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Non-perturbative studies of gauge theories: their renormalisation and hierarchies of scales

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NON-PERTURBATIVE STUDIES

OF GAUGE THEORIES:-

THEIR RENORMALISATION AND

HIERARCHIES OF SCALES

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ABSTRACT

Two aspects of gauge theories are studied in the non-perturbative regime; firstly, using a set of pre-determined, approximate renormalised Feynman rules, the divergent parts of the $O(\alpha_s)$ virtual graphs of the process $e^+e^- \to q\bar{q}$ are determined to explicitly test whether multiplicative renormalisation is preserved by these rules.

The calculation is performed using dimensional regularisation in $2(2 - \epsilon)$ dimensional Euclidean space, where the divergences appear as $1/\epsilon^n$ poles as $\epsilon \to 0$. Though the corrections to both the fermion-photon vertex and to the final state self energy are shown to have $1/\epsilon$ singularities, the coefficients of these are quite different. This mis-match in singular behaviour signals the breakdown of multiplicative renormalisation, which, in turn, implies that the physical process is not guaranteed to be finite and the rules used are inadmissable as a set of consistent Feynman rules.

The second investigation is to solve numerically the Schwinger-Dyson equation for the fermion propagator in QED in three (Euclidean) dimensions. The aim being to study the scale of dynamical mass generation.

To control infrared divergences the 1/N (flavour) expansion is used and to close the equation vertex and gauge propagator are approximated by their lowest order forms in 1/N. Numerical solutions for the fermion self energy and wavefunction renormalisation are determined. The latter is found not to be suppressed by $O(1/N)$, contrary to the expectation of Appelquist et al, and the coupled equation for these functions has to be solved.

It is then found that a mass scale is dynamically generated and that a scale hierarchy between it and the dimensionful coupling, α, of many orders of magnitude exists (typically $m/\alpha \sim 10^{-7}$ for $N_F = 5$). Thus showing, albeit in a simplified 'toy' model, how large scale hierarchies can 'naturally' occur in gauge theories with spontaneously broken symmetries.

This Thesis is dedicated to my parents

DECLARATION

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

The research described in chapters V and VI has been carried out in collaboration with Dr. Mike Pennington and has been published as a Brookhaven National Laboratory preprint;

'Hierarchy of scales in three dimensional QED ', February 1988.

(Submitted to Phys. Rev.)

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CHAPTER I INTRODUCTION

I.1 INTRODUCTION

The underlying theory on which the work in this thesis is based is Quantum Chromodynamics (QCD) [Ref.I.1]. This is the gauge theory that is generally accepted as describing the strong, or colour, force, responsible for the interaction of quarks, believed to be the constituents of the particles known as Hadrons. Together with the Weinberg-Salam model of the electroweak interaction [Ref.I.2], it is the full extent of our experimentally testable knowledge of the quantum world of elementary particles. Figure [I.1] graphically represents this knowledge and the role played by QCD. The solid lines indicate generally accepted, state-of-the-art, physics, and the dashed lines represent conjecture or experimentally unproven theory. A horizontal left to right axis can be viewed as a (non-linear) increase in energy, and correspondingly an, approximate chronological history.

electrostatics

Maxwell's
theory

magnetism

Weinberg-Salam
model

Weak interaction

Grand Unified
Theories

Strong interaction——QCD

Theory of
everything
(Superstrings?)

Newton   Einstein
Classical gravity ——————— Quantum gravity ——————
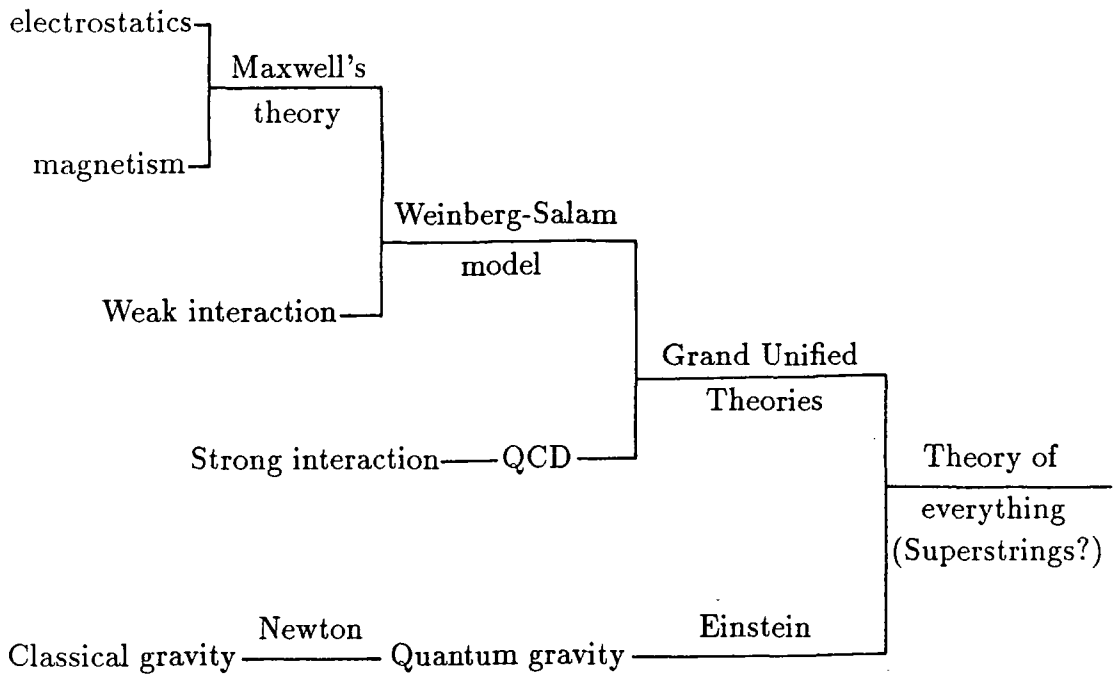
Figure I.1   The four forces of nature

1

As the figure shows, in the strive to get a 'theory of everything' to explain the four known forces of nature, QCD is essentially an experimentally substantiated theory of the strong interaction in isolation from the other three interactions. (Although some grand unified theories (GUT's) unifying QCD with the electroweak interaction do exist, for example the SU(5) model [Ref.I.3], but so far all have failed experimental verification). Consequently we will have little to say about these other interactions and will concentrate on the strong, or hadronic, sector in isolation.

The work presented in this thesis will be placed within the framework of QCD in later chapters. In this chapter we present a review of QCD; outlining the main turning points in its historical development, and the features and failings that have led to its sucesses and future developments. From here we are in a position to tackle the aims of this thesis.

I.2 THE QUARK MODEL

QCD, as we shall see, is the theory of how the constituents of the hadrons, the quarks, interact. The development of this theory took several decades and was originally preceded by a non-dynamical model developed from a need to "book-keep" the growing number of 'elementary' particles, making no attempt to describe the interactions of the hadrons. This model became known as the Quark, or Parton model [Ref.I.4]. In the context of a full dynamical model of QCD, the Quark model set up the kinematics.

In the following sections we review the development of the Quark model illustrating why such a model was necessary and how it provided a simple account of all the known hadrons. The quark concept was introduced in 1964, but we start with a discussion of the pre-quark days to provide a motivation and background.

I.2.1 Pre-quark physics

Up until the 1930's the history of science had kept the number of elementary particles to around four. The earth, wind, fire and water of the Greek philosophers became, after many centuries of effort, the photon, electron, proton and neutron of early 20th century atomic physics. The photon was the carrier of Maxwell's electromagnetic field and the electron, proton and neutron were the constituents of atoms. The elementary particles were few in number and all had some sort of explanation.

Despite the subsequent introduction of three new particles: the positron (1932) as the anti-electron proposed by Dirac, the neutrino, postulated by Fermi as the carrier of the 'missing' momentum in the weak β-decay of the neutron, $n \rightarrow p + e^- + \bar{\nu}$, and the pions ($\pi^{\pm}, \pi^0$) predicted by Yukawa in 1935 as the mediators of the strong force, order still prevailed and seven fundamental particles was an acceptable number.

However, in 1947, two, far reaching discoveries were made. Firstly the penetrating particles observed in cosmic rays, believed to be Yukawa's pions were shown to decay into electrons and neutrinos: a decay inconsistent with pions. The muons, as they became known, were a new type of particle. 207 times as heavy as the electron, but otherwise duplicating the role of the latter. A redundant, free gift from nature? A still more confusing event of the same year was the discovery of the "V-events" in cosmic rays [Ref.I.5]. These were interpreted as the decay of still heavier (than the neutron) particles. Two such events were originally seen and these 'strange' new particles were collectively called the strange hyperons.

By now three of the ten known elementary particles (muons and hyperons) were unexplained. The confusion was complicated, in 1953, when the first of the particle accelerators was built. Now, for the first time, subatomic particles could be produced and studied in laboratories, independent of cosmic rays

[Ref.I.6]. Another hyperon was subsequently seen at the accelerator and the decay modes of these strange particles was established as,

$$\Lambda \rightarrow p + \pi^-$$

$$\Sigma^+ \rightarrow p + \pi^0 \quad \text{or} \quad n + \pi^+$$

$$\Xi \rightarrow \pi^- + \Lambda \rightarrow \pi^- + p + \pi^-.$$

Despite this progress, the hyperons remained mystery particles appearing to obey strange physical laws:

I.2.2 Strange physics

The main mystery of the hyperons, and the similar, but lighter, newly discovered kaons, or strange particles as they collectively became known, was that they were copiously produced but lived inordinate lengths of time ($\sim 10^{-9}$s) on the strong interaction time scale ($\sim 10^{-23}$s). The copious production indicating a strong interaction process should, by the principle of reversibility, also predict a strong interaction decay giving the strange particles lifetimes of the order of 10^{-20}s. Some unknown principle was suppressing the expected decays of these particles.

The problem was formalised by M. Gell-Mann and K. Nishijima by introducing a new additive quantum number called strangeness (S) [Ref.I.7]. Gell-Mann proposed, at the time in an ad-hoc way, that the strange particles had non-zero strangeness quantum number, and by demanding that the strong interaction conserves strangeness, the strong decay of the hyperons, for example, into non-strange nucleons and pions is forbidden. By allowing the weak interaction to violate strangeness enables the hyperons to decay weakly and thus accounts for the disparity in time scales of production and decay. Similar arguments would, however, also seem to forbid the strong production of these particles.

But, if the strange particles were always produced in pairs (associated production) [Ref.I.8] with each particle having opposite strangeness quantum number, the overall strangeness of a strong production interaction would be conserved. For example

$$\pi^+ + n \rightarrow K^+ + \Lambda$$
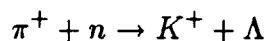
is an allowed strong interaction if (say) the K^+ is assigned S=+1, and Λ S=-1.

By introducing the strange quantum number and associated production, the disparity of the time scales of the strange particles was, at least formalised, if not fully explained. (The assignment of strangeness was carried out by hand, guided only by the need to forbid various reactions.)

Not only was this 'strange physics' presenting problems, but the number of elementary particles was increasing. With, the by now, handful of strange particles there were around twenty or so particles. It was begining to look doubtful if all could be truly elementary. At the very least some sort of order, or book-keeping was now needed. Again it was Gell-Mann who proposed the method, which proved to be the final step before the introduction of the quarks. To elucidate this ordering we must introduce the concept of isospin multiplets [ref.I.9]:

I.2.3   Particle multiplets

When the hadronic world was populated only by the neutron and proton, it was known that the strong interaction did not distinguish between these two particles. Only their electromagnetic properties allowed them to be separated. This symmetry of the strong interaction was expressed by putting the two particles into a particle doublet, collectively known as a nucleon:

$$N \equiv \begin{pmatrix} \mathrm{p} \\ \mathrm{n} \end{pmatrix}$$

The strong interaction treats the nucleon as a single entity and does not distinguish between the components. To accommodate the electromagnetic differences the doublet is assigned a new quantum number, Isospin, equal to I=1/2. The proton is then assigned I_3=+1/2, and the neutron I_3=-1/2. (I_3 is the third component of the isospin vector.) The isospin, therefore, is essentially a measure of the charge of the members of the doublet.

With the subsequent discovery of the pions ($\pi^{\pm}$, π^0), it was found that they too were indistinguishable by the strong interaction. Consequently, the pions were placed in a triplet of isospin I=1 with I_3= 1,0,-1 for the π^+, π^0, π^- respectively. Collecting particles into multiplets in this way provided a shorthand notation for the hadrons. Specifying the isospin and charge centre (the average charge) of a given multiplet completely specifies the charges of all the particles in the multiplet.

With the discovery of the strange particles similar multiplets were searched for. Gell-Mann realised that the heavy strange particles could also be fitted into multiplets, but with the charge centre shifted from the +1/2 centre of the neucleon doublet. Similarly, the light strange particles could be fitted into a multiplet, but this time with centre shifted relative to the zero centre of the pion triplet. In doing so the strange baryons (hyperons) and mesons (kaons) obeyed the empirical rule

$$S - \text{Baryons (Mesons)} = 2 \times \text{displacement of charge centre relative to } +1/2 \ (0).$$

(1.1)

Further to this rule, plots of I_3 against S for all particles of the same spin-parity showed up a deeper 'supermultiplet' order shown in [Fig.I.2], from which the idea of quarks was a direct consequence.

S

n   ⊢0   p

—— Σ⁻ ———————— Λ⁰Σ⁰ ———————— Σ⁺ —→ I_3

-1   1

Ξ⁻   ⊢-2   Ξ⁰

$J^p = 1/2^+$ baryon octet

S

Δ⁻   Δ⁰   ⊢0   Δ⁺   Δ⁺⁺

———— Σ⁻ ——— Σ⁰ ——— Σ⁺ ——— → I_3

-3/2   -1/2   1/2   3/2

Ξ⁻   ⊢-2   Ξ⁰

ω⁻ -3
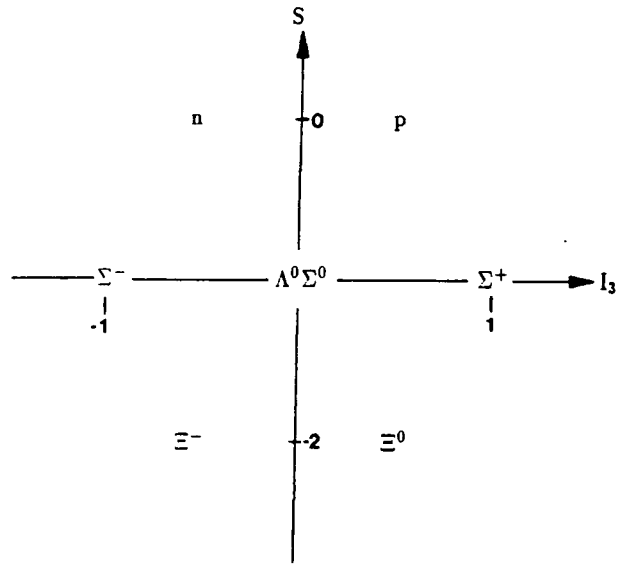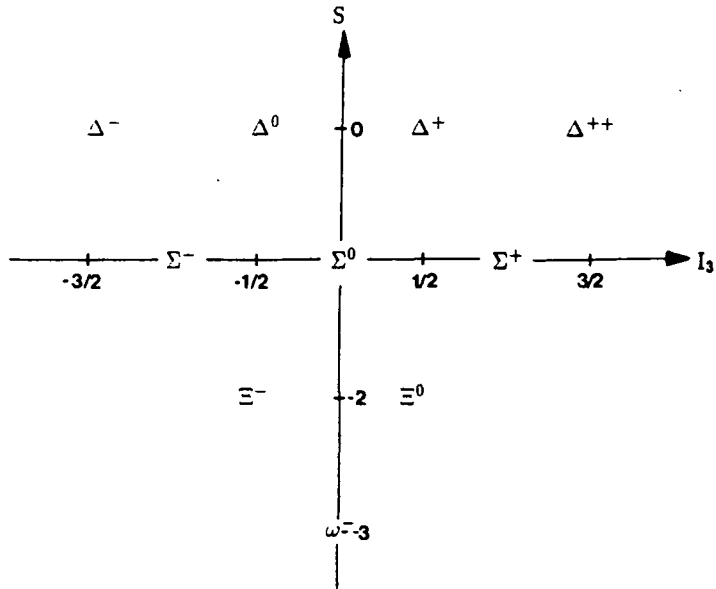
$J^p = 3/2^+$ baryon decuplet

S

—u ———————— d —→ I_3

-1/2   1/2

s -1
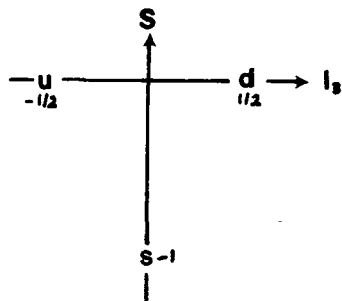
Fundamental quark flavour representation of SU(3)

Figure I.2   Particle multiplets

I.2.4 SU(3) quark representation

From the plots of [Fig.I.2] (and similar ones for the meson supermulti-plets) Gell-Mann and Ne'eman, in 1961 [Ref.I.10], realised that these supermul-tiplets were equivalent to the octet and decuplet representations of the group SU(3). However, at the time, the Ω^- in the $3/2^+$ decuplet had not been discov-ered. The SU(3) interpretation therefore had some predictive content, and the subsequent discovery of this particle gave considerable experimental credance to the idea.

The octet and decuplet representation of SU(3) are reducible. This im-plies that there is a fundamental representation out of which the reducible ones can be constructed [Ref.I.11]. Having recognised the group as SU(3), the 3 tells us that the fundamental matrix representation of the group is made of 3×3 ma-trices that therefore must act on a three component vector, or group space. The fundamental multiplet is therefore a triplet. Gell-Mann called the components of this vector, or triplet, quarks. In particular up, down and strange "flavours" as the three components [Ref.I.12]. The fundamental triplet is shown in [Fig.I.2].

The higher representations, observed in the hadron spectrum, were now considered as different combinations of the fundamental quark representation, generically denoted by q. In particular, the combinations

$$qqq \equiv 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

$$q\bar{q} \equiv 3 \otimes \bar{3} = 8 \oplus 1, \tag{1.2}$$

seemed to fit the observed baryon and meson multiplets respectively. The baryons were proposed to contain three quarks, and the mesons a quark- anti-quark pair. All the known hadrons could then be accounted for as these qqq or $q\bar{q}$ combinations of the three u,d,s quarks.

To summarise the development of the quark model (as it became known)

so far; the increase in number of 'elementary' hadrons found in cosmic rays or particle accelerators, needed explaining. The unusual decay behavior of the strange particles lead to the introduction of the strange quantum number, S. Analogous to the isospin nucleon doublet and pion triplet of the non-strange particles, the hyperons and kaons were also grouped into multiplets, but with their charge centre shifted relative to the non-strange multiplets. Equation [1.1] was then proposed to define the strange quantum number of the multiplets. A deeper order amongst the particles was revealed when the third component of isospin was plotted against strangeness, as shown in [Fig.I.2], and the representations of SU(3) were recognised. The fundamental representation of this group was said to act on the three (u,d,s) quark flavours. Although not considered real particles, the quarks provided a remarkably economical, ordered way of accounting for all the observed hadrons.

However there were still some unanswered problems of this scheme:

i) Why were no free quarks observed, and why were only qqq, $q\bar{q}$ combinations realised in nature?

ii) To explain the observed hadrons quarks must be fermions (1/2 integer spin). Measurement of the Ω^-, with quark content SSS, showed it to be in the zero angular momentum state. Consequently all the three quarks would then be in the same quantum state, explicitly violating the inviolate Fermi-Dirac statistics observed by all fermions.

iii) As yet the quark model was just a neat mathematical way of representing the observed particles. It had no deeper theoretical footing. Indeed, the quarks, at this time, were not thought of as real particles at all, they were merely mathematical abstractions.

The first two of these problems were solved, in 1964, at the expense of introducing a new quantum number (that increased the number of quarks three-

fold) and another ad-hoc rule governing the allowed observable particles. The third problem could then be addressed, and armed with some new experimental data, discussed in section I.2.6, from which our goal, QCD, was developed. We start, in the next section with the introduction of new quantum number.

I.2.5   Quarks are Red.....Blue.....or Green!

The new quantum number was the concept of colour [Ref.I.13], introduced by O. Greenberg in 1964. If each quark flavour could 'exist' in three different colour states, Red, Blue and Green, say, then the three strange quarks of the Ω^- could be accounted for as each having a different colour. Therefore, each would occupy a different quantum state and the Ω^- would be consistent with Fermi-Dirac statistics.

Another piece of experimental evidence for the existence of three colours was the so-called R-ratio,

$$R \equiv \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

At high energy this ratio is expected to be constant and proportional to the sum of the squared quark charges,

$$R \propto \sum_q e_q^2.$$

For three (and later 4 and 5) quark flavours, using this relation, R was found to be a factor of 3 too small compared to the experimentally measured value. By proposing three quark colours, the number of each quark flavour is tripled and consequently so is R, to agree with experiment. (It should be pointed out here that this argument is only valid if the quarks are real constituents inside the hadrons. This was only later shown to be so, but we mention this argument for colour here for completeness.)

Tripling the number of quarks by introducing the concept of colour would seem, in turn, to increase the number of hadrons. Each hadron with red quarks would have corresponding blue and green partners. These colour partners were not, however, seen, or wanted to describe the observed hadrons. The ad-hoc rule invoked to explain the missing extra hadrons was that only colour singlets (in the language of group theory) are observable. Baryons were made colour singlets, or 'colourless', by having one of each colour, qqq = red, blue, green='white'. While the mesons were colourless by having three colour anti-colour pairs, one of each colour, again giving 'white', or a colour singlet.

By introducing the concept of colour to explain the statistics of the Ω^- we were then forced to introduce the, ad-hoc, colour singlet rule. This in turn then answers the first of the problems of the quark concept; why no free quarks were seen. Coloured quarks in isolation violate the 'only colour singlets are observable' rule. The qqq, $q\bar{q}$ combinations of quarks observed are also explained as the simplest possible colourless colour singlet combinations.

Analogous to the SU(3) of flavour proposed by Gell-Mann, a similar SU(3) of colour, was now proposed to formalise the rule [Ref.I.14]. Here the fundamental representation did not act on flavour (u,d,s) but a colour vector, (R,B,G). If the q, $\bar{q}$ labels in [equ.1.2] are now taken to be colour labels, all the physically observed particles are in the singlets of the decomposition of the RHS. The higher (8,10) representations are not observable. Interchange of the colours, or equivalently a rotation in colour space, maps a given singlet particle into itself, in contrast to rotating in flavour space that maps one member of a multiplet into another, different member. Rotating in colour space has no physically observable effects. The colour symmetry is an exact, unbroken internal symmetry of the strong interaction. (The flavour symmetry is only approximate, the non-degenerate u,d,s masses (and charges) break the flavour symmetry and allows us to distinguish between particles in a given multiplet. It is the breaking of the

flavour symmetry that gives the rich variety to the hadron world.)

By introducing an exact SU(3) colour symmetry we have explained, or at least formalised the first two of the problems we set out of the quark concept. However, we have still relied upon the ad-hoc colour singlet rule. Further, at the time the quarks were also only thought of as mathematical abstractions, convenient to represent the real hadrons and were just a neat book-keeping system. However, all this changed in the late 1960's as the result of a classic series of experiments performed at the Stanford Linear Accelerator (SLAC), that were to completely change the interpretation of the quark idea, which we discuss in the next section.

I.2.6 Deep inelastic scattering (DIS) data

The particle accelerators that, as always, set the pace for theoretical advances usually provide the evidence to either knock-down or substantiate a theory. The DIS experiments at SLAC [Ref.I.15] provided the latter for the quark model. The experiments, schematically shown in [Fig.I.3], collided an electron into a proton. The latter broke up on impact and created a shower of hadronic debris. The electron, however, being a truly elementary particle, survived the impact and was used as a 'tag' to measure the scattering proccess. The electron and proton interact via the exchange of a virtual photon. If the latter has a small enough wavelength (high enough energy) it will probe the internal structure, if any, of the proton. By using a photon probe, the internal structure of a hadronic particle is being investigated by its electromagnetic properties.

The revolutionary results of the SLAC experiments were that the scattering angle, θ, of the incident electron was consistent with the scattering off free, spin-half, point-like objects within the proton. In analogy to the famous Rutherford α-particle experiments fifty years earlier, in a significant number of events the incident electron was scattered through a large angle. This was inter-

preted as the electron scattering off a hard scattering centre within the proton. The events where the scattering angle was small were similarly interpreted as the electron passing through the proton without encountering one of the hard scattering centres.
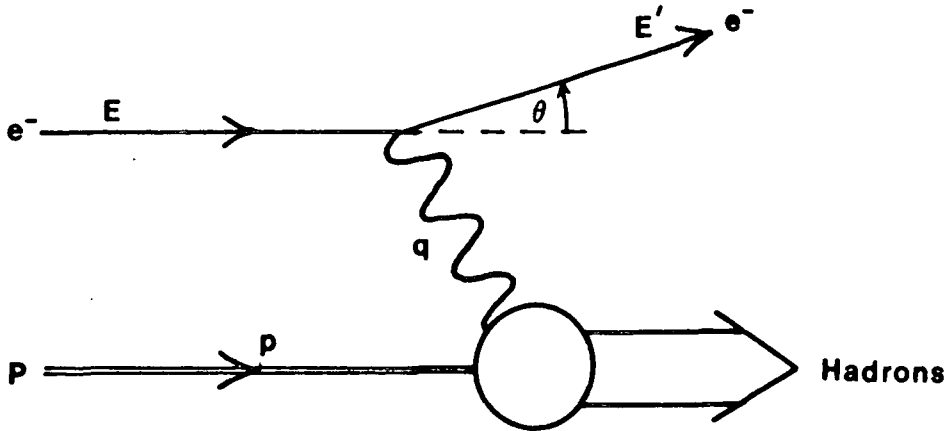
Figure I.3   Electron-proton DIS event

The actual evidence for point-like scattering inside the proton came from the observed Bjorken scaling [Ref.I.16] obeyed by the structure functions of the proton. The latter are functions to describe the interaction of a photon with a proton. As no theory exists where |proton> and |Hadronic debris> are states, the exact form of the proton structure function cannot be theoretically determined. However, if the photon interacts with free point-like objects within the proton the high energy behaviour of the structure functions, W_1 and W_2 can be shown to be, [Ref.I.16]

$$\nu W_2 = \sum_i Q_i^2$$

$$W_1 = \frac{\nu W_2}{2Mx}$$

13

where we adopt the standard notation,

$$M \quad = \quad \text{proton mass}$$

$$Q_i = \text{point} - \text{like (quark) charge}$$

$$\nu = E - E' = p \cdot q / M$$

$$x = -\frac{q^2}{2M\nu}. \quad (0 \leq x \leq 1).$$

The important point of these equations is that the combination νW_2 is independent of momentum, q^2, and is said to exhibit Bjorken scaling. This behaviour was observed in the experiments.

The parton model of Feynman [Ref.I.17] interpreted the free point-like scattering centres as the quarks that essentially instantaneously interacted with the photon, and only on much greater time scales did the subsequent break-up and hadronisation of the proton occur. As such, these two processes could be treated independently; the proton-photon interaction being described in terms of the free quarks interacting with the photon, followed by, the unexplained, hadronisation.

For the first time there was strong experimental evidence that the quarks really did exist inside a proton. They were no longer mathematical entities and, although still not observed in isolation, were now thought of as 'real' particles.

Now the quarks where considered as real constituents of hadrons, all known hadrons (by now in the hundreds) could be explained in terms of the 3(colour)×3(flavour) quarks as fundamental particles. The later addition of two new quark flavours to explain the new heavy meson specrum [Ref.I.18] (The charm (1974) and the bottom (1977) quarks) increased the number of quarks to 3×5=15, but were easily accommodated into the quark model by introducing charm and bottom quantum numbers analogous to the strangeness. The as yet

undiscovered, but predicted, top quark, will likewise be readily accommodated into the model when (if) it is found.

For completeness, [Fig.I.4] shows the quark property table as it exists today. From this table all the known hadrons and their properties can be constructed as qqq or $q\bar{q}$ combinations of these quarks. (In the table C and B are the charm and bottom quantum numbers of the quarks.)

Flavour	Spin	Charge	I_3	S	C	B
up	1/2	+2/3	1/2	0	0	0
down	1/2	-1/3	-1/2	0	0	0
strange	1/2	-1/3	0	-1	0	0
charm	1/2	+2/3	0	0	1	0
bottom	1/2	-1/2	0	0	0	-1

Figure I.4   **Quark property table**

Despite having made such great progress in describing the rich structure of the hadronic spectrum, the parton model still contains ad-hoc rules. A still more serious short-fall is that it is a purely kinematic model, making no attempt to describe the dynamics, or interactions of the quarks. As the new fundamental constituents of the hadrons the question of the quark dynamics is clearly an important one. To attempt to describe the dynamics the quark model discussed above must be embedded into space-time. It is to this problem that we now turn in the following section.

I.3 DYNAMICAL QUARKS

In this section we review the development of a dynamical quark theory. Having, in the previous, section sketched how quarks became accepted as the fundamental constituents of hadrons, we are now in a position to try and con-

struct a field theory of the quark fields. In doing so we hope to be able to answer some of the outstanding problems of the quark model, for example, the hadronisation and confinement of quarks. Although we should point out immediately, that neither of these questions have in fact been answered by the theory that has been developed, namely QCD. Nevertheless, the latter is still the generally accepted field theory of the strong interaction and has consequently been the subject of much study since its original conception by C.N. Yang and R. Mills in 1954 [Ref.I.19].

Throughout the development of the quark model, the description of the electromagnetic interaction as a gauge theory [Ref.I.20] (QED) was providing spectacular agreement with experiment. It seemed natural for the strong interaction to follow in the wake of such success. The gauge theory of colour, our goal ,QCD, was the result.

Before explicitly discussing the description of quarks in a gauge theory we outline in the following section, the general ideas of gauge theories.

I.3.1 Global and local symmetries

Galileo was the first person to realise the invariance of physical laws of nature. An observer on land and one on a moving ship would both deduce the same force (of gravity) by observing a cannon ball fall on the ship.

Ever since, invariance, or symmetries of the laws of nature have been one of the foundation stones of physics. Perhaps the most famous symmetry is that of space-time invariance embodied at the heart of Einstein's theory of general relativity. In words, this says that there is no prefered direction in space-time. Mathematically this is expressed as the invariance under the group of space-time rotations, the Lorentz group.

The nucleon isospin doublet discussed in section I.2.3 is an example of a symmetry of the strong interaction. The inability of this interaction to dis-

tinguish between the two components of the doublet implies that a rotation in isospin space, effectively mixing the two (n,p) states, will have no physically observable consequences.

In field theory such symmetries are expressed as the invariance of the lagrangian under group transformations of the fields. The simplest example of such a theory is QED, the field theory of a free Dirac elecrton, represented by a field $\Psi(x)$. The appropriate lagrangian here is:

$$L_0 = \bar{\Psi}(x)(i\gamma_\mu \partial_\mu - m)\Psi(x) \qquad (1.3)$$

Clearly L_0 is invariant under a constant phase change of the field,

$$\Psi(x) \rightarrow \Psi'(x) = e^{i\alpha}\Psi(x). \qquad (1.4)$$

However, L_0 is only invariant if α is a constant, having the same values at all space-time points. The symmetry is therefore said to be global. Now, having seen the global symmetry of L_0, but with nothing to determine the actual, global, value of α, we should therefore be able to demand that α can be chosen at will, at all different space-time points. In making α a function of x the symmetry a said to be a local symmetry [Ref.I.21].

From the derivative term $\partial_\mu \Psi(x)$ of L_0 we immediately see that the lagrangian in [equ.1.3] does not have this local symmetry. However we can impose it by defining the covariant derivative, D_μ, to replace the standard space-time derivative, ∂_μ. In particular,

$$D_\mu \Psi(x) \equiv (\partial_\mu + ieA_\mu(x))\Psi(x).$$

where e is a free parameter, and $A_\mu(x)$ is a new vector field, called the gauge field. Re-defining L_0 as

$$L'_0 = \bar{\Psi}(x)\gamma_\mu(\partial_\mu + ieA_\mu(x))\Psi(x) - m\bar{\Psi}(x)\Psi(x) \tag{1.5}$$

we can impose local invariance of the lagrangian by defining the gauge field to transform under the phase change as:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{1.6}$$

By using the transformations of equations [1.4] and [1.6] the lagrangian of [equ.1.5] is locally, or gauge, symmetric and $\alpha(x)$ is a function of x.

As yet the new gauge field, $A_\mu(x)$, is not a real dynamical field since there is no kinetic term in L'_0 involving the derivatives of $A_\mu(x)$. However, using standard notation, we can add such a contribution to the lagrangian that is invariant under the field transformations as,

$$L'_0 \rightarrow \bar{\Psi}(x)(i\gamma_\mu\partial_\mu + ie\gamma_\mu A_\mu(x))\Psi(x) - m\bar{\Psi}(x)\Psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.7}$$

with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{1.8}$$

So, in [equ.1.7], with the new, dynamical gauge field we have recovered a locally invariant lagrangian. In doing so we have sketched QED, one of the simplest, and most successful example of the gauge principle that states:

Interactions of fields arise from the requirement that a theory is locally, or gauge invariant.

In the case of QED we see that the $e\bar{\Psi}(x)\gamma_\mu A_\mu(x)\Psi(x)$ term has coupled the electron field to the gauge field, with coupling strength e, explicitly introducing an electron interaction as a result of demanding gauge invariance.

As a simple gauge theory QED was extensively developed by Feynman and others [ref.I.20] with remarkable success. Such a theory can be readily applied to quarks. Simply $\Psi(x)$ can be taken as the quark field since both electrons and quarks are point like, spin-half fermions. From this substitution we immediately derive the electromagnetic properties of the quarks. However we are more interested in their strong interaction properties. Guided by the gauge principle we must find a new local symmetry of the strong interaction that will dictate the strong interaction dynamics of the quarks. This symmetry is the topic of the next section.

I.3.2 Gauging colour

We have already discussed two symmetries of the strong interaction; the approximate flavour symmetry (broken by the non-degenerate quark masses), and the exact colour symmetry. The exactness of the latter indicates that it is this, colour symmetry that is to be made local, or gauged, in analogy with the nucleon doublet, this local colour symmetry is the freedom to assign the constituents of a hadron colours on a local basis. All we need ensure is that a proton, for example, has one quark of each colour. The colour attributed to each quark is totally un-measurable, and therefore arbitrary.

We have already seen, in the quark model, that the colour symmetry group is SU(3). While the U(1) symmetry of QED was represented by the invariance under the transformation of [equ.1.4], SU(3) symmetry is represented by invariance under

$$\Psi(x) \rightarrow \Psi'(x) = e^{i\theta(x)_a \lambda_a/2}\Psi(x) \tag{1.9}$$

where the $\lambda_a/2$ are called the generators of SU(3) [Ref.1.11]. For SU(3) there are eight such generators. The standard representation is the set of eight 3×3

Gell-Mann matrices (given in most text books). As previously noted, the vector on which this representation acts is defined in terms of the three colours:

$$\Psi \equiv \begin{pmatrix} \text{Red quark} \\ \text{Blue quark} \\ \text{Green quark} \end{pmatrix}$$

This matrix representation embodies the crucial difference between the $SU(3)_{colour}$ and $U(1)_{electromagnetism}$. In the latter, two transformations commute, and $U(1)$ is said to be abelian. In $SU(3)$, however, two transformations do not commute since

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = i f_{abc} \frac{\lambda_c}{2}$$

where f_{abc} are the structure constants of $SU(3)$.

The important consequence of this non-commutivity of the group generators is that to maintain gauge invariance in the lagrangian of [equ.1.3] we must redefine the covariant derivative and gauge fields as: [Ref.I.22]

$$D_\mu \Psi(x) \equiv (\partial_\mu - ig\theta_a \frac{\lambda_a}{2} A_\mu)\Psi(x) \qquad (1.10a)$$

$$A_\mu^a(x) \rightarrow A_\mu^{a'}(x) = A_\mu^a(x) + f_{abc}\theta^b(x)A_\mu^c(x) - \frac{1}{g}\partial_\mu\theta^a(x) \qquad (1.10.b)$$

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g f_{abc}A_\mu^b(x)A_\nu^c(x) \qquad (1.10.c)$$

We note that there are now eight gauge fields (a, b, c = colour indices=1...8) and that the last term of [equ.1.10c] indicates that the gauge field couples to itself, or the gauge field carries a colour index and therefore has colour charge. Before, in QED, no such term existed in [equ.1.8] and the gauge field, identified as the photon had no electric charge. It is purely a carrier of the electromagnetic field. The non-abelian colour fields, however, are at the same time carriers *and* sources of the colour field and therefore have self interactions.

This is the crucial, general difference between abelian and non-abelian theories that gives the latter a much richer, and more complex content.

With the definitions of [equ.1.10] we now have the colour gauge invariant lagrangian that describes the dynamical interactions of the quarks: QCD.

$$L_{QCD} = \sum_k \bar{\Psi}_k(x)(i\gamma_\mu D_\mu - m_k)\Psi_k(x) - \frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}) \qquad (1.11)$$

(The trace in the last term represents the contracting of colour indices of $F_a^{\mu\nu}$ using the identity $Tr[\lambda_a, \lambda_b] = 8\delta_{ab}$)

To summarise, we have incorporated the SU(3) colour symmetry of the static quark model of section I.2., into a gauge symmetry of the lagrangian. By the gauge principle this symmetry uniquely defines the dynamics, or interactions of the coloured (strong interaction) fields. The eight coloured, gluon, gauge fields are the carriers, or exchange particles of the colour interaction that describes the quark (and gluon) interactions.

Having introduced field interactions, the concept of a free isolated quark is unreal, as a quark propagating in free space will be constantly interacting with the gauge fields. In particle language, the quark will continually emit and absorb virtual gluon-anti-gluon pairs. The most widely used techniques for accommodating these interactions is called perturbation theory [Ref.I.23], which we briefly discuss in the following section.

I.3.3   Perturbation theory

In QED the interaction strength is very weak. In terms of [equ.1.7] the coupling parameter, e, is very small. Consequently electrons interact with photons only very weakly and the effect of such interactions can be seen as a small perturbation on the free electron field.

From the lagrangian the Feynman rules [Ref.I.24] for the bare field prop-

agators and vertices can be derived. Using these rules, Feynman diagrams can be constructed. In perturbation theory, such diagrams are ordered according to their power of the coupling (in QED, e). Diagrams with high powers of the coupling are treated as negligible due to the smallness of the coupling, and only the lower order diagrams (the easiest to calculate) are considered. In QED, the smallness of e (or rather of $\alpha \equiv e^2/4\pi \simeq 1/137$) ensures that the first few terms of the perturbation expansion are a good approximation to the full, all orders, result. (It is this fact that gives QED such good agreement with experiment)

The obvious question is whether such a scheme can be applied to QCD. We immediately come up against an apparent dilemma: at small distances, within a hadron, the DIS experiments showed the quarks to be essentially free particles, implying g (the coupling in [equ.1.10]) is small. However, at large distances, greater than the typical hadron radius, say 1fm, the coupling must be very strong to explain confinement of the quarks and the strong interaction between the hadrons, many orders of magnitude stronger than the electromagnetic interaction. Thus there appears to be two regimes, and only in the former is perturbation theory applicable. Two points are in order here:

i) The inapplicability of perturbation theory at large distances (or equivalently small momenta) explains why the phenonema of hadronisation and confinement are so poorly understood.

ii) Confined to small distances, perturbative QCD describes the dynamics of quarks in a regime where they are not observed as real particles. As such it is the theory of an 'un-physical' regime that we use to deduce the behaviour of hadrons at distances where the theory explicitly breaks down!

How this apparent momentum dependence of the coupling comes about is a direct consequence of the field interactions. The form of the dependence can be formalised by considering what are called the renormalisation group equations, which we discuss in the following section.

I.3.4 Renormalisation group equation

The fact that the coupling, g, is momentum dependent to be able to accommodate both confinement and the freedom of the quark model, can be seen because higher order loop corrections to the bare coupling, g_b, will, in general, be momentum dependent. Consequently, it appears that the form of the momentum dependence would depend on the order of perturbation theory being considered. However, the necessity to renormalise a theory, to account for the field interactions and expurgate any infinities that arise in loop integrals, allows a 'leading orders' momentum dependence to be determined, via the renormalisation group equation [Ref.I.25].

The foundation of this equation is that infinities, arising from loop integrals, can be regularised by introducing an ultra-violet cut-off, κ. Having done so, we can demand that all Feynman diagrams combine in such a way that a given physical Green's function is independent of κ. The theory of multiplicative renormalisation, discussed in chapter II, ensures that the physical Green's functions, Γ, can be related to the bare ones, Γ_b, by a factor Z that represents the effects of the field interactions,

$$\Gamma(p, g, \mu) = Z\Gamma(p, g_b, \kappa)$$

where p represents the external momenta, g the couplings and μ a renormalisation scale introduced to balance the dimensions of κ. Physically μ is the momentum scale at which the renormalised coupling is defined such that, $g = g(\mu)$ and the coupling is explicitly a function of this renormalisation scale μ. Demanding the invariance of the Green's function to κ, or equivalently μ, can be expressed as the Stückelberg-Peterman equation [ref.I.26],

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma(g)\right)\Gamma(p, g, \mu) = 0 \tag{1.12}$$

where

$$\beta(g) = \mu\left(\frac{\partial g}{\partial \mu}\right)_{g_b,\kappa} = -\kappa\left(\frac{\partial g}{\partial \kappa}\right)_{g_b,\mu} \qquad (1.13)$$

$$\gamma(g) = \mu\left(\frac{\partial ln(Z)}{\partial \mu}\right)_{g_b,\kappa}. \qquad (1.14)$$

These equations ensure that a given Green's function is independent of the renormalisation scale μ. In particular, [equ.1.13] governs the momentum behavior of the coupling constant, g. Dimensional analysis demands that a scaling in μ is compensated for by a scaling in p:

$$G(pe^t, g, \mu) = G(p, g, \mu e^{-t})$$

Now, [equ.1.13] governs the change in g that cancels a scaling of μ. If the solution to this equation is generically $\bar{g}(t)$, then the previous eqaution becomes,

$$G(pe^t, g, \mu) = G(p, \bar{g}(t), \mu)$$

and we see that as the momenta, p, changes the coupling runs.

Using [equ.1.13] and the leading order perturbative correction to the quark-gluon coupling, the lowest order approximation to β(g) can be determined. Applying this result back into [equ.1.13] allows the leading logarithms of all orders to be summed to get the leading logarithm dependence of the coupling $\alpha \equiv g^2/4\pi$ as [Ref.I.26]

$$\alpha(Q^2) = \frac{4\pi}{\beta_0 ln(Q^2/\Lambda^2)} \qquad (1.15)$$

Λ is a scale parameter introduced as a consequence of the independence to μ. β_0 is a parameter characteristic of a given theory [ref.I.28]. For QCD β_0 is given by

$$\beta_0 = 11 - \frac{2N_f}{3}$$

where N_f is the number of quark flavours. For $N_f \leq 16$ β_0 is greater than zero. This implies that as $Q^2 \to \infty$, $\alpha \to 0$. This property is called asymptotic freedom [Ref.I.29] and explains why, at short distances (equivalent to large momentum) the quarks are essentially free. We have thus been able to theoretically justify this assumption made in the Parton model.

Conversely, as $Q^2 \to \Lambda^2$ the coupling diverges, and perturbation theory becomes useless. The colour force grows in strength and may well do so to permanently confine quarks.

I.3.5 Summary

Having arrived at the lagrangian of QCD in [equ.1.11] as the lagrangian of the colour interaction, we have, in the previous two sections, discussed two important aspects of the theory; the application of perturbation theory as a means of calculating the field interactions, and asymptotic freedom that ensures that in the large momentum regime the perturbation expansion is a good approximation.

In an effort to probe the non-perturbative regions several theoretical methods exist. One such example are the Schwinger-Dyson equations which make an attempt to sum all orders of perturbation theory. The work in this thesis is based on such equations and are discussed in more detail in following chapters. Consequently we finish this introduction having outlined the theory QCD as a gauge theory describing the colour interaction between quarks and gluons, commonly cast in a perturbative expansion at high energy.

One final comment is in order here; as pointed out earlier, we have concentrated on QCD and have completely ignored the other three forces of nature,

except for a brief mention of QED. This is because the work in this thesis is concerned with the strong interaction in isolation from the other interactions. (Although the second project deals with QED_3, this is only as a tractable limit of QCD_3, and it is the latter that we are really interested in.)

I.4 THIS THESIS

This thesis describes two projects. Both make attempts to investigate the non-perturbative regime via the Schwinger-Dyson equations.

The first project investigates the applicability of multiplicative renormalisation, and consequently the physical usefulness of, specific, truncated, solutions of the Schwinger-Dyson equations. This is done by explicitly calculating the divergent parts of two Feynman diagrams, that, according to multiplicative renormalisation, should be equal.

The second project involves the solution of the Schwinger-Dyson equation for the fermion propagator in three space-time dimensions. The aim here is to investigate the generation of scale hierarchies in gauge theories and the applicability of such a model to the study of finite-temperature gauge theories.

CHAPTER II MUTIPLICATIVE RENORMALISATION

II.1 INTRODUCTION TO RENORMALISATION

Ever since the very early days of quantum field theory, divergences have been an ever present feature. If anything is to be subsequently learned from a theory, such divergences must be first isolated and then removed. (Although a minority school of thought, with the noteable inclusion of P.A.M Dirac, believes that the very existence of infinities is a result of our feable attempts at describing nature. In effect they are a 'no-go sign', their existence signalling the failure of quantum field theory as a consistent, useful theory). However, returning to the main school of thought, these infinities do exist and consequently must be removed from any observable. This is the role of renormalisation theory [Ref.II.1].

Having digested the previous paragraph introducing renormalisation as a necessary procedure to expurgate infinities from a theory, we must immediately add a qualifying statement. Renormalisation is not solely motivated by the need to control infinities. It has its own, independent physical founding, irrespective of the existence of divergences. In an interacting field theory the interactions of the fields shift, or renormalise, the parameters in the 'bare' lagrangian. For example, the bare mass, m_0, of a field is shifted by the effect of the interactions of the field to an observable mass, m. (This is analogous to the effective mass of an electron inside a crystal being different to the bare mass of a free electron. The interaction of the electron with the crystal lattice effectively changing the response of the particle to, for example, an electric field which manifests itself as changing the observable mass).

If infinities do arise in the calculation of the field interactions then m_0 would have to be suitably divergent so as to cancel the interaction divergences to yield a finte, observable mass. As we can't 'switch off' the interactions we

can never measure m_0 and are therefore not overly concerned that it is formally divergent. Only physical observables, or renormalised quantities need be finite. Clearly, in a finite theory, with no infinities, the shift in parameters (the renormalisation) is finite and both the bare and renormalised quantities are finite, although, still only the latter is actually measurable.

The implementation of a renormalisation scheme changes the fundamental 'building blocks' of a theory from the bare quantities appearing in the lagrangian to the renormalised ones, from which physical observables can be derived. In particular, the Feynman rules derived from the lagrangian are replaced by renormalised, or 'dressed' rules which include, implicitly, the interactions of the fields. For example, the renormalised fermion propagator, shown in [Fig.II.1], is the sum of the bare propagator and all the higher order corrections to the two point function. (Renormalised functions are represented by 'blobs' on the bare quantity).

If we want to calculate a given process, automatically including all higher order corrections, we have now to use the renormalised Feynman rules and evaluate the relevant irreducible diagrams. That is, diagrams that can't be reduced by the removal of vertex or propagator corrections.

The removal of infinities arising from higher order corrections is achieved by adjusting the parameters of the lagrangian to cancel the divergences. This is done by adding counter terms to the lagrangian [Ref.II.2], for example to a mass term we add a counter term to get,

$$-\frac{m^2}{2}A^2 \quad + \quad \frac{\delta m^2}{2}A^2. \tag{2.1}$$

The addition of the counter term introduces no new fields or interactions into the theory; it is of the same form as the mass term. It provides a formal way of representing the interaction renormalisations and a way to cancel the infinities. δm is adjusted, 'by hand' to explicitly cancel the divergences arising from the
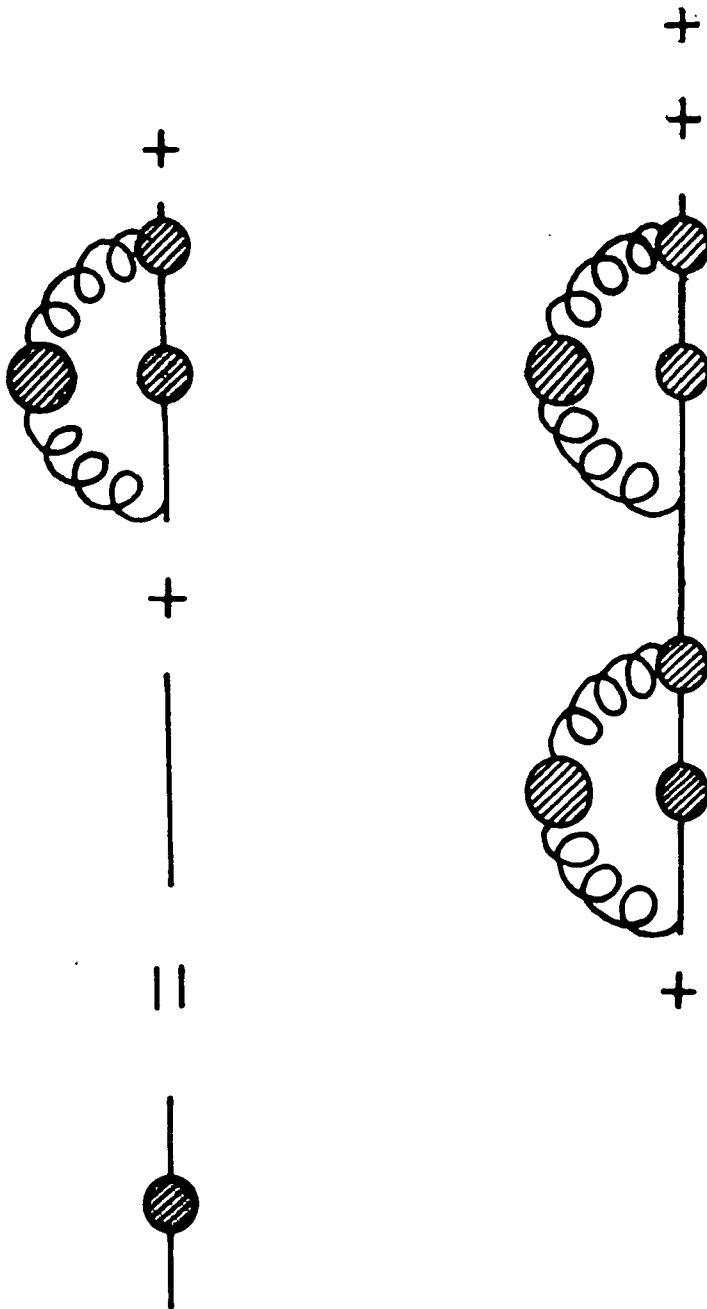
Figure II.1   Renormalised fermion propagator

bare lagrangian interactions.

An alternative approach, which we discuss in the next section, is to view the renormalisation shift in parameters as a multiplicative re-scaling of the bare lagrangian. It is this, multiplicative renormalisation, that we are concerned with in this work. In particular we are interested in the multiplicative renormalisation of the graphs relevant to the physical proccess $e^+e^- \to q\bar{q}$. We therefore give, in the following section, an introduction to the ideas of multiplicative renormalisation.

II.2 MULTIPLICATIVE RENORMALISATION

So far we have introduced the concept of renormalisation as a way to represent interactions of fields, including any possible infinities that might appear, and more specifically the effect this has on physical observables.

The renormalisation of a theory implies a shift in the parameters of the lagrangian, while introducing no new types of term. (The latter would change the actual theory being considered and the new terms would themselves, need renormalising, as would, ad infinitum, any new terms introduced in so doing). As such, the shift in parameters can be represented as a multiplicative re-scaling. For example, the kinetic term of a field, ψ_0, and it's counter term, can written as

$$\frac{1}{2}\partial\psi_0^2 + \frac{\delta Z}{2}\partial\psi_0^2 \quad = \quad \frac{(1+\delta Z)}{2}\partial\psi_0^2 \quad \equiv \quad \frac{Z}{2}\partial\psi_0^2. \tag{2.2}$$

The factor of Z, called the wavefunction renormalisation, can be absorbed by a re-scaled definition of the field,

$$\psi_0(x) \to \psi(x) = Z^{-1/2}\psi_0(x). \tag{2.3}$$

With this re-scaled field [equ.2.2] is of the same form as the origonal, bare,

kinetic term,

$$\frac{1}{2}\partial\psi_0^2 \rightarrow \frac{1}{2}\partial\psi^2. \tag{2.4}$$

Analogous arguments yield mass and coupling constant renormalisation re-scaling of the bare quantities. Not surprisingly, this representation of renormalisation is called multiplicative renormalisation [Ref.II.3].

Having absorbed the counter terms, and their implicit infinities, into the re-scaled quantities we can now write down the renormalised lagrangian in terms of the latter (by simply replacing the bare fields, couplings and masses by their renormalised values) and the resulting Green's functions will be finite. The theory has been renormalised. (We note that it is not always possible to carry out such a procedure and the infinities cannot be controlled. Such theories are said to be non-renormalisable [Ref.II.4] and are of very little physical interest. No finite observables can be derived from them).

It is an important theorem of renormalisation that the renormalisation prescription can be used to render a (renormalisable) theory finite order by order in perturbation theory. That is, if the perturbation expansion is truncated at some, finite order, the renormalisation factors can be determined to self-consistently make the theory finite [Ref.II.5].

This being so, the (regularised) values of the counter terms, or equivalently the renormalisation factors, clearly depend on the order of perturbation theory being used. In principle, the renormalised quantities include effects from all higher order corrections. In practice, however, only a finite number of corrections are calculated. Thus, if only one loop graphs are considered the renormalisation factors only have to absorb one loop divergences. To two loops, both one and two loop divergences have to be cancelled and so on to any arbitrary order.

One of the strongest consequences of multiplicative renormalisation is that the scaling factor of a given Green's function only depends on the number, and type, of irreducible vertices and external particle legs. Each of which carry factors of Z_λ, Z_ψ respectively (Z_λ is the coupling constant renormalisation). Internal reducible propagators and vertices don't contribute renormalisation factors since they are implicitly included in the factors coming from the external functions. Indeed, it is the very function of the latter to represent the 'internal' corrections to the Green's function.

This implies that, as far as the renormalisation of the strong interaction is concerned, both the graphs of [Fig.II.2a,b] have the same external legs and no vertices. (The boson in [Fig.II.2b] is a photon, not a strong interaction particle). Consequently they have the same renormalisation factors to absorb their divergences arising from higher order corrections. Equivalently, we can say that on renormalising they have the same divergent behavior. Of course, if we were including the renormalisation of the electromagnetic interactions the photon in [Fig.II.2b] would contribute as an external leg and the graphs would no longer have this property of equal divergent behaviour.

In this work we evaluate the one loop corrections to these graphs, shown in [Fig.II.2c,d], using a specific set of renormalised Feynman rules. The aim is to explicitly test whether multiplicative renormalisation survives when using these rules. The immediate reaction to this is that, as already noted, multiplicative renormalisation holds to any, and therefore all, orders of perturbation theory. Using full, renormalised Feynman rules is therefore bound to preserve multiplicative renormalisation and our motivation for calculating the graphs might appear redundant.

Indeed this would be the case if the functions we use really were the full renormalised ones that sum all orders of perturbation theory. However, in reality, the renormalised functions are obtained from solutions of the
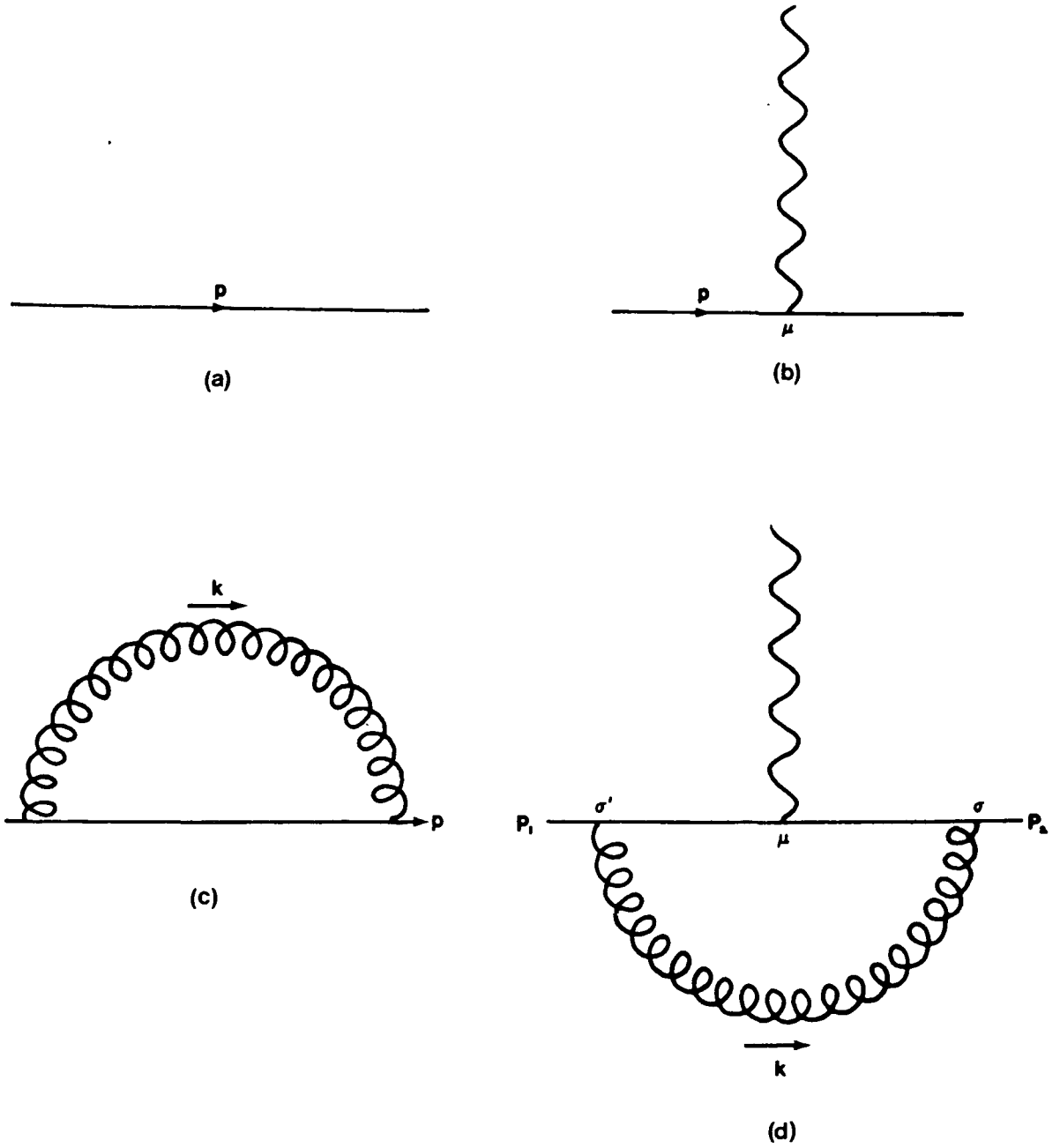
Fig.II.2 Two point fermion graphs

a] tree level fermion propagator

b] tree level fermion-photon vertex

c] one loop propagator correction

d] one loop vertex correction

Schwinger-Dyson equations [Ref.II.6]. To solve these exactly would involve the solution of an infinite set of coupled integral equations. Further, this infinite set is nested; the n-point function is given interms of the (n+1)-point function. This nesting generates, in a perturbative sense, the infinity of loops involved in an n-point function. A consequence of this infinite hierarchy is that no consistent renormalisation programme has been developed, and it is not known how to renormalise the equations exactly [Ref.II.7]. A tractable alternative, putting aside the renormalisation difficulties, is to truncate the set of equations to leave a closed, finite system of soluble equations. The effect of this truncation amounts to ignoring some Feynman graphs when determining the Green's functions. Only a semi-infinite number of graphs are included in the renormalisation scheme. Consequently, it is no longer clear to what, infinite, order of perturbation theory the equations are solved. This uncertainty in the order to which the Green's functions are renormalised in turn makes it unclear whether they are then consistent and admit multiplicative renormalisation. If this is not the case it might imply that such truncated functions are not admissable and their use would not render physical processes (in our case $e^+e^- \rightarrow q\bar{q}$) finite.

To summarise, we are not so much intersted in the determination of a set of renormalised Feynman rules in this work, but rather the consequence of their use in the calculation of the graphs relevant to a physical process. This is the main objective of this work. However, before turning to this, we introduce the concept of dimensional regularisation, used throughout this work to formally manipulate infinities. Then, to serve as an introduction, we present the calculation of the graphs using bare, unrenormalised, propagator and vertex functions to explicitly show that, as predicted by multiplicative renormalisation, using a consistent set of Feynman rules, the graphs do in fact have the same divergent behaviour.

II.3 DIMENSIONAL REGULARISATION

Upto now we have banded about the term 'infinities' and performed various cancellations of such terms with scant regard to the fact that to handle such numbers is non-trivial. Infinity is a very special number. In doing so we have brushed aside a very important aspect of renormalisation theory; the formal manipulation of infinities can only be done by first regularising them in some way. That is the divergence must be expressed as the behaviour of a function in a particular limit. Away from this limit the function is finite, and well behaved and can be manipulated without any problem. At the end of a calculation the limit is invoked and, having renormalised the potentially divergent functions away, the resulting function is finite.

For example, the divergent integral

$$I \equiv \int \frac{d^4k}{(k^2 + L)^2} = 2\pi^2 \int_0^\infty \frac{dk\,k^3}{(k^2 + L)^2} \tag{2.5}$$

can be regularised by cutting the integral off at some momentum Λ and defining

$$\tilde{I} \equiv 2\pi^2 \int_0^\Lambda \frac{dk\,k^3}{(k^2 + L)^2} = \pi^2 \left\{ ln\left(\frac{\Lambda^2 + L}{L}\right) + \frac{\Lambda^2}{\Lambda^2 + L}\right\}.$$

Then, as we take the limit $\Lambda \to \infty$, $\tilde{I} \to I$. $\tilde{I}$ is the regularised divergent integral.

This cut-off regularisation is one of a number of schemes to regularise divergent quantities [Ref.II.8]. In this work we use the scheme of dimensional regularisation [Ref.II.9]. This follows from the observation that I in [equ.2.5] is only ultra-violet divergent in four dimensions. If, formally, we take the dimensionality of space-time, n, to be a variable

$$n \equiv 2\omega \equiv 2(2 - \epsilon) \tag{2.6}$$

the previous integral is only divergent in the limit $n \to 4$ (or equivalently, $\epsilon \to 0$).

Re-evaluating [equ.2.5] in n-dimensional space-time gives,

$$I_n = \int \frac{d^n k}{(k^2 + L)^2} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{k dk^{n-1}}{(k^2 + L)^2}. \tag{2.7}$$

Using the identity

$$\int_0^\infty \frac{dy y^b}{(y^2 + L)^a} = \frac{\Gamma(\frac{b+1}{2})\Gamma(a - \frac{b+1}{2})}{2\Gamma(a)L^{a+\frac{b+1}{2}}} \tag{2.8}$$

we can write [equ.2.7] as

$$I_n = \frac{2\pi^{2-\epsilon}}{\Gamma(2 - \epsilon)} \frac{\Gamma(2 - \epsilon)\Gamma(\epsilon)}{2\Gamma(2)L^\epsilon}. \tag{2.9}$$

The divergence of this expression in the limit n→ 4 is contained in the divergence of $\Gamma(\epsilon)$ as $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \Gamma(\epsilon) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Gamma(1 + \epsilon) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[1 - \nu_E \epsilon + O(\epsilon^2)]$$

where ν_E is the Euler-Mascaroni constant. We see that the divergence is formally expressed as a $1/\epsilon$ pole as $\epsilon \to 0$. For $\epsilon \neq 0$ the divergence is regularised and can be manipulated. Only at the end of a calculation do we take the limit $\epsilon \to 0$.

The idea of regularising divergences in this way, by continuing the dimensions of space-time gained great support when t'Hooft [Ref.II.10] showed the procedure was gauge invariant. This, along with formulating the axioms of dimensional regularisation [Ref.II.11] is highly non-trivial. For this work we only need to use the idea of regulating divergences in such a way and will have nothing more to say on the theoretical formulation of the scheme. In a similar way to how we arrived at [equ.2.9] all the results in Appendix A can be obtained. Having thus provided a means of regulating divergent quantities we now, as promised, use this technique to evaluate the graphs of [Fig.II.c,d] using bare, unrenormalised Feynman rules.

II.4 MULTIPLICATIVE RENORMALISATION USING BARE FEYNMAN RULES

As discussed above, since the graphs of [Fig.II.2c] and [Fig.II.2d] have the same (strong interaction) external legs, multiplicative renormalisation tells us that they have the same multiplicative renormalisation factor, Z_ϕ, that absorbs the divergences of the two graphs. Equivalently this implies that the graphs have the same, regularised divergences. As promised, in this section we present the calculation of the two graphs using dimensional regularisation and show that, in the form of $1/\epsilon$ poles the divergences are, indeed, the same in the graphs. (Once the photon vertex graph of [Fig.II.2d] is contracted with a momentum vector $e^{-1}p_\mu$ to balance the dimensionality of the diagrams, as the Ward identity of QED tells us we must do [Ref.II.12])

$$\frac{-i}{\not{p}+m}$$

$$\frac{1}{k^2}\left\{\delta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right\}$$

$$-ig\mu^{2-\omega}\gamma_\mu$$

$$-ie\gamma_\mu\mu^\varepsilon$$
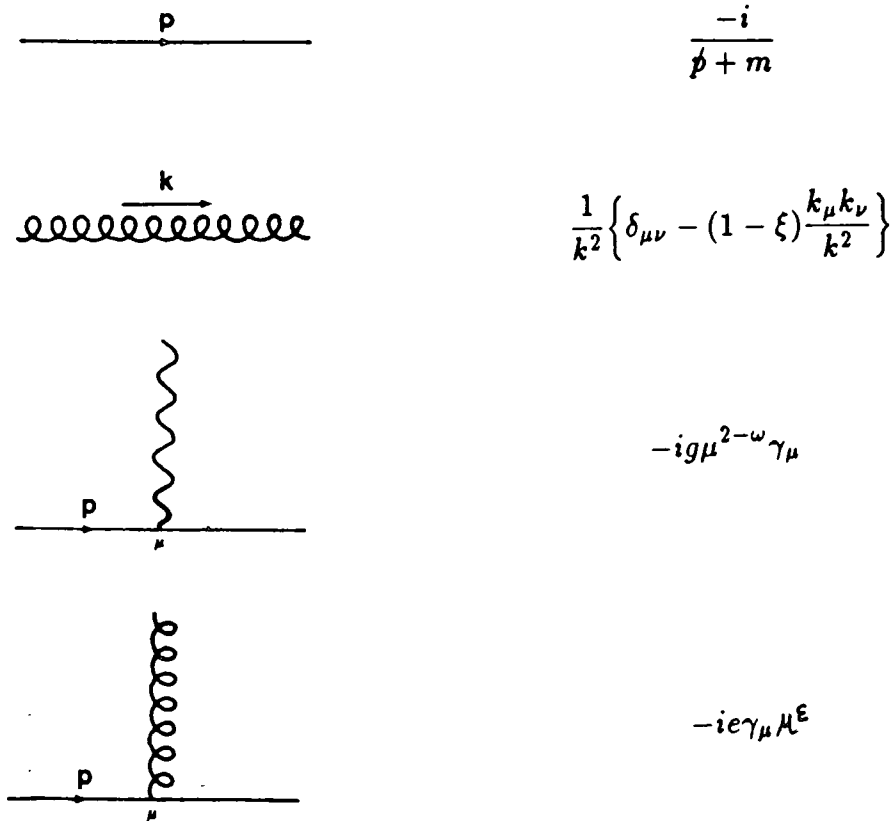
Figure II.3   Unrenormalised Feynman rules

First we consider the self energy graph of [Fig.II.2c]. Using the bare, unrenormalised Feynman rules in Euclidean space, given in [Fig.II.3]. In the Feynman gauge ($\xi=1$), the self energy is given by,

$$\Sigma(p) = -(g\mu^{2-\omega})^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \gamma_\mu \frac{(-i)}{(\not{p} - \not{k})} \gamma_\nu \frac{\delta_{\mu\nu}}{k^2}$$

where the dimensionful factor of $\mu^{2(2-\omega)}$ is needed to ensure the dimensionality of the coupling is correct.

Using the anticommutator of Euclidean gamma-matrices, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ it follows that $\not{a}\not{a} = -a^2$ and the above expression becomes,

$$\Sigma(p) = -i(g\mu^{2-\omega})^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\mu(\not{p} - \not{k})\gamma^\mu}{k^2(p-k)^2}.$$

Feynman parameterising this equation using the identity of [equ.A.1] gives

$$\Sigma(p) = -i(g\mu^{2-\omega})^2 \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\mu(\not{p} - \not{k})\gamma^\mu}{[k^2(1-x) + x(p-k)^2]^2},$$

which, defining $k' \equiv k - px$ which shifts the origin and does not effect the measure, leads to

$$\Sigma(p) = -i\frac{(g\mu^{2-\omega})^2}{(2\pi)^{2\omega}} \int_0^1 dx \int d^{2\omega}k' \frac{\gamma_\mu[\not{p}(1-x) - \not{k}']\gamma^\mu}{[k'^2 + L]^2}$$

where,

$$L \equiv p^2 x(1-x).$$

We can drop the terms odd in k from this equation since

$$\int d^n k\, k_\mu \equiv 0. \tag{2.10}$$

This follows because having integrated over n dimensional k space there is nothing left to carry the Lorentz index, μ. Thus, using [equ.A.5] the k' integral can be done leaving,

$$\Sigma(p) = -\frac{ig^2\mu^{4-2\omega}}{(4\pi)^\omega}\frac{\Gamma(1-\epsilon)}{\epsilon}\int_0^1 dx\frac{(1-x)}{L^\epsilon}\gamma_\mu\not{p}\gamma^\mu.$$

Now $\gamma^\mu\not{p}\gamma_\mu = 2(1-\epsilon)\not{p}$ which allows us to write,

$$\Sigma(p) = -\frac{2ig^2}{16\pi^2}\not{p}\frac{\Gamma(1-\epsilon)}{\epsilon}(1-\epsilon)\int_0^1 dx\frac{(1-x)}{[x(1-x)p^2/4\pi\mu^2]^\epsilon}.$$

Performing the x integral using [equ.A.2] yields,

$$\Sigma(p) = -\frac{2ig^2\not{p}}{16\pi^2}\frac{\Gamma(1-\epsilon)(1-\epsilon)}{\epsilon}\left(\frac{p^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)},$$

and extracting the $1/\epsilon$ poles as we take the limit $\epsilon \to 0$ we get,

$$\Sigma(p)_{singular} \quad = \quad -\frac{ig^2\not{p}}{16\pi^2\epsilon}. \tag{2.11}$$

Having thus extracted the singular part of the graph of [Fig.II.2c] we now do the same thing for the "vertex" graph of [Fig.II.2d], where, using the Feynman rules of [Fig.II.3] in the Feynman gauge

$$\Gamma_\rho = -(e\mu^\epsilon)(g\mu^\epsilon)^2\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\gamma_\sigma\frac{1}{(\not{p}+\not{k})}\gamma_\rho\frac{1}{(\not{k}+\not{q})}\frac{\delta_{\sigma\sigma'}}{k^2}\gamma_{\sigma'},$$

which immediately gives, after Feynman parameterising,

$$\Gamma_\rho = -\frac{2i(e\mu^\epsilon)(g\mu^\epsilon)^2}{(2\pi)^{2\omega}}\int_0^1 dx\int_0^{1-x} dy\int\frac{d^{2\omega}k\gamma_\sigma(\not{p}+\not{k})\gamma_\rho(\not{k}+\not{q})\gamma^\sigma}{[k^2+x(p^2+2p\cdot k)+y(q^2+2q\cdot k)]^3}.$$

This time we shift the origin by defining $k' \equiv k + px + qy$, which gives,

$$\Gamma_\rho = -2i(e\mu^\varepsilon)(g\mu^\varepsilon)^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k'}{(2\pi)^{2\omega}} \gamma_\sigma [\slashed{k}' - \slashed{q}y + \slashed{p}(1-x)]$$

$$\times \frac{\gamma_\rho [\slashed{k}' - \slashed{p}x + \slashed{q}(1-y)]\gamma^\sigma}{[k'^2 + x(1-x)p^2 + y(1-y)q^2 - 2p\cdot qxy]}.$$

Again, keeping only even terms in k' leaves,

$$\Gamma_\rho = -2i(e\mu^\varepsilon)(g\mu^\varepsilon)^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\sigma [\slashed{k}\gamma_\rho\slashed{k} + [\slashed{p}(1-x) - \slashed{q}y]}{[k^2 + L]^3}$$

$$\times \gamma_\rho [\slashed{q}(1-y) - \slashed{p}x]]\gamma^\sigma \tag{2.12}$$

where now,

$$L \equiv x(1-x)p^2 + y(1-y)q^2 - 2p\cdot qxy.$$

To evaluate this expression we now split Γ_ρ into the contribution quadratic in k in the numerator and the remainder as $\Gamma_\rho^{(1)}$, $\Gamma_\rho^{(2)}$ respectively;

$$\Gamma_\rho = \Gamma_\rho^{(1)} + \Gamma_\rho^{(2)}$$

where,

$$\Gamma_\rho^{(1)} \equiv -2i(e\mu^\varepsilon)(g\mu^\varepsilon)^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\sigma\slashed{k}\gamma_\rho\slashed{k}\gamma_\sigma}{[k^2 + L]^3},$$

and

$$\Gamma_\rho^{(2)} \equiv -2ie\mu^\varepsilon(g\mu^\varepsilon)^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\gamma_\sigma [\slashed{p}(1-x) - \slashed{q}y]\gamma_\rho [\slashed{q}(1-y) - \slashed{p}x]\gamma_\sigma}{[k^2 + L]^3}.$$

Now, under the integral

$$\int d^n k \not{k} \gamma_\mu \not{k} f(k^2, L) = \gamma_a \gamma_\mu \gamma_b \int d^n k \, k_a k_b f(k^2, L)$$

and the a,b indices can only be carried, after integration, by δ_{ab} so, using [equ.2.10], we can write,

$$\int d^n k \, k_a k_b f(k^2, L) \cong \delta_{ab} I. \tag{2.13}$$

Multiplying this equation by δ_{ab} we get,

$$\int d^n k \, k^2 f(k^2, L) = nI = 2(2 - \epsilon)I.$$

Hence, substituting for I into the previous equation we see that

$$\int d^n k \, k_a k_b f(k^2, L) = \frac{\delta_{ab}}{2(2 - \epsilon)} \int d^n k \, k^2 f(k^2, L)$$

or,

$$\int d^n k \not{k} \gamma_\mu \not{k} f(k^2, L) = \frac{\gamma_a \gamma_\mu \gamma_a}{2(2 - \epsilon)} \int d^n k \, k^2 f(k^2, L)$$

$$= \frac{(1 - \epsilon)\gamma_\mu}{(2 - \epsilon)} \int d^n k \, k^2 f(k^2, L). \tag{2.14}$$

Using this result $\Gamma_\rho^{(1)}$ can be re-cast as,

$$\Gamma_\rho^{(1)} = -2i(e\mu)(g\mu)^2 \frac{(1 - \epsilon)}{(2 - \epsilon)} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \frac{\gamma_\sigma \gamma_\rho \gamma^\sigma k^2}{[k^2 + L]^3}.$$

We can now contract the gamma matrices using [equ.B.1] to obtain

$$\Gamma_\rho^{(1)} = -4i(e\mu)(g\mu)^2 \frac{(1 - \epsilon)^2}{(2 - \epsilon)} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \frac{\gamma_\rho k^2}{[k^2 + L]^3}.$$

Evaluating the k integral using [equ.A.3] yields,

$$\Gamma_\rho^{(1)} = -2ie\mu^\epsilon (g\mu^\epsilon)^2 \gamma_\rho \frac{(1-\epsilon)^2(2-\epsilon)\Gamma(1+\epsilon)}{(4\pi)^{2(2-\epsilon)}\epsilon(2-\epsilon)}$$

$$\times \int_0^1 dx \int_0^{1-x} dy \frac{1}{[p^2x(1-x) + q^2y(1-y) - 2p\cdot qxy]^\epsilon}$$

We now have to extract the pole contributions from this double integral. Using the expansion

$$a^\epsilon = 1 - \epsilon ln(a) + O(\epsilon^2)$$

we can expand the denominator such that $\Gamma_\rho^{(1)}$ can be written

$$-2ieg^2\mu^\epsilon \frac{\gamma_\rho(1-\epsilon)^2}{(4\pi)^2}\frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dx \int_0^{1-x} dy$$

$$\times \left\{ 1 - \epsilon ln\left[\frac{p^2x(1-x) + q^2y(1-y) - 2p\cdot qxy}{4\pi\mu^2}\right]\right\}$$

The first term in the curly brackets can be trivially integrated, and using the expansion

$$\frac{(1-\epsilon)^2}{\epsilon}\Gamma(1+\epsilon) \approx \frac{1}{\epsilon} - 2 - \nu_E + O(\epsilon)$$

where ν_E is the Euler-Mascaroni constant, we get

$$\Gamma_\rho^{(1)} = -2ieg^2\mu^\epsilon \frac{\gamma_\rho}{(4\pi)^2}\frac{1}{2}\left\{\frac{1}{\epsilon} - 2 - \nu_E\right.$$

$$-2\int_0^1 dx \int_0^{1-x} dy ln\left(\frac{p^2x(1-x) + q^2y(1-y) - 2p\cdot qxy}{4\pi\mu^2}\right)\Bigg\}$$

since $\int_0^1 dx x ln(x)$ is finite the remaining integral is also finite (and independent of ϵ), and the pole contribution to $\Gamma_\rho^{(1)}$ is given by the $\frac{1}{\epsilon}$ pole of the first term,

$$\Gamma_\rho^{(1)}|_{singular} = -\frac{ieg^2\mu^\epsilon\gamma_\rho}{16\pi^2\epsilon} \tag{2.15}$$

Now, returning to the second term in the decomposition, $\Gamma_\rho^{(2)}$, the k integral can be performed using [equ.A.4] to yield,

$$\Gamma_\rho^{(2)} = -\frac{ie\mu^\epsilon(g\mu^\epsilon)^2\Gamma(1+\epsilon)}{(4\pi)^{(2-\epsilon)}} \int_0^1 dx \int_0^{1-x} dy \frac{\gamma_\sigma[\not{p}(1-x) - \not{q}y]\gamma_\rho[\not{q}(1-y) - \not{p}x]\gamma^\sigma}{[p^2x(1-x) + q^2(1-y) - 2p\cdot qxy]^{1+\epsilon}} \tag{2.16}$$

This integral is finite in the limit $\epsilon \rightarrow 0$. We can see this by factorising the denominator as $(y - \lambda_+)(y - \lambda_-)$ where $\lambda_\pm$ is a constant and noting that the y integral then gives, at worst, $ln(y - \lambda_\pm)$ terms when combined with terms in the numerator. The x integral of these logarithms can similarly be recast in the form of $\int dx ln(x - \rho_\pm)$ which, again, are finite. Thus $\Gamma_\rho^{(2)}$ has no divergent terms and is completely finite. The only situation when divergences do occur is when $p^2 = q^2 = (p + q)^2 \equiv 0$, the mass-shell condition, when the denominator $\equiv 0$ for all x, y. These infrared divergences are avoided by demanding $p^2, q^2 < 0$.

Hence the only (UV) divergent contributions to the total graph comes from $\Gamma_\rho^{(1)}$ in [equ.2.15]. Comparing this equation with [equ.2.11], the divergent part of [Fig.II.2.c], we immediately see

$$e\Sigma(p)|_{singular} = p^\mu\Gamma_\mu|_{singular} \tag{2.17}$$

where the factors of e and p^μ come from a QED Ward identity [Ref.II.12], that relates the fermion-photon vertex to the fermion propagator.

We have now explicitly calculated the divergent parts of the two irreducible graphs of [Fig.II.2c,d] and shown, as predicted, they are the same. This

implies the same counter terms would be necessary to cancel both divergences. Or, in the language of multiplicative renormalisation, the same re-scaling of the fields and couplings will render both graphs finite. The calculation was to one loop level, so effectively we have determined the one loop counter terms.

Having demonstrated multiplicative renormalisation in this specific example we can now turn to the main aim of this work. To re-calculate the graphs using (approximate) renormalised propagator and vertex functions. Before presenting the details of this calculation we give, in the following section, a brief summary of the functions we use and their origin.

II.5 THE RENORMALISED PROPAGATOR AND VERTEX FUNCTIONS

As indicated earlier, the renormalised functions are arrived at solutions of the Scwinger-Dyson equations. The nested nature of the equations (the n-point function depending on the (n+1)-point function, ad infinitum) means that although formally well defined, a consistent renormalisation scheme for the equations does not exist. Moreover, in order be able to solve, for instance, for the 2 and 3 point functions, as in the present case, certain approximations have to be made. It is both the nested nature of these equations and these approximations that then make it unclear whether multiplicative renormalisation is preserved, and whether, using these, or *any* other, approximate functions will render physical cross sections finite

We now review the Schwinger-Dyson equations to elucidate the functions we use, given in [Fig.II.5].

II.5.1 The Schwinger-Dyson equations

The essence of the Schwinger-Dyson equations is that, in principle, they allow all orders of perturbation theory to be summed in a given Green's func-

tion (although they can also be derived completely independently of perturbation theory and are, in the truest sense of the word, fully non-perturbative [Ref.II.13]). In chapter V we give, in relation to another problem, a schematic derivation of the equation for the fermion propagator. We now present a more rigorous discussion of the equations based on the functional methods of field theory [Ref.II.14].

Starting from a generating functional, Z[J], of a source $J_\mu^a(x)$ (a is a colour index), defined in terms of the action S[A,J];

$$Z[J] \equiv N \int [dA_\mu^a] exp(iS[A_\mu^a, J_\mu^a]), \qquad (2.18),$$

(N is a normalising factor), the connected n-point Green's functions $G_n(A_\mu(x_1)....A_\mu(x_n))$ are given by functional derivatives of $ln(Z[J])$;

$$G_n(A_\mu(x_1)...A_\mu(x_n)) = \frac{1}{Z}\left[\prod_{\alpha=1}^{n}\left\{\frac{1}{i}\frac{\delta}{\delta J_\mu^{a\alpha}(x_\alpha)}\right\}iW[J]\right]_{J=0}, \qquad (2.19)$$

where $W[J]$ is defined by $W[J] \equiv -iln(Z[J])$. Using the relation between the two point Green's function and the propagator $\Delta_{\mu\nu}^{ab}$;

$$< A_\mu^a(x)A_\nu^b(y) >= \Delta_{\mu\nu}^{ab}(x,y)+ < A_\mu^a(x) >< A_\nu^b(y) >, \qquad (2.20)$$

equation [2.18] can be differentiated three times to yield, for the three point function

$$< A_\mu^a x)A_\nu^b(y)A_\sigma^c(z) >= \frac{1}{i}\frac{\delta\Delta_{\mu\nu}^{bc}(y,z)}{\delta J_\mu^a(x)} + \Delta_{\mu\nu}^{ab}(x,y) < A_\sigma^c(z) >$$

$$+\Delta_{\mu\sigma}^{ac}(x,z) < A_\nu^b(y) > +\Delta_{\nu\sigma}^{bc}(y,z) < A_\mu^a(x) >$$

$$+ < A_\mu^a(x) >< A_\nu^b(y) >< A_\sigma^c(z) > . \qquad (2.21)$$

Invariance of the action, $\delta S = 0$, yields the current conservation equation

$$-D_\nu^{ab} F^{b\mu\nu}(x) = J^{a\mu}(x), \qquad (2.22)$$

where

$$D_\nu^{ab} \equiv \partial_\nu \delta^{ab} - g f_{abc} A_\nu^c(x)$$

and f_{abc} are the structure constants of the gauge group.

Combining equations [2.20] and [2.21], after somewhat lengthy, but straightforward, manipulation yields the Schwinger-Dyson equation for the gauge boson,

$$(\delta_{\mu\nu} - \partial_\mu \partial_\nu) < A_\nu^a(x) > + \frac{g}{2} \Gamma_{\mu\nu\sigma}^{oabc}(x,y,z) \{ \Delta_{\nu\sigma}^{bc}(y,z) + < A_\nu^b(y) >< A_\sigma^c(z) > \}$$

$$- \frac{g^2}{6} \Gamma_{\mu\nu\sigma\tau}^{0abcd}(x,y,z,w) \left\{ \frac{\delta \Delta_{\sigma\tau}^{cd}(y,z)}{i\delta J_\nu^b(w)} + 3\delta_{\sigma\tau}^{cd}(y,z) < A_\nu^b(w) > \right.$$

$$\left. + < A_\nu^b(w) >< A_\sigma^c(y) >< A_\tau^d(z) > \right\} = J_\mu^a(x). \qquad (2.23)$$

with $\Gamma_{\mu\nu\sigma}^{0abc}(x,y,z)$ and $\Gamma_{\mu\nu\sigma\tau}^{0abcd}(x,y,z,w)$ the bare three and four point functions respectively.

Repeated differentiation of this equation will yield the higher order equations which form an infinite set of coupled equations for all the n point functions. In general [equ.2.23], and it's derivatives, relate the n point function to the next two in the hierarchy (ie. the n+1 and n+2 functions);

$$\Gamma^{(n)} = G(\Gamma^{(2)}, \Gamma^{(3)}, \ldots \ldots \Gamma^{(n+1)}, \Gamma^{(n+2)}). \qquad (2.24)$$

Further, G can be separated into two terms, G_1 and G_2, where only G_2 contains $\Gamma^{(n+2)}$,

$$\Gamma^{(n)} = G_1(\Gamma^{(2)}.....\Gamma^{(n+1)}) + G_2(\Gamma^{(2)}....\Gamma^{(n+1)}\Gamma^{(n+2)}). \qquad (2.25)$$

To be able to solve this infinite set of equations one obvious approximation is to truncate the series at some value of n. Then one needs to know, or at least approximate, the (n+1) and (n+2) functions to close the set. To this end one could drop the function G_2 in [equ.2.25] for the n point function, which eliminates one of the unknowns explicitly. Then only $\Gamma^{(n+1)}$ need be given to be able to start solving the set of equations for $\Gamma^{(n)}$. Information about the longitudinal part, at least, of the n+1 function can be gained by using the Slavnov-Taylor identities [Ref.II.15]. These are obtained by repeated differentiation of the current conservation equation [2.22]. They relate the (n+1) functions to the n point ones. However they only yield information about the longitudinal part of the functions, that is they are relations for $P^\mu\Gamma_\mu^{(n)}$. The transverse part of the function (satisfying $P^\mu\Gamma_T^{(n)} \equiv 0$) is totally unconstrained. To eliminate this degree of freedom various approximations/ansatz have been used by different workers, although not discussed here (eg. The Mandelstam approximation [Ref.II.16], or setting $\Gamma^{(n)} \equiv \Gamma_L^{(n)}$ [Ref.II.17])

Having made the above truncations and approximations it is then, in principle, possible to solve the resulting closed set of equations [Ref.II.13]. In this work we are not concerned with the (considerable) complexities and details of such a programme and use, without further explanation, the functions arrived at by A.D. Worrall in his PhD.thesis [Ref.II.18], given here in [Fig.II.5], who used the strategy outlined above. What is important to this work is the effect of truncating the series. To illustrate this we consider the equation for the two point gauge boson function, diagramatically represented (neglecting ghost fields) in [Fig.II.4]. The last three terms on the right hand side include the four point

function $\Gamma^{(4)}$, and are represented in, and only in, the function $G_2(\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)})$ in the decomposition

$$\Gamma^{(2)} = G_1(\Gamma^{(2)}, \Gamma^{(3)}) + G_2(\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}). \qquad (2.26)$$

From [Fig.II.4] we see that by omitting G_2, it is the higher order graphs that are dropped. This can be justified qualitatively by the fact that the last two irreducible graphs in the equation are $O(\alpha_s^2)$ whereas the others are $O(\alpha_s)$ or less. For $\alpha_s \ll 1$ the last two contributions can be safely dropped. The tadpole diagram in [Fig.II.4], being independent of momentum, can only have the tensor structure $\delta_{\mu\nu}$. Thus, in projecting out the transverse part of the equation the tadpole diagram gives no contribution.

Having thus reduced the equation to a form involving only the one loop diagram, and doing a similar thing for the fermion equation, they can then be solved, using the Slavnov-Taylor identities, to give the functions of [Ref.II.18] presented here in [Fig.II.5]. The renormalised propagators and vertices displayed in this figure are not exact analytic expressions but are rather simplified representations embodying the essential features of the numerical solutions to the truncated Schwinger-Dyson equations arrived at by A.D. Worrall. The bare rules of [Fig.II.3] are a trivial approximation to these forms with F(p)$\equiv$ G(p)$\equiv$1.

Clearly, from [Fig.II.4], the approximations briefly discussed above omit some graphs. Although the blobs on the internal propagators and vertices indicate an infinite summation of Feynman graphs. The omission of some graphs implies that the solutions to the equations are not a self-consistent summation of all possible graphs and that, in reality it is a semi-infinite summation that is represented in the solutions. Even though it is possible to make the truncation gauge invariant, it is not clear whether the resulting equations can be renormalised consistently. The nested nature of the equations prohibits a straightforward renormalisation scheme for the full equations and consequently

obscures the renormalisation of the truncated set.

Having thus equipped ourselves with a set of, albeit approximate, renormalised Feynman rules we now set out to re-calculate the singular parts of the graphs of [Figs.II.2c,d], the main task of this work.
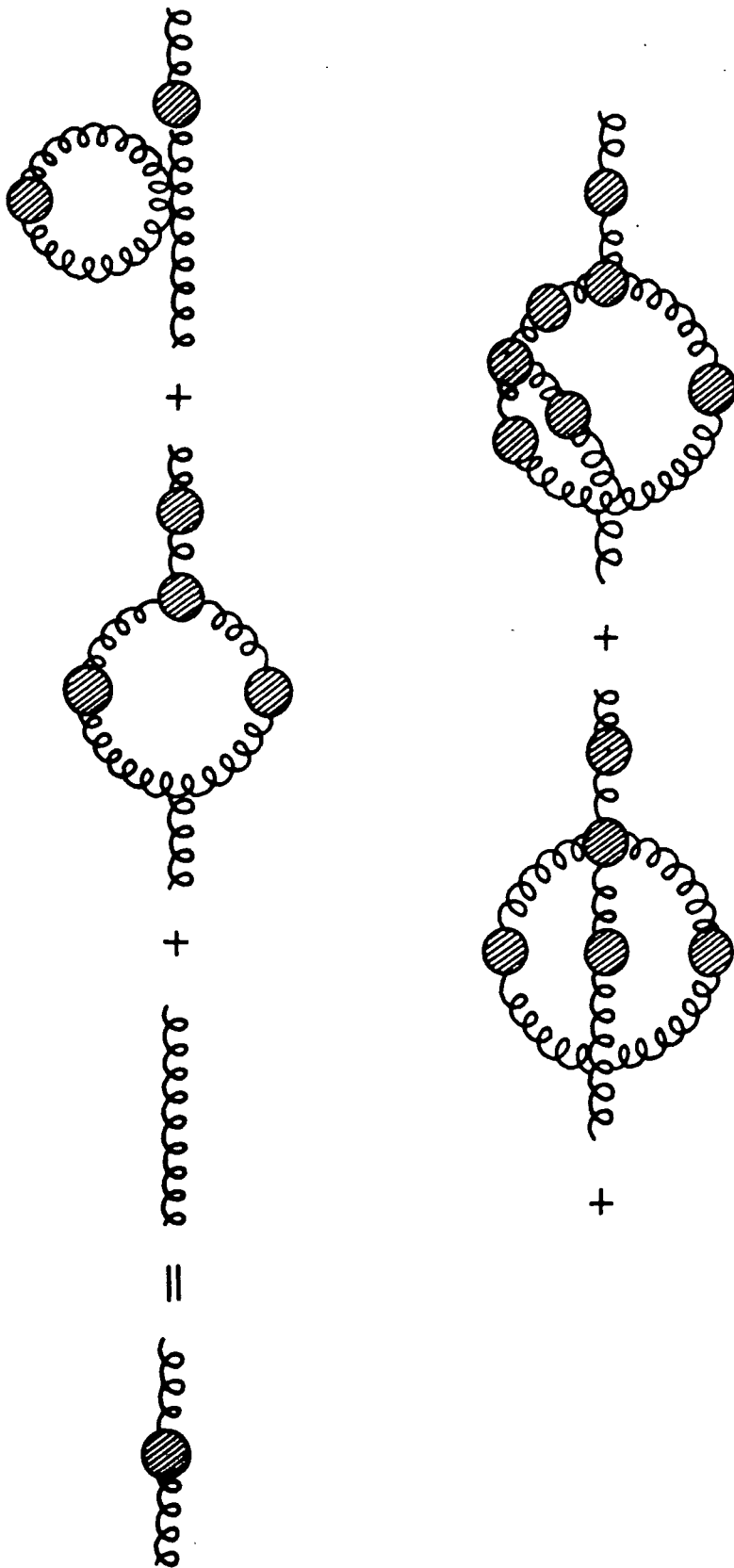
Figure II.4   The gauge propagator
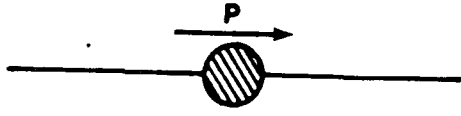
$$\frac{-iF(p^2)}{\not{p} + m}$$
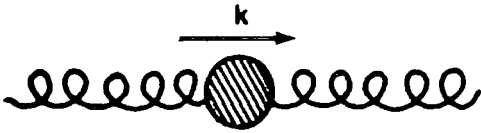
$$\frac{G(k^2)}{k^2}\left\{\delta_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}\right\}$$
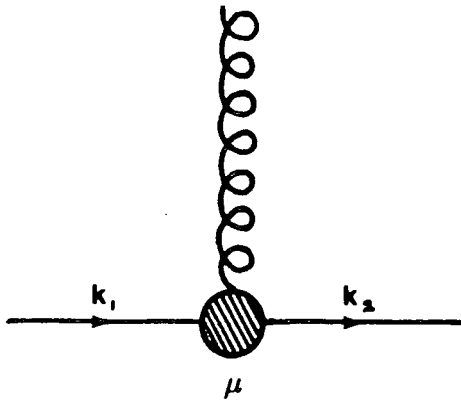
$$-ig(\mu)^{2-\omega}\left\{\frac{1}{2}\left[\frac{1}{F(k_1^2)} + \frac{1}{F(k_2^2)}\right]\gamma_\mu \right.$$
$$\left. +\frac{1}{2}\left[\frac{1}{F(k_1^2)} - \frac{1}{F(k_2^2)}\right]\frac{(k_1 + k_2)_\mu(\not{k}_1 + \not{k}_2)}{(k_2^2 - k_2^2)}\right\}$$

$$F(p^2) \equiv \frac{ap^2}{(p^2 + p_o^2)}$$

$$G(k^2) \equiv \gamma\left(1 + \frac{\sigma}{k^2}\right)$$

Figure II.5   **Renormalised feynman rules**

CHAPTER III THE FERMION SELF ENERGY GRAPH

III.1 CALCULATION OF THE FERMION SELF ENERGY GRAPH USING RENORMALISED FEYNMAN RULES

In this, and the following chapter we use the renormalised Feynman rules of [Fig.II.5] to evaluate the singular parts of the fermion self energy graph of [Fig.II.2c], and the fermion-photon vertex graph of [Fig.II.2d] respectively.

As previously, the calculation will be performed in $n \equiv 2(2 - \epsilon)$ dimensional Euclidean space, where divergences will occur only when $\epsilon \to 0$.

III.1.1 The self energy graph

With the long term aim of calculating the physical cross-section for $e^+e^- \to 2,3$ jets in mind, rather than just evaluating the self-energy graph of [Fig.II.2c] we 'add on' a photon vertex to give the, physically more relevant $O(g_s^2)$ $\gamma^* \to q\bar{q}$ graph of [Fig.III.1].

This graph has the same singularities as the self-energy graph since the photon is added outside the closed loop. It is an one-particle-reducible addition that, usually, would be cut off the diagram.

Further, in calculating the cross-section for $e^+e^- \to q\bar{q}$ the graph of [Fig.III.1] is contracted with the amplitude for $q\bar{q} \to \gamma^*$ to give a contribution to the |amplitude|^2. We can achieve this, again without introducing any additional divergences, by evaluating the graph of [Fig.III.2]. The dashed line through the fermion lines in this figure indicates that it is to be evaluated by the Cutkosky rules [Ref.III.1], where these fermion momenta are on shell. ie

$$p_i^2 \quad = \quad E_i^2 + \vec{p}_i^2 \quad = \quad m_i^2 \qquad (3.1)$$

with this condition the only loop integral is over the momentum variable k, which is just the self energy part of the graph.
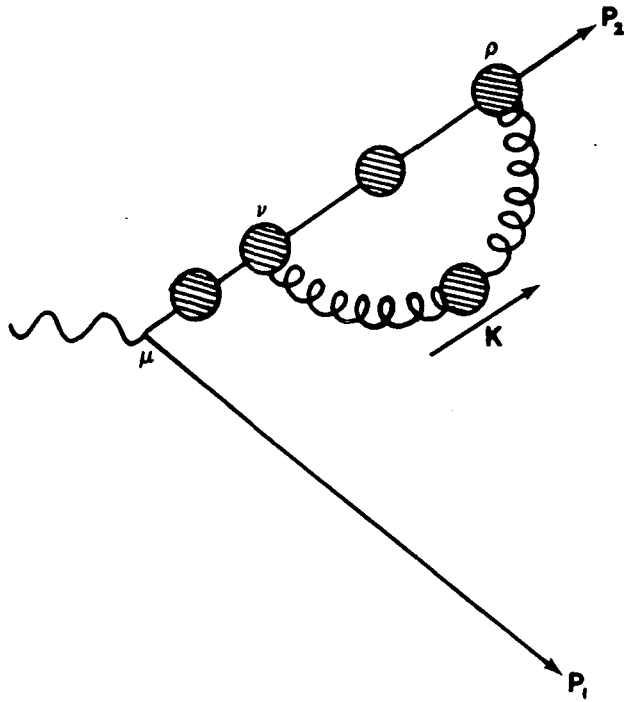
Figure III.1 $\gamma^* \to q\bar{q}$
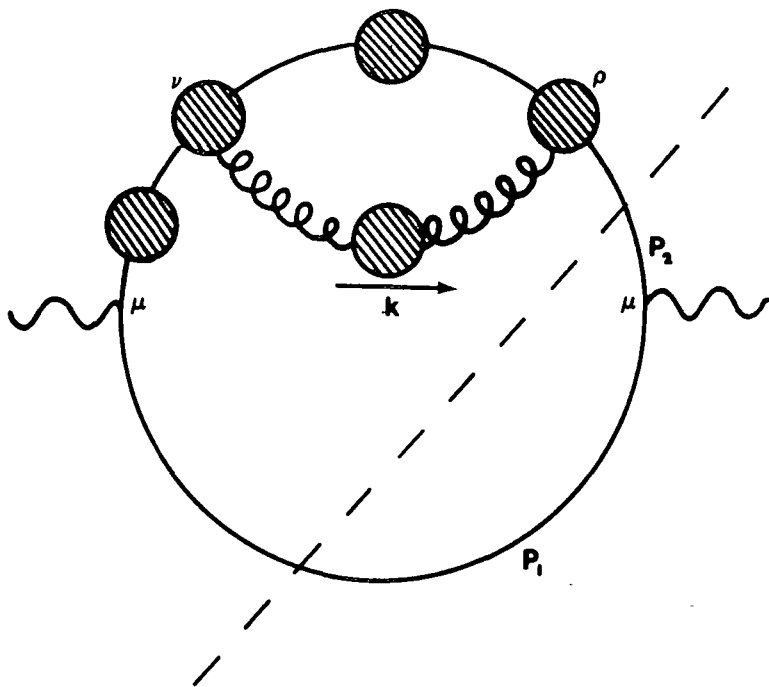
Figure III.2 $|e^+ e^- \to q\bar{q}|^2$

Thus [Fig.III.2] still has the same singular behaviour as the original graph of [Fig.II.2c]. It is to [Fig.III.2] that we now turn on the grounds of it's relevance to the physical process.

Using the Feynman rules of [Fig.II.5] the spin averaged (over the photon polarisations) amplitude for [fig.III.2] is

$$|\overline{m}|^2 \quad = \quad \frac{C_2(F)\bar{u}(P_2)}{2} \int \frac{d^n k}{(2\pi)^n} \Gamma_\rho[-p_2, (k-p_2)] S_F(k-p_2) \Gamma_\nu[(k-p_2), -p_2]$$

$$\times S_F(-p_2)(-ie\gamma_\mu)u(p_1)\Delta_{\nu\rho}(k^2)\bar{u}(p_1)(-ie\gamma_\mu)u(p_2) \tag{3.2}$$

where the $u(p)$ are the full fermion wavefunctions, $C_2(F)$ is the colour factor associated with the fermion loop, and e is the electromagnetic charge of the fermions.

Now, using renormalised functions, the completeness relation for massless Dirac spinors becomes

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) \quad = \quad F(p^2)\not{p}. \tag{3.3}$$

Using this relation twice in [equ.3.2] gives

$$|\overline{m}|^2 \quad = \quad \frac{-e^2 C_2(F)F(p_1^2)F(p_2^2)}{2} \int \frac{d^n k}{(2\pi)^n} Tr\{\not{p}_2 \Gamma_\rho[-p_2, (k-p_2)] S_F(k-p_2)$$

$$\times \Gamma_\nu[(k-p_2), -p_2] S_F(-p_2)\gamma_\mu \Delta_{\nu\mu}(k^2)\not{p}_1 \gamma_\mu\}. \tag{3.4}$$

Defining the variables,

$$A \quad = \quad \frac{1}{2}\left[\frac{1}{F[(-p_2)^2]} + \frac{1}{F[(k-p_2)^2]}\right]$$

$$B \;=\; \frac{1}{2}\left[\frac{1}{F[(k-p_2)^2]} - \frac{1}{F[(-p_2)^2]}\right]\frac{1}{[(k-p_2)^2 - (-p_2)^2]}, \qquad (3.5)$$

and using the identity of [equ.B.1], [equ.3.4] can be written as

$$|\overline{m}|^2 = C_2(F)(ge)^2(\mu^2)^{2-\omega}(1-\epsilon)F(p_1^2)F(p_2^2)\int\frac{d^n k}{(2\pi)^n}\frac{F[(k-p_2)^2]}{(k-p_2)^2}\frac{F(p_2^2)}{p_2^2}\frac{G(k^2)}{k^2}$$

$$\times Tr\{\not{p}_2[A\gamma_\mu + B(\not{k}-2\not{p}_2)(k-2p_2)_\mu](k-p_2)[A\gamma_\mu + B(\not{k}-2\not{p}_2)(k-2p_2)_\mu]\not{p}_2\not{p}_1\},$$
$$(3.6)$$

where we use the notation $n \equiv 2\omega \equiv 2(2-\epsilon)$ interchangably for ease of exposition.

To evaluate this last expression we find it convenient to break it up into two pieces: the A^2 term and the $(AB + BA + B^2)$ term. As shown in [Fig.II.5] $G(k^2)$ is the sum of two terms, one singular at $k=0$ and the other a constant. Consequently, in the following calculations, we insert two forms of $G(k^2)$ in turn (the equations being linear in G) and add the resulting terms with appropriate coefficients, given in [Fig.II.5]. First we input the constant term, $G(k^2)=1$, and secondly the singular, $G(k^2)= k^{-2}$, term. The latter contributions to the amplitude will be denoted by a subscript $G(k^2)=k^{-2}$ and the former by $G(k^2)=1$ subscripts.

First we consider the piece proportional to A^2.

III.2 THE A² CONTRIBUTION TO $|\overline{m}|^2$.

Extracting the A^2 term from [equ.3.6] gives,

$$|\overline{m}|^2{}_{A^2} \equiv \lambda F(p_1^2)\frac{F^2(P_2^2)}{p_2^2}\int\frac{d^n k}{(2\pi)^n}\frac{F[(k-p_2)^2]}{(k-p_2)^2}\frac{G(k^2)}{k^2}A^2 Tr[\not{p}_2\gamma_\mu(\not{k}-\not{p}_2)\gamma_\mu\not{p}_2\not{p}_1],$$
$$(3.7)$$

where λ is defined by

$$\lambda \;=\; C_2(F)(ge)^2(\mu^2)^{2-\omega}(1-\epsilon) \quad .$$

Using the gamma-matrix identities of equations [B.1] and [B.2] the trace can be straightforwardly evaluated as,

$$2(1-\epsilon)2^{2-\epsilon}[2p_1 \cdot p_2(k \cdot p_2 - p^2) - p^2(k \cdot p_1 - p_1 \cdot p_2)],$$

and from [equ.3.5] we get,

$$A^2 = \frac{1}{4F^2[(k-p_2)^2]F^2[(p_2)^2]}\left\{F^2(p_2^2) + 2F(p_2^2)F[(k-p_2)^2] + F^2[(k-p_2)^2]\right\} \quad .$$

Combining these last two expressions in [equ.3.7] now gives,

$$|\overline{m}|^2_{A^2} = \lambda(1-\epsilon)2^{1-\epsilon}\frac{F(p_1^2)}{p_2^2}\int\frac{d^n k}{(2\pi)^n}G(k^2)\frac{[2p_{12}(k \cdot p_2 - p_2^2) - p_2^2(k \cdot p_1 - p_{12})]}{k^2(k-p_2)^2}$$

$$\times\left[\frac{F^2(p_2^2)}{F[(k-p_2)^2]} + 2F(p_2^2) + F[(k-p_2)^2]\right] \tag{3.8}$$

where $p_{12} \equiv p_1 \cdot p_2$.

To evaluate this equation we first set $G(k^2) = 1$ and determine the three terms of the last bracket separately. Then the procedure will be repeated with $G(k^2) = k^{-2}$.

III.2.1 The $G(k^2) = 1$ contribution to $|\overline{m}|^2_{A^2}$

Setting $G(k^2) = 1$ and using the explicit form of $F[(k-p_2)^2]$ from [Fig.II.5], the $F^2(p_2^2)/F[(k-p_2)^2]$ term of [equ.3.8] is

$$J_1^1 \equiv 2^{1-\epsilon}\lambda(1-\epsilon)\frac{F(p_1^2)F(p_2^2)}{ap_2^2}\int\frac{d^nk}{(2\pi)^n}\frac{[2p_{12}(k\cdot p_2 - p_2^2) - p_2^2(k\cdot p_1 - p_{12})]}{k^2(k-p_2)^4}$$

$$\times[(k-p_2)^2 + p_0^2], \tag{3.9}$$

where the superscript on the left hand side indicates the $G(k^2) = 1$ contribution, and the subscript indicating the first term of [equ.3.8].

Defining

$$\rho_1 \equiv 2^{1-\epsilon}\lambda(1-\epsilon)\frac{F(p_1^2)F^2(p_2^2)}{ap_2^2(2\pi)^n}$$

and Feynman parameterising the denominator using the identity of [equ.A.1] gives,

$$J_1^1 = 2\rho_1\int_0^1 dx\, x\int d^nk\frac{[2p_{12}(k\cdot p_2 - p_2^2) - p_2^2(k\cdot p_1 - p_{12})][(k-p_2)^2 + p_0^2]}{x(k-p_2)^2 + (1-x)k^2]^3}.$$

Changing the integration variable to

$$k' \equiv k - p_2 x \tag{3.10}$$

which amounts to a shift in the origin and does not affect the measure allows the integral to be written as,

$$J_1^1 = 2\rho_1\int_0^1 dx\, x\int d^nk'\frac{[2p_{12}(k'\cdot p_2 - p_2^2(1-x)) - p_2^2(k'\cdot p_1 - p_{12}(1-x))]}{[k'^2 + x(1-x)p_2^2]^3}$$

$$\times[(k'-p_2(1-x))^2 + p_0^2]. \tag{3.11}$$

By noting that the denominator is now only a function of k'^2 (from henceforth k' will be written as k) and that in dimensional regularisation

$$\int d^n k F(k^2) k_\mu \equiv 0,$$

the only terms that contribute to [equ.3.11] are those pieces of the numerator even in k'. Keeping only these terms [equ.3.11] becomes

$$J_1^1 = 2\rho_1 \int_0^1 dx \, x \int \frac{d^n k}{[k^2 + L]^3} \{ -2p_{12}(k \cdot p_2)^2 2(1 - x)$$

$$-p_{12} p_2^2 (1 - x)[k^2 + p_2^2(1 - x)^2 + p_o^2] + 2p_2^2(k \cdot p_1)(k \cdot p_2)(1 - x) \}, \qquad (3.12)$$

where

$$L \equiv x(1 - x)p_2^2.$$

Now using [equ.2.12], the dot products of k and p_i can be written, under the k integral, as

$$(k \cdot p_1)(k \cdot p_2) \to \frac{k^2 p_{12}}{n} \quad , \quad (k \cdot p_2)^2 \to \frac{k^2 p_2^2}{n} \qquad (3.13)$$

giving, after a little rearranging,

$$J_1^1 = -2\rho_1 p_{12} p_2^2 \int_0^1 dx \, x \int \frac{d^n k}{[k^2 + L]^3} \left\{ \frac{(2 + n)}{n}(1 - x)k^2 + (1 - x)^3 p_2^2 + (1 - x)p_o^2 \right\}.$$

$$(3.14)$$

We can now perform the $d^n k$ integral using equations [A.3] and [A.4] to obtain

$$J_1^1 = -2\rho_1 p_{12} p_2^2 \int_0^1 dx\, x(\pi)^{2-\epsilon} \frac{\Gamma(\epsilon)}{2} \left[\frac{(3-\epsilon)(1-x)}{[x(1-x)p_2^2]^\epsilon} + \frac{\epsilon[p_2^2(1-x)^3 + p_o^2(1-x)]}{[x(1-x)p_2^2]^{1+\epsilon}} \right].$$

The x-dependence of all the terms in this equation is of the form $x^a(1-x)^b$, so the remaining x integral can be done using [equ.A.2] to yield

$$J_1^1 = -2\rho_1 p_{12} p_2^2 (\pi)^{2-\epsilon} \frac{\Gamma(\epsilon)}{2} \left[(3-\epsilon)(p_2^2)^{-\epsilon} \frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)} \right.$$

$$\left. + \frac{\epsilon}{(p_2^2)^{1+\epsilon}} \left(p_2^2 \frac{\Gamma(3-\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-2\epsilon)} + p_o^2 \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \right) \right].$$

To extract the poles from this equation we expand around $\epsilon = 0$ the identities of equations [B.6], [B.7], [B.8], so that

$$J_1^1 = -\rho_1 p_{12} p_2^2 \pi^2 [1 - \epsilon ln(\pi p_2^2)] \left(\frac{1}{\epsilon} - \nu_E \right) \left\{ \frac{(3 - \epsilon)}{[\cdots} \qquad \frac{\nu_E]}{]} \right.$$

$$\left. + \frac{\epsilon}{p_2^2} \left[p_2^2 \frac{(2-\epsilon)[1-\epsilon(1-\nu_E)](1+\epsilon\nu_E)}{[(6-\epsilon(7-\nu_E)]} + \frac{p_o^2(1+\epsilon\nu_E)^2}{[1-2\epsilon(1-\nu_E)]} \right] \right\}$$

where ν_E is the Euler-Mascaroni constant. (The apparently dimensionful argument of the logarithm combines with the $(\mu^2)^\epsilon$ in ρ_1 to give $[1 - \epsilon ln(\pi p_2^2/\mu^2)]$).

Keeping only the $1/\epsilon^n$ poles as $\epsilon \to 0$ yields for the singular parts of J_1^1

$$J_1^1|_{singular} = -\frac{\rho_1 p_{12} p_2^2 \pi^2}{2\epsilon} = -\frac{C_2(F)(ge)^2 F(p_1^2)F(p_2^2)p_{12}}{16\pi^2 a\epsilon} \qquad (3.15)$$

Having completed the calculation of the singular parts of the first term of [equ.3.8] we now return to this equation to evaluate the singular parts of the $2F(p_2^2)$ term of $|\overline{m}|_{A^2}^2$. The $F(p_2^2)$ can be extracted from under the integral to give

$$J_2^1 \equiv \rho_2 \int \frac{d^n k}{k^2 (k - p_2)^2} [2p_{12}(k \cdot p_2 - p_2^2) - p_2^2(k \cdot p_1 - p_{12})]$$

where

$$\rho_2 \equiv 2^{2-\epsilon}(1 - \epsilon)\lambda \frac{F(p_1^2)F(p_2^2)}{p_2^2(2\pi)^n}.$$

Feynman parameterising the denominator of this integral and changing variables in exactly the same way as previously, we now get,

$$J_2^1 = \rho_2 \int_0^1 dx \int \frac{d^n k}{(k^2 + L)^2} [2p_{12}(k \cdot p_2 - p_2^2(1 - x)) - p_2^2[k \cdot p_2 - p_{12}(1 - x)]],$$

and again, keeping even terms in k only, then leaves,

$$J_2^1 = \rho_2 \int_0^1 dx \int d^n k \frac{-p_{12}p_2^2(1 - x)}{(k^2 + L)^2}. \tag{3.16}$$

This expression can be integrated over $d^n k$ using [equ.A.5] as

$$J_2^1 = -\rho_2 p_{12} p_2^2 \pi^{n/2} \frac{\Gamma(n/2 - 1)}{(2 - n/2)} \int_0^1 dx \frac{(1 - x)}{[x(1 - x)p_2^2]^{(2-n/2)}},$$

and we can now perform the x integral as before leaving, after expressing n in terms of ϵ ,

$$J_2^1 = -\rho_2 p_{12} p_2^2 \pi^{2-\epsilon} \frac{\Gamma(1 - \epsilon)}{\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(2 - \epsilon)}{\Gamma(3 - 2\epsilon)},$$

which on expanding about $\epsilon = 0$, and keeping only the singular $1/\epsilon$, leaves for the singular part

$$J_2^1|_{singular} - \frac{\rho_2 p_{12} p_2^2 \pi^2}{2\epsilon} = \frac{-C_2(F)(ge)^2 F(p_1^2)F(p_2^2)p_{12}}{8\epsilon \pi^2}. \tag{3.17}$$

Finally from [equ.3.8], the last term, proportional to $F(k - p_2^2)$, is evaluated in a similar way. The Feynman parameterisation is exactly the same as in the previous term giving, after discarding terms odd in k,

$$j_3^1 = -\rho_3 \int_0^1 dx p_{12} p_2^2 (1 - x) \int \frac{d^n k}{[k^2 + L']^2} \tag{3.18}$$

where

$$\rho_3 \equiv 2^{1-\epsilon} \lambda (1 - \epsilon) \frac{F(p_1^2)a}{(2\pi)^{2(2-\epsilon)} p_2^2}$$

and now,

$$L' \equiv p_o^2 x + p_2^2 x (1 - x). \tag{3.19}$$

Performing the k integral we get,

$$j_3^1 = -\rho_3 p_{12} p_2^2 \int_0^1 dx (1 - x) \frac{\Gamma(1 - \epsilon)}{\epsilon} [p_o^2 x + x(1 - x)p_2^2]^{-\epsilon}.$$

To perform the x integral we expand the square bracket using [equ.B.8], as

$$[p_o^2 x + x(1 - x)p_2^2]^{-\epsilon} = (-1^{-\epsilon})\{1 - \epsilon ln[-xp_o^2(1 + (1 - x)p_2^2/p_o^2)]\}.$$

Integrating the first term immediately, gives

$$j_3^1 = -\rho_3 p_{12} p_2^2 \pi^{2-\epsilon} \frac{\Gamma(1 - \epsilon)}{\epsilon} \left\{ \frac{1}{2} - \epsilon \int_0^1 dx (1 - x)[lnx + ln[-p_o^2 - (1 - x)p_2^2]] \right\}.$$

For $p_2^2, p_o^2 < 0$ the remaining integral is finite and the explicit factors of ϵ in the numerator and denominator cancel. Hence the singular part of j_3^1 is given by the first term of this equation,

$$J_3^1|_{singular} = -p_3 p_{12} p_2^2 \frac{\Gamma(1-\epsilon)}{2\epsilon} = -\frac{C_2(F)(ge)^2 p_{12} F(p_1^2)a}{16\pi^2\epsilon} \qquad (3.20)$$

This completes the evaluation of [equ.3.8] with $G(k^2) = 1$. Combining J_1^1, J_2^1, J_3^1, given by equations [3.15], [3.17] and [2.20] respectively, we get, for the $G(k^2) = 1$ contribution to $|\overline{m}|_{A^2}^2$

$$|\overline{m}|_{A^2}^2|_{G(k^2)=1} = -\frac{C_2(F)(ge)^2 p_{12} F(p_1^2)}{16\pi^2\epsilon} \left\{ \frac{F^2(p_2^2)}{a} + 2F(p_2^2) + a \right\}$$

or, using [Fig.II.5] to express $F(p_2^2)$,

$$|\overline{m}|_{A^2}^2|_{G(k^2)=1} = -\frac{C_2(F)(ge)^2 p_{12} F(p_1^2)}{16\pi^2\epsilon(p_0^2 + p_2^2)^2} a[4p_2^4 + 4p_o^2 p_2^2 + p_o^4]. \qquad (3.21)$$

In the limit $F \equiv G \equiv 1$ the renormalised functions we are using here reduce to the bare ones of [Fig.II.3]. In this limit $A \to 1$ and $B \to 0$ in [equ.3.5]. As such it is the A^2 term, in [equ.3.21], that will give whole bare answer. Taking this limit in [equ.2.21] yields

$$|\overline{m}|_{bare}^2 = -\frac{C_2(F)(ge)^2 p_{12}}{4\pi^2\epsilon}. \qquad (3.22)$$

Previously, in chapter II, we evaluated the bare self energy (equation [2.11]). To compare this to [equ.3.22] we must first embed the bare self energy in the amplitude of [Fig.III.2] as discussed at the begining of this chapter. That is we must add on a photon-fermion vertex and external wavefunctions to the isolated self energy (Σ) of [equ.2.11], and then contract this with the graph of $\gamma^* \to q\bar{q}$. Doing this we immediately obtain

$$-i|\overline{m}|^2 = \frac{C_2(F)}{2} Tr\{\bar{u}(p_2)(ie\gamma_\mu)S_F^0(p_1)\Sigma(p_1)u(p_1)\bar{u}(p_1)(ie\gamma_\mu)u(p_2)\}$$

where $C_2(F)$ is a fermion loop colour factor and the factor of 1/2 comes from a spin averaging over initial spin states. $S_F^0(p_1)$ is the bare fermion propagator given in [Fig.II.3]. Substituting in the expresssions for S_F^0 and Σ and contracting wavefunctions using the completeness relation

$$\sum_s \bar{u}(p)^s u(p)^s = \not{p}$$

yields

$$-i|\overline{m}|^2 = \frac{i(eg)^2 C_2(F)}{16\pi^2\epsilon} Tr\{\not{p}_2\gamma_\mu\not{p}_1\gamma_\mu\}.$$

Evaluating the trace then gives

$$|\overline{m}|^2 = -\frac{(eg)^2 C_2(F)p_{12}}{4\pi^2\epsilon} \tag{3.23}$$

which, as it should be, is identical to [equ.3.22], the bare limit of the renormalised calculation.

Having, thus obtained the $G(k^2) = 1$ contribution to the A^2 term of $|\overline{m}|^2$, we now go back to [equ.3.8] and set $G(k^2) = k^{-2}$. In a similar way to the above calculation we now determine the $G(k^2) = k^{-2}$ contribution to $|\overline{m}|^2_{A^2}$.

III.2.2 The G(k^2)=k^{-2} contribution to $|\overline{m}|^2_{A^2}$

Setting $G(k^2) = k^{-2}$ explicitly increases the overall factor of k in the denominator to k^4. This increases the power of $[k^2 + L]$ by one, after Feynman parameterising. There are also additional factors of x and $(1-x)$ from [equ.A.1]

but the numerator of the $d^n k$ integral is unchanged. Consequently we can use some of the previous intermediate steps, omitting some of the algebra.

As before, we consider the three terms of [equ.3.8] separately, but now with $G(k^2) = k^{-2}$. The first term, analogous to [equ.3.9], becomes

$$j_1^{k^{-2}} \equiv 2^{1-\epsilon}\lambda(1-\epsilon)\frac{F(p_1^2)F(p_2^2)}{p_2^2 a} \int \frac{d^n k}{(2\pi)^n} \frac{[2p_{12}(k \cdot p_2 - p_2^2) - p_2^2(k \cdot p_1 - p_{12})]}{k^4(k-p_2)^4}$$

$$\times [(k-p_2)^2 + p_o^2],$$

and in the same way as we arrived at [equ.3.14] we now get,

$$j_1^{k^{-2}} = -\rho_1 p_2^2 p_{12}\frac{\Gamma(4)}{\Gamma(2)} \int_0^1 dx x(1-x) \int \frac{d^n k}{[k^2+L]^4} \left\{ \frac{(2+n)}{n}(1-x)k^2 \right.$$

$$\left. +(1-x)^3 p_2^2 + (1-x)p_o^2 \right\}.$$

and using the integrals of equations [A.6] and [A.7] we can perform the k integrals leaving

$$j_1^{k^{-2}} = -\rho_1 p_{12} p_2^2 \frac{\Gamma(4)}{\Gamma^2(2)} \frac{\pi^{n/2}}{6} \int_0^1 dx x(1-x) \left\{ \frac{(2+n)}{2} \frac{(1-x)\Gamma(3-n/2)}{L^{3-n/2}} \right.$$

$$\left. +\frac{[(1-x)^3 p_2^2 + (1-x)p_o^2]\Gamma(4-n/2)}{L^{4-n/2}} \right\},$$

we can immediately integrate this expression using [equ.A.2] to obtain, in terms of ϵ,

$$j_1^{k^{-2}} = -\rho_1 p_{12} p_2 \pi^{2-\epsilon} \frac{\Gamma(4)}{(p_2^2)^\epsilon} \frac{\Gamma(1+\epsilon)}{6} \left\{ \frac{(2+n)}{n} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)} \right.$$

$$+\frac{(1+\epsilon)}{p_2^2}\left[p_2^2\frac{\Gamma(3-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)}+p_o^2\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\right]\Big\}.$$

The first term is finite for $\epsilon\to 0$ and the last two terms give the divergent $1/\epsilon$ pole, via the $\Gamma(-\epsilon)$ in the numerator,

$$j_1^{k^{-2}}\to\frac{\rho_1 p_{12}\pi^2}{\epsilon p_2^2}(p_2^2+p_o^2)$$

which, substituting for ρ_1 immediately gives,

$$j_1^{k^{-2}}=\frac{C_2(F)p_{12}(eg)^2}{8a\pi^2\epsilon p_2^4}(p_2^2+p_o^2)F(p_1^2)F^2(p_2^2). \tag{3.24}$$

Now, from [equ.3.15], we can immediately write for the $2F(p_2^2)$ term of [equ.3.8]

$$j_2^{k^{-2}}\equiv-2\rho_2 p_{12}p_2^2\int_0^1 dx(1-x)^2\int\frac{d^n k}{[k^2+L]^3}.$$

We can integrate this expression, in the now familiar way, to obtain

$$j_2^{k^{-2}}=-\rho_2 p_{12}p_2^2\pi^{2-\epsilon}\frac{\Gamma(1+\epsilon)}{(p_2^2)^{1+\epsilon}}\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}$$

which, keeping only the singular parts gives,

$$j_2^{k^{-2}}\to\frac{\rho_2 p_{12}p_2^2\pi^2}{2p_2^2\epsilon}\quad=\frac{C_2(F)(eg)^2 p_{12}}{8\pi^2 p_2^2\epsilon}F(p_1^2)F(p_2^2). \tag{3.25}$$

Finally, we consider the $F[(k-p_2)^2]$ term: from [equ.3.18] we get,

$$j_3^{k^{-2}}\equiv-2\rho_3 p_2^2 p_{12}\int_0^1 dx(1-x)^2\int\frac{d^n k}{[k^2+L']^3}$$

where now, as in [equ.3.19],

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$$L' \equiv p_o^2 + p_2^2 x(1 - x).$$

The k integral is performed, leaving

$$J_3^{k^{-2}} = -\rho_3 p_{12} p_2^2 \pi^{2-\epsilon} \Gamma(1 + \epsilon) \int_0^1 dx \frac{(1 - x)^2}{[p_o^2 x + x(1 - x)p_2^2]^{1+\epsilon}}.$$

The denominator of this expression can be written as

$$x^{1+\epsilon}[p_o^2 + (1 - x)p_2^2]$$

since the square bracket is finite and integrable for $\epsilon = 0$ and contributes no $1/\epsilon$ singularities. That is the only singularities come from the common factor of $x^{-1-\epsilon}$. So,

$$J_3^{k^{-2}} = -\rho_3 p_{12} p_2^2 \pi^{2-\epsilon} \Gamma(1 + \epsilon) \int_0^1 dx \frac{(1 - x)^2 x^{-1-\epsilon}}{[p_o^2 + (1 - x)p_2^2]}.$$

To perform this integral we expand the square bracket of the denominator as

$$\frac{1}{(p_o^2 + (1 - x)p_2^2)} = \frac{1}{(p_o^2 + p_2^2)} \sum_{i=0}^{\infty} C_i^{-1} x^i \left(\frac{p_2^2}{p_o^2 + p_2^2}\right)^i$$

where C_i^{-1} are the bionomial coefficients. Inserting this expression into the previous equation yields

$$J_3^{k^{-2}} = -\frac{\rho_3 p_{12} p_2^2 \pi^{2-\epsilon} \Gamma(1 + \epsilon)}{(p_o^2 + p_2^2)} \int_0^1 dx \sum_{i=0}^{\infty} dx (1 - x)^2 x^{-1-\epsilon+i} \left(\frac{p_2^2}{p_o^2 + p_2^2}\right)^i.$$

We can now change the order of the integration and summation and integrate this expression term by term. However, we immediately note that the only

divergent term will come from the $i = 0$ contribution of the sum. With $C_0^{-1} = 1$ we therefore get

$$J_3^{k^{-2}}\big|_{singular} = -\frac{\rho_3 P_{12} p_2^2 \pi^{2-\epsilon} \Gamma(1+\epsilon)}{(p_o^2 + p_2^2)} \int_0^1 dx(1-x)^2 x^{-1-\epsilon}$$

$$= -\frac{\rho_3 p_{12} p_2^2 \pi^{2-\epsilon} \Gamma(1+\epsilon)}{(p_o^2 + p_2^2)} \frac{\Gamma(3)\Gamma(-\epsilon)}{\Gamma(3-\epsilon)}.$$

Expanding the gamma functions about $\epsilon = 0$ now leaves

$$J_3^{k^{-2}} = \frac{\rho_3 p_{12} p_2^2 \pi^2}{\epsilon(p_o^2 + p_2^2)} = \frac{C_2(F)(eg)^2 F(p_1^2) a p_{12}}{8\epsilon\pi^2(p_o^2 + p_2^2)}. \tag{3.26}$$

Having, now evaluated all three terms of [equ.3.8] we can add the contributions, given by equations [3.24], [3.25] and [3.26], to get, for the singular part of the $G(k^2) = k^{-2}$ contribution to $|\overline{m}|_{A^2}^2$

$$|\overline{m}|_{A^2}^2\big|_{G(k^2)=1/k^2} = \frac{C_2(F)(eg)^2 p_{12} F(p_1^2)}{8\epsilon\pi^2} \left\{ \frac{F^2(p_2^2)(p_o^2 + p_2^2)}{a p_2^4} + \frac{F(p_2^2)}{p_2^2} + \frac{a}{(p_o^2 + p_2^2)} \right\}.$$

Expanding the bracket almost immediately gives ,

$$|\overline{m}|_{A^2}^2\big|_{G(k^2)=1/k^2} = \frac{3 C_2(F)(eg)^2 p_{12} F(p_1^2) a}{8\pi^2\epsilon(p_o^2 + p_2^2)}. \tag{3.27}$$

So, finally adding equations[3.21] and [3.27], with the latter weighted by σ (from [Fig.II.5]), we can write down the full singular contribution from the A^2 term of [equ.3.6] as,

$$|\overline{m}|_{A^2}^2 = -\frac{C_2(F)(eg)^2 p_{12}\gamma a}{16\pi^2\epsilon(p_o^2 + p_2^2)} \left\{ \frac{[4p_2^4 + p_o^2 p_2^2 + p_o^4]}{(p_o^2 + p_2^2)} - 6\sigma \right\}. \tag{3.28}$$

Having thus far calculated the singular parts of the term proportional to A^2 in [equ.3.6] we now turn to the remaining terms of $|\overline{m}|^2$ which are calculated in a similar way in the next section.

·III.3 THE AB AND B² TERMS OF $|\overline{m}|^2$

If we consider the remaining terms of [equ.3.6] together as

$$|\overline{m}|^2_{AB} \equiv \lambda \frac{F(p_1^2)F(p_2^2)}{p_2^2} \int \frac{d^n k}{(2\pi)^n} \frac{F[(k-p_2)^2]G(k^2)}{(k-p_2)^2 k^2} Tr \Big\{ \not{p}_2 \not{p}_1 \not{p}_2$$

$$\times [AB\gamma_\mu(\not{k}-\not{p}_2)(\not{k}-2\not{p}_2)(k-2p_2)_\mu + BA(\not{k}-2\not{p}_2)(k-2p_2)_\mu(\not{k}-\not{p}_2)\gamma_\mu$$

$$+B^2(k-2p_2)_\mu(\not{k}-2\not{p}_2)(\not{k}-\not{p}_2)(\not{k}-2\not{p}_2)(k-2p_2)_\mu]\Big\}, \qquad (3.29)$$

the trace can be immediately simplified to

$$Tr[2AB + B^2(k-2p_2)^2][\not{p}_2\not{p}_1\not{p}_2(\not{k}-2\not{p}_2)(\not{k}-\not{p}_2)(\not{k}-2\not{p}_2)].$$

Now, from [equ.3.5] and [Fig II.5], after a little algebra we can write A and B as,

$$B = \frac{-p_o^2}{2ap_2^2(k-p_2)^2} \quad , \quad A = \frac{[p_2^2(k-p_2)^2 + p_o^2(p_2^2 + (k-p_2)^2]}{ap_2^2(k-p_2)^2},$$

from which it is straightforward to show,

$$[2AB + B^2(k-2p_2)^2] = \frac{-p_o^2}{4a^2 p_2^4 (k-p_2)^4}[p_o^2 k^2 + 4p_2^2 + 4p_2^2(k-p_2)^2].$$

Inserting these results into [equ.3.29] now yields,

$$|\overline{m}|^2_{AB} \equiv -\frac{\lambda F(p_1^2) F^2(p_2^2) p_o^2}{4a p_2^6 (2\pi)^n} \int d^n k \frac{G(k^2)}{k^2} Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k} - 2\not{p}_2)(\not{k} - \not{p}_2)(\not{k} - 2\not{p}_2)]$$

$$\times \frac{[p_o^2 k^2 + 4 p_2^2 (k - p_2)^2]}{(k - p_2)^4 [(k - p_2)^2 + p_o^2]}. \qquad (3.30)$$

As in $|\overline{m}|^2_{A^2}$ we first solve this equation with $G(k^2) = 1$. Having done this, and making use of some of these results, we repeat the calculation with $G(k^2) = k^{-2}$

III.3.1 The $G(k^2) = 1$ contribution to $|\overline{m}|^2_{AB}$

For brevity we define the variable α as,

$$\alpha \equiv -\frac{\lambda F(p_1^2) F^2(p_2^2) p_o^2}{4a p_2^6 (2\pi)^n}.$$

We first consider the piece of [equ.3.30] proportional to $p_o^2 k^2$. That is $J^1_{p_o^2}$ where, again the superscript 1 indicates the $G(k^2) = 1$ contribution,

$$J^1_{p_o^2} \equiv \alpha p_o^2 \int d^n k \frac{Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k} - 2\not{p}_2)(\not{k} - \not{p}_2)(\not{k} - 2\not{p}_2)]}{(k - p_2)^4 [(k - p_2)^2 + p_o^2]}. \qquad (3.31)$$

Feynman parameterising in the usual way gives,

$$J^1_{p_o^2} = 2\alpha p_o^2 \int_0^1 dx (1 - x) \int d^n k \frac{Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k} - 2\not{p}_2)(\not{k} - \not{p}_2)(\not{k} - 2\not{p}_2)]}{[(k - p_2)^2 + x p_o^2]^3},$$

and shifting the origin in k space by defining $k' \equiv (k - p_2)$ we obtain,

$$J^1_{p_o^2} = 2\alpha p_o^2 \int_0^1 dx (1 - x) \int d^n k \frac{Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k}' - \not{p}_2)\not{k}'(\not{k}' - \not{p}_2)]}{[(k' + L]^3},$$

where, now $L \equiv xp_o^2$. As usual, we keep only terms even in k', the trace can then be evaluated using [equ.B.4] and the previous equation reduces to,

$$J_{p_o^2}^1 = 2p_o^2 \alpha 2^{1+n/2} \, p_{12} p_2^2 \int_0^1 dx (1-x) \int \frac{d^n k k^2}{[k^2 + L]^3}.$$

Performing the k integral then leaves,

$$J_{p_o^2}^1 = 2^{3-\epsilon} p_{12} p_2^2 p_o^2 \alpha \pi^{2-\epsilon} \frac{(2-\epsilon)\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dx \frac{(1-x)}{[xp_o^2]^\epsilon},$$

which can be immediately integrated over x to give,

$$2^{3-\epsilon} p_{12} p_2^2 p_o^2 \alpha \pi^{2-\epsilon} \frac{(2-\epsilon)}{\epsilon} \frac{\Gamma(1+\epsilon)}{(p_o^2)^\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2)}{\Gamma(3-\epsilon)}$$

and a pole contribution of,

$$J_{p_o^2}^1|_{singular} = \frac{8p_o^2 p_{12} p_2^2 \pi^2 \alpha}{\epsilon} \quad = -C_2(F)(eg)^2 F(p_1^2) F^2(p_2^2) p_o^4 p_{12} 8 a \pi^2 p_2^4 \epsilon. \quad (3.32)$$

Now returning to [equ.3.30] and the remaining term, proportional to p_2^2,

$$J_{p_2^2}^1 \equiv 4p_2^2 \alpha \int \frac{d^n k}{k^2(k-p_2)^2} \frac{Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k}-2\not{p_2})(\not{p}-\not{p_2})(\not{k}-2\not{p_2})]}{[(k-p_2)^2 + p_o^2]},$$

can be written, using partial fractions, as

$$J_{p_2^2}^1 \equiv \frac{4\alpha p_2^2}{p_o^2} \{I(0, p_2^2) - I(p_o^2, p_2^2)\} \quad (3.33)$$

with

$$I(p_o^2, p_2^2) \equiv \int \frac{d^n k Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k}-2\not{p_2})(\not{p}-\not{p_2})(\not{k}-2\not{p_2})]}{k^2[(k-p_2)^2 + p_o^2]}. \quad (3.34)$$

Feynman parameterising this last expression as,

$$I(p_o^2, p_2^2) = \int_0^1 dx \int \frac{d^n k \, Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k} - 2\not{p_2})(\not{k} - \not{p_2})(\not{k} - 2\not{p_2})]}{[k^2 + x(p_o^2 + p_2^2 - 2k \cdot p_2)]^2},$$

we shift the origin by $k' \equiv k - p_2 x$ to give the denominator as

$$[k'^2 + L]^2 \equiv [k'^2 + x(1 - x)p_2^2 + xp_o^2]^2,$$

and, keeping only even terms in k' in the trace, after somewhat lengthy but straightforward manipulation of gamma matrices, the numerator becomes

$$2^{n/2} p_2^2 p_{12}[k^2(3 - x) + p_2^2(1 - x)(2 - x)^2]$$

$$+ 2^{1+n/2}(1 - x)k \cdot p_2[2p_{12}k \cdot p_2 - p_2^2 k \cdot p_1].$$

Using [equ.3.13] to express the dot products in k gives,

$$I(p_o^2, p_2^2) = \frac{2^{n/2} p_{12} p_2^2}{n} \int_0^1 dx \int \frac{d^n k}{[k^2 + L]^2} \Big[k^2[3n + 2 - (2 + n)x] + np_2^2(1 - x)(2 - x)^2 \Big],$$

$$(3.35)$$

which we can integrate over k in the usual way to yield,

$$I(p_o^2, p_2^2) = \frac{2^{n/2} p_{12} p_2^2}{n} \frac{\pi^{n/2} \Gamma(n/2 - 1)}{(2 - n/2)} \int_0^1 dx \Big[-\frac{n[3n + 2 - (2 + n)x]}{2(n/2 - 1)} L^{1-\epsilon}$$

$$+ \frac{np_2^2(1 - x)(2 - x)^2}{L^\epsilon} \Big].$$

The remaining integral over x is finite for $\epsilon \to 0$ and we can set $\epsilon \equiv 0$ everywhere except for the $(2 - n/2) = \epsilon$ in the denominator to obtain,

$$I(p_o^2, p_2^2) = \frac{2^2 \pi^2 p_{12} p_2^2}{4\epsilon} \int_0^1 dx \Big[2[(2+n)x - (3n+2)][xp_o^2 + x(1-x)p_2^2]$$

$$+ 4p_2^2(1-x)(2-x)^2 \Big].$$

The last term can be integrated trivially to immediately give,

$$I(p_o^2, p_2^2) = \frac{\pi^2 p_{12} p_2^2}{\epsilon} \Big[\frac{17p_2^2}{3} + 2 \int_0^1 dx [(2+n)x - (3n+2)][xp_o^2 + x(1-x)p_2^2] \Big]. \quad (3.36)$$

The other combination of (p_o^2, p_2^2) required in [equ.3.33] is $(0, p_2^2)$. Setting $p_o^2 \equiv 0$ in the previous equation thus gives

$$I(0, p_2^2) = \frac{\pi^2 p_{12} p_2^2}{\epsilon} \Big[\frac{17p_2^2}{3} + 2 \int_0^1 dx [(2+n)x - (3n+2)]x(1-x)p_2^2 \Big]. \quad (3.37)$$

Rather than evaluating the last two equations separately we notice that in [equ.3.33] we only need their difference. Consequently [equ.3.37] cancels with the identical term in [equ.3.36] leaving,

$$I(0, p_2^2) - I(p_o^2, p_2^2) = \frac{2\pi^2 p_{12} p_2^2}{\epsilon} \int_0^1 dx \, x p_o^2 [(3n+2 - (2+n)x].$$

This, trivially, yields,

$$I(0, p_2^2) - I(p_o^2, p_2^2) = \frac{10\pi^2 p_2^2 p_o^2}{\epsilon}, \quad (3.38)$$

and combining [equ.3.33] and [equ.3.38] now gives,

$$J^1_{p^2_2} = 4\alpha p^2_2 \frac{10\pi^2 p^2_2 p^2_o}{\epsilon} = -\frac{10 C_2(F)(eg)^2 F(p^2_1) F^2(p^2_2) p^2_o p_{12}}{16 a p^2_2 \pi^2 \epsilon}. \qquad (3.39)$$

· So, combining equations [3.32] and [3.39] we get, for the $G(k^2) = 1$ contribution to the $[2AB + B^2(k - p^2_2)]$ term,

$$|\overline{m}|^2_{AB}|_{G(k^2)=1} = -C_2(F)(eg)^2 \frac{F(p^2_1) F^2(p^2_2) p_{12} p^2_o}{8 a \pi^2 p^4_2 \epsilon} \left[p^2_o + 5 p^2_2 \right]. \qquad (3.40)$$

Now, finally for this graph we have to calculate the $G(k^2) = k^{-2}$ contribution to the $[2AB + B^2(k - 2p_2)]$ term as follows:

III.3.2 The $G(k^2) = k^{-2}$ contribution to $|\overline{m}|^2_{AB}$

Again, splitting [equ.3.30] into two parts we return first to the term proportional to p^2_o. So, setting $G(k^2) = k^{-2}$, [equ.3.31] becomes,

$$J^{k^{-2}}_{p^2_o} \equiv \alpha p^2_o \int d^n k \frac{Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k} - 2\not{p_2})(\not{k} - \not{p_2})(\not{k} - 2\not{p_2})]}{k^2(k - p_2)^4[(k - p_2) + p^2_o]}. \qquad (3.41)$$

Expressing the denominator as the sum of partial fractions gives,

$$J^{k^{-2}}_{p^2_o} = \frac{\alpha}{p^2_o} \int d^n k Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k} - 2\not{p_2})(\not{k} - \not{p_2})(\not{k} - 2\not{p_2})]$$

$$\times \left[\frac{p^2_o}{k^2(k - p_2)^4} - \frac{1}{k^2(k - p_2)^2} + \frac{1}{k^2[(k - p_2)^2 + p^2_o]} \right],$$

where we note that the last two terms are just

$$\frac{-\alpha}{p^2_o}[I(0, p^2_2) - I(p^2_o, p^2_2)]$$

of [equ.3.34]. So, using the result of [equ.3.38] we immediately get,

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$$J_{p_o^2}^{k^{-2}} = \alpha \int \frac{d^n k}{k^2(k-p_2)^4} Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k} - 2\not{p}_2)(\not{k} - \not{p}_2)(\not{k} - 2\not{p}_2)] - \frac{10\alpha p_{12} p_2^2 \pi^2}{\epsilon}. \quad (3.42)$$

Defining the remaining integral as I, and Feynman parameterising in the usual way gives,

$$I \equiv 2\alpha \int_0^1 dx\, x \int d^n k \frac{Tr[\not{p}_2 \not{p}_1 \not{p}_2 (\not{k} - 2\not{p}_2)(\not{k} - \not{p}_2)(\not{k} - 2\not{p}_2)]}{[(k - xp_2)^2 + x(1-x)p_2^2]^3}.$$

Using the same shift in variables as in $I(p_o^2, p_2^2)$ (*ie* $k' \equiv k - p_2 x$), the numerator can be directly lifted from [equ.3.35] to give, with $L' \equiv x(1-x)p_2^2$,

$$I = 2\alpha \int_0^1 dx\, x \int \frac{d^n k'}{[k' + L']^3} \frac{2^{n/2} p_{12} p_2^2}{n}[k'^2[3n + 2 - (2+n)x] + np_2^2(1-x)(2-x)^2].$$
$$(3.43)$$

Integrating over k' yields,

$$I = \alpha \frac{2^{2-\epsilon} p_{12} p_2^2}{(2-\epsilon)} \int_0^1 dx \frac{\pi^{2-\epsilon} \Gamma(1+\epsilon)}{2}\left[(2-\epsilon)\frac{[6(2-\epsilon) + 2 - 2(1 + (2-\epsilon))x]}{\epsilon L'^\epsilon}\right.$$

$$\left. + \frac{2(2-\epsilon)p_2^2(1-x)(2-x)^2}{L'^{(1+\epsilon)}}\right].$$

The last term is finite for $\epsilon = 0$ and can be dropped, leaving potential pole terms coming from,

$$I \to 2^{2-\epsilon} \alpha p_{12} p_2^2 \pi^{2-\epsilon} \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dx\, x \frac{[3(2-\epsilon) + 1 - x[1 + (2-\epsilon)]]}{[x(1-x)p_2^2]^\epsilon},$$

which can now be integrated over x yielding,

$$I = 2^{2-\epsilon} \alpha p_{12} p_2^2 \pi^{2-\epsilon} \frac{\Gamma(1+\epsilon)}{\epsilon} \Big\{ [3(2-\epsilon)+1] \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)}$$

$$-(3-\epsilon) \frac{\Gamma(3-\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-2\epsilon)} \Big\},$$

which after a little rearranging leaves,

$$I = \frac{10 \alpha p_{12} p_2^2 \pi^2}{\epsilon}. \tag{3.44}$$

Substituting this result into [equ.3.42] we immediately see that the pole terms exactly cancel in $J_{p_2^2}^{k^{-2}}$, which thus contributes nothing to $|\overline{m}|_{AB}^2|_{G(k^2)=k^{-2}}$.

The final contribution to $|\overline{m}|_{G(k^2)=k^{-2}}^2$ remaining to be calculated is the piece of [equ.3.30] proportional to p_2^2,

$$J_{p_2^2}^{k^{-2}} \equiv 4\alpha p_2^2 \int d^n k \frac{Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k}-2\not{p_2})(\not{k}-\not{p_2})(\not{k}-2\not{p_2})]}{k^4(k-p_2)^2[(k-p_2)^2+p_o^2]},$$

which can be written in terms of partial fractions as

$$J_{p_o^2}^{k^{-2}} = \frac{4\alpha p_2^2}{p_o^2} \{ \tilde{I}(0,p_2^2) - \tilde{I}(p_o^2,p_2^2) \}, \tag{3.45}$$

where,

$$\tilde{I}(p_o^2,p_2^2) \equiv \int d^n k \frac{Tr[\not{p_2}\not{p_1}\not{p_2}(\not{k}-2\not{p_2})(\not{k}-\not{p_2})(\not{k}-2\not{p_2})]}{k^4[(k-p_2)^2+p_o^2]}.$$

From [equ.3.35] we can instantly write,

$$\tilde{I}(p_o^2,p_2^2) = 2 \int_0^1 dx (1-x) \int \frac{d^n k}{[k^2+L]^3} \frac{2^{n/2} p_{12} p_2^2}{n} \{ k^2[3n+2-(2+n)x]$$

$$+np_2^2(1-x)(2-x)^2\},$$

with $L \equiv xp_o^2 + x(1-x)p_2^2$. Using equations [A.3] and [A.4] we can perform the k integral to get,

$$\tilde{I}(p_o^2,p_2^2) = \frac{2^{1-\epsilon}p_{12}p_2^2\pi^{2-\epsilon}\Gamma(1+\epsilon)}{(2-\epsilon)} \int_0^1 dx(1-x)\left\{[6(2-\epsilon)+2-[2+2(2-\epsilon)]x]\frac{(2-\epsilon)}{\epsilon L^\epsilon}\right.$$

$$\left.+\frac{2(2-\epsilon)p_2^2(1-x)(2-x)^2}{L^{1+\epsilon}}\right\}. \tag{3.46}$$

Considering the terms in the first square bracket only, the contribution to $\tilde{I}(0,p_2^2) - \tilde{I}(p_o^2,p_2^2)$ is,

$$\frac{2^{2-\epsilon}p_{12}p_2^2\pi^{2-\epsilon}\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dx(1-x)[3(2-\epsilon)+1-(3-\epsilon)x]$$

$$\times\left\{[x(1-x)p_2^2]^{-\epsilon} - [xp_o^2+x(1-x)p_2^2]^{-\epsilon}\right\}. \tag{3.47}$$

Using [equ.B.8], the last bracket of this expression can be expanded as,

$$[1 - \epsilon ln[x(1-x)p_2^2] - 1 + \epsilon ln[xp_o^2 + x(1-x)p_2^2] + O(\epsilon^2)]$$

and we can see that the result is $O(\epsilon)$ which cancels the ϵ in the denominator of [equ.3.47] such that the $L^{-\epsilon}$ terms of [equ.3.46] contribute no pole pieces to $J_{p_2^2}^{k^{-2}}$.

Hence the only pole terms in $\tilde{I}(0,p_2^2) - \tilde{I}(p_o^2,p_2^2)$ come from the second term of [equ.3.46],

$$2^{2-\epsilon}p_{12}p_2^4\pi^{2-\epsilon}\Gamma(1+\epsilon) \int_0^1 dx(1-x)^2(2-x)^2$$

$$\times \left\{ \frac{1}{[x(1-x)p_2^2]^{1+\epsilon}} - \frac{1}{[xp_o^2 + x(1-x)p_2^2]^{1+\epsilon}} \right\}.$$

The factors of $(1 - x)$ in the numerator ensure that the similar terms in the denominator are nullified and as such the only divergent poles come from the explicit $x^{1+\epsilon}$ in the denominator. Extracting these factors we can, in the remainder of the expression set ϵ identically equal to zero and get,

$$\tilde{I}(0,p_2^2) - \tilde{I}(p_o^2,p_2^2) \rightarrow 4\pi^2 p_{12} p_2^2 \int_0^1 dx (1-x)^2 (2-x)^2 x^{-1-\epsilon}$$

$$\times \left\{ \frac{1}{(1-x)p_2^2} - \frac{1}{[(1-x)p_2^2 + p_o^2]} \right\}.$$

Combining the two terms of the curly bracket gives

$$4\pi^2 p_{12} p_2^2 p_o^2 \int_0^1 dx \frac{(1-x)[1 + (1-x)]^2 x^{-1-\epsilon}}{[p_o^2 + (1-x)p_2^2]}$$

which, on expanding the denominator in a binomial expansion yields,

$$4\pi^2 p_{12} p_2^2 \int_0^1 dx [(1-x) + 2(1-x)^2 + (1-x)^3] x^{-1-\epsilon} \sum_{i=0} C_{-1}^i \left\{ \left(\frac{p_2^2}{p_o^2} \right) (1-x) \right\}^i.$$

We can now integrate this expression term by term as

$$4\pi^2 p_{12} p_2^2 \sum_{i=0} C_{-1}^i \left(\frac{p_2^2}{p_o^2} \right)^i \Gamma(-\epsilon) \left\{ \frac{\Gamma(i+2)}{\Gamma(i+2-\epsilon)} + \frac{2\Gamma(i+3)}{\Gamma(i+3-\epsilon)} + \frac{\Gamma(i+4)}{\Gamma(i+4-\epsilon)} \right\}.$$

Expanding around $\epsilon = 0$ and recognising the sum as $(1 + p_2^2/p_o^2)^{-1}$ gives, for the pole pieces,

$$\tilde{I}(0, p_2^2) - \tilde{I}(p_o^2, p_2^2) \rightarrow -\frac{16 p_{12} p_2^2 p_o^2 \pi^2}{\epsilon(p_o^2 + p_2^2)}.$$

Substituting this into [equ.45] we get,

$$J_{p_2^2}^{k^{-2}}\Big|_{singular} = -\frac{4\alpha p_{12} p_2^4 16}{\epsilon(p_o^2 + p_2^2)} = -\frac{C_2(F)(eg)^2 F(p_1^2) F^2(p_2^2) p_{12} p_o^2}{\epsilon a p_2^2 \pi^2 (p_o^2 + p_2^2)}. \tag{3.48}$$

This last expression is now the pole terms of $|\overline{m}|^2_{G(k^2)=1/k^2}$ (since the pieces of [equ.3.27] with $G(k^2) = k^{-2}$ given in [equ.3.41] were not singular).

Thus, adding [equ.3.40] and [equ.3.48] with appropriate weighting of the terms we get the complete expression for the singular parts of $|\overline{m}|^2_{AB}$ as,

$$|\overline{m}|^2_{AB} = -\frac{C_2(F)(eg)^2 F(p_1^2) F^2(p_2^2) p_{12} p_o^2 \gamma}{8\pi^2 \epsilon p_2^4 a}\left\{5p_2^2 + p_o^2 - \frac{8\sigma p_2^2}{(p_o^2 + p_2^2)}\right\}. \tag{3.49}$$

Finally, we just have to add [equ.3.28] and [equ.3.49] to get the full, singular contribution to the |amplitude|^2 of [Fig.III.2] as,

$$|\overline{m}|^2 = -\frac{C_2(F)(eg)^2 F(p_1^2) p_{12} \gamma a}{16\pi^2 \epsilon (p_o^2 + p_2^2)^2}\left\{[4p_2^4 + 14p_2^2 p_o^2 + 3p_o^4] - \sigma\frac{[6p_o^4 + 6p_2^4 + 28p_2^2 p_o^2]}{(p_o^2 + p_2^2)}\right\}. \tag{3.50}$$

In the limit $F \equiv g \equiv 1$ ($a, \gamma \equiv 1$, $\sigma, p_0 \equiv 0$), which reduces the renormalised functions to the bare ones, this last expression should yield [equ.3.23] evaluated earlier as the bare |amplitude|^2 of [Fig.III.2]. Setting $\gamma = a = 1$ and $\sigma = p_0^2 = 0$ immediately yields [equ.3.23] and, as it had to, the renormalised amplitude reduces to the bare one, with a single $1/\epsilon$ pole.

However we really want to compare the result in [equ.3.50] with the full, renormalised graph of [Fig.II.2d]. It is to this graph that we now turn in the following chapter.

CHAPTER IV THE FERMION-PHOTON VERTEX GRAPH

IV.1 THE FERMION-PHOTON VERTEX GRAPH

Having previously incorporated the self energy graph of [Fig.II.2c] into the expression for the |amplitude|^2 for the physical proccess $e^+e^- \to q\bar{q}$, we must now do the same for the fermion-photon vertex graph of [Fig.II.2d]. That is contract this graph with the graph for $q\bar{q} \to \gamma^*$. However, [Fig.II.2d], in the language of $\gamma^* \to q\bar{q}$ has one more internal fermion propagator than [Fig.III.1]. This increases the complexity of the integrand under the loop integral. However, since there are only two final particle states $(q\bar{q})$, in [Fig.II.2d] the contraction with $q\bar{q} \to \gamma^*$ is straightforward, and we therefore found it convenient to evaluate the graph of [Fig.II.2d] before contracting it with $q\bar{q} \to \gamma^*$. (Rather than the other way round, as previously when calculating the contribution from [Fig.II.2c])

Hence, for the amplitude of [Fig.II.2d.] we get,

$$-im_\mu = C_2(F) \int \frac{d^n k}{(2\pi)^n} \bar{u}(p_2)\Gamma_\rho[-p_2,(k-p_2)]S_F(k-p_2)(-ie\gamma_\mu)$$

$$\times S_F(k+p_1)\Gamma_\nu[(k+p_1),p_1)]\Delta_{\nu\rho}(k^2)u(p_1). \tag{4.1}$$

Defining

$$A \equiv \frac{1}{2}\left[\frac{1}{F[(k-p_2)^2]} + \frac{1}{F(p_2^2)}\right] \qquad B \equiv \frac{1}{2}\left[\frac{1}{F[(k-p_2)^2]} - \frac{1}{F(p_2^2)}\right]\frac{1}{[(k-p_2)^2 - p_2^2]}$$

$$C \equiv \frac{1}{2}\left[\frac{1}{F[(k+p_1)^2]} + \frac{1}{F(p_1^2)}\right] \qquad D \equiv \frac{1}{2}\left[\frac{1}{F[(k+p_1)^2]} - \frac{1}{F(p_1^2)}\right]\frac{1}{[(k+p_1)^2 - p_1^2]}$$

the Feynman rules of [Fig.II.5] can be recast in a form such that [equ4.1] can be written,

$$-im_\mu = -iC_2(F)eg^2(\mu^2)^{2-n/2} \int \frac{d^nk}{(2\pi)^n} \frac{G(k^2)}{k^2} \frac{F[(k-p_2)^2]}{(k-p_2)^2} \frac{F[(k+p_1)^2]}{(k+p_1)^2} \bar{u}(p_2)$$

$$\times [A\gamma_\nu + B(\not{k}-2\not{p_2})(k-2p_2)_\nu](\not{k}-\not{p_2})\gamma_\mu(\not{k}+\not{p_1})[C\gamma^\nu + D(\not{k}+2\not{p_1})(k+2p_1)^\nu]u(p_1).$$

$$(4.2)$$

Again, this equation is linear in $G(k^2)$, so we can consider the constant and k^{-2} singular parts of $G(k^2)$ separately. Also, as previously, we find it convenient to break the previous equation into pieces and evaluate each piece in turn, first setting $G(k^2)=1$, followed by $G(k^2) = k^{-2}$, adding the results at the end.

IV.2 THE AC CONTRIBUTION TO -im_μ

We extract the piece proportional to AC from the last equation as

$$-im_{AC} \equiv -iC_2(F)(eg)^2(\mu^2)^{2-n/2} \int \frac{d^nk}{(2\pi)^n} \frac{G(k^2)}{k^2} \frac{F[(k+p_1)^2]}{(k+p_1)^2} \frac{F[(k-p_2)^2]}{(k-p_2)^2}$$

$$\times AC\bar{u}(p_2)\gamma^\nu(\not{k}-\not{p_2})\gamma_\mu(\not{k}+\not{p_1})\gamma_\nu u(p_1). \qquad (4.3)$$

Now, using the explicit form for $F(p^2)$,given in [Fig.II.5],

$$F(x) \equiv \frac{ax}{(x+p_o^2)}$$

AC can be written, after some rearranging, as

$$AC = \frac{1}{4a^2}\left\{4 + \frac{2p_o^2(p_1^2+p_2^2)}{p_1^2 p_2^2} + \frac{p_o^4}{p_1^2 p_2^2} + \frac{p_o^2(2p_2^2+p_o^2)}{p_2^2(k+p_1)^2}\right.$$

$$+\frac{p_o^2(2p_1^2+p_o^2)}{p_1^2(k-p_2)^2}+\frac{p_o^4}{(k+p_1)^2(k-p_2)^2}\Bigg\},$$

which, on inserting into [equ.4.3] we get,

$$m_{AC}=C_2(F)\frac{(eg)^2(\mu^2)^{2-n/2}}{4}\int\frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}\frac{\bar{u}(p_2)\gamma^\nu(\not k-\not p_2)\gamma_\mu(\not k+\not p_1)\gamma_\nu u(p_1)}{[(k+p_1)^2+p_o^2][(k-p_2)^2+p_o^2]}$$

$$\times\Bigg\{4+\frac{2p_o^2(p_1^2+p_2^2)}{p_1^2 p_2^2}+\frac{p_o^4}{p_1^2 p_2^2}+\frac{p_o^2(2p_2^2+p_o^2)}{p_2^2(k+p_1)^2}+\frac{p_o^2(2p_1^2+p_o^2)}{p_1^2(k-p_2)^2}+\frac{p_o^4}{(k+p_1)^2(k-p_2)^2}\Bigg\}.$$

Expanding the denominator using partial fractions gives, after a re-grouping of terms

$$m_{AC}=C_2(F)\frac{(eg)^2(\mu^2)^{2-n/2}}{4}\int\frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}\bar{u}(p_2)\gamma^\nu(\not k-\not p_2)\gamma_\mu(\not k+\not p_1)\gamma_\nu u(p_1)$$

$$\times\Bigg\{\left(1+\frac{p_o^2(p_1^2+p_2^2+p_o^2)}{p_1^2 p_2^2}\right)\frac{1}{[(k+p_1)^2+p_o^2)][(k-p_2^2)+p_o^2]}$$

$$+\frac{(p_o^2+p_2^2)}{p_2^2}\frac{1}{(k+p_1)^2[(k-p_2)^2+p_o^2]}+\frac{(p_o^2+p_1^2)}{p_1^2}\frac{1}{(k-p_2)^2[(k+p_1)^2+p_o^2]}$$

$$+\frac{1}{(k+p_1)^2(k-p_2)^2}\Bigg\}. \tag{4.4}$$

We note that all the terms of this equation are of the form

$$I_\mu(q_o^2,p_o^2)\equiv\int d^n k\frac{G(k^2)}{k^2}\frac{\bar{u}(p_2)\gamma^\nu(\not k-\not p_2)\gamma_\mu(\not k+\not p_1)\gamma_\nu u(p_1)}{[(k+p_1)^2+q_o^2][(k-p_2)^2+p_o^2]}, \tag{4.5}$$

so defining $I(q_o^2, p_o^2)$ by this equation, [equ.4.4] can be written

$$m_{AC} = \frac{C_2(F)(eg)^2(\mu^2)^{2-n/2}}{4(2\pi)^n}\left\{[1 + \frac{p_o^2(p_o^2 + p_1^2 + p_2^2)}{p_1^2 p_2^2}]I_\mu(p_o^2, p_o^2)\right.$$

$$\left. + \frac{(p_o^2 + p_2^2)}{p_2^2}I_\mu(0, p_o^2) + \frac{(p_o^2 + p_1^2)}{p_1^2}I_\mu(p_o^2, 0) + I_\mu(0, 0)\right\}. \tag{4.6}$$

Before evaluating $I_\mu(q_o^2, p_o^2)$ we note that, as it should be, the above expression is symmetric under the interchange $p_1 \leftrightarrow p_2$ and $q_o^2 \leftrightarrow p_o^2$. (The graph of [Fig.II.2d] is symmetric under the interchange $q \leftrightarrow \bar{q}$) Also, that on setting $p_o^2 = q_o^2 \equiv 0$ the curly bracket gives $4I_\mu(0, 0)$. The 4 cancels with the 4 in the denominator and, setting $G(k^2) = 1$ we are left with

$$m_{AC} = \frac{C_2(F)(eg)^2(\mu^2)^{2-n/2}}{(2\pi)^n}I_\mu(0, 0), \tag{4.7}$$

which, as expected, is just the result obtained by using bare propagators and vertices. (Γ_ρ of [equ.2.12] with $\mu \leftrightarrow \rho$.)

Now, returning to evaluate $I_\mu(q_o^2, p_o^2)$ we first set $G(k^2) = 1$

IV.2.1 The G(k^2)=1 contribution to m_{AC}

Setting $G(k^2) = 1$ in [equ.4.5] gives,

$$I_\mu^1(q_o^2, p_o^2) \equiv \int d^n k \frac{\bar{u}(p_2)\gamma_\nu(\slashed{k} - \slashed{p}_2)\gamma_\mu(\slashed{k} + \slashed{p}_1)\gamma^\nu u(p_1)}{k^2[(k + p_1)^2 + q_o^2][(k - p_2)^2 + p_o^2]}. \tag{4.8}$$

where the superscript indicates the $G(k^2) = 1$ contribution.

To evaluate this expression we use the same techniques as used in chapter III. Feynman parameterising the denominator and shifting the origin by defining,

$$k' \equiv k + p_1 x - p_2 y$$

gives, keeping only even terms in k',

$$I^1_\mu(q^2_o, p^2_o) = 2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k}{[k^2 + L]^3} \bar{u}(p_2) \gamma_\nu [\not{k} \gamma_\mu \not{k}$$

$$-(\not{p}_2(1-y) + \not{p}_1 x)\gamma_\mu(\not{p}_1(1-x) + \not{p}_2 y)]\gamma^\nu u(p_1),$$

where L is defined by

$$L \equiv p^2_1 x(1-x) + p^2_2 y(1-y) + q^2_o x + p^2_o y + 2p_{12} xy. \tag{4.9}$$

We can now contract the γ^ν in the numerator using [equ.B.2] to give the numerator of the integrand as,

$$2\bar{u}(p_2)[(1-\epsilon)\not{k}\gamma_\mu\not{k} - (1-x)(1-y)\not{p}_1\gamma_\mu\not{p}_2 + \epsilon xy \not{p}_1\gamma_\mu\not{p}_2]u(p_1).$$

where we have used the Dirac equation for massless particles

$$\not{p} u(p) = \bar{u}(p)\not{p} \equiv 0 \tag{4.10}$$

to eliminate some of the terms. Finally, under the integral, the k dependence of the first term of the numerator can be written

$$\int d^n k \not{k} \gamma_\mu \not{k} = \frac{(1-\epsilon)\gamma_\mu}{(2-\epsilon)} \int d^n k \, k^2$$

and, in the remaining terms the γ_μ can be extracted as

$$\bar{u}(p_2)\not{p}_1\gamma_\mu\not{p}_2 u(p_1) = 2p_{12}\bar{u}(p_2)\gamma_\mu u(p_1).$$

Using these last two identities we arrive at

$$I_\mu^1(q_o^2, p_o^2) = 2^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k}{[k^2 + L]^3} \bar{u}(p_2)\gamma_\mu u(p_1) \times$$

$$\times \left\{ \frac{(1-\epsilon)^2}{(2-\epsilon)} k^2 - 2p_{12}[(1-x)(1-y) - \epsilon xy] \right\}. \qquad (4.11)$$

We note that we have exracted the loop integral as a multiplicative factor to the tree graph of $\gamma^* \to q\bar{q}$, proportional to $\bar{u}(p_2)\gamma_\mu u(p_1)$, which can easily be contracted with the amplitude for $q\bar{q} \to \gamma^*$ at the end of the calculation.

Now, using equations [A.3] and [A.4] we can immediately perform the k integral to get,

$$I_\mu^1(q_o^2, p_o^2) = \Delta_\mu \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{(1-\epsilon)^2}{\epsilon L^\epsilon} - \frac{2p_{12}[(1-x)(1-y) - \epsilon xy]}{L^{1+\epsilon}} \right\}, \qquad (4.12)$$

where, for brevity we define Δ_μ as

$$\Delta_\mu \equiv 2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^{2-\epsilon}\Gamma(1+\epsilon).$$

The first term of [equ.4.12] is evaluated by factorising the denominator as

$$L^{-\epsilon} = [p_1^2 x(1-x) + q_o^2 x + p_2^2 y(1-y) + p_o^2 y + 2p_{12}xy]^{-\epsilon}$$

$$\equiv [-(y - \lambda_+)(y - \lambda_-)]^{-\epsilon}$$

which, in terms of partial fractions gives,

$$L^{-\epsilon} = (\lambda_+ + \lambda_-)^{-\epsilon} \left[\frac{1}{(y - \lambda_+)} - \frac{1}{(y - \lambda_-)} \right]^\epsilon.$$

So the first term of [equ.4.12] becomes,

$$I_\mu^1(q_o^2, p_o^2)_1 \equiv \Delta_\mu \int_0^1 dx \int_0^{1-x} dy \frac{(1-\epsilon)^2}{\epsilon} (\lambda_+ + \lambda_-)^{-\epsilon}$$

$$\times \left\{ 1 + \epsilon ln \left(\frac{1}{y - \lambda_+)} - \frac{1}{(y - \lambda_-)} \right) \right\},$$

where the subscript 1 indicates the first term. The logarithm in the second term of this expression gives a finite integral and the explicit ϵ's in the numerator and denominator cancel so that the singular contributions of the above expression all come from

$$I_\mu^1(q_o^2, p_o^2)_1 \to \frac{\Delta_\mu (1-\epsilon)^2}{\epsilon} \int_0^1 dx \int_0^{1-x} dy (\lambda_+ + \lambda_-)^{-\epsilon}.$$

Expanding this integral as,

$$\Delta_\mu (1-\epsilon)^2 \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{\epsilon} - ln(\lambda_+ + \lambda_-) \right\},$$

again the logarithm gives a finite integral leaving the singular part of $I_\mu^1(q_o^2, p_o^2)_1$ as,

$$I_\mu^1(q_o^2, p_o^2)_1 = \frac{\Delta_\mu}{2\epsilon}. \tag{4.13}$$

We note that the above result is true irrespective of the values of q_o^2, p_o^2, that is

$$I_\mu^1(0, p_o^2)_1 = I_\mu^1(p_o^2, p_o^2)_1 = I_\mu^1(0, 0)_1 = \frac{\Delta_\mu}{2\epsilon}. \tag{4.15}$$

Now, returning to the second term of [equ.4.12], defined by

$$I_\mu^1(q_o^2, p_o^2)_2 \equiv -2p_{12} \Delta_\mu \int_0^1 dx \int_0^{1-x} dy [(1-x)(1-y) - \epsilon xy]$$

$$\times \frac{1}{[p_1^2 x(1-x) + p_2^2 y(1-y) + q_o^2 x + p_o^2 y + 2p_{12}xy]^{1+\epsilon}}.$$

We see that this term is of the same form as the $\Gamma_\rho^{(2)}$ term of [equ.2.16] calculated earlier using bare propagators and vertices except now there are additional $q_o^2 x$ and $p_o^2 y$ terms in the denominator. However these extra terms don't create any new singularities in the integral so, as before with $\Gamma_\rho^{(2)}$ we see that $I_\mu^1(q_o^2, p_o^2)_2$ is finite.

Hence the singular part of [equ.412] is given by [equ.4.13],

$$I_\mu^1(q_o^2, p_o^2) = \frac{\Delta_\mu}{2\epsilon} = \frac{\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2}{\epsilon}. \tag{4.16}$$

In passing we note that, since [equ.4.16] is independent of p_o^2, q_o^2, substituting this result into [equ.4.7] we get,

$$m_{0AC} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)}{16\pi^2\epsilon}$$

which is exactly the result obtained from the calculation using bare functions, as it should be!

So, substituting [equ.4.16] into [equ.4.6] we get, for the singular part of m_{AC} with $G(k^2) = 1$,

$$m_{AC}\big|_{G(k^2)=1} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)}{16\pi^2\epsilon}\left\{1 + \frac{p_o^2[2p_1^2 + 2p_2^2 + p_o^2]}{4p_1^2 p_2^2}\right\}. \tag{4.17}$$

Now we return to [equ.4.5] and set $G(k^2) = k^{-2}$

IV.2.2 The $G(k^2) = k^{-2}$ contribution to m_{AC}

Setting $G(k^2) = k^{-2}$ in [equ.4.5] gives,

$$I_\mu^{k^{-2}}(q_o^2, p_o^2) \equiv \int d^n k \frac{\bar{u}(p_2)\gamma_\nu(\not k - \not p_2)\gamma_\mu(\not k + \not p_1)\gamma^\nu u(p_1)}{k^4[(k+p_1)^2 + q_o^2][(k-p_2)^2 + p_o^2]}. \qquad (4.18)$$

Feynman parameterising gives, from [equ.4.11], with L given by [equ.4.9],

$$I_\mu^{k^{-2}}(q_o^2, p_o^2) = 3!2 \int_0^1 dx \int_0^{1-x} dy (1 - x - y) \int \frac{d^n k}{[k^2 + L]^4} \bar{u}(p_2)\gamma_\mu u(p_1)$$

$$\times \left\{ \frac{(1-\epsilon)^2 k^2}{(2-\epsilon)} - 2p_{12}[(1-x)(1-y) - \epsilon xy] \right\}$$

which we integrate to give,

$$3!2\bar{u}(p_2)\gamma_\mu u(p_1) \frac{\pi^{2-\epsilon}\Gamma(1+\epsilon)}{6} \int_0^1 dx \int_0^{1-x} dy (1 - x - y)$$

$$\times \left\{ \frac{(1-\epsilon)^2}{L^{1+\epsilon}} - 2p_{12}(1+\epsilon)\frac{[(1-x)(1-y) - \epsilon xy]}{L^{2+\epsilon}} \right\}. \qquad (4.19)$$

Initially, considering the first term only, the relevant integral is,

$$I_1(q_o^2, p_o^2) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(1 - x - y)}{[p_1^2 x(1-x) + p_2^2 y(1-y) + q_o^2 x + p_o^2 y + 2p_{12}xy]^{1+\epsilon}}.$$

We shall show that this is finite when $\epsilon \to 0$, by simply setting $\epsilon = 0$ and evaluating it. To do this we consider the relevant combinations of (q_o^2, p_o^2) in turn. First we consider the case of $q_o^2 = 0$ and any p_o^2. It is then clearly conveinient to introduce the variable z, where $y \equiv (1-x)z$, so that

$$I_1(0, p_o^2) = \int_0^1 dx \int_0^1 dz \frac{(1-z)}{[p_1^2 x + p_2^2 z(1 - z(1-x)) + p_o^2 z + 2p_{12}xz]} \qquad (4.21)$$

and we can now trivially interchange the order of the x, z integrations to give simply

$$I_1(0,p_o^2) = \int_0^1 dz(1-z) \int_0^1 dx \frac{1}{[p_o^2 x + p_2^2 z(1-z) + x(p_1^2 + 2p_{12}z + p_2^2 z^2)]}$$

$$= \int_0^1 \frac{dz(1-z)}{p_1^2 + 2p_{12}z + p_2^2 z^2} ln\left(\frac{p_1^2 + p_2^2 z + p_o^2 z + 2p_{12}z}{p_o^2 z + p_2^2 z(1-z)}\right) \qquad (4.22)$$

which for $p_1^2 \neq 0$ is obviously finite since $\int_0 dzln(z)$ is finite. We note parenthetically that the apparent asymmetry between the treatment of p_o, q_o (ie. which is zero or not) is illusory. If $p_o^2 = 0$ for any q_o^2 then the one would interchange the order of the x, y integrals in [équ.4.19] first and let $x = (1-y)z$ instead. The answer is always symmetric under the interchange $p_o^2 \leftrightarrow q_o^2$, $p_1 \leftrightarrow p_2$. Thus this integral is always finite for p_1^2, p_2^2 non-zero just as expected from equations [4.18] and [4.19] being both ultraviolet and infrared finite.

When $p_0, q_o^2 \neq 0$, then there is no question that the singularity structure of [equ.4.19] is unchanged by setting $p_1^2 = p_2^2 = 0$ which greatly simplifies the calculation and is numerically accurate for $|p_1^2|, |p_2^2| \ll p_o^2, q_o^2$. Then with $p_1^2 = p_2^2 = 0$

$$I_1(q_o^2, p_o^2) = \int_0^1 dx \int_0^{1-x} dy \frac{1-x-y}{[q_o^2 x + y(p_o^2 + 2p_{12}x)]}$$

$$= \int_0^1 \frac{dx}{p_o^2 + 2p_{12}x}\left[\left(1 - x + \frac{q_o^2 x}{p_o^2 + 2p_{12}x}\right)ln\left(1 + \frac{(1-x)(p_o^2 + 2p_{12}x)}{xq_o^2}\right) - (1-x)\right] \qquad (4.23)$$

Again it is clear that this is finite since $\int_0^1 dxln(x)$ is finite and no other denominators vanish for $p_o^2, q_o^2 \neq$. Consequently, $I_1(p_o^2, p_o^2)$ is finite with no ϵ poles.

Having computed the first term of [equ.4.19] with the relevant combinations of (q_o^2, p_o^2) we now return to the second term of this equation:

$$I_2(q_o^2, p_o^2) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)[(1-x)(1-y) - \epsilon xy]}{[p_1^2 x(1-x) + p_2^2 y(1-y) + q_o^2 x + p_o^2 y + 2p_{12} xy]^{2+\epsilon}}$$
(4.24)

Again we consider the case when either or both of p_o^2, q_o^2 is zero when we cannot simplify [equ.4.24] by setting $p_1^2 = p_2^2 = 0$. Let $q_o^2 = 0$ and as in equations [4.21] and [4.22] we set $y = (1-x)z$ and then

$$I_2(0, p_o^2) = \int_0^1 dx(1-x)^{1-\epsilon} \int_0^1 \frac{dz(1-z)[1-z+(1-\epsilon)xz]}{[p_1^2 x + p_2^2 z(1-z) + p_2^2 xz + p_o^2 z + 2p_{12} xz]^{2+\epsilon}}$$
(4.25)

Since we are interested in the limit $\epsilon \to 0$, we write the numerator as $(1-x)(1-z)(1-z+xz) + O(\epsilon)$, noting that the $O(1)$ and $O(\epsilon)$ integrals have the same singularity structure. Intechanching the x, z integrations,

$$I_2(0, p_o^2) = \int_0^1 dz(1-z) \int_0^1 dx \frac{(1-x)(1-z+xz) + O(\epsilon)}{[a+bx]^{2+\epsilon}}$$

where $a \equiv p_o^2 z + p_2^2 z(1-z)$ and $b \equiv p_1^2 + p_2^2 z + 2p_{12} z$. It is then straight forward to deduce that

$$
\begin{aligned}
I_2(0, p_o^2) = \int_0^1 \frac{dz(1-z)}{b} \Bigg\{ & \left(1 + \frac{a}{b}\right)\left(1 - z\left(1 + \frac{a}{b}\right)\right) \frac{[a^{-1-\epsilon} - (a+b)^{-1-\epsilon}]}{1+\epsilon} \\
& + \frac{1}{b}\left(-1 + 2z\left(1 + \frac{a}{b}\right)\right) \frac{[a^{-\epsilon} - (a+b)^{-\epsilon}]}{\epsilon} \\
& + \frac{z}{b^3} \frac{[a^{1-\epsilon} - (a+b)^{1-\epsilon}]}{1-\epsilon} \Bigg\}(1 + O(\epsilon))
\end{aligned}
$$
(4.26)

Since $p_1^2 \neq 0, p_o^2 \neq 0$, b never vanishes in the region of integration (provided for instance $p_1^2, p_2^2, 2p_{12}$ have the same sign), thus the only singularities that could

occur must arise at the end-points of integration. a, of course, has an explicit factor of z which makes the part of the first term in the integrand with $a^{-1-\epsilon}$ divergent. This we shall explicitly evaluate shortly. However, all the other terms in [equ.4.26] are finite. the last term with the $(1-\epsilon)$ factor is the most obvious having only positive powers of z. the middle term with the explicit $1/\epsilon$ looks superficially divergent from this factor, but this is cancelled by the numerator since

$$\lim_{\epsilon 0} \frac{[a^{-\epsilon} - (a+b)^{-\epsilon}]}{\epsilon} = ln\left(\frac{a+b}{a}\right)$$

which again gives a finite contribution. All that remains is to discuss the first term in [equ.4.26]

$$\int_0^1 \frac{dz(1-z)}{b}\left(1+\frac{a}{b}\right)\left(1-z\left(1+\frac{a}{b}\right)\right)\frac{[a^{-1-\epsilon}-(a+b)^{-1-\epsilon}]}{1+\epsilon}(1+O(\epsilon)) \quad (4.27)$$

now, of course, when $\epsilon \to 0$, the a and (a+b) do not cancel and any singularity can only occur from the $a^{-1-\epsilon}$ term with its explicit $z^{-1-\epsilon}$, since $\int_0^1 dz z^{-1-\epsilon} = \Gamma(-\epsilon)/\Gamma(1-\epsilon) = -1/\epsilon$. By inspection, the singular piece is contained in

$$\int_0^1 \frac{dz}{b} a^{-1-\epsilon}(1+O(z))(1+O(\epsilon)) \quad (4.28)$$

$$= \int_0^1 dz z^{-1-\epsilon}(p_1^2 + p_2^2 z + 2p_{12}z)(p_o^2 + p_2^2 - p_2^2 z)^{-1-\epsilon}(1+O(z))(1+O(\epsilon))$$

Since any singularity is clearly of $O(1/\epsilon)$, the $O(z)$ and $O(\epsilon)$ terms must give finite contributions. The main part can then be usefully rewritten in the following way:

$$\int_0^1 dz z^{-1-\epsilon}\left\{(p_o^2 + p_2^2 - p_2^2 z)^{-1-\epsilon}[(p_1^2 + p_2^2 z + 2p_{12}z)^{-1} - (p_1^2)^{-1}]\right.$$

$$+(p_1^2)^{-1}[(p_o^2 + p_2^2 - p_2^2 z)^{-1-\epsilon} - (p_o^2 + p_2^2)^{-1}] + (p_1^2)^{-1}(p_o^2 + p_2^2)^{-1}\Big\}$$

Noting that the first square bracket is proportional to z since

$$[(p_1^2 + p_2^2 z + 2p_{12})^{-1} - (p_1^2)^{-1}(p_1^2 + p_2^2 z + 2p_{12})^{-1}(p_2^2 + 2p_{12})z$$

this term, when combined with the explicit $z^{-1-\epsilon}$, gives a finite integral. Similarly the second square bracket can be shown to be finite leaving the only singular parts of $I_2(0, p_o^2)$ coming from the $(p_1^2)^{-1}(p_o^2 + p_2^2)^{-1}$ term. This is easily integrated to give

$$I_2(o, p_o^2) = \frac{-1}{\epsilon p_1^2(p_o^2 + p_2^2)} \tag{4.29}$$

Thus the second term of [equ.4.19] has the singular parts

$$I_{\mu 2}^{k^{-2}}(0,0) = \frac{2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2 2p_{12}}{\epsilon p_1^2 p_2^2}$$

$$I_{\mu 2}^{k^{-2}}(0,p_o^2) = \frac{2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2 2p_{12}}{\epsilon p_1^2(p_o^2 + p_2^2)} \tag{4.30}$$

$$I_{\mu 2}^{k^{-2}}(p_o^2,0) = \frac{2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2 2p_{12}}{\epsilon p_1^2(p_o^2 + p_2^2)}$$

Lastly, we have to evaluate $I_2(p_o^2, p_o^2)$, the singularity structure of which is unaltered by simply setting $p_1^2 = p_2^2 = 0$ then with $a = xp_o^2$, $b = (p_o^2 + 2p_{12})$ we get

$$I_2(p_o^2, p_o^2) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(1 - x - y)[(1-x)(1-y) - \epsilon xy]}{[a + by]^{2+\epsilon}} \tag{4.31}$$

Or, in terms of the variable $w \equiv a + by$,

$$I_2(p_o^2, p_o^2) = \int_0^1 dx \int_a^{a+b(1-x)} \frac{dw}{b^3 w^{2+\epsilon}} [b(1-x)+a-w][-(w-a)(1-x+\epsilon)+(1-x)b].$$

Now, the terms of order $w^{-\epsilon}$ integrate to give terms of order $w^{1-\epsilon}$ which are in turn finite for $\epsilon = 0$. Hence we can drop terms in the numerator quadratic in w. Similarly the terms proportional to $w^{-1-\epsilon}$ give, on integration,

$$\frac{-1}{\epsilon}[(a + b(1 - x))^{-\epsilon} - a^{-\epsilon}],$$

which, on expanding the two terms in the $\epsilon \to 0$ limit using [equ.B.8], yields, on substituting for a and b,

$$-ln\left[\frac{xp_o^2 + (1 - x)(p_o^2 + 2p_{12}x)}{xp_o^2}\right].$$

Integrating this term over x gives a finite result with no $1/\epsilon$ poles.

Consequently the only potentially divergent terms of [equ.4.29] come from the part of the numerator of order $w^{2-\epsilon}$, that is,

$$I_2(p_o^2, p_o^2) \to \int_0^1 \frac{dx}{b^3} \int_a^{a+b(1-x)} dw[b(1 - x) + a][a(1 - x + \epsilon) + (1 - x)b]w^{-2-\epsilon}.$$

Further, the divergences will only arise from the contribution of the lower limit of the w integration since, at the upper limit $w^{-1-\epsilon}$ is finite for $0 \le x \le 1$. Hence, integrating over w and dropping the contribution from the upper limit leaves,

$$I_2(p_o^2, p_o^2) \to \frac{1}{(1 + \epsilon)} \int_0^1 \frac{dx}{b^3} [b(1 - x) + a][a(1 - x + \epsilon) + (1 - x)b]a^{-1-\epsilon}.$$

The cause of any divergences is not the b^3 in the denominator since the non-zero value of p_o^2 ensures $b \neq 0$, but rather it is the $a^{-1-\epsilon}$ in the numerator. Terms in the square brackets of the numerator that kill the $a^{-1-\epsilon}$ are consequently finite, leaving the potentially divergent term as,

$$I_2(p_o^2, p_o^2) \rightarrow \frac{1}{(1+\epsilon)} \int_0^1 \frac{dx}{b^3} b^2 (1-x)^2 a^{-1-\epsilon},$$

or, expanding a and b in terms of x,

$$I_2(p_o^2, p_o^2) = \frac{1}{(1+\epsilon)} \int_0^1 dx \frac{(1-x)^2 (xp_o^2)^{-1-\epsilon}}{(p_o^2 + 2p_{12}x)}.$$

Using partial fractions this expression can now be written,

$$\frac{1}{(1+\epsilon)p_o^4} \int_0^1 dx (1-x)^2 (xp_o^2)^{-\epsilon} \left\{ \frac{1}{x} - \frac{2p_{12}}{(p_o^2 + 2p_{12}x)} \right\}.$$

The second term is finite for $\epsilon \rightarrow 0$ leaving only the first term contributing to the divergent part as,

$$\frac{(p_o^2)^{-\epsilon}}{(1+\epsilon)p_o^4} \frac{\Gamma(3)\Gamma(-\epsilon)}{\Gamma(3-\epsilon)}.$$

Expanding this expression for $\epsilon \rightarrow 0$ finally gives,

$$I_{\mu 2}^{k^{-2}}(p_o^2, p_o^2) = \frac{2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2 2p_{12}}{\epsilon p_o^4}. \tag{4.32}$$

We can thus deduce a general formula from equations [4.30] and [4.32] for the singular part of $I_\mu^{k^{-2}}$, remembering I_1, [equ.4.20], is finite:

$$I_\mu^{k^{-2}}(q_o^2, p_o^2) = \frac{2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2 2p_{12}}{\epsilon(p_1^2 + q_o^2)(p_2^2 + p_o^2)}. \tag{4.33}$$

We can now substitute this into [equ.4.6] to give the complete singular part of the $G(k^2) = k^{-2}$ contribution to $-im_{AC}$:

$$m_{AC}\big|_{G(k^2)=1/k^2} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)}{32\pi^2}\frac{2p_{12}}{\epsilon}$$

$$\left\{\left[1 + \frac{p_o^2(p_o^2 + p_1^2 + p_2^2)}{p_1^2 p_2^2}\right]\frac{1}{(p_1^2 + p_o^2)(p_2^2 + p_o^2)}\right.$$

$$\left.+\frac{(p_o^2 + p_2^2)}{p_2^2}\frac{1}{p_1^2(p_2^2 + p_o^2)} + \frac{(p_o^2 + p_1^2)}{p_1^2}\frac{1}{p_2^2(p_1^2 + p_o^2)} + \frac{1}{p_1^2 p_2^2}\right\}$$

$$= \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)}{32\pi^2}\frac{2p_{12}.4}{\epsilon p_1^2 p_2^2} \tag{4.34}$$

Adding this equation to [equ.4.17] with appropriate weighting, given in [Fig.II.5], gives the full m_{AC} contribution as,

$$m_{AC} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)\gamma}{16\pi^2\epsilon}\left\{\frac{(2p_1^2 + p_o^2)(2p_2^2 + p_o^2)}{4p_1^2 p_2^2} + \frac{4\sigma p_{12}}{p_1^2 p_2^2}\right\} \tag{4.35}$$

Which completes the calculation of the singular parts of the terms of [equ.4.2] proportional to AC. We consider next the term of [equ.4.2] proportional to AD.

IV.3 THE AD CONTRIBUTION TO -im_μ

The terms in [equ.4.2] proportional to AD are

$$m_{AD} \equiv C_2(F)eg^2(\mu^2)^\epsilon \int \frac{d^n k}{(2\pi)^n} \frac{G(k^2)}{k^2} \frac{F[(k-p_2)^2]}{(k-p_2)^2} \frac{F[(k+p_1)^2]}{(k+p_1)^2}$$

$$\times \bar{u}(p_2)(\not{k}+2\not{p}_1)(\not{k}-\not{p}_2)\gamma_\mu(\not{k}+\not{p}_1)(\not{k}+2\not{p}_1)u(p_1)AD. \tag{4.36}$$

Writing the gamma matrices of the numerator as

$$\bar{u}(p_2)(\not{k}+2\not{p}_1)(\not{k}-\not{p}_2)\gamma_\mu(\not{k}+\not{p}_1)[(\not{k}+\not{p}_1)+\not{p}_1]u(p_1),$$

and using the Dirac equation [equ.4.10], we get

$$-(k+p_1)^2\bar{u}(p_2)(\not{k}+2\not{p}_2)(\not{k}-\not{p}_2)\gamma_\mu u(p_1).$$

Now,

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$$AD = \frac{1}{4}\left[\frac{1}{F[(k-p_2)^2]} + \frac{1}{F(p_2^2)}\right]\left[\frac{1}{F[(k+p_1)^2]} - \frac{1}{F(p_1^2)}\right]\frac{1}{[(k+p_1)^2 - p_1^2]},$$

which, can be written, using the explicit form of $F(p^2)$ as, after a little algebra,

$$AD = \frac{-1}{4a^2}\left\{\frac{p_o^2(p_o^2 + 2p_2^2)}{p_1^2 p_2^2 (k+p_1)^2} + \frac{p_o^4}{p_1^2(k+p_1)^2(k-p_2)^2}\right\}. \tag{4.37}$$

Thus, combining these last two results in [equ.4.36], yields,

$$m_{AD} = \frac{C_2(F)eg^2(\mu^2)^\epsilon}{4}\int\frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}\left\{\frac{p_o^2(p_o^2 + 2p_2^2)}{p_1^2 p_2^2} + \frac{p_o^4}{p_1^2(k-p_2)^2}\right\}$$

$$\times\frac{\bar{u}(p_2)(\slashed{k} + 2\slashed{p}_1)(\slashed{k} - \slashed{p}_2)\gamma_\mu u(p_1)}{[(k+p_1)^2 + p_o^2][(k-p_2)^2 + p_o^2]}.$$

Combining the curly bracket and the denominator as

$$p_o^2\left[\frac{(p_o^2 + 2p_2^2)}{p_1^2 p_2^2[(k+p_1)^2 + p_o^2][(k-p_2)^2 + p_o^2]}\right.$$

$$\left.+\frac{1}{p_1^2}\left(\frac{1}{(k-p_2)^2} - \frac{1}{[(k-p_2)^2 + p_o^2]}\right)\frac{1}{[(k+p_1)^2 + p_o^2]}\right], \tag{4.38}$$

we can write the previous equation as,

$$m_{AD} = \frac{C_2(F)eg^2(\mu^2)^\epsilon p_o^2}{4(2\pi)^n}\left\{\frac{(p_o^2 + p_2^2)}{p_1^2 p_2^2}J_\mu(p_o^2, p_o^2) + \frac{1}{p_1^2}J_\mu(p_o^2, 0)\right\} \tag{4.39}$$

where,

$$J_\mu(q_o^2, p_o^2) \equiv \int\frac{d^n k}{k^2}\frac{G(k^2)\bar{u}(p_2)(\slashed{k} + 2\slashed{p}_1)(\slashed{k} - \slashed{p}_2)\gamma_\mu(p_1)}{[(k+p_1)^2 + q_o^2][(k-p_2)^2 + p_o^2]}. \tag{4.40}$$

To evaluate $J_\mu(q_o^2, p_o^2)$ we first, as usual, set $G(k^2) = 1$.

IV.3.1 The G(k²)=1 contribution to m_{AD}

· Setting $G(k^2) = 1$ in [equ.4.40] and Feynman parameterising yields,

$$J_\mu^1(q_o^2, p_o^2) = 2 \int_0^1 dx \int_0^{1-x} dy \int d^n k \, \bar{u}(p_2)(\not{k} + 2\not{p}_1)(\not{k} - \not{p}_2)\gamma_\mu u(p_1)$$

$$\times \frac{1}{[k^2(1 - x - y) + x[(k + p_1)^2 + q_o^2] + y[(k - p_2)^2 + p_o^2]]^3} \tag{4.41}$$

where, as usual, the superscript 1 indicates the $G(k^2) = 1$ contribution.

The denominator of [equ.4.40] is exactly the same as [equ.4.8], so defining $k' \equiv k + p_1 x - p_2 y$ the denominator of the previous expression becomes

$$[k'^2 + L]^3,$$

with L defined in [equ.4.9]. In terms of the variable k' the numerator becomes, keeping even terms in k' only,

$$\bar{u}(p_2)[\not{k}'\not{k}' - (2 - x)\not{p}_1(x\not{p}_1 + (1 - y)\not{p}_2)]\gamma_\mu u(p_1).$$

Setting $p_1^2 \equiv 0$ and $\not{p}_1\not{p}_2 \to -2p_{12}$ between the spinors gives,

$$\bar{u}(p_2)(-k'^2 + (2 - x)(1 - y)2p_{12})\gamma_\mu u(p_1),$$

so that [equ.4.41] becomes,

$$J_\mu^1(q_o^2, p_o^2) = 2 \int_0^1 dx \int_0^{1-x} dy \int d^n k \frac{[-k^2 + (2 - x)(1 - y)2p_{12}]}{[k^2 + L]^3} \bar{u}(p_2)\gamma_\mu u(p_1). \tag{3.42}$$

Performing the k integral using equations [A.3] and [A.4] leads to,

$$J_\mu^1(q_o^2, p_o^2) = -2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^{2-\epsilon}\frac{\Gamma(1+\epsilon)}{2}\int_0^1 dx \int_0^{1-x} dy \left\{ \frac{(2-\epsilon)}{\epsilon L^\epsilon} \right.$$

$$\left. -\frac{(2-x)(1-y)2p_{12}}{L^{1+\epsilon}} \right\}. \tag{3.43}$$

The first term is exactly the same as the first term of the AC contribution in [equ.4.12], but multiplied not by Δ_μ but $-\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2(2-\epsilon)/(1-\epsilon)^2$. Hence, from [equ.4.13] we get, for the first term in [equ.4.43],

$$\frac{-\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2}{\epsilon}. \tag{4.44}$$

To evaluate the second term of [equ.4.43] we, again, consider the relevant combinations of (q_o^2, p_o^2) in turn.

First, setting $q_o^2 \neq 0, p_o^2 = 0$ from [equ.4.9], with $p_i^2 = 0$, L becomes

$$L = p_o^2 x + 2p_{12}xy.$$

Hence the relevant integral for the second term of [equ.4.43] is,

$$J' \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(2-x)(1-y)}{[p_o^2 x + 2p_{12}xy]^{1+\epsilon}}.$$

The y integral is done by changing variables to $z \equiv p_o^2 + 2p_{12}y$ giving,

$$J' = -\int_0^1 dx \frac{(2-x)x^{-1-\epsilon}}{(2p_{12})^2}\left\{ \frac{(2p_{12}+p_o^2)}{\epsilon}[[p_o^2 + 2p_{12}(1-x)]^{-\epsilon} - (p_o^2)^{-\epsilon}] \right.$$

$$\left. +\frac{1}{(1-\epsilon)}[[p_o^2 + 2p_{12}(1-x)]^{1-\epsilon} - (p_o^2)^{1-\epsilon}] \right\}.$$

Rearranging this equation as,

$$J' = -\int_0^1 dx \frac{(2-x)x^{-1-\epsilon}}{(2p_{12})^2} \left\{ \frac{(2p_{12} + p_o^2 - \epsilon 2p_{12}x)}{\epsilon(1-\epsilon)}[p_o^2 + 2p_{12}(1-x)]^{-\epsilon} \right.$$

$$\left. - \frac{(p_o^2)^{-\epsilon}(2p_{12} + p_o^2 - 2p_{12}\epsilon)}{\epsilon(1-\epsilon)}, \right.$$

we can set $\epsilon = 0$ in the exponents of the curly bracket of the integrand to get,

$$J' = -\int_0^1 dx \frac{(2-x)x^{-1-\epsilon}}{\epsilon(1-\epsilon)(2p_{12})^2}[2p_{12} + p_o^2 - \epsilon 2p_{12}x - (2p_{12} + p_o^2 - 2p_{12}\epsilon)]$$

$$= -\frac{1}{2p_{12}(1-\epsilon)}\int_0^1 dx(2-x)x^{-1-\epsilon}(1-x).$$

We can now perform the remaining integral in this expression as,

$$J' = \frac{-1}{2p_{12}(1-\epsilon)}\left[\frac{\Gamma(2)\Gamma(-\epsilon)}{\Gamma(2-\epsilon)} + \frac{\Gamma(3)\Gamma(-\epsilon)}{\Gamma(3-\epsilon)} \right] \rightarrow \frac{2}{2p_{12}\epsilon}.$$

Hence, combining this equation with [equ.2.44], gives [equ.4.43] with $q_o^2 \neq 0, p_o^2 = 0$ as,

$$J_\mu^1(q_o^2, 0) = -\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2\left[\frac{1}{\epsilon} - \frac{2p_{12}}{2p_{12}\epsilon} \right] \equiv 0 \qquad (4.45)$$

and the singular pieces cancel leaving only a finite contribution.

Now we have to consider $q_o^2 = p_o^2 \neq 0$. Again the first term of [equ.4.43] is given by [equ.4.44]. For the second term L is now given by

$$L \equiv p_o^2 x + y(p_o^2 + 2p_{12}x)$$

and the integral is

$$J' \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(2-x)(1-y)}{[p_o^2 x + y(p_o^2 + 2p_{12}x)]^{1+\epsilon}}.$$

The y integral is done by changing variables to

$$z \equiv p_o^2 x + y(p_o^2 + 2p_{12}x) \equiv a + by, \qquad (4.46)$$

to obtain,

$$J' = \int_0^1 dx \frac{(2-x)}{\epsilon(1-\epsilon)b^2} \left\{ \frac{(a+b)}{-\epsilon} [[a+b(1-x)]^{-\epsilon} - a^{-\epsilon}] \right.$$

$$\left. -\frac{1}{(1-\epsilon)} [[a+b(1-x)]^{1-\epsilon} - a^{1-\epsilon}] \right\}.$$

As before, rearranging as

$$J' = -\int_0^1 dx \frac{(2-x)}{\epsilon(1-\epsilon)b^2} \left\{ [a+\epsilon bx][a+b(1-x)]^{-\epsilon} - [a+b-\epsilon b]a^{-\epsilon} \right\},$$

we can, again, set $\epsilon = 0$ in the exponents of the curly bracket to get,

$$J' = \frac{-1}{(1-\epsilon)} \int_0^1 dx \frac{(2-x)(1-x)}{(p_o^2 + 2p_{12}x)}.$$

The ϵ dependence has now dropped out and we are left with a finite integral. That is the second term of [equ.4.43] has no poles for $q_o^2 = p_o^2 \neq 0$. Thus from [equ.4.44] we immediately get,

$$J_\mu^1(p_o^2, p_o^2) = -\frac{\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2}{\epsilon}. \qquad (4.47)$$

Hence, combining equations [4.45] and [4.47] in [equ.4.39] we get, for m_{AD} with $G(k^2) = 1$,

$$m_{AD}|_{G(k^2)=1} = -\frac{C_2(F)eg^2 p_o^2}{64\pi^2}\frac{(p_o^2 + p_2^2)}{\epsilon p_1^2 p_2^2}\bar{u}(p_2)\gamma_\mu u(p_1). \qquad (4.48)$$

Having determined the $G(k^2) = 1$ contribution to m_{AD} we now re-trace the above calculation to evaluate the $G(k^2) = k^{-2}$ part.

IV.3.2 The $G(k^2)=k^{-2}$ contribution to m_{AD}

We use again the decomposition of [equ.4.39], but now J_μ is given, from [equ.4.42] by, (with L defined by [equ.4.9]),

$$J_\mu^{k^{-2}}(q_o^2, p_o^2) \equiv 3!\bar{u}(p_2)\gamma_\mu u(p_1)\int_0^1 dx\int_0^{1-x} dy(1 - x - y)$$

$$\times \int d^n k\frac{[-k^2 + (2 - x)(1 - y)2p_{12}]}{[k^2 + L]^4}.$$

Integrating over k in the usual way gives,

$$J_\mu^{k^{-2}}(q_o^2, p_o^2) = 3!\pi^{2-\epsilon}\bar{u}(p_2)\gamma_\mu u(p_1)\frac{\Gamma(1 + \epsilon)}{6}\int_0^1 dx\int_0^{1-x} dy(1 - x - y)\left\{\frac{-(2 - \epsilon)}{L^{1+\epsilon}}\right.$$

$$\left.+\frac{(1 + \epsilon)(2 - x)(1 - y)2p_{12}}{L^{2+\epsilon}} \qquad (4.49)\right.$$

The first term here is exactly the same as the first term of the $G(k^2) = k^{-2}$ contribution to the AC term in [equ.4.19]. Hence from the discussion immediately before [equ.4.25] we see that for $q_o^2 = p_o^2 \neq 0$ this term has no singular parts. And from [equ.4.23], which is the symmetric partner to $(p_o^2, 0)$ and has the same value, we get for $(p_o^2, 0)$ the contribution of the first term of [equ.4.49] as,

$$\frac{-1}{2p_{12}\epsilon}\left\{1 - \frac{(2p_{12} + p_o^2)}{2p_{12}}ln\left(\frac{2p_{12} + p_o^2}{p_o^2}\right)\right\}. \qquad (4.50)$$

We must now consider the second term of [equ.4.49] for the relevant combinations of (q_o^2, p_o^2). We start with $(p_o^2, 0)$ and the integral,

$$\tilde{J}^{k^{-2}}(p_o^2, 0) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)(2-x)(1-y)}{[p_o^2 x + 2p_{12}xy]^{2+\epsilon}}.$$

Transforming to the variable $z \equiv p_o^2 + 2p_{12}y$ we get, after some algebra,

$$\tilde{J}^{k^{-2}}(p_o^2, 0) = \int_0^1 dx \frac{(2-x)x^{-2-\epsilon}}{(2p_{12})^3} \int_{p_o^2}^{2p_{12}(1-x)+p_o^2} dz \Big\{ (2p_{12} + p_o^2)(2p_{12}(1-x) + p_o^2)$$

$$-z[2p_o^2 + 2p_{12}(2-x)] + z^2 \Big\} z^{-2-\epsilon}.$$

Integrating this expression term by term gives,

$$\tilde{J}^{k^{-2}}(p_o^2, 0) = \int_0^1 dx \frac{(2-x)x^{-2-\epsilon}}{(2p_{12})^3}$$

$$\times \Bigg\{ \frac{(2p_{12} + p_o^2)(2p_{12}(1-x) + p_o^2)}{-(1+\epsilon)} \Big\{ [2p_{12}(1-x) + p_o^2]^{-1-\epsilon} - (p_o^2)^{-1-\epsilon} \Big\}$$

$$- \Big\{ \frac{2p_o^2 + 2p_{12}(2-x)}{-\epsilon} \Big\} \Big\{ [2p_{12}(1-x) + p_o^2]^{-\epsilon} - (p_o^2)^{-\epsilon} \Big\}$$

$$+ \Big\{ \frac{[2p_{12}(1-x) + p_o^2]^{1-\epsilon} - (p_o^2)^{1-\epsilon}}{1-\epsilon} \Big\} \Bigg\}.$$

The overall factor of $x^{-2-\epsilon}$ gives a $1/\epsilon$ (but not $1/\epsilon^2$) pole via $\Gamma(-1-\epsilon)$, so in the first and third curly brackets of the last equation we can therefore set $\epsilon \equiv 0$ and get, after combining terms,

$$\tilde{J}^{k^{-2}}(p_o^2, 0) = \int_0^1 dx \frac{(2-x)x^{-2-\epsilon}}{(2p_{12})^3} \Big\{ \frac{2p_{12}(1-x)(2p_{12} + 2p_o^2)}{p_o^2}$$

$$-\left\{\frac{2p_o^2 + 2p_{12}(2-x)}{\epsilon}\right\}\left\{[2p_{12}(1-x) + p_o^2]^{-\epsilon} - (p_o^2)^{-\epsilon}\right\}\right].$$

The first term can be integrated straightforwardly using [equ.A.2] giving,

$$\tilde{J}^{k^{-2}}(p_o^2, 0) = \frac{3(2p_{12} + p_o^2)}{\epsilon p_o^2 (2p_{12})^2} - \int_0^1 dx \frac{(2-x)x^{-2-\epsilon}}{\epsilon(2p_{12})^3}\left\{2p_o^2 + 2p_{12}(2-x)\right\}$$

$$\times\left\{[2p_{12}(1-x) + p_o^2]^{-\epsilon} - (p_o^2)^{-\epsilon}\right\}. \tag{3.51}$$

Expanding the last curly bracket of the remaining integral using [equ.B.8], the explicit ϵ in the denominator cancels leaving, for the integral,

$$\int_0^1 dx \frac{(2-x)x^{-2-\epsilon}}{(2p_{12})^3}[2p_o^2 + 2p_{12}(2-x)]ln\left[\frac{(p_o^2 + 2p_{12})}{p_o^2}\left(1 - \frac{2p_{12}x}{(p_o^2 + 2p_{12})}\right)\right].$$

Since the overall factor of $x^{-2-\epsilon}$ is quadratically divergent for $\epsilon \to 0$ we only have to keep the first term of the expansion, about $x = 0$, of the logarithm

$$\frac{1}{(2p_{12})^3}ln\left(\frac{p_o^2 + 2p_{12}}{p_o^2}\right)\int_0^1 dx(2-x)x^{-2-\epsilon}[2p_o^2 + 2p_{12}(2-x)]$$

$$+\frac{1}{(2p_{12})^3}\int_0^1 dx(2-x)x^{-2-\epsilon}[2p_o^2 + 2p_{12}(2-x)]\left\{\frac{-2p_{12}x}{(p_o^2 + 2p_{12})}\right\}.$$

We can now integrate this expression to give

$$\frac{1}{(2p_{12})^3}ln\left(\frac{p_o^2 + 2p_{12}}{p_o^2}\right)\left\{2(p_o^2 + 2p_{12})\left[\frac{-2}{(1+\epsilon)} + \frac{1}{\epsilon}\right] - 2p_{12}\left[\frac{-1}{\epsilon} - \frac{1}{(1-\epsilon)}\right]\right\}$$

$$-\frac{1}{(2p_{12})^2(p_o^2+2p_{12})}\left\{2(p_o^2+2p_{12})\left[\frac{-2}{\epsilon}-\frac{1}{(1-\epsilon)}\right]-2p_{12}\left[\frac{2}{(1-\epsilon)}-\frac{1}{(2-\epsilon)}\right]\right\},$$

or, keeping only $1/\epsilon$ poles,

$$\frac{[2p_o^2+3(2p_{12})]}{\epsilon(2p_{12})^3}ln\left(\frac{p_o^2+2p_{12}}{p_o^2}\right)+\frac{4(p_o^2+2p_{12})}{\epsilon(2p_{12}(p_o^2+2p_{12})},$$

which, in [equ.4.51] gives,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,0)=\frac{1}{\epsilon}\left\{\frac{[3(2p_{12})-p_o^2]}{(2p_{12})^2p_o^2}-\frac{[2p_o^2+3(2p_{12})]}{(2p_{12})^3}ln\left(\frac{p_o^2+2p_{12}}{p_o^2}\right)\right\}. \tag{4.52}$$

Now, setting $p_o^2=q_o^2\neq0$ gives,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2)=\int_0^1dx\int_0^{1-x}dy\frac{(1-x-y)(2-x)(1-y)}{[p_o^2x+p_o^2y+2p_{12}x]^{2+\epsilon}},$$

and using the change of variables of [equ.4.46] we get,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2)=\int_0^1dx(2-x)\int\frac{dz}{b^3z^{2+\epsilon}}[b(1-x)-(z-a)][b-(z-a)]$$

which gives, after some algebra and integrating over z,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2)=\int_0^1dx\frac{(2-x)}{b^3}\left[(a+b)[a+b(1-x)]\frac{z^{-1-\epsilon}}{-(1+\epsilon)}\right.$$

$$\left.-[a+b+b(1-x)]\frac{z^{-\epsilon}}{-\epsilon}+\frac{z^{1-\epsilon}}{1-\epsilon}\right]_a^{a+b(1-x)}. \tag{4.53}$$

Now, the term proportional to $z^{-\epsilon}$ gives, on substituting in the limits and expanding using [equ.B.8],

$$\frac{-1}{\epsilon}\left[z^{-\epsilon}\right]_a^{a+b(1-x)} \to ln\left(\frac{a+b(1-x)}{a}\right)$$

which gives a finite result when integrated over x. (The explicit factor of b^3 in the denominator of [equ.4.53] is always non-zero due to the p_o^2 term of $b \equiv p_o^2 + 2p_{12}x$.)

Similarly the term in [equ.4.53] proportional to $z^{1-\epsilon}$ is non-singular for $\epsilon \to 0$ and can be dropped. Thus the only potential divergent $1/\epsilon$ pole in [equ.4.53] comes from the first term,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2) \to \frac{-1}{(1+\epsilon)} \int \frac{dx}{b^3}\left[(2-x)(a+b)[a+b(1-x)]z^{-1-\epsilon}\right]_a^{a+b(1-x)}.$$

Further the divergence only comes from the bottom limit since $a + b(1-x)$ is non-zero for all $0 \le x \le 1$. Hence

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2) \to \frac{1}{(1+\epsilon)} \int_0^1 dx \frac{(2-x)(a+b)[a+b(1-x)]}{b^3 a^{1+\epsilon}}$$

which gives, after expanding the numerator,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2) = \frac{1}{(1+\epsilon)} \int \frac{dx}{b^3}(2-x)[a^{1-\epsilon} + b(1-x)a^{-\epsilon} + ba^{-\epsilon} + b^2(1-x)a^{-1-\epsilon}].$$

As $b \ne 0$ ($\forall x : 0 \le x \le 1$) any divergence occurs in the terms of order $a^{-1-\epsilon}$ (or smaller powers). Hence the first three terms in the previous expression are finite and are dropped, giving

$$\tilde{J}_\mu^{k^{-2}}(p_o^2,p_o^2) = \frac{1}{(1+\epsilon)} \int_0^1 dx \frac{(2-x)(1-x)(p_o^2x)^{-1-\epsilon}}{[p_o^2 + 2p_{12}x]}.$$

Using partial fractions this equation becomes,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2, p_o^2) = \frac{-(p_o^2)^{-1-\epsilon}}{(1+\epsilon)p_o^2} \int_0^1 dx(1-x)(2-x) \left[\frac{2p_{12}x^{-\epsilon}}{[p_o^2 + 2p_{12}x]} - \frac{x^{-\epsilon}}{x} \right],$$

where the first term is finite for $\epsilon \to 0$ leaving,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2, p_o^2) = \frac{1}{(1+\epsilon)(p_o^2)^{2+\epsilon}} \int_0^1 dx(1-x)(2-x)x^{-1-\epsilon}$$

which can be immediately integrated to give, dropping finite terms,

$$\tilde{J}_\mu^{k^{-2}}(p_o^2, p_o^2) = \frac{-2}{\epsilon(p_o^2)^2}. \tag{3.54}$$

Having thus considered the relevant combinations of (q_o^2, p_o^2) we can now combine equations [4.50] and [4.52] into [equ.4.49] for $q_o^2 \neq 0, p_o^2 = 0$ to get,

$$J_\mu^{k^{-2}}(p_o^2, 0) = \frac{\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2}{\epsilon} \left\{ \frac{-1}{(2p_{12}} \left[1 - \frac{(2p_{12} + p_o^2)}{2p_{12}} ln\left(\frac{2p_{12} + p_o^2}{p_o^2} \right) \right] \right.$$

$$\left. + \left[\frac{[3(2p_{12}) - p_o^2]}{p_o^2} - \frac{(2p_o^2 + 3(2p_{12}))}{2p_{12}} ln\left(\frac{p_o^2 + 2p_{12}}{p_o^2} \right) \right] \right\}.$$

Or, combining terms

$$J_\mu^{k^{-2}}(p_o^2, 0) = \frac{\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2}{\epsilon} \left\{ \frac{[3(2p_{12}) - 2p_o^2]}{p_o^2} - \frac{[p_o^2 + 2(2p_{12})]}{2p_{12}} ln\left(\frac{p_o^2 + 2p_{12}}{p_o^2} \right) \right\}. \tag{4.55}$$

Similarly, from [equ.4.54] we get for $q_o^2 = p_o^2 \neq 0$,

$$J_\mu^{k^{-2}}(p_o^2, p_o^2) = \frac{-2\bar{u}(p_2)\gamma_\mu u(p_1)\pi^2(2p_{12})}{\epsilon p_o^4}. \tag{4.56}$$

Substituting these last two equations into [equ.4.39] now yields the $G(k^2) = k^{-2}$ contribution to m_{AD} as,

$$m_{AD}|_{G(k^2)=k^{-2}} = \frac{c_2(F)eg^2 p_o^2 \bar{u}(p_2)\gamma_\mu u(p_1)}{64\epsilon\pi^2}\left\{ \frac{-2(2p_{12}(p_o^2 + 2p_{12})}{p_1^2 p_2^2 p_o^4}\right.$$

$$\left. + \frac{1}{2p_{12}p_1^2}\left[\frac{[3(2p_{12}) - 2p_o^2]}{p_o^2} - \frac{[p_o^2 + 2(2p_{12})]}{2p_{12}}ln\left(\frac{p_o^2 + 2p_{12}}{p_o^2}\right)\right]\right\}. \tag{4.57}$$

Finally, adding this result to the $G(k^2) = 1$ contribution, [equ.4.48], with apropriate weighting, gives the full AD contribution to the amplitude as,

$$m_{AD} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)p_o^2\gamma}{64\epsilon\pi^2}\left\{ \frac{-(p_o^2 + p_2^2)}{p_1^2 p_2^2} + \sigma\left\{\frac{-2(2p_{12})(p_o^2 + p_2^2)}{p_1^2 p_2^2 p_o^4}\right.\right.$$

$$\left.\left. + \frac{1}{2p_{12}p_1^2}\left[\frac{[3(2p_{12}) - 2p_o^2]}{p_o^2} - \frac{[2(2p_{12}) + p_o^2]}{2p_{12}}ln\left(\frac{2p_{12} + p_o^2}{p_o^2}\right)\right]\right\}\right\} \tag{4.58}$$

Now we have to return to [equ.4.2] and evaluate the term proportional to BC.

IV.4 THE BC CONTRIBUTION TO $-im_\mu$

From [equ.4.2] we get, for the contribution to the amplitude proportional to BC,

$$m_{BC} = C_2(F)eg^2(\mu^2)^\epsilon \int \frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}\frac{F[(k - p_2)^2]}{(k - p_2)^2}\frac{F[(k + p_1)^2]}{(k + p_1)^2}$$

$$\times \bar{u}(p_2)(\not{k} - 2\not{p_2})(\not{k} - \not{p_2})\gamma_\mu(\not{k} + \not{p_1})(\not{k} - 2\not{p_2})u(p_1)BC. \tag{4.59}$$

BC is just *AD* with $p_1 \leftrightarrow -p_2$, so from [equ.4.37] we immediately get

$$BC = \frac{-1}{4a^2}\left\{\frac{p_o^2(p_o^2 + 2p_1^2)}{p_1^2 p_2^2(k - p_2)^2} + \frac{p_o^4}{p_2^2(k + p_1)^2(k - p_2)^2}\right\}$$

which, in turn, immediately gives, analagous to [equ.4.39],

$$m_{BC} = \frac{C_2(F)eg^2(\mu^2)^\epsilon}{4(2\pi)^n}\left\{\frac{p_o^2(p_1^2 + p_o^2)}{p_1^2 p_2^2}\tilde{I}_\mu(p_o^2, p_o^2) + \frac{p_o^2}{p_2^2}\tilde{I}_\mu(0, p_o^2)\right\} \qquad (4.60)$$

where now

$$\tilde{I}_\mu(q_o^2, p_o^2) \equiv \int d^n k \frac{G(k^2)}{k^2} \frac{\bar{u}(p_2)\gamma_\mu(\not{k} + \not{p}_1)(\not{k} - 2\not{p}_2)u(p_1)}{[(k + p_1)^2 + q_o^2][(k - p_2)^2 + p_o^2]} \qquad (4.61)$$

As before we first set $G(k^2) = 1$.

IV.4.1 The G(k²)=1 contribution to m_{BC}

Setting $G(k^2) = 1$ in [equ.4.61] and Feynman parameterising yields

$$\tilde{I}_\mu(q_o^2, p_o^2) = 2\int_0^1 dx \int_0^{1-x} dy \int d^n k \bar{u}(p_2)\gamma_\mu(\not{k} + \not{p}_1)(\not{k} - 2\not{p}_2)u(p_1)$$

$$\times \frac{1}{[k^2(1 - x - y) + y[(k + p_1)^2 + q_o^2] + x[(k - p_2)^2 + p_o^2]]^3}. \qquad (4.62)$$

Defining the shifted variable

$$k' \equiv k + p_1 y - p_2 x$$

the denominator becomes

$$[k'^2 + L]^3$$

where

$$L \equiv y(1 - y)p_1^2 + x(1 - x)p_2^2 + 2p_{12}xy + q_o^2 y + p_o^2 x. \qquad (4.63)$$

In terms of k' the numerator, keeping only even terms in k', now becomes,

$$-\bar{u}(p_2)\gamma_\mu u(p_1)[k'^2 - 2p_{12}(2 - x)(1 - y)].$$

Combining these last two equations in [equ.4.62] leads to

$$\tilde{I}_\mu(q_o^2, p_o^2) = -2\bar{u}(p_2)\gamma_\mu u(p_1) \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k}{[k^2 + L]^3}[k^2 - 2p_{12}(2 - x)(1 - y)]. \qquad (4.64)$$

We note that this integral is identical to that in [equ.4.42] (the analogous equation for the AD term) except that the L is now defined by [equ.4.63], not ,as previously by [equ.4.9]. However, we see that [equ.4.63] is identical to [equ.4.9] with $q_o^2 \leftrightarrow p_o^2$ and $p_1^2 \leftrightarrow p_2^2$. Further, noting that in [equ.4.60] we have to determine $\tilde{I}_\mu(0, p_o^2)$ whereas in [equ.4.39] it was $J_\mu(p_o^2, 0)$ that was relevant, we have swapped the arguments of the functions. Thus if we make the interchange,

$$q_o^2 \leftrightarrow p_o^2$$

$$p_1^2 \leftrightarrow p_2^2$$

in the AD term we immediately get the BC contribution to m. That is m_{BC} is given by [equ.4.58] with $p_1^2 \leftrightarrow p_2^2$, repeated here as,

$$m_{BC} = -\frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)p_o^2\gamma}{64\epsilon\pi^2}\left\{-\frac{(p_o^2+p_1^2)}{p_1^2 p_2^2} + \sigma\left\{\frac{2(2p_{12})(p_o^2+p_2^2)}{p_1^2 p_2^2 p_o^4}\right.\right.$$

$$\left.\left. +\frac{1}{2p_{12}p_1^2}\left[\frac{[3(2p_{12})-2p_2^2]}{p_o^2} - \frac{[2(2p_{12})+p_o^2]}{2p_{12}}ln\left(\frac{2p_{12}+p_o^2}{p_o^2}\right)\right]\right\}\right\}. \qquad (4.65)$$

Now, finally we have to evaluate the piece of [equ.4.2] proportional to BD which we consider in the next section.

IV.5 THE BD CONTRIBUTION TO -im_μ

From [equ.4.2] we get

$$m_{BD} \equiv C_2(F)eg^2(\mu^2)^\epsilon \int \frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}\frac{a^2 BD\bar{u}(p_2)(k-2p_2)(k+2p_1)}{[(k+p_1)^2+p_o^2][(k-p_2)^2+p_o^2]}$$

$$\times(\not{k}-2\not{p}_2)(\not{k}-\not{p}_2)\gamma_\mu(\not{k}+\not{p}_1)(\not{k}+2\not{p}_1)u(p_1). \qquad (4.66)$$

Using the Dirac equation [equ.4.10] the numerator can be written as

$$a^2 BD(k-2p_2)(k+2p_1)(k-p_2)^2(k+p_1)^2\bar{u}(p_2)\gamma_\mu u(p_1). \qquad (4.67)$$

Now, from the definitions of B, D and using the explicit form of F in [Fig.II.5],

$$BD = \frac{p_o^4}{4a^2}\left[\frac{1}{p_2^2(k-p_2)^2}\right]\left[\frac{1}{p_1^2(k+p_1)^2}\right]$$

which, with [equ.4.67] gives,

$$m_{BD} = \frac{C_2(F)eg^2(\mu^2)^\epsilon p_o^4\bar{u}(p_2)\gamma_\mu u(p_1)}{4p_1^2 p_2^2}\int\frac{d^n k}{(2\pi)^n}\frac{G(k^2)}{k^2}$$

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$$\times \frac{(k - 2p_2)(k + 2p_1)}{[(k + p_1)^2 + p_o^2][(k - p_2)^2 + p_o^2]}. \tag{4.68}$$

As always we first consider the $G(k^2) = 1$ term.

IV.5.1 The G(k^2)=1 contribution to m_{BD}

Setting $G(k^2) = 1$ in the last equation yields,

$$m^1_{BD} \equiv \frac{C_2(F)eg^2(\mu^2)^\epsilon p_o^4 \bar{u}(p_2)\gamma_\mu u(p_1)}{(2\pi)^n 4p_1^2 p_2^2} \int \frac{d^n k}{k^2} \frac{(k - 2p_2)(k + 2p_1)}{[(k + p_1)^2 + p_o^2][(k - p_2)^2 + p_o^2]}. \tag{4.69}$$

Feynman parameterising this equation gives the relevant integral as,

$$I \equiv 2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k(k - 2p_2)(k + 2p_1)}{[k^2(1 - x - y) + x[(k + p_1)^2 + p_o^2] + y[(k - p_2)^2 + p_o^2]]^3}.$$

Shifting the origin by defining $k' \equiv k + xp_1 - yp_2$ the denominator becomes

$$[k' + L]^3$$

where

$$L \equiv p_1^2 x(1 - x) + p_2^2 y(1 - y) + p_o^2 x + p_o^2 y + 2p_{12}xy \tag{4.70}$$

and the numerator gives, after a little algebra and dropping terms odd in k,

$$(k - 2p_2)(k + 2p_1) \to k'^2 - p_{12}[xy + (2 - x)(2 - y)] - p_2^2 y(2 - y) - p_1^2 x(2 - x).$$

Using the mass shell condition $p_i^2 u(p_i) \equiv 0$ the last two terms are zero, leaving

$$I = 2 \int_0^1 dx \int_0^{1-x} dy \int d^n k \frac{[k^2 - 2p_{12}(2 - x - y + xy)]}{[k^2 + L]^3}.$$

On integrating over k using equations [A.3] and [A.4] we get

$$I = 2 \int_0^1 dx \int_0^{1-x} dy \pi^{2-\epsilon} \frac{\Gamma(1+\epsilon)}{2} \left\{ \frac{(2-\epsilon)}{\epsilon L^\epsilon} - \frac{2p_{12}(2 - x - y + xy)}{L^{1+\epsilon}} \right\}. \quad (4.71)$$

The first term is exactly the same integral as the first term of the *AC* contribution in [equ.4.12] giving, from [equ.4.13],

$$I_1 \equiv \frac{\pi^{2-\epsilon} \Gamma(1+\epsilon)(2-\epsilon)}{2\epsilon} \quad (4.72)$$

The integral relevant to the second term of [equ.4.71] is,

$$I_2 \equiv \int_0^1 dx \int_0^{1-x} dy \frac{(2 - x - y + xy)}{L^{1+\epsilon}} \quad (4.73).$$

from L in [equ.4.70] we see that for $p_o^2 \neq 0$ the bare singularities at $x, y = 1$ are killed off by the p_o^2 terms. Hence the only possible singularities ($L \to 0$) are at x and/or y zero. At these points the p_o^2 and p_i^2 terms are of the same form as the p_o^2 terms. Thus the form of the singularities (if any) can be obtained by dropping the p_i^2 terms and only considering the p_o^2 contributions. As we shall see, the latter don't, in fact, give any singularities and the p_i^2 terms can be dropped without omitting any singular terms. So, writing

$$L \equiv (a + by) = xp_o^2 + y(p_o^2 + 2p_{12}x)$$

and transforming variables by defining $z \equiv a + by$ [equ.4.73] becomes, after some rearranging

$$I_2 = \int_0^1 dx \int \frac{dz\, z^{-1-\epsilon}}{b^2} [(1-x)a + (2-x)b - (1-x)z]$$

The third term of the square bracket is immediately finite (the b^2 in the denominator is always non-zero and causes no divergences), and is dropped, leaving

$$I_2 = \int \frac{dx}{b^2} [(1-x)a + (2-x)b] \left\{ \frac{[a+b(1-x)]^{-\epsilon} - a^{-\epsilon}}{-\epsilon} \right\}.$$

Expanding the curly bracket as

$$\left\{ \frac{[a+b(1-x)]^{-\epsilon} - a^{-\epsilon}}{-\epsilon} \right\} = -ln \left\{ \frac{a+b(1-x)}{a} \right\} + O(\epsilon)$$

we see that the argument of the logarithm is well behaved and I_2 is finite with no pole pieces.

Hence the second term of [equ.4.72] is finite and it is only the first term that gives the pole contribution to I Substituting this into [equ.4.69] we immediately get

$$m_{BD}|_{G(k^2)=1} = \frac{C_2(F)eg^2 p_o^4 \bar{u}(p_2)\gamma_\mu u(p_1)}{64\pi^2 \epsilon p_1^2 p_2^2}. \tag{4.74}$$

Finally, we re-trace this last calculation but now setting $G(k^2) = k^{-2}$.

IV.5.2 The $G(k^2)=k^{-2}$ contribution to m_{BD}

Setting $G(k^2) = k^{-2}$ we immediately get, from [equ.4.68]

$$m_{BD}^{k^{-2}} = \frac{C_2(F)eg^2(\mu^2)^\epsilon p_o^2 \bar{u}(p_2)\gamma_\mu u(p_1)}{4p_1^2 p_2^2 (2\pi)^n} \int \frac{d^n k}{k^4} \frac{(k-2p_2)(k+2p_1)}{[(k+p_1)^2 + p_o^2][(k-p_2)^2 + p_o^2]}. \tag{4.75}$$

Feynman parameterising in exactly the same way as in the $G(k^2) = 1$ contribution, we now get, for the integral,

$$I \equiv 3! \int_0^1 dx \int_0^{1-x} dy (1 - x - y) \int \frac{d^n k}{[k^2 + L]^4} [k^2 - 2p_{12}(2 - x - y + xy)] \quad (4.76)$$

where L is defined by [equ.4.70]. Integrating over k using equations [A.5] and [A.6] leads to

$$I = 3! \int_0^1 dx \int_0^{1-x} dy (1 - x - y) \frac{\pi^{2-\epsilon} \Gamma(1 + \epsilon)}{6}$$

$$\times \left\{ \frac{(2 - \epsilon)}{L^{1+\epsilon}} - \frac{2p_{12}(2 - x - y + xy)(1 + \epsilon)}{L^{2+\epsilon}} \right\}.$$

Again the first term is exactly the integral of the first term of the AC contribution with $q_o^2 = p_o^2$: that is [equ.4.21] which was shown previously to have no poles. Thus, the relevant part of the previous equation is now

$$I = -2p_{12} \Gamma(2 + \epsilon) \pi^{2-\epsilon} \int_0^1 dx \int_0^{1-x} dy \frac{(1 - x - y)(2 - x - y + xy)}{L^{2+\epsilon}}. \quad (4.77)$$

As before, putting the quadratic momentum terms in L on mass-shell, ($p_i^2 = 0$), and defining

$$L \equiv a + by = [xp_o^2 + y(2p_{12}x + p_o^2)]$$

the y integral can be performed by changing variables to $z \equiv a + by$ to yield, after a little algebra,

$$I = -2p_{12} \Gamma(2 + \epsilon) \pi^{2-\epsilon} \int_0^1 \frac{dx}{b^3} \left[[b(2 - x) + a(1 - x)][a + b(1 - x)] \frac{z^{-1-\epsilon}}{-(1 + \epsilon)} \right.$$

$$-[b(2-x) + 2a(1-x) + b(1-x)^2]\frac{z^{-\epsilon}}{-\epsilon} + \frac{(1-x)z^{1-\epsilon}}{(1-\epsilon)}\Bigg]_a^{a+b(1-x)} .$$

Now, as previously in [equ.4.53], the only divergent term comes from the $z^{-1-\epsilon}$ part of this equation,

$$I \to \frac{2p_{12}\Gamma(2+\epsilon)\pi^{2-\epsilon}}{(1+\epsilon)} \int_0^1 \frac{dx}{b^3}[a+b(1-x)][a(1-x)+(2-x)b]$$

$$\times \left\{ [a+b(1-x)]^{-1-\epsilon} - a^{-1-\epsilon} \right\}.$$

$[a + b(1-x)] = p_o^2 + 2p_{12}(1-x)$ and is non-zero for $0 \le x \le 1$ so that the first term of the curly bracket is finite and yields no pole pieces. Dropping it leaves,

$$I \to \frac{-2p_{12}\Gamma(2+\epsilon)\pi^{2-\epsilon}}{(1+\epsilon)} \int_0^1 \frac{dx}{b^3}[a+b(1-x)][a(1-x)+b(2-x)]a^{-1-\epsilon}.$$

Any factor of a in the square brackets will kill off the divergence of the overall $a^{-1-\epsilon}$ giving a finite contribution. Hence, dropping such terms leaves

$$I \to \frac{-2p_{12}\Gamma(2+\epsilon)\pi^{2-\epsilon}}{(1+\epsilon)} \int_0^1 dx \frac{(1-x)(2-x)(p_o^2 x)^{-1-\epsilon}}{[p_o^2 + 2p_{12}x]}$$

which can be expressed, using partial fractions as,

$$I = \frac{2p_{12}\Gamma(2+\epsilon)\pi^{2-\epsilon}}{(1+\epsilon)(p_o^2)^{2+\epsilon}} \int_0^1 dx(1-x)(2-x)(p_o^2)^{-2-\epsilon}\left\{ \frac{2p_{12}x^{-\epsilon}}{(p_o^2 + 2p_{12}x)} - x^{-1-\epsilon} \right\}.$$

The first term is finite for $\epsilon \to 0$ leaving,

$$I \to \frac{-2p_{12}\Gamma(2+\epsilon)\pi^{2-\epsilon}}{(1+\epsilon)(p_o^2)^{2+\epsilon}} \int_0^1 dx(1-x)(2-x)x^{-1-\epsilon}.$$

Integrating this equation, dropping finite terms now yields,

$$I \to \frac{4p_{12}\pi^2}{\epsilon(p_o^2)^2}. \tag{4.78}$$

Substituting this last equation into [equ.4.75] immediately gives the $G(k^2) = k^{-2}$ contribution to m_{BD} as,

$$m_{BD}|_{G(k^2)=k^{-2}} = \frac{c_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)2p_{12}}{64\pi^2 p_1^2 p_2^2 \epsilon}. \tag{4.79}$$

So, having determined the $G(k^2) = k^{-2}$ contribution, we can add this result to the $G(k^2) = 1$ term of [equ.4.74] to get the complete BD contribution as,

$$m_{BD} = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_1)\gamma}{64\pi^2 p_1^2 p_2^2 \epsilon}\left\{p_o^4 + 2\sigma(2p_{12})\right\}. \tag{4.80}$$

The previous equation is the final term needed for the amplitude of [Fig.II.2d] and adding equations [4.35], [4.58] and [4.80] we get the full expression for the amplitude of [FigII.2d] as,

$$-im_\mu = \frac{C_2(F)eg^2\bar{u}(p_2)\gamma_\mu u(p_2)\gamma}{64\pi^2\epsilon}\left\{4 + \frac{2p_o^2(2p_1^2 + 2p_2^2 + p_o^2)}{2p_1^2 p_2^2} + \frac{16\sigma p_{12}}{p_1^2 p_2^2}\right.$$

$$-\frac{p_o^2(p_2^2 + p_o^2)}{p_1^2 p_2^2} - p_o^2\sigma\left\{\frac{4p_{12}(p_2^2 + p_o^2)}{p_1^2 p_2^2 p_o^4} - \frac{1}{2p_{12}p_1^2}\left[\frac{[3(2p_{12}) - 2p_o^2]}{p_o^2}\right.\right.$$

$$\left.\left.-\frac{[2(2p_{12}) + p_o^2]}{2p_{12}}ln\left(\frac{2p_{12} + p_o^2}{p_o^2}\right)\right]\right\}$$

$$-\frac{p_o^2(p_1^2 + p_o^2)}{p_1^2 p_2^2} - p_o^2\sigma\left\{\frac{4p_{12}(p_1^2 + p_o^2)}{p_1^2 p_2^2 p_o^4} - \frac{1}{2p_{12}p_2^2}\left[\frac{[3(2p_{12}) - 2p_o^2]}{p_o^2}\right.\right.$$

$$-\frac{[2(2p_{12})+p_o^2]}{2p_{12}}ln\left(\frac{2p_{12}+p_o^2}{p_o^2}\right)\Big]\Big\}$$

$$+\frac{[p_o^4+2\sigma(2p_{12})]}{p_1^2 p_2^2}\Bigg\} \tag{4.81}$$

Before constructing the |amplitude|² from this equation we can take the limit $F \equiv G \equiv 1$ which should reduce it to the vertex graph using bare Feynman rules, as given in [equ.2.12]. Taking this limit in [equ.4.81] (equivalent to $p_o^2 \equiv \sigma \equiv 0$) ensures only the first term survives as

$$-im_\mu \rightarrow -\frac{iC_2(F)eg^2}{16\pi^2\epsilon}\bar{u}(p_2)\gamma_\mu u(p_1)$$

which, as expected, is just the bare result of [equ.2.12] sandwiched between the external wavefunctions.

Now, to compare this (horrible!) amplitude with the |amplitude|² of [Fig.III.2] we must contract the above with the amplitude for $\gamma^* \rightarrow q\bar{q}$, (and average over initial spin states). Hence the $\bar{u}(p_2)\gamma_\mu u(p_1)$ factor contracts with $e\bar{u}(p_2)\gamma_\mu u(p_1)$ of the amplitude for $\gamma^* \rightarrow q\bar{q}$ to give,

$$\frac{e}{2}Tr[\bar{u}(p_2)\gamma_\mu u(p_1)\bar{u}(p_1)\gamma_\mu u(p_2)]$$

which, using $\sum u(p)\bar{u}(p) = F(p^2)\not{p}$ twice yields

$$-2(2p_{12}F(p_1^2)F(p_2^2)e.$$

Hence the |amplitude|² is now,

$$|\bar{m}|^2 = \frac{C_2(F)2p_{12}F(p_1^2)F(p_2^2)(eg)^2\gamma}{32\pi^2\epsilon}\left\{4+\frac{p_o^2(p_1^2+p_2^2+p_o^2)}{p_1^2 p_2^2}+\frac{20\sigma p_{12}}{p_1^2 p_2^2}\right.$$

$$-p_o^2\sigma\left\{\frac{4p_{12}(2p_o^2+p_1^2+p_2^2)}{p_1^2p_2^2p_o^4}-\frac{1}{2p_{12}p_1^2}\left[\frac{[6p_{12}-2p_o^2]}{p_o^2}-\frac{[p_o^2+4p_{12}]}{2p_{12}}ln\left(\frac{p_o^2+2p_{12}}{p_o^2}\right)\right]\right.$$

$$\left.-\frac{1}{2p_{12}p_2^2}\left[\frac{[6p_{12}-2p_o^2]}{p_o^2}-\frac{[p_o^2+4p_{12}]}{2p_{12}}ln\left(\frac{p_o^2+2p_{12}}{p_o^2}\right)\right]\right\}\right\}. \qquad (4.82)$$

As a guide to where all the terms in this equation come from, the follwing table gives the equation numbers of the terms in the decomposition of the |amplitude|^2;

$-im$ = (4.2)	$-im_{AC}$ (4.6)		$-im_{AD}$ (4.39)		$-im_{BC}$ (4.59)		$-im_{BD}$ (4.68)	
	G=1	G=k^{-2}	G=1	G=k^{-2}	G=1	G=k^{-2}	G=1	G=k^{-2}
	(4.8) (4.17)	(4.18) (4.34)	(4.41) (4.48)	(4.49) (4.57)			(4.69) (4.74)	(4.75) (4.79)
(4.82)	(4.35)		(4.58)		(4.65)		(4.80)	

<div align="center">

Table IV.1   The decomposition of -im

</div>

where the numbers in brackets are the relevant equation numbers, and the pairs of equations are, top and bottom, the integral and its evaluation respectively.

IV.6 CONCLUSIONS

In section II.4 of chapter II we showed that using bare, unrenormalised Feynman rules, the two graphs of figures [II.2c] and [II.2d] had the same divergent behaviour. A prediction we made from multiplicative renormalisation.

The main aim of this work was to repeat the calculation using a set of (approximate) analytic representations of the renormalised Feynman rules, given in [Fig.II.5], obtained from the truncated solutions of the Schwinger-Dyson equations. In particular, we set out to test explicitly, by calculation, whether the

approximations used in deriving the functions, discussed in section II.5, preserve multiplicative renormalisability. A failure to do so implying either that the truncations of the equations, or the equations themselves not admitting a consistent renormalisation (or both), and that the use of the renormalised functions is inadmissable without further study of the renormalisation programme.

The relevance of the graphs of figures [II.2c] and [II.2d] to the physical process $\gamma^* \rightarrow q\bar{q}$ made it more useful to construct the contributions of these graphs to the |amplitude|^2 of this process, rather than evaluate the graphs in isolation. This was possible because the divergent behaviour is unaffected by doing this contraction, since the divergences arise from internal loops, independent of the external wavefunctions.

Having extracted the singular parts of the two contributions, given in equations [3.43] and [4.82], we can now make the following statements in conclusion:

1)   Comparing the divergent behaviour of figures [II.2c] and [II.2d] (equations [3.43] and [4.82]) we see that they are no longer of the same form. Though the vertex correction amplitude and the self energy both have $1/\epsilon$ divergences, these do not have equal and opposite coefficients.

2)   In the limit $G(k^2) \equiv F(p^2) \equiv 1$ (equivalent to $p_o^2 \equiv \sigma \equiv 0$, $a \equiv \gamma \equiv 1$) which reduces the renormalised functions of [Fig.II.5] to the bare ones of [Fig.II.3], we see that adding equations [3.43] and [4.71] gives

$$|\overline{m}|^2(\gamma^* \rightarrow q\bar{q})|_{\text{singular}} = -\frac{C_2(F)(eg)^2 p_{12}}{4\pi^2\epsilon} + \frac{C_2(F)(eg)^2 p_{12}}{4\pi^2\epsilon} \equiv 0,$$

and as expected (and explicitly shown in section II.4) the ultra-violet divergences exactly cancel. The uncancelled $1/\epsilon^2$ pole in [equ.4.82] arises soley from the infrared divergent terms with $G(k^2) = k^{-2}$ and consequently these vanish in the

limit $G(k^2) \equiv 1$ $(\sigma \equiv 0)$.

3)   The mis-match of the divergent behaviour instantly signals the breakdown of multiplicative renormalisation. As discussed earlier there are two possible 'theoretical' causes for this loss of renormalisability; firstly, it is not known how to renormalise the full, infinite nested hierarchy of Schwinger-Dyson equations. That is, independently of any further approximations needed to solve the equations, a consistent renormalisation programme for the full infinite set has not been developed. Moreover, even just for the two point functions alone, problems arise as discussed on page 481 of [Ref.II.7]. Clearly such a programme is needed before the Schwinger-Dyson equations provide a sound basis for studying the non-perturbative regime.

The second possible cause of the breakdown of renormalisability are the truncations of the full Schwinger-Dyson equations necessary to yield a closed set of soluble equations. For example, dropping G_2 in [equ.2.25] means that only a semi-infinite number of graphs are summed as an approximation to *all* graphs. It is possible that, by omitting graphs in the determination of the functions $F(k^2)$ and $G(k^2)$, then the two graphs we considered, having different configurations of propagators and vertices, will effectively [A be evaluated to different, inconsistent orders of perturbation theory. The order by order cancellation of divergences is then inherently lost. Consequently, all orders of perturbation theory, or at least a self consistently renormalisable truncation, must be used with the Schwinger-Dyson equations to determine an admissable set of renormalisation functions.

Naively it is plausible that the omitted graphs in, for example [Fig.II.2c], might produce additional loop integrals and hence factors of $1/\epsilon$ that might have the numerical coefficients needed to ensure cancellation of the divergences between the graphs. Necessarily this is intimately connected to the specific truncations used.

To summarise, the breakdown in renormalisation can have one, or both,

of the causes outlined above; either the full infinite set of equations, although formally well defined, is inconsistent until a satisfactory renormalisation programme is developed. Or, (and) the truncations then required to close this set are, themselves, not consistent. Of course, any such solutions of the Schwinger-Dyson equations go beyond perturbation theory and so inevitably lose the relatively straightforward order by order renormalisability of the perturbative approach.

Both the possible fatalities outlined above are 'theoretical' ones inherent in the form of the equations. A further 'numerical' problem could lie in the simple analytic representations used in [Fig.II.5] not being sophisticated enough to quantitatively describe the renormalisation functions, despite giving the correct form over a finite region of momentum [Ref.II.18]. Thus an oversimplified parameterisation could affect the numerical coefficients of the poles.

The very nature of this work, a consistency check by direct calculation, and the three possible causes of the failure of this check means that, apart from saying that the renormalised rules of [Fig.II.5] are inadmissable and are not guaranteed to yield finite physical cross sections, we can say little more about how each of the possible fatalities affects the results. Having shown that a combination of the failings dicussed above conspire to produce an inadmissable set of rules, the next step would be to isolate each failing and investigate its affect. Clearly the first, and most important point is to understand the renormalisation of the full Scwinger-Dyson equations. If it is found that no consistent renormalisation scheme exists, beyond perturbation theory, then the hope of using the equations, independent of any further truncations and parameterisations, to yield renormalised functions is lost straight away. Only if such a scheme is developed, and the full set of equations can be renormalised consistently, can we be confident in using the equations as a way of investigating the non-perturbative aspects of QCD.

CHAPTER V QED$_3$

V.1 INTRODUCTION

As discussed in the introduction, QCD is the accepted theory of the strong interaction. One of it's important features is asymptotic freedom; at large momentum (or equivalently small distances) the coupling constant becomes very small and a perturbative expansion in the coupling can be usefully employed. However, at the opposite extreme, at small momentum, it is believed that the coupling grows and the perturbative expansion becomes useless.

An infrared divergent coupling implies that particles coupled to the gauge field are confined: isolated quarks, for example, would not exist since this would require an infinite amount of energy to overcome the diverging coupling strength.

Phenonemon, such as confinement, requiring a knowledge of the small momentum regime clearly have to be investigated by some non-perturbative method. [Ref.V.1]. One such approach is to turn to the use of simplified "toy" models which, although not a realistic description of the real world, are tractable and contain enough "real-life" features to warrant further investigation; the hope being that the simplified mathematics of such models allows general principles and mechanisms to become more transparent, therefore, giving insight into how similar principles come about in more realistic, but less tractable models.

It is the purpose of this work to consider such a "toy" model, QED in three (Euclidean) space-time dimensions (QED$_3$). The motivation for studying this is two-fold: first as a model to investigate the so called gauge hierarchy problem [Ref.V.2]; how is it that a theory with only one dimensionful parameter, at tree level, can accommodate a hierarchy of mass scales, once quantum corrections are included, with the scales differing by many orders of magnitude? In four dimensions this is the problem of explaining why, starting from a grand unified group (SU(5) for example) with an inherent mass scale of the order of 10^{15}GeV,

the strong interaction scale is of order 1GeV and the weak-interaction scale is order 10^2GeV. It is, however, inconceivable that there are no other mass-scales in the intervening thirteen orders of magnitude (the so-called desert region [Ref.V.3]) and that the breaking of the unified group leads to such light weak gauge-bosons.

Secondly, it will be shown that such a three dimensional theory becomes effective in the description of a "realistic" four dimensional gauge theory at equilibrium with a finite temperature heat bath [Ref.V.4]. Such theories are relevant to the possible deconfinement of the strong interaction fields. In a way analogous to heating a bar magnet and restoring rotational symmetry by destroying correlations between spins, it is believed that symmetries, spontaneously broken at zero temperature, might be unbroken at high temperatures. In particular damping of long range correlations between confined fields might lead to deconfinement of strong interaction particles [Ref.V.4a].

Finite temperature gauge theories are also of cosmological interest [Ref.V.5] since it is generally believed that the early universe existed at much higher temperature than that which prevails today. Such theories are also relevant to the study of heavy ion collisions where, for very short times, effective high temperatures exist. We have, however, nothing more to say concerning these types of investigations in this work.

After discussing in more detail the motivation and relevance of three-dimensional theories the main task of this work is to calculate, in a non-perturbative way, the fermion self energy function. The aim being to investigate, by direct calculation, how very different scales can be generated by quantum corrections, and whether our theory spontaneously generates a mass.

V.2 SCALE HIERARCHIES IN GAUGE THEORIES

In this section the ideas of hierarchies of vastly differing mass scales that seemingly occur in certain gauge theories are discussed. In particular the famous SU(5) scale hierarchy [Ref.V.6] illustrates the, mainly aesthetic, problem of satisfactorily accommodating such large hierarchies.

The similarities to QED$_3$ will then be discussed illustrating how this, apparently oversimplified, model can tell us something about real hierarchies.

V.2.1 The scale hierarchy problem in SU(5)

The electroweak model of Weinberg and Salam [Ref.I.2] went some way to unifying two of the forces of nature. Although unification is somewhat misleading since their model still contains two distinct couplings. True unification would describe the, apparently, differing interaction strengths in terms of one, unified coupling. Even more ambitious schemes would also attempt to include the strong interactions in the unification.

The existence of one unified coupling is made qualitatively plausible by asymptotic freedom of the strong interaction: the gluon coupling falls monotonically as a function of increasing momentum. Thus some, presumably large, momentum scale exists where the strong coupling is of the same order as the weak and electromagnetic interaction strengths which, although functions of momentum, vary more slowly. At this scale the usual distinction between quarks and leptons, based on the former enjoying the strong interaction while the latter not, is lost. All the particles feel one unified force.

The simplest way of achieving such unification of the coupling constants is to introduce a grand unified group, G [Ref.V.7], large enough to contain as subgroups the accepted SU(3) and SU(2)×U(1) groups of the low energy strong and electroweak theories,

$$G \supset SU(3) \times U(1). \tag{5.1}$$

Georgi and Glashow [Ref.V.8] were the first to argue that SU(5) is the smallest simple group that contains the decomposition of [equ.5.1] and satisfies certain physical constraints.

Having chosen a single group, G, with a single unified coupling this symmetry must then be spontaneously broken to arrive at the SU(3)×U(1) low energy symmetries of the strong and electromagnetic interactions. If we further want to retain the Weinberg-Salam model of the electroweak interactions there must be at least two separate levels of symmetry breaking, the first breaking G down to SU(3)×SU(2)×U(1) and secondly the Weinberg-Salam breaking of SU(2)×U(1) to U(1) of electromagnetism;

$$G \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1). \qquad (5.2)$$

Motivated by trying to explain the obvious disparity in the strength between the strong and electroweak interactions at ordinary energies (ie.*much* less than the unification scale), the success of such models hinges on the understanding of the spontaneous symmetry breaking, wherein lies the problem of scale hierarchies: the two scales of spontaneous symmetry breaking differ by thirteen or more orders of magnitude [Ref.V.3]. The second stage of symmetry breaking in [equ.5.2] occurs at a scale set by the mass of the $W^{\pm}$ bosons [Ref.V.9], at the order of 100 GeV. However, the first stage, breaking G to $SU(3) \times SU(2) \times U(1)$ is forced to occur at some "superlarge" energy for the following reasons [Ref.V.3]:

The momentum dependence of running couplings is a relatively soft logarithmic one [Ref.V.10],

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{\{1 + (\beta_0/4\pi)\alpha(\mu^2)ln(Q^2/\mu^2)\}}. \qquad (5.3)$$

where β_0 is a theory dependent constant and μ is a renormalisation point. The logarithmic dependence ensures the couplings run relatively slowly, requiring

many orders of magnitude for the running to compensate for the large low-energy disparity in coupling strengths. Each of the three, low energy interactions obey equations of the form of [equ.5.3] with only the beta function, β_0, being interaction dependent [Ref.V.10]. If M_x is the momentum scale where the couplings are unified [equ.5.3] can be re-written,

$$\frac{1}{\alpha_i(Q^2)} - \frac{1}{\alpha(M_x^2)} = -\frac{(\beta_0)_i}{4\pi} ln\left(\frac{Q^2}{M_x^2}\right). \qquad (5.4)$$

where $i = 1, 2, 3$ for the three interaction couplings, α. Eliminating $\alpha(M_x^2)$ (and the mixing between SU(2) and U(1) couplings) we can relate the strong and electromagnetic couplings at low energy (α, α_s respectively) to the unification scale M_x, using $(\beta_0)_s = 11 - 2N_f/3$ and $(\beta_0)_{em} = -4/3$

$$\frac{\alpha(Q^2)}{\alpha_s(Q^2)} = \frac{3}{8}\left\{1 - \frac{11\alpha}{2\pi} ln\left(\frac{M_x^2}{Q^2}\right)\right\}. \qquad (5.5)$$

Taking α/α_s as typically $1/30$ at $Q^2 = 10$ GeV^2 gives M_x of the order of 10^{16}GeV. Only above this scale is the gauge group G unbroken and the three forces unified; the first stage of symmetry spontaneous breaking in [equ.5.2] occurs at momentum of order 10^{16}GeV.

Further evidence that M_x is of order 10^{16}GeV comes from the physical constraints of proton decay: for G unbroken, leptons become indistinguishable from quarks and lepton and baryon number violating interactions would be expected, for example, proton decay [Ref.V.11], diagramatically represented in [Fig.V.1].

Such decays would be mediated by massive gauge bosons X,Y. (The massless gauge bosons of the group G becoming massive via a Higgs mechanism [Ref.V.12]). Experimental limits on the proton lifetime, $\tau_p \geq 10^{32}$ years [Ref.V.13], demand that the bosons X,Y are extremely heavy, $M_x \geq 10^{14}$GeV. The X,Y masses set the scale for the first stage of symmetry breaking and agree,

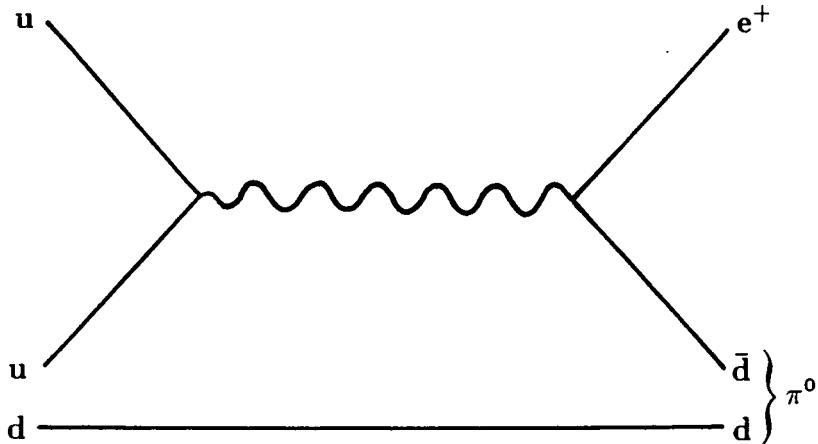qualitatively, with the previous argument for $M_x \gg M_w$.

Figure V.1 Heavy boson mediated proton decay

It is perhaps difficult to justify that only after 14 orders of magnitude does the second stage of symmetry breaking set in. Questions such as

"Why does nature pick two such vastly differing scales? And what (if any) is the physical significance of this hierarchy?"

seem unavoidable. We would feel more comfortable if the two scales were of roughly the same order, or at least connected in some way by some intermediate scales to fill the "desert" between these two scales, to avoid such nagging questions.

Spontaneous chiral symmetry breaking (SCSB) [Ref.V.14] in QED₃ is a simplified, tractable model in which scale hierarchies resulting from spontaneous symmetry breakdown can be studied. The tree level theory has one scale; that of the dimensionful coupling . With quantum corrections the dynamical generation of a mass, by SCSB, provides the means of spontaneously generating a second scale. If the scale of SCSB is very different from the 'intrinsic' scale of the coupling there is a spontaneously generated hierarchy analogous to the SU(5)

case outlined above. Although there is only one stage of symmetry breaking in QED$_3$ the scale already present in the tree level theory mimics another stage by providing a mass scale.

By studying QED$_3$ we aim to show how, given one scale (analogous to M_x or M_w in SU(5)) a symmetry can be spontaneously broken at another, very different scale. By calculation of the fermion self energy we show that there does exist a significant hierarchy in scales in QED$_3$ (although, perhaps not one of fourteen or so orders of magnitude), explicitly demonstrating that such hierarchies can occur "naturally" and, if not completely answering questions on the origins of scale differences, at least making them more plausible and easy to digest.

Having discussed the first motivation for studying QED$_3$ we now turn to the second.

V.3 GAUGE THEORIES AT FINITE TEMPERATURE

The second motivation for considering the model, QED$_3$, is that in an appropriate limit it becomes relevant to the study of QCD$_4$ at finite temperature [Ref.V.15]. As hinted at earlier, the latter might yield information on the (possible) deconfinement of strong interaction fields of finite temperature QCD$_4$. The commonly believed mechanism for deconfinement in these theories is the decoupling theorem of Appelquist and Carazzone [Ref.V.16] which tells us that, at finite temperature the correlation between fields of mass m is exponentially damped according to

$$\lim_{|\vec{x}|\to\infty} < \phi(t,\vec{x})\phi(0,\vec{0}) > \sim e^{-m|\vec{x}|}.$$

If the strong interaction gauge fields (the gluons) can acquire a mass in some way, the long range correlations between two, widely separated gluons would

become damped according to this equation. In the limit of infinite separation the gluons would be totally uncorrelated and would not know of each others existence; they would become deconfined.

In the following section we outline how a three dimensional model becomes effective in the description of QCD$_4$ at finite temperature. The model is a three dimensional gauge field coupled to scalars and fermions. The gauge field being the space components of the original four dimensional gauge field with the time component of the latter coupling like a scalar. The authors of [Ref.V.15] argue that all fermions decouple from the model as a consequence of their non-zero ground state energy, and that scalars (including the time component of the four dimensional gauge field) spontaneously generate a temperature dependent mass at one loop level,

$$m \sim \sqrt{\alpha} T$$

and likewise decouple. Having done this we are left with a pure three dimensional gauge theory and the question of deconfinement is re-cast as the spontaneous generation of a mass for this gauge theory.

In this work we consider the spontaneous mass generation of a fermion, not gauge, field, that is, we do not realise the limit $|\vec{x}| \to \infty$ where the fermions have decoupled. However, the hope is that the study of the realisation of a mass for any field (in our case the fermion) is a useful prototype for a similar process in the gauge field case.

As promised, in the next section we show how three dimensional gauge theories describe their four dimensional counterparts at finite temperature. In paticular we see how, in a given representation, the Feynman rules are essentially the zero temperature rules and that the time component of the original gauge field does, in fact, couple as a massless scalar.

V.3.1 Feynman rules at finite temperature

Starting from an ensemble of finite temperature Green's functions (analogous to a statistical physics ensemble) we show that by mapping time to an imaginary, discrete interval [Ref.V.17] the Feynman rules at finite temperature are essentially the same as those at zero temperature. In this "imaginary time" representation the change in boundary conditions at finite temperature leaves the standard Feynman rules essentially unchanged [Ref.V.17].

For a Hamiltonian H and fields $\phi(x)$ the finite temperature ensemble of Green's functions is defined by

$$G_\beta(x_1.....x_j) \equiv \frac{Tre^{-\beta H}T\phi(x_1).....\phi(x_j)}{Tre^{-\beta H}}. \qquad (5.6)$$

Where β is the inverse temperature (strictly speaking $\beta^{-1} = K_B T$, but we normalise the Boltzman constant, K_B, to be 1). The T in [equ.5.6] stands for the usual time ordering of operators.

Continuing the time arguments, x_{i0}, to the interval [Ref.V.17],

$$0 \leq ix_{i0} \leq \beta, \qquad (5.7)$$

the boundary conditions for this representation can be obtained from [equ.5.6]. Using generic short hand notation $< T\phi(x)\phi(y) >$ for the RHS of [equ.5.6] the "time ordering" for imaginary time is defined by

$$< T\phi(x)\phi(y) > = < \phi(x)\phi(y) > \qquad ix_0 > iy_0$$

$$= \pm < \phi(y)\phi(x) > \qquad iy_0 > ix_0$$

(The minus sign appearing from the anticommutivity of fermion fields).

For the imaginary time interval $[0, -i\beta]$ of [equ.5.7] we have from [equ.5.6],

$$(Tre^{-\beta H} G_\beta(x,y))|_{x=0} = Tre^{-\beta H}\phi(y_0,\underline{y})\phi(0,\underline{x}).$$

Inserting the unit operator $1 = e^{-\beta H}e^{\beta H}$ and cyclically permuting the arguments of the trace gives,

$$(Tre^{-\beta H} G_\beta(x,y))|_{x=0} = Tr\left\{e^{-\beta H}e^{\beta H}\phi(x_0,\underline{x})e^{-\beta H}\phi(y_0,\underline{y})\right\}$$

$$= Tr\left\{e^{-\beta H}\phi(-i\beta,\underline{x})\phi(y_0,\underline{y})\right\}$$

$$= \pm(Tre^{-\beta H} G_\beta(x,y))|_{x_0=-i\beta}. \qquad (5.8)$$

Equation [5.8] is an (anti-) periodicity condition on finite temperature Green's functions,

$$G_\beta(x,y)|_{x_0=0} = \pm G_\beta(x,y)|_{x_0=-i\beta}. \qquad (5.9)$$

Exploiting this imaginary time periodicity gives rise to a Fourier sum in the Fourier transform representation of $G_\beta(x,y)$;

$$G_\beta(x,y) = \left(\frac{-1}{i\beta}\right)\sum_{n=-\infty}^{\infty} e^{-i\omega_n x_0}\int\frac{d^3p}{(2\pi)^3}\left(\frac{-1}{i\beta}\right)\sum_{n'=-\infty}^{\infty} e^{i\omega_{n'}y_0}\int\frac{d^3p'}{(2\pi)^3}$$

$$\times e^{-ip\cdot x}e^{ip'\cdot y}G_\beta(\omega_n,p,\omega_{n'},p'), \qquad (5.10)$$

where $\omega_n = 2\pi n/(-i\beta)$, n odd (even) for fermionic (bosonic) fields.

Noting that $G(x,y) = G(x-y)$, ie a function only of coordinate differences, and inverting [equ.5.10], we can write, with $\vec{t} \equiv (\vec{x}-\vec{y}), \quad \vec{s} \equiv \vec{y}$

$$G_\beta(\omega_n, \vec{p}, \omega_{n'}, \vec{p'}) = \int_0^{-i\beta} dx_0 e^{i\omega_n x_0} \int_0^{-i\beta} dy_0 e^{-i\omega_{n'} y_0} \int d^3\vec{t} \int d^3\vec{s}$$

$$\times e^{-i\vec{p}\cdot\vec{t}} e^{-i(\vec{p'}-\vec{p})\cdot\vec{s}} G_\beta(\omega_n, \omega_{n'}, \vec{t}). \tag{5.11}$$

Performing the $\vec{s}$ integral to give a δ-function and using a similar transformation for the discrete $\omega_{n,n'}$ leads to

$$G_\beta(\omega_n, \vec{p}, \omega_{n'}, \vec{p'}) = \int_0^{-i\beta} dx_0 e^{i\omega_n x_0} \delta_{n,n'} (2\pi)^3 \delta^{(3)}(\vec{p'} - \vec{p}) \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} G_\beta(\omega_n, \vec{x}) \tag{5.12}$$

or, in momentum space,

$$G_\beta(\omega_n, \vec{p}, \omega_{n'}, \vec{p'}) = -i\beta \delta_{n,n'} \delta^{(3)}(\vec{p'}, \vec{p}) G_\beta(\omega_n, \vec{p}). \tag{5.13}$$

Inserting this result into [equ.5.10] and performing the trivial δ- function integrals leaves,

$$G_\beta(x - y) = \left(\frac{-1}{i\beta}\right) \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-ip(x-y)} G_\beta(\omega_n, \vec{p}). \tag{5.14}$$

Taking, for example, the fields as scalars satisfying a standard Klien-Gordon equation

$$(\Box_x + m^2) G_\beta(x - y) = -i\delta(x - y)$$

where the 'box' operator is defined by,

$$\Box_x \equiv \frac{\partial^2}{\partial t^2} - \vec{\nabla}_x^2$$

and substituting [equ.4.14] into the field equation leads to

$$\left(\frac{-1}{i\beta}\right)\sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}(\Box_x + m^2)e^{ip(x-y)}G_\beta(\vec{p},\omega_n) = -i\delta^{(4)}(x-y)$$

$$= \left(\frac{-1}{i\beta}\right)\sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}e^{ip(x-y)}(-p^2 + m^2)G_\beta(\vec{p},\omega_n). \tag{5.15}$$

Using the integral (and discrete sum) representation of the δ-function on the RHS gives

$$G_\beta(\omega_n,\vec{p}) = \frac{-1}{(-p^2 + m^2)}. \tag{5.16}$$

This is an identical expression to the standard zero-temperature two-point scalar Green's function. We have thus given an illustration of how, in the imaginary time representation, [equ.5.7], Feynman graphs are evaluated in the same way as in zero temperature theories. Except that [equ.5.9] tells us that the imaginary time is periodic which implies from [equ.5.13] that the four dimensional δ-functions and momenta integrals become three dimensional, with the latter accompanied by a discrete sum, [equ.5.10].

In summary, the finite temperature Feynman rules are modified from the zero temperature case by,

$$p_0 \rightarrow ip_0$$

$$\delta^{(4)} \rightarrow \frac{\beta}{2\pi i}\delta_{nn'}\delta^{(3)}(\vec{p}) \tag{5.17}$$

$$\int d^4p \rightarrow \frac{2\pi i}{\beta}\int d^3\vec{p}\sum_{n=-\infty}^{\infty} \quad ; n = \text{even(odd)} : \text{bosons(fermions)}.$$

The last of these three equations already hints at a three dimensional theory. The integral is a (3 dimensional) Feynman integral while the sum, over a discrete set of energy modes is analogous to the sum over colours in a Feynman loop integral leading to an overall multiplicative colour factor. However, this on its own, is not enough to give a truly three dimensional effective theory. As it stands all [equ.5.17] tells us is that we only integrate over the space components and sum over a discrete set of time components of fields that are still four dimensional (ie have four components) . In particular the pure gauge part of QCD$_4$,

$$L_{gauge} = -\frac{1}{4} Tr G^{\mu\nu} G_{\mu\nu}, \qquad (5.18)$$

involves all four components of the gauge field A_μ via

$$G_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \qquad (5.19)$$

where $\mu, \nu = 1, 2, 3, 4$. However, in the static limit the contribution of the electric ($\mu, \nu = 0$) fields to the lagrangian density is

$$G_{0i}^2 \sim |D_i A_0|^2$$

$$\equiv |(\partial_i - ig A_i) A_0|^2. \qquad (5.20)$$

That is the electric components couple as scalars to the magnetic part of the gauge field. Decomposing the gauge lagrangian of [equ.5.18] as

$$G_{\mu\nu} G^{\mu\nu} = G_{0i} G^{0i} + G_{i0} G^{i0} + G_{ij} G^{ij},$$

(where $i, j = 1, 2, 3$) and treating the first two terms as a scalar field, A_0 , coupling to a three component gauge field A_i as in [equ.5.20], the pure gauge

part of the static theory becomes, with [equ.5.17], an effective three dimensional gauge theory coupled to massless scalars [Ref.V.15].

Our supposition, immediately following [equ.5.17], has proved correct. By using the imaginary time representation of [equ.5.7] the four dimensional theory becomes equivalent to a three dimensional one with an infinite number of modes of the fields (the sum in [equ.5.17]). This summation over modes is analogous to an infinite number of colours in conventional QCD.

Strictly speaking QCD$_3$ is relevant to finite temperature QCD$_4$. However, as discussed in the following section, the abelian model QED$_3$ serves as a tractable prototype. Particularly as at large N (number of colours) the non-abelian interactions of QCD$_3$ are suppressed by order N^{-1} and the abelian, QED$_3$-like interactions dominate [Ref.V.18]. In this limit QED$_3$ with N flavours of fermions becomes a relevant model in the investigation of QCD$_4$ with N colours at finite temperature.

Having motivated the study of QED$_3$ on the grounds of being a model relevant to scale hierarchies and finite temperature effects in the, more realistic theory of QCD in four dimensions we now turn, in the next section, to discuss the model in more detail.

V.4 THE MODEL: QED$_3$

In this section we introduce our model QED$_3$ in detail, illustrating why we are forced to abandon the familiar perturbative loop expansion and instead turn to the large N (number of flavours) expansion. In this non-perturbative expansion, the lowest order Schwinger-Dyson equation [Ref.V.19] for the fermion self energy is set up, the solution of which is the main task of this work and subsequent sections.

With N flavours of massless fermion fields, ψ, the lagrangian density for QED in three space time dimensions is

$$L_{QED} = \sum_{k=1}^{N} \bar{\psi}_k (i\not{\partial} - e\not{A})\psi_k - \frac{1}{4}F_{ij}^2.$$ (5.21)

where the coupling, e, has dimensions of mass, and

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

$i, j = 1, 2, 3$.

Although [equ.5.21] permits a two dimensional representation of the Dirac gamma matrices, $\gamma_i = \sigma_i$, ($\sigma_i \equiv$ Pauli matrices), in such a representation there is no other 2×2 anti-commuting matrix to play the role of γ_5 [Ref.V.19]. Consequently [equ.5.21] has no chiral symmetry transformation,

$$\psi \rightarrow \psi' \equiv e^{i\alpha\gamma_5}\psi,$$ (5.22)

and the massless theory has no more (or less) symmetry than the massive theory. Since we aim to solve for the fermion self energy and in the language of spontaneously generating scale hierarchies, this can be seen as analogous to spontaneously breaking chiral symmetry, we therefore want to allow for the possibility of SCSB in our model. We achieve this by using the four dimensional representation,

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \quad , \quad \gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \quad , \quad \gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$ (5.23)

with two further anti-commuting matrices playing the role of γ_5, giving [equ.5.22] two Chiral transformations;

$$\psi \rightarrow e^{i\alpha\gamma_3} \qquad ; \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\psi \to e^{i\alpha\gamma_5} \qquad ; \gamma_5 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad\qquad (5.24)$$

Theories in three dimensions have dimensionful couplings, that is they are superenormalisable [Ref.V.20]. Perturbative expansions in the superenormalisable coupling of such theories with massless fields are beset by infrared divergences [Ref.V.21] and are said to be infrared sensitive. On dimensional grounds the dimensionless perturbation parameter must be proportional to the ratio of the (dimensionful) coupling divided by some momentum scale. In the absence of any mass to provide this scale, high order diagrams have high orders of momentum in the denominator leading to infrared divergences in perturbative loop integrals. For example the two loop contribution to the fermion self energy of [Fig.V.3] can be seen to be logarithmically infrared divergent. To see this we have first to compute the one loop correction to the boson propagator as follows.

V.4.1 The one loop contribution to the gauge propagator

Expressing the diagram of [Fig.V.2] in terms of the Euclidean fermion propagator $S_F(p)$, given in [Fig.II.3], and the fermion-boson vertex, $-ie\gamma_\mu$, gives the one loop gauge polarisation tensor, $\pi_{\mu\nu}$, as

$$\pi_{\mu\nu}(p) \equiv (-ie)^2 Tr \int \frac{d^3k}{(2\pi)^3} \gamma_\mu S_F(k+p)\gamma_\nu S_F(k). \qquad\qquad (5.25)$$

The transversality of $\pi_{\mu\nu}$ allows us to extract the tensor structure and write this expression as

$$\pi_{\mu\nu}(p) \equiv I(p)(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}). \qquad (5.26)$$
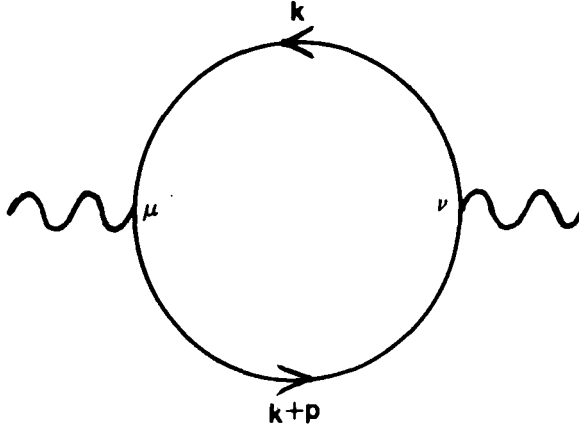

Figure V.2   The one loop boson correction

To determine $I(p)$ we multiply both sides of this equation by $P_{\mu\nu} \equiv \delta_{\mu\nu} + p_\mu p_\nu/p^2$ and take the trace to obtain

$$I(p) = \frac{-e^2}{2(2\pi)^3} Tr \int d^3k \frac{[(k \cdot p)^2 - k^2 p^2]}{k^2(k + p)^2}.$$

Evaluating the trace, remembering that we are using a 4×4 representation of the gamma matrices, leads straightforwardly to

$$I(p) = \frac{-4e^2}{(2\pi)^3 p^2} \int d^3k \frac{[(k \cdot p)^2 - p^2 k^2]}{k^2(k + p)^2}. \qquad (5.27)$$

This integral is performed by introducing the Feynman parameterisation

$$\int \frac{d^3k}{k^2(k + p)^2} \equiv \int_0^1 dx \int \frac{d^3k}{[k^2(1 - x) + (k + p)^2 x]^2},$$

and defining the variable $L \equiv p^2 x(1 - x)$ allows [equ.5.27] to be written in terms of the shifted parameter $k' \equiv k + px$ as

$$I(p) = \frac{-4e^2}{(2\pi)^3 p^2} \int_0^1 dx \int d^3k' \frac{[(k' \cdot p)^2 - p^2 k'^2]}{[k'^2 + L]^2}.$$

Under the integral $(p \cdot k')^2$ becomes $p^2 k'^2 / 3$ which gives

$$I(p) = \frac{8e^2}{3(2\pi)^3} \int_0^1 dx \int d^3k' \left\{ \frac{1}{(k'^2 + L)} - \frac{L}{(k'^2 + L)^2} \right\}.$$

The k' integral can now be done using the identity

$$\int \frac{d^n k}{(k^2 + L)^A} \equiv \frac{\pi^{n/2} \Gamma(A - n/2)}{\Gamma(A) L^{(A-n/2)}}$$

to give

$$I = \frac{8e^2}{3(2\pi)^3} \int_0^1 dx (-3\pi^2) L^{1/2}.$$

The remaining x integral can now be performed giving

$$I = \frac{-e^2 q}{8}.$$

Inserting this result into [equ.5.26] then gives, for the one loop gauge polarisation tensor

$$\pi_{\mu\nu} = \frac{-e^2 p}{8} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \tag{5.28}$$

We have presented this calculation in some detail since use of this result will be made in following sections. For now, we use it to show how the two loop fermion self energy contribution of [Fig.V.3] is infrared divergent.

V.4.2 The two loop contribution to the fermion self energy

In terms of the fermion and boson propagators, $S_F(p)$ and $D^0_{\mu\rho}(p)$, and the polarisation tensor $\pi_{\mu\nu}$ of [equ.5.28], the graph in [Fig.V.3] is given by

$$(-ie)^2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S_F(q-p) \gamma_\nu D^0_{\nu\sigma}(q) \pi_{\sigma\mu}(q) D^0_{\mu\rho}(q), \qquad (5.29)$$
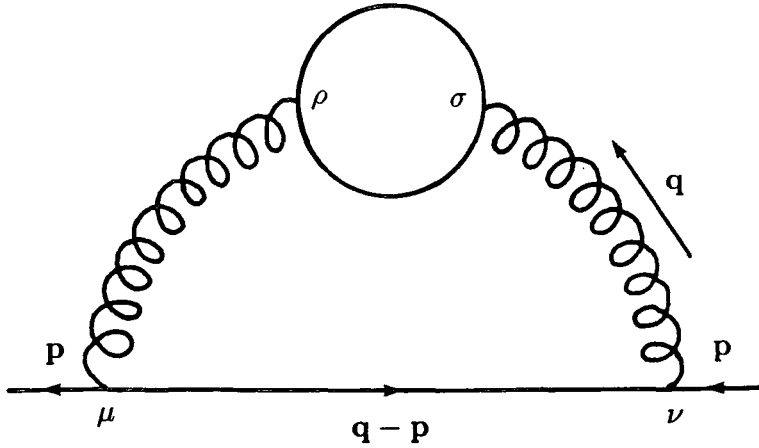

Figure V.3   A two loop fermion self energy contribution

and using the Feynman rules of [Fig.II.3] in the Feynman gauge this expression becomes

$$(-ie)^2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu \frac{(\slashed{q} - \slashed{p})}{(q-p)^2} \gamma_\nu \frac{(\delta_{\nu\sigma}q^2 - q_\nu q_\sigma)}{q^4} \frac{(-e^2 q)}{8} \frac{(\delta_{\sigma\rho}q^2 - q_\sigma q_\rho)}{q^4},$$

and we can immediately see that the ten powers of q in the denominator ensure that this expression behaves like

$$\int \frac{dq}{q} \sim ln(q)$$

and the two loop graph of [Fig.V.3] is logarithmically infrared divergent.

This fatal sickness of the perturbative loop expansion forces an alternative dimensionless expansion parameter to be employed. The only dimensionless quantity in [equ.5.21] is N, the number of flavours. For N large, 1/N is a small dimensionless parameter in which a perturbative expansion can be used that is infrared safe; this so called "1/N expansion" [Ref.V.22] is discussed in the next section.

V.5 THE 1/N EXPANSION IN QED₃

For N flavours of fermions a useful expansion in 1/N is possible if N is large. Graphs are then ordered according to their power of N, not the power of coupling constant. The zero dimensions of N avoids the infrared divergences associated with coupling expansions.

For graphs to have sensible large N limit the coupling, e, must be re-defined as $e/\sqrt{N}$. To illustrate this we consider the one-loop diagram of [Fig.V.2]: in an analogous way to colour factors in QCD there is a "flavour factor" associated with virtual fermion loops. The fermion in the loop of [Fig.V.2] can be one of N possible colours. Summing over these possibilities leads to a multiplicative factor of N.

In the N→ ∞ limit such a factor causes the graph to diverge. However, redefining the coupling constant to be $e/\sqrt{N}$ the two vertices in the graph contribute factors of $N^{1/2}$ to the denominator. Combining these with the combinatoric flavour factor in the numerator gives the graph a smooth, non-divergent, large N limit,

$$(e/\sqrt{N})^2 N = e^2.$$

With the re-defined coupling constant a given graph has an associated power of N, determined by counting the number of closed fermion loops and

vertices. The power of N can only be zero or negative since fermion loops contributing positive powers to the numerator always have at least two vertices killing off the positive power. So with all graphs having a sensible, well defined large N limit a perturbative expansion in 1/N is made possible, the "1/N expansion".

By grouping graphs by their power of N we can sum an infinite number of diagrams at any given order in N. For example the graph of [Fig.V.2] is zeroth order in N. This graph can be added any number of times to another, arbitary diagram without changing the power of N of the latter. If a graph is of some order of N, then so are all the graphs with [Fig.V.2] insertions giving an infinite number of contributions to this order. By summing up an infinite number of Feynman graphs at each order in N we ensure that the divergences of individual graphs, necessarily encountered in the loop expansion, cancel, as they must do, to give finite Green's functions. Although we still have to expand in powers of N the infinite summations to each power take us beyond finite-order perturbation theory and the associated divergences.

The following argument shows, as promised, that any non-abelian diagram, involving gauge self interactions, is suppressed relative to a similar non-abelian one with no self interactions:

A non-abelian graph can always be obtained from an abelian one by either adding gauge propagators or replacing a fermion line by a gauge propagator. In the first case there are two additional vertex factors of $N^{-1/2}$ that are not compensated by additional closed fermion loop factors, and in the second case the removal of a fermion loop decreases the colour factor by one power of N. In both cases the non-abelian graph is supressed by order N^{-1} compared to the related Abelian graph. To leading order in N this implies that only Abelian diagrams contribute and, as stated in the previous section, QCD$_3$ with N colours becomes QED$_3$ with N flavours in the limit N→ ∞. This is then the relevant

model for QCD$_4$ in this limit.

For the leading order(s) results to be meaningful, N must be large (infinite for the leading order to become exact). Naively there are three fermion colours in the real world and 1/3 is not terribly small. However, the true expansion parameter is probably not simply 1/N, but c/N, where c is some, possibly small number reviving the reliability of the expansion. A similar situation occurs in the usual perturbative expansion of QED$_4$. The expansion parameter here is not e^2, the coupling constant, but $e^2/(4\pi)$. So though e^2 is small $e^2/(4\pi)$ is even smaller making the perturbative expansion very reliable.

In the context of finite temperature field theory the limit of N truly tending to infinity can be reconciled by referring to the third equation of [equ.5.17]. Here the infinite sum over N discrete energy modes is analogous to summing over an infinite number of flavours. In this respect the leading (zeroth) order, in 1/N, result becomes exact when discussing finite temperature QCD.

Massless QED$_3$ is finite and tractable in the 1/N expansion and is the model we use. Although our model is non-perturbative in the sense described above, there are still an infinite number of orders of N to be summed. This summation is achieved by using the Schwinger-Dyson equations [Ref.V.27]. In doing so we, in principle, sum up the infinite number of graphs contributing to a given Green's function by first summing (infinite) subsets of divergent graphs in the 1/N expansion and then summing these finite subsets. In particular, in the next chapter, we set up and solve (in given approximations) the Schwinger-Dyson equation for the fermion self energy.

CHAPTER VI THE FERMION SELF ENERGY

VI.1 THE SCHWINGER-DYSON EQUATION FOR THE FERMION SELF ENERGY

At the expense of having to solve a set of coupled integral equations, the Schwinger-Dyson equations, at least in principle, allow full renormalised propagators and vertices to be calculated. They provide a way of summing all Feynman diagrams and would, if soluble, give results not only true to all orders in perturbation theory, but, as discussed in chapter II, are independent of perturbation theory, and fully non-perturbative as well [Ref.VI.1]. However, such an ambitious objective is insoluble as it stands and approximations, effectively only summing (infinite) subsets of Feynman diagrams, have to be made.

Having seen in the previous chapter how we were forced to use the $1/N$ expansion for QED$_3$ we now utilise the Schwinger-Dyson equation for the fermion propagator to sum over orders of the $1/N$ expansion. We still require certain approximations to make the equation tractable and in this work we solve the equation using the lowest orders (in $1/N$) vertex and gauge propagator. The resulting, simplified equation is solved by a numerical, iterative procedure.

The Schwinger-Dyson equation is derived now, in the following way: in Euclidean space, with the gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ the inverse fermion propagator, $S_F^{-1}(p)$, can be written

$$-iS_F^{-1}(p) = \not{p}[1 + A(p)] + \Sigma(p) \tag{6.1}$$

where $\Sigma(p)$ is the self energy and $A(p)$ is the wavefunction renormalisation. The latter can be identified with the renormalisation function $F(p^2)$ of chapters II, III and IV (We use here the notation of $A(p)$ to be consistent with Appelquist et al [Ref.VI.2]). As we shall see, the chirally symmetric solution with massless

fermions, ie $\Sigma(p) \equiv 0$, is always a possible solution and then [equ.6.1] reduces to $S_F(p) = i\{[1 + A(p)]\not{p}\}^{-1}$ which is the full fermion propagator of [Fig.II.5] with $F(p^2) \equiv 1 + A(p)$.

The full fermion propagator on the LHS of [equ.6.1] can be decomposed in terms of one particle-irreducible parts, $\tilde{\Sigma}(p)$ of [Fig.VI.1] as shown in [Fig.II.1].
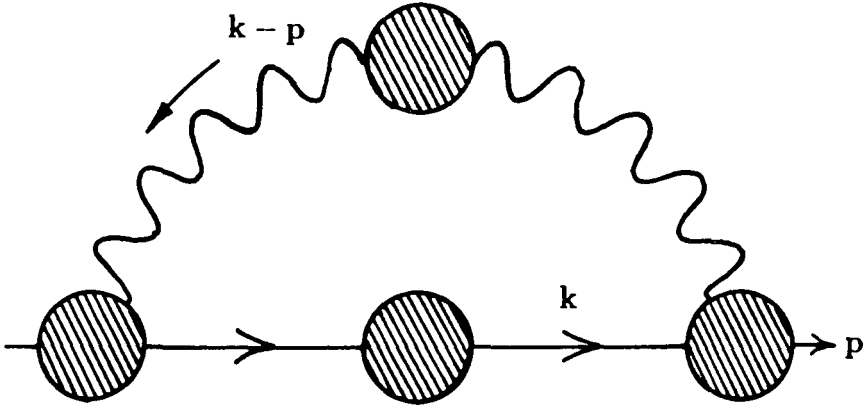

Figure VI.1   The renormalised fermion self energy

We can write the equation of [Fig.II.1], with $S_F^0(p)$ the bare, unrenormalised propagator as

$$S_F(p) = S_F^0(p) + S_F^0(p)\tilde{\Sigma}(p)S_F^0(p) + S_F^0(p)\tilde{\Sigma}(p)S_F^0(p)\tilde{\Sigma}(p)S_F^0(p) + ...$$

$$= S_F^0(p) + S_F^0\tilde{\Sigma}(p)S_F(p), \tag{6.2}$$

from which immediately follows,

$$S_F^{-1}(p) - S_F^{0\,-1}(p) = -\tilde{\Sigma}(p). \tag{6.3}$$

Inserting the expression for $S_F^{-1}(p)$ and $S_F^{0^{-1}}(p)$ from [equ.6.1] (the latter with $\Sigma(p) \equiv A(p) \equiv 0$) gives

$$\not{p}A(p) + \Sigma(p) = i\tilde{\Sigma}(p). \tag{6.4}$$

The self energy contribution of [Fig.VI.1] can be evaluated in terms of the gauge propagator $D_{\mu\nu}(k)$ and vertex function Γ_μ, as

$$\tilde{\Sigma}(p) = \frac{\alpha}{N} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu}(k - p)\Gamma_\mu S_F(k)\Gamma_\nu \tag{6.5}$$

where the coupling constant, e, extracted from the vertices defines $\alpha \equiv e^2/N$ (N=number of flavours). Combining the last two equations with [equ.6.1] we arrive at the Schwinger-Dyson equation for the fermion self enegy,

$$\not{p}A(p) + \Sigma(p) = \frac{-\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{D_{\mu\nu}(k - p)\Gamma_\mu[\not{k}(1 + A(k)) - \Sigma(k)]\Gamma_\nu}{[k^2(1 + A(k))^2 + \Sigma^2(k)]}. \tag{6.6}$$

It is this equation that we evaluate to yield the self energy $\Sigma(p)$. However we immediately note that we have only one equation (although later we split it into two) with four unknowns; $A(p), \Sigma(p), D_{\mu\nu}(k - p), \Gamma_\mu$. The ideal solution to this problem would be to derive the analogous Schwinger-Dyson equations for the vertex function and gauge propagator and solve the resulting set of coupled integral equations– a very difficult problem. Here a simpler alternative is employed, this is to approximate two of the unknowns by known functions. In particular, in the first step of an expansion in 1/N we can use the lowest order vertex and gauge propagator,

$$\Gamma_\mu^0 = i\gamma_\mu$$

$$D^0_{\mu\nu}(k) = \frac{(\delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu)}{[k^2 + \bar{\alpha}k]} \tag{6.7}$$

where $\bar{\alpha}$ is defined by $\alpha/8$. The second of these equations can be obtained from the 1-loop gauge propagator of [equ.5.28]. Analogous to [equ.6.3] the leading order equation for the gauge propagator (shown in [Fig.II.4]) is

$$D^{-1}_{ij}(k) = D^0_{ij}{}^{-1}(k) - \pi_{ij}(k)$$

where the gauge polarisation tensor $\pi_{ij}(k)$ is given by N times [equ.5.28] (The factor of N comes from the combinatoric factor of N possible fermion flavours in the virtual fermion loop). Writing $D_{ij}(k) \equiv D(k)(\delta_{ij} - \hat{k}_i \hat{k}_j)$ we can extract the tensor structure of the above equation leaving

$$D^{-1}(k) = k^2 + \frac{Ne^2 k}{8}.$$

which, in terms of $\alpha = Ne^2$ gives the second of [equ.6.7]. In the spirit of the large N expansion it is hoped that higher order 1/N corrections to the above functions will be significantly suppressed [ref.VI.2].

Inserting these approximate functions into [equ.6.6] eliminates two of the unknowns giving,

$$\not{p}A(p) + \Sigma(p) = \frac{\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{[\delta_{\mu\nu}(k-p)^2 - (k-p)_\mu(k-p)_\nu]}{(k-p)^4[1 + \bar{\alpha}/(k-p)]}$$

$$\times \frac{\gamma_\mu[\not{k}(1 + A(k)) - \Sigma(k)]\gamma_\nu}{[k^2(1 + A(k))^2 + \Sigma^2(k)]}. \tag{6.8}$$

We still have two unknown functions $A(p), \Sigma(p)$ in one equation. However, decomposing the RHS into terms proportional to the unit matrix and those

proportional to the Dirac gamma matrices we can equate them separately to the second and first terms respectively on the LHS and effectively resolve the equation into two distinct, but coupled, equations soluble for the two unknowns. To this end the numerator of [equ.6.8] can be written

$$\Sigma(k)\{(\not k - \not p)(\not k - \not p) - (k-p)^2 \gamma_\mu \gamma_\mu\} - [1 + A(k)]\{(\not k - \not p)\not k(\not k - \not p) - (k-p)^2 \gamma_\mu \not k \gamma_\mu\}.$$

Contracting the gamma matrices using $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ and $\not a \not a = -a^2$ gives

$$2\Sigma(k)(k-p)^2 + [1 + A(k)]2(k^2 - k \cdot p)(\not k - \not p),$$

where the first term is proportional to the unit matrix and the second is proportional to the gamma matrices. Equating these terms to the similar ones on the LHS of [equ.6.8] gives the two equations,

$$\Sigma(p) = \frac{2\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{(k-p)^2[1 + \bar\alpha/(k-p)][k^2(1 + A(k))^2 + \Sigma^2(k)]}. \quad (6.9)$$

$$\not p A(p) = \frac{2\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{[1 + A(k)](\not k - \not p)(k^2 - k \cdot p)}{(k-p)^4[1 + \bar\alpha/(k-p)][k^2(1 + A(k))^2 + \Sigma^2(k)]}. \quad (6.10)$$

and we have two, coupled, integral equations for $A(p)$ and $\Sigma(p)$

To solve these we initially set $A(p) \equiv 0$ on the RHS of these equations to obtain the lowest order, in an iterative procedure, result for the self energy, Σ_0, from [equ.6.9]. This, on insertion into [equ.6.10], then yields the lowest order wavefunction renormalisation. However, at this stage we will find that this iterative procedure becomes unstable and computer-time expensive due to the size of $A(p)$, which, contrary to the arguments of Appelquist et al in [Ref.VI.2] is

not suppressed by $O(1/N)$. Consequently we can only obtain qualitative results from the next iteration of the self energy, and are forced to return to the coupled set of equations [6.9] and [6.10] to obtain reliable, quantitative results.

We describe the decoupled solutions first as it allows us to outline the procedure of solution, which is readily applied to the full coupled case. Also, the qualitative results are instructive.

Having set up the equations, we now turn to the first step of their solution, evaluating Σ_0.

VI.2 THE ZEROTH ORDER SOLUTION FOR $\Sigma(p)$

Appelquist et al. [Ref.VI.2] argued that the wavefunction renormalisation could be ignored completely in [equ.6.9] since it is generated perturbatively and would be of order N^{-1}. They argue that all three dimensional theories are superenormalisable and have a finite number of divergent graphs. In the case of QED_3 *all* graphs are ultra violet finite [Ref.VI.3]. Thus renormalisation counter terms, and therefore the wavefunction renormalisation, are finite. Further, since the latter is generated by interactions of at least $O(e)$, in the limit $\alpha \equiv e^2 N \rightarrow$ constant and N tending to infinity, terms of $O(e)$ are suppressed by $O(N^{-1})$. Thus the factor of $[1+A(k)]$ in [equ.6.9] is $[1+O(N^{-1})]$ and can be approximated by unity.

With such an argument Appelquist et al. effectively decouple the equations leaving a single equation for $\Sigma(p)$. However, in section VI.3 where we evaluate the wavefunction renormalisation we show that it is of order 1 for a significant range of momentum and that the argument of Appelquist is invalid. Further discussion of this will be delayed until the expressions for A(p) are presented. For the time being we note that setting A(p)$\equiv 0$ in [equ.6.9] is only valid as the first step of an iterative procedure, and not on the grounds of it being negligible compared with unity.

Thus, dropping A(k) in [equ.6.9] decouples the wavefunction renormali-sation and leaves the following, non-linear integral equation for the fermion self energy,

$$\Sigma(p) = \frac{2\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{[1 + \bar{\alpha}/(k-p)][k^2 + \Sigma^2(k)](k-p)^2}. \qquad (6.11)$$

In a frame defined by

$$p \equiv (0, 0, p)$$

$$k \equiv k[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta] \qquad (6.12)$$

the angular integral, over solid angle $d\Omega$, can be extracted as,

$$\int \frac{d\Omega}{(k-p)^2[1 + \bar{\alpha}/(k-p)]} = \int_{-1}^{1} \frac{2\pi dx}{[A - Bx][1 + \bar{\alpha}\sqrt{A - Bx}]}$$

where $x \equiv \cos\theta$, $A \equiv p^2 + k^2$, $B \equiv 2pk$. Changing variables to $y \equiv A - Bx$ and then $z \equiv \sqrt{y}$ allows the integral to be evaluated as

$$\frac{4\pi}{B} \int_{\sqrt{A-B}}^{\sqrt{A+B}} \frac{dz}{(\bar{\alpha} + z)} = \frac{2\pi}{pk} ln\left\{ \frac{\bar{\alpha} + p + k}{\bar{\alpha} + |p - k|} \right\}.$$

With this result, and the scalar integral still to do [equ.6.11] reduces to,

$$\Sigma(p) = \frac{4\bar{\alpha}}{N\pi^2 p} \int_0^\infty \frac{dk\, k\Sigma(k)}{[k^2 + \Sigma^2(k)]} ln\left\{ \frac{\bar{\alpha} + p + k}{\bar{\alpha} + |p - k|} \right\}. \qquad (6.13)$$

Since the coupling, $\bar{\alpha}$, has dimensions of mass and we are interested in the size of this scale relative to that of the self energy, $\Sigma(p)$, the relevant dimensionless quantity is $\Sigma(p)/\alpha$ (or equivalently $\Sigma(p)/\bar{\alpha}$). Defining the variables

$$U \equiv \frac{p}{\bar{\alpha}}, \qquad V \equiv \frac{k}{\bar{\alpha}} \qquad (6.14)$$

[equ.6.13] can be recast in terms of dimensionless quantities as,

$$\frac{\Sigma(\bar{\alpha}U)}{\bar{\alpha}} = \frac{4}{N\pi^2 U} \int_0^\infty \frac{dV V [\Sigma(\bar{\alpha}V)/\bar{\alpha}]}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} ln\left\{\frac{1+U+V}{1+|U-V|}\right\} \qquad (6.15)$$

Before setting about solving this equation we first note some of its properties: firstly, clearly $\Sigma(p)$ identically equal to zero, the chiral symetric solution, is one possibility. However we are seeking non-zero solutions. Secondly, from the integrand of this equation we note that the $\Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2$ in the denominator ensures that the equation is non-linear. However, for large V this factor becomes negligible and the equation becomes approximately linear and, as a result, scale invariant. That is, at large V the overall scale of $\Sigma/\bar{\alpha}$ cancels on both sides of the equation and is therefore arbitrary. As such, it is the smaller momentum region $V \sim \Sigma(\bar{\alpha}V)/\bar{\alpha}$ that determines the overall scale of $\Sigma(p)/\bar{\alpha}$.

The non-linearity of the equation leads us to a numerical method of solution. In particular we adopt an iterative procedure where the output of one iteration is used as the input on the RHS of [equ.6.15] in the subsequent one. A standard trapezoidal method of integration was employed to integrate over a finite region of $V_{min} \leq V \leq V_{max}$. This truncation of the semi-infinite range of V in [equ.6.15] is admissable since, by expanding the logarithm in both asymptotic limits we see the integrand converges to zero,

$$\text{Integrand} \quad \lim_{v \to 0} \quad \frac{V}{\Sigma(\bar{\alpha}V)/\bar{\alpha}} \frac{V}{(1+U)}[1 + O(V^3)] \to 0$$

$$\lim_{v \to \infty} \quad \frac{\Sigma(\bar{\alpha}V)/\bar{\alpha}}{V} \frac{U}{(1+V)}[1 + 0(1+V)^{-1}] \to 0,$$

and a suitable choice of (V_{min}, V_{max}) ensures a negligible contribution to the integral is omitted. Having thus restricted the integration region to a finite range suitable for numerical integration the RHS of [equ.6.15] can be evaluated at values of $V_{min} \leq U \leq V_{max}$ using the previous iteration to provide values of $\Sigma(\bar{\alpha}V)/\bar{\alpha}$.

Rather than simply using the output of one iteration directly as the input to the next, the latter was given by the average of the input and output of the previous iteration. That is the input of the nth iteration was given by,

$$(\text{Input})_n = \tfrac{1}{2}\Big\{(\text{Input})_{n-1} + (\text{Output})_{n-1}\Big\}$$

By using this average we reduce the difference between inputs of successive iterations, which increases the stability of the convergence to the required solution. Large differences, particularly in the neighbourhood of the true solution, can cause undesirable oscillations between iterations that upset the smooth convergence to the desired solution.

Within restrictions imposed by computer-time limitations, such an iterative procedure can be repeated over many iterations to arbitrary accuracy. To give a quantitative measure of the difference between two iterations a Chi-squared defined by

$$\chi^2 \equiv \sum_U \left\{1 - \frac{\text{input}(U)}{\text{output}(U)}\right\}^2,$$

where the sum is over values of U, was used. In practice the sum over U was restricted to values of small U $(V) \sim 1$ where the equation is non-linear and the overall scale of $\Sigma(\bar{\alpha}U)/\bar{\alpha}$ is determined. The assumption being that this region, as the non-linear regime that determines the overall scale, is the important one for obtaining good convergence (small χ^2). This ensures that the iterations do, in fact, converge to the true solution that satisfies [equ.6.15], and not one at a

'false' scale that might satisfy the equation over a small region.

The only input required in the procedure described above is an initial input function for $\Sigma(\bar{a}V)/\bar{a}$ to be used in the first iteration. Having supplied this the subsequent iterations are 'self-contained' and require no further input. To determine this initial function we return to [equ.6.15] in the asymptotic region $U \ll 1$ where the integral equation can be recast as a second order differential equation. Given the (approximate) overall scale, $\Sigma(0)/\bar{a}$, this differential equation can be used to 'track' out a solution for $\Sigma(\bar{a}U)/\bar{a}$ as U increases in small steps away from zero. Such a solution, with modifications discussed below, was used to provide an initial input function for the first iteration. In the following section we now turn to the details of obtaining the differential equation and this asymptotic solution.

VI.2.1 The U≪1 approximation to $\Sigma(\bar{a}u)/\bar{a}$

With the modulus of $U - V$ appearing in the argument of the logarithm in [equ.6.15], it is natural to divide the integral into two parts, $V \leq U$, $V \geq U$. Expanding the logarithm in each region using

$$\lim_{x \to 0}\left\{ ln\left(\frac{1+x}{1-x}\right)\right\} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + + +\right) \qquad (6.16)$$

yields,

$$\frac{\Sigma(\bar{a}U)}{\bar{a}} = \frac{8}{N\pi^2 U}\left\{ \int_0^u \frac{dV V \Sigma(\bar{a}V)/\bar{a}}{[V^2 + \Sigma^2(\bar{a}V)/\bar{a}^2]}\left\{ \frac{V}{1+U} + O\left(\frac{V}{1+U}\right)^3\right\} \right.$$

$$\left. + \int_u^\infty \frac{dV \Sigma(\bar{a}V)/\bar{a}}{[V^2 + \Sigma^2(\bar{a}V)/\bar{a}^2]}\left\{ \frac{U}{1+V} + O\left(\frac{U}{1+V}\right)^3\right\}\right\} \qquad (6.17)$$

For $U \ll 1$ the asymptotic form of this equation is well approximated by keeping only the first term in each region,

$$\frac{\Sigma(\bar{\alpha}U)}{\bar{\alpha}} \simeq \frac{8}{N\pi^2 U} \left\{ \int_0^u \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} \frac{V}{(1+U)} \right.$$

$$\left. + \int_u^\infty \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} \frac{U}{(1+V)} \right\}.$$

Taking the derivative of this equation with respect to U we can express the integral equation as a differential equation as follows,

$$\frac{d}{dU}[\Sigma(\bar{\alpha}U)/\bar{\alpha}] = \frac{8}{N\pi^2} \left\{ \frac{d}{dU}[U(1+U)]^{-1} \int_0^u \frac{dV V^2 \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} + \right.$$

$$\left. + \frac{1}{U(1+U)} \frac{d}{dU} \int_0^u \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} + \frac{d}{dU} \int_u^\infty \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} \frac{U}{(1+V)} \right\}.$$

The last two terms cancel leaving

$$\frac{d}{dU}[\Sigma(\bar{\alpha}U)/\bar{\alpha}] = \frac{-8}{N\pi^2} \frac{(2U+1)}{[U(1+U)]^2} \int_0^u \frac{dV V^2 \Sigma(\bar{\alpha})/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]}.$$

Rearranging, and taking the second derivative with respect to U yields

$$\frac{d}{dU} \left\{ \frac{U^2(1+U)^2}{1+2U} \frac{d}{dU} \left(\frac{\Sigma(\bar{\alpha}U)}{\bar{\alpha}} \right) \right\} = \frac{-8}{N\pi^2} \frac{d}{dU} \int_0^u \frac{dV V^2 \Sigma(\bar{\alpha}v)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]}.$$

In the asymptotic region $U \ll 1$ this reduces to the following, non-linear, second order differential equation,

$$\frac{d}{dU} \left\{ U^2 \frac{d}{dU}[\Sigma(\bar{\alpha}U)/\bar{\alpha}] \right\} = \frac{-8}{N\pi^2} \frac{U^2 \Sigma(\bar{\alpha}U)/\bar{\alpha}}{[U^2 + \Sigma^2(\bar{\alpha}U)/\bar{\alpha}^2]}. \tag{6.18}$$

Defining the variable $x \equiv U^2$, this last equation can be recast as

$$2x \frac{d^2 \Sigma / \bar{\alpha}}{dx^2} = \frac{-4}{N\pi^2} \frac{\Sigma / \bar{\alpha}}{[x^2 + \Sigma^2 / \bar{\alpha}^2]} - 3 \frac{d\Sigma / \bar{\alpha}}{dx} \tag{6.19}$$

Assuming $\Sigma(0)/\bar{\alpha}$ and $d\Sigma/dx|_{x=0}$ are known, a solution to [equ.6.19] can be determined by stepping out from $x = 0$. Although remembering this equation is only valid for $U \ll 1$ the solution is also strictly only valid in this region.

To obtain the initial boundary conditions we set $x=0$ in [equ.6.19]. The coefficient of the second derivative vanishes to give

$$\frac{4}{N\pi^2} \frac{1}{\Sigma(0)/\bar{\alpha}} = 3 \left(\frac{d\Sigma(x)/\bar{\alpha}}{dx} \right)_{x=0} \tag{6.20}$$

which determines the first derivative in terms of $\Sigma(0)$. The latter is totally unconstrained. Consequently there is a solution to [equ.6.19] for every value of $\Sigma(0)/\bar{\alpha}$, and as an initial input $\Sigma(0)$ can be chosen arbitrarily. (Of course the closer to the real value of $\Sigma(0)$ the quicker the iterative procedure will converge). Armed with these boundary conditions the solution to [equ.6.19] can be mapped out. By taking small numerical steps away from $x = 0$ we can track out the evolution of the derivatives of the function, and hence $\Sigma/\bar{\alpha}$ by expressing it as a Taylor expansion in terms of it's derivatives,

$$\Sigma(x + \delta) = \Sigma(x) + \delta \Sigma'(x) + \frac{\delta^2}{2!} \Sigma''(x) \ + \ +$$

$$= \Sigma(x) + \frac{\delta}{2} \left\{ \Sigma'(x) + [\Sigma'(x) + \delta \Sigma''(x)] \right\}$$

$$\simeq \Sigma(x) + \frac{\delta}{2} \left\{ \Sigma'(x) + \Sigma'(x + \delta) \right\}, \tag{6.21}$$

and similarly for the first derivative,

$$\Sigma'(x + \delta) \simeq \Sigma'(x) + \frac{\delta}{2} \left\{ \Sigma''(x) + \Sigma''(x + \delta) \right\}. \tag{6.22}$$

To evaluate $\Sigma'(x + \delta)$ we need $\Sigma''(x + \delta)$. This is obtained from [equ.6.19] with $\Sigma(x + \delta)$ and $\Sigma'(x + \delta)$ approximated by the first two terms of their Taylor expansions,

$$\Sigma(x + \delta) \simeq \Sigma(x) + \delta\Sigma'(x)$$

$$\Sigma'(x + \delta) \simeq \Sigma'(x) + \delta\Sigma''(x).$$

Having thus determined $\Sigma''(x + \delta)$ we can, using equations [6.21] and [6.22], determine $\Sigma'(x + \delta)$ and hence $\Sigma(x + \delta)$ itself. These quantities are then used in the re-evaluation of the equations for $x \rightarrow x + \delta$. In this way the evolution of $\Sigma(U)/\bar{a}$ can be built up for $U \geq 0$. In [Fig.VI.2] we show the results of this procedure, for the somewhat arbitrary value of N=2.6 (following Appelquist et al.).

This solution of [equ.6.19] is an approximate solution to the full integral equation [equ.6.17] in the asymptotic region $U \ll 1$. The break-down of the solution outside this region is seen by the curve in [Fig.VI.2] crossing the x axis and becoming negative, whereas inspection of [equ.6.17] shows the self energy is positive-definite. Wherever the solution went negative it was, by hand, set to zero to obtain the initial input function for the iterative solution of the full integral equation.

Having provided an initial input function the 'self contained' iterative procedure discussed in the previous section can be used to obtain a solution of [equ.6.17] to arbitary accuaracy. In [Fig.VI.3] and [Fig.VI.4] we show such a solution, again for N=2.6. The first logarithmic plot showing the behaviour over several decades of momentum U, while the second, linear plot shows the asymptotic, small U behavior.

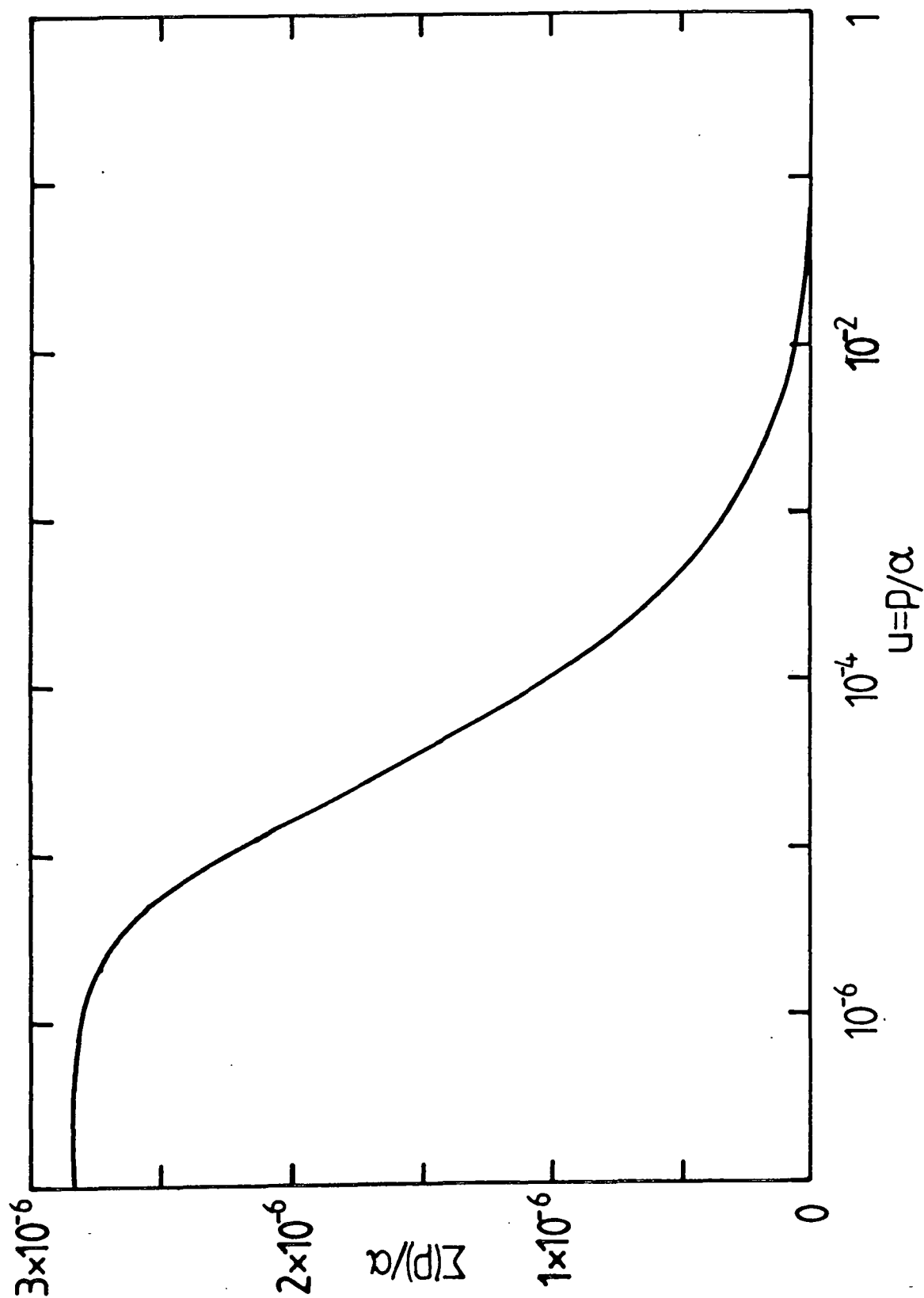We have thus far presented a discussion and the results of determining

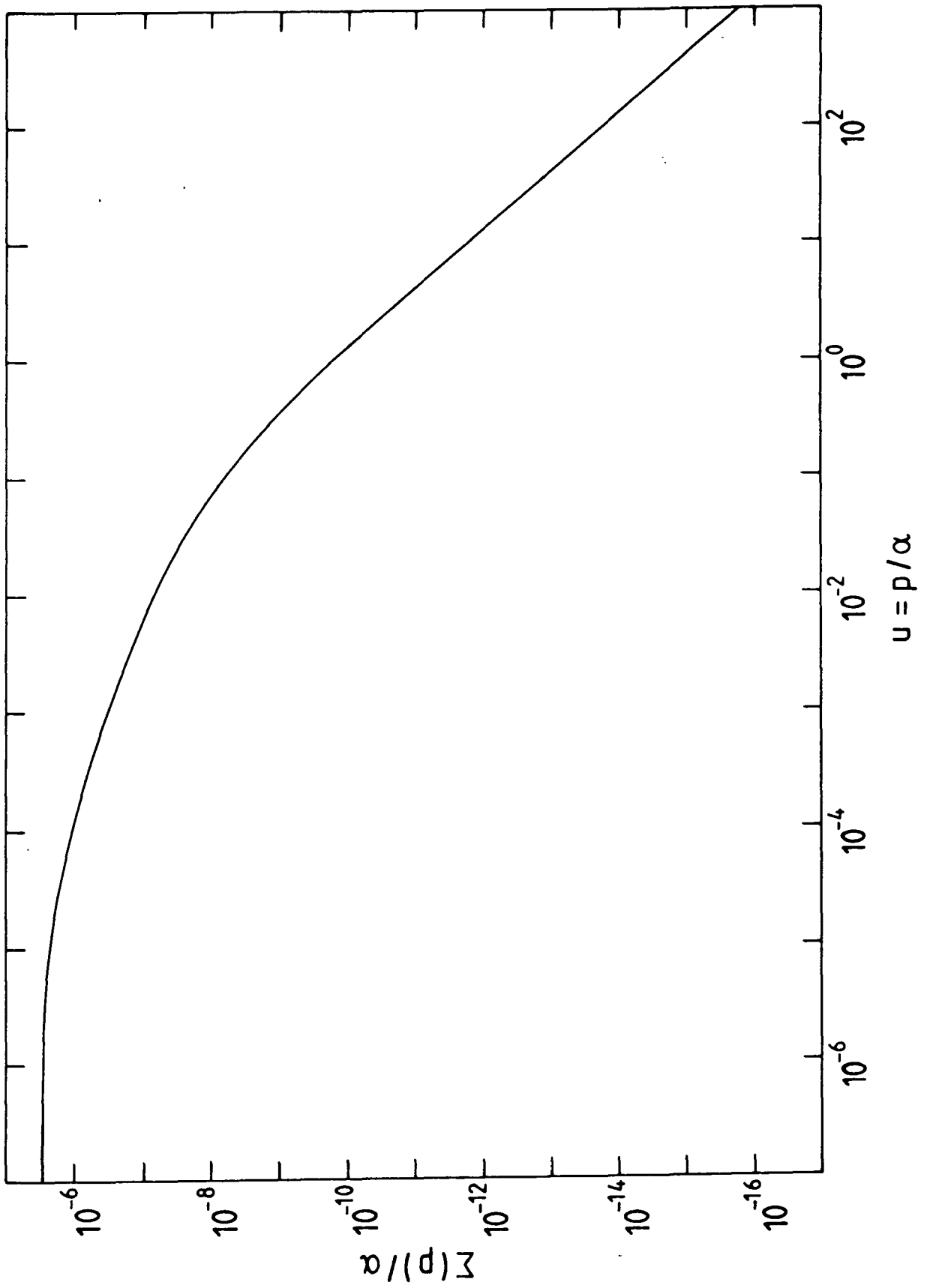Figure VI.2 $\Sigma(\alpha U)/\alpha$ from differential equation

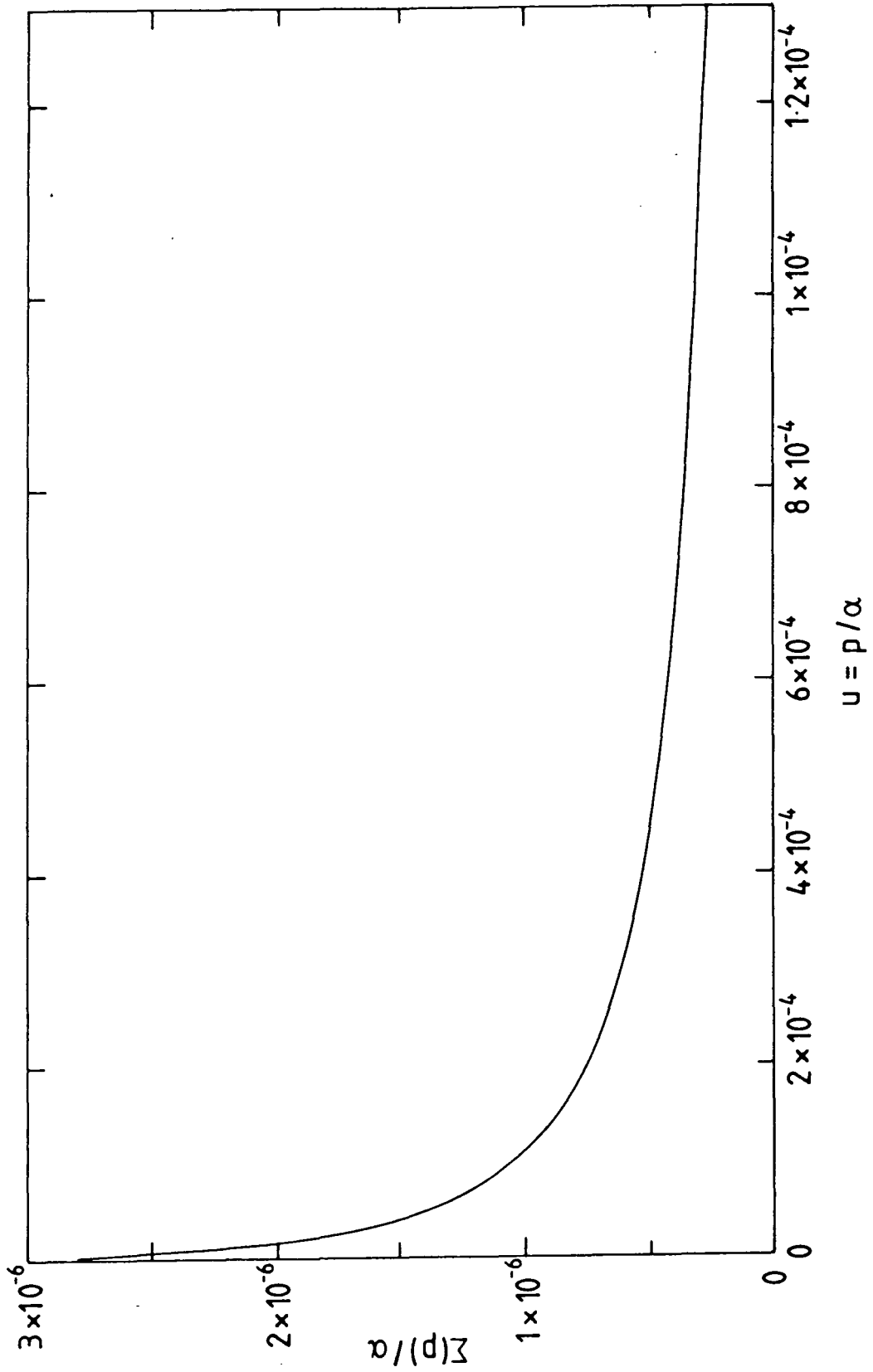Figure VI.3   Logarithmic plot of $\Sigma(\alpha U)/\alpha$ against U

Figure VI.4   Linear plot of $\Sigma(\alpha U)/\alpha$ against U

the first iteration of the self energy as a function of momentum $U = p/\alpha$ for the single value of N=2.6. In the following section we discuss the N dependence of the self energy and how, from this one value of N, solutions for other values can be determined.

VI.2.2 The N dependence of the self energy

The dependence of the fermion self energy on N, the number of flavours, is determined by repeating the iterative solution of the integral equation, [equ.6.15], each time taking a small step in N away from the previous solution. Now, the initial input function is not provided by an approximate solution to the differential equation [equ.6.19], but by the full solution of the integral equation obtained for the previous value of N. In this way we can track out solutions for different N values, shown in [Fig.VI.5].

The y axis of this figure has been labelled m/α where m is the dynamically generated mass of the fermion. We can justify this notation by identifying the mass as the pole of the propagator of [equ.6.1],

$$m \equiv \frac{\Sigma(m)}{1 + A(m)}$$

Having set $A(p) \equiv 0$, this equation reduces to $m = \Sigma(m)$. Now, noting from [Fig.VI.4], that $\Sigma \sim$ constant for small p, the solution to this equation is $m = \Sigma(0)$ and $\Sigma(0)$ can be identified with the fermion mass.

Starting as we did, from N=2.6 care had to be taken when increasing N because the overall scale, $\Sigma(0)/\alpha$ drops off very rapidly. The increment in N had to be small enough to ensure the scale was not drastically altered between iterations so that the initial input function would be reliable. For this reason we were restricted to N$\leq$ 3.

In this and the previous section we have discussed the first iteration to
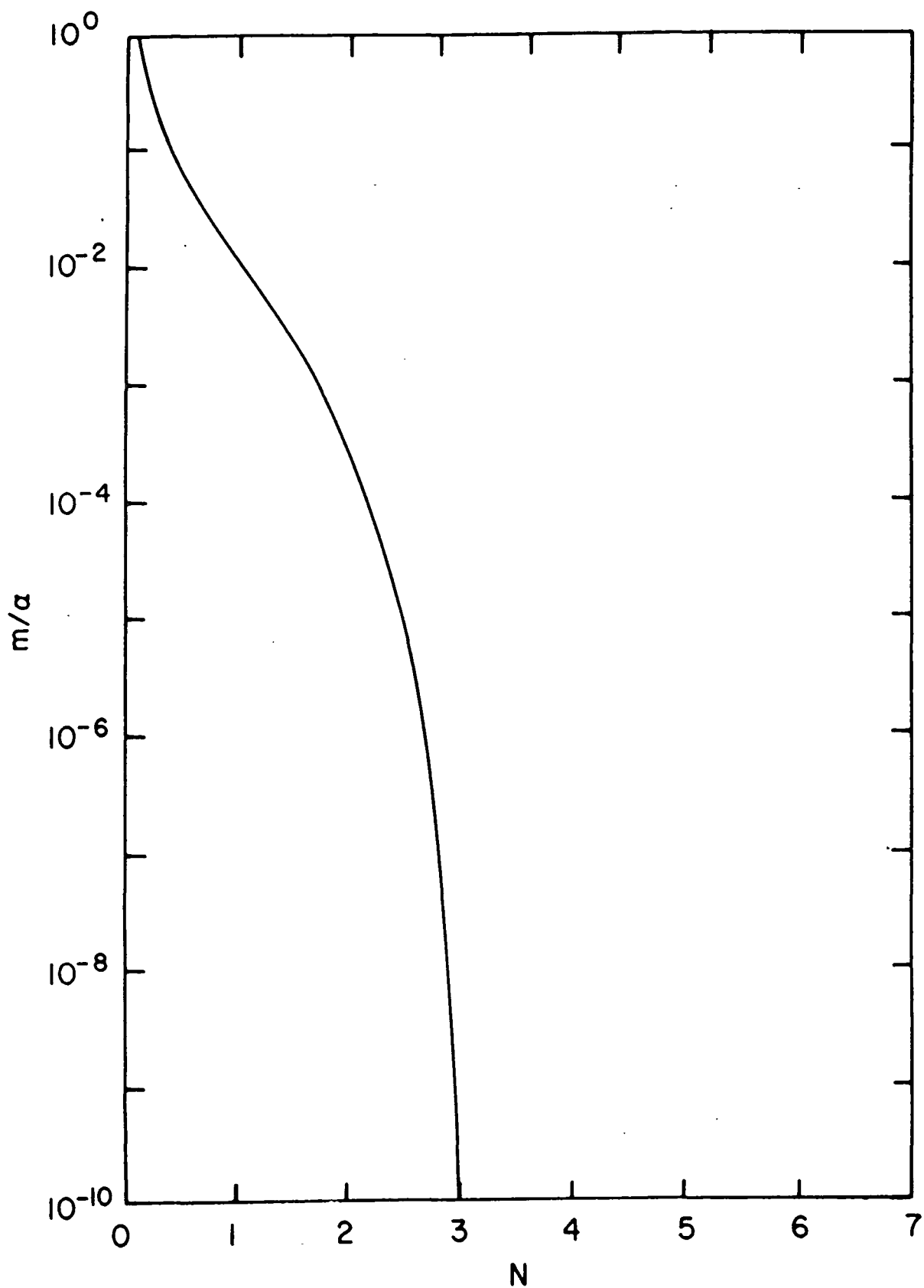
Figure VI.5   The N dependence of $\Sigma(0)/\alpha$

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the self energy as functions of both momentum and N. We now return to the coupled set of equations, [6.9] and [6.10] and use our results so-far to determine the wavefunction renormalisation A(p). This is the subject of the following sections.

VI.3   A FIRST ITERATION TO THE WAVEFUNCTION RENORMALISATION

As mentioned previously, Appelquist et al. effectively decoupled equations [6.9] and [6.10] by arguing that A(p), being perturbatively generated, is suppressed by $O(N^{-1})$ and can effectively be set to zero. In this and the following section we solve [equ.6.10] for A(p), using the previously determined self energy function, and show that A(p) is not, in fact, suppressed by $O(N^{-1})$ and cannot be decoupled as Appelquist et al. argue. Rather it must be determined to allow further iterations of the self energy to be evaluated.

In the spirit of the iterative procedure a first approximation to A(p) is obtained from [equ.6.10] by setting $A(p) \equiv 0$ on the RHS,

$$\not{p} A(p) = \frac{16\bar{\alpha}}{N} \int \frac{d^3k}{(2\pi)^3} \frac{(\not{k} - \not{p})(k^2 - k \cdot p)}{(k-p)^4 [1 + \bar{\alpha}/(k-p)][k^2 + \Sigma^2(k)]}. \tag{6.23}$$

We note that the gamma matrix dependence of the RHS of this equation, after integration can only be carried by $\not{p}$. The latter is explicitly extracted by mutiplying the equation by $\not{p}$ and taking the trace of both sides as

$$Tr[\not{p}\not{p}A(p)] = \frac{16\bar{\alpha}}{N} \int \frac{d^3k}{(2\pi)^3} \frac{Tr[\not{p}\not{k} - \not{p}\not{p}](k^2 - k \cdot p)}{(k-p)^4 [1 + \bar{\alpha}/(k-p)][k^2 + \Sigma^2(k)]}$$

giving, cancelling the $\not{p}$ on both sides,

$$p^2 A(p) = \frac{16\bar{\alpha}}{N} \int \frac{d^3k}{(2\pi)^3} \frac{(p \cdot k - p^2)(k^2 - k \cdot p)}{(k-p)^4 [1 + \bar{\alpha}/(k-p)][k^2 + \Sigma^2(k)]}. \tag{6.24}$$

In the frame defined by [equ.6.12] we can perform the angular integral as follows

VI.3.1   The angular integral

We can factorise the angular dependent part of [equ.6.24] as

$$A(p) \equiv \frac{16\bar{\alpha}}{Np^2} \int_0^\infty \frac{dk k^2}{(2\pi)^3} \frac{1}{[k^2 + \Sigma^2(k)]} I(k,p) \qquad (6.25)$$

where $I(k,p)$ contains the angular integral and is defined by

$$I(k,p) \equiv \int \frac{d\Omega (k^2 - k \cdot p)(k \cdot p - p^2)}{(k-p)^4 [1 + \bar{\alpha}/(k-p)]}.$$

Adopting the definitions of A, B, x of section VI.2 the terms of the numerator can be recast using,

$$|k - p| = [k^2 - 2kp\cos\theta + p^2]^{1/2}$$

$$= [A - Bx]^{1/2}$$

giving

$$(k^2 - k \cdot p)(k \cdot p - p^2) = \frac{1}{4}(2k^2 - Bx)(Bx - 2p^2)$$

$$= \frac{-1}{4}[(A - Bx)^2 + (B^2 - A^2)]$$

so that

$$I(k,p) = \frac{-1}{2\pi} \int_{-1}^1 dx \frac{[(A - Bx)^2 + B^2 - A^2]}{[A - Bx]^2 [1 + \bar{\alpha}/\sqrt{A - Bx}]}$$

$$= \frac{-\pi}{2} \int_{-1}^1 \frac{dx}{[1 + \bar{\alpha}/\sqrt{A - Bx}]} - \frac{\pi(B^2 - A^2)}{2} \int_{-1}^1 \frac{dx}{[A - Bx][1 + \bar{\alpha}/\sqrt{A - Bx}]}$$

$$\qquad (6.26)$$

The first term is evaluated with the change of variables $y \equiv \sqrt{A - Bx} + \bar{\alpha}$ as,

$$\frac{-\pi}{2} \int_{-1}^{1} \frac{dx\sqrt{A - Bx}}{[1 + \bar{\alpha}/\sqrt{A - Bx}]} = \frac{-\pi}{2}\left\{2 + \frac{2}{B}\int_{\sqrt{A+B}+\bar{\alpha}}^{\sqrt{A-B}+\bar{\alpha}} \frac{dy(y - \bar{\alpha})}{y}\right\}$$

$$= \frac{-\pi}{B}\left\{B + \bar{\alpha}[\sqrt{A - B} - \sqrt{A + B}] + \bar{\alpha}^2 ln\left(\frac{\sqrt{A + B} + \bar{\alpha}}{\sqrt{A - B} + \bar{\alpha}}\right)\right\}. \qquad (6.27)$$

The second term of [equ.6.26] is

$$\frac{-\pi(B^2 - A^2)}{2} \int_{-1}^{1} \frac{dx}{(A - Bx)^2}\left\{1 - \frac{\bar{\alpha}}{[\bar{\alpha} + \sqrt{A - Bx}]}\right\}$$

$$= \frac{-\pi(B^2 - A^2)}{2}\left\{\frac{2}{(A^2 - B^2)} - \bar{\alpha}\int_{-1}^{1} \frac{dx}{(A - Bx)^2[\bar{\alpha} + \sqrt{A - Bx}]}\right\}. \qquad (6.28)$$

With a change of variables, $z \equiv \sqrt{A - Bx}$ the remaining integral becomes

$$\frac{2\bar{\alpha}}{B}\int_{\sqrt{A+B}}^{\sqrt{A-B}} \frac{dz}{z^3(\bar{\alpha} + z)} = \frac{2\bar{\alpha}}{B}\left[\frac{-1}{2\bar{\alpha}z^2} + \frac{1}{\bar{\alpha}^2 z} - \frac{1}{\bar{\alpha}^3}ln\left(\frac{\bar{\alpha} + z}{z}\right)\right]_{\sqrt{A+B}}^{\sqrt{A-B}}$$

$$= \frac{2}{B}\left\{\frac{-B}{(A^2 - B^2)} + \frac{1}{\bar{\alpha}}\left(\frac{\sqrt{A + B} - \sqrt{A - B}}{\sqrt{A - B}\sqrt{A + B}}\right) - \frac{1}{\bar{\alpha}^2}ln\left(\frac{(\bar{\alpha} + \sqrt{A - B})\sqrt{A + B}}{(\bar{\alpha} + \sqrt{A + B})\sqrt{A - B}}\right)\right\}.$$

Thus [equ.6.28] is

$$\frac{-\pi(B^2 - A^2)}{B}\left\{\frac{1}{\bar{\alpha}^2}\left(\frac{\sqrt{A + B} - \sqrt{A - B}}{\sqrt{A + B}\sqrt{A - B}}\right) - \frac{1}{\bar{\alpha}^3}ln\left(\frac{(\bar{\alpha} + \sqrt{A - B})\sqrt{A + B}}{(\bar{\alpha} + \sqrt{A + B})\sqrt{A - B}}\right)\right\}. \qquad (6.29)$$

Combining equations [6.27] and [6.29] and translating back to (p, k) variables, we can write the angular integral as,

$$I(k,p) = \frac{-\pi}{2pk}\left\{2pk + \bar{\alpha}[|p-k| - (p+k)] + \bar{\alpha}^2 ln\left(\frac{\bar{\alpha}+p+k}{\bar{\alpha}+|p-k|}\right)\right.$$

$$\left. - \frac{(p^2-k^2)^2}{\bar{\alpha}}\left(\frac{p+k-|p-k|}{|p^2-k^2|}\right) + \frac{(p^2-k^2)^2}{\bar{\alpha}^2}ln\left(\frac{(\bar{\alpha}+|p-k|)(p+k)}{(\bar{\alpha}+p+k)(|p-k|)}\right)\right\}$$

which, after a bit of rearranging, becomes

$$I(k,p) = \frac{-\pi}{2pk}\left\{2pk + \frac{[\bar{\alpha}^2 + |p^2 - k^2|][|p-k| - (p+k)]}{\bar{\alpha}}\right.$$

$$\left. + \frac{(p^2-k^2)^2}{\bar{\alpha}^2}ln\left(\frac{p+k}{|p-k|}\right) + \frac{[\bar{\alpha}^4 - (p^2-k^2)^2]}{\bar{\alpha}^2}ln\left(\frac{p+k+\bar{\alpha}}{|p-k|+\bar{\alpha}}\right)\right\}. \qquad (6.30)$$

Having thus performed the angular integral in [equ.6.25] we are left with a scalar integral equation for $A(p)$,

$$A(p) = -\frac{8\bar{\alpha}\pi}{Np^3}\int_0^\infty \frac{dk}{(2\pi)^3}\frac{k}{[k^2+\Sigma^2(k)]}\left\{2pk + \frac{(\bar{\alpha}^2 + |p^2 - k^2|)(|p-k| - (p+k))}{\bar{\alpha}}\right.$$

$$\left. + \frac{(p^2-k^2)^2}{\bar{\alpha}^2}ln\left(\frac{p+k}{|p-k|}\right) + \frac{[\bar{\alpha}^4 - (p^2-k^2)^2]}{\bar{\alpha}^2}ln\left(\frac{p+k+\bar{\alpha}}{|p-k|+\bar{\alpha}}\right)\right\} \qquad (6.31)$$

It is this equation that we now have to evaluate in the next section.

VI.3.2 The scalar integral

To perform the scalar integral we first transform to the dimensionless variables of [equ.6.14],

$$A(U) = \frac{-1}{N\pi^2 U^3}\int_0^\infty \frac{dVV}{[V^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]}\left\{2UV + (1+|U^2-V^2|)(|U-V| - (U+V))\right.$$

$$+(U^2 - V^2)^2 ln\left(\frac{U+V}{|U-V|}\right) + (1 - (U^2 - V^2)^2) ln\left(\frac{1+U+V}{1+|U-V|}\right)\right\} \qquad (6.32)$$

. Noting that the $\Sigma(k)$ dependence is in the form $\Sigma(\bar{\alpha}V)/\bar{\alpha}$, which is exactly the dimensionless quantity calculated in the previous sections, we see that there is no explicit dependence on the arbitrary coupling α in this equation.

At first glance, individual terms of this equation appear to diverge in the asymptotic limit $V \to \infty$. However, as we shall see, the divergent terms exactly cancel leaving a nicely convergent integrand.

As before, the integrand is divided into two parts appropriate to the modulus $|U - V|$ in the integrand. To overcome the semi-infinite range of integration the variable of integration in both parts of the integral is mapped onto a finite interval by the following changes of variables,

$$0 \leq V \leq U \qquad ; \qquad x \equiv V/U$$

$$U \leq V \leq \infty \qquad ; \qquad x \equiv U/V \qquad (6.33)$$

In both regions the variable V is mapped onto the unit range of x,

$$0 \leq x \leq 1.$$

Having written the integrand as in [equ.6.32], the separate terms appear to diverge as $V \to \infty$. Using [equ.6.33], these apparent divergences are translated into the asymptotic behaviour of the terms in the second region as $x \to 0$

Writing [equ.6.26] as the sum of the contributions from the two regions, I_1, I_2 respectively, gives, in terms of the new variable x,

$$A(U) = I_1 + I_2 \qquad (6.34)$$

where

$$I_1 \equiv \frac{-1}{N\pi^2 U} \int_0^1 \frac{dx\,x}{[x^2 + \Sigma^2(\bar{\alpha}Ux)/\bar{\alpha}^2]} \left\{ 2x - \frac{2x[U + U^2(1 - x^2)]}{U} \right.$$

$$\left. + U^2(1 - x^2)^2 ln\left(\frac{1 + x}{1 - x}\right) + \frac{[1 - U^4(1 - x^2)^2]}{U^2} ln\left(\frac{1 + U + Ux}{1 + U - Ux}\right) \right\} \qquad (6.35)$$

and

$$I_2 \equiv \frac{-1}{N\pi^2 U} \int_0^1 \frac{dx}{x} \frac{1}{[U^2 + x^2\Sigma^2(\bar{\alpha}U/x)/\bar{\alpha}^2]} \left\{ \frac{2}{x} - \frac{2[x^2 + U^2(1 - x^2)^2]}{Ux^2} \right.$$

$$\left. + \frac{U^2(1 - x^2)^2}{x^4} ln\left(\frac{1 + x}{1 - x}\right) + \frac{[x^4 - U^4(1 - x^2)^2]}{U^2 x^4} ln\left(\frac{1 + U + U/x}{1 + U/x - U}\right) \right\}. \qquad (6.36)$$

Although we have mapped the integral onto a finite range, suitable for numerical integration we are still faced with the problem of the terms in I_2 as $x \to 0$.

VI.3.3   The cancellation of the apparently divergent terms in I_2 (U)

The problem of the integrand of [equ.6.26] apparently diverging as k, or euivalently V, tends to infinity has been mapped onto the asymptotic behaviour of I_2 as $x \to 0$. in this limit, explicit factors of x in the denominator make individual terms of [equ.6.36] diverge.

However, expanding the logarithms in this equation using the Taylor expansion of [equ.6.16] the divergent terms exactly cancel leaving the convergent asymptotic integrand

$$\frac{8U^2}{3N\pi^2} \frac{x}{[U^2 + x^2\Sigma^2(\bar{a}U/x)/\bar{a}^2]} \{1 + O(x^2)\}. \qquad (6.37).$$

We see that the integrand does, infact, converge as a result of divergent terms cancelling. However, a numerical computer evaluation would still run into problems; the individual terms would still have to be evaluated and then cancelled. In the limit this would require the computer to cancel infinite numbers to give a resultant zero. Even as the terms are finite, but large, the computer manipulations start to become unreliable and the results meaningless. To avoid running into such problems a cut-off, x_{min}, was employed to truncate the integration region to $x_{min} \leq x \leq 1$, excluding the problematic $x \to 0$ region. The size of the cut-off was determined by plotting the integrands of equations [6.36] and [6.37], as evaluated by the computer. For x decreasing the two curves start to come closer as the higher order terms in [equ.6.37] become less significant and the first order result is a good approximation to the unexpanded integrand of [equ.6.36]. Ideally as x tends to zero the two functions would converge to each other. In practice, as x decreases further the curves start to grow apart again as the numerical cancellations in [equ.6.36] start to become unreliable. The value of x where the functions are closest together is x_{min}; the smallest value of x where the computer manipulations are reliable. We found a cut-off of $x = 5 \times 10^{-2}$ was necessary.

Using such a cut-off allows the contribution from $x_{min} \leq x \leq 1$ to be evaluated numerically, using a Simpson's rule technique. The remaining contribution from $x \leq x_{min}$ is still to be evaluated analytically as follows.

VI.3.4   The analytic contribution from $x \leq x_{min}$

From [equ.6.37], taking the leading term only, the contribution from $x \leq$

x_{min} to I_2 is

$$\tilde{I}_2 \equiv \frac{-8U^2}{3N\pi^2} \int_0^{x_{min}} \frac{dx}{[U^2 + x^2\Sigma^2(\bar{\alpha}U/x)/\bar{\alpha}^2]} \frac{x}{(U+x)^2}.$$

In the limit $U \gg x^2\Sigma^2(\bar{\alpha}U/x)/\bar{\alpha}^2$, valid from [Fig.VI.3] in the regions where $\Sigma/U \leq 1$, this reduces to

$$\tilde{I}_2 \simeq \frac{-8}{3N\pi^2} \int_0^{x_{min}} \frac{dx\, x}{(U+x)^2}$$

which can be straightforwardly integrated to give

$$\tilde{I}_2 = \frac{-8}{3N\pi^2} \left\{ ln\left(\frac{U + x_{min}}{U}\right) - \frac{x_{min}}{(U + x_{min})} \right\}. \tag{3.38}$$

Now we have to evaluate the similar $x \leq x_{min}$ contribution to I_1. That is,

$$\tilde{I}_1 \equiv \frac{-1}{N\pi^2U} \int_0^{x_{min}} \frac{dx\, x}{[x^2 + \Sigma^2(\bar{\alpha}Ux)/\bar{\alpha}^2]} \left\{ 2x - \frac{2x[1 + U^2(1 - x^2)]}{U} \right.$$

$$\left. +U^2(1 - x^2)^2 ln\left(\frac{1+x}{1-x}\right) + \frac{[1 - U^4(1 - x^2)^2]}{U^2} ln\left(\frac{1 + U + Ux}{1 + U - Ux}\right) \right\}.$$

Performing a similar expansion of the logarithms as for $\tilde{I}_2$, after some algebra, the order x terms in the curly bracket of the above integrand cancel. The leading terms come from the order x^3 terms. In particular,

$$\lim_{x \to 0} \tilde{I}_1 \to \frac{-1}{N\pi^2U} \int \frac{dx\, x^4}{[x^2 + \Sigma^2(\bar{\alpha}Ux)/\bar{\alpha}^2]} \left\{ \frac{2U}{3(1+U)^2}[4 + 5U + 10U^2 + 5U^3] \right.$$

$$\left. -3U^2 + O(x) \right\}.$$

The large power of x in the numerator ensures this integral is of, at least, order x^3. Hence, in comparison with the $o(x_{min})$ contribution of [equ.6.37], $\tilde{I}_1$ can be ignored as a negligible contribution. (This was given further validity by taking $\Sigma^2/\bar{\alpha}^2$ as some appropriate constant value and evaluating the above integral explicitly. It was, as expected, found to be several orders down on the magnitude of $\tilde{I}_2$.)

Having numerically evaluated the contributions from $x_{min} \leq x \leq 1$ in equations [6.35] and [6.36] and having analytically approximated the contributions from $x \leq x_{min}$, [equ.3.37], they can be added to give the full solution to [equ.3.32]. Figure [VI.6] shows a plot of the solution as a function of U for, the now familiar value of N=2.6.

It is convenient for later to represent the output A(U) by a simple analytic expression. This is obtained in the following section having first noted that,

VI.3.5   Discussion

The immediate conclusion from [Fig.VI.6] is that for three or more orders of magnitude in U the value of $A(p)$ is a significant fraction of one. This immediately casts doubt on the validity of the argument of Appelquist et al. of ignoring the wavefunction renormalisation in [equ.6.9]. However, this does not necessarily imply that the solution obtained in this approximation, [Fig.VI.4], is incorrect. The relevant part of the denominator dependent on $A(U)$ is, in terms of the variable k,

$$[k^2(1 + A(k))^2 + \Sigma^2(k)].$$

The size of $A(k)$ only becomes important if the dominant contribution to the integral comes from the region $k \geq \Sigma(k)$. Only then will $A(k)$ have a significant effect on the solution of [equ.6.9]. Whether it does or not is discussed in section
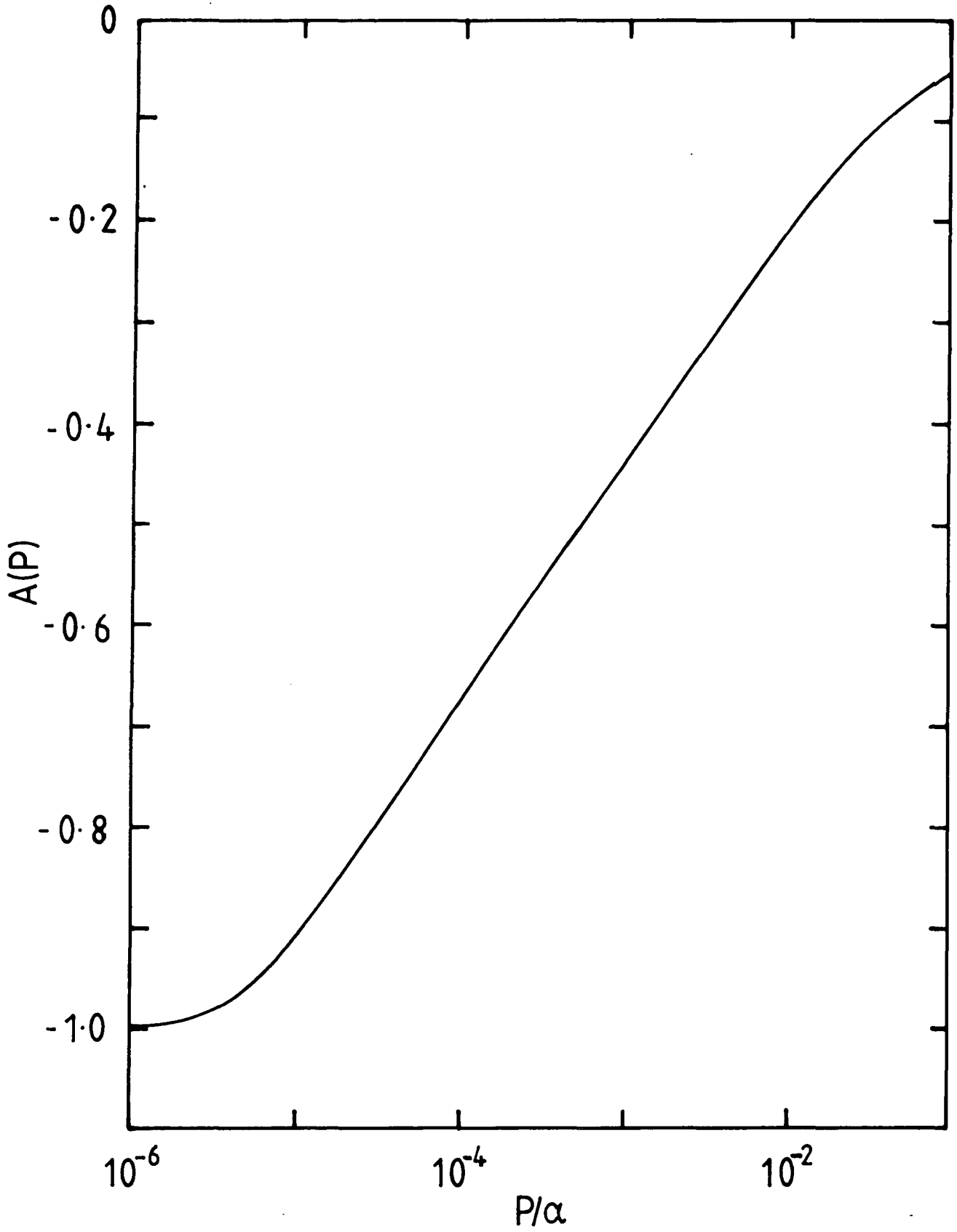
Figure VI.6   The wavefunction renormalisation

VI.5 where a full discussion of the validity of the Appelquist approximation and it's consequences is presented. For the time being we accept the results in [Fig.VI.6] as the first iteration of $A(p)$, which we use in [equ.6.9] to consider the next iteration of the self energy function. For this purpose we found it convenient to use an approximate analytic expression for $A(p)$. This is the subject of the next section.

VI.4   AN ANALYTIC EXPRESSION FOR THE WAVEFUNCTION RENORMALISATION

Having determined a numerical representation of $A(p)$, [Fig.VI.6], and found it was not negligible we now seek an approximate, analytic expression. To do this we return to [equ.6.32] and expand the logarithms, which necessarily restricts the solution to $U \ll 1$. However, over this range the analytic solution was found to approximate the numerical one very well. Having such an analytic expression is valuable as it aids our later discusion of the N dependence of A(p).

So, transforming to the dimensionless variables of [equ.6.14], A(p) is given by [equ.6.32]. As noted in section VI.2, the region that sets the scale of $\Sigma(k)$ is $\Sigma(k) \sim k$. From [Fig.VI.4] we see that in this region $\Sigma(k)$ has only a small k dependence. Consequently, in the region $V \leq \Sigma(\bar{\alpha}V)$ where Σ becomes relevant to the integrands of $A_{1,2}$, we can approximate $\Sigma/\bar{\alpha}$ by a constant value, $\sigma \sim \Sigma(0)/\bar{\alpha}$;

$$\frac{\Sigma^2}{\bar{\alpha}^2} = \sigma^2 = \text{constant}$$

and, as previously, splitting the integrand into two regions with common boundary $U = V$, we get

$$A(U) = A_1 + A_2$$

where

$$A_1 \equiv \frac{1}{N\pi^2 U^3} \int_0^U \frac{dV\,V}{[V^2 + \sigma^2]} I_1(U,V)$$

$$A_2 \equiv \frac{1}{N\pi^2 U^3} \int_U^\infty \frac{dV}{V[V^2 + \sigma^2]} I_2(U,V)$$

and $I_1(U,V)$, $I_2(U,V)$ given by the curly bracket of [equ.6.32] in the two regions respectively.

To evaluate these integrals we will have to expand the logarithms in the integrand and consequently will be restricted to the region $U < 1$. In $I_1(U,V)$ with $V < U < 1$ we get, expanding the logarithms,

$$I_1(U,V) = \frac{8V^3}{3} - \frac{5UV^3}{3} + O(V^4)$$

keeping only the first term gives

$$A_1(U) = \frac{8}{3N\pi^2 U^3} \int_0^U \frac{dV\,V^4}{[V^2 + \sigma^2]} = \frac{8}{3N\pi^2 U^3} \left[\frac{V^3}{3} - \sigma^2 V + \sigma^3 \tan^{-1}\left(\frac{V}{\sigma}\right) \right]_0^U$$

which, substituting in the limits immediately yields

$$A_1(U) = \frac{8}{3N\pi^2} \left\{ \frac{1}{3} - \frac{\sigma^2}{U^2} + \frac{\sigma^3}{U^3} \tan^{-1}\left(\frac{U}{\sigma}\right) \right\} + O(U).$$

This result can be simplified in the regions $1 \gg U \gg \sigma$ where

$$A_1(U) \simeq \frac{8}{9N\pi^2} \qquad ; 1 \gg U \gg \sigma, \tag{6.39}$$

and $1 \gg \sigma \gg U$ where we can expand the inverse tangent as

$$\frac{\sigma^3}{U^3} \tan^{-1}\left(\frac{U}{\sigma}\right) - \frac{\sigma^2}{U^2}$$

$$= \frac{\sigma^3}{u^3}\left(\frac{U}{\sigma} - \frac{U^3}{3\sigma^3} + \frac{U^5}{5\sigma^5} + + +\right) - \frac{\sigma^2}{U^2} \to \frac{-1}{3}$$

so that

$$A_1(U) \simeq \frac{8}{9N\pi^2}O(U^2/\sigma^2) \simeq 0 \qquad ; 1 \gg \sigma \gg U. \qquad (6.40)$$

Now, in the second region, $V \geq U$, we again expand the logarithms in $I_2(U,V)$ to obtain

$$I_2(U,V) \simeq \frac{8U^3}{3(1+V)^2} - \frac{2U^5V^3}{(1+V)^5} + ...$$

which, keeping only the first term again, yields

$$A_2(U) = \frac{8}{3N\pi^2}\int_U^\infty \frac{dV\,V}{[V^2+\sigma^2](1+V)^2}.$$

Now, by re-casting the integrand as

$$A_2(U) = \frac{8}{3N\pi^2}\int_U^\infty dV\left\{\frac{2\sigma^2}{(1+\sigma^2)(V^2+\sigma^2)} + \frac{V(1-\sigma^2)}{(1+\sigma^2)(V^2+\sigma^2)}\right.$$

$$\left. - \frac{(1-\sigma^2)}{(1+\sigma^2)^2(1+V)} - \frac{1}{(1+\sigma^2)(1+V)^2}\right\},$$

the separate terms can be integrated to give

$$A_2(U) = \frac{8}{3N\pi^2}\left[\frac{2\sigma}{(1+\sigma^2)}\tan^{-1}\left(\frac{V}{\sigma}\right) + \frac{(1-\sigma^2)}{2(1+\sigma^2)^2}ln(\sigma^2+V^2)\right.$$

$$\left. - \frac{(1-\sigma^2)}{(1+\sigma^2)^2}ln(1+V) - \frac{1}{(1+\sigma^2)(1+V)}\right]_U^\infty.$$

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Substituting in the limits and dropping terms of order $\sigma \ll 1$ leaves

$$A_2(U) \simeq \frac{8}{3N\pi^2}\left[\frac{-1}{2}ln(U^2 + \sigma^2) - 1\right].$$

Now, adding this expression to those for $A_1(U)$ in the two regions yields,

$$A(U) = \frac{8}{3N\pi^2}\left\{\frac{1}{2}ln\left(\frac{1}{U^2 + \sigma^2}\right) - \frac{2}{3}\right\} \qquad 1 \gg U \gg \sigma.$$

$$A(U) = \frac{8}{3N\pi^2}\left\{\frac{1}{2}ln\left(\frac{1}{U^2 + \sigma^2}\right) - 1\right\} \qquad 1 \gg \sigma \gg U. \qquad (6.41)$$

And we have an analytic expression for the wavefunction renormalisation, in the region $U \ll 1$.

From this expression we can see why, contrary to the argument of Appelquist et al, A(p) is not $O(N^{-1})$ and suppressed. The latter argue that the explicit N^{-1} on the RHS of [equ.6.10] is uncancelled and consequently suppresses the size of A(p). However, we now see from [equ.6.57] that for $\sigma \gg U$ (or $\Sigma \gg U$) A(p) behaves as

$$A(p) = \frac{8}{3N\pi^2}ln\left(\frac{1}{\sigma}\right) \sim \frac{1}{N}ln\left(\frac{\bar{\alpha}}{\Sigma(0)}\right).$$

From [Fig.VI.5] we see that $\Sigma(0)/\bar{\alpha}$ drops off with N exponentially (or faster). Equivalently $ln[\bar{\alpha}/\Sigma(0)]$ grows like N (at least), and cancels the explicit N in the last equation. The integral in [equ.6.10] has compensated for the explicit 1/N and A(p) is not suppressed, as argued by Appelquist et al in [Ref.VI.2]. The latter relied on the perturbative nature of the 1/N expansion. However, in reality, the Schwinger-Dyson equations are truly non-perturbative and have a more complicated, and less transparent behaviour.

Having computed the wavefunction renormalisation and found that it is not negligible we must now consider the effect this has on the self energy.

The rigorous treatment would be to return to equations [6.9] and [6.10] and solve them as a coupled set. However, before considering this procedure we can qualitatively see the effect of a non-zero A(p) by considering the next iteration of [equ.6.9] using the result for A(p).

VI.5   THE SECOND ITERATION OF THE FERMION SELF ENERGY

So far we have solved the equations [6.9] and [6.10] "in series "to obtain the first iterations to the self energy and wavefunction renormalisation respectively. The next step in this procedure is to return to [equ.6.9] and determine the next iteration of the self energy, using our results so far as input on the RHS.

Having performed the angular integral exactly as before, the analogue of [equ.6.15] is now

$$\frac{\Sigma(\bar{\alpha}U)}{\bar{\alpha}} = \frac{4}{N\pi^2 U} \int_0^\infty \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2(1 + A(\bar{\alpha}))^2 + \Sigma^2(\bar{\alpha}V)/\bar{\alpha}^2]} ln\left(\frac{1 + U + V}{1 + |U - V|}\right) \quad (6.42)$$

where the wavefunction renormalisation on the RHS is given by our results of the previous section. Although the solution of this equation is not a rigorous step in the solution of equations [6.9] and [6.10] as a coupled set, it does allow us to obtain a qualitative estimate of the effect of a non-zero wavefunction renormalisation. We achieve this by setting A(p) in [equ.6.58] equal to a constant mean value, $A(\bar{\alpha}V) \sim < A >$, so the equation becomes

$$\frac{\Sigma(\bar{\alpha}U)}{\bar{\alpha}} = \frac{4}{\bar{N}\pi^2 U} \int_0^\infty \frac{dV V \Sigma(\bar{\alpha}V)/\bar{\alpha}}{[V^2 + \Sigma^2(\bar{\alpha}V)/[\bar{\alpha}^2(1+ < A >)^2]]} ln\left(\frac{1 + U + V}{1 + |U - V|}\right)$$

$$(6.43)$$

where $\bar{N} \equiv N[1+ < A >]^2$ is an effective value of N. Dividing both sides of this equation by $[1+ < A >]$ we get [equ.6.15] with N replaced by $\bar{N}$ and $\Sigma/\bar{\alpha}$ by $\Sigma/[\bar{\alpha}(1+ < A >)]$.

From [Fig.VI.6] we see that a rough estimate of $< A >$ is of the order -1/2, so that $\Sigma/[\bar{\alpha}(1+ < A >)] \sim 2\Sigma/\bar{\alpha}$. Hence, from [equ.6.43] we see that the scale of $2 \times \Sigma/\bar{\alpha}$ is set by the solution of the first iteration of the self energy at the effective value of $\bar{N} \equiv N[1+ < A >]^2$. For an N of 2.6 used earlier this implies an $\bar{N}$ of ~ 0.65 which, from the curve of [Fig.VI.5] shows that the scale is enhanced by several orders of magnitude. The curve of this figure becomes a lot flatter. (Although the intercept of the y axis does not significantly change since, for $N \to 0$, $\bar{N}$ also tends to zero, or $\bar{N} \to N$.)

By this oversimplified argument we have shown that a non-zero wavefunction renormalisation has a pronounced effect on the self energy. Consequently we have taken our iterative method as far as is useful. To obtain reliable quantitative results we have to return to equations [6.9] and [6.10] and solve them as a truly coupled system. This is the subject of the following sections.

VI.6 THE COUPLED EQUATIONS

The N dependence of $A(p)$, discussed in section VI.5, implies that the only correct procedure is to solve equations [6.9] and [6.10] together as a coupled system. Such a procedure was carried out, as discussed in the work of M.R. Pennington and the author in [Ref.VI.4]. Here we present the main results of the coupled system.

The N dependence of $A(p)$ also implies that higher, than 1/N, order terms are included in $A(p)$. Consequently, to be consistent, we must include higher order terms in the vertex function (Remembering, up to now we have been using the bare vertex of [equ.6.7]; $i\gamma_\mu$). To achieve this we use the general fermion-photon that satisfies the Ward identity, given in [Ref.VI.5];

$$\Gamma_\mu(k,p) = \frac{\gamma_\mu}{2}[2 + A(k) + A(p)] + \frac{1}{2}\frac{[A(k) - A(p)]}{(k^2 - p^2)}(\not{k} + \not{p})(k + p)_\mu$$

$$-\frac{[\Sigma(k) - \Sigma(p)]}{(k^2 - p^2)}(k+p)_\mu + \Gamma_\mu^T(k,p).$$

The transverse part of this vertex, $\Gamma_\mu^T(k,p)$, satisfying $(k-p)_\mu \Gamma_\mu^T(k,p) \equiv 0$, is, following [Ref.VI.6], believed to be unimportant in the infrared regime and is dropped. Further, having already seen that it is the $k \sim p$ region that determines the scale of $\Sigma(p)$, we can set $k = p$ in the above expression to get

$$\Gamma_\mu(k,p) \simeq \gamma_\mu[1 + A(k)] \tag{6.44}$$

as an approximation to the full vertex function. Inserting this function at one (not both, to avoid double counting) of the vertices of [equ.6.6] yields, analogous to equations [6.13] and [6.31],

$$\Sigma(p) = \frac{4\bar{\alpha}}{\pi^2 Np} \int_0^\infty \frac{dk\,k\Sigma(k)[1 + A(k)]}{[k^2(1 + A(k))^2 + \Sigma^2]} ln\left(\frac{k+p+\bar{\alpha}}{|k-p| + \bar{\alpha}}\right) \tag{6.45}$$

and

$$A(p) = \frac{-8\bar{\alpha}\pi}{p^3 N} \int_0^\infty \frac{dk[1 + A(k)]^2}{[k^2(1 + A(k))^2 + \Sigma^2(k)]} \times$$

$$\times \left\{ 2pk + \frac{[\bar{\alpha}^2 + |p^2 - k^2|][|p - k| - (p+k)]}{\bar{\alpha}} \right.$$

$$\left. + \frac{(p^2 - k^2)}{\bar{\alpha}^2} ln\left(\frac{p+k}{|p-k|}\right) + \frac{[\bar{\alpha}^4 - (p^2 - k^2)^2]}{\bar{\alpha}^2} ln\left(\frac{p+k+\bar{\alpha}}{|p-k| + \bar{\alpha}}\right) \right\}. \tag{6.46}$$

Adopting the same iterative procedure used in solving [equ.6.13], these two equations can be similarly solved as a coupled set.

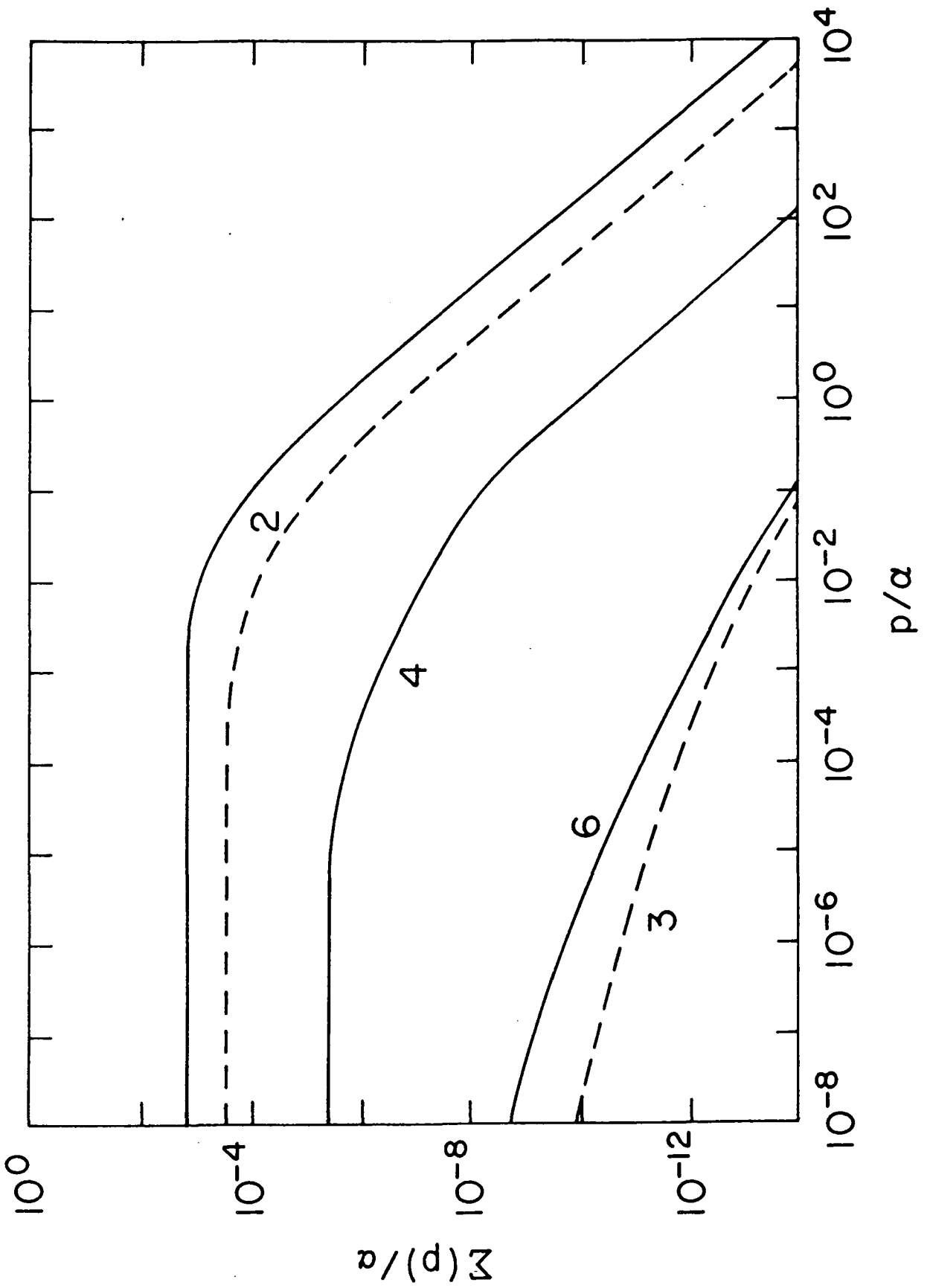In [Fig.VI.7] we show the solution thus obtained for $\Sigma(p)/\alpha$ at three

Figure VI.7 $\Sigma(p)/\alpha$ **Vs.** p

different values of N. The dashed lines of this figure are from the previous, decoupled solution of [equ.6.13], included here to show the effect of the over simplified approximations of the latter. The solid curve of [Fig.VI.8] shows the N dependence of the mass (determined from the pole of [equ.6.1]). Again, the dashed curve is the decoupled solution. As argued in section VI.5, the full coupled solution of $\Sigma(0)/\alpha$ falls off with N at a slower rate. We also note now, that for large N the dependence is exponential, agreeing with Pisarski in [Ref.VI.7]. The value of the slope is -3.7, which is almost exactly $-3\pi^2/8$. That is,

$$m \simeq m_0 e^{-3\pi^2 N/8}$$

We also note that we have only reproduced this exponential of Pisarski by including the wavefunction renormalisation in the solution of the self energy.

In [Fig.VI.9] we show the wavefunction renormalisation for three different values of N. For p<m, with m given in [Fig.VI.8], A(p) is seen to be nearly constant. This is because, analogous to [equ.6.41], we now get, with $\Sigma(0)$ replaced by m,

$$A(0) \simeq \frac{8}{3\pi^2 N} ln\left(\frac{8m}{\alpha}\right) = -1 + O(1/N).$$

The important point here is that the slope $-3.7 \simeq -3\pi^2/8$ ensures that A(p)+1 $\sim O(1/N)$ and is positive for all N. This is essential, since $A(p) + 1$ is just the inverse of the full wavefunction normalisation, $F(p^2)$, which from [equ.3.3] (and chapter II), must be positive definite. The above equation ensures this.

For p>m, the non-zero $\Sigma(k)$ in [equ.6.46] rapidly becomes irrelevant and can effectively be set to zero. In doing so the factors of $(1+A(k))$ in the numerator and denominator cancel. This being so the only N dependence left in [equ.6.46] is the explicit 1/N, and A(p) is suppressed by order 1/N. This
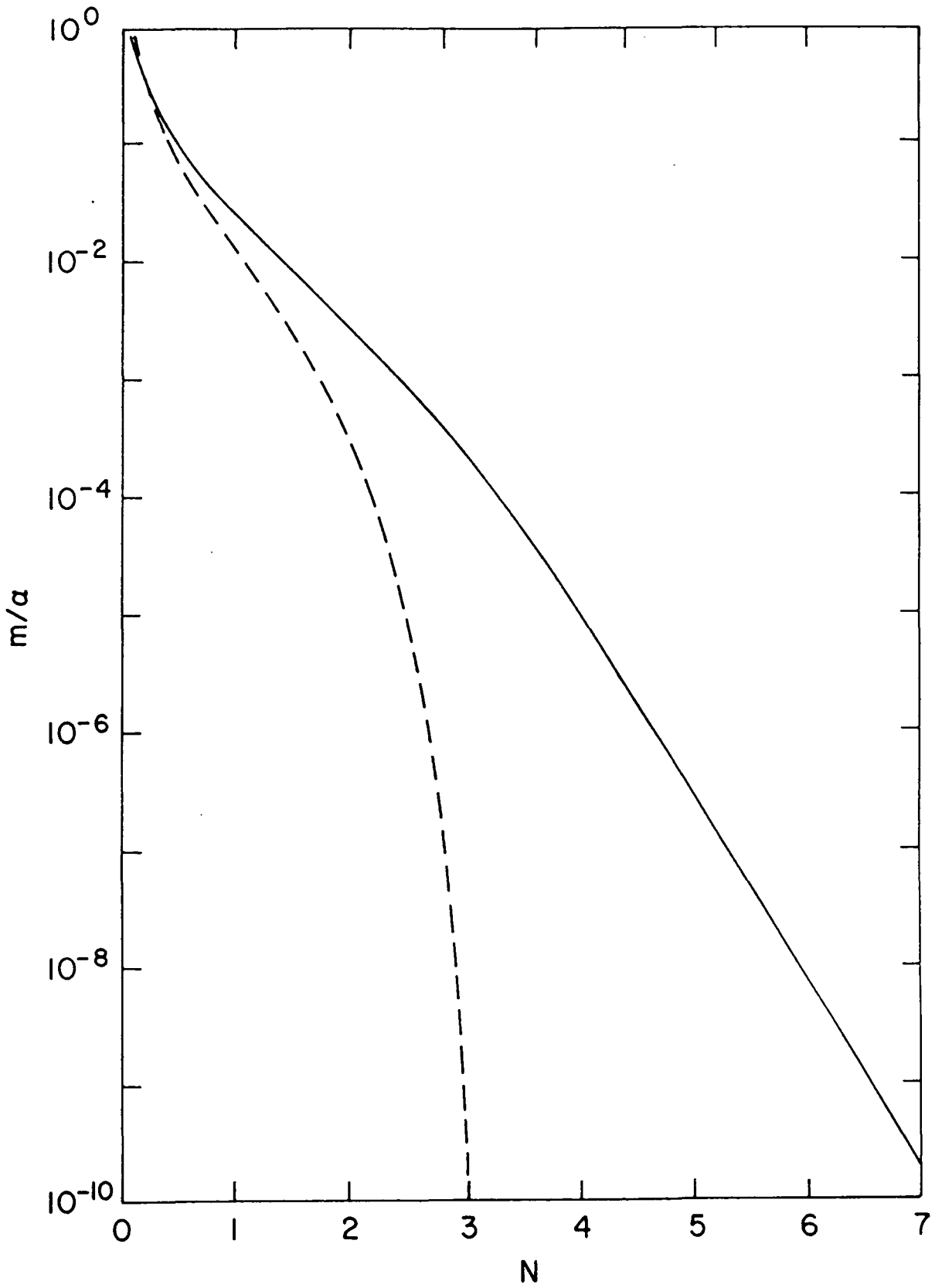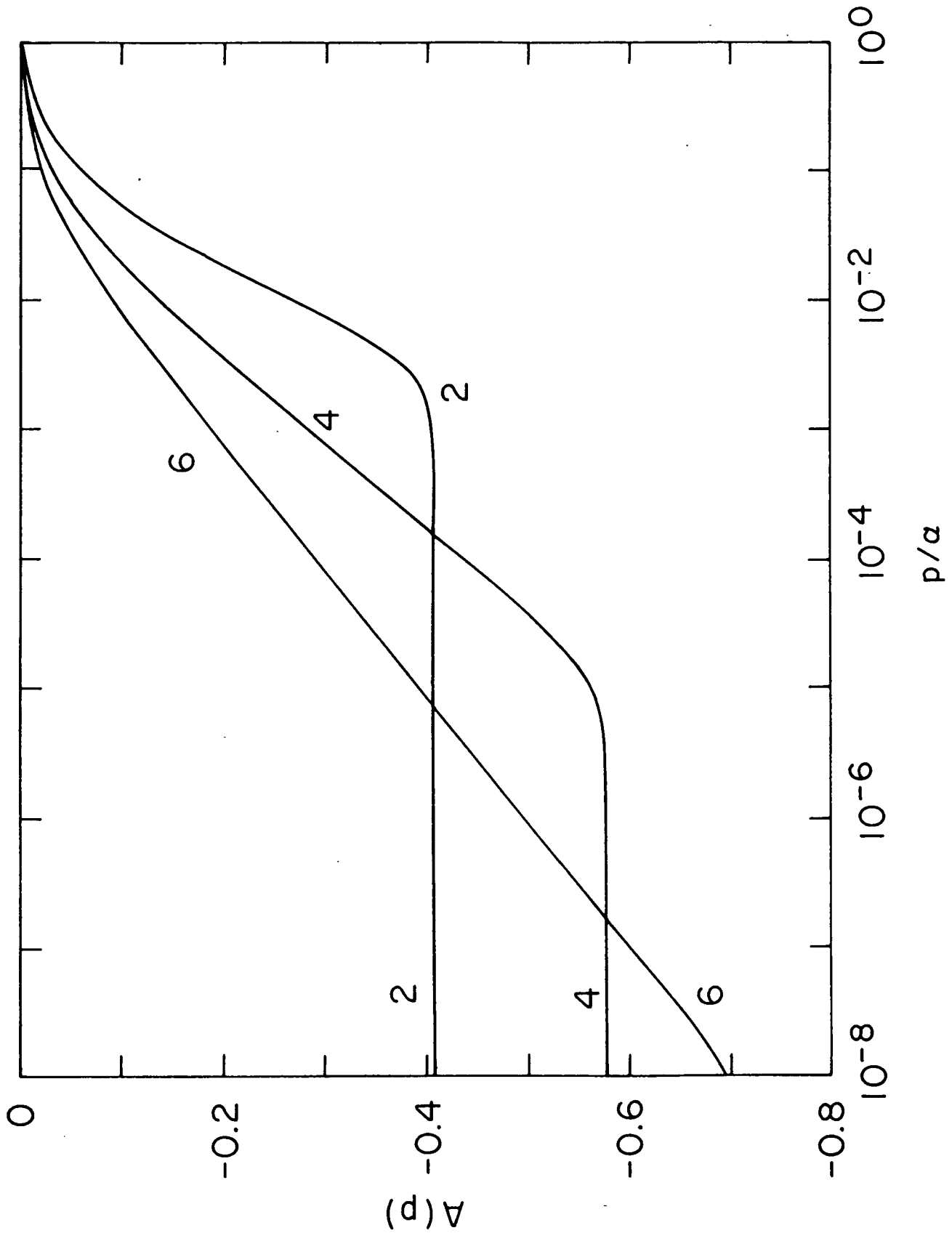
Figure VI.8 m/α **Vs.** N

Figure VI.9   The Wavefunction renormalisation

behaviour can be seen in the figure by noting that the rising parts of the curves are in the ratio $1/2 : 1/4 : 1/6$, which is exactly as expected for a $1/N$ behaviour.

Having obtained these solutions to the coupled equation [6.9] and [6.10], that can be trusted quantitatively, we can now make the following concluding remarks.

VI.7   CONCLUSIONS

Having motivated the study of QED$_3$ in a $1/N$ expansion, we set out to solve the Schwinger-Dyson equation for the self energy. Faced with two coupled integral equations for the wavefunction renormalisation and the self energy, we initially decoupled them (on the assumption thast the wavefunction renormalisation was going to be small, of order $1/N$) and solved for the self energy; reproducing, and extending the results of Appelquist et al. Then, being in a position to obtain the lowest order solution for the wavefunction renormalisation we discovered by doing so, that A(p) is not, as expected, negligible; a result of the non-perturbative nature of the equations.

A simple qualitative argument then showed that this would have a significant effect on the next iteration of $\Sigma(p)$. If, after each iteration of Σ, we then evaluate A(p) to find it has a large effect on the next iteration of Σ, the method becomes unstable and computer-time expensive. Consequently, to make our qualitative results quantitatively reliable, we were forced to solve the equations as a coupled set (by adopting the same procedure as used in the decoupled case). Having done so we find that:

1)   The fermion does dynamically generate a mass.

2)   The hierarchy of m/α can be of many orders of magnitude and, indeed, increase exponentially with N.

The mass can be seen to be a dynamically chiral symmetry breaking mass

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because of the rapid damping of m for large momentum [Ref.VI.8]. At large momentum, the mass is zero, and the chiral symmetry of the tree level theory is intact. At smaller momentum, the mass grows and the chiral symmetry is spontaeously broken. This large momentum behaviour of the self energy in [Fig.VI.7] can be seen qualitatively by returning to the uncoupled solution, in particular, the large ($p \gg \alpha$) momentum behaviour of the differential equation [6.18]. A series solution to this equation can readily be obtained as,

$$\Sigma(p) = \frac{B}{p^2}\left\{1 + \frac{b\alpha}{p} + +\right\} \qquad (p \gg \alpha)$$

The other solution to this second order equation behaves like

$$\Sigma_1(p) = C\left\{1 + \frac{c\alpha}{p} + +\right\}$$

which is like a bare mass (a constant) and is inadmissable since there was no such bare mass term in [equ.6.1].

The relevant mass to the possible deconfinement of the gluon fields in QCD_4 at finite temperature is that of the boson field. To investigate this the Schwinger-Dyson equation for the gauge boson propagator would have had to be included in the set of equations. In this work we used the lowest order (in $1/N$) gauge propagator and consequently gained no further information on its behaviour. However, by studying this simplified system, we have shown that a dynamical mass can be generated for the fermion field. As a result, we can't say anything concrete about deconfinement except that in this work we have set up the stage and completed a preliminary step in the eventual solution of the complete coupled system of the fermion *and* boson Schwinger-Dyson equations.

With respect to the investigation of scale hierarchies, we have demonstrated that large hierarchies can be generated dynamically. For example, for N=5, we see, from [Fig.VI.8], a scale difference of $m/\alpha \sim 10^{-7}$, where

m is the mass of the fermion and α plays the role of the unification scale, M_x. In a true SU(5) model M_x is also a dynamically generated scale (that of SU(5)→SU(3)×SU(2)×U(1)). By identifying this with α in QED$_3$, a scale intrinsic to the tree level theory, we are completely ignoring the mechanism of the generation of the scale M_x. We only investigate the hierarchy between it and the new scale (m) generated by the spontaneous breakdown of chiral symmetry (Analogous to the spontaneous symmetry breakdown of SU(2)×U(1)→U(1)).

One final word of caution is relevant here to the use of our simplified 'toy' model (QED$_3$). The boson propagator of [equ.6.7] is softened in the infrared regime by N fermion loops. However, as we saw in the renormalised functions of chapter II, $G(k^2)$ contains a k^{-2} term which hardens (makes more singular) the infrared behaviour of the boson propagator. Our argument that non-abelian interactions in QCD$_3$ are supressed by 1/N to yield QED$_3$ should consequently be treated with care when applying our results to more realistic, non-abelian theories, ie QCD$_3$ relevant to finite temperature QCD$_4$.

APPENDIX A: $n \equiv 2\omega \equiv 2(2 - \epsilon)$ DIMENSIONAL EUCLIDEAN INTEGRALS

$$\frac{1}{a_1 a_2 .... a_n} \equiv (n - 1)! \int_0^1 dx_1 \int_0^1 dx_2 ..... \int_0^1 dx_n \frac{\delta(1 - x_1 - x_2 .... - x_n)}{[a_1 x_1 + a_2 x_2 + ... a_n x_n]^n}.$$

$$\frac{1}{a_1^2 a_2 .... a_n} \equiv n! \int_0^1 dx_1 \int_0^1 dx_2 ..... \int_0^1 dx_n \frac{x_1 \delta(1 - x_1 - x_2 .... - x_n)}{[a_1 x_1 + a_2 x_2 + ... a_n x_n]^{(n+1)}}. \quad (A.1)$$

$$\int_0^1 dx \, x^n (1 - x)^m = \frac{\Gamma(1 + n)\Gamma(1 + m)}{\Gamma(2 + n + m)}. \quad (A.2)$$

$$\int \frac{d^n k \, k^2}{[k^2 + L]^3} = \pi^{2-\epsilon} \frac{(2 - \epsilon)\Gamma(1 + \epsilon)L^{-\epsilon}}{2\epsilon}. \quad (A.3)$$

$$\int \frac{d^n k}{[k^2 + L]^3} = \pi^{2-\epsilon} \frac{\Gamma(1 + \epsilon)}{2L^{1+\epsilon}}. \quad (A.4)$$

$$\int \frac{d^n k}{[k^2 + L]^2} = \pi^{2-\epsilon} \frac{\Gamma(1 - \epsilon)}{\epsilon L^\epsilon}. \quad (A.5)$$

$$\int \frac{d^n k}{[k^2 + L]^4} = \pi^{2-\epsilon} \frac{\Gamma(2 + \epsilon)}{6L^{2+\epsilon}}. \quad (A.6)$$

$$\int \frac{d^n k \, k^2}{[k^2 + L]^4} = \pi^{2-\epsilon} \frac{(2 - \epsilon)\Gamma(1 + \epsilon)}{6L^{1+\epsilon}}. \quad (A.7)$$

$$\int \frac{d^n k \, k^2}{[k^2 + L]^2} = \pi^{2-\epsilon} \frac{(2 - \epsilon)\Gamma(1 - \epsilon)L^{1-\epsilon}}{\epsilon(1 - \epsilon)}. \quad (A.8)$$

APPENDIX B: GAMMA MATRIX IDENTITIES

$$\gamma_\mu \not{a} \gamma_\mu = 2(1 - \epsilon)\not{a}. \qquad (B.1)$$

$$\gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\delta \gamma_\nu = 2\gamma_\delta \gamma_\beta \gamma_\alpha - 2\epsilon \gamma_\alpha \gamma_\beta \gamma_\delta. \qquad (B.2)$$

$$Tr[\not{a}\not{b}\not{c}\not{d}] = 2^{2-\epsilon}[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)]. \qquad (B.3)$$

$$Tr[\not{a}\not{b}] = -2^{2-\epsilon} a \cdot b. \qquad (B.4)$$

$$\Gamma(n + 1) = n\Gamma(n). \qquad (B.5)$$

$$\Gamma(m + n\epsilon) = \Gamma(m)[1 + n\epsilon\psi(m) + O(\epsilon^2)]. \qquad (B.6)$$

$$\psi(1) = -\nu_E. \qquad \psi(2) = 1 - \nu_E. \qquad \psi(3) = \frac{3}{2} - \nu_E. \qquad (B.7)$$

$$a^\epsilon = 1 - \epsilon ln(a) + O(\epsilon^2). \qquad (B.8)$$

B1

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