Self - duality and extended objects

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SELF - DUALITY AND EXTENDED OBJECTS

by

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To Deborah,

Andrew and James

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Preface

The work presented in this thesis was carried out between October 1986 and September 1989 in the Department of Mathematical Sciences at the University of Durham, under the supervision of Dr. E.F.Corrigan.

The material in this thesis has not been submitted previously for any degree in this or any other university.

No claim of originality is made for the material reviewed in chapters 1 and 3. The material in chapter 2 has been published in Physics Letters **B 226** (1989) 244 . The material in chapter 4 is claimed to be substantially original work.

I should like to thank Ed Corrigan for his help and encouragement over the period of this work. I should also like to thank the other members of the department for many useful conversations. Finally, I acknowledge and thank the Science and Engineering Research Council for financial support.

ABSTRACT

In 1986 Polyakov published his theory of rigid string. I investigate the instantons associated with the consequent new fine structure of strings in four dimensional Euclidean space-time. I reduce the self-dual equation of rigid string instantons to a simple form and show that (p,q) torus knots satisfy the equation, thus forming an interesting new class of solutions. I calculate by computer the world-sheet self-intersection number of the first few such closed knotted strings and derive a very simple formula for the self-intersection number of a torus knot. I consider an interpretation in terms of the first Chern number and discover the empirical formula $Q = q - p$ for the instanton number, Q, of torus knots and links.

In 1987 Biran, Floratos and Savvidy pioneered an approach for constructing self-dual equations for membranes. I present some new solutions for self-dual membranes in three dimensions. In 1989 Grabowski and Tze pointed out a new class of exceptional immersions for which self-dual equations can be constructed and for which there are no known non-trivial solutions. By analogy with (p,q) torus knots, I describe an algorithm for generating a class of potential solutions of self-dual lumps in eight dimensions. I show how these come to within a single sign change of solving all the required constraints and come very close to solving all the 32 self-dual (4;8)-brane equations.

CHAPTER 1

A REVIEW OF STRING FINE STRUCTURE

In some sense, strings lead not only to
unification of interactions but to
unification of ideas.
A.M.Polyakov

§1.1 Introduction

The S-matrix theory of elementary particle physics reached a crescendo in 1968 when Veneziano found a unified description of duality which seemed to be in the spirit of the bootstrap philosophy. The idea that S-matrix theory might be fundamental then began to wane with the construction of a string theoretic derivation of the Veneziano formula. With the advent of QCD in 1973, interest in both S-matrix theory and hadronic string theory diminished considerably. String theory, however, was elevated in 1974 to the status of a potential theory of everything, including gravity, because quantised string at the Planck scale ($\sim 10^{-33}$cms), which is way beyond the hadronic scale ($\sim 10^{-13}$cms), contains a massless spin 2 state in its spectrum, which Scherk and Schwarz interpreted as a graviton.

Orthodox string theory (see Green, Schwarz and Witten [1]) rests on the physical principle, first proposed by Nambu (1970) and Goto (1971), that nature prefers to minimise the area of the 2-dimensional string world-sheet. The theory, only clearly recognised as a theory of relativistic string by 1973, was quantised and found in 1972 to be ghost-free in less than or equal to 26 dimensions. It was anomaly-free only in 26 dimensions. Then the theory was made supersymmetric, first on the world-sheet in 1971, and then in space-time in 1981. Quantum superstring was found to be anomaly-free only in 10 dimensions. Bosonic string and superstring with N=1 supersymmetry were combined in 1985 to make 'heterotic string' which is Lorentz invariant, tachyon-free and consistent only for gauge groups SO(32) or $E_8 \otimes E_8$. Since this symmetry is easily large enough to incorporate the Standard Model symmetry, $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, the stringy theory of everything, for the first time, began to make some contact with elementary particle phenomenology via compactification scenarios.

A number of unorthodox string theories have been developed along the way. Chan and Paton [2] attached quarks to the ends of open strings in an attempt to include their quantum numbers. Chodos and Thorn [3] investigated the possibility of associating a finite rest mass density with the relativistic string. Kikkawa et al. [4] discussed a model of string with a massive particle at each end. More recently, Freund and Olson [5] have developed a non-Archemedean string theory in which the coordinates on the string world-sheet are taken to be p-adic numbers.

The unorthodox string theory with fine structure, which we are going to look at in the first two chapters of this thesis, was introduced by Polyakov [6,7]. It involves giving string a rigidity by adding an extrinsic curvature term to the orthodox action. This leads to smooth and creased string world-sheet phases. It has been argued by Ambjørn and Durhuus [8] that "regularised bosonic string needs extrinsic curvature". According to Polyakov, "it is conceivable, though not proved, that QCD is described by this new string theory".

The physical appeal of the addition of rigidity to the orthodox string model means that the Polyakov string theory has had a long prehistory involving, for example, discussion by Saito et al. [9] of the statistical mechanical theory of stiff chains (of elongated molecules such as polymers) and discussion by Peliti and Leibler [10] of thermal fluctuations of lipidic membranes (such as red blood cells). The latter argue that there is a low temperature phase in which the membrane surface is rigid and flat and a high temperature phase in which the membrane surface appears crumpled. This resembles the phase structure of the rigid string world-sheet.

We now consider some of the consequences demonstrated by Polyakov of adding an extrinsic curvature term to the string theory action. First we show how to derive the relevant curvature relations. Then we introduce rigid string and rigid particle actions.

§1.2 Extrinsic Geometry

The reparametrisation invariant distance between two points P and Q on a surface $\mathbb{M}^2$ parametrised by curvilinear coordinates $\xi^a = (\xi^1,\xi^2) = (\tau,\sigma)$ and embedded in a flat higher dimensional space is $\int_P^Q ds$, where

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dX^\mu dX^\nu \\
&= \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} d\xi^a d\xi^b \equiv g_{ab}\, d\xi^a d\xi^b \,. \qquad (1.2.1)
\end{aligned}
$$

This is the first fundamental quadratic form associated with the surface and, writing $\partial_a \equiv \partial/\partial\xi^a$,

$$
g_{ab}(X) \equiv \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \qquad (1.2.2)
$$

is the induced metric tensor.

Consider the case $\mu = 1,2,3$; that is $\mathbb{M}^2 \hookrightarrow \mathbb{R}^3$. There is a second fundamental quadratic form connected with such an embedding. It relates to the vertical drop associated with translations on the surface and it can be defined as

$$
\begin{aligned}
ds\,d\vartheta &= -\eta_{\mu\nu} dX^\mu dN^\nu \\
&= -\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial N^\nu}{\partial \xi^b} d\xi^a d\xi^b \equiv K_{ab} d\xi^a d\xi^b \,, \qquad (1.2.3)
\end{aligned}
$$

where ds is an infinitesimal translation on the surface and $d\vartheta$ is the resulting angular departure from the tangent plane. $N^{\mu}(\xi)$ is the unit vector perpendicular to the tangent plane at any point $X^{\mu}(\xi)$ on the surface, so that the vectors $(\partial_{\tau}X^{\mu},\ \partial_{\sigma}X^{\mu},\ N^{\mu})$, shown in figure 1.2.1, form a moving triad basis. In three dimensions

$$N = \frac{\partial_{\tau}X \wedge \partial_{\sigma}X}{|\partial_{\tau}X \wedge \partial_{\sigma}X|} . \tag{1.2.4}$$

We define

$$K_{ab}(X,N) \equiv -\eta_{\mu\nu}\partial_{a}X^{\mu}\partial_{b}N^{\nu} \tag{1.2.5}$$

as the extrinsic curvature tensor. Dividing equation (1.2.3) by equation (1.2.1) gives the normal curvature $\kappa = \frac{d\vartheta}{ds}$ which can be regarded as a function of $\lambda = \frac{d\sigma}{d\tau}$, the slope of a line on the surface. The extreme values of $\kappa(\lambda)$ are used in the definitions† of mean curvature, $\frac{1}{2}(\kappa_{max} + \kappa_{min})$, and total or Gaussian curvature, $\kappa_{max}\kappa_{min}$. Note that the total curvature of a cylinder, or a corrugated sheet, or a plane with a straight crease in it, is zero since in these cases $\kappa_{min} = 0$. Expressing $\partial_{a}\partial_{b}X^{\mu}$ in the moving triad $(\partial_{\tau}X^{\mu},\ \partial_{\sigma}X^{\mu},\ N^{\mu})$ leads to the Gauss equations which are differential equations relating the components of g_{ab} and K_{ab} (see e.g. [11]),

$$\partial_{a}\partial_{b}X^{\mu} = \Gamma^{c}_{ab}\partial_{c}X^{\mu} + K_{ab}N^{\mu} . \tag{1.2.6}$$

† These are the standard definitions to be found in, for example, Struik [11]. Note that the mean curvature has dimensions $[L^{-1}]$ while total curvature has dimensions $[L^{-2}]$.

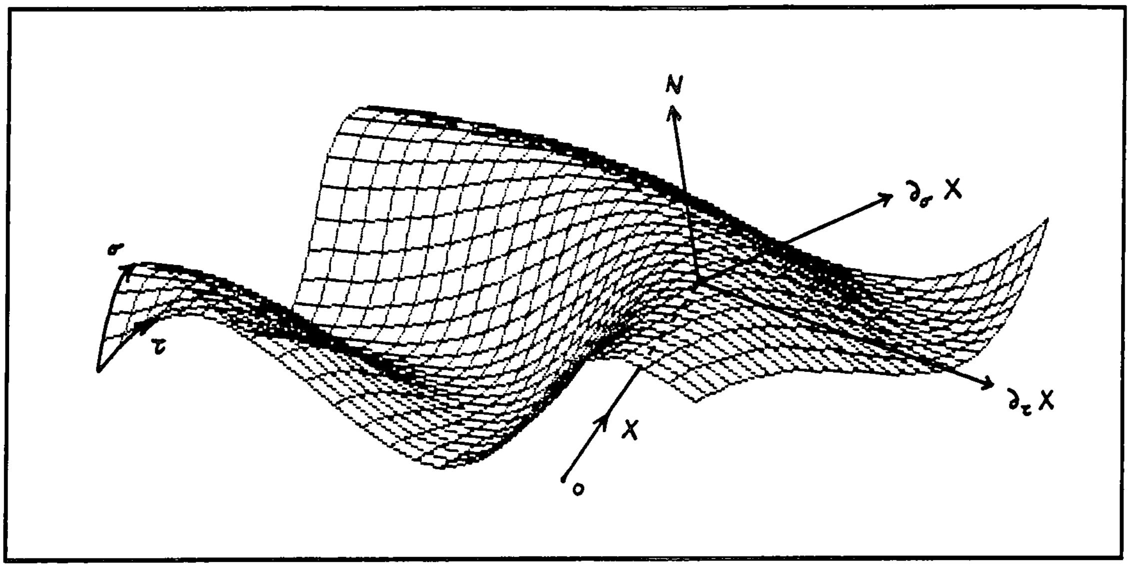

Figure 1.2.1 An arbitrary surface showing the moving triad basis

Since $\partial_a N^\mu$ lie in the tangent plane to the surface, they can be written as linear combinations of $\partial_\tau X^\mu$ and $\partial_\sigma X^\mu$. The coefficients can be expressed in terms of the components of g_{ab} and K_{ab} giving the Weingarten equations [11]

$$\partial_a N^\mu = -K_{ab} g^{bc} \partial_c X^\mu \; . \tag{1.2.7}$$

Applying the identity $\partial_a(\partial_b \partial_c X^\mu) \equiv \partial_b(\partial_a \partial_c X^\mu)$ to the Gauss equations (1.2.6) and using the Weingarten relations (1.2.7) leads to the Codazzi-Mainardi equations [11]

$$\partial_a K_{bc} - \Gamma^d_{ba} K_{dc} = \partial_c K_{ba} - \Gamma^d_{bc} K_{da} \; , \tag{1.2.8}$$

where Γ^a_{bc} are the usual Christoffel symbols defined by

$$\Gamma^{a}_{\ bc}(g) = \frac{1}{2} g^{ad}\left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}\right) . \tag{1.2.9}$$

Using the equations of Gauss (1.2.6) and Weingarten (1.2.7) in the usual expression for the intrinsic curvature scalar associated with the surface,

$$R = g^{ab} R^{c}_{\ acb}$$

$$= g^{ab}\left(\partial_c \Gamma^{c}_{\ ab} - \partial_b \Gamma^{c}_{\ ac} + \Gamma^{e}_{\ ab}\Gamma^{c}_{\ ec} - \Gamma^{e}_{\ ac}\Gamma^{c}_{\ cb}\right) , \tag{1.2.10}$$

gives

$$R(g) = \left(K^{a}_{\ a}\right)^2 - \left(K^{a}_{\ b}K^{b}_{\ a}\right) , \tag{1.2.11}$$

which relates intrinsic to extrinsic curvature.

All these equations can be generalised to the case of a surface embedded in a four dimensional Euclidean space, $\mathbb{M}^2 \hookrightarrow \mathbb{R}^4$. Here the space complementary to the surface is two dimensional, so we must introduce two normals at each point $N^{A\mu}(\xi)$; $A = 1,2$; $\mu = 1,2,3,4$. They are chosen to be orthogonal to the tangent space vectors,

$$N^{A\mu}\partial_a X^{\mu} = 0 , \tag{1.2.12}$$

and mutually orthonormal,

$$N^{A\mu}N^{B\mu} = \delta^{AB} . \qquad (1.2.13)$$

(Since the background is flat there is no need to distinguish covariant from contravariant (Greek) indices.) There are then two extrinsic curvature tensors, $K^{A}{}_{ab}$, defined at each point in $\mathbb{M}^2$. Expressing $\partial_a\partial_b X^\mu$ in the moving tetrad basis $(\partial_\tau X^\mu,\ \partial_\sigma X^\mu,\ N^{1\mu},\ N^{2\mu})$ gives the generalised Gauss equations [6]

$$\partial_a\partial_b X^\mu = \Gamma^c{}_{ab}\partial_c X^\mu + K^A{}_{ab}N^{A\mu} , \qquad (1.2.14)$$

and generalised Weingarten equations [12]

$$\partial_a N^{A\mu} = -\left(N^{A\nu}\partial_a N^{B\nu}\right)N^{B\mu} - K^A{}_{ab}g^{bc}\partial_c X^\mu . \qquad (1.2.15)$$

The relation between the curvature scalar of the Riemannian manifold $\mathbb{M}^2$ and the extrinsic curvature tensor associated with its embedding is now [6]

$$R = \left(K^{Aa}{}_a\right)^2 - K^{Aa}{}_b K^{Ab}{}_a . \qquad (1.2.16)$$

This relationship plays a vital role in the development of the rigid string model which we shall now describe.

§1.3 Rigidity

String theory normally begins with the Nambu-Goto action

$$S_1 = \mu \iint d^2\xi \sqrt{g} , \tag{1.3.1}$$

where μ has dimensions of force, $[ML^{-1}]$ (taking $c = 1$), and is interpreted as the constant string world-sheet surface tension. The Lagrangian

$$\begin{aligned} \sqrt{g} &\equiv \sqrt{Det\ g_{ab}} \\ &= \left[(\partial_\tau X)^2 (\partial_\sigma X)^2 - (\partial_\tau X \cdot \partial_\sigma X)^2 \right]^{1/2} \end{aligned} \tag{1.3.2}$$

is the area density of a parallelogram with sides $\partial_\tau X d\tau$ and $\partial_\sigma X d\sigma$. Therefore $dA = \sqrt{g}\, d\tau d\sigma$ defines an element of area of the string world-sheet. Taking the string action to be $\propto \iint dA$ implies a principle of least world-sheet area analagous to the particle action $\propto \int ds$ implying a principle of shortest world-line length.

However, S_1 can also be viewed as a cosmological term in the action for the metric tensor field g_{ab} on $\mathbb{M}^2$, μ being the cosmological constant. From this point of view the Einstein term $\propto \iint d^2\xi \sqrt{g}\, R$ should also be added. In the case of a two dimensional manifold, $\mathbb{M}^2$, the integrand is a total divergence and the integral is the Euler characteristic, $\chi(p) = 2\text{-}2p$, where p is the (constant) genus of $\mathbb{M}^2$. So the Einstein term does not influence free string dynamics.

Polyakov [6] noticed that the individual terms on the right hand side of (1.2.16) are not total divergences, although they are related by a total divergence which makes them equivalent under integration. This leads to the scale invariant generalisation of the Nambu-Goto action (1.3.1)

$$\begin{aligned} S &= S_1 + S_2 \\ &= \mu \iint d^2\xi \, \sqrt{g} + \rho \iint d^2\xi \, \sqrt{g} \, K^{Aa}{}_b \, K^{Ab}{}_a \,. \end{aligned} \tag{1.3.3}$$

The new constant, ρ, has dimensions [ML] ($N^{A\mu}$ are unit vectors, $\mu = 1,\ldots,D$; $A = 1,\ldots,D\text{-}2$). ρ is interpreted as string world-sheet rigidity because it measures the opposition to extrinsic world-sheet bending. S_2 has the dynamical role of distinguishing smooth world-sheets of a given area from creased world-sheets of the same area because, while S_1 is the same in both cases, S_2 is small for smooth and large for creased world-sheets.

S_2 can be rewritten as

$$S_2 = \rho \iint d^2\xi \, \sqrt{g} \, g^{ab} \, \partial_a t^{\mu\nu} \, \partial_b t^{\mu\nu} \,, \tag{1.3.4}$$

where

$$t^{\mu\nu} = \frac{\varepsilon^{ab}}{\sqrt{g}} \partial_a X^\mu \partial_b X^\nu . \tag{1.3.5}$$

This can be demonstrated by substituting (1.3.5) into (1.3.4) and then using (1.2.14) and the orthogonality properties of $N^{A\mu}$ to get an expression for $K^{Aa}{}_b K^{Ab}{}_a$. Alternatively, S_2 can be written as

$$S_2 = \rho \iint d^2\xi \sqrt{g}\, g^{ab}\, \nabla_a N^{A\mu}\, \nabla_b N^{A\mu} , \tag{1.3.6}$$

where

$$\nabla_a N^{A\mu} = \partial_a N^{A\mu} + \left(N^{A\nu} \partial_a N^{B\nu}\right) N^{B\mu} \tag{1.3.7}$$

$$= - K^A{}_{ab} g^{bc} \partial_c X^\mu \tag{1.3.8}$$

from (1.2.15), as can be verified by substituting (1.3.8) into (1.3.6).

By analogy with rigid string, we can introduce rigid point particles (see e.g. Pavšič [13]). We write the action as

$$\begin{aligned} S &= S_1 + S_2 \\ &= m \int d\tau \sqrt{g} + \lambda^2 \int d\tau \frac{1}{\sqrt{g}} \left(\frac{dt^\mu}{d\tau} \right)^2 , \end{aligned} \tag{1.3.9}$$

where

$$t^{\mu} = \frac{1}{\sqrt{g}} \frac{dX^{\mu}}{d\tau} . \tag{1.3.10}$$

Compare this with (1.3.4) and (1.3.5). S_1 is the usual relativistic particle action, which scales as [L], while S_2 is also reparametrisation invariant and scales as $[L^{-1}]$, a curvature. λ^2 has dimensions $[ML^2]$, an inertia.

§1.4 Progress with Rigid String

Curtright et al. [12] investigated the consequences of rigidity on classical motions of string. They found that open rigid string has identical solutions to orthodox open string but that closed rigid string has new solutions which correspond to string wrapped a number of times round a circle which becomes more oblate with increasing angular momentum (an ellipse rotating about its minor axis). The string tension is balanced by the centripetal acceleration and rigidity. They evaluated the angular momentum and energy of these solutions and found that, for small angular velocity, the Regge trajectories are non-linear. In the case of zero angular momentum, and hence zero centripetal acceleration, they found a finite energy static circular 'hoop' solution where the string tension exactly balances the string rigidity.

Braaten and Zachos [14] analysed the stability of this static hoop of radius $\sqrt{\rho/\mu}$ and energy $4\pi\sqrt{\rho\mu}$. They showed that it is unstable to radial perturbations because the vibration modes have complex frequencies. Numerically, they found that the radius of the circular string either rapidly grows to ∞ or decreases to zero. They concluded that the static hoop solution is not the ground state solution to the closed rigid string.

A number of authors believe that color flux tubes in QCD can be described effectively by the theory of rigid strings. Bagán [15] compared the rigid string model with QCD at intermediate energies and found good agreement for a particular choice of the rigidity constant ρ. Others [16-18] have calculated the static quark potential. Kogut et al. [19] have performed lattice calculations to investigate confinement.

There has been much interest in the application of rigid string theory to the theory of random surfaces (reviewed in [20]) because spikes encountered in the triangulation of the bosonic string world-sheet are smoothed out by the addition of an extrinsic curvature term.

There have been a number of attempts to advance beyond the Polyakov rigid string. Alonso et al. [21] introduced rigid superstrings (or supersprings [22]) by supersymmetrising the Polyakov rigid action in the light-cone gauge. Lindström et al. [23] have proposed a Weyl invariant string with rigidity term as well as supersymmetric versions of the theory [24]. Ichinose [25] has performed semi-classical quantisation of the theory while Itoi and Kubota [26] have shown how BRST quantisation can be achieved by reducing the Lagrangian to a simpler form.

Viswanatha and Zhou [27] discovered a new invariance of the extrinsic curvature term, which they call H-invariance. Itoi and Kubota [28] discovered an action which is equivalent to string theory with extrinsic curvature and keeps its gauge invariance but does not have its higher order derivatives. Itoi [29] added a new scale invariant extrinsic torsion term to the rigid string which makes the $N^{A\mu}$ as well as the X^{μ} into dynamical variables.

For future reference, we note that Lindström [30] has given a derivation of the rigid string action by starting from a membrane action and compactifying, keeping some dependence of the string on the compactified coordinate.

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CHAPTER 2

RIGID STRING INSTANTONS AND TORUS KNOTS

One is tempted to propose the trefoil knot as an emblem of our universe.
G.Burde & H.Zieschang

§2.1 Instantons

Consider a rigid particle on a closed path in two Euclidean dimensions; $\mu = 1,2$. Noticing from (1.3.10) that $t^2 = 1$, we can write the action (1.3.9) as

$$S = \int d\tau \; m \sqrt{g} \left(t^\mu \mp \frac{\lambda \; \varepsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}} \frac{dt^\nu}{d\tau} \right)^2$$

$$\pm \int d\tau \; 2\lambda \sqrt{m} \; \varepsilon^{\mu\nu} t^\mu \frac{dt^\nu}{d\tau} . \qquad (2.1.1)$$

So the action is bounded by a topological part, the second part of (2.1.1), and this bound is attained for

$$t^\mu = \pm \frac{\lambda \; \varepsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}} \frac{dt^\nu}{d\tau} . \qquad (2.1.2)$$

Solving for X^μ and picking a convenient origin gives

$$X^{\mu} = \pm \frac{\lambda\ \varepsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}} \frac{dX^{\nu}}{d\tau} , \tag{2.1.3}$$

which describes a circle radius $\lambda/\sqrt{m}$. This finite action solution is an instanton (anti-instanton) with parametric solution

$$X^{\mu} = \frac{\lambda}{\sqrt{m}} \left(Cos\ \vartheta(\tau),\ Sin\ \vartheta(\tau) \right) , \tag{2.1.4}$$

which represents closed self-intersecting world-lines on a plane, where ϑ is an arbitrary function of τ.

The second part of (2.1.1) can be shown to be topological by proving that its variation is zero by using the fact that δt^{μ} is parallel to $dt^{\mu}/d\tau$. This topological part of the action gives the algebraic total number of times the particle world-line intersects itself. This can be demonstrated by substituting (2.1.4) into the topological part of the action and finding that it reduces to $\vartheta(2\pi)$ - $\vartheta(0)$.

We can apply similar arguments to the rigid string action. In particular (1.3.4) can be expressed as

$$S_2 = \rho \iint d^2\xi\ \sqrt{g}\ g^{ab} \left\{ \tfrac{1}{2}\ \partial_a\left(t^{\mu\nu} \mp {}^{*}t^{\mu\nu} \right) \partial_b\left(t^{\mu\nu} \mp {}^{*}t^{\mu\nu} \right) \pm \partial_a {}^{*}t^{\mu\nu}\ \partial_b t^{\mu\nu} \right\} , \tag{2.1.5}$$

where

$$ {}^{*}t^{\mu\nu} \equiv \tfrac{1}{2}\, \varepsilon^{\mu\nu\rho\lambda}\, t^{\rho\lambda} \; . \tag{2.1.6}$$

S_2 is bounded by a topological part to the action and this bound is attained for

$$ \partial_a \left(t^{\mu\nu} \mp {}^{*}t^{\mu\nu} \right) = 0 \; . \tag{2.1.7}$$

Consider the (instanton) equation (2.1.7) with negative sign. We can find [1] (see also Wheater [2]) instanton solutions of

$$ t^{\mu\nu} - {}^{*}t^{\mu\nu} = c^{\mu\nu} \; , \tag{2.1.8}$$

where $c^{\mu\nu}$ are the constants from integration of (2.1.7).

We choose a Euclidean conformal gauge in which

$$ \partial_\tau X^\mu \, \partial_\sigma X^\mu = 0 \tag{2.1.9}$$

and

$$ \left(\partial_\tau X \right)^2 = \left(\partial_\sigma X \right)^2 \; . \tag{2.1.10}$$

Then equation (2.1.8) becomes, from (2.1.6) and (1.3.5),

$$ \begin{aligned} &\left(\partial_\tau X^\mu \, \partial_\sigma X^\nu - \partial_\tau X^\nu \, \partial_\sigma X^\mu \right) \\ &- \tfrac{1}{2}\, \varepsilon^{\mu\nu\rho\lambda} \left(\partial_\tau X^\rho \, \partial_\sigma X^\lambda - \partial_\tau X^\lambda \, \partial_\sigma X^\rho \right) \\ &= c^{\mu\nu} \left(\partial_\sigma X \right)^2 \; . \end{aligned} \tag{2.1.11}$$

Contracting equation (2.1.11) with $\partial_\tau X^\mu$ gives

$$\partial_\sigma X^\nu = c^{\mu\nu} \, \partial_\tau X^\mu \, . \qquad (2.1.12)$$

Contracting (2.1.11) with $\partial_\sigma X^\mu$ gives

$$\partial_\tau X^\nu = - \, c^{\mu\nu} \, \partial_\sigma X^\mu \, . \qquad (2.1.13)$$

Combining (2.1.12) and (2.1.13) gives

$$c^{\mu\nu} \, c^{\rho\mu} = - \, \delta^{\nu\rho} \, . \qquad (2.1.14)$$

Also, since $t^{\mu\nu}$ and ${}^*t^{\mu\nu}$ are antisymmetric, so is $c^{\mu\nu}$ from (2.1.8):

$$c^{\mu\nu} = - \, c^{\nu\mu} \, . \qquad (2.1.15)$$

A solution to (2.1.14) and (2.1.15) is

$$c^{\mu\nu} = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{pmatrix} , \qquad (2.1.16)$$

in which $\alpha^2 = \beta^2 = 1$.

Substituting this into (2.1.11) we find that $\beta = -\alpha$. Then (2.1.12) and (2.1.13) give

$$\left.\begin{aligned} \partial_\tau X^1 &= -\alpha\ \partial_\sigma X^2 , \\ \partial_\tau X^2 &= \alpha\ \partial_\sigma X^1 , \\ \partial_\tau X^3 &= \alpha\ \partial_\sigma X^4 , \\ \partial_\tau X^4 &= -\alpha\ \partial_\sigma X^3 . \end{aligned}\right\} \qquad (2.1.17)$$

These are the rigid string instanton equations. Note that any X^μ satisfying (2.1.17) automatically satisfies the Euclidean string constraint equations (2.1.9) and (2.1.10). By differentiating (2.1.17) we see that the string equation of motion

$$\partial_\tau \partial_\tau X^\mu + \partial_\sigma \partial_\sigma X^\mu = 0 \qquad (2.1.18)$$

is satisfied automatically.

Take $\alpha = 1$ and notice that the first two relationships in (2.1.17) are the Cauchy-Riemann relations for an analytic function

$$F(z) = X^2(z) + iX^1(z) , \qquad (2.1.19)$$

where $z = \tau + i\sigma$. Similarly the last two relationships in (2.1.17) are the Cauchy-Riemann relations for

$$G(z) = X^3(z) + iX^4(z) . \qquad (2.1.20)$$

Thus we can generate an instanton solution from any two complex analytic functions F and G as

$$X^{\mu} = \left(Im\ F,\ Re\ F,\ Re\ G,\ Im\ G \right) . \tag{2.1.21}$$

As in the case of the particle in two dimensions, where the topological part of the action was interpreted as giving the algebraic number of self-intersections of the Euclidean particle world-line, so in the rigid string case, the topological part of the action can, according to Polyakov, be interpreted as the algebraic number of self-intersections of the string world-sheet. For instantons, S_2 of (2.1.5) becomes

$$S_2 = \rho \iint d^2\xi \sqrt{g}\ g^{ab}\ \partial_a {}^*t^{\mu\nu}\ \partial_b t^{\mu\nu} . \tag{2.1.22}$$

Polyakov [3] gives the self-intersection number of the surface as

$$\nu\left(\mathbb{M}^2\right) = \frac{S_2}{2\pi\rho} = \frac{1}{2\pi} \iint d^2\xi \sqrt{g}\ g^{ab}\ \partial_a {}^*t^{\mu\nu}\ \partial_b t^{\mu\nu} , \tag{2.1.23}$$

which Mazur and Nair [4] write, using (1.3.5), (2.1.6) and (1.2.14), as

$$\nu\left(\mathbb{M}^2\right) = \frac{1}{\pi} \iint d^2\xi\ \ g^{cd}\ \varepsilon^{ab}\ \varepsilon_{AB}\ K^A{}_{ac}\ K^B{}_{bd} . \tag{2.1.24}$$

We shall use this last formula for the self-intersection number in our discussion in §2.2 of the first Chern number which is associated with a rigid string world-sheet in four dimensional space-time.

§2.2 Torus Knots

We are interested to find an example of a self-dual string world-sheet with a finite non-zero number of self-intersections. Taking $F \propto G$ gives $\nu = 0$, so this would be too trivial.

However, a string with a knot in it would seem to offer hope of providing an example of a world-sheet with a few undeniable self-intersections, the number increasing with the complexity of the knot.

Begin by considering the simplest possible knot, the trefoil of figure 2.2.1(A). This can be generated from

$$u^2 + v^3 = 0 \quad ; \quad (u,v) \in \mathbb{C}^2 \ . \tag{2.2.1}$$

Consider two orthogonal complex planes with arbitrary points u on the first and v on the second. If we consider the set of pairs (u,v) for which the relationship $u^2 + v^3 = 0$ holds, then the intersection of this set of points with a small sphere, S^3_ε , centered at the origin of $\mathbb{C}^2$, forms a trefoil knot (see e.g. Milnor [5]).

As the S^3_ε sphere gets a little bigger so the knot gets bigger, but as S^3_ε gets smaller the knot contracts to a point giving us a singularity at the origin. It will be by integrating over such singularities that our self-intersection number, ν, will take on integral values.

The solution of $u^2 + v^3 = 0$ in terms of a complex parameter z can be written as

$$u = z^3 \quad ; \quad v = - z^2 \, . \tag{2.2.2}$$

The trefoil can therefore be specified in terms of two analytic functions, z^3 and $-z^2$.

Take the functions in (2.2.2) as our required functions F and G in (2.1.21), where $z = \tau + i\sigma$.

$$X^\mu = \left(Im\ z^3,\ Re\ z^3,\ Re\ -z^2,\ Im\ -z^2 \right) ,$$

$$\therefore \quad X^\mu = \left(3\tau^2\sigma - \sigma^3,\ \tau^3 - 3\tau\sigma^2,\ \sigma^2 - \tau^2,\ -2\tau\sigma \right) , \tag{2.2.3}$$

$$\left.\begin{aligned} \therefore \quad \partial_\tau X^\mu &= \left(6\tau\sigma,\ 3(\tau^2 - \sigma^2),\ -2\tau,\ -2\sigma \right) , \\ \& \quad \partial_\sigma X^\mu &= \left(3(\tau^2 - \sigma^2),\ -6\tau\sigma,\ 2\sigma,\ -2\tau \right) . \end{aligned}\right\} \tag{2.2.4}$$

The string constraints and equation of motion are satisfied by (2.2.3).

Calculate $\sqrt{g}$ and $\partial_\tau X^\mu \partial_\sigma X^\nu$ and hence find $t^{\mu\nu}$ and $\partial_a t^{\mu\nu}$. Since we are considering instanton solutions for which, from (2.2.7),

$$\partial_a t^{\mu\nu} = \partial_a {}^*t^{\mu\nu} , \tag{2.2.5}$$

and since we are working in the Euclidean conformal gauge, the expression for the self-intersection number simplifies to

$$\nu = \frac{1}{2\pi} \iint d^2\xi \left(\partial_a t^{\mu\nu} \right)^2 . \tag{2.2.6}$$

For the trefoil we find that

$$\left(\partial_a t^{\mu\nu} \right)^2 = \frac{288}{\left(9(\tau^2+\sigma^2) + 4 \right)^2} . \tag{2.2.7}$$

Integrating this over all $0 \leq \sigma < 2\pi$ for a closed string and $-\infty \leq \tau \leq \infty$, we find

$$\nu(\text{trefoil}) = 4 . \tag{2.2.8}$$

The construction of the trefoil based on the complex curve $u^2 + v^3 = 0$ can be generalised to the construction of the infinity of (p,q) torus knots based on the complex curves $u^p + v^q = 0$, where p and q are chosen to be relatively prime. Thus the trefoil is a (2,3) torus knot.

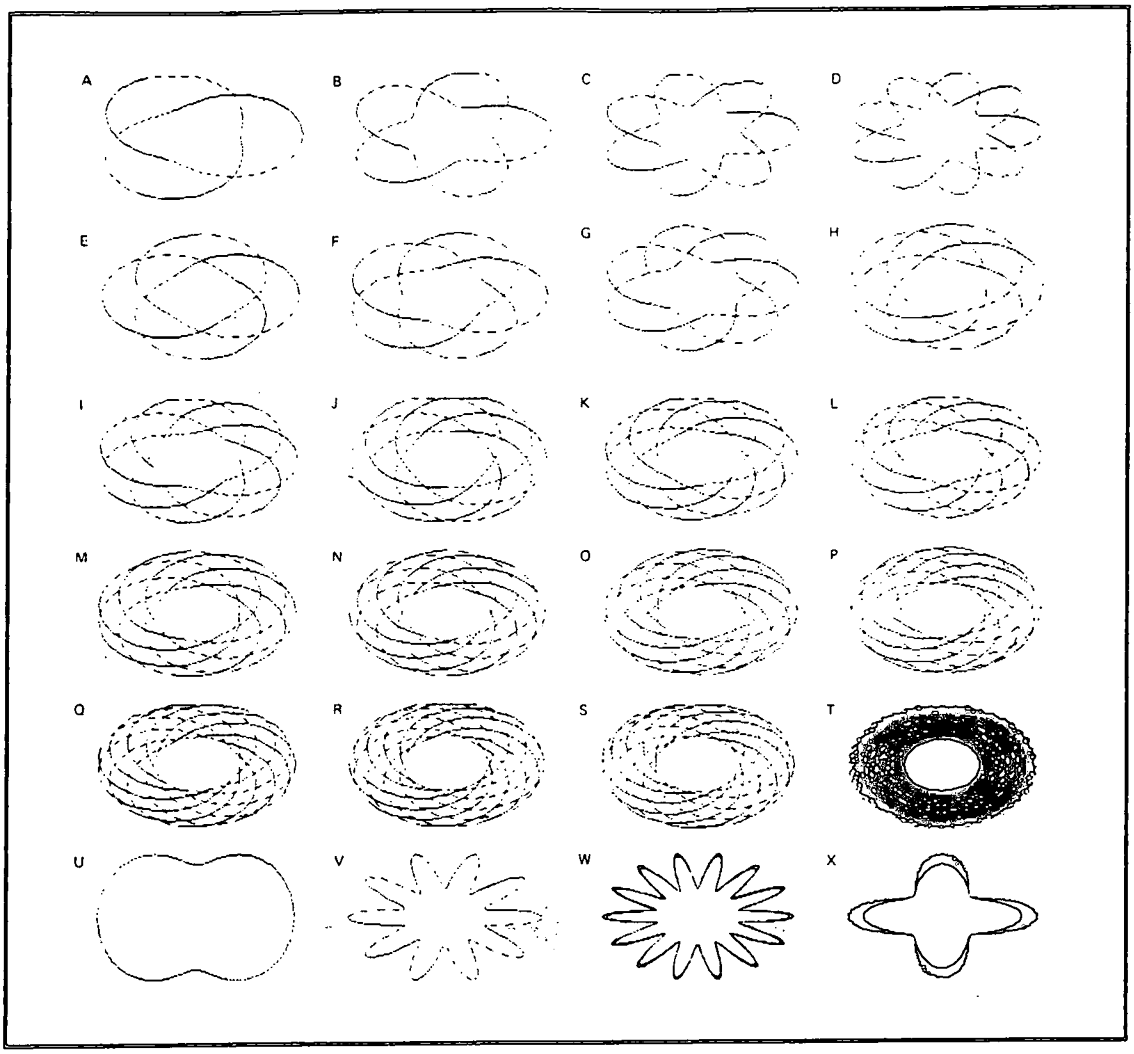

Figure 2.2.1 (p,q) torus knots and links formed by plotting the set of points (x,y,z) which lie on a torus of major radius R and minor radius r, where $x = (R+r\mathrm{Cos}2\pi tq)\mathrm{Cos}2\pi tp$, $y = (R+r\mathrm{Cos}2\pi tq)\mathrm{Sin}2\pi tp$ and $z = r\mathrm{Sin}2\pi tq$. t is a parameter which takes a discrete number of values between 0 and 1. Hidden lines going round the back of the torus are dotted.

(A) (2,3)knot	(B) (2,5)knot	(C) (2,7)knot	(D) (2,9)knot
(E) (3,4)knot	(F) (3,5)knot	(G) (3,7)knot	(H) (4,5)knot
(I) (4,7)knot	(J) (5,6)knot	(K) (5,7)knot	(L) (5,8)knot
(M) (6,7)knot	(N) (7,8)knot	(O) (7,9)knot	(P) (8,9)knot
(Q) (8,11)knot	(R) (9,10)knot	(S) (9,11)knot	(T) (503,634)knot
(U) (2,4)link	(V) (3,30)link	(W) (7,98)link	(X) (51,204)link

Following the calculation for the trefoil, we take for the (2,5) torus knot

$$X^{\mu} = \left(Im\ z^5,\ Re\ z^5,\ Re\ -z^2,\ Im\ -z^2 \right) . \tag{2.2.9}$$

This gives

$$\nu(2,5) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{2\pi} d\tau d\sigma \ \frac{7200\ (\tau^2 + \sigma^2)^2}{\left(25(\tau^2+\sigma^2)^3 + 4 \right)^2} \tag{2.2.10}$$

$$= \int_{0}^{\infty} dr \ \frac{7200\ r^5}{(25\ r^6 + 4)^2} \tag{2.2.11}$$

$$\therefore \quad \nu(2,5) = 12 . \tag{2.2.12}$$

Similarly,

$$\nu(2,7) = \frac{1}{2\pi} \iint d\tau d\sigma \ \frac{39200\ (\tau^2 + \sigma^2)^4}{\left(49(\tau^2+\sigma^2)^5 + 4 \right)^2} \tag{2.2.13}$$

$$\therefore \quad \nu(2,7) = 20 . \tag{2.2.14}$$

Results for some other knots, computed using program 2.2.1, are shown in table 2.2.1 .

```
% Integrand for Winding No. of (p,q) Torus Knot.3/4/89
%------------------------------------------------------
% Put in p and q for (p,q) torus before running.
% p:=?; q:=?;
operator x; operator s; depend x,s(1),s(2);
z:=s(1)+i*s(2); zq:=z**q; zp:=-z**p;
x(1):=coeffn(zq,i,1); x(2):=coeffn(zq,i,0);
x(3):=coeffn(zp,i,0); x(4):=coeffn(zp,i,1);
matrix dx(2,4);
  for a:=1:2 do
    for m:=1:4 do
      dx(a,m):=df(x(m),s(a))$
roots:=for m:=1:4 sum dx(1,m)**2$
matrix tmn(4,4);
  for m:=1:4 do
    for n:=1:4 do
      tmn(m,n):=dx(1,m)*dx(2,n)$
tmn:=tmn-tp(tmn)$ tmn:=tmn/roots$
matrix d1tmn(4,4);
  for m:=1:4 do
    for n:=1:4 do
      d1tmn(m,n):=df(tmn(m,n),s(1))$
matrix d2tmn(4,4);
  for m:=1:4 do
    for n:=1:4 do
      d2tmn(m,n):=df(tmn(m,n),s(2))$
  for a:=1:4 do
    for b:=1:4 do
      d1tmn(a,b):=d1tmn(a,b)**2$
  for a:=1:4 do
    for b:=1:4 do
      d2tmn(a,b):=d2tmn(a,b)**2$
matrix dtmn(4,4);
dtmn:=d1tmn+d2tmn$
operator temp;
  for m:=1:4 do
    temp(m):=for n:=1:4 sum dtmn(m,n)$
integrand:=for m:=1:4 sum temp(m)$
on gcd; integrand; end;
%------------------------------------------------------
```

Program 2.2.1 REDUCE program for calculating the integrand for the self-intersection number of a (p.q) torus knot.

p	q	$\int dr$ integrand	ν
2	3	$288r/(9r^2+4)^2$	4
2	5	$7200r^5/(25r^6+4)^2$	12
2	7	$39200r^9/(49r^{10}+4)^2$	20
2	9	$127008r^{13}/(81r^{14}+4)^2$	28
3	4	$1152r/(16r^2+9)^2$	4
3	5	$7200r^3/(25r^4+9)^2$	8
3	7	$56448r^7/(49r^8+9)^2$	16
4	5	$3200r/(25r^2+16)^2$	4
4	7	$56448r^5/(49r^6+16)^2$	12
5	6	$7200r/(36r^2+25)^2$	4
5	7	$39200r^3/(49r^4+25)^2$	8
5	8	$115200r^5/(64r^6+25)^2$	12
6	7	$14112r/(49r^2+36)^2$	4
7	8	$25088r/(64r^2+49)^2$	4
7	9	$127008r^3/(81r^4+49)^2$	8
8	9	$41472r/(81r^2+64)^2$	4
8	11	$557568r^5/(121r^6+64)^2$	12
9	10	$64800r/(100r^2+81)^2$	4
9	11	$313632r^3/(121r^4+81)^2$	8

Table 2.2.1 Self-intersection number, ν, for a (p,q) torus knot rigid string instanton.

Looking at these results, the general form for $\nu(p,q)$ would seem to be given by the empirical formula

$$\nu(p,q) = 4\int_0^\infty dr \; \frac{2(q-p)^2 q^2 p^2 r^{2(q-p-1)+1}}{(q^2 r^{2(q-p)} + p^2)^2} . \tag{2.2.15}$$

So we find

$$\nu(p,q) = 4\,(q - p)\ , \tag{2.2.16}$$

which represents an infinite hierarchy of knotted instantons.

Notice that that ν always turns out to be a multiple of 4. Mazur and Nair [4] argue, from a different point of view, that $\nu = 4c_1$ where c_1 is the first Chern number and $c_1 \in \mathbb{Z}$.

In the case of a two dimensional manifold embedded in four dimensions we have a special situation. The curvature 2-form defined on $\mathbb{M}^2$, integrated over the two dimensions of $\mathbb{M}^2$, gives an integer, the Euler characteristic. Also, since the co-dimension is two, the curvature associated with the embedding connection 2-form integrated over $\mathbb{M}^2$ gives another integer, the first Chern number.

Write the covariant derivative of the normal vectors as

$$\begin{aligned} \nabla_a N^{A\mu} &\equiv \partial_a N^{A\mu} + A^{AB}{}_a N^{B\mu} \\ &= -\,K^{Ab}{}_a\, \partial_b X^{\mu}\ , \end{aligned} \tag{2.2.17}$$

where $A^{AB}{}_a$ is an $SO(2)$ connection for the parallel transport of the normal vectors on $\mathbb{M}^2$, and the covariant derivative of the tangent vectors

$$t^{\mu}_a \equiv \partial_a X^{\mu} \tag{2.2.18}$$

in the form

$$D_a t^\mu_b \equiv \partial_a t^\mu_b - \Gamma^c_{ab} t^\mu_c$$
$$= K^A_{ab} N^{A\mu} , \qquad (2.2.19)$$

from (1.2.14).

We see from (1.3.7) that

$$A^{AB}_a = N^{A\mu} \partial_a N^{B\mu} . \qquad (2.2.20)$$

From this connection we can derive the field strength tensor

$$F^{AB}_{ab} = \nabla_a N^{A\mu} \nabla_b N^{B\mu} - \nabla_a N^{B\mu} \nabla_b N^{A\mu} . \qquad (2.2.21)$$

The first Chern number is then defined in [4] as

$$c_1 \equiv \frac{1}{2\pi} \int_{M^2} tr\ F = \frac{1}{8\pi} \int d^2\xi\ \varepsilon^{AB}\ \varepsilon^{ab}\ F^{AB}_{ab} . \qquad (2.2.22)$$

Using (2.2.17), (2.2.21) and (2.1.24) gives

$$c_1 = \frac{\nu}{4} , \qquad (2.2.23)$$

which leads to the suggestion [4] that we use $Exp\left(i\vartheta\frac{\nu}{4}\right)$ to represent the effect of ϑ vacua.

§2.3 Links

When p and q are not relatively prime then the complex curve given by the reducible polynomial $u^p + v^q = 0$ represents a number of closed linked strings. For example, $u^2 + v^2 = 0$ represents two closed hoops lying on the surface of a torus and linking together once, like two links in a chain. In general, *p* tells the number of closed strings involved and *q/p* tells the number of times each wraps round the body of the torus, as can be seen from figure 2.2.1(U-X).

We calculate winding number, ν, for a few simple links and obtain the results in table 2.3.1 .

p	q	$\int dr$ integrand	ν/4
2	2	0	0
2	4	$128r^2/(4r^2+1)^2$	2
2	6	$1152r^6/(9r^8+1)^2$	4
3	3	0	0
3	6	$288r^4/(4r^6+1)^2$	3
4	4	0	0
4	6	$1152r^2/(9r^4+4)^2$	2
6	8	$4608r^2/(16r^4+9)^2$	2
8	10	$12800r^2/(25r^4+16)^2$	2

Table 2.3.1 Self-intersection number,V, for (p,q) linked rigid string instantons.

From these results we infer the formula

$$\nu(p,q) = 4 \int_0^{\infty} dr \; \frac{2(q-p)^2 \, \hat{q}^2 \, \hat{p}^2 \, r^{2(q-p-1)+1}}{(\hat{q}^2 \, r^{2(q-p)} + \hat{p}^2)^2} \; , \tag{2.3.1}$$

where

$$\hat{p} = \frac{p}{GCD\ (p,q)}$$

and (2.3.2)

$$\hat{q} = \frac{q}{GCD\ (p,q)} \; .$$

Cancelling out the greatest common divisors we again find

$$\nu(p,q) = 4\ (q - p)\ , \tag{2.3.3}$$

contrary to an assertion in [6], which states that (2.2.15) does not hold true for links.

The interpretation of ν in the case of links is rather different to that of knots. For links, some of the integers contributing to $\nu/4$ come from the previous knot singularities but now there are other contributions from higher order contacts between surfaces. Thus a (4,6) link gets its contributions from two copies of the trefoil, while a (2,4) link gets its contributions from a second order contact. A (p,p) link only involves point contact which contributes nothing to ν.

Links are also represented by complex curves such as

$$v^2 + u^2 + u^3 = 0 \tag{2.3.4}$$

or

$$v^2 u + u^3 + v^3 = 0 \ . \tag{2.3.5}$$

These give more complicated integrands which are much harder to integrate. For example, (2.3.4) gives

$$\nu = \frac{1}{2\pi} \iint d\tau d\sigma \, \frac{32\left(9(\tau^2 + \sigma^2)^2 + 6(\tau^2 - \sigma^2) + 1\right)}{\left(9(\tau^2 + \sigma^2)^2 + 2(5\sigma^2 - \tau^2) + 1\right)^2} \ . \tag{2.3.6}$$

Notice that the denominator of the integrand of (2.3.6) cannot be zero for real τ and, therefore, (2.3.4) will have finite self-intersection number. We leave further analysis of such links to future researches.

We close this chapter with an allusion to a recent paper by 't Hooft [7] in which he shows that certain gravitational instanton solutions describe actual physical particles like solitons, but which are unstable and decay into large numbers of ordinary particles.

Experience shows that knots are usually highly stable. Knotted rigid string instantons are likely to be quite stable and might in some way represent elementary particles, perhaps along the lines of Jehle [8] who considered torus knots of quantised flux. Jehle claims that this can solve the problem of quark fractional charges, by addition of spinning (about the straight torus axis) angular momentum and whirling (about the circular internal torus axis) angular momentum, and that the strangeness quantum number can be identified with the unknotting number of the knot. It would be interesting to know the energies of the very simplest set of knots (not necessarily torus knots) to compare with the known elementary particle mass spectrum. One might speculate that the universe is gradually becoming more knotted, and the structure (of protons, galaxies etc.) which we see developing around us is associated with the stability of knots in strings. One might even entertain as elementary (i.e. fundamental) a 'cotton wool' model of matter.

§2.4 References for Chapter 2

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CHAPTER 3

A REVIEW OF BOSONIC AND SUPER P - BRANES

I propose that one should allow
the electron to have, in general,
an arbitrary shape and size.
P.A.M.Dirac

§3.1 Brane Dynamics

Classical membrane theory can describe the motion of an elastic, perfectly flexible rubber sheet stretched on the $x - y$ plane under tension **T** which is given gentle oscillations by a force whose component in the z direction is F. Analysis of the equilibrium of an infinitesimal surface area element leads to the equation of motion

$$\frac{\partial}{\partial x}\left(\mathbf{T}\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathbf{T}\frac{\partial z}{\partial y}\right) = \rho\frac{\partial^2 z}{\partial t^2} - F(t,x,y) , \qquad (3.1.1)$$

where ρ is the mass per unit area of the rubber sheet. This equation of motion can be derived from the action

$$S = \int_{t_1}^{t_2}\iint dtdxdy \left\{ \frac{\mathbf{T}}{2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] - \frac{\rho}{2}\left(\frac{\partial z}{\partial t}\right)^2 - F.z \right\} , \qquad (3.1.2)$$

which encapsulates the physics of simple membranes.

If T is constant and F is zero then (3.1.1) reduces to

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} , \qquad (3.1.3)$$

where

$$c = \sqrt{\frac{\mathrm{T}}{\rho}} . \qquad (3.1.4)$$

Relativistic membrane theory begins here if we take c to be the speed of light.

If we give our membrane a rectangular boundary of sides a and b then a solution of (3.1.3) is

$$z(t,x,y) = Cos(2\pi ft)\ Sin\left(\frac{m\pi x}{a}\right)\ Sin\left(\frac{n\pi y}{b}\right) , \qquad (3.1.5)$$

with frequency

$$f = \frac{c}{2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) . \qquad (3.1.6)$$

A pair of natural numbers, (m,n), labels a normal mode of vibration of membrane with a rectangular boundary.

If we give our membrane a circular boundary at radius a then a solution of (3.1.3) is

$$z(t,r,\vartheta) = J_m(nr)\ Cos(m\vartheta)\ Cos(nct)\ , \tag{3.1.7}$$

with boundary condition $J_m(na) = 0$, and where the Bessel function $J_m(\eta)$ is

$$J_m(\eta) = \sum_{k=0}^{\infty} \frac{(-1)^k\ (\eta/2)^{2k+m}}{k!\ \Gamma(m+k+1)}\ . \tag{3.1.8}$$

One whole and one natural number, (m,n), label a normal mode of vibration of an ideal rubber drum. A closed bubble would involve the spherical harmonics, et cetera.

The new theory of membranes as elementary extended objects begins with a generalisation of general relativity. In general relativity a particle moves along a world-line. In curvilinear coordinates the distance ds between neighbouring points is given by

$$\begin{aligned} ds^2 &= -\ G_{\mu\nu}\ dx^\mu dx^\nu \\ &= -\ G_{\mu\nu}\ \frac{dx^\mu}{d\tau}\ \frac{dx^\nu}{d\tau}\ d\tau^2\ , \end{aligned} \tag{3.1.9}$$

where the elements of the metric tensor $G_{\mu\nu}$ are, in general, functions of these coordinates. The particle (0-brane) action is then

$$S = - mc \int ds$$

$$= - mc \int d\tau \sqrt{- G_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}} \ , \qquad (3.1.10)$$

where τ signifies a reparametrisation of the world-line.

String theory generalises (3.1.10) to the string (1-brane) Nambu-Goto action

$$S = - \kappa \iint dA$$

$$= - \kappa \iint d^2\xi \sqrt{- \, Det \; G_{\mu\nu} \frac{\partial X^{\mu}}{\partial \xi^{a}} \frac{\partial X^{\nu}}{\partial \xi^{b}}} \ , \qquad (3.1.11)$$

in which the integrand now represents a reparametrisation invariant area element of the string world-sheet, and a,b = 1,2 . Membrane theory generalises (3.1.11) to the membrane (2-brane) action, first written down by Dirac [1],

$$S = - \mathbf{T} \iiint dV$$

$$= - \mathbf{T} \iiint d^3\xi \sqrt{- \, Det \; G_{\mu\nu} \frac{\partial X^{\mu}}{\partial \xi^{a}} \frac{\partial X^{\nu}}{\partial \xi^{b}}} \ , \qquad (3.1.12)$$

which describes the minimal immersion of a three dimensional membrane world-volume into a higher dimensional space-time with metric $G_{\mu\nu} = G_{\mu\nu}(X)$, as illustrated in figure 3.1.1 .

Action (3.1.12) generalises to the p-brane (i.e. p dimensionally extended object) action

$$S = - \mathbf{T} \int d^d\xi \sqrt{- Det\, G_{\mu\nu}\, X^{\mu}_{\ ,a}\, X^{\nu}_{\ ,b}}\ , \qquad (3.1.13)$$

where $d = p + 1$ and $a,b = 1,2,\ldots,d$. Also $X^{\mu}_{\ ,a} \equiv \dfrac{\partial X^{\mu}}{\partial \xi^{a}}$.

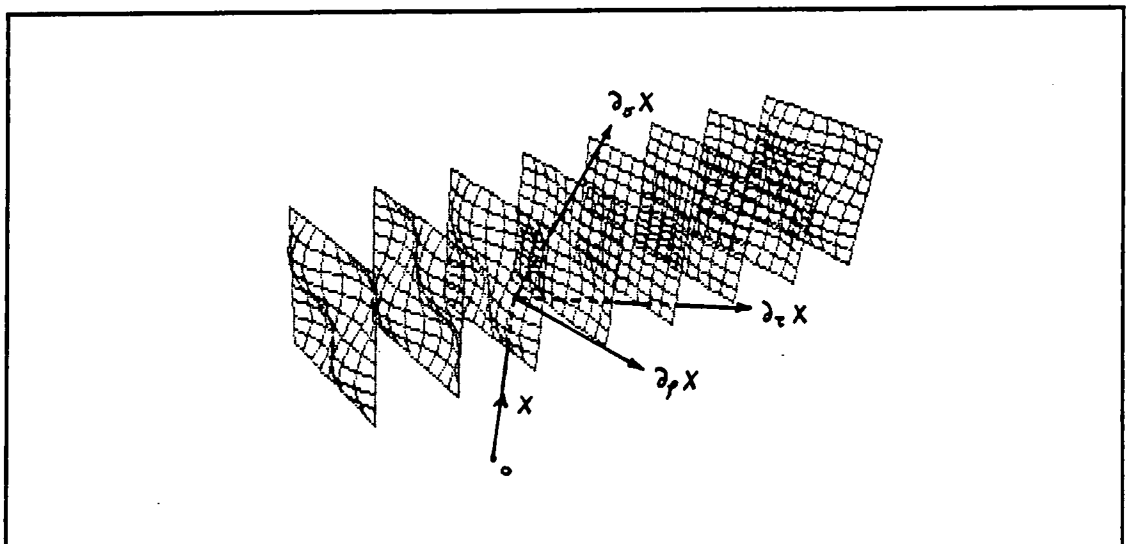

Figure 3.1.1 World-volume illustrated by equal time interval snapshots of a square membrane vibrating in normal mode (2,2), showing the 3 tangents at some point X.

Variation of (3.1.13) gives the generalised geodesic equation

$$g^{ab} \left\{ \partial_a X^{\mu}_{\ ,b} - \Gamma^{c}_{ab} X^{\mu}_{\ ,c} + \Gamma^{\mu}_{\nu\lambda} X^{\nu}_{\ ,a} X^{\lambda}_{\ ,b} \right\} = 0\ . \qquad (3.1.14)$$

In the case of a flat Minkowski space-time background this equation of motion reduces to the wave equation

$$\Delta X^{\mu} \equiv \frac{1}{\sqrt{-g}}\, \partial_{a}\left(\sqrt{-g}\; g^{ab} X^{\mu}{}_{,b} \right) = 0 , \qquad (3.1.15)$$

in which $g_{ab} \equiv X^{\mu}{}_{,a} X_{\mu,b}$ and $g \equiv Det\; g_{ab}$.

From (3.1.12) one can find the momentum current density associated with the membrane motion,

$$P^{1}{}_{\mu} \equiv - \frac{\delta L}{\delta X^{\mu}{}_{,1}} = \mathrm{T} \sqrt{-g}\; g^{1a} X_{\mu,a} \cdot \qquad (3.1.16)$$

and hence find constants of the motion defined by

$$\left.\begin{aligned} P_{\mu} &\equiv \iint P^{1}{}_{\mu}\; d\xi^{2} d\xi^{3} \\ &\text{and} \\ J_{\mu\nu} &\equiv \iint \left(X_{\mu} P^{1}{}_{\nu} - X_{\nu} P^{1}{}_{\mu} \right) d\xi^{2} d\xi^{3} . \end{aligned}\right\} \qquad (3.1.17)$$

Reparametrisation invariance of the world-volume leads to the local identities

$$\left.\begin{array}{l} \left(P^1_{\ \mu} \right)^2 + \mathbf{T}^2 \, Det\left(X^{\mu}_{\ ,i} X_{\mu,j} \right) \equiv 0 \\ \text{and} \\ P^1_{\ \mu} \, X^{\mu}_{\ ,i} \equiv 0 \ , \end{array}\right\} \quad (3.1.18)$$

where i,j = 2,3 . (See (4.1.5 & 6) for a more general statement.)

In the same way as the ends of open relativistic string are found to move at the speed of light, Laziev and Savvidy [2] have shown that the boundary of open relativistic membrane moves at the speed of light.

Collins and Tucker [3] have analysed membrane which is partially closed to form a cylinder. It is described with the usual parametrisation by

$$X^{\mu} = \left(\tau, \ r(\tau) \ Cos \ \sigma, \ r(\tau) \ Sin \ \sigma, \ \rho \right) . \quad (3.1.19)$$

They derived

$$g_{ab} = \begin{pmatrix} 1-\dot{r}^2 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad (3.1.20)$$

which leads to harmonic motion for r, the radius of the cylinder,

$$\ddot{r}(\tau) = -\frac{1}{r_0^{\,2}} \, r(\tau) , \quad (3.1.21)$$

where r_0 is the maximum radius. Collins and Tucker also analysed completely closed spherical membranes in four dimensions described by

$$X^{\mu} = \left(\tau,\ r(\tau)\ Sin\ \sigma\ Cos\ \rho,\ r(\tau)\ Sin\ \sigma\ Sin\ \rho,\ r(\tau)\ Cos\ \sigma \right), \tag{3.1.22}$$

which gives induced metric

$$g_{ab} = \begin{pmatrix} 1-\dot{r}^2 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2 Sin^2\sigma \end{pmatrix}. \tag{3.1.23}$$

The equation of motion of the radius is

$$\dot{r}(\tau) = \frac{1}{r_0^{\,2}} \left(r_0^{\,4} - r^4(\tau) \right)^{1/2}, \tag{3.1.24}$$

which gives the pulsating solution depicted in figure 3.1.2 .

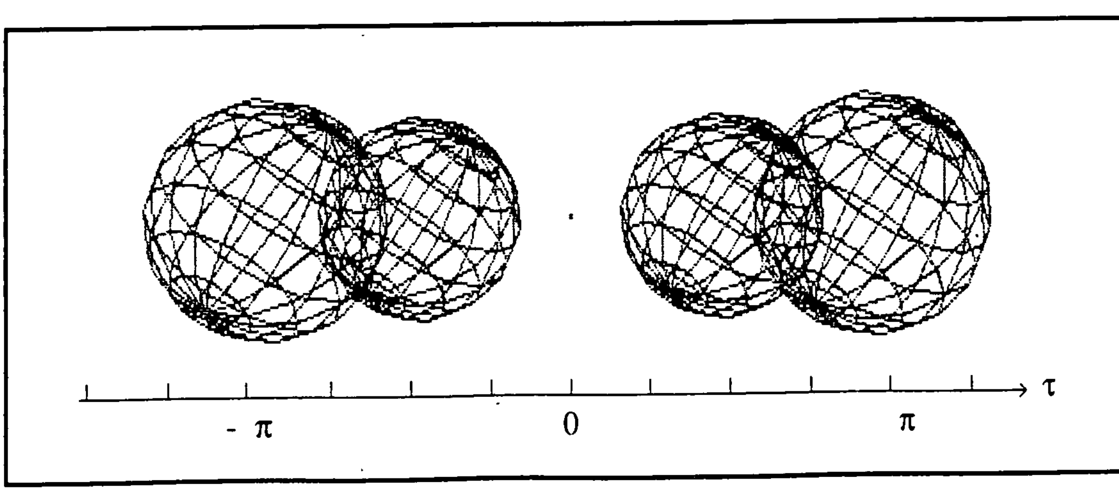

Figure 3.1.2 One period of the pulsating, closed spherical membrane of Collins and Tucker.

Sawhill [4] has taken this spherical membrane and considered it in a non-flat background described by the Schwartzschild metric

$$G_{\mu\nu} = Diag \left\{ 1-\beta,\ -(1-\beta)^{-1},\ -r^2,\ -r^2\ Sin^2\vartheta \right\}, \tag{3.1.25}$$

where $\beta = 2GM/r^2$ (M is the membrane mass resulting from its non-zero surface tension and G is Newton's gravitational constant). He found the equation of motion

$$-4r\ddot{r}(\beta-1) + 8\dot{r}^2(\beta-1) + \beta\dot{r}^3 + \beta(\beta-1)^2\dot{r} - 8(\beta-1)^3 = 0\ , \tag{3.1.26}$$

which reduces to (3.1.24) when $\beta = 0$. A spherical membrane described by (3.1.26) does not collapse to a point but asymptotically approaches its own self-induced event horizon, giving it a certain stability.

Recently, there has been a lot of excitement concerning discoveries about the symmetry of closed membranes. The first step was made by Hoppe [5] who showed that, after light cone gauge fixing of relativistic membrane, there still remains a residual symmetry which is equivalent to preservation of the area of the membrane. Hoppe then showed that this gave the Hamiltonian a symmetry which approximated to $SU(N)$ when N approached the limit $N \longrightarrow \infty$. Thus, he proved that the group of diffeomorphisms of the sphere, $SDiff(S^2)$, can be approximated by $SU(N \longrightarrow \infty)$. Floratos et al. [6] demonstrated how the gauge fields

$A_\mu{}^a(x)$ of an $SU(N)$ Yang-Mills theory become functions $A_\mu(x,\vartheta,\varphi)$ whereby ϑ and φ parametrise an 'internal' sphere at a point.

The second major step forward in the investigation of membrane symmetries was made by Floratos and Iliopoulos [7] and Antoniadis et al. [8] who investigated infinitesimal diffeomorphisms of a toroidal membrane where the variables σ_1 and σ_2 parametrise the angles of two circles in $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$. They found that they could write the generators of the group of infinitesimal area-preserving diffeomorphisms of the surface of a torus, $SDiff(\mathbb{T}^2)$, as

$$L_{n_1,n_2} = i\, h_{n_1,n_2} \left(n_2 \frac{\partial}{\partial\sigma_1} - n_1 \frac{\partial}{\partial\sigma_2} \right) , \qquad (3.1.27)$$

where

$$h_{n_1,n_2} = Exp\ i(n_1\sigma_1 + n_2\sigma_2) \quad ; \quad n_1,n_2 \in \mathbb{Z} , \qquad (3.1.28)$$

together with

$$P_i = \frac{\partial}{\partial\sigma_i} \quad ; \qquad i = 1,2 . \qquad (3.1.29)$$

The generators (3.1.27 & 29) satisfy the algebra

$$\left.\begin{aligned} \left[L_{n_1,n_2} , L_{m_1,m_2} \right] &= (n_1 m_2 - m_1 n_2)\, L_{n_1+m_1,n_2+m_2} , \\ \left[P_i , L_{n_1,n_2} \right] &= n_i \, L_{n_1,n_2} , \\ \left[P_i , P_j \right] &= 0 . \end{aligned}\right\} \qquad (3.1.30)$$

They then proceeded to show that algebra (3.1.30) has a Virasoro subalgebra wherein the Virasoro generators are written as linear combinations of the $L_{n,m}$'s. Hence they made contact with string theory.

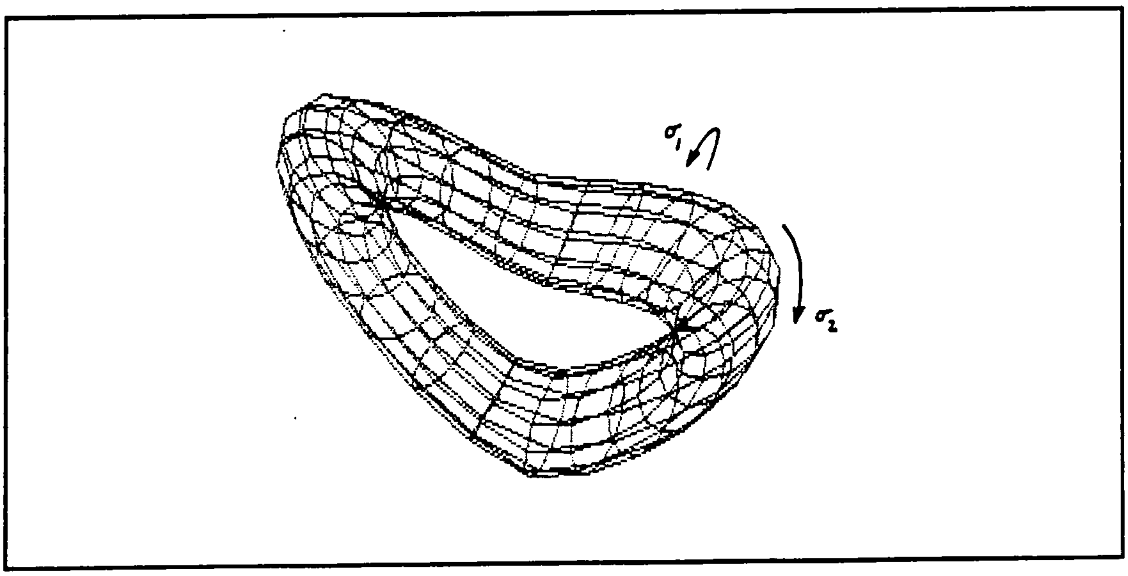

Figure 3.1.3 Snapshot of an area preserving diffeomorphism of a toroidal membrane.

§3.2 Quantisation

Since the equation of motion of a membrane is non-linear in any gauge, it cannot be quantised canonically. One way of proceeding is to perform a semi-classical quantisation. We give a formal schematic outline of the steps involved in this method.

1. Find a stable classical solution, X_{cl}, and expand around it with small fluctuations, Y:

$$X^{\mu} = X^{\mu}_{cl} + \frac{1}{\sqrt{T}} Y^{\mu} .$$

This approximation becomes exact in the limit $\mathbf{T} \longrightarrow \infty$.

2. Substitute X into the usual action to obtain an action to second order in quantum fluctuations:

$$S \approx S_{cl} + \int d\tau d\sigma d\rho \sqrt{-g_{cl}}\, K^{ab}_{\mu\nu}\, Y^{\mu}_{,a} Y^{\nu}_{,b} .$$

3. Find the (linear) equation of motion for Y.

4. Find the general solution, e.g.

$$Y = y_0 + p\tau + \frac{1}{\sqrt{2}} \sum_{m,n=1}^{\infty} \frac{1}{\omega_{mn}} e^{i(m\sigma+n\rho)} \left[\alpha^{\dagger}_{mn} e^{i\omega_{mn}\tau} + \alpha_{-m-n} e^{-i\omega_{mn}\tau} \right] .$$

5. Find P, the conjugate momentum to Y.

6. Impose equal τ commutation relations, e.g.

$$\left[P^{\mu}(\sigma,\rho), Y^{\nu}(\sigma',\rho') \right] = -(2\pi)^2 i\, \delta^{\mu\nu}\, \delta(\sigma-\sigma')\, \delta(\rho-\rho') .$$

7. Find the commutation relations of the expansion coefficients, e.g.

$$\left[\alpha_{mn}, \alpha^{\dagger}_{m'n'} \right] = \frac{mn}{\omega_{mn}} \delta_{mm'} \delta_{nn'} ,$$

and

$$\left[P^{\mu}, y^{\mu}_{0} \right] = - i \, \delta^{\mu\nu} .$$

8. Find the constraint equations, e.g.

$$\phi_{a} \equiv P \, . \, Y_{,a} \approx 0 .$$

9. Impose the condition that they annihilate physical states.

$$\phi_{a} | \, \text{phys} > = 0 .$$

10. Look at the spectrum of states.

Using this method, Kikkawa and Yamasaki [9] arrived at a conclusion concerning the existence of a critical dimension and massless gauge bosons which seemed rather unfortunate for the proponents of fundamental membrane. Their argument, in outline, runs as follows.

In string theory, the dimensions of the string tension constant, κ, are $[MT^{-1}]$. Therefore, on dimensional grounds, one can derive uniquely a classical expression for the spin J in terms of the mass m (total energy), the speed of light c and the string tension κ, viz

$$J = A \, \kappa^{-1} (mc)^{2} , \tag{3.2.1}$$

where A is a dimensionless number. This implies a linear Regge trajectory (J v m^2). A quantum mechanical analysis introduces a new constant h and (3.2.1) becomes

$$J = A \kappa^{-1}(mc)^2 + B h , \qquad (3.2.2)$$

where B is a dimensionless number. This second term causes the Regge trajectory to be shifted by an amount corresponding to the ground state energy (Casimir energy) of the string. A quantum consistent theory for bosonic string requires that $B \in \mathbb{Z}$. Brink and Nielsen [10] found, after a novel process of regularisation using the Riemann ζ-function, that for open string, $B = \frac{1}{24}(D-2)$, which means that the bosonic string is quantum consistent in D = 26 dimensional space-time, the critical dimension.

Kikkawa and Yamasaki attempt a similar calculation of the Casimir energy of bosonic membrane. Dimensional analysis leads uniquely to

$$J = A\, T^{-1/2} (mc)^{3/2} + B h \qquad (3.2.3)$$

for membrane, which implies non-linear Regge trajectories. To calculate B they stabilize open membrane by rotating it simultaneously in two independent planes, $X^1 - X^2$ and $X^3 - X^4$. (Thus they require $D \geq 5$.) Applying the semi-classical quantisation method to this stable classical solution, they calculate the Casimir energy and, after a rather involved and delicate calculation, they find, for the leading trajectory of massless membrane, that

$$B = -\left\{(D\text{-}4)\left[\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\sqrt{(n+|l|+\tfrac{1}{4})(n+\tfrac{1}{4})-\tfrac{1}{16}}\right]\right.$$

$$\left.+\left[\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\sqrt{(n+|l|+\tfrac{3}{4})(n+\tfrac{3}{4})-\tfrac{7}{16}}\right]\right\}. \qquad (3.2.4)$$

By using a new generalised ζ-function regularisation method, followed by numerical integration, they argue that (3.2.4) gives

$$B = (0.1392569...)D - 1.1717121... \; . \qquad (3.2.5)$$

Thus a spin 1 massless state (gauge boson) requires a space-time of dimension $D = 11.387989...$ while a spin 2 massless state (graviton) requires a space-time of dimension $D = 18.568962...$. They conclude that no massless state can be expected in the membrane model and hence that it can't be a unification model.

They did leave open the possibility that the introduction of supersymmetry might allow massless states in integer dimensions. We shall mention later that one of the current objections to supermembrane is that there are *too many* (a continuum of) massless states.

However, their conclusion regarding bosonic membrane has been questioned. For example, Kikkawa and Yamasaki introduce an ultraviolet cutoff parameter which effectively gives the membrane a thickness. This new dimensionful parameter might be able to be used to weaken the above vital dimensional arguments. Also, it has been suggested that string borders open membrane which again introduces a new dimensionful parameter. Alternatively, it has been argued that membrane of a different topology from that considered above might contain the massless states being sought.

One could also envisage [11] the following criticism. Kikkawa and Yamasaki derive the expression

$$J = \frac{2}{3}\sqrt{\frac{pq}{\mathbf{T}}}\left(E^{2/3} - \frac{3}{2} H^{(Q)} E^{1/2} \right), \tag{3.2.6}$$

where p and q are relatively prime integers describing membrane revolving p times in the $X^1 - X^2$ plane and q times in the $X^3 - X^4$ plane in one period. $H^{(Q)}$ is supposed to represent the Hamiltonian operator associated with the fluctuation part of the action, and calculation of the vacuum expectation value of $H^{(Q)}$, $\langle 0 | H^{(Q)} | 0 \rangle$, is used to derive (3.2.4). Dimensional analysis gives

$$H^{(Q)} \propto \mathbf{T}^{\alpha}\, c^{(\frac{1}{2} - 3\alpha)}\, m^{(1 - 3\alpha)}\, h^{2\alpha}. \tag{3.2.7}$$

Without a representative mass, $\alpha = \frac{1}{3}$, leaving an expression for $H^{(Q)}$ involving $h^{2/3}$ which is suspect because it seems most unlikely that this basic physical quantity should be given in terms of cube roots of a universal constant. In particular, it is hard to see how this could arise from a perturbative approach. In any event, it seems that the no-go conclusion of Kikkawa and Yamasaki has not been universally accepted and work on fundamental bosonic membranes continues.

One advance of great importance in the BRST method of quantisation to be described shortly, as well as in the supersymmetric discussion later in this chapter, is an alternative form of the *p*-brane action which has the identities (3.1.18) built in. The alternative form for the particle action (3.1.10) is

$$S[X(\tau);g(X)] = -\tfrac{1}{2}\int d\tau\,\sqrt{-g}\;g^{-1}\left(G_{\mu\nu}\,\frac{dX^{\mu}}{d\tau}\,\frac{dX^{\nu}}{d\tau} + gm^{2}\right), \tag{3.2.8}$$

in which g is taken to be a new independent field. Varying the action with respect to g leads to the equation $p^2 = m^2$. Substituting back the equation of motion for g into (3.2.8) returns the original expression for the action, (3.1.10).

Similarly for massless string, the alternative to the Nambu-Goto action in (3.1.11) is the Polyakov action

$$S[X^{\mu}(\xi);g^{ab}(X)] = -\tfrac{1}{2}\,\kappa\int d^2\xi\,\sqrt{-g}\;g^{ab}\,X^{\mu}{}_{,a}\,X_{\mu,b}\,. \tag{3.2.9}$$

It has been shown by Howe and Tucker [12] for membrane, and by Sugamoto

[13] for the general p-brane alternative action, that one must add a cosmological term giving

$$S[X^{\mu}(\xi);g^{ab}(X)] = -\frac{1}{2}\mathbf{T}_p \int d^{p+1}\xi \sqrt{-g} \left\{(1-p)+ \frac{1}{2}g^{ab}X^{\mu}{}_{,a}X_{\mu,b} \right\} . \qquad (3.2.10)$$

The cosmological term disappears for string because $p = 1$. This is the form of the action which is to be compared, in the case of a Minkowski space-time metric, with the simple action of (3.1.2).

Requiring the variation of the action with respect to the world-body metric, $\delta S/\delta g^{ab}$, to be zero leads to an expression for the stress-energy tensor

$$T_{ab} = \frac{1}{2}\, g_{ab} \left\{ (1-p) + g^{cd}X^{\mu}{}_{,c}X_{\mu,d}\right\} - X^{\mu}{}_{,a}X_{\mu,b} = 0 \ , \qquad (3.2.11)$$

from which the p-brane local identities now follow as constraints on the motion. If the cosmological term is not added in (3.2.10) then contracting the resulting T_{ab} with g^{ab} gives

$$\frac{1}{2}\left(X^{\mu}{}_{,a} \right)^2 (p + 1) = \left(X^{\mu}{}_{,a} \right)^2 ,$$

which is inconsistent unless $p = 1$.

By introducing this new independent metric on the p-brane world-body we have introduced a tensor with $\frac{1}{2}(p - 1)(p - 2)$ components. From reparametrisations of the world-body we have $(p + 1)$ independent gauge invariances which leaves $\frac{1}{2}p(p + 1)$ components of g_{ab} free. In the case of string this one free function, ϕ, can be chosen so that g_{ab} is written

$$g_{ab} = e^{\phi} \eta_{ab} , \tag{3.2.13}$$

where η_{ab} is the two dimensional flat Minkowski metric. It then happens that the Polyakov action (3.2.9) can be considerably simplified because

$$\sqrt{-g}\; g^{ab} = e^{\phi} \sqrt{-\eta}\; e^{-\phi} \eta^{ab} = \eta^{ab} . \tag{3.2.14}$$

Thus the action (3.2.9) is independent of ϕ (i.e. has Weyl invariance) and this fact is responsible for much of the progress which has been made in string theory.

This simplification is not possible in the case of $p > 1$. We shall mention two important gauge choices in the case of membranes. Here there are three free functions in g_{ab} and these can be chosen to lie on the diagonal. A diagonal gauge choice has obvious scope for simplifying problems and an example will be mentioned in the next chapter. Another gauge choice, often called the Hoppe gauge but which is really more restrictive than a gauge choice, is specified by

$$g^{11} + Det\, g^{ij} = 0 \quad ; \quad g^{1i} = 0 , \tag{3.2.15}$$

in which $i,j = 2,3$. In two dimensions this reduces to the Weyl gauge because

$$g_{ab} = \frac{-1}{g^{11}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e^{\phi}\, \eta^{ab} , \tag{3.2.16}$$

as in (3.2.12).

We now outline one non-historical way to understand the modern method of quantisation due to Becchi, Rouet, Stora and Tyutin. One starts from the recognition that any gauge invariant theory necessarily has variables which do not correspond to true degrees of freedom: gauge invariance implies non-physical degrees of freedom. Using the Fadeev-Popov method of adding ghosts and anti-ghosts to the action one can reduce the number of degrees of freedom. This procedure does not change the path integral (i.e. the physics) although it does expand the Fock space. In order to avoid overcounting in the path integral it is necessary to gauge fix the action (i.e. completely specify a gauge). This leads to an effective action which can be written as

$$S_{\text{Effective}} = S + S_{\text{F-P ghosts}} + S_{\text{Gauge fix}} . \tag{3.2.17}$$

This effective action is no longer gauge invariant but it does have a residual symmetry, called BRST symmetry, which reflects the generalised Ward identities.

Given an action with a local symmetry, one can derive a Noether current and hence a conserved charge, Q. In the case under discussion, this charge generates the BRST symmetry transformation. The crucial property of Q is that $Q^2 = 0$. The final step in the BRST procedure is to use this nilpotent BRST charge, Q, to project out the original theory from the expanded Fock space by the condition on physical states that

$$Q \mid phys > \; = \; 0 \; . \qquad (3.2.18)$$

In the next section we shall discuss an application of this quantisation method to bosonic membrane theory.

§3.3 Some of the Recent Progress

Quantisation of the membrane action ((3.2.10) with $p = 2$), according to the BRST method, has been performed by Fujikawa and Kubo [14] (see also [15]). Using the Hoppe gauge, they integrate out the auxiliary fields and eliminate the world-volume metric, using its equation of motion to derive an effective action

$$S_{\mathrm{eff}} = \int d^3\xi \left\{ \tfrac{1}{2} \left(X^{\mu}{}_{,0} \right)^2 - \tfrac{1}{2}\, Det\, h_{ij} + i\bar{C}_0 \left(C^0{}_{,0} - C^i{}_{,i} \right) + i\bar{C}_i C^i{}_{,0} \right\} ,$$

(3.3.1)

in which

$$h_{ij} \equiv X^{\mu}{}_{,i} X_{\mu,j} + i\bar{C}_i C^0{}_{,j} + \bar{C}_j C^0{}_{,i}$$

and where $i,j = 1,2$. C^a and $\bar{C}_a$ are Fadeev-Popov ghosts and antighosts ($a = 0,1,2$). The charge associated with the residual BRST symmetry of S_{eff} they find to be

$$Q = \int d^2\xi \left\{ C^0 \left[\tfrac{1}{2} \left(X^{\mu}{}_{,0} \right)^2 + \tfrac{1}{2}\, Det\, h_{ij} \right] + C^i X^{\mu}{}_{,i} X_{\mu,0} \right.$$

$$\left. - i\bar{C}_0 \left[C^i C^0{}_{,i} + C^0 C^i{}_{,i} \right] - i\bar{C}_i C^j C^i{}_{,j} \right\} . \qquad (3.3.2)$$

By analogy with string theory, Fujikawa and Kubo introduce generalised Virasoro generators from the definition

$$L_a \equiv \left\{ Q \, , \mathcal{C}_a \right\} . \tag{3.3.3}$$

They find the following (anti)commutation relations amongst these operators.

$$\left.\begin{aligned} \left[Q \, , X^{\mu} \right] &= -iC^a X^{\mu}{}_{,a} \, , \\ \left\{ Q \, , C^a \right\} &= -iC^b C^a{}_{,b} \, , \\ \left\{ Q \, , \mathcal{C}_0 \right\} &= H + i\partial_0 \left(\mathcal{C}_a C^a \right) \, , \\ \left\{ Q \, , Q \right\} &= 0 \, , \qquad \text{etc} \ldots \end{aligned}\right\} \tag{3.3.4}$$

where $\partial_0 \equiv \partial/\partial\xi^0 \equiv \partial/\partial\tau$ and the Hamiltonian density, H, is

$$H = \tfrac{1}{2} \left[X^{\mu}{}_{,0} X_{\mu,0} + Det \, h_{ij} \right] + i\mathcal{C}_0 C^i{}_{,i} \, . \tag{3.3.5}$$

We have the required nilpotency of Q from the last anti-commutation relation in (3.3.4). This condition leads to the critical dimension of 26 in the case of string. For membrane however, by calculating the commutation relations explicitly, Fujikawa and Kubo find that there are no anomalous terms in the L_0 commutation relations and hence there is apparently no critical dimension (although see [16] for a more general analysis of this question). From (3.3.3) and the second last anticommutation relation in (3.3.4) we have

$$L_0 = H + i\partial_0 \left(\mathcal{C}_a C^a \right) \tag{3.3.6}$$

and thus the first generalised Virasoro constraint for BRST invariant states (arising from reparametrisation invariance):

$$< L_0 > = < H + i\partial_0 \left(\mathcal{C}_a C^a \right) > = 0 \; . \tag{3.3.7}$$

This constraint, together with the techniques of semi-classical quantisation, are used by Kubo [17] to derive a critical dimension of 27 for bosonic membrane. In bosonic membrane theory a critical dimension does not seem to follow from consistency requirements, as it does in bosonic string theory, but could follow from reality requirements (i.e. the existence of massless gauge bosons). Kubo shows that in a narrow membrane limit the Virasoro constraint (3.3.7) reduces to the same form as for bosonic string. Since string in 26 dimensions has massless vector bosons, in the adjoint representation of a simply-laced Lie group, Kubo argues that $D = 27$ is singled out for membrane.

Kubo begins by compactifying N of the D membrane coordinates on an N-torus, $\mathbb{T}^N$, formed from an integral lattice, Γ, by

$$\mathbb{T}^N = \mathbb{R}^N / \Gamma \; . \tag{3.3.8}$$

He then considers the zero modes of X^μ and regards them as the background solution

$$X_{bg} = x + \mathbf{T}p\tau + \sum_N W_i\sigma^i \ , \tag{3.3.9}$$

where the allowed winding numbers, W_i, lie on the lattice, Γ, and ξ^i = (τ,σ^1,σ^2) . In order to remove zero mass degeneracy, Kubo compactifies on $\mathbb{T}^N \times \mathbb{S}^1$ rather than $\mathbb{T}^N$ and divides the allowed momenta into left moving (p) and right moving ($\tilde{p}$) parts.

Kubo adds quantum fluctuations about this stable background:

$$X = X_{bg} + \frac{1}{\sqrt{\mathbf{T}}} Y \ , \tag{3.3.10}$$

and, from the resulting new contribution to the action, he obtains an expression for Y of the form

$$Y = \sum_{m\neq 0} \left\{ E(m)^{-1/2} a(m)\, e^{-i(E\tau + m.\sigma)} + \text{h.c.} \right\} + \sum_{m\neq 0} \left(\frac{2\pi m_1}{a_1} \right) \left\{ i\tau E(m)^{-1/2} b(m) + E(m)^{-3/2} b(m)^{\dagger} \right\} e^{-m.\sigma} \tag{3.3.11}$$

in which the energy is given by

$$E(m) = \left[\left(\frac{2\pi m_1}{a_1} \right) \mathbf{T}^{-1/2} (p - \tilde{p})^2 + \frac{2\pi m_2}{a_2} \right]^{1/2} , \tag{3.3.12}$$

which might be compared with (3.1.6). Also

$$m.\sigma \equiv \frac{2\pi m_1}{a_1}\,\sigma^1 + \frac{2\pi m_2}{a_2}\,\sigma^2 \,. \tag{3.3.13}$$

Imposing equal time commutation relations on the X's leads to

$$\left[\, a^{\mu}(m) \;,\; a^{\nu}(m') \,\right] = \delta(m - m')\,\eta^{\mu\nu} \,, \tag{3.3.14}$$

and similarily for $b(m)$. Inserting X into the first Virasoro condition (3.3.7) gives an expression of the form

$$\begin{aligned} &< p^2 + \tilde{p}^2 + 2\pi a_2 \mathrm{T} \Bigg\{ \sum_{m_1 \neq 0} |m_1| \left(2a(m)^{\dagger}.a(m) + b(m).b(-m) \right) \\ &- \frac{a_1}{2\pi}(D - 3)\left(\frac{a_1 a_2}{\pi (p - \tilde{p})^{1/2}} \right) \left[2\sum_{m,\,n=1}^{\infty} \left(a_1^2 m^2 + \frac{a_2^2 n^2}{p - \tilde{p}} \right)^{-3/2} \right. \\ &\left. + \left(a_1^{-3} + a_2^{-3}(p - \tilde{p})^{3/2} \right) \sum_{n=1}^{\infty} n^{-3} \right] \Bigg\} > = 0 \,. \end{aligned} \tag{3.3.15}$$

The crucial step in the argument is to take the narrow membrane limit, which is defined as the limit in which the ratio of the length of one side of the membrane, a_2, to the length of the other, a_1, becomes zero. In this limit, together with the infinite tension limit, $\mathrm{T} \longrightarrow \infty$, keeping $a_2\mathrm{T}$ constant, and truncating oscillators with n modes, Kubo shows that the Virasoro constraint reduces to that of compactified bosonic string in the conformal gauge. Thus he claims that bosonic membrane has a critical dimension of 27 because of a reality rather than a consistency condition.

In this section we have attempted to give an overview of some of the ways in which bosonic membrane theory is currently developing.

§3.4 Classification of Super p-Branes

Ordinary numbers commute while Grassmann numbers anticommute. In canonical quantum theory, bosonic fields satisfy commutation relations while fermionic fields satisfy anticommutation relations. A simple supernumber, z, has a body consisting of an ordinary (commuting) real number part, x, and a soul consisting of a Grassmann anticommuting number part, θ. Thus $z = (x,\theta)$ defines a point in a simple superspace $\mathbb{R}^1_c \times \mathbb{R}^1_a$, which represents the commuting and anticommuting parts separately.

A superfield is a function ranging over supernumbers. It can be used to represent the (anti)commutation relations of both bosonic and fermionic fields at once. For example, the position of a superparticle in superspace, with 4 real ($m = 1,2,3,4$) and 4 Grassmann ($\mu = 1,2,3,4$) dimensions, at some time τ, could be given by

$$z^M(\tau) = \left(x^m(\tau) \ , \ \theta^\mu(\tau) \right) . \tag{3.4.1}$$

All labels now have two parts, e.g. $M = (m,\mu)$. Roman capitals label superspace coordinates, Roman small letters label the usual commuting space coordinates and Greek small letters label the new anticommuting space coordinates. Early alphabet are tangent space components, middle alphabet are space components. (These conventions represent a change from the conventions used up until this point in the thesis.)

Wess and Zumino [17] have demonstrated how to give a geometrical formulation of supergravity in superspace. They introduce a supervielbein, $E_M^{\ A}(z)$, $A = (a,\alpha)$, which transform world tensors into tangent space tensors and has submatrices $E_m^{\ a}$, $E_\mu^{\ \alpha}$, $E_m^{\ \alpha}$ and $E_\mu^{\ a}$. Similarily, they introduce the superconnection $\Phi_{MA}^{\ \ B}$. Both the supervielbein and the superconnection can be written as coefficients of 1-forms:

$$\left.\begin{aligned} E^A &\equiv dz^M\, E_M^{\ A} \ , \\ \Phi_A^{\ B} &\equiv dz^M\, \Phi_{MA}^{\ \ B} \ . \end{aligned}\right\} \qquad (3.4.2)$$

Using $d \equiv dz^M(\partial/\partial z^M)$, the supercurvature 2-form is

$$R_A^{\ B} \equiv \tfrac{1}{2}\, dz^N \wedge dz^M\, R_{MNA}^{\ \ \ \ B} = d\Phi_A^{\ B} + \Phi_A^{\ C} \wedge \Phi_C^{\ B} \qquad (3.4.3)$$

and the supertorsion 2-form is

$$T^A \equiv \tfrac{1}{2}\, dz^N \wedge dz^M\, T_{MN}^{\ \ \ A} = dE^A + E^B \wedge \Phi_B^{\ A} \ . \qquad (3.4.4)$$

Torsion is usually zero in ordinary space but is not zero in superspace.

With the above geometrical interpretation, a Lagrangian with a supersymmetry can be viewed as a description of an arrangement of bosons and fermions in superspace such that there is a supertransformation which transforms bosons into fermions and vice versa without changing the action. If we take $D = 11$ supergravity, which has a local supersymmetry, then the curved superspace to be considered has 11 x's and $2^{\lfloor D/2 \rfloor}=32$ θ's (the minimum size of spinors in D dimensions). The notation $\lfloor D/2 \rfloor$ means round down $\frac{D}{2}$ to the nearest integer. So $m = 1,2,....,11$ and $\mu = 1,2,....,32$ which gives the superspace $\mathbb{R}^{11}_a \times \mathbb{R}^{32}_c$. It turns out that in the superspace formulation of $D = 11$ supergravity, the equations of motion can be written as restrictions on the components of the supertorsion (3.4.4). We shall see the importance of this for super p-branes soon.

Now trace the evolution of the action from (3.2.9), written

$$S = -\kappa \int d^2\xi \, \frac{1}{2} \sqrt{-g} \; g^{ij} E_i{}^a E_j{}^b \eta_{ab} \,, \tag{3.4.5}$$

where

$$E_i{}^a = \partial_i X^m(\xi) \; E_m{}^a(X) \tag{3.4.6}$$

and $E_m{}^a$ is the vielbein, so that

$$G_{mn} = E_m{}^a E_n{}^b \eta_{ab} \,. \tag{3.4.7}$$

The $D = 10$ space-time supersymmetric Green-Schwarz action [18] has the same form as (3.4.5) except that

$$E_i^{\ a} = \partial_i z^M(\xi)\ E_M^{\ a}(z)\ , \tag{3.4.8}$$

reflecting the introduction of fermionic coordinates ($\partial_i \equiv \partial/\partial\xi^i$). In order to incorporate the superparticle local fermionic κ-symmetry defined in (3.5.1), Witten [19] added a Wess-Zumino term to the Green-Schwarz action to give the superstring action

$$S = -\kappa \int d^2\xi \left\{ \tfrac{1}{2}\sqrt{-g}\ g^{ij}\ E_i^{\ a} E_j^{\ b} \eta_{ab} + \tfrac{1}{2}\ \varepsilon^{ij} E_i^{\ A} E_j^{\ B} \mathbf{A}_{BA} \right\}\ , \tag{3.4.9}$$

where $\mathbf{A}_{BA}$ are the components of a super 2-form. Hughes et al. [20] were the first to show that, despite beliefs to the contrary, fermionic κ-symmetry can be generalised from string. They extended it to super (4;6)-brane.

The superstring action, (3.4.9), was extrapolated by Bergshoeff et al. [21] to the supermembrane action,

$$S = -\mathbf{T}\int d^3\xi \left\{ \tfrac{1}{2}\sqrt{-g}\ \left[g^{ij}\ E_i^{\ a} E_j^{\ b} \eta_{ab} - 1 \right] + \tfrac{1}{3!}\varepsilon^{ijk} E_i^{\ A} E_j^{\ B} E_k^{\ C} \mathbf{A}_{CBA} \right\}\ , \tag{3.4.10}$$

and hence to super p-branes generally ($p = d - 1$), i.e.

$$S = -\mathbf{k}_d \int d^d\xi \left\{ \tfrac{1}{2}\sqrt{-g}\ \left[g^{ij} E_i^{\ a} E_j^{\ b} \eta_{ab} - (d-2) \right] + \tfrac{1}{d!}\ \varepsilon^{i_1 i_2 \ldots i_d}\ E_{i_1}^{\ A_1}\ E_{i_2}^{\ A_2} \ldots E_{i_d}^{\ A_d}\ \mathbf{A}_{A_d \ldots A_2 A_1} \right\}\ , \tag{3.4.11}$$

where the $(p + 1)$-form $\mathbf{A}$ can be regarded as the potential for a closed $(p + 2)$-form field strength $\mathbf{F}$, where $\mathbf{F} = d\mathbf{A}$. Evans [22] has derived an explicit formula for $\mathbf{A}$ for any p in the case of a flat D dimensional superspace and Azcárraga and Townsend [23] have shown that $\mathbf{F}$ is a member of the $(p+2)^{\text{th}}$ Chevalley-Eilenberg cohomology (supertranslation) group of the superspace.

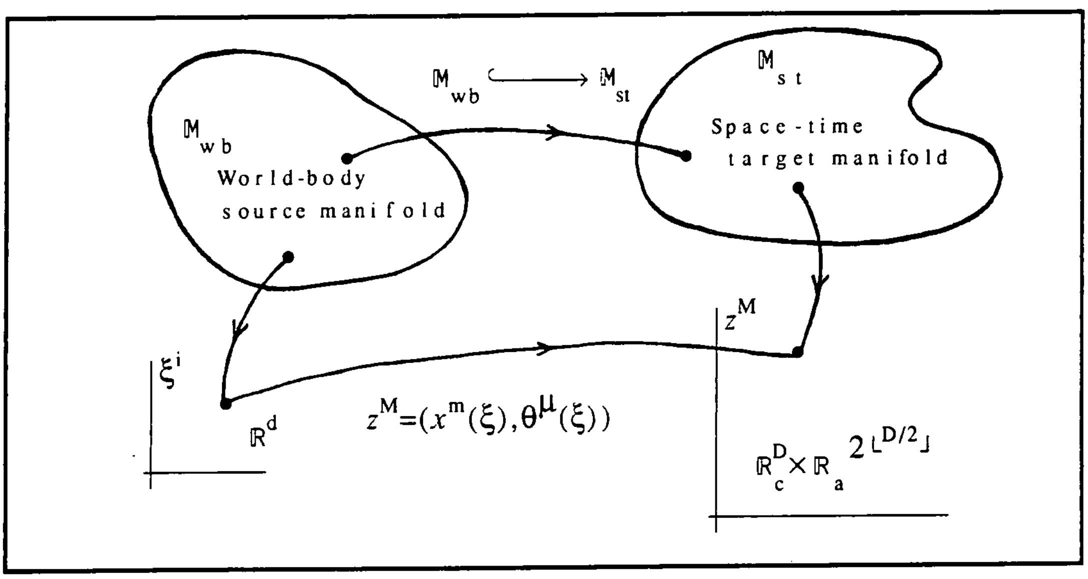

Figure 3.4.1 Schematic picture of the embedding of a d-dimensional world-body into a D-dimensional space-time.

A formal version of the action (3.4.11) may be written $S \sim \int (*1+\mathbf{A})$ and a formal diagram of the embedding may be written

$$\begin{array}{ccc} \mathbb{M}_{wb} & \hookrightarrow & \mathbb{M}_{st} \\ \cap & & \cap \\ \mathbb{R}^d & \xrightarrow{x(\xi)} & \mathbb{R}^D \\ \| & & \\ \mathbb{R}^d & \xrightarrow{\theta(\xi)} & \mathbb{R}^{2^{\lfloor D/2 \rfloor}} \end{array} \quad . \tag{3.4.12}$$

Using the Bianchi identity, $d\mathbb{F} = 0$, one can show that the condition for the existence of a closed $(p + 2)$-form reduces to a gamma matrix identity (e.g. (3.5.15)) and hence to the condition that the number of physical (transverse) bosonic and fermionic degrees of freedom should match. Achúcarro et al. [24] express this condition as

$$D - d = \frac{n\,N}{4} \quad , \tag{3.4.13}$$

where N is the number of space-time supersymmetries and n is the (real) dimension of the (possibly chiral) spinor representation of the D dimensional Lorentz group. So nN is the total number of components of $(\theta^{1\mu}, \theta^{2\mu}, \ldots, \theta^{N\mu})$. The divisor of 4 comes from a halving of the number of fermionic degrees of freedom as a result of fermionic κ-symmetry and another halving from the requirement that the spinors be self-conjugate. They then show that (3.4.12) is only satisfied, for $d < D$, in 12 cases. These 12 possible p-brane (i.e. $(d;D)$-brane‡) theories are plotted in figure 3.4.3 .

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‡ The use of a semi-colon in this context distinguishes our notation from that of some other authors who call this a (p,D)-brane.

The four sequences are labeled $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ to reflect the fact that they have, respectively, 1, 2, 4 and 8 bosons and fermions, which seems potentially of deep significance [25].

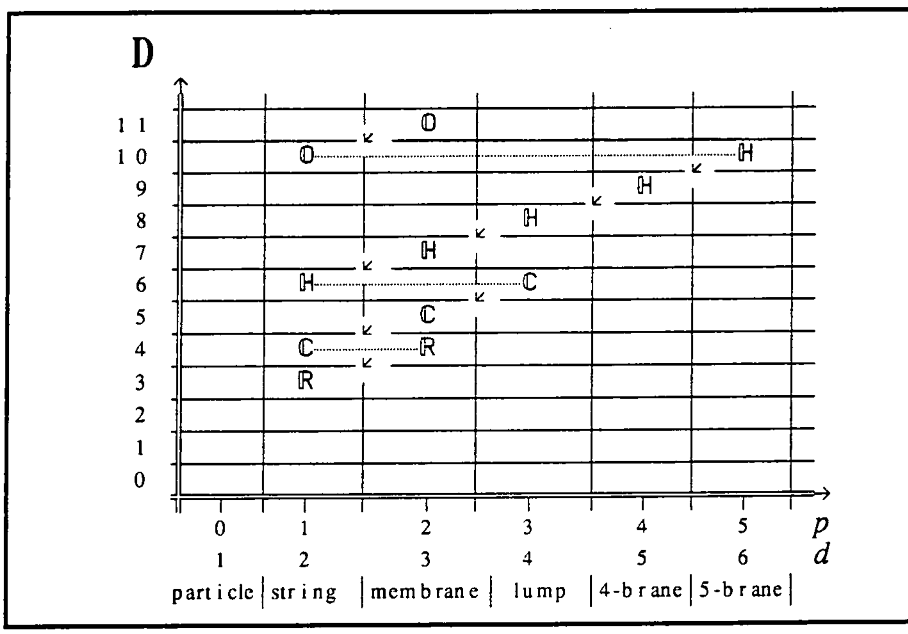


Figure 3.4.2 Plot of the 4 sequences of allowed (d:D)-brane theories

Believing that a theory of everything should predict the signature of space-time as well as its dimensionality, Blencowe and Duff [26] investigated the situation with S space and T time dimensions in space-time and s space and t time dimensions in the world-body ($s \leq S, t \leq T$) and found various generalisations of figure 3.4.2 . The only visible change in the action (3.4.11) is that $\sqrt{-g}$ is replaced by $\sqrt{(-1)^t g}$. No new theories appear on figure 3.4.2 , but various versions of each theory, with $d = s+t$ and $D = S+T$, become possible. For example, from considerations of femionic κ-symmetry, restricting to N = 1 flat superspace and assuming super-Poincaré invariance, they found, for the allowable values of S and T, those plotted in figure 3.4.3 .

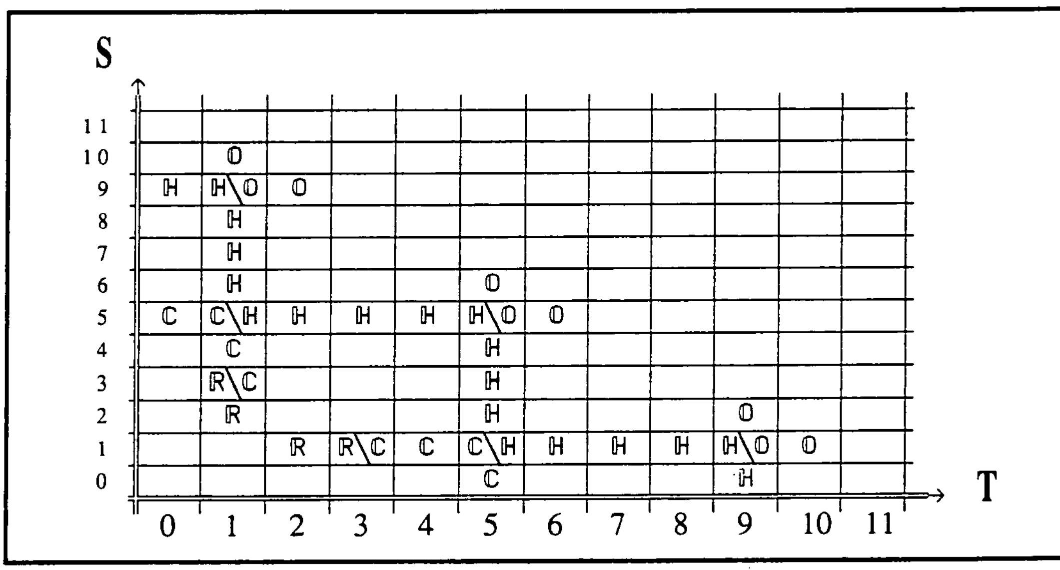


Figure 3.4.3 Plot of allowed super (t,s;T,S)-brane theories, assuming super-Poincare invariance.

The original supersymmetric version of string theory involved a supersymmetry on the world-sheet and is called spinning string theory. Only much later was a space-time supersymmetric version found, called superstring theory. It was shown to be essentially equivalent in the light cone gauge to spinning string (via the GSO projection). Similarily with membrane, the earliest attempt to introduce supersymmetry was as spinning membrane by Howe and Tucker [27]. They showed that world-volume supersymmetrisation of (3.2.10), with p=2, requires the three dimensional supergravity fields to satisfy certain constraints which makes the world-volume metric, which was added in as a free auxiliary field, no longer independent. This makes the bosonic sector of the supersymmetrised Howe-Tucker action inequivalent to the Dirac action. If the supergravity fields are left unconstrained then supersymmetrisation requires that the Einstein-Hilbert term $\sqrt{-g}\,R$ be added, which again makes the bosonic sector inequivalent to the Dirac action. Bergshoeff et al. [28] proposed a no-go theorem to the effect that no spinning version of the Dirac action is possible.

However, Lindström and Roček [29] have recently introduced a new alternative *p*-brane action which is Weyl invariant (i.e. invariant under rescalings of g^{ij}) for any *p*:

$$S = -T_p \int d^{p+1}\xi \, \sqrt{-g} \left(g^{ij} X^m_{,i} X^n_{,j} G_{mn}(X) \right)^{\frac{p+1}{2}} . \qquad (3.4.14)$$

They show that this action can be used to construct spinning *p*-branes. The auxiliary fields remain auxiliary and can be eliminated by their equation of motion to give, in the case of membrane, an action which is equivalent to the Dirac action in the bosonic sector.

But Abraham et al. [30] show that the Weyl invariance of (3.4.14) for $p > 1$ implies that there is no possibility of space-time conformal invariance. They call this an "exclusion principle" between space-time and world-body conformal invariance. This would seem to imply a new no-go theorem for $p > 1$ namely that spinning *p*-branes and super *p*-branes cannot be reconciled. Probably for this reason, interest seems to have temporarily moved to consideration of supersymmetric spinning particle actions and generalisations thereof.

§3.5 Super (3;11)-Brane

Supermembrane in 11 dimensions is the progenitor of all other super p-branes in the sense that all others can be obtained from it by simultaneous dimensional reduction (labelled ↙ in figure 3.4.2) and duality transformations of **A** (labelled ——). Duff et al. [31] have shown by the Kaluza-Klein method how to carry out dimensional reduction of super (3;11)-brane to yield type IIA super (2;10)-brane. They make a partial gauge choice by identifying the ξ^3 ($\equiv \rho$) dimension of the membrane with the X^{11} dimension of the space-time. Then they compactify this dimension on a circle, throwing away all massive modes on space-time and the world-volume. This done, the supermembrane equation of motion reduces to that of the superstring. Generally, we can say, for any $k \in \mathbb{Z}$,

$$\Big(\exists \text{ Super}(d;D)\text{-Brane} \Big) \wedge \Big(\exists\, k{:}0 < k < p \Big) \Rightarrow \exists \text{ Super}(d\text{-}k;D\text{-}k)\text{-Brane.}$$

In other words, given an allowed super $(d;D)$-brane, if there is an integer between 0 and $p = d\text{-}1$ then there also exists an allowed super $(d\text{-}k;D\text{-}k)$-brane.

The super (3;11)-brane has the following known symmetries.

1. Global super-Poincaré invariance.

$$\begin{aligned}\delta z^{M} &\equiv (\delta X^{m} ,\ \delta\theta^{\mu}) \\ &= (l^{m}{}_{n} X^{n} + a^{m} ,\ \tfrac{1}{4} l_{mn} \Gamma^{mn} \theta^{\mu}) .\end{aligned}$$

2. World-volume diffeomorphism invariance.

$$\delta z^{M} = \eta^{i}(\xi)\ \partial_{i} z^{M} \ ,$$

$$\delta g_{ij} = \eta^{k} \partial_{k} g_{ij} + 2 \partial_{(i} \eta^{k} g_{j)k}$$

3. Superspace diffeomorphism invariance.

$$\delta z^{M} = - K^{M}(z) \ ,$$

$$\delta E_{M}{}^{A} = K^{N} \partial_{N} E_{M}{}^{A} + \partial_{M} K^{N} E_{N}{}^{A} \ ,$$

$$\delta \mathbf{A}_{MNP} = K^{Q} \partial_{Q} \mathbf{A}_{MNP} + \partial_{M} K^{Q} \mathbf{A}_{QNP} + \partial_{N} K^{Q} \mathbf{A}_{QPM} + \partial_{P} K^{Q} \mathbf{A}_{QMN} \ .$$

4. Superspace 3-form gauge invariance.

$$\delta \mathbf{A}_{MNP} = \partial_{(M}\ \Sigma_{NP)} ,$$

and the discrete transformation $\mathbf{A}_{MNP} \longrightarrow - \mathbf{A}_{MNP}$ together with an odd number of space or time reflections.

5. Global space-time supersymmetry.

$$\begin{aligned}\delta z^{M} &\equiv (\delta X^{m} \ ,\ \delta\theta^{\mu}) \\ &= (i \bar{\epsilon}_{\mu}\ \Gamma^{m}\ \theta^{\mu} \ ,\ \epsilon^{\mu}) .\end{aligned}$$

In a physical gauge some of the rigid space-time supersymmetry survives and becomes rigid world-volume supersymmetry. This allows transformation of the 8 bosonic and 8 femionic degrees of freedom into one another.

6. Local world-volume fermionic κ-invariance (Siegel symmetry).

$$\delta E^{A} \equiv (\delta z^{M} E_{M}{}^{a} \ , \ \delta z^{M} E_{M}{}^{\alpha}) \\ = \left(0 \ , \ (1 + \tilde{\Gamma})^{\alpha}{}_{\beta} \ \kappa^{\beta}(\xi) \right) , \tag{3.5.1}$$

where $\kappa^{\beta}(\xi)$ is a 32 component spinor in space-time and a scalar on the world-volume.

$$\tilde{\Gamma}^{\alpha}{}_{\beta} \equiv \frac{1}{3! \sqrt{-g}} \varepsilon^{ijk} E_{i}{}^{a} E_{j}{}^{b} E_{k}{}^{c} (\Gamma_{abc})^{\alpha}{}_{\beta} , \tag{3.5.2}$$

where

$$\Gamma_{abc} \equiv \Gamma_{[a} \Gamma_{b} \Gamma_{c]} . \tag{3.5.3}$$

It was to ensure this symmetry that the Wess-Zumino-Witten term had to be added in (3.4.9). One can show by symmetry arguments that

$$\tilde{\Gamma}^{2} = 1$$

i.e.

$$\tilde{\Gamma}^{\alpha}{}_{\gamma} \tilde{\Gamma}^{\gamma}{}_{\beta} = \delta^{\alpha}{}_{\beta} \tag{3.5.4}$$

which allows one to define orthogonal idempotent projection operators, $\frac{1}{2}(1 \pm \tilde{\Gamma})$, which can be used to remove $\frac{1}{2}$ of the femionic degrees of freedom, as was needed in the derivation of (3.4.13).

Consider closed membrane.. Varying the super (3;11)-brane action (3.4.10) with respect to g^{ij} gives

$$\delta L = \tfrac{1}{2}\,\delta\sqrt{-g}\;g^{ij}E_i{}^aE_j{}^b\eta_{ab} + \tfrac{1}{2}\sqrt{-g}\;\delta g^{ij}E_i{}^aE_j{}^b\eta_{ab} - \tfrac{1}{2}\,\delta\sqrt{-g}\ , \tag{3.5.5}$$

and using

$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\;g_{ij}\delta g^{ij}\ , \tag{3.5.6}$$

together with

$$\frac{\delta S}{\delta g^{ij}} = 0\ , \tag{3.5.7}$$

leads to

$$g_{ij} = E_i{}^aE_j{}^b\eta_{ab}\ , \tag{3.5.8}$$

which is the equation of motion for the metric tensor.

Varying with respect to z^M gives, in the bosonic sector, as one may expect from (3.1.14),

$$\partial_i(\sqrt{-g}\;g^{ij}X^m{}_{,j}) + \Gamma^m_{np}\sqrt{-g}\;g^{ij}X^n{}_{,i}X^p{}_{,j} = \tfrac{1}{3!}\varepsilon^{ijk}X^n{}_{,i}X^p{}_{,j}X^q{}_{,k}\mathbf{F}^m{}_{npq}\ , \tag{3.5.9}$$

where

$$\mathbf{F}_{mnpq} \equiv 4\,\partial_{[m}\mathbf{A}_{npq]}\ . \tag{3.5.10}$$

Varying with respect to E^A and g^{ij} gives

$$\delta L = \tfrac{1}{2}\,\delta\sqrt{-g}\;g^{ij}E_i{}^aE_j{}^b\eta_{ab} + \tfrac{1}{2}\sqrt{-g}\;\delta g^{ij}E_i{}^aE_j{}^b\eta_{ab}$$
$$+ \sqrt{-g}\;g^{ij}\;\delta E_i{}^a\;E_j{}^b\eta_{ab} + 3\varepsilon^{ijk}\delta E_i{}^A\;E_j{}^BE_k{}^C\mathbf{A}_{CBA}$$
$$+ \varepsilon^{ijk}E_i{}^AE_j{}^BE_k{}^C\;\delta\mathbf{A}_{CBA} - \tfrac{1}{2}\,\delta\sqrt{-g}\;, \qquad (3.5.11)$$

where

$$\delta E_i{}^A = \delta\left(\partial_i z^M E_M{}^A\right)$$
$$= \partial_i(\delta E^A) + \delta z^N(\partial_i z^M)\left[\frac{\delta E_M{}^A}{\delta z^N} - \left(-1\right)^{MN}\frac{\delta E_N{}^A}{\delta z^M}\right]$$
$$= \partial_i(\delta E^A) - \delta z^N E_i{}^M\;T_{MN}{}^A + \delta z^N E_i{}^M\Phi_{MN}{}^A \qquad (3.5.12)$$

from (3.4.4). Demanding invariance under Siegel symmetry (6) yields constraints such as

$$T_{\alpha\beta}{}^a = -2i\left(\Gamma^a\right)_{\alpha\beta} \qquad (3.5.13)$$

and

$$\mathbf{F}_{ab\alpha\beta} = -\frac{i}{3}\left(\Gamma_{ab}\right)_{\alpha\beta}\;. \qquad (3.5.14)$$

Using $d\mathbf{F} = 0$ with (3.5.14) leads to one of the Γ-matrix identities used to derive (3.4.13):

$$\left(\Gamma^{ab}\right)_{(\alpha\beta}\left(\Gamma^b\right)_{\gamma\delta)} = 0\;. \qquad (3.5.15)$$

Constraint (3.5.13) is a constraint on the torsion of superspace.

the magical aspect of super (3;11)-brane, found by Bergshoeff et al. [21] and which has given impetous to much of the research in supermembranes, is the discovery that the constraints on the torsion in superspace, required for Siegel symmetry of the action, are exactly equivalent to the constraints on the torsion in superspace implied by the equations of motion of $D = 11$ supergravity. This replicates the intimate connection between background and foreground found in super (2;10)-brane theory and gives $D = 11$ supergravity a much needed raison d'être.

There has been very little discussion of self-dual super p-branes in the literature. It may be that the conditions adopted by Zumino [32] in his discussion of super Yang-Mills could be adapted for super p-branes. In particular we might consider

$$\left.\begin{aligned} \mathbf{F} &= * \, \mathbf{F} \\ \text{with} \qquad & \\ \theta &= \tfrac{1}{2} \left(1 + \tilde{\Gamma} \right) \theta \end{aligned}\right\} \qquad (3.4.16)$$

to extend self-duality to the fermionic sector. By the first equation in (3.4.16) we mean something of the form of (4.4.3). We shall not pursue this here, although we shall pursue self-dual bosonic p-branes in the next chapter.

Although many aspects of the subject are omitted from this survey of current research in super (3;11)-brane, we note that compactification schemes of supermembranes are being considered (e.g. [34]), area-preserving diffeomorphisms of supermembrane are being investigated (e.g. [35]), and a debate is currently underway between those who argue that supermembranes have a continuous mass spectrum, and are therefore inherently unstable [36], and those who argue that backgrounds can be found which give supermembrane a stable vacuum and a discrete spectrum of massive states [37]. This last work joins the growing literature on (super) singletons. We shall now look a little closer at the issue of the mass spectrum.

§3.6 Mass Spectrum and Related Problems

A number of approaches have been proposed to extract information about the quantum theory of super (3;11)-branes. We consider semi-classical quantisation, path integral and group theoretical methods.

Mezincescu et al. [37] followed the approach of Kikkawa and Yamasaki for the case of supermembrane. They closed the flat open membrane by making a pancake of two open discs and joining their edges. Without a rigidity term in the action the crease is non-singular. To stabilize it, they rotated it simultaneously in the $X^1 - X^2$ and $X^3 - X^4$ planes. They considered fluctuations, now involving fermionic coordinates, about the stable classical solution X_{cl}:

$$X = X_{cl} + (Y,\chi) \tag{3.6.1}$$

and calculated the Casimir energy. Like Kikkawa and Yamasaki, they found that massless states could not exist in integer dimensions and so concluded that supermembrane could not be a unification theory either. Their calculation has been criticised on the grounds that it was performed in a non-supersymmetric background which does not allow the mode by mode cancellation of the vacuum energy which is required.

Again using the method of semi-classical quantisation, Duff et al. [38] stabilised super (3;11)-brane by wrapping it on a torus by

$\mathbb{R}^{11} \longrightarrow \mathbb{R}^9 \times \mathbb{T}^2$. Fluctuations now include fermionic components as in (3.6.1). For the bosonic sector the results followed the steps of §3.2 . For the fermionic sector, step 4 now includes

$$\chi = S_{00} + \sum e^{i(n\sigma+m\rho)} \left[\frac{m - in}{\omega_{mn}} S^{\dagger}_{mn} e^{i\omega_{mn}\tau} + S_{-m-n} e^{-i\omega_{mn}\tau} \right] \tag{3.6.2}$$

and step 7 includes

$$\left\{ S_{00}^{\alpha} , S_{00}^{\dagger\,\beta} \right\} = 2\, \delta^{\alpha\beta} . \tag{3.6.3}$$

The spectrum is generated by terms of the form

$$\alpha^{\dagger}_{m_1 n_1} \ldots \alpha^{\dagger}_{m_i n_i} S^{\dagger}_{p_1 q_1} \ldots S^{\dagger}_{p_j q_j} \mid \mathrm{vac} > . \tag{3.6.4}$$

By this means they found that the boson and fermion contributions to the vacuum energy cancel, which is encouraging as regards the existence of massless states.

Bergshoeff et al. [39] consider path integration as a way to get beyond semi-classical quantisation. This approach has to take into account world-volumes of arbitrary genus. The integration therefore has to be separated into an integration over all metrics for a 3-volume of definite topology followed by a sum over all topologies i.e.

$$Z \sim \sum_{\text{all topologies}} \left(\int_{\text{a topology}} D[g]\, e^{iS} \right) . \qquad (3.6.5)$$

However, the classification of all compact 3-manifolds is an unsolved problem. The classification of all compact 2-manifolds has been achieved in terms of the simply connected homogeneous Riemannian 2-manifolds $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{B}^2$ which have positive, zero and negative curvature respectively. All compact 2-manifolds can be obtained by quotients of these with freely acting discrete isometry groups, Γ. It has been conjectured [40] that there is an analogous list for 3-manifolds, $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{B}^3$, $\mathbb{S}^2\times \mathbb{R}$, $\mathbb{B}^2\times \mathbb{R}$, $\overline{SL(2;\mathbb{R})}$, *Nil* and *Sol*, from which all compact 3-manifolds can be obtained by quotienting with Γ, and possibly involving surgical sums. *Nil* is the nilpotent Heisenberg group and *Sol* is the solvable group of real matrices of the form $\begin{pmatrix} a & 1 & b \\ 0 & 1/a & c \\ 0 & 0 & 1 \end{pmatrix}$ with a positive.

Despite this problem, progress with the path integral method might still be possible because massless states only need to exist in one particular topology. We note that Ho and Hosotani [41] have developed a bosonic membrane field theory defined as a theory of surface functionals. Considering only toroidal membrane, they have found an exact solution involving massless states and an equally spaced mass-squared spectrum.

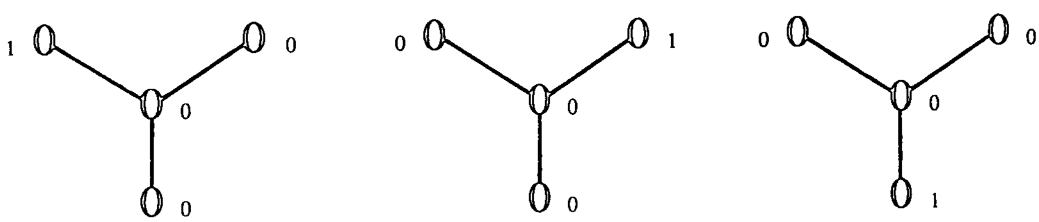
Bars, Pope and Sezgin [42] have employed arguments of group theory to obtain definitive results. Given that space-time supersymmetry remains unbroken non-perturbatively (proved post hoc in [43] and [44]) they conclude immediately that there are massless particles because the ground state energy is zero. The quantum numbers of these particles are determined by quantising the zero modes. These zero modes are the degrees of freedom of a completely collapsed membrane (effectively a superparticle with the embedding of figure 3.4.1 reducing to equation (3.4.1)). They use the light cone gauge so that the physical degrees of freedom can be counted directly without analysing constraints. In $D = 11$ a fermionic state has 32 degrees of freedom. But after Siegel projection and an appropriate gamma matrix representation, these can be shown to reduce to 16 real degrees of freedom. The 16 real fermionic zero modes satisfy a Clifford algebra similar to (3.6.3). This algebra is realised on $2^{D-3} = 256$ states with $2^{D-4} = 128_B$ bosons and 128_F fermions. These states are classified under the transverse $SO(D\text{-}2) = SO(9)$ group as

$$[\ \mathbf{44} \oplus \mathbf{84}\ ]_B \oplus \mathbf{128}_F\ , \tag{3.6.6}$$

which is the $D = 11$ supergravity supermultiplet including a graviton (**44**). Thus the mass spectrum of supermembrane coincides with the allowed background fields. Similar arguments apply to the zero modes of the closed superstring yielding the $D = 10$ supergravity supermultiplet. Repeating this analysis for all the allowed super p-branes shows that only these two super p-brane theories, the octonionic sequence of figure 3.4.2 , contain a graviton in their spectrum. Their conclusion is that $D = 11$ is the critical dimension for supermembranes.

A second argument from group theory concerns the presence of anomalies in the classically allowed super p-brane theories. Bars [45] bases his argument on the assertion that, in the light cone quantisation of an $SO(1,D\text{-}1)$ covariant theory, closure of the Lorentz generators implies that the massive states should reassemble into complete representations of the little group $SO(D\text{-}1)$. If this fails for the first massive level in any topological sector then the theory in question breaks Lorentz invariance and therefore has incurable anomalies.

The case of closed super (1,1;1,9)-brane (see figure 3.4.3 for interpretation) has manifest $SO(8)$ symmetry in the light cone gauge. Modes are divided into left movers (L) and right movers (R). The three 8-dimensional representations of $SO(8)$, with Dynkin labels



have vector (v), spinor (s) and antispinor ($\bar{s}$) states with contents 8v, 8s and 8$\bar{s}$ respectively. The first massive states are those from the stringy version of (3.6.4) of the form

$$\alpha_1^\dagger\, \bar{\alpha}_1^\dagger \quad \text{or} \quad S_1^\dagger\, \bar{S}_1^\dagger \quad \text{or} \quad \alpha_1^\dagger\, \bar{S}_1^\dagger \quad \text{or} \quad \bar{\alpha}_1^\dagger\, S_1^\dagger \quad |\mathrm{vac}\rangle\ , \tag{3.6.7}$$

so the first massive level has symmetry

$$\left[\mathbf{8}_V^L\, \mathbf{8}_V^R \oplus \mathbf{8}_S^L\, \mathbf{8}_{\bar{S}}^R \oplus \mathbf{8}_V^L\, \mathbf{8}_{\bar{S}}^R \oplus \mathbf{8}_V^R\, \mathbf{8}_S^L \right] |vac>$$

$$= \left\{ \left[\mathbf{8}_V \oplus \mathbf{8}_S \right]_L \otimes \left[\mathbf{8}_V \oplus \mathbf{8}_{\bar{S}} \right]_R \right\} |vac> . \qquad (3.6.8)$$

The vacuum itself has the symmetry of the $D = 10$ supergravity supermultiplet which can be written

$$|vac> = | \mathbf{L} > \otimes | \mathbf{R} > = \left[\mathbf{8}_V \oplus \mathbf{8}_{\bar{S}} \right]_L \otimes \left[\mathbf{8}_V \oplus \mathbf{8}_S \right]_R . \qquad (3.6.9)$$

Therefore left movers (and similarly for right movers) have symmetry

$$\left[\mathbf{8}_V \oplus \mathbf{8}_S \right] \otimes \left[\mathbf{8}_V \oplus \mathbf{8}_{\bar{S}} \right]$$

$$= \left[\mathbf{1}_V \oplus \mathbf{28}_V \oplus \mathbf{35}_V \oplus \mathbf{8}_V \oplus \mathbf{56}_V \right] \oplus \left[\mathbf{8}_{\bar{S}} \oplus \mathbf{8}_S \oplus \mathbf{56}_{\bar{S}} \oplus \mathbf{56}_S \right]$$

$$= [\, \mathbf{44} \oplus \mathbf{84} \,]_B \oplus \mathbf{128}_F \ , \qquad (3.6.10)$$

which is a complete representation of $SO(9)$, as required.

Bars then investigates the situation for the case of the first excited states of super (1,2;1,10)-brane compactified on a torus. Again it has manifest $SO(8)$ symmetry in the light cone gauge. The states are now

$$\alpha^{\dagger}_{10}\, \alpha^{\dagger}_{-10} \quad \text{or} \quad S^{\dagger}_{10}\, \bar{S}^{\dagger}_{-10} \quad \text{or} \quad \alpha^{\dagger}_{10}\, \bar{S}^{\dagger}_{-10} \quad \text{or} \quad \alpha^{\dagger}_{-10}\, S^{\dagger}_{10}\, |\text{vac}\rangle \tag{3.6.11}$$

The symmetry in this case is, using (3.6.6),

$$\begin{aligned} &\left(\left[\mathbf{8}_V \oplus \mathbf{8}_S \right] \otimes \left[\mathbf{8}_V \oplus \mathbf{8}_{\bar{S}} \right] \right) |\text{vac}\rangle \\ &= \left([\, \mathbf{44} \oplus \mathbf{84}\,]_B \oplus \mathbf{128}_F \right) \otimes \left([\, \mathbf{44} \oplus \mathbf{84}\,]_B \oplus \mathbf{128}_F \right) \\ &= \mathbf{2^{15}}_{\mathbf{B}} + \mathbf{2^{15}}_{\mathbf{F}} \end{aligned} \tag{3.6.12}$$

and forms a complete representation of $SO(10)$ as required.

Similar arguments for all other p-branes have failed in some topology. Bars concludes that apart from $D = 10$ superstring, only $D = 11$ supermembrane might be quantum consistent. In some sense, supermembrane contains superstring and therefore there is reason to consider quantised super (3;11)-brane as, possibly, the unique theory of everything.

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CHAPTER 4

SELF-DUAL QUATERNIONIC LUMPS IN OCTONIONIC SPACE-TIME

Time and space are modes by which we think
and not conditions in which we live.
A.Einstein

§4.1 Introduction to Self-Dual p-Branes

Moving from strings to p-branes has given extra vitality to discussion of the geometry of minimal immersions. The second order equations of motion of p-branes are in general highly non-linear and, as such, are hard to solve. However, Biran, Floratos and Savvidy [1] have pioneered an approach for constructing self-dual equations for membranes which are first order and have been solved in particular cases.

It has been noticed [2] that there are new classes of exceptional geometries for which self-dual equations can be constructed. This was noticed independently recently by Grabowski and Tze [3]. This chapter describes self-dual p-branes in various backgrounds, in particular self-dual membranes (2-branes) and lumps (3-branes) immersed in the exceptional geometries, concentrating especially on the case of a 3-brane (lump) in 8 dimensions, which we call a (4;8)-brane.

4. *Self-Dual Quaternionic Lumps in Octonionic Space-Time*

By analogy with (p,q) torus knots invoked in chapter 2 in the solution of self-dual (2;4)-brane [4], we introduce the quaternionic counterpart as a proposal for the formulation of specific solutions of the self-dual (4;8)-brane equation. (No non-trivial solutions are currently known.) We describe an algorithm for generating this class of potential solutions which we call 'quots' (quaternionic knots). Although tantalizingly close to satisfying the 32 self-dual equations, we argue that some new idea is required before this ansatz will yield the infinite hierarchy of exact solutions analogous to that in [4].

The action of a $(d;D)$-brane is given by the generalised Einstein-Nambu-Goto volume integral

$$S = \mathbf{T} \int \sqrt{g}\; d^{d}\xi \;, \tag{4.1.1}$$

where

$$g_{ab} = \frac{\partial X^{\mu}}{\partial \xi^{a}} \frac{\partial X^{\nu}}{\partial \xi^{b}} \eta_{\mu\nu} \equiv X^{\mu}{}_{,a} X_{\mu,b} \;, \tag{4.1.2}$$

with $\mu,\nu = 0,1,\ldots,D\text{-}1$ and $a,b = 0,1,\ldots,d\text{-}1$, unless D and d are odd in which case we count from 1 to D and 1 to d respectively. The fundamental constant of the theory, $\mathbf{T}$, has dimensions $[ML^{1-d}]$, with the speed of light $c = 1$.

The $(d;D)$-brane is taken to be closed so that no boundary conditions need to be considered.

We consider the case of a flat Euclidean background space where $\eta_{\mu\nu} = Diag\ (1,1,...,1)$. We also introduce the conjugate momentum

$$P^{a\mu} \equiv \frac{\delta L}{\delta X_{\mu,a}} = \mathbf{T} \sqrt{g}\ g^{ab}\ X^{\mu}_{\ ,b}\ . \tag{4.1.3}$$

The equation of motion (see (3.1.14)) then reduces to

$$\partial_a P^{a\mu} = 0\ . \tag{4.1.4}$$

The appropriate identities can be written

$$P^{a}_{\ \mu} X^{\mu}_{\ ,b} = L\ \delta^{a}_{\ b} = \mathbf{T} \sqrt{g}\ \delta^{a}_{\ b}\ , \tag{4.1.5}$$

which corresponds to the vanishing of the Hamiltonian, and

$$P^{a}_{\ \mu} P^{b\mu} = \mathbf{T}^2\ g\ g^{ab}\ , \tag{4.1.6}$$

which is the extended object generalisation of $p^2 = m^2$ for relativistic particles.

§4.2 Self-Dual (d;d)-Branes

Consider (2;2)-brane (string in two dimensional space-time) and define self-dual (2;2)-brane by

$$\tilde{P}^{a}{}_{\mu} = \mathbf{T}\ \varepsilon^{ab}\varepsilon_{\mu\nu}X^{\nu}{}_{,b}\ , \tag{4.2.1}$$

where ε^{ab} is the two dimensional permutation symbol fixed by

$$\varepsilon^{01} \equiv +1\ . \tag{4.2.2}$$

Recall that the two dimensional permutation tensor ϵ^{ab} is defined by

$$\epsilon^{ab} \equiv \frac{\varepsilon^{ab}}{\sqrt{g}}\ . \tag{4.2.3}$$

Also note that $\sqrt{\eta} = 1$ and therefore that

$$\epsilon^{\mu\nu} = \varepsilon^{\mu\nu} \tag{4.2.4}$$

in our particular case. $\tilde{P}^{a}{}_{\mu}$ is thus a world-sheet tensor density which satisfies (4.1.4), since $\partial_{a}X^{\nu}{}_{,b}$ is symmetric in a and b while ε^{ab} is antisymmetric.

Expanding $\sqrt{g}$ explicitly for the (2;2)-brane gives

$$\sqrt{g} = X^0{}_{,0}X^1{}_{,1} - X^1{}_{,0}X^0{}_{,1} \,, \tag{4.2.5}$$

while expanding $\tilde{P}^a{}_\mu X^\mu{}_{,b}$ gives

$$\tilde{P}^a{}_\mu X^\mu{}_{,b} = \mathbf{T}\ \varepsilon^{ac}\varepsilon_{\mu\nu}X^\nu{}_{,c}X^\mu{}_{,b} = \mathbf{T}\begin{pmatrix} \left(X^0{}_{,0}X^1{}_{,1}-X^1{}_{,0}X^0{}_{,1}\right) & 0 \\ 0 & \left(X^1{}_{,1}X^0{}_{,0}-X^0{}_{,1}X^1{}_{,0}\right)\end{pmatrix} \tag{4.2.6}$$

showing that constraint (4.1.5) is satisfied. Substituting (4.2.1) into (4.1.6) gives

$$\begin{aligned}\tilde{P}^a{}_\mu \tilde{P}^{b\mu} &= \mathbf{T}^2\ \varepsilon^{ac}\varepsilon_{\mu\nu}X^\nu{}_{,c}\varepsilon^{bd}\varepsilon^\mu{}_\rho X^\rho{}_{,d} \\ &= \mathbf{T}^2\ \varepsilon^{ac}\varepsilon^{bd}g_{cd} = \mathbf{T}^2\ \mathrm{A}dj\ g_{ab} \\ &= \mathbf{T}^2\ g\ g^{ab}\,. \end{aligned} \tag{4.2.7}$$

So constraint (4.1.6) is satisfied. The self-dual equation is therefore $P^a{}_\mu = \tilde{P}^a{}_\mu$, that is, from (4.1.3) and (4.2.1),

$$X^\mu{}_{,a} = \epsilon_a{}^b\epsilon^\mu{}_\nu X^\nu{}_{,b}\,. \tag{4.2.8}$$

Similarily it can be shown [5] that self-dual $(d;d)$-branes can be defined generally by

$$\mathcal{P}^{a_1}_{\mu_1} = \mathbf{T} \frac{1}{(d\text{-}1)!} \varepsilon^{a_1 a_2 a_3 .. a_d} \varepsilon_{\mu_1 \mu_2 \mu_3 .. \mu_d} X^{\mu_2}_{,a_2} X^{\mu_3}_{,a_3} .. X^{\mu_d}_{,a_d} . \tag{4.2.9}$$

This satisfies the equation of motion by symmetry arguments. It also satisfies the constraints from the properties of contracted products of permutation tensors and by the definition of determinant in terms of permutation tensors.

The covariant self-dual (3;3)-brane equation is therefore

$$X^{\mu_1}_{,a_1} = \frac{1}{2} \in_{a_1}^{a_2 a_3} \in^{\mu_1}_{\mu_2 \mu_3} X^{\mu_2}_{,a_2} X^{\mu_3}_{,a_3} . \tag{4.2.10}$$

In a gauge in which g_{ab} is diagonal, Biran et al. simplify (4.2.10) to

$$E^{\mu_1}_{a_1} = \frac{1}{2} \varepsilon_{a_1 a_2 a_3} \varepsilon^{\mu_1 \mu_2 \mu_3} E^{\mu_2}_{a_2} E^{\mu_3}_{a_3} , \tag{4.2.11}$$

where

$$E^{\mu}_{a} \equiv \frac{X^{\mu}_{,a}}{\sqrt{X_{,a}^{2}}} \tag{4.2.12}$$

and show that

$$X(\xi) = R(\xi^1)\left\{ Cos\Phi(\xi^3)\ Cos\Psi(\xi^2),\ Cos\Phi(\xi^3)\ Sin\Psi(\xi^2),\ Sin\Phi(\xi^3) \right\} \tag{4.2.13}$$

and

$$X(\xi) = \left\{ \left[R + r(\xi^1)\ Cos\Phi(\xi^3) \right] Cos\Psi(\xi^2),\ \left[R + r(\xi^1)\ Cos\Phi(\xi^3) \right] Sin\Psi(\xi^2),\ r(\xi^1)\ Sin\Phi(\xi^3) \right\} \tag{4.2.14}$$

are solutions of (4.2.11).

Since any orthogonal transformation of these solutions will give new solutions, we have verified that, for example,

$$X(\xi) = \left\{ R(\xi^1)\ S(\xi^2)\ Cos\Theta(\xi^3),\ R(\xi^1)\ S(\xi^2)\ Sin\Theta(\xi^3),\ \frac{1}{2} \left(R(\xi^1)^2 - S(\xi^2)^2 \right) \right\} \tag{4.2.15}$$

and

$$X(\xi) = \lambda \left\{ Sinh\Theta(\xi^1)\ Sin\Phi(\xi^2)\ Cos\Psi(\xi^3),\ Sinh\Theta(\xi^1)\ Sin\Phi(\xi^2)\ Sin\Psi(\xi^3),\ Cosh\Theta(\xi^1)\ Cos\Phi(\xi^2) \right\} \tag{4.2.16}$$

are also solutions of (4.2.11).

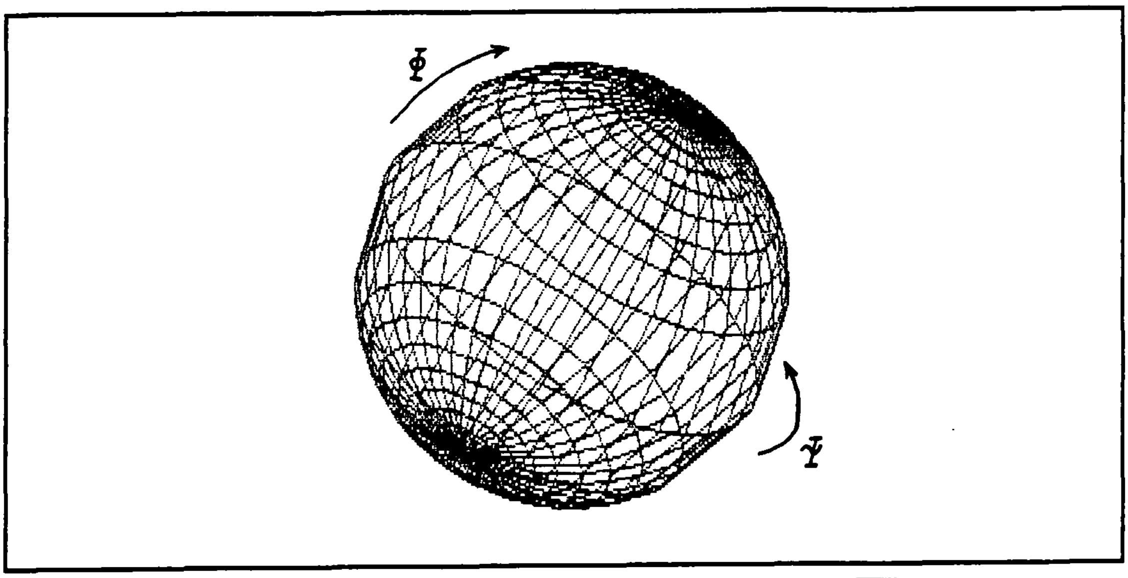

Figure 4.2.1 Solution (4.2.13) for $R(\xi^1) = \xi^1$, $\Psi(\xi^2) = \xi^2$, $\Phi(\xi^3) = \sqrt{\xi^3}$.

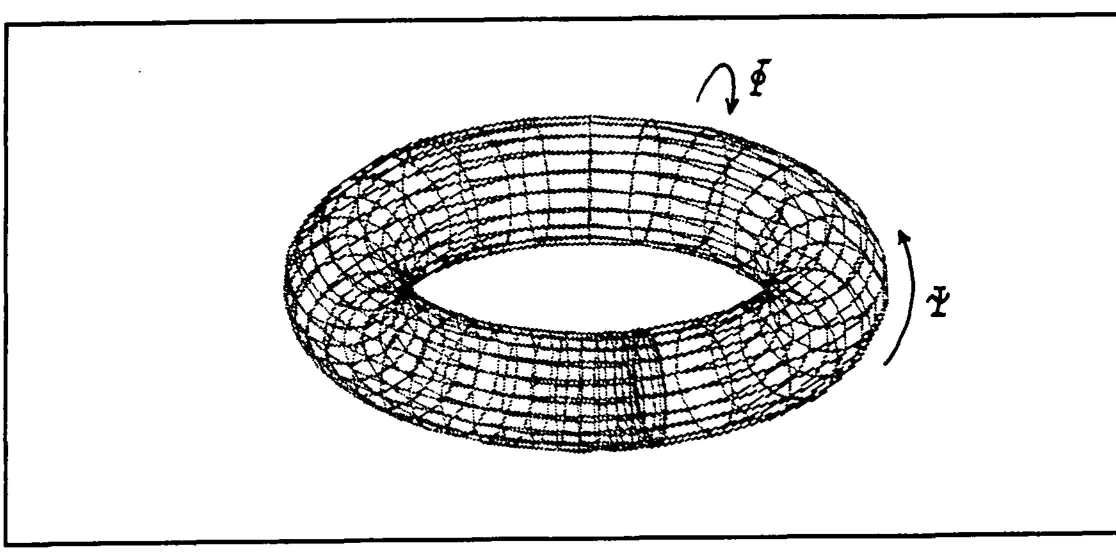

Figure 4.2.2 Solution (4.2.14) for $r(\xi^1)=\xi^1$, $\Psi(\xi^2)=2\pi Sin\xi^2$, $\Phi(\xi^3)=\xi^3$.

§4.3 Self-Dual (d;D)-Branes

The self-dual equation of rigid string instantons (2.1.17) [4], self-dual (2;4)-branes, can be derived by defining

$$\mathcal{P}^{a}{}_{\mu} = \mathbf{T}\ \varepsilon^{ab} J_{\mu\nu} X^{\mu}{}_{,b}\ , \tag{4.3.1}$$

where an almost complex structure has been imposed on space-time by

$$J_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{4.3.2}$$

(4.3.1) automatically solves (4.1.4) and (4.1.6), but only satisfies (4.1.5) on condition that

$$\sqrt{g} = J_{\mu\nu} X^{\mu}{}_{,0} X^{\nu}{}_{,1}\ , \tag{4.3.3}$$

as is the case if $X^{\mu}{}_{,a}$ has the symmetry of

$$X^{\mu}{}_{,a} \sim \begin{pmatrix} A & -B \\ B & A \\ C & D \\ -D & C \end{pmatrix} , \tag{4.3.4}$$

which is true in, for example, the case of

$$X^{\mu} = \left(\tau^{3}- 3\tau\sigma^{2},\ 3\tau^{2}\sigma - \sigma^{3},\ \sigma^{2}- \tau^{2},\ -2\tau\sigma \right) , \tag{4.3.5}$$

where $\tau \equiv \xi^0$ and $\sigma \equiv \xi^1$ (c.f. (2.2.3)).

Another interesting example of a self-dual $(d;D)$-brane ($d < D$) has been discussed by Grabowski and Tze [3]. This is the first exceptional case (see [6]), a (3;7)-brane (see also [7] for a self-dual (3;5)-brane). Define

$$\tilde{P}^a{}_\mu = \mathbf{T}\, \tfrac{1}{2}\, \varepsilon^{abc} C_{\mu\nu\lambda} X^\nu{}_{,b} X^\lambda{}_{,c} \,, \tag{4.3.6}$$

where $C_{\mu\nu\lambda}$ are the octonion structure constants. (4.3.6) satisfies (4.1.4) because of the complete antisymmetry of $C_{\mu\nu\lambda}$. (4.3.6) also satisfies (4.1.6) as can be demonstrated using the identity

$$C_{\mu\nu\lambda} C^{\rho\sigma\lambda} = \delta_\mu{}^\rho \delta_\nu{}^\sigma - \delta_\mu{}^\sigma \delta_\nu{}^\rho + H_{\mu\nu}{}^{\rho\sigma} \,, \tag{4.3.7}$$

where $\mu,\nu,.. = 1,2,...,7$ and $H_{\mu\nu}{}^{\rho\sigma}$ is a completely antisymmetric rank 4 tensor which quantifies the non-associativity of octonion multiplication in a similar way that $C_{\mu\nu\lambda}$ quantifies their non-commutativity.

However, (4.1.5) is only satisfied if

$$\sqrt{g} = C_{\mu\nu\lambda} X^\mu{}_{,1} X^\nu{}_{,2} X^\lambda{}_{,3} \,. \tag{4.3.8}$$

Consider the self-dual (3;7)-brane equation resulting from (4.3.6) and (4.1.3),

$$X^{\mu}{}_{,a} = \frac{1}{2} \in_a{}^{bc} C^{\mu}{}_{\nu\lambda} X^{\nu}{}_{,b} X^{\lambda}{}_{,c} \ . \tag{4.3.9}$$

Contracting (4.3.9) with $X_{\mu}{}^{,a}$ gives (4.3.8). The right side of (4.3.8) can be regarded as a generalised Jacobian for immersions.

As a first step towards solving (4.3.9), we shall consider how to find solutions of (4.3.8). One approach to obtaining a solution to (4.3.8) is to take the self-dual (3;3)-brane solutions (4.2.13 - 16) as trivial (3;7)-brane solutions. $C_{\mu\nu\lambda}$ is invariant under G_2 transformations. Taking a general matrix $G_{\mu\nu} \in G_2$, then

$$G_{\mu}{}^{\rho} G_{\nu}{}^{\sigma} G_{\lambda}{}^{\tau} C_{\rho\sigma\tau} = C_{\mu\nu\lambda} \ . \tag{4.3.10}$$

Substituting this into (4.3.8) gives

$$\sqrt{g} = C_{\rho\sigma\tau} \left(G_{\mu}{}^{\rho} X^{\mu}{}_{,1} \right) \left(G_{\nu}{}^{\sigma} X^{\nu}{}_{,2} \right) \left(G_{\lambda}{}^{\tau} X^{\lambda}{}_{,3} \right) \ . \tag{4.3.11}$$

Thus applying a general transformation of the seven dimensional representation of the exceptional Lie group G_2 to our trivial (3;7)-brane solutions will give the solutions in a form which might be more interesting. We shall not pursue this approach here but we shall make a few more remarks about $C_{\mu\nu\lambda}$.

A basis for octonion multiplication is defined as *admissible* if it is such that

$$|O_1||O_2| = |O_1O_2| \quad ; \quad O_i \in \mathbb{O} , \tag{4.3.12}$$

where

$$|O| \equiv \sqrt{\left(ScO\right)^2 + \left(VecO \, . \, VecO\right)} \tag{4.3.13}$$

is the *norm* of octonion O. Equation (4.3.12) is equivalent to the condition that

$$\left(VecO_1 \wedge VecO_2\right)^2 = Det\left(VecO_i \, . \, VecO_j\right) , \tag{4.3.14}$$

where i,j=1,2. The caret symbol signifies the cross product in 7 dimensions which is defined by $C_{\mu\nu\lambda}$ (see [8]) and the dot signifies the 7 dimensional dot product. Both these products are a consequence of octonion multiplication. *VecO*, $O \in \mathbb{O}$, is similar to *ImZ*, $Z \in \mathbb{C}$, and selects the 7 dimensional vector part of O. (For a quaternion $Q \in \mathbb{H}$, *VecQ* selects the 3 dimensional vector part of Q.) *ScO* is similar to *Rez* and selects the scalar part of O.

The Moufang identity,

$$C_{\rho\sigma\tau} = \tfrac{1}{3} C_{\mu\rho}{}^{\nu} C_{\nu\sigma}{}^{\lambda} C_{\lambda\tau}{}^{\mu} , \tag{4.3.15}$$

is also only valid in an admissible basis, as is the equivalent identity,

$$C_{\rho\sigma\tau} = \tfrac{1}{4} H_{\rho\sigma}{}^{\mu\nu} C_{\mu\nu\tau} , \qquad (4.3.16)$$

which follows by application of (4.3.7) to (4.3.15).

A particular basis is specified by a list of 7 triples each involving 3 of the integers 1 to 7 without repetition. Cyclic rotations of a triple give an equivalent triple. The quaternionic analogue is the basis for the usual cross product in 3 dimensions. This cross product is based on the SO(3) invariant tensor ϵ_{abc} which is characterised by a single triple involving the integers 1 to 3. The triple 123 is equivalent to 312 or 231 and specifies

$$\epsilon_{abc} = \sqrt{g}\ \varepsilon_{abc} \qquad (4.3.17)$$

by defining $\varepsilon_{123} \equiv +1$, which uniquely implies, by complete antisymmetry, all of the other components of ε_{abc}. The only alternative basis for quaternions is the triple 132 which signifies the distinction between left and right handed coordinate systems in 3 dimensions.

For octonions there are 480 different choices of basis, 240 clockwise and 240 anti-clockwise. We call a basis *anti-clockwise* if it can be represented on the diagram in figure 4.3.1 where A to G are to be identified one to one with the integers 1 to 7 in some order.

Each of the 7 triangles represents one of the 7 triples, the arrow indicates the order. If, for example, A to G are identified with 1 to 7 respectively then figure 4.3.1 represents the basis $\begin{vmatrix} 134 \\ 245 \\ 356 \\ 467 \\ 571 \\ 612 \\ 723 \end{vmatrix}$.

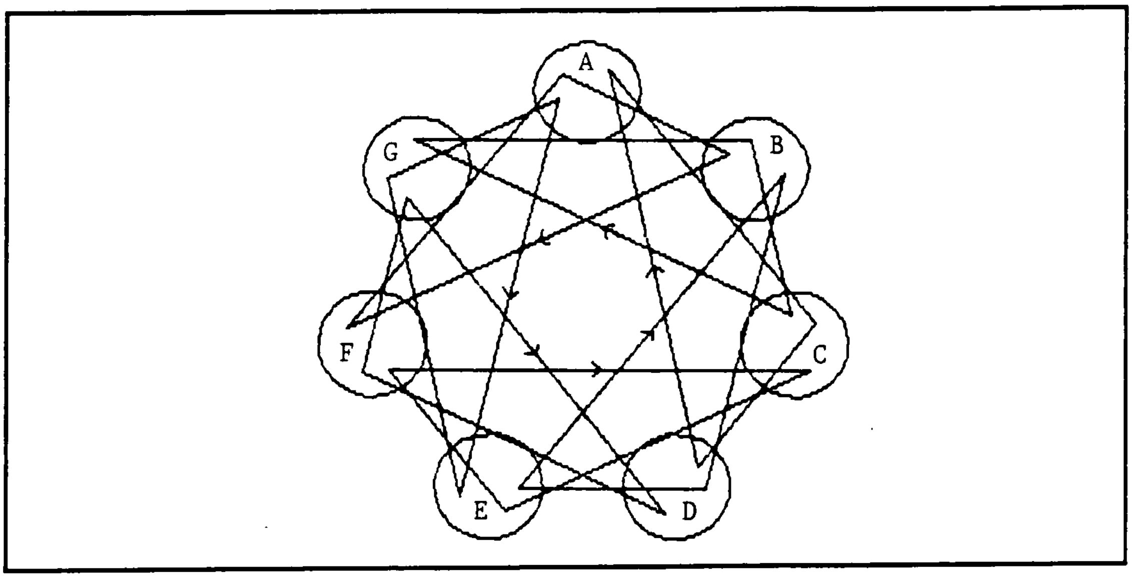

Figure 4.3.1 Representation of the 240 anti-clockwise bases of octonionic multiplication.

This is interpreted as meaning that $C_{134} = 1$ along with the other two even permutations. $C_{413} = C_{341} = 1$. The three odd permutations are then -1. $C_{143} = C_{314} = C_{431} = -1$. This assignment process is repeated for each of the 7 triples and all other elements of $C_{\mu\nu\lambda}$ are set to zero. There are, on the face of it, 7! ways of placing 1 to 7 on figure 4.3.1 . However the starting position is arbitrary so 7! should be divided by 7. Also, since cyclic permutations are irrelevant, each basis is equivalent to two others generated by cyclically permuting the entire columns of the basis. Figure 4.3.1 therefore represents $\frac{7!}{7.3} = 240$ distinct bases. Changing the directions of all the arrows on figure 4.3.1 gives the 240 clockwise bases.

In summary, any admissible basis can be written such that each of the integers 1 to 7 appears one and only one time in each column, and each row does not have repetitions. The other important characteristic of an admissible basis is that no two rows have more than one integer in common. This rules out as inadmissible a basis such as $\begin{vmatrix} 123 \\ 234 \\ 345 \\ 456 \\ 567 \\ 671 \\ 712 \end{vmatrix}$.

An alternative way of representing bases by a triangle and circle does not allow one to distinguish readily an admissible basis, figure 4.3.2 , from an inadmissible one, figure 4.3.3 .

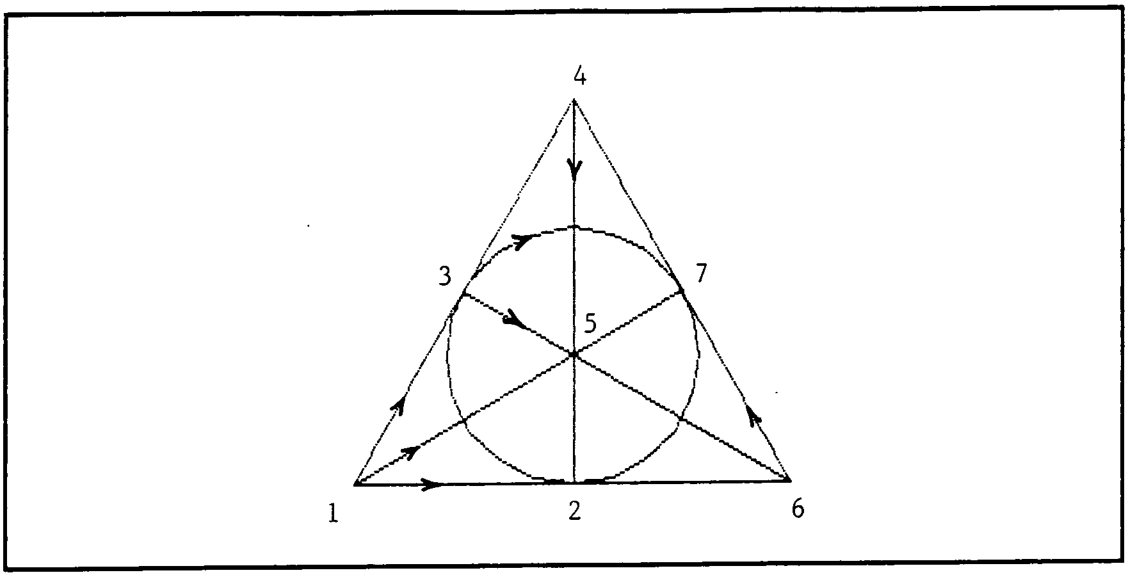

Figure 4.3.2 An admissible basis.

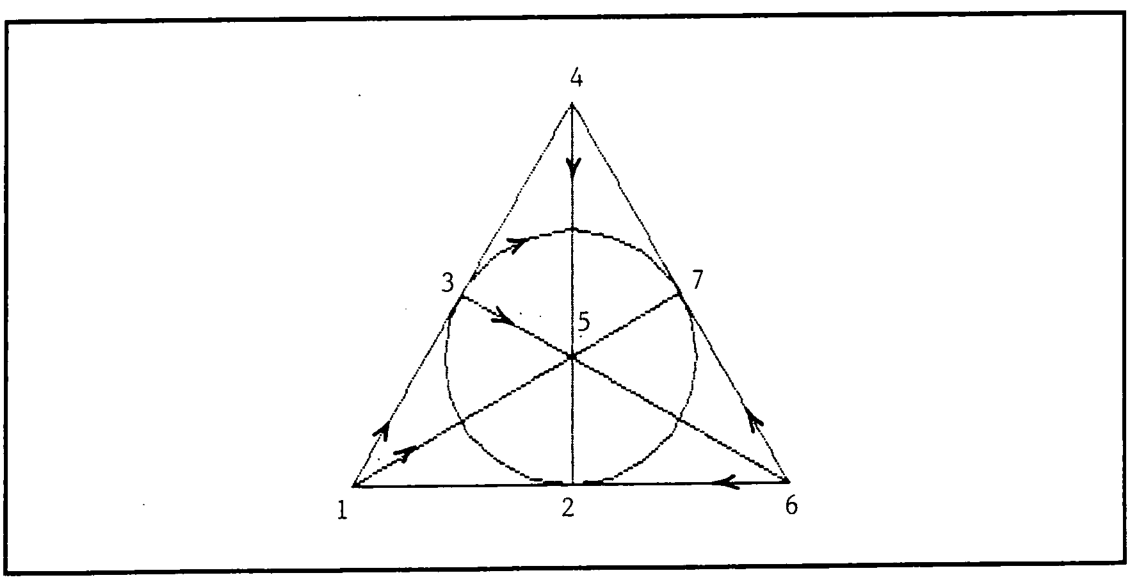

Figure 4.3.3 An inadmissible basis.

§4.4 Self-Dual (4;8)-Brane

Since quaternions have embeddings in octonions, $\mathbb{H} \hookrightarrow \mathbb{O}$, it is natural to generalise the (3;7)-brane, seen as $Vec\mathbb{H} \hookrightarrow Vec\mathbb{O}$, to (4;8)-brane. This involves the second fundamental geometry with an exceptional automorphism group. For self-dual (4;8)-brane, we write

$$\tilde{P}^{a}{}_{\mu} = \mathrm{T}\,\frac{1}{3!}\,\varepsilon^{abcd}\;T_{\mu\nu\rho\sigma}\;X^{\nu}{}_{,b}X^{\rho}{}_{,c}X^{\sigma}{}_{,d}\;. \tag{4.4.1}$$

The completely antisymmetric tensor $T_{\mu\nu\rho\sigma}$ was introduced in [9]. It can be defined, making use of (4.3.7), from

$$\left.\begin{aligned} T_{0\mu\nu\rho} &= \pm C_{\mu\nu\rho}\,, \quad \text{etc...} \\ T_{\mu\nu\rho\sigma} &= H_{\mu\nu\rho\sigma}\,, \end{aligned}\right\} \tag{4.4.2}$$

where $\mu,\nu = 1,2,\ldots,7$ but $\mu,\nu \neq 0$. Choosing the positive sign defines a self-dual tensor, $T_{\mu\nu\rho\sigma}$ $(\mu,\nu = 0,1,2,\ldots.,7)$, while choosing the negative sign defines an anti-self-dual tensor. Changing between anti-clockwise and clockwise bases of $C_{\mu\nu\rho}$ has an equivalent effect so we can take the positive sign in (4.4.2) without loss provided we consider both clockwise and anti-clockwise bases.

Consider the 1024 'doubly self-dual' equations analogous to self-dual Yang-Mills

$$F^{\mu\nu}_{ab} = \frac{1}{4} \in_{ab}{}^{cd} T^{\mu\nu}{}_{\rho\sigma} F^{\rho\sigma}_{cd} , \tag{4.4.3}$$

where

$$F^{\mu\nu}_{ab} \equiv X^{\mu}{}_{,[a} X^{\nu}{}_{,b]} . \tag{4.4.4}$$

Contracting (4.4.3) with $X_{\nu}{}^{,b}$ gives

$$X^{\mu}{}_{,a} = \frac{1}{3!} \in_{a}{}^{bcd} T^{\mu}{}_{\nu\rho\sigma} X^{\nu}{}_{,b} X^{\rho}{}_{,c} X^{\sigma}{}_{,d}, \tag{4.4.5}$$

which are the 32 self-dual equations arising from (4.4.1) and (4.1.3).

Contracting (4.4.5) with $X_{\mu}{}^{,a}$ gives

$$\sqrt{g} = T_{\mu\nu\rho\sigma} X^{\mu}{}_{,0} X^{\nu}{}_{,1} X^{\rho}{}_{,2} X^{\sigma}{}_{,3} , \tag{4.4.6}$$

which is the condition necessary for (4.4.1) to satisfy constraint (4.1.5). The equation of motion (4.1.4) and the second constraint (4.1.6) are automatically satisfied by (4.4.1) without further conditions.

§4.5 (p,q) Quots

We wish to find a solution of (4.4.5). First we look for a solution of (4.4.6). Consider the quaternionic equation

$$U^p + V^q = 0 \quad ; \quad U,V \in \mathbb{H} . \tag{4.5.1}$$

A solution of (4.5.1) is given by

$$U = K^q \quad , \quad V = -K^p \quad ; \quad K \in \mathbb{H} . \tag{4.5.2}$$

Take the case where

$$K = t + xi + yj + zk \equiv t + \underline{r} \tag{4.5.3}$$

and p=2, q=3, then

$$\left. \begin{aligned} U &= (t^3 - 3tr^2) + (3t^2 - r^2)\underline{r} \quad , \\ &\text{and} \\ V &= (r^2 - t^2) - 2t\underline{r} \quad . \end{aligned} \right\} \tag{4.5.4}$$

Call this quaternion analogy of a (2,3) torus knot, a (2,3) quot. Now construct an octonion, X, by catenating U to V, inserting a dimensionful constant (with dimensions of length), **L**, (which we shall henceforth set to 1) to avoid dimensional problems:

$$ {}^{pq}X = (U \, , \, \mathbf{L}^{q/p} \, V) \, . \tag{4.5.5} $$

Then consider ${}^{pq}X$ as a potential solution of (4.4.5) or (4.4.6). Writing ${}^{pq}X^{\mu} = (ScU, VecU, ScV, VecV)$ then

$$ {}^{23}X^{\mu} = \left((t^3 - 3tr^2), (3t^2 - r^2)\underline{r}, (r^2 - t^2), -2t\underline{r} \right) . \tag{4.5.6} $$

Let us first compute $\sqrt{g}$ from this ${}^{23}X$. We find

$$ \sqrt{{}^{23}g} = (t^2 + r^2)\left[9(t^2 + r^2) + 4\right]\left[(3t^2 - r^2)^2 + (2t)^2\right] . \tag{4.5.7} $$

For ${}^{23}X$ to have a chance of solving (4.4.6), $Det\ {}^{23}g_{ab}$ must be a perfect square. (4.5.7) shows that ${}^{23}X$ does satisfy this non-trivial criterion.

The left side of (4.4.6) does not appear to depend upon the basis chosen for octonion multiplication whereas the right side certainly does, from (4.4.2). Therefore ${}^{23}X$ could only be expected to satisfy (4.4.6) in one particular choice of basis. What basis should we use?

In fact we have made an implicit choice of basis in our construction of ${}^{23}X$. Defining a complex number as two real numbers catenated together is equivalent to saying that the first is the real part and the second is the imaginary part of the complex number. Symbolically;

$$\mathbb{C} \equiv (\mathbb{R}_1, \mathbb{R}_2) \equiv (\mathbb{R}_1 + i\mathbb{R}_2) \ . \tag{4.5.8}$$

However, the situation is more involved in the case of quaternions because the real and imaginary parts are now themselves complex numbers.

$$\begin{aligned} \mathbb{H} \equiv (\mathbb{C}_1, \mathbb{C}_2) \equiv (\mathbb{C}_1 + j\mathbb{C}_2) &\equiv \left\{(\mathbb{R}_{11} + i\mathbb{R}_{12}) + j(\mathbb{R}_{21} + i\mathbb{R}_{22})\right\} \\ &= (\mathbb{R}_{11} + i\mathbb{R}_{12} + j\mathbb{R}_{21} - k\mathbb{R}_{22}) \ , \end{aligned} \tag{4.5.9}$$

in the usual 123 basis. Note the appearance of the minus sign. To avoid this we shall choose to put the basis element to the right of the coefficient. (Although we could have used j in (4.5.8).) So our understanding of how we have derived an octonion from two quaternions is, symbolically,

$$\begin{aligned} \mathbb{O} \equiv (\mathbb{H}_1, \mathbb{H}_2) \equiv (\mathbb{H}_1 + \mathbb{H}_2 l) &\equiv \left\{(\mathbb{C}_{11} + \mathbb{C}_{12} j) + (\mathbb{C}_{21} + \mathbb{C}_{22} j) l\right\} \\ = \Big\{\Big[(\mathbb{R}_{111} + \mathbb{R}_{112} i) + (\mathbb{R}_{121} + \mathbb{R}_{122} i) j\Big] &+ \Big[(\mathbb{R}_{211} + \mathbb{R}_{212} i) + (\mathbb{R}_{221} + \mathbb{R}_{222} i) j\Big] l\Big\} \ . \end{aligned} \tag{4.5.10}$$

Further, we choose the convention that we evaluate innermost brackets first. It is this we are taking to be our octonion. Therefore we are implicitly taking $ij = k$ (i.e. 123), $il = m$ (i.e. 145), $jl = n$ (i.e. 246) and $(ij)l = kl = o$ (i.e. 347). The only admissible basis with those assignments is $\begin{vmatrix} 123 \\ 246 \\ 365 \\ 451 \\ 572 \\ 617 \\ 734 \end{vmatrix}$. Changing the signs of various coefficients of ${}^{23}X$ is equivalent to choosing different implicit bases.

Calculating the right hand side of (4.4.6) in this basis gives

$$
\begin{aligned}
{}^{23}TXXXX &\equiv T_{\mu\nu\rho\sigma}\, {}^{23}X^{\mu}_{,0}\, {}^{23}X^{\nu}_{,1}\, {}^{23}X^{\rho}_{,2}\, {}^{23}X^{\sigma}_{,3} \\
&= (t^2 + r^2)\left[9(t^2 + r^2) + 4\right]\left[(3t^2 - r^2)^2 - (2t)^2\right] . \qquad (4.5.11)
\end{aligned}
$$

Comparing with (4.5.7) we see that the only difference is a single sign.

For the case of a (2,5) quot we have

$$
{}^{25}X = \left[(t^5 - 10t^3r^2 + 5tr^4),\ (5t^4 - 10t^2r^2 + r^4)\underline{r}\ ,\ (r^2 - t^2),\ -2t\underline{r}\ \right] . \qquad (4.5.12)
$$

We find

$$
\sqrt{{}^{25}g} = (t^2 + r^2)\left[25(t^2 + r^2)^3 + 4\right]\left\{(5t^4 - 10t^2r^2 + r^4)^2 + (2t)^2\right\} , \qquad (4.5.13)
$$

while, in the chosen basis,

$$ {}^{25}TXXXX = (t^2+r^2)\left(25(t^2+r^2)^3+4\right)\left\{(5t^4-10t^2r^2+r^4)^2-(2t)^2\right\} . \tag{4.5.14}$$

These again only differ by a single sign. Similarily,

$$ \sqrt{{}^{34}g} = (t^2+r^2)^2\left(16(t^2+r^2)^2+9\right)\left\{\left[4t(t^2-r^2)\right]^2 \oplus (3t^2-r^2)^2\right\} , \tag{4.5.15}$$

and

$$ \sqrt{{}^{35}g} = (t^2+r^2)^2\left(16(t^2+r^2)+9\right)\left\{(5t^4-10t^2r^2+r^4)^2 \oplus (3t^2-r^2)^2\right\} , \tag{4.5.16}$$

where $\oplus$ identifies the offending sign.

```
%SDHOB35              Self-Dual (H;O)-Brane (3,5) Quot                       24/7/89
%_______________________________________________________________________________
OPERATOR X$ OPERATOR S$ DEPEND X,S(1),S(2),S(3),S(4)$
OPERATOR XDOTX$ XDOTX:=S(2)**2+S(3)**2+S(4)**2$
OPERATOR VECF$ VECF:=5*S(1)**4-10*XDOTX*S(1)**2+XDOTX**2$
X(1):=S(1)**5-10*XDOTX*S(1)**3+5*S(1)*XDOTX**2$% \
X(2):=VECF*S(2)$                                          %  :  5
X(3):=VECF*S(3)$                                          %  : K
X(4):=VECF*S(4)$                                          % /     (ScU,VecU,ScV,VecV)
X(5):=3*S(1)*XDOTX-S(1)**3$                               % \
X(6):=(XDOTX-3*S(1)**2)*S(2)$                             %  :   3
X(7):=(XDOTX-3*S(1)**2)*S(3)$                             %  : -K
X(8):=(XDOTX-3*S(1)**2)*S(4)$                             % /-------------------------
MATRIX Xma(8,4)$ FOR m:=1:8 DO FOR a:=1:4 DO Xma(m,a):=DF(X(m),S(a))$
MATRIX Gab(4,4)$ Gab:=(TP Xma)*Xma$
S(1):=t$ S(2):=x$ S(3):=y$ S(4):=z$ %--------------------------------------------
ARRAY Eabcd(4,4,4,4)$ FOR a:=1:4 DO FOR b:=1:4 DO FOR c:=1:4 DO FOR d:=1:4 DO
        Eabcd(a,b,c,d):=(1/12)*(b-a)*(c-b)*(d-c)*(c-a)*(d-b)*(d-a)$
MATRIX Bs(7,3)$ %************* B A S I S ************************************
   Bs:=MAT((1,2,3),(2,4,6),(3,6,5),(4,5,1),(5,7,2),(6,1,7),(7,3,4))$   %*******
ARRAY Cijk(7,7,7)$ FOR k:=1:7 DO
  <<Cijk(Bs(k,1),Bs(k,2),Bs(k,3)):=1; Cijk(Bs(k,2),Bs(k,3),Bs(k,1)):=1;
    Cijk(Bs(k,3),Bs(k,1),Bs(k,2)):=1; Cijk(Bs(k,1),Bs(k,3),Bs(k,2)):=-1;
    Cijk(Bs(k,2),Bs(k,1),Bs(k,3)):=-1;Cijk(Bs(k,3),Bs(k,2),Bs(k,1)):=-1>>$
%============  Hijkl = DilDjk - DikDjl + CijhCklh  ========================
MATRIX Dij(7,7)$ FOR a:=1:7 DO Dij(a,a):=1$
ARRAY DijDkl(7,7,7,7)$ FOR i:=1:7 DO FOR j:=1:7 DO FOR k:=1:7 DO FOR l:=1:7 DO
            DijDkl(i,j,k,l):=Dij(i,j)*Dij(k,l)$
ARRAY CijhCklh(7,7,7,7)$ FOR i:=1:7 DO FOR j:=1:7 DO FOR k:=1:7 DO FOR l:=1:7 DO
         CijhCklh(i,j,k,l):=FOR h:=1:7 SUM Cijk(i,j,h)*Cijk(k,l,h)$
ARRAY Hijkl(7,7,7,7)$ FOR i:=1:7 DO FOR j:=1:7 DO FOR k:=1:7 DO FOR l:=1:7 DO
       Hijkl(i,j,k,l):=DijDkl(i,l,j,k)-DijDkl(i,k,j,l)+CijhCklh(i,j,k,l)$
%===  Tmnrs = Hijkl ; T1nrs = Tmn1s = Cijk ; Tm1rs = Tmnr1 = -Cijk ==========
ARRAY Tmnrs(8,8,8,8)$ FOR i:=1:7 DO FOR j:=1:7 DO FOR k:=1:7 DO FOR l:=1:7 DO
                 Tmnrs(1+i,1+j,1+k,1+l):=Hijkl(i,j,k,l)$
                FOR i:=1:7 DO FOR j:=1:7 DO FOR k:=1:7 DO
 <<Tmnrs(1,1+i,1+j,1+k):=Cijk(i,j,k)$ Tmnrs(1+i,1,1+j,1+k):=-Cijk(i,j,k)$
   Tmnrs(1+i,1+j,1,1+k):=Cijk(i,j,k)$ Tmnrs(1+i,1+j,1+k,1):=-Cijk(i,j,k)>>$
ARRAY TXXX(8,4,4,4)$ FOR m:=1:8 DO FOR b:=1:4 DO FOR c:=1:4 DO FOR d:=1:4 DO
    TXXX(m,b,c,d):=FOR n:=1:8 SUM FOR r:=1:8 SUM FOR s:=1:8 SUM
            Tmnrs(m,n,r,s)*Xma(n,b)*Xma(r,c)*Xma(s,d)$
MATRIX Pam(4,8)$ FOR a:=1:4 DO FOR m:=1:8 DO Pam(a,m):=(1/6)*FOR b:=1:4 SUM
    FOR c:=1:4 SUM FOR d:=1:4 SUM Eabcd(a,b,c,d)*TXXX(m,b,c,d)$
TXXXX:=FOR m:=1:8 SUM FOR n:=1:8 SUM FOR r:=1:8 SUM FOR s:=1:8 SUM
         Tmnrs(m,n,r,s)*Xma(m,1)*Xma(n,2)*Xma(r,3)*Xma(s,4)$
MATRIX PaXb(4,4)$ PaXb:=Pam*Xma$
MATRIX PaPb(4,4)$ PaPb:=Pam*TP Pam$
G:=DET Gab$ rG:=SQRT G$
% Checks ...........................................................
PaXb(1,2); PaXb(1,3); PaXb(2,3);
PaXb(1,1)-TXXXX; PaXb(1,1)-rG;
PaPb-G*1/Gab;
ON FACTOR;
TXXXX-rG; TXXXX; rG; END$
%_______________________________________________________________________________
```

Program 4.5.1 REDUCE program to calculate ${}^{35}TXXXX$ and $\sqrt{{}^{35}g}$ and to check equation of motion and constraints.

These results enable us to formulate the general empirical result that

$$\sqrt{{}^{pq}g} = (t^2 + r^2)^{p-1}\left(q^2(t^2+r^2)^{q-p} + p^2\right)\left\{ f(q)^2 + f(p)^2 \right\},$$

(4.5.17)

where

$$f(n) = \sum_{j=1}^{\lceil n/2 \rceil} (-1)^{j-1} \binom{n}{2j-1} t^{n-2j+1}\, r^{2(j-1)} . \qquad (4.5.18)$$

in which $\lceil n/2 \rceil$ means round up to the nearest integer. The corresponding expression for ${}^{pq}TXXXX$ requires only the last + sign in (4.5.17) to be changed. Note that $Det\ {}^{pq}g_{ab}$ is always a perfect square. Changing, for example, the signature of the space-time metric was found, in the cases considered, to destroy this necessary condition. Notice ${}^{pp}TXXXX = 0$. Also note, if $f(p) = 0$ then ${}^{pq}X$ is a solution of (4.4.6).

The formulae of (4.5.17 & 18) enabled us to correctly predict

$$\sqrt{{}^{27}g} = (t^2 + r^2)\left(49(t^2+r^2)^5 + 4\right)\left\{(7t^6 - 35t^4r^2 + 21t^2r^4 - r^6)^2 \oplus (2t)^2\right\} .$$

(4.5.19)

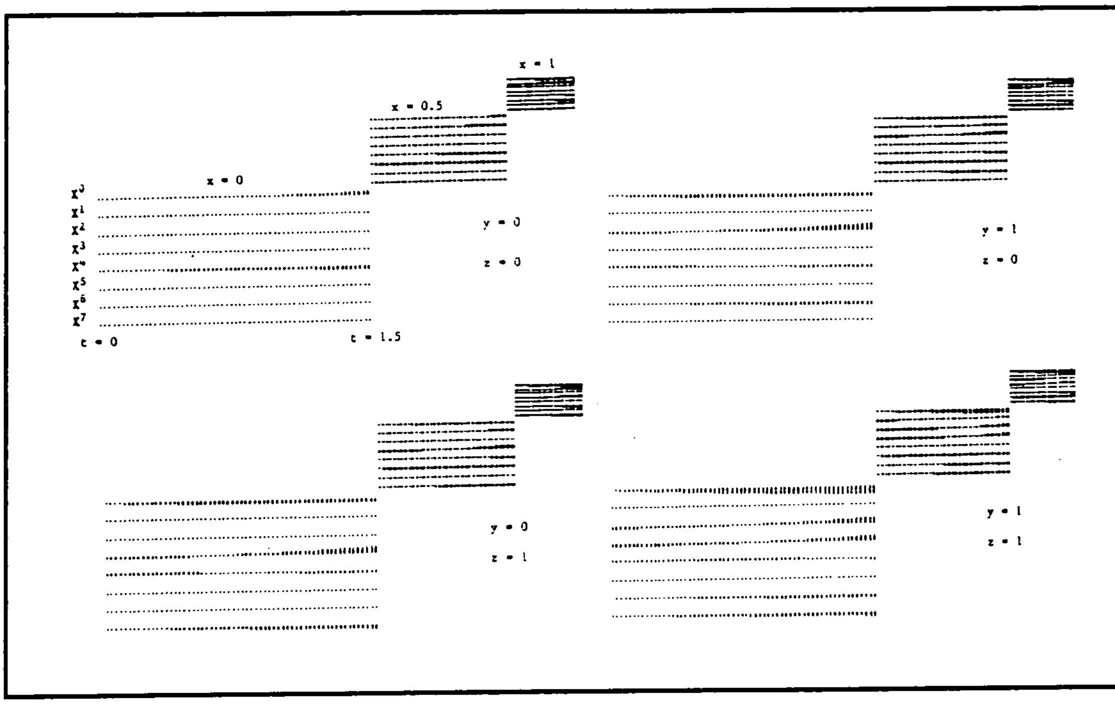

Figure 4.5.1 A presentation of the (2,3) quot of equation (4.5.6) using a general program introduced in [10], intended for data of an arbitrary number of dimensions.

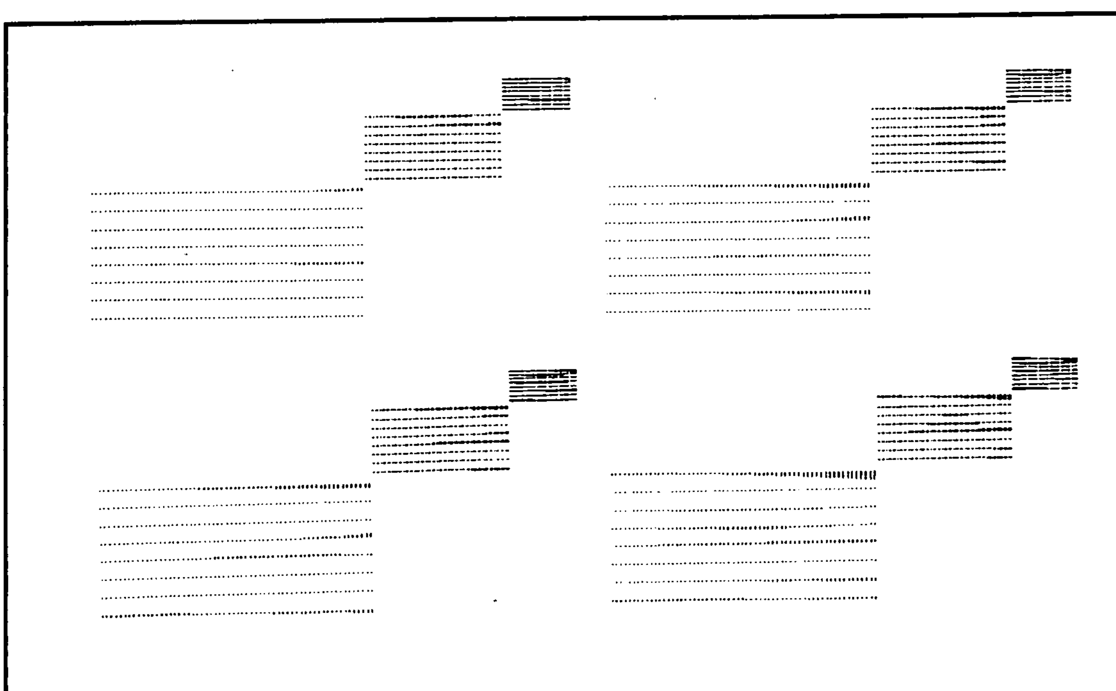

Figure 4.5.2 A presentation of a (3,4) quot.

Obviously one would hope that changing to another basis, or taking the anti-self-dual version of $T_{\mu\nu\rho\sigma}$, or rearranging the rows and columns of $X^{\mu}{}_{,a}$, or multiplying some of the rows and columns of $X^{\mu}{}_{,a}$ by -1 , all of which leave $\sqrt{g}$ unchanged, would enable one to alter the offending sign in *TXXXX*. We argue that this is not possible. Firstly note that all of these variables are absorbed in the 'change of basis' variable.

Take ${}^{23}X^{\mu}{}_{,a}$ and write it out explicitly.

$$
{}^{23}X^{\mu}{}_{,a} = \begin{pmatrix}
3t^2 - 3r^2 & -6tx & -6ty & -6tz \\
6tx & 3t^2 - 3x^2 - y^2 - z^2 & -2xy & -2xz \\
6ty & -2xy & 3t^2 - x^2 - 3y^2 - z^2 & -2yz \\
6tz & -2xz & -2yz & 3t^2 - x^2 - y^2 - 3z^2 \\
-2t & 2x & 2y & 2z \\
-2x & -2t & 0 & 0 \\
-2y & 0 & -2t & 0 \\
-2z & 0 & 0 & -2t
\end{pmatrix} . \tag{4.5.20}
$$

${}^{23}TXXXX$ picks one entry from each column for the non-zero components of $T_{\mu\nu\rho\sigma}$ and multiplies them together with a ± 1 in front depending upon the basis chosen. Notice that $\sqrt{{}^{23}g}$ in (4.5.7) contains terms $+16t^4$, $+4x^6$, $+4y^6$ and $+4z^6$. There is only one way to obtain these terms in ${}^{23}TXXXX$:

$$+16t^4 \;\Rightarrow\; T_{4567} = +1 \qquad \therefore\; H_{4567} = +1\ , \tag{4.5.21}$$

$$+4x^6 \;\Rightarrow\; T_{5423} = -1 \qquad \therefore\; H_{4523} = +1\ , \tag{4.5.22}$$

$$+4y^6 \;\Rightarrow\; T_{6143} = -1 \qquad \therefore\; H_{4163} = +1\ , \tag{4.5.23}$$

$$+4z^6 \;\Rightarrow\; T_{7124} = -1 \qquad \therefore\; H_{4127} = +1\ . \tag{4.5.24}$$

Using (4.3.7), (4.5.21 & 22) imply that $C_{451} = C_{671} = C_{231}$ and (4.5.23 & 24) imply that $C_{415} = C_{635} = C_{275}$. We require an admissible basis to exist containing $\begin{vmatrix} 451 \\ 671 \\ 231 \\ 653 \\ 257 \\ \cdots \\ \cdots \end{vmatrix}$ which of necessity requires one of the unknown rows to be 437 or 473 because both missing rows must contain a 4 and the remaining four elements must be taken from 2,3,6 and 7 since every number has to appear three times in total. Now, rotating the rows into standard form in which there is no repetition in the columns, we find, fixing the first row as 145, only two possibilities, $\begin{vmatrix} 145 \\ 716 \\ 231 \\ 653 \\ 572 \\ \cdots \\ \cdots \end{vmatrix}$ and $\begin{vmatrix} 145 \\ 671 \\ 312 \\ 536 \\ 257 \\ \cdots \\ \cdots \end{vmatrix}$ both of which have 4,3 and 7 in the same column. Therefore it is impossible to add 437 or 473 as a new row without introducing repetition in the central column. Therefore no basis can make (4.5.11) equal to (4.5.7).

Thus we have shown that the infinite set of (p,q) quots very nearly solve (4.4.6), but for a single sign in (4.5.17). There are many possible ways of altering terms on the left or the right hand side of (4.4.6) but there seems to be no obvious way of correcting the problematic sign. The aim is to find a solution to the 32 equations of

(4.4.5). We have presented an algorithm which is so close to solving (4.4.6) that we believe there is a way to make it solve (4.4.6) and (4.4.5).

To see how close ${}^{23}X$, for example, can come to solving (4.4.5), we use program 4.5.1 to compare $\tilde{P}^{a\mu}$ of equation (4.2.1) with $P^{a\mu}$ of (4.1.3), which is exactly equivalent to (4.4.5). We find that, overall, there is good agreement in the admissible clockwise basis $\begin{vmatrix} 123 \\ 246 \\ 365 \\ 437 \\ 514 \\ 671 \\ 752 \end{vmatrix}$. For example, in the 24 terms in the polynomial of $\tilde{P}^{11}$ and in the polynomial of P^{11} only a single term is different:

$$\begin{aligned}\tilde{P}^{11} = {} & 27t^6 - 45t^4x^2 - 45t^4y^2 - 45t^4z^2 + 12t^4 + 21t^2x^4 + 42t^2x^2y^2 \\ & + 42t^2x^2z^2 - 12t^2x^2 + 21t^2y^4 + 42t^2y^2z^2 - 12t^2y^2 \\ & + 21t^2z^2 \oplus 4t^2z^2 - 3x^6 - 9x^4y^2 - 9x^4z^2 - 9x^2y^4 \\ & - 18x^2y^2z^2 - 9x^2z^4 - 3y^6 - 9y^4z^2 - 9y^2z^4 - 3z^6 \, .\end{aligned} \tag{4.5.25}$$

The only difference between (4.5.25) and P^{11} is that $\oplus$ is a minus sign in P^{11} and the coefficient is 12 not 4. Similarily, the only difference between $\tilde{P}^{12}$ and P^{12} is that $\tilde{P}^{12}$ has one extra term, $8tyz$. The other 11 terms agree exactly. The same happens for $\tilde{P}^{13}$ and P^{13}. $\tilde{P}^{13}$ has one extra term, $-8txz$, the other 11 agree. For the other 29 equations, there is again overall agreement, but it is not exact. We believe that there is some way to make quots solve (4.4.6), and that when this is accomplished then they will also satisfy (4.4.5) exactly.

We also believe our expression for the right hand side of (4.4.6) rather than the left hand side because, in general, it factorizes more than the left hand side.

If this infinite hierarchy of self-dual (4;8)-brane solutions can be made to work properly, then it might yield an exact mathematical classification of 'arrangements' analogous to the torus knot classification of paths. The corresponding (3;7)-brane solutions might, similarily, classify foams.

§4.6 References for Chapter 4

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