Unitary models in two dimensions

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UNITARY MODELS IN TWO DIMENSIONS

by

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CONTENTS

ABSTRACT

"Unitary Models in Two Dimensions"

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Unitary models in two dimensions are classes of low dimensional theories which provide us with a convenient theoretical laboratory for studying various aspects of the theory of elementary particles. In this thesis, purely bosonic $U(N)$ sigma models with the Wess-Zumino-Witten (WZW) term in two-dimensional Euclidean space and the supersymmetric (Susy) $U(N)$ σ models with and without this term are discussed. Particular attention is paid to the classical solutions of the equations of motion of these models. Due to the integrability of these models, we can associate with them a Lax-pair formalism. We observe that solutions of the Lax-pair equations of the $U(N)$ σ model provide us with solutions of the $U(N)$ σ model with the WZW-term. This is also the case for solutions of the Susy $U(N)$ σ model with the WZW-term which can be constructed from solutions of the Lax-pair equations of the Susy $U(N)$ σ model. We present also some explicit solutions of the Susy $U(N)$ σ model without the WZW-term.

Many properties of the constructed solutions for both the purely bosonic and Susy models are explored. In particular, we calculate the values of the action for some solutions and study the stability properties of these solutions and find that all the constructed solutions of these models correspond to the saddle points of the action. Finally we consider the linearized fermion equations in the fixed background of a bosonic field. Special attention is paid to the case when the background field is given by a solution of the $U(N)$ σ model with and/or without the WZW-term. Some classes of solutions of this problem are presented and their properties are discussed. We observe that a class of these solutions is related to the components of the energy-momentum tensor of the purely bosonic σ model and prove that some of these solutions are traceless.

DECLARATION

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1986 and December 1989. This material has not been submitted previously for any degree in this or any other university.

No claim of originality is made for chapter 2; the work in chapter 3, 4 and 5 is claimed as original, except where authors have been specifically acknowledged in the text.

The work in chapter 3 and chapter 4 has been published in two papers by the author in collaboration with B. Piette and W. J. Zakrzewski [1,2]; while the work in chapter 5, undertaken by the author with W. J. Zakrzewski, is available as a Durham University preprint [3], and has been submitted for publication to Zeitschrift für Physik C.

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1. INTRODUCTION.

It is generally believed that non-abelian gauge theories are likely to play an important role in any field theoretical description of the theory of elementary particles. For example, weak and electromagnetic interactions are described by such a theory, and it is generally felt that this is also the case for strong interactions. These theories, in the case of an $SU(2)$ symmetry group, are defined in terms of a Lagrangian density L given by

$$L = \text{Tr}\, F_{\mu\nu} F_{\mu\nu}, \tag{1.1}$$

with $\mu, \nu = (1, 2, 3, 4)$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{1.2}$$

and where A_μ is a vector function of Euclidean four-dimensional space-time with values in $SU(2)$.

Most quantities in non-abelian gauge theories are given in terms of functional integrations

$$\int D[A_\mu]\; e^{-\int d^4x\, L(A_\mu)}\, O(A_\mu). \tag{1.3}$$

Hence one of the major difficulties in making progress with these theories is the lack of understanding of how to perform many of these integrations. One approach is to attempt to calculate these integrals numerically. However, the results of such numerical attempts are encouraging, but unavoidably involve various approximations, making the results inconclusive. The expression in equation (1.3) involves the fields defined in Euclidean space. In fact the original theory is defined in Minkowski space, but then the expressions like (1.3) involve terms which oscillate very wildly. Thus to improve the calculability of the expressions like

(1.3) we continue the original theory to Euclidean space and so consider (1.3) in Euclidean space where the contribution of large fields A_μ is exponentially suppressed. Hence all discussion in this thesis will be of the fields, their equations of motion etc. all in Euclidean space.

When one tries to calculate functional integrations like (1.3) analytically, one finds that the only viable approach available in many cases is based on an expansion around the stationary points of the action of the theory and then perturbation theory of the resultant effective theory. Thus one has to determine first the stationary points of the action. They are provided by seeking solutions of the Euclidean equations of motion (the Euler-Lagrange equations) of the theory. For this reason we shall perform all our studies in this thesis in Euclidean space. The Euler-Lagrange equations corresponding to (1.1) are given by

$$D_\mu F_{\mu\nu} \equiv \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0. \tag{1.4}$$

These equations, when written in terms of the gauge potential A_μ, are second order highly nonlinear partial differential equations. However, due to the Bianchi identity

$$D_\mu \, {}^*F_{\mu\nu} = 0, \tag{1.5}$$

where

$${}^*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{1.6}$$

and $\epsilon_{\mu\nu\alpha\beta}$ is the totally anti-symmetric 4-tensor (with $\epsilon_{1234} = +1$). It is well known that a subclass of solutions of the Euler-Lagrange equations (1.4) is provided by the solutions of the first order equations

$$F_{\mu\nu} = \pm \, {}^*F_{\mu\nu}, \tag{1.7}$$

known as the "self-duality" equations. These equations can be thought of as resulting from requiring the Lagrangian density to be equal to the modulus of the topological charge density of the theory, that is, imposing the additional constraint

$$L = \pm \mathcal{Q}, \tag{1.8}$$

where

$$\mathcal{Q} = \mathrm{Tr}\ F_{\mu\nu}\, {}^{*}F_{\mu\nu} \tag{1.9}$$

is the topological charge density.

The most interesting solutions of these equations are those for which the action is finite, as it is only for them that the perturbation theory of fluctuations around them can be set up. All finite action solutions of equations (1.7) have been implicitly determined by Atiyah et al. [4]. In the case of the plus (minus) sign in (1.7), the corresponding finite action solutions are called instantons (anti-instantons). They correspond to local minima of the action. Hence, these solutions are stable under small fluctuations.

The question which arises now is whether there are any further solutions of (1.4) which are of finite action and which are not solutions of (1.7)? This is a difficult problem and so far nobody has found such non-instanton solutions. Nevertheless if they exist they presumably would also have to be included in any stationary point calculation of (1.3). Moreover, also the task of calculating the fluctuations about the instanton solutions of the equations of motion has turned out to be a hard mathematical problem.

Due to this complexity of non-abelian gauge theories in four dimensions, some people have turned their attention to models in lower dimensions which exhibit some features of the four-dimensional theory, but where the relevant calculations are simpler to perform. In two (Euclidean) dimensions, several classes of such models have been proposed, namely the $O(N)$ nonlinear sigma models [5], the principal chiral models [6], the $\mathbb{C}P^{N-1}$ nonlinear σ models [7-10] and its non-abelian generalizations – the complex Grassmannian models [11]. All of these models are good examples of field theories with a non-trivial dynamical content which has a geometric origin and, as is well known, they are in many aspects closely related to non-abelian gauge theories. Some of the properties of these models which they share with non-abelian gauge theories are: at the classical level, their geometrical nature, the non-trivial topological structure of the space of field configurations and their conformal invariance, and at the quantum level, the phenomena of dynamical mass generation and of asymptotic freedom.

The interest in two-dimensional sigma models has considerably increased over the last few years as the models have become a laboratory for testing many ideas in particle physics. (For more details on this we refer the reader to the recent book by Zakrzewski [12]). Also it has been realised that the origin of many properties of the strings and superstrings is intrinsingly tied to the two-dimensional nature of their manifold and that in the low energy limit the physics of the strings can be described by an effective sigma model. In addition, being governed by a nonlinear Lagrangian, the fields of σ models are being used to test many ideas associated with monopole or skyrmion scattering. On the other hand, from the purely mathematical point of view σ models provide interesting examples of harmonic maps, and as such are clearly interesting in themselves. For such low dimensional theories, it is important to understand their classical properties as completely as possible and in particular to find all possible stationary points of the action, *i.e.* the classical solutions of the equations of motion.

One of the most attractive features of these models is that classically they provide examples of integrable systems in two dimensions. Namely, it is known that the nonlinear equations of motion are precisely the compatibility conditions for a certain linear system of first-order partial differential equations (Lax-pair) containing one free parameter. Thus they possess an infinite number of local [5,11,13] and/or nonlocal [14] conservation laws. These conserved quantities generate an infinite dimensional algebra of dynamical symmetries (the Kac-Moody algebra)

[15]. Among the models which possess such properties are the principal chiral models [6] taking their values in Lie groups and the symmetric space σ models defined on the Riemannian symmetric space [11,13] G/H, where G is a Lie group and H a certain subgroup. The latter includes the $O(N)$ σ model [5], the $\mathbb{C}P^{N-1}$ model [7], and the complex Grassmannian models [11].

The analogies between nonlinear sigma models and non-abelian gauge theories suggest that they can and should be extended to incorporate fermionic matter fields. One natural way of doing this is by coupling the bosons and the fermions supersymmetrically [16,17]. It has been proved that the above mentioned classical integrability of the model continues to hold in that case too [18]. On the other hand, for the $\mathbb{C}P^{N-1}$ model [19], and the Grassmannian model [20], the integrability is preserved when bosons and fermions are coupled minimally rather than supersymmetrically. Moreover, the general criteria for classical integrability of the principal chiral models with fermions was given by Abdalla and Forger [21].

Most of the studies of various sigma models have been concerned with the cases when the target manifolds were either group spaces or some coset spaces. Various properties of such models have been studied and many finite action classical solutions derived and their properties examined in detail. In chapter 2 we present a detailed formulation of various types of purely bosonic σ models in two Euclidean dimensions and also of their Susy extensions. The defining equations and the general instanton and non-instanton solutions are presented for each type of both purely bosonic and Susy models. Many properties of these solutions are discussed and it is shown that any solution of the purely bosonic $\mathbb{C}P^{N-1}$ model, and Grassmannian model which is neither instanton nor anti-instanton in nature is necessarily unstable under small perturbations. However, all non-trivial solutions of the principal chiral model are shown to be unstable. Also, a useful reformulation of these models using projectors is presented. Moreover, in this chapter we discuss also the equations of a fermion in the fixed background of a bosonic $\mathbb{C}P^{N-1}$ and Grassmannian solution.

Recently, motivated by some observations from the geometric properties of the strings, the interest has shifted towards sigma models with the so-called WZW-term [22]. This term was first introduced by Wess and Zumino [23] who, in the context of the chiral theory of pions, showed that it represents the effects of flavour anomalies. More recently it was reintroduced by Polyakov and Wiegmann [24], and Witten [25], who studied various effects associated with its presence in different models. In addition, Witten [22] has pointed out that as this term breaks some reflection symmetries of the manifold, it should be included in any low energy approximation to QCD based on Skyrme ideas [25] in which the original Skyrme model possesses too many symmetries. Recently, the effects associated with the WZW-term have also been studied from a more geometrical point of view. In fact, Braaten et al. [26] have shown that the additional contribution to the equations of motion from the WZW-term can be interpreted as the contribution from the "torsion" of the target manifold space. In this thesis, we will not need to use this interpretation as we will be concerned with trying to solve the equations of motion of both the purely bosonic and Susy $U(N)$ sigma models with the WZW-term and luckily we do not need to have to solve the unitarity constraints. As the purely bosonic and Susy two-dimensional σ models are known to be integrable, recently it has been shown that the purely bosonic $U(N)$ σ models with the WZW-term [27] and their Susy extension models [28] are also integrable. Thus one can associate with them a Lax-pair formalism and so they possess an infinite number of conservation laws.

In chapter 3 we discuss the two-dimensional, purely bosonic, $U(N)$ σ models with the WZW-term. From now on we will refer to these models as the "WZW-σ models". We construct explicit finite action classical solutions of these models. We also show how to relate solutions of the WZW-σ model to the solutions of the

Lax-pair equations for the corresponding σ model without the WZW-term. Also, we demonstrate that the solutions of the $U(N)$ σ model discussed in chapter 2 and the solutions of the WZW-σ model are related and can be derived from each other. Then we describe the construction of these solutions and study some of their properties. In particular, we compute the value of the action of the WZW-σ model corresponding to some of these solutions, as we show that the action of a solution of the WZW-σ model is finite if the action of the corresponding solution of the $U(N)$ σ model is also finite. In addition, we also study the stability properties of these solutions under small fluctuations around them and show that all these solutions have the same number of negative modes as the corresponding solutions of the model without the WZW-term. Thus they are the saddle points of the action and so the solutions are unstable.

The real, more physical models, should include fermions and so it would be interesting to check how many of the properties found for the purely bosonic $U(N)$ σ model do survive the addition of fermions. A convenient way of including fermions into σ models consists of extending these models to become Susy. Chapter 4 therefore deals with studying the Susy extension of the $U(N)$ σ models with and without the WZW-term in two dimensions. As we use these terms very frequently in this thesis we will refer to the Susy σ models without the WZW-term as the "Susy σ models" and those with this term as the "Susy WZW-σ models".

In chapter 4 we define these models and derive their equations of motion. Then we construct general Susy solutions of the Susy σ models. The procedure of this construction is similar to the one used in the purely bosonic σ model [29]. We also derive solutions of the Susy WZW-σ model from the solutions of the Lax-pair problem for the Susy σ model. The rest of the chapter studies some properties of these solutions. We calculate the value of the action for some of these solutions and show that they are related to those of the purely bosonic model. Precisely, they are given by the laplacian of a logarithm of a function depending only on the bosonic part of the theory, and therefore, we see that there is no fermionic contribution to the action. Finally we study the stability properties of these solutions and find that all solutions of both the Susy σ models and the Susy WZW-σ models have the same number of negative modes and that these negative modes are exactily the same as those of the purely bosonic model. Thus we find that the addition of fermions to the purely bosonic $U(N)$ σ models does not introduce any further instabilities but that it leads only to the appearance of further zero modes.

The Susy generalisation of σ models in two-dimensional Euclidean space provides interesting examples of boson-fermion interactions. In chapter 5 we investigate classical solutions of a boson-fermion model based on the Susy WZW-σ models. In particular we look at the linearized fermion equations with a fixed background bosonic field. We construct some classes of solutions of this system for which the background field is a solution of the $U(N)$ σ model with and/or without the WZW-term. Then, as usual, we study some properties of these solutions. In particular, we compute the energy-momentum tensor of the purely bosonic $U(N)$ σ model and show that its components are related to a class of the obtained solutions. In addition, we prove that some classes of these solutions are traceless.

We finish this thesis with chapter 6 which is devoted to comments and conclusions. We summarise the obtained results of this thesis, and discuss briefly their possible applications.

2. Formulation of Sigma Models in Two Dimensions.

2.1 Pure Bosonic Sigma Models

In this section, we introduce the pure bosonic sigma models in two-dimensional Euclidean space, where the dynamical variables are bosonic fields, *i.e.* commuting objects. We start with the $U(N)$ σ models, also called "principal chiral models". These models, first discussed by Zakharov and Mikhailov [6], can be defined in terms of the Lagrangian density

$$L = \frac{1}{4} \operatorname{Tr} \partial_\mu Q^\dagger \partial_\mu Q, \tag{2.1}$$

where the matrix field $Q(x_1, x_2)$ belongs to the group of $N \times N$ unitary matrices $U(N)$, and $\mu = x_1,\, x_2$ and '$\dagger$' denotes the hermitian conjugation. As $Q \in U(N)$, we see that $Q^\dagger = Q^{-1}$ and so that Q satisfies the constraint

$$Q^\dagger Q = Q Q^\dagger = 1. \tag{2.2}$$

The above-given Lagrangian is invariant under

$$Q \longrightarrow Q' = h^{-1} Q\, k, \tag{2.3}$$

where h and k are constant unitary matrices. Hence, the real invariance of the model is $U(N) \times U(N)$.

The Euler-Lagrange equations for this model (classical equations of motion) follow, as usual, from the condition that the action

$$S = \int L\, d^2x \tag{2.4}$$

is extremal and so are given by

$$\partial_\mu (Q^\dagger \partial_\mu Q) = 0, \tag{2.5}$$

together with the constraint (2.2). We seek solutions of these equations for which the action is finite. This condition comes from the requirement of quantisation in terms of path integrals. The condition of the finiteness of the action effectively compactifies the two-dimensional Euclidean space thus allowing us to take over the results derived in the case when the basic space is given by S^2. This compactification introduces topology and is directly responsible for the discrete values of the action.

As demonstrated by Din and Zakrzewski [30], it is convenient to change the Euclidean variables (x_1, x_2) to holomorphic and antiholomorphic variables:

$$x_{\pm} = x_1 \pm i x_2. \tag{2.6}$$

In terms of these variables the Lagrangian density can then be rewritten as

$$L = \frac{1}{2} \operatorname{Tr} (\partial_+ Q^\dagger \, \partial_- Q + \partial_- Q^\dagger \, \partial_+ Q), \tag{2.7}$$

where $\partial_\pm$ denotes partial derivatives with respect to $x_\pm$, and the equations of motion become

$$\partial_+(Q^\dagger \partial_- Q) + \partial_-(Q^\dagger \partial_+ Q) = 0. \tag{2.8}$$

If the field Q is restricted further by $Q = Q^\dagger$, then it corresponds to a grassmannian field. Before we introduce the grassmannian models, we present the so-called $O(N)$ σ models [5,31]. The basic fields of these models are the N-component real unit vectors q^a ; $a = 1, \cdots, N$ which are functions of the two-dimensional space locally parametrised by x_1 and x_2. The Lagrangian density for this theory is given by

$$L = \partial_\mu q \cdot \partial_\mu q, \qquad \text{where} \quad \mu = x_1, x_2 \tag{2.9}$$

together with the constraint

$$q \cdot q = 1. \tag{2.10}$$

The Euler-Lagrange equations corresponding to (2.9) are

$$\partial_\mu \partial_\mu q + (\partial_\mu q \cdot \partial_\mu q) q = 0. \tag{2.11}$$

Again the two-dimensional space is compactified by requiring the solutions to be of finite action.

Belavin and Polyakov [32], and Woo [33] have shown that the $O(N)$ σ models have stable instanton solutions only for the case when $N = 3$. The stability of the $O(3)$ instantons can be shown to be based on topology as the solutions are characterized by different values of a conserved topological number. However, for $N > 3$, there is no corresponding non-trivial topological quantity for the $O(N)$ solutions, and Din and Zakrzewski [31] have shown that its absence makes all non-trivial solutions unstable.

As we mentioned above, a subclass of the $U(N)$ models consists of those for which $Q = Q^\dagger$. They correspond to embeddings of grassmannian models [11]. To introduce these models, Zakrzewski [34] considered their field taking values in the complex Grassmann manifold $G(M, N)$, the manifold of M-dimensional complex sub-spaces $\mathbb{C}^M$ in $\mathbb{C}^N$, passing through the origin. This manifold can be written as a quotient space

$$G(M, N) = \frac{U(N)}{U(M) \times U(N-M)}. \tag{2.12}$$

Let $Q(x_1, x_2)$ be an element of $U(N)$ which can be decomposed as

$$Q = (Z, Y), \tag{2.13}$$

where

$$Z = (Z_1 \cdots Z_M), \qquad Y = (Z_{M+1} \cdots Z_N) \tag{2.14}$$

in which each $Z_i \, (i = 1, \cdots, N)$ is an N-component column vector. Then the unitarity of Q implies that the vectors Z_i are orthonormal to each other

$$Z_i^\dagger \, Z_k \;=\; \delta_{ik}. \tag{2.15}$$

The grassmannian models are defined by considering the $N \times M$ matrix Z (the analogue of q^a vector of the $O(N)$ σ models), as a dynamical variable with $1 \leq M < N$, together with the constraint

$$Z^\dagger \, Z \;=\; 1_M, \tag{2.16}$$

where 1_M denotes the $M \times M$ unit matrix. The Lagrangian density of the model is defined as

$$L \;=\; \mathrm{Tr}\; (D_\mu Z)^\dagger \, D_\mu Z, \tag{2.17}$$

where the covariant derivative D_μ is defined by

$$D_\mu \Psi \;=\; \partial_\mu \Psi \;-\; \Psi \, A_\mu \tag{2.18}$$

in which A_μ is a composite gauge potential defined by

$$A_\mu \;=\; Z^\dagger \partial_\mu Z, \qquad A_\mu^\dagger \;=\; -A_\mu. \tag{2.19}$$

The Lagrangian density (2.17) is invariant under global $U(N)$ transformations

$$Z \quad \longrightarrow \quad Z' \;=\; h \, Z, \tag{2.20}$$

where h is constant unitary matrix, and also under local $U(M)$ transformations

$$Z \quad \longrightarrow \quad Z' \;=\; Z \, K, \tag{2.21}$$

where $K = K(x_1, x_2) \in U(M)$.

The equations of motion of the grassmannian models are given by

$$D_\mu D_\mu Z + Z\,(D_\mu Z)^\dagger\, D_\mu Z = 0, \tag{2.22}$$

together with the constraint (2.16).

In the special case when $M = 1$, the grassmannian model defined above is called the $\mathbb{C}P^{N-1}$ model, as the complex Grassmann manifold in this case is equivalent to the complex projective space. The $\mathbb{C}P^{N-1}$ model, as first discussed by Eichenherr [7], Cremmer and Scherk [8], Golo and Perelomov [9] and D'Adda et al. [10], is therefore described by an N-component complex vector, together with the normalization condition $Z^\dagger\, Z = |Z|^2 = 1$.

The $\mathbb{C}P^{N-1}$ model possesses abelian $U(1)$ symmetry, and its composite gauge field A_μ is a purely imaginary function of the two Euclidean dimensions. The simplest $\mathbb{C}P^{N-1}$ model corresponds to $N = 2$. Moreover, as D'Adda et al. [10] observed, the $\mathbb{C}P^1$ model is equivalent to the $O(3)$ σ model. To exhibit this equivalence, it suffices to take

$$q^i = Z^\dagger\, \sigma_i\, Z, \tag{2.23}$$

where Z is now a two-component $\mathbb{C}P^1$ field and σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then simple algebraic manipulations show that

$$(D_\mu Z)^\dagger\, D_\mu Z \longrightarrow \partial_\mu q\, \partial_\mu q,$$

$$\text{and} \qquad |Z|^2 = 1 \longrightarrow q \cdot q = 1.$$

Hence the two theories are classically equivalent.

In terms of the complex variables $x_\pm$, the Lagrangian density (2.17) can be rewritten as

$$L = 2\,\mathrm{Tr}\,[(D_+Z)^\dagger D_+Z + (D_-Z)^\dagger D_-Z], \tag{2.24}$$

where

$$D_\pm = \partial_\pm - Z^\dagger \partial_\pm Z \tag{2.25}$$

and the corresponding equations of motion (2.22) become

$$D_-D_+Z + Z\,(D_+Z)^\dagger (D_+Z) = 0, \tag{2.26}$$

or equivalently

$$D_+D_-Z + Z\,(D_-Z)^\dagger (D_-Z) = 0. \tag{2.27}$$

At this stage, by analogy with

$$F_{\mu\nu} = \pm\, {}^*F_{\mu\nu}$$

of non-abelian gauge theories, a subclass of solutions of (2.26) is provided by the solutions of the self-duality (and antiself-duality) equations

$$D_\pm Z = 0. \tag{2.28}$$

It is easy to show that all solutions of (2.28) are also solutions of (2.26) or (2.27) but, being first order in derivatives, are easier to solve. In fact, as in non-abelian gauge theories, equations (2.28) are associated with the existence of a topological charge density

$$\mathcal{Q} = 2\,\mathrm{Tr}\,[(D_+Z)^\dagger D_+Z - (D_-Z)^\dagger D_-Z]. \tag{2.29}$$

Moreover, they can be considered as having come from the requirement that

$$L = \pm\,\mathcal{Q}. \tag{2.30}$$

The finite action solutions of the self-duality (antiself-duality) equations

$D_- Z = 0$ ($D_+ Z = 0$) are known as the instanton (anti-instanton) solutions. These solutions, in the $\mathbb{C}P^{N-1}$ case have been given by D'Adda et al. [10]; and for the general grassmannian models they have been given by Macfarlane [35]. In the $\mathbb{C}P^{N-1}$ case, as discussed by D'Adda et al. [10], the general instanton solution is given by

$$Z = \frac{f(x_+)}{|f(x_+)|}, \tag{2.31}$$

where f is an N-component vector whose components are holomorphic functions, *i.e.* depending on x_+ only. For the general anti-instanton solution, the components of f are antiholomorphic functions, *i.e.* depending on x_-. The finiteness of the action imposes conditions on the components of f; namely, they have to be rational functions of their argument. However, gauge invariance shows that it is sufficient to consider only polynomial components of f (with no overall factors). Moreover, it can be shown that

$$S = 2\pi k, \tag{2.32}$$

where k is the degree of the polynomial components of f, and is called the instanton number. Notice that all instanton (anti-instanton) solutions mentioned above are stable. This stability is guaranteed by the existence of the topological charge.

In analogy with (2.31), an instanton solution of the general grassmannian model is obtained from a set of M linearly independent holomorphic vectors $f_1 \cdots f_M$, properly orthonormalised in order to satisfy the constraint (2.16). Following Din and Zakrzewski [30], this orthonormalisation can be performed by considering a $N \times M$ matrix $\check{Z}$ consisting of $f_1 \cdots f_M$ and defining a $M \times M$ hermitian matrix $M = \check{Z}^\dagger \check{Z}$. This matrix M is positive definite and invertible because of the linear independence of the vectors. Hence $M^{\frac{1}{2}}$ and $M^{-\frac{1}{2}}$ exist and are unique. Then, as Din and Zakrzewski [30] have shown,

$$Z = \check{Z}\,(M)^{-\frac{1}{2}} \tag{2.33}$$

is a simple generalization of (2.31) and satisfies both the instanton equations of motion and the constraint. For the anti-instantons, the vectors $f_1 \cdots f_M$ have to be antiholomorphic.

Clearly, in contradistinction to the four-dimensional Yang-Mills theories, the form of all solutions to the self-duality equations (2.28) is very simple and explicit. The question now arises whether there exist solutions of finite action other than those corresponding to instantons and anti-instantons.

Before we answer this question, let us reformulate the model, as discussed for example by Sasaki [36] and by Zakrzewski [34]. This formulation is based on the introduction of an $N \times N$ projection matrix $\mathbb{P}$, defined by

$$\mathbb{P} = Z\,Z^{\dagger} = \sum_{i=1}^{M} Z_i\,Z_i^{\dagger}, \tag{2.34}$$

where $\mathbb{P}$ is a hermitian projector, and so satisfies

$$\mathbb{P} = \mathbb{P}^{\dagger} = \mathbb{P}^2. \tag{2.35}$$

The Lagrangian density (2.17) can be rewritten as

$$L = \frac{1}{2}\,\mathrm{Tr}\,(\partial_\mu \mathbb{P}\,\partial_\mu \mathbb{P}) \tag{2.36}$$

and the Euler-Lagrange equations (2.22) become

$$[\partial_\mu \partial_\mu \mathbb{P}, \mathbb{P}] = 0. \tag{2.37}$$

In terms of the complex variables $x_\pm$, equations (2.36) and (2.37) take the form

$$L = 2\,\mathrm{Tr}\,(\partial_+ \mathbb{P}\,\partial_- \mathbb{P}), \tag{2.38}$$

and

$$[\partial_+\partial_-\mathbb{P}, \mathbb{P}] = 0. \tag{2.39}$$

Moreover, in this formulation, the first-order self-duality equations become

$$\partial_-\mathbb{P}\,\mathbb{P} = 0 \qquad \text{and} \qquad \mathbb{P}\partial_-\mathbb{P} = 0, \tag{2.40}$$

or equivalently

$$\mathbb{P}\partial_+\mathbb{P} = 0 \qquad \text{and} \qquad \partial_+\mathbb{P}\,\mathbb{P} = 0. \tag{2.41}$$

The $\mathbb{C}P^{N-1}$ model is specified within this formulation by the requirement that

$$\text{rank}\ \mathbb{P} = 1.$$

Next we return to the question of non-instanton solutions of (2.26). All finite action solutions of the equations of motion (2.26), for the $\mathbb{C}P^{N-1}$ models, have been found by Din and Zakrzewski [30,37], who showed that all these solutions are derivable from the instanton solutions. Their construction arose out of a work by Borchers and Garber [38], who considered a similar problem in the case of the $O(N)$ σ models. The construction of Din and Zakrzewski starts with the consideration of a vector field $g \in \mathbb{C}^N$, which is nonzero. Then they define an operator P_+ by

$$P_+ g = \partial_+ g - g\,\frac{g^\dagger \partial_+ g}{|g|^2}, \tag{2.42}$$

and its repeated action by

$$P_+^k g = P_+(P_+^{k-1} g), \tag{2.43}$$

where

$$P_+^0 g \equiv g. \tag{2.44}$$

Then Din and Zakrzewski [30,37] show that

$$Z = \frac{P_+^k f}{|P_+^k f|}, \qquad \partial_- f = 0 \tag{2.45}$$

solves the $\mathbb{C}P^{N-1}$ equations of motion (2.26), where $k = 0, 1, \cdots, N-1$. Moreover, any solution of the equations of motion is of this form for an arbitrary rational analytic vector f. Taking $k = 0$ in equation (2.45) we recover the instanton solutions of D'Adda et al. [10], given by (2.31), and for $k = N-1$ we get anti-instantons. For any other choice of k within the above specified range, new classes of solutions are obtained.

Next we investigate some classes of finite action solutions for the more general grassmannian model as constructed by Sasaki [36]. His construction depends on the consideration of M linearly independent holomorphic N-component column vectors

$$f_1, f_2, \cdots, f_M, \qquad \partial_- f_i = 0, \tag{2.46}$$

and it leads to the introduction of another set of N-component column vectors $(f_{M+1}, f_{M+2}, \cdots, f_N)$ by

$$\begin{aligned} f_{M+1} &= \partial_+ f_1, \qquad f_{M+2} = \partial_+ f_2, \qquad \cdots \quad, f_{2M} = \partial_+ f_M, \\ f_{2M+1} &= \partial_+^2 f_1, \qquad f_{2M+2} = \partial_+^2 f_2, \qquad \cdots \quad, f_{3M} = \partial_+^2 f_M, \\ & \qquad\qquad \cdots \quad, f_N. \end{aligned} \tag{2.47}$$

The vectors $f_1, \cdots, f_N$ are assumed to be linearly independent. Then he orthonormalises these vectors by the Gramm-Schmidt procedure keeping their order and obtains

$$e_1, e_2, \cdots, e_N. \tag{2.48}$$

To be more precise, the Gramm-Schmidt procedure gives

$$e_1 = \frac{f_1}{|f_1|},$$

$$e_i = \frac{g_i}{|g_i|}, \qquad \text{with} \quad g_i = f_i - \sum_{k=1}^{i-1} e_k \, (e_k^\dagger \cdot f_i), \quad i = 2, \cdots, N.$$

Next he considers sequences of M consecutive orthonormal vectors defined by

$$\begin{aligned} Z_1 &= (e_1, e_2, \cdots, e_M), \\ Z_2 &= (e_2, e_3, \cdots, e_{M+1}), \\ &\vdots \\ Z_{N-M+1} &= (e_{N-M+1}, \cdots, e_N). \end{aligned} \tag{2.49}$$

Then he [36] shows that $Z_j \, (j = 1, 2, \cdots, N - M + 1)$ satisfies the equations of motion (2.26) in the case of $G(M, N)$ model. In particular, Z_1 is an instanton and Z_{N-M+1} is an anti-instanton. In terms of the projection formulation, this is equivalent to stating that the projection matrix

$$\mathbb{P}_j = Z_j \, Z_j^\dagger = \sum_{k=j}^{M+j-1} e_k \, e_k^\dagger \tag{2.50}$$

solves equation (2.39). The solutions constructed above are called "generic" solutions, as they depend on the maximal number of holomorphic functions. However, the bosonic $G(N, M)$ model is known to have other types of solutions called "degenerate" and "reducible" [39]. The degenerate solutions are constructed in the same way as the generic solutions except that the number of the input holomorphic vectors is less than M. The reducible solutions are obtained from the same set of orthonormal vectors (2.48) but the condition that Z should consist of M consecutive vectors is relaxed in a rather specific way [39].

For the special case of $M = 1$, the orthonormal vectors $e_1, \cdots, e_N$ are obtained from

$$f, \partial_+ f, \partial_+^2 f, \cdots, \partial_+^{N-1} f$$

by the Gramm-Schmidt procedure. In this case, each e_i satisfies the equations of the $\mathbb{C}P^{N-1}$ model.

Din and Zakrzewski [30,37,40] have studied various properties of the above mentioned solutions. They computed the corresponding action and topological charge densities and showed that only the instanton and anti-instanton solutions are relative minima of the action. In fact, in both the $\mathbb{C}P^{N-1}$ [30,37] and the general grassmannian [40] case, they proved that all non-instanton (or non-anti-instanton) solutions do not correspond to local minima of the action and so are unstable.

Returning to the principal chiral model, it has been known for some time that all solutions of grassmannian models are also solutions of the two-dimensional $U(N)$ chiral models (as the grassmannian subspace is totally geodesic in $U(N)$) [41]. To see this, we write

$$Q = (1 - 2\mathbb{P}), \tag{2.51}$$

for some projector $\mathbb{P}$. Then our Lagrangian density (2.1) becomes

$$L = \mathrm{Tr}\,(\partial_\mu \mathbb{P}\, \partial_\mu \mathbb{P}), \tag{2.52}$$

and the equations of motion (2.5) become

$$[\partial_\mu \partial_\mu \mathbb{P}, \mathbb{P}] = 0. \tag{2.53}$$

However, as we have shown before

$$\mathbb{P} = Z\, Z^\dagger \tag{2.54}$$

solves equation (2.53), where Z is a grassmannian solution. Thus the grassmannian solution is also a solution of the $U(N)$ chiral model.

Until quite recently not much was known about other solutions of the $U(N)$ chiral models. However, in an interesting paper [42], Uhlenbeck suggested how further solutions can be found. Namely, she proved a theorem showing that all classical solutions of the chiral model are of the form

$$Q_\ell = K(1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_\ell), \tag{2.55}$$

where ℓ is an integer (which Uhlenbeck called the "uniton" number), K is a constant matrix and R_i's are projectors which satisfy some first order differential equations.

Uhlenbeck's theorem [42] provides a convenient way of generating new solutions from the old ones; namely, one writes

$$Q = Q_0(1 - 2R). \tag{2.56}$$

Then, as Uhlenbeck has shown, Q satisfies the equations of motion if Q_0 does and if the projector R satisfies the equations

$$\begin{aligned} RA_-(1 - R) &= 0, \\ (1 - R)[\partial_- R + A_- R] &= 0, \end{aligned} \tag{2.57}$$

where

$$A_\pm = \frac{1}{2} Q_0^\dagger \partial_\pm Q_0.$$

If $Q_0 = K$, equation (2.57) becomes

$$\partial_- R R = 0,$$

that is, the self-duality equations for the instantons of the grassmannian models. For $Q_0 \neq K$ we obtain more general solutions, which include non-instanton solutions of grassmannian models and also non-grassmannian solutions.

Moreover, as was shown by Uhlenbeck, all finite action solutions of the $U(N)$ model can be constructed from the constant solutions by adding to them less than N unitons. As a consequence, for $N = 2$, the most general solutions of the $U(2)$ are the one-uniton solutions, *i.e.* the instanton solutions of the $\mathbb{C}P^1$ model. A further consequence of the Uhlenbeck construction of solutions, which is easy to prove, is that [43] all ℓ-uniton solutions possess an important property, namely their A_-^ℓ is given by

$$A_-^\ell = \sum_{i=1}^{\ell} \partial_- R_i. \tag{2.58}$$

The main aspect of the Uhlenbeck construction is that it reduces the problem of solving the equations of motion to having to solve a first order non-linear partial differential equation coupled with a nonlinear algebraic equation. This last equation admits two obvious solutions, namely

$$RA_- = 0 \qquad \text{and} \qquad A_-(1 - R) = 0.$$

This last observation was exploited by Wood [44], who called unitons which satisfy these equations "basic" and "antibasic" respectively. Moreover, in that paper [44], Wood showed that any uniton factor $(1 - 2R)$ corresponding to a given solution Q_0 can be factorised as

$$(1 - 2R) = (1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_k)$$

for some $k \leq N$, where $(1 - 2R_1)$ is a basic uniton factor for Q_0 and $(1 - 2R_i)$ are basic uniton factors for the solutions

$$Q_i = Q_0(1 - 2R_1) \cdots (1 - 2R_{i-1}).$$

Recently, Piette and Zakrzewski [45,29] have performed an explicit construction of all finite action classical solutions of the $U(3)$, $U(4)$ and of some solutions of the $U(N)$ chiral models. Their construction is based on Uhlenbeck's theorem. Here we will present these solutions for the $U(3)$ model and discuss the general construction of solutions for the $U(N)$ model. To give explicit forms of these solutions, in the $U(3)$ case, we consider an orthogonal sequence consisting of the vectors $P_+^k f$, where f is an analytic vector and $P_+^k f$ are defined as in (2.42)-(2.44). These vectors satisfy the following properties [34]:

$$\begin{aligned} &1. \quad (P_+^i f)^\dagger (P_+^j f) = 0 \qquad \text{if} \quad i \neq j, \\ &2. \quad \partial_-(P_+^k f) = -P_+^{k-1} f \, \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2}, \\ &3. \quad \partial_+\Big(\frac{P_+^k f}{|P_+^k f|^2}\Big) = \frac{P_+^{k+1} f}{|P_+^k f|^2}, \\ &4. \quad P_+^N f = 0. \end{aligned} \qquad (2.59)$$

Next we introduce the notation that, if V is a vector then the corresponding projector is denoted by $P(V)$, *i.e.*

$$P(V) = \frac{V\,V^\dagger}{|V|^2}. \qquad (2.60)$$

Also, we introduce the notation

$$P_0 = P(f), \;\cdots\;, P_k = P(P_+^k f). \qquad (2.61)$$

Then, as we know from Uhlenbeck's theorem that for $U(N)$ the largest uniton number is less than N, Piette and Zakrzewski [45] have shown that all solutions of the $U(3)$ model correspond to either one- or two-uniton solutions and so, up to a multiplication from the left by an arbitary constant matrix, are given by

$$\begin{aligned} Q_1 &= (1 - 2P_0), \\ Q_2 &= (1 - 2(P_0 + P_1)), \\ Q_3 &= (1 - 2(P_0 + P_1))(1 - 2P(V_1)). \end{aligned} \qquad (2.62)$$

where V_1 is given by

$$V_1 = \alpha f + \beta P_+^2 f,$$

with α and β holomorphic, *i.e.* functions of x_+.

For the $U(4)$ model [45], all solutions can be expressed as either one-, two- or three-uniton solutions. For more detailed description of these solutions, we refer the reader to [45]. In addition, let us observe that all fields of the $U(3)$ model can be embedded into the $U(4)$ model, and so we see that the $U(3)$ solutions are automatically also solutions of the $U(4)$ model.

Wood's factorisation theorem tells us that to construct all solutions of the $U(N)$ model, we have to add successive basic unitons to the one-uniton solution. However, this appears to be a very difficult task. Even in the $U(4)$ case Piette and Zakrzewski [45,29] have found that the construction of the general three-uniton configurations was rather difficult to perform. Nevertheless, they observed that all solutions correspond to configurations which can be obtained by the addition of a general basic uniton to the already known general grassmannian solutions [34,46].

To discuss this observation we, first of all, consider a general matrix V and the associated projector $P(V)$. When V is of maximal rank, this projector is given by

$$P(V) = V(V^\dagger V)^{-1}V^\dagger. \tag{2.63}$$

Then we observe that the one-uniton solutions of the $U(N)$ model are of the form

$$Q = K(1 - 2R_1), \tag{2.64}$$

where R_1 satisfies

$$(1 - R_1)\partial_- R_1 = 0. \tag{2.65}$$

These solutions are the so-called instanton solutions of grassmannian models that have been known for some time [34]. The most general solutions for R_1 of this class are given by

$$R_1 = P(F), \tag{2.66}$$

where F is a holomorphic matrix (*i.e.* whose entries are functions of only x_+) of maximal rank.

Having determined the most general form of the one-uniton solution we can now construct a two-uniton solution by adding a basic uniton to the one-uniton configuration. Thus following Piette and Zakrzewski [29], we construct an orthogonal holomorphic basis sequence of DZ type [46]. This construction can be summarised as follows (for a detailed description of the construction see [29,46]). We start by taking a set of linearly independent holomorphic matrices

$$F_1, F_2, \cdots, F_{2r+1}, \tag{2.67}$$

which are of maximal rank and which satisfy

$$(1 - P(F))\partial_+ F_i = 0, \tag{2.68}$$

where F is a matrix constructed out of matrices $F_1, F_2, \cdots, F_{i+1}$ by putting them side-to-side. Then we construct an orthogonal holomorphic basis sequence

$$Y_1 = F_1, \qquad Y_i = \Big(1 - \sum_{j=1}^{i-1} P(Y_j)\Big) F_i, \tag{2.69}$$

where by construction, all the Y_i's have the natural holomorphic normalisation, which means that they satisfy

$$Y_i^\dagger \partial_- Y_i = 0 \tag{2.70}$$

and in addition they also satisfy

$$Y_i^\dagger \partial_\pm Y_j = 0, \tag{2.71}$$

for all i, j such that $|i-j| \geq 2$. Then

$$Q = (1 - 2R_1), \tag{2.72}$$

where

$$R_1 = \sum_{i=1}^{r} P(Y_{2i}), \tag{2.73}$$

is a grassmannian solution of the $U(N)$ model. Observe that, when $r = 1$ and Y_1 is empty, we recover the instanton solutions described before.

To add a basic uniton to these solutions, we have to split each set Y_i into various subsets

$$Y_i = (V_i, U_i, W_i, I_i), \tag{2.74}$$

chosen in such a way that

$$M_i^{\dagger} \partial_{\pm} N_j = 0, \qquad i \neq j \tag{2.75}$$

where M_i and N_j stand any set from V_i, U_i or W_i. Then the general basic uniton R_2 which we can add to this grassmannian solution so that the resultant two-uniton solution takes the form

$$Q = (1 - 2R_1)(1 - 2R_2), \tag{2.76}$$

is given by [29]

$$\begin{aligned} R_2 &= \sum_{i=1}^{2r+1} P(V_i) + P(U), \\ U &= \sum_{i=1}^{2r+1} U_i \, a_i, \end{aligned} \tag{2.77}$$

and where a_i's are holomorphic matrices of maximal rank. Note that

$$V_i^{\dagger} \partial_{-} V_j = 0, \qquad U^{\dagger} \partial_{-} V_i = 0$$

is satisfied for all i and j. Observe that if all a_i's but one vanish, the solution becomes a grassmannian solution. In other words, the above non-grassmannian $U(N)$ solutions interpolate between many different grassmannian solutions.

Now that we have discussed the solutions of the $U(N)$ σ model, it is interesting to state some of their properties established by Piette and Zakrzewski [47]. In particular, they have calculated the values of the action of the solutions and studied their stability. First let us point out the fact [43] that for a general solution corresponding to ℓ-unitons

$$Q = (1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_\ell), \tag{2.78}$$

the value of the action is given by

$$S = \sum_{i=1}^{\ell} \breve{q}_i, \tag{2.79}$$

where $\breve{q}_i$ is the topological charge corresponding to the projector R_i given by

$$\breve{q}_i = \int d^2x \, \mathrm{Tr}\, (\partial_+ R_i \, R_i \, \partial_- R_i - \partial_- R_i \, R_i \, \partial_+ R_i). \tag{2.80}$$

Moreover, if $P(V)$ is a projector for which V is a maximal rank matrix with the natural holomorphic normalisation, *i.e.* $V^\dagger \partial_- V = 0$, then [47]

$$\breve{q}(P(V)) = \int d^2x \, \partial_+ \partial_- \ln \, det \, |V|^2 = 2\pi k, \tag{2.81}$$

where k is the leading power of $det\,|V|$ at infinity.

It is now straightforward to show that [47]:

$$\begin{aligned} \breve{q}(R_1) &= \int d^2x \left[\sum_{i=1}^{r} \partial_+ \partial_- \ln \, det \, |Y_{2i}|^2 \right], \\ \breve{q}(R_2) &= \int d^2x \left[\sum_{i=1}^{2r+1} \partial_+ \partial_- \ln \, det \, |V_i|^2 + \partial_+ \partial_- \ln \, det \, |U|^2 \right], \end{aligned} \tag{2.82}$$

are the topological charges corresponding to the projectors R_1 and R_2 given by (2.73) and (2.77) respectively.

Piette et al. [43] have proved that all the non-constant solutions of the $U(N)$ σ model are unstable. In fact they showed that the operator of the fluctuation around these solutions possesses at least one negative mode, thus showing their instability.

We finish this section by observing that the above procedure of constructing solutions of the $U(N)$ model has also given us solutions of the $SU(N)$ chiral model. In the later case we have to choose appropriately the arbitrary constant matrix K in (2.55).

2.2 Supersymmetric Sigma Models

So far we have introduced only bosonic models. However, in most physical applications, fermions are also important. In field theories, fermions are described by anticommuting spinor fields. A convenient and frequently used method of including fermions into sigma models is that which renders the theory supersymmetric (Susy). In this case the most convenient way of proceeding is through the use of superspace. As the inclusion of fermions to the $U(N)$ σ models will represent one of our main topics under consideration later on in this thesis, here we restrict ourselves to the discussion of the Susy $\mathbb{C}P^{N-1}$ and grassmannian models.

The Susy $\mathbb{C}P^{N-1}$ models have been constructed by D'Adda et al. [16] in analogy with the case of Susy $O(N)$ σ models [17]. To define them, we follow closely the conventions of ref. [48]. Thus we introduce a two-dimensional superspace $(x_1, x_2, \theta_1, \theta_2)$, where the anticommuting θ_1 and θ_2 are two components of a real Grassmannian spinor θ. Notice that the word 'grassmannian' is used here to denote that the corresponding quantities are anticommuting and not that they are elements of a grassmannian space. Then we consider a superfield which, in this case, is still given by an N-component vector

$$\Phi_\alpha(x_1, x_2, \theta_1, \theta_2) = Z_\alpha(x_1, x_2) + i\,\theta_j \chi^j_\alpha(x_1, x_2) + i\,\theta_1\theta_2\, F_\alpha(x_1, x_2), \qquad (2.83)$$

where Z_α is a pure bosonic $\mathbb{C}P^{N-1}$ field, χ^j_α is a two-component (anticommuting) spinor and F_α is an auxiliary scalar field.

We introduce also the two-dimensional γ-matrices defined by

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma_1\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, we introduce the supercovariant derivative

$$\check{D} = \check{\partial} - (\Phi^\dagger \check{\partial}\Phi), \tag{2.84}$$

where $\check{\partial}$ is given by

$$\check{\partial} = \partial_\theta + i\not{\partial}\theta = \begin{pmatrix} \partial_{\theta_1} + i(\theta_1\partial_{x_1} + \theta_2\partial_{x_2}) \\ \partial_{\theta_2} + i(\theta_1\partial_{x_2} - \theta_2\partial_{x_1}) \end{pmatrix}, \tag{2.85}$$

with $\partial_{\theta_i} = \frac{\partial}{\partial\theta_i}$ and $\partial_{x_i} = \frac{\partial}{\partial x_i}$. Observe that the operators $\check{D}$ and $\check{\partial}$ are fermionic in nature and so in all calculations we have to apply their anticommuting properties.

The action of the Susy $\mathbb{C}P^{N-1}$ model is given by [16,48]:

$$S = \int d^2x d\theta_1 d\theta_2 \, (\check{D}\Phi)^\dagger \gamma_5 \check{D}\Phi, \tag{2.86}$$

where, as in the purely bosonic model, Φ satisfies

$$\Phi^\dagger\Phi = 1. \tag{2.87}$$

In [16] D'Adda et al. have performed the integration over θ_1 and θ_2, and eliminated all auxiliary fields using their equations of motion ending up with

$$\begin{aligned} S = \int d^2x \Big[& (D_\mu Z_\alpha)^\dagger D_\mu Z_\alpha - i(\psi^\dagger_\alpha \gamma^\mu D_\mu \psi_\alpha) \\ & + \frac{1}{4}[(\psi^\dagger_\alpha\psi_\alpha)^2 + (\psi^\dagger_\alpha\gamma_5\psi_\alpha)^2 - (\psi^\dagger_\alpha\gamma_\mu\psi_\alpha)^2]\Big], \end{aligned} \tag{2.88}$$

where $\psi_\alpha = \chi_\alpha - Z_\alpha(Z^\dagger_\beta \chi_\beta)$, with the constraints

$$Z^\dagger_\alpha Z_\alpha = 1, \qquad Z^\dagger_\alpha \psi_\alpha = \psi^\dagger_\alpha Z_\alpha = 0, \tag{2.89}$$

and where $D_\mu Z_\alpha = \partial_\mu Z_\alpha - (Z^\dagger_\beta \partial_\mu Z_\beta)\, Z_\alpha$.

The action (2.88) is invariant under the following gauge transformations

$$\begin{aligned} Z_\alpha \quad &\longrightarrow \quad Z'_\alpha = e^{i\Lambda(x_1,x_2)} Z_\alpha, \\ \psi_\alpha \quad &\longrightarrow \quad \psi'_\alpha = e^{i\Lambda(x_1,x_2)} \psi_\alpha, \end{aligned} \tag{2.90}$$

and under the Susy transformations

$$\begin{aligned} \delta Z_\alpha &= i\epsilon\psi_\alpha, \\ \delta\psi_\alpha &= -\frac{1}{2} i\epsilon Z_\alpha\, \psi^\dagger_\beta \psi_\beta + \frac{1}{2} i\gamma_5 \epsilon Z_\alpha\, (\psi^\dagger_\beta \gamma_5 \psi_\beta) \\ &\quad + \gamma_\mu \epsilon\, [D_\mu Z_\alpha - \frac{1}{2} i Z_\alpha\, (\psi^\dagger_\beta \gamma_\mu \psi_\beta)]. \end{aligned} \tag{2.91}$$

As in the purely bosonic case discussed in the previous section, D'Adda et at. [16] have pointed out that the Susy $\mathbb{C}P^1$ model is identical to the Susy $O(3)$ σ model. This can be shown as easily as in the non-Susy case by defining a field

$$q^i = \Phi^\dagger\, \sigma_i\, \Phi, \qquad i = 1, 2, 3.$$

Then the action (2.86) and the constraint (2.87) written in terms of q^i become the action and the constraint of the Susy $O(3)$ σ model.

Let us now reformulate the Susy $\mathbb{C}P^{N-1}$ model by changing our variables to the complex variables

$$x_\pm = x_1 \pm i x_2.$$

Also, we introduce the similar conventions for the spinors θ and χ given by

$$\theta_\pm = \theta_1 \pm i\theta_2, \qquad \chi_\pm = \frac{1}{2}(\chi_1 \pm i\chi_2).$$

In this formulation, our superfield Φ_α becomes

$$\Phi = Z + i(\theta_+\chi_- + \theta_-\chi_+) + \frac{1}{2}\theta_-\theta_+F. \tag{2.92}$$

The constraint (2.87) thus implies

$$\begin{aligned} Z^\dagger \cdot Z &= 1, \\ Z^\dagger \cdot \chi_\pm + \chi_\mp^\dagger \cdot Z &= 0, \\ F^\dagger \cdot Z + Z^\dagger \cdot F &= 2(\chi_+^\dagger \cdot \chi_+ - \chi_-^\dagger \cdot \chi_-), \end{aligned}$$

where the "dot" denotes the scalar product in the $\mathbb{C}P^{N-1}$ space.

The action of the model now becomes

$$S = 2\int d^2x d\theta_+ d\theta_- \left[(\check{D}_+\Phi)^\dagger \cdot (\check{D}_+\Phi) - (\check{D}_-\Phi)^\dagger \cdot (\check{D}_-\Phi)\right], \tag{2.93}$$

where the supercovariant derivative is

$$\check{D}_\pm = \check{\partial}_\pm - (\Phi^\dagger \cdot \check{\partial}_\pm\Phi),$$

with the generators of the Susy transformations given by:

$$\check{\partial}_\pm = -i\partial_{\theta_\pm} + \theta_\pm\partial_\pm.$$

The equations of motion, corresponding to (2.93), are given by

$$\check{D}_+\check{D}_-\Phi + (\check{D}_-\Phi)^\dagger \cdot (\check{D}_-\Phi)\,\Phi = 0. \tag{2.94}$$

The self-dual solutions of these equations are given by the solutions of

$$\check{D}_-\Phi = 0, \tag{2.95}$$

while $\check{D}_+\Phi = 0$ corresponds to the antiself-duality case.

A solution of (2.95) is easily seen to be

$$\Phi = \frac{w}{|w|}, \tag{2.96}$$

where the scalar superfield w depends only on x_+ and θ_+, *i.e.* is given by a holomorphic superfield,

$$w(x_+, \theta_+) = f(x_+) + i\theta_+ g(x_+). \tag{2.97}$$

As in the purely bosonic $\mathbb{C}P^{N-1}$ case, it is sufficient to consider f and g which are N-vectors of only polynomials in x_+ without common roots. Of course the function $g(x_+)$ is a grassmann function of its variable. Using (2.92), (2.96) and (2.97), Din et al. [48] have presented explicit expressions for the fields Z, $\chi_\pm$ and F for the Susy $\mathbb{C}P^{N-1}$ instanton solutions. Moreover, they found that, for these solutions, the fermionic part of the action vanishes and that the bosonic part is the same as for the instantons of the pure bosonic $\mathbb{C}P^{N-1}$ model.

To find Susy solutions of the nonlinear equations (2.94), Din et al. [48] have generalised the algorithm for obtaining the general classical solutions of the purely bosonic $\mathbb{C}P^{N-1}$ model. In their procedure one starts with a holomorphic superfield $w(x_+, \theta_+)$ and constructs a sequence of further holomorphic superfields $\check{\partial}_+ w$, $\check{\partial}_+^2 w$, $\cdots$, $\check{\partial}_+^k w$ and then Gramm-Schmidt orthonormalises them, obtaining a sequence of superfields which solve the equations of motion. However, there is a problem in this approach; namely, the superfields $\check{\partial}_+^i w$ for odd i's are spinorial and thus do not belong to the space from which one would like to construct a basis.

To solve this problem, Din et al. [48] have introduced a set of constant grassmannian (anticommuting) variables $\varepsilon_\pm^{(k)} = \varepsilon_1^{(k)} \pm i\varepsilon_2^{(k)}$, and then considered the sequence

$$w,\ \varepsilon_+ \check{\partial}_+ w,\ \check{\partial}_+^2 w,\ \varepsilon_+^{(2)} \check{\partial}_+^3 w,\ \cdots. \tag{2.98}$$

These vectors are now all bosonic, although, some of them are just grassmann even, *i.e.* are not given by ordinary functions. Din et al. [48] have circumvented this by treating such even elements as usual c-numbers. This is an additional assumption, but an assumption which leads to reasonable results (*i.e.* allows us to obtain expressions which are solutions of the equations of motion). Then proceeding as before, *i.e.* Gramm-Schmidt orthonormalising the first N-independent vectors in the sequence (2.98) one obtains [48]:

$$\hat{e}^{i+1} = \frac{(\hat{h}^i)^\dagger \cdot \hat{h}^{i+1}}{(\hat{h}^i)^\dagger \cdot \hat{h}^i}, \qquad i = 0, \cdots N-1, \tag{2.99}$$

where

$$\hat{h}^i = w \wedge \varepsilon_+ \breve{\partial}_+ w \wedge \breve{\partial}_+^2 w \wedge \varepsilon_+^{(2)} \breve{\partial}_+^3 w \wedge \cdots \wedge a_i \breve{\partial}_+^i w,$$

and where

$$a_i = \begin{cases} 1, & i = 2p; \\ \varepsilon_+^{(p)}, & i = 2p-1. \end{cases}$$

Then one can easily show that [48] for every k

$$\Phi^k = \frac{\hat{e}^k}{|\hat{e}^k|} \tag{2.100}$$

is a solution of the equations of motion (2.94).

Also in ref. [48], some properties of these solutions were studied. In particular it was shown there that the value of the action corresponding to the solution Φ^k is given by

$$S_k = \int d^2x \left[2\partial_+\partial_- \ln |\hat{e}^k|^2 + 4 \sum_{i=0}^{k-1} \partial_+\partial_- \ln |\hat{e}^i|^2 \right]_{\theta=0}. \tag{2.101}$$

This is always an integer multiple of 2π which can be found explicitly in terms of the structure of w.

We can now briefly consider the problem of a fermion in the fixed background of a bosonic $\mathbb{C}P^{N-1}$ solution [48,49]. This problem comes from the action of the Susy $\mathbb{C}P^{N-1}$ model by droping all nonlinear terms in ψ_α and $\psi^\dagger_\alpha$ in the equations of motion corresponding to (2.88). In this case one obtains the so-called background Dirac-like equation:

$$(\not{D}\psi)_\alpha - Z_\alpha(Z^\dagger_\beta \not{D}\psi_\beta) = 0, \tag{2.102}$$

together with the constraint

$$Z^\dagger_\alpha Z_\alpha = 1, \qquad Z^\dagger_\alpha \psi_\alpha = \psi^\dagger_\alpha Z_\alpha = 0.$$

The fermion solutions of (2.102) for the case of the background field Z_α being an instanton or anti-instanton solution were reported in [16]. For a general background field Z_α, such solutions were given by Din and Zakrzewski in [49].

Next we turn our attention to the Susy grassmannian σ model. This model is a generalisation of the Susy $\mathbb{C}P^{N-1}$ model discussed above. However, here, the fields Z, χ and F are $N \times M$ matrices and so is the superfield $\Phi(x_1, x_2, \theta_1, \theta_2)$, with the constraint $\Phi^\dagger\Phi = 1_M$. Using the same conventions as before, the action of the Susy grassmannian model is defined by [50,51]

$$S = \int d^2x \, d\theta_1 d\theta_2 \ \mathrm{Tr}\,[(\check{D}\Phi)^\dagger \gamma_5 \check{D}\Phi]. \tag{2.103}$$

Fujii et al. [50,51] have performed the integration over θ's and eliminated the auxiliary field F and obtained

$$\begin{aligned} S = \int d^2x \ \mathrm{Tr}\,[\,&2(D_+Z)^\dagger (D_+Z) + 2(D_-Z)^\dagger (D_-Z) \\ &- 4i(\psi^\dagger_- D_+\psi_+ + \psi^\dagger_+ D_-\psi_-) + 4(\psi^\dagger_+\psi_+\psi^\dagger_-\psi_- - \psi^\dagger_+\psi_-\psi^\dagger_-\psi_+)\,], \end{aligned} \tag{2.104}$$

together with the constraint

$$Z^\dagger Z = 1_M, \qquad Z^\dagger\psi_\pm = 0, \tag{2.105}$$

where $\psi_\pm = \frac{1}{2}(\psi_1 \pm i\psi_2)$, and $D_\pm = \partial_\pm - Z^\dagger\partial_\pm Z$.

As in ref. [51], the equations of motion can be obtained by minimizing the action (2.104) with the constraint term

$$S_c = \int d^2x \ \mathrm{Tr}\ [\lambda(Z^\dagger Z - 1) + \mu Z^\dagger \psi_+ + \psi_+^\dagger Z \mu^\dagger + \nu Z^\dagger \psi_- + \psi_-^\dagger Z \nu^\dagger], \quad (2.106)$$

where the Lagrange multiplier fields λ, μ, ν are $M \times M$ matrices and λ is hermitian and bosonic whereas μ and ν are fermionic. The Euler-Lagrange equations read:

$$\begin{aligned} D_+ D_- Z + Z(D_- Z)^\dagger (D_- Z) + i(\psi_+ \psi_-^\dagger D_+ Z + \psi_- \psi_+^\dagger D_- Z) & \\ - i\,[\,(D_+ Z)\psi_-^\dagger \psi_+ + (D_- Z)\psi_+^\dagger \psi_-] &= 0, \\ (1 - Z Z^\dagger) D_+ \psi_+ + i(\psi_- \psi_+^\dagger \psi_+ - \psi_+ \psi_+^\dagger \psi_-) &= 0, \\ (1 - Z Z^\dagger) D_- \psi_- + i(\psi_+ \psi_-^\dagger \psi_- - \psi_- \psi_-^\dagger \psi_+) &= 0. \end{aligned} \quad (2.107)$$

As in the Susy $\mathbb{C}P^{N-1}$ model, if we neglect the second and third order terms in the fermion field ψ, we obtain the equation of the pure bosonic grassmannian model, studied in the last section, and the linearized Dirac equations

$$(1 - Z Z^\dagger) D_\pm \psi_\pm = 0, \quad (2.108)$$

together with the constraint (2.105). Fujii et al. [50] and Zakrzewski [34] have obtained fermion classical solutions for equations (2.108) in which Z is a pure bosonic grassmannian solution, namely

$$Z = Z_j = (e_j, e_{j+1}, \cdots, e_{M+j-1}), \qquad j = 1, 2, \cdots, N - M + 1,$$

(see equation (2.49) for more detail). Defining the $N \times N$ projection matrices

$$P_j = Z_j Z_j^\dagger = \sum_{k=j}^{M+j-1} e_k e_k^\dagger, \qquad Q_j = \sum_{k=1}^{j-1} e_k e_k^\dagger, \qquad R_j = \sum_{k=M+j}^{N} e_k e_k^\dagger,$$

$$P + Q + R = 1,$$

the fermion solutions are easily found to be [50,34]

$$\psi_+ \;=\; Q_j\,\Gamma_+\,a_j^\dagger, \qquad \psi_- \;=\; R_j\,\Gamma_-\,a_j^{-1}, \tag{2.109}$$

where Γ_+ and Γ_- are $N \times M$ (anti) holomorphic matrices ($\partial_+\Gamma_+ = 0$, $\partial_-\Gamma_- = 0$), and the $M \times M$ matrix a_j is defined by

$$a_j \;=\; Z_j^\dagger F_j, \qquad \text{with} \quad F_j \;=\; (f_j,\, f_{j+1},\, \cdots, f_{M+j-1}), \tag{2.110}$$

i.e. a_j is the transformation matrix between the holomorphic functions and the orthonormal vectors and as such is non-singular. Observe that, at this stage, the fermionic character of the field ψ is irrelevant; however, if one wants to endow ψ with the fermionic character, this can be done by treating the arbitrary functions $\Gamma_\pm$ as grassmann numbers.

Fujii et al. [51] have given explicit solutions of the coupled nonlinear boson-fermion equations (2.107), under the same assumption as in the $\mathbb{C}P^{N-1}$ case introduced by Din et al. [48]; namely, that the fermion field can be treated as a commuting (c-number) field. Their construction [51] started with $2M$ linearly independent N-component holomorphic vectors $f_1, f_2, \cdots, f_{2M}$ with $\partial_- f_i = 0$; then they constructed another set of N-component vectors $f_{2M+1}, f_{2M+2}, \cdots, f_N$ defined as

$$\begin{aligned} f_{2M+1} \;=\; \partial_+ f_1, \qquad f_{2M+2} \;=\; \partial_+ f_2, \quad \cdots \quad, f_{3M} \;=\; \partial_+ f_M, \\ \cdots \quad, f_{2M+i} \;=\; \partial_+ f_i, \quad \cdots \quad, f_N. \end{aligned}$$

Supposing that the vectors $f_1, \cdots, f_N$ are linearly independent, they orthonormalized them by the Gramm-Schmidt procedure and obtained a basis of $\mathbb{C}^N$:

$$e_1,\, e_2,\, \cdots, e_N, \qquad e_i^\dagger \cdot e_k \;=\; \delta_{ik}.$$

Next they considered $3M$ consecutive vectors from this set and groups them into the following three $N \times M$ matrices

$$
\begin{aligned}
Z_{(-1)} &= (e_{j-M},\, e_{j-M+1},\, \cdots, e_{j-1}), \\
Z_{(0)} &= (e_{j},\, e_{j+1},\, \cdots, e_{j+M-1}), \\
Z_{(1)} &= (e_{j+M},\, e_{j+M+1},\, \cdots, e_{j+2M-1}),
\end{aligned}
\tag{2.111}
$$

where $1 \leq j - M$ and $N \geq j + 2M - 1$. Then Fujii et al. [51] showed that a generic solution of equations (2.107) is given by

$$
\begin{aligned}
Z &= Z_{(0)}, \\
\psi_+ &= c_+ \, Z_{(-1)} \, (a^\dagger_{(-1)})^{-1} \, a^\dagger_{(0)}, \\
\psi_- &= c_- \, Z_{(1)} \, a_{(1)} \, a^{-1}_{(0)},
\end{aligned}
\tag{2.112}
$$

in which $c_\pm$ are arbitrary complex constants satisfying $c_+ \bar{c}_- = -i$. The non-singular $M \times M$ matrices $a_{(k)}$, $k = -1, 0, 1$ can be defined in a way similar to (2.110) and they satisfy $D_+ a^\dagger_{(k)} = 0$ and $D_- a^{-1}_{(k)} = 0$.

3. The $U(N)$ σ Models with the Wess-Zumino-Witten Term.

In the previous chapter, we reviewed the basic formulations of the grassmannian and $U(N)$ σ models in two-dimensions. We showed how the instanton and non-instanton solutions can be constructed for these models and we stated some of the most important properties of these solutions. In the present chapter, we shall study the two dimensional $U(N)$ WZW-σ model. We shall limit ourselves to constructing explicit finite action classical solutions of this model. We will show how to relate solutions of this model to the solutions of the Lax-pair system for the corresponding sigma model without the WZW-term. Then we will describe the construction of these solutions and study their properties. We will compute the value of the action of the WZW-σ model for some of these solutions, and then prove that all solutions of this model have the same number of negative modes as the corresponding solutions of the model without the WZW-term, and so, as a consequence [43], that they are all unstable.

First let us introduce the WZW-term. As we said in the introduction, this term was first introduced by Wess and Zumino [23], then reintroduced by Polyakov and Wiegmann [24] and Witten [25]. It represents a topological term. The inclusion of this term naturally involves a space with one extra dimension and such that the physical space is its boundary. In our model, this term is given by a three-dimensional integral [25,27] which is locally (but not globally) a total divergence. The definition of the WZW-term requires, however, that we extend the field configuration $g(x_1, x_2) \in U(N)$ to a field configuration $\breve{g}(x_1, x_2, t) \in U(N)$ which depends on an additional variable t which satisfies $0 \leq t \leq 1$. Following Witten [25] we choose the boundary conditions of this extension to be such that

$$\breve{g}(x_1, x_2, 1) = g(x_1, x_2) \qquad \text{and} \quad \breve{g}(x_1, x_2, 0) = K,$$

where K is a constant unitary matrix. More details of our continuation will be given later on.

For reasons which will become clearer later we choose to write the action of the WZW-σ models as

$$S = \frac{1}{4} \int d^2x \ \mathrm{Tr} \, (\partial_\mu g^\dagger \, \partial_\mu g) + \frac{i\lambda}{6} \int d^3x \ \mathrm{Tr} \ (\breve{g}^\dagger \partial_\mu \breve{g} \, \breve{g}^\dagger \partial_\nu \breve{g} \, \breve{g}^\dagger \partial_\rho \breve{g}) \epsilon^{\mu\nu\rho}, \tag{3.1}$$

where λ is a purely imaginary parameter and g and $\breve{g}$ are unitary matrices:

$$g^\dagger g = \breve{g}^\dagger \breve{g} = 1. \tag{3.2}$$

The first term in (3.1) is the standard $U(N)$ σ model action of the field g (which corresponds to Q of the previous chapter). The second contribution in (3.1) is the WZW topological term. The variation of the action (3.1) under $g \to g + \delta g$ can be easily shown to be

$$\delta S = \int d^2x \ \mathrm{Tr} \ \Big[g^\dagger \delta g [\frac{1}{2} \partial_\mu (g^\dagger \partial_\mu g) - \frac{i\lambda}{2} \epsilon^{\mu\nu} \partial_\mu (g^\dagger \partial_\nu g)] \Big]. \tag{3.3}$$

The equations of motion of the WZW-σ model therefore do not depend on our extension and are given by

$$\partial_\mu (g^\dagger \partial_\mu g) - i\lambda \, \epsilon^{\mu\nu} \partial_\mu (g^\dagger \partial_\nu g) = 0, \tag{3.4}$$

together with the constraint (3.2).

If we now perform the change of variables

$$x_\pm = x_1 \pm i x_2, \tag{3.5}$$

the action of the WZW-σ model (3.1) can be rewritten as

$$\begin{aligned} S =& \frac{1}{2} \int d^2x \ \mathrm{Tr} \ (\partial_+ g^\dagger \, \partial_- g + \partial_- g^\dagger \, \partial_+ g) \\ &- \lambda \int d^3x \ \mathrm{Tr} \ [(\partial_+ \breve{g}^\dagger \, \partial_- \breve{g} - \partial_- \breve{g}^\dagger \, \partial_+ \breve{g}) \, \breve{g}^\dagger \, \dot{\breve{g}}], \end{aligned} \tag{3.6}$$

where the "dot" denotes the partial derivative with respect to t. Also, the equations of motion (3.4) become

$$(1 - \lambda)\partial_+(g^\dagger\partial_- g) + (1 + \lambda)\partial_-(g^\dagger\partial_+ g) = 0. \tag{3.7}$$

Clearly, when $\lambda = 0$ the equations reduce to the well studied equations of the model without the WZW-term.

As we will discuss relations between solutions of both models, we adopt the convention that the fields and the solutions of the model with the WZW-term will be denoted by g and those of the model without this term by Q. Clearly, the solutions of the model without the WZW-term satisfy:

$$\partial_+(Q^\dagger\partial_- Q) + \partial_-(Q^\dagger\partial_+ Q) = 0, \tag{3.8}$$

together with the constraint $Q^\dagger Q = 1$. In the next section we will show how to relate the solutions of (3.7) to the solutions of the Lax-pair system for the corresponding sigma model without the WZW-term.

3.1 Solutions of the WZW-σ models

Despite the fact that one of the original motivations for the inclusion of the WZW-term in the Lagrangian has been to reduce some of it symmetries [25] many of these symmetries remain [26,27] and if we consider only the classical version of the theory and its equations of motion, the functional space of the solutions of the equations of motion exhibits many symmetries [27], which can be exploited in the construction of these solutions. For example, it is well known that if Q is a solution of (3.8) then $Q^\dagger$ is also its solution. This symmetry can be extended to a property of g's *i.e.* for all λ's: if $g(\lambda)$ is a solution of (3.7) then so is $g(-\lambda)^\dagger$.

The two-dimensional σ models are known to be integrable [6,52] as we can associate with them a Lax-pair formalism (also called the Hilbert-Riemann problem in some papers [53]) and so they possess an infinite number of conservation laws. Recently [27], it has been shown that the WZW-σ models are also integrable. In what follows we will show that this is not unexpected, as we will demonstrate that the solutions of both models are related, and can be derived from each other.

To do this, let us consider the equations of the Lax-pair problem for the $U(N)$ σ model:

$$\partial_+\Psi = \Psi\frac{2A_+}{1+\lambda}, \qquad \partial_-\Psi = \Psi\frac{2A_-}{1-\lambda}, \tag{3.9}$$

where

$$A_\pm = \frac{1}{2}Q^\dagger\partial_\pm Q, \tag{3.10}$$

and where $\Psi(x_+, x_-, \lambda)$ is a $N \times N$ matrix-valued function, and λ is an additional complex parameter. To claim that (3.9) is a linear system of equations for the $U(N)$ σ model, we need to show that the integrability conditions for Ψ imply the equations of motion for the $U(N)$ σ model. To see this, we multiply the first equation of (3.9) by $(1-\lambda)$ and differentiate it with respect to x_-, and multiply the second equation by $(1+\lambda)$ and differentiate it with respect to x_+, then subtract the results and get

$$(1-\lambda)[(\partial_-\Psi)A_+ + \Psi\partial_-A_+] - (1+\lambda)[(\partial_+\Psi)A_- + \Psi\partial_+A_-] = 0. \tag{3.11}$$

Puting (3.9) in (3.11), and then performing some simple manipulations we obtain

$$\Psi\Big[\lambda(\partial_+A_- + \partial_-A_+) + \Big(\partial_+A_- - \partial_-A_+ + 2[A_+, A_-]\Big)\Big] = 0. \tag{3.12}$$

We see that the integrability of Ψ for arbitrary λ implies that

$$\begin{aligned} \partial_+A_- + \partial_-A_+ &= 0, \\ \text{and} \quad \partial_+A_- - \partial_-A_+ + 2[A_+, A_-] &= 0. \end{aligned} \tag{3.13}$$

These are just the equations of motion and the curvatureless conditions of $A_\pm$ for the $U(N)$ σ model.

The unitarity of Q provides us with some further conditions on Ψ [54], namely that

$$\Psi^{\dagger -1}(\lambda) = \Psi(-\lambda^*). \qquad (3.14)$$

As a consequence, we see that if λ is imaginary, then Ψ is unitary and (3.9) can be rewritten as

$$(1 + \lambda)\Psi^\dagger \partial_+ \Psi = 2A_+, \qquad (1 - \lambda)\Psi^\dagger \partial_- \Psi = 2A_- \qquad (3.15)$$

from which we see that

$$(1 + \lambda)\partial_-(\Psi^\dagger \partial_+ \Psi) + (1 - \lambda)\partial_+(\Psi^\dagger \partial_- \Psi) = 2(\partial_- A_+ + \partial_+ A_-). \qquad (3.16)$$

Thus, as Q satisfies (3.8),

$$g = \Psi$$

satisfies (3.7). This means that the solutions of the Lax-pair problem for the σ model have provided us with a simple way of constructing solutions of the WZW-σ model.

Note that we could have chosen to construct the Lax-pair problem for our system in a different way; namely, we could have chosen

$$\partial_+ \varphi = \frac{2\,B_+}{1 + \lambda}\,\varphi, \qquad \partial_- \varphi = \frac{2\,B_-}{1 - \lambda}\,\varphi, \qquad (3.17)$$

where

$$B_\pm = \frac{1}{2}\,\partial_\pm Q\,Q^\dagger. \qquad (3.18)$$

Once again, as $Q^\dagger$ is a solution of (3.8), it is easy to show that $\varphi(-\lambda)$ satisfies (3.7) for the imaginary values of λ.

Before we proceed any further let us show that there is a simple relationship between the solutions of (3.9) and those of (3.17). To see this we observe that we can always take

$$\varphi = Q X^{-1} \tag{3.19}$$

for some matrix X. Inserting this expression into (3.17) and multiplying it from the left by $Q^\dagger$ and from the right by X we find that

$$\partial_+ X = X \frac{2A_+}{1 + \frac{1}{\lambda}} \qquad \partial_- X = X \frac{2A_-}{1 - \frac{1}{\lambda}}, \tag{3.20}$$

thus showing that

$$\varphi(\lambda) = Q\,\Psi^{-1}(\lambda^{-1}). \tag{3.21}$$

In fact, we can consider $\varphi^\dagger$ as a solution of (3.9), in which Q is replaced by $Q^\dagger$.

In addition we can show how any solution of the WZW-σ model can be transformed into a solution of the usual σ model. Asuming that g is a solution of (3.7) we define

$$C_\pm = (1 \pm \lambda) g^\dagger \partial_\pm g. \tag{3.22}$$

It is then straightforward to show that

$$\partial_+ C_- + \partial_- C_+ = 0, \qquad C_-^\dagger = -C_+. \tag{3.23}$$

These two equations imply [6,54] that we can set

$$C_\pm = Q^\dagger \partial_\pm Q, \tag{3.24}$$

where $Q^\dagger Q = 1$, and then Q is a solution of (3.8), which, however, in general does depend on λ. We see that each solution of the WZW-σ model can be related to a solution of the usual σ model with the relation provided by the solutions of the Lax-pair equations for the σ model.

We may now ask if any solution g of (3.7) is automatically also a solution of one of the Lax-pair equations (3.9) or (3.17). This question is equivalent to asking whether $C_\pm$ does not depend on λ. As such, this question is not well posed, as, in reality, we have here a family of solutions parametrised by λ.

Take, for example, $Q(\beta)$, a family of solutions of the $U(N)$ σ model where β is some parameter. Computing $\Psi(\beta)$ provides us with a family of solutions of the WZW-σ model. If we now decide to choose $\beta = \lambda$, then $C_\pm = (1 \pm \lambda) g^\dagger \partial_\pm g$ will depend on λ, in agreement with the claim above.

What we should be asking is then: given a solution $g_0(\lambda_0)$ of the WZW-σ model for a fixed value λ_0 of λ, is it possible to extend it to a function $g'(\lambda)$, such that g' is a solution of both (3.7) and of either the Lax-pair (3.9) or (3.17) for all values of λ, in which $A_\pm$ or respectively $B_\pm$ are independent of λ, and such that $g'(\lambda_0) = g_0(\lambda_0)$. To do this we simply solve (3.9) for $A_\pm = g_0^\dagger \partial_\pm g_0 \, (1 \pm \lambda_0)$ or (3.17) for $B_\pm = \partial_\pm g_0 \, g_0^\dagger \, (1 \pm \lambda_0)$. Then $\Psi_0(\lambda)$ or $\varphi_0(-\lambda)$ is the required solution.

3.2 Solutions of the Lax-pair Equations for the $U(N)$ σ Models

Before constructing any solutions of the Lax-pair equations (3.9) or (3.17) given in the previous section, we briefly recall Uhlenbeck's [42] construction of solutions of the $U(N)$ sigma model discussed in the previous chapter. There we showed that according to Uhlenbeck's theorem [42], any solution Q_ℓ of (3.8) can be factorised as

$$Q_\ell = K(1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_\ell), \tag{3.25}$$

where K is a constant matrix, ℓ is an integer (the uniton number), and R_i's are projectors which can be constructed by induction, as they satisfy

$$\begin{aligned} R_\ell A_-^{\ell-1}(1 - R_\ell) &= 0, \\ (1 - R_\ell)(\partial_- R_\ell + A_-^{\ell-1} R_\ell) &= 0, \end{aligned} \tag{3.26}$$

where

$$A_-^\ell = \frac{1}{2} Q_\ell^\dagger \partial_- Q_\ell. \tag{3.27}$$

From this construction we can deduce, that if we define (by induction)

$$K_\ell = K_{\ell-1}(1 - a R_\ell), \tag{3.28}$$

where a is a complex number (which does not depend on ℓ) and R_i's are the projectors which satisfy the equations given above, then

$$K_\ell^{-1} = (1 - b R_\ell)\, K_{\ell-1}^{-1}, \tag{3.29}$$

where b is a complex number, which satisfies

$$a + b - a\, b = 0. \tag{3.30}$$

As a result we can now prove that

$$\begin{aligned} K_\ell^{-1} \partial_- K_\ell &= b A_-^\ell \\ K_\ell^{-1} \partial_+ K_\ell &= a A_+^\ell. \end{aligned} \tag{3.31}$$

We prove this by induction. The result is trivialy true for the constant factor and

$$\begin{aligned} K_\ell^{-1} \partial_- K_\ell =& \, b\, [\, \partial_- R_\ell (1 - a R_\ell) + A_-^{\ell-1} \\ & - b R_\ell A_-^{\ell-1} - a A_-^{\ell-1} R_\ell + ab R_\ell A_-^{\ell-1} R_\ell] \\ =& \, b\, [\, A_-^{\ell-1} + \partial_- R_\ell - a\Big((1 - R_\ell)\partial_- R_\ell + A_-^{\ell-1} R_\ell - R_\ell A_-^{\ell-1}\Big)], \end{aligned} \tag{3.32}$$

but from (3.26), we have

$$(1 - R_\ell)\partial_- R_\ell + A_-^{\ell-1} R_\ell - R_\ell A_-^{\ell-1} = 0. \qquad (3.33)$$

Thus

$$K_\ell^{-1}\partial_- K_\ell = bA_-^\ell. \qquad (3.34)$$

A similar calculation proves also the other equation for K_ℓ.

As an immediate consequence, we see that if we choose

$$a = \frac{2}{1+\lambda}, \qquad b = \frac{2}{1-\lambda}, \qquad (3.35)$$

the condition (3.30) is satisfied and

$$\Psi = K_\ell \qquad (3.36)$$

is a solution of (3.9) corresponding to Q_ℓ.

Moreover, as this solution is unique up to a multiplication by a constant matrix, we see that we have thus solved (3.9) completely.

We would like to point out that our expression for K_ℓ agrees with expressions given in ref. [53]. In that paper, various solutions of the grassmannian models were studied and the solutions of the corresponding Lax-pair equations were derived in an explicit form. As the grassmannian space is totally geodesic in $U(N)$ these grassmannian solutions are also solutions of the $U(N)$ model and so the results of ref. [53] provide us with various solutions of the WZW-σ models. However, as we now know from Uhlenbeck's equations, all grassmannian solutions, when considered as solutions of the $U(N)$ models, are in fact contained in the expression (3.25) for some ℓ and some choice of projectors R_i. Thus we see that the interesting dependence on λ observed in ref. [53] comes from the product of factors a and the properties of our projectors R_i.

Let us point out at this stage that our solutions have already been given in the Uhlenbeck paper [42]. In that paper Uhlenbeck, using a different but related parameter λ, introduced auxiliary functions E_λ, which she then used to prove her factorisation theorem for the ordinary chiral model. These functions E_λ correspond, in fact, to our functions Ψ.

We finish this section by pointing out that due to the relation (3.21) the solutions to (3.17) are given by

$$\varphi = Q_\ell K_\ell^{-1}, \tag{3.37}$$

where the parameter b appearing in K_ℓ^{-1}, as defined in (3.29), is given by

$$b = \frac{-2\lambda}{1-\lambda}. \tag{3.38}$$

3.3 Properties of the Solutions of the WZW-σ Models

As we have shown in the previous sections, the solutions of the $U(N)$ WZW-σ model can be obtained very easily from the solutions of the $U(N)$ σ model. It would then be interesting to see whether the properties of both sets of solutions are similar. In particular, is the action of the solutions of the model with the WZW-term also quantised, and can it be related to the topological charges of the projectors R_i's out of which the solutions are constructed? Another interesting question would be to determine whether the solutions are also unstable, or if, on the contrary, the WZW-term stablises them.

Before we can answer these questions, we have to find a convenient way of computing explicitly the contribution of the WZW-term. As this term is given by a three-dimensional integral, we know [25,27], as we have already said earlier in this chapter, that we have to extend our field $g(x_1, x_2)$ to a field $\breve{g}(x_1, x_2, t)$, defined over the three-dimensional space locally parametrised by x_1, x_2 and t, with $0 \leq t \leq 1$ and such that

$$\breve{g}(x_1, x_2, t) = \begin{cases} g(x_1, x_2), & \text{at } t = 1; \\ 1, & \text{at } t = 0. \end{cases} \tag{3.39}$$

This extension shows that our cylinder, with an infinitely large base described by x_1, x_2 and "height" t, is being mapped by $\breve{g}$ into a full-two sphere in $U(N)$, the surface of which is nothing else but the field $g(x_1, x_2)$. To perform this explicitly, let us first consider a one-uniton-like field configuration

$$g(x_+, x_-) = (1 - a\,R), \tag{3.40}$$

where $a = \frac{2}{1+\lambda}$. For our extension we can now use

$$\begin{aligned} \breve{g}(x_+, x_-, t) &= (1 - a\,R)^t \\ &= (1 - a(t)\,R) \\ &= e^{i\alpha t R}, \end{aligned} \tag{3.41}$$

where

$$a(t) = (1 - e^{i\alpha t}) \tag{3.42}$$

with

$$i\alpha = \ln(1 - a) = \ln\left(\frac{\lambda - 1}{\lambda + 1}\right). \tag{3.43}$$

The extension for other configurations is given by induction, *i.e.* for

$$g_\ell(x_+, x_-) = g_{\ell-1}(x_+, x_-)\,(1 - a\,R_\ell), \tag{3.44}$$

we use

$$\begin{aligned} \breve{g}_\ell(x_+, x_-, t) &= \breve{g}_{\ell-1}(x_+, x_-, t)\,(1 - a\,R_\ell)^t \\ &= K_\ell(a(t)). \end{aligned} \tag{3.45}$$

To extend solutions corresponding to φ, we apply the same method by raising each factor of both Q and Ψ to the power t.

To study properties of our solutions of the WZW-σ model, let us first of all calculate explicitly the values of the action (3.1) for these solutions. We start by considering the solution of (3.7), constructed by solving (3.9) for the one-uniton field configuration

$$Q_1 = (1 - 2R_1), \tag{3.46}$$

namely:

$$g_1 = (1 - a\,R_1), \tag{3.47}$$

where

$$a = \frac{2}{1 + \lambda}, \tag{3.48}$$

For this solution, according to our extension, we have

$$\breve{g}_1 = (1 - a(t)\,R_1), \tag{3.49}$$

where $a(t)$ is given by (3.42), and from Uhlenbeck's construction, the projector R_1 satisfies

$$\partial_- R_1\, R_1 = 0. \tag{3.50}$$

Note that had we solved (3.17) instead of (3.9), we would have obtained the same solution. Now when $\lambda = 0$, we recover the one-uniton solution of the usual $U(N)$ σ model for which, as we mentioned in the previous chapter, the action is given by [47]

$$\begin{aligned} S_0 &= 4 \int d^2x \; \mathrm{Tr}\,(\partial_- R_1\, \partial_+ R_1) \\ &= 4\,\breve{q}(R_1), \end{aligned} \tag{3.51}$$

where

$$\breve{q}(R_1) = \int d^2x \; \mathrm{Tr}\,(\partial_+ R_1\, R_1 \partial_- R_1 - \partial_- R_1\, R_1 \partial_+ R_1) \tag{3.52}$$

is the topological charge corresponding to the projector R_1. For nonvanishing values of λ, we can apply the relations (3.31) for this solution and it is then easy to show that the action of the WZW-σ model (3.6) is given by

$$S = - \int d^2x \; |a|^2 \; \mathrm{Tr} \; (A^1_- \, A^1_+) \\ + \lambda \int d^3x \; i \, \alpha \, |a(t)|^2 \; \mathrm{Tr} \; [(A^1_+ \, A^1_- \; - \; A^1_- \, A^1_+) R_1]. \tag{3.53}$$

If we now insert the values of $A^1_\pm$ for our case, which are given by

$$A^1_- = \partial_- R_1, \qquad A^1_+ = -(A^1_-)^\dagger, \tag{3.54}$$

then, equation (3.53) becomes

$$S = \int d^2x \, |a|^2 \; \mathrm{Tr} \; (\partial_- R_1 \, \partial_+ R_1) \\ - \int d^3x \, i\alpha\lambda \, |a(t)|^2 \; \mathrm{Tr} \; [(\partial_+ R_1 \, \partial_- R_1 \; - \; \partial_- R_1 \, \partial_+ R_1) R_1].$$

Using the properties of the projector R_1 we get

$$S = \left[\, |a|^2 + \int_0^1 dt \, i\alpha\lambda \, |a(t)|^2 \right] \int d^2x \; \mathrm{Tr} \; (\partial_- R_1 \, \partial_+ R_1). \tag{3.55}$$

From (3.51), this equation becomes

$$S = \frac{1}{4} \left[\, |a|^2 + \int_0^1 dt \, i\alpha\lambda \, |a(t)|^2 \right] S_0, \tag{3.56}$$

where S_0 is the action of the corresponding solution of the usual σ model. Writing $\lambda = ik$ and using relations between a, α and λ, it is easy to perform the t integration and obtain

$$S = h(k) \, S_0, \tag{3.57}$$

where

$$h(k) = 1 - k \; arctg(k^{-1}). \tag{3.58}$$

It is important to note that in (3.42) α is defined only up to the addition of an integer multiple of 2π and, as a consequence, the value of (3.55) is not uniquely defined. This well known fact [25,27] comes from the possibility of choosing different extensions of g, which in our case correspond to different choices of the arbitrary constant in α. In (3.57) and in (3.58) this freedom is hidden in the choice of the phase of *arctg* in $h(k)$. If we fix this phase by choosing $0 \leq \tan^{-1}(k^{-1}) \leq \pi$, then $h(k)$ is monotonic in k. As k varies from $-\infty$ to $+\infty$, then $h(k)$ decreases from ∞ to 0, and for $k = 0$, $h(k) = 1$.

Observe that if we now consider a solution of (3.7), constructed by solving (3.17) instead of (3.9), for the one-uniton solution Q_1, then, according to (3.37), we will find that

$$g_1 = (1 - 2R_1)(1 - b\,R_1), \tag{3.59}$$

where b takes the same form as in (3.38). Moreover, the action of this solution is exactly the same as in the previous solution (3.47).

The computation of the value of the action for more general solutions is far more complicated. If we consider, for example, the solution constructed from a two-uniton solution

$$Q_2 = (1 - 2R_1)(1 - 2R_2), \tag{3.60}$$

where we have taken $K = 1$, by using the Lax-pair (3.9), we have

$$g_2 = (1 - a\,R_1)(1 - a\,R_2), \tag{3.61}$$

and

$$\breve{g}_2 = (1 - a(t)\,R_1)(1 - a(t)\,R_2), \tag{3.62}$$

where $a = \frac{2}{1+\lambda}$ and $a(t) = 1 - e^{i\alpha t}$. Then, from Uhlenbeck's construction, the projectors R_1 and R_2 satisfy, in addition to (3.50),

$$\begin{aligned} R_2\partial_- R_1\,(1 - R_2) &= 0, \\ (1 - R_2)(\partial_- R_2 + \partial_- R_1\,R_2) &= 0. \end{aligned} \tag{3.63}$$

For $\lambda = 0$, the action of the solution is given by

$$S_0 = 4\Big(\check{q}(R_1) + \check{q}(R_2)\Big), \tag{3.64}$$

where

$$\check{q}(R_i) = \int d^2x \;\mathrm{Tr}\;(\partial_+ R_i\, R_i \partial_- R_i - \partial_- R_i\, R_i \partial_+ R_i) \tag{3.65}$$

is the topological charge corresponding to R_i, which can be computed explicitly for any given solution [47]. For nonvanishing values of λ we can use the relations (3.31) and obtain for the WZW-σ model action:

$$\begin{aligned} S = &- \int d^2x\, |a|^2 \;\mathrm{Tr}\;(A^2_- A^2_+) \\ &+ \lambda \int d^3x\; i\alpha\, |a(t)|^2 \;\mathrm{Tr}\Big[(A^2_+ A^2_- - A^2_- A^2_+) \\ &\qquad \times [R_2 + (1 - a^*(t)R_2)R_1(1 - a(t)R_2)]\Big]. \end{aligned} \tag{3.66}$$

Inserting the corresponding $A^2_\pm$, which are now given by

$$A^2_- = \partial_- R_1 + \partial_- R_2, \qquad A^2_+ = -(A^2_-)^\dagger, \tag{3.67}$$

and as before, writing $\lambda = ik$, and using the relations between a, α and λ, it is then easy to show that

$$\begin{aligned} S =& h(k)\, S_0 \\ &+ \alpha\, k \int d^3x\, |a(t)|^2 \;\mathrm{Tr}\;(\partial_+ R_2\, \partial_- R_2\, R_1 - \partial_- R_2\, \partial_+ R_2\, R_1 - \partial_- R_1\, \partial_+ R_2). \end{aligned} \tag{3.68}$$

If R_2 is a basic uniton [44,29], then it satisfies

$$R_2 \partial_- R_1 = 0,$$

while for an antibasic uniton we have

$$\partial_- R_1 (1 - R_2) = 0.$$

Thus we find that

$$S = h(k) S_0 + \alpha k \int_0^1 dt\, |a(t)|^2 \int d^2x\, \partial_+ \partial_- \text{ Tr } (R_2 R_1). \tag{3.69}$$

However, as

$$\int d^2x\, \partial_+ \partial_- \text{ Tr } (R_1 R_2) = 0 \tag{3.70}$$

for any two projectors, we conclude that

$$S = h(k) S_0, \tag{3.71}$$

which generalises the result obtained for the one-uniton like configuration in the sense that the action of the solution of the $U(N)$ σ model with the WZW-term is given by the action of the corresponding solution of the usual $U(N)$ σ model multiplied by $h(k)$.

We now consider a solution constructed from the two-uniton solution Q_2 of the σ model, given by (3.60), but derived by using the second Lax-pair equations (3.17). In this case, according to (3.37), we find

$$g_2 = (1 - 2 R_1)(1 - 2 R_2)(1 - b R_2)(1 - b R_1), \tag{3.72}$$

where b is the same as in (3.38). The action of this solution can be shown to be equal to the action of our previous solution (3.61) and so is given by (3.68), or (3.71) when our unitons are basic or antibasic.

Unfortunately, so far, we have been unable to find a corresponding result for the more general case. However, it is easy to show that if the action of Q, a solution of the $U(N)$ σ model without the WZW-term, is finite, the action of g, the corresponding solution of the $U(N)$ WZW-σ model, is also finite. To prove this, we consider

$$g_\ell = \prod_{i=1}^{\ell}(1 - a R_i), \tag{3.73}$$

for which

$$\check{g}_\ell = \prod_{i=1}^{\ell}(1 - a(t) R_i), \tag{3.74}$$

where a and $a(t)$ are the same as before. In this case, the action (3.6) takes the form

$$\begin{aligned} S =& \frac{4}{(1-\lambda)(1+\lambda)} \int d^2x \ \mathrm{Tr}\ [A_-^{\ell\dagger} A_-^{\ell}] \\ & - \lambda \int d^3x \ \mathrm{Tr} \left[|a(t)|^2 \, [A_-^{\ell\dagger} A_-^{\ell} - A_-^{\ell} A_-^{\ell\dagger}] \, \check{g}_\ell^\dagger \, \dot{\check{g}}_\ell \right], \end{aligned} \tag{3.75}$$

where

$$A_-^\ell = \frac{1}{2} Q_\ell^\dagger \partial_- Q_\ell. \tag{3.76}$$

At this stage, let us prove by induction that for our expression (3.74), we have

$$\check{g}_\ell^\dagger \, \dot{\check{g}}_\ell = i\alpha \sum_{j=1}^{\ell} \check{R}_j, \tag{3.77}$$

where

$$\check{R}_j = (1 - a(t) R_\ell)^\dagger \cdots (1 - a(t) R_{j+1})^\dagger R_j (1 - a(t) R_{j+1}) \cdots (1 - a(t) R_\ell), \tag{3.78}$$

for $j \le \ell - 1$, and $\check{R}_\ell = R_\ell$. To do this, it is easy to check that the result is true for the one- and two-uniton solutions. Then assuming that it is true for the

ℓ-uniton solution, we have to prove that it is also true if we add another uniton factor. Thus we take

$$g \,=\, g_\ell\,(1\,-\,a\,R), \qquad \breve{g} \,=\, \breve{g}_\ell\,(1\,-\,a(t)\,R). \tag{3.79}$$

In this case, we find that

$$\begin{aligned}
\breve{g}^\dagger\,\dot{\breve{g}} &= (1\,-\,a(t)\,R)^\dagger \breve{g}_\ell^\dagger\,\dot{\breve{g}}_\ell(1\,-\,a(t)\,R)\,-\,(1\,-\,a^*(t)\,R)R\,\frac{da(t)}{dt} \\
&= i\alpha\Big((1\,-\,a(t)\,R)^\dagger \sum_{j=1}^{\ell} \breve{R}_j\,(1\,-\,a(t)\,R)\,+\,R\Big) \\
&= i\alpha \sum_{j=1}^{\ell+1} \breve{R}_j,
\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}
\breve{R}_j &= (1\,-\,a(t)\,R)^\dagger(1\,-\,a(t)\,R_\ell)^\dagger \cdots (1\,-\,a(t)\,R_{j+1})^\dagger\,R_j \\
&\quad\times\,(1\,-\,a(t)\,R_{j+1}) \cdots (1\,-\,a(t)\,R_\ell)(1\,-\,a(t)\,R),
\end{aligned} \tag{3.81}$$

for all $j \leq \ell$, and $\breve{R}_{\ell+1} = R$, thus showing that (3.77) is valid for the general ℓ-uniton solutions (3.74). Returning to the finiteness problem, we see that as

$$0 \leq \int d^2x \ \mathrm{Tr}\ (A_-^{\ell\,\dagger}\,A_-^\ell\,\breve{R}_j) \leq \int d^2x \ \mathrm{Tr}\ (A_-^{\ell\,\dagger}A_-^\ell) \tag{3.82}$$

for all R_j, the action (3.75) must be finite if Q_ℓ is a finite action solution of the $U(N)$ σ model.

One of the most important properties of any solution is its stability when subjected to small fluctuations. To study this property for our solutions g of the WZW-σ model, we follow the procedure of Piette et al. [43] who investigated the stability properties of the usual $U(N)$ σ model (without the WZW-term). We consider a small fluctuation by assuming that the field g' in the neighbourhood of g can be written in the form

$$g' = g\, e^{\epsilon X}, \tag{3.83}$$

where ϵ is small and the matrix field X satisfies

$$X^{\dagger} = -X, \tag{3.84}$$

in order that $g' \in U(N)$. Substituting (3.83) into the action of the WZW-σ model (3.1) and calculating only to second order in ϵ, it is easy to see that

$$\begin{aligned} S(g') = S(g) &+ \frac{\epsilon}{2} \int d^2x \; \mathrm{Tr} \left[\Big(\partial_\mu (g^\dagger \partial_\mu g) \;-\; i\lambda\, \epsilon^{\mu\nu} (\partial_\mu g^\dagger \partial_\nu g) \Big)\, X \right] \\ &+ \frac{\epsilon^2}{4} \Big[\int d^2x \; \mathrm{Tr}\, [\partial_\mu X^\dagger\, \partial_\mu X \;-\; g^\dagger \partial_\mu g\, (X \partial_\mu X \;-\; \partial_\mu X\, X)] \\ &+ 2i\lambda \int d^3x \; \mathrm{Tr}\, \{ \breve{g}^\dagger \partial_\nu \breve{g}\, \breve{g}^\dagger \partial_\rho \breve{g} \Big(\frac{1}{2} \breve{X}^2\, \breve{g}^\dagger \partial_\mu \breve{g} \;+\; \frac{1}{2} \breve{g}^\dagger \partial_\mu \breve{g}\, \breve{X}^2 \;+\; \frac{1}{2} \partial_\mu \breve{X}^2 \\ &- \breve{X}\, \breve{g}^\dagger \partial_\mu \breve{g}\, \breve{X} \;-\; \breve{X} \partial_\mu \breve{X} \Big) \;+\; \Big(\partial_\mu \breve{X} \;+\; \breve{g}^\dagger \partial_\mu \breve{g}\, \breve{X} \;-\; \breve{X}\, \breve{g}^\dagger \partial_\mu \breve{g} \Big) \\ &\times \Big(\partial_\nu \breve{X} \;+\; \breve{g}^\dagger \partial_\nu \breve{g}\, \breve{X} \;-\; \breve{X}\, \breve{g}^\dagger \partial_\nu \breve{g} \Big)\, \breve{g}^\dagger \partial_\rho \breve{g}\, \} \epsilon^{\mu\nu\rho} \Big], \end{aligned} \tag{3.85}$$

where $\breve{g}(x_1, x_2, t)$ and $\breve{X}(x_1, x_2, t)$ are the extension of $g(x_1, x_2)$ and $X(x_1, x_2)$ respectively, defined over the three-dimensional space. However, from the equations of motion of the WZW-σ model we see that the first order terms in ϵ vanish. Then the lowest nonvanishing terms of

$$\delta S \;=\; S(g') - S(g)$$

are given by:

$$
\begin{aligned}
\delta S =& \frac{\epsilon^2}{4}\Big[\int d^2x \ \mathrm{Tr}\ [\partial_\mu X^\dagger \partial_\mu X \ - \ g^\dagger \partial_\mu g\, (X\partial_\mu X \ - \ \partial_\mu X\, X)] \\
&+ \ 2i\lambda \int d^3x \ \mathrm{Tr}\ [\, \partial_\mu \check{X}\, \partial_\nu \check{X}\, \check{g}^\dagger \partial_\rho \check{g} \\
&- \ \frac{1}{2}(\partial_\mu \check{X}\, \check{X} \ - \ \check{X}\partial_\mu \check{X})\check{g}^\dagger \partial_\nu \check{g}\, \check{g}^\dagger \partial_\rho \check{g}]\, \epsilon^{\mu\nu\rho}\Big] \\
=& \frac{\epsilon^2}{4}\Big[\int d^2x \ \mathrm{Tr}\ [\, \partial_\mu X^\dagger \partial_\mu X \ - \ g^\dagger \partial_\mu g\, (X\partial_\mu X \ - \ \partial_\mu X\, X)] \\
&+ \ i\lambda \int d^3x\, \partial_\mu \ \mathrm{Tr}\ [\, (\check{X}\partial_\nu \check{X} \ - \ \partial_\nu \check{X}\, \check{X})\check{g}^\dagger \partial_\rho \check{g}]\, \epsilon^{\mu\nu\rho}\Big].
\end{aligned}
\tag{3.86}
$$

Next we integrate by parts, and change the variables to the complex coordinates $x_\pm$ and obtain

$$
\begin{aligned}
\delta S =& \frac{\epsilon^2}{2}\int d^2x \ \mathrm{Tr}\ [\, \partial_- X^\dagger \partial_+ X \ + \ \partial_+ X^\dagger \partial_- X \\
&- \ (1 \ + \ \lambda)\, g^\dagger \partial_+ g\, (X\, \partial_- X \ - \ \partial_- X\, X) \\
&- \ (1 \ - \ \lambda)\, g^\dagger \partial_- g\, (X\, \partial_+ X \ - \ \partial_+ X\, X)\,].
\end{aligned}
\tag{3.87}
$$

As we have seen before, g, a solution of the WZW-σ model is related to Q, a solution of the usual σ model by

$$
g^\dagger \partial_\pm g \ = \ \frac{Q^\dagger \partial_\pm Q}{1 \ \pm \ \lambda}. \tag{3.88}
$$

Thus, (3.87) becomes

$$
\begin{aligned}
\delta S =& \frac{\epsilon^2}{2}\int d^2x \ \mathrm{Tr}\ [\, \partial_- X^\dagger \partial_+ X \ + \ \partial_+ X^\dagger \partial_- X \\
&- \ Q^\dagger \partial_+ Q\, (X\partial_- X \ - \ \partial_- X\, X) \ - \ Q^\dagger \partial_- Q\, (X\partial_+ X \ - \ \partial_+ X\, X)\,],
\end{aligned}
\tag{3.89}
$$

which, in turn, can be written in the x_1, x_2 variables as

$$
\delta S \ = \ \frac{\epsilon^2}{4}\int d^2x \ \mathrm{Tr}\ [\, \partial_\mu X^\dagger \partial_\mu X \ - \ Q^\dagger \partial_\mu Q\, (\partial_\mu X^\dagger X \ - \ X^\dagger \partial_\mu X)\,]. \tag{3.90}
$$

In fact, this is nothing else but the stability equation for the σ model without the WZW-term for the solution Q studied by Piette et al. [43]. In other words, the negative fluctuation modes for g are exactly the same as the negative modes for Q and, as all solutions of the σ model are unstable [43], so are the solutions of the WZW-σ model.

For solutions which are unstable, it is interesting to know the number of directions of instability, the so-called negative modes of the fluctuation operator around a given solution. Clearly to find these negative modes of fluctuation, we need to find X such that δS is negative. Piette et al. [43] have shown how to do this in the case of the grassmannian embeddings of the $SU(N)$ chiral models. They considered fluctuations which are hermitian, $X^\dagger = X$. In this case

$$Q = A(1 - 2\mathbb{P}), \tag{3.91}$$

where $\mathbb{P}$ is a grassmannian projector satisfies

$$\begin{aligned} \mathbb{P} = \mathbb{P}^\dagger &= \mathbb{P}^2, \\ \mathbb{P}\partial_+\mathbb{P} &= 0, \\ \partial_+\mathbb{P}\,\mathbb{P} &= \partial_+\mathbb{P}, \end{aligned} \tag{3.92}$$

and A is a constant matrix, so chosen that $\det Q = 1$. Then they showed that [43] for

$$X = \mathbb{P}\,K\,\mathbb{P} - \frac{1}{N}\,\mathrm{Tr}\,\mathbb{P}\,K, \tag{3.93}$$

where K is a constant matrix, which can be written as

$$K = V\,V^\dagger, \tag{3.94}$$

with V is an N-component vector,

$$\delta S = -\frac{1}{2}\int d^2x\,|V^\dagger\partial_+\mathbb{P}\,V|^2\,(1 + \frac{1}{N}), \tag{3.95}$$

which clearly shows that this is a negative mode.

Let us proceed further and show that some negative modes of the fluctuation operator are given by solutions of the associated background Dirac problem. To see this we observe that, as we will see later, the associated background Dirac problem reduces to

$$\partial_+\psi_+ + [\frac{1}{2}Q^\dagger\partial_+Q, \psi_+] = 0, \quad (3.96)$$

$$\partial_-\psi_- + [\frac{1}{2}Q^\dagger\partial_-Q, \psi_-] = 0, \quad (3.97)$$

where $\psi_\pm$ denotes the helicity eigenstates of a spinor ψ (more detailes will be given later). Notice that, if ψ_+ solves (3.96), then a solution of (3.97) is given by $\psi_- = \pm(\psi_+)^\dagger$.

If following ref. [43] we seek fluctuations which are hermitian, we can take for X a hermitian solution of (3.96), which, as we said above, is also a solution of (3.97). On the other hand, if we seek antihermitian fluctuations, we can take for X the antihermitian solution of (3.96), which also solves (3.97).

To claim that such fluctuations are indeed negative modes, we observe that equation (3.89) can be rewritten as

$$\delta S = \epsilon^2 \int d^2x \ \mathrm{Tr} \Big(\partial_-X^\dagger \partial_+X + \partial_-X^\dagger \, [\frac{1}{2}Q^\dagger\partial_+Q, X] - \partial_+X \, [\frac{1}{2}Q^\dagger\partial_-Q, X]\Big). \quad (3.98)$$

Then if we substitute

$$\partial_+X = - [\frac{1}{2}Q^\dagger\partial_+Q, X], \qquad \partial_-X = - [\frac{1}{2}Q^\dagger\partial_-Q, X], \quad (3.99)$$

we get

$$\delta S = -\epsilon^2 \int d^2x \ \mathrm{Tr} \Big((\partial_+X)^\dagger \partial_+X\Big), \quad (3.100)$$

which clearly is negative definite, and so $X = \psi_+ = \pm(\psi_-)^\dagger$ provides a negative mode of fluctuation. The problem of finding solutions of the background Dirac problem will be considered in the next chapters.

We finish this chapter by observing that the grassmannian models can be considered as special cases of the corresponding chiral models in which the basic unitary matrix-valued fields Q are also hermitian, $Q^\dagger = Q$. However, as we saw in the previous chapter, in the construction of multi-uniton solutions to the chiral model we have found that one-uniton solutions do satisfy this condition. This is also true for the multi-uniton solutions

$$Q = K \prod_{i=1}^{\ell} (1 - 2 R_i), \tag{3.101}$$

for which the projectors R_i's commute with each other and K is a constant unitary and hermitian matrix. These properties do not generalise to the solutions of the WZW-σ models. To see this observe that

$$(g(\lambda))^\dagger = g(\lambda^*) \neq g(\lambda), \tag{3.102}$$

even for the one-uniton solutions. This is not surprising, as one can not add the WZW-term to grassmannian models. To have models with such a term the sigma models must take values in the group and not in a coset space.

4. The Supersymmetric $U(N)$ σ Models with the Wess-Zumino-Witten Term.

So far we have studied only bosonic $U(N)$ σ models with the WZW-term. We constructed classical solutions for these models and we found that these solutions are related to, and in fact can be derived from, the solutions of the Lax-pair problem for the $U(N)$ σ model without the WZW-term. Moreover, studying the stability properties of the WZW-σ model we have found that they are not altered by the addition of the WZW-term. In fact the fluctuation determinant is independent of this term.

The real, more physical models, should include fermions. A convenient way of including fermions into σ models discussed before, consists of extending these models to become Susy. In this chapter, we will study the Susy extension of the $U(N)$ σ model with and without the WZW-term. These are the so-called "Susy σ models" and "Susy WZW-σ models". It is interesting to check how many of the properties found for the purely bosonic $U(N)$ σ model do survive the addition of fermions. In particular, will the addition of fermions make the model less stable in the case with and without the WZW-term?

This chapter is organized as follows. In the next section, we introduce the Susy WZW-σ models and derive their equations of motion. The Susy σ models will be introduced as a special case from the Susy WZW-σ models. In section 4.2 we generalise Uhlenbeck's factorisation [42] to the Susy σ models. We construct general Susy solutions of the Susy σ models by following a procedure similar to the one used in the purely bosonic σ model [29]. In section 4.3 we use solutions of the Lax-pair problem for the Susy σ model to derive solutions of the Susy WZW-σ model. In the last section we study some properties of these solutions. We calculate the value of the action for some of these solutions and show that they are related to those of the purely bosonic model. Then we discuss the stability properties of these solutions and prove that all these solutions are unstable.

4.1 Formulation of the Susy WZW-σ Models

Recall that in the Susy procedure, in addition to the standard coordinates x_1, x_2 one has to introduce also anticommuting (Grassmannian) coordinates θ_1, θ_2 which are two components of a real Grassmannian spinor θ. Also, the bosonic field $g(x_1, x_2)$ is replaced by a scalar superfield $\Phi(x_1, x_2, \theta_1, \theta_2)$ defined on the superspace with coordinates x_1, x_2, θ_1, θ_2. Such a procedure of supersymmetrisation for the $U(N)$ σ model with the WZW-term was discussed in some detail by Abdalla et al. [55] and Di Vecchia et al. [56]. We will adopt it too, but here, for convenience, we choose to define our superfield Φ as:

$$\Phi(x_1, x_2, \theta_1, \theta_2) = g(x_1, x_2)[1 + i\theta_1\psi_2(x_1, x_2) - i\theta_2\psi_1(x_1, x_2) + i\theta_1\theta_2 F(x_1, x_2)], \tag{4.1}$$

where ψ_1 and ψ_2 are two components of a complex anticommuting spinor field ψ (which anticommutes with θ_i) and F is an auxiliary scalar field. The hermitian conjugate of Φ is

$$\Phi^\dagger = [1 + i\theta_1\psi_2^\dagger - i\theta_2\psi_1^\dagger + i\theta_1\theta_2 F^\dagger]g^\dagger.$$

On Φ we impose the constraint

$$\Phi^\dagger\,\Phi = \Phi\,\Phi^\dagger = 1. \tag{4.2}$$

This constraint, when expanded in power series in θ_i, implies the following conditions on fields g, ψ_i and F:

$$\begin{aligned} g^\dagger g &= 1, \\ \psi_i^\dagger &= -\psi_i, \\ F + F^\dagger &= i(\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1). \end{aligned} \tag{4.3}$$

For the action of the Susy WZW-σ model we take [55,56]

$$S = \frac{1}{8}\int d^2x\, d^2\theta \ \mathrm{Tr}\ \epsilon^{\mu\nu}(D_\mu\Phi^\dagger D_\nu\Phi) - \frac{k}{4}\int d^3x\, d^2\theta \ \mathrm{Tr}\ [(D_\mu\check{\Phi}^\dagger D_\mu\check{\Phi})\check{\Phi}^\dagger \dot{\check{\Phi}}], \tag{4.4}$$

where k is a real parameter and the superfields Φ and $\check{\Phi}$ are unitary matrices and so satisfy $\Phi^\dagger\Phi = \check{\Phi}^\dagger\check{\Phi} = 1$. As in the pure bosonic model, the 'dot' denotes partial derivative with respect to the additional variable t. The supercovariant derivatives D_μ are defined by

$$\begin{pmatrix} D_{x_1} \\ D_{x_2} \end{pmatrix} = \partial_\theta + i\,\partial\!\!\!/\,\theta = \begin{pmatrix} \partial_{\theta_1} + i(\theta_1\partial_{x_1} + \theta_2\partial_{x_2}) \\ \partial_{\theta_2} + i(\theta_1\partial_{x_2} - \theta_2\partial_{x_1}) \end{pmatrix}, \tag{4.5}$$

where $\partial_{\theta_i} = \frac{\partial}{\partial\theta_i}$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and where we have introduced the two-dimensional γ-matrices defined by

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first term in (4.4) is the Susy extension of the $U(N)$ σ model action of the superfield Φ. The second contribution in (4.4) is the Susy WZW-term in which the matrix-valued superfield $\Phi(x_1, x_2, \theta_1, \theta_2)$ has been extended to $\check{\Phi}(x_1, x_2, t, \theta_1, \theta_2)$, where the additional variable t satisfies $0 \le t \le 1$. As in the previous chapter, following Witten [25], we choose the boundary conditions of this extension to be such that

$$\check{\Phi}(x_1, x_2, 1, \theta_1, \theta_2) = \Phi(x_1, x_2, \theta_1, \theta_2) \qquad \text{and} \qquad \check{\Phi}(x_1, x_2, 0, \theta_1, \theta_2) = K,$$

where K is a constant unitary matrix.

It is convenient to change the Euclidean variables x_1, x_2 to the holomorphic and antiholomorphic variables

$$x_\pm = x_1 \pm ix_2.$$

Also, we introduce the similar conventions for the spinor components θ_i, *i.e.* we define

$$\theta_{\pm} = \theta_1 \pm i\theta_2,$$

and we write the helicity components of the two-component spinor ψ as

$$\psi_{\pm} = \frac{1}{2}(\psi_1 \pm i\psi_2).$$

In terms of these conventions, our superfield (4.1) and its hermitian conjugate can be rewritten as

$$\Phi = g\,[1 - \theta_+\psi_- + \theta_-\psi_+ + \frac{1}{2}\theta_-\theta_+F], \tag{4.6}$$

and

$$\Phi^\dagger = [1 - \theta_+\psi_+^\dagger + \theta_-\psi_-^\dagger + \frac{1}{2}\theta_-\theta_+F^\dagger]g^\dagger. \tag{4.7}$$

Also, the constraint (4.2) implies that

$$\begin{aligned} g^\dagger g &= 1, \\ \psi_\pm^\dagger &= -\psi_\mp, \\ F + F^\dagger &= 2(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-). \end{aligned} \tag{4.8}$$

Moreover, we define

$$D_{\pm} = \frac{1}{2}(D_{x_1} \mp iD_{x_2}),$$

which, using the above expressions, can be written as

$$D_{\pm} = \partial_{\theta_\pm} + i\theta_\pm\partial_\pm, \tag{4.9}$$

where $\partial_{\theta_\pm} = \frac{\partial}{\partial\theta_\pm}$ and $\partial_\pm = \frac{\partial}{\partial x_\pm}$. From the above definitions, one can easily show that

$$\begin{aligned} D_{x_1}D_{x_1} &= i\partial_{x_1}, & D_{x_2}D_{x_2} &= -i\partial_{x_1}, \\ D_+D_+ &= i\partial_+, & D_-D_- &= i\partial_-, \\ \{D_+, D_-\} &= 0, & D_\pm^\dagger &= -D_\mp. \end{aligned} \tag{4.10}$$

According to the standard rules of Grassmannian integration, we have

$$\int d\theta_2 d\theta_1 \; \theta_1\theta_2 \; = \; \int d\theta_+ d\theta_- \; \theta_-\theta_+ \; = \; 1, \tag{4.11}$$

and from the relation $\theta_\pm = \theta_1 \pm i\theta_2$, we have

$$\theta_-\theta_+ \; = \; 2i\theta_1\theta_2. \tag{4.12}$$

Equations (4.11) and (4.12) thus imply

$$d^2\theta \; = \; d\theta_2 d\theta_1 \; = \; 2i \, d\theta_+ d\theta_- .$$

At this stage, the action of the Susy WZW-σ model (4.4) can be rewritten as

$$\begin{aligned} S =& \frac{1}{2}\int d^2x \, d\theta_+ d\theta_- \; \mathrm{Tr} \, (D_+\Phi^\dagger \, D_-\Phi \; - \; D_-\Phi^\dagger \, D_+\Phi) \\ & - \; ik \int d^3x \, d\theta_+ d\theta_- \; \mathrm{Tr} \, \Big[(D_+\check{\Phi}^\dagger \, D_-\check{\Phi} \; + \; D_-\check{\Phi}^\dagger \, D_+\check{\Phi}) \, \check{\Phi}^\dagger \, \dot{\check{\Phi}} \Big]. \end{aligned} \tag{4.13}$$

Next we perform the integration over the anticommuting variables in the action (4.4). Thus, we see that only the coefficient of the quadratic term in the expansion of the Lagrangian in (4.4) in powers of θ contributes to S. Also, we eliminate the auxiliary field F (using its equations of motion) and after a couple of pages of algebra, we find that

$$S = S_0 + \frac{1}{4}\int d^2x \ \mathrm{Tr}\ [i\psi^\dagger \partial\!\!\!/ \psi + i\psi^\dagger g^\dagger \partial\!\!\!/ g\,\psi + \frac{1}{4}(\psi^\dagger\gamma_5\psi)^2 + \frac{k^2}{4}(\psi^\dagger\psi)^2 - ik\,\psi^\dagger\gamma_5\gamma^\mu\psi\, g^\dagger\partial_\mu g\,], \tag{4.14}$$

where

$$S_0 = \frac{1}{4}\int d^2x \ \mathrm{Tr}\ (\partial_\mu g^\dagger\,\partial_\mu g) - \frac{k}{6}\int d^3x \ \mathrm{Tr}\ \epsilon^{\mu\nu\rho}(g^\dagger\partial_\mu g\, g^\dagger\partial_\nu g\, g^\dagger\partial_\rho g). \tag{4.15}$$

Notice that S_0 is exactly the action of the purely bosonic model defined in the previous chapter with $\lambda = ik$. Clearly, when $k = 0$, S reduces to the action of the Susy σ model without the WZW-term.

The equations of motion corresponding to (4.4) can be easily shown not to depend on our extension used in the definition of the WZW-term, but be given by

$$\epsilon^{\mu\nu} D_\mu(\Phi^\dagger D_\nu\Phi) - k\, D_\mu(\Phi^\dagger D_\mu\Phi) = 0, \tag{4.16}$$

which can be rewritten in the form

$$(1 - \lambda)D_+(\Phi^\dagger D_-\Phi) - (1 + \lambda)D_-(\Phi^\dagger D_+\Phi) = 0, \tag{4.17}$$

where $\lambda = ik$. Again, it is clear that when $k = 0$, the equations reduce to the equations of motion of the Susy σ model

$$D_+(\phi^\dagger D_-\phi) - D_-(\phi^\dagger D_+\phi) = 0, \tag{4.18}$$

where ϕ denotes the corresponding superfield which is required to satisfy the constraint $\phi^\dagger\phi = 1$.

Observe that if we define

$$A_{\pm} = \frac{1}{2}\Phi^{\dagger} D_{\pm}\Phi,$$

which satisfy the zero-curvature condition

$$D_{+}A_{-} + D_{-}A_{+} + 2\{A_{+}, A_{-}\} = 0, \tag{4.19}$$

and consider

$$C_{\pm} = (1 \pm \lambda) A_{\pm},$$

then equation (4.17) becomes

$$D_{+}C_{-} - D_{-}C_{+} = 0. \tag{4.20}$$

However, from (4.17) and (4.19), it is easy to see that

$$D_{+}C_{-} + D_{-}C_{+} + 2\{C_{+}, C_{-}\} = 0. \tag{4.21}$$

This means that there must exist a matrix superfield G which satisfies the constraint $G^{\dagger}G = 1$ and is related to C by

$$C_{\pm} = \frac{1}{2} G^{\dagger} D_{\pm}G.$$

This matrix superfield G can then be expressed in terms of the component fields g, ψ_i and F. Moreover, equations (4.20) and (4.21) together guarantee that the Susy WZW-σ model is integrable [28].

Returning to (4.17), we see that the equations of motion can be resolved into the components

$$
\begin{aligned}
\partial_+\psi_+ + \frac{1}{2}(1+\lambda)\,[g^\dagger\partial_+g\,,\,\psi_+] - \frac{i}{4}(1+\lambda)\,(\psi_-F + F^\dagger\psi_-) &= 0\\
\partial_-\psi_- + \frac{1}{2}(1-\lambda)\,[g^\dagger\partial_-g\,,\,\psi_-] + \frac{i}{4}(1-\lambda)\,(\psi_+F + F^\dagger\psi_+) &= 0\\
(1-\lambda)\,[\partial_+(g^\dagger\partial_-g) + i\partial_+(\psi_+\psi_+)] + (1+\lambda)\,[\partial_-(g^\dagger\partial_+g) + i\partial_-(\psi_-\psi_-)] &= 0,
\end{aligned}
\tag{4.22}
$$

with F given by

$$
F = (1+\lambda)\psi_+\psi_- - (1-\lambda)\psi_-\psi_+.
$$

This shows that they describe a rather complicated system of coupled bosonic and fermionic fields. It is easy to see that [28] the last equation in (4.22) corresponds to the conservation of the Noether currents $J_\pm$:

$$
\partial_+J_- + \partial_-J_+ = 0,
$$

where

$$
\begin{aligned}
J_- &= (1-\lambda)(g^\dagger\partial_-g + i\psi_+\psi_+),\\
J_+ &= (1+\lambda)(g^\dagger\partial_+g + i\psi_-\psi_-).
\end{aligned}
$$

In the next two sections, we will present solutions of the Susy σ model equations of motion (4.18) and of the equations of motion of the Susy WZW-σ model (4.17).

4.2 Solutions of the Susy σ Model

To construct solutions of the $U(N)$ Susy σ model, we will follow a procedure very similar to the one used to construct classical solutions of the purely bosonic $U(N)$ σ model [29], which we discussed in the second chapter. First we need to prove a generalisation of Uhlenbeck's factorisation for the Susy σ model. To do this we assume that ϕ_0 is a given solution of (4.18) and that R is a projector which satisfies

$$\begin{aligned} RA^0_-(1 - R) &= 0, \\ (1 - R)(D_-R + A^0_-R) &= 0, \end{aligned} \tag{4.23}$$

where

$$A^0_- = \frac{1}{2}\phi_0^\dagger D_-\phi_0.$$

Then, the factorisation states that

$$\phi_1 = \phi_0(1 - 2R) \tag{4.24}$$

is another solution of (4.18). To prove this it is sufficient to note that

$$\begin{aligned} A^1_- &= \frac{1}{2}\phi_1^\dagger D_-\phi_1 = A^0_- + D_-R, \\ A^1_+ &= (A^1_-)^\dagger = A^0_+ - D_+R. \end{aligned}$$

Thus

$$D_+A^1_- - D_-A^1_+ = D_+A^0_- - D_-A^0_+ + D_+D_-R + D_-D_+R.$$

As ϕ_0 is a solution of (4.18) and $\{D_+, D_-\} = 0$, we see that the right hand side vanishes and that ϕ_1 satisfies the equations of motion (4.18). Notice that we have proved that solutions can be constructed in this way. However, we don't know whether this procedure gives us all solutions of these equations. In the purely bosonic case, when we restrict ourselves to solutions of finite action, this is guaranteed by Uhlenbeck's theorem but we don't know whether her theorem can be extended to cover also the Susy case.

Applying the above mentioned procedure several times enables us to construct new solutions from any given one by successively adding new projector factors corresponding to Uhlenbeck's unitons; here, in what follows, we will call them Susy unitons.

We are now ready to construct classes of solutions for the $U(N)$ Susy σ models. The simplest of these are the Susy one-uniton solutions (Susy self-dual solutions); namely, those obtained from (4.24) with ϕ_0 a constant (we choose $\phi_0 = 1$), *i.e.*

$$\phi_1 = 1 - 2R, \tag{4.25}$$

where

$$(1 - R)D_- R = 0. \tag{4.26}$$

The simplest solutions of this equation are those for which

$$R = P(V) = V(V^\dagger V)^{-1} V^\dagger, \tag{4.27}$$

where V is a maximal rank matrix superfield which satisfies

$$D_- V = 0. \tag{4.28}$$

The solutions of (4.28) are of the form

$$V = V_0(x_+) + i\theta_+ V_-(x_+), \tag{4.29}$$

where V_0 is a holomorphic maximal rank bosonic matrix and V_- is any holomorphic fermionic matrix. Such solutions were first considered by D'Adda et al. [16] and Din et al. [48] in their studies of Susy grassmannian σ models.

To construct more general solutions of the $U(N)$ Susy σ-models, we follow the steps used in ref. [29] by adding Susy unitons to the Susy one-uniton solution constructed above. So a Susy ℓ-uniton solution will be of the type

$$\phi_\ell = K(1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_\ell), \tag{4.30}$$

where K is a constant matrix, R_i's are projectors which satisfy (4.23) for the corresponding $A_-^{i-1} = \frac{1}{2}\phi_{i-1}^\dagger D_- \phi_{i-1}$ and ℓ represents the Susy uniton number.

The method used in ref. [29] consists of constructing a set of mutually embedded vector subspaces equivalent to the so-called holomorphic basis sequence of DZ type [46]. Applying this technique, we will first construct some Susy non-self-dual grassmannian solutions. Some of these solutions will be different from those discussed by Din et al. [48]. Then, we will add an additional Susy uniton to these solutions and construct some Susy non-grassmannian $U(N)$ solutions.

Before constructing such solutions, let us give a simple example of a Susy non-self-dual $\mathbb{C}P^{N-1}$ solution. Consider the matrix

$$F = \Big(f \,,\, \partial_+ f \Big) + i\theta_+ \Big(\partial_+ f \,,\, \partial_+^2 f \Big)\Gamma, \tag{4.31}$$

where f is an holomorphic bosonic vector $(\partial_- f = 0)$ and where $(f, \partial_+ f)$ stands for a $N \times 2$ matrix whose first column is f, and Γ is given by

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix},$$

where Γ_{11}, Γ_{12} and Γ_{22} are fermionic constants. If $\Gamma_{11}\Gamma_{12} = \Gamma_{11}\Gamma_{22} = 0$ and, in addition, if

$$V = f + i\theta_+ \Gamma_{11}\, \partial_+ f,$$

then

$$\begin{aligned} D_+ V &= i\, \partial_+ f\, (\theta_+ + \Gamma_{11}) \\ &= i\, F \begin{pmatrix} 0 \\ \theta_+ + \Gamma_{11} \end{pmatrix}. \end{aligned}$$

Thus as $P(F)P(V) = P(V)$, we see that

$$(1 - P(F))D_\pm P(V) = 0. \tag{4.32}$$

We can now take

$$\phi_0 = (1 - 2P(F)), \qquad \text{and} \qquad R = P(V). \tag{4.33}$$

Then as $D_- F = D_- V = 0$, it is easy to check that (4.23) is satisfied and so

$$\phi_1 = (1 - 2P(F))(1 - 2P(V)) = \Big(1 - 2(P(F) - P(V))\Big) \qquad (4.34)$$

is a new solution of (4.18). This is actually a Susy $\mathbb{C}P^{N-1}$ solution of the $U(N)$ model spanned by a regular vector (*i.e.* a vector which possesses a projector)

$$u = (1 - P(V))\,[\partial_+ f + i\theta_+(\Gamma_{12}\,\partial_+ f + \Gamma_{22}\,\partial_+^2 f)].$$

To generalise the above construction, we start from a regular superholomorphic matrix

$$V_1 = F_1(x_+) + i\theta_+ G_1(x_+), \qquad \text{with} \quad D_- V_1 = 0,$$

where F_1 is a maximal rank bosonic matrix and G_1 a fermionic matrix. Next we construct another superholomorphic regular matrix

$$V_2 = \Big(F_1(x_+)\,,\; F_2(x_+)\Big) + i\theta_+\Big(G_1(x_+)\,,\; G_2(x_+)\Big),$$

such that

$$D_+ V_1 = V_2\,\omega,$$

for some fermionic holomorphic matrix ω:

$$\omega = \omega_0 + i\theta_+\omega_-.$$

This condition imposes some restrictions on F_2 and G_2, namely:

$$\begin{aligned} iG_1 &= \Big(F_1\,,\; F_2\Big)\omega_0, \\ \partial_+ F_1 &= \Big(G_1\,,\; G_2\Big)\omega_0 + \Big(F_1\,,\; F_2\Big)\omega_-. \end{aligned} \qquad (4.35)$$

Thus we can put, in full generality,

$$G_2 = \Big(F_1\ ,\ F_2\Big)A_2 + G_2',$$

for some matrix A_2 and for some G_2' orthogonal to F_1 and F_2:

$$\Big(F_1\ ,\ F_2\Big)^\dagger G_2' = 0.$$

It is easy to check that the restriction (4.35) implies that F_2 must be such that F_1 and F_2 together span G_1. Moreover, $\partial_+ F_1$ must be spanned by F_1, F_2 and G_2'. Given the matrix V_1, it is easy to construct V_2 which satisfies (4.35) and which is such that

$$P(V_2)\,P(V_1) = P(V_1).$$

For such a V_2, we see that

$$(1 - P(V_2))D_\pm P(V_1) = 0,$$

which by (4.23) and (4.26) implies that

$$\phi = (1 - 2P(V_2))(1 - 2P(V_1)) \tag{4.36}$$

is a solution of (4.18). As $P(V_2)$ and $P(V_1)$ commute, we see that ϕ is grassmannian ($\phi^\dagger = \phi$).

Applying the method described above, it is easy to construct a set of superholomorphic matrices

$$V_1\,, V_2\,, \cdots\,, V_k$$

with $D_- V_i = 0$ for all i, such that

$$P(V_i)\,P(V_j) = P(V_j)$$

for all $i \geq j$ and

$$(1 - P(V_i))D_+P(V_j) = 0 \qquad i > j,$$

and which moreover satisfy

$$(1 - P(V_i))D_-P(V_j) = 0 \qquad i \geq j.$$

These matrices are equivalent to what is called in refs. [29,46] a holomorphic basis sequence of DZ type. Defining

$$\phi_i = (1 - 2P_k)(1 - 2P_{k-1}) \cdots (1 - 2P_i), \tag{4.37}$$

as in the purely bosonic case [29], we prove by induction that

$$A_-^i = \sum_{j=i}^{k} D_-P(V_j),$$

and

$$P(V_{i-1})A_-^i = 0 \qquad \text{for all} \quad i.$$

From Uhlenbeck's factorisation, we can now conclude that each $P(V_i)$ is a Susy version of a basic [44] uniton for the solution ϕ_{i-1}; and, ϕ_i, given by (4.37), is a solution of (4.18).

All the solutions constructed so far are grassmannian solutions of the Susy $U(N)$ σ models, but they are different from those constructed by Din et al. [48] as all our expressions are given by well defined projectors.

To construct non-grassmannian solutions, we must add a further Susy uniton to the solution (4.37). To do this let us first define, for all j such that $1 \leq j \leq \ell$, the "largest" matrix Y_j such that

$$Y_j = V_j a_j,$$

where each a_j is a rectangular matrix and V_j's are so chosen that

$$(1 - P(V_j))D_+Y_j = 0.$$

Next we construct

$$\varphi_j = (1 - P(V_{j-1}))Y_j,$$

and taking

$$W = \sum_{j=1}^{\ell+1} \varphi_j b_j,$$

for an arbitrary regular superholomorphic matrix b_j, we find that

$$\check{\phi}^{\ell} = \prod_{j=\ell}^{1}(1 - 2P(V_j))(1 - 2P(W)) \tag{4.38}$$

is a non-grassmannian solution of (4.18). The proof of this statment is similar to the one given in the purely bosonic case [29]. Thus first we observe that

$$\varphi_j^{\dagger} D_- P(V_k) = 0 \qquad \text{for all} \quad j\,,k$$

and so that

$$\varphi_j^{\dagger} A_-^{\ell} = 0 \qquad \text{for all} \quad j,$$

thus demonstrating that the first equation in (4.23) is satisfied. Moreover, as

$$D_- P(V_k)\varphi_j = 0 \qquad \text{for all} \quad k \neq j-1,$$

we see that

$$A_-^{\ell}\, \varphi_j = P(V_{j-1})D_-P(V_{j-1})Y_j.$$

Then few lines of algebra show that the second equation of (4.23) is also satisfied as [29]

$$(1 - P(W))(D_-W + A_-^\ell W) = -\sum_{i=1}^{\ell+1}\Big(1 - P(W) - P(V_{i-1})\Big)D_-P(V_{i-1})Y_i a_i = 0,$$

which completes the proof.

4.3 Solutions of the Susy WZW-σ Model

As we said in previous chapters many two-dimensional models are known to be integrable, as one can associate with them linear systems (the Lax-pair formalism) and so they possess an infinite number of conservation laws. In ref. [28] it was shown that the Susy WZW-σ models possess these properties.

As in the purely bosonic model discussed in the previous chapter it is easy to see that the solutions of the Susy WZW-σ model are very closely related to the solutions of such a linear system. To see this, consider the Lax-pair equations for the Susy $U(N)$ σ model:

$$D_+\Psi = \Psi\frac{2A_+}{1+\lambda}, \qquad D_-\Psi = \Psi\frac{2A_-}{1-\lambda}, \tag{4.39}$$

where

$$A_\pm = \frac{1}{2}\phi^\dagger D_\pm\phi,$$

and where $\Psi(x_+, x_-, \theta_+, \theta_-, \lambda)$ is a $N \times N$ matrix-valued superfield, and λ is an additional complex parameter. Then, exactly as in the purely bosonic case, the unitarity of ϕ provides us with some further conditions on Ψ [54], and if λ is imaginary, then Ψ is unitary. Moreover it is easy to see that, if ϕ satisfies (4.18),

$$\Phi = \Psi \tag{4.40}$$

satisfies (4.17). This shows that the solutions of the Lax-pair problem for the Susy $U(N)$ σ model provide us with a simple way of constructing solutions of the Susy WZW-σ model.

Notice that, like in the previous chapter, we could have chosen to construct the Lax-pair problem in the form

$$D_+\chi = \frac{2B_+}{1+\lambda}\chi, \qquad D_-\chi = \frac{2B_-}{1-\lambda}\chi, \tag{4.41}$$

where

$$B_\pm = \frac{1}{2} D_\pm\phi\,\phi^\dagger.$$

Then as $\phi^\dagger$ is a solution of (4.18), it is easy to show that $\chi(-\lambda)$ satisfies (4.17) for the imaginary value of λ.

Moreover, as in the previous chapter, it is easy to prove that the solutions Ψ of (4.39) are related to the solutions χ of (4.41) by

$$\chi(\lambda) = \phi\,\Psi^{-1}(\frac{1}{\lambda}). \tag{4.42}$$

Following the construction discussed in the previous chapter, we define

$$K_\ell = K_{\ell-1}(1 - a\,R_\ell), \tag{4.43}$$

where a is a complex number (which does not depend on ℓ) and R_i's are the projectors which satisfy (4.23). Then

$$K_\ell^{-1} = (1 - b\,R_\ell)\,K_{\ell-1}^{-1}, \tag{4.44}$$

where b is the complex number, which satisfies

$$a + b - a\,b = 0. \tag{4.45}$$

As a result we can now deduce that

$$\begin{aligned} K_\ell^{-1}\,D_-K_\ell &= b\,A_-^\ell, \\ K_\ell^{-1}\,D_+K_\ell &= a\,A_+^\ell, \end{aligned} \tag{4.46}$$

where

$$A_\pm^\ell = \frac{1}{2}\phi_\ell^\dagger D_\pm \phi_\ell. \tag{4.47}$$

The proof is exactly the same as the one given in the previous chapter for the purely bosonic case.

As an immediate consequence, we see that if we choose

$$a = \frac{2}{1+\lambda}, \qquad b = \frac{2}{1-\lambda},$$

the condition (4.45) is satisfied and

$$\Psi = K_\ell \tag{4.48}$$

is a solution of (4.39) corresponding to ϕ_ℓ.

4.4 Properties of the Susy Solutions

After we have shown how to construct solutions of the Susy $U(N)$ σ model with and without the WZW-term, it is interesting to study some of the properties of these solutions. In particular, we would like to compute the values of the action corresponding to these solutions. It would then be interesting to check whether these values are related to the values of the action of the purely bosonic $U(N)$ σ models with and without the WZW-term, studied in the previous chapters, and whether they can be related to the topological charges of the projectors R_i's which appear in the construction of the solutions. Moreover, we would like to know how the values of the Susy action are related to the properties of the superholomorphic matrices V_i's. A further interesting point would be to determine whether these solutions are unstable.

Before we proceed to do this, let us recall that in the purely bosonic $U(N)$ σ model, the value of the action of a ℓ-uniton solution is given by the sum of the topological charges of all the projectors corresponding to each uniton. To check whether this property can be extended to the Susy models discussed above, we consider the action of the Susy $U(N)$ σ model

$$\begin{aligned} S_0 &= \frac{1}{2} \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr}\ (D_+\phi^\dagger \, D_-\phi \ - \ D_-\phi^\dagger \, D_+\phi) \\ &= -4 \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr}\ (A_+ \, A_-), \end{aligned} \tag{4.49}$$

where

$$A\pm \ = \ \frac{1}{2}\phi^\dagger D_\pm \phi.$$

Then we look at a Susy solution constructed by the factorisation procedure

$$\phi \ = \ \phi_0 \, (1 \ - \ 2R), \tag{4.50}$$

where ϕ_0 is a given solution of the Susy σ model and R is a projector which satisfies (4.23). We recall from the previous sections that for this solution

$$\begin{aligned} A_+ &= A_+^0 \ - \ D_+ R, \\ A_- &= A_-^0 \ + \ D_- R, \end{aligned}$$

where $A_\pm^0 = \frac{1}{2}\phi_0^\dagger D_\pm \phi_0$. For the solution (4.50), the action (4.49) becomes

$$S_0 \ = \ -4 \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr}\ [A_+^0 \, A_-^0 \ - D_+R \, D_-R + A_+^0 D_-R - D_+R \, A_-^0]. \tag{4.51}$$

Using (4.23) and the trace properties, it is easy to see that

$$\begin{aligned} \mathrm{Tr}\ D_+R \, A_-^0 \ &= \ - \, \mathrm{Tr}\ D_+R \, (1 \ - \ R) D_-R, \\ \mathrm{Tr}\ A_+^0 D_-R \ &= \ \mathrm{Tr}\ D_+R \, (1 \ - \ R) D_-R. \end{aligned}$$

Then a few lines of algebra show that

$$S_0 = S_0(\phi_0) + 4 \int d^2x \, d\theta_+ d\theta_- \, \check{\mathcal{Q}}, \tag{4.52}$$

where $S_0(\phi_0)$ is the action of the Susy σ model for the solution ϕ_0 and where

$$\check{\mathcal{Q}} = \mathrm{Tr} \, (D_+R \, R \, D_-R + D_-R \, R \, D_+R) \tag{4.53}$$

is the Susy topological charge density of the projector R. At this stage we conclude that, as in the purely bosonic model, the action of a Susy ℓ-uniton solution can be computed by adding the Susy topological charge densities in superspace of all projectors defining each uniton and so the action is given by

$$S_0 = 4 \int d^2x \, d\theta_+ d\theta_- \sum_{i=1}^{\ell} \check{\mathcal{Q}}_i, \tag{4.54}$$

where the Susy topological charge density $\check{\mathcal{Q}}_i$ is given by

$$\check{\mathcal{Q}}_i = \mathrm{Tr} \, (D_+R_i \; R_i \, D_-R_i + D_-R_i \; R_i \, D_+R_i). \tag{4.55}$$

Let us now calculate the value of the action (4.49) for a Susy ℓ-uniton solution

$$\phi_\ell = K \, (1 - 2R_1)(1 - 2R_2) \cdots (1 - 2R_\ell). \tag{4.56}$$

As we have seen, all we need to do is to compute the topological charge density (4.55) for every projector R_i in (4.56). To do this we write

$$R_i = P(V_i) = V_i (V_i^\dagger V_i)^{-1} V_i^\dagger, \tag{4.57}$$

and normalise V_i in such a way that

$$V_i^\dagger D_- V_i = 0. \tag{4.58}$$

Then we find that

$$\check{Q}_i = -\operatorname{Tr}\left[D_+V_i^\dagger\, D_-V_i\,|V_i|^{-2} + D_-V_i^\dagger\,(1 - P(V_i))D_+V_i\,|V_i|^{-2}\right]$$
$$= \operatorname{Tr}\, D_+\Big(D_-|V_i|^2\; |V_i|^{-2}\Big).$$

Thus (4.54) becomes

$$S_0 = 4\int d^2x\, d\theta_+ d\theta_-\; \operatorname{Tr}\, D_+\Big(D_-|V_i|^2\; |V_i|^{-2}\Big). \tag{4.59}$$

Performing the θ integration using the property that for any matrix-valued superfield X

$$\int d\theta_+ d\theta_-\, X = \partial_{\theta_+}\partial_{\theta_-} X = D_+D_-X\,|_{\theta_+=\theta_-=0}, \tag{4.60}$$

we find that

$$S_0 = -4\int d^2x\; \operatorname{Tr}\left[D_+D_+\,[\,D_-D_-|V_i|^2\; |V_i|^{-2} - D_-|V_i|^2\, D_-|V_i|^{-2}\,]\right]_{\theta_+=\theta_-=0}.$$

If we now write

$$V_i = V_i^0 + i\theta_+V_i^- + i\theta_-V_i^+ + \theta_-\theta_+V_i^1,$$

and use the properties (4.10) and the fact that for any invertible matrix A

$$\partial_\mu\, \operatorname{Tr}\,(\ln\, det\,(A)\,) = \operatorname{Tr}\,(\partial_\mu A\; A^{-1}), \tag{4.61}$$

then we end up with

$$\begin{aligned} S_0 &= 4\int d^2x\, \partial_+\partial_-\; \ln\, det\,|V_i^0|^2 \\ &= \int d^2x\, \partial_\mu\partial_\mu\; \ln\, det\,|V_i^0|^2, \end{aligned} \tag{4.62}$$

which is, in fact, the action corresponding to the purely bosonic projector

$$P_i^0 = V_i^0(V_i^{0\dagger}V_i^0)^{-1}V_i^{0\dagger}.$$

In the particular case corresponding to the Susy one-uniton solution given by

(4.25),(4.26) the action of the Susy σ model is given by

$$\begin{aligned} S_0 &= -4\int d^2x d\theta_+ d\theta_- \ \mathrm{Tr}\ (D_- R\, D_+ R) \\ &= -4\int d^2x \left[D_+ D_- \ \mathrm{Tr}\ (D_- R\, D_+ R)\right]_{\theta_+=\theta_-=0}, \end{aligned} \tag{4.63}$$

where we have used the property (4.60). If we now reexpress the projector R in terms of the superholomorphic matrix V, which is given by (4.27)-(4.29), we find that

$$\begin{aligned} S_0 &= -4\int d^2x \left[D_+ D_- \ \mathrm{Tr}\ [D_- V^\dagger (1 - P(V)) D_+ V \ |V|^{-2}]\right]_{\theta_+=\theta_-=0} \\ &= -4\int d^2x \ \mathrm{Tr}\ \left[D_+ D_+ [D_- D_- |V|^2 \ |V|^{-2} - D_- |V|^2 D_- |V|^{-2}]\ \right]_{\theta_+=\theta_-=0}, \end{aligned}$$

and so that

$$S_0 = \int d^2x \ \partial_\mu \partial_\mu \ \ln\ det|V_0|^2. \tag{4.64}$$

Of course this is the value of the action for the corresponding solution of the purely bosonic $U(N)$ σ model [47]. We see that there is no fermionic contribution to the action. In fact, this result has already been obtained by Din et al. [48] for the case of the Susy $\mathbb{C}P^{N-1}$ model.

Next, we calculate the action of the Susy WZW-σ model defined by (4.13). To do this, we have to find a convenient way of computing explicitly the contribution of the Susy WZW-term. As we have already mentioned before, for the extension of the superfield Φ to $\check{\Phi}$ we will follow the procedure given in the previous chapter for the purely bosonic model.

We start by considering the solution of (4.17) constructed by solving (4.39) for the Susy one-uniton configuration

$$\phi = (1 - 2R),$$

namely:

$$\Phi = (1 - a R), \tag{4.65}$$

where

$$D_- R R = 0 \qquad \text{and} \qquad a = \frac{2}{1 + \lambda} .$$

For our extention we can now take

$$\begin{aligned} \check{\Phi}(x_+, x_-, t, \theta_+, \theta_-) &= (1 - a R)^t \\ &= (1 - a(t) R) \\ &= e^{i\alpha t R}, \end{aligned} \tag{4.66}$$

where

$$a(t) = (1 - e^{i\alpha t}),$$

with

$$i\alpha = \ln (1 - a) = \ln \frac{\lambda - 1}{\lambda + 1} .$$

It is now easy to show that for this solution, the action of the Susy WZW-σ model (4.13) is given by

$$\begin{aligned} S &= - \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr} \ (D_- R \, D_+ R) \, |a|^2 \\ &\quad + k \int d^3x \, d\theta_+ d\theta_- \ \alpha \, |a(t)|^2 \ \mathrm{Tr} \ [(D_+ R \, D_- R + D_- R \, D_+ R) R] \\ &= - \left[|a|^2 - \int_0^1 dt \ \alpha k \, |a(t)|^2 \right] \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr} \ (D_- R \, D_+ R). \end{aligned} \tag{4.67}$$

Using the relations between a and α, and writing $\lambda = ik$, we can perform the t integration and obtain

$$\begin{aligned} S &= - 4 \, h(k) \int d^2x \, d\theta_+ d\theta_- \ \mathrm{Tr} \ (D_- R \, D_+ R) \\ &= h(k) \, S_0, \end{aligned} \tag{4.68}$$

where S_0 is the action of the corresponding solution of the Susy σ model without the WZW-term given by (4.63) or (4.64), and where

$$h(k) = 1 - k \tan^{-1}(k^{-1}). \tag{4.69}$$

Notice that $h(k)$ is exactly the same factor as in the purely bosonic WZW-σ model established in the previous chapter. As we have already mentioned in that chapter, it is important to note that as in the purely bosonic case the value of α is defined only up to the addition of an integer multiple of 2π and, as a consequence, the value of (4.67) is not uniquely defined.

We see that the action of the solution of the Susy WZW-σ model is given by the action of the corresponding solution of the Susy σ model multiplied by $h(k)$. Moreover, as we have shown above, the value of the action of the solution of the Susy WZW-σ model is given by the action of the corresponding solution of the purely bosonic $U(N)$ σ model multiplied by the same factor $h(k)$.

Next we study one of the most important properties of our Susy solutions, that is their stability under small fluctuations. Following the procedure of ref. [43], which we used in the previous chapter for the purely bosonic case, we consider

$$\Phi' = \Phi\, e^{\epsilon X}, \tag{4.70}$$

where Φ is a solution of the equations of motion of the Susy WZW-σ model (4.17), and where X is an antihermitian $(X^\dagger = -X)$ matrix-valued superfield. Then it is easy to check that the first order terms in ϵ vanish because of the equations of motion and that the lowest nonvanishing terms of

$$\delta S = S(\Phi') - S(\Phi)$$

are given by:

$$\delta S = \frac{i\,\epsilon^2}{4}\int d^2x\, d^2\theta\ \mathrm{Tr}\Big[2D_+X\,D_-X + (1+\lambda)\Phi^\dagger D_+\Phi\,(XD_-X - D_-X\,X) - (1-\lambda)\Phi^\dagger D_-\Phi\,(XD_+X - D_+X\,X)\Big], \tag{4.71}$$

where $\lambda = ik$. However, as we have shown before, the solutions Φ of the Susy WZW-σ model are related to the solutions ϕ of the Susy σ model without the WZW-term by

$$(1 \pm \lambda)\,\Phi^\dagger D_\pm\Phi = \phi^\dagger D_\pm\phi. \tag{4.72}$$

Using this property we find that (4.71) takes the form

$$\delta S = \epsilon^2\int d^2x\, d\theta_+ d\theta_-\ \mathrm{Tr}\Big[D_-X\,D_+X - \frac{1}{2}\phi^\dagger D_+\phi\,(XD_-X - D_-X\,X) + \frac{1}{2}\phi^\dagger D_-\phi\,(XD_+X - D_+X\,X)\Big]. \tag{4.73}$$

Let us point out that had we started with a Susy model without the WZW-term (*i.e.* considered (4.13) with $k = 0$) and used the corresponding equations of motion of the model we would have got exactly the same stability equation (4.73). Thus we see that the negative fluctuation modes for Φ are exactly the same as the negative modes for ϕ.

Let us now prove that, as for the purely bosonic model [43], all the Susy ℓ-uniton solutions, given by (4.56), are unstable. To see this, we consider the field configuration given by

$$\phi_\ell^{(t)} = \prod_{i=1}^{\ell}(1 - 2R_i)^t = \prod_{i=1}^{\ell}\Big(1 - a(t)R_i\Big), \tag{4.74}$$

where

$$a(t) = (1 - e^{i\pi t}).$$

Notice that for $t = 1$ this corresponds to (4.56). If we now compute the action of the Susy σ model without the WZW-term for this field configuration we find that

$$S(\phi_\ell^{(t)}) = \frac{1}{2}\int d^2x\, d\theta_+ d\theta_-\ \mathrm{Tr}\left(D_+\phi_\ell^{(t)\dagger}\, D_-\phi_\ell^{(t)} - D_-\phi_\ell^{(t)\dagger}\, D_+\phi_\ell^{(t)}\right).$$

Using relation (4.46), it is easy to show that

$$S(\phi_\ell^{(t)}) = S(\phi_\ell)\,\frac{|a(t)|^2}{4} = S(\phi_\ell)\left(\frac{1-\cos(\pi t)}{2}\right). \tag{4.75}$$

As this function decreases as we move away from $t = 1$, we can conclude that all Susy ℓ-uniton solutions of the Susy σ model are unstable.

To find the corresponding negative modes, we first perform the integration over θ's in equation (4.73). Thus we consider

$$\phi = Q\,[\,1 - \theta_+\psi_- + \theta_-\psi_+ + \frac{1}{2}\theta_-\theta_+F\,], \tag{4.76}$$

Also, we consider a general fluctuation X in the form

$$X = X_0 + \theta_+X_- + \theta_-X_+ + \theta_-\theta_+X_1. \tag{4.77}$$

The condition $X^\dagger = -X$ thus implies

$$X_0^\dagger = -X_0,$$
$$X_\pm^\dagger = X_\mp,$$
$$X_1^\dagger = -X_1.$$

Then a few pages of calculation shows that the expression for δS, equation (4.73), after the integration over θ's becomes

$$\begin{aligned}
\delta S =& \epsilon^2 \int d^2x \ \mathrm{Tr} \Big[-\partial_- X_0 \, \partial_+ X_0 \ - \ \partial_- X_0 \ [(\frac{1}{2} Q^\dagger \partial_+ Q \ + \ \frac{i}{2} \psi_- \psi_-), X_0] \\
& - \ \partial_+ X_0 \ [(\frac{1}{2} Q^\dagger \partial_- Q \ + \ \frac{i}{2} \psi_+ \psi_+), X_0] \\
& + \ i \partial_- X_- \, X_- \ + \ 2i \, (\frac{1}{2} Q^\dagger \partial_- Q \ + \ \frac{i}{2} \psi_+ \psi_+) \, X_- X_- \\
& + \ i \partial_+ X_+ \, X_+ \ + \ 2i \, (\frac{1}{2} Q^\dagger \partial_+ Q \ + \ \frac{i}{2} \psi_- \psi_-) \, X_+ X_+ \\
& + \ \frac{i}{2} \psi_- \{ \, [X_0, \partial_- X_-] + [X_-, \partial_- X_0] \, \} \ - \ \frac{i}{2} \psi_+ \{ \, [X_0, \partial_+ X_+] + [X_+, \partial_+ X_0] \, \} \\
& + \ \frac{i}{2} \, \{ \, \partial_- \psi_- \, [X_0, X_-] \ - \ \partial_+ \psi_+ \, [X_0, X_+] \, \} \\
& - \ \frac{1}{2} \, (\psi_+ \psi_- \ + \ \psi_- \psi_+)(X_+ X_- \ + \ X_- X_+) \\
& + \ X_1 X_1 \ - \ X_1 (\psi_- X_+ \ + \ X_+ \psi_- \ + \ \psi_+ X_- \ + \ X_- \psi_+) \Big] .
\end{aligned} \tag{4.78}$$

Clearly, in general, this expression is rather complicated but if we choose our fluctuation X to be independent of θ's, *i.e.*

$$X_- \ = \ X_+ \ = \ X_1 \ = \ 0,$$

then δS takes the form

$$\begin{aligned}
\delta S \ = \ - \, \epsilon^2 \int dx^2 \ \mathrm{Tr} \Big[\partial_- X_0 \, \partial_+ X_0 \ + \ \partial_- X_0 \, [\, (\frac{1}{2} Q^\dagger \partial_+ Q + \frac{i}{2} \psi_- \psi_-) \, , \, X_0] \\
+ \ \partial_+ X_0 \, [\, (\frac{1}{2} Q^\dagger \partial_- Q + \frac{i}{2} \psi_+ \psi_+) \, , \, X_0] \Big] .
\end{aligned} \tag{4.79}$$

But even this expression is difficult to study in full generality. Thus, to simplify the calculations even further let us look at the simplest Susy one uniton solution and the corresponding purely bosonic solution:

$$\phi \ = \ 1 \ - \ 2R, \qquad Q \ = \ 1 \ - \ 2\mathbb{P}, \tag{4.80}$$

with the projectors R and $\mathbb{P}$ defined by

$$R = h(h^\dagger h)^{-1} h^\dagger , \qquad \mathbb{P} = f(f^\dagger f)^{-1} f^\dagger,$$

where

$$h = f(x_+) + i\theta_+ g(x_+).$$

From (4.76) and (4.80) we deduce that

$$\begin{aligned} \psi_- &= 2i(1 - \mathbb{P})\, g(f^\dagger f)^{-1} f^\dagger, \\ \psi_+ &= 2i\, f(f^\dagger f)^{-1} g^\dagger (1 - \mathbb{P}). \end{aligned} \tag{4.81}$$

As a consequence we see that

$$\psi_+ \psi_+ = \psi_- \psi_- = 0,$$

and equation (4.79) now becomes

$$\delta S = \frac{\epsilon^2}{4} \int d^2x \ \mathrm{Tr}\, [\partial_\mu X_0^\dagger\, \partial_\mu X_0 - Q^\dagger \partial_\mu Q\, (\partial_\mu X_0^\dagger X_0 - X_0^\dagger\, \partial_\mu X_0)], \tag{4.82}$$

which again reproduces the stability equation for the purely bosonic models established in the previous chapter. This shows that any negative mode for the bosonic one uniton solution is also a negative mode for the corresponding Susy one uniton solution. In general we expect further negative modes. However, we expect these modes to be purely bosonic in nature, *i.e.* we believe that the fermionic fluctuations do not lead to any new instabilities. All our attempts to find such instabilities have failed, as they have always given zero modes of the fluctuation operator. The general proof of this property is still an open question.

5. Solutions of a Boson-Fermion Model Based on the Susy WZW-σ Model.

So far we have studied classical solutions of the Susy $U(N)$ σ models with and without the WZW-term. We have found that the addition of fermions to the purely bosonic $U(N)$ σ models does not introduce any further instabilities but that it leads only to the appearance of further zero modes.

The Susy versions of sigma models provide interesting examples of interacting systems of bosons and fermions. In this chapter, we will investigate classical solutions of one such model. This model is based on the Susy version of the $U(N)$ σ model with the WZW-term studied in the previous chapter. It describes a coupled system of boson and fermion fields with their interactions fixed by the requirement of supersymmetry.

We recall from the previous chapter that the equations of motion of the Susy WZW-σ model can be resolved into the components

$$\begin{aligned}
\partial_+\psi_+ + \frac{1}{2}(1+\lambda)\,[g^\dagger\partial_+ g,\psi_+] - \frac{i}{4}(1+\lambda)\,(\psi_- F + F^\dagger\psi_-) &= 0,\\
\partial_-\psi_- + \frac{1}{2}(1-\lambda)\,[g^\dagger\partial_- g,\psi_-] + \frac{i}{4}(1-\lambda)\,(\psi_+ F + F^\dagger\psi_+) &= 0,\\
(1-\lambda)[\,\partial_+(g^\dagger\partial_- g) + i\partial_+(\psi_+\psi_+)\,] + (1+\lambda)[\,\partial_-(g^\dagger\partial_+ g) + i\partial_-(\psi_-\psi_-)\,] &= 0,
\end{aligned} \tag{5.1}$$

where

$$F = (1+\lambda)\psi_+\psi_- - (1-\lambda)\psi_-\psi_+.$$

In fact, the equations given above describe a rather complicated system of coupled bosonic and fermionic fields. To understand this system better, we consider the linearized fermion equations, *i.e.* the equations for fermions in the fixed background of a bosonic field g, which satisfies its own, purely bosonic, equations of motion. These equations come from (5.1) by neglecting all higher than linear terms in $\psi_\pm$, and so are given by

$$\partial_+\psi_+ + (1+\lambda)\,[\frac{1}{2}g^\dagger\partial_+ g, \psi_+] = 0, \tag{5.2}$$

$$\partial_-\psi_- + (1-\lambda)\,[\frac{1}{2}g^\dagger\partial_- g, \psi_-] = 0, \tag{5.3}$$

and

$$(1-\lambda)\partial_+(g^\dagger\partial_- g) + (1+\lambda)\partial_-(g^\dagger\partial_+ g) = 0, \tag{5.4}$$

together with the constraints

$$g^\dagger g = 1, \qquad \psi_\pm^\dagger = -\psi_\mp. \tag{5.5}$$

Clearly, equation (5.4) shows that for g we should take a solution of the purely bosonic $U(N)$ σ-model with the WZW-term discussed in chapter three.

It is important to notice that had we considered the Susy σ model without the WZW-term, *i.e.* with the action given by

$$S = \frac{1}{2}\int d^2x\, d\theta_+ d\theta_-\ \mathrm{Tr}\,(D_+\phi^\dagger D_-\phi - D_-\phi^\dagger D_+\phi),$$

we would have got the following equations of motion:

$$\begin{aligned}
\partial_+\psi_+ + \frac{1}{2}[Q^\dagger\partial_+ Q, \psi_+] - \frac{i}{4}(\psi_- F + F^\dagger\psi_-) &= 0,\\
\partial_-\psi_- + \frac{1}{2}[Q^\dagger\partial_- Q, \psi_-] + \frac{i}{4}(\psi_+ F + F^\dagger\psi_+) &= 0,\\
\partial_+(Q^\dagger\partial_- Q + i\psi_+\psi_+) + \partial_-(Q^\dagger\partial_+ Q + i\psi_-\psi_-) &= 0,
\end{aligned} \tag{5.6}$$

where

$$F = \psi_+\psi_- - \psi_-\psi_+.$$

If we now neglect all higher than linear terms in $\psi_\pm$ we get the corresponding linearized fermion equations:

$$\partial_+\psi_+ + \frac{1}{2}[Q^\dagger\partial_+ Q, \psi_+] = 0, \tag{5.7}$$

$$\partial_-\psi_- + \frac{1}{2}[Q^\dagger\partial_-Q, \psi_-] = 0, \qquad (5.8)$$

and

$$\partial_+(Q^\dagger\partial_-Q) + \partial_-(Q^\dagger\partial_+Q) = 0, \qquad (5.9)$$

together with the constraints

$$Q^\dagger Q = 1, \qquad \psi_\pm^\dagger = -\psi_\mp. \qquad (5.10)$$

Equation (5.9) shows that Q must be a solution of the purely bosonic $U(N)$ σ model without the WZW-term, *i.e.* this time Q does not depend on λ.

In the next section, we will present some classes of solutions of the above linearized Dirac-like equations (5.2)-(5.3) for which g solves (5.4). As a particular case, we will look at the corresponding solutions of (5.7)-(5.8) for which Q solves (5.9). In the following section, we will study some properties of these solutions and show that a class of the obtained solutions are related to the components of the energy-momentum tensor of the purely bosonic $U(N)$ σ model. In addition, we will prove that some classes of these solutions are traceless.

5.1 Fermion Solutions

Before we discuss solutions of the linearized Dirac-like equations, let us recall briefly the procedure of constructing solutions of the WZW-σ model given in chapter three. In that procedure, following the original ideas of Uhlenbeck [42], we have mentioned that any solution g_ℓ of (5.4) can be factorised as

$$g_\ell = K(1 - aR_1)(1 - aR_2)\cdots(1 - aR_\ell), \qquad (5.11)$$

where K is a constant matrix, ℓ is an integer (the uniton number), R_i's are projectors which satisfy

$$\begin{aligned} R_\ell A_-^{\ell-1}(1 - R_\ell) &= 0, \\ (1 - R_\ell)(\partial_- R_\ell + A_-^{\ell-1} R_\ell) &= 0, \end{aligned} \tag{5.12}$$

and where $A_\pm^\ell$ are given by

$$A_-^\ell = \frac{1}{b} g_\ell^\dagger \partial_- g_\ell, \qquad A_+^\ell = \frac{1}{a} g_\ell^\dagger \partial_+ g_\ell, \tag{5.13}$$

where a and b are complex numbers which satisfy

$$a + b - ab = 0. \tag{5.14}$$

In the limit $a = b = 2$ we see that g_ℓ, a solution of the WZW-σ model, becomes Q_ℓ, a solution of the $U(N)$ σ model without the WZW-term, *i.e.* a solution of (5.9). Moreover, these solutions possess an important property, namely their A_-^ℓ is given by [43]

$$A_-^\ell = \sum_{i=1}^{\ell} \partial_- R_i\,. \tag{5.15}$$

We are now ready to construct some classes of fermion solutions for the systems (5.2)-(5.3) and (5.7)-(5.8) corresponding to the above bosonic solutions. To start off let us observe that had we considered grassmannian bosonic solutions (*i.e.* those g_ℓ for which $g_\ell^\dagger = g_\ell$) then we have to require that $a = b$. However, from (5.14) we see that this requirment leads to $a = b = 2$ and that this corresponds to the solutions Q_ℓ of the model without the WZW-term. This claim agrees with what we stated at the end of chapter three, namely

$$(g(\lambda))^\dagger = g(\lambda^*) \neq g(\lambda).$$

Thus, since there are no nontrivial grassmannian solutions for (5.4), there are no corresponding fermion solutions for the linearized equations (5.2)-(5.3).

However, for the grassmannian solutions Q_ℓ of the purely bosonic $U(N)$ σ model without the WZW-term, it is easy to see that

$$\psi_+ = \psi_- = iQ_\ell \tag{5.16}$$

solve the linearized equations (5.7)-(5.8). The simplest solution of this class is a one-uniton solution (we drop the irrelevant constant matrix K):

$$Q_1 = (1 - 2R_1) \tag{5.17}$$

with the projector R_1 defined by

$$R_1 = f(f^\dagger f)^{-1} f^\dagger \tag{5.18}$$

for some holomorphic N-component vector $f(x_+)$.

However, in the previous chapter, in which we studied the stability properties of the Susy solutions of the Susy σ model with and without the WZW-term, we derived expressions for $\psi_\pm$, namely

$$\begin{aligned} \psi_- &= 2i(1 - R_1)\, g(f^\dagger f)^{-1} f^\dagger, \\ \psi_+ &= 2i\, f\, (f^\dagger f)^{-1} g^\dagger (1 - R_1), \end{aligned} \tag{5.19}$$

where the projector R_1 is defined as in (5.18), and $g = g(x_+)$ is a grassmannian (anticommuting) N-vector of polynomials in x_+. Then, a few lines of calculation show that the expressions (5.19) solve the linearized equations (5.7)-(5.8) corresponding to the one-uniton solution (5.17).

Next we present more general classes of solutions of the linear systems (5.2)-(5.3) and (5.7)-(5.8). To construct them we consider

$$\psi_+ = A_-, \qquad \psi_- = A_+, \tag{5.20}$$

where $A_\pm$ are given by (5.13) with

$$a = \frac{2}{1+\lambda}, \qquad b = \frac{2}{1-\lambda}.$$

In this case, equations (5.2) and (5.3) become

$$\begin{aligned} \partial_+ A_- + [A_+, A_-] &= 0, \\ \partial_- A_+ + [A_-, A_+] &= 0. \end{aligned}$$

However, adding these two equations together we get

$$\partial_+ A_- + \partial_- A_+ = 0,$$

which is the equation of motion (5.4) of the purely bosonic $U(N)$ WZW-σ model. Since $(A_\pm)^\dagger = -A_\mp$, we see that $\psi_\pm$ given by (5.20) satisfy the constraint (5.5) and solve (5.2)-(5.3).

Notice that the solutions (5.20) satisfy (5.2)-(5.3) up to an overall multiplication of $\psi_+\,(\psi_-)$ by a function of $x_-\,(x_+)$ respectively. Notice also that the expressions (5.20) in which we can set $a = b = 2$ in the definitions of $A_\pm$ would solve the linearized fermion equations (5.7)-(5.8).

We are now ready to construct further solutions of the equations (5.2)-(5.3) and (5.7)-(5.8). To do this we introduce a method of generating new solutions from the old ones. Thus we consider

$$\psi_+ = A_- \psi_+^{(1)}, \qquad \psi_- = A_+ \psi_-^{(1)}. \tag{5.21}$$

Then it is easy to see that $\psi_\pm$ given by (5.21) satisfy (5.2)-(5.3) if $\psi_\pm^{(1)}$ do. For $\psi_\pm^{(1)}$ we can take

$$\psi_+^{(1)} = A_-, \qquad \psi_-^{(1)} = A_+, \tag{5.22}$$

which, as have shown above, are solutions of (5.2)-(5.3). Thus we find that

$$\psi_+ = A_- A_- , \qquad \psi_- = A_+ A_+ \tag{5.23}$$

are new solutions of (5.2)-(5.3). Repeating this procedure many times we find that

$$\psi_+ = (A_-)^n , \qquad \psi_- = (A_+)^n \tag{5.24}$$

satisfy equations (5.2)-(5.3) for all integer values of n. However, to satisfy the constraint (5.5), we must multiply each solution by an appropriate constant for each n. Notice that the expressions (5.24) also solve (5.7)-(5.8), where we have to set $a = b = 2$ in the definitions of $A_\pm$.

5.2 Properties of Fermion Solutions

Before we study various properties of the above constructed solutions, let us calculate first the energy-momentum tensor $T_{\mu\nu}$ of the purely bosonic $U(N)$ σ model without the WZW-term, *i.e.* the model defined in terms of the Lagrangian density

$$L = \frac{1}{4} \,\mathrm{Tr}\, \partial_\mu Q^\dagger \,\partial_\mu Q. \tag{5.25}$$

For this model, it is easy to see that the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{1}{4} \,\mathrm{Tr}\, [\, \partial_\mu Q^\dagger \,\partial_\nu Q + \partial_\nu Q^\dagger \,\partial_\mu Q - \delta_{\mu\nu} \,\partial_\rho Q^\dagger \,\partial_\rho Q \,]. \tag{5.26}$$

Clearly, $T_{\mu\nu}$ satisfies

$$T_{\mu\mu} = 0, \qquad T_{\mu\nu} = T_{\nu\mu}, \qquad \partial_\mu T_{\mu\nu} = 0,$$

i.e. the energy-momentum tensor of the model is traceless, symmetric and conserved. If we now introduce the complex variables $x_\pm$, we find that

$$T = T_{x_1 x_1} + i T_{x_1 x_2} = \text{Tr}\, \partial_- Q^\dagger\, \partial_- Q. \tag{5.27}$$

However, we know that $A_\pm$ of the model are given by

$$A_\pm = \frac{1}{2} Q^\dagger \partial_\pm Q.$$

Thus, equation (5.27) can be rewritten as

$$T = -4\ \text{Tr}\ A_-\, A_-. \tag{5.28}$$

From the above discussion, we see that the components of the energy-momentum tensor of the $U(N)$ σ model are related to our solutions of the corresponding linearized fermion background equations of the same model.

To investigate some properties of these solutions let us consider first the one-uniton solution given by

$$Q_1 = (1 - aR_1) \qquad \text{with} \qquad R_1 = f\,(f^\dagger f)^{-1} f^\dagger. \tag{5.29}$$

For this solution, according to (5.15), it is easy to see that

$$A_- = \partial_- R_1 = \frac{f\,(P_+ f)^\dagger}{|f|^2}, \qquad (A_-)^2 = 0, \tag{5.30}$$

where the operator P_+ is defined in the same manner as in chapter two. In that chapter, we reported that all vectors $P_+^k f$ were orthogonal to each other. Thus for the one-uniton solution (5.29) we find that

$$\text{Tr}\ (A_-)^n = 0, \qquad \text{for all} \quad n. \tag{5.31}$$

Next we consider a simple two-uniton solution given by

$$Q_2 = (1 - aR_1)(1 - aR_2), \tag{5.32}$$

where R_1 is defined as above, and where

$$R_2 = R_1 + \frac{(P_+f)(P_+f)^\dagger}{|P_+f|^2}.$$

Then, due to (5.15), we see that for this solution

$$\begin{aligned} A_- &= \partial_- R_1 + \partial_- R_2 = \frac{(P_+f)(P_+^2f)^\dagger}{|P_+f|^2} + \frac{f(P_+f)^\dagger}{|f|^2}, \\ (A_-)^2 &= \frac{f(P_+^2f)^\dagger}{|f|^2}, \\ (A_-)^3 &= 0. \end{aligned} \tag{5.33}$$

This shows that for our simple two-uniton solution (5.32), the property (5.31) is also satisfied for all n.

Given these results we may wonder whether this property holds for any number of unitons. Let us try to prove this property by induction on the number of uniton factors in a given solution. We have proved above that this property holds for a one- and (some) two-uniton solutions. Next we assume that any ℓ-uniton solution A_- satisfies

$$\mathrm{Tr}\,(A_-)^n = 0. \tag{5.34}$$

Then the induction requires the proof that adding to this solution a uniton factor given by $(1 - aR)$ where R is a projector which satisfies (5.12), the result is still true, *i.e.* that

$$\mathrm{Tr}\,(A_- + \partial_- R)^n = 0. \tag{5.35}$$

To prove (5.35) we, first of all, expand $(A_- + \partial_- R)^n$. As in general A_- and $\partial_- R$ do not commute we prove first the following lemmas:

Lemma (5.2.1)

If R is a basic uniton (antibasic uniton) [29], then

$$(R_-)^n A_- = \begin{cases} (-1)^p A_-(R_-A_-)^p, & \text{if } n = 2p; \\ (-1)^p (R_-A_-)^{p+1}, & \text{if } n = 2p+1, \end{cases} \tag{5.36}$$

for all p, where $R_- = \partial_- R$.

Proof: Let us consider the case when R is a basic uniton. In this case we have

$$R A_- = 0. \tag{5.37}$$

For antibasic uniton we would have

$$A_-(1 - R) = 0.$$

From (5.12) and (5.37), it is easy to see that

$$(R_-)^2 = -(A_- R R_- + R_- A_- R), \tag{5.38}$$

which, in turn, implies that

$$(R_-)^2 A_- = -A_- R_- A_-. \tag{5.39}$$

The proof is now straightforward; we apply (5.39) sufficiently many times to the left hand side of (5.36). For example, if $n = 2p$, we have

$$\begin{aligned} (R_-)^{2p} A_- &= -(R_-)^{2p-2} A_- R_- A_- = (R_-)^{2p-4} A_- (R_- A_-)^2 \\ &= \cdots = (-1)^p A_- (R_- A_-)^p. \end{aligned}$$

Lemma (5.2.2)

For the projector R

$$\mathrm{Tr}\,(R_-)^n \;=\; \begin{cases} 2(-1)^p\,\mathrm{Tr}\,(R_-A_-)^p, & \text{if } n = 2p; \\ 0, & \text{if } n = 2p+1. \end{cases} \tag{5.40}$$

Proof: Using (5.38), it is easy to verify that

$$\mathrm{Tr}\,(R_-)^n \;=\; \begin{cases} -2\,\mathrm{Tr}\;R(R_-)^{2p-1}A_-, & n = 2p; \\ -2\,\mathrm{Tr}\;R(R_-)^{2p}A_-, & n = 2p+1. \end{cases} \tag{5.41}$$

Then when $n = 2p+1$, we can use (5.36) and find that

$$\mathrm{Tr}\,(R_-)^n \;=\; -2\;\mathrm{Tr}\;R(R_-)^{2p}A_- \;=\; 2(-1)^{p+1}\;\mathrm{Tr}\;RA_-\,(R_-A_-)^p \;=\; 0,$$

where we have used (5.37). On the other hand, when $n = 2p$, also we use (5.36) and (5.37) and find that

$$\mathrm{Tr}\,(R_-)^n \;=\; -2\;\mathrm{Tr}\;R(R_-)^{2p-1}A_- \;=\; 2(-1)^p\;\mathrm{Tr}\;RR_-A_-\,(R_-A_-)^{p-1}.$$

Then from the fact that

$$RR_-A_- \;=\; (R_- \;-\; R_-R)A_- \;=\; R_-A_-,$$

we get the required result.

We are now ready to present our expansion

Proposition

For a general (non-commuting) A_- and R_- which satisfy the lemmas above

$$(A_- + R_-)^n \;=\; \sum_{k=1}^{n-1}\Big[\,(A_-)^{n-k}\,(R_-)^k + R_-\,(A_-)^{n-k}\,(R_-)^{k-1}\Big] + (A_-)^n + (R_-)^n, \tag{5.42}$$

for all n.

Proof: Before we give a proof of this expansion, let us mention that we have guessed the form of (5.42) by applying REDUCE to expand $(A_- + R_-)^n$ for a few values of n.

We prove our expansion by induction. It is clearly true for $n = 1,2$ and 3. Assuming that it is true for n, we have to show that if we multiply (5.42) corresponding to n by another factor $(A_- + R_-)$, we get the same expansion (5.42) in which n has to be replaced by $(n+1)$. To do this, we calculate

$$\begin{aligned}(A_- + R_-)^n(A_- + R_-) &= \sum_{k=1}^{n-1}\Big[(A_-)^{n-k}(R_-)^{k+1} + R_-(A_-)^{n-k}(R_-)^k\Big] \\ &+ \sum_{k=1}^{n-1}\Big[(A_-)^{n-k}(R_-)^kA_- + R_-(A_-)^{n-k}(R_-)^{k-1}A_-\Big] \\ &+ (A_-)^{n+1} + (R_-)^{n+1} + (A_-)^nR_- + (R_-)^nA_-.\end{aligned} \tag{5.43}$$

Then few lines of algebra show that this expression can be rewritten as

$$\begin{aligned}(A_- + R_-)^n(A_- + R_-) &= \sum_{k=1}^{n}\Big[(A_-)^{n-k+1}(R_-)^k + R_-(A_-)^{n-k+1}(R_-)^{k-1}\Big] \\ &+ \sum_{k=1}^{n-1}\Big[(A_-)^{n-k}(R_-)^kA_- + R_-(A_-)^{n-k}(R_-)^{k-1}A_-\Big] \\ &+ (A_-)^{n+1} + (R_-)^{n+1} + (R_-)^nA_- - R_-(A_-)^n.\end{aligned} \tag{5.44}$$

If we now use our lemma (5.2.1) and perform some simple manipulations, it is easy to show that

$$\sum_{k=1}^{n-1}\Big[(A_-)^{n-k}(R_-)^kA_- + R_-(A_-)^{n-k}(R_-)^{k-1}A_-\Big] + (R_-)^nA_- - R_-(A_-)^n = 0. \tag{5.45}$$

Looking at (5.45) and (5.44) we observe that the remaining terms of (5.44) are nothing else but the expansion (5.42) for $(n+1)$ thus completing the proof.

Having proved these two lemmas and established our expansion of the non-commuting A_- and R_-, we are now ready to complete the proof that (5.35) is satisfied for all n. To do this, we observe that from (5.42) we have

$$\mathrm{Tr}\,(A_- + R_-)^n = \mathrm{Tr}\left(2\sum_{k=1}^{n-1}(R_-)^k(A_-)^{n-k} + (A_-)^n + (R_-)^n\right). \qquad (5.46)$$

However, from (5.34) we see that the second term on the right hand side of (5.46) vanishes and from (5.39) we find that

$$\mathrm{Tr}\,\sum_{k=1}^{n-1}(R_-)^k(A_-)^{n-k} = \begin{cases} -(-1)^p\,\mathrm{Tr}\,(R_-A_-)^p, & \text{if } n = 2p; \\ 0, & \text{if } n = 2p+1. \end{cases} \qquad (5.47)$$

Finally, from lemma (5.2.2) and equation (5.47), we find that the remaining terms of (5.46) mutually cancel and so that (5.35) is satisfied for all n.

So far we have proved (5.35) only for A_- which correspond to the solutions in which all uniton factors involve only unitons which are either basic or antibasic. However, Wood [44] has shown that all uniton factors can be factorised further into finite products of uniton factors involving only basic (or antibasic) unitons. Hence our result holds in full generality.

6. CONCLUSION.

In this thesis we have studied models which are known to possess many properties in common with four-dimensional non-abelian gauge theories; namely the two-dimensional sigma models. As these models are two-dimensional, all relevant calculations are simpler to perform than in four-dimensional non-abelian gauge theories. All studies in this thesis have been performed in Euclidean space, as we think of these calculations as providing the first step towards the quantisation of the theory in terms of path integrals. Because of this we have considered only the finite action classical solutions of the Euler-Lagrange equations of the theory. We see that, in contradistinction to the four-dimensional non-abelian gauge theories case, the solutions of the self-duality equations (instantons) and of the non-self-duality equations (non-instantons) for these models can be constructed very easily and explicitly.

One of the main topics of this thesis was the construction of finite action classical solutions of the sigma models for which the target manifolds are group spaces; namely, the purely bosonic $U(N)$ WZW-σ models and the Susy $U(N)$ σ models with and without the WZW-term. However, for completeness, in chapter 2, we presented a brief review of the well known $\mathbb{C}P^{N-1}$ and grassmannian models and their Susy extensions, in which case the fields take values in coset spaces, and also of the purely bosonic $U(N)$ σ model (principle chiral model). In particular, we presented the Euler-Lagrange equations for these models and their general instanton and non-instanton solutions and showed that the instanton and anti-instanton solutions of the $\mathbb{C}P^{N-1}$ and grassmannian models are relative minima of the action, and so are stable. Moreover, we also showed that all non-trivial solutions of the $U(N)$ σ model do not correspond to minima of the action, and so being saddle points of the action are unstable under small fluctuations. Also in chapter 2 we looked at the linearized fermion equations in the fixed background of bosonic $\mathbb{C}P^{N-1}$ and grassmannian fields.

Then we constructed finite action classical solutions of the Euclidean two-dimensional $U(N)$ WZW-σ models. We showed that these solutions are related to and in fact can be derived from the solutions of the Lax-pair problem for the corresponding $U(N)$ σ model without the WZW-term. In addition, we showed that each solution of the WZW-σ model can be related to a solution of the usual σ model with the connection being provided by the solutions of the Lax-pair equations for the σ model. To study properties of these solutions, we had to find a way of computing the contribution of the WZW-term. As this term is given by a three-dimensional integral, we followed Witten [25] and extended our field $g(x_1, x_2)$ to a field $\breve{g}(x_1, x_2, t)$ defined over the three-dimensional space locally parametrised by x_1, x_2 and t with $0 \leq t \leq 1$ and such that $\breve{g}(x_1, x_2, 1) = g(x_1, x_2)$ and $\breve{g}(x_1, x_2, 0) = 1$. Next we computed the values of the action of the WZW-σ model for some of our solutions. We found that these values are given by the action of the corresponding solutions of the usual σ model multiplied by some function $h(k)$ given by

$$h(k) \;=\; 1 \;-\; k\;\tan^{-1}(k^{-1}). \tag{6.1}$$

Moreover, studying the stability properties of the solutions of the WZW-σ model we found that the WZW-term does not stablise them but that they have the same number of negative modes as the corresponding solutions of the model without the WZW-term, and so, as a consequence, that they are all also unstable.

It is worth adding at this point that the action and the equations of motion of the $U(N)$ WZW-σ model, when formulated in Euclidean space in terms of the complex coordinates $x_\pm = x_1 \pm ix_2$, are given by

$$\begin{aligned} S =& \frac{1}{2}\int d^2x \;\mathrm{Tr}\;(\partial_+ g^\dagger\,\partial_- g \;+\; \partial_- g^\dagger\,\partial_+ g) \\ &-\lambda\int d^3x\;\mathrm{Tr}\;[(\partial_+\breve{g}^\dagger\,\partial_-\breve{g}\;-\;\partial_-\breve{g}^\dagger\,\partial_+\breve{g})\,\breve{g}^\dagger\;\dot{\breve{g}}\,], \end{aligned} \tag{6.2}$$

and

$$(1 - \lambda)\partial_+(g^\dagger\partial_- g) + (1 + \lambda)\partial_-(g^\dagger\partial_+ g) = 0, \tag{6.3}$$

together with the constraint $g^\dagger g = 1$. If we want to formulate this model in Minkowski space in terms of light-cone coordinates $x_\pm = x_1 \pm x_2$, then the Minkowskian WZW-σ model takes the same form, with the same expression for the action and the same equations of motion as in Euclidean space. The only difference lies in the nature of the variables. Strictly speaking, the WZW-term has to be defined in Euclidean space. Thus, if we deal with the Minkowskian model we have to use the equations of motion obtained from the action in Euclidean space after continuing them back to Minkowski space using the substitution (analytic continuation)

$$x_2 \longrightarrow i\,x_2. \tag{6.4}$$

Of course it is clear from the above discussion that the Minkowskian and Euclidean $U(N)$ σ model without the WZW-term are indistinguishable too with the same action and equations of motion.

We should add a few words of explanation at this point on the reality of our action in Euclidean space. The original Lagrangian, introduced in Minkowski space, has a real coefficient of its WZW-term, *i.e.* λ has to be real in the Minkowskian action (6.2). This implies that the two terms in the Euclidean action will have different reality conditions. In consequence, except for the special values $\lambda = \pm 1$, as discussed by Witten [25], the equations of motion mix the reality conditions and so are more restrictive than in the case without the WZW-term. To overcome this problem we can either continue analytically in λ to λ being purely imaginary (keeping the g field unitary) or keep λ real and accept non-unitary solutions for g (in this case $g^\dagger$ should be interpreted as g^{-1}). This is essentially the approach adopted in this thesis where the solutions of the Euclidean model were seeked in which λ was continued to purely imaginary values (of course, our solutions can also be considered as solutions for λ real (except $\lambda = \pm 1$), then the fields are non-unitary and $g^\dagger$ should be interpreted as g^{-1}).

The question then arises whether a solution of one type of the Euclidean or Minkowskian model will give a solution of the corresponding model of the other type after the substitution (6.4). To consider this problem we notice that after the substitution (6.4), in general, the matrix g will no longer be unitary. Indeed, this substitution does not commute with the constraint $g^\dagger g = 1$. So if, for real x_1, x_2, one has $g^\dagger(x_1, x_2) = g^{-1}(x_1, x_2)$, then in general $g^\dagger(x_1, ix_2) \neq g^{-1}(x_1, ix_2)$. Thus the substitution (6.4) transforms a solution of the Euclidean model into a solution of the Minkowskian model and vice versa only for the $SL(N)$ and $GL(N)$ models. So if one wants to construct classical solutions of the $U(N)$ models, they have to be studied separately for both types of models.

To consider fermion effects in sigma models, we studied Susy extensions of the $U(N)$ chiral models with and without the WZW-term in two dimensions. We exhibited some classical solutions of these models and discussed their properties. We showed that, as in the purely bosonic WZW-σ models, the solutions of the Susy WZW-σ model can be derived from the solutions of the Lax-pair equations for the corresponding Susy σ model. We generalised the procedure due to Uhlenbeck [42] and used it to construct some of these solutions in an explicit form. Computing the value of the action of the Susy model for some of these solutions we have found that they are related to those of the purely bosonic model, and so, are given by the laplacian of a logarithm of an appropriate function.

One of the most important properties of the solutions is their stability and so we looked at this problem for our Susy solutions. We showed that the solutions of the Susy WZW-σ model have the same number of negative modes as the corresponding solutions of the Susy σ model. Naively, we would expect solutions in a bigger space to be less stable but we did not find this to be the case. It appears that the inclusion of fermions does not introduce any further instabilities but, as expected, it leads only to the appearace of further zero modes. We have no proof of these claims but having looked at various cases we are reasonably convinced that they are true. This problem is still unsolved.

The obtained results show that the introduction of fermions does not modify the theory in an unexpected way. This is true at least for our solutions. We found that the introduction of fermions does alter the bosonic part of the theory very much as expected. In fact, for most solutions, the fermionic contributions to the equations of motion cancel. On the other hand the fermionic part does depend crucially on the specific form of the bosonic fields but its expression can be derived quite easily. Thus we would expect that any quantisation of the full theory would reproduce all effects of its purely bosonic part modified by the usual effects associated with the existence of fermions.

The Susy extension of sigma models provide interesting examples of boson-fermion interactions. The final topic of this thesis was the study of a Euclidean two-dimensional model which describes a coupled set of interacting boson and fermion fields whose nonlinear interactions are fixed by the requirement of supersymmetry. We constructed some classes of solutions of the linearized fermion equations in the fixed background of a bosonic field. The background field was taken to be a solution of the $U(N)$ σ model with and/or without the WZW-term. We studied various properties of these solutions and showed that a class of the obtained solutions is related to the components of the energy-momentum tensor of the purely bosonic $U(N)$ σ model. In addition, we have proved that some classes of the constructed solutions are traceless.

Even though our solutions were obtained for the linearized equations, some of them also solve the full equations. In particular, this is the case for the solutions

$\psi_+ = (A_-)^n$ and $\psi_- = (A_+)^n$, where n is the uniton number of the background field. The reason for this is that for these solutions the additional nonlinear contributions to the equations of motion vanish.

Let us point out that the problem of solving linear equations may appear to be relatively straightforward. However, it is easy to check that as our linearized equations contain $\psi_\pm$ fields which are matrix-valued and as the equations involve their commutators, the problem is far from trivial. Of course we could have tried to seek solutions for which the commutators vanish or even such that the $\psi_\pm$ fields are proportional to the unit matrix. However such solutions would have tended to possess singularities or would grow linearly at infinity. Moreover, they would not have depended on the properties of the background field. Our solutions of the linearized equations are intrinsically tied to the properties of the background fields, are regular everywhere and, at least for the simplest cases discussed in the text, vanish at infinity. Thus they belong to the class of fields one would consider in the study of the Atiyah-Singer theorem. In fact this theorem involves solutions of the linearized equations which are normaliseable on the sphere, and so vanish at infinity. This condition is satisfied by our solutions. Thus they encode the information about the topological properties of the purely bosonic and/or Susy models.

Finally, as was shown in ref. [43], our solutions of the linearized equations provide us with some negative modes of the fluctuation operator of a purely bosonic theory.

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