\{ \emptyset b,G \}$$

which with the constraint matrix above gives the following fundamental brackets:

(46) $$\{ Xi,Xj \}^* = 0$$

(47) $$\{ Pi,Pj \}^* = \frac{XiPj - XjPi}{\underline{X}.\underline{X}}$$

(48) $$\{ X_i,P_j \}^* = \delta_{ij} - \frac{X_iX_j}{X.X}$$

We remark at this stage that the R.H.S. of (47) has given us the SO(3) Casimir operator. In this example for N = 1, that is, quantisation on the circle lying in Rn, it is instructive to show how the 'phase space reduction' idea of § 1.3.6 can be understood. In this example it can be seen very simply by choosing the basis variables of the system to be $(\theta;p)$ where:

(49a) $X1 = \text{Cos}\theta$ (49b) $P1 = -p\ \text{Sin}\theta$

(49c) $X2 = \text{Sin}\theta$ (49d) $P2 = p\ \text{Cos}\theta$

with $0 < \theta < 2\pi$, $p \in R$.

Defining the bracket $\{ f,g \}r$ to be:

(50) $$\{ f,g \}r := \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial\theta}\frac{\partial f}{\partial p}$$

where $f = f(Xi(\theta,p);Pj(\theta,p))$ and similarly $g = g(Xi(\theta,p),Pj(\theta,p))$.

Direct calculation shows that:

(51) $\{ Xi,Xj \}^* = \{ Xi,Xj \}r$

(52) $\{ Xi,Pj \}^* = \{ Xi,Pj \}r$

(53) $\{ Pi,Pj \}^* = \{ Pi,Pj \}r$

That is the reduced bracket (50) in the variables $(\theta;p)$ is equivalent to a Dirac bracket in the variables $(Xi;Pj)$ from (46,47,48).

1.4.0 The Quantisation of Classical Theories.

Introduction.

In this section we briefly describe the quantisation of classical theories by the so called 'group theoretical' approach of C.Isham et al.[36,37,38]. Once again the emphasis is not so much to give a comprehensive account of these ideas, but simply to explain them sufficiently to motivate the considerations of Part II. At the local level at least, the group theoretical approach to quantisation is essentially the same as the 'Dirac bracket method' of quantising systems whose associated classical phase space is some constrained hypersurface. An important difference is that the G.T. approach leads one to give a new interpretation to the 'iħ' in the Heisenberg algebra, as being the result of the existence of a 2-cocycle in the translation group on Rn [37] (the graded emulation of this notion is discussed in § 2.5). Throughout this brief look at the quantisation process, we concentrate on a particular approach to quantum theory - that it arises as a result 'doing something' to the classical theory. By this we mean that it is assumed that one already has a well developed classical theory from which to construct the quantum theory through the devising of some sort of 'quantisation map'. This is not the only path between a quantum theory and the classical theory though. For example, in the case of Superstring theory one thinks of the classical theory as resulting from the first order terms in the quantum string [54]. We do not discuss this approach to quantisation here, concentrating wholly on the canonical path.

1.4.1 The Basic Postulates of Quantum Theory.

In the standard approach to quantum mechanics followed in for example [9,24,53,60], physical states of a system are described by unit rays of vectors in some complex Hilbert space H. (A unit ray Γ in Hilbert space is the set of vectors $\{\mu\Gamma\}$, where $\{ \Gamma \in H : |\Gamma| = 1 \}$, $\mu = \exp(i\alpha)$, and where $|\Gamma|$ is the norm of Γ in H. This 1-1 correspondance between the physical states of the system and rays in Hilbert space comes about through the way in which the theory is interpreted, which is in a probabilistic manner. The transition between one state represented by unit ray ø and another Γ, is numerically equal to the following:

(1) $$\text{Probability: } ø \longrightarrow \Gamma = |ø.\Gamma|$$

That is, the square of the modulus of the inner product between any two vectors, ø ∈ ø and Γ ∈ Γ, one from each of the rays ø and Γ respectively. The space of rays in Hilbert space is a projective space: the quotient space obtained by dividing the Hilbert space H by the circle S(2). Hilbert space is also a linear space, so the quantum theory incorporates a principle of superposition. That is if {μΓ} and {δø} are two rays representing two states of the system, then $\theta = \alpha\Gamma+\beta ø$ is another vector in H space and so the ray {τθ} represents another state of the system (where $\alpha,\beta,\delta,\mu,\tau \in C$). We remark that while the vectors θ and τθ both represent the same state of the system, in general $\theta' = \alpha(\mu'\Gamma) + \beta(\delta' ø)$ represents a different state. This means that while overall phase changes are unimportant (representing 'movement' within a ray), changes in relative phase within a state consisting of a superposition of other states are important and do change the overall state of the system. Physical quantities (observables) associated with the system (such as energy, angular momentum etc) appear in quantum theory as linear operators on Hilbert space. What value does one obtain by measuring such an observable? The quantum principle states that a measurement will result in an eigenvalue of the linear operator associated with that particular observable. Thus, for an operator to represent an observable on H space, it must be hermitian (to ensure the reality of its eigenvalues). If the system is in a state ø which consists of superposition of eigenvectors of some observable, then the probability of measuring a particular eigenvalue of an operator O, say, is (where we assume all H space vectors are normalised):

(2) (Probability of measuring αi) = $|\theta i.\phi|$

where O: H --->H and $O.\theta i = \alpha i\theta i$ (no sum over i) with $\alpha i \in R$.
Hence, over a successive number of measurements with each time the system starting from the same state ø, the average value of the observable represented by the operator O will be:

(3) $$\phi.O\ \phi$$

A state (where we now use the terms 'state','ray' and 'vector' interchangeably unless otherwise stated) develops in time through the action of an hermitian operator U(t,to). That is if øt represents the state ø at time t, then:

(4) $$\phi t = U(t,to).\phi to$$

We may write:

(5) $$Ut = exp(-itH')$$

where H' is the self adjoint Hamiltonian operator for the system.

Eigenvalues of the Hamiltonian operator give the energy of the system, which is to say:

(6) $$H'\phi = E\phi$$

A symmetry of the Hamiltonian H' of a quantum system is said to have been generated, when a collection of hermitian operators {Ti} on the Hilbert space H commute with the Hamiltonian [9,53]:

(7) $$[\ H',Ti\ ] = 0 \quad , \quad for\ i = 1,...,k$$

If Ti,Tj commute with the Hamiltonian operator then, by the Jacobi identity, so does [Ti,Tj]. It follows that the collection of k operators that commute with the Hamiltonian H', together with the commutator bracket operation form a k-dimensional Lie algebra on H. It is easy to prove the following theorem by Ehrenfest on time development in quantum mechanical systems [26]. If O is an hermitian operator on some Hilbert space H, and H' is the Hamiltonian operator, then:

(8) $$\frac{d(\phi t.O.\phi t)}{dt} = \frac{1}{i}\ \phi t.[\ O,H'\ ]\phi t$$

Indeed, if we transfer the time dependence of the system from the state vectors to the operators - the so called Heisenberg picture - then we may write this expression as:

(9) $$\frac{dQ(t)}{dt} = \frac{1}{i} [T(t), H']$$

The operators for position $\{Q_i\}$ and momentum $\{P_i\}$ do not commute in quantum theory and thus it is impossible to find an eigenstate of both these operators simultaneously - a measurement of position destroys momentum information and vice versa. The following commutation relations are imposed on the operators $\{Q_i\}$ and $\{P_j\}$:

(10) $$[Q_i, Q_j] = 0 = [P_i, P_j] \text{ , and } [Q_i, P_j] = ih\, \delta_{ij}$$

These are Heisenberg's canonical commutation relations (C.C.R.). Using the Baker-Cambell-Hausdorff formulae we can exponentiate these relations to obtain Weyl relations (or Heisenberg group) corresponding to the C.C.R.'s (10):

(11) $$\begin{aligned} V\alpha . V\alpha' &= V\alpha+\alpha' \\ U\beta . U\beta' &= U\beta+\beta' \\ V\alpha . U\beta &= \langle \alpha,\beta \rangle U\beta . V\alpha \end{aligned}$$

where $V\alpha := \exp(i\alpha_j P_j)$, $U\beta := \exp(i\beta_j Q_j)$, $\langle \alpha,\beta \rangle := \exp(i\alpha_j\beta_j)$ and where $\alpha_i, \beta_j \in R$.

Quantum mechanics on Rn is basically about finding irreducible representations of this group (11). Heisenberg's 'matrix mechanics' is the problem of finding an irreducible representation of the algebra (10) that diagonalises the Hamiltonian operator $H'(Q_i,P_j)$ and solves the eigenvalue problem, whilst Schroedingers 'wave mechanics' is about solving the differential equation that results from the substitution of a particular representation of (10) - the Schroedinger representation - into the energy eigenvalue equation $H'\phi = E\,\phi$. The Stone-Von Neumann theorem [57,61] states that that any other integrable representation of the algebra (10) must be equivalent to the Schroediger representation:

(12) $$\begin{aligned} \exp(i\beta_j Q_j) &: u(x) \longrightarrow \exp(i\beta_j x_j)u(x) \\ \exp(i\alpha_j P_j) &: u(x) \longrightarrow u(x + \alpha) \end{aligned}$$

Thus, quantum mechanics on Rn has an essentially unique solution [37].

1.4.2 Quantising a classical theory.

Although, in many fundamental ways, quantum mechanics is utterly different from classical mechanics, both in the space in which it acts and the types of statements it makes, many key relationships within the theory appear to have counterparts in classical mechanics. Take, for example, the similarity between Ehrenfest's theorem from ∮ 1.4.1 and equation (52) for the time development of a classical observable in ∮ 1.0.6; or the condition (53) that a collection of classical phase space functions generates a symmetry of the Hamiltonian and the condition (7) above that [H,Ti] = 0. Consider also the strong resemblence between the fundamental Poisson brackets (17) in ∮ 1.0.3 and the Heisenberg commutation relations (10) in ∮ 1.4.1. Clearly, knowing more about the structure of this powerful link between the two theories might make it possible to consider more general theoretical settings. For instance, in a quantum theory based on a classical theory whose phase space is some manifold (for example a 2-sphere), what algebra will then replace the Heisenberg algebra and will there be a similar correspondence between it and some group algebra on the classical phase space? The 'group theoretic' approach to quantisation tackles these questions by appealing to the symmetry group (if there is one) of the underlying classical phase space. It shows how to construct from it the 'quantum group', irreducible representations of which will form the quantum theory in the same way that irreducible representations of the Heisenberg group are the basis of quantum mechanics on Rn [37].

Firstly what constitutes quantisation of a classical theory? The process Dirac and others orginated was that to each classical observable $f \in F(M)$ defined on some phase space M is associated a self adjoint linear operator $\underset{\sim}{f}$ on Hilbert space $\underset{\sim}{H}$ such that [36]:

(A) The map: $f \dashrightarrow \underline{f}$ is real, linear. That is:

$$af + bg \dashrightarrow a\underline{f} + b\underline{g} , \quad \text{where } a,b \in R$$

(B) The unit operators correspond: $1 \dashrightarrow \underline{1}$.

(C) $G(f) \dashrightarrow G(\underline{f})$, where G is some function of $f \in F(M)$.

(D) $\{ f,g \} \dashrightarrow \frac{1}{ih} [\underline{f},\underline{g}]$

In the event that there are constraints inherent in the classical theory and its Lagrangian is singular, then the correspondence suggested by (D) comes badly adrift because the constraints of the theory are not preserved through time. To rectify this problem Dirac proposed that (D) should be replaced by:

(D′) $\{ f,g \}^* \dashrightarrow 1 [\underline{f},\underline{g}]$,

where $\{ , \}^*$ represents the Dirac bracket, cf § 1.3.

A quantisation scheme based around this correspondence does not have the above mentioned trouble because, as we know, a Dirac bracket between any function and a constraint is identically zero and therefore valid throughout the time evolution of the system. We will return to this later.

The objective of G.T. quantisation scheme [37] is to find for a general configuration space the Lie group whose algebra corresponds to the role the Heisenberg algebra plays when carrying out quantisation on Rn. But where then does the Heisenberg algebra come from? Clearly it is just the algebra one obtains by taking the Poisson bracket of the (special) set of classical observables, the co-ordinate functions (q_i,p_j):

$$\{ q_i,q_j \} = 0 = \{ p_i,p_j \}$$

(13) $$\{ q_i,p_j \} = \delta_{ij} , \quad \text{for } i,j = 1,\dots,n$$

$$\{ c,q_i \} = 0 = \{ c,p_j \}$$

where $q_i,p_j \in F(M)$ and $c \in R$: the constant functions on phase space M. Put in a more general way, it appears that quantisation is based on finding irreducible representations of some finite dimensional Lie group, whose Lie algebra is some sub-algebra of the infinite dimensional Lie algebra of functions on phase space, with the Poisson bracket as the combination principle. What makes the set of observable (q_i, p_j) special is that any function on R^n (or indeed on some local co-ordinate patch of a general manifold) may be expressed in terms of them. Any sub-algebra of the infinite dimensional Lie algebra of functions which is used to construct a quantum theory around, must reflect this property in some way [37]. The fundamental question is - how does the quantum group of this sub-algebra relate to the classical phase space? This question is answered in [37] through appeal to the case of quantum theory on R^n. Briefly the argument goes as follows. Previously we defined the operators $\underline{V}_\alpha$ and $\underline{U}_\beta$ which make up the Heisenberg group (11). These operators act on the position and momentum operators $\underline{Q}_i$ and $\underline{P}_j$ as follows:

$$\underline{V}_\alpha . \underline{Q}_i . \underline{V}_\alpha = \underline{Q}_i - \alpha_i \underline{1} \qquad (14)$$

$$\underline{U}_\beta . \underline{P}_i . \underline{U}_\beta = \underline{P}_i + \beta_i \underline{1}$$

Where $\underline{1}$ is the unit operator

Clearly the quantum operators $\underline{V}_\alpha, \underline{U}_\beta$ representing the Heisenberg group (11) are acting in a fashion reminiscent of the generators of the group of abelian transalations of the classical variables $(q_i;p_j)$. This suggests that (see [37] for the full argument) the group whose algebra will play the same role as the Heisenberg algebra on some more general non-linear phase space, will be a Lie transformation group of that phase space. This Lie group must also be symplectic, because its algebra must be capable to being realised by some sub-algebra of the infinite dimensional Lie algebra of classical observables on phase space, which has the P.B. as a combination principle. We may define a (left acting) Lie transformation group as a Lie group G, whose elements $g \in G$ may act on some underlying manifold M in such a way that the map, $g: p \longrightarrow g.p$ for $p \in M$ and $g \in G$, is a differentiable map. Also for the identity element $i \in G$ we have $ip = p$, and for any $g,g' \in G$ then $g(g'p) = (gg')p$ for all $p \in M$. A good example of a Lie transformation group is the action of SO(3) on the 2-Sphere, where the three generators of SO(3) generate rotations round the sphere. Barring some global considerations this quantisation principle is in essence close to that of Dirac's for constrained systems. As we know from § 1.3, the Dirac bracket is none other than a Poisson bracket in a reduced set of variables, with these variables being the unconstrained variables in some local patch of the underlying phase space of the theory. When we substitute Dirac brackets for commutator brackets, what we are really trying to do is to represent the 'generalised canonical group' of transformations that preserve the Dirac bracket, by linear operators on some Hilbert space. However, because it is not generally possible to move canonically from one set of reduced variables (that transform the Dirac bracket into looking like a P.B.) to another adjacent set, the group theoretical approach to quantisation is far better suited to tackling problems which have a global element to them. One situation where these two approaches will exactly match each other, is in the case where the constraint functions of the Dirac theory themselves form a function group. This is because under these circumstances generalised canonical transformations are also canonical transformations. Given then that the group which will provide us with our quantum algebra is a group of symplectic transformations of the underlying classical phase space, which classical observables do we choose to quantise? The group theoretical programme [37] requires that one choose the set of classical observables which make a Poisson bracket realisation of the algebra of the symplectic group. For this to make sense, the (assumed finite) subset of the infinite set of all smooth functions on phase space, must be sufficiently large as to play the same role as the (qi;pj) in managing to generating all observables, in that particular local co-ordinate patch of the phase space. This requirement demands that the action of the Lie transformation group G on M must be <u>transitive</u> [37]. That is, given any $p, p' \in M$ then there must exist an element $g \in G$ such that $p' = g.p$. This means that using G and a single point $p \in M$, one may reach any other point in the phase space, which ensures that the set of observables which realise the Lie algebra of G is large enough locally to express <u>any</u> observable on M as a function of this special set [37].

Following [37] we can now list the steps for the 'group theoretical' quantisation of a classical theory which has some general manifold M as its phase space:

(A) Find a Lie transformation group G which acts transitively and symplectically (so it preserves the natural symplectic 2-form on M) on M. For the sake of simplicity we assume that this Lie group has finite dimension N, say. Clearly, G has some Lie algebra associated with it L(G):

(15) $$[\, T_i, T_j \,] = C_{ij}^{\;\;k}\, T_k \quad , \quad \text{where} \quad \{T_i\} \in L(G) \text{ and}$$

with $i,j,k = 1,\ldots,N$, and where $C_{ij}^{\;\;k}$ are the structure constants of the Lie algebra L(G) of G.

(B) Find a finite subset of the set of all smooth functions on phase space M which produce a Poisson bracket realisation of the above algebra L(G). That is, find a correspondence:

(16) $$T_i \longrightarrow f_i \quad , \quad \text{for } i = 1,\ldots,N$$

such that:

(17) $$\{\, f_i, f_j \,\} = C_{ij}^{\;\;k}\, f_k$$

(C) 'Quantise' the set of functions $\{f_i\}$, by mapping them onto a set of hermitian operators $\{\underline{f}_i\}$ which act on some Hilbert space H, and which obey the same commutation relations:

(18) $$f_i \longrightarrow \underline{f}_i \qquad \text{for } i = 1,\ldots,N$$

where:

(19) $$[\, \underline{f}_i, \underline{f}_j \,] = c_{ij}^{\;\;k}\, \underline{f}_k$$

(D) Find irreducible representations of this algebra.

Points (A) through (D) outline in 'physics type' language, the basic steps of the mathematically precise quantisation procedure described by C.Isham in [37]. Put in more mathematically correct language steps (A) to (D) involves the following:

(A′) Finding a Lie transformation group G which acts transitively and symplectically on M. That is if ϕ is a smooth diffeomorphism of M induced by the action of the group G, then $\phi^*W = W$ where W is the natural symplectic 2-form on M, and ϕ^* is the pullback operator associated with the diffeomorphism ϕ.

Taking the N dimensional Lie algebra L(G) associated with this group and using each generator of the algebra to determine a vector field on M. This is achieved by exponentiating each generator of the algebra to obtain a 1-parameter family of diffeomorphisms, which then determine (by the process discussed in § 1.2.3) an associated set of N locally Hamiltonian vector fields ΓTi.

That is:

(20) $$\Gamma : Ti \longrightarrow \Gamma Ti \quad , \quad i = 1,\ldots,N$$

where $Ti \in L(G)$ and $\Gamma Ti \in$ {Locally Hamiltonian vector fields on M}. For the group theoretical quantisation procedure to succeed we need fully Hamiltonian vector fields on M [37]. This requirement along with the necessity that the map Γ be one-to-one, in order to create an isomorphism between the algebra of generators and their corresponding H.V.F.'s, sets further constraints on the group G and the phase space M. To ensure the map Γ is 1-1, action of G on M must be almost effective, meaning that if there exists an element $g \in G$ such that $gp = p$ for all $p \in M$, then it must neccessarily imply that $g \in D$, where D is a discrete subgroup of G.

(B′) Finding a map - technically known as the 'Souriau momentum map'(see [55]), which maps the algebra L(G) into the space of observables on M :

(21) $$P : Ti \in L(G) \longrightarrow P(Ti) = fi \in F(M) \ , \text{ for } i = 1,\ldots,N$$

Where the map P is linear and also a Lie algebra isomorphism, meaning that the collection of functions $\{P(T_i) = f_i\}$ realise the Lie algebra L(G) of G under Poisson bracket combination. This requirement uncovers another possible upset in the quantisation process, namely the algebra L(G) might have what is known as a non-trivial 2-cocycle attached to it. Abstractly a 2-cocycle is a skew-symmetric map Z, say, which maps a pair of elements in the Lie algebra L(G) of G into the reals R:

(22) $$Z(A,B) = -Z(B,A) \in R \text{ , with } A,B \in L(G).$$

The map also satisfies the Jacobi identity:

(23) $$Z(A,[B,C]) + Z(C,[A,B]) + Z(B,[C,A]) = 0$$

Basically these objects come about through the possibility of re-defining the functions $P(T_i)$ in (21) up to the addition of an arbitary constant (where we recall that kernel of the map between the set of observables on M and their associated H.V.F.'s is the space of constant functions). In simple language this means that a real number Z_{ij}, say, might appear on the R.H.S. of (17):

(24) $$\{ f_i,f_j \} = C_{ij}^{k} f_k + Z_{ij} \quad \text{, with } Z_{ij} \in R$$

If this happens then there is a problem, because clearly the functions $\{P(T_i)\}$ are no longer realising the algebra L(G) of G. To try and circumvent this problem we are free to add arbitrary constants to the the functions $P(T_i)$ in an attempt to cancel out the numbers Z_{ij}, by the following redefinition:

(25) $$P(T_i) = f_i \longrightarrow f'_i = f_i + Z_i \text{ , where } Z_i \in R .$$

In this case we require:

(26) $$C_{ij}^{k} Z_k + Z_{ij} = 0$$

Under what conditions for the structure constants C_{ij}^{k} are we assured of being able to solve (26) for the numbers Z_i? Stated like this it is fairly clear that we can do this when the Cartan-Killing form of the Lie algebra L(G) is non-singular and so we are able to invert (26). This is the case when the group G is semi-simple.

In summary it is certainly possible to find a map P:L(G)--->F(M) which has the properties we desire, in the circumstances that the second cohomology group of L(G) vanishes. An example of this occurs when the group G is semi-simple.

The number of sets of functions {P(Ti)} that one can find which satisfy our requirements is actually labelled by the elements of the first cohomology group of L(G) [37]. An example of a situation when this group is trivial is once again the case when the group G is semi-simple [37]. Thus, when G is semi-simple essentially there is a unique choice of the set of functions P(Ti).

But suppose we want to quantise a dynamical system with a phase space whose transitively acting Lie group G does have a non-trivial 2-cocycle? In this event the 'quantum group' of the theory is not the Lie group G of symmetries of the classical phase space M, but rather it is this group centrally extended. Basically this means we actually incorporate the non-removeable 2-cocycle into the algebra we are trying to realise, by enlarging the group G to include a central term (this means that the algebra of the new group G', say, now includes a term that commutes with all the generators of the algebra, in just the same manner as the constant functions commute with all functions on phase space under Poisson bracket combination).

We finish off this section with an example of the group theoretical approach to quantum mechanics for the most frequently studied case: when the underlying phase space is simply Euclidean space Rn. As it turns out this is a case in point when the second cohomology group of the space is non-trivial, which results in the algebra of the transformation group having to be centrally extended. This results in the familiar ih term in Heisenberg algebra [37].

1.4.3 The group theoretical approach to the quantisation of Rn.

Here the underlying configuration space of our theory is Euclidean space Rn. The phase space is therefore just the trivial bundle of cotangent spaces, which here is the direct product of Rn with itself: Rn ⊗ Rn. Because of the uncomplicated nature of this phase space we are able to define co-ordinates $(q_1,\ldots,qn;p_1,\ldots,pn)$ which can be used through the space. Similarly it follows that there is a globally well-defined Poisson bracket given by the usual expression (14) in ∮ 1.0.3 expanded in the above co-ordinates. What then is the relevant Lie group of transitive and symplectic transformations on the phase space? The obvious choice is the abelian group of translations [37], which acts on the phase space as follows:

$$g(a1,\ldots,an;b1,\ldots,bn)(\ldots,qi,\ldots;\ldots,pj,\ldots) \longrightarrow (\ldots,qi+ai,\ldots;\ldots,pj+bj,\ldots) \qquad (27)$$

where $g \in G$ and for $i,j = 1,\ldots,n$. Also for $g,g' \in G$ we have:

$$g(g'p) = (gg')p = (g'g)p = g'(gp) \qquad (28)$$

The Lie algebra corresponding to this group is simply the set of unit basis vectors in R2n:

$$Ea = (0,\ldots,0,\underset{\text{position } a}{1},0,\ldots,0)\ , \quad \text{for } a = 1,\ldots,2n \qquad (29)$$

Vectors in this Lie algebra are combined by vector addition and so clearly we have:

$$[\,Ea,Eb\,] := Ea.Eb - Eb.Ea = 0 \qquad (30)$$

The programme now demands that we must realise this algebra by 2n independent functions on phase space, using the Poisson bracket as the Lie bracket. Thus we must find $fa = fa(qi;pj) \in F(M)$ such that:

$$\{\, fa,fb \,\} = 0 \quad , \quad \text{for all } a = 1,\ldots,n\ . \qquad (31)$$

Given that we need 2n independent functions, the best we can do is the correspondence:

(32) $(\ldots,E_a,\ldots) \longrightarrow (\ldots,f_a,\ldots) = (\ldots,q_i,\ldots;\ldots,p_j,\ldots)$

where $a = 1,\ldots,2n$ and $i = 1,\ldots,n$.

Clearly this does not realise the algebra (30) because of the cross terms:

(33) $$\{ f_i,f_{j+n} \} = \{ q_i,p_j \} = \delta_{ij} \quad , \quad i,j = 1,\ldots,n$$

This problem is a consequence of a non-trivial 2-cocycle in the Lie transformation algebra (30) we are using. It is not possible to remove it by redefinition of the functions f_i because of the abelian nature of the group (27). The remedy to this problem, as indicated above, is to incorporate the 2-cocycle into the algebra (30), and make this centrally extended algebra the one around which to base the quantum mechanics of the system. Changing notation and calling generators:

(34) $$E_i := Q_i \;, \text{ and } E_{i+n} := P_i \quad , \quad \text{for } i = 1,\ldots,n$$

The vector space of the centrally extended Lie algebra is now the space R^n+R^n+R, with a typical element $\mathbf{A} \in L(G)$ of the algebra being $(a_1,\ldots,a_n;b_1,\ldots,b_n;c) \in R^n+R^n+R$. In terms of the generators $\mathbf{Q}_i$ and $\mathbf{P}_j$, we have a typical element of $L(G)$ as:

(35) $$\mathbf{A} = \sum_i a_i\mathbf{Q}_i + b_i\mathbf{P}_i + c\mathbf{1} \quad , \quad \text{with } \mathbf{A} \in L(G) \quad , \quad i = 1,\ldots,n$$

and where the $2n+1$-tuple $\mathbf{1} = (0,\ldots,0,\ldots,1)$. (Note that now $\mathbf{Q}_i$ and $\mathbf{P}_j$ are also $2n+1$-tuples defined like E_a in (29) except now with an extra 0 in the $2n+1$ slot). The Lie bracket of this centrally extended algebra in terms of the generators $\mathbf{Q}_i$ and $\mathbf{P}_j$ is [37]:

(36) $$\begin{aligned} [\, \mathbf{Q}_i,\mathbf{Q}_j \,] &= 0 \\ [\, \mathbf{P}_i,\mathbf{P}_j \,] &= 0 \\ [\, \mathbf{Q}_i,\mathbf{P}_j \,] &= Z(\mathbf{Q}_i,\mathbf{P}_j) = \delta_{ij}\, \mathbf{1} \end{aligned}$$

for $i,j = 1,\ldots,n$. For two general elements $\mathbf{A},\mathbf{B} \in L(G)$ of the algebra the Lie bracket is:

(37) $$[\, \mathbf{A},\mathbf{B} \,] = \sum_i (a_i.b'_i - a'_i.b_i)\mathbf{1}$$

With $\mathbf{A} = \sum_i a_i\mathbf{Q}_i + b_i\mathbf{P}_i + c\mathbf{1}$, and $\mathbf{B} = \sum_i a'_i\mathbf{Q}_i + b'_i\mathbf{P}_i + c'\mathbf{1}$.

Clearly we may realise this algebra by observables on phase space with the Poisson bracket as the Lie bracket, by the correspondance:

(38) $\underline{Q}i$ ---> qi , $\underline{P}i$ ---> pi , $\underline{1}$ ---> 1 where i = 1,...,n.

The final step is to quantise these classical observables to obtain familiar Heisenberg algebra, by performing the following map into the space of linear operators on Hilbert space [37]:

(39) qj ---> -i$\underline{q}$j , pj ---> -i$\underline{p}$j , 1 ---> -ih$\underline{1}$

Here the under-lining now represents operators on Hilbert space and where the scaling factor h, Planck´s constant, comes about through some implicit scaling in (38) which we have not shown here.

What this example demonstrates is that we may interpret the ih$\underline{1}$ in the Heisenberg algebra, as coming about through the existance of a non-trivial second cohomology class for the Lie algebra of the group of abelian translations on Rn [37]. This is quite a satisfactory result as it is reminiscent of the appearance of anomalous terms in, for example, current algebras [39,65].

The above example concludes this rather telegraphic look at the group theoretical approach to quantisation contained in [37]. It also concludes the first, introductory, part of this thesis. Now, except where cited, the work is claimed as original from § 2.1 onwards.

PART II

2.0.0 Graded Analogues of Classical Concepts.

Introduction

In the sections comprising Part II the original work of this thesis is presented. The work consists of attempting to find consistent graded analogues to the various pieces of classical machinery which were introduced in the first half of this thesis, the ultimate goal in mind being the inclusion of fermion like variables into C.Isham's group theoretical approach to quantisation [37] reviewed earlier. Although in the end this task is not fully achieved, the pursuit of the goal raises a number of interesting points which we feel are sufficient to justify the undertaking.

Before starting the author would like to state that the initial stages of the work presented here, namely § 2.1, were carried out without the knowledge of a particularly important reference [22]. Subsequently further references, particularly the work of Marjorie Batchelor [10,11] and also additional work by F.A Berezin et al [15] were brought to the author's attention*. This work has a direct bearing on the considerations of § 2.1 and, on discovery of [22], there was no point in pursuing further investigations in this area. Earlier knowledge of the more accessible and rigorous approach to supermanifold theory presented in [22] for example, might have significantly increased the progress made towards producing a credible graded generalisation to C.Isham's work in [37]. We commence this half of the thesis by making some introductory remarks and statements about graded algebras, and their uses in constructing graded analogues to classical mechanics. For a review of the uses of graded algebras in physics see [21].

* The author wishes to thank Professor Isham for bringing these references to his attention.

2.0.1 Z_2 graded algebras.

The work that follows relies extensively on the properties of so called 'graded' objects. This refers to members of a Z_2 graded algebra A, which as a vector space is the direct sum of two sectors, A_0 and A_1. If $a_i \in A_i$ and $a'_j \in A_j$, then:

(1) $$a_i.a'_j = (-1)^{ij} a'_j.a_i \quad \text{for } i,j = 0,1$$

Typically we refer to the members of A_0 as being equivalently; 'even', 'commuting' or 'boson like', and denoting them by the Roman alphabet. Similarly members of A_1 are refered to as being equivalently; 'odd', 'anticommuting' or 'fermion like', and usually denoted by the Greek alphabet. An element of A which lies either fully in A_0 or in A_1 is called homogeneneous. For a more detailed exposition of graded vector spaces see [22]. An example of a Z_2 graded algebra is a Grassmann algebra which has a set of N generators θ_α, for $\alpha = 1,\dots,N$, such that:

(2) $$\theta_\alpha.\theta_\beta = -\theta_\beta.\theta_\alpha \quad , \quad \theta_\alpha.\theta_\alpha = 0 \quad , \text{ for all } \alpha,\beta = 1,\dots,N$$

We denote this algebra G_N. Clearly all polynomials of even order: 1, $\theta_\alpha.\theta_\beta$, $\theta_\alpha.\theta_\beta.\theta_\delta.\theta_\mu$,...etc (where we have included 1) are members of A_0, and all polynomials of odd order: θ_α, $\theta_\alpha.\theta_\beta.\theta_\delta$,...etc are members of A_1.

2.0.2 Graded analogues of classical mechanics.

In a series of papers R.Casalbuoni et al. [8,17,18] developed a version of classical mechanics which incorporated the use of Z_2 graded variables as the fundamental variables of the theory. There is a number of reasons why such a theory might be of interest, not the least of them being that it allows the study of graded variables in physics in a very familiar classical setting. However, the reason why it is of interest to us here is because the group theoretical quantisation programme described in [37] requires the existence of a classical theory, the symmetries of which will provide a basis for determining the group of the quantum theory. Casalbuoni's work is useful because it provides just such a theory, however unfortunately it is not stated in a way that makes a generalisation of the group theoretical approach to quantisation straightforward. The only other approach that was avalable on this type of graded analogue of classical mechanics was a short section at the end of B.Kostant's work on graded manifold theory [40]. Structurally this seemed as far removed from [37] as Casalbuoni's work did, and the language it was set in - algebraic geometry - was mathematically tougher to follow (as it turns out the type of approach that seems best suited to provide a convincing generalisation of [37] is the one adopted by B.De Witt in [22], where the 'supermanifold' is defined. However at this stage the author wasn't aware of this work). It was decided to use Casalbuoni's work as a basis for the approach we take here, and to couch it in a formalism more suited to our needs with regard to producing a graded generalisation of [37]. Rather than giving a long review of the results and conclusions aired in [8,17,18], we decide to limit ourselves to a brief but sufficient sketch of this approach to Z_2 graded classical mechanics, a theory which Casalbuoni et al. named 'pseudomechanics'.

2.0.3 Some elements of pseudomechanics.

The Lagrangian formulation of pseudomechanics takes place on a configuration space C described by a set of n real co-ordinates $\{q_i\}$ and a further N co-ordinates $\{\theta_\alpha\}$ which form a G_N Grassmann algebra. Thus the 'position' of a pseudoparticle in the configuration space C is described by the (n+N)-tuple $(..,q_i,..;..,\theta_\alpha,..)$ with $i = 1,...,n$ and $\alpha = 1,...,N$. These position variables are functions of some real time parameter $t \in R$, and this leads to the concept of generalised velocity co-ordinates $(..,q_i,.;..,\theta_\alpha,..)$. Thus we may completly describe the dynamics of a pseudoparticle using the generalised position and velocity co-ordinates $(.,q_i,.;.,q_j,.;..,\theta_\alpha,.;.,\theta_\beta,.)$, with $i,j = 1,..,n$ and $\alpha,\beta = 1,..,N$.

Following [18] one may now define a pseudomechanical Lagrangian function $L = L(q_i(t), q_j(t); \theta_\alpha(t), \theta_\beta(t))$ which typically will look like:

$$(3) \qquad L = i \sum_\alpha \theta_\alpha \theta_\alpha + \sum_i m_i(q_i) - V(q,\theta)$$

Where $V(q,\theta)$ is a potential term. Defining the action S as:

$$(4) \qquad S := \int L dt$$

Then the principle of least action leads us to the following Euler Lagrange equations of motion [18]:

$$(5) \qquad m_i \frac{dq_i}{dt} = - \frac{\partial V}{\partial q_i} , \quad (\text{ no sum over}) \; i = 1,\ldots,n$$

$$(6) \qquad \frac{d\theta_\alpha}{dt} = -i \frac{\partial V}{\partial \theta_\alpha} , \qquad \text{and} \quad \alpha = 1,\ldots,N$$

As with the case of standard classical mechanics we now define the Hamiltonian function H, and reformulate our theory in a Z graded generalisation of phase space. We define:

$$(7) \qquad p_i := \frac{\partial L}{\partial \dot{q}_i} , \qquad \pi_\alpha := \frac{\partial L}{\partial \dot{\theta}_\alpha}$$

where the p_i commute and the π_α anti-commute. Then the Hamiltonian is defined:

$$(8) \qquad H := \sum_{i,\alpha} (\dot{q}_i p_i + \dot{\theta}_\alpha \pi_\alpha) - L$$

Once again employing the principle of least action as in § 1.2 we are lead to Hamilton's equations [18]:

$$(9) \qquad \frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i} , \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad i = 1,\ldots,n$$

$$(10) \qquad \frac{d\pi_\alpha}{dt} = - \frac{\partial H}{\partial \theta_\alpha} , \quad \frac{d\theta_\alpha}{dt} = - \frac{\partial H}{\partial \pi_\alpha} \qquad \alpha = 1,\ldots,N$$

Where the Hamiltonian function $H = H(q,p;\theta,\pi)$ is even and differentiation is with respect to the real time parameter t. Notice that there is a relative sign difference between the equations of the even sector and a similarity of sign between the equations of the odd sector. We will return to this point shortly.

We now introduce the reader to the graded Poisson bracket defined in [17], a structure which leads naturally to the notion of the infinitesimal graded canonical transformation. We will make extensive use of this device in § 2.1.

2.0.4 The graded Poisson bracket.

From the graded version of Hamilton's equations (9) and (10) we can see that the natural variables of this type of graded phase space are $(..,q_i,..;..,p_j.;..,\theta_\alpha,.;..,\pi_\beta,.)$ where the q_i,p_j are the natural bosonic phase space co-ordinates as before, and the θ_α, π_β are conjugate pairs of anticommuting odd co-ordinates, which together with the q_i,p_j make up the full Z_2 graded phase space.

By considering the time derivative of some general function $F = F(q,p;\theta,\pi)$ defined on graded phase space, we can arrive at the expression for the graded Poisson bracket by following the same steps as we do in § 1.0.3. In fact one finds [18]:

$$\frac{dF(q,p;\theta,\pi)}{dt} = \{ F,H \}' + \frac{\partial F}{\partial t} \qquad (11)$$

Where the $\{ \ , \ \}'$ represents the graded Poisson bracket, which is defined as follows:

$$\{ F,G \}' := \sum_i \left(\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} \right) - \sum_\alpha \left(\frac{\partial G}{\partial \theta_\alpha}\frac{\partial F}{\partial \pi_\alpha} + \frac{\partial G}{\partial \pi_\alpha}\frac{\partial F}{\partial \theta_\alpha} \right) \qquad (12)$$

Where $F = F(q,p;\theta,\pi)$ and $G = G(q,p;\theta,\pi)$ are two functions on the graded phase space. As it turns out [17], this definition (12) is only viable for the case when the function G is even (commutative). To deal with the other alternatives, we require our graded bracket to satisfy an additional condition. We demand that if μ is some odd parameter, then:

$$\mu \{ E,O \}' = \{ \mu E,O \}' = \{ E,\mu O \}' \qquad (13)$$

Where E is some even function of $(q_i,p_j;\theta_\alpha,\pi_\beta)$ and O is some odd function. That is, we demand that the graded bracket forms an algebra over a Grassmann ring [17].

This extra condition (13) means that we may now define the additional graded Poisson brackets as follows [17]:

(14) $$\{ O,E \}' := \sum_i \left(\frac{\partial O}{\partial qi}\frac{\partial E}{\partial pi} - \frac{\partial E}{\partial qi}\frac{\partial O}{\partial pi} \right) - \sum_\alpha \left(\frac{\partial O}{\partial \theta\alpha}\frac{\partial E}{\partial \pi\alpha} + \frac{\partial E}{\partial \theta\alpha}\frac{\partial O}{\partial \pi\alpha} \right)$$

(15) $$\{ E,O \}' := \sum_i \left(\frac{\partial E}{\partial qi}\frac{\partial O}{\partial pi} - \frac{\partial O}{\partial qi}\frac{\partial E}{\partial pi} \right) + \sum_\alpha \left(\frac{\partial E}{\partial \theta\alpha}\frac{\partial O}{\partial \pi\alpha} + \frac{\partial O}{\partial \theta\alpha}\frac{\partial E}{\partial \pi\alpha} \right)$$

(16) $$\{ Oa,Ob \}' := \sum_i \left(\frac{\partial Oa}{\partial qi}\frac{\partial Ob}{\partial pi} + \frac{\partial Ob}{\partial qi}\frac{\partial Oa}{\partial pi} \right) - \sum_\alpha \left(\frac{\partial Oa}{\partial \theta\alpha}\frac{\partial Ob}{\partial \pi\alpha} + \frac{\partial Ob}{\partial \theta\alpha}\frac{\partial Oa}{\partial \pi\alpha} \right)$$

These definitions lead to the following properties of the graded bracket under the interchange of the various object functions:

(17) $$\{ Ea,Eb \}' = - \{ Eb,Ea \}'$$

(18) $$\{ E,O \}' = - \{ O,E \}'$$

(19) $$\{ Oa,Ob \}' = \{ Ob,Oa \}'$$

Where O, Oa, Ob are odd; E, Ea, Eb even. This completes the definition of the graded Poisson bracket (some additional properties of the bracket are stated in appendix A). We now look at the group of fixed transformations which preserve the structure of this bracket, that is, the group of 'infinitesimal graded canonical transformations' (I.Gr.C.T.).

2.0.5 The infinitesimal graded canonical transformations.

We define the group of graded canonical transformations in a similar way as we did before in § 1.0.4, only this time making the generating function graded even. Thus we have the following definition for a graded canonical transformation (Gr.C.T.):

Definition

A graded canonical transformation is a transformation of the fundamental variables $(qi;\dot{q}j;\theta\alpha,\dot{\theta}\beta)$ such that the new and old Lagrangian functions differ by at most a total time derivative of some arbitrary function f, say. That is, we have a transformation:

(20) $$(qi,\dot{q}j;\theta\alpha,\dot{\theta}\beta) \longrightarrow (q'i,\dot{q}'j;\theta'\alpha,\dot{\theta}'\beta)$$

(21) $$L(q,\dot{q};\theta,\dot{\theta}) \longrightarrow L'(q',\dot{q}';\theta',\dot{\theta}')$$

where $i,j = 1,\ldots,n$ and $\alpha,\beta = 1,\ldots,N$, and such that:

(22) $$L'(q',\dot{q}';\theta',\dot{\theta}') = L(q,\dot{q};\theta,\dot{\theta}) - \frac{df(q,\dot{q}';\theta,\dot{\theta}')}{dt}$$

By differentiation this produces the following relationships [18]:

(23) $$\frac{dp'_i}{dt} = -\frac{\partial f}{\partial q'_i}, \quad \frac{dp_i}{dt} = \frac{\partial f}{\partial q_i} \qquad i = 1,\ldots,n$$

(24) $$\frac{d\pi'_\alpha}{dt} = -\frac{\partial f}{\partial \theta_\alpha}, \quad \frac{d\pi_\alpha}{dt} = \frac{\partial f}{\partial \theta_\alpha} \qquad \alpha = 1,\ldots,N$$

Using (8) this is equivalent to:

(25) $$H'(q',p';\theta',\pi') = H(q,p;\theta,\pi) + \frac{\partial f}{\partial t}$$

One can show [18] that the class of I.Gr.C.T. which do not change the graded character of the fundamental variables may all be generated by even functions defined on graded phase space. That is if:

(26) $$q_i = q'_i + \delta q_i \quad , \quad p_i = p'_i + \delta p_i$$

(27) $$\theta_\alpha = \theta'_\alpha + \delta\theta_\alpha \quad , \quad \pi_\alpha = \pi'_\alpha + \delta\pi_\alpha$$

where $\delta q_i, \delta p_i$ commute and $\delta\theta_\alpha, \delta\pi_\alpha$ anticommute, then:

(28) $$H = H' + \frac{\partial F}{\partial t}$$

with $F = F(q,p;\theta,\pi)$ and is even, and $F = \sum_{i,\alpha} (\delta q_i p_i + \delta\theta_\alpha \pi_\alpha) - f$.
The variation of a general function on graded phase space under the transformation (26,27) is:

(29) $$G(q,p;\theta,\pi) \longrightarrow G + \delta G \ , \quad \text{where} \quad \delta G = \{ G,F \}'$$

Where $\{\ ,\ \}'$ represents the graded Poisson bracket.

The group theory of these infinitesimal transformations is discussed in detail in [18]. What the work there demonstrates is that I.Gr.C.T. are those transformations which preserve the graded equivalent of the symplectic form in ∮ 1.0.5, which is the ortho-symplectic form [18]:

(30) $$\Gamma = \begin{pmatrix} A_{ij} & 0 \\ 0 & B_{\alpha\beta} \end{pmatrix}$$

where:

(31) $$(A)_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & 0 & \\ & & \vdots & & \end{pmatrix} \quad , \quad (B)_{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \\ 0 & 0 & -1 & 0 & \\ & & \vdots & & \end{pmatrix}$$

Thus the group which plays the same role in pseudomechanics as the symplectic group does in classical mechanics, is the super Lie group OSp(n|N), that is the group of orthosymplectic transformations which preserve the orthosymplectic form (30)[48].

For further details of Casalbuoni's work on pseudomechanics the reader should consult the references given earlier in this section, as we now leave the introductory material as it stands and start on the original contributions of this thesis.

2.1.0 Further Investigation into the Graded Poisson Bracket.

Introduction

In this section we investigate some further properties of the graded Poisson bracket (G.P.B.) with a view to determining to what extent Casalbuoni's definition of the bracket (12,14,15,16) is consistant, when it is used in the construction of graded generalisations of some of the structures introduced in Part I. The first structure we attempt to apply Z_2 grading to is the Hamiltonian vector field of § 1.2.3. What we discover is that a single graded version of this vector field is not enough to enable the reproduction of basic properties that a generalisation should possess, in order to be consistent with the definition of the G.P.B. This leads to the introduction of the notion of 'left acting' and 'right acting' vector fields, which are a fundamental feature of the geometry of spaces which involve anticommuting variables. Although we arrive at this conclusion independently, and purely through the study of graded analogues to classical 'phase space', this claim is supported elsewhere [15,22], and in a far more rigorous setting than we use here. We start by trying to find a consistent Z_2 graded Hamiltonian vector field.

2.1.1 A Z_2 graded analogue of the Hamiltonian vector field.

In § 1.2.3 of this thesis, we saw that there is a natural geometrical way in which to imagine the application of a regular infinitesimal canonical transforamtion (I.C.T.) to some dymamical system. This is through the integral curves of some suitably smooth and well defined Hamiltonian vector field (H.V.F.) on the phase space of the system. The H.V.F. is the differential operator associated with some well defined function $f \in F(M)$ on phase space, which maps some other well defined function $g \in F(M)$ onto the Poisson bracket between itself and f. Thus, if $f,g \in F(M)$ then:

(1) $$Hf : g \longrightarrow \{ g,f \}$$

Where Hf is the Hamiltonian vector field associated with $f \in F(M)$. In terms of the natural co-ordinate basis (.,qi,..;.,pj,..) on phase space M, the H.V.F. has the following form:

(2) $$Hf = \sum_i \frac{\partial f}{\partial pi}\frac{\partial}{\partial qi} - \frac{\partial f}{\partial qi}\frac{\partial}{\partial pi} \quad , \qquad i = 1,\ldots,n.$$

The first aspect of the definition in ∮ 2.0 of the graded Poisson bracket we wish to explore, is to descover if it is possible to consider this bracket definition as arising in a similar manner from some graded analogue of the H.V.F.

Rather than delving deeply into the precise nature of the space which is being dealt with here, we decide to proceed in a more pedestrian manner using the constructions introduced in Part I as a guide. In ∮ 2.0 we saw that there is a graded version of phase space which is described by graded variables ($qi,pj;\theta\alpha,\pi\beta$), with $i,j = 1,\ldots,n$ and $\alpha,\beta = 1,\ldots,N$, and where the ($qi,pj$) commute and the ($\theta\alpha,\pi\beta$) anticommute. Although we haven't defined exactly what this means in a mathematical sense, we assume that these variables represent some form of local expression for the natural co-ordinates of an object we call a 'super phase space' (this approach is made rigorous in [22]). We assume that it is possible to express this super phase space in a globally well defined manner, as it is for a normal symplectic manifold, but at this stage we don't know how to do that. All we have are local co-ordinate expansions. Vectors in this approach we assume, for time being, are derivative operators with respect to the natural super phase space (S.P.S.) variables ($qi,pj;\theta\alpha,\pi\beta$), as they are on a local co-ordinate patch of a normal manifold (we will say more about this later though). Using only this limited machinery is it now possible to construct a convincing Z_2 graded generalisation to the H.V.F.? Any generalisation to the H.V.F. we produce must reflect the above property (1) of Hf in some way. That is, if we denote the graded Hamiltonian vector field (G.H.V.F.) associated with some Z_2 graded function f on S.P.S. by Φf, then we demand that:

(3) $$\Phi f : g \longrightarrow \{ f, g \}'$$

Where now f,g are both Z_2 graded functions and { , }´ represents the graded Poisson bracket defined in § 2.0. Thus we require that the G.H.V.F. associated with the graded function f must map any graded function g on S.P.S. to the graded Poisson bracket between f and g. This seems to be a reasonable request to make of any candidate G.H.V.F.

Looking at the expression for the graded Poisson bracket in § 2.0.4 the natural guess to make for the G.H.V.F. is the following:

(4) $$\Phi f := \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} \right) - \sum_\alpha \left(\frac{\partial f}{\partial \pi_\alpha}\frac{\partial}{\partial \theta_\alpha} + \frac{\partial f}{\partial \theta_\alpha}\frac{\partial}{\partial \pi_\alpha} \right)$$

Where the first term in brackets is the normal H.V.F.(2) contributed from the even co-ordinates (qi,pj), and the second term is the odd extension associated with the co-ordinates (θα,πβ). If this is used as a definition for the G.H.V.F. Φf, does it then satisfy the requirement (3) consistently for all possible choices of character for the graded functions f and g ? We now examine the separate cases which occur:

A) Suppose we take f to be an even generator function on S.P.S. and g to be even also. Then:

$$\Phi f(g) := \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) - \left(\frac{\partial f}{\partial \pi_\alpha}\frac{\partial g}{\partial \theta_\alpha} + \frac{\partial f}{\partial \theta_\alpha}\frac{\partial g}{\partial \pi_\alpha} \right)$$

$$= \left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) + \left(\frac{\partial g}{\partial \theta_\alpha}\frac{\partial f}{\partial \pi_\alpha} - \frac{\partial f}{\partial \theta_\alpha}\frac{\partial g}{\partial \pi_\alpha} \right)$$

$= + \{ g,f \}'$, since the f, g anticommute.

B) Now take the case when f is even, but g is odd:

$$\Phi f(g) := \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) - \left(\frac{\partial f}{\partial \theta_\alpha}\frac{\partial g}{\partial \pi_\alpha} + \frac{\partial f}{\partial \theta_\alpha}\frac{\partial g}{\partial \pi_\alpha} \right)$$

$$= \left(\frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - \left(\frac{\partial g}{\partial \theta_\alpha} \frac{\partial f}{\partial \pi_\alpha} + \frac{\partial f}{\partial \theta_\alpha} \frac{\partial g}{\partial \pi_\alpha} \right)$$

$$= + \{ g,f \}'$$

So when the generator function f is even, the definition (4) for the G.H.V.F. Φf is satisfactory. Unfortunately difficulties occur when we take the generator function f to be odd. To see this:

C) When generator function f is odd, but g is even:

(5) $$\Phi f(g) := \ldots = \left(\frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - \left(\frac{\partial g}{\partial \theta_\alpha} \frac{\partial f}{\partial \pi_\alpha} + \frac{\partial f}{\partial \theta_\alpha} \frac{\partial g}{\partial \pi_\alpha} \right)$$

and finally,

D) With both generator function f and g odd:

(6) $$\Phi f(g) := \ldots = - \left(\frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - \left(\frac{\partial f}{\partial \theta_\alpha} \frac{\partial g}{\partial \pi_\alpha} + \frac{\partial g}{\partial \theta_\alpha} \frac{\partial f}{\partial \pi_\alpha} \right)$$

We can see now that both (C) and (D) are a departure from equation (3), because the relative signs between the first and second terms of (5) and (6) have changed, when compared to the corresponding definitions (15) and (16) in § 2.0 for the {Even,Odd}′ and {Odd,Odd}′ brackets respectively. It is not surprising that we have encountered some difficulty when dealing with a generator function whose character is odd. This is because an anticommuting generator function actually changes the character of the variable on which it operates, since:

(7) $$\{\text{Even},\text{Odd}\}' = \text{Odd}$$

This means that I.Gr.C.T. generated by these functions mix the odd and even sectors of the super phase space, in action similar to the odd generators of a supersymmetry algebra [5]. All this leads us to conclude that definition of the G.H.V.F. we have used in (4) is only a suitable candidate when the generator function is even. When the generator function is odd, the character of the function being operated on is effecting the outcome of the signs in the final expression. Since it is neccessary that a successful candidate G.H.V.F. must be independent of the character of the object function which it operates on, it follows that definition given in (4) will not suffice.

Before making another guess at the G.H.V.F. for the case of an odd generator function, first we look a little more carefully at what is meant by differentiation of, or by, graded objects. This is covered in Berezin's book [12] and which we follow here. In this, it is made clear that two equivalent definitions of differentiation by Grassmann like objects are possible. Since in general a function of Grassmann variables may be thought of as being a finite power expansion in those variables, all we need consider the differentiation of is a term like:

(8) $A\alpha\beta\ldots\mu(\theta\alpha\theta\beta\ldots\theta\mu)$, with P terms, where $\alpha,\beta,\mu = 1,\ldots,N\geq P$

$A\alpha\beta\ldots\mu$ is a real valued function of the even fundamental S.P.S. variables $(\ldots,qi,\ldots,pj,\ldots)$, and which is totally antisymmetric in the indices $\alpha,\beta,\ldots,\mu$.

The differentiation of an expression like (8) by a Grassmann variable, $\theta\delta$ say, may be approached in two ways [12]. Either by defining 'differentiation to the right', thus:

(9) $$\frac{\overset{>}{\partial}}{\partial\theta\delta}(A\alpha\beta\ldots\tau\mu\theta\alpha\theta\beta\ldots\theta\tau\theta\mu) := A\alpha\ldots\mu(\delta\alpha\delta\ \theta\beta\ldots\theta\mu - \delta\beta\delta\ \theta\alpha\ldots\theta\mu + \ldots\ldots + (-1)^{P-1}\delta\mu\delta\ \theta\alpha\ldots\theta\tau)$$

That is, you permute each successive term to the extreme left hand side of the pollynomial, and then replace it by a Kronecker delta. Or alternatively, you can define 'differentiation to the left' by:

(10) $$(A\alpha\beta\ldots\tau\mu\theta\alpha\theta\beta\ldots\theta\tau\theta\mu)\frac{\overset{<}{\partial}}{\partial\theta\delta} := A\alpha\ldots\mu(\delta\mu\delta\ \theta\alpha\ldots\theta\tau - \delta\tau\delta.\theta\alpha\ldots\theta\mu + \ldots\ldots + (-1)^{P-1}\delta\alpha\delta\ \theta\beta\ldots\theta\mu)$$

This time successive terms have been permuted to the extreme right hand side and then replaced by a Kronecker delta. We illustrate these two processes by a simple example. Suppose we only have two Grassmann elements θ and θ, then:

(11)
$$\frac{\overset{>}{\partial}}{\partial\theta_1}(\theta_1\theta_2) = \theta_2 \quad , \quad (\theta_1\theta_2)\frac{\overset{<}{\partial}}{\partial\theta_1} = -\theta_2$$
$$\frac{\overset{>}{\partial}}{\partial\theta_2}(\theta_1\theta_2) = -\theta_1 \quad , \quad (\theta_1\theta_2)\frac{\overset{<}{\partial}}{\partial\theta_2} = \theta_1$$

That is here left and right derivatives differ by a sign.

In fact we can write down the general relationship between the left and right derivative. It is:

(12) $$\frac{\overrightarrow{\partial} F}{\partial A} \equiv \frac{(-1)^{|F||A|}}{(-1)^{|A|}} F \frac{\overleftarrow{\partial}}{\partial A}$$

Where $|F|$, $|A|$ are the characters of F and A respectively. When a graded function F is even, then $|F| = 0$. When F is odd then $|F| = 1$.

Note that the only case which does incorporate a sign change is when F is even and A is odd. That is $|F| = 0$, $|A| = 1$.

Now that we have a better understanding of what is meant by the differentiation in the context of Z_2 graded variables, we return to the problem of defining a graded version of the H.V.F. What we discover is that it is precisely the sign changing property of the left-handed derivative in the case of two even functions which is required to define the G.H.V.F. for the case when the generating function is odd. In fact the definition of the G.H.V.F. applicable to this situation is the following:

(13) $$f\Phi := \left(\frac{\overleftarrow{\partial}}{\partial q_i} f \frac{\overleftarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i} f \frac{\overleftarrow{\partial}}{\partial q_i}\right) - \left(\frac{\overleftarrow{\partial}}{\partial \theta_\alpha} f \frac{\overleftarrow{\partial}}{\partial \pi_\alpha} + \frac{\overleftarrow{\partial}}{\partial \pi_\alpha} f \frac{\overleftarrow{\partial}}{\partial \theta_\alpha}\right)$$

where we have introduced some new notation. The generator function f now appears to the left of Φ to indicate it is odd and that we are now dealing with vector fields which act to the left. To that this does solve the problem, we calculate:

A) When the generator function f is odd, and g is even:

$$(g)f\Phi := \left(g\frac{\overleftarrow{\partial}}{\partial q_i} f \frac{\overleftarrow{\partial}}{\partial p_i} - g\frac{\overleftarrow{\partial}}{\partial p_i} f \frac{\overleftarrow{\partial}}{\partial q_i}\right) - \left(g\frac{\overleftarrow{\partial}}{\partial \theta_\alpha} f \frac{\overleftarrow{\partial}}{\partial \pi_\alpha} + g\frac{\overleftarrow{\partial}}{\partial \pi_\alpha} f \frac{\overleftarrow{\partial}}{\partial \theta_\alpha}\right)$$

(14) $$= \left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right) + \left(\frac{\partial g}{\partial \theta_\alpha}\frac{\partial f}{\partial \pi_\alpha} + \frac{\partial g}{\partial \pi_\alpha}\frac{\partial f}{\partial \theta_\alpha}\right)$$

$$= \{ g,f \}' \quad \text{as required.}$$

B) When generator f and g are both odd:

$$(g)f\Phi := \left(g\frac{\overleftarrow{\partial}}{\partial q_i} f \frac{\overleftarrow{\partial}}{\partial p_i} - g\frac{\overleftarrow{\partial}}{\partial p_i} f \frac{\overleftarrow{\partial}}{\partial q_i}\right) - \left(g\frac{\overleftarrow{\partial}}{\partial \theta_\alpha} f \frac{\overleftarrow{\partial}}{\partial \pi_\alpha} + g\frac{\overleftarrow{\partial}}{\partial \theta_\alpha} f \frac{\overleftarrow{\partial}}{\partial \pi_\alpha}\right)$$

(15) $= \ldots = \{ g,f \}$ as required.

So, in order to incorporate the character changing properties of an odd generator embodied by equation (7), we must introduce the notion of differentiation to the left and 'left handed' vector fields. This device naturally takes care of the graded character of the function being operated on, as a candidate G.H.V.F. should. Thus, in attempting to generalise the idea of the Hamiltonian vector field to a Z_2 graded environment, we have come across an important structural clue as to the nature of differential geometry on this type of graded space. Vectors, it appears, come in two distinct types: ones built from derivatives which act to the right, and ones built from derivatives which act to the left. Clearly the identity (12) enables one to move between these two spaces when the variables are homogeneous. In order to construct graded Hamiltonian vector fields on these spaces, both left and right acting vectors are required. The right acting vectors are employed for G.H.V.F.s associated with even, commuting, generator functions and which we denote by Φf, and left acting vectors are employed for G.H.V.F.s associated with odd, anticommuting, generator functions, which we denote by $f\Phi$. Although we came across these idea's independently they were not wholly without precedent. Berezin [13] introduces two types of 1-form on his supermanifold, and the existance of both left and right generators to super Lie algebra's [5] again indicated that this was the right direction to take. Later discovery of De Witt's work [22] confirmed this.

In the next section we start to explore the properties of the G.H.V.F.s (4) and (13), to discover to what extent Φf and $f\Phi$ play a role similar to their classical counterpart Hf in ∯ 1.2.3.

2.1.2 <u>Properties of the G.H.V.F.s Φf and $f\Phi$.</u>

In ∯ 1.2.3 we saw that key property of H.V.F.'s is that they form a Lie algebra whose combination principle is a Poisson bracket. This is one way to think of the Poisson bracket between two functions: as resulting from the Lie commutator between the two corresponding H.V.F.'s, since we have from § 1.2.3:

$$(16) \qquad [\, Hf, Hg \,] = -\, H\{f,g\}$$

where $f,g \in F(M)$ and Hf, Hg are the H.V.F.'s associated with f and g. How then does this relationship generalise to the case of the G.H.V.F.s Φf and $f\Phi$? Let us examine the case of the right acting field Φf first.

2.1.3 The Right-acting field Φf.

We first must define the graded commutator between two right-acting graded vector fields V and W, say. Since a right-acting field by definition maps a graded function onto another graded function by action on the right, we may appeal to the earlier definition of the commutator bracket in § 1.1.1 and define:

$$(17) \qquad [\, \overrightarrow{V}, \overrightarrow{W} \,]'(f) := \overrightarrow{V}(\overrightarrow{W}f) - \overrightarrow{W}(\overrightarrow{V}f)$$

Where V and W are graded vector fields, f is a graded function and [,]' represents the graded commutator brackets. In component form this is:

$$(18) \qquad \left[\, V_i \frac{\partial}{\partial a_i}, W_j \frac{\partial}{\partial a_j} \,\right]' = \sum_{i,j} \left(V_i \frac{\partial W_j}{\partial a_i} \frac{\partial}{\partial a_j} - \frac{\epsilon (-1)^{|V_i||W_j|}}{(-1)^{|a_i||a_j|}} W_j \frac{\partial V_i}{\partial a_j} \frac{\partial}{\partial a_i} \right)$$

Where $V = V_i$, $W = W_j$ and $\epsilon = (-1)^{(|W_j|+|V_i|+|a_i|+|a_j|)}$.

Note: $\epsilon = 1$ when V and W are both right-acting G.H.V.F.s.

The factor in front of the second part of the expression arises by the requirement that the terms which are second order in derivatives must cancel. For the situation when V and W are two right-acting G.H.V.F.s Φf and Φg, say, we have:

$$(19) \qquad [\, \Phi f, \Phi g \,]' =$$

$$= \left[\frac{\overrightarrow{\partial} f}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial q_i} - \frac{\overrightarrow{\partial} f}{\partial q_i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overrightarrow{\partial} f}{\partial \pi_\alpha} \frac{\overrightarrow{\partial}}{\partial \theta_\alpha} - \frac{\overrightarrow{\partial} f}{\partial \theta_\alpha} \frac{\overrightarrow{\partial}}{\partial \pi_\alpha} \,,\, \frac{\overrightarrow{\partial} g}{\partial q_i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overrightarrow{\partial} g}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial q_i} - \frac{\overrightarrow{\partial} g}{\partial \theta_\alpha} \frac{\overrightarrow{\partial}}{\partial \pi_\alpha} - \frac{\overrightarrow{\partial} g}{\partial \pi_\alpha} \frac{\overrightarrow{\partial}}{\partial \theta_\alpha} \right]'$$

Let us now look at the coefficients of the operator $\frac{\overrightarrow{\partial}}{\partial \theta_\alpha}$; there are eight contributions to it (and omitting the arrows):

$$\left(\frac{\partial g}{\partial p_j}\frac{\partial^2 f}{\partial q_j \partial \pi_\alpha} + \frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial p_i \partial \pi_\alpha}\right) - \left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q_i \partial \pi_\alpha} + \frac{\partial g}{\partial q_j}\frac{\partial^2 f}{\partial p_j \partial \pi_\alpha}\right) + ..$$

(20)

$$..+ \left(\frac{\partial f}{\partial \theta_\beta}\frac{\partial^2 g}{\partial \pi_\beta \partial \pi_\alpha} - \frac{\partial g}{\partial \pi_\beta}\frac{\partial^2 f}{\partial \theta_\beta \partial \pi_\alpha}\right) + \left(\frac{\partial f}{\partial \pi_\beta}\frac{\partial^2 g}{\partial \theta_\beta \partial \pi_\alpha} - \frac{\partial g}{\partial \theta_\beta}\frac{\partial^2 f}{\partial \pi_\beta \partial \pi_\alpha}\right)$$

Collecting toghether and rearranging the first and second, third and fourth, terms of (20) we see that this does indeed come to:

(21) $$+ \frac{\overset{>}{\partial}}{\partial \pi_\alpha}\{ f,g \}' \frac{\overset{>}{\partial}}{\partial \theta_\alpha}$$

Without verifying the other terms here, one can carry on and prove the following result:

(22) $$[\Phi f,\Phi g]' = - \Phi\{f,g\}'$$

A version of this result occurs in Kostant's review [40], though there the underlying space is substantially different from the one we employ here. It might seem somewhat of a miracle that two suċh similar looking results occur in such diverse theoretical background. The reason for this will be discussed at the end of this section. We now continue by examining the case of the graded commutator between two left-acting fields $f\Phi$ and $g\Phi$.

2.1.4 <u>The Left-acting field fΦ.</u>

If $\overset{<}{V}$ and $\overset{<}{W}$ are now two left-acting vector fields (denoted by the arrows), then we may define the graded commutator bracket between them as being:

(23) $$(f)[\overset{<}{V},\overset{<}{W}]' := (f\overset{<}{V})\overset{<}{W} - (f\overset{<}{W})\overset{<}{V}$$

where f is some Z_2 graded function on super phase space. By doing a similar calculation to (20) one may verify that the following relationship holds:

(24) $$(h)[f\Phi,g\Phi]' \equiv - \Phi\{f,g\}'(h)$$

Where; $|f| = |g| = 1$, $|h| = 0,1$ and f, g, h are Z_2 graded functions on S.P.S. Notice that because the graded Poisson bracket of the odd functions f and g is itself an <u>even</u> function, it appears on the right hand side of the Φ. What this means is that the relsionshiop (24) has set up an isomorphism between the spaces of left-acting and right-acting vector fields. Thus we may write:

(25) $$[\, f\Phi, g\Phi \,]' \approx -\, \Phi\{f,g\}'$$

Which is the left-acting equivalent of the right-acting equality given in (22).

We now turn our attention to the final possibility, which is the graded commutator bracket between a left-acting and a right-acting G.H.V.F.

2.1.5 Left-acting and Right-acting fields.

This is the most interesting case in that it implies an interaction between the odd and even sectors, which is needed in a non trivial way for supersymmetry to be possible. First of all we must define the graded commutator bracket between a left-acting and a right-acting vector field. As it turns out this is simpler than might at first be expected. If $\overset{>}{V}$ is some right-acting vector field, and $\overset{<}{W}$ is some field acting to the left, then we may define the graded commutator brackets between these two fields as follows:

(26) $$(h)[\, \overset{>}{V}, \overset{<}{W} \,]' := (\overset{>}{V}h)\overset{<}{W} - \overset{>}{V}(h\overset{<}{W})$$

Where $|h| = 0, 1$, is some Z_2 graded function on S.P.S. If we now set $\overset{>}{V} = \Phi f$ for $|f| = 0$, and $\overset{<}{W} = g\Phi$ for $|g| = 1$, then we may compute the bracket:

$$(h)[\, \Phi f, g\Phi \,]' \quad \text{for } |h| = 0,1$$

After some calculation similar to (20), but made more tricky by the presence of both left and right acting derivatives - for example the terms corresponding to the coefficients of $\overset{<}{\frac{\partial}{\partial\theta}}, \overset{>}{\frac{\partial}{\partial\theta}}$ for the case $|h| = 0$ above are:

$$+ \overset{>}{\frac{\partial f}{\partial p}} \overset{<}{\frac{h\partial}{\partial\theta}} \overset{>}{\frac{\partial^2 g}{\partial q}} \overset{<}{\frac{\partial}{\partial\pi}} - \overset{>}{\frac{\partial f}{\partial q}} \overset{<}{\frac{h\partial}{\partial\theta}} \overset{>}{\frac{\partial^2 g}{\partial p}} \overset{<}{\frac{\partial}{\partial\pi}} - \overset{>}{\frac{\partial^2 f}{\partial\pi}} \overset{<}{\frac{\partial}{\partial q}} \overset{>}{\frac{\partial h}{\partial\theta}} \overset{<}{\frac{g\partial}{\partial p}} + \overset{>}{\frac{\partial f}{\partial\pi}} \overset{<}{\frac{\partial}{\partial p}} \overset{>}{\frac{\partial h}{\partial\theta}} \overset{<}{\frac{g\partial}{\partial q}}$$

$$- \overset{>}{\frac{\partial^2 f}{\partial\pi}} \overset{<}{\frac{\partial}{\partial\theta}} \overset{>}{\frac{\partial h}{\partial\theta}} \overset{<}{\frac{g\partial}{\partial\pi}} + \overset{>}{\frac{\partial f}{\partial\pi}} \overset{<}{\frac{h\partial}{\partial\theta}} \overset{>}{\frac{\partial^2 g}{\partial\theta}} \overset{<}{\frac{\partial}{\partial\pi}} - \overset{>}{\frac{\partial^2 f}{\partial\pi}} \overset{<}{\frac{\partial}{\partial\pi}} \overset{>}{\frac{\partial h}{\partial\theta}} \overset{<}{\frac{g\partial}{\partial\theta}} + \overset{>}{\frac{\partial f}{\partial\theta}} \overset{<}{\frac{h\partial}{\partial\theta}} \overset{>}{\frac{\partial^2 g}{\partial\pi}} \overset{<}{\frac{\partial}{\partial\pi}}$$

We find that the following relationship holds true:

(27) $(h)[\ \Phi f,g\Phi\ ]' := (\Phi f(h))g\Phi - \Phi f((h)g\Phi) \equiv (h)\{f,g\}'\Phi$

for any Z_2 graded function h, where $|h| = 0,1$ and where $|f| = 0$ and $|g| = 1$.

These three relationships (22),(25) and (27) represent the the generalisation of the H.V.F. relationship (1) to case of the G.H.V.F. The fact that one is able to do this makes the need for the presence of left and right acting vector fields on this type of super phase space uncontrived. It also makes it possible to carry out non-trivial supersymmetry transformations on the S.P.S., in a way that preserves the graded Poisson bracket, but we will discuss this in ∮ 2.4.2. In the next section we explore the properties of the graded Poisson bracket under a change of variables, with a view to generalising the 'function of a function' rule (32) in ∮ 1.0.5. As we will see, once again we are lead to the natural employment of left-acting and right-acting derivatives in deriving the graded generalisations to this classical relationship.

2.1.6 A graded identity.

Introduction

In ∮ 1.0.5 a well-known Poisson bracket identity was stated, namely:

(28) $$\{\ F,G\ \} \equiv \frac{\partial F}{\partial \Gamma a}\{\ \Gamma a,\Gamma b\ \}\frac{\partial G}{\partial \Gamma b}\quad,\quad \text{where } \Gamma a = \Gamma a(q;p).$$

This is an important identity and which is used, for example, in the proof of the function group theorems [27]. In this section we investigate the graded generalisation of this identity and show that it has an elegant description in terms of left and right acting derivatives. And that one is naturally led to introduce a "left-right tensor" similar to those that appear in [22]. This leads to the concept of the a graded generalised Poisson bracket.

The graded Taylor's theorem

To prove a graded version of (28) we must consider what it is we mean by a graded function of a graded functions and, in particular, whether it possible to use a graded version of Taylor's theorem. These questions are covered more fully in [22], or for a rigorous treatment see A.Roger's paper [49]. Here we confine ourselves to some elementary considerations.

Suppose we have some collection of n Z_2 graded functions f_a, where $a = 1,\dots,n$, $f_a = f_a(q_i;p_j;\theta_\alpha;\pi_\beta)$, with $\alpha,\beta = 1,\dots,n' \geq n$ and where $(\theta_\alpha,\pi_\beta)$ together form a $G_{n',n'}$ Grassmann algebra. Let us now consider functions $F(f_a)$ of these functions (f_a). How may we differentiate $F(f_a)$ with respect to the variables $(\theta_\alpha,\pi_\alpha)$? It can be shown [49] that the 'function of a function' rule for differentiation works in the graded case, provided we can split up the functions f_a into odd and even sectors. That is, we reorder the (f_a) into the disjoint sets $(f_s;\phi_{s'})$ where $s = 1,\dots,N$; $s' = 1,\dots,N'$ with $n = N + N'$, and where $|f_s| = 0$, $|\phi_{s'}| = 1$, for all s, s'.

Thus we have:

(29) $$F(f_a) \equiv F'(f_s;\phi_{s'})$$

and where $f_s = f_s(q_i;p_j;\theta_\alpha;\pi_\beta)$, $\phi_{s'} = \phi_{s'}(q_i;p_j;\theta_\alpha;\pi_\beta)$.

Graded function of a function differentiation then becomes:

(30) $$\frac{\partial F(f_s;\phi_{s'})}{\partial\theta_\alpha} = \frac{\partial f_s}{\partial\theta_\alpha}\frac{\partial F}{\partial f_s} + \frac{\partial\phi_{s'}}{\partial\theta_\alpha}\frac{\partial F}{\partial\phi_{s'}}$$

Where it is important that this same ordering is used throughout the calculation of these expressions. It is now possible to calculate the graded version of Taylor's theorem. Because the functions $(\phi_{s'})$ are odd, they form a $G_{N'}$ subalgebra of the Grassmann algebra $G_{n',n'}$, so typically a function $F(f_s;\phi_{s'})$ will have the form:

(31) $$F(f_s;\phi_{s'}) = F_{s'}\phi_{s'} + F_{s't'}(\phi_{s'}.\phi_{t'}) +\dots+ F_{s'\dots u'}(\phi_{s'}\dots\phi_{u'})$$

Where the functions: $F_{s'}$; $F_{s't'}$; ...; $F_{s'\dots t'}$, are all Z_2 graded even functions of $f_s(q_i;p_j;\theta_\alpha;\pi_\beta)$ totally antisymmetric in the indices $s',t',\dots,u'$, up to a highest order n' in $f_{s'}$. A normal Taylor expansion may now be carried out on each successive even term - because each f_s is an even function of $(q_i;p_j;\theta_\alpha,\pi_\beta)$, it may potentially involve a completely 'Grassmann free' leading term: the purely bosonic sector contribution.

We now have enough machinery to calculate a graded version of the identity (28).

Calculating the graded identity

We are now in a position to calculate the graded version of the identity (28). Consider a collection of Z_2 graded functions:

(32) $$E_i = E_i(f_s;\phi_s') \ , \ \Gamma_j = \Gamma_j(f_s;\phi_s') \ , \quad \text{where } i,j = 1,2$$

Also $|E_i|,|f_s| = 0$ and $|\Gamma_i|,|\phi_s'| = 1$, with the functions f_s, ϕ_s' being those defined above. After much tedious calculation, with great attention having to be paid to the signs of the various terms, we obtain the following identities:

(33) $$\{E_1,E_2\}' = \frac{\partial E_1}{\partial f_s}\frac{\partial E_2}{\partial f_t}\{f_s,f_t\}' + \frac{\partial E_1}{\partial \phi_s'}\frac{\partial E_2}{\partial f_s}\{f_s,\phi_t'\}' - \frac{\partial E_1}{\partial f_s}\frac{\partial E_2}{\partial \phi_t'}\{f_s,\phi_t'\}' - \frac{\partial E_1}{\partial \phi_s}\frac{\partial E_2}{\partial \phi_t'}\{\phi_s',\phi_t'\}'$$

(34) $$\{E,\Gamma\}' = \frac{\partial E}{\partial f_s}\frac{\partial \Gamma}{\partial f_s}\{f_s,f_t\}' + \frac{\partial E}{\partial f_s}\frac{\partial \Gamma}{\partial \phi_t'}\{f_s,\phi_t'\}' - \frac{\partial E}{\partial \phi_t}\frac{\partial \Gamma}{\partial f_s}\{f_s,\phi_t'\}' - \frac{\partial E}{\partial \phi_s}\frac{\partial \Gamma}{\partial \phi_t'}\{\phi_s',\phi_t'\}'$$

(35) $$\{\Gamma_1,\Gamma_2\}' = \frac{\partial \Gamma_1}{\partial f_s}\frac{\partial \Gamma_2}{\partial f_t}\{f_s,f_t\}' + \frac{\partial \Gamma_1}{\partial f_s}\frac{\partial \Gamma_2}{\partial \phi_t'}\{f_s,\phi_t'\}' + \frac{\partial \Gamma_1}{\partial \phi_t}\frac{\partial \Gamma_2}{\partial f_s}\{f_s,\phi_t'\}' + \frac{\partial \Gamma_1}{\partial \phi_s}\frac{\partial \Gamma_2}{\partial \phi_t'}\{\phi_s',\phi_t'\}$$

Which as it stands it does not have a great deal in common in a structural sense with the identity (28). However, writing (33,34,35) thus:

(36) $$\{E_1,E_2\}' = \frac{\partial E_1}{\partial f_s}\{f_s,f_t\}'\frac{\partial E_2}{\partial f_t} - \frac{\partial E_1}{\partial \phi_s'}\{\phi_s',f_t\}'\frac{\partial E_2}{\partial f_t} + \frac{\partial E_1}{\partial f_s}\{f_s,\phi_t'\}'\frac{\partial E_2}{\partial \phi_t'} - \frac{\partial E_1}{\partial \phi_s'}\{\phi_s',\phi_t'\}'\frac{\partial E_2}{\partial \phi_t'}$$

(37) $$\{E,\Gamma\}' = \frac{\partial E}{\partial f_s}\{f_s,f_t\}'\frac{\partial \Gamma}{\partial f_t} + \frac{\partial E}{\partial f_s}\{f_s,\phi_t'\}'\frac{\partial \Gamma}{\partial \phi_t'} - \frac{\partial E}{\partial \phi_s'}\{\phi_s',f_t\}'\frac{\partial \Gamma}{\partial f_t} - \frac{\partial \Gamma}{\partial \phi_s'}\{\phi_s',\phi_t'\}'\frac{\partial \Gamma}{\partial \phi_t'}$$

(38) $$\{\Gamma_1,\Gamma_2\}' = \frac{\partial \Gamma_1}{\partial f_s}\{f_s,f_t\}'\frac{\partial \Gamma_2}{\partial f_t} + \frac{\partial \Gamma_1}{\partial f_s}\{f_s,\phi_t'\}'\frac{\partial \Gamma_2}{\partial \phi_t'} + \frac{\partial \Gamma_1}{\partial \phi_s'}\{\phi_s',f_t\}'\frac{\partial \Gamma_2}{\partial f_s} + \frac{\partial \Gamma_1}{\partial \phi_s'}\{\phi_s',\phi_t'\}'\frac{\partial \Gamma_2}{\partial \phi_t'}$$

It now becomes clear that in order to bring the troublesome minus signs into line, we must use left-acting derivatives for the left hand side term on each of the above brackets. This works because the only sign changes occur when an odd derivative acts on an even function. Using this device we obtain the expressions:

$$\{E_1, E_2\}' = \frac{E_1\overleftarrow{\partial}}{\partial fs}\{fs, ft\}'\frac{\overrightarrow{\partial}E_2}{\partial ft} + \frac{E_1\overleftarrow{\partial}}{\partial \text{ø}s'}\{\text{ø}s', ft\}'\frac{\overrightarrow{\partial}E_2}{\partial fs}$$

(39)

$$+ \frac{E_1\overleftarrow{\partial}}{\partial fs}\{fs, \text{ø}t'\}'\frac{\overrightarrow{\partial}E_2}{\partial \text{ø}t'} + \frac{E_1\overleftarrow{\partial}}{\partial \text{ø}s'}\{\text{ø}s', \text{ø}t'\}'\frac{\overrightarrow{\partial}E_2}{\partial \text{ø}t'}$$

... and so on.

Now all the signs are homogeneous and positive, which enables us to write (39) in a similar manner to (28). We do this by defining the variable (ha) := (fs,øs'), where a = 1,...,N+N' = n. Now we may write:

(40)

$$\{F, G\}' \equiv \frac{F\overleftarrow{\partial}}{\partial ha}\{ha, hb\}'\frac{\overrightarrow{\partial}G}{\partial hb}$$

which is the graded version of (28). The beauty of this expression is that the graded characters of the various functions which appear in the expression do not have to be included in an explicit manner. All the signs present in the expansions (33,34,35) are now taken care of by the left-acting derivative. Thus we see once again that both left-acting and right-acting derivatives occur naturally when one makes statements in a Z_2 graded environment. We explore some of the implications of this in the next short section.

2.1.7 Structures on Z_2 graded phase space.

Introduction

The last three sections represent an attempt to interpret the graded Poisson bracket defined by Casalbuoni [17] in a more 'geometrical' way - geometrical in the sense that the construction seeks to mimic the standard differential geometry of symplectic spaces. These considerations were made in a background as free as possible from technical jargon, in the hope that the underlying structure of the space under investigation would become clearer as the work continued. Progress towards this end did seem to be being made. What started out as essentially a guess as to the form of the graded Hamiltonian vector field, became fairly plausible through the next two sections. The time seemed to be right to try for a more general theory as to the nature of this type of space. Below we confine ourselves to just some preliminary discussion as to where we believed our investigations to be leading - the work in [22] supports that this speculation was along the right lines.

Vectors and Tensors

In § 1.1.1 we saw how a useful way of looking at vectors on a manifold is as linear combinations of partial derivatives, taken with respect to the local co-ordinates on that particular patch of the manifold. That is, if $\{f^i\}$ for i = 1,..N is some local co-ordinate system on a manifold M, then a general vector field V defined on that patch may be expressed as the linear combination:

$$\underline{V} = V^i(f)\frac{\partial}{\partial f^i} \tag{41}$$

Where the real function $V^i(f)$ are known as the components of the contravarient vector $\underline{V}$. However, what the work of the last three sections indicates is that for a graded generalisation of these ideas to be viable, both types of graded differentiation are required - this leads to the notion of right-acting and left-acting vectors which we denote respectivly as follows:

$$\overset{>}{V} = V^a(h)\frac{\overset{>}{\partial}}{\partial h^a}\ , \quad \overset{<}{V} = \frac{\overset{<}{\partial}}{\partial h^a}\,{}^aV(h)\ , \quad \text{where } a = 1,\ldots,n \tag{42}$$

and where the functions $\{h^a\}$ are a set similar to the one employed in the last part of § 2.1.6 above. The Z_2 graded functions $V^a(h)$ and ${}^aV(h)$ are the components of the right-acting and left-acting vectors $\overset{>}{\underline{V}}$ and $\overset{<}{\underline{V}}$ respectively, with $|{}^aV|$, $|V^a|$ equalling either 0 or 1. Under a change of basis $\{h^a\}$ ---> $\{h'^a\}$, the co-ordinates of the vectors $\underline{V}$ and $\underline{V}$ would transform as expected:

$$(43)\quad {}^{a}V(h) \longrightarrow {}^{a'}V(h') = h'^{a}\frac{\overleftarrow{\partial}}{\partial h^{b}}\,{}^{b}V\ ,\qquad V_{a}(h) \longrightarrow V'_{a}(h') = V_{b}\frac{\overrightarrow{\partial} h'_{a}}{\partial h^{b}}$$

for a,b = 1,...,n. Examples of these left-acting and right-acting vector fields are provided by the previously introduced G.H.V.F.s $f\Phi$ and Φf. For more rigorous development of this type of vector space see [22].

Another direction in which the work of the previous sections indicates a notational development might be made, is the introduction of what we call the left-right tensor. This is an object H, say, the components of which have indices which correspond to both left and right action. For example the components of H with respect to the basis $\{h^a\}$ might be:

$$(44)\qquad {}^{a}H_{b}(h)\ ,\qquad \text{with}\quad a,\, b = 1,\dots,n$$

The existence of an object of this nature is not inconceivable, in fact, we already have an example of just such a left-right tensor in the graded Poisson bracket. We can see this quickly by employing the identity (40). If we define the components ${}^{\alpha}H_{\beta}$ as follows:

$$(45)\qquad {}^{a}H_{b} := \{\, h^a, h^b \,\}'$$

where the functions $(\dots,h^a,\dots) := (\dots,q^i,\dots;\dots,p_j,\dots;\dots,\theta^\alpha,\dots;\dots,\pi_\beta,\dots)$ are the fundamental co-ordinate variables, then the fundamental graded Poisson brackets and the structure of the identity (40) ensure that (45) defines a bona fide left-right tensor of the type we suggest. In this instance the left-right tensor (45) is none other than the orthosymplectic form (30) in ∮ 2.0.5 [18]. That is we have:

$${}^{a}H_{b} = \Gamma^{ab} := \begin{pmatrix} A^{ij} & 0 \\ 0 & B^{\alpha\beta} \end{pmatrix}$$

where the matrices A and B are those defined in ∮ 2.0.5. Using this left-right tensor the possibility of defining a 'generalised graded Poisson bracket' arises. This is envisaged to be the graded counter-part to the generalised Poisson bracket of ∮ 1.3.4, and would take the general form:

$$\{ A,B \}'' := A\frac{\overleftarrow{\partial}}{\partial h^a}\, {}^aH^b \frac{\overrightarrow{\partial}}{\partial h^b}B \qquad (46)$$

where the h^a, $a = 1,...,n$, are now some general (ordered: even first, odd second) graded set of variables, and ${}^aH^b$ is some graded matrix with the same block symmetries as those described in ∮ 2.3.6. We then would demand that the bracket { , }'' satisfies the graded Jacobi identities (see Apendix A), and determine what conditions that places on the elements of ${}^aH^b$. Although we have not proved this, we feel sure that the graded Dirac bracket introduced by Casalbuoni in [17] is an example of just such a graded generalised bracket. What would be interesting to find out is whether it is possible to reduce the graded Dirac bracket of [17] to the form of a graded Poisson bracket in a reduced number of graded variables. This would require a graded generalisation of the function group theorems of ∮ 1.3.6, and represents an obvious target for further research.

This concludes this brief look at possible further Z_2 graded structures on graded phase space - a topic which for the most part is well covered in [22]. However, De Witt spends little time on the development of the graded phase space aspects of his work, so the questions concerning graded function groups, to the best of the author's knowledge, still remain unanswered. We do some work on these objects in ∮ 2.3.

2.2.0 A Class of Canonical Transformations.

Introduction

In Part I the idea of the canonical transformation was explored in some detail. We recall that the canonical transformation may be regarded either from an 'active' point of view as being associated with a movement along a path in phase space parameterised by time. Or, it may be thought of in a 'passive' way as the transformation between different sets of canonical co-ordinates in some patch of the phase space.

In this section we view the graded canonical transformations from a passive viewpoint, as being a transformation between two canonical co-ordinate sets which preserves the graded Poisson bracket. In addition to the matrix transformations briefly described at the start of Part II, we demonstrate that there exists an infinite class of such transformations, characterised by an arbitrary function. Furthermore we demonstrate for the simplest case, the G1,1 algebra, that these transformations may be made to form a group provided that various conditions are imposed on the bosonic generator functions. Another way of putting this is to say that the collection of graded Hamiltonian vector fields associated with the generator functions of the transformations, can be made to satisfy the graded Jacobi identities under certain conditions. These conditions are studied.

2.2.1 The simplest case: G1,1.

We start this investigation by considering only the most simple of super phase spaces, namely the space in which there are two bosonic conjugate variables $(q;p)$ and two fermionic Grassmann co-ordinates $(\theta;\pi)$ which are also conjugate. These variables satisfy the following fundamental graded Poisson bracket relations:

(1) $$\{ q,q \}' = 0 = \{ p,p \}' , \quad \{ q,p \}' = 1$$

(2) $$\{ p,\theta \}' = 0 = \{ q,\theta \}' , \quad \{ p,\pi \}' = 0 = \{ q,\pi \}'$$

(3) $$\{ \theta,\pi \}' = -1$$

Where the minus sign in (3) is by earlier convention, and $\{ \, , \, \}'$ represents the graded Poisson bracket. The question we now ask is what freedom exists to redefine the variables $(q;p,\theta;\pi)$? That is, is it possible to find a new set of four variables, two even, two odd, such that the relations (1,2,3) are still satisfied? Since we are dealing with a G1,1 system, these new functions must have the following form:

(4) $$P = A(q,p) + B(q,p)\theta.\pi$$

(5) $$Q = a(q,p) + b(q,p)\theta.\pi$$

(6) $$\Gamma = X(q,p)\theta + Y(q,p)\pi$$

(7) $$\Phi = x(q,p)\theta + y(q,p)\pi$$

Where P,Q are even, and Γ,Φ are odd, and are obtained by power expansion in the Grassmann variables, followed by separation of odd and even sectors. We now demand that the set $(Q;P;\Phi;\Gamma)$ satisfy the following conditions:

(8) $$\{ Q,P \}' = 1 \;,\ldots,\; \{ \Phi,\Gamma \}' = -1$$

that is just the conditions $(q;p,\theta;\pi)$ satisfied in (1,2,3). If we now calculate out these brackets and set the coefficients of the terms of the various Grassmann contributions to zero, we obtain the following conditions on the set of functions $(A,a,\ldots,Y,y)$:

(9)	$\{ P,Q \}' = -1$	implies	$\{ A,a \} = -1$
		and	$\{ B,a \} + \{ A,b \} = 0$
(10)	$\{ P,\Gamma \}' = 0$	implies	$\{ A,X \} - BX = 0$
		and	$\{ A,Y \} + BY = 0$
(11)	$\{ Q,\Phi \}' = 0$	implies	$\{ a,x \} - bx = 0$
		and	$\{ a,y \} + by = 0$
(12)	$\{ \Gamma,\Phi \}' = -1$	implies	$\{ X,y \} + \{ x,Y \} = 0$
		and	$Yx + Xy = -1$
(13)	$\{ \Gamma,\Gamma \}' = 0$	implies	$XY = 0$
(14)	$\{ \Phi,\Phi \}' = 0$	implies	$xy = 0$

Where the antisymmetry of the graded bracket for even objects ensures that the other relations are satisfied identically.

We now impose the following ansatz on the equations (9,..,14) which effectively reduces the "mixing" of the canonical co-ordinates to come from the fermionic sector:

$$A(q,p) = p \quad , \quad a(q,p) = q \tag{15}$$

This now reduces the conditions (9) and (10) to the following:

$$\frac{\partial B}{\partial p} + \frac{\partial b}{\partial q} = 0 \tag{16}$$

$$\frac{\partial X}{\partial q} + BX = 0 \tag{17}$$

$$\frac{\partial Y}{\partial q} - BY = 0 \tag{18}$$

If we now set the function Y = 0, (18) is satisfied identically, and we may solve the equations (16) and (17) to obtain the following new canonical set $(Q;P;\Phi;\Gamma)$:

$$P = p - \frac{1}{X}\frac{\partial X}{\partial q}\,\theta.\pi \tag{19}$$

$$Q = q + \frac{1}{X}\frac{\partial X}{\partial p}\,\theta.\pi \tag{20}$$

$$\Gamma = X\theta \tag{21}$$

$$\Phi = \frac{\pi}{X} \tag{22}$$

Where X = X(q,p) is an arbitrary function of (q;p). We see that the function X is acting as a type of conformal factor scaling on the Grassmann sector, where as in the even sector a non-trivial mixing of the even and odd co-ordinates has occurred. By placing X = 1 we may return the set $(Q;P;\Phi;\Gamma)$ back to the original set $(q;p;\theta;\pi)$.

If we now expand this transformation infinitesimally about the identity we get:

$$X = 1 + \mu h(q,p) \quad , \quad \text{where } \mu \ll 1 \tag{23}$$

$$P = p - (1 - \mu h + O(\mu\ ))\frac{\partial(1 + \mu h)}{\partial q}\theta.\pi$$

$$= p - \mu\frac{\partial h}{\partial q}\,\theta.\pi + O(\mu\ ) \tag{24}$$

(25) $$Q = q + \mu\frac{\partial h}{\partial q}\,\theta.\pi$$

(26) $$\Gamma = (1 + \mu h)\theta$$

(27) $$\Phi = (1 - \mu h)\pi$$

That is:

(28) $$\frac{\partial P}{\partial \mu} = \frac{\partial h}{\partial q} \quad , \quad \frac{\partial Q}{\partial \mu} = -\frac{\partial h}{\partial p}$$

(29) $$\frac{\partial \Gamma}{\partial \mu} = h \quad , \quad \frac{\partial \Phi}{\partial \mu} = - h$$

Equations (28) and (29) give the generators. Under the conditions of the ansatz (15) equations (19,...,22) are the only possibilities for the G1,1 system.

There are two observations that can be made straight away about the new canonical set (19,...,22):

a) There is a 'reflection' symmetry to the equations in the sense that the discrete transformation:

(30) $$q \longleftrightarrow p \quad , \quad \theta \longleftrightarrow -\theta \quad , \quad \pi \longleftrightarrow -\pi \quad , \quad X \longleftrightarrow -X$$

leaves the set unchanged. Also,

b) it is possible to extend the bosonic sector arbitrarily, by the addition of an index to all the bosonic co-ordinates involved. That is:

$$P \longrightarrow P_i = p_i - \frac{1}{X}\frac{\partial X}{\partial q_i}\,\theta.\pi$$ and so on, where now

$$\{ P_i, Q_j \}' = -\delta_{ij}.$$

Because of the heavily constrained nature of the G1,1 system it is actually possible to "invert" the relations (19,...,22) in the following manner. If we call:

(31) $$P = p - F(q,p)\theta.\pi$$

(32) $$Q = q + G(q,p)\theta.\pi$$

We have:

(33) $$\theta.\pi = \Gamma.\Phi$$

(34) $$P = p - F(q,p)\Gamma.\Phi$$

(35) $$Q = q + G(q,p)\Gamma.\Phi$$

Thus:

(36) $$p = P + F(Q - G(q,p)\Gamma.\Phi,\ P + F(q,p)\Gamma.\Phi)\ \Gamma.\Phi$$

(37) $$q = Q - G(Q - G(q,p)\Gamma.\Phi,\ P + F(q,p)\Gamma.\Phi)\ \Gamma.\Phi$$

Because of the anti-commuting nature of Γ,Φ, a normally infinite process of substitution is cut off straight away, to give by Taylor expansion the following expressions for the set $(q;p;\theta;\pi)$:

(38) $$p = P + F(Q,P)\Gamma.\Phi$$

(39) $$q = Q - G(Q,P)\Gamma.\Phi$$

(40) $$\theta = \frac{\Gamma}{X(Q,P)}$$

(41) $$\pi = X(Q,P)\Phi$$

Where the θ term is fixed by the π term. This means that we may use the $(Q;P;\Gamma;\Phi)$ to define a graded canonical Lagrange bracket of the type introduced in Part I, for which the set $(q;p;\theta;\pi)$ give the fundamental bracket relations when the functions F and G are defined as in (19) and (20).

As mentioned at the beginning of this section, so far all we have is an infinite set of canonical transformations, we don´t yet have a group structure. In order to raise these transformations to form a group we must impose certain conditions on the bosonic functions taking part. To see how this comes about we perform the following calculations. We combine tranformations involving different generator functions with each other:

(42) $$\{ P_x,P_y \}' = \frac{\partial}{\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial q} - \frac{1}{X}\frac{\partial X}{\partial q} \right)\theta.\pi$$

(43) $$\{ Q_x,Q_y \}' = \frac{\partial}{\partial p}\left(\frac{1}{Y}\frac{\partial Y}{\partial p} - \frac{1}{X}\frac{\partial X}{\partial p} \right)\theta.\pi$$

(44) $$\{ P_x,Q_y \}' = -1 + \left(\frac{\partial}{\partial p}\left(\frac{1}{X}\frac{\partial X}{\partial q}\right) - \frac{\partial}{\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial p}\right)\right)\theta.\pi$$

(45) $$\{ \Gamma_x,Q_y \}' = -\frac{\partial X}{\partial p}\,\theta + \frac{X}{Y}\frac{\partial Y}{\partial q}\,\pi$$

(46) $$\{ \Gamma_x,P_y \}' = \frac{\partial X}{\partial q}\,\theta - \frac{X}{Y}\frac{\partial Y}{\partial q}\,\pi$$

(47) $$\{ \Phi_x,Q_y \}' = -\frac{1}{XY}\frac{\partial Y}{\partial p}\,\theta + \frac{1}{X}\frac{\partial X}{\partial p}\,\pi$$

(48) $$\{ \Phi x, Py \}' = \frac{1}{XY}\frac{\partial Y}{\partial q}\theta - \frac{1}{X}\frac{\partial X}{\partial q}\pi$$

(49) $$\{ \Phi y, \Gamma x \}' = -\frac{X}{Y} - \frac{1}{Y}\{X,Y\}\theta.\pi$$

(50) $$\{ \Gamma x, \Gamma y \}' = 0$$

(51) $$\{ \Phi x, \Phi y \}' = 0$$

What these relationships are is the components of the action of the various G.H.V.F.s on each other. For example if we take the case of the generator function Px, then the G.H.V.F. associated with this function is right-acting and of the form:

(52) $$HPx = \left(1 - \frac{\partial}{\partial p}\left(\frac{1}{X}\frac{\partial X}{\partial q}\right)\theta.\pi\right)\frac{\partial}{\partial q} - \left(1 - \frac{\partial}{\partial q}\left(\frac{1}{X}\frac{\partial X}{\partial p}\right)\theta.\pi\right)\frac{\partial}{\partial p} - \frac{1}{X}\frac{\partial X}{\partial q}\theta\frac{\partial}{\partial \theta} + \frac{1}{X}\frac{\partial X}{\partial q}\pi\frac{\partial}{\partial \pi}$$

And if we now look at the graded commutator bracket we obtain equations (42), (43), and (44):

(53) $$[HPx, HPy]' = \frac{\partial^2}{\partial p\,\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial q} - \frac{1}{X}\frac{\partial X}{\partial q}\right)\theta.\pi\frac{\partial}{\partial q} - \frac{\partial}{\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial q} - \frac{1}{X}\frac{\partial X}{\partial q}\right)\theta.\pi\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial q} - \frac{1}{X}\frac{\partial X}{\partial q}\right)\theta\frac{\partial}{\partial \theta} - \frac{\partial}{\partial q}\left(\frac{1}{Y}\frac{\partial Y}{\partial q} - \frac{1}{X}\frac{\partial X}{\partial q}\right)\pi\frac{\partial}{\partial \pi}$$

and so on. In order to raise these transformations to the level of a group we must test that the graded Hamiltonian vector fields associated with the bosonic generator function X, namely the set (HPx, HQx, HΦx, HΓx), must satisfy the graded Jacobi identities when combined with the similar sets corresponding to different generators Y and Z, say. That is, we must check:

(54) $$[HPx,[HPy,HPz]']' + [HPz,[HPx,HPy]']' + [HPy,[HPz,HPx]']' = 0$$

and so on, for all possible combinations of functions in X, Y and Z. On calculating the various brackets we find that they all vanish identically <u>except</u> two. They are the brackets associated with the G.P.B. combination of the following functions:

(55) $$\{ Qx, \{ Qy, \Phi z \}'\}' + \text{Cyclic Perms.} = Z\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial p^2} - \frac{1}{X}\frac{\partial^2 X}{\partial p^2}\right)\theta$$

(56) $$\{ Px, \{ Py, \Gamma z \}'\}' + \text{Cyclic Perms.} = \frac{1}{Z}\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial q^2} - \frac{1}{X}\frac{\partial^2 X}{\partial q^2}\right)\pi$$

Thus, in order for the graded Jacobi identities to be satisfied, we demand that X, Y and Z meet the following requirements:

(57) $$Z \neq 0$$

(58) $$\frac{1}{Y}\frac{\partial^2 Y}{\partial p^2} - \frac{1}{X}\frac{\partial^2 X}{\partial p^2} = 0$$

(59) $$\frac{1}{Y}\frac{\partial^2 Y}{\partial q^2} - \frac{1}{X}\frac{\partial^2 X}{\partial q^2} = 0$$

To solve these conditions, first we separate the variables and look for solutions of the form:

(60) $$X(q,p) = A(q)B(p)$$

(61) $$Y(q,p) = V(q)W(p)$$

Substituting (60) and (61) into (58) and (59) gives the following differential conditions on the functions A,V,B and W:

(62) $$\frac{1}{W}\frac{d^2 W}{dp^2} - \frac{1}{B}\frac{d^2 B}{dp^2} = 0$$

(63) $$\frac{1}{V}\frac{d^2 V}{dq^2} - \frac{1}{A}\frac{d^2 A}{dq^2} = 0$$

Let us now look for solutions of (62) and (63) of the following form:

(64) $$W(p) = F(B(p))$$

(65) $$V(q) = G(A(q))$$

for some well behaved functions F and G. Turning now to (62) (solving (62) and (63) being essentially the same problem) we show how to generate solutions to this equation. Substituting (64) into (62) and collecting terms yeilds the following equation:

(66) $$\frac{F''}{F' - \frac{F}{B}} = \frac{-B_{pp}}{(B_p)^2}$$

where the dashes on the left represent differentiation w.r.t. B, and where the subscript p on the right represents differentiation w.r.t. p. For the moment let us write this equation as follows:

(67) $$\frac{F''}{F' - \frac{F}{B}} = H(B) = \frac{-B_{pp}}{(B_p)^2}$$

where H is some function of B. This then leads to the two equations:

$$F'' - H(B)\left(F' - \frac{F}{B}\right) = 0 \tag{68}$$

$$B_{pp} = -H(B)(B_p)^2 \tag{69}$$

Taking (68) first, we notice that a particular solution is F = B. This particular intergral has been obvious from the outset in (58) and (59). Using the methods from the theory of linear differential equations, we know that another particular intergral will be:

$$F = B.\int \frac{1}{B^2}\,\mathrm{Exp}\left(\int H(B)dB\right)dB \tag{70}$$

so the general solution will have the form:

$$F = cB + eB.\int \frac{1}{B^2}\,\mathrm{Exp}\left(\int H(B)dB\right)dB \tag{71}$$

for some constants c and e. For the case of equation (69) we have:

$$\frac{B_{pp}}{B_p} = -B_p\,H(B)\ . \tag{72}$$

Intergrating both sides we obtain:

$$B_p = \mathrm{Exp}\left(-\int H(B)dB\right) \tag{73}$$

By substituting (73) back into (71) we may now eliminate H(B) from the expression for F, giving a general solution of the following:

$$F(B) = cB + eB.\int \frac{1}{B^2}\,dp \tag{74}$$

where B is some unspecified function of p. Doing a similar analysis for equation (62) and the q dependence, gives us the following general solution in seperable variable form for the functions (60) and (61):

$$X(q,p) = A(q)B(p) \tag{75}$$

$$Y(q,p) = \left(aA + bA\int \frac{1}{A^2}\,dq\right)\left(cB + eB\int \frac{1}{B^2}\,dp\right) \tag{76}$$

where a, b, c and e are constants. Thus we see that the function X(q,p) is still arbitrary but, in order for the graded Jacobi identities to be satisfied, it will dictate the form of the function Y(q,p). Suppose, for example $X(q,p) = kq^n p^m$. Then (76) would give us the form of equation Y(q,p) as being:

$$Y(q,p) = \left(aq^n + \frac{b}{q^{n-1}}\right)\left(cp^m + \frac{d}{p^{m-1}}\right)$$

2.2.2 The G2,2 system.

As one increases the number of Grassmann variables which make up the odd sector of the space, the number of constraint equations increases in an alarming fashion! So much so, in fact, that to think of actually solving the constraints for a Gn,n system for $n > 2$ by hand is unrealistic. Below we examine the G2,2 system, following a similar path to that taken in the earlier example. As before we set out by writing down the general form of the new set of canonical variables. In the case of the G2,2 algebra the fermionic variables are $(\theta_1, \theta_2; \pi_1, \pi_2)$, where $(\theta\alpha; \pi\beta)$ satify:

(77) $$\{ \theta\alpha, \pi\beta \}' = -\delta\alpha\beta$$

(78) $$\{ \theta\alpha, \theta\beta \}' = 0 = \{ \pi\alpha, \pi\beta \}' \qquad \text{for } \alpha, \beta = 1,2$$

For the moment we assume for simplicity that the bosonic sector consists merely of the single canonical conjugate pair $(q;p)$. We denote the new, 'twisted', canonical set by $(Q;P;\Gamma\alpha;\Phi\beta)$ with $\alpha, \beta = 1,2$, and as before we demand that these new variables satisfy the following conditions:

(79) $$\{ P, Q \}' = -1$$

(80) $$\{ P, \Gamma\alpha \}' = 0 = \{ Q, \Gamma\alpha \}' , \quad \{ P, \Phi\alpha \}' = 0 = \{ Q, \Phi\alpha \}'$$

(81) $$\{ \Gamma\alpha, \Gamma\beta \}' = 0 = \{ \Phi\alpha, \Phi\beta \}' , \quad \{ \Gamma\alpha, \Phi\beta \}' = -\delta\alpha\beta$$

The requirement that P,Q are even and $\Gamma\alpha, \Phi\beta$ are odd variables implies that the general form these Z_2 graded functions take is as follows:

(82) $$P = A + B.\theta_1.\theta_2 + C\alpha\beta.\theta\alpha.\pi\beta + D.\pi_1.\pi_2 + E.\theta_1.\theta_2.\pi_1.\pi_2$$

(83) $$Q = a + b.\theta_1.\theta_2 + c\alpha\beta.\theta\alpha.\pi\beta + d.\pi_1.\pi_2 + e.\theta_1.\theta_2.\pi_1.\pi_2$$

(84) $$\Gamma\alpha = U\alpha\beta.\theta\beta + V\alpha\beta.\pi\beta + W\alpha\beta.\theta_1.\theta_2.\pi\beta + X\alpha\beta.\pi_1.\pi_2.\theta\beta$$

(85) $$\Phi\alpha = u\alpha\beta.\theta\beta + v\alpha\beta.\pi\beta + w\alpha\beta.\theta_1.\theta_2.\pi\beta + x\alpha\beta.\pi_1.\pi_2.\theta\beta$$

Computing the brackets (79),(80),(81) yields the following conditions the set of functions (A,a........,X,x) of (q,p) must satisfy: From brackets (79) we obtain:

(86) $\{ A,a \} = -1$

(87) $\{ A,b \} + \{ B,a \} + B(c_1^1 + c_2^2) - b(C_1^1 + C_2^2) = 0$

(88) $\{ A,d \} + \{ D,a \} + D(c_1^1 + c_2^2) - d(C_1^1 + C_2^2) = 0$

(89) $\{ A,c_1^1 \} + \{ C_1^1 ,a \} + C^1\alpha.c\alpha 1 - c_1\alpha.C\alpha^1 + Bd - bD = 0$

(90) $\{ A,c_2^2 \} + \{ C_2^2 ,a \} + C^2\alpha.c\alpha 2 - c_2\alpha.C\alpha^2 + Bd - bD = 0$

(91) $\{ A,c_2^1 \} + \{ C_2^1 ,a \} + C_2\alpha.c\alpha^1 - c_2\alpha.C\alpha^1 = 0$

(92) $\{ A,c_1^2 \} + \{ C_1^2 ,a \} + C_1\alpha.c\alpha^2 - c_1\alpha.C\alpha^2 = 0$

(93) $\{ A,e \} + \{ B,d \} + \{ D,b \} + \{ E,a \} + ..$

$..+ E(c_1^2 + c_2^1 - c_1^1 - c_2^2) - e(C_1^2 + C_2^1 - C_1^1 - C_2^2) = 0$

From the (78) bracket we get $\{ P,\Gamma\alpha \}' = 0$, which implies:

(94) $\{ A,U\alpha\beta \} - C\delta\beta.U\alpha\delta - B.V\alpha\mu = 0$

(95) $\{ A,V\alpha\beta \} - C\beta\delta.V\alpha\delta - D.U\alpha\mu = 0$

where $\mu,\beta = 1,2 \; ; \; 2,1$

(96) $\{ A,W\alpha\beta \} + \{ B,V\alpha\beta \} + \{ C\beta^2,U\alpha\beta \} - \{C\alpha^1,U\alpha^2 \} - ..$

$..- B.X\alpha\mu + C\beta\delta.W\alpha\delta + C_2^2.W\alpha\beta + C_1^1.W\alpha\beta - E.U\alpha\mu = 0$

(97) $\{ A,X\alpha\beta \} + \{ D,U\alpha\beta \} + \{ C_1\beta,V\alpha_2 \} - \{ C_2\beta,V\alpha_1 \} - ..$

$..- D.W\alpha\mu - C\delta\beta.X\alpha\delta + C_2^2.X\alpha\beta + C_1^1.X\alpha\beta - E.V\alpha\mu = 0$

Finally the (81) bracket $\{ \Gamma\alpha,\Gamma\beta \}' = 0$ gives us:

(98) $U\alpha\mu.V\beta\mu + U\beta\mu.V\alpha\mu = 0$

(99) $\{ U\alpha^1,U\beta^2 \} + \{ U\beta^1,U\alpha^2 \} - U\alpha\mu.W\beta\mu - U\beta\mu.W\alpha\mu = 0$

(100) $\{ V\alpha_1,V\beta_2 \} + \{ V\beta_1,V\alpha_2 \} - V\alpha\mu.X\beta\mu - V\beta\mu.X\alpha\mu = 0$

(101) $\{ U\alpha^1,V\beta_1 \} + \{ U\beta^1,V\alpha_1 \} - U\alpha^2.X\beta^1 - U\beta^2.X\alpha^1 + ..$

$..+ V\alpha_2.W\beta_1 + V\beta_2.W\alpha_1 = 0$

(102) $\{ U\alpha^2,V\beta_2 \} + \{ U\beta^2,V\alpha_2 \} + U\alpha^1.X\beta^2 + U\beta^1.X\alpha^2 - ..$

$..- V\alpha_1.W\beta_2 - V\beta_1.W\alpha_2 = 0$

(103) $\{ U\alpha^1,V\beta_2 \} + \{ U\beta^1,V\alpha_2 \} + U\alpha^1.X\beta^1 + U\beta^1.X\alpha^1 + ..$

$..+ V\alpha_2.W\beta_2 + V\beta_2.W\alpha_2 = 0$

(104) $\{ U\alpha^2,V\beta_1 \} + \{ U\beta^2,V\alpha_1 \} - U\alpha^2.X\beta^2 - U\beta^2.X\alpha^2 - ..$

$..- V\alpha_1.W\beta_1 - V\beta_1.W\alpha_1 = 0$

(105) $\{ U\alpha^1,X\beta^2 \} + \{ X\beta^1,U\alpha^2 \} + \{ V\alpha_1,V\beta_2 \} + \{ W\beta_1,V\alpha_2 \} + ..$

$...+ \{ W\alpha_1,V\beta_2 \} + \{ U\beta^1,X\alpha^2 \} + \{ X\alpha^1,U\beta^2 \} + W\beta\mu.X\alpha\mu = 0$

One can read the other conditions straight off by changing case appropriately, the only other change being in condition (98), which for the bracket $\{ \Gamma\alpha, \Phi\beta \}'$ reads:

(106) $$U\alpha\mu . v\beta\mu + u\beta\mu . V\alpha\mu = -\delta\alpha\beta$$

We now attempt to solve the above set of constraints. Proceeding as before we impose an ansatz to simplify the conditions. However, unlike the G1,1 case, there are now several possibilities. As it turns out the ansatz which generalises the G1,1 result is the following:

(107) $$P = p + C\alpha\beta . \theta\alpha . \pi\beta + \tfrac{1}{2} E(\theta\alpha . \pi\alpha)$$

(108) $$Q = q + c\alpha\beta . \theta\alpha . \pi\beta + \tfrac{1}{2} e(\theta\alpha . \pi\alpha)$$

(109) $$\Gamma\alpha = U\alpha\alpha . \theta\alpha$$

no sum over α

(110) $$\Phi\alpha = v\alpha\alpha . \pi\alpha$$

Without stating the calcuation here, this leads to the following generalisation of the G1,1 result:

(111) $$P = p - \frac{1}{X}\frac{\partial X}{\partial q}(\theta\alpha . \pi\alpha) - \frac{1}{2X}\{ \frac{\partial X}{\partial q}, X \}(\theta\alpha . \pi\alpha)$$

(112) $$Q = q + \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha . \pi\alpha) + \frac{1}{2X}\{ \frac{\partial X}{\partial p}, X \}(\theta\alpha . \pi\alpha)$$

(113) $$\Gamma\alpha = X\theta\alpha$$

(114) $$\Phi\alpha = \frac{\pi\alpha}{X}$$

As before X is an arbitrary function of (q,p); acting as a conformal scaling factor in the fermionic sector. We now explicitly demonstrate how the various terms combine in the case of the $\{ Q,P \}'$ bracket:

$\{ Q,P \}' =$

(115) $$1 + \left(\{ \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha . \pi\alpha), p \} - \{ q, \frac{1}{X}\frac{\partial X}{\partial q}(\theta\alpha . \pi\alpha) \} \right) + ..$$

(116) $$.. + \left(\{ \frac{1}{X} \{ \frac{\partial X}{\partial p}, X \}(\theta\alpha . \pi\alpha), p \} - \{ q, \frac{1}{X} \{ \frac{\partial X}{\partial q}, X \}(\theta\alpha . \pi\alpha) \} \right) - ..$$

(117) $$.. - \{ \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha . \pi\alpha), \frac{1}{X}\frac{\partial X}{\partial q}(\theta\beta . \pi\beta) \} - ..$$

(118) $$.. - \{ \frac{1}{X}\frac{\partial X}{\partial p}(\theta\alpha . \pi\alpha), \frac{1}{X} \{ \frac{\partial X}{\partial q}, X \}(\theta\alpha . \pi\alpha) \} - ..$$

(119) $$.. - \{ \frac{1}{X} \{ \frac{\partial X}{\partial p}, X \}(\theta\alpha . \pi\alpha), \frac{1}{X}\frac{\partial X}{\partial q}(\theta\alpha . \pi\alpha) \} - ..$$

$$(120) \qquad \ldots - \left(\left\{ \frac{1}{X} \left\{ \frac{\partial X}{\partial p}, X \right\}(\theta\alpha.\pi\alpha), \frac{1}{X} \left\{ \frac{\partial X}{\partial q}, X \right\}(\theta\alpha.\pi\alpha) \right\} \right)$$

The cancellations between brackets (115) to (120) work as follows: Bracket (115) is (almost trivially) zero. The first term in bracket (116) gives the following contributions:

$$\left(-\frac{1}{X}\frac{\partial X}{\partial q} \left\{ \frac{\partial X}{\partial p}, X \right\} - \frac{1}{2X}\left\{ X, \frac{\partial^2 X}{\partial q \partial p} \right\} - \frac{1}{2X}\left\{ \frac{\partial X}{\partial q}, \frac{\partial X}{\partial p} \right\} \right)(\theta\alpha.\pi\alpha)$$

Clearly the second of these terms cancels with those produced by the second term in bracket (116), while the remaining two terms cancel with the term:

$$- \left\{ \frac{1}{X}\frac{\partial X}{\partial p}, \frac{1}{X}\frac{\partial X}{\partial q} \right\}(\theta\alpha.\pi\alpha)$$

of bracket (117). We now have no further trouble, as the Grassmann terms ensure that additional terms in (118,119,120) and the second term in (117) are all identically zero. Thus the non-trivial cancellation is that occurring between the terms in bracket (116) and (117). One can carry on and verify the other graded canonical commutation relations in a similar manner. As before we have the freedom of adding a bosonic index to p and q to give, in general for a G2,2 system:

$$Pi = pi - \frac{1}{X}\frac{\partial X}{\partial qi}(\theta\alpha.\pi\alpha) - \frac{1}{2X}\left\{ \frac{\partial X}{\partial qi}, X \right\}(\theta\alpha.\pi\alpha)$$

$$Qi = qi + \frac{1}{X}\frac{\partial X}{\partial pi}(\theta\alpha.\pi\alpha) + \frac{1}{2X}\left\{ \frac{\partial X}{\partial pi}, X \right\}(\theta\alpha.\pi\alpha)\ .$$

$$\Gamma\alpha = X(qi,pj)\theta\alpha$$

$$\Phi\alpha = \frac{\pi\alpha}{X(qi,pj)}$$

We now go on to generalise this result to the Gn,n system for arbitary $n \geq 2$.

2.2.3 Generalisation to the Gn,n system.

Having seen how the above class of canonical transformations generalised up from the G1,1 system to the G2,2, it is now straight forward to determine how these transformations work for higher orders of Grassmannian variables. Taking note of the cancellation between the terms (114) and (115) of the previous section, we find that the G2,2 generalisation of the above result is as follows:

(121)

$$P_i = p_i - \frac{1}{X}\frac{\partial X}{\partial q_i}(\theta\alpha.\pi\alpha) - \frac{1}{2X}\{\frac{\partial X}{\partial q_i},X\}(\theta\alpha.\pi\alpha) - \frac{1}{2.3X}\{\{\frac{\partial X}{\partial q_i},X\},X\}(\theta\alpha.\pi\alpha)$$

(122)

$$Q_i = q_i + \frac{1}{X}\frac{\partial X}{\partial p_i}(\theta\alpha.\pi\alpha) + \frac{1}{2X}\{\frac{\partial X}{\partial p_i},X\}(\theta\alpha.\pi\alpha) + \frac{1}{2.3X}\{\{\frac{\partial X}{\partial p_i},X\},X\}(\theta\alpha.\pi\alpha)$$

and where as before the function X acts as a conformal factor on the fermionic sector:

(123) $$\Gamma\alpha = X(q_i,p_j)\theta\alpha$$

(124) $$\Phi\beta = \frac{\pi\beta}{X(q_i,p_j)}$$

where $\alpha,\beta = 1,2,3$

It is not hard now to see how the Gn,n result works. Indeed we obtain:

(125)

$$P_i = p_i - \frac{1}{X}\frac{\partial X}{\partial q_i}(\theta\alpha.\pi\alpha) - \ldots - \frac{1}{N!X}\{\ldots\{\frac{\partial X}{\partial q_i},X\},X\},\ldots,X\}(\theta\alpha.\pi\alpha)$$

(126)

$$Q_i = q_i + \frac{1}{X}\frac{\partial X}{\partial p_i}(\theta\alpha.\pi\alpha) + \ldots + \frac{1}{N!X}\{\ldots\{\frac{\partial X}{\partial p_i},X\},X\},\ldots,X\}(\theta\alpha.\pi\alpha)$$

with

(127) $$\Gamma\alpha = X(q_i,p_j)\theta\alpha$$

(128) $$\Phi\alpha = \frac{\pi\beta}{X(q_i,p_j)}$$

where $\alpha,\beta = 1,\ldots,n$

The various cancelations between terms work in a similar manner to the G2,2 example. It is interesting to wonder how one might introduce further functions into this canonical symmetry in order to make the fermionic sector transformations more interesting. The author feels this might be possible to do if one demands that the arbitrary functions themselves satisfies various conditions. For example, a fermionic sector with individual conformal factors on the Grassmann terms, thus:

(129) $\Gamma\alpha = X\alpha(q,p)\theta\alpha$

(130) $\Phi\alpha = \dfrac{\pi\alpha}{X\alpha(q,p)}$ no sum over α, for $\alpha = 1,\ldots,n$

might form part of a canonical set which satisfied (79,80,81), if one imposed that the arbitary functions $X\alpha(q,p)$, $\alpha = 1,\ldots,n$ themselves formed a commutative function group and satisfied:

(131) $\{ X\alpha, X\beta \} = 0$, for $\alpha,\beta = 1,\ldots,n$.

This condition ensures that (129) and (130) satify the conditions (81) for $\alpha,\beta = 1,\ldots,n$, however, finding the even members of the set to satify the remaining conditions (79) and (80) is not easy. Never the less this does seem a fairly promising avenue for future enquiry. Take, for instance the example of the G2,2 algebra. There exists a 'special case' solution of the algebra suggested above as follows:

(132) $Q = q + \frac{1}{F}\frac{\partial F}{\partial p}(\theta_1 . \pi_1) + \frac{1}{G}\frac{\partial G}{\partial p}(\theta_2 . \pi_2)$

(133) $P = p$

(134) $\Gamma_1 = F\theta_1$

(135) $\Gamma_2 = G\theta_2$

(136) $\Phi_1 = \dfrac{\pi_1}{F}$

(135) $\Phi_2 = \dfrac{\pi_2}{G}$

Where the functions F and G are independant and $F = F(p)$, $G = G(p)$. Because F and G are both functions of p the condition (131) is satisfied identically. Clearly, a similar graded canonical set is possible if F and G are both made functions of q instead, and equations (132) and (133) are adjusted to become:

(138) $Q = q$

(139) $P = p - \frac{1}{F}\frac{\partial F}{\partial q}(\theta_1 . \pi_1) - \frac{1}{G}\frac{\partial G}{\partial q}(\theta_2 . \pi_2)$

The above graded canonical transformation is similar to a pairing up of two G1,1 transformations described earlier in § 2.2.1. This is really as far as this subclass of the functions satisfying (131) will take us. The case of F = F(q) and G = G(p), say, is quite heavily constrained by the condition (81), and making the jump to the general case described by (79),(80) and (81) is not easy to do. It may be that this simple conformal structure in the fermionic sector described by (127) and (128) can not be made to work without the presence of additional higher order odd terms, but proving this is difficult.

We now continue our investigations into Z_2 graded structures associated with the graded Poisson bracket, by studying a Z_2 graded generalisation of the function group introduced in ∯ 1.3.6.

2.3.0 Graded Function Groups.

Introduction

In ∯ 1.3.6 of the introduction, two theorems on structures known as function groups were quoted. The importance of these theorems lies in their telling us that for a classical system which incorporates constraints, locally at least, we may always find a reduced set of variables which satisfy the canonical brackets [27]. A consequence of this is that locally we can always choose a co-ordinate system which makes the Dirac bracket look like a Poisson bracket in a reduced number of varibles [44] (we assume the constraints of the system are 'second class'). Unfortunately this transformation is in general not canonical, so in many situations it is not practical to employ this co-ordinate system [44].

Potentially, at least, function groups might be of considerable interest, in that the subclass of the linearised groups covers the Poisson bracket realisations of the Lie algebras. By this we mean that a general function group is of the form:

(1) $$\{ fa,fb \} = Fab(f)$$

of which a Poisson bracket realisation of a Lie algebra:

(2) $$\{ fa,fb \} = Cabc\ fc$$

is a linear example. There might also be global features to these objects which are of interest, with seemingly little work having been done recently on the subject. However, as far as the author knows, little or no work at all has been done on the the graded generalisation of these ideas. Casalbuoni [18] defines a graded version of the Dirac bracket, but does not develop the underlying notion of the existance of a graded function group, or it being possible to produce a generalisation of the classical theorems quoted in § 1.3.6 to include anticommuting variables. Although it seems unlikely that this area has gone unstudied, it should be remembered that the discovery of supersymmetry [46,62] and its subsequent development did not follow a graded generalisation of the canonical quantisation path. With the Poisson bracket associated with a classical theory being mapped over to the commutator bracket of the quantum theory under construction. For this reason a systematic study of graded function groups may have been overlooked.

We start this study by taking a look at the simplest example of the graded function group, that being the case when there is only one conjugate pair of fermionic variables: the G1,1 system.

2.3.1 The G1,1 system function group.

We start with a system of four canonical variables $(q;p;\theta;\pi)$, two even and two odd, and we ask what possibilities exist for graded function groups of rank two. That is to say we are interested in a set of functions $\Phi\alpha$ with $\alpha = 1,2$ such that:

$$\{ \Phi\alpha,\Phi\beta \}' = F\alpha\beta(\Phi) \qquad (3)$$

Where the function $F\alpha\beta$ is graded (anti)symmetric in the indices $\alpha,\beta = 1,2$. In the next section we look at the choice of one even and one odd function for the functions $\Phi\alpha$.

2.3.2 The 'even-odd' G1,1 function group.

For this example we have one even and one odd function making up the functions $\Phi\alpha$, which are denoted E and Γ respectively. That is we have:

$$E = E(q,p;\theta,\pi) \; , \; |E| = 0 \; , \quad \text{and} \quad \Gamma = \Gamma(q,p;\theta,\pi) \; , \; |\Gamma| = 1$$

and where the functions E, Γ satify:

(4) $$\{ E,\Gamma \}' = F(E,\Gamma) \quad , \quad |F(E,\Gamma)| = 1$$

(5) $$\{ \Gamma,\Gamma \}' = G(E,\Gamma) \quad , \quad |G(E,\Gamma)| = 0$$

where clearly antisymmetry of the 'even-even' bracket implies:

(6) $$\{ E,E \}' \equiv 0$$

What possibilities are exist then for equations (4) and (5) ? Since by assumption we have only one odd function Γ, power series expansion immediately implies that they take the following form:

(7) $$\{ E,\Gamma \}' = F(E)\,\Gamma$$

(8) $$\{ \Gamma,\Gamma \}' = G(E)$$

However, the functions F and G are not independent as we must ensure that the graded Jacobi identities are satisfied. That is:

(9) $$\{ E,\{ \Gamma,\Gamma \}'\}' + \{ \Gamma,\{ \Gamma,E \}'\}' - \{ \Gamma,\{ E,\Gamma \}'\}' = 0$$

Substituting (7) and (8) into (9) gives:

(10) $$\{ E,G(E) \}' - \{ \Gamma,F(E)\Gamma \}' - \{ \Gamma,F(E)\Gamma \}' = 0$$

By employing (40) of ∲ 2.1.6 this implies:

(11) $$\{ \Gamma,F(E)\Gamma \}' = F(E)\,\{ \Gamma,\Gamma \}' - \{ F(E),\Gamma\}\,\Gamma = 0$$

But since $\{ F(E),\Gamma \}' \approx \Gamma$ the second term must vanish, leaving us with the following condition:

(12) $$F(E)G(E) = 0$$

Thus we see there are two possibilities for the form of equations (7) and (8). Either:

(A)

(13) $$\{ E,\Gamma \}' = F(E)\,\Gamma$$

(14) $$\{ \Gamma,\Gamma \}' = 0$$

or

(B)

(15) $$\{ E,\Gamma \}' = 0$$

(16) $$\{ \Gamma,\Gamma \}' = G(E)$$

So requiring that the brackets (7) and (8) satisfy the graded Jacobi identities imposes great restrictions on the form that they may take. Having determined this form, we now examine the explicit structure of the functions E and Γ.

Power expanding E and Γ in terms of their Grassmann sector gives the following:

(17) $$E = a(q,p) + b(q,p)\theta.\pi$$

(18) $$\Gamma = \alpha(q,p)\theta + \beta(q,p)\pi$$

where the functions a,b,α,β are all graded even and real valued. Dealing first with possibility (A), substitution of (18) into (14) yields the following:

$$\{ \alpha\theta + \beta\pi,\alpha\theta + \beta\pi \}' = 2\{ \alpha\theta,\beta\pi \}' = 2(\{ \alpha,\beta \}\theta.\pi - \alpha\beta) = 0$$

which is only satisfied when $\alpha\beta = 0$.

Let us choose $\beta = 0$, that is we give the function Γ the following form:

(19) $$\Gamma = \alpha(q,p)\theta$$

Substituting (17) and (19) into (13) now gives us:

$$\{ a + b\theta.\pi,\alpha\theta \}' = \{ a,\alpha \}\theta + b\alpha\{ \theta.\pi,\theta \} =$$
$$= (\{ a,\alpha \} - b\alpha)\theta = F(a + b\theta.\pi)\theta = F(a)\alpha\theta$$

where we have Taylor expanded the second last term. Thus we now have the following relationship between the even component functions:

(20) $$\alpha F(a) = \{ a,\alpha \} - b\alpha$$

That is:

$$b(q,p) = \{ a,\ln\alpha \} - F(a)$$

Thus for equations (17) and (18) to satisfy (13) and (14), we require the functions $E(q,p;\theta,\pi)$ and $\Gamma(q,p;\theta,\pi)$ to be of the following form:

(21) $$E = a + (\{ a,\ln\alpha \} - F(a))\theta.\pi$$

(22) $$\Gamma = \alpha\theta$$

Before dealing with the second possibility (B), let us briefly review the steps that led to equations (21) and (22). Recall that we made two initial choices. Firstly that we were dealing with a graded phase space consisting of two even and two odd variables. And secondly that the function group under consideration consisted of one even and one odd function. Having made those choices, the requirement that the functions which make up the group must satisfy the graded Jacobi identities dictates that the group may take on two possible forms (A) or (B). Looking at form (A) in more detail, employing expansions in powers of the graded variables, we were able to 'solve' the graded function group up to the presence of two arbitary even, real functions of (q,p). This was done at the expense of making one further choice, which amounted to fixing the form of the odd sector of the function group, resulting in equation (19). Choosing the only other possibility inherent in the condition results in an equivalent pair of equations to (21) and (22), and is not of great significance. We now continue our analysis and deal with the other possible form (B) of the 'even-odd' G1,1 function group.

We approach case (B) in a similar manner to case (A). Substituting (18) into (16) we get:

(23) $$\{ \alpha\theta + \beta\pi, \alpha\theta + \beta\pi \}' = 2(\{ \alpha, \beta \}\theta.\pi - \alpha.\beta) = G(E)$$

First let us look at the form of G(E). Let us assume G(x) may be expressed as some power series in x, thus:

(24) $$G(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_n x_n^n + \ldots$$

Where we do not specify finiteness at this stage. Substituting expression (17) for x now gives us:

(25) $$G(E) = \mu_0 + \mu_1(a + b\theta.\pi) + \ldots + \mu_n(a + b\theta.\pi)^n + \ldots$$

Looking at the nth term in this expansion we have the following:

(26) $$(a + b\theta.\pi)^n = a^n + na^{n-1} b\, \theta.\pi$$

So the function G(E) has the form:

$$G(E) = (\mu_0 + \mu_1 a + \ldots + \mu_n a^n + \ldots) + \tag{27}$$
$$+ b(\mu_0 + 2\mu_1 a + \ldots n\mu_n a^{n-1} + \ldots)\theta.\pi$$

If the function G(x) is expressible as a finite polynomial, then we may write (27) above as:

$$G(E) = G(a) + bG'(a)\theta.\pi \tag{28}$$

If G(x) is not a finite polynomial, then there exist pathological functions that would make the step between (27) and (28) erroneous, however, for most well behaved infinite polynomials (28) will still hold.

We may now employ (28) in (23) to give:

$$-\alpha.\beta + \{\alpha,\beta\}\theta.\pi = \tfrac{1}{2}(G(a) + bG'(a)\theta.\pi) \tag{29}$$

Clearly this implies:

$$\alpha.\beta = -\tfrac{1}{2}G(a) \tag{30}$$

$$\{\alpha,\beta\} = \tfrac{1}{2}bG'(a) \tag{31}$$

Turning now to (15) we have:

$$\{a + b\theta.\pi, \alpha\theta + \beta\pi\}' = 0 \tag{32}$$

which gives us:

$$\{a,\alpha\}' = \alpha.b \tag{33}$$

$$\{a,\beta\}' = -\beta.b \tag{34}$$

Thus we have four equations (30),(31),(33) and (34), linking the functions α,β,a and b. Using equations (30) and (31) we may determine β and b:

$$\beta = -\frac{G(a)}{2\alpha} \tag{35}$$

$$b = \frac{1}{\alpha}\{a,\alpha\}' \tag{36}$$

Equations (33) and (34) are now satisfied identically by β and b. Substitution of the above expressions for β and b into the expansions (17) and (18) for the functions E and Γ give us the following general forms of these variables:

$$E = a + \{a, \ln\alpha\}'\theta.\pi \tag{37}$$

$$\Gamma = \alpha\theta - \frac{G(a)\pi}{2\alpha} \tag{38}$$

We remark that once again, after analysis similar in content to that which we did for group (A), we have been led to a general expression for the functions which comprise the function group (B). Again, after the group has been chosen, there exists two arbitrary real valued functions α and a of the even phase space variables (q,p) which characterise the group functions E and Γ. The above analysis completes the second of the only two possibilities (A) and (B) for a graded function group consisting of one even and one odd function, constructed from two even and two odd conjugate phase space variables $(q,p;\theta,\pi)$.

Before leaving this example it is interesting to compare the form of the above function groups and their associated functions with the work we did earlier in ∮ 2.2 on graded canonical transformations. This is best done in a column, with the bracket relations on the L.H.S. and the functions which satisfy those brackets on the R.H.S. Observe, from ∮ 2.2.1 we have:

(1)

$$\{ Q,P \}' = 1 \ , \ \{ Q,Q \}' = 0 = \{ P,P \}' \quad ; \quad Q = q + \{ q,\ln X \}\theta.\pi$$

$$\{ \Gamma,\Phi \}' = -1 \ , \ \{ Q,\Gamma \}' = 0 = \{ P,\Gamma \}' \quad ; \quad P = p + \{ p,\ln X \}\theta.\pi$$

$$\{ \Gamma,\Phi \}' = 0 \ , \ \{ \Gamma,\Gamma \}' = 0 = \{ \Phi,\Phi \}' \quad ; \quad \Gamma = X\theta \ , \quad \Phi = \frac{\pi}{X}$$

And from the above work on the (A) and (B) type function groups we have:

(2)

$$\{ E,\Gamma \}' = F(E)\Gamma \quad ; \quad E = a + (\{ a,\ln\alpha \} - F(a))\theta.\pi$$

$$\{ \Gamma,\Gamma \}' = 0 \quad ; \quad \Gamma = \alpha\theta$$

(3)

$$\{ E,\Gamma \}' = 0 \quad ; \quad E = a + \{ a,\ln\alpha \}'\theta.\pi$$

$$\{ \Gamma,\Gamma \}' = G(E) \quad ; \quad \Gamma = \alpha\theta - \frac{G(a)\pi}{2\alpha}$$

We can see that in a sense groups (2) and (3) represent possible 'compressions' of the fundamental bracket relations (1). Notice how the function a(q,p) in (2) and (3) now plays the role that the variables (q,p) played in (1) - this is showing the mixing in the purely bosonic part of the function group. The other arbitrary function $\alpha(q,p)$ in (2) and (3), is playing a role similar to the $X(q,p)$ in (1), which represents a conformal rescaling of the fermionic sector. Notice in (3) that the function G has caused a mixing of the fermionic variables θ and π. Because this is only a first example, it is hard to understand clearly what is going on here. A necessary first step is to analyse fully all the possible graded canonical transformations of the type found in ∯ 2.2, as the freedom introduced by the arbitrary functions associated with these symmetries seem to be present throughout function group type reductions in the number of graded variables.

We now carry on with our analysis of the G1,1 graded function groups by next looking at the possibility of reducing the two even and two odd graded canonical variables down to two odd functions.

2.3.3 The 'odd-odd' G1,1 function group.

The next possibility we examine is that of the two pairs of graded canonical variables $(q,p;\theta,\pi)$ being used to construct a graded function group consisting of two odd functions, which we denote by $\Phi_1 = \Phi_1(q,p;\theta,\pi)$ and $\Phi_2 = \Phi_2(q,p;\theta,\pi)$, where $|\Phi_1| = 1$ and $|\Phi_2| = 1$. In this situation equation (3) gives us the following:

(39) $$\{ \Phi_1 ,\Phi_1 \}' = A(\Phi_1 ,\Phi_2)$$

(40) $$\{ \Phi_1 ,\Phi_2 \}' = B(\Phi_1 ,\Phi_2)$$

(41) $$\{ \Phi_2 ,\Phi_2 \}' = C(\Phi_1 ,\Phi_2)$$

where the function $B(\Phi_1,\Phi_2)$ is required to be symmetric under the interchange of Φ_1 and Φ_2.

As before we must first check to see what restrictions the graded Jacobi identities place on us. With only two odd functions we have:

(42) $$\{ \Phi_1 ,\{ \Phi_2 ,\Phi_2 \}'\}' + 2\{ \Phi_2 ,\{ \Phi_1 ,\Phi_2 \}'\}' = 0$$

(43) $$\{ \Phi_2 ,\{ \Phi_1 ,\Phi_1 \}'\}' + 2\{ \Phi_1 ,\{ \Phi_2 ,\Phi_1 \}'\}' = 0$$

Looking at (39),(40),(41) again, it is clear that since we only have two odd functions in the system, the only possibilities for the functions $A(\Phi_1,\Phi_2)$, $B(\Phi_1,\Phi_2)$ and $C(\Phi_1,\Phi_2)$ are to be of the following form:

$$A(\Phi_1,\Phi_2) = a\ \Phi_1.\Phi_2 \tag{44}$$

$$B(\Phi_1,\Phi_2) = \tfrac{1}{2}b(\Phi_1.\Phi_2 - \Phi_2.\Phi_1) \tag{45}$$

$$C(\Phi_1,\Phi_2) = c\ \Phi_1.\Phi_2 \tag{46}$$

Where a,b,c are real constants.

Substituting (44),(45) and (46) into (42) gives us the following conditions on the constants a,b,c:

$$b.c = 0 \quad , \quad a.c + 2b = 0 \tag{47}$$

Unfortunately, doing the same substitution with (43) yields the further condition:

$$a.b = 0 \tag{48}$$

Thus the only possibility for the form of the brackets (39,40,41) is the trivial group:

$$\{\Phi_1,\Phi_1\}' = 0 \tag{49}$$

$$\{\Phi_1,\Phi_2\}' = 0 \tag{50}$$

$$\{\Phi_2,\Phi_2\}' = 0 \tag{51}$$

As before we make the following Grassmann expansions:

$$\Phi_1 = \alpha_1\theta + \beta_1\pi \tag{52}$$

$$\Phi_2 = \alpha_2\theta + \beta_2\pi \tag{53}$$

With $\alpha_1,\alpha_2,\beta_1,\beta_2$ being functions of (q,p). We now substitute these expressions into equations (49),(50) and (51).

This gives us the following conditions on the functions α ,α and β ,β :

$$\alpha_1.\beta_1 = 0 \tag{54}$$

$$\alpha_2.\beta_2 = 0 \tag{55}$$

$$\alpha_1.\beta_2 + \alpha_2.\beta_1 = 0 \tag{56}$$

$$\{\alpha_1,\beta_2\} + \{\alpha_2,\beta_1\} = 0 \tag{57}$$

Clearly, various choices are possible here. For example take the case of:

$$\alpha_2 = 0 \tag{58}$$

$$\beta_1 = 0 \tag{59}$$

Then (56) would imply:

(60) $$\alpha_1 . \beta_2 = 0$$

Which amounts to either:

(61) $$\Phi_1 = 0 \quad , \quad \Phi_2 = \beta_2 \pi$$

or

(62) $$\Phi_1 = \alpha_1 \theta \quad , \quad \Phi_2 = 0$$

One can quickly see that another possibility is:

(63) $$\Phi_1 = \alpha_1 \theta \quad , \quad \Phi_2 = \alpha_2 \theta$$

and so on. Thus we conclude that the requirements of satifying the graded Jacobi identities place severe restrictions on the form and solutions of the ´odd-odd´ graded function group. All that remains to complete the analysis of this class of graded function groups, is to examine the case of the reduction to two even functions.

2.3.4 The ´even-even´ G1.1 graded function group.

We now consider the final case of the graded canonical set $(q,p;\theta,\pi)$ being reduced down to two even functions, which are denoted $E_1(q,p;\theta,\pi)$ and $E_2(q,p;\theta,\pi)$, with $|E_1| = 0 = |E_2|$. The function group equation (3) in this case becomes:

(64) $$\{ E_1 , E_2 \}' = F(E_1 . E_2)$$

Where the function F is anti-symmetric under interchange of E_1, E_2. The Jacobi identities are satisfied identically for only two even functions, so we have no conditions on the function F that way. Some progress can be made by assuming the function $F(E_1, E_2)$ has the following form:

(65) $$F(E_1,E_2) = A(E_1)B(E_2) - A(E_2)B(E_1)$$

Then if we place:

(66) $$E_1 = a_1 + b_1 \theta . \pi$$

(67) $$E_2 = a_2 + b_2 \theta . \pi$$

we have from before:

(68) $$A(E_1) = A(a_1) + b_1 A'(a_1) \theta . \pi$$

(69) $$A(E_2) = A(a_2) + b_2 A'(a_2) \theta . \pi$$

(70) $$B(E_1) = B(a_1) + b_1 B'(a_1) \theta . \pi$$

(71) $$B(E_2) = B(a_2) + b_2 B'(a_2) \theta . \pi$$

where we assume the functions A and B are suitably well behaved, and upon substitution in (64) gives us:

(72) $$\{ a_1, a_2 \} = A(a_1)B(a_2) - A(a_2)B(a_1)$$

(73) $$\{ a_1, b_2 \} + \{ b_1, a_2 \} = b_2(A(a_1)B'(a_2) - A(a_2)B'(a_1)) + + b_1(A'(a_1)B(a_2) - A'(a_2)B(a_1))$$

The first condition (72) tells us that the functions a_1, a_2, must themselves form a bosonic function group with the function F. The second equation (73) is a condition that the functions b_1, b_2 must satisfy for E_1, E_2 to also satisfy (64). By employing a power expansion argument similar to the one we used above, it is clear that <u>any</u> well behaved function F will produce a condition similar to (72). That is, the zeroth order term in Grassmann variables of the functions E_1 and E_2 will always satisfy (72). The additional functions b_1, b_2 will then have to be chosen to satisfy the further constraints which occur. Thus we see that the 'even-even' graded function group for the G1,1 system is by far the least constrained of the three possibilities, and nearest in form to the standard function group defined in Part I. An example of an 'even-even' graded function group which physically might be of interest is the case when the two even functions E_1, E_2 form a conjugate pair. That is:

(74) $$\{ E_1, E_2 \}' = 1$$

Under these circumstances, using the expansions (66) and (67) for E_1 and E_2, we obtain the following conditions on the functions a_1, a_2, b_1, b_2:

(75) $$\{ a_1, a_2 \} = 1$$

(76) $$\{ a_1, b_2 \} + \{ b_1, a_2 \} = 0$$

If we suppose that $b_1 = b_1(a_1, a_2)$ and $b_2 = b_2(a_1, a_2)$, then clearly the following functions satisfy (74):

(77) $$E_1 = a_1 + (ka_1 + f(a_2))\theta.\pi$$

(78) $$E_2 = a_2 - (ka_2 + g(a_1))\theta.\pi$$

Where f and g are arbitrary functions of a and a respectively, and k is some real constant. A particularly obvious choice for the functions a_1 and a_2 which satisfies (75) is the following:

(79) $$E = q + (kq + f(p))\theta.\pi$$

(80) $$E = p - (kp + g(q))\theta.\pi$$

This concludes this brief look at the simplest examples of the idea of the graded function group, which we believe serves to show that the idea does make sense and should be investigated systematically. At this stage we do not have enough examples to be able to realistically hope to project what the general principle is lying behind these objects, we feel certain that the graded permutation group will play a role in this theory somewhere, though this is just speculation. Again, it is not clear if anything fundamentally new will come out of such a study. However so long as objects such as the graded Dirac bracket are in use by the physics community, then the structure of graded function groups should be understood.

We now leave this subject and go on to investigate the problem of realising a general super Lie algebra using graded functions defined on a super phase space of the type we have been dealing with, where the graded Poisson bracket is used to define the Lie combination operation.

2.4.0 Graded Poisson Bracket Realisations.

Introduction

In Part I it was made clear that a necessary element that is required before being able to carry out the group theoretical quantisation program in [37], is understanding how to realise a general Lie algebra using phase space observables and the Poisson bracket as the Lie combination rule. What this amounts to locally is that, given some Lie algebra $[T_i, T_j] = C_{ij}^k T_k$, we may find well defined functions on phase space F_i such that $\{ F_i, F_j \} = C_{ij}^k F_k$. Technically such a correspondance is known as a 'Souriau momentum map' and has many interesting properties [55]. In this section we examine a graded generalisation of this idea, with particular attention being paid to the graded equivalent of the cocycles of Part I. What finding the graded generalisation of the momentum map described above comes down to is, given some super Lie algebra L which has both commuting and anticommuting generators, we must find a correspondance between these generators and the space of odd and even functions on the graded phase space, which realises the algebra under graded Poisson bracket combination. That is, if the super algebra has the following form:

(1) $$[E_i, E_j]' = C_{ij}^{k} E_k$$ with $i,j,k = 1,..,n$

(2) $$[E_i, O_\alpha]' = D_{i\alpha}^{\beta} O_\beta$$ and $\alpha,\beta = 1,..,N$

(3) $$[O_\alpha, O_\beta]' = F_{\alpha\beta}^{k} E_k$$

Where E_i represents the even (commuting) generators, and O_α the odd (anti-commuting) generators. And where C_{ij}^k, $D_{i\alpha}^\beta$, $F_{\alpha\beta}^k$ are the structure constants of the super algebra, with $[\ ,\ ]'$ representing the graded commutator bracket which is defined as follows:

(4) $$[A,B]' := A.B - (-1)\ B.A$$

where $|E_i| = 0$, $|O_\alpha| = 1$.

Also we require that the brackets (1,2,3) satisfy the graded Jacobi identities (see appendix A), which means that the structure constants of the algebra C, D and F in (1,2,3) satisfy:

$$C_{ij}^{m} C_{km}^{n} + C_{ki}^{m} C_{jm}^{n} + C_{jk}^{m} C_{im}^{n} = 0 \tag{5}$$

$$F_{\beta\mu}^{l} D_{\alpha l}^{\delta} + F_{\alpha\beta}^{l} D_{\mu l}^{\delta} + F_{\mu\alpha}^{l} D_{\beta l}^{\delta} = 0 \tag{6}$$

$$C_{ik}^{l} F_{\alpha\beta}^{k} + D_{\alpha i}^{\mu} F_{\mu\beta}^{l} + D_{\beta i}^{\mu} F_{\mu\alpha}^{l} = 0 \tag{7a}$$

$$C_{ij}^{k} D_{\alpha k}^{\delta} + D_{j\alpha}^{\mu} D_{i\mu}^{\delta} + D_{\alpha i}^{\mu} D_{j\mu}^{\delta} = 0. \tag{7b}$$

We wish to find a correspondence:

(8) $E_i \longrightarrow f_i(q,p;\theta,\pi)$, $O_\alpha \longrightarrow f_\alpha(q,p;\theta,\pi)$

where $|f_i| = 0$, for $i = 1,..,n$, and $|f_\alpha| = 1$, for $\alpha = 1,...,N$, and such that the functions f_i, f_α satisfy the following:

$$\{ f_i,f_j \}' = C_{ij}^{k} f_k \tag{9}$$

$$\{ f_i,f_\alpha \}' = D_{i\alpha}^{\beta} f_\beta \tag{10}$$

$$\{ f_\alpha,f_\beta \}' = F_{\alpha\beta}^{k} f_k \tag{11}$$

with $i,j,k = 1,..,n$ and $\alpha,\beta = 1,...,N$

If this is possible to do one says that the superalgebra has been truly realised. Unfortunately in general this is not possible to do. In carrying out this process in Part I on a straight forward Lie algebra, we saw how central terms, called cocycles, appear and which have to be removed if a true realisation is to be made. A similar occurence arises in the graded case, which we describe in the next section.

2.4.1 <u>Central terms in graded Poisson bracket realisations</u>.

The local central extension which can occur in graded Poisson bracket realisations, appear on the right hand side of equations (9), (10) and (11) in the following manner:

$$\{ f_i,f_j \}' = C_{ij}^{k} f_k + z_{ij} \tag{12}$$

$$\{ f_i,f_\alpha \}' = D_{i\alpha}^{\beta} f_\beta + z_{i\alpha} \tag{13}$$

$$\{ f_\alpha,f_\beta \}' = F_{\alpha\beta}^{k} f_k + z_{\alpha\beta} \tag{14}$$

Where $|z_{ij}| = 0 = |z_{\alpha\beta}|$, $|z_{i\alpha}| = 1$. Also the central terms have the following symmetry properties:

(15) $$z_{ij} = -z_{ji}$$

(16) $$z_{i\alpha} = -z_{\alpha i}$$

(17) $$z_{\alpha\beta} = z_{\beta\alpha}$$

in accord with the symmetries of the graded Poisson bracket. The first property we prove about these central terms is the following:

Proposition 1

The central term $z_{i\alpha}$ in equation (13) is identically zero.

Proof

The central term $z_{i\alpha}$ arises from a graded (skew)symmetric object Z(A,B), which we call a graded 2-cocycle [59], and where A,B are generators of the superalgebra defined by equations (1), (2) and (3). We have:

(18) $$z_{i\alpha} \equiv Z(E_i,O_\alpha) = - Z(O_\alpha,E_i) = -z_{\alpha i}$$

However, since the graded poisson bracket is bilinear in Grassmann numbers, for consistancy we also have:

(19) $$Z(\epsilon O_\alpha,E_i) = Z(O_\alpha,\epsilon E_i) = \epsilon Z(O_\alpha,E_i) , \quad \text{where } |\epsilon| = 1$$

This implies:

$$\epsilon Z(E_i,O_\alpha) = Z(E_i,\epsilon O_\alpha) = Z(\epsilon O_\alpha,E_i) = \epsilon Z(O_\alpha,E_i) = - \epsilon Z(E_i,O_\alpha)$$

$$\Rightarrow \quad z_{i\alpha} = -z_{i\alpha} \quad \Rightarrow \quad z_{i\alpha} \equiv 0 \qquad \text{as claimed.}$$

Where we have used the fact that Z(A,B) is graded (skew)symmetric. Thus, requiring $z_{i\alpha}$ to be antisymmetric in (i,α) is inconsistent with the bilinearity of the graded Poisson bracket. This is a well known result, see for example [5] on central extensions to superalgebras.

What other conditions do the these objects satisfy? By applying the super Jacobi identities to the brackets (12), (13) and (14) we obtain the following identities:

(20) $$C_{ij}^{\ \ l} z_{kl} + C_{ki}^{\ \ l} z_{jl} + C_{jk}^{\ \ l} z_{il} = 0$$

(21) $$F_{\alpha\beta}^{\ \ j} z_{ij} + D_{\alpha i}^{\ \ \mu} z_{\beta\mu} + D_{\beta i}^{\ \ \mu} z_{\alpha\mu} = 0$$

Equation (20) is just the standard result similar to the one we encountered in ∮ 1.4.2, which is just the condition that the 2-cocycle is closed. However (21) is associated with the graded extension. We notice that (20) and (21) are satisfied identically by the application of the super Jacobi identities (5,6,7), if z_{ij} and $z_{\alpha\beta}$ are of the following form:

(22) $$z_{ij} = C_{ij}^{k} z_k$$

(23) $$z_{\alpha\beta} = F_{\alpha\beta}^{k} z_k$$

Where $|z_k| = 0$, for $i,j,k = 1,\ldots,n$.

This is equivalent to the situation in ∮ 1.4.2 where the 2-cocycle is the exterior derivative of a 1-coboundary. In Part I it was shown that it is possible under the conditions of the Lie group having a non-degenerate Killing form, to invert equation (22) (that is the group is semisimple). Under these cicumstances it is possible to 'remove' the central terms from (12) and (14), by the addition of $- z_k$ to the functions f_k. The question arises as to whether it possible to discover a similar remedy in the graded case. That is, is it in general possible to invert equation (22)? If we can, then the graded cocycle is removeable. In studying this problem, let us first assume that the bosonic part of the super algebra is semi-simple and compact. That is:

(24) $$C_{im}^{n} C_{jn}^{m} = \delta_{ij} \quad : \quad \text{the Kronecker delta}$$

The super Cartan-Killing forms are [45]:

(25) $$g_{ij} = \delta_{ij} - D_{i\alpha}^{\beta} D_{j\beta}^{\alpha} = - g_{ji}$$

(26) $$g_{\alpha\beta} = D_{\alpha i}^{\mu} F_{\beta\mu}^{i} - D_{\beta i}^{\mu} F_{\alpha\mu}^{i} = - g_{\beta\alpha}$$

(27) $$g_{\alpha i} = g_{i\alpha} = 0$$

We are required to invert the equation:

(28) $$z_{ij} = C_{ij}^{k} z_k$$

The first observation we make is that to raise and lower indices we should use the full, supersymmetric, Cartan-Killing forms. For this reason we make the further assumption that the supersymmetric Cartan-Killing forms g_{ij}, $g_{\alpha\beta}$ have well behaved inverses. That is:

$$g_{ij}\, g^{jk} = \delta_i^{\,k} \equiv \delta_{ik} \tag{29}$$

$$g_{\alpha\beta}\, g^{\beta\tau} = \delta_\alpha^{\,\tau} \tag{30}$$

Because we are using the full Killing forms (25) and (26) we note that now it will not follow automatically that $C_{ijk} \equiv C_{ij}^{\,k}$ as in the standard theory. In fact, if we define:

$$C'_{ijk} := g_{km}\, C_{ij}^{\,m} \tag{31}$$

then we may state the following proposition:

<u>Proposition II</u>

The tensor C'_{ijk} is totally anti-symmetric in indices.

<u>Proof</u>

Since by definition:

$$C'_{ijk} := g_{km}\, C_{ij}^{\,m} = (\, C_{kr}^{\,s}\, C_{ms}^{\,r} - D_{k\alpha}^{\,\beta}\, D_{m\beta}^{\,\alpha}\,)\, C_{ij}^{\,m} \tag{32}$$

the first expresssion in the bracket gives the standard anti-symmetric result for the bosonic algebra. For the second term we have:

$$- D_{k\alpha}^{\,\beta}\, D_{m\beta}^{\,\alpha}\, C_{ij}^{\,m} = D_{k\alpha}^{\,\beta}\,(D_{\beta i}^{\,\mu}\, D_{j\mu}^{\,\alpha} + D_{j\beta}^{\,\mu}\, D_{i\mu}^{\,\beta}\,)$$

$$= D_{k\alpha}^{\,\beta}\, D_{j\beta}^{\,\mu}\, D_{i\mu}^{\,\alpha} - D_{k\alpha}^{\,\beta}\, D_{i\beta}^{\,\mu}\, D_{j\mu}^{\,\alpha}$$

Where we have used the super Jacobi identity (7b). Manifestly this is anti-symmetric in i and j, interchanging k and j we get:

$$D_{j\alpha}^{\,\beta}\, D_{k\beta}^{\,\mu}\, D_{i\mu}^{\,\alpha} - D_{j\alpha}^{\,\beta}\, D_{i\beta}^{\,\mu}\, D_{k\mu}^{\,\alpha} = -\,(D_{k\mu}^{\,\alpha}\, D_{j\alpha}^{\,\beta}\, D_{i\beta}^{\,\mu} - D_{k\beta}^{\,\mu}\, D_{i\mu}^{\,\alpha}\, D_{j\alpha}^{\,\beta}\,)$$

which completes the proof.

We also note that:

$$g^{kl}\, C'_{ijl} = C_{ij}^{\,k} \tag{33}$$

We are now able to invert the expression (28) since:

$$z_{ij} = C_{ij}^{\,k}\, z_k = C'_{ijk}\, z^k = C'_{kij}\, z^k \tag{34}$$

raising j on both sides we get:

(35) $$z_i^{\ j} = C_{ki}^{\ \ j} z^k$$

and now multiplying through by $C_{mj}^{\ \ i}$ gives us:

(36) $$C_{mj}^{\ \ i} z_i^{\ j} = C_{mj}^{\ \ i} C_{ki}^{\ \ j} z^k = \delta_{mk} z^k = z^m$$

That is:

(37) $$z_m = g_{mk}\, g^{lj}\, C_{kj}^{\ \ i} z_{il}$$

So we see that although we move through an intermediate state and deal with the non-standard C′ijk tensor, the property of anti-symmetry is sufficient to ensure the inversion is still possible, albeit under certain assumptions. We are now able to prove the the following proposition, which goes part way to generalising that stated in § 1.4.2:

Proposition III

Under conditions (24),(29) and (30), and also the caveat below, the central extentions z_{ij} and $z_{\alpha\beta}$ in equation (12) and (14) may be removed by a redefinition of functions f_i and f_α.

Proof

Redefine f_i by the addition of $-z_m$ from (37). The super Jacobi indentities (5) to (7) then ensure that the f_i satisfy (9) and (11). However there is a caveat:

Caveat

There is a further assumption underlying this. It is that given $z_{ij} = C_{ij}^{\ \ k} z_k$, then the only possibility for $z_{\alpha\beta}$ such that equations (22) and (23) are satisfied identically, is that the components $z_{\alpha\beta}$ are of the form $z_{\alpha\beta} = F_{\alpha\beta}^{\ \ k} z_k$. Subject to this caveat the proof is complete. What this amounts to is showing that the only solution of:

(38) $$D_{i\alpha}^{\ \ \beta} A_{\mu\beta} + D_{i\mu}^{\ \ \beta} A_{\alpha\beta} = 0 \quad , \quad \text{where} \quad A_{\alpha\beta} = z_{\alpha\beta} - F_{\alpha\beta}^{\ \ k} z_k$$

is that:

(39) $$A_{\alpha\beta} \equiv 0$$

It seems reasonable to hope that this requirement on the $D_{i\alpha}^{\ \ \beta}$ is already covered by demanding g_{ij} has a well defined inverse, though the author does not know a proof of this. Until this point is cleared up equation (38) represents a further condition on the types of super algebras which permit the removal of cocycles.

2.4.2 Graded Poisson bracket realisations.

Having looked at the question of central extensions we move on to the question first posed in the introduction to § 2.4, namely, is it possible to find a graded version of the momentum map (8). That is a correspondence such that (9), (10) and (11) are satisfied. In this section we generalise one classical realisation of this problem. This result appears in a less general form in [8]. It is well known classical result that, given some Lie algebra

(40) $$[\, E_i, E_j \,] = C_{ij}^{\;k} E_k$$

then the following map realises this algebra by observables on phase space, under Poisson bracket combination:

(41) $$E_i \;{-}{-}{-}{>}\; f_i := C_{ij}^{\;k} q^j p_k$$

It is easy to verify by employing the Jacobi identity that:

(42) $$\{\, f_i, f_j \,\} = C_{ij}^{\;k} f_k$$

What is the graded generalisation of this result? The following map is found to work:

(43) $$E_i \;{-}{-}{-}{>}\; f_i := C_{ij}^{\;k} q^j p_k + D_{i\alpha}^{\;\beta} \pi_\beta . \theta^\alpha$$

(44) $$O_\alpha \;{-}{-}{-}{>}\; f_\alpha := F_{\alpha\beta}^{\;k} p_k \theta^\beta + D_{\alpha i}^{\;\beta} q^i \pi_\beta$$

We demonstrate this explicitly for the {even, odd} case:

$$\{\, f_i, f_\alpha \,\}' = ((C_{im}^{\;k} p_k)(F_{\alpha\mu}^{\;m} \theta^\mu) - (D_{\alpha m}^{\;\mu} \pi_\mu)(C_{ij}^{\;m} q^j)) -$$

$$- ((D_{i\mu}^{\;\beta} \pi_\beta)(D_{\alpha j}^{\;\mu} q^j) + (F_{\alpha\mu}^{\;j} p_j)(D_{i\beta}^{\;\mu} \theta^\beta))$$

$$= D_{i\alpha}^{\;\mu} (F_{\beta\mu}^{\;l} p_l \theta^\beta + D_{j\mu}^{\;\beta} q^j \pi_\beta)$$

$$= D_{i\alpha}^{\;\mu} f_\mu$$

Where we have used the super Jacobi identity (7). We can see clearly that (43) and (44) are the natural generalisation to (41) if we state them in a more unified form using graded matrices similar to those introduced by Casalbuoni in [18]. Setting:

$$(45) \qquad (Q^a) = (q^i;\theta^\alpha) \quad , \quad (P_b) = (p_j;\pi_\beta)$$

with $i,j = 1,\ldots,n$, $\alpha,\beta = 1,\ldots,N$, and $a,b = 1,\ldots,n+N$. And defining the matrix M to be:

$$(46) \qquad (M)_{ab} := \begin{bmatrix} C & D \\ D & F \end{bmatrix}$$

Where C,D,F are the structure constants of the super algebra (1,2,3), then we may write the maps (43) and (44) as:

$$(47) \qquad (G_a) \longrightarrow (f_a) = (P_a)M_{ab}(Q^b)$$

Where $(G_a) = (E_i;O_\alpha)$ and $(f_a) = (f_i;f_\alpha)$. This is the natural form in which to see the generalisation of the map (41).

Another way that the map defined by (43) and (44) may be understood is by the use of the graded Hamiltonian vector fields of § 2.1. If we form the G.H.V.F.s associated with the functions f_i and f_α, thus:

$$(48) \qquad \Phi f_i := C_{ij}^{m}\, q^{j} \frac{\overrightarrow{\partial}}{\partial q^m} - C_{im}^{j}\, p_j \frac{\overrightarrow{\partial}}{\partial p_m} - D_{i\alpha}^{\beta}\, \theta^{\alpha} \frac{\overrightarrow{\partial}}{\partial \theta^\beta} - D_{i\mu}^{\alpha}\, \pi_\alpha \frac{\overrightarrow{\partial}}{\partial \pi_\mu}$$

$$(49) \qquad {}_\alpha f\Phi := \frac{\overleftarrow{\partial}}{\partial q^m} F_{\alpha\beta}^{m}\, \theta^{\beta} - \frac{\overleftarrow{\partial}}{\partial p_m} D_{\alpha m}^{\beta}\, \pi_\beta - \frac{\overleftarrow{\partial}}{\partial \theta^\beta} D_{\alpha k}^{\beta}\, q^k - \frac{\overleftarrow{\partial}}{\partial \pi_\mu} F_{\alpha\mu}^{k}\, p_k$$

Then employing (27) from § 2.1 and using the properties of f_i and f_α, then the following holds:

$$(50) \qquad [\, \Phi f_i, \Phi f_j \,]' = C_{ij}^{k}\, \Phi f_k$$

$$(51) \qquad [\, \Phi f_i, {}_\alpha f\Phi \,]' = D_{i\alpha}^{\beta}\, {}_\beta f\Phi$$

$$(52) \qquad [\, {}_\alpha f\Phi, {}_\beta f\Phi \,]' = F_{\alpha\beta}^{k}\, \Phi f_k$$

Thus we have realised the super algebra (1,2,3) by the use of graded Hamiltonian vector fields and the graded commutator bracket.

Before we leave this subject there is one further point to make. In the event that the super algebra in question is of a type which has C ≡ 0 and F ≡ 0, then the functions fi and fα may be adjusted to the following:

$$(53) \qquad f'_i = p_i + D_{i\alpha}{}^{\beta}\, \pi_\beta\, \theta^\alpha$$

$$(54) \qquad f'_\alpha = \pi_\alpha + D_{i\alpha}{}^{\beta}\, q^i\, \pi_\beta$$

This is the realisation which appears in [8].

2.5.0 Group Theoretical approach to Quantisation of Graded Systems.

Introduction

In ∮ 1.4.3 we explain briefly some of the ideas behind the group theoretical approach to quantisation, a full account of which is given by C.Isham in [36,37,38]. As we explained previously, the motivation for this thesis came from a desire to extend this approach to quantisation to include systems which incorporate anticommuting variables in a natural way. To be able to construct a rigorous general theory which achieves this, requires considerable familiarity with the differential geometry of graded phase spaces. However, it is possible to examine the idea from a less ambitious angle. Because at this stage we lack a fully developed theory of graded manifolds, instead we decide to employ a 'flat' graded parameter space of the type frequently used by physicists (and confusingly known as superspace) on which to illustrate some of the ideas we have been descussing. The actual superspace we use is just a flat coset space obtained from a completely abelian graded algebra, which is designed to parallel the example in ∮ 1.4.3 on the group theoretical approach to the quantisation of Rn drawn from [37].

We begin this section by carrying out what is essentially a verbal exercise of inserting words like 'graded' and 'super-symmetric' into the programme described in [37] and which we reviewed in ∮ 1.4.2. After some descussion as to what this might mean, we illustrate how this approach to graded quantisation might look when applied to a 'flat' configuration space of mixed real and Grassmann variables, which is the direct product $R_n \otimes G_N$ of an n-dimensional real space R_n with an N-dimensional Grassmann algebra G_N.

2.5.1 The graded quantisation programme.

The central idea of the 'group theoretical' approach to quantisation is that the quantum operators of the theory come about by a 'quantisation map' from a special set of observables appearing in the classical theory. The observables which comprise this set are those which realise the algebra of the classical transformation symmetry group (or a central extension of this group) associated with the classical phase space under quantisation. This approach to quantisation gives a nice interpretation of the 'ih' in the standard Heisenberg canonical commutation relations, as being a kind of anomaly: a failure of realisation between the classical symmetry of the phase space and the natural observables of the system. In an outline form we might express the generalisation of this quantisation procedure as a correspondence between the columns of the following chart:

Standard Bosonic programme	Bosonic/Fermionic programme
Classical mechanics with phase space being some symplectic manifold with symmetry.	Pseudomechanics on a graded phase space being some ortho-symplectic supermanifold with supersymmetry
Find some transitively acting Lie group on the phase space, which links all the points in the phase space to one another by the action of the group.	Find some transitive acting Lie super group on S.P.S. which links all points in S.P.S. by the action of the supergroup.

Realise the algebra of this group by some subset of the set of well defined observables on phase space, using the Poisson bracket as the Lie combination operation. If this is not possible to do with this algebra, use the central extension.	Realise the algebra of this super group by some subset of the set of graded observables using the graded Poisson bracket as the Lie combination operation. If this is not possible to do with this superalgebra, use the graded central extension.
Carry out a 'quantisation map' from this set of classical observables to a corresponding set of linear operators in some Hilbert space which realise the quantum algebra.	Carry out a 'quantisation map' from this set of pseudo-classical observables to a corresponding set of graded linear operators on super Hilbert space which realise the quantum algebra.
Complete the process by finding irreducible representations of the quantum algebra on Hibert space.	Find irreducible representations of super algebra on super Hibet space.

These then are the words. Below we discuss some of the problems one encounters when actually trying to associate meaning with the graded programme suggested above.

2.5.2 Discussion of the graded quantisation programme.

The first point to consider is that a pseudomechanical phase space is not as intuitive an idea as a standard classical phase space whose local co-ordinate functions are all real and commutative. Faced with some classical theory, it is likely that one will have a good intuitive grasp of the form the classical phase space will take (though we note that this is not always the case. For example the phase space of the classical string is not straight forward to envisage), or a notion at least of some of the symmetries the space might possess. However, when the configeration space of the theory involves anticommuting variables as it does in the case here, this type of intuition is far harder to come by in even the simplest of examples. The lack of easy visualisation of the variables makes graded theories intrinsically harder to grasp. For this reason the link between the way the graded phase space is expressed and the group of symmetries of the space, if indeed there is one, might be hard to find. In fact often in the physics literature superspace is actually defined by use of the symmetry group, by identifying it as the coset space between some known supergroup and its even Lie subgroup [32,41]. A theory constructed in this manner essentially reverses the first two of the correspondence steps above, making the link between the two steps a definition. Two examples of popular superspaces which have been constructed in this way are [41]: (A) Super Minkowski space and (B) Super De-sitter space. Looking at these two spaces in turn, in the case of Super Minkowski space we have:

(A) The quotient is taken between the Super-Poincare group and the Lorentz group. The resulting space is necessarily flat and has four bosonic co-ordinates $X\mu$, $\mu = 1,\ldots,4$, and four fermionic co-ordinates $\theta\alpha$, $\alpha = 1,\ldots,4$. This type of superspace can be decomposed into a direct product of the bosonic and fermionic parameter spaces, thus:

Superspace $S = B \otimes F$, where B = Bosonic configeration space
and F = Fermionic parameter space.

Where we have:

(1) $$S = (g/h) \otimes (G/g)$$

where $g \leq h$ = {even subgroup of supergroup G}. In the case of super Minkowski space this decomposition works as follows:

(2) Minkowski superspace $M = (g/h) \otimes (G/g)$

where; G = Super Poincare group, g = Poincare group , h = Lorentz group. Thus:

(3) M = (Minkowski 4-space) $\otimes$ (Fermionic parameter space)

where M has co-ordinates:

(4) $(X_a) = (X_\mu;\theta_\alpha)$, $a = 1,\dots,8$

with $X_\mu \in R^4$, $\mu = 1,\dots,4$, and θ_α form a G4 Grassmann algebra.

(B) For the case of Super de Sitter space we have:

(5) $$S = G/h = B \otimes F = \frac{OSp(N;4)}{SL(2,C) \otimes O(N)}$$

Where the bosonic and fermionic parameter subspaces are:

(6) $$B = \frac{Sp(4;R)}{SL(2,C)} \approx \frac{O(3,2)}{O(3,1)}$$

which is the standard de Sitter space, and:

(7) $$F = \frac{OSp(N;4)}{Sp(4;R) \otimes O(N)}$$

as the Fermionic sector.

For a superspace defined in this way one would, by definition, automatically have a transitively acting Lie supergroup on the superspace. So for this sub-class of graded spaces, the link between the symmetry of the space of the space itself is by construction trivial. We note that we would want our notion of a supermanifold to be a far more general construction than the type of superspaces defined above, and hence we would need to understand what is meant by the 'transitive action' of a supergroup on a supermanifold in a wider context. For the spaces defined as above the standard group definition suffices.

To have successfully completed the first two parts of the correspondance we suggest above, we must have furnished ourselves with some sort of graded phase space, which has a local co-ordinate expansion which looks something like $(q_i,p_j;\theta_\alpha,\pi_\beta)$, where the (q_i,p_j) commute and the $(\theta_\alpha,\pi_\beta)$ anticommute. Also we must have some supergroup S which moves us (transitively) round the space in some manner to be defined. Having obtained this group, we can start the process of realising the algebra of this group, or if necessary a central extension of it, by functions on the graded phases space. These functions determine which observables to map over to become linear operators on super Hilbert space, from which the quantum theory of the system is determined through irreducible representations. If the correspondence suggested above does indeed work out, then we would expect that the quantum algebra of these linear operators to be either the graded Lie algrebra associated with the supergroup S, or a central extension of this algegra if there is an obstruction analagous to the cocycles of § 1.4.2 present, and which we discussed earlier in § 2.4.1.

The programme we outline above represents how we expect a graded generalisation of the group theoretical approach to quantisation to work in a broad sense. To give a more detailed exposition of such a programme would require far more machinery than we have developed here. Instead, we work through an example which applies these ideas to a superspace of a type commonly employed by physicists, that of the coset space between a super group and its even Lie subgroup. The example we use is designed to parallel the example given in § 1.4.3 of the group theoretical approach to the quantisation of Rn, which was taken from the work of C.Isham in [37].

2.5.3 The group theoretical approach to the quantisation of Rn ⊗ GN.

The natural generalisation to the example in § 1.4.3, is the quantisation of the configuration space Rn ⊗ GN. That is, a space consisting of n commuting real co-ordinate functions xi, and N anticommuting Grassmann parameters θα. By analogy to the case of Rn, translations about the space may be affected by the action of the graded abelian Lie transformation group on some arbitary point in the space (because we have not defined an abstract superspace yet, this is really the definition of $R^n \otimes G^N$: a coset space between the group of graded abelian translations and its even subgroup). The algebra of this graded abelian group is:

(8) $$[E_i,E_j]' = 0$$

(9) $$[E_i,O_\alpha]' = 0$$

(10) $$[O_\alpha,O_\beta]' = 0$$

Where E_i, $i = 1,...,n$ are the even generators, and O_α, $\alpha = 1,..,N$ are the odd generators. Also $[\, , \,]'$ represents the graded commutator bracket. To see what the group action associated with this algebra is on some point in the coset space, we exponentiate the generators. An element g of this group may be expressed in following way:

(11) $$g(x^i,\theta^\alpha) = \exp(x^iE_i + \theta^\alpha O_\alpha)$$

Because all the grade commutation relations are zero, use of the graded B-C-H formulae gives us the group action on a point in the the space $R^n \otimes G^N$:

(12) $$g(x^i,\theta^\alpha).(y^i,\pi^\alpha) \longrightarrow (y^i + x^i, \theta^\alpha + \pi^\alpha).$$

So the graded commutation relations (8),(9) and (10) produce a group action on a point in $R^n \otimes G^N$ which is the natural generalisation of the example in § 1.4.3 of the action of the abelian group of translations on R^n. This then deals with the structure of the configuration space. To obtain the full super phase space we take the direct product of the configuration space with the 'momentum sector', thus:

$$\text{Super Phase Space} = (\text{Configuration space}) \otimes (\text{Momentum Sector})$$
$$= (R^n \otimes G^N) \otimes (R^n \otimes G^N) = R^{2n} \otimes G^{2N}$$

The full graded abelian translation group is obtained by exponentiating the following algebra:

(13) $$[E_i,E_j]' = 0 \;,\quad [F_i',F_j']' = 0 \;,\quad [E_i,F_j']' = 0$$

(14) $$[E_i,O_\alpha]' = 0 \;,\quad [F_i',M_\alpha']' = 0 \;,\quad [O_\alpha,F_j']' = 0$$

(15) $$[O_\alpha,O_\beta]' = 0 \;,\quad [M_\alpha',M_\beta']' = 0 \;,\quad [O_\alpha,M_\beta']' = 0$$

where (E_i, O_α), with $i = 1,..,n$ and $\alpha = 1,..,N$ are the configuration space generators from above. And $(F_{i'}, M_{\alpha'})$, with $i' = 1,..,n$ and $\alpha' = 1,..,N$ are the generators associated with translations in the momentum sector of the super phase space. Writing:

(16) $(A_s) = (E_i, F_{i'})$, with $s = 1,...,2n$

(17) $(H_\mu) = (O_\alpha, O_{\alpha'})$, with $\mu = 1,...,2N$

then we may write the brackets (13),(14) and (15) as:

(18) $[\, A_s, A_t \,]' = 0$

(19) $[\, A_s, H_\mu \,]' = 0$

(20) $[\, H_\mu, H_\delta \,]' = 0$

This then is the algebra of the group of pseudo-classical graded abelian translation of the super phase space $R^n \otimes G^N$. Following the comments earlier in this section, it is this algebra that we are required to find a true graded Poisson bracket realistion of to be able to continue with the group theoretical approach to quantisation.

Specifically we must find a map from the space of generators of the algebra into the space of graded observables on super phase space, which preserves the algebra (18), (19) and (20), with the graded commutator bracket being mapped over to the graded Poisson bracket. However, as we know from the previous discussion in § 2.3, a realisation of the algebra (18), (19), (20) will involve central terms. That is, in general a map from the space of generators into the space of graded observables, thus:

(21) $A_s \longrightarrow f_s(q_i, p_j; \theta_\alpha, \pi_\beta)$, where $|f_s| = 0$ for $s = 1,...,2n$

(22) $H_\mu \longrightarrow g_\mu(q_i, p_j; \theta_\alpha, \pi_\beta)$, where $|g_\mu| = 1$ for $\mu = 1,...,2N$

will necessarily produce central terms as follows:

(23) $\{\, f_s, f_t \,\}' = z_{st}$

(24) $\{\, f_s, g_\mu \,\}' = 0$

(25) $\{\, g_\mu, g_\varnothing \,\}' = z_{\mu\varnothing}$

These central terms are indicative of a non-trivial graded 2-cocycle in the graded algebra (18,19,20), which is not possible to remove by the addition of constant terms to the functions f_s and g_μ, because the structure constants of the algebra are all zero. Explicitly this goes as follows. Let us try to realise the algebra (18,19,20) by a set of functions on super phase space. The simplest functions we can use are the graded canonical variables themselves ($q_i, p_j; \theta_\alpha, \pi_\alpha$), with the natural correspondance:

(26) $$E_i \longrightarrow p_i \quad , \quad O_\alpha \longrightarrow \pi_\alpha$$

(27) $$F_{i'} \longrightarrow q_i \quad , \quad M_{\alpha'} \longrightarrow \theta_\alpha$$

(Note that the prime in (26,27) is just a tag which is used to indicate the momentum sector.)

Clearly the mapping (26) and (27) does not produce a graded Poisson bracket realisation of the algebra (18,19,20), because of the cross terms which result in the numbers z_{st} and $z_{\mu\phi}$ being of the following form for this choice of the functions f_s and g_μ:

(28) $$z_{st} = \delta_{is}.\delta_{i't}$$

(29) $$z_{\mu\phi} = \delta_{\alpha\mu}.\delta_{\alpha'\phi}$$

Although the mapping (26) and (27) of the generators A_s and H_μ to the functions f_s and g_μ is only one possible choice, in general any other choice will also result in the numbers z_{st} and $z_{\mu\phi}$ being non-zero. That is, it is impossible to realise the algebra (18,19,20) as it stands by graded observables on super phase space. Following the discussion of the subject earlier, this is a signal to us that we are dealing with the wrong algebra - in other words the group about which the quantum theory is built is different from the super group of Lie transformations of the underlying psuedo-classical phase space. This is similar to the idea of the 'anomaly'. Basically an anomaly occurres when a classical symmetry is broken at the quantum level [39]. Here we have a somewhat similar situation: it is not possible to base a quantum theory round a (psuedo) classical symmetry and therefore that symmetry is not present at the quantum level. In fact, the graded algebra we should be dealing with is the central extension of the (psuedo) classical algebra, which in the case of (13), (14) and (15) is the following:

(30) $[E_i, E_j]' = 0$, $[F_i', F_j'] = 0$, $[E_i, F_j']' = T_{ij}'$

(31) $[E_i, O_\alpha]' = 0$, $[F_i', M_\alpha'] = 0$, $[O_\alpha, F_j']' = 0$

(32) $[O_\alpha, O_\beta]' = 0$, $[M_\alpha', M_\beta']' = 0$, $[O_\alpha, M_\beta']' = T_{\alpha\beta}'$

(33) $[T_{ij}', \text{any generator}]' = 0 = [T_{\alpha\beta}', \text{any generator}]'$

where $|T_{ij}'| = 0$ and $|T_{\alpha\beta}'| = 0$, and in this notation may be expressed:

(34) $T_{ij}' = \delta_{ij}$, $T_{\alpha\beta}' = \delta_{\alpha\beta}$

This algebra is now realised by the mapping (26) and (27), so it does form a suitable basis round which to form a quantum theory. The hermiticity of operators in quantum mechanics deems a final alteration to the algebra (30,31,32,33) appropriate before it may used to construct a quantum theory around:

(35) $T_{ij}' = ih\ \delta_{ij}$, $T_{\alpha\beta}' = ih\ \delta_{\alpha\beta}$.

Thus, brackets (30) to (33) together with the choise of central extensions (35) constitute the familiar quantum algebra of the $R^n \otimes G^N$ system (for example see [17]).

This concludes this short look at the group theoretical approach to quantisation in a graded setting, the purpose of which really is to show that such an idea can make sense. For a fuller treatment of this subject considerably more graded machinery is required.

2.6.0 The O(3) Supersymmetric Sigma model: Quantum Mechanics on a Sphere with Fermions.

Introduction

In this final section of the thesis we discuss the natural generalisation to include fermions of the 'particle on a sphere' model looked at in ∮ 1.3.8. We derive this model in a simple way from considerations similar to those present in papers by E. Witten [63,64], J.Barcelos-Neto et al [6,7] and M.Spiegelglas [56]. And we calculate the Dirac brackets of this theory. During this work, reference [6] came to the attention of the author, and which highlights an interesting difference between the work we do here and that done in [6] and [56]. The model dealt with in these references does not include various extra constraints, which in the approach we take here seem unavoidable. As we shall see, one of the consequences these additional constraints have is to put in question how, if at all, the model that we present here is supersymmetric. From these considerations it would appear that further work is required to determine the role of the secondary constraints in the supersymmetry of this model.

2.6.1 O(3) supersymmetric quantum mechanics in (0+1) dimensions.

Following [6,56,63], we introduce the standard O(3) supersymmetric quantum mechanics model, making as few as possible assumptions as to the nature of the variables concerned, stating the transformations under which the model is supersymmetric [6]. We then introduce a primary constraint analogous to the 'motion on a 2-sphere' constraint used in ∮ 1.3.8 and, after determining all the associated secondary constraints, calculate the Dirac brackets.

The Model

As in [6,56] we start with a general superfield $\Phi i(\theta,\bar{\theta},t)$, where $i = 1,2,3$, and θ, $\bar{\theta}$ are two independent Grassmannian variables such that:

(1) $$\theta^2 = 0 = \bar{\theta}^2 \quad , \quad \theta.\bar{\theta} + \bar{\theta}.\theta = 0$$

Furthermore, we have the following well known super covariant derivatives:

(2) $$D = \frac{\partial}{\partial \theta} - i\bar{\theta}\frac{\partial}{\partial t}$$

(3) $$D = \frac{\partial}{\partial \bar{\theta}} - i\theta\frac{\partial}{\partial t}$$

And a general superspace action integral:

(4) $$A = \int dt\,d\theta\,d\bar{\theta}\ L(\Phi i, D\Phi i, \bar{D}\Phi i)$$

which is invarient under two supersymmetry transformations:

(5) $$\delta\Phi i(t,\theta,\bar{\theta}) := \epsilon\left(\frac{\partial}{\partial \theta} + i\bar{\theta}\frac{\partial}{\partial t}\right)\Phi i(t,\theta,\bar{\theta})$$

(6) $$\delta\Phi i(t,\theta,\bar{\theta}) := \bar{\epsilon}\left(\frac{\partial}{\partial \bar{\theta}} + i\theta\frac{\partial}{\partial t}\right)\Phi i(t,\theta,\bar{\theta})$$

where ϵ and $\bar{\epsilon}$ are two independant anti-commuting parameters. We now choose the Lagrangian to be:

(7) $$L(\Phi i, D\Phi i, \bar{D}\Phi) = -\tfrac{1}{2}\sum_i \bar{D}\Phi i . D\Phi$$

and we expand the superfield Φi in powers of θ, $\bar{\theta}$ thus:

(8) $$\Phi i(t,\theta,\bar{\theta}) = Xi(t) + \phi i(t)\theta + \bar{\phi} i(t)\bar{\theta} + Fi(t)\theta.\bar{\theta}$$

We make the following remarks at this stage:

(A) The $Xi(t)$ is a real, 3-component, time-dependent field which forms the bosonic co-ordinates of our (super) configuration space. In components $(Xi(t)) = (x(t), y(t), z(t))$.

(B) The $\phi i(t)$ is a complex valued, 3-component, anti-commuting time dependent field. It therefore has 6 independent anti-commuting components with $\bar{\phi} i(t)$ being the complex conjugate field.

Substituting the superfield expansion (8) into the Langrangian (7), and carrying out the integration over the Grassmanian variables we obtain the following expression for the action in component form [6]:

(9) $$A = \int dt\left(\tfrac{1}{2}(Xi)^2 + \tfrac{1}{2}i(\bar{\phi} i.\dot{\phi} i - \dot{\bar{\phi}} i.\phi i) + \tfrac{1}{2}(Fi)^2\right)$$

Also substituting the expansion (8) for Φi into the two supersymmetry transformations (5) and (6) gives us the SUSY transformations in component form [6]:

(10) $\delta X_i = \epsilon \phi_i$, $\bar{\delta} X_i = - \bar{\epsilon}\bar{\phi}_i$

(11) $\delta \phi_i = 0$, $\bar{\delta}\phi_i = \bar{\epsilon}(F_i + i\dot{X}_i)$

(12) $\delta\bar{\phi}_i = \epsilon(F_i - i\dot{X}_i)$, $\bar{\delta}\bar{\phi}_i = 0$

(13) $\delta F_i = i\epsilon\dot{\phi}_i$, $\bar{\delta} F_i = i\bar{\epsilon}\dot{\bar{\phi}}_i$

This is straight supersymmetric quantum mechanics without any additional features or assumptions, and with the $F_i(t)$, $i = 1,2,3$, acting as real valued auxiliary fields.

Now, following [6,56,63], we impose the supersymmetric equivalent of 'the motion on a sphere' constraint of ∮ 1.3.8, by demanding that:

(14) $$\Phi_i.\Phi_i = R^2$$

where $R \in \mathbb{R}$ and we use the Einstein summation convension.
This is a natural generalisation to the $\underline{X}.\underline{X} = R^2$ condition that we encountered in ∮ 1.3.8, and a quite reasonable primary constraint. By looking at components of the various Grassmann variables we obtain four conditions that the component fields must obey:

(15) $$X_i.X_i = R^2$$

This is purely the reemergence of the 'motion on a sphere' constraint from ∮ 1.3.8 now relevant to the bosonic sector of this theory.

(16) $X_i.\phi_i = 0$: the θ component.

(17) $X_i.\bar{\phi}_i = 0$: the $\bar{\theta}$ component.

(18) $F_i.X_i + \phi_i.\bar{\phi}_i = 0$: the $\theta\bar{\theta}$ component

Let us look at the effect of the SUSY transformations (10) to (13) on the constraints (16) to (18). Under the δ transformations the four variations are:

(19) $$\delta(X_i.X_i) = 2X_i.\delta X_i = 2\epsilon\phi_i.X_i = 0$$

(20) $$\delta(X_i.\phi_i) = \delta X_i.\phi_i + X_i.\delta\phi_i = \epsilon\phi_i.\phi_i = 0$$

(21) $$\delta(X_i.\bar{\phi}_i) = \delta X_i.\bar{\phi}_i + X_i.\delta\bar{\phi}_i = \tfrac{1}{2}\,\epsilon\,\frac{d}{dt}(X_i.X_i - R^2)$$

(22) $$\delta(F_i.X_i + \phi_i.\bar{\phi}_i) = \delta F_i.X_i + F_i.\delta X_i + \delta\phi_i.\bar{\phi}_i + \phi_i.\delta\bar{\phi}_i$$
$$= i\epsilon\,\frac{d}{dt}(\phi_i.X_i)$$

Where we have used (12) in (22), and employ the Einstein summation convention. The $\bar{\delta}$ transformations work in a similar manner, except for (22), which instead becomes:

$$\bar{\delta}(Fi.Xi + \phi i.\bar{\phi}i) = \ldots = i\epsilon \frac{d(\bar{\phi}i.Xi)}{dt} \qquad (23)$$

We will return to these variations later; at this stage we confine ourselves to saying that the requirement that primary constraints (16) to (18) are preserved under the SUSY transformations depends on the constraints (16) to (17) being preserved in time.

Following [63] the argument now goes that because there are no derivatives of F appearing in the constraints (16) to (18) or in the Lagrangian in (9), it is possible to eliminate the auxiliary field F from it, yeilding the Lagrangian encountered in [56]:

$$L = \tfrac{1}{2} M(\dot{X}i)^2 + \tfrac{1}{4}i(\bar{\phi}i.\dot{\phi}i - \dot{\bar{\phi}}i.\phi i) + \frac{M}{8R^2}(\bar{\phi}i.\phi i)^2 \qquad (24)$$

This Lagrangian together with the constraints (16) to (17) give the essential elements of the theory we will be considering here. As can be seen, it has been derived in a straightforward manner by imposing a natural superfield constraint (14) on a standard super-symmetric quantum mechanics lagrangian. We have made no assumptions as to the nature of the fields ϕi and $\bar{\phi}i$ (for instance in the 1+1 dimensional sigma model of [6,63] these fields are majorana spinors), and our superfield constraint (14) is compatible with the transformations (10) through (13) providing we ensure that these constraints are preserved throughout time. Since we are interested essentially in 'Dirac quantisation', we must now move into a Hamiltonian formalism, and use that setting to examine any further constraints of the theory.

2.6.2 The Hamiltonian formalism.

We define the bosonic and fermionic conjugate momenta as follows:

$$Pi := \frac{\partial L}{\partial \dot{X}i} \qquad (25)$$

$$\pi i := \frac{\partial L}{\partial \dot{\phi}i} \qquad \text{for } i = 1,2,3 \qquad (26)$$

(27) $$\bar{\pi}i := \frac{\partial L}{\partial \dot{\bar{\phi}}i}$$

Clearly (25) is well defined, however because our Lagrangian (24) is linear in fermionic kinetic terms, we are led to the following fermionic constraints:

(28) $$\pi i + \tfrac{1}{4}i\bar{\phi}i = 0 \quad , \qquad \text{for } i = 1,2,3$$

(29) $$\bar{\pi}i + \tfrac{1}{4}i\phi i = 0 \quad . \qquad \text{for } i = 1,2,3$$

As a result of these constraints, the Hamiltonian associated with the Lagrangian (24) contains no fermionic momenta terms, but just the bosonic momenta and the fermionic potential term, thus:

(30) $$H = \frac{1}{2M}(Pi.Pi) - \frac{M}{8R^2}(\bar{\phi}i.\phi i)^2$$

At this stage the degrees of freedom are 6 bosonic and 6 fermionic (where we include constraints (28) and (29) in the counting), which balance as expected. The key point about Lagrangians linear in kinetic terms is to realise that this feature must be incorporated from the start, by straight away using the Dirac brackets formed from the constraints (28) and (29), and employing these brackets as the natural Poisson brackets of the system. If one does not do this it is tempting to believe that extra secondary constraints are required to ensure that (28) and (29) remain valid under the time development of the system. This is erroneous and may be avoided by incorporating these linear fermionic constraints from the start, and thus removing all mention of the fermionic momenta πi and $\bar{\pi}i$ right from the outset.

Doing this then, the fundamental Poisson brackets we will be using are the following:

(31)
$$\{ Xi,Pj \}' = \delta ij \quad , \quad \{ Xi,Xj \}' = 0 \quad , \quad \{ Pi,Pj \}' = 0$$
$$\{ \bar{\phi}i,\phi j \}' = i\delta ij \quad , \quad \{ \phi i,\phi j \}' = 0 \quad , \quad \{ \bar{\phi}i,\bar{\phi}j \}' = 0$$

We now have everything that is required to be able to determine any secondary constraints that might be associated with the primary constraints (15),(16) and (17). We do this by demanding that the primary constraints remain valid over all values of the time parameter t. Using (31) we obtain the following:

(32) $$\{ Xi.Xi-R^2,H \}' = \frac{2}{M}(Xi.Pi)$$

(33) $$\{ \phi i.Xi,H \}' = \frac{1}{M}(Pi.\phi i) - \tfrac{1}{4}(Xi.\phi i)(\phi i.\bar{\phi} i)$$

(34) $$\{ \bar{\phi} i.Xi,H \}' = \frac{1}{M}(Pi.\bar{\phi} i) - \tfrac{1}{4}(Xi.\bar{\phi} i)(\phi i.\bar{\phi} i)$$

The second two terms in (33) and (34) are proportional to (16) and (17) respectively and so may be disregarded. However, we see that the requirement that (32),(33) and (34) are preserved throughout the systems time development imposes three new secondary constraints on us:

(35) $$Xi.Pi = 0$$

(36) $$Pi.\phi i = 0$$

(37) $$Pi.\bar{\phi} i = 0$$

Further action of the Hamiltonian on (35), (36) and (37) fails to produce any new constraints, so (35) through to (37) represents the sum total. At this point an interesting question comes to light, the answer to which is not obvious to the author at the present time and which deserves further investigation. The question simply is: can we expect the secondary constraints (35) to (37) to be preserved by the supersymmetry transformations (10) to (13)? The variations in the secondary constraints produced by the supersymmetry transformations δ are given by the following:

(38) $$\delta(Xi.Pi) = \delta Xi.Pi + Xi.\delta Pi = \epsilon(\phi i.Pi + Xi.\dot{\phi} i)$$

(39) $$\delta(Pi.\phi i) = \delta Pi.\phi i + Pi.\delta\phi i = \epsilon\dot{\phi} i.\phi i$$

(40) $$\delta(Pi.\bar{\phi} i) = \delta Pi.\bar{\phi} i + Pi.\delta\bar{\phi} i = \epsilon(\phi i.\bar{\phi} i + Pi.(Fi - i\dot{X} i))$$

It is clear that using the equations of motion two out of the three variations associated with the transformations δ may be made to vanish, but to satisfy the third does not seem possible. As can be seen above, variation (40) for the transformations δ does not seem to want to vanish. It may be that it is asking too much for the secondary constraints to be compatible with the supersymmetry

transformations, or it could be that the model as we have described it is not truly supersymmetric. These questions need to be answered and certainly deserve further investigation at some later date.
We now move on and calculate the graded Dirac brackets of our model. However, before we do that, it is worth making a few observations about graded matrices and their inversion.

2.6.3 The graded matrix

The graded matrix, which is a matrix incorporating both odd and even variables, has the following block form:

(41)
$$\left|\begin{array}{c|c} \text{Even} & \text{Odd} \\ \hline \text{Odd} & \text{Even} \end{array}\right|$$

By way of illustration we take the simplest possible case of the graded 2 x 2 matrix for an example. Let:

(42)
$$G = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix}$$

Where $|a| = |b| = 0$, $|\alpha| = |\beta| = 1$. One may then verify that the following matrix is the inverse of G:

$$G = \begin{bmatrix} \frac{1}{a}\left(1 + \frac{\alpha . \beta}{a . b} \right) & -\frac{\alpha}{a . b} \\ -\frac{\beta}{a . b} & \frac{1}{b}\left(1 - \frac{\alpha . \beta}{a . b} \right) \end{bmatrix}$$

Thus the Grassmann elements somewhat obscure how the general case will work. As it turns out, the constraints matrix we are required to invert has a large number of zero entries, which removes much of the difficulty. However, it is useful to show how one might tackle the general case.

Assuming our graded matrix has the block form:

(43)
$$M = \left[\begin{array}{c|c} A & \alpha \\ \hline \beta & B \end{array}\right]$$

Where α and β are matrices with purely odd components, and A and B matrices of purely even, then the block inverse to M has the following form:

(44) $$M = \begin{bmatrix} C & \mu \\ \delta & D \end{bmatrix}$$

where:

(45) $$C = (A - \alpha B^{-1} \beta)^{-1}$$

(46) $$D = (B - \beta A^{-1} \alpha)^{-1}$$

(47) $$\mu = - A^{-1} \alpha (B - \beta A^{-1} \alpha)^{-1}$$

(48) $$\delta = - B^{-1} \beta (A - \alpha B^{-1} \beta)^{-1}$$

Thus we see that we are never faced with the bewildering task of inverting a matrix made up of purely odd components, but only even and even products of odd matrices.

For our purposes it is useful to see how matrices M which have the following block form look upon inversion. If we have:

(49) $$A = - A^{T} \quad , \quad \alpha = - \beta^{T} \quad , \quad B = B^{T}$$

Where the superscript 'T' represents 'transpose', then substitution of (44) into $M.M^{-1} = I = M^{-1}.M$ quickly yields the following (note that by virtue of the definition these symmetries (49) are always present in a graded constraints matrix):

(50) $$C = -C^{T} \quad , \quad \mu = \delta^{T} \quad , \quad D = D^{T}$$

Notice how the sign of the inverted odd block μ does not change upon transposition.

Calculation of any graded inverse may now be carried out using the relation (45) to (48) - as a technical point it is possible to use a software package which incorporates anticommuting variables (for example REDUCE) by employing the Cayley-Hamilton theorem which we demonstrate in appendix B. We now have all the machinery need to calculate the graded Dirac brackets of our model.

2.6.4 The graded Dirac brackets.

One of the original motivations of this thesis was to make a more detailed investigation into the nature of the graded function group; a subject which in the classical case has a strong bearing on the possible forms the Dirac bracket may take. To see how these ideas translate to the situation where the fundamental variables of the theory are Z_2 graded, is a worthy topic of investigation because of its possible bearing on the question of dealing with constraints in supersymmetric theories. Problems concerning graded constraints have dogged these models from the outset [5,29,32], and so discovering exactly under what circumstances the reduction of graded phase spaces may be carried out might prove to be useful. Although a start has been made to answering some of these questions, it has become clear that a far better understanding of the nature of the supermanifold is neccessary before one can hope to fully solve these problems. Some work has been done by Casalbuoni [18] on the graded Dirac bracket, however the reduction properties the bracket should possess were not demonstrated there. There seems little doubt that the graded Dirac bracket is an example of the 'generalised graded Poisson bracket' introduced in § 2.1.7, but further work is required to fully determine its true nature. Below, we use the definition given by Casalbuoni in [18], to calculate the graded Dirac brackets of the fundamental variables in our model. We will see that the inclusion of the constraint (17) produces a substantial departure in the form of the brackets from the corresponding expressions in J.Barcelos-Neto et al's paper [6].

The functional form of the Dirac bracket

Rather than repeating the construction of [18] here, we simply state that one can construct a bracket, analogous in form to the standard Dirac bracket, using real and Grassmann variables as the fundamental variables of the theory. It has the following form:

$$\{ A,B \}^* := \{ A,B \}' - \{ A,\Phi_a \}'(C^{-1})_{ab} \{ \Phi_b,B \}' \quad (51)$$

Where A and B are functions defined over some graded phase space, and the $\{\Phi_a\}$ are a set of ordered graded constraints, even constraints coming first and odd, with the indices a,b running over both odd and even functions in the manner introduced in § 2.1.6. The matrix $(C^{-1})_{ab}$ is the inverse of the matrix of constraints, and is defined:

(52) $$(C^{-1})ab := (\{ \Phi a,\Phi b \}') = (Cab)^{-1}$$

The ordering of the odd and even sectors of the indices a,b ensures that the matrix (Cab) and its inverse ($C^{-1}ab$) have the form (50) and therefore make sense as graded matrices. To calculate the Dirac bracket of the fundamental variables of the model, the matrix of constraints and its inverse must be determined. We do this below.

2.6.5 The matrix of constraints.

The constraints of our theory {Φa} form a 6-tuple, which we order as follows:

(53) $$(\Phi a) = (Xi.Xi, Xi.Pi; \phi i.Xi, \phi i.Pi, \bar{\phi} i.Xi, \bar{\phi} i.Pi)$$

with even constraints leading. Of the thirty six contibutions to the constraints matrix, happily fourteen are outright zero. These come from, for example, the graded brackets between the following constraints:

(54) $$\{ Xi.Xi, Xj.Xj \}' = 0 \quad , \quad \{ \phi i.Xi, \phi i.Pi \}' = 0$$

and so on (where we have used the fundamental graded Poisson bracket relations (31)). A further twelve brackets give purely constraint type terms on the R.H.S. These are for example:

(55) $$\{ Xi.Xi, Pi.\phi i \}' = 2Xi.\phi i \quad , \quad \{ Xi.Xi, Pi.\bar{\phi} i \}' = 2Xi.\bar{\phi} i$$

Which we may set to zero by virtue of the constraints themselves. Finally we have the remaining ten non-trivial contributions which are the following:

(56) $$\{ Xi.Xi, Xj.Pj \}' = 2Xi.Xi \quad , \quad \{ \phi i.Xi, \bar{\phi} j.Xj \}' = iXi.Xi$$

(57) $$\{ \phi i.Pi, \bar{\phi} j.Xj \}' = \phi i.\bar{\phi} i + iXi.Pi = \phi i.\bar{\phi} i \ ,$$

(58) $$\{ \phi i.Pi, \phi j.Pj \} = iPj.Pj$$

Bearing in mind that the graded Poisson bracket has the symmetries given earlier, we now have all the contributions required to form the matrix. Thus we have:

(59)

$$C_{ab} = \begin{pmatrix} 0 & 2Xi.Xi & 0 & 0 & 0 & 0 \\ -2Xi.Xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & iXi.Xi & -\bar{\phi}i.\phi i \\ 0 & 0 & 0 & 0 & \bar{\phi}i.\phi i & iPi.Pi \\ 0 & 0 & iXi.Xi & \bar{\phi}i.\phi i & 0 & 0 \\ 0 & 0 & -\bar{\phi}i.\phi i & iPj.Pj & 0 & 0 \end{pmatrix}$$

where the symmetries of the graded bracket ensure that this matrix (59) is of the sort type described by (49). Because of the large number of zeros it is now a simple task to calculate the required inverse. Calling:

(60) $$\Sigma = (\bar{\phi}i.\phi i) - (Xi.Xi)(Pj.Pj)$$

we have for the inverse C ab:

(61)

$$C\ ab = \begin{pmatrix} 0 & \frac{-1}{Xi.Xi} & 0 & 0 & 0 & 0 \\ \frac{1}{Xi.Xi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{iPj.Pj}{\Sigma} & -\frac{\bar{\phi}i.\phi i}{\Sigma} \\ 0 & 0 & 0 & 0 & \frac{\bar{\phi}i.\phi i}{\Sigma} & \frac{iXi.Xi}{\Sigma} \\ 0 & 0 & \frac{iPj.Pj}{\Sigma} & \frac{\bar{\phi}i.\phi i}{\Sigma} & 0 & 0 \\ 0 & 0 & -\frac{\bar{\phi}i.\phi i}{\Sigma} & \frac{iXi.Xi}{\Sigma} & 0 & 0 \end{pmatrix}$$

We see that because of the lack of odd terms the constraints matrix has simply decoupled into a direct sum of the two even sectors. We can now employ (51) and calculate the graded Dirac brackets for the fundamental variables of our theory. For the purposes of the calculation it is most convenient to form the following four graded 6-tuples obtained by taking the graded Poisson brackets between the fundamental variables of the theory and the vector (53) made up of the constraints (Φa). They are:

(62) $$Qi := \{ Xi, \Phi \}' = (0, Xi, 0, \phi i, 0, \bar{\phi} i)$$

(63) $$Ui := \{ Pi, \Phi \}' = (-2Xi, -Pi, -\phi i, 0, -\bar{\phi} i, 0)$$

(64) $$\Gamma i := \{ \phi i, \Phi \}' = (0, 0, 0, 0, Xi, Pi)$$

(65) $$\Gamma i := \{ \phi i, \Phi \}' = (0, 0, Xi, Pi, 0, 0)$$

Where in this notation we have suppressed the constraint index 'a' in (53) by making Φ a vector. The 'i' index is of course just the normal i = 1,2,3, label from before. Also we may define a 'Grassmann transpose', denoted τ, of these row vectors as the column vectors whose components are:

(66) $$(Qi^{\tau})a = -(Qi)a \quad , \quad (Ui^{\tau})a = -(Ui)a$$
$$(\Gamma i^{\tau})a = (\Gamma i)a \quad , \quad (\Gamma i^{\tau})a = (\Gamma i)a$$

In this notation we may now write the graded Dirac bracket of the fundamental variables of the theory as follows:

(67) $$\{ Xi,Xj \}^* = - Qi.C^{-1}.Qj$$

(68) $$\{ Xi,Pj \}^* = \delta ij - Qi.C^{-1}.Uj$$

...and so on. Substituting (62) gives us explicitly the ten graded Dirac brackets:

(69) $$\{ Xi,Xj \}^* = i\frac{(Xk.Xk)}{\Sigma}(\bar{\phi} i.\phi j + \phi i.\bar{\phi} j)$$

(70) $$\{ Xi,Pj \}^* = \delta ij - \frac{Xi.Xj}{(Xk.Xk)} + \frac{(\bar{\phi} k.\phi k)}{\Sigma}(\bar{\phi} i.\phi j - \phi i.\bar{\phi} j)$$

(71) $$\{ Xi,\phi j \}^* = - \frac{1}{\Sigma}((\phi k.\bar{\phi} k)\phi i.Xj + i(Xk.Xk)\phi i.Pj)$$

(72) $$\{ Xi,\phi j \}^* = \frac{1}{\Sigma}((\bar{\phi} k.\phi k)\bar{\phi} i.Xj - i(Xk.Xk)\bar{\phi} i.Pj)$$

(73) $$\{ Pi,\phi j \}^* = \frac{1}{\Sigma}(i(Pk.Pk)\phi i.Xj - (\bar{\phi} k.\phi k)\phi i.Pj)$$

(74) $$\{ Pi,\phi j \}^* = \frac{1}{\Sigma}(i((Pk.Pk)\bar{\phi} i.Xj + (\bar{\phi} k.\phi k)\bar{\phi} i.Pj)$$

(75) $$\{ Pi,Pj \}^* = \frac{1}{(Xk.Xk)}(Pi.Xj -Xi.Pj) + i\frac{(Pk.Pk)}{\Sigma}(\bar{\phi} i.\phi j + \phi i.\bar{\phi} j)$$

(76) $$\{ \phi i,\phi j \}^* = 0$$

(77) $$\{ \bar{\phi} i,\phi j \}^* = 0$$

(78) $$\{ \phi i,\bar{\phi} j \}^* = i\delta ij - \frac{i}{\Sigma}((Pk.Pk)Xi.Xj + (Xk.Xk)Pi.Pj) + \frac{(\bar{\phi} k.\phi k)}{\Sigma}(Pi.Xj - Xi.Pj)$$

These are all the graded Dirac brackets of our model.

2.6.6 Concluding comments.

In conclusion to this section we make the following remarks. It is clear that this model is different form that of [6] and [56] throught the inclusion of the 'extra constraint' $\bar{\phi}k.Xk = 0$, an addition which results in the Dirac brackets of the theory having further terms not present in these papers. The full ramifications of adding this constraint are not yet clear, as it appears that by so doing, the supersymmetry of the model is being put into question. And yet the inclusion this constraint appears to be the natural thing to do. This is a question which needs further investigation. There are other avenues along which additional studies might proceed, however. This model could be an ideal candidate on which to try out the group theoretical approach to quantisation in non-trivial background. An investigation along these lines would first seek to determine what super group is relevant to the S.P.S. of this model. The algebra of this group would have O(3) as a bosonic sub-algebra - this is because of the form of the constraints matrix, which is O(3) invarient in the bosonic sector. Also, finding a new set of variables which reduced the graded Dirac brackets (69) through (78) to graded Poisson brackets similar to the classical example given in ∮ 1.3.8, and thus locally reducing the graded phase space, is another interesting path of inquiry. One can imagine useing the non-trivial topology of the sphere to try and induce a 'twist' into the quantum theory, similar to the classical twisted representations studied in [ ]*. These are potentially highly rewarding areas to investigate, from the point of view of understanding the underlying structure of all graded theories, and we feel represent worthy subjects for further research.

* The author wishes to thank Professor Isham and Dr E. Corrigan for their help with these ideas.

Appendix A

Further useful properties of the graded Poisson bracket include the following [18]:

(a) The super Jacobi identities:

(1) $\{ E1,\{ E2,E3 \} \} + \{ E2,\{ E3,E1 \} \} + \{ E3,\{ E1,E2 \} \} = 0$

(2) $\{ O1,\{ O2,O3 \} \} + \{ O2,\{ O3,O1 \} \} + \{ O3,\{ O1,O2 \} \} = 0$

(3) $\{ E1,\{ E2,O1 \} \} + \{ E2,\{ O1,E1 \} \} + \{ O1,\{ E1,E2 \} \} = 0$

(4) $\{ E1,\{ O1,O2 \} \} + \{ O1,\{ O2,E1 \} \} - \{ O2,\{ E1,O1 \} \} = 0$

(b) The product rules:

(5) $\{ E1.E2,E3 \} = E2.\{ E1,E3 \} + \{ E1,E2 \}.E3$

(6) $\{ O1.O2,E1 \} = O1.\{ O2,E1 \} + \{ O1,E1 \}.O2$

(7) $\{ O1.E1,E2 \} = O1.\{ E1,E2 \} + \{ O1,E2 \}.E1$

(8) $\{ E1.E2,O1 \} = E1.\{ E2,O1 \} + \{ E1.O1 \}.E2$

(9) $\{ E1.O1,O2 \} = E1.\{ O1,O2 \} - \{ E1,O2 \}.O1$

(10) $\{ O1.O2,O3 \} = O1.\{ O2,O3 \} - \{ O1,O3 \}.O2$

Where we have $|Ei| = 0$, $i = 1,2,3$, and $|O\alpha| = 1$ for $\alpha = 1,2,3$.

Appendix B

The Cayley-Hamilton theorem

Let A be some n X n square matrix over some field k:

$$A = \begin{matrix} a11 & \cdots & a1n \\ \vdots & & \vdots \\ an1 & \cdots & ann \end{matrix}$$

then the characteristic matrix C of A is defined as:

(1) $$C = t\,1n - A$$

where $t \in R$ is a real parameter and 1n is the n X n identity matrix. The characteristic polynomial Cp of A is defined as:

(2) $$Cp = Det(C)$$

The Cayley-Hamilton theorem states that every matrix is a zero of its characteristic polynomial. This finds elegant expression in terms of the following determinant [28]. For some n X n matrix A:

$$Det \begin{vmatrix} A^n & A^{n-1} & \cdots & A & 1 \\ Tr(A^n) & Tr(A^{n-1}) & \cdots & Tr(A) & n \\ Tr(A^{n-1}) & \cdots & \cdots & n-1 & 0 \\ Tr(A^{n-2}) & \cdots & n-2 & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ Tr(A) & 1 & 0 \cdots & 0 & 0 \end{vmatrix} \equiv 0$$

This is a useful device to use in conjunction with an algebraic package like REDUCE, because it gives expression to the inverse of A as a power series. For example in the case of n = 4 we have:

$$Det(A)A^{-1} = -A^3 + Tr(A)A^2 - \frac{1}{2}((Tr(A))^2 - Tr(A^2))A - \frac{1}{6}(3(Tr(A^2)Tr(A) - 2Tr(A^3) - (Tr(A))^3)1$$

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