The quenching of nonlinear oscillations by dither

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Abstract

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Chapter I discusses the background of the problem. Chapter II uses small parameter theory to examine the mechanism of quenching in certain special cases. In Chapter III an interesting aspect of quenching is discussed in some depth for a special 2-dimensional problem. Chapter IV discusses sufficient conditions for quenching to occur in general systems.

The Quenching of Nonlinear Oscillations by Dither

by

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Contents

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Durray-Shahwar

July 1990

Abstract

The suppression of unwanted vibrations in a system by the injection of a high-frequency dither is a well-known engineering technique. Very little has been published about the theory of this method and what has been published has often been lacking in mathematical rigour. This work is an attempt to correct this situation.

Chapter I discusses the background of the problem. Chapter II uses small parameter theory to examine the mechanism of quenching in certain special cases. In Chapter III an interesting aspect of quenching is discussed in some depth for a special 2-dimensional problem. Chapter IV discusses sufficient conditions for quenching to occur in general systems.

Chapter I

ANALYSIS OF A LINEAR PLANT

1.1 Model of a Physical System.

A quantitative understanding of the functioning of any system is not possible without a mathematical model. Such models must be able to predict the behaviour of the system once the initial conditions and external influences are known. The mathematical structure of differential equations is most suited for this purpose.

Let us think of a set of variables which describe the state of the system, and collect them in a 'state vector' denoted by $\mathbf{x}(t)$. Now if at a particular time the values of the state variables are known, we should be able to predict the future behaviour of the system using our mathematical model, provided that we also know the external influences acting on the system. These external influences can be collected into an 'input vector' $\mathbf{u}(t)$.

The argument above implies that the differential equations should be of the first order in time derivative. In a convenient notation these can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t). \tag{1.1}$$

In general the function $\mathbf{f}$ is nonlinear. Such a representation of the physical system is called the state-space description.

Another representation of the physical system is the input-output description. In this picture one has to define another set of quantities called the output variables. These are generally a subset of the state variables and describe aspects of the system's behaviour that can be measured and controlled. These are collected in an 'output vector' $\mathbf{y}(t)$,

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, t), \tag{1.2}$$

where $\mathbf{h}$ is another function which can be nonlinear. Consequently $\mathbf{x}$ can be eliminated from equations (1.1) and (1.2), and a differential equation relating output

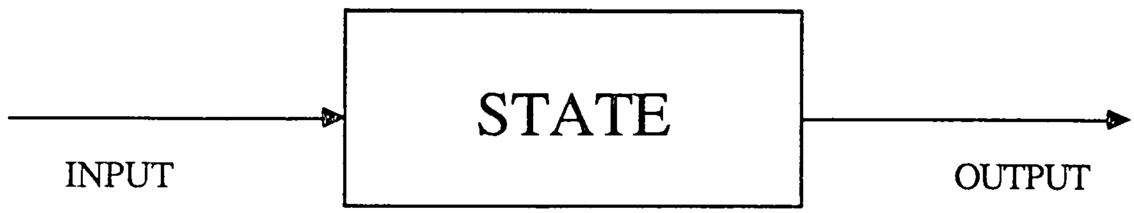


Figure 1.1: Block representation of a system.

directly to the input can be written down. Diagrammatically a system can be pictured as in Figure 1.1 .

From the point of view of mathematics, the ability to accurately predict the future state of a system places conditions on the differential equations of the form of (1.1). Specifically it is required that solutions exist and are unique. For differential equations of the form under discussion, a theorem due to Picard which guarantees the existence and uniqueness of solutions is stated by Cronin [4,p.14] as follows.

Theorem 1-1. Picard's Theorem: *Let $\mathcal{D}$ be an open set in $(\mathbf{x}, t)$ space. Let $(\mathbf{x}_o, t_o)$ be in $\mathcal{D}$, and a, b be positive real numbers such that the set*

$$\mathcal{R} = \{(\mathbf{x}, t) \mid |t - t_o| \leq a, |\mathbf{x} - \mathbf{x}_o| \leq b\}$$

is contained in $\mathcal{D}$. Suppose that a function $\mathbf{f}$ is defined and is continuous on $\mathcal{D}$ and satisfies a Lipschitz condition with respect to $\mathbf{x}$ on $\mathcal{R}$. Let

$$\mathcal{M} = \max_{(\mathbf{x},t)\in\mathcal{R}} |\mathbf{f}(\mathbf{x}, t)|$$

and

$$\mathcal{A} = \min\,[a, \frac{b}{\mathcal{M}}].$$

Then the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

has a unique solution on $(t_o - \mathcal{A}, t_o + \mathcal{A})$*, such that* $\mathbf{x}(t_o) = \mathbf{x}_o$ *. This solution is such that*

$$|\mathbf{x}(t) - \mathbf{x}_o| \leq \mathcal{M}\mathcal{A}$$

for all t *in the interval* $(t_o - \mathcal{A}, t_o + \mathcal{A})$.

Picard's theorem places no bound or limitation on the domain of the solution. It merely states that the interval $(t_o - \mathcal{A}, t_o + \mathcal{A})$ is contained in the domain of solution $\mathbf{x}(t)$. Thus for a particular equation the solution may be defined for all real t even though the theorem only guarantees existence on a finite interval. This indeed is the case for 'well behaved' linear systems [4,p.67].

Ideally a physical system should operate in a steady state. In the state-space representation this means that if the external influences or the input is constant then the system's state should not change. This can be expressed as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}_o, \mathbf{u}_o, t)\\ &= 0,\end{aligned} \tag{1.3}$$

where we have denoted the constant input as $\mathbf{u}_o$ and $\mathbf{x}_o$ is the state of the system at initial time t_o.

In actual practice, during the operation of a system, a small fluctuation in the input $\mathbf{u}_o$ may cause the system to move away from the steady state $\mathbf{x}_o$ and therefore the output changes. For this reason the system design has to provide a mechanism to control the perturbation so that the system keeps on operating at a steady state. In other words the system has to be controlled.

The key concept in control devices is that of feedback. The feedback is motivated by a 'controller', and deviations from a steady state output activate a

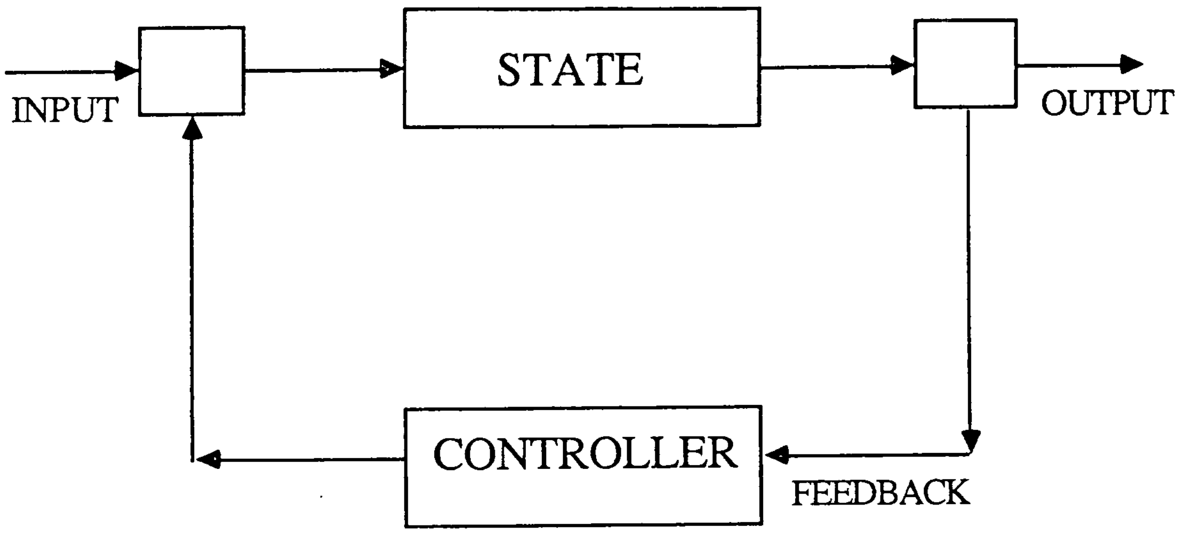


Figure 1.2: A system with a feedback.

correcting signal which forces the system back to the desired steady state. Figure 1.2 gives a block diagram for the situation described above.

1.2 A Linear Plant.

A system which is to be controlled is called a plant. A plant is said to be linear if the state variable $\mathbf{x}(t)$ is related to the input $\mathbf{u}(t)$ through a linear differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1.4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices which in general depend on time. If $\mathbf{x}$ is of dimension n and $\mathbf{u}$ is of dimension r, then $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix. Further, the output variable $\mathbf{y}$ can be written quite generally as

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{1.5}$$

In the special case where $\mathbf{A}$ and $\mathbf{B}$ are time independent we have what is called a linear autonomous plant. One can, without loss of much generality drop the term $\mathbf{Du}$ in equation (1.5) as it is simply proportional to the input and hence independent of the system dynamics. Further, one can, in certain cases assume the output of the system to be the entire state, that is

$$\mathbf{y}(t) = \mathbf{x}(t),$$

in which case the output equation (1.5) is omitted.

Thus for the steady state of the linear autonomous plant (1.4) we have

$$\mathbf{A}\mathbf{x}_\circ + \mathbf{B}\mathbf{u}_\circ = 0 \tag{1.6}$$

or if $\mathbf{A}$ has an inverse

$$\mathbf{x}_\circ = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}_\circ \tag{1.7}$$

Assuming that at time t, the output (or state) has changed to $\mathbf{x}(t)$, one can write an equation for the deviation

$$\vec{\xi} = \mathbf{x}(t) - \mathbf{x}_\circ(t)$$

as

$$\dot{\vec{\xi}} = \mathbf{A}\vec{\xi} + \mathbf{B}[\mathbf{u}(t) - \mathbf{u}_\circ]. \tag{1.8}$$

In case the input does not change, that is $\mathbf{u}(t) = \mathbf{u}_\circ$ then

$$\dot{\vec{\xi}} = \mathbf{A}\vec{\xi}. \tag{1.9}$$

For such a homogeneous linear differential equation, the trivial solution

$$\vec{\xi} = 0 \tag{1.10}$$

represents an equilibrium or steady state.

1.3 Stability.

Qualitatively an undisturbed motion $\vec{\xi}$ of equation (1.9), which depicts an autonomous system, is considered to be stable if the application of a small disturbance results in a (disturbed) motion which remains close to the unperturbed motion for all later times. If for small disturbances the effect on the motion tends to disappear, the undisturbed motion is called 'asymptotically stable'. Further, if regardless of the magnitude of the perturbation the effect tends to disappear, the undisturbed motion is said to be 'asymptotically stable in the large'.

The linearity of equation (1.9) ensures that if the condition

$$\vec{\xi}(t) \longrightarrow 0 \quad as \quad t \longrightarrow +\infty, \tag{1.11}$$

holds the solution $\vec{\xi} \equiv 0$ is asymptotically stable.

For a linear autonomous system straightforward arguments (see for example Jordan and Smith [9,p.227]) lead to the conclusion that the dynamic behaviour of the system is determined by the eigenvalues λ of the matrix $\mathbf{A}$. As regards stability of a system one can state a theorem as follows.

Theorem 1-2. *Let $Re(\lambda)$ denote the real part of λ, the eigenvalue of the constant matrix $\mathbf{A}$ in*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

The solution $\mathbf{x} = 0$ is asymptotically stable if and only if

$$Re(\lambda) < 0$$

for all eigenvalues λ.

(For a detailed discussion on this aspect the reader is referred to Cronin [4,p.156], Hayashi [8,p.33] and Sanchez [14,p.73].)

Note that the foregoing theorem has reduced the problem of determining the stability of the system to the problem of studying the real part of the eigenvalues of the matrix **A**. This in itself is not a simple problem especially if the system is of high dimensions.

The problem of stability of a system of differential equations was considered by A. M. Liapunov. In his famous second method, as given by Zubov [18,p.14], Liapunov defined a quadratic function, known as Liapunov function. This is a generalisation of the concept of energy in a conservative dynamical system where the energy decreases to zero for an equilibrium or stable state. In the matrix notation being followed, the properties attributed to the Liapunov functions can be satisfied by quadratic forms of matrices (see for example Barnett and Storey [2,p.71,77]).

Theorem 1-3. *For the system*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

the solution

$$\mathbf{x} = 0$$

is asymptotically stable if there exists a symmetric positive definite matrix **P** *such that*

$$\mathbf{Q} = -(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})$$

is positive definite.

Proof:

If

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$$

are eigenvalues of **P** then

$$\lambda_1 \geq \frac{V(\mathbf{x})}{|\mathbf{x}|^2} \geq \lambda_n > 0 \qquad (1.12)$$

for nonzero **x**, where we have written the Liapunov function

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}.$$

Similarly if

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$$

are eigenvalues of $\mathbf{Q}$ then

$$\mu_1 \geq \frac{-\mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}}{|\mathbf{x}|^2} \geq \mu_n > 0. \tag{1.13}$$

If $\mathbf{x}(t)$ is any solution then from equation (1.13) we get

$$\begin{aligned}\frac{dV(\mathbf{x})}{dt} &= \frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) \\ &= \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}\end{aligned}$$

that is

$$\dot{V}(\mathbf{x}) \leq -\mu_n|\mathbf{x}|^2.$$

Also using equation (1.12) we can write

$$\begin{aligned}\dot{V}(\mathbf{x}) &\leq \frac{-\mu_n}{\lambda_1}\lambda_1|\mathbf{x}|^2 \\ &\leq \frac{-\mu_n}{\lambda_1}(\mathbf{x}^T\mathbf{P}\mathbf{x})\end{aligned}$$

and thus we have

$$\dot{V}(\mathbf{x}) + \frac{\mu_n}{\lambda_1}V(\mathbf{x}) \leq 0$$

or

$$\frac{d}{dt}[e^{+\mu_n t/\lambda_1}V(\mathbf{x})] \leq 0$$

for all t.

Thus we conclude that the term in the square brackets is a monotonically decreasing function of t which implies that

$$e^{\mu_n t/\lambda_1}V(\mathbf{x}(t)) \leq V(\mathbf{x}(0)).$$

Now it is straightforward to write

$$|\mathbf{x}(t)| \leq \sqrt{\frac{\lambda_1}{\lambda_n}}\,|\mathbf{x}(0)|\,e^{-\mu_n t/2\lambda_1}.$$

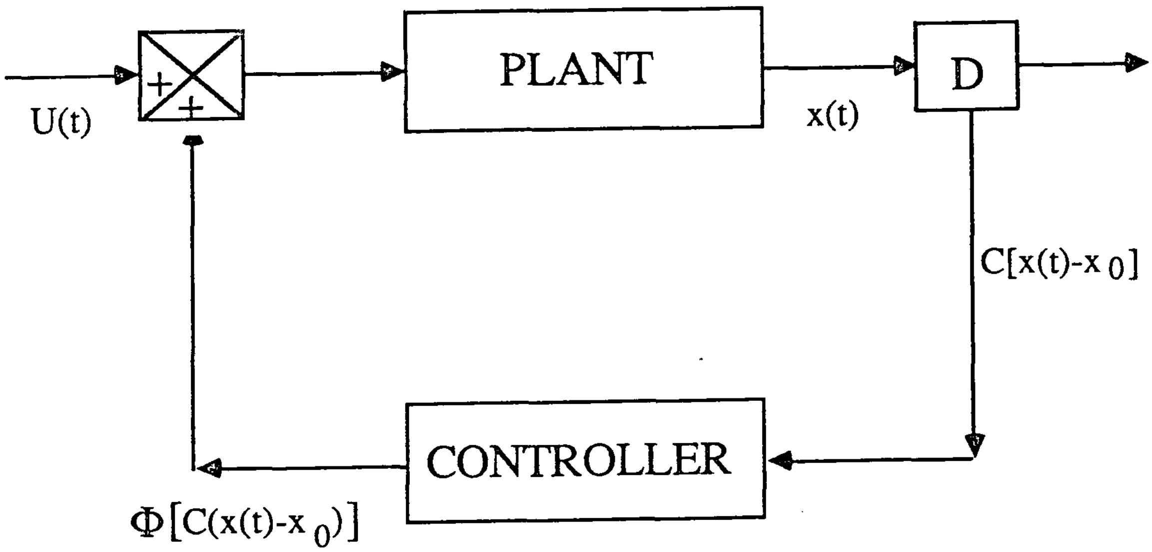


Figure 1.3: Block diagram of feedback control of a plant.

Since the exponential decreases for increasing time we conclude that

$$|\mathbf{x}(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty$$

which is the condition required.

We note here that $\kappa(\mathbf{P}) = \lambda_1/\lambda_n$ is called the **condition number** of $\mathbf{P}$.

1.4 Feedback Control of Unstable Plant.

In general the steady state of a system, that is the equilibrium solution to the mathematical model involved is not stable. Stability is then achieved by involving a feedback as was shown in Figure 1.2.

For simplicity let us consider a linear system modelled by equation (1.4), that is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

with

$$\mathbf{x} \in \Re^n$$

$$\mathbf{u} \in \Re^r.$$

The steady state would then satisfy equation (1.6). Consider now Figure 1.3 which represents a plant with an external input $\mathbf{u}(t)$ and output $\mathbf{x}(t)$. It is desired to operate the system at the steady state $(\mathbf{u}_\circ, \mathbf{x}_\circ)$. The block diagram depicts three stages in the feedback control. At the output a measuring device measures the deviation from the steady state output $\mathbf{x}_\circ$, generally giving a value

$$\mathbf{C}(\mathbf{x}(t) - \mathbf{x}_\circ),$$

where $\mathbf{C}$ is a constant matrix of dimension, say $s \times n$ to the controller. The controller in turn adds to (or subtracts from) the input a function which is continuous and may be written as

$$\Phi[\mathbf{C}(\mathbf{x}(t) - \mathbf{x}_\circ)].$$

The function Φ in effect maps the real space of dimension s to the r-dimensional real space.

The addition of the feedback signal Φ changes our plant equation (1.4) to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}[\mathbf{u} + \Phi[\mathbf{C}(\mathbf{x}(t) - \mathbf{x}_\circ)]] \qquad (1.15).$$

The equation for the deviation $\vec{\xi} = \mathbf{x}(t) - \mathbf{x}_\circ$ then becomes

$$\dot{\vec{\xi}} = \mathbf{A}\vec{\xi} + \mathbf{B}[(\mathbf{u} - \mathbf{u}_\circ) + \Phi(\mathbf{C}\vec{\xi})]. \qquad (1.16)$$

Once again we can limit to a constant input condition

$$\mathbf{u}(t) = \mathbf{u}_\circ$$

for all

$$t \geq t_\circ$$

to get

$$\dot{\vec{\xi}} = \mathbf{A}\vec{\xi} + \mathbf{B}\Phi(\mathbf{C}\vec{\xi}). \qquad (1.17)$$

Now it is clear that the plant will be stable provided that the steady state solution

$$\vec{\xi} = 0$$

is an asymptotically stable solution of equation (1.17).

It is to be noticed that the system of equations is no longer necessarily linear. However the investigation of stability via quadratic forms (Liapunov functions) discussed earlier can be extended to a general $(n \times n)$ system of the form

$$\dot{\vec{\xi}} = \mathbf{f}(t, \vec{\xi}) \qquad (1.18)$$

with the requirement that Picard's condition (Theorem 1-1) holds throughout the real space $\Re \times \Re^n$.

Theorem 1-4. *If there exists a positive real constant ϵ and a constant real symmetric and positive definite $n \times n$ matrix $\mathbf{P}$ with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

such that

$$(\mathbf{f}(\vec{\xi}, t))^T \mathbf{P}\vec{\xi} + \vec{\xi}^T \mathbf{P}\mathbf{f}(\vec{\xi}, t) + 2\epsilon\vec{\xi}^T \mathbf{P}\vec{\xi} \leq 0 \qquad (1.19)$$

for all points

$$(t, \vec{\xi}) \in \Re \times \Re^n,$$

then every solution $\vec{\xi}(t)$ of equation (1.18) satisfies

$$|\vec{\xi}(t)| \leq |\vec{\xi}(t_o)| \sqrt{\frac{\lambda_1}{\lambda_n}}\, e^{\epsilon(t_o - t)}$$

for all t later than t_o. Hence

$$|\vec{\xi}(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

The proof of this assertion runs parallel to the proof of Theorem 1-3. An alternative statement of the theorem and proof can be found in Cronin [4,p.195] and a discussion is also given by Porter [13,p.63–65]. The above theorem has an important corollary which we now state.

Corollary. *Suppose that the Jacobian matrix*

$$\mathbf{J}(t,\vec{\xi}) = \frac{\partial \mathbf{f}(t,\vec{\xi})}{\partial \vec{\xi}} \tag{1.20}$$

exists and has continuous elements. If $\mathbf{f}(t,0) = 0$ *for all* t *and there exists a constant (real) number* $\epsilon > 0$ *and a constant, symmetric positive definite matrix* $\mathbf{P}$ *such that*

$$(\mathbf{J}(t,\vec{\xi}))^T\mathbf{P} + \mathbf{P}\mathbf{J}(t,\vec{\xi}) + 2\epsilon\mathbf{P} \leq 0 \tag{1.21}$$

for all points $(t,\vec{\xi})$. *Then* $\mathbf{f}(t,\vec{\xi})$ *satisfies equation (1.19) and hence* $\vec{\xi} = 0$ *is an asymptotically stable solution.*

Proof:

We can write

$$\mathbf{f}(t,\vec{\xi}) = \mathbf{f}(t,\vec{\xi}) - \mathbf{f}(t,0)$$

or introducing a scalar θ with

$$0 \leq \theta \leq 1$$

$$\mathbf{f}(t,\vec{\xi}) = \int_0^1 \frac{d}{d\theta}\mathbf{f}(t,\theta\vec{\xi})d\theta$$

which can be written using equation (1.20) and the chain rule as

$$\mathbf{f}(t,\vec{\xi}) = \int_0^1 \mathbf{J}(t,\theta\vec{\xi})\vec{\xi}d\theta.$$

Multiplying both sides by $\vec{\xi}^T\mathbf{P}$ we have,

$$\vec{\xi}^T\mathbf{P}\mathbf{f}(t,\vec{\xi}) = \int_0^1 \vec{\xi}^T\mathbf{P}\mathbf{J}(t,\theta\vec{\xi})\vec{\xi}d\theta. \tag{1.22}$$

Now if we take its transpose and add the two we can write

$$\mathbf{f}^T(t,\vec{\xi})\mathbf{P}\vec{\xi} + \vec{\xi}^T\mathbf{P}\mathbf{f}(t,\vec{\xi}) + 2\epsilon\vec{\xi}^T\mathbf{P}\vec{\xi} = \int_0^1 \vec{\xi}^T[\mathbf{J}^T(t,\theta\vec{\xi})\mathbf{P} + \mathbf{P}\mathbf{J}(t,\theta\vec{\xi}) + 2\epsilon\mathbf{P}]\vec{\xi}\, d\theta. \tag{1.23}$$

The integrand on the right hand side is negative semidefinite and hence the condition of the theorem follows.

Now let us consider again the feedback control equation (1.17). In practice a systems designer will desire the controller $\mathbf{B}\Phi$ to have

$$\Phi(0) = 0 \tag{1.24}$$

and $Re(\lambda) < 0$ for all eigenvalues λ of the Jacobian matrix

$$\mathbf{J}_f(0) = \mathbf{A} + \mathbf{B}\mathbf{J}_\Phi(0)\mathbf{C}. \tag{1.25}$$

Equations (1.24) and (1.25) ensure that $\vec{\xi} = 0$ is a locally asymptotically stable solution of equation (1.17). However, significantly it does not ensure that

$$\lim_{t\to\infty} |\vec{\xi}(t)| = 0$$

for every solution $\vec{\xi}(t)$ of equation (1.17).

Theorem 1-5. Ultimate Boundedness Theorem: *Suppose that there exist positive constants ϵ and $r_\circ$ and a constant positive definite matrix* $\mathbf{P}$ *such that equation (1.19) holds for all $(t, \vec{\xi})$ with*

$$|\vec{\xi}| \geq r_\circ.$$

If $r_1 \geq r_\circ$ then every solution $\vec{\xi}(t)$ of equation (1.18) with

$$|\vec{\xi}(t_\circ)| \leq r_1$$

satisfies the following conditions:

1. $|\vec{\xi}(t)| \leq r_1\sqrt{\kappa(\mathbf{P})}$ *for all $t \geq t_\circ$.*

2. $|\vec{\xi}(t)| \leq r_\circ\sqrt{\kappa(\mathbf{P})}$ *for all* t *when*

$$t \geq t_\circ + \frac{1}{\epsilon}\log[\frac{r_1}{r_\circ}\sqrt{\kappa(\mathbf{P})}].$$

That is every solution has

$$|\vec{\xi}(t)| \leq r_\circ\sqrt{\kappa(\mathbf{P})}$$

for all sufficiently large t.

[Here $\kappa(\mathbf{P})$ is the condition number of the matrix $\mathbf{P}$ and the number $r_\circ\sqrt{\kappa(\mathbf{P})}$ is called an ultimate bound of the solution of equation (1.18).]

Proof:

From Theorem 1-4 we can see that if a solution $\vec{\xi}(t)$ of equation (1.18) is such that for

$$t_\circ \leq t \leq t_1$$

we have

$$|\vec{\xi}(t)| \geq r_\circ,$$

then in the same interval

$$\begin{aligned} |\vec{\xi}(t)| &\leq |\vec{\xi}(t_\circ)|\sqrt{\kappa(\mathbf{P})}\, e^{\epsilon(t_\circ - t)} \\ &\leq r_1\sqrt{\kappa(\mathbf{P})}\, e^{\epsilon(t_\circ - t)}\ . \end{aligned}$$

This would give the contradiction

$$|\vec{\xi}(t_1)| < r_\circ$$

if

$$t_1 > t_\circ + \frac{1}{\epsilon}\log[\frac{r_1}{r_\circ}\sqrt{\kappa(\mathbf{P})}].$$

Hence we conclude that the solution $\vec{\xi}(t)$ must enter the ball

$$|\vec{\xi}| \leq r_\circ \tag{1.26}$$

within a time interval

$$\Delta t = \frac{1}{\epsilon} \log[\frac{r_1}{r_o} \sqrt{\kappa(\mathbf{P})}]. \tag{1.27}$$

Further during this time interval the solution $\vec{\xi}$ of equation (1.18) is bounded by

$$|\vec{\xi}(t)| \leq r_1 \sqrt{\kappa(\mathbf{P})}. \tag{1.28}$$

In case the function $\vec{\xi}$ leaves the region (1.26) again we take t_o to be the time at which it does so. Then $|\vec{\xi}(t_o)| = r_o$ and the above argument shows that it can remain outside the region (1.26) for a time no longer than (1.27) and satisfies

$$|\vec{\xi}(t)| \leq r_o \sqrt{\kappa(\mathbf{P})}. \tag{1.29}$$

throughout this interval.

1.5 Periodic Solutions and Hunt.

We have seen above that provided the solution $\vec{\xi}(t)$ to equation (1.18) lies in a small spherical region of radius r_1 at some initial time t_o (and the conditions of Theorem 1-5 hold), it will stay bounded for all later times t. At this point it is of interest to consider the existence of periodic solutions. For autonomous ordinary differential equations of second order, the Poincare–Bendixon theorem (see for example Hartman [7,p.151]) lays down the conditions for the existence of periodic solutions. It has been shown that this theorem can be extended under additional conditions to higher order systems [16,17]. A further question which can be asked at this stage is about the stability of the periodic solution or the closed trajectory [18].

The existence of a periodic solution for the feedback control equation (1.17) has important consequences. Suppose that equation (1.17) has a periodic solution or in other words a closed trajectory, say Γ which lies completely outside the region $\vec{\xi} \leq r_1$ (see Figure 1.4). Such an isolated closed trajectory is called a limit cycle. In case all nearby trajectories approach it asymptotically then it is called a stable limit cycle. It is possible that some error in design of the plant or a perturbation

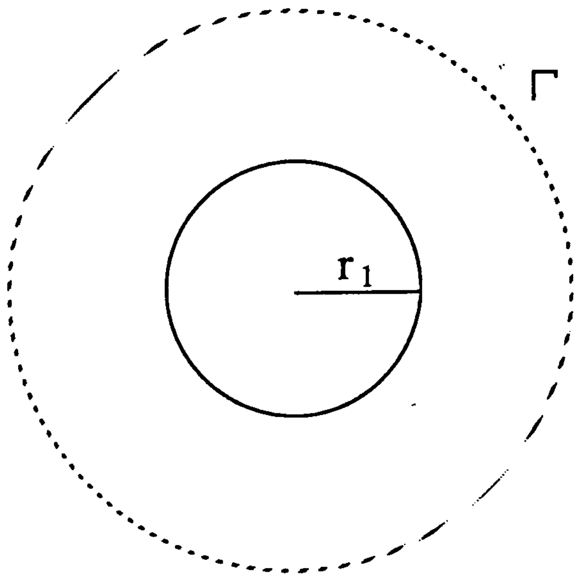


Figure 1.4: A limit cycle Γ.

in the input results in driving $\vec{\xi}(t)$ onto (or near to) Γ. In such an event the output (or state) of the plant acquires an oscillatory character

$$\mathbf{x}(t) = \mathbf{x}_o + \vec{\xi}(t) \tag{1.30}$$

which is not desired. These so called "self-oscillations" in a physical system are dangerous and need to be "hunted" and overcome.

It was observed by Oldenberger [9] that the 'hunt' or self-oscillation of some non-linear systems can be reduced or even completely eliminated by injecting into the control element an additional high frequency sinusoidal signal.

1.6 Dither.

The high frequency signal used to stabilize the plant is called 'dither'. The block diagram (Figure 1.5) shows its injection just before the control device. The

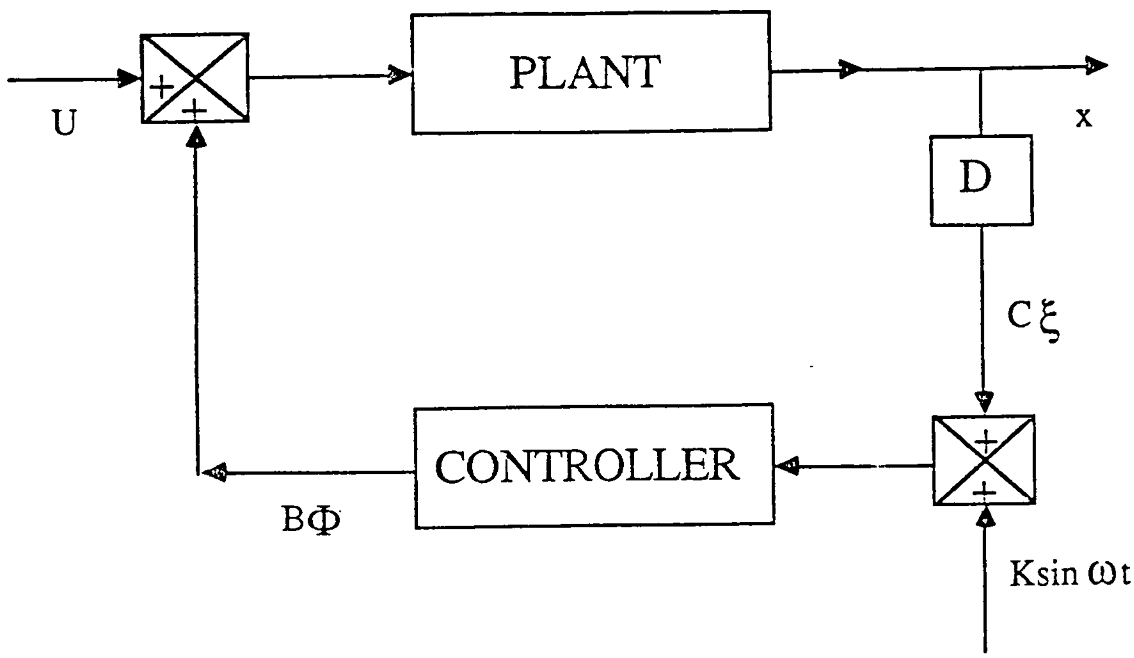


Figure 1.5: Introduction of dither to a feedback control plant.

feedback control equation (1.17) is now modified to

$$\frac{d\vec{\xi}}{dt} = \mathrm{A}\vec{\xi} + \mathrm{B}\Phi(\mathrm{C}\vec{\xi} + \mathbf{k}\sin\omega t), \tag{1.31}$$

where we have a constant vector $\mathbf{k}$ ($\in \Re^s$) and a large constant frequency ω. The aim of introducing the high frequency dither is to ensure that the condition (1.24) holds for all solutions of equation (1.31). This is attempted by appropriately choosing the vector $\mathbf{k}$ and frequency ω. If success is achieved the hunt or the parasitic oscillation is said to have been quenched.

In addition to the sinusoidal dither mentioned above other forms are also applied. Cook [3,p.149] gives details of various types of dither signals used. The problem then is to study the response of a nonlinear system to a high frequency input. This is the aim of engineering studies such as those by Oldenberger and Boyer [12] and Simpson and Power [15].

Engineering studies, such as in referances [12] and [15], focus on analysing the response of a system using an approximate method. The method used is called the describing function approach. The main idea is that since the behaviour is periodic, one can use Fourier or Laplace transforms. Further as the oscillatory character is usually dominated by a small number of frequencies one can approximate the input-output relation by a function known as the describing function. The method is also outlined by Cook [3,p.49] and extensively applied by Atherton [1]. The method has been widely used to predict nonlinear effects such as the excitation of limit cycles. However it can be misleading, particularly for low frequencies and larger time periods.

Our aim in this work is to make use of the qualitative theory of differential equations and the method of averaging, which we will now describe, to undertake a more rigorous study of quenching the limit cycle.

1.7 Method of Averaging.

Consider the system given by equation (1.31). Let us define a function

$$\Psi_{\mathbf{k}}(\vec{\xi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\vec{\xi} + \mathbf{k}\sin\theta)\, d\theta \tag{1.34}$$

for every $\vec{\xi}$ and $\mathbf{k}$ in the real space of dimension s. Then the autonomous system

$$\frac{d\vec{\hat{\xi}}}{dt} = \mathbf{A}\vec{\hat{\xi}} + \mathbf{B}\Psi_{\mathbf{k}}(\mathbf{C}\vec{\hat{\xi}}) \tag{1.35}$$

is called the averaged version of equation (1.31).

Theorem 1-6. *If $\vec{\xi}(t)$ and $\vec{\hat{\xi}}(t)$ are solutions of equations (1.31) and (1.35) respectively such that at initial time t_o*

$$\vec{\xi}(t_o) = \vec{\hat{\xi}}(t_o),$$

then for any finite number T, there exists a number $\omega_o(T)$ such that the distance $\|[\vec{\xi}(t) - \vec{\hat{\xi}}(t)]\|$ is of order T/ω, throughout the interval

$$t_o \leq t \leq t_o + T$$

provided that $\omega \geq \omega_o(T)$.

From the above theorem it can be concluded that if every solution of the autonomous system (1.35) satisfies the condition of asymptotic stability, that is

$$|\bar{\vec{\xi}}(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \tag{1.36}$$

then the solutions of equation (1.31) will follow those of the averaged version into a small neighbourhood, say δ of the origin and will remain in this neighbourhood thereafter provided that ω is sufficiently large.

As mentioned earlier there are various types of dither signals injected into the control device of a plant. In a general form a dither can be denoted by $\mathbf{k}p(\omega t)$ where $p(t)$ is some continuous real function periodic in a variable t with period denoted by λ. Thus a general dither equation is

$$\frac{d\vec{\xi}}{dt} = \mathbf{A}\vec{\xi} + \mathbf{B}\Phi[\mathbf{C}\vec{\xi} + \mathbf{k}p(\omega T)] \tag{1.37}$$

and the corresponding averaged equation is

$$\frac{d\bar{\vec{\xi}}}{dt} = \mathbf{A}\bar{\vec{\xi}} + \mathbf{B}\Psi_{\mathbf{k}}(\mathbf{C}\bar{\vec{\xi}})$$

where

$$\Psi_{\mathbf{k}}(\mathbf{y}) = \frac{1}{\lambda}\int_0^{\lambda} \Phi(\mathbf{y} + \mathbf{k}p(\theta))\, d\theta \tag{1.38}$$

for all $\mathbf{y}$ and $\mathbf{k}$ in $\Re^s$.

1.8 Some Formal Computations of Averages.

In this section we suppose that y, k are real numbers and that $\Phi(y)$ is an analytic function whose Taylor expansion has an infinite radius of convergence. We also suppose that

$$\Psi_{\mathrm{k}}(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(y + \mathrm{k}\sin\theta)\, d\theta$$

Expanding $\Phi(y + \mathrm{k}\sin\theta)$ by Taylor's theorem we get

$$\Phi(y + \mathrm{k}\sin\theta) = \Phi(y) + \sum_{i=1}^{\infty} \frac{\Phi^{(i)}(y)}{i!} (\mathrm{k}\sin\theta)^i. \tag{1.39}$$

Integrating both sides of equation (1.39) we get

$$\Psi_{\mathrm{k}}(y) = \Phi(y) + \sum_{i=1}^{\infty} \frac{\Phi^i(y)}{i!} \mathrm{k}^i \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^i\theta \, d\theta \tag{1.40}$$

We know that for odd i

$$\int_{-\pi}^{\pi} \sin^i\theta \, d\theta = 0,$$

and for even i $[i = 2r]$ we have

$$\int_{-\pi}^{\pi} \sin^{2r}\theta \, d\theta = \frac{(2r)!}{2^{2r}\,(r!)^2}. \tag{1.41}$$

Substituting back in equation (1.40) we get

$$\Psi_{\mathrm{k}}(y) = \Phi(y) + \sum_{r=1}^{\infty} \frac{\Phi^{(2r)}(y)}{(r!)^2} [\mathrm{k}/2]^{2r} \tag{1.42}$$

(1.) Now consider the special case when

$$\Phi(y) = \sin y.$$

Then

$$\Phi^{(2r)}(y) = (-1)^r \sin y,$$

and equation (1.42) reduces to

$$\begin{aligned} \Psi_{\mathrm{k}}(y) &= (\sin y)\{1 + \sum_{r=1}^{\infty} \frac{(-1)^r}{(r!)^2} [\mathrm{k}/2]^{2r}\} \\ &= (\sin y) J_o(\mathrm{k}) \end{aligned} \tag{1.43}$$

where $J_o(\mathrm{k})$ denotes the Bessel function of order zero.

Similarly for the case $\Phi(y) = \cos y$ we deduce that

$$\Psi_\mathrm{k}(y) = (\cos y) J_o(\mathrm{k}).$$

(2.) Now we take

$$\Phi(y) = \frac{1}{C^2 + y^2}$$

so that

$$\Psi_\mathrm{k}(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{C^2 + (y + \mathrm{k} \sin\theta)^2} \, d\theta, \tag{1.44}$$

where C, k, and y are real constants with $C \neq 0$. The integral can be written as a contour integral in the complex plane as

$$I = \oint_c \frac{dz}{iz[C^2 + [y + \frac{\mathrm{k}(z - \frac{1}{z})}{2i}]^2]}, \tag{1.46}$$

where $z = e^{i\theta}$ and the contour c is the circle $|z| = 1$. The integrand has singularities at the roots of the equation

$$-4C^2 z^2 + (\mathrm{k} z^2 + 2iyz - \mathrm{k})^2 = 0 \tag{1.47}$$

which further yeilds

$$\begin{aligned} z^2 + \frac{2(iy - C)z}{\mathrm{k}} - 1 = 0 \\ z^2 + \frac{2(-iy - C)z}{\mathrm{k}} - 1 = 0 \end{aligned} \tag{1.48}$$

where we change z to $-z$ in the second equation.

Thus the integral can be written as

$$I = \frac{-4}{i\mathrm{k}^2} \oint_c \frac{z dz}{(z - \alpha_1)(z - \alpha_2)(z + \bar{\alpha}_1)(z + \bar{\alpha}_2)} \tag{1.49}$$

where we have denoted the roots of equations (1.48) by α_1,α_2,$\bar{\alpha}_1$ and $\bar{\alpha}_2$ respectively. It can be evaluated using the Cauchy theorem which requires the sum of the residues at isolated singularities inside the contour c, in this case a circle $|z| = 1$.

Since

$$|\alpha_1\alpha_2| = +1$$

one of the roots, say α_1 lies inside the contour while the other will not contribute to the integral. Similarly $\bar{\alpha}_1$ will contribute while $\bar{\alpha}_2$ will not. Thus for the integral we have

$$I = \frac{-8\pi}{K^2}[\frac{\alpha_1}{(\alpha_1-\alpha_2)(\alpha_1+\bar{\alpha}_1)(\alpha_1+\bar{\alpha}_2)} + \frac{\bar{\alpha}_1}{(\bar{\alpha}_1+\alpha_1)(\bar{\alpha}_1+\alpha_2)(\bar{\alpha}_1-\bar{\alpha}_2)}]. \quad (1.50)$$

Note that we have

$$(z+\bar{\alpha}_1)(z+\bar{\alpha}_2) = z^2 + \frac{2(iy+C)}{\mathrm{k}}z - 1,$$

where we now substitute $z = \alpha_1$ to get

$$(\alpha_1+\bar{\alpha}_1)(\alpha_1+\bar{\alpha}_2) = \frac{4C\alpha_1}{\mathrm{k}} \quad (1.51)$$

and taking the complex conjugate we have also

$$(\bar{\alpha}_1+\alpha_1)(\bar{\alpha}_1+\alpha_2) = \frac{4C\bar{\alpha}_1}{\mathrm{k}}. \quad (1.52)$$

Substituting equations (1.51)and (1.52) back in the results (1.50) we can write

$$\begin{aligned} I &= \frac{-8\pi}{\mathrm{k}^2}[\frac{\alpha_1}{(\alpha_1-\alpha_2)(\frac{4C}{\mathrm{k}}\alpha_1)} + \frac{\bar{\alpha}_1}{(\bar{\alpha}_1-\bar{\alpha}_2)(\frac{4C}{\mathrm{k}}\bar{\alpha}_1)}] \\ &= -\frac{2\pi}{\mathrm{k}C}[2Re(\frac{1}{\alpha_1+\frac{1}{\alpha_1}})] \end{aligned} \quad (1.53)$$

which is solely in terms of the root α_1 of the first of equation (1.48). Using the fact that the roots α_1 and α_2 satisfy

$$\alpha_1+\alpha_2 = \frac{-2(iy-C)}{\mathrm{k}}, \quad \alpha_1\alpha_2 = -1,$$

we deduce that

$$\alpha_1 + \frac{1}{\alpha_1} = \frac{\pm 2\sqrt{\mathrm{k}^2+(iy-C)^2}}{\mathrm{k}}, \quad (1.54)$$

and hence

$$I = -\frac{4\pi}{\mathrm{k}C}[Re\frac{\mathrm{k}}{\pm 2\sqrt{\mathrm{k}^2 + (iy - C)^2}}]. \tag{1.55}$$

Let us put

$$\omega = (\mathrm{k}^2 + C^2 - y^2 - 2iyC)$$

and

$$\cos\theta = \frac{\mathrm{k}^2 + C^2 - y^2}{|\omega|}.$$

Then using trigonometric relations we get

$$I = \pm\frac{2\pi}{C}\frac{\cos\theta/2}{\sqrt{|\omega|}}. \tag{1.56}$$

Thus

$$\Psi_{\mathrm{k}}(y) = \frac{1}{\sqrt{2}C}\frac{\sqrt{A+B}}{A},$$

with

$$A = (B^2 + 4C^2y^2)^{\frac{1}{2}}, \qquad B = (\mathrm{k}^2 + C^2 - y^2).$$

When $y = 0$ we get

$$\Psi_{\mathrm{k}}(0) = \frac{1}{C\sqrt{(\mathrm{k}^2 + C^2)}}. \tag{1.57}$$

(3.) When equation (1.57) is differentiated partially with respect to C we can differentiate equation (1.44) under the integral sign to get

$$-\frac{\mathrm{k}^2 + 2C^2}{2C^3(\mathrm{k}^2 + C^2)^{\frac{-3}{2}}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} - 2C[C^2 + (\mathrm{k}\sin\theta)^2]^{-2}\, d\theta.$$

When divided by $2C$ this shows that if

$$\Phi(y) = \frac{1}{(C^2 + y^2)^2}$$

then

$$\Psi_{\mathrm{k}}(0) = \frac{\mathrm{k} + 2C^2}{2C^3(\mathrm{k}^2 + C^2)^{\frac{3}{2}}}. \tag{1.59}$$

1.9 Equations with Generalised Dither.

With $y, \mathrm{k} \in \Re$ we now discuss dither of the form $\mathrm{k}p(\omega t)$. Let us restrict to functions $p(t)$ which satisfy:

i. $p(t)$ is continuous, λ periodic real function of t.

ii. $p(t)$ is twice differentiable and $\ddot{p}(t)$ is continuous in $\Re$.

iii. The equation $\dot{p}(t) = 0$ has only a finite number N of roots in the period interval $0 \leq t \leq \lambda$.

As an example the above conditions are obviously satisfied by $p(t) = \sin t$. We shall now propose

Lemma 1-1. *Suppose that a function*

$$f : \Re \longrightarrow \Re$$

has continuous derivatives and we write

$$f_{\mathrm{k}}(y) = \frac{1}{\lambda}\int_0^{\lambda} f(y + \mathrm{k}p\theta)\, d\theta$$

for all k. *If* $|f(y)|$ *and* $|f'(y)|$ *are both bounded in* $\Re$ *and the function* $p(t)$ *satisfies the three conditions listed above then*

$$f'_{\mathrm{k}}(y) = \frac{1}{\lambda}\int_0^{\lambda} f'(y + \mathrm{k}p(\theta))\, d\theta$$

satisfies

$$f'_{\mathrm{k}}(y) \longrightarrow 0 \quad \text{as} \quad \mathrm{k} \longrightarrow \infty$$

uniformly for $y \in \Re$.

Proof

Since $|f'(y)|$ and $|f(y)|$ are both bounded in $\Re$ therefore for some constants $\mathcal{L}$ and $\mathcal{M}$,

$$|f(y)| \leq \mathcal{M}$$

$$|f'(y)| \leq \mathcal{L}$$

for all y.

Since $\dot{p}(t) = 0$ has only a finite number N of roots in the period interval $0 \leq t \leq \lambda$, each of these can be enclosed in a small open interval such that the union $\mathcal{E}$ of these non-overlapping intervals has total length less than $\epsilon\lambda/2\mathcal{L}$. Then $[0, \lambda] - \mathcal{E}$ consists of $N + 1$ closed intervals $[a_\nu, b_\nu]$ with $\nu = 1, 2, \ldots, N$.

Thus we can write

$$f_{\mathrm{k}}'(y) = \frac{1}{\lambda}\Big[\int_{\mathcal{E}} f'(y + \mathrm{k}p(\theta))\, d\theta + \sum_{\nu=1}^{N+1} \int_{a_\nu}^{b_\nu} f'(y + \mathrm{k}p(\theta))\, d\theta\Big]. \tag{1.60}$$

Since $|f'(y)| \leq \mathcal{L}$ we have

$$\begin{aligned} \Big|\int_{\mathcal{E}} f'(y + \mathrm{k}p(\theta))d\theta\Big| &\leq \int_{\mathcal{E}} \mathcal{L} d\theta \\ &\leq \frac{1}{2}\mathcal{E}\lambda. \end{aligned}$$

Focus now on the second term of equation (1.60). Since $|\dot{p}(\theta)|$ is continuous and nonzero in each closed interval $[a_\nu, b_\nu]$ therefore there exists a constant h such that

$$h \geq \frac{1}{|\dot{p}(\theta)|}$$

for all θ in $[0, \lambda] - \mathcal{E}$. Integration by parts gives

$$\begin{aligned} \int_{a_\nu}^{b_\nu} f'(y + \mathrm{k}p(\theta))\, d\theta &= \Big[\frac{f(y + \mathrm{k}p(\theta))}{\mathrm{k}\dot{p}(\theta)}\Big]_{a_\nu}^{b_\nu} \\ &+ \int_{a_\nu}^{b_\nu} \frac{\ddot{p}(\theta)}{\mathrm{k}\dot{p}(\theta)^2} f(y + \mathrm{k}p(\theta))\, d\theta\,. \end{aligned} \tag{1.61}$$

Since $|f| \leq \mathcal{M}$ and $\frac{1}{|\dot{p}(\theta)|} \leq h$ this gives

$$\begin{aligned} \Big|\int_{a_\nu}^{b_\nu} f'(y + \mathrm{k}p(\theta))\Big| &\leq \frac{2\mathcal{M}h}{|\mathrm{k}|} \\ &+ \int_{a_\nu}^{b_\nu} \frac{|\ddot{p}(\theta)|\mathcal{M}h^2}{|\mathrm{k}|}\, d\theta\,. \end{aligned}$$

This, equations (1.60) and (1.61) now give

$$|f_k'(y)| \leq \frac{\epsilon}{2} + \frac{2\mathcal{M}h(N+1)}{\lambda k} + \frac{\mathcal{M}h^2C}{\lambda k}, \tag{1.62}$$

where

$$C = \int_0^\lambda |\ddot{p}(\theta)|\, d\theta.$$

If

$$k_1 = \frac{[4(N+1)h + 2h^2C]\mathcal{M}}{\lambda\epsilon}$$

then equation (1.62) gives

$$|f_k'(y)| < \epsilon,$$

for all k with k > k_1. That is

$$f_k' \longrightarrow 0 \quad \text{as} \quad |k| \longrightarrow \infty$$

uniformly for $y \in \Re$. This establishes Lemma 1-1.

If we take

$$f(y) = \frac{1}{C}\arctan\frac{y}{C}$$

then $f'(y) = 1/(C^2 + y^2)$ and these satisfy the hypothesis of Lemma 1-1 because $|f(y)| < (\pi/2C)$ and $|\dot{f}(y)| \leq 1/C^2$. So Lemma 1-1 gives

$$\int_0^\lambda \frac{d\theta}{C^2 + (y + kp(\theta))^2} \longrightarrow 0 \quad \text{as} \quad |k| \longrightarrow \infty \tag{1.63}$$

uniformly for $y \in \Re$ provided that $p(t)$ satisfies the three conditions listed above. Now we are in a position to prove the following theorem.

Theorem 1-7. *Suppose $f : \Re^s \longrightarrow \Re^r$ is continuous and $\mathbf{k} \in \Re^s$ so that we can write*

$$f_{\mathbf{k}}(\mathbf{y}) = \frac{1}{\lambda} \int_0^\lambda \mathbf{f}(\mathbf{y} + \mathbf{k}p(\theta))\, d\theta.$$

If $\mathbf{f}(\mathbf{y}) \to 0$ as $|\mathbf{y}| \to \infty$ then $f_{\mathbf{k}}(\mathbf{y}) \to 0$ as $|\mathbf{k}| \to \infty$ uniformly for $\mathbf{y} \in \Re^s$.

Proof

For any given ϵ we must prove that there exists a number $k_1(\epsilon)$ independent of $\mathbf{y}$, such that $|f_{\mathbf{k}}(\mathbf{y})| < \epsilon$ for all $\mathbf{y}, \mathbf{k} \in \Re^s$ with $|\mathbf{k}| \geq k_1(\epsilon)$.

Since $\mathbf{f}(\mathbf{y}) \to 0$ as $|\mathbf{y}| \to \infty$ there exists a number C such that $|\mathbf{f}(\mathbf{y})| < (\epsilon/2)$ for all $\mathbf{y}$ with $|\mathbf{y}| \geq C$.

Let $\mathcal{M}$ be the maximum value of the continuous function $|\mathbf{f}(\mathbf{y})|$ for all $\mathbf{y}$ in a bounded closed ball $|\mathbf{y}| \leq C$, then

$$|\mathbf{f}(\mathbf{y})| \leq \frac{\epsilon}{2} + \frac{\mathcal{M}(C^2 + a^2)}{|\mathbf{y}|^2 + a^2}$$

for all $\mathbf{y} \in \Re^s$ and $a > 0$ is a constant. Further

$$|f_{\mathbf{k}}(\mathbf{y})| \leq \frac{1}{\lambda} \int_0^\lambda |f(\mathbf{y} + \mathbf{k}p(\theta))|\, d\theta$$

$$\leq \frac{\epsilon}{2} + \frac{\mathcal{M}(C^2 + a^2)}{\lambda} \int_0^\lambda \frac{1}{a^2 + |\mathbf{y} + \mathbf{k}p(\theta)|^2}\, d\theta.$$

We know that

$$a^2 + |\mathbf{y} + \mathbf{k}p(\theta)|^2 \geq a^2 + (Z + |\mathbf{k}|p(\theta))^2$$

where $Z = |\mathbf{k}|^{-1}(\mathbf{k} \cdot \mathbf{y})$ and therefore

$$|f_{\mathbf{k}}(\mathbf{y})| \leq \frac{\epsilon}{2} + \frac{\mathcal{M}(a^2 + C^2)}{\lambda} \int_0^\lambda \frac{d\theta}{a^2 + (Z + |\mathbf{k}|p(\theta))^2}.$$

By equation (1.63) we can choose a number k_1 independent of Z such that

$$\left| \int_0^\lambda \frac{d\theta}{a^2 + (Z + |\mathbf{k}|p(\theta))^2} \right| \leq \frac{\epsilon\lambda}{2\mathcal{M}(a^2 + C^2)}$$

for all $Z, \mathbf{k}$ with $|\mathbf{k}| \geq k_1$ and therefore

$$\begin{aligned} |f_{\mathbf{k}}(\mathbf{y})| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

for all $\mathbf{k}, \mathbf{y} \in \Re^s$ with $|\mathbf{k}| \geq k_1$. Hence the conclusion that

$$f_{\mathbf{k}}(\mathbf{y}) \to 0 \quad \text{as} \quad \mathbf{k} \to \infty$$

uniformly for $\mathbf{y} \in \Re^s$. This establishes Theorem 1-7.

Chapter II

THE USE OF SMALL PARAMETERS

2.1 The Theory of Small Parameters.

A classical problem in the study of differential equations is the analysis of a system given as

$$\dot{\mathbf{x}} = \lambda \mathbf{F}(t, \mathbf{x}, \lambda)\,, \tag{2.1}$$

where λ is a parameter usually considered to be small. It is clear from the above equation that when λ approaches zero we would expect a solution $\mathbf{x}(t)$ to approach a constant value. Let us define a finite interval $\mathcal{E}$ in $(t, \mathbf{x}, \lambda)$ space,

$$\mathcal{E} = \{(t, \mathbf{x}, \lambda) :\; 0 \leq t \leq a\,, |\mathbf{x}| \leq b, -c \leq \lambda \leq c\} \tag{2.2}$$

where our solution $\mathbf{x}(t)$ is approximately constant. Further we demand that Picard's theorem (Theorem 1.1) is satisfied and

$$\begin{aligned} \mathcal{M} &= \max|\mathbf{F}(t, \mathbf{x}, \lambda)| \quad \text{for} \quad (t, \mathbf{x}, \lambda) \in \mathcal{E} \\ \delta &= \min\,[c\,, \frac{b}{2a\mathcal{M}}] > 0\,. \end{aligned} \tag{2.3}$$

The above result allows us to state a theorem (for example Sanchez [14,p.136]).

Theorem 2-1. Small Parameter Theorem: *If $|\lambda|$ is less than a small positive number δ then the solution $\mathbf{x}(t)$ of the differential equation (2.1) with $\mathbf{x}(0) = 0$ can be extended throughout the region $0 \leq t \leq a$ and satisfies*

$$\mathbf{x}(t) \leq |\lambda|\, a\mathcal{M}$$

throughout this interval. In other words the magnitude $|\mathbf{x}(t)|$ is of order λ throughout the time interval $[0, a]$.

The above discussion can be generalised to the case of periodic solutions. Consider that the equation

$$\dot{\mathbf{x}} = \mathbf{G}(t, \mathbf{x}, \lambda) \tag{2.4}$$

has a periodic solution for a fixed value of the parameter λ, say $\lambda = 0$. An important question then is : Does equation (2.4) have a periodic solution for small $|\lambda|$? , and further is this periodic solution sufficiently 'close' to the given periodic solution?

If equation (2.4) models an autonomous system, the appearance of periodic solutions for small $|\lambda|$ is called the **bifurcation** of periodic solutions. As we shall be concerned with autonomous systems, it is appropriate here to illustrate the general method involved (see also Jordan and Smith [9,p.101]).

2.2 Harmonic Oscillator with a Small Perturbation.

Let us consider an autonomous system in two dimensions,

$$\begin{aligned} \frac{dx}{dt} &= y - \lambda\, f(x,y) \\ \frac{dy}{dt} &= -x - \lambda\, g(x,y) \end{aligned} \tag{2.5}$$

where λ is a small positive parameter, and $f(x,y)$, $g(x,y)$ are polynomials in x and y. When the small parameter λ is taken to be zero, equation (2.5) reduces to the familiar harmonic system,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x, \end{aligned} \tag{2.6}$$

whose trajectory is given by a circle of radius r_o,

$$x^2 + y^2 = r_o^2. \tag{2.7}$$

For small λ we intuitively expect the trajectories of the system (2.5) to be near the circle (2.7) and they may be simple closed curves or spirals.

In order to further analyse the behaviour of trajectories we transform to polar coordinates to write

$$\begin{aligned}\dot{r} &= (x\dot{x} + y\dot{y})\, r^{-1} \\ &= -\lambda[f(r\cos\theta, r\sin\theta)\cos\theta + g(r\cos\theta, r\sin\theta)\sin\theta],\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\dot{\theta} &= (x\dot{y} - y\dot{x})\, r^{-2} \\ &= -1 + \lambda r^{-1}\, [f(r\cos\theta, r\sin\theta)\sin\theta - g(r\cos\theta, r\sin\theta)\sin\theta].\end{aligned} \tag{2.9}$$

Thus we get

$$\frac{dr}{d\theta} = \lambda \frac{du}{d\theta}$$

where we have substituted

$$r = r_o + \lambda u(\theta)$$

with constant r_o. It is clear that for small λ

$$\frac{du(\theta)}{d\theta} = f(r\cos\theta, r\sin\theta)\cos\theta + g(r\cos\theta, r\sin\theta)\sin\theta + O(\lambda). \tag{2.11}$$

We can further write

$$u(\theta) = U(r_o, \theta) + v(\theta),$$

where

$$U(r_o, \theta) = \int_0^{\theta} [f_o\cos\phi + g_o sin\phi]\, d\phi, \tag{2.12}$$

with

$$f_o = f(r_o\cos\phi, r_o\sin\phi).$$

And also

$$\frac{du}{d\theta} = \frac{dU}{d\theta} + \frac{dv}{d\theta}. \tag{2.13}$$

Comparing equations (2.11) and (2.13) we conclude that $dv/d\theta$ is of order λ where λ is small. We use the small parameter theorem (Theorem 2.1) to say that $v(\theta)$ is of order λ in the region $0 \leq \theta \leq 2\pi$ provided that

$$v(0) = 0.$$

The polar equation of the trajectory can thus be written as

$$\begin{aligned} r(\theta) &= r_o + \lambda\, u(\theta) \\ &= r_o + \lambda\, U(r_o, \theta) + O(\lambda^2) \end{aligned} \tag{2.14}$$

in $0 \leq \theta \leq 2\pi$ with $r(0) = r_o$.

From equation (2.9) we find that θ decreases as t increases. Further if

$$r(2\pi) - r_o > 0 \tag{2.15}$$

the trajectory spirals inwards and if

$$r(2\pi) - r_0 < 0 \tag{2.16}$$

the trajectory spirals outwards. Now, since

$$r(2\pi) - r_o = \lambda[U(r_o, 2\pi) + O(\lambda)] \tag{2.17}$$

we can restate the conditions (2.15) and (2.16) as conditions on $U(r_o, 2\pi)$. In other words if

$$U(r_o, 2\pi) > 0$$

the trajectory spirals inwards and if

$$U(r_o, 2\pi) < 0$$

they spiral outwards, provided that λ is sufficiently small and positive.

It is easy to observe from equations (2.12) and (2.13) that if r_1 is a root of the polynomial equation

$$U(r, \theta) = 0$$

at which

$$\frac{dU}{dr} \neq 0$$

then a closed trajectory exists within a distance of order λ of the circle

$$x^2 + y^2 = {r_1}^2.$$

2.3 Applications.

In the following we will make use of the method of averaging outlined in Chapter I and the small parameter approach given above to analyse two model systems. First we consider a system modelled by a second order equation:

$$\ddot{x} + \epsilon a\dot{x} + \epsilon\Psi_{\mathrm{k}}(\dot{x}) + x = 0 \tag{2.18}$$

with

$$\Phi(y) = y^5 - by^3 + cy$$

where a, b, c and ϵ are positive and ϵ is small. We wish to find the radii of closed trajectories and examine what happens to them as the number k increases from zero.

Recall that

$$\begin{aligned}\Psi_{\mathrm{k}}(y) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(y + \mathrm{k}\sin\theta)\, d\theta \\ &= \Phi(y) + \sum_{r=1}^{\infty} \frac{\Phi^{2r}(y)}{(r!)^2}(\mathrm{k}/2)^{2r},\end{aligned} \tag{2.19}$$

where we have used equation (1.42). Calculating the derivatives involved and rearranging we get

$$\Psi_{\mathrm{k}}(y) = y^5 + y^3\,(5\mathrm{k}^2 - b) + y(\frac{15}{8}\mathrm{k}^4 - \frac{3}{2}b\mathrm{k}^2 + c). \tag{2.20}$$

We can write equation (2.18) as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - \epsilon[y^5 + y^3\,(5\mathrm{k}^2 - b) + 5(\frac{15}{8}\mathrm{k}^4 - \frac{3}{2}b\mathrm{k}^2 + a + c)],\end{aligned} \tag{2.21}$$

and a comparison with equation (2.5) yields

$$f(x,y) = 0$$

$$g(x,y) = y^5 + y^3(5k^2 - b) + y(a + c + \frac{15}{8}k^4 - \frac{3}{2}bk^2).$$

Further to analyse the behaviour of trajactories we transform to polar coordinates and follow the procedure outlined in the example above. We have

$$U(r) = \int_{-\pi}^{\pi} [r^5 \sin^6\phi + r^3 \sin^4\phi(5k^2 - b) + r\sin^2\phi(a + c + \frac{15}{8}k^4 - \frac{3}{2}bk^2)]\, d\phi$$
$$= \frac{\pi r}{8}[5r^4 + 6r^2(5k^2 - b) + 8(a + c + \frac{15}{8}k^4 - \frac{3}{2}bk^2)]. \tag{2.22}$$

For a closed periodic trajectory with radius r we require

$$U(r) = 0$$

which can be solved for r^2 and gives two roots

$$r^2 = \frac{-3(5k^2 - b) \pm \sqrt{9(5k^2 - b) - 5[8(a + c) + 15k^4 + 2bk^2]}}{5}. \tag{2.23}$$

Let us first consider the case when k is zero, then equation (2.23) reduces to

$$r^2 = \frac{+3b \pm \sqrt{9b^2 - 40(a + c)}}{5}$$

and the roots are real if

$$b^2 > \frac{40}{9}(a + c). \tag{2.24}$$

Further if

$$(a + c) \geq 0$$

then both roots are positive, which means that we have two closed trajectories. In case

$$(a + c) < 0$$

then one root will be positive and the other negative. The negative root corresponds to an imaginary radius, so we will have one periodic trajectory.

For nonzero k, we can write the discriminant in equation (2.23) as

$$D = \frac{15}{2}b^2 - 40(a+c) + 6(5\mathrm{k}^2 - \frac{1}{2}b)^2 \tag{2.25}$$

and observe that it decreases in the interval $0 < \mathrm{k}^2 < b/10$ and increases for $\frac{b}{10} < \mathrm{k}^2 < \infty$. The minimum value of the discriminant is thus at $\mathrm{k}^2 = \frac{b}{10}$, and for real roots we require

$$b^2 > \frac{16}{3}(a+c). \tag{2.26}$$

For this condition the roots (2.23) and their average are plotted as a function of k^2 in Figure 2.1 for given values of the parameters. From equations (2.24) and (2.26) it is of interest to focus on the region

$$\frac{16}{3}(a+c) > b^2 > \frac{40}{9}(a+c). \tag{2.27}$$

In this region we again plot the roots r_1^2, r_2^2 and their average as a function of k^2 in Figure 2.2. We note that for large k^2 the radii squared of the orbits become negative — the dither has been quenched.

We next consider a special equation of the form

$$\ddot{x} + \epsilon f(\dot{x}) + x = 0 \tag{2.28}$$

which can be written as

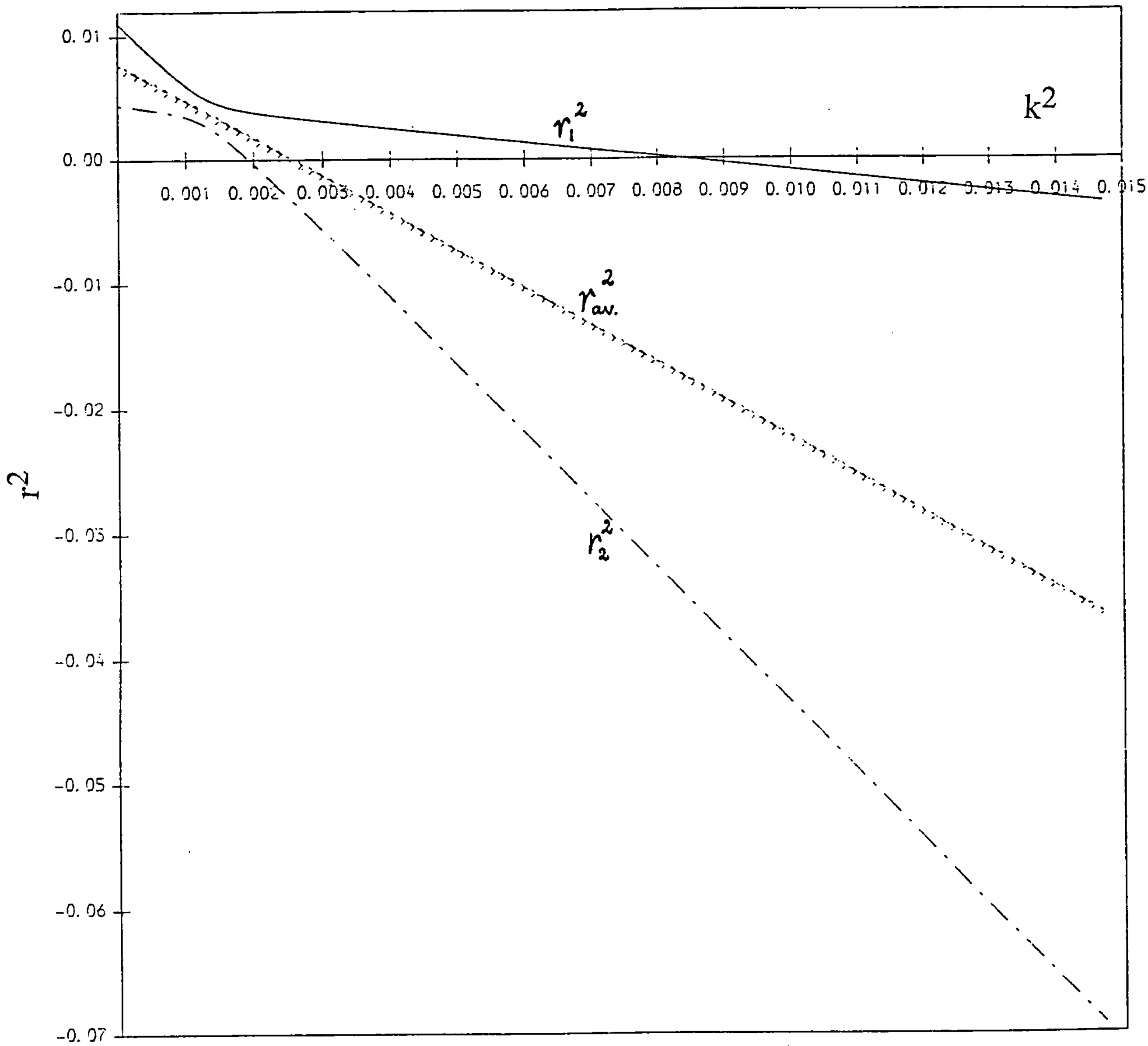
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon f(y)\,. \end{aligned} \tag{2.29}$$

In the case when

$$f(y) = ay - \sin y \tag{2.30}$$

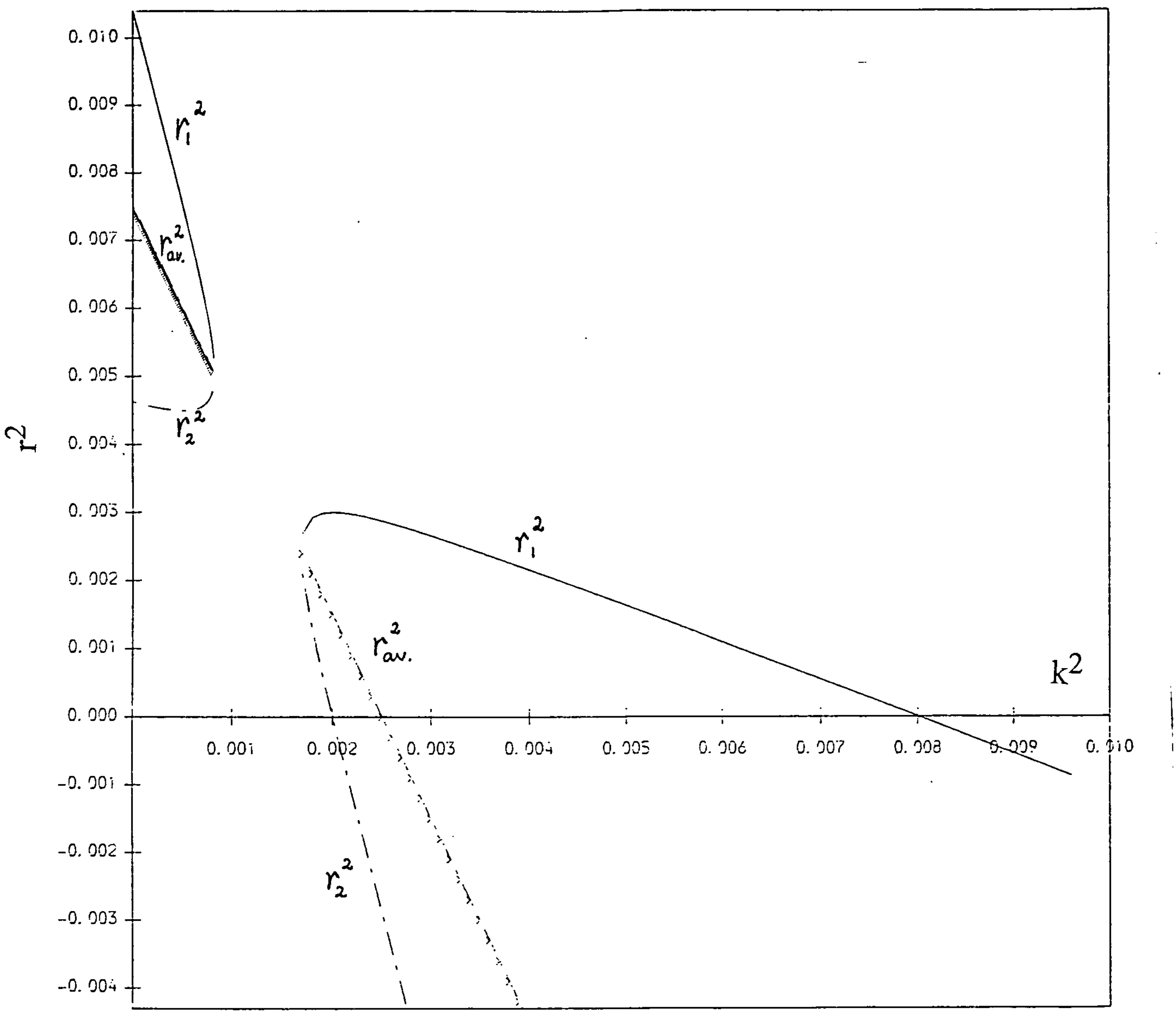
we can write the averaged version of equation (2.29) as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon f_{\mathrm{k}}(y), \end{aligned} \tag{2.31}$$



a = 0.00001
b = 0.0128
c = 0.00002

Figure 2.1: The real roots of equation (2.18) with $b^2 > 16(a+c)/3$.



a = 0.00001
b = 0.0128
c = 0.00002

Figure 2.2: The real roots in the region given by equation (2.27).

where equation (1.43) gives

$$f_{\mathrm{k}}(y) = ay - \sin y \, J_{\mathrm{o}}(\mathrm{k}), \tag{2.32}$$

$J_{\mathrm{o}}(\mathrm{k})$ denoting a Bessel function. Hence for our 2×2 system (2.31) we get

$$\begin{aligned} g(x,y) &= 0 \\ f(x,y) &= ay - J_{\mathrm{o}}(\mathrm{k})\sin y, \end{aligned} \tag{2.33}$$

and we can now apply the method of small parameters to write

$$U(r) = \int_{-\pi}^{\pi} [ar\sin^2\phi - \sin(r\sin\phi)(\sin\phi)J_{\mathrm{o}}(\mathrm{k})]\, d\phi \tag{2.34}$$

which can be written as

$$U(r) = \frac{ar\pi}{2} - J_{\mathrm{o}}(\mathrm{k}) \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(r)^{2\nu+2}}{(2\nu+1)!} \int_{-\pi}^{\pi} (\sin\phi)^{2\nu+2}\, d\phi, \tag{2.35}$$

where we have used the expansion

$$\sin\alpha = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(\alpha)^{2\nu+1}}{(2\nu+1)!}, \tag{2.36}$$

which after integration gives

$$U(r) = \frac{ar\pi}{2} - J_{\mathrm{o}}(\mathrm{k})\pi \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(2r)^{2\nu+1}(2\nu+2)}{(2)^{2\nu+2}((\nu+1)!)^2}. \tag{2.37}$$

It is easy to show that

$$\begin{aligned} \frac{dJ_{\mathrm{o}}(r)}{dr} &= -\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(2\nu+2)(r)^{2\nu+1}}{((\nu+1)!)^2 2^{2\nu+2}} \\ &\equiv J_{\mathrm{o}}'(r) \end{aligned} \tag{2.38}$$

and hence

$$U(r) = \frac{ar\pi}{2} + J_{\mathrm{o}}(\mathrm{k})J_{\mathrm{o}}'(r)\pi. \tag{2.39}$$

Further as

$$J_1(r) = -J_o'(r)$$

we have

$$U(r) = \frac{ar\pi}{2} - \pi J_o(\mathrm{k}) J_1(r). \tag{2.40}$$

Closed trajectories occur when $U(r) = 0$ or

$$\frac{a}{2J_o(\mathrm{k})} = \frac{J_1(r)}{r}. \tag{2.41}$$

To get the radii of the closed trajectories we plot

$$Z = \frac{J_1(r)}{r} \tag{2.42}$$

versus r. Next we draw the line (see Figure 2.3)

$$Z' = \frac{a}{2J_o(\mathrm{k})} \tag{2.43}$$

for different values of k. The points of intersection give us the radii of the closed trajectories. Since $J_1(r)$ is bounded it is clear that for large values of r we have $U(r) > 0$ (assuming that a is positive). This condition means that all trajectories spiral in from infinity and wind up to the outermost closed trajectory. On the other hand if we have

$$\frac{a}{2J_o(\mathrm{k})} < \frac{1}{2} \tag{2.44}$$

then for small values of r we get $U(r) < 0$, and thus the trajectories spiral outwards from origin to the first closed trajectory. Else if for small r

$$\frac{a}{2J_o(\mathrm{k})} > \frac{1}{2}$$

then $U(r)$ is positive and the trajectories would spiral inwards to origin because in this case there are no closed trajectories.

Next we wish to analyse the behaviour of the closed trajectories as the value of the parameter k increases from zero. As in Figure 2.3 let us fix the value of a

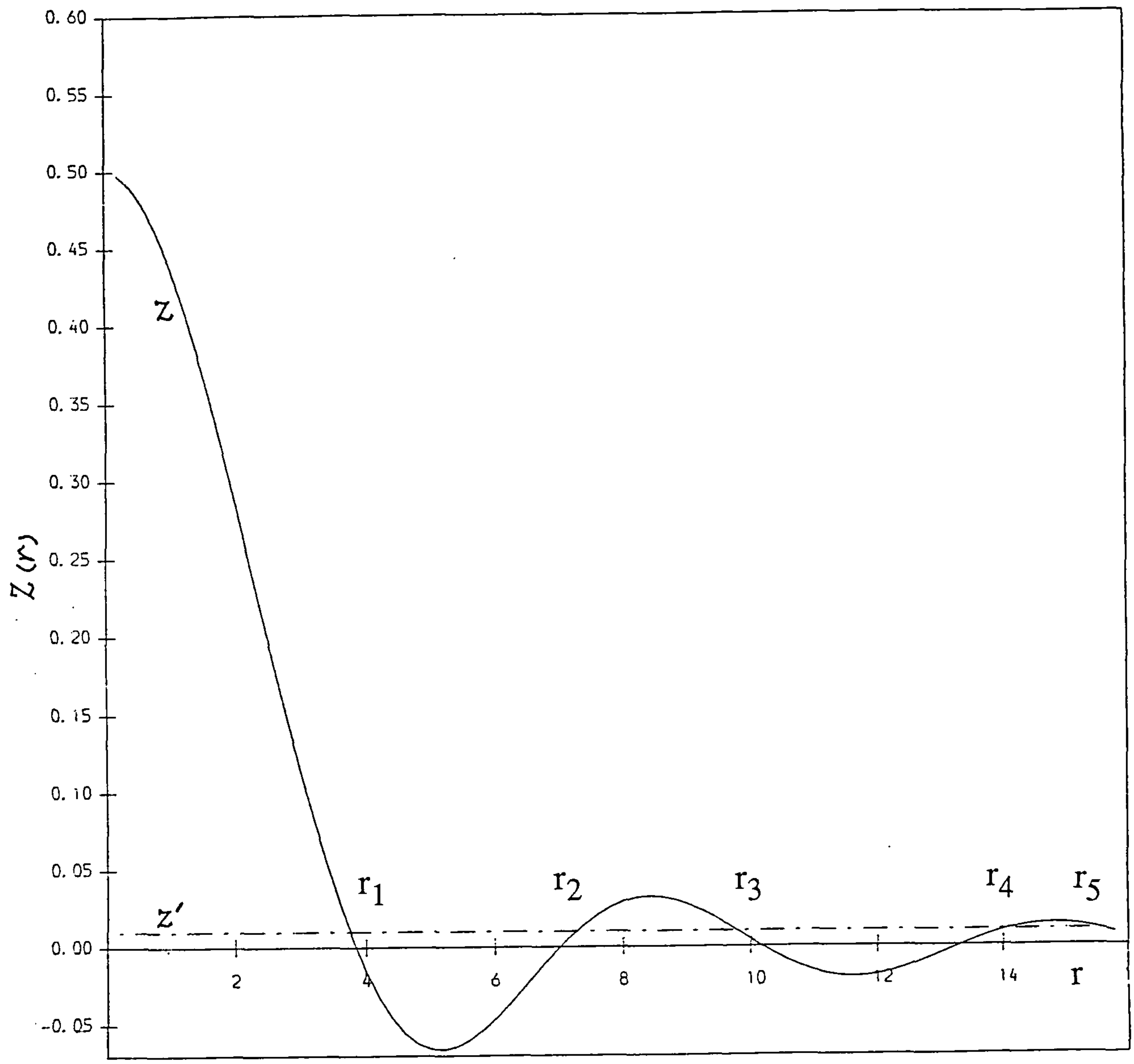


Figure 2.3: The function Z alongwith the line Z'. Here k = 0 and $a = 0.02$.

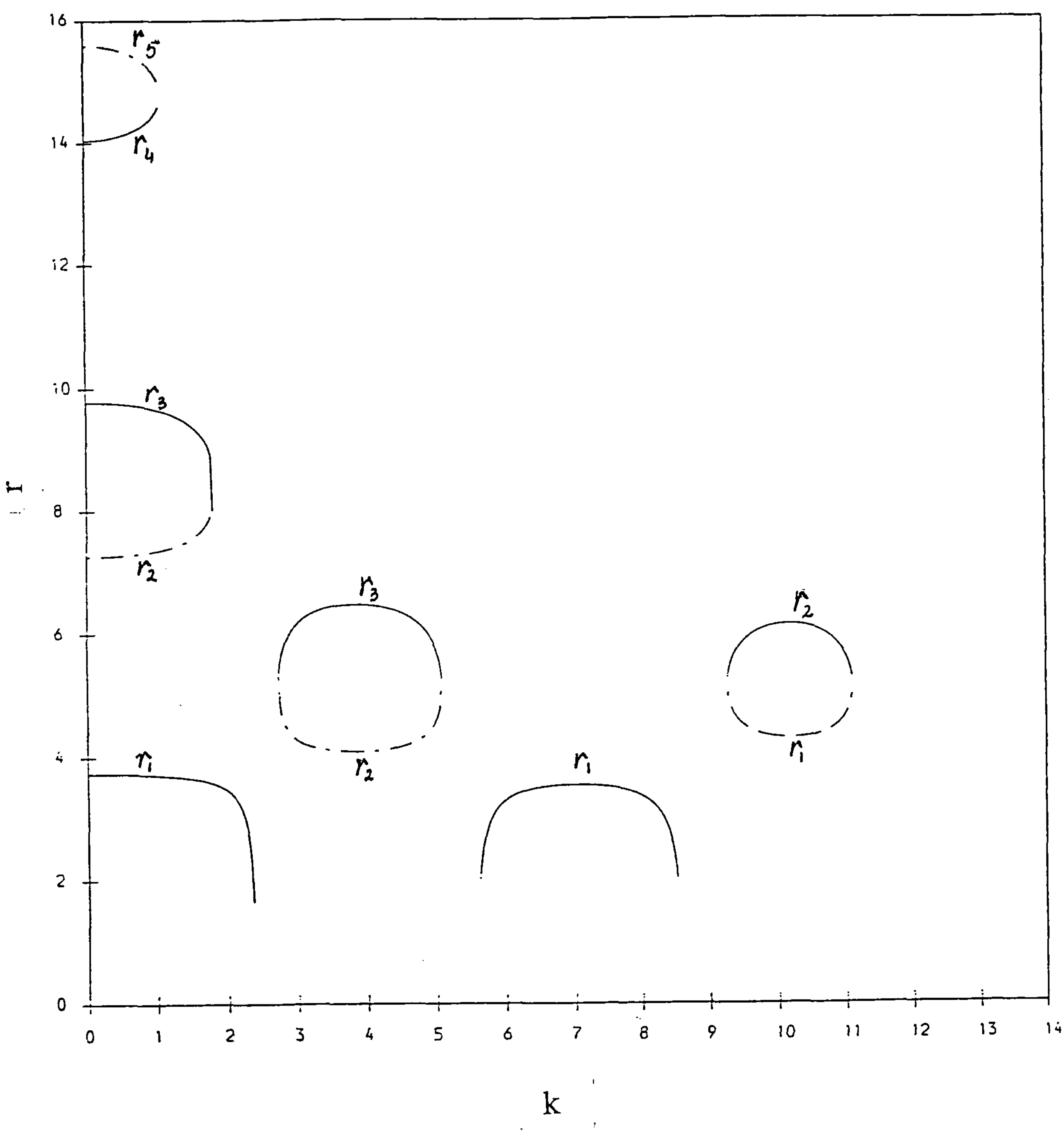


Figure 2.4: The radii r of closed trajectories plotted against k.

as 0.02. First we take k to be zero. From Figure 2.3 we conclude that there are five points where the line (2.43) intersects the curve (2.42). These are denoted as $r_1, r_2, r_3, r_4,$ and r_5. In Figure 2.4 we plot the radii r of the closed trajectories against k. It maybe noted that for large enough values of k, $J_o(\mathrm{k})$ is small and either the condition (2.45) holds or

$$\frac{a}{2J_o(\mathrm{k})} < -0.069,$$

and $U(r) > 0$, so that all trajectories spiral in from infinity to the origin. In other words by making k sufficiently large we can make the averaged equation asymptotically stable in the large.

Chapter III

HARD AND SOFT QUENCHING

3.1 Hard and Soft Quenching.

In Chapter II above we studied quenching of unwanted oscillations by the injection of a sinusoidal dither. The systems considered can be represented by equations of the form

$$\ddot{x} + \epsilon F(\dot{x}) + x = 0, \tag{3.1}$$

with a small parameter ϵ. We considered the functions

(i) $F(y) = ay - \sin y$ with a constant and positive.

(ii) $F(y) = y^5 - by^3 + cy$ with b and c constant and positive.

In both cases we found that as the amplitude k of the sinusoidal dither signal increased the oscillations were finally quenched. In fact the closed trajectories in the phase plane disappear and reappear later but the final stage of quenching involved a stable closed trajectory shrinking down to the critical point. Thereafter no closed trajectory appears for any large value of k. This sequence of events is known as 'soft quenching'.

A different phenomena was observed by Oldenberger and Boyer in the work reported in J. Hale [6,p.150] . They considered a third order equation

$$\frac{d\ddot{x}}{dt} + 2\ddot{x} + \dot{x} + \Phi(x) = 0, \tag{3.2}$$

where $\Phi(x)$ is a real function sketched in Figure 3.1 . The system has a self excited oscillation which is asymptotically stable and the problem is to quench this oscillation by replacing $\Phi(x)$ by $\Phi(x + \text{k}\sin\omega t)$, where k and ω are large. The observation of Oldenberger and Boyer was that as k increased above a certain finite value the oscillations disappear. However just before its disappearance

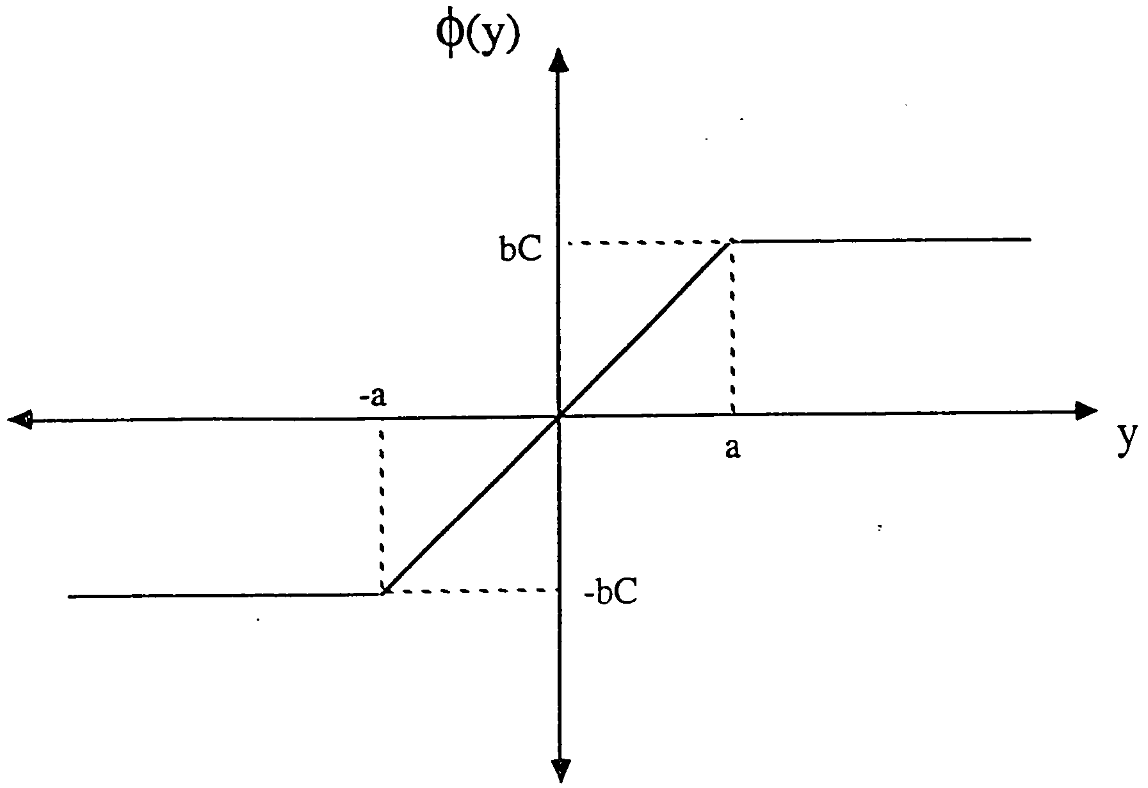


Figure 3.1: The function $\Phi(x)$ used in equation (3.2).

the amplitude of the oscillation was large. Hence this phenomena is called 'hard quenching'.

Using the non-rigorous method of describing functions, they concluded that before the disappearance of the unwanted self-oscillation the system also had an unstable oscillation very close to the stable one. Once the oscillations disappear the zero solution becomes asymptotically stable.

In other words, the phenomena observed by Oldenberger and Boyer was different from soft quenching. In this case the final stage of quenching involved a stable closed trajectory in $\Re^3$ which remained at a finite distance from the critical point (the origin) as the dither amplitude k increased. As k increases an unstable closed trajectory expands out from the origin and destroys the stable trajectory

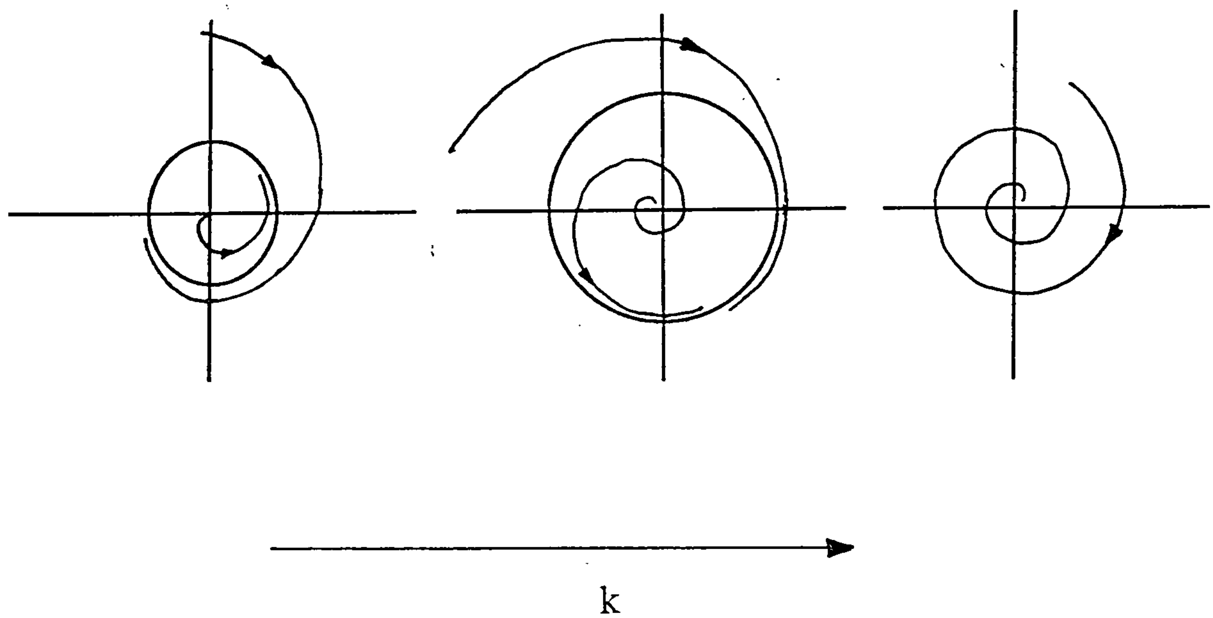


Figure 3.2: As k increases an unstable closed trajectory expands out from the origin and destroys the stable trajectory.

(See Figure 3.2). One could reverse the sequence and say that with the variation of k a stable critical point "bifurcates" into a stable limit cycle and an unstable singular point. The bifurcation effect of this type was studied by Andronov in his famous bifurcation theorem.

3.2 Andronov's Bifurcation Theorem.

Let us consider a 2×2 autonomous system

$$\begin{aligned} \dot{x} &= \mu x + y + p(x,y,\mu) \equiv P(x,y,\mu) \\ \dot{y} &= -x + \mu y + q(x,y,\mu) \equiv Q(x,y,\mu), \end{aligned} \tag{3.3}$$

where μ is a real parameter and p and q are real functions which are analytic at the point (0,0,0) .

Theorem 3-1. *If the analytic functions $p(x,y,\mu)$ and $q(x,y,\mu)$ satisfy the conditions*

(i) $p(0,0,\mu) = q(0,0,\mu) = 0$ *for all* μ, *that is* $(0,0)$ *is a critical point for each* μ , *and*

(ii) The critical derivatives of the functions p and q with respect to x and y are all zero at $(0,0)$ *for each* μ,
then there exists a constant $\mu_1 > 0$ *and a disc* $D_1 \subset \Re^2$ *containing the origin* $(0,0)$ *such that one and only one of the following statements is true.*

(A) For $-\mu_1 < \mu \leq 0$ *the system (3.3) has no closed trajectory in* D_1 *but a stable closed trajectory* Γ_μ *expands from the origin as* μ *increases from* 0 *to* μ_1.

(B) There exists a closed trajectory Γ_μ *in* D_1 *which shrinks down to the origin as* μ *increases from* $-\mu_1$ *to zero and* D_1 *contains no closed trajectory for* $0 \leq \mu < \mu_1$.

(C) D_1 *contains no closed trajectories when* $\mu \neq 0$ *and* $-\mu_1 < \mu < \mu_1$, *but every trajectory in* D_1 *is closed when* $\mu = 0$.

The three cases (A), (B) and (C) are depicted in Figure 3.3 . (We omit the proof but refer the interested reader to Minorsky [10,p.169] . In general it is very complicated to find out which of the paths is followed in a particular case. Sometimes it is possible to arrive at a decision by making use of the Bendixon negative criterion (see for example Davies and James[5,p.88]).

As an example consider the scalar equation

$$\ddot{x} - 2\mu\dot{x} + (1+\mu^2)x + (\sin\dot{x})^3 = 0 \tag{3.4}$$

which can be reduced to the 2×2 system

$$\begin{aligned} \dot{x} &= \mu x + y \\ \dot{y} &= -x + \mu y - (\sin(y+\mu x))^3 \end{aligned} \tag{3.5}$$

using

$$y = \dot{x} - \mu x.$$

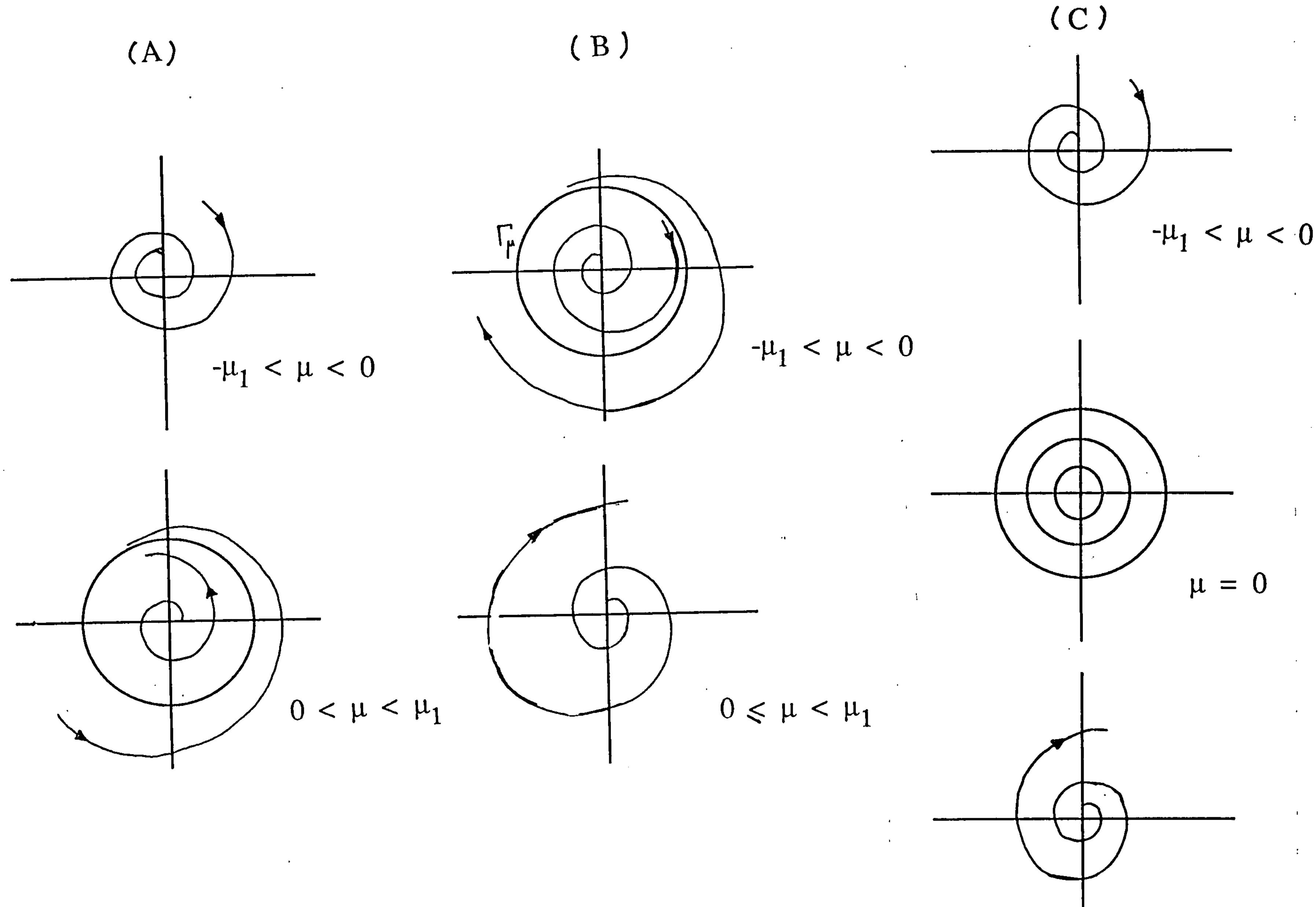


Figure 3.3: The various possible trajectories given by Andronov's bifurcation theorem.

This is of the form of equation (3.3) with

$$\begin{aligned} p(x,y,\mu) &= 0 \\ q(x,y,\mu) &= -[\sin(y+\mu x)]^3 \end{aligned} \tag{3.6}$$

and the conditions (i) and (ii) of Andronov's theorem are satisfied. Thus according to Andronov's theorem one of the results (A), (B) or (C) holds. In such a situation the negative criterion of Bendixon helps. Now

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2\mu - 3\sin^2(y+\mu x)\cos(y+\mu x). \tag{3.7}$$

Since

$$\sin^2\alpha\cos\alpha > 0 \quad \text{for} \quad \frac{-\pi}{2} < \alpha < \frac{\pi}{2},\ \alpha \neq 0, \tag{3.8}$$

we conclude that

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} < 0 \tag{3.9}$$

in the strips (see Figure 3.4),

$$0 < y + \mu x < \frac{\pi}{2} \quad \text{and} \quad \frac{-\pi}{2} < y + \mu x < 0 \tag{3.10}$$

provided that

$$\mu \leq 0.$$

Thus Bendixon's negative criterion tells us that if $\mu \leq 0$ then there is no closed trajectory in the shaded strip of Figure 3.4. Thus the possibilities (B) and (C) are excluded and (A) occurs, that is a stable closed trajectory bifurcates from the origin as μ increases from zero.

In general it is not easy to determine a suitable change of coordinates to reduce the equations of the system to the form of equation (3.3). We can however restate Andronov's theorem in a form which makes it easier to use in practice. Let us generally write

$$\begin{aligned} \dot{x} &= P(x,y,\mu) \\ \dot{y} &= Q(x,y,\mu). \end{aligned} \tag{3.11}$$

Then we can write the alternative form of Andronov's bifurcation theorem as follows.

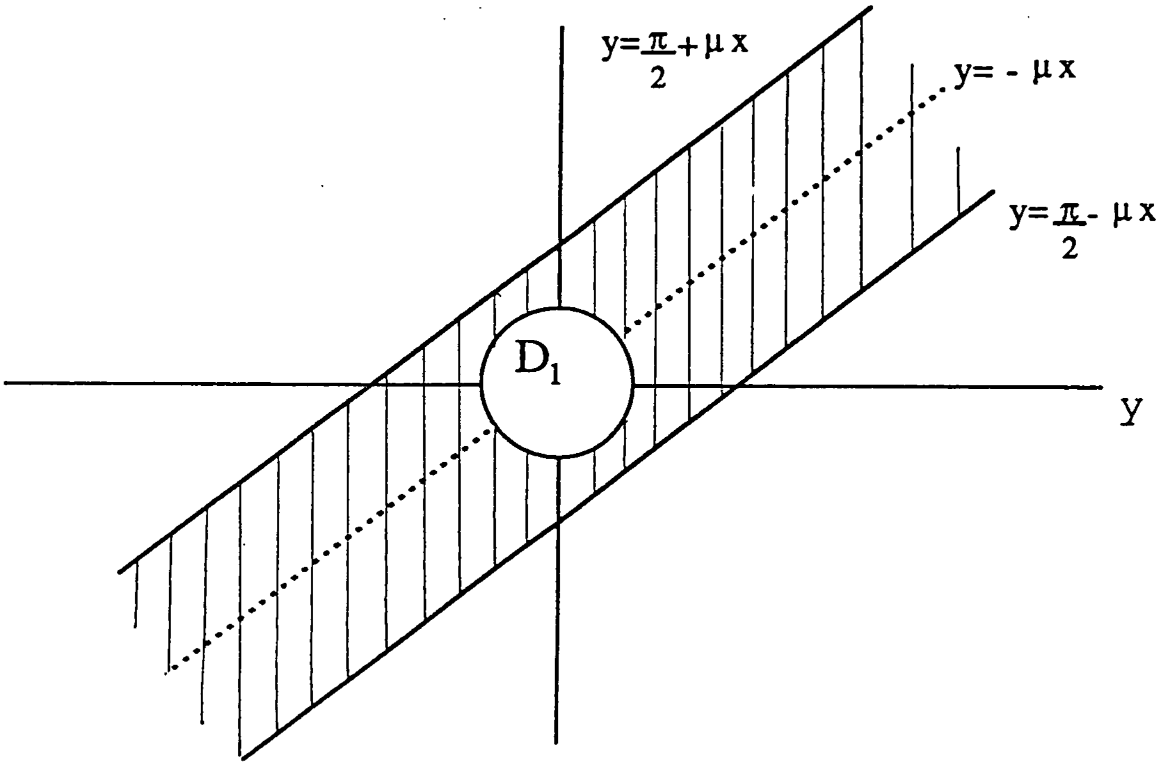


Figure 3.4: If $\mu \le 0$ then there is no closed trajectory in the shaded strip.

Theorem 3-2. *If the analytic functions $P(x,y,\mu)$ and $Q(x,y,\mu)$ satisfy :*

i. $P(0,0,\mu) = Q(0,0,\mu) = 0$ *for all* μ,

ii. the Jacobian matrix at the origin $\mathrm{J}(0,0)$ *has complex eigenvalues* $\alpha(\mu) \pm i\beta(\mu)$ *with*

$$\alpha(0) = 0$$

$$\beta(0) > 0,$$

iii. and the transversality condition

$$0 < \alpha'[= d\alpha(\mu)/d\mu]_{\mu=0},$$

then there exists a constant $\mu_1 > 0$ and a disc D_1 containing the origin $(0,0)$ such that one and only one of the statements (A), (B) or (C) above is true.

Proof

Since $\mathbf{J}(0,0)$ has complex eigenvalues $\alpha \pm i\beta$, therefore there exists an invertible 2×2 matrix $\mathbf{M}$ such that

$$\mathbf{M}^{-1}\mathbf{J}(0,0)\mathbf{M} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{3.12}$$

If we write equation (3.3) as

$$\begin{gathered} \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu) \\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} P(x,y,\mu) \\ Q(x,y,\mu) \end{pmatrix} \end{gathered} \tag{3.13}$$

then the substitution

$$\mathbf{x} = \mathbf{M}\mathbf{z}$$

gives

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{M}^{-1}\mathbf{F}(\mathbf{M}\mathbf{z}, \mu) \\ &\equiv \mathbf{G}(\mathbf{z},\mu). \end{aligned} \tag{3.14}$$

The Jacobian matrices are thus related as

$$\mathbf{J}_{\mathbf{G}}(\mathbf{z}) = \mathbf{M}^{-1}\mathbf{J}_{\mathbf{F}}(\mathbf{M}\mathbf{z})\mathbf{M} \tag{3.15}$$

and so

$$\begin{aligned} \mathbf{J}_{\mathbf{G}}(0) &= \mathbf{M}^{-1}\mathbf{J}_{\mathbf{F}}(0)\mathbf{M} \\ &= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \end{aligned} \tag{3.16}$$

Now if

$$\mathbf{z} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

then we can write equation (3.14) as

$$\dot{\mathbf{z}} = \mathbf{J}_{\mathbf{G}}(0)\mathbf{z} + \begin{pmatrix} p \\ q \end{pmatrix} \tag{3.17}$$

where at $\mathbf{z} = 0$

$$\begin{pmatrix} p \\ q \end{pmatrix} = 0. \tag{3.18}$$

Thus we get from equations (3.16) and (3.17)

$$\begin{aligned} \dot{\xi} &= \alpha\xi + \beta\eta + p(\xi, \eta, \mu) \\ \dot{\eta} &= -\beta\xi + \alpha\eta + q(\xi, \eta, \mu). \end{aligned} \tag{3.19}$$

and that at $\mathbf{z} = 0$

$$p(0, 0, \mu) = q(0, 0, \mu) = 0$$

which meets the first condition of Theorem 3-1.

From equation (3.16) one can easily verify that at the origin

$$\frac{\partial p}{\partial \xi} = \frac{\partial p}{\partial \eta} = \frac{\partial q}{\partial \xi} = \frac{\partial q}{\partial \eta} = 0$$

for each μ, which meets the second condition of Theorem 3-1.

However equations (3.19) are not quite of the form of equation (3.3). The form is achieved by introducing

$$T = \beta(\mu)t$$

to get

$$\begin{aligned} \dot{\xi} &= \epsilon\xi + \eta + \frac{1}{\beta}p(\xi, \eta, \mu) \\ \dot{\eta} &= -\xi + \epsilon\eta + \frac{1}{\beta}q(\xi, \eta, \mu) \end{aligned} \tag{3.20}$$

where

$$\epsilon = \frac{\alpha(\mu)}{\beta(\mu)}.$$

And as

$$\alpha(0) = 0$$

$$\beta(0) > 0$$

we conclude that ϵ is zero when μ is zero. Further

$$\frac{d\epsilon}{d\mu} = \frac{\beta(\mu)\alpha'(\mu) - \alpha(\mu)\beta'(\mu)}{[\beta(\mu)]^2} \tag{3.21}$$

and thus

$$\left(\frac{d\epsilon}{d\mu}\right)_{\mu=0} = \frac{\alpha'(0)}{\beta(0)} > 0, \tag{3.22}$$

which can be solved to get $\mu(\epsilon)$ which is strictly increasing in some interval $-\epsilon_0 < \epsilon < \epsilon_0$. Then replacing μ in equations (3.19) by $\mu(\epsilon)$ will give equations of the form of (3.3) with ϵ as a parameter. Thus the conclusions (A), (B) and (C) of the Andronov theorem follow from equations (3.20). Also as the linear mapping $\mathbf{x} = \mathbf{M}\mathbf{z}$ transforms the trajectories of the system (3.20) in the (ξ, η) plane to the trajectories of system (3.11) in the (x, y) plane, it follows that (A),(B) and (C) also hold for the trajectories of equation (3.11). This completes the proof of the alternative form of Theorem 3-1.

Restating the Andronov theorem does not in any way help us to decide which of the possibilities actually occurs. It just makes its use easier and straightforward. In deciding the eventual course of the system one may use Bendixon's negative criterion.

3.3 The Rayleigh Equation.

To illustrate the whole procedure we will consider a second order system analogous to equation (3.2) given as,

$$\ddot{x} + a\dot{x} + x - \Phi(\dot{x}) = 0. \tag{3.23}$$

In this equation a is a positive constant and $\phi(y)$ is a differentiable real function with the condition that there exists a constant m such that

$$m \geq |\Phi(y)| \quad \text{for} \quad -\infty < y < \infty. \tag{3.24}$$

The phase system can therefore be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - ay + \Phi(y). \end{aligned} \tag{3.25}$$

The first thing is to ensure that the system has a proper closed trajectory. For this purpose let us assume that the dither is $\mathrm{k}p(t)$, where the function $p(t)$ is

(i) continuous, λ periodic and real.

(ii) twice differentiable and $\ddot{p}$ is continuous in $\Re$ and

(iii) the equation $\dot{p}(t) = 0$ has only a finite number 'N' of roots in the period interval $0 \le t \le \lambda$.

The averaged equation can then be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - ay + \Psi_{\mathrm{k}}(y), \end{aligned} \tag{3.26}$$

with

$$\Psi_{\mathrm{k}}(y) = \frac{1}{\lambda}\int_0^{\lambda} \Phi(y + \mathrm{k}p(\theta))d\theta. \tag{3.27}$$

From equation (3.24) we have

$$|\Psi_{\mathrm{k}}(y)| \le m \quad \text{for all} \quad y \in \Re, \tag{3.28}$$

and the only critical point of equation (3.26) is

$$x = \Psi_{\mathrm{k}}(0) \; ; \; y = 0.$$

Now the Jacobian matrix of the system (3.26) is

$$\mathbf{J}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & -a + \Psi'_{\mathrm{k}}(y) \end{pmatrix}, \tag{3.29}$$

(where $\Psi'_{\mathrm{k}}(y) = \frac{d}{dy}\Psi_{\mathrm{k}}(y)$) and thus the eigenvalues of $\mathbf{J}(\Psi_{\mathrm{k}}(0), 0)$ are

$$\lambda = \frac{1}{2}[(\Psi'_{\mathrm{k}}(0) - a) \pm \sqrt{(a - \Psi'_{\mathrm{k}}(0))^2 - 4}]. \tag{3.30}$$

These have positive real parts and therefore the critical point is an unstable focus or node when

$$\Psi'_{\mathrm{k}}(0) > a. \tag{3.31}$$

To demonstrate the existence of a closed trajectory we proceed by shifting the critical point to the origin by substituting

$$z = \Psi_k(0) - x,$$

which changes equation (3.26) to

$$\begin{aligned} \dot{z} &= -y \\ \dot{y} &= z - F(y), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} F(y) &= ay - \Psi_k(y) + \Psi_k(0) \\ &= \int_0^y f(\eta) d\eta, \end{aligned} \tag{3.33}$$

with

$$f(\eta) = a - \Psi_k'(\eta).$$

Notice that equations (3.32) represent the Lienard's equation [5,p.96]

$$\ddot{y} + f(y)\dot{y} + y = 0. \tag{3.34}$$

Recalling equation (3.28) and the fact that 'a' is a positive constant we get

$$\begin{aligned} \lim_{y \to \infty} F(y) &= +\infty \\ \lim_{y \to -\infty} F(y) &= -\infty . \end{aligned} \tag{3.35}$$

Theorems by Levinson and also Smith and Dragilev [10,p.103] then establish the existence of at least one closed curve, Γ_c $(\Gamma_c \subset \Re^2)$ encircling the origin, in the phase plane. It can further be demonstrated (see for example Davies and James [5,p.97]) that all the trajectories of equation (3.32) outside of Γ_c will spiral inwards towards the closed curve, and the trajectories inside Γ_c will spiral outward towards it. A corollary to the Levinson theorem then assures the existence of a closed trajectory of equation (3.32) inside Γ_c provided that the critical point at

$(0,0)$ is unstable. In other words this means that $\Psi'_k > a$. In case we now choose k to be zero then

$$\Psi_k(y) = \Phi(y),$$

and our averaged equation (3.26) reduces to the original system. Thus we deduce that the system given in equation (3.25) has a closed trajectory encircling its critical point $(\Phi(0), 0)$ provided that $\Phi'(0) > a$ and the condition (3.24) holds. Thus quenching of the closed trajectories of the system (3.25) does not occur while $\Psi'_k(0) > a$.

We now recall Lemma 1-1 which states that if there is a real constant m satisfying the condition (3.24) and further there also exists a real constant m_1 such that

$$m_1 \geq |\Phi'(y)|, \tag{3.36}$$

for all $y \in \Re$, then

$$\Psi'_k(y) \longrightarrow 0 \quad \text{as} \quad |k| \longrightarrow \infty \tag{3.37}$$

uniformly. This ensures that there exists a real positive number k_0 such that if

$$|k| \geq k_0$$

then

$$|\Psi'_k| < a \text{ for all } y \in \Re.$$

Now an application of Bendixon's negative criterion to equation (3.26) gives

$$\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} &= \Psi'_k(y) - a \\ &< 0, \end{aligned} \tag{3.38}$$

for all $y \in \Re$. Thus it is safe to conclude that no closed trajectory exists when $|k|$ is greater than k_0. All trajectories come in from infinity and tend to the critical point $(\Psi_k(0), 0)$ which is stable.

Summarizing it can be stated that if for the Rayleigh equation (3.23) the conditions (3.24) and (3.26) hold then all closed trajectories of the averaged system (3.26) are quenched for $|k| \geq k_0$.

We now prove that

$$k\frac{\partial}{\partial k}[\Psi_k'(0)] < 0 \tag{3.39}$$

for all nonzero k, provided that $\Phi''(y)$ is continuous and

$$y\Phi''(y) < 0 \tag{3.40}$$

for all real and nonzero y. Notice that from equation (3.27) we can write

$$\frac{\partial^2 \Psi_k(y)}{\partial k \partial y} = \frac{1}{\lambda}\int_0^\lambda \frac{\partial^2 \Phi(y + kp(\theta))}{\partial k \partial y}\, d\theta, \tag{3.41}$$

and

$$\frac{\partial \Psi'(y)}{\partial k} = \frac{1}{\lambda}\int_0^\lambda p(\theta)\Phi''(y + kp(\theta))\, d\theta$$

which gives

$$\begin{aligned} k\frac{\partial}{\partial k}[\Psi_k'(0)] &= \frac{1}{\lambda}\int_0^\lambda kp(\Phi''(kp(\theta))\, d\theta \\ &< 0 \end{aligned} \tag{3.42}$$

because of equation (3.40). Thus the statement (3.39) is true.

Further we show that if $p(t)$ and $\Phi(y)$ are odd functions and $\Phi(y)$ satisfies the conditions in equations (3.24), (3.36) and (3.40) and also

$$\Phi'(0) > a,$$

then there is one and only one number

$$k(a) > 0,$$

such that a periodic orbit of system (3.25) bifurcates from its critical point as k varies across $k(a)$.

As $p(t)$ and $\Phi(y)$ are odd therefore $\Psi_k(y)$ is an odd function of y, thus

$$\Psi_k(0) = 0,$$

and the critical point of the system (3.25) is at the origin for all k. The system's Jacobian matrix has eigenvalues (at the origin) given by

$$\frac{1}{2}[(\Psi_k'(0) - a) \pm \sqrt{(\Psi_k'(0) - a)^2 - 4}], \tag{3.43}$$

which are complex when

$$(\Psi_k'(0) - a)^2 < 4, \tag{3.44}$$

and their real part

$$\alpha = \frac{1}{2}[\Psi_k'(0) - a] \tag{3.45}$$

is zero when

$$\Psi_k'(0) = a. \tag{3.46}$$

As k increases from zero to infinity the number $\Psi_k'(0)$ decreases continuously to zero. Now because of equation (3.39) the equation (3.46) has one and only one positive root $k(a)$. If we change from the parameter k to μ where

$$\mu = k(a) - k, \tag{3.47}$$

then the eigenvalues

$$\alpha(\mu) \pm i\beta(\mu)$$

at the critical point have

$$\alpha(0) = 0 \quad , \quad \beta(0) = 1 \tag{3.48}$$

and

$$\begin{aligned}(\frac{d\alpha(\mu)}{d\mu})_{\mu=0} &= -\frac{\partial}{\partial k}[\frac{1}{2}\Psi_k'(0)]_{k(a)} \\ &> 0.\end{aligned} \tag{3.49}$$

The above discussion has established the validity of the three hypotheses of the alternative form of the Andronov's theorem (Theorem 3-2) and therefore one of the statements (A), (B) or (C) of Theorem 3-1 must hold as μ is varied across zero (that is as k is varied across $k(a)$).

3.4 A Special Case.

Consider the situation where

$$p(t) = \sin t, \tag{3.50}$$

and

$$\Phi(y) = \frac{b}{C}\arctan\frac{y}{C}. \tag{3.51}$$

Such functions often occur in the theory of control. It follows that

$$\Phi'(y) = \frac{b}{C^2 + y^2}. \tag{3.52}$$

Now

$$|\Phi(y)| < \frac{\pi b}{2C} \tag{3.53a}$$

and

$$|\Phi'(y)| < \frac{b}{C^2}, \tag{3.53b}$$

for all y. Further from equation (3.27) one gets for the averaged version

$$\begin{aligned} |\Psi_k'(y)| &< \frac{1}{2\pi}\int_0^{2\pi} \frac{b}{C^2}\, d\theta \\ &= \frac{b}{C^2}. \end{aligned} \tag{3.54}$$

As equation (3.54) holds for all $y \in \Re$, we can write

$$|\Psi_k'(0)| < \frac{b}{C^2}.$$

From equation (3.37) we recall that as k increases from zero to infinity, $\Psi_k'(0)$ decreases continuously from

$$\begin{aligned} \Psi_0'(0) &= \Phi'(0) \\ &> a \end{aligned}$$

to zero. This implies that the equation

$$\Psi_k'(0) = a \tag{3.55}$$

has one and only one positive root $\mathrm{k}(a)$, with

$$|a| < \frac{b}{C^2}, \quad \text{or} \quad b > aC^2. \tag{3.56}$$

From equation (3.52) we can write

$$\Phi'(y + \mathrm{k}\sin\theta) = \frac{b}{C^2 + (y + \mathrm{k}\sin\theta)^2} \tag{3.57}$$

and thus

$$\Psi'_{\mathrm{k}}(y) = \frac{b}{2\pi}\int_0^{2\pi} \frac{1}{C^2 + (y + \mathrm{k}\sin\theta)^2}\, d\theta. \tag{3.58}$$

Putting in y to be zero and using equation (1.57) we get

$$\Psi'_{\mathrm{k}}(0) = \frac{b}{C\sqrt{\mathrm{k}^2 + C^2}}. \tag{3.59}$$

Using equation (3.55) and writing

$$\mathrm{k} = \mathrm{k}(a)$$

we find

$$\mathrm{k}(a) = \frac{\pm\sqrt{b^2 - a^2C^4}}{aC}. \tag{3.60}$$

Finally from equations (3.58) and (1.55) we can write

$$\Psi'_{\mathrm{k}}(y) = \frac{b}{C} Re[\omega^{-1/2}] \tag{3.61}$$

with

$$\omega = \mathrm{k}^2 + C^2 - y^2 - 2iCy. \tag{3.62}$$

Let us write $\theta = \arg\omega$. Then

$$\begin{aligned} \Psi'_{\mathrm{k}}(y) &= \frac{b}{C} Re\,[|\omega|^{\frac{-1}{2}} \exp\{\frac{-i\theta}{2}\}] \\ &= \frac{b}{C}\cos\frac{\theta}{2}\,|\omega|^{\frac{-1}{2}}. \end{aligned} \tag{3.63}$$

Further as

$$|\cos\frac{\theta}{2}| \leq 1$$

we have

$$|\Psi'_{\mathrm{k}}(y)| \leq \frac{b}{C|\omega|^{\frac{1}{2}}}.$$

Since

$$|\omega|^2 = [y^2 + C^2 - \mathrm{k}^2]^2 + 4\mathrm{k}^2C^2,$$

we have

$$|\omega|^{\frac{1}{2}} \geq (2\mathrm{k}C)^{\frac{1}{2}}$$

and hence conclude that

$$|\Psi'_{\mathrm{k}}(y)| \leq \frac{b}{\sqrt{2|\mathrm{k}|C^3}} \tag{3.64}$$

for all y. Then equation (3.38) shows that quenching occurs for

$$|\mathrm{k}| \geq k_0$$
$$= \frac{b^2}{a^2C^3}.$$

3.5 Conditions for Hard Quenching.

We have seen above that if $p(t)$ and $\Phi(y)$ are odd functions and the conditions given by equations (3.24), (3.36) and (3.40) are satisfied then there exists one and only one positive $\mathrm{k}(a)$ such that a periodic orbit of the averaged system (3.26) bifurcates from the critical point as k varies across the value $\mathrm{k}(a)$. In this section we wish to focus on the conditions to ensure that hard quenching occurs. Let us assume that the conditions listed above are satisfied. Then we state:

Theorem 3-3. *If*

$$\Psi'''_{\mathrm{k}(a)}(0) > 0$$

$$[\Psi'''_{\mathrm{k}(a)}(0) < 0]$$

then an unstable [stable] closed trajectory of (3.26) expands [shrinks] from [to] the critical point as k *increases from [to] the value* $\mathrm{k}(a)$.

Proof

As $\Psi_k(y)$ and hence $\Psi''_k(y)$ is odd we have

$$\Psi''_k(0) = 0$$

for every k. Applying Bendixon's negative criterion to the system (3.26) we get

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \Delta_k(y)$$

where

$$\Delta_k(y) = \Psi'_k(y) - a.$$

In the case $\Psi'''_{k(a)}(0)$ is positive we have

$$\begin{aligned}\Psi'_{k(a)}(0) &= a\\ \Psi''_{k(a)}(0) &= 0\\ &< \Psi'''_{k(a)}(0)\end{aligned}$$

and therefore

$$\Delta_{k(a)}(y) > \Delta_{k(a)}(0) = 0,$$

for all non zero y in some interval $-h < y < h$. Also from equation (3.39)

$$\begin{aligned}0 &> \frac{\partial}{\partial k}[\Psi'_k(0)]\\ &= \frac{\partial}{\partial k}\Delta_k(0).\end{aligned}$$

It follows from continuity that this holds for all (y, k) in some small rectangle with centre at the point (0,k(a)). The mean value theorem then gives

$$\begin{aligned}\Delta_k(y) &= [\Delta_k(y) - \Delta_{k(a)}(y)] + [\Delta_{k(a)}(y) - \Delta_{k(a)}(0)]\\ &= [(k - k(a))(\frac{\partial \Delta_k(y)}{\partial k})_{k_1}] + [\Delta_{k(a)}(y) - \Delta_{k(a)}(0)],\end{aligned}$$

for $k_1$lying between k and k(a). This is positive for for

$$\begin{aligned}k(a) - \delta < k &\leq k(a)\\ y &\in (-h, h).\end{aligned} \tag{3.66}$$

That is

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \Delta_{\mathrm{k}}(y)$$
$$> 0,$$

for the conditions given in equation (3.66). Hence there are no closed trajectories in the strip $-h < y < h$ and Andronov's theorem (Theorem 3-1) then gives a closed trajectory near the critical point for

$$\mathrm{k}(a) < \mathrm{k} < \mathrm{k}(a) + \delta.$$

Then $\mathrm{k} > \mathrm{k}(a)$ implies

$$\Psi_{\mathrm{k}}'(0) < \Psi_{\mathrm{k}(a)}(0)$$
$$= a.$$

By equation (3.43) we have a stable critical point and an unstable closed trajectory.

In the case when

$$\Psi_{\mathrm{k}}'''(0) < 0$$

we have

$$\Delta_{\mathrm{k}(a)}(y) < \Delta_{\mathrm{k}(a)}(0)$$
$$= 0$$

for all non zero y in $(-h, h)$. From equation (3.39) and continuity

$$\frac{\partial \Delta_{\mathrm{k}}(y)}{\partial \mathrm{k}} < 0$$

for all (y, k) in some small rectangle with centre at $(0, \mathrm{k}(a))$. The mean value theorem gives k_1 between k and $\mathrm{k}(a)$ such that

$$\Delta_{\mathrm{k}}(y) = (\mathrm{k} - \mathrm{k}(a))(\frac{\partial \Delta_{\mathrm{k}}(y)}{\partial \mathrm{k}})_{k_1} + [\Delta_{\mathrm{k}(a)}(y) - \Delta_{\mathrm{k}(a)}(0)]$$

which is negative for

$$\mathrm{k}(a) \leq \mathrm{k} < \mathrm{k}(a) + \delta$$
$$y \in (-h, h),$$

and gives

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \Delta_k(y)$$
$$< 0$$

in the strip $-h < y < h$. Andronov's theorem (Theorem 3-1) then gives a small closed trajectory around the critical point when

$$k(a) - \delta < k < k(a).$$

For $k < k(a)$ we have

$$\Psi'_k(0) > \Psi'_{k(a)}(0) = 0.$$

By equation (3.43) the critical point is unstable and the closed trajectory is stable.

This completes the proof of Theorem 3-3. In this theorem we observe that the condition $\Psi'''_{k(a)}(0) > 0$ ensures that hard quenching occurs. It is to be noted, however, that it is a condition on the funtion $\Psi_k(y)$ which is generally not known. There is a case of sufficient general interest for which one can work out $\Psi'''_k(0)$. The function $\Phi(y)$ given in equation (3.51) occurs quite often in control theory. From equations (3.52) and (3.27)

$$\Psi'_k(y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{b}{C^2 + (y + k \sin\theta)^2}\, d\theta. \tag{3.67}$$

It follows that

$$\Psi'_k(y) = \frac{b}{C} Re\Big([k^2 + C^2]^{\frac{-1}{2}} [1 + \frac{2iC}{k^2 + C^2} y - \frac{1}{k^2 + C^2} y^2]^{\frac{-1}{2}}\Big) \tag{3.68}$$

which yields

$$\Psi'_k(y) = \frac{b}{C\sqrt{k^2 + C^2}} [1 - \frac{(k^2 - 2C^2)}{2(k^2 + C^2)^2} y^2]. \tag{3.69}$$

Thus we find

$$\Psi_k(0) = \frac{b}{C\sqrt{k^2 + C^2}}, \tag{3.70}$$

and

$$\Psi'''_k(0) = \frac{b(k^2 - 2C^2)}{2(k^2 + C^2)^{5/2}}. \tag{3.71}$$

From equations (3.55) and (3.70) we find

$$a = \frac{b}{C\sqrt{\mathrm{k}^2(a) + C^2}}$$

or

$$\mathrm{k}(a) = \frac{\sqrt{b^2 - a^2C^4}}{aC}, \tag{3.72}$$

and the condition for hard quenching to occur can be written as

$$\frac{b(\mathrm{k}^2(a) - 2C^2)}{C(\mathrm{k}^2(a) + C^2)^{5/2}} > 0$$

which gives alongwith equation (3.72) the requirement that

$$a < \frac{b}{\sqrt{3}C^2}. \tag{3.73}$$

As stated earlier we do not in general know the function $\Psi_{\mathrm{k}}(y)$. It is therefore useful to derive conditions for a general odd function $\Phi(y)$ which is known.

Theorem 3-4. *Suppose that $p(t) = \sin t$ and that $\Phi(y)$ is an odd function which satisfies equation (3.40), and further*

$$0 < \Phi'''(y)y < \frac{\delta}{y^4} \quad \text{for all} \quad y \geq \sigma \tag{3.74}$$

$$\Phi''(y) \to 0, \Phi'(y) \to 0, \Phi(y) \to l \quad \text{as} \quad y \to +\infty, \tag{3.75}$$

where δ, σ, l are positive constants. Then $\Psi'''_{\mathrm{k}}(0)$ is positive for all k greater than k_2 where

$$k_2 = \max\left(\frac{5\sigma}{3}, \frac{765}{162}\frac{\delta}{l}\right). \tag{3.76}$$

Proof

By successive integration of equation (3.74) over the interval (y, ∞) we get

$$\begin{aligned} \frac{\delta}{3y^3} &= \int_y^\infty \frac{\delta}{\eta^4}\,d\eta \\ &> \int_y^\infty \Phi'''(\eta)\,d\eta = -\Phi''(y). \end{aligned} \tag{3.77}$$

Using equation (3.75)

$$\begin{aligned}\frac{\delta}{6y^2} &= \int_y^\infty \frac{\delta}{3\eta^3}\,d\eta \\ &> \int_y^\infty -\Phi''(\eta)\,d\eta = \Phi'(y),\end{aligned} \tag{3.78}$$

and

$$\begin{aligned}\frac{\delta}{6y} &= \int_y^\infty \frac{\delta}{6\eta^2} \\ &> \int_y^\infty \Phi'(\eta)\,d\eta = l - \Phi(y),\end{aligned} \tag{3.79}$$

all provided that

$$y \geq 0.$$

Now

$$\Psi_{\mathrm{k}}'''(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi'''(\mathrm{k}\sin\theta)\,d\theta,$$

and as $\Phi(y)$ is odd $\Phi'''(y)$ is even and we can write

$$\Psi_{\mathrm{k}}'''(0) = \frac{2}{\pi}\int_0^{\mathrm{k}} \frac{\Phi'''(u)}{\sqrt{\mathrm{k}^2 - u^2}}\,du, \tag{3.80}$$

where we have written

$$u = \mathrm{k}\sin\theta.$$

This can be broken up and rewritten as

$$\frac{\pi}{2}\Psi_{\mathrm{k}}'''(0) = \int_0^L \frac{\Phi'''(u)}{\sqrt{\mathrm{k}^2 - u^2}} + \int_L^{\mathrm{k}} \frac{\Phi'''(u)}{\sqrt{\mathrm{k}^2 - u^2}}\,du, \tag{3.81}$$

where

$$\sigma \leq L < \mathrm{k}.$$

Because $\Phi'''(u) > 0$ for $u \geq \sigma$ and $\Phi''(u)$ is odd, we get

$$\frac{\pi}{2}\Psi_k'''(0) \geq \frac{\Phi''(L)}{\sqrt{k^2 - L^2}} + \int_0^L \frac{-u\Phi''(u)}{[k^2 - u^2]^{(3/2)}}\, du \,. \tag{3.81}$$

Since $-u\Phi''(u) > 0$ by equation (3.40) this integral exceeds

$$k^{-3} \int_0^L -u\Phi''(u)\, du = k^{-3}[\Phi(L) - L\Phi'(L)].$$

Using this and equations (3.77), (3.78) and (3.79) we get

$$\begin{aligned} \frac{\pi}{2}\Psi_k'''(0) &\geq \frac{\Phi''(L)}{\sqrt{k^2 - L^2}} + \frac{1}{k^3}[\Phi(L) - L\Phi'(L)], \\ &> \frac{-\delta}{3L^3\sqrt{k^2 - L^2}} + \frac{1}{k^3}[l - \frac{\delta}{3L}]. \end{aligned} \tag{3.83}$$

If we choose

$$L = \frac{3}{5}k \,,\ k > \frac{5\sigma}{3}$$

then $\sigma \leq L \leq k$ and the second of equations (3.83) gives

$$\frac{\pi}{2}\Psi_k'''(0) > \frac{1}{k^3}[l - \frac{\delta}{k}(\frac{805}{324})]. \tag{3.84}$$

This gives $\Psi_k'''(0) > 0$ provided that $k > 805\,\sigma/324\,l$ and

$$k_2 = \max[\frac{5\,\sigma}{3}, \frac{805\,\delta}{324\,l}]. \tag{3.85}$$

Which brings us to the end of the proof. Three corollaries follow.

Corollary 3-1. *If the conditions of Theorems 3-3 and 3-4 hold and*

$$0 < a < \Psi_{k_2}'(0), \tag{3.86}$$

then hard quenching occurs.

Proof

We know that bifurcation occurs when

$$a = \Psi'_{\mathrm{k}}(0).$$

It follows from equation (3.40) that $\Psi'_{\mathrm{k}}(0)$ is a decreasing function of k so the condition (3.86) implies that the root $\mathrm{k}(a)$ of the above equation is greater than k_2. Then Theorem 3-4 gives

$$\Psi'''_{\mathrm{k}(a)}(0) > 0$$

and bifurcation at $\mathrm{k}(a)$ is unstable and hard quenching follows.

Corollary 3-2. *If the conditions on $\Phi(y)$ of Theorems 3-3 and 3-4 are satisfied and*

$$\Phi'(y) \geq \frac{b}{C^2 + y^2}$$

for all y then hard quenching occurs provided that

$$0 < a < \frac{b}{C\sqrt{C^2 + \mathrm{k}_2^2}}$$

where k_2 is given in equation (3.85).

Proof

The proof follows from Corollary 3-1 if we can prove that

$$\Psi'_{k_2}(0) > a > 0.$$

So it is sufficient to prove that

$$\Psi'_{k_2}(0) \geq \frac{b}{C\sqrt{C^2 + \mathrm{k}_2^2}}. \tag{3.87}$$

Recall that we can write

$$\hat{\Phi}'(y) = \frac{b}{C^2 + y^2},$$

when

$$\hat{\Phi}(y) = \frac{b}{C} \arctan \frac{y}{C}.$$

Now equation (3.59) gives

$$\begin{aligned}
\Psi'_{\mathrm{k}}(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi'(\mathrm{k}\sin\theta)\, d\theta \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\Phi}'(\mathrm{k}\sin\theta)\, d\theta \\
&= \frac{b}{C\sqrt{C^2 + \mathrm{k}^2}}.
\end{aligned}$$

Therefore equation (3.87) holds and Corollary 3-2 follows from Corollary 3-1.

Corollary 3-3. *If the conditions of $\Phi(y)$ of Theorems 3-3 and 3-4 are satisfied then hard quenching occurs for*

$$0 < a < \frac{2}{\pi} \frac{\Phi(k_2)}{k_2}. \tag{3.88}$$

Proof:

Since $\Phi(y)$ is odd,

$$\begin{aligned}
\Psi'_{\mathrm{k}}(0) &= \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \Phi'(\mathrm{k}\sin\theta)\, d\theta \\
&= \frac{2}{\pi} \int_{0}^{\mathrm{k}} \frac{\Phi'(u)}{\sqrt{\mathrm{k}^2 - u^2}}\, du.
\end{aligned}$$

From equations (3.40) and (3.75) we deduce that $\Phi'(y) \geq 0$ for $y \geq 0$. Hence,

$$\Psi'_{\mathrm{k}}(0) \geq \frac{2}{\pi} \int_{0}^{\mathrm{k}} \frac{\Phi'(u)}{\mathrm{k}}\, du = \frac{2\Phi(\mathrm{k})}{\pi \mathrm{k}}.$$

If equation (3.88) holds then $0 < a < \Psi'_{k_2}(0)$. This implies that $\mathrm{k}(a) > k_2$ because $\Psi'_{\mathrm{k}}(0)$ is a decreasing function of k by equation (3.39). Hence $\Psi'''_{\mathrm{k}(a)}(0) > 0$ by Theorem 3-4. Then hard quenching occurs by Theorem 3-3.

Chapter IV

GENERAL THEORY OF QUENCHING

4.1 Systems with Generalised Dither.

Let us recall the general feedback control equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\Phi(\mathbf{C}x) \tag{4.1}$$

where in general

$$\Phi : \Re^s \longrightarrow \Re^r,$$

and the averaged equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\Psi_{\mathbf{k}}(\mathbf{C}x). \tag{4.2}$$

At the end of Chapter I a dither of the form $\mathbf{k}p(\omega t)$, where $p(t)$ is some continuous λ periodic function, was discussed. The feedback control equation is then modified to read

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\Phi(\mathbf{C}x + \mathbf{k}p(\omega t)). \tag{4.3}$$

The function $p(t)$ was required to satisfy the following three conditions :

I. $p(t)$ is a continuous, λ periodic, real function of t.

II. $p(t)$ is twice differentiable and $\ddot{p}(t)$ is continuous in $\Re$.

III. the equation $\dot{p}(t) = 0$ has only a finite number N of roots in the period interval $0 \leq t \leq \lambda$.

Subsequently Lemma 1-1 and Theorem 1-7 were stated and proved. Here we wish to further extend the analysis of generalized dither.

From the averaged function

$$\Psi_{\mathbf{k}}(\mathbf{y}) = \frac{1}{\lambda}\int_0^\lambda \Phi(\mathbf{y} + \mathbf{k}p(\theta))d\theta \tag{4.4}$$

and if Φ has continuous partial derivatives in $\Re^s$ we get the Jacobian matrix

$$\mathbf{J}_{\Psi_{\mathbf{k}}}(\mathbf{y}) = \frac{1}{\lambda}\int_0^\lambda \mathbf{J}_{\Phi}(\mathbf{y} + \mathbf{k}p(\theta))\, d\theta. \tag{4.5}$$

Theorem 4-1. *Suppose that there exist continuous $r \times s$ matrices $\mathbf{G}(\mathbf{y})$ and $\mathbf{H}(\mathbf{y})$ such that*

(i) $\mathbf{J}_{\Phi}(\mathbf{y}) = \mathbf{G}(\mathbf{y}) + \mathbf{H}(\mathbf{y})$ *for all* $\mathbf{y} \in \Re^s$,

(ii) the spectral norm $|\mathbf{H}(\mathbf{y})| \longrightarrow 0$ *as* $|\mathbf{y}| \longrightarrow \infty$,

(iii) there exists a positive constant ϵ and a positive, constant and symmetric matrix $\mathbf{P}$ *so that*

$$\mathbf{P}\mathbf{N}(\mathbf{y}) + \mathbf{N}^T(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P} \leq 0 \quad \textit{for all} \quad \mathbf{y} \in \Re^s$$

where

$$\mathbf{N}(\mathbf{y}) = \mathbf{A} + \mathbf{B}\mathbf{G}(\mathbf{y})\mathbf{C}.$$

If $p(t)$ satisfies the conditions I, II and III then there exists a number k_2 independent of $\mathbf{y}$ *such that*

$$\mathbf{P}\mathbf{M}_{\mathbf{k}}(\mathbf{y}) + \mathbf{M}_{\mathbf{k}}^T(\mathbf{y})\mathbf{P} + \epsilon\mathbf{P} \leq 0 \tag{4.6}$$

for all $\mathbf{y}$ *, and* $\mathbf{k}$ *with* $|\mathbf{k}| \geq k_2$*, where*

$$\mathbf{M}_{\mathbf{k}}(\mathbf{y}) = \mathbf{A} + \mathbf{B}\mathbf{J}_{\Psi_{\mathbf{k}}}(\mathbf{y})\mathbf{C}.$$

Proof:

If $\mathbf{G}_{\mathbf{k}}(\mathbf{y})$, $\mathbf{H}_{\mathbf{k}}(\mathbf{y})$ are the averaged versions of $\mathbf{G}(\mathbf{y})$, $\mathbf{H}(\mathbf{y})$ then equation (4.5) gives

$$\mathbf{J}_{\Psi_{\mathbf{k}}}(\mathbf{y}) = \mathbf{G}_{\mathbf{k}}(\mathbf{y}) + \mathbf{H}_{\mathbf{k}}(\mathbf{y})$$

If

$$\mathbf{N_k(y)} = \mathbf{A} + \mathbf{BG_k(y)C}$$

then hypothesis (iii) gives

$$\begin{aligned} \mathbf{PN_k} + \mathbf{N_k}^T\mathbf{P} + 2\epsilon\mathbf{P} &= \frac{1}{\lambda}\int_0^\lambda (\mathbf{PN} + \mathbf{N}^T\mathbf{P} + 2\epsilon\mathbf{P})\, d\theta \\ &\leq 0. \end{aligned} \tag{4.7}$$

We can write

$$\mathbf{PM_k(y)} + \mathbf{M_k}^T\mathbf{(y)P} + \epsilon\mathbf{P} = (\mathbf{PN_k(y)} + \mathbf{N_k}^T\mathbf{(y)P} + 2\epsilon\mathbf{P}) + \mathbf{S},$$

with

$$\mathbf{S} = \mathbf{PBH_k(y)C} + \mathbf{C}^T\mathbf{H_k}^T\mathbf{(y)B}^T\mathbf{P}^T - \epsilon\mathbf{P}.$$

As equation (4.7) holds we only need to prove that $\mathbf{S} \leq 0$.

With $\mathbf{x} \in \Re^s$,

$$\begin{aligned} \mathbf{x}^T\mathbf{Sx} &= 2\mathbf{x}^T\mathbf{PBH_k(y)Cx} - \epsilon\mathbf{x}^T\mathbf{Px} \\ &\leq 2|\mathbf{x}|^2|\mathbf{P}||\mathbf{B}||\mathbf{H_k(y)}||\mathbf{C}| - \epsilon\lambda_n|\mathbf{x}|^2, \end{aligned} \tag{4.8}$$

where λ_n is the least eigenvalue of $\mathbf{P}$.

Since hypothesis (ii) gives

$$\lim_{|\mathbf{y}|\to\infty} |\mathbf{H(y)}| = 0,$$

it follows from Theorem 1-7 that

$$\lim_{|\mathbf{k}|\longrightarrow\infty} |\mathbf{H_k(y)}| = 0$$

uniformly for all $\mathbf{y} \in \Re^s$. So there exists a number k_2 independent of $\mathbf{y}$ such that

$$|\mathbf{H_k(y)}| < \frac{\epsilon\lambda_n}{2|\mathbf{P}||\mathbf{B}||\mathbf{C}|}$$

for all $\mathbf{y}, \mathbf{k} \in \Re^s$ with $|\mathbf{k}| \geq k_2$. Then

$$\begin{aligned}\mathbf{x}^T\mathbf{S}\mathbf{x} &\leq |\mathbf{x}|^2[2|\mathbf{P}||\mathbf{B}||\mathbf{C}||\mathbf{H}_\mathbf{k}(\mathbf{y})| - \epsilon\lambda_n] \\ &\leq 0\end{aligned}$$

for all $\mathbf{y}, \mathbf{k} \in \Re^s$ with $|\mathbf{k}| \geq k_2$, and it follows that

$$\mathbf{P}\mathbf{M}_\mathbf{k}(\mathbf{y}) + \mathbf{M}_\mathbf{k}{}^T(\mathbf{y})\mathbf{P} + \epsilon\mathbf{P} < 0, \tag{4.9}$$

for all $\mathbf{y}, \mathbf{k} \in \Re^s$ with $|\mathbf{k}| \geq k_2$. Thus Theorem 4-1 is established. A corollary follows.

Corollary 4-1. *Suppose that $\Phi(\mathbf{y})$ and $p(t)$ are both odd functions which also satisfy the hypotheses of Theorem 4-1, then every solution $\mathbf{x}(t)$ of the averaged equation (4.2) satisfies*

$$|\mathbf{x}(t)| \leq \sqrt{\kappa(\mathbf{P})}|\mathbf{x}(t_0)|e^{\frac{1}{2}\epsilon(t-t_0)} \tag{4.10}$$

for all $t \geq t_0$.

[In particular this gives $|\mathbf{x}(t)| \longrightarrow 0$ as $t \longrightarrow \infty$ for every solution of equation (4.2).]

Proof:

As $\Phi(\mathbf{y})$ and $p(t)$ are odd, equation (4.4) gives

$$\Psi_\mathbf{k}(0) = \frac{1}{\lambda}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \Phi(\mathbf{k}p(\theta))\, d\theta = 0.$$

This and equation (4.6) imply (4.10) by the corollary to Theorem 1-5.

Corollary 4-2. *In the special case $r = s = 1$ Theorem 4-1 holds when hypothesis (ii) is replaced by* *(ii') the function $H(y)$ and $\int_0^y H(\eta)\,d\eta$ are both bounded in $\Re$.*

Proof:

The only use made of hypothesis (ii) in the proof of Theorem 4-1 was to show that

$$\mathbf{H}_\mathbf{k}(\mathbf{y}) \longrightarrow 0 \quad \text{as} \quad |\mathbf{k}| \longrightarrow \infty$$

uniformly for all $\mathbf{y} \in \Re^s$. When $r = s = 1$, $H(y)$ and $H_\mathbf{k}(y)$ are real functions of a real variable y and we can deduce that

$$H_\mathbf{k}(y) \longrightarrow 0 \quad \text{as} \quad |\mathbf{k}| \longrightarrow \infty$$

uniformly for $y \in \Re$, by substituting

$$f(y) = \int_0^y H(\eta)\,d\eta \tag{4.11}$$

in Lemma 1-1. Then

$$f'(y) = H(y)$$

and $|f(y)|, |f'(y)|$ are bounded in $\Re$ by hypothesis (ii').

Note that Theorem 4-1 is useful in practice but it does not tell us how to choose $\mathbf{G}(\mathbf{y})$ and $\mathbf{H}(\mathbf{y})$ in the general case. The following theorem avoids this difficulty.

Theorem 4-2. *Suppose that*

$$\Phi : \Re^s \longrightarrow \Re^r$$

has continuous partial derivatives in $\Re^s$. Also suppose that there exist positive constants ϵ, δ and a constant symmetric matrix $\mathbf{P}$ ($\mathbf{P} > 0$), such that

$$\mathbf{PM}(\mathbf{y}) + \mathbf{M}^T(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P} \leq 0 \tag{4.12}$$

for all $\mathbf{y} \in \Re^s$ with $|\mathbf{y}| \geq \delta$ where

$$\mathbf{M}(\mathbf{y}) = \mathbf{A} + \mathbf{B}\mathbf{J}_\Phi\mathbf{C}. \tag{4.13}$$

If $p(t)$ satisfies conditions I, II, and III, then there exists a constant k_2 such that

$$\mathbf{P}\mathbf{M}_{\mathbf{k}}(\mathbf{y}) + \mathbf{M}_{\mathbf{k}}{}^{T}(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P} < 0$$

for all $\mathbf{y}, \mathbf{k} \in \Re^s$ *with* $|\mathbf{k}| \geq k_2$.

Proof:

Define a continuous function

$$\sigma : \Re \longrightarrow \Re$$

as follows:

$$\sigma(\xi) = \begin{cases} 1 & \text{for all} \quad \xi \leq \delta \\ 1 + \delta - \xi & \text{for} \quad \delta < \xi < \delta + 1 \\ 0 & \text{for all} \quad \xi \geq \delta + 1 \end{cases} \tag{4.14}$$

Now choose any vector $\mathbf{b}$ with $|\mathbf{b}| > \delta$ so that equation (4.12) is satisfied for $\mathbf{M}(\mathbf{b})$ and define

$$\mathbf{G}(\mathbf{y}) = \sigma(|\mathbf{y}|)\mathbf{J}_{\Phi}(\mathbf{b}) + [1 - \sigma(|\mathbf{y}|)]\mathbf{J}_{\Phi}(\mathbf{y}), \tag{4.15}$$

where $\mathbf{b}$ is kept constant. As $\sigma(\xi)$ is continuous in $\Re$ so $\mathbf{G}(\mathbf{y})$ is continuous in $\Re^s$. Using the definition of $\mathbf{N}(\mathbf{y})$ as given in Theorem 4-1 and substituting for $\mathbf{G}(\mathbf{y})$ from equation (4.15) we get

$$\mathbf{N}(\mathbf{y}) = \sigma(|\mathbf{y}|)\mathbf{M}(\mathbf{b}) + [1 - \sigma(|\mathbf{y}|)]\mathbf{M}(\mathbf{y}), \tag{4.16}$$

with $\mathbf{M}(\mathbf{y})$ as given in equation (4.13). Hence

$$\begin{aligned} \mathbf{P}\mathbf{N}(\mathbf{y}) + \mathbf{N}^T(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P} =& \sigma[\mathbf{P}\mathbf{M}(\mathbf{b}) + \mathbf{M}^T(\mathbf{b})\mathbf{P} + 2\epsilon\mathbf{P}] \\ &+ (1 - \sigma)[\mathbf{P}\mathbf{M}(\mathbf{y}) + \mathbf{M}^T(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P}]. \end{aligned} \tag{4.17}$$

From equations (4.12) and (4.14) we get

$$\mathbf{P}\mathbf{N}(\mathbf{y}) + \mathbf{N}^T(\mathbf{y})\mathbf{P} + 2\epsilon\mathbf{P} \leq 0, \tag{4.18}$$

for all $\mathbf{y} \in \Re^s$, which is hypothesis (iii) of Theorem 4-1.

Now define

$$\begin{aligned} \mathbf{H}(\mathbf{y}) &= \mathbf{J}_\Phi(\mathbf{y}) - \mathbf{G}(\mathbf{y}) \\ &= \sigma(|\mathbf{y}|)[\mathbf{J}_\Phi(\mathbf{y}) - \mathbf{J}_\Phi(\mathbf{b})]. \end{aligned} \tag{4.19}$$

From equation (4.14) we have

$$\mathbf{H}(\mathbf{y}) = 0$$

for all $\mathbf{y}$ with

$$|\mathbf{y}| \geq \delta + 1.$$

Hence $|\mathbf{H}(\mathbf{y})| \longrightarrow 0$ as $|\mathbf{y}| \longrightarrow \infty$, that is to say that $\mathbf{H}(\mathbf{y})$ satisfies hypotheses (i) and (ii) of Theorem 4-1 and thus the conclusion of Theorem 4-2 follows.

The following Corollary has the same proof as Corollary 4-1,

Corollary 4-3. *Suppose $\Phi(\mathbf{y})$ and $p(t)$ are odd functions which also satisfy the hypothesis of Theorem 4-2, then every solution $\mathbf{x}(t)$ of the averaged equation (4.2) satisfies the result (4.10) and therefore*

$$|\mathbf{x}(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty$$

for every solution.

[Remark: In the above Corollary the fact that $\Phi(\mathbf{y})$ is odd is used to prove that $\Psi_\mathbf{k}(0) = 0$ for all $\mathbf{k}$. However this assumption ($\Phi(\mathbf{y})$ is odd) is a severe restriction. When we abandon this requirement we may have $\Psi_\mathbf{k}(0) \neq 0$. However if we assume that

$$\Phi(\mathbf{y}) + \Phi(-\mathbf{y}) \rightarrow 0 \quad \text{as} \quad |\mathbf{y}| \rightarrow +\infty$$

and that $p(t)$ is odd, then we can deduce that

$$\Psi_\mathbf{k}(0) \longrightarrow 0 \quad \text{as} \quad |\mathbf{k}| \longrightarrow +\infty.]$$

Proof:

For any function $U(\theta)$ which is continuous in

$$-a \leq \theta \leq a$$

we have

$$\int_{-a}^{a} [U(\theta) + U(-\theta)]\, d\theta = 2 \int_{-a}^{a} U(\theta)\, d\theta. \tag{4.20}$$

Substituting $a = \lambda/2$ and $U(\theta) = \Phi(\mathbf{k}p(\theta))$ and using

$$g(\mathbf{y}) = \Phi(\mathbf{y}) + \Phi(-\mathbf{y}) \quad \text{and} \quad p(\theta) = p(-\theta)$$

we get

$$\begin{aligned} 2\lambda\Psi_{\mathbf{k}}(0) &= \int_{-\lambda/2}^{\lambda/2} g(\mathbf{k}p(\theta))\, d\theta \\ &\longrightarrow 0 \quad \text{as} \quad |\mathbf{k}| \longrightarrow \infty \end{aligned}$$

from Theorem 1-7.

Corollary 4-4. *Suppose that $p(t)$ is odd,the hypotheses of Theorem 4-1 or 4-2 hold and*

$$\Phi(\mathbf{y}) + \Phi(-\mathbf{y}) \longrightarrow 0 \quad \text{as} \quad |\mathbf{y}| \longrightarrow \infty,$$

then for each $\eta > 0$ there exists a number $k_3 \geq k_2$ such that if $|\mathbf{k}| \geq k_3$ then every solution of equation (4.2) ultimately enters the ball

$$S_\eta(0) = [\mathbf{x} \in \Re^n : |\mathbf{x}| < \eta] \tag{4.21}$$

and remains in it thereafter. Furthermore if $|\mathbf{x}(t)| \geq \eta$ for $t_0 < t < T$ then

$$|\mathbf{x}(T)| \leq \sqrt{\kappa(\mathbf{P})}\, |\mathbf{x}(t_0)|\, e^{\frac{1}{4}\epsilon(t_0 - T)}. \tag{4.22}$$

Proof:

We can choose k_3 so that

$$|\Psi_{\mathbf{k}}(0)| < \frac{\epsilon\eta}{4|\mathbf{B}|\, |\kappa(\mathbf{P})|^2} \tag{4.23}$$

for all $|\mathbf{k}| \geq k_3$. Without loss of generality we can also choose $k_3 \geq k_2$. If

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{B}\Psi_{\mathbf{k}}(\mathbf{C}\mathbf{x})$$

then

$$\mathbf{J_f}(\mathbf{x}) = \mathbf{A} + \mathbf{B}\mathbf{J}_{\Psi_k}(\mathbf{Cx})\mathbf{C}$$
$$= \mathbf{M_k}(\mathbf{Cx}).$$

Recall that

$$\begin{aligned} \mathbf{f}(\mathbf{x}) - \mathbf{f}(0) &= \int_{\theta=0}^{\theta=1} \frac{d}{d\theta}\mathbf{f}(\theta\mathbf{x})\, d\theta \\ &= \int_0^1 \mathbf{J_f}(\theta\mathbf{x})\, \mathbf{x}\, d\theta \end{aligned} \tag{4.24}$$

and so

$$\mathbf{f}(\mathbf{x}) = \int_0^1 \mathbf{M_k}(\theta\mathbf{Cx})\, \mathbf{x}\, d\theta + \mathbf{f}(0). \tag{4.25}$$

Multiplying throughout by $2\mathbf{x}^T\mathbf{P}$ and adding $\epsilon\mathbf{x}^T\mathbf{Px}$ we get

$$\begin{aligned} \mathbf{x}^T\mathbf{Pf}(\mathbf{x}) + \mathbf{f}^T(\mathbf{x})\mathbf{Px} + \epsilon\mathbf{x}^T\mathbf{Px} =& \mathbf{x}^T\mathbf{Pf}(0) + \mathbf{f}^T(0)\mathbf{Px} \\ &+ \int_0^1 \mathbf{x}^T[\mathbf{PM_k} + \mathbf{M_k}^T\mathbf{P} + \epsilon\mathbf{P}]\mathbf{x}\, d\theta. \end{aligned} \tag{4.26}$$

Since now $|\mathbf{k}| \geq k_3 \geq k_2$, from Theorem 4-1 or 4-2 we get

$$\mathbf{P}^T\mathbf{M_k}(\theta\mathbf{Cx}) + \mathbf{M_k}^T(\theta\mathbf{Cx}) + \epsilon\mathbf{P} \leq 0$$

and hence

$$\begin{aligned} \mathbf{x}^T\mathbf{Pf}(\mathbf{x}) + \mathbf{f}^T(\mathbf{x})\mathbf{Px} + \epsilon\mathbf{x}^T\mathbf{Px} &\leq \mathbf{x}^T\mathbf{Pf}(0) + \mathbf{f}^T(0)\mathbf{Px} \\ &\leq 2|\mathbf{x}|\,|\mathbf{P}||\mathbf{f}(0)| \end{aligned} \tag{4.27}$$

for all $\mathbf{x} \in \Re^n$. If λ_n is the least eigenvalue of $\mathbf{P}$ then

$$\mathbf{x}^T\mathbf{Px} \geq \lambda_n|\mathbf{x}|^2$$

and we can write

$$\mathbf{x}^T\mathbf{Pf}(\mathbf{x}) + \mathbf{f}^T(\mathbf{x})\mathbf{Px} + \frac{1}{2}\epsilon\mathbf{x}^T\mathbf{Px} \leq \frac{1}{2}\epsilon\lambda_n|\mathbf{x}|\left(\frac{4|\mathbf{P}|\,|\mathbf{f}(0)|}{\epsilon\lambda_n} - |\mathbf{x}|\right). \tag{4.28}$$

Hence we have

$$\mathbf{x}^T\mathbf{P}\mathbf{f}(\mathbf{x}) + \mathbf{f}^T(\mathbf{x})\mathbf{P}\mathbf{x} + \frac{1}{2}\epsilon\mathbf{x}^T\mathbf{P}\mathbf{x} \leq 0 \tag{4.29}$$

for $\mathbf{x}$ with $|\mathbf{x}| \geq r_0$ where

$$r_0 = \frac{4\,|\mathbf{P}|\,|\mathbf{f}(0)|}{\epsilon\lambda_n}.$$

From the theorem on ultimate boundedness (Theorem 1-5) this ensures that every solution $\mathbf{x}(t)$ of equation (4.2) ultimately enters the spherical ball of radius $r_0\sqrt{\kappa(\mathbf{P})}$ and remains in it thereafter.

The condition (4.22) also follows when

$$|\mathbf{x}(t)| \geq r_0 \quad \text{for} \quad t_0 \leq t \leq T.$$

Now using $\kappa(\mathbf{P}) = |\mathbf{P}|\lambda_n^{-1}$ and

$$\mathbf{f}(0) = \mathbf{B}\Psi_{\mathbf{k}}(0)$$

we can write

$$r_0 \leq \frac{4\kappa(\mathbf{P})}{\epsilon}\,|\mathbf{B}\Psi_{\mathbf{k}}(0)|.$$

This and equation (4.23) give

$$\begin{aligned}\kappa(\mathbf{P})r_0 &\leq \frac{4\kappa^2(\mathbf{P})\,|\mathbf{B}|\,|\Psi_{\mathbf{k}}(0)|}{\epsilon} \\ &< \eta.\end{aligned}$$

Now

$$S_{\kappa(\mathbf{P})r_0}(0) \subset S_\eta(0)$$

and so the conclusion of the corollary is established.

4.2 Justification of the Method of Averaging

The main concern of this thesis is with the behaviour of the solutions of the dithered equation (4.3). Since Corollaries 4-3, 4-4 only describe the behaviour of solutions of the averaged equation (4.2) we need to consider how closely these are followed by corresponding solutions of equation (4.3). Let us write equation (4.3) briefly as $\dot{\mathbf{x}} = f(t, \mathbf{x})$, where

$$f(t, \mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{B}\Phi(\mathbf{C}\mathbf{x} + \mathbf{k}p(\omega t)). \tag{4.31}$$

Since $p(t + \lambda) = p(t)$ this satisfies $f(t + \sigma, \mathbf{x}) = f(t, \mathbf{x})$, where $\sigma = \lambda/\omega$. Let us also write equation (4.2) briefly as $\dot{\mathbf{u}} = g(\mathbf{u})$ where

$$g(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}\Psi_{\mathbf{k}}(\mathbf{C}\mathbf{u}). \tag{4.32}$$

From equation (4.4) we deduce that

$$\begin{aligned} g(\mathbf{u}) &= \frac{1}{\sigma} \int_0^{\sigma} f(t, \mathbf{u})\, dt \\ &= \frac{1}{\sigma} \int_b^{b+\sigma} f(t, \mathbf{u})\, dt, \end{aligned} \tag{4.33}$$

for all real b.

Assuming that $\Phi(\mathbf{y})$ and $\partial\Phi(\mathbf{y})/\partial\mathbf{y}$ are continuous in $\Re^s$, there exist constants $\mathcal{M}$ and $\mathcal{L}$ such that

$$|f(t, \mathbf{x})| \leq \mathcal{M} \tag{4.34}$$

for all $t, \mathbf{x} \in \Re \times S_{2r}(0)$, and

$$|f(t, \mathbf{x}_1) - f(t, \mathbf{x}_2)| \leq \mathcal{L}\,|\mathbf{x}_1 - \mathbf{x}_2| \tag{4.35}$$

for all $t \in \Re$, and all $\mathbf{x}_1, \mathbf{x}_2 \in S_{2r}(0)$. Further by making $\mathcal{M}, \mathcal{L}$ sufficiently large we can suppose that

$$|g(\mathbf{u})| \leq \mathcal{M} \tag{4.36}$$

for all $\mathbf{u} \in S_r(0)$ and

$$|g(\mathbf{u}_1) - g(\mathbf{u}_2)| \leq \mathcal{L}(\mathbf{u}_1 - \mathbf{u}_2) \tag{4.37}$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in S_r(0)$.

If a solution $\mathbf{x}(t)$ of

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

is in the spherical ball $S_{2r}(0)$ for $t_1 \le t \le t_2$ then we find that

$$|\mathbf{x}(t_2) - \mathbf{x}(t_1)| \le \mathcal{M}(t_2 - t_1). \tag{4.38}$$

And by similar arguments if a solution $\mathbf{u}(t)$ of the averaged equation (4.2) lies in the spherical ball $S_r(0)$ for a time $t_1 \le t \le t_2$ it can be shown that

$$|\mathbf{u}(t_2) - \mathbf{u}(t_1)| \le \mathcal{M}(t_2 - t_1). \tag{4.39}$$

Theorem 4-3. *Suppose that* $\omega \ge \omega_0$, *where*

$$\omega_0(T) = \frac{2\mathcal{M}\lambda}{r}(1 + e^{\mathcal{L}T}\mathcal{L}T) \tag{4.40}$$

and suppose that a solution $\mathbf{u}(t)$ *of the averaged equation (4.2) has* $\mathbf{u}(t) \in S_r(0)$ *for* $0 \le t \le T$. *If* $\mathbf{x}(t)$ *is the solution of equation (4.3) having*

$$\mathbf{x}(0) = \mathbf{u}(0)$$

then

$$\mathbf{x}(t) \in S_{2r}(0) \quad \textit{for} \quad 0 \le t \le T$$

and

$$|\mathbf{x}(t) - \mathbf{u}(t)| \le \frac{h}{\omega} \quad \textit{for} \quad 0 \le t \le T$$

where

$$h = \mathcal{M}\lambda(2 + e^{\mathcal{L}T}\mathcal{L}T). \tag{4.41}$$

[Note: The importance of the theorem lies in the fact that $\frac{h}{\omega}$ can be made as small as we please by taking ω sufficiently large.]

Proof:

Let $\Sigma(\nu)$ denote the statement that

$$\mathbf{x}(t) \in S_{2r}(0) \quad \text{for} \quad 0 \le t \le \nu\sigma \tag{4.42}$$

and that

$$|\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)| \leq e^{\mathcal{L}\sigma\nu}\mathcal{L}\mathcal{M}\sigma^2\nu. \tag{4.43}$$

We will prove by induction that $\Sigma(\nu)$ is true for all integers ν with $0 \leq \nu \leq \frac{T}{\sigma}$.

Since $\mathbf{x}(0) = \mathbf{u}(0)$, the statement $\Sigma(0)$ is obviously true. We now assume that $\Sigma(\nu)$ is true and deduce from it that $\Sigma(\nu+1)$ is true provided $\nu + 1 \leq \frac{T}{\sigma}$.

As $\Sigma(\nu)$ affirms equation (4.42) we need to verify that $\mathbf{x}(t) \in S_{2r}(0)$ for $\nu\sigma \leq t \leq (\nu+1)\sigma$. Now we can write

$$\mathbf{x}(t) = [\mathbf{x}(t) - \mathbf{x}(\nu\sigma)] + [\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)] + \mathbf{u}(\nu\sigma). \tag{4.44}$$

Using equations (4.38), (4.43) and $\mathbf{u}(\nu\sigma) \leq r$ we get

$$|\mathbf{x}(t)| \leq \mathcal{M}(t - \sigma\nu) + e^{\mathcal{L}\sigma\nu}\mathcal{L}\mathcal{M}\sigma^2\nu + r. \tag{4.45}$$

Using $t \leq \sigma(\nu+1) \leq T$ and $\sigma = \lambda/\omega < \lambda/\omega_0$, we get

$$\begin{aligned} |\mathbf{x}(t)| &\leq \mathcal{M}\sigma + e^{\mathcal{L}\sigma\nu}\mathcal{L}\mathcal{M}(\sigma\nu)\sigma + r \\ &\leq \frac{\lambda}{\omega_0}\mathcal{M} + e^{\mathcal{L}T}\mathcal{L}\mathcal{M}T\frac{\lambda}{\omega_0} + r. \end{aligned} \tag{4.46}$$

Using equation (4.40) this reduces to $|\mathbf{x}(t)| = \frac{3}{2}r$.

So we find that $\mathbf{x}(t)$ never reaches the boundary of $S_{2r}(0)$ during the interval $\nu\sigma \leq t \leq (\nu+1)\sigma$ and hence we can safely conclude that $\mathbf{x}(t) \in S_{2r}(0)$ throughout the interval

$$0 \leq t \leq (\nu+1)\sigma.$$

Using equation (4.33) we can write

$$\begin{aligned} \mathbf{x}(\sigma + \nu\sigma) - \mathbf{x}(\nu\sigma) &= \int_{\nu\sigma}^{\sigma+\nu\sigma} f(t, \mathbf{x})\, d\mathbf{x} \\ &= \int_{\nu\sigma}^{\sigma+\nu\sigma} f(t, \mathbf{x}(\nu\sigma))\, dt + \int_{\nu\sigma}^{\sigma+\nu\sigma} [f(t, \mathbf{x}(t)) - f(t, \mathbf{x}(\nu\sigma))]\, dt \\ &= \sigma g(\mathbf{x}(\nu\sigma)) + E_1. \end{aligned} \tag{4.47}$$

Using equations (4.35) and (4.38) we get

$$|E_1| \leq \int\limits_{\nu\sigma}^{\sigma+\nu\sigma} \mathcal{L}\mathcal{M}(t-\sigma\nu)\,dt \\ = \frac{\mathcal{L}\mathcal{M}}{2}\sigma^2. \tag{4.48}$$

Similarly for $\mathbf{u}(t)$ one gets

$$\mathbf{u}(\sigma+\nu\sigma) - \mathbf{u}(\nu\sigma) = \sigma g(\mathbf{u}(\nu\sigma)) + E_2 \tag{4.49}$$

and

$$|E_2| \leq \frac{\mathcal{L}\mathcal{M}}{2}\sigma^2. \tag{4.50}$$

From equations (4.47) and (4.49) we now get using (4.37)

$$|\mathbf{x}((\nu+1)\sigma) - \mathbf{u}((\nu+1)\sigma)| \leq |\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)| + \sigma\mathcal{L}|\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)| + |E_1| + |E_2|. \tag{4.51}$$

Using equations (4.48), (4.50) and (4.43) we get

$$|\mathbf{x}((\nu+1)\sigma) - \mathbf{u}((\nu+1)\sigma)| \leq e^{\mathcal{L}(\sigma+\nu\sigma)}\mathcal{L}\mathcal{M}\sigma^2(\nu+1). \tag{4.52}$$

The above proves that $\Sigma(\nu+1)$ is true provided that $\Sigma(\nu)$ is true. Therefore $\Sigma(\nu)$ holds for all integers ν satisfying

$$0 \leq \nu \leq \frac{T}{\sigma}.$$

For any t in the interval $0 \leq t \leq T$ there exists an integer ν such that

$$\nu\sigma \leq t \leq (\nu+1)\sigma.$$

Then

$$\mathbf{x}(t) - \mathbf{u}(t) = [\mathbf{x}(t) - \mathbf{x}(\nu\sigma)] + [\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)] + [\mathbf{u}(\nu\sigma) - \mathbf{u}(t)] \tag{4.53}$$

and so

$$|\mathbf{x}(t) - \mathbf{u}(t)| \leq |\mathbf{x}(\nu\sigma) - \mathbf{u}(\nu\sigma)| + 2\mathcal{M}(\sigma), \tag{4.54}$$

where we have used equations (4.38) and (4.39). Using equation (4.43), $\sigma\nu \leq T$, and $\sigma = \lambda/\omega$ we get

$$|\mathbf{x}(t) - \mathbf{u}(t)| \leq \frac{h}{\omega}, \tag{4.55}$$

where h is defined in equation (4.41). This concludes the proof of Theorem 4-3.

To see how Theorem 4-3 can be used to justify the method of averaging we observe that if the conditions of Corollary 4-4 are satisfied then T can be chosen in equation (4.22) to make $|\mathbf{u}(t)| < \frac{1}{3}|\mathbf{u}(t_0)|$ for every solution $\mathbf{u}(t)$ of equation (4.2) such that $|\mathbf{u}(t)| \geq \eta$ for $t_0 \leq t \leq T$. It then follows from Theorem 4-3 that the solution $\mathbf{u}(t)$ of equation (4.3) having $\mathbf{x}(t_0) = \mathbf{u}(t_0)$ satisfies

$$|\mathbf{x}(T)| < \frac{1}{3}|\mathbf{x}(t_0)| + \frac{h}{\omega} < \frac{1}{2}|\mathbf{x}(t_0)|$$

provided that $\omega > \omega_0$ and $\omega > 6h/\eta$. From this we deduce by iteration that every solution $\mathbf{x}(t)$ of equation (4.3) must ultimately enter the ball $S_{3\eta/2}(0)$ and remain in it thereafter. That is, for sufficiently large ω all solutions of the dithered equation (4.3) will be quenched into a small neighbourhood of the steady state. In this way, the study of the averaged equation can yield the desired result for the dithered system.

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