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Abstract

In this thesis we study the non-perturbative behaviour of the fermion propagator in an Abelian gauge theory, namely four dimensional, quenched QED –where by quenched we mean that we neglect the effect of the fermion loops in the boson propagator. What is of primary interest is the dynamical generation of mass.

In order to carry out this study we need to make use of the Schwinger-Dyson equations, which are the field equations of the theory. For the investigation of the fermion propagator, the form of the three point interaction is of critical importance. We study the usual ansatz, the Ball-Chiu form, for the three point function, that is obtained from the Ward-Takahashi identities, and improve upon it. This is done by making use of the powerful constraints that Multiplicative Renormalizability place upon the theory in the perturbative (high energy) region.

We initially study the theory in the massless case, for simplicity, where we find that using our improved ansatz we can obtain an exact, non-perturbative solution for the renormalised wave function.

Moving on, we then study the theory in the massive case –where we have a brief interlude to look at the ladder approximation. We solve the theory in the case where there is a finite cutoff and reproduce the well-known critical coupling point. We then consider the case where there is an infinite cutoff, when we find **no** critical coupling. We discuss and explain the differences. Returning to our improved ansatz for the fermion-boson vertex we solve the renormalised theory for both the wavefunction and mass function and find that there is **no** critical coupling. In doing this having a form for the fermion-boson vertex that satisfies both the Ward-Takahashi identity and Multiplicative Renormalizability is essential. These

studies suggest that full QED may turn out to be a theory without a critical coupling and thus be free of phase changes.

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DECLARATION

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

The research in chapters two to four has been carried out in collaboration with Dr. M.R. Pennington and is partly summarised in the papers,

D.C. Curtis and M.R. Pennington, Phys. Rev. D42 (1990) 4165.

D.C. Curtis and M.R. Pennington, Phys. Rev. D44 (1991) 536.

Also in collaboration with Drs. M.R. Pennington and D.A. Walsh, in the paper:

D.C. Curtis, M.R. Pennington and D.A.Walsh, Phys. Lett. B249 (1990) 528.

which is relevant to chapter two.

Contents

Appendicies:

CHAPTER ONE

BACKGROUND

1.1 *Introduction* :

Since earliest times, human beings have sought to understand the physical world in which they exist, whether for the purpose of divination or out of philosophical interest. In Europe, the Greeks are seen as the first people to think about the underlying mechanisms (or laws) of physics. Their theories about the matter in the universe tend to come within one of two broad schools of thought. In the first school there was no void and no micro-structure. ANAXAGORAS postulated that in every volume there existed POTENTIALITIES and the properties of the object was due to the predominating potentialities within it much like a mixture of pigments in a given colour of paint. In a way it can be seen as a very crude field theory where the total field is the sum of the orthogonal fields that make it up. ARISTOTLE believed that all matter was made up of the four elements earth, water, fire and air. The properties of an object were dependent upon the concentrations of the various elements within it . He also allowed for one element to be changed into another. It was this that led alchemists of the Middle Ages to attempt the transmutation of the base metals (Iron and lead) to the noble metals (gold and silver) using the philosophers stone (the ultimate meta-physical catalyst for an alchemist).

In the second school there was a void and micro-structure. This was the invention of LEUCIPPUS and was developed by DEMOCRITUS. In these theories all there exists is void and atoms which cannot be broken down further. The atoms continue in their motion and are only effected by collisions with other atoms. DEM-

OCRITUS attempted to account for all physical, biological and psychological occurrences using the atomist theory, and as such was the first attempt at a theory of everything (T.O.E.). PLATO [1] saw the void as a geometrical matrix upon which geometrical shapes (matter) existed. Whilst the second school might be seen as closer to what we believe in today they tried to fit the universe into their plan for it rather than the other way round. Indeed Plato believed that experimental knowledge of the world held no merit. Aristotle on the other hand was well aware of the need for empirical investigation and that theories must bow to experimental results. In this way he has more in common with modern science than does Plato who is more in tune with philosophy. Indeed two of Aristotle's students , THEOPHRASTUS and STRAPO, used his investigative methods to improve his theories and develop postulates that had something akin to micro-structure and void.

The biggest problem that all of the Greek thinkers had was trying to make quantitative statements and indeed it wasn't until the invention of calculus in the 17th century that the impetus was put back into physics.

In the 16th and 17th centuries, exploration and empire building by the Europeans lead to a great need for both optical equipment and a theory for navigation and surveying. Hence a boom in the investigation of the properties and form of light occured. With the publication of OPTICKS[2], NEWTON sort to set out his theories on the nature of light by experimental observations of refraction, reflection, interference and diffraction. In his work he set out his belief in the corpuscular theory of light (that light is made up of a beam of particles) even though his results on diffraction explicitly violated this and indeed were very suggestive of a wave form for light. Indeed Newton held a dogged disbeleaf in the possibility of a wave form for light as he didn't believe it would explain the rectilinear propagation of light. This was even though CHRISTIAN HUYGENS had shown[3] that this was possible using an envelope of secondary waves from an extended source. Huygens showed that he could account for reflection and refraction with his wave envelope but unfortunately didn't go on to look at diffraction. However his revolutionary idea, of considering what had until then been considered to be a particle, had far reaching effects. For is not today's particle physics based upon the idea that the particles are all wave functions (packets) and in solid state physical objects such as the phonon are waves moving through the crystal structure.

The wave theory of light was not however universally excepted straight away. Indeed for the next century it was the corpuscular theory that was generally accepted within the scientific community (and totally by the general public) even though it was opposed by such people as Huygens, Hooke, Leibnitz and Euler. This acceptance of corpuscular theory was in main due to two things:–

Firstly, the near demi-god stature of Newton, to criticise him about any small aspect of his work was to criticise the whole – the infallible creator of mechanics.

Secondly, the difficulty that the supporters of the wave theory had in calculating experimentally observed results. (The ideas of phase difference and integration of wavelets simply just hadn't been thought of as yet). In the face of these difficulties the meta-physical solutions offered by corpuscular theory was an attractive alternative. Experimentation was not however so moribund and the body of data was considerably extended during the century.

It wasn't until 1802 that the theoretical break-throughs that wave theory needed really started to happen. In this year a physician by the name of THOMAS YOUNG published a work in which he explained the occurence of Newton's rings using wave theory and the important concept of two waves of light cancelling and strengthening each other[4]; the interference principle had been stated. At the same time a French engineer called AUGUSTIN FRESNEL started an exacting experimental and theoretical study of the properties of light. In a series of works from 1815-24[5] he more or less explained all the effects of reflection, refraction, diffusion and diffraction of light, bringing in such concepts as phase angle of polarised light, integration of wavelets and light as a transverse wave. The theory at this time still used the concept of an ether in which the waves travelled. The problem with having an ether is that in such a fluid medium it is natural to presume that the waves are longitudinal. The experiment of FIZEAU (1849) which showed that light travels faster in air than in a liquid showed that wave theory was the correct theory and thus the problem of the ether must in some way be conceptual.

In 1873 after working on electricity and magnetism MAXWELL proposed that light was in fact electromagnetic waves that are propagated in the context of electric and magnetic fields[6] (thus removing the problem of the ether). Thus visible light is just a small slice of the spectrum of electro-magnetic (E.M) radiation. (Indeed the usage of the term visible light can be misinterpreted to mean that the visible wavelengths carry some sort of pigment with them whereas of course colour is a physiological/psychological effect due to the limitations of the cones in the retina and the interpretation of the optic nerve sense data by our psyche[7]).

It is worthwhile at this point to write down Maxwell's equations of E.M. radiation. They fall into two groups, the Inhomogeneous Maxwell Equations (I.M.E.):–

$$\begin{aligned} \underline{\nabla}.\underline{E} &= \rho \\ \underline{\nabla}\times\underline{B} - \partial_t\underline{E} &= \underline{j} \\ \Rightarrow \partial_t\rho + \underline{\nabla}.\underline{j} &= 0 \end{aligned}$$

and the Homogeneous Maxwell Equations (H.M.E.):–

$$\begin{aligned} \underline{\nabla}.\underline{B} &= 0 \\ \underline{\nabla}\times\underline{E} + \partial_t\underline{B} &= 0 \end{aligned}$$

where $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields respectively and $\underline{j}$ and ρ are the electric charge density and electric current density respectively. We've taken $c^2 = 1$ and absorbed ε_0 into ρ and μ_0 into $\underline{j}$.

By inspection of H.M.E. it can be seen that we can rewrite the electric and magnetic fields in terms of new fields (the electromagnetic potentials):-

$$\underline{E} = -\underline{\nabla}\phi - \partial_t \underline{A}$$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

Then under the following transformation it is easy to see that $\underline{E}$ and $\underline{B}$ (ie. physical observables) are left invariant by.

$$A^{\mu}(x) \rightarrow A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}\lambda(x), \qquad \partial^{\mu} = (\partial^t, -\underline{\nabla}) \tag{1.1}$$

(Greek indices refer to 4-D space-time coordinates and run over 0,1,2,3)

Continuing on we then define the Faraday (or E.M. field) tensor as

$$F_{\mu\nu} = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)$$

This object is anti-symmetric and in terms of the magnetic and electric fields is given by:-

$$F_{00} = 0,\ F_{0i} = E_i,\ F_{i0} = -E_i,\ F_{ij} = \epsilon_{ijk}B_k$$

(Roman indices refer to 3-D space coordinates and run over 1,2,3) and as $F_{\mu\nu}$ can be written soley in terms of the fields $\underline{E}$ and $\underline{B}$ it is manifestly invariant under the transformation (1.1).

It is now possible to write Maxwell's equtions in a more compact form:-

HME:-

$$\partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} + \partial_{\lambda}F_{\mu\nu} = 0 \tag{1.2}$$

IME:-

$$\partial_{\mu}F^{\mu\nu} = j^{\nu} \tag{1.3}$$

where we have constructed the 4-vector j^{ν} out of the electric charge/current densities viz $j^{\nu} = (\rho, \underline{j})$. Substituting into eq. (1.3) our definition for the Faraday tensor we reach an equation which has an interesting property:–

$$\partial^2 A^\nu(x) - \partial^\nu(\partial_\mu A^\mu(x)) = j^\nu \tag{1.4}$$

The thing about this equation is that if we transform the initial electromagnetic potentials, $A^\mu(x)_I$, using a specific choice of $\lambda(x)$, say $\chi(x)$, such that the new value $A^\mu(x)$ gives $\partial_\mu A^\mu(x) = K = const.$ then it acquires a very simple form:–

$$\partial^2 A^\nu(x) = j^\nu \tag{1.5}$$

with the subsidary condition that

$$\partial^2\chi(x) + \partial_\mu A^\mu(x)_I = K$$

These are two inhomogeneous massless Klein-Gordon equations. There is still a residual degree of freedom in our choice of $\chi(x)$ and to get rid of this we further choose $\chi(x)$ such that

$$\partial^2\chi(x) = K. \qquad \text{ie. } \partial_\mu A^\mu(x)_I = 0$$

Thus we have a 4-D equation for $A^\nu(x)$,(1.5), with two subsidiary conditions. This means that $A^\nu(x)$ has two degrees of freedom – just as we expect for a transversely polarised wave. It is usual to take $K = 0$ for simplicity and $\partial.A = 0$ is known as the Lorentz condition.

As an aside it is of interest to note that HME (1.2) follows immediately from the definition of $F_{\mu\nu}$ and is in the form of a Bianchi identity. IME (1.4) can be obtained from the Euler-Lagrange equation for a Lagrangian

$$\mathcal{L} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + j.A \tag{1.6}$$

as

$$\begin{aligned} 0 = & \ \partial^\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial^\mu A^\nu(x))}\right) - \frac{\delta \mathcal{L}}{\delta A^\nu(x)} \\ = & \ \partial^\mu(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) - j_\nu \\ = & \ \partial^2 A^\nu(x) - \partial_\nu(\partial . A) - j_\nu . \end{aligned}$$

With his work on black body radiation, Max Plank introduced the idea of quantised energy levels[8]. Einstein then developed this to explain the photoelectric effect by postulating that radiation is emitted and absorbed in quanta or photons[9] (quantised wave packets).

1.2 *Gauge Theories* :

In many areas of physics it is common to come across the situation where physical results are independent of an overall change in phase of the basic wave functions or fields of the theory. For instance in forced vibration theory it is the relative difference between the phase of the driving and forced oscillators that is important, not the position of the reference point in the cycle (ie the overall phase). Such an invariance of a theory under a change of phase is known as a global symmetry as the phase angle θ is constant for all points in space-time.

In 1929[10] Weyl looked at the effect on a theory of making the phase angle space-time dependent $\theta \rightarrow \theta(x)$. This is called a local or gauge symmetry transformation as the value of the phase angle is dependent upon the local value of the space-time coordinates. Theories that are invariant under such transformations are known as gauge theories. They give rise to new interactions, which, not surprisingly, are called gauge interactions. It is possible to split gauge theories into two classes depending upon whether the structure constants c_{ijk} of the basis group $\{T_i\}$ of their underlying Lie algebra are identically equal to zero (abelian) or not (non-abelian), where

$$[T_i, T_j] = c_{ijk} T_k .$$

1.2a *Abelian Gauge Theories* :

Let us now consider the Dirac equation for a free electron field $\varphi(x)$ that transforms under a U(1) Lie group transformation:–

$$\mathcal{L}_0 = \overline{\varphi}(x)(i\not{\partial} - m)\varphi(x)$$

where $\overline{\varphi}(x) = \varphi^{\dagger}(x)\gamma^0$ and $\not{\partial} = \gamma^{\mu}\partial_{\mu}$. Now $U(1)$ is of dimension one and is such that $\forall\ u(x) \in U(1)\quad u(x)^{\dagger}u(x) = 1$. It is easy to see that in the fundamental representation the transform of $\varphi(x)$ is written as

$$\varphi(x) \xrightarrow{u(x)} \mathcal{D}(u(x))\varphi(x) = u(x)\varphi(x)$$

where $\mathcal{D}(u)$ is the representation of a group element u for a local/gauge $U(1)$ symmetry

$$\mathcal{D}(u(x)) = u(x) = exp(-i\theta(x))$$

which is trivially abelian. In order for physics to be invariant under this gauge symmetry we need the Lagrangian to be invariant under this transformation. Clearly

$$\begin{aligned} m\overline{\varphi}(x)\varphi(x) \xrightarrow{u(x)} & m\overline{\varphi}(x)\mathcal{D}^{-1}(u(x))\mathcal{D}(u(x))\varphi(x) \\ =& m\overline{\varphi}(x)\varphi(x) \end{aligned}$$

but the derivative term is not however invariant as:–

$$\begin{aligned} \overline{\varphi}(x)\not{\partial}\varphi(x) \xrightarrow{u(x)} & \overline{\varphi}(x)\mathcal{D}^{-1}(u)\not{\partial}(\mathcal{D}(u)\varphi(x)) \\ =& \overline{\varphi}(x)\not{\partial}\varphi(x) + \overline{\varphi}(x)\mathcal{D}^{-1}(u)(\not{\partial}\mathcal{D}(u))\varphi(x) \\ =& \overline{\varphi}(x)\not{\partial}\varphi(x) - i\overline{\varphi}(x)(\not{\partial}\theta(x))\varphi(x). \end{aligned}$$

So we need to replace ∂_{μ} by a new object D_{μ}, the gauge-covariant derivative, which will make the derivative term invariant, ie

$$D_\mu\varphi(x) \rightarrow [D_\mu\varphi(x)]' = \mathcal{D}(u)D_\mu\varphi(x).$$

To investigate this we shall write the gauge-covariant derivative as:-

$$D_\mu = (\partial_\mu + igA_\mu)$$

where $A_\mu(x)$ is the gauge field and as we are dealing with free electrons $g = e$ the electric charge. For invariance to be fulfilled we require that:–

$$\begin{aligned}
[(\partial_\mu + ieA_\mu)\varphi(x)]' &= \mathcal{D}(u)(\partial_\mu + ieA_\mu)\varphi(x) \\
\Rightarrow exp(-i\theta(x))(\partial_\mu + ieA_\mu)\varphi(x) &= (\partial_\mu + ieA_\mu)'\varphi(x)' \\
&= (\partial_\mu + ieA'_\mu)\mathcal{D}(u)\varphi(x) \\
&= (\partial_\mu + ieA'_\mu)exp(-i\theta(x))\varphi(x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow ieA_\mu &= ieA'_\mu - i\partial_\mu\theta(x) \\
A'_\mu &= A_\mu + \frac{1}{e}\partial_\mu\theta(x).
\end{aligned}$$

Remarkably this gauge transformation is the same as the Poincaré transformation (1.1) on the electromagnetic potential with $\theta(x) = e\lambda(x)$ and so we identify this A_μ as the electromagnetic potential. Our Lagrangian now reads:–

$$\begin{aligned}
\mathcal{L} &= \overline{\varphi}(x)(i\not{D} - m)\varphi(x) \\
&= \overline{\varphi}(x)(i\not{\partial} - m)\varphi(x) - A_\mu e\gamma^\mu\overline{\varphi}(x)\varphi(x).
\end{aligned}$$

As we have introduced a new field A_μ we also have to introduce a kinetic term for it , but what form should this take ? We expect that as A_μ is a 4-vector it will be quadratic in derivatives, but what will it be specifically ? We are guided in our choice simply because we have associated A_μ with the electromagnetic potential and so we wish its Euler-Lagrange equation to give the IME , eq (1.4) . Now we know that a Lagrangian for A_μ given by eq (1.6) gives us IME and also the form of $F_{\mu\nu}$ thus defined also gives the HME (eq (1.2)) trivially. These $F_{\mu\nu}$ being independent of the gauge transformation. So associating $e\gamma^\mu\overline{\varphi}(x)\varphi(x)$with j^μ in eq (1.6) we propose adding a term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian to give

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu e\gamma^\mu\overline{\varphi}(x)\varphi(x) + \overline{\varphi}(x)(i\partial\!\!\!/ - m)\varphi(x)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.7}$$

Should we also introduce a mass term $m_A A.A$ into the Lagrangian for this new field? The answer is no, because any such term would not be invariant under the gauge transformation for A_μ. Hence as we'd expect the field associated with our electromagnetic potential is massless. We have thus arrived in a moderately painless way at the Lagrangian for the quantum interaction of electrons and the electromagnetic field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\varphi}(x)(iD\!\!\!\!/ - m)\varphi(x),$$

this is known as QED.

We shall now note some points about this Lagrangian:–

(1)
$$\begin{aligned}[D_\mu, D_\nu]\varphi(x) &= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\varphi(x) - e^2[A_\mu, A_\nu]\varphi(x) \\ &= ieF_{\mu\nu}\varphi(x)\end{aligned} \tag{1.8}$$

as in an abelian theory $[A_\mu, A_\nu] = 0$

(2) There is no gauge field self-interaction term in the Lagrangian (as the Lie algebra is abelian – the gauge field has no $U(1)$ quantum number/coupling constant) and so, in the absence of electrons, the theory is a free-field theory.

1.2b *Non-abelian Gauge Theories* :

In 1954 Yang and Mills extended Weyl's original work to include non-abelian gauge theories[11]. For an example let us remember that in baryon physics the Pauli exclusion principle demands that from the existence of the $\Delta^{++}(= uuu)$, for example, that the quarks have an internal colour $SU(3)$ symmetry (Lie group). So let us consider the free, three component Lagrangian in the fundamental representation:–

$$q \xrightarrow{u} \mathcal{D}(u)q = exp(-i\varepsilon_a(x)\lambda^a/2)q$$

the λ_a being the Gell-Mann matrices which express the non-abelian nature of $SU(3)$. In the fundamental representation:–

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c.$$

Setting the number of quark flavours to one, for simplicity, the initial Lagrangian is then the same as before:–

$$\mathcal{L}_I = \overline{q}(i\not{\partial} - m)q$$

again it is the derivative term that breaks the gauge invariance of the Lagrangian and so we again replace ∂_μ by D_μ where now we write:–

$$D_\mu = \partial_\mu + i\frac{g}{2}A_\mu$$

where in the $SU(3)$ Lie algebra $A_\mu = A_{a\mu}\lambda_a$. Requiring

$$D_\mu q \xrightarrow{u} (D_\mu q)' = \mathcal{D}(u)D_\mu q$$

we get

$$\begin{aligned}\mathcal{D}(u)(\partial_\mu + i\frac{g}{2}A_\mu)q &= D'_\mu q' \\ &= (\partial_\mu + i\frac{g}{2}A'_\mu)\mathcal{D}(u)q\end{aligned}$$

$$\Rightarrow \quad i\frac{g}{2}A'_\mu \mathcal{D}(u) = \mathcal{D}(u)i\frac{g}{2}A_\mu - \partial_\mu(\mathcal{D}(u))$$
$$A'_\mu = \mathcal{D}(u)A_\mu\mathcal{D}^{-1}(u) + \frac{i2}{g}\partial_\mu(\mathcal{D}(u))\mathcal{D}^{-1}(u)$$

The question now is what derivative term shall we use? If we try using the Faraday tensor as defined in eq.(1.7) we end up with a kinematic term that transforms as follows:–

$$F_{\mu\nu}F^{\mu\nu} \rightarrow \left[\partial_\mu\{\mathcal{D}A_\nu\mathcal{D}^{-1} + \frac{i2}{g}(\partial_\nu\mathcal{D})\mathcal{D}^{-1}\} - \partial_\nu\{\mathcal{D}A_\mu\mathcal{D}^{-1} + \frac{i2}{g}(\partial_\mu\mathcal{D})\mathcal{D}^{-1}\}\right]$$
$$\bullet\left[\partial^\mu\{\mathcal{D}A^\nu\mathcal{D}^{-1} + \frac{i2}{g}(\partial^\nu\mathcal{D})\mathcal{D}^{-1}\} - \partial^\nu\{\mathcal{D}A^\mu\mathcal{D}^{-1} + \frac{i2}{g}(\partial^\mu\mathcal{D})\mathcal{D}^{-1}\}\right]$$

which by inspection of the g^0 terms on the R.H.S. is not gauge invariant. In order to find a sensible definition for $F_{\mu\nu}$ we look back at the abelian case and try to find a general definition for it that is independent of the particular gauge fields we are using. Equation (1.8) is one such candidate, it gives $F_{\mu\nu}$ in terms of the covariant derivatives only and transforms in the following way:–

$$\frac{ig}{2}F_{\mu\nu}q \rightarrow \frac{ig}{2}(F_{\mu\nu}q)' = ([D_\mu, D_\nu]q)' = ([D_\mu D_\nu - D_\nu D_\mu]q)'$$

Now D_μ is constructed so that $D'_\mu\mathcal{D}(u) = \mathcal{D}(u)D_\mu$. Therefore

$$\frac{ig}{2}F'_{\mu\nu}\mathcal{D}(u)q = D'_\mu\mathcal{D}(u)D_\nu q - D'_\nu\mathcal{D}(u)D_\mu q = \mathcal{D}(u)D_\mu D_\nu q - \mathcal{D}(u)D_\nu D_\mu q$$
$$= \mathcal{D}(u)[D_\mu, D_\nu]q$$
$$\Rightarrow \frac{ig}{2}F'_{\mu\nu} = \mathcal{D}(u)\frac{ig}{2}F_{\mu\nu}\mathcal{D}(u)^{-1}$$

This is more amenable to being formed into a gauge invariant object. Indeed

$$tr(F'_{\mu\nu}F^{\mu\nu\prime}) = tr(F_{\mu\nu}F^{\mu\nu}) \tag{1.9}$$

is gauge invariant. This definition for $F_{\mu\nu}$ also obeys a non-abelian version of the

H.M.E. as from the Jacobi identity for a Lie group :–

$$\begin{aligned} 0 =&\ [D_\mu,[D_\nu,D_\sigma]] + [D_\nu,[D_\sigma,D_\mu]] + [D_\sigma,[D_\mu,D_\nu]] \\ \Rightarrow 0 =&\ D_\mu F_{\nu\sigma} + D_\nu F_{\sigma\mu} + D_\sigma F_{\mu\nu} \end{aligned}$$

Projecting $F_{\mu\nu}$ onto the Lie algebra of $SU(3)$ we find that:–

$$\begin{aligned} F_{\mu\nu} = F_{\mu\nu a}\lambda^a \to F'_{\mu\nu} =&\ F'_{\mu\nu b}\lambda^b = \mathcal{D}(u)F_{\mu\nu b}\lambda^b\mathcal{D}(u)^{-1} \\ =&\ \mathcal{D}(u)\lambda^b\mathcal{D}(u)^{-1}F_{\mu\nu b} \\ =:&\ \lambda^a\mathcal{D}^A{}_a{}^b(u)F_{\mu\nu b} \end{aligned}$$

where $\mathcal{D}^A{}_a{}^b(u)$ is the adjoint representation of the Lie group. So $F_{\mu\nu}$ transforms according to the adjoint representation of the group

$$F_{\mu\nu a} \to F'_{\mu\nu a} = \mathcal{D}^A{}_a{}^b(u)F_{\mu\nu b}.$$

Now for the Gell-Mann matrices $tr(\lambda^a\lambda^b) = 2\delta^{ab}$ therefore we have that

$$\begin{aligned} tr(F_{\mu\nu}F^{\mu\nu}) =&\ tr(\lambda^a\lambda^b)F_{\mu\nu a}F_b^{\mu\nu} = 2\delta^{ab}F_{\mu\nu a}F_b^{\mu\nu} \\ =&\ 2F_{\mu\nu a}F_a^{\mu\nu}. \end{aligned}$$

So $F_{\mu\nu a}F_a^{\mu\nu}$ is gauge invariant from (1.9) . Looking back to the abelian case it is natural for us to choose $-\frac{1}{4}F_{\mu\nu a}F_a^{\mu\nu}$ as the kinetic term for the gauge fields and so our Lagrangian becomes:–

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu a}F_a^{\mu\nu} + \overline{q}(i\not{D} - m)q. \tag{1.10}$$

Although this looks generically the same as the abelian Lagrangian it is dramatically different. This is because the kinematic term of the gauge field has the form:–

$$
\begin{aligned}
\frac{ig}{2}F_{\mu\nu} &= [D_\mu, D_\nu] = [\partial_\mu + \frac{ig}{2}A_\mu, \partial_\nu + \frac{ig}{2}A_\nu] \\
&= \frac{ig}{2}\partial_\mu A_\nu - \frac{ig}{2}\partial_\nu A_\mu + \left(\frac{ig}{2}\right)^2 [A_\mu, A_\nu] \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{ig}{2}[A_\mu, A_\nu] \\
&= \lambda^a(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}) + \frac{ig}{2}[\lambda^b, \lambda^c]A_{\mu b}A_{\nu c} \\
&= \lambda^a(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + \frac{ig}{2}2if_a^{bc}A_{\mu b}A_{\nu c}) \\
ie. \qquad F_{\mu\nu a} &= \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - gf_a^{bc}A_{\mu b}A_{\nu c}
\end{aligned}
$$

Thus in contrast to note (2) at the end of the abelian gauge theory section there are gauge self interactions in non-abelian theories and so in the absence of the spinor fields (quarks here) the theory is NOT a free field theory. The particular theory that we have been looking at in this section is called QCD.

1.3 *Quantisation of Gauge Theories* [11] :

To give a thorough explaination of the different approaches to quantising gauge theories would take up the rest of this thesis on its own. Even then it is doubtful whether full justice would be done to the field. Thus I will assume that the reader has an adequate working knowledge of this subject and just make some passing comments that are pertinent to the work in the rest of the thesis.

In the canonical quantisation method the operators are taken as $A_\mu(x)$ and their conjugate momenta $\pi_\mu(x) = \delta\mathcal{L}/(\delta(\partial A_\mu))$, their commutation relations are taken as inputs to the theory. Because there are only two real degrees of freedom for the gauge fields we then need some constraints to be applied to the operators. Examples of this are taking $\underline{\nabla}.\underline{A} = 0$(Radiation gauge) or $A_3(x) = 0$ (Axial gauge) these however sacrifice manifest Lorentz covariance. A more popular formulation is to keep the explicit Lorentz covariance, allowing the extra degrees of freedom to give a Hilbert space with indefinite metric, then decouple the unphysical states from the theory by use of a constraint applied to the final states. This is done by demanding that $\partial.A(x) \mid \varphi\rangle = 0$ (Gupta-Bleuler method), this comes from the Lorentz gauge condition, $\partial.A(x) = 0$, of classical electromagnetism. Though it is a weaker constraint than the classical one, it is more than adequate for the quantum theory. The common theme to all of these gauges is that at some point in the theory a condition is needed to remove the two spurious degrees of freedom from the gauge field. Thus when we turn to look at the more theoretically powerful method of Feynman Path Integrals it should not surprise us that we will need a gauge constraint.

We shall look at the path integral formulation for an abelian theory (QED) with no fermion fields. (For a non-abelian theory the process is substantially harder and requires the inclusion of Faddeev-Popov Ghost fields [11,12]). With this simplification our path integral has the form:–

$$Z[J_\mu] = N \int \mathcal{D}A^\mu exp[iS] \tag{1.11}$$

where the action $S[A] = \int(\mathcal{L}(A_\mu) - J^\mu A_\mu)d^4x$ in four dimensions. (N = Normalisation factor such that $Z[0] = 1$). $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and J^μ is the source term. Then :–

$$\begin{aligned} Z[J_\mu] &= \int \mathcal{D}A^\mu exp[i \int d^4x\{-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J^\mu A_\mu\}] \\ &= \int \mathcal{D}A^\mu exp[i \int d^4x\{-\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) - J^\mu A_\mu\}] \\ &= \int \mathcal{D}A^\mu exp[i \int d^4x\{\frac{1}{2}A_\mu(y)K^{\mu\nu}(x-y)A_\nu(x) - J^\mu A_\mu\}] \end{aligned}$$

where

$$K^{\mu\nu}(x-y) = \delta(x-y)[g^{\mu\nu}\partial_x^2 - \partial_x^\mu \partial_x^\nu] \tag{1.12}$$

and we define the inverse by:–

$$\int d^4y K^{\mu\nu}(x-y)K^{-1}_{\nu\alpha}(y-z) = g^\mu_\alpha \delta(x-z)$$

Now

$$\frac{1}{2}A_\mu(y)K^{\mu\nu}(x-y)A_\nu(x) - \frac{1}{2}J^\mu(x)A_\mu(x) - \frac{1}{2}J^\mu(y)A_\mu(y)$$
$$= \frac{1}{2}\left(A_\mu(y) - \int d^4y J^\alpha(z)K^{-1}_{\alpha\mu}(y-z)\right) K^{\mu\nu}(x-y)$$
$$\bullet \left(A_\nu(x) - \int d^4x K^{-1}_{\nu\alpha}(z-x)J^\alpha(z)\right)$$
$$-\frac{1}{2}\int d^4y J^\nu(x)K^{-1}_{\nu\alpha}(y-x)J^\alpha(y).$$

Redefining $A'_\mu(x) = A_\mu(x) - \int d^4x K^{-1}_{\mu\nu}(z-x)J^\alpha(z)$ and assuming $\mathcal{D}A'_\mu(x) = \mathcal{D}A_\mu(x)$ we have:–

$$Z[J_\mu] = \int \mathcal{D}A'^\mu exp\Big[i\int d^4x\{\frac{1}{2}A'_\mu(y)K^{\mu\nu}(x-y)A'_\nu(x)$$
$$-\frac{1}{2}\int d^4y J^\nu(y))K^{-1}_{\nu\mu}(x-y)J^\mu(x)\}\Big]$$
$$\sim exp\Big[-\frac{i}{2}\int d^4x d^4y J^\nu(y)K^{-1}_{\nu\mu}(y-x)J^\mu(x)\Big].$$

K^{-1} thus joins two gauge field sources and is therefore the gauge propagator:–

$$J^\nu(y) \overset{K^{-1}_{\nu\mu}(y-x)}{- - - - - - - -} J^\mu(x)$$

Now

$$\int d^4y K^{\mu\nu}(x-y)K^{-1}_{\nu\alpha}(y-z) = g^\mu_\alpha\delta(x-z). \tag{1.13}$$

The Fourier transform of $K^{-1}_{\nu\alpha}(y-z)$ is

$$\sim \int \frac{d^4k}{(2\pi)^4}e^{-ik.(y-z)}D_{\nu\alpha}(k)$$

where as the gauge field is a vector field, $D_{\nu\alpha}(k)$ must have the following form:–

$$D_{\nu\alpha}(k) = [a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]$$

then (1.13) becomes, using (1.12) :-

$$\int d^4y\delta(x-y)[g^{\mu\nu}\partial_y^2 - \partial_y^\mu\partial_y^\nu]\int\frac{d^4k}{(2\pi)^4}e^{-ik.(y-z)}[a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]$$
$$= g_\alpha^\mu\int\frac{d^4k}{(2\pi)^4}e^{-ik.(x-z)}$$

$$\Rightarrow \int\frac{d^4k}{(2\pi)^4}\int d^4y\delta(x-y)[-k^2g^{\mu\nu} + k^\mu k^\nu][a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]e^{-ik.(y-z)}$$
$$= g_\alpha^\mu\int\frac{d^4k}{(2\pi)^4}e^{-ik.(x-z)}$$

$$\begin{aligned}\Rightarrow g_\alpha^\mu &= [-k^2g^{\mu\nu} + k^\mu k^\nu][a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]\\ &= a(k^2)[-k^2g_\alpha^\mu + k^\mu k_\alpha] + b(k^2)[-k^2k^\mu k_\alpha + k^2k^\mu k_\alpha]\\ &= a(k^2)[-k^2g_\alpha^\mu + k^\mu k_\alpha]\end{aligned}$$

which contradicts the setup. It is thus seen that (1.13) does not exist. This is because the functional integration is over all gauge fields, including those only related by the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$. What we want to do then is to fix the gauge fields to those in only one specific gauge, we look at only one point on each fibre in the fibre space of the gauge fields. For example if we add a gauge fixing term $\mathcal{L}_{GF} = -(\partial.A)^2/2$ to the Lagrangian then we get

$$\begin{aligned}\int d^4x\mathcal{L}' &= \int d^4x(\mathcal{L} + \mathcal{L}_{GF})\\ &= \int d^4x[\frac{-1}{2}(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu\partial_\nu A^\nu) - J^\mu A_\mu]\\ &= \int d^4x[\frac{1}{2}A_\nu\partial^2A^\nu - \frac{1}{2}A_\nu\partial_\mu\partial^\nu A^\mu + \frac{1}{2}A_\nu\partial^\nu\partial_\mu A^\mu - J^\mu A_\mu]\\ &= \int d^4x[\frac{1}{2}A_\nu\partial^2A^\nu - J^\mu A_\mu]\\ &= \int d^4x[\frac{-1}{2}(\partial_\mu A^\nu)^2 - J^\mu A_\mu]\end{aligned}$$

which gives the following Euler-Lagrange equation:–

$$-\partial^2A^\nu + J^\nu = 0$$

(The IME with the Lorentz gauge condition $\partial.A = 0$). In order to get the more general condition $\partial.A = \ const.$ we write $\mathcal{L}_{GF} = \ -(\partial.A)^2/(2\xi)$ where ξ is called the covariant gauge parameter. We can then return to the path integral to calculate the photon/gauge propagator that is now generated. However, it should be noted that as the path integral now stands (1.11) it is not formally well defined [13]. This is because the poles of the propagator in momentum space exist on the energy axis in Minkowski space-time. In order to make the path integral well defined in Minkowski space-time it is neccesary to deform the path of integration slightly, this is equivalent to taking the poles off the axis by an amount $\pm i\varepsilon$. Alternatively, one can make the path integral well defined by Wick rotating the time axis into the complex plane so that it lies along the pure imaginary axis $t \longrightarrow +i\tau$. We are now in Euclidean space-time with metric (-,-,-,-). As with most texts on path integrals [11,13] we shall engage in our formal manipulations of the theory in Minkowski space-time and Wick rotate to Euclidean space-time only when we wish to calculate some physics. It is, however, an important point to remember that particle physics is only really well defined in Euclidean space-time. We leave this technical point to return to the calculation of the gauge propagator. The path integral is now:–

$$
\begin{aligned}
Z[J_\mu] &= \int \mathcal{D}A^\mu exp[i \int d^4x\{\mathcal{L} + \ \mathcal{L}_{GF} - J^\mu A_\mu\}] \\
&= \int \mathcal{D}A^\mu exp[i \int d^4x\{\frac{1}{2}A_\mu \partial^2 A^\mu - \ \frac{1}{2}(1 - \frac{1}{\xi})A_\nu \partial^\nu \partial^\mu A_\mu - J^\mu A_\mu\}] \\
&= \int \mathcal{D}A^\mu exp[i \int d^4x\{\frac{1}{2}A_\mu(y)K^{\mu\nu}(x-y)A_\nu(x) - J^\mu A_\mu\}]
\end{aligned}
$$

where

$$K^{\mu\nu}(x-y) = \ \delta(x-y)[g^{\mu\nu}\partial^2 - \ (1 - \frac{1}{\xi})\partial^\mu \partial^\nu]$$

and from before we know that the inverse $K^{-1}_{\nu\mu}(x-y)$ is the gauge propagator and is given by:–

$$\int d^4y K^{\mu\nu}(x-y)K^{-1}_{\nu\alpha}(y-z) = \ g^\mu_\alpha \delta(x-y).$$

Using the Fourier transformation of $K^{-1}_{\nu\alpha}(y-z)$ as before we have that for the Fourier transformation of the propagator $D_{\mu\nu}(k) = a(k^2)g_{\mu\nu} + b(k^2)k_\mu k_\nu$

$$\int d^4y\delta(x-y)[g^{\mu\nu}\partial_y^2 - (1-\frac{1}{\xi})\partial_y^\mu\partial_y^\nu]\int \frac{d^4k}{(2\pi)^4}e^{-ik.(y-z)}[a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]$$
$$= g_\alpha^\mu \int \frac{d^4k}{(2\pi)^4}e^{-ik.(x-z)}$$

$$\Rightarrow g_\alpha^\mu = [-k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu][a(k^2)g_{\nu\alpha} + b(k^2)k_\nu k_\alpha]$$
$$= a(k^2)[-k^2g_\alpha^\mu + (1-\frac{1}{\xi})k^\mu k_\alpha] - \frac{1}{\xi}b(k^2)k^2k^\mu k_\alpha$$

Hence

$$D_{\mu\nu}(k) = \frac{-1}{k^2}[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}]$$

where, as an aside, we note that:–

$\xi = 1$ is known as the Feynman gauge.

$\xi = 0$ is known as the Landau gauge.

When we include the fermion fields once again we have in the Lagrangian a coupling $\overline{\varphi}A_\mu\varphi$ between the two types of field that leads to the gauge (and fermion) propagator being affected. The gauge propagator has the form:–

$$\frac{-\mathcal{G}(k^2)}{k^2}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}] + \xi\frac{k_\mu k_\nu}{k^4}$$

where, if the coupling e is small, $\mathcal{G}(k^2)$ has a perturbative series with the leading term being 1 (the result without fermions). By perturbative we mean that due to the 'smallness' of e any term $O(e^n)$ is strictly smaller than any term $O(e^{n+1})$. This allows us to expand any function in ascending powers of e up to an order ,say, N plus a remainder $\mathcal{R}(N+1)$, that decreases with increasing N. We can also find the form for the fermion propagator by using a similar method, this time however we use Grassmann variables (anti-commuting) $\eta(x), \overline{\eta}(x)$ as source terms. Our fermion source term part then being $\overline{\eta}(x)\varphi(x) + \overline{\varphi}(x)\eta(x)$. In a perturbative expansion we get the following form for the fermion propagator:–

$$\frac{F(k^2)}{\not{k} - \Sigma(k^2)} \tag{1.14}$$

where the leading terms of $\mathcal{F}(k^2)$ and $\Sigma(k^2)$ are 1 and m respectively.

The perturbative expansion parameter that is used in QED is $\alpha = e^2/4\pi$. As with all gauge theories it can be shown from the renormalisation group equation (see section 2.5a) that α varies with energy. In QED however the variation of α is remarkably small for a very wide range of perturbative energies. The figure that is often quoted for α is $1/137$ which is a very small number and so any perturbative series will be very well ordered and appear to converge very quickly. This has lead to the calculations in QED being amongst the most accurate theoretical predictions of experimental results in the whole of physics! For instance the anomalous magnetic moment of the electron is known to an accuracy of 2 parts in 10^6 [14]. Quite clearly we must present a compelling arguement to justify extending the study of QED to cover the whole non-perturbative energy region. Indeed to obtain this justification we need to discuss an experiment in nuclear physics.

1.4 *The Pauli equation and GSI* :

Within electrodynamics in nuclear physics there is the concept of critical charge. In order to see where this comes from we return to our QED Lagrangian (1.10) and look at the Euler-Lagrange equation for the φ spinor field:–

$$(i\not{D} - m)\varphi(x) = 0$$

which is known as the Dirac equation. Because in the area of nuclear physics we are interested in, the nuclei studied are heavy we can work in the non-relativistic limit. We are then able to make some reasonable approximations/assumptions. Firstly as we have heavy non-relativistic objects their velocities will be small compared to their masses and so we can consider the system to be made up of steady state solutions, $\varphi(x_\mu) = \varphi(\underline{x})_E e^{-i(E+m)x_0}$, $(x_0 = t)$ [15]. The Dirac equation is then:–

$$0 = (i\partial\!\!\!/ - eA\!\!\!/ - m)\varphi(\underline{x})_E e^{-i(E+m)x_0} \qquad (1.15)$$

We then use the COULOMB field representation for the gauge field in which $A_\mu = (Ze/(4\pi \mid \underline{x} \mid), \underline{0})$ where Z is the atomic number of the large nucleus we consider scattering off. Eq. (1.15) then simplifies to:–

$$0 = \left(i\underline{\partial\!\!\!/} + \gamma_0\left(E + m - \frac{Z\alpha}{\mid \underline{x} \mid}\right) - 1(m)\right)\varphi(\underline{x})_E \qquad (1.16)$$

We can write the 4-component spinor $\varphi(\underline{x})_E$ in terms of two 2-component spinors viz:–

$$\varphi(\underline{x})_E = \begin{pmatrix} \chi(\underline{x}) \\ \eta(\underline{x}) \end{pmatrix}$$

and with the usual notation for the 4×4 representation for the gamma matrices, namely:–

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

The σ_i are the Pauli matrices, which obey the following equation:–

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon^{ijk}\sigma_k$$

Substituting these into (1.16) we obtain the two following equations:–

$$-i\underline{\sigma}.\underline{\nabla}\eta(\underline{x}) + \left(E - \frac{Z\alpha}{\mid \underline{x} \mid}\right)\chi(\underline{x}) = 0$$
$$+i\underline{\sigma}.\underline{\nabla}\chi(\underline{x}) + \left(E + 2m - \frac{Z\alpha}{\mid \underline{x} \mid}\right)\eta(\underline{x}) = 0.$$

Now in a non-relativistic theory we have that $2m \gg E - (Z\alpha/ \mid \underline{x} \mid)$ so we have an identity for $\eta(\underline{x})$ from the second equation that when we substitute into the first equation gives us:–

$$0 = \frac{-1}{2m}\sigma_i\sigma_j\partial_i\partial_j\chi(\underline{x}) + \left(E - \frac{Z\alpha}{\mid \underline{x} \mid}\right)\chi(\underline{x})$$

which leads to:–

$$E\chi(\underline{x}) = \frac{1}{2m}\nabla^2\chi(\underline{x}) + \frac{Z\alpha}{\mid \underline{x} \mid}\chi(\underline{x})$$

the Pauli Equation. We get from this that we now have in this model an effective coupling $Z\alpha$ and if $Z > \frac{1}{\alpha} = 137$ we have what is known as super-critical charge and a pertubation expansion in $Z\alpha$ is no longer valid. This calculation can be redone for two heavy colliding nuclei and we get that the effective coupling is now $Z\alpha$ where $Z = Z_1 + Z_2$ where Z_1 and Z_2 are the atomic numbers of the two nuclei .

So what happens experimentally in such a situation ? Now this question is interesting enough for experimental nuclear physicists to want to find out about. A great deal of work on this problem has been done in a series of experiments carried out at GESELLSCHAFT FÜR SCHWERIONENFORSCHUNG (GSI) in Darmstadt, Germany[16]. In the first of a series of experiments, beams of U^{238} were fired at Cm^{248} foil, thus giving a total $Z = 188$. Due to the large delta-e^- background that exists in the experiment only e^+ were looked for. (The delta-e^- background is due to the atomic electrons in the Cm^{248} foil being knocked out by the U^{238} beam. In low energy nuclear scattering experiments they leave squiggly 'delta' shaped tracks in the detectors.) They saw a peak over the background that isn't explicable by Rutherford theory. They suggested that this could be due to the formation of a meta-stable giant dinuclear system. In the second experiment they tested this suggestion by looking for the narrow e^+- peak for various values of supercritical Z, Z_{SC}. This is because such a dinuclear system would give a peak position that scales as Z^A[16], where $A = O(10)$. What they found was that the peak position stayed fixed, ruling out the original suggestion. By analysing the Doppler broadening expected for projectile-like, recoil-like and centre-of-mass-like emission sources and comparing to their results, they showed that the e^+-peak came from a centre-of-mass source. They then put forward a new suggestion that a new neutral particle could be the source. This however would give back-to-back e^+e^--pair emission. In the third experiment they also looked for e^- emissions though the removal of the delta-e^- background complicated things. They found back-to-back e^+e^- emission peaks at $380 \pm 15keV$ that didn't vary as Z_{SC} was varied. Using Monte Carlo generated data they showed that the peaks cannot be explained by internal effects of the colliding systems but could be explained by the decay of a neutral particle moving slowly in the centre-of-mass.

It has been suggested[17] that such an effect could be explained if at some value for the effective coupling $\alpha' = Z\alpha$ there is a change of phase (to " the strong coupling phase", see sections 1.5 and 2.3). The vacuum for this new phase is made up of e^+e^- bound pairs with negative energy. When the two nuclei with Z_{SC} collide a region of this new phase can exist in the extremely high electromagnetic fields present. This region of new phase is stationary in the centre-of-mass frame. With the moving away of the nuclei after the collision, the electromagnetic fields rapidly decrease to leave the strong coupling phase region existing as a quasi-stable vacuum which then decays to the weak coupling vacuum releasing mono-energetic back-to-back e^+e^- pairs, with energy independent of the particular value of Z_{SC}. This concept of a strong coupling vacuum made up of e^+e^- bound pairs is not new to physics, Cooper pairs in superconductivity have been around for some time. With these experimental results and possible theoretical schenario I hope to have justified why non-perturbative QED is worthwhile studying even though perturbative QED gives many experimental results to astounding accuracy. In the search for a strong coupling phase to explain the GSI experimental results we need to find a value for the coupling above which a theory with bare mass $m = 0$ in its Lagrangian can still generate a non-zero dynamical mass $\Sigma(p^2)$ in the fermion propagator (1.14) (chiral symetry is broken) and below which it cannot generate a non-zero dynamical mass, (chiral symetry is unbroken). Such a coupling will be called *a critical coupling.*

A number of techniques have been developed to look at the non-perturbative behaviour of gauge theories. The first we shall look at is :-

1.5 *Lattice Gauge Theory Approach* :

The idea behind lattice gauge theory is to take the path integral of the theory under consideration and put it on a discretised lattice of space-time points, with distance a between neighbouring points, calculate the theoretical predictions you are interested in and then decrease a by a small amount and continue the process. In this way predictions of results for the continum theory can be made. It wasn't until the advent of the Wilson plaquette[18], and great strides in the quality of algorithms and machine architecture to combat critical slowing down as $a \to 0$ that lattice approaches could hope to tackle gauge theories in a non-perturbative way. We shall now discuss the work and results of just one typical group that are involved in using lattice gauge theories to look at QED[19-22]. We shall follow the development of their work over a period of time in order to give a coherent discussion, they are headed by Dagotto, Kocić and Kogut (DKK). (We could of course have chosen anyone of a number of groups). In lattice calculations what we want to do is to put $m = 0$ in our path integral and calculate the value of the dynamical mass (described by the condensate $\langle\overline{\varphi}\varphi\rangle(m = 0)$). However because any lattice field we use will be finite in size, we cannot put $m = 0$. For any very small m and finite lattice, the numerical system will undergo tunnelling which will cause $\langle\overline{\varphi}\varphi\rangle(m)$ to vanish. What is done then is to take the value of the condensate for a series of small, decreasing m and extrapolate the value of the condensate to $\langle\overline{\varphi}\varphi\rangle(0)$ [19]. What they find for example in QED3 (with $1/N$ expansion [23]) is that there is a critical number of flavours $N_c = 3.5 \pm 0.5$ above which $\langle\overline{\varphi}\varphi\rangle(0) = 0$ and below which $\langle\overline{\varphi}\varphi\rangle(0) > 0$. For the situation where $N > N_c$ they find that $\beta = 1/a$ cannot be taken to infinity with a non-zero $\langle\overline{\varphi}\varphi\rangle(0)$. $\langle\overline{\varphi}\varphi\rangle(0)$ obeys a mean-field behaviour with respect to β and goes as $A(\beta_c - \beta)^{1/2}$ for $\beta \leq \beta_c \approx 0.24$[20] (c.f. magnetic susceptibility for $T < T_c$ using the Brillouin method, but with $T \sim 1/\beta$). This means that for $N > N_c$ as we try to go to the continuum theory, $\langle\overline{\varphi}\varphi\rangle(0)$ undergoes a simple phase change as β goes past β_c and becomes identically zero. This means that in QED3, using the $1/N$ expansion, we can write the dynamical mass as [19]:–

$$m_{dym} \approx e^2 N exp[-2\pi/\sqrt{N_c/N - 1}]$$

There is a correspondence that can be drawn between QED3 and quenched QED4[19] that allows us to write in quenched QED4:–

$$m_{dym} \approx \Lambda exp[-\pi/\sqrt{\alpha/\alpha_c - 1}]$$

(where Λ is the momentum cut off.) So N_c corresponds to $1/\alpha_c$ and we get a dynamical mass for coupling $> \alpha_c$. This looks all well and good but there are some fundamental problems with lattice gauge theories that put a big question mark over the whole approach.

Firstly the errors involved in extrapolating the condensate to the zero mass case are not really under control. For instance Taylor expanding the condensate in terms of the mass we get:–

$$\langle\overline{\varphi}\varphi\rangle(m) = \langle\overline{\varphi}\varphi\rangle(0) + m\langle\overline{\varphi}\varphi\rangle(0)' + \frac{m^2}{2}\langle\overline{\varphi}\varphi\rangle(0)'' + \ldots$$

If we extrapolate the condensate to its value at $m = 0$ linearly ($\langle\overline{\varphi}\varphi\rangle_{lin}(0)$) or quadratically ($\langle\overline{\varphi}\varphi\rangle_{quad}(0)$) then , as long as the values of non-zero m from which we extrapolate are small, our answers for $\langle\overline{\varphi}\varphi\rangle_{lin}(0)$ and $\langle\overline{\varphi}\varphi\rangle_{quad}(0)$ should be in good agreement with each other. What we find, however, is that they are not [19]. Factors of five difference are common, as are differences of sign! Moreover, the steepness of the slope of $\langle\overline{\varphi}\varphi\rangle(m)$ means that any small error in the numerical calculation, because of finite machine accuracy, will give an amplified error in $\langle\overline{\varphi}\varphi\rangle(0)$ (fig 1.1). As DKK say "predictions of chiral condensates are subject to considerable systematic uncertainties". This means that it might well be impossible to tell the difference between a small dynamical mass and zero dynamical mass. For instance, the usual values for m used are $O(10^{-2})$ and the smallest used are $O(10^{-4})$, remembering m cannot be too small otherwise numerical tunnelling will set in. If the systematic errors are $\approx 1\%$ then dynamical masses up to $O(10^{-6})$ can't be distinguished from zero. Now from analytic work on QED3 (see for example page 111 Ref.[23]) it is believed that for $N = 3.5$ $m_{dyn} = O(10^{-15})$ (with no sign of a critical point), well below the resolution possible on the lattice. Secondly on the plots of $\langle\overline{\varphi}\varphi\rangle$ vs.β that show the mean-field behaviour there is a residual 'tail' for $\beta > 0.24$ (fig 1.2). This is usually set to zero using the justification of systematic accuracy or a convenient parameterising of two or three points in the tail is done which shows that it goes to zero quickly. Whereas it is not clear whether the tail actually carries on to some very small asymptotic value as $\beta \to \infty$. Indeed in a work with $m \simeq O(10^{-3})$ [21] it is claimed that $\beta_c = 0.257 \pm 0.001$ ($\approx 10\%$ higher than before). Thirdly the finite size of the lattice can have a tremendous effect. Work on quenched QED4 [22] on a series of lattices with M^3 sites ($M = 8, 16, 24, 32, 48, 64, 80$) has shown that the extrapolated values of $\langle\overline{\varphi}\varphi\rangle(0)$ are lifted from zero (to machine accuracy) to some non-zero value and continue to increase with M (fig 1.3). Moreover the value of β_c increases as M increases with no asymptotic value behaviour for β_c being discernable (fig 1.4). So it could be that β_c and N_c are just artefacts of using some value of the lattice size M and that as $M \to \infty$ β_c and N_c both tend to infinity. In conclusion, the systematic errors are too big to see really small dynamical masses if they exist and instead calculations 'see' the finite size of the lattice.

1.6 *Continuum Approach* :

It was in an effort to liberate the theory from the problems associated with lattice gauge theories that a number of groups using a number of different methods have looked at the continuum (analytic) theory and attempted to make non-perturbative calculations. The various methods used by the groups involved in continuum calculations are all based upon the use of the Schwinger-Dyson equations (S-DE's), which are the fully non-perturbative Green's function equations for the gauge theory, and the Ward-Takahashi identities (W-TI's), the non-perturbative relationships between Green's functions. The S-DE's are a direct consequence of any given field dying off asymptotically in any direction in space-time, and the W-TI's come from the path integral (physics) being invariant under gauge transformations. In order to be able to see how these equations are generated we need to extend slightly our previous discussion on path integrals. We need to see how the path integral is related to the Green's functions of the theory and introduce the concept of the EFFECTIVE ACTION. For the sake of clarity we will consider a scalar theory (so we do not get bogged down in Grassmann algebra and the consideration of three fields $\varphi, \overline{\varphi}$ and A_μ) instead of looking at QED or QCD. ϕ^3 is an example of a scalar theory, though in what follows we will be more interested in the source term $J(x)\phi(x)$ than in the interactions, which we will largely ignore. We now look at the path integral:–

$$Z[J] = N \int \mathcal{D}\phi exp\left[i \int d^4x(\mathcal{L}(x) + J(x)\phi(x))\right].$$

When we discuss the Green's functions of the theory it will be easier to use a canonical version for the path integral. To obtain a canonical version we note that given the 'correct' normalisation factor N then in the non-interacting (free) theory:–

$$Z_0[J] = \langle 0 \mid T\{exp\left[i \int d^4x J(x)\phi(x)\right]\} \mid 0\rangle$$

where T is the time ordered product function and the zero on $Z_0[J]$ stands for the free theory. Together with the following identities:-

$$\langle 0 \mid T\{A[\phi]exp[i\int d^4x J(x)\phi(x)]\} \mid 0\rangle = A[-i\frac{\partial}{\partial J}]Z_0[J]$$

and

$$Z[J] = exp\left[i\int d^4x \mathcal{L}_I\left(-i\frac{\partial}{\partial J}\right)\right]Z_0[J].$$

(where $\mathcal{L}_I$ is the interaction Lagrangian) we get:–

$$Z[J] = \langle 0 \mid T\{exp\left[i\int d^4x\,(\mathcal{L}_I(\phi) + J(x)\phi(x))\right]\} \mid 0\rangle$$

which is the canonical version of the path integral. The Green's functions of the theory are defined in terms of the canonical time ordered bra and ket notation. This is a historical hold-over from quantum mechanics from where the physical meaning of the Green's functions in particle theory comes. The definition of the Green's functions are given in the Heisenburg Picture (H), in which the states stay constant with respect to time whilst the operators change in time:–

$$i\mathcal{G}_n(x_1,\ldots,x_n) := \langle_H 0 \mid T\{\phi(x_1)\ldots\phi(x_n)\} \mid 0\rangle_H$$

Field theory, however, is formulated in terms of the Dirac (Interaction) Picture (D), in which both states and operators vary. In order to redefine the Green's functions in terms of path integrals (Dirac picture) we make recourse to the Gell-Mann–Low formula:–

$$\begin{aligned}&\langle_H 0 \mid T\{\phi(x_1)\ldots\phi(x_n)\} \mid 0\rangle_H \\ &\quad = \frac{\langle_D 0 \mid T\{\phi_D(x_1)\ldots\phi_D(x_n)exp[i\int d^4y\mathcal{L}_I(\phi_D(y))]\} \mid 0\rangle_D}{\langle_D 0 \mid T\{exp[i\int d^4y\mathcal{L}_I(\phi_D(y))]\} \mid 0\rangle_D}.\end{aligned}$$

All our previous work on path integrals has been done in the Dirac picture and so we will now return to taking states to be implicitly in this picture. We can now see that :-

$$i\mathcal{G}_n(x_1,\ldots,x_n) \;=\; \frac{1}{Z[0]}(-i)^n \frac{\partial^n Z[J]}{\partial J(x_1)\ldots\partial J(x_n)}\Bigg|_{J(x_i)=\,0\forall i} \qquad (1.17)$$

This is not the end of the story as eq (1.17) has some interesting properties that we don't really want in a physically meaningful object. To see this we pull out the ϕ^3 interaction term in our scalar theory and look at the second order Green's function $i\mathcal{G}_2(x_1,x_2)$ (usually written $i\Delta(x_1-x_2)$)

$$\begin{aligned}
i\Delta(x_1-x_2) &= \langle 0 \mid T\{\phi(x_1)\phi(x_2)exp\left[i\int d^4y\mathcal{L}_I(\phi(y))\right]\} \mid 0\rangle \\
&= \langle 0 \mid T\{\phi(x_1)\phi(x_2)[1+\; i\lambda\int d^4y\phi^3(y) \\
&\quad -\; \lambda^2\int d^4yd^4z\phi^3(y)\phi^3(z)+\; O(\lambda^3)]\} \mid 0\rangle \\
&= i\Delta_0(x_1-x_2)+\; i\lambda\int d^4y\langle 0 \mid T\{\phi(x_1)\phi(x_2)\phi^3(y)\} \mid 0\rangle \\
&\quad -\; \lambda^2\int d^4yd^4z\langle 0 \mid T\{\phi(x_1)\phi(x_2)\phi^3(y)\phi^3(z)\} \mid 0\rangle +\; O(\lambda^3)
\end{aligned}$$

where $i\Delta_0(x_1,x_2) \;=\; \langle 0 \mid \; T\{\phi(x_1)\phi(x_2)\} \mid 0\rangle$, the Green's function in a non-interacting theory. By writing

$$\phi^3(a)=\lim_{w_i\to a}\phi(w_1)\phi(w_2)\phi(w_3)$$

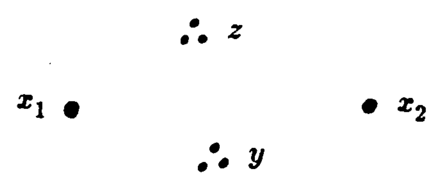
we are able to make sensible progress with the second and third terms on the RHS:-

$$\begin{aligned}
i\Delta(x_1-x_2) =\; & i\Delta_0(x_1-x_2) \\
&+ i\lambda\int d^4y\lim_{w_i\to y}\langle 0 \mid T\{\phi(x_1)\phi(x_2)\phi(w_1)\phi(w_2)\phi(w_3)\} \mid 0\rangle \\
&-\lambda^2\int d^4yd^4z\lim_{w_i\to y}\lim_{v_i\to z}\langle 0 \mid T\{\phi(x_1)\ldots\phi(v_3)\} \mid 0\rangle +\; O(\lambda^3).
\end{aligned}$$

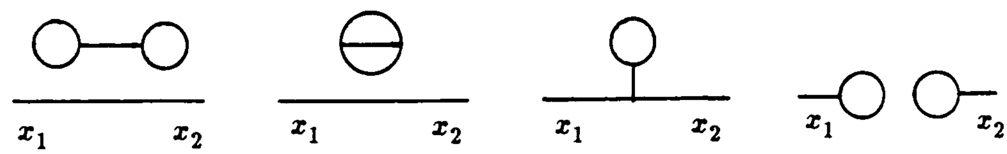
Wick's Theorem[24] tells us how to combine pairs of fields within a time ordered set of fields. For example:–

$$T\{\phi(x_1)\phi(x_2)\} = \,:\phi(x_1)\phi(x_2): + \,i\Delta_0(x_1-x_2)$$

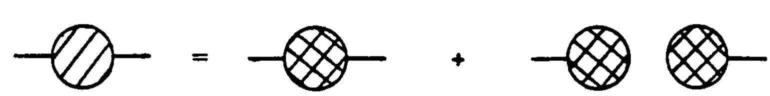
where $:\phi(x_1)\phi(x_2):$ is the normal ordered product (all creation operators are put to the left of the annihilation operators). The term of order λ on the RHS is of an odd order in ϕ and therefore no matter how we combine pairs of fields we always end up with at least one normal ordered field, thus this term must give zero as $\langle 0 \mid : \ldots : \mid 0 \rangle = 0$. The term of order λ^2 is even in ϕ and therefore it is possible to combine pairs of fields in such a way that we end up with some combinations with no normal ordered fields. Applying the bra and ket we get non-zero objects from the λ^2 term of the form $\Delta\ \Delta\ \Delta$. To get all the correct combinations of fields it is easier to work diagramatically, where we can also take the $w_i \to y$ and $v_i \to z$ limits. We start off with:–



and by exhaustively joining any two points with each other and letting $w_i \to y$ and $v_i \to z$ we have that the only possible diagrams are:–



So even if we can factorise out the vacuum bubble terms (which we can do as they exponentiate to give us:– the full set of diagrams = diagrams without vacuum bubble terms $\times \exp(\, i\phi)$. Where $i\phi$ = the set of all connected vacuum bubble terms, $\phi \in \Re$), we see pictorially our equation for the Green's function is not connected:–



where  represents the Green's function, and  represents connected diagram contributions. We can write the pictorial equation as:–

$$i\mathcal{G}(x_1, x_2) = i\mathcal{G}_c(x_1, x_2) + i\mathcal{G}_c(x_1) i\mathcal{G}_c(x_2) \tag{1.18}$$

where the subscript c denotes a connected diagram. The unconnected diagrams don't really make sense in a physical theory where we are trying to define a propagator (or for any n point Green's function where we want all the fields to interact, ie be connected). What we want is a definition in which we don't have these unconected diagrams. We are able to do this in a simple way if, instead of considering the path integral $Z[J]$, we work with the functional $W[J]$ defined by:–

$$W[J] = -i \log Z[J].$$

We can see that:–

$$\left.\frac{\partial W}{\partial J_1}\right|_{J=0} = \left.\frac{1}{i}\frac{1}{Z}\frac{\partial Z}{\partial J_1}\right|_{J=0} = i\mathcal{G}(x_1) = i\mathcal{G}_c(x_1)$$

as a one point function is connected. (We have used $J_1 = J(x_1)$). Indeed (taking $J = 0$ as implicit now):–

$$\begin{aligned}
\frac{1}{i}\frac{\partial^2 W}{\partial J_1 \partial J_2} &= \frac{1}{i^2}\frac{1}{Z}\frac{\partial^2 Z}{\partial J_1 \partial J_2} - \frac{1}{i^2}\left(\frac{1}{Z}\frac{\partial Z}{\partial J_1}\right)\left(\frac{1}{Z}\frac{\partial Z}{\partial J_2}\right) \\
&= i\mathcal{G}(x_1, x_2) - i\mathcal{G}_c(x_1) i\mathcal{G}_c(x_2) \\
&= i\mathcal{G}_c(x_1, x_2)
\end{aligned}$$

from eq (1.18). In general it can be seen that

$$i\mathcal{G}_c(x_1, \ldots, x_n) = \left(\frac{1}{i}\right)^{n-1} \frac{\partial^n W}{\partial J_1 \ldots \partial J_n}.$$

We also define the CLASSICAL ACTION $\phi_c(x)$ by the following differential equation:–

$$\phi_c(x) := \frac{\partial W}{\partial J}.$$

This then lets us define what will become a very useful object known as the EFFECTIVE ACTION, $\Gamma(\phi_c)$:–

$$W[J] = \Gamma(\phi_c) + \int d^4x J(x)\phi_c(x). \tag{1.19}$$

So we see that $\Gamma(\phi_c)$ is just the Legendre transform of $W[J]$.

As an aside we note that differentiating (1.19) with respect to J we see that $\Gamma(\phi_c)$ is independent of J, viz:–

$$\phi_c = \frac{\partial W}{\partial J} = \frac{\partial \Gamma}{\partial J} + \phi_c$$

From (1.19) then we have the useful identities:–

$$\frac{\partial W}{\partial J} = \phi_c \quad , \quad \frac{\partial \Gamma}{\partial \phi_c} = -J.$$

This means that

$$\begin{aligned} i\mathcal{G}_c(x_1,x_2) = \frac{\partial^2 W}{\partial J_1 \partial J_2} = \frac{\partial \phi_1}{\partial J_2} \quad &\text{and} \quad \frac{\partial^2 \Gamma}{\partial \phi_1 \partial \phi_2} = -\frac{\partial J_1}{\partial \phi_2} \\ \Rightarrow \qquad \int \frac{\partial^2 W}{\partial J_1 \partial J_2} \frac{\partial^2 \Gamma}{\partial \phi_2 \partial \phi_3} d^4x_2 &= -\,\delta(x_1 - x_3) \qquad (1.20) \\ \text{So} \quad \Gamma(x_1,x_2) = \frac{1}{i}\frac{\partial^2 \Gamma}{\partial \phi_1 \partial \phi_2} &= i\mathcal{G}_c^{-1}(x_1,x_2). \end{aligned}$$

Thus we see that $\Gamma(\phi)$, the Legendre transform of $W[J]$, may be related to the inverse of G_c as W is related to G_c. Proceeding to differentiate (1.20) with respect to J_4 we get:-

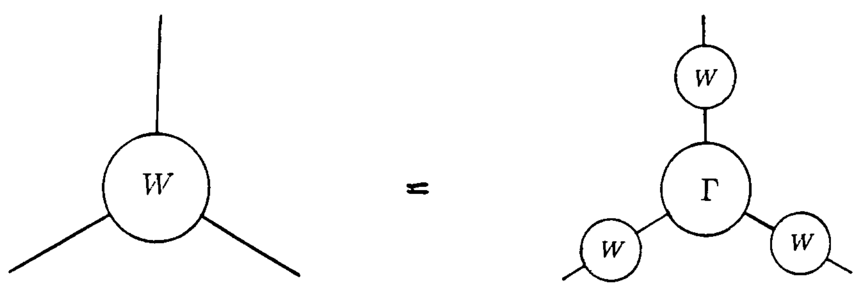
$$0 = \int d^4x_2 \frac{\partial^3 W}{\partial J_1 \partial J_2 \partial J_4} \frac{\partial^2 \Gamma}{\partial \phi_2 \partial \phi_3} + \int d^4x_2 d^4x_5 \frac{\partial^2 W}{\partial J_1 \partial J_2} \left(\frac{\partial^3 \Gamma}{\partial \phi_2 \partial \phi_3 \partial \phi_5} \right) \frac{\partial^2 W}{\partial J_5 \partial J_4} \tag{1.21}$$

as $\partial\phi_5/\partial J_4 = \partial^2 W/\partial J_5 \partial J_4$. Multiplying by $\int d^4x_3 \partial^2 W/\partial J_3 \partial J_4$ and using eq (1.20) ,(1.21) becomes:-

$$\frac{\partial^3 W}{\partial J_1 \partial J_6 \partial J_4} = \int d^4x_2 d^4x_5 d^4x_3 \frac{\partial^2 W}{\partial J_3 \partial J_6} \frac{\partial^2 W}{\partial J_1 \partial J_2} \frac{\partial^2 W}{\partial J_5 \partial J_4} \frac{\partial^3 \Gamma}{\partial \phi_2 \partial \phi_3 \partial \phi_5}$$

$$\Rightarrow i\mathcal{G}_c(x_1, x_6, x_4) = \int d^4x_2 d^4x_5 d^4x_3 \mathcal{G}_c(x_3, x_6) \mathcal{G}_c(x_1, x_2) \mathcal{G}_c(x_5, x_4) \frac{\partial^3 \Gamma}{\partial \phi_2 \partial \phi_3 \partial \phi_5}$$

which can be expressed diagramatically as:-



and

$$\Gamma(x_1, \ldots, x_n) := \left. \frac{1}{i} \frac{\partial^n \Gamma}{\partial \phi_1 \ldots \partial \phi_n} \right|_{\phi=0}$$

is the proper (or one particle irreducible) vertex, the inverse propagator for $n = 2$. With these new tools we can proceed to investigate how the Schwinger-Dyson equations and Ward-Takahashi identities are generated.

1.6a *Schwinger-Dyson Equations* [25] :

The S-DE's come about because in quantum theory all the fields are taken to die off towards infinity in any direction. This means that if we integrate a field over a space-time sphere with radius tending to infinity we get zero. Then using the divergence theorem we have that the derivative of the path integral with respect to a field of the theory is zero. ie :–

$$0 = \int \prod_{i=1}^{n} \mathcal{D}\phi_i \frac{\partial}{\partial \phi_j(x)} exp[i \int d^4z(\mathcal{L}(z) + \sum_{i=1}^{n} \phi_i X_i)]$$

where ϕ_i are the fields of the theory, X_i their source terms and ϕ_j the particular field we are considering. If we choose to consider QED then we have :–

$$\begin{aligned} 0 &= \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \frac{\partial}{\partial \phi_j} exp[i \int d^4z(\mathcal{L}(z) + J^\mu A_\mu + \overline{\eta}(z)\varphi(z) + \overline{\varphi}(z)\eta(z))] \\ &= \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \frac{\partial}{\partial \phi_j} exp[iS] \end{aligned}$$

with $\mathcal{L}(z) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\varphi}(i\not{D} - m)\varphi - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$ and $\phi_j = \{A_\mu, \varphi, \overline{\varphi}\}$. As it is always instructive to work through a simple example to demonstrate how a generic method works we will consider $\phi_j = A_\mu$ and see how this leads us to a field equation for QED. As the method does not depend upon any ordering or 'throwing away of small terms' (eg by taking e to be small) it is valid for all momenta, field and charge sizes. Thus we have a method that gives us non-perturbative field equations. We then start from :–

$$\begin{aligned} 0 &= \int \mathcal{D}A_\rho \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \frac{\partial}{\partial A_\mu(x)} exp[iS] \\ &= \int \mathcal{D}A_\rho \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \Big[\frac{\partial}{\partial A_\mu(x)} i \int d^4z(\mathcal{L}(z) + J^\nu A_\nu \\ &\qquad + \overline{\eta}(z)\varphi(z) + \overline{\varphi}(z)\eta(z)) \Big] exp[iS]. \end{aligned}$$

When considering the action of derivatives on $Z[J, \eta, \overline{\eta}]$ we see that

$$A_\mu \equiv \frac{\partial}{i\partial J_\mu} \, , \; \varphi \equiv \frac{\partial}{i\partial \overline{\eta}} \, , \; \overline{\varphi} \equiv - \frac{\partial}{i\partial \eta}.$$

Now

$$\frac{\partial}{\partial A_\mu(x)}\int d^4z J^\nu(z)A_\nu(z) = \int d^4z J^\mu(z)\delta(x-z) = J^\mu(x)$$

and $$\frac{\partial}{\partial A_\mu(x)}\int d^4z\mathcal{L}(z) = [\partial_x^2 g_{\mu\nu} - (1-\xi)\partial_{x\mu}\partial_{x\nu}]A^\nu(x) - e\overline{\varphi}(x)\gamma_\mu\varphi(x).$$

So $$0 = \left[(\partial^2 g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu)\left(\frac{-i\partial}{\partial J_\nu}\right) - e\left(\frac{i\partial}{\partial\eta}\right)\gamma_\mu\left(\frac{-i\partial}{\partial\overline{\eta}}\right) + J_\mu\right] Z[J,\eta,\overline{\eta}].$$

Also putting $Z = exp\ iW$ we find that the equation becomes :–

$$0 = \left[J_\mu + (\partial^2 g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu)\frac{\partial}{i\partial J_\nu} + e\left(\frac{\partial}{i\partial\eta}\right)\gamma_\mu\left(\frac{\partial}{i\partial\overline{\eta}}\right)\right] exp[iW]$$

$$\Rightarrow 0 = J_\mu + \left[\partial^2 g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\right]\frac{\partial W}{\partial J_\nu} - ie\frac{\partial}{\partial\eta}\left(\gamma_\mu\frac{\partial W}{\partial\overline{\eta}}\right) + e\frac{\partial W}{\partial\eta}\gamma_\mu\frac{\partial W}{\partial\overline{\eta}} \tag{1.22}$$

Using the preparatory work done earlier in this section, we can write the Legendre transform of W :–

$$W[J_\mu,\eta,\overline{\eta}] = \Gamma[A_\mu,\varphi,\overline{\varphi}] + \int d^4x(J^\mu A_\mu + \overline{\varphi}\eta + \overline{\eta}\varphi)$$

with

$$\begin{aligned} A_\mu &= \frac{\partial W}{\partial J^\mu}, \quad -\overline{\varphi} = \frac{\partial W}{\partial\eta}, \quad \varphi = \frac{\partial W}{\partial\overline{\eta}} \\ J^\mu &= -\frac{\partial\Gamma}{\partial A_\mu}, \quad \eta = -\frac{\partial\Gamma}{\partial\overline{\varphi}}, \quad \overline{\eta} = \frac{\partial\Gamma}{\partial\varphi} \end{aligned} \tag{1.23}$$

and the subsidiary results :–

$$\frac{\partial^2 W}{\partial\eta\partial\overline{\eta}} = \frac{\partial\varphi}{\partial\eta}, \quad \frac{\partial^2\Gamma}{\partial\varphi\partial\overline{\varphi}}, \quad -\frac{\partial\eta}{\partial\varphi}$$

Using these we find that equation (1.22) becomes :–

$$0 = -\frac{\partial\Gamma}{\partial A^\mu(x)} + [\partial^2 g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu]A^\nu(x) - e\overline{\varphi}(x)\gamma_\mu\varphi(x) - ie\gamma_\mu\left(\frac{\partial^2\Gamma}{\partial\overline{\varphi}(x)\partial\varphi(x)}\right)^{-1} \tag{1.24}$$

From the first term of the RHS of (1.24) we see that in order to obtain a field equation for say the inverse photon propagator $\partial^2\Gamma/(i\ \partial A_\mu(x)\partial A_\nu(y))\big|_{A=0=\varphi=\overline{\varphi}}$ we can differentiate eq (1.24) with respect to $A_\nu(y)$ and set $A = \varphi = \overline{\varphi} = 0$ we then have that :-

$$\frac{\partial^2\Gamma(0)}{\partial A^\mu(x)A^\nu(y)} = [\partial^2 g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu]\delta(x-y)$$
$$- ie\int d^4z_1 d^4z_2\gamma_\mu\left(\frac{\partial^2\Gamma(0)}{\partial\overline{\varphi}(x)\partial\varphi(z_1)}\right)^{-1}\frac{\partial^3\Gamma(0)}{\partial A^\nu(y)\partial\overline{\varphi}(z_1)\partial\varphi(z_2)}\left(\frac{\partial^2\Gamma(0)}{\partial\overline{\varphi}(z_2)\partial\varphi(x)}\right)^{-1}$$

where $\partial^n\Gamma(0) = \partial^n\Gamma\big|_{A=\varphi=\overline{\varphi}=0}$, the second term on the RHS comes about due to functional differentiation and the first term on the RHS is equal to the inverse of the non-interacting Green's function $i\Delta_0(x,y)$. Then using the generalised form for the proper vertex for vector and fermi fields:-

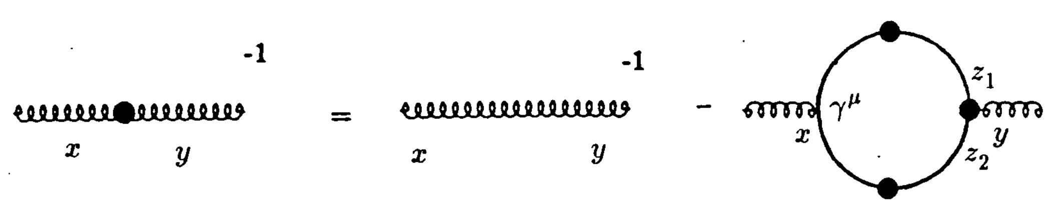
$$\Gamma^{(n,m)\mu_1,\dots\mu_n}(x_1,\dots,x_n;y_1,\dots,y_m)$$
$$= i\frac{\partial^{n+2m}\Gamma(0)}{\partial A_{\mu_1}(x_1)\dots\partial A_{\mu_n}(x_n)\partial\overline{\varphi}(y_1)\dots\partial\overline{\varphi}(y_m)\partial\varphi(y_1)\dots\partial\varphi(y_m)}$$

($n =$ no. bosons, $m =$ no. fermi pairs) we have the following equation:-

$$-i\Gamma^{(2,0)\mu,\nu}(x,y) = \left(i\Delta_0^{\mu\nu}(x,y)\right)^{-1}$$
$$-ie\gamma_\mu\left(-i\Gamma^{(0,1)}(x,z_1)\right)^{-1}\left(-i\Gamma^{(1,1)\nu}(y,z_1,z_2)\right)\left(-i\Gamma^{(0,1)}(z_2,x)\right)^{-1}$$

$$\Rightarrow \Gamma^{(2,0)\mu,\nu}(x,y) = \Delta_0^{\mu\nu-1}(x,y) + ie\gamma_\mu\Gamma^{(0,1)-1}(x,z_1)\Gamma^{(0,1)-1}(z_2,x)\Gamma^{(1,1)\nu}(y,z_1,z_2)$$

as $\Gamma^{(2,0)}$ and $\Gamma^{(0,1)}$ are the inverse propagators for the photon and fermion respectively we find that diagramatically the equation is :-



(where we have picked up the minus sign from the definition of the fermi loop). This equation is known as the S-DE for the photon propagator.

If, however, instead of A_μ we use $\phi_j = \varphi$ we obtain the S-DE for the fermion propagator, which diagramatically looks like :-

Differentiating with an arbitary number of fields we generate and infinite set of S-DE's for the Green's functions $\Gamma^{(n,m)}$. We note that from the diagrams for the propagators [2 ($= n + 2m$) point functions] that they are given in terms of Green's function up to and including 3 point functions. Indeed upon analysing the general S-DE's it can be seen that an N point Green's function is given in terms of functions up to and including $N + 2$ point Green's functions. This means that we have an infinite set of nested equations. Clearly this is going to present some difficulty in solving these equations. However before we think about how to tackle this problem we shall study another set of complimentary field equations that will help us in this task.

1.6b *Ward–Takahashi Identities* [26] :

As discussed in section 1.1, in a gauge theory, physical objects should be unchanged under a gauge transformation. So we can demand that our physical theory (eg QED) be gauge independent and see what restrictions this places upon the fields (and hence the Green's functions) of the theory. We go about this by studying the constraints that we must have for the path integral Z (Heisenburg vacuum) to be independent of a gauge transformation. As with when we investigated the S-DE's the technique we use involves no ordering or approximations using a smallness of e argument and so the equations obtained are non-perturbative. When the theory in question is QED the constraints are known as Ward–Takahashi Identities[27]. So for QED with a path integral :-

$$Z[J,\eta,\overline{\eta}] = \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}\overline{\varphi} e^{iS} \ , \quad S = \int \mathcal{L}_{eff} d^4 z$$

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\varphi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\varphi - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \overline{\eta}\varphi + \overline{\varphi}\eta$$

under the infinitesimal gauge transformation (where we only consider the transformation to $O(\Lambda)$)

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \ , \ \varphi \rightarrow \varphi - ie\Lambda\varphi \ , \ \overline{\varphi} \rightarrow \overline{\varphi} + ie\Lambda\overline{\varphi}$$

the first two terms of the effective Lagrangian are invariant under the transformation and so (assuming that $\mathcal{D}A'_\mu \equiv \mathcal{D}A_\mu$, $\mathcal{D}\varphi' \equiv \mathcal{D}\varphi$ and $\mathcal{D}\overline{\varphi}' \equiv \mathcal{D}\overline{\varphi}$) we find that :–

$$Z[J,\eta,\overline{\eta}] \xrightarrow{\Lambda} Z'[J,\eta,\overline{\eta}] = \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}\overline{\varphi} e^{iS+\ i\delta S}$$

where

$$\begin{aligned} \delta S &= \int d^4x \Big[-\frac{1}{\xi}(\partial^\mu A_\mu)(\partial^\nu \partial_\nu \Lambda) + J^\mu \partial_\mu \Lambda + \overline{\eta}(-ie\Lambda\varphi) + ie\Lambda\overline{\varphi}\eta \Big] \\ &= \int d^4x \Lambda(x) \Big[-\frac{1}{\xi}\partial^2(\partial^\mu A_\mu) - \partial_\mu J^\mu - ie\overline{\eta}\varphi + ie\overline{\varphi}\eta \Big] \end{aligned}$$

Therefore:–

$$\begin{aligned} Z'[J,\eta,\overline{\eta}] = \int \mathcal{D}A_\mu \mathcal{D}\varphi \mathcal{D}\overline{\varphi} e^{iS} \Big[1 + i \int d^4x \Lambda(x) \{ -\frac{1}{\xi}\partial^2(\partial^\mu A_\mu) - \partial_\mu J^\mu \\ - ie\overline{\eta}\varphi + ie\overline{\varphi}\eta \} \Big] \end{aligned}$$

and as we demand gauge invariance of the theory this means that $Z' = Z$. Thence we have that :–

$$0 = i \int d^4x \Lambda(x) \Big[-\frac{1}{\xi}\partial^2(\partial^\mu A_\mu) - \partial_\mu J^\mu - ie\overline{\eta}\varphi + ie\overline{\varphi}\eta \Big] Z[J,\eta,\overline{\eta}]$$

This must be true for all gauge transformations $\Lambda(x)$ and as such $\Lambda(x)$ is an arbitary function, giving us that the rest of the integrand is identically equal to zero :–

$$
\begin{aligned}
0 &= \left[-\frac{1}{\xi}\partial^2(\partial^\mu A_\mu) - \partial_\mu J^\mu - ie\overline{\eta}\varphi + ie\overline{\varphi}\eta \right] Z[J,\eta,\overline{\eta}] \\
&= \left[\frac{i}{\xi}\partial^2(\partial^\mu \frac{\partial}{\partial J^\mu} - \partial_\mu J^\mu - e\overline{\eta}\frac{\partial}{\partial \overline{\eta}} + e\eta\frac{\partial}{\partial \eta} \right] Z[J,\eta,\overline{\eta}]
\end{aligned}
$$

where we have used $A_\mu \equiv \frac{\partial}{i\partial J_\mu}$, $\varphi \equiv \frac{\partial}{i\partial \overline{\eta}}$, $\overline{\varphi} \equiv -\frac{\partial}{i\partial \eta}$. Substituting in $Z = exp\ iW$ we have that :–

$$
0 = -\frac{1}{\xi}\partial^2(\partial^\mu \frac{\partial W}{\partial J^\mu}) - \partial_\mu J^\mu - ie\overline{\eta}\frac{\partial W}{\partial \overline{\eta}} + ie\eta\frac{\partial W}{\partial \eta}
$$

with the Legendre transformation (1.23) and its associated identities we immediately get the result :–

$$
0 = -\frac{1}{\xi}\partial^2\partial^\mu A_\mu(x) + \partial_\mu \frac{\partial \Gamma}{\partial A_\mu(x)} + ie\varphi(x)\frac{\partial \Gamma}{\partial \varphi(x)} - ie\overline{\varphi}(x)\frac{\partial \Gamma}{\partial \overline{\varphi}(x)} \tag{1.25}
$$

which is the generating functional equation for the WHOLE class of W-TI's for QED.

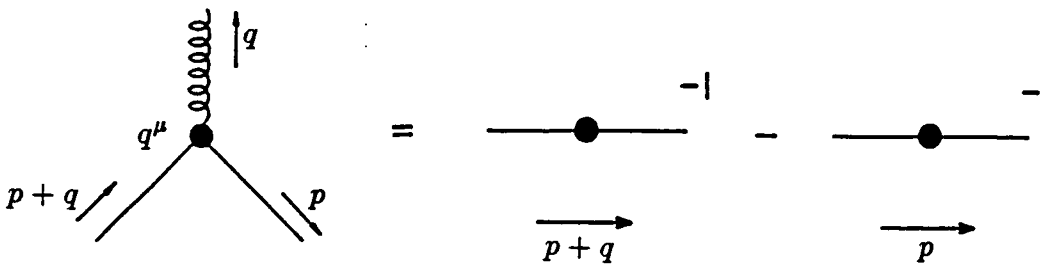
We shall continue, as an example, by looking at the constraints we obtain when we differentiate with respect to $\overline{\varphi}(x_1)$ and $\varphi(y_1)$ (We, of course, evaluate this at the point $\overline{\varphi} = \varphi = A_\mu = 0$). Equation (1.25) becomes

$$
-\partial_{x\mu}\frac{\partial^3\Gamma(0)}{\partial \overline{\varphi}(x_1)\partial \varphi(y_1)\partial A_\mu(x)} = ie\delta(x-x_1)\frac{\partial^2\Gamma(0)}{\partial \overline{\varphi}(x)\partial \varphi(y_1)} - ie\delta(x-y_1)\frac{\partial^2\Gamma(0)}{\partial \overline{\varphi}(x_1)\partial \varphi(x)}
$$

Taking the Fourier transform of this we have :–

$$
q^\mu \Gamma_\mu(p,q,p+q) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{1.26}
$$

where $S_F^{-1}(p+q)$ is the inverse fermion propagator, q_μ the 4-momentum on the incoming photon and p_μ the 4-momentum on an incoming fermion. Diagramatically this is written as :–

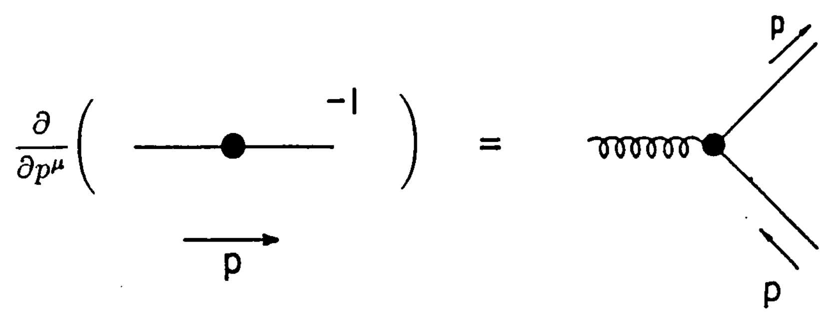


This is the W-TI for the 3 point vertex Γ_μ. It relates the 3 point vertex to the 2 point fermion propagator. An infinite set of W-TI's can be generated from (1.25) relating N point functions to lower point functions $M(< N)$

Taking the $q_\mu \to 0$ limit of (1.26) we obtain the older identity known as the Differential Ward Identity (∂ W-I) :-

$$\frac{\partial S_F^{-1}}{\partial p^\mu} = \Gamma_\mu(p, 0, p)$$

which diagramatically is :-



This implies that the insertion of a photon, with zero momentum, into a fermion propagator is equivalant to differentiation of the inverse propagator. It may seem from this that the W-TI's help with our problem of solving the infinitely nested S-DE's, as using the W-TI's for $N+2$ and $N+1$ point functions we can rewrite the S-DE for an N point function in terms of functions upto and including N and no higher (whereas before we had to go upto $N+2$ point functions). Thus starting from the S-DE's for the 2 point functions we can iteratively solve for each N up to an arbitary value. However, a slightly more detailed look at (1.26) reveals that it does not tell us everything about Γ_μ. This is because Γ_μ is contracted with q_μ which acts as a Killing vector on Γ_μ masking all information about parts Γ_T of Γ that are transverse to q_μ and allowing us access only to the parts Γ_L that are longitudinal to q_μ

We now look at two approaches that are commonly adopted in order to attempt to solve the S-DE's.

1.7 *Spectral Representation* [28–34] :

The first, and possibly theoretically more powerful, method is to write the propagators of the theory in terms of spectral representations. In order to explain what we mean by this we shall look at QCD and its confinement, an area where this method has had some notable theoretical success. From experimental observation and theoretical bias it is believed that quarks (and hence gluons) are not free observable objects in QCD (they don't appear in the S matrix) and as such can be compared to longitudinal and scalar photons in QED. In QED the common method to work with this is to use the Gupta–Bleuler method (see section 1.3) which involves the use of an indefinite metric and a condition on the physical states. On a detailed examination of the BRS transformation properties of QCD (a description of which can be found in Refs. [29] and [30]) we find that the generator of the BRS transformation Q_B (known as the BRS charge) is nilpotent (ie $Q_B^2 = 0$). This means that the irreducible representations of the BRS transformation vector space are singlet and doublet subspaces only. Q_B is also Hermitian and gives the BRS transformation of an arbitary field ϕ in the following way:- $\delta\phi = i[Q_B, \phi]_\pm$, where $\pm$ indicates commutator for non-ghost field ϕ and anti-commutator for ghost field ϕ. Within the BRS transformation set there is also a further 'local' conserved charge, Q_C, that satisfies $i[Q_C, \phi] = N\phi$, N =ghost number of ϕ . Q_C generates a scale (not a phase) transformation and is Hermitian. Q_B and Q_C obey $i[Q_C, Q_B] = Q_B$ (i) (which together with $Q_B^2 = 0$ (ii) defines the BRS Algebra). The BRS vector space, $\mathcal{V}$, can then be decomposed into two orthogonal irreducible reps. namely $\mathcal{V} = \mathcal{V}_S \bigoplus \mathcal{V}_D$ where:–

$\mathcal{V}_S$ is the BRS singlet space. $\forall\ |s\rangle \in \mathcal{V}_S$, $Q_B|s\rangle = 0 = iQ_C|s\rangle$ and $\langle s|s\rangle > 0$, $\mathcal{V}_S$ has a (+)ve definite metric.

$\mathcal{V}_D$ is the BRS doublet space and $\forall\ |w\rangle \in \mathcal{V}_D$ $\langle w|w\rangle = 0$ [28],$\mathcal{V}_D$ has zero norm. $\mathcal{V}_D$ can be further decomposed:–

$\mathcal{V}_D = \mathcal{V}_p \bigoplus \mathcal{V}_d$ ($\mathcal{V}_p$ =parent subspace, $\mathcal{V}_d$ =daughter subspace) and $\forall\ |p\rangle \in \mathcal{V}_p$ $\exists\ |d\rangle \in \mathcal{V}_d$ such that $\langle p|d\rangle \neq 0$. Also $\exists 1-1$ correspondence between $\mathcal{V}_p$ and $\mathcal{V}_d$ given by $Q_B|p\rangle = |d\rangle$ such a pair of states $\{|p_1\rangle, |d_1\rangle\}$ is called a doublet. If $iQ_C|p_1\rangle = N|p_1\rangle$ then condition (i) of the BRS Algebra $\Rightarrow iQ_C|d_1\rangle = (N+1)|d_1\rangle$.

The condition on the physical states is (like in the Gupta-Bleuler method) $Q_B|phys\rangle = 0$ [28] and we can easily define the physical subspace, $\mathcal{V}_{phys}$ of $\mathcal{V}$ to be $\{|f\rangle : |f\rangle \in \mathcal{V} \text{and} Q_B|f\rangle = 0\}$. Then from the stated properties of the subspaces $\mathcal{V}_{phys} = \mathcal{V}_S \bigoplus \mathcal{V}_d$ and then $|x\rangle$, $|x'\rangle \in \mathcal{V}_{phys}$ have the form $|x\rangle = |d\rangle + |s\rangle$ and $|x'\rangle = |d'\rangle + |s'\rangle$. From the zero norm of $\mathcal{V}_d$ and the orthogonality of $\mathcal{V}_S$and $\mathcal{V}_D$ we get:–

$$\langle x|x'\rangle = \langle s|s'\rangle = \langle x|P(\mathcal{V}_S)|x'\rangle$$

where $P(\mathcal{V}_S)$ is the projection operator onto the singlet subspace. Thus the physically observable states (those that enter the S matrix) are those in $\mathcal{V}_S$, hence we can consider BRS doublet states as CONFINED. This is because non-confined, asymptotic fields are realised in perturbation (S matrix) theory and confined, non-asymptotic fields are not. We will use this as our definition of confined/unconfined gluons. The gluon propagator, $iD^{ab}_{\mu\nu}(k)$, is, of course, the Fourier transform of $\langle 0|T\{A^a_\mu(x)A^b_\nu(y)\}|0\rangle$, which in the Landau gauge is transverse and has the form

$iD^{ab}_{\mu\nu}(k) = i\delta^{ab}(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})D(k^2)$ where $D(k^2)$ is normalised at a point $\mu^2 > 0$ such that $k^2 D(k^2)\big|_{k^2=\mu^2} = 1$. Now for our own seemingly perverse reasons (that will become clear later) we wish to define a similar object using the projection operator $P(\mathcal{V}_S)$. We want to conserve the transversality of the resultant object, so we can't use the obvious $\langle 0|T\{A^a_\mu(x)P(\mathcal{V}_S)A^b_\nu(y)\}|0\rangle = \langle 0|T\{A^{pa}_\mu(x)A^{pb}_\nu(y)\}|0\rangle$, A^{pa} is A^a projected onto $\mathcal{V}_S$, which is no longer necessarily transverse. Hence we define $A^{pa}_{\mu\nu} = \partial_\mu A^{pa}_\nu - \partial_\nu A^{pa}_\mu$ and look at $\langle 0|T\{A^{pa}_{\mu\nu}(x)A^{pb}_{\rho\lambda}(y)\}|0\rangle$ (which is transverse in the sense that if either of the pairs $\mu\nu$ or $\rho\lambda$ are contracted with k's we get zero). The Fourier transform is now:–

$$iG^{pab}_{\mu\nu,\rho\lambda}(k) = i\delta^{ab}(k_\mu k_\rho g_{\nu\lambda} - k_\mu k_\lambda g_{\nu\rho} - k_\nu k_\rho g_{\mu\lambda} + k_\nu k_\lambda g_{\mu\rho})D^p(k^2)$$

where $D^p(k^2)$ is normalised in the same way as $D(k^2)$[32]. (So if $D^p(k^2) \neq 0$ gluons are NOT confined, from our earlier definition). We also define the dimensionless variable $R(k^2/\mu^2, g) = k^2 D(k^2, \mu^2, g)$ where $g(k^2/\mu^2, g_0)$ is the effective coupling and is such that $g(1, g_0) = g_0 =$ the bare coupling that occurs in the Lagrangian. g is a Renormalisation Group invariant and obeys the following Renormalisation Group Equation(R-GE) (see section 2.5a):–

$$\mathcal{D}g = 0$$

where

$$\mathcal{D} = u\frac{\partial}{\partial u} - \beta(g_0)\frac{\partial}{\partial g_0^2}$$

and

$$u = k^2/\mu^2 \ , \ \beta(g_0) = u\frac{\partial g_0^2}{\partial u} = g_0^4(\beta_0 + g_0^2\beta_1 + \ldots)$$

which has the solution:–

$$u = exp\left[\int_{g_0^2}^{g^2} dx \beta^{-1}(x)\right]$$

(for small g_0 and g). Now QCD is known experimentally to be asymptotically free, which means that $g \rightarrow 0$ as $k^2 \rightarrow \infty$ (Minkowski space-time). For this to happen we must have that $\beta_0 < 0$ (in order to make $g = 0$ a U.V. fixed point). $\beta_0 = -(11 - \frac{2}{3}N_f)/(16\pi^2)$ in all renormalisation schemes and gauges and so for $\beta_0 < 0$ we must have the well known result $N_f \leq 16$, where N_f is the number of quark flavours. Because of asymptotic freedom we have for large momentum

$$\lim_{k^2 \rightarrow +\infty} \{(\log(k^2/\mu^2)\; g^2(k^2/\mu^2, g_0)\} = |\beta_0^{-1}| \tag{1.27}$$

$D(k^2)$ also obeys a R-GE, namely:–

$$(\mathcal{D} + \gamma(g_0))\, D(k^2) = 0$$

where the integral of $\gamma(g_0)$ is the anomalous dimension. $\gamma(g_0) = g_0^2(\gamma_1 + g_0^2\gamma_2 + \ldots)$ and γ_1 is dependant on the gauge (but independent of the renormalisation scheme). In the Landau gauge:–

$$\gamma_1 = -\frac{1}{16\pi^2}(13 - \frac{4}{3}N_f).$$

Solving the R-GE and substituting for R we get that

$$R(k^2/\mu^2, g) = R(1, g) exp\left[\int_{g_0^2}^{g^2} dx\gamma(x)\beta^{-1}(x)\right]$$

(for small g_0 and g). $R(1, g) = 1$ from normalisation. Now the integrand is singular at $x = 0$ and so we split it into two parts using the expansions for $\gamma(x)$ and $\beta(x)$ that we have:–

$$\gamma(x)\beta^{-1}(x) = \gamma_1\beta_0^{-1}x^{-1} + \tau(x)$$

where $\tau(x)$ is integrable at $x = 0$. Then our equation for $R(k^2/\mu^2, g)$ becomes:–

$$R(k^2/\mu^2, g) = \left(\frac{g^2}{g_0^2}\right)^{\frac{\gamma_1}{\beta_0}} exp\left[\int_{g_0^2}^{g^2} dx\tau(x)\right].$$

Then using (1.27) we get

$$R(k^2/\mu^2, g)_{k^2 \to +\infty} \sim C_V \left(\log k^2/\mu^2\right)^{-\gamma_1/\beta_0} \tag{1.28}$$

where

$$C_V = \left(g_0^2 |\beta_0|\right)^{-\gamma_1/\beta_0} exp\left[\int_{g_0^2}^{g^2} dx \tau(x)\right] > 0$$

and hence:–

$$D(k^2, \mu^2, g_0)_{k^2 \to +\infty} \sim C_V k^{-2} \left(\log k^2/\mu^2\right)^{-\gamma_1/\beta_0}. \tag{1.29}$$

In order for us to check whether we can write the propagator in terms of a spectral representation we need to study the asymptotic behaviour of the propagator along the cut on the (+)ve real axis (the cut is due to the physical thresholds). Because in this case there is no cut along the (−)ve real axis, in order to write the propagator in terms of a spectral function we need to show that $\lim_{k^2 \to \infty} D =$ finite, this is called the SUGAWARA–KANAZAWA conditions[33]. Indeed from (1.29) we can see that $\lim_{k^2 \to \infty} D = 0$ which is the Sugawara–Kanazawa condition for D to be written in the form of an unsubtracted KALLEN–LEHMANN SPECTRAL REPRESENTATION[33], viz:–

$$D(k^2, \mu^2, g_0) = \int_0^\infty dk'^2 \, \frac{\rho(k'^2)}{k'^2 - k^2} \tag{1.30}$$

where $\rho(k^2) = ImD(k^2)/\pi$. Indeed upon inspection of (1.28) we can see that for $\gamma_1/\beta_0 > 0$ (ie $N_f \leq 9$) $\lim_{k^2 \to \infty} R = 0$ and so R can be written as an unsubtracted spectral function in its own right giving:–

$$R(k^2/\mu^2, g) = \int_0^\infty dk'^2 \, \frac{k'^2 \rho(k'^2)}{k'^2 - k^2}. \tag{1.31}$$

Substituting (1.30) into (1.31) using the original definition of R in terms of D we have that:–

$$\int_0^\infty dk'^2 \, \frac{k^2 \rho(k'^2)}{k'^2 - k^2} = \int_0^\infty dk'^2 \, \frac{k'^2 \rho(k'^2)}{k'^2 - k^2}$$

$$\Rightarrow \quad \int_0^\infty dk'^2 \ \rho(k'^2) = 0 \tag{1.32}$$

the super convergence relation for $\rho(k'^2)$.

Now D (and hence ρ) is defined on an indefinite metric and so (1.32) doesn't help us much. However there is a long proof, using the Lehmann-Symanzik-Zimmerman (LSZ) reduction formula [31], that tells us that D and D^p obey the same R-GE and so the results we have above hold true for their projected counterparts:–

$$\begin{aligned} D^p(k^2, \mu^2, g_0) = \int_0^\infty dk'^2 \frac{\rho(k'^2)}{k'^2 - k^2} \qquad \forall N_f \\ \text{and} \int_0^\infty dk'^2 \rho^p(k'^2) = 0 \qquad N_f \leq 9 \end{aligned} \tag{1.33}$$

(1.33) does now help us as D^p (and hence ρ^p) are defined with (+)ve semi-definite metric, thus $\rho^p(k^2) \geq 0$. Combining this result with (1.33) it is found that we must have $\rho^p(k^2) = 0 \ \forall k^2$ and $N_f \leq 9$ giving $D^p(k^2, \mu^2, g_0) = 0 \ \forall N_f \leq 9$, but from our definition of D^p and our work on the BRS singlet/doublet subspaces we immediately see that for $0 \leq N_f \leq 9$ we have confined gluons. (Lower limit as $N_f < 0 \Rightarrow\Leftarrow$ physics). This is a sufficient but not necessary condition. However, the work of Nishijima (see for example the last sections in [30] and [31]) has shown it is highly probable that it is in fact a necessary condition. Using the reasonable assumption that gluon confinement is equivalent to quark confinement, we have confinement for $0 \leq N_f \leq 9$ (which is a much tighter bound on N_f than that of $0 \leq N_f \leq 16$ from asymptotic freedom).

Whilst this theoretical result seems to bode well for the spectral representation technique, attempts to compute $\rho(k^2)$ (and hence D) non-perturbatively[34] have run into enormous problems. Indeed it seems that this technique is numerically intractable.

In order to attempt a numerically tractable technique we turn to our third and final method (we devote the rest of this thesis to this method):–

1.8 *The Functional Substitution method* [35] :

In this method the infinite set of nested S-DE's are truncated down to the two point Green's functions by employing ansatze for the relevant three point Green's functions. The tensor properties of the two point Green's functions are known from the spin of the fields they represent and thus they can be written as general functions of momentum squared, multiplied by tensor structures. So, for instance, in QED the only two point Green's functions are the fermion and photon propagators, $S_F(p)$ and $\Delta_{\mu\nu}(p)$ respectively, which have the following forms:–

$$\begin{aligned} S_F(p) &\sim \frac{F(p^2)}{\not p - \Sigma(p^2)} \\ \Delta_{\mu\nu}(p) &\sim \frac{\mathcal{G}(p^2)}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^4} \end{aligned} \tag{1.34}$$

$\mathcal{F}(p^2) =$ the fermion wavefunction, $\Sigma(p^2) =$ the fermion mass function and $\mathcal{G}(p^2) =$ the photon function.

Note that there is no function with the covariant gauge parameter, ξ, part, this comes directly from the Ward-Takahashi Identity for the photon propagator:–

$$p^\mu \left(\Delta(p)\right)^{-1}_{\mu\nu} = \frac{p_\nu p^2}{\xi}$$

from (1.34):–

$$\left(\Delta(p)\right)^{-1}_{\mu\nu} = \frac{p^2}{\mathcal{G}(p^2)}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{1}{\xi}p_\mu p_\nu$$

hence the function in front of ξ in $\Delta(p)_{\mu\nu}$ is identically equal to 1.

In order for the S-DE's for the two point Green's functions and the ansatze for the three point Green's functions to form a closed set (ie. in any way to be potentially solvable) the ansatze can only be made up of the functions $\mathcal{F}(p^2)$, $\Sigma(p^2)$, $\mathcal{G}(p^2)$ and various tensors. (In QED there is only one three point Green's function, the fermion-fermion-photon function). We then substitute (1.34) and the ansatze into the S-DE's for the two point Green's functions and solve for the functions, using some numerical method.

The main technical problems with this method are:–

(1) Choosing a sensible ansatz (which will be dealt with in the next chapter)

(2) The art of solving the equations numerically (which we will look at in the subsequent chapters).

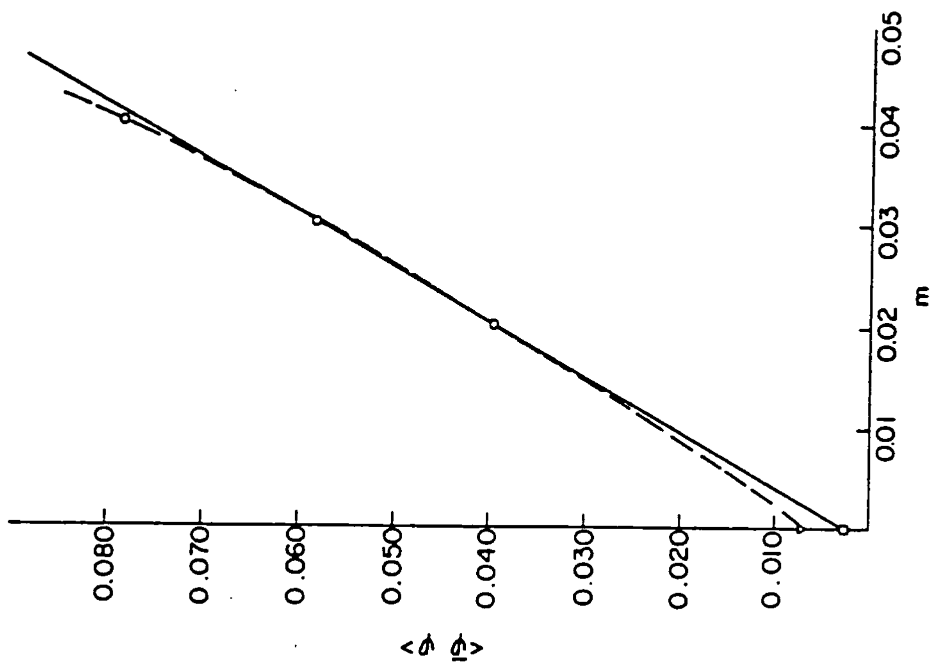


Fig. 1.1 Linear and quadratic fits to $\langle\varphi\varphi\rangle(m)$ vs. m, from Ref. 19.

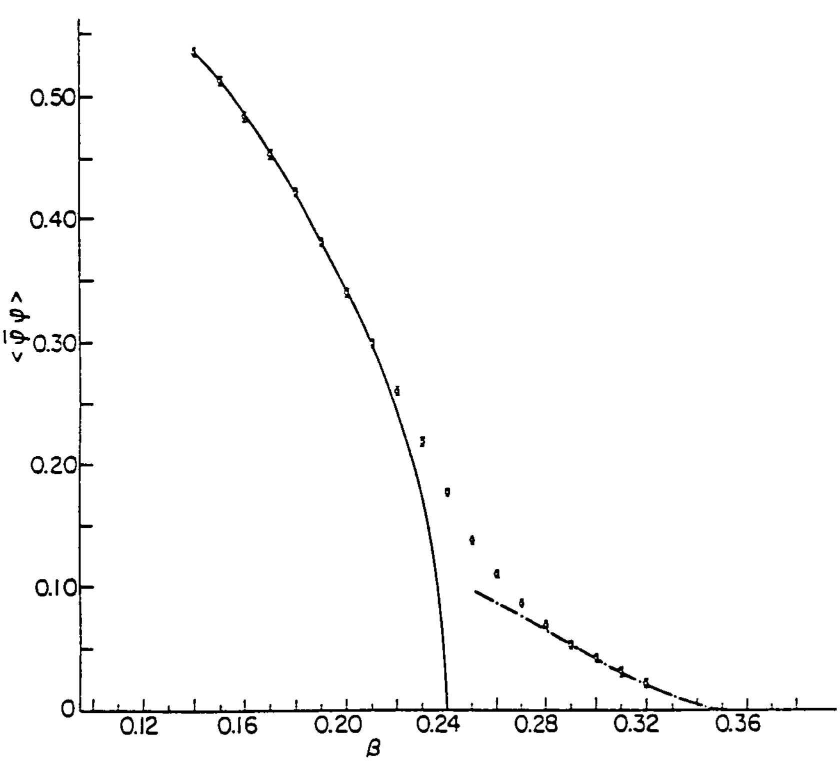


Fig. 1.2 Plot of $\langle\varphi\varphi\rangle$ (0) vs. β showing the tail on the mean field behaviour, from Ref. 20.

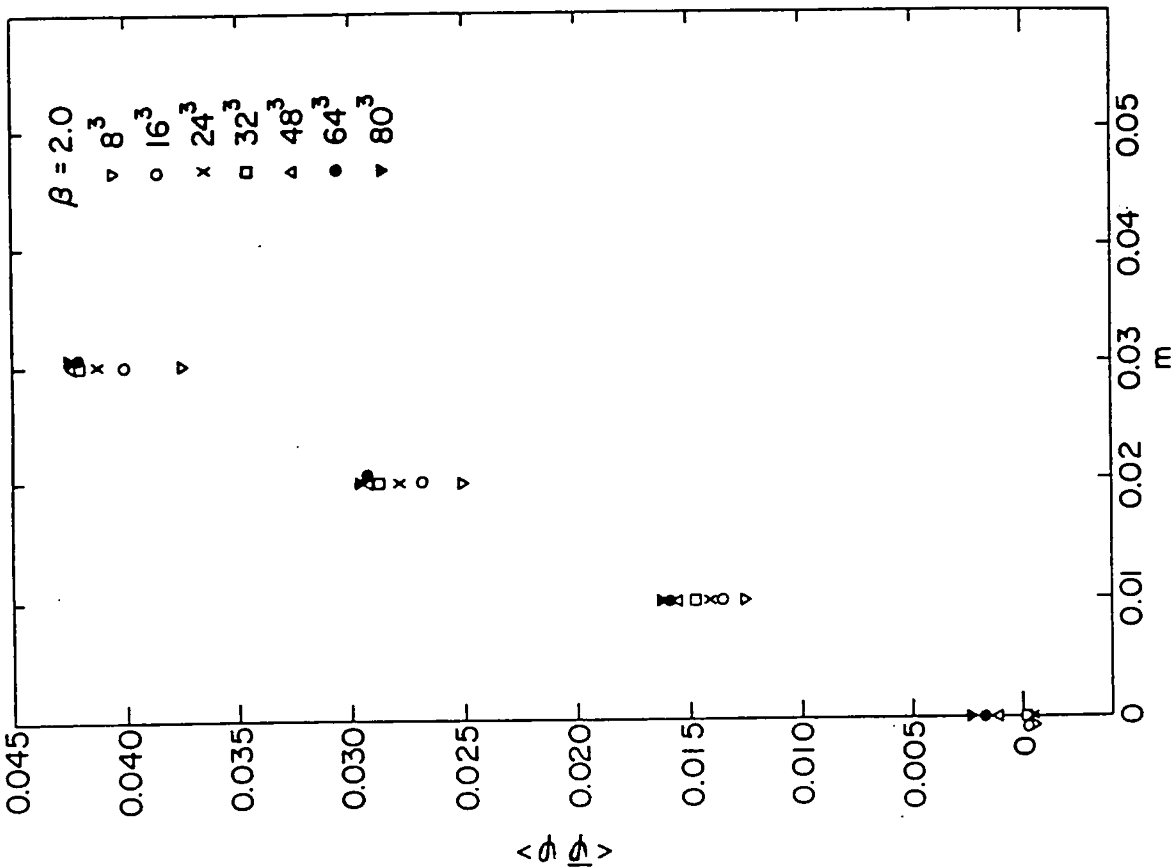


Fig. 1.3 Plot of $\langle\varphi\varphi\rangle(m)$ vs. m for varying lattice sizes, which shows the lifting of $\langle\varphi\varphi\rangle(0)$ due to finite lattice effects, from Ref. 22.

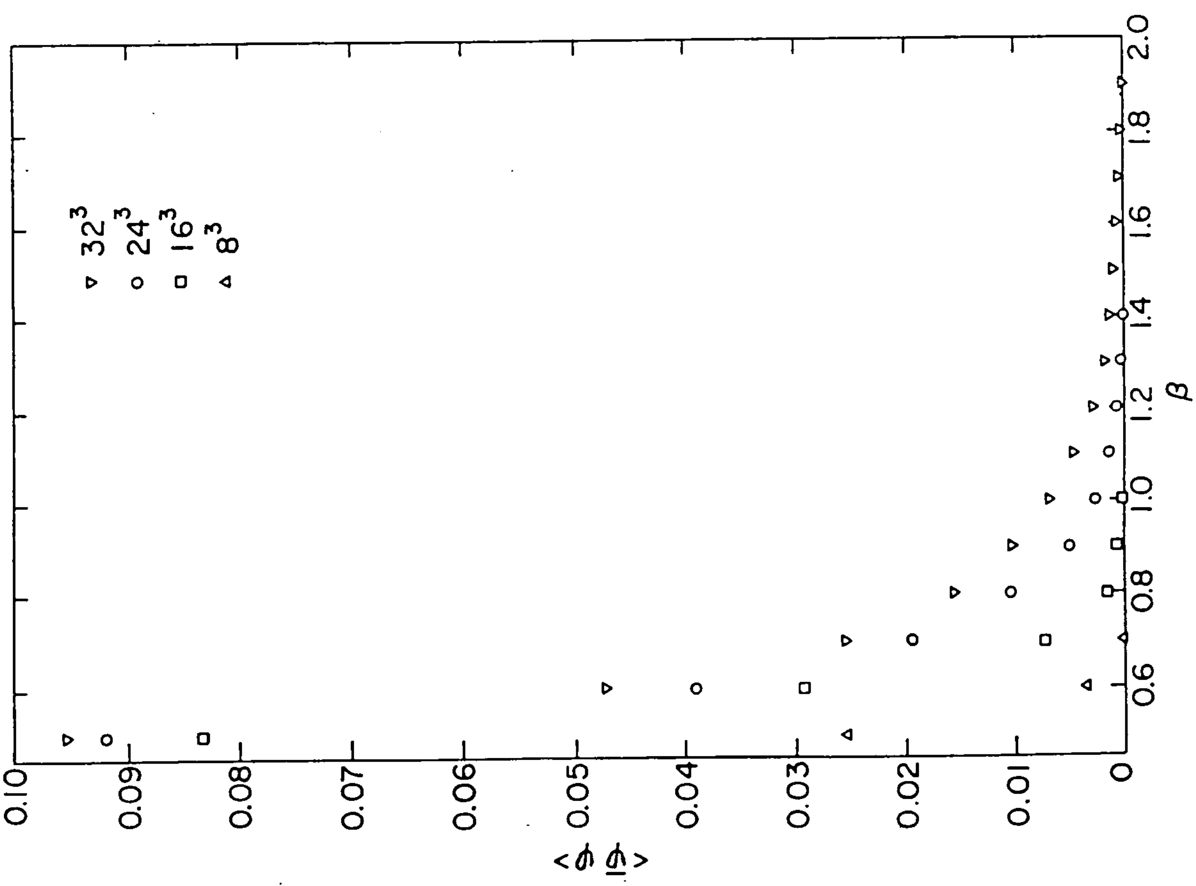


Fig. 1.4 The behaviour of $\langle\varphi\varphi\rangle(m)$ vs. β with varying lattice sizes, from Ref. 22.

CHAPTER TWO

THE ANSATZ

2.1 *Introduction* :

We begin this chapter by studying the various ansatze that are presently popular. We shall then make use of the multiplicative renormalisation (M.R.) of the theory (QED) to guide us in proposing a new ansatz for the three point vertex. In the functional substitution method, as the full set of S-DE's are truncated to just the two point Green's functions plus an ansatz for the three point function a full understanding of the consequences of the chosen ansatz is clearly of extreme importance as it is THE major input into the theory and therefore the point at which the study can be most contaminated.

2.2 *Conventions* :

Before we undertake calculations we need to state our conventions. The truncated S-DE's we are looking at are shown diagramatically in fig. 2.1, and are written in integral form as:–

$$\begin{aligned} \Delta_{\mu\nu}^{-1}(p) &= \Delta_{\mu\nu}^{0\,-1}(p) + \frac{e^2 N_f}{(2\pi)^4} \int_M d^4k\,\gamma^\mu\, S_F(k)\,\Gamma^\nu(k,p)\, S_F(q) \\ S_F^{-1}(p) &= S_F^{0\,-1}(p) - \frac{e^2}{(2\pi)^4} \int_M d^4k\,\gamma^\mu\, S_F(k)\,\Gamma^\nu(k,p)\,\Delta_{\mu\nu}(q) \end{aligned} \tag{2.1}$$

where the integral $\int_M$ has Minkowski measure, $q = k - p$ and the superscript 0 denotes the bare quantities. S_F is the full fermion propagator and $\Delta_{\mu\nu}$ the full photon propagator. They are given by:–

$$S_F(p) = i\left(\frac{F(p^2)}{\not{p} - \Sigma(p^2)}\right)$$

$$\Delta^{\mu\nu}(q) = -i\left[\frac{\mathcal{G}(q^2)}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi\frac{q^\mu q^\nu}{q^4}\right]$$

and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Γ^ν is of course the fermion-boson three point vertex and is given by the input ansatz. N_f is the number of fermion flavours. When we use these equations to calculate objects we shall make a Wick rotation to Euclidean space-time. When we do this we let $x_0 \rightarrow +ix_0$ and $w_0 \rightarrow +iw_0$, ($w = k$ or p) then $\int_M d^4x \rightarrow +i\int_E d^4x$. In order to be able to make a Wick rotation to link Euclidean and Minkowski space-times there cannot be any mass poles of the theory in the complex momentum region of w_0 that the rotation sweeps through. If there are such poles then we cannot identify Minkowski space-time with Euclidean space-time. However, as was pointed out in section 1.3, the path integral (and hence the theory) is only well defined in Euclidean space-time and that is where we will calculate objects. We gave the equations and propagators in Minkowski space-time as that is the form in which most people are accustomed to seeing the Feynman rules. Indeed when the Euclidean Feynman rules are written down the i's from the Wick rotation are often dropped explicitly and taken to be there implicitly.

We now undertake a comparitive study of some ansatze that are in common use:–

2.3 *Bare ansatz and the quenched approximation* :

A very popular ansatz at the moment is the Bare Vertex in which $\Gamma^\mu = \gamma^\mu$. This is of course just the lowest order (extreme perturbative) form for the vertex. This has been used by the Kiev group [36,37] extensively in the quenched approximation.

The rationale behind the quenched approximation is that in the propagator equations (2.1) for N_f equal mass fermion flavours, the variable N_f only explicitly appears in front of the fermion loop integral in the equation for the photon propagator and as such we can treat it mathematically as a free variable. In the quenched approximation we then set $N_f = 0$ which simplifies the set of S-DE's to just the S-DE for the fermion (fig. 2.2), as then $\mathcal{G}(q^2) = 1$. We shall refer to quenched QED as qQED. Using a bare vertex in qQED is also known as quenched planar QED.

In their work, the Kiev group substitute $\Gamma^\mu = \gamma^\mu$ into the equation for the fermion and then project out the γ^μ (1) parts by multiplying by $\not p$ (1) and tracing giving the equations for the fermion wavefunction and mass as:–

$$F^{-1}(p^2) = 1 - \frac{\alpha_0}{4\pi^3 p^2} \int_E^{\Lambda^2} \frac{d^4k F(k^2)}{k^2 + \Sigma^2(k^2)} \frac{1}{q^2} \Bigg[- 2k.p + \frac{1}{q^2} \left((k^2 + p^2)k.p - 2k^2p^2 \right) (\xi - 1) \Bigg]$$

$$\frac{\Sigma(p^2)}{F(p^2)} = \frac{\alpha_0}{4\pi^3} \int_E^{\Lambda^2} \frac{d^4k F(k^2)\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \frac{1}{q^2} [3 + \xi]$$

which on performing the angular integrals becomes

$$\begin{aligned} F^{-1}(p^2) &= 1 + \frac{\alpha_0 \xi}{4\pi} \int^{\Lambda^2} \frac{dk^2 F(k^2)}{k^2 + \Sigma^2(k^2)} [\theta(p^2 - k^2)\frac{k^4}{p^4} + \theta(k^2 - p^2)] \\ \frac{\Sigma(p^2)}{F(p^2)} &= \frac{(3 + \xi)\alpha_0}{4\pi} \int^{\Lambda^2} \frac{dk^2 F(k^2)\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} [\theta(p^2 - k^2)\frac{k^2}{p^2} + \theta(k^2 - p^2)] \end{aligned} \tag{2.2}$$

where Λ^2 is the U.V. cut-off needed to regularise the integral (see section 2.5),

$$\theta(w^2) = \begin{cases} 1 & w^2 \geq 0; \\ 0 & w^2 < 0. \end{cases}$$

the bare mass, m_0, has been taken equal to zero and the angular integral calculations are done elsewhere (see Appendix A). Working in the Landau gauge ($\xi = 0$) we then have that:–

$$\begin{aligned} F(p^2) &= 1 \quad \forall p^2 \\ \Sigma(p^2) &= \frac{3\alpha_0}{4\pi} \int^{\Lambda^2} \frac{dk^2 \Sigma(k^2)}{k^2 + \Sigma^2(k^2)} [\ \theta(p^2 - k^2)\frac{k^2}{p^2} + \theta(k^2 - p^2)]. \end{aligned} \tag{2.3}$$

The dynamical mass that they obtain from this is:–

$$\Sigma \approx 4\ \Lambda \exp\left[\frac{-\Theta}{\sqrt{\frac{\alpha_0}{\alpha_c} - 1}}\right] \quad \text{for} \quad \alpha_0 > \alpha_c$$

where Θ is a positive constant and $\Sigma = 0$ for $\alpha_0 < \alpha_c$. In the region $\alpha_0 > \alpha_c$, as we take $\Lambda^2 \to \infty$ (the continuum limit) the dynamical mass, Σ, (energy gap) diverges. This is an indicator that the energy of the ground state is not bounded from below giving a collapse of the wavefunction. (When the Bethe–Salpeter wavefunction of the system has an infinite set of zeros)[38]. To remove the collapse phenomenon, as $\Lambda^2 \to \infty$, they make the bare parameter α_0 depend upon the cut off Λ in such a way as to make $\Sigma(\Lambda^2 \to \infty)$ finite, viz:–

$$\alpha_0(\Lambda^2) = \alpha_c + \frac{\pi^2 \alpha_c}{\log^2(4\Lambda/\Sigma)} \overset{\Lambda^2 \to \infty}{\longrightarrow} \alpha_c \ (= \frac{\pi}{3})$$

This is not the usual way of renormalising the coupling constant (ie $\alpha(\mu) = Z_3(\mu/\Lambda)\alpha_0(\Lambda)$). We see that α_c is now acting as a U.V. fixed point in the massive phase as $\alpha_0 \to \alpha_{c\,+}$, whilst in the massless phase α_0 is a constant[36].

Then defining the β-function as $\beta(\alpha) := \Lambda \frac{d\alpha(\Lambda)}{d\Lambda}$ we get

$$\beta(\alpha_0) = \begin{cases} -2\ \alpha_c(\alpha_0/\alpha_c - 1)^{3/2}\ \Theta^{-1} & \alpha_0 > \alpha_c; \\ 0 & \alpha_0 < \alpha_c. \end{cases}$$

(Analysis of unquenched QED has also been carried out [39] using a coupling $\alpha_0 N$, see for example [23], when they find a critical number of flavours, N_c, above which there is no mass and below which there exists mass).

One should, however, have some major reservations about this ansatz:–

(1) The ansatz only obeys the W-TI in the Landau gauge in a theory with

$\Sigma(p^2) =$ const. as with $\Gamma^\mu = \gamma^\mu$:-

$$q^\mu \Gamma^\mu = \not{k} - \not{p}$$
$$S_F^{-1}(k) - S_F^{-1}(p) = \frac{\not{k}}{F(k^2)} - \frac{\not{p}}{F(p^2)} - \frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}.$$

These only allow $q^\mu \Gamma^\mu = S_F^{-1}(k) - S_F^{-1}(p)$ when $F(k^2) = 1 = F(p^2)$ (which only occurs in the Landau gauge for the bare ansatz, (2.2)) and $\Sigma(k^2) = \Sigma(p^2)\ \forall k^2, p^2$. In a gauge theory it seems highly suspect to use an ansatz for Γ^μ that only stands a chance of satisfying the gauge invarience of the theory in one gauge. Gauge invariance demands that $F(p^2) = 1$ (which is satisfied with $\xi = 0$) and $\Sigma(p^2) =$ const. which is not possible from (2.3). Indeed how can we expect our 'physical' outputs to be gauge independent when our input (ansatz) isn't!

(2) With a form like $\Gamma^\mu = \gamma^\mu$ we cannot include any non-perturbative behaviour into the vertex. Indeed it isn't even possible to include the usual perturbative corrections that are needed for high energy phenomenology. This seems very strange. To comment on a quote from ref. [36] "The appearance of such a point α_c in the ladder approximation is caused by the dynamics which cannot be obtained in perturbation theory" the question has to be asked where do these non-perturbative affects have a chance of entering as the form $\Gamma^\mu = \gamma^\mu$ is a very strong suppressor of any affects, perturbative or non-perturbative. For example it is widely known that if we undertake a perturbative calculation of the dynamical mass with a non-zero current mass, m_0, then even if we undertake an all orders resummation using the Renormalisation Group equations (see later in this chapter, section 2.5a) then we still only get an answer that is of the form:-

$$\Sigma(p^2) = m_0 X(p^2)$$
$$X(p^2) \approx \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha^n A_n B_{m,n} \log^m(p^2/\Lambda^2)$$

which always gives $m_0 = 0 \Rightarrow \Sigma(p^2) = 0$, a perturbative result. This prompts the comment "how can anything non-perturbative come out of the theory when only perturbative things have been put in".

(3) In the work of the Kiev group they have found that in the dynamical mass generation phase ($\alpha_0 > \alpha_c$) that the electrons condense to form bound state tachyons ('Cooper pairs'). This is a QED version of hadronisation/confinement and as such you would expect $F(p^2)$ at some point to equal zero, but $F(p^2) = 1 \; \forall p^2$ and this is a problem.

(4) Finally, if we look at (2.2) with $\xi \neq 0$ and attempt to renormalise it in the usual multiplicitive way (for qQED $F_R(p^2/\mu^2) = Z(\mu^2/\Lambda^2)F(p^2/\Lambda^2)$, $\alpha_R = \alpha_0$ and $\xi_R = \xi$ see section 2.5) then we find that there is no way of succeeding in multiplicitavely renormalising the theory, which means that we don't really know if there is a continuum theory associated with the bare vertex.

2.4 *The longitudinal part of the vertex and the Ball–Chiu ansatz* :

In order to attempt to find a better ansatz than the bare one, we start off by saying that any ansatz must first and foremost obey the W-TI for all gauges (after all we are dealing with gauge theories here!):–

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p).$$

Now q_μ acts as a Killing vector on the transverse vertex space and as such we can split the vertex into two parts, the longitudinal part, Γ_L, and the transverse part, Γ_T, then:–

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p) \tag{2.4}$$

where $q_\mu \Gamma^\mu_L(k,p) = S_F^{-1}(k) - S_F^{-1}(p)$ and $q_\mu \Gamma^\mu_T(k,p) = 0$. It seems that there is an infinite uncertainty in Γ_L as we can put arbitrary amounts of Γ_T into it and still satisfy (2.4). Indeed one example would be to write Γ_μ as:–

$$\Gamma^\mu(k,p) = \Gamma^\mu_T(k,p) + \frac{q^\mu}{q^2}\left(S_F^{-1}(k) - S_F^{-1}(p)\right) \tag{2.5}$$

which satisfies equations (2.4) [40]. This form with

$$\Gamma^\mu_T(k,p) = \left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\gamma^\nu\beta(k,p)$$

has recently been popularised[41]. It is however singular when $q^2 = 0$ but $q^\mu \neq 0^\mu$. In affect this means that the ∂W-I is not non-trivially satisfied:–

$$\Gamma_\mu(p,p) = \frac{\partial S_F^{-1}(p)}{\partial p_\mu}$$

To see this we Taylor expand $S_F^{-1}(k)$:–

$$S_F^{-1}(k) = S_F^{-1}(p) + (k-p)_\nu \frac{\partial S_F^{-1}(p)}{\partial p_\nu} + \ldots$$

Then (2.5) becomes:–

$$\begin{aligned}\Gamma^\mu(k,p) &= \Gamma^\mu_T(k,p) + \frac{q^\mu q^\nu}{q^2}\frac{\partial S_F^{-1}(p)}{\partial p_\nu} + O(q)\\ &= \Gamma^\mu_T(k,p) + \frac{q^\mu q^\nu}{q^2}\Gamma^\nu(p,p) + O(q)\end{aligned}$$

and when $k^\mu \to p^\mu$ we find that necessarily

$$\Gamma^\mu_T(k,p) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Gamma_\nu(k,p)$$

and we are left with the trivial identity $\Gamma_\mu(k,p) = \Gamma_\mu(k,p)$. Hence we can see that the form in ref. [41] is incorrect. Things are not all bad news though as combining the ∂W-I with the form for the fermion propagator, $S_F^{-1}(p) = \left(\not p - \Sigma(p^2)\right)/F(p^2)$ we get

$$\Gamma^\mu(p,p) = \frac{\gamma^\mu}{F(p^2)} + 2p^\mu \not p \frac{\partial F^{-1}(p^2)}{\partial p^2} - 2p^\mu \frac{\partial}{\partial p^2}\left(\frac{\Sigma(p^2)}{F(p^2)}\right)$$

which when we use the k,p symmetry of the vertex let's us write this in a natural way:–

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p)$$

with

$$\begin{aligned}\Gamma^\mu_L(k,p) = \frac{\gamma^\mu}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) &+ \frac{(k+p)^\mu(\not k + \not p)}{2(k^2-p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \\ &- \frac{(k+p)^\mu}{(k^2-p^2)}\left(\frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}\right)\end{aligned}$$

and

$$\Gamma^\mu_T(p,p) = 0. \tag{2.6}$$

This form for Γ_L is known as the Ball–Chiu vertex[42] and we shall denote it by Γ_{BC}. Then:–

$$\Gamma^\mu(k,p) = \Gamma^\mu_{BC}(k,p) + \Gamma^\mu_T(k,p)$$

where $q_\mu \Gamma^\mu_{BC}(k,p) = S_F^{-1}(k) - S_F^{-1}(p)$, $q_\mu \Gamma^\mu_T(k,p) = 0$ and $\Gamma^\mu_T(p,p) = 0$. Thus using ∂W-I we have the ONLY way of splitting Γ into Γ_L and Γ_T. From physics, the vertex as a whole must be free from kinematic singularities. From the lack of kinematic singularities in Γ_{BC} we have then that Γ_T is also without kinematic singularities.

Whilst the W-TI and ∂W-I give us a UNIQUE, known form for $\Gamma_L(k,p)$ they don't actually give us the form for $\Gamma_T(k,p)$. The usual thing to do then is to use the Minimal Ball-Chiu ansatz, where we put $\Gamma_T(k,p) = 0$ leaving $\Gamma^\mu(k,p) = \Gamma^\mu_{BC}(k,p)$. This is not completely satisfactory, but at least our ansatz obeys the W-TI and it's a starting point. Indeed using this ansatz in massive 3 dimensional QED (which is related to qQED4) it has been found that there is no critical point N_c ($\equiv 1/\alpha_c$)[23], which is QUALITATIVELY different from the results with $\Gamma^\mu = \gamma^\mu$ (and lattice results to date). In massless QCD it has been used to find solutions for $F(p^2)$ and $\mathcal{G}(p^2)$ which give confinement[43]. However it would be nice to try to find what $\Gamma_T(k,p)$ is and see what improvements, if any, we could obtain to these results.

2.5 *Multiplicative Renormalisation and* $\Gamma_T(k,p)$:

When we make calculations from the initial Lagrangian (called the bare Lagrangian) we find that the integrals that we end up with are, in the main, divergent. For example $\int_{p^2}^{\infty} dk^2/k^2$ is logarithmically divergent. In order for us to be able to tackle this problem, and hopefully find a theoretical way of removing these divergences, we first of all need to REGULATE them. That is we introduce some parameter that makes the integral finite except for some value of the parameter where the integral diverges again. For instance, in dimensional regularisation the dimension of space time is increased by ε which makes the integral finite except for when $\varepsilon = 0$. Another example is cut off regularisation when the range of integration is cut off at a finite value Λ^2, the integral is then finite except in the limit $\Lambda^2 \to \infty$. It is this regularisation method that we shall use as it is easier to look at numerically and as we deal only with one loop calculations it is a valid method (with two or more loops the cut-off method breaks gauge invariance and so cannot be used without care). We are now able to look at how to remove these regularised divergences, this is known as renormalising the theory. The usual method of renormalising theories is to use Multiplicative Renormalisation (M.R.), where we introduce renormalised fields/Green's functions and their associated Z functions. So for QED we have

$$\varphi_R(p^2,\mu^2) = Z_2^{-1/2}(\mu^2,\Lambda^2)\varphi(p^2,\Lambda^2),$$

$$A_R^\mu(p^2,\mu^2) = Z_3^{-1/2}(\mu^2,\Lambda^2)A^\mu(p^2,\Lambda^2),$$

$$\Gamma_R^\mu(p,k,\mu) = Z_1(\mu,\Lambda)\Gamma^\mu(p,k,\Lambda)$$

$$\text{and thus} \quad e_R(\mu) = \frac{Z_2 Z_3^{1/2}}{Z_1}\, e,$$

where μ^2 is the renormalisation point where we define objects to have a specified value. This gives us the renormalised Lagrangian $\mathcal{L}_R(\varphi_R, \overline{\varphi}_R, A_R)$.

The advantages of MR over other renormalisation schemes are:–

(1) it is difficult to envisage a different sensible renormalisation scheme. For example additive schemes would have to be very specifically chosen in order to go through the calculations and cancel the divergences at the end (in MR this is easily done as the renormalisation is 'bolted' onto the field). Indeed in Bogoliubov–Parasiuk–Hepp (BPH) renormalisation[45,46], of the perturbative theory, where corrections are added into the bare Lagrangian to remove divergences of a given order α^n, starting with $O(\alpha)$, one finds that for a renormalisable theory the correction terms must be of the form of the original interaction terms of the bare Lagrangian. This means that as we remove the divergences order by order in α^n we build up terms of the form $Y(\alpha)\times$ interaction term, where $Y(\alpha)$ is a perturbative series in α^n (The proof that this renormalises the theory is called Hepp's Theorem[46]). We can then identify the series $Y(\alpha)$ with a Z factor and thus rewrite the Green's functions as renormalised Green's functions. Thus we see that BPH renormalisation is equivalent to perturbative MR, and we have not gained a new method of renormalisation. In fact MR is only rigorously known to work in the perturbative region but it is believed to work non-perturbatively as how else can we renormalise a theory ? It is also believed to work because:–

(2) MR directly leads to the R-GE's, which give us such experimentally observable phenomena as scaling in QCD. So how do we get the R-GE's ?

2.5a *The Renormalisation Group Equations* :

Let us consider QED, then the renormalised Green's functions are given by:–

$$\Gamma_R^{\mu_1,\ldots,\mu_n(n,m)}(p_1,\ldots,p_n;q_1,\ldots,q_n;g_R,\xi_R,\mu)\Bigg|_{p_i^2=\ \mu^2} =$$

$$Z(p_1,\ldots,p_n;q_1,\ldots,q_n;g_R,\xi_R,\mu)$$
$$\times\ \Gamma^{\mu_1,\ldots,\mu_n(n,m)}(p_1,\ldots,p_n;q_1,\ldots,q_n;g,\xi,\Lambda)\Bigg|_{p_i^2=\ \mu^2}$$

where n = number of photon fields and m = number of fermion/anti-fermion fields. Now W-TI give us that $Z = Z_A^{n/2}\ Z_\varphi^{m/2}$ and so (taking $p_i^2 = \mu^2$ as read):–

$$\begin{aligned}\mu\frac{d}{d\,\mu}\Gamma_R^{\mu_1,\ldots,\mu_n(n,m)} &= \left(\mu\frac{\partial}{\partial\,\mu}+\ \mu\frac{\partial g}{\partial\,\mu}\frac{\partial}{\partial\,g}+\ \mu\frac{\partial\xi}{\partial\,\mu}\frac{\partial}{\partial\xi}\right)\Gamma_R^{\mu_1,\ldots,\mu_n(n,m)}\\ &= \mu\frac{d}{d\,\mu}\left(exp\big[\frac{n}{2}\log Z_A\big]exp\big[\frac{m}{2}\log Z_\varphi\big]\Gamma^{\mu_1,\ldots,\mu_n(n,m)}\right)\\ &= \left(\frac{n}{2}\mu\frac{d\ \log Z_A}{d\,\mu}+\ \frac{m}{2}\mu\frac{d\ \log Z_\varphi}{d\,\mu}\right)\ Z_A^{\frac{n}{2}}\ Z_\varphi^{\frac{m}{2}}\Gamma^{\mu_1,\ldots,\mu_n(n,m)}.\end{aligned}$$

Therefore we obtain:–

$$\left(\mu\frac{\partial}{\partial\,\mu}+\ \beta(g)\frac{\partial}{\partial\,g}-\ \frac{n}{2}\gamma_A(g)-\ \frac{m}{2}\gamma_F(g)+\ \delta(g)\frac{\partial}{\partial\xi}\right)\Gamma_R^{\mu_1,\ldots,\mu_n(n,m)} =\ 0 \qquad (2.7)$$

where

$$\beta(g) =\ \mu\frac{\partial g}{\partial\,\mu}\ ,\ \delta(g) =\ \mu\frac{\partial\xi}{\partial\,\mu}\ ,\ \gamma_A(g) =\ \mu\frac{d\ \log Z_A}{d\,\mu}\ ,\ \gamma_F(g) =\ \mu\frac{d\ \log Z_\varphi}{d\,\mu}$$

This is known as the (Stueckelberg–Peterman[47]) Renormalisation Group equation (R–GE). The R-GE can be used to write series for the Green's functions for a wide range of momenta (renormalisation flows), they also allow us to resum the expressions we get for the first few orders in α to infinite order expressions in α (renormalisation improved resumation). For example, in some massless ϕ^m type theory, if we look at the n^{th} order Green's function, Γ^n, with, for example, dimension D then dimensional analysis gives:–

$$\Gamma^{(n)}(p_i, g, \mu) = \mu^D f(\frac{p_i p_j}{\mu^2}, g)$$

f a dimensionless function. Then if we let $p_i \to p_i e^t$

$$\Rightarrow \Gamma^{(n)}(p_i e^t, g, \mu) = \mu^D f(\frac{p_i p_j}{\mu^2} e^{2t}, g)$$

$$\Rightarrow \left[\mu\frac{\partial}{\partial\mu} + \frac{\partial}{\partial t}\right]\Gamma^{(n)}(p_i e^t, g, \mu) = D\mu^D f + \mu^D\left[\mu\frac{\partial}{\partial\mu} + \frac{\partial}{\partial t}\right] f = D\mu^D f$$

$$\text{ie} \qquad \left[\mu\frac{\partial}{\partial\mu} + \frac{\partial}{\partial t} - D\right]\Gamma^{(n)}(p_i e^t, g, \mu) = 0$$

and the R-GE is:–

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma^{(n)}(p_i e^t, g, \mu) = 0$$

$$\Rightarrow \left[-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + D - n\gamma(g)\right]\Gamma^{(n)}(p_i e^t, g, \mu) = 0,$$

the Inhomogeneous Callan–Symanzik Equation (if $D = 0$ it is the Homogeneous Callan–Symanzik Equation). Putting:–

$$\Gamma^{(n)}(p_i e^t, g, \mu) = \exp\left[tD + n\int_0^g \frac{\gamma(x)}{\beta(x)}dx\right] F^{(n)}(p_i e^t, g, \mu)$$

leads to

$$\left(-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right) F^{(n)}(p_i e^t, g, \mu) = 0. \tag{2.8}$$

Now

$$\beta(\overline{g}) = \frac{\partial}{\partial t}\overline{g}(t) \text{ where } \overline{\mu} = \mu e^t \Rightarrow t = \int_{\overline{g}(0)=g}^{\overline{g}(t)} \frac{dx}{\beta(x)}$$

differentiating with respect to g we find that:–

$$
\begin{aligned}
0 &= \frac{\partial \overline{g}(t)}{\partial g}\frac{1}{\beta(\overline{g}(t))} - \frac{1}{\beta(g)} \\
\Rightarrow 0 &= \left(-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right)\overline{g}(t)
\end{aligned}
\tag{2.9}
$$

(2.8) and (2.9) then tell us that $F^{(n)}$ must depend upon t and g via $\overline{g}(t)$ only and therefore $F^{(n)}(p_i e^t, g, \mu) = F^{(n)}(p_i, \overline{g}(t), \mu)$. Now

$$
\begin{aligned}
\exp\left[n\int_0^g \frac{\gamma(x)}{\beta(x)}dx\right] &= \exp\left[n\int_0^{\overline{g}(t)} \frac{\gamma(x)}{\beta(x)}dx + n\int_{\overline{g}(t)}^{g}\frac{\gamma(x)}{\beta(x)}dx\right] \\
&= H\left(\overline{g}(t)\right)\exp\left[-n\int_0^t \gamma(\overline{g}(t))dx\right]
\end{aligned}
$$

$$
\Rightarrow \qquad \Gamma^{(n)}(p_i e^t, g, \mu) = \Gamma^{(n)}(p_i, \overline{g}(t), \mu)\exp\left[tD - n\int_0^t \gamma(\overline{g}(t))dx\right]
$$

tD is called the canonical dimension and $-n\int_0^t \gamma(\overline{g}(t))dx$ the anomalous dimension. It can be shown that:–

$$
\begin{aligned}
\beta(g) &= -\frac{\beta_0}{16\pi^2}\overline{g}(\mu)^3 + O(g^5) \\
\gamma(g) &= \frac{\gamma_0}{16\pi^2}\overline{g}(\mu)^2 + O(g^4).
\end{aligned}
$$

So to lowest order perturbation theory

$$
\Gamma^{(n)}(p_i e^t, g, \mu) = \Gamma^{(n)}(p_i, \overline{g}(t), \mu)\left(\frac{q}{\mu}\right)^D\left(\frac{g(q)}{g(\mu)}\right)^{n\gamma_0/\beta_0} \qquad \text{where } \frac{q}{\mu} = e^t
$$

(a standard result that can be found in any postgraduate lecture course).

In a recent paper by Brown and Dorey[48] it was suggested that a further check on ansatze would be to calculate the results that they gave in perturbation theory, up to some level, and check to see whether the forms given are MR. What we will do in the following sections is to attempt to turn this the other way round. Instead of ansatze being checked against MR we attempt to use MR to build an ansatz (ie to investigate the transverse part of the vertex). As the R-GE's are perturbative equations the form of our ansatz will, of course, only be well known in this region, however using the known symmetries of the vertex and the fact that it must be made up of $F(p^2)$ and $\Sigma(p^2)$ we can generate a non-perturbative ansatz. Due to the complexities of dealing with the R-GE's for full QED we look at the R-GE for qQED, in which instance we are left with $\xi_R = \xi$ and $e_R = e$ (as WTI $\Rightarrow Z_1 = Z_2$ and qQED $\Rightarrow Z_3 = 1$). Thus the R-GE's, for massless qQED, are:–

$$\left(\mu\frac{\partial}{\partial\mu} - \frac{m}{2}\gamma_m(e)\right)\Gamma_R^{\mu_1,\ldots,\mu_n(n,m)} = 0. \tag{2.10}$$

We now investigate the constraints that MR imposes on qQED.

2.5b *How MR restricts the forms for $F(p^2)$ and $\Sigma(p^2)$:*

In qQED the renormalised fields are $\varphi_R(p^2,\mu^2) = Z_2^{-1/2}(\mu^2,\Lambda^2)$, $A_R^\mu = A^\mu$, $e_R = e$ and $\xi_R = \xi$, so for the renormalised wavefunction we have

$$F_R(p^2,\mu^2) = Z_2^{-1}(\mu^2,\Lambda^2)F(p^2,\Lambda^2). \tag{2.11}$$

We write these functions as a perturbative series. In fact we shall start off by writing them as leading log series, where we only have terms $\alpha^n \log^n$, this is permissible as these are the most divergent terms present in the series, qQED being only logarithmically divergent. The results we obtain aren't affected by sub-leading logs (as terms $\alpha^n \log^{n-p}$ can only affect terms that have equal or greater p, where n can vary) and the working is much easier to follow. We start off by writing the series to

$O(\alpha^2)$:–

$$\begin{aligned}
F(p^2,\Lambda^2) &= 1+\ \alpha A_1\log\frac{p^2}{\Lambda^2}+\ \alpha^2 A_2\log^2\frac{p^2}{\Lambda^2}\\
Z_2^{-1}(\mu^2,\Lambda^2) &= 1+\ \alpha B_1\log\frac{\mu^2}{\Lambda^2}+\ \alpha^2 B_2\log^2\frac{\mu^2}{\Lambda^2}\\
F_R(p^2,\mu^2) &= 1+\ \alpha C_1\log\frac{p^2}{\mu^2}+\ \alpha^2 C_2\log^2\frac{p^2}{\mu^2}.
\end{aligned}$$

Then (2.11) tells us that:–

$$\begin{aligned}
&1+\ \alpha\left(A_1\log\frac{p^2}{\Lambda^2}+\ B_1\log\frac{\mu^2}{\Lambda^2}\right)\\
&\qquad+\ \alpha^2\left(A_2\log^2\frac{p^2}{\Lambda^2}+\ A_1B_1\log\frac{p^2}{\Lambda^2}\log\frac{\mu^2}{\Lambda^2}+\ B_2\log^2\frac{\mu^2}{\Lambda^2}\right)\\
&=\ 1+\ \alpha\left(C_1\log\frac{p^2}{\Lambda^2}-\ C_1\log\frac{\mu^2}{\Lambda^2}\right)\\
&\qquad+\ \alpha^2\left(C_2\log^2\frac{p^2}{\Lambda^2}-\ 2C_2\log\frac{p^2}{\Lambda^2}\log\frac{\mu^2}{\Lambda^2}+\ C_2\log^2\frac{\mu^2}{\Lambda^2}\right)\\
&\Rightarrow A_1 =\ C_1 =\ -B_1\ ,\ A_2 =\ B_2 =\ C_2 =\ \frac{A_1^2}{2!}
\end{aligned}$$

on equating powers of both $\log(p^2/\Lambda^2)$ and α. We then carry out an inductive proof, assuming that

$$A_n =\ C_n =\ \frac{X^n}{n!} =\ (-1)^n B_n$$

then:–

$$\begin{aligned}
F(p^2,\Lambda^2) &= \sum_{n=0}^{N}\alpha^n\frac{X^n}{n!}\log^n\frac{p^2}{\Lambda^2}\ +\ \alpha^{N+1}A_{N+1}\log^{N+1}\frac{p^2}{\Lambda^2}\\
Z^{-1}(\mu^2,\Lambda^2) &= \sum_{n=0}^{N}\alpha^n\frac{(-X)^n}{n!}\log^n\frac{\mu^2}{\Lambda^2}\ +\ \alpha^{N+1}B_{N+1}\log^{N+1}\frac{\mu^2}{\Lambda^2}\\
F_R(p^2,\mu^2) &= \sum_{n=0}^{N}\alpha^n\frac{X^n}{n!}\log^n\frac{p^2}{\mu^2}\ +\ \alpha^{N+1}C_{N+1}\log^{N+1}\frac{p^2}{\mu^2}.
\end{aligned}$$

Looking at (2.11) for $O(\alpha^{N+1})$ we have:–

$$
\begin{aligned}
&\alpha^{N+1} C_{N+1} \log^{N+1} \frac{p^2}{\mu^2} \\
&= \sum_{n=1}^{N} \alpha^n \frac{X^n}{n!} \log^n \frac{p^2}{\Lambda^2} \alpha^{N+1-n} \frac{-X^{N+1-n}}{(N+1-n)!} \log^{N+1-n} \frac{\mu^2}{\Lambda^2} \\
&\quad + \alpha^{N+1} A_{N+1} \log^{N+1} \frac{p^2}{\Lambda^2} + \alpha^{N+1} B_{N+1} \log^{N+1} \frac{\mu^2}{\Lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{X^{N+1}\alpha^{N+1}}{(N+1)!} \sum_{n=0}^{N+1} \log^n \frac{p^2}{\Lambda^2} (-\log \frac{\mu^2}{\Lambda^2})^{N+1-n} \frac{(N+1)!}{(N+1-n)!n!} \\
&\quad + \alpha^{N+1} \log^{N+1} \frac{\mu^2}{\Lambda^2} \left[B_{N+1} - \frac{(-X)^{N+1}}{(N+1)!} \right] \\
&\quad + \alpha^{N+1} \log^{N+1} \frac{p^2}{\Lambda^2} \left[A_{N+1} - \frac{X^{N+1}}{(N+1)!} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{X^{N+1}\alpha^{N+1}}{(N+1)!} \left(\log \frac{p^2}{\Lambda^2} - \log \frac{\mu^2}{\Lambda^2} \right)^{N+1} \\
&\quad + \alpha^{N+1} \log^{N+1} \frac{\mu^2}{\Lambda^2} \left[B_{N+1} - \frac{(-X)^{N+1}}{(N+1)!} \right] \\
&\quad + \alpha^{N+1} \log^{N+1} \frac{p^2}{\Lambda^2} \left[A_{N+1} - \frac{X^{N+1}}{(N+1)!} \right]
\end{aligned}
$$

$$
\Rightarrow \qquad C_{N+1} = A_{N+1} = \frac{X^{N+1}}{(N+1)!} = (-1)^{N+1} B_{N+1}
$$

the MR constraints on the leading log coefficients of the series.

We now look at the series including next-to-leading log terms, we use the results already obtained from the discussion of leading logs. As part of our renormalisation scheme we impose the normalisation:–

$$
F_R(p^2, \mu^2) = F(p^2, \Lambda^2)\Big|_{\mu^2 = \Lambda^2}
$$

ie $Z_2(\Lambda^2, \Lambda^2) = 1$ which has the affect of giving F_R and F the same form.

$$F(p^2, \Lambda^2) = \sum_{n=0}^{\infty} \alpha^n \frac{A^n}{n!} \left(\log^n \frac{p^2}{\Lambda^2} + B_n \log^{n-1} \frac{p^2}{\Lambda^2} \right) \qquad B_0 = 0$$

$$Z^{-1}(\mu^2, \Lambda^2) = \sum_{n=0}^{\infty} \alpha^n \frac{(-A)^n}{n!} \left(\log^n \frac{\mu^2}{\Lambda^2} + C_n \log^{n-1} \frac{\mu^2}{\Lambda^2} \right) \qquad C_0 = 0$$

$$F_R(p^2, \mu^2) = \sum_{n=0}^{\infty} \alpha^n \frac{A^n}{n!} \left(\log^n \frac{p^2}{\mu^2} + B_n \log^{n-1} \frac{p^2}{\mu^2} \right)$$

We know that the leading log parts satisfy MR, (2.11), from before, so we concentrate on the next to leading logs at $O(\alpha^{N+1})$:–

$$\frac{A^{N+1}}{(N+1)!} B_{N+1} \log^N \frac{p^2}{\mu^2} = \frac{A^{N+1}}{(N+1)!} B_{N+1} \sum_{m=0}^{N} \frac{N!(-1)^{N-m}}{m!(N-m)!} \log^m \frac{p^2}{\Lambda^2} \log^{N-m} \frac{\mu^2}{\Lambda^2}$$
$$= \sum_{m=0}^{N+1} \frac{A^m}{m!} \frac{(-A)^{N+1-m}}{(N+1-m)!} \Bigg[B_m \log^{m-1} \frac{p^2}{\Lambda^2} \log^{N+1-m} \frac{\mu^2}{\Lambda^2}$$
$$+ C_{N+1-m} \log^m \frac{p^2}{\Lambda^2} \log^{N-m} \frac{\mu^2}{\Lambda^2} \Bigg]$$

$$\Rightarrow 0 = \log^m \frac{p^2}{\Lambda^2} \log^{N-m} \frac{\mu^2}{\Lambda^2} \Bigg[\frac{N!(-1)^{N-m}}{(N+1)!m!(N-m)!} B_{N+1}$$
$$- B_{m+1} \frac{(-1)^{N-m}}{(N-m)!(m+1)!} - C_{N+1-m} \frac{(-1)^{N+1-m}}{(N+1-m)!m!} \Bigg]$$

$$\Rightarrow 0 = \frac{(N+1-m)}{(N+1)} B_{N+1} - \frac{(N+1-m)}{(m+1)} B_{m+1} + C_{N+1-m}$$

putting $m = 0$, $N = k$ gives $C_{k+1} = B_1(k+1) - B_{k+1}$
and $m = 1$, $N = k$ gives $C_{k+1} = \frac{1}{2} B_2(k+1) - B_{k+2} \frac{k+1}{k+2}$

$$\Rightarrow \qquad B_{k+2} = \frac{k+2}{2} B_2 - (k+2) B_1 + \frac{k+2}{k+1} B_{k+1}$$
$$\Rightarrow \qquad B_{k+2} = \frac{1}{2} B_2 (k+2)(k+1) - B_1 (k+2) k \tag{2.12}$$

by iterative substitution. So we have the MR constraints for next to leading log coefficients.

It can be seen that even though we can , in principle, find the MR constraints for the coefficients of any order of non-leading log each order has to be worked out individually and as we get further away from leading logs the amount of effort involved increases rapidly. We shall leave the MR constraints at just the two we have calculated.

If we look at the perturbative series for $\Sigma(p^2)$ (we must have m_0 non-zero here else we shall get the trivial $\Sigma = 0$ and no useful information) we have in the same way:–

$$\Sigma(p^2,\Lambda^2) = m_0 \sum_{n=0}^{\infty} \alpha^n \frac{A^n}{n!}\left(\log^n \frac{p^2}{\Lambda^2} + B_n \log^{n-1}\frac{p^2}{\Lambda^2}\right) \qquad B_0 = 0$$

$$Z_\Sigma^{-1}(\mu^2,\Lambda^2) = \sum_{n=0}^{\infty} \alpha^n \frac{(-A)^n}{n!}\left(\log^n \frac{\mu^2}{\Lambda^2} + C_n \log^{n-1}\frac{\mu^2}{\Lambda^2}\right) \qquad C_0 = 0$$

$$\Sigma_R(p^2,\mu^2) = \sum_{n=0}^{\infty} \alpha^n \frac{A^n}{n!}\left(\log^n \frac{p^2}{\mu^2} + B_n \log^{n-1}\frac{p^2}{\mu^2}\right)$$

with the same constraints on the B_k's as for F. We now look at what perturbative results we get from the various ansatze.

2.5c *The Gauge dependence of the fermion mass*[49] :

(i) We start off by looking at the bare ansatz, $\Gamma^\mu = \gamma^\mu$, in the S-DE's (2.2), but with a non-zero current mass (as we are working perturbatively). We look at the equations using the Born series, starting off with the lowest order answer and then repeatedly substituting into the integral to generate up the higher order answers. We work perturbatively keeping only the leading log results and neglecting m_0 with respect to k or p. The equations we end up with are then:–

$$F^{-1}(p^2) = 1 + \frac{\alpha_0 \xi}{4\pi}\int_{p^2}^{\Lambda^2} \frac{dk^2 F(k^2)}{k^2}$$

$$\frac{\Sigma(p^2)}{F(p^2)} = m_0 + \frac{(3+\xi)\alpha_0}{4\pi}\int_{p^2}^{\Lambda^2} \frac{dk^2 F(k^2)\Sigma(k^2)}{k^2}.$$

Starting with the equation for F and the lowest order solution $F(p^2) = 1$, after repeated substitution we obtain:–

$$F(p^2) = 1 + \frac{\alpha_0 \xi}{4\pi} \log \frac{p^2}{\Lambda^2} + \frac{3}{2}\xi^2 \left(\frac{\alpha_0}{4\pi}\right)^2 \log^2 \frac{p^2}{\Lambda^2} + \frac{5}{2}\xi^3 \left(\frac{\alpha_0}{4\pi}\right)^3 \log^3 \frac{p^2}{\Lambda^2} + \ldots \quad (2.13)$$

From the equation for Σ and starting from $\Sigma(p^2) = m_0$ we obtain:–

$$\Sigma(p^2) = m_0 \left[1 - \frac{3\alpha_0}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{9}{2} - 3\xi\right) \left(\frac{\alpha_0}{4\pi}\right)^2 \log^2 \frac{p^2}{\Lambda^2} \right. \\ \left. + \left(-\frac{9}{2} + 9\xi - 4\xi^2\right) \left(\frac{\alpha_0}{4\pi}\right)^3 \log^3 \frac{p^2}{\Lambda^2} + \ldots \right]. \quad (2.14)$$

(ii) For the minimal Ball-Chiu ansatz $\left(\Gamma^\mu = \Gamma^\mu_{BC}\right)$ from (B.8 and B.9) in Appendix B we find that the perturbative equations we have are:–

$$F^{-1}(p^2) = 1 - \frac{\alpha_0}{4\pi} \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \left[\frac{3}{4} - \left(\xi + \frac{3}{4}\right) \frac{F(k^2)}{F(p^2)} \right]$$

$$\frac{\Sigma(p^2)}{F(p^2)} = m_0 + \frac{\alpha_0}{4\pi} \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \left[\frac{3}{2}\Sigma(k^2) + \xi \frac{F(k^2)}{F(p^2)} \Sigma(p^2) + \frac{3}{2} \frac{F(k^2)}{F(p^2)} \Sigma(k^2) \right]$$

again starting from $F(p^2) = 1$ and $\Sigma(p^2) = m_0$ we find:–

$$F(p^2) = 1 + \frac{\alpha_0 \xi}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{\xi^2}{2} - \frac{3\xi}{8}\right) \left(\frac{\alpha_0}{4\pi}\right)^2 \log^2 \frac{p^2}{\Lambda^2} \\ + \left(\frac{\xi^3}{6} - \frac{3\xi^2}{8} + \frac{3\xi}{16}\right) \left(\frac{\alpha_0}{4\pi}\right)^3 \log^3 \frac{p^2}{\Lambda^2} + \ldots \quad (2.15)$$

and

$$\Sigma(p^2) = m_0 \left[1 - \frac{3\alpha_0}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{9}{2} + \frac{3\xi}{8}\right) \left(\frac{\alpha_0}{4\pi}\right)^2 \log^2 \frac{p^2}{\Lambda^2} \right. \\ \left. + \left(-\frac{9}{2} - \frac{3\xi}{16} - \frac{\xi^2}{8}\right) \left(\frac{\alpha_0}{4\pi}\right)^3 \log^3 \frac{p^2}{\Lambda^2} + \ldots \right]. \quad (2.16)$$

(iii) With the *real* vertex, what ever it is, we would repeat the Born process substituting in higher and higher order values of Γ , F and Σ to solve the equations.

We don't know what the vertex is, but we do know that its perturbative expansion starts off as $\Gamma^\mu = \gamma^\mu + O(\alpha)$, and so we can calculate F and Σ to $O(\alpha)$ (where we get the same result as for both the bare and Ball-Chiu ansatze):–

$$F(p^2) = 1 + \frac{\alpha_0 \xi}{4\pi} \log \frac{p^2}{\Lambda^2} + O(\alpha^2)$$
$$\Sigma(p^2) = m_0 \left[1 - \frac{3\alpha_0}{4\pi} \log \frac{p^2}{\Lambda^2} + O(\alpha^2) \right].$$

However from the MR leading log constraints we know that for the real MR vertex:–

$$F(p^2) = 1 + \frac{\alpha_0 \xi}{4\pi} \log \frac{p^2}{\Lambda^2} + \frac{\xi^2}{2} \left(\frac{\alpha_0}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} + \frac{\xi^3}{6} \left(\frac{\alpha_0}{4\pi} \right)^3 \log^3 \frac{p^2}{\Lambda^2} + O(\alpha^4) \tag{2.17}$$

$$\Sigma(p^2) = m_0 \left[1 - \frac{3\alpha_0}{4\pi} \log \frac{p^2}{\Lambda^2} + \frac{9}{2} \left(\frac{\alpha_0}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} - \frac{9}{2} \left(\frac{\alpha_0}{4\pi} \right)^3 \log^3 \frac{p^2}{\Lambda^2} + O(\alpha^4) \right]. \tag{2.18}$$

It is the mass that is the physical object and hence it should be the mass that is independent of the gauge. If we compare the three results from the bare, Ball-Chiu and real vertices writing the results as

$$\Sigma(p^2) = m_0 \left[1 + \frac{\alpha_0}{4\pi} A_0 \log \frac{p^2}{\Lambda^2} + \left(\frac{\alpha_0}{4\pi} \right)^2 (B_0 + B_1 \xi) \log^2 \frac{p^2}{\Lambda^2} \right.$$
$$\left. + \left(\frac{\alpha_0}{4\pi} \right)^3 (C_0 + C_1 \xi + C_2 \xi^2) \log^3 \frac{p^2}{\Lambda^2} + \ldots \right]$$

we have

	A_0	B_0	B_1	C_0	C_1	C_2
Bare	-3	$+9/2$	-3	$-9/2$	$+9$	-4
Ball–Chiu	-3	$+9/2$	$+3/8$	$-9/2$	$-3/16$	$-1/8$
Real	-3	$+9/2$	0	$-9/2$	0	0

Table 2.1. Coefficients in the leading log. expansion of $\Sigma(p^2)$.

and the coefficients for α^2 and α^3 are plotted in fig. 2.3. As we can see the results from the bare vertex vary dramatically with the gauge. This shows how unsuitable the bare vertex is with regard to the gauge invarience of physical objects (as studied in section 2.3). The Ball-Chiu ansatz gives much better results, but they are still not what is expected from the real vertex, especially the α^2 coefficients. It is in the difference between the Ball-Chiu and the *real* results that the affects of the transverse vertex lie and it is in order to understand the transverse part that we move on to look at the perturbative expansion for the vertex.

2.5d *The Vertex (perturbative)* :

In order to be able to get any sort of idea about Γ_T at all we look at the perturbative expansion of the vertex. We wish to study the affects that the vertex has on the fermion mass/wavefunction to leading logs.. In order to do this we need to calculate those parts of Γ that affect the leading log. parts of the propagator. From a quick look at the S-DE for the propagator (2.1) we can see that in the leading log. approx. the integral in the equation is of the form:–

$$\int \frac{d^4k}{k^4}(\not{k} + m_0)\Gamma(k,p)$$

and so using the fact that $\int$ odd powers $= 0$ we see that the leading log. results for the fermion are affected by terms $O(k^0)$ and $O(k^{-1})$ in Γ. The checks that we carried out on the results from the various ansatze in the last section showed their first differences from each other at $O(\alpha^2)$, which corresponds to knowing Γ to $O(\alpha)$. So the first clues to Γ_T should appear at $O(\alpha)$ in Γ and so we start off by calculating the $O(k^0)$ and $O(k^{-1})$ contributions to Γ up to $O(\alpha)$, with $k^2 \gg p^2 \gg m^2$. The perturbative expansion for Γ (S-DE) to this order is shown in fig. 2.4, and can be written as:–

$$-ie\Gamma^\mu(k,p) = -ie\gamma^\mu - ie\Lambda^\mu$$

where

$$-ie\Lambda^{\mu} = \int_M d^4w(-ie\gamma^{\rho})S_F(k-w)(-ie\gamma^{\mu})S_F(p-w)(-ie\gamma^{\nu})\Delta_{\nu\rho}(w)$$

with

$$\Delta_{\nu\rho}(w) = \frac{-i}{w^2}[\delta_{\nu\rho} + (\xi - 1)\frac{w_\nu w_\rho}{w^2}]$$

and

$$S_F(q) = i\frac{\not{q} + m}{q^2 - m^2}.$$

We can split the integral into four discrete parts:–

I_1 : odd number of γ–matrices and $\delta_{\nu\rho}$ part of the photon propagator.

I_2 : odd number of γ–matrices and $w_\nu w_\rho$ part of the photon propagator.

I_3 : even number of γ–matrices and $\delta_{\nu\rho}$ part of the photon propagator.

I_4 : even number of γ–matrices and $w_\nu w_\rho$ part of the photon propagator.

For historical reasons we shall call I_1 and I_2 the massless parts and I_3 and I_4 the massive parts. We now look at each one in turn:–

(a)I_1:

$$I_1 = -\int_M \frac{d^4w\ e^3}{(2\pi)^4}\left(\frac{\gamma^\nu(\not{k} - \not{w})\gamma^\mu(\not{p} - \not{w})\gamma^\rho + m^2\gamma^\nu\gamma^\mu\gamma^\rho}{w^2[(k-w)^2 - m^2][(p-w)^2 - m^2]}\right)\delta_{\nu\rho}.$$

Wick rotating we get:–

$$\begin{aligned}
I_1 &= \frac{-i\ e^3}{(2\pi)^4}\int_E \frac{d^4w\gamma^\nu(\not{k} - \not{w})\gamma^\mu(\not{p} - \not{w})\gamma^\nu}{w^2[(k-w)^2 + m^2][(p-w)^2 + m^2]} \\
&= \frac{-i\ e^3}{(2\pi)^4}(-2)\int \frac{d^4w(\not{p} - \not{w})\gamma^\mu(\not{k} - \not{w})}{w^2[(k-w)^2 + m^2][(p-w)^2 + m^2]} \\
&= \frac{4i\ e^3}{(2\pi)^4}\int^{\Lambda} d^4w\int_0^1 dx\int_0^{1-x} dy[(1-x)(1-y)\not{p}\gamma^\mu\not{k} - y(1-y)\not{p}\gamma^\mu\not{p} \\
&\qquad - x(1-x)\not{k}\gamma^\mu\not{k} + xy\not{k}\gamma^\mu\not{p} - \frac{1}{2}w^2\gamma^\mu]\ /[w^2 + L]^3
\end{aligned}$$

using the Feynman integral rule with:–

$$L = k^2x(1-x) + p^2y(1-y) - 2k.pxy + m^2x + m^2y$$

(see (C.1) Appendix C). Now from (C.2 and C.3):–

$$\int^{\Lambda} \frac{d^4w}{[w^2+L]^3} = \frac{\pi^2}{2L} \quad \text{and} \quad \int^{\Lambda} \frac{d^4w\ w^2}{[w^2+L]^3} = \pi^2 \log\frac{\Lambda^2}{L} - \frac{3\pi^2}{2}.$$

Therefore

$$\begin{aligned} I_1 = \frac{i\,e^3}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy \Bigg[&\big[(1-x)(1-y)\not{p}\gamma^\mu\not{k} - y(1-y)\not{p}\gamma^\mu\not{p} - x(1-x)\not{k}\gamma^\mu\not{k} \\ &+ xy\not{k}\gamma^\mu\not{p}\,\big]\ /2L - \frac{1}{2}\gamma^\mu(\log\frac{\Lambda^2}{L} - \frac{3}{2})\Bigg] \\ = \frac{i\,e^3}{4\pi^2}\Bigg[\frac{1}{2}\int_0^1 dx \int_0^1 dz\ &(1-x)\big[-x(1-x)\not{k}\gamma^\mu\not{k} + xz(1-x)\not{k}\gamma^\mu\not{p} \\ &+ (1-x)(1-z+xz)\not{p}\gamma^\mu\not{k} \\ &- z(1-x)(1-z+xz)\not{p}\gamma^\mu\not{p}\big]\ /L \\ &- \frac{1}{2}\gamma^\mu\int_0^1 dx\int_0^1 dz\ (1-x)(\log\frac{\Lambda^2}{L} - \frac{3}{2})\Bigg] \end{aligned}$$

where $y = z(1-x)$. Now the only parts we want are those that give $O(k^0 \log k^2)$ or $O(k^{-1}\log k^2)$ as these give the leading logs in S-DE for the fermion propagator. Using the results of Appendix C, (C.7) and $0 = \int dx\ x^n(1-x)/L \quad n = 1,2,3,\ldots$ (C.8), we see that

$$I_1 \overset{k^2\to\infty}{=} \frac{i\,e^3}{16\pi^2}\left[\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{\not{p}\gamma^\mu\not{k}}{k^2}\log\frac{k^2}{p^2}\right].$$

(b)I_2:

$$I_2 = \frac{-i\, e^3(\xi - 1)}{(2\pi)^4} \int_E \frac{d^4w \not{w}(\not{k} - \not{w})\gamma^\mu(\not{p} - \not{w})\not{w}}{w^4[(k - w)^2 + m^2][(p - w)^2 + m^2]}$$

$$= \frac{-i6\, e^3(\xi - 1)}{(2\pi)^4} \int d^4w \int_0^1 dx \int_0^{1-x} dy(1 - x - y)(\not{w} + \not{k}x + \not{p}y)$$

$$(\not{k}(1 - x) - \not{p}y - \not{w})\gamma^\mu(\not{p}(1 - y) - \not{k}x - \not{w})(\not{w} + \not{k}x + \not{p}y)/(w^2 + L)^4$$

from the Feynman integral rule for a repeated term in the denominator and L the same as before.

$$I_2 = \frac{-i6\, e^3(\xi - 1)}{(2\pi)^4} \left(I_{2,4} + I_{2,2} + I_{2,0}\right)$$

where $I_{2,i} = w^i \times fn(x, y)$ and from (C.4-C.6):–

$$\int \frac{d^4w\; w^4}{[w^2 + L]^4} = \pi^2 \left[\log \frac{\Lambda^2}{L} - \frac{11}{6}\right], \quad \int \frac{d^4w\; w^2}{[w^2 + L]^4} = \frac{\pi^2}{3L}$$

$$\text{and} \quad \int \frac{d^4w}{[w^2 + L]^4} = \frac{\pi^2}{6L^2}.$$

Then

$$I_{2,4} = \pi^2 \int_0^1 dx \int_0^1 dz(1 - x)^2(1 - z)\gamma^\mu \left[\log \frac{\Lambda^2}{L} - \frac{11}{6}\right]$$

$$\begin{aligned}
I_{2,2} = \pi^2 \int_0^1 dx \int_0^1 dz \frac{(1 - x)^2}{3L} \Big[& - \not{p}\gamma^\mu \not{k} \frac{1}{2}(1 - x)(1 - z + 2zx) \\
& - k^\mu \not{p}(1 - 2x)(1 - x)z - \not{p}\not{k}\gamma^\mu(1 - x)^2 z \\
& + \frac{1}{2}\not{k}\gamma^\mu \not{k}x\,(1 - 2x) - k^\mu \not{k}x\,(1 - 2x) \\
& - \gamma^\mu \not{p}\not{k}x\,(1 - z + xz) - \not{k}\gamma^\mu \not{p}zx\,(1 - x) \\
& - \gamma^\mu k^2 x\,(1 - 2x) - p^\mu \not{k}x\,(1 - 2z + 2zx)\Big](1 - z)
\end{aligned}$$

$$\begin{aligned}
I_{2,0} = \pi^2 \int_0^1 dx \int_0^1 dz \frac{(1 - x)^2}{6L^2} \Big[& \gamma^\mu \not{p}\not{k}k^2x^2(1 - x)(1 - z + zx) - \gamma^\mu k^4 x^3(1 - x) \\
& - \gamma^\mu \not{k}\not{p}k^2x^2z(1 - x)^2 + \not{k}\not{p}\gamma^\mu k^2x^3z(1 - x) \\
& - \not{p}\not{k}\gamma^\mu k^2x^2z(1 - x)^2\Big](1 - z)
\end{aligned}$$

(where we have dropped terms down in order of k and so cannot give const$\times \log k^2$ terms).

Using the results in Appendix C, (C.7-C.12), we have that at the relevant order:–

$$I_{2,4} = \frac{1}{6}\gamma^\mu \pi^2 \log \frac{\Lambda^2}{k^2}$$

$$I_{2,2} = \frac{\pi^2}{3}\int_0^1 dz\left[-\frac{1}{2}(1-z)^2 \not{p}\gamma^\mu \not{k} - k^\mu \not{p} z(1-z) - \not{p}\not{k}\gamma^\mu z(1-z)\right]\frac{1}{k^2}\log\frac{k^2}{p^2}$$

$$= \frac{\pi^2}{18k^2}\left[-\not{p}\gamma^\mu\not{k} - k^\mu\not{p} - \not{p}\not{k}\gamma^\mu\right]\log\frac{k^2}{p^2}$$

$$I_{2,0} = 0$$

putting this all together we obtain:–

$$I_2 = \frac{i\, e^3(\xi - 1)}{16\pi^2}\left[\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{1}{3k^2}\left(\not{p}\gamma^\mu\not{k} + k^\mu\not{p} + \not{p}\not{k}\gamma^\mu\right)\log\frac{k^2}{p^2}\right]$$

$$= \frac{i\, e^3(\xi - 1)}{16\pi^2}\left[\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{k^\mu\not{p}}{k^2}\log\frac{k^2}{p^2}\right].$$

(c)I_3:

$$I_3 = \frac{i\, e^3 m}{(2\pi)^4}\int_E \frac{d^4w\left[\gamma^\nu(\not{k} - \not{w})\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu(\not{p} - \not{w})\gamma^\nu\right]}{w^2\left[(k-w)^2 + m^2\right]\left[(p-w)^2 + m^2\right]}$$

$$= \frac{i\, e^3 m}{(2\pi)^4}\int_E \frac{d^4w(k^\mu + p^\mu - 2w^\mu)}{w^2\left[(k-w)^2 + m^2\right]\left[(p-w)^2 + m^2\right]}$$

$$= \frac{i8\, e^3 m}{(2\pi)^4}\int d^4w \int_0^1 dx \int_0^{1-x} dy \frac{\left[k^\mu(1-2x) + p^\mu(1-2y) - 2w^\mu\right]}{(w^2 + L)^3}$$

$$= \frac{i\, e^3\, 4m\pi^2}{(2\pi)^4}\int_0^1 dz \int_0^1 dx \frac{(1-x)\left[k^\mu(1-2x) + p^\mu(1-2z+2zx)\right]}{L}.$$

From Appendix C, (C.8 and C.9), to the order required:–

$$I_3 = \frac{i\, e^3 m}{4\pi^2}\int_0^1 dz \frac{k^\mu}{k^2}\left[\log\frac{k^2}{p^2}\right]$$

$$= \frac{i\, e^3 m}{4\pi^2}\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}.$$

(d)I_4:

$$I_4 = \frac{i\,e^3 m(1-\xi)}{(2\pi)^4} \int_E \frac{d^4w\left[\not w(\not k - \not w)\gamma^\mu \not w + \not w\gamma^\mu(\not p - \not w)\not w\right]}{w^4\left[(k-w)^2+m^2\right]\left[(p-w)^2+m^2\right]}$$

$$= \frac{i\,e^3 m(1-\xi)6}{(2\pi)^4} \int d^4w \int_0^1 dx \int_0^{1-x} dy(1-x-y)(\not w + \not k x + \not p y)$$

$$\left[\left[(1-x)\not k - \not p y - \not w\right]\gamma^\mu + \gamma^\mu\left[(1-y)\not p - \not k x - \not w\right]\right]$$

$$(\not w + \not k x + \not p y)/(w^2+L)^4$$

$$= \frac{i\,e^3 m(1-\xi)6}{(2\pi)^4} \int d^4w \int_0^1 dx \int_0^{1-x} dy(1-x-y)\frac{w^2C+D}{(w^2+L)^4}$$

where to the relevant orders, in k:–

$$C = k^\mu(1-3x)$$

$$D = \gamma^\mu \not k k^2 x^2 - 2k^\mu k^2 x^3.$$

Then, using (C.4 and C.5):–

$$I_4 = \frac{i\,e^3 m(1-\xi)\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^1 dz(1-x)^2(1-z)\left(\frac{2C}{L}+\frac{D}{L^2}\right)$$

$$= I_{4,C} + I_{4,D}$$

where $I_{4,C}$ and $I_{4,D}$ are the C and D parts respectively. From Appendix C, (C.8-C.10):–

$$I_{4,C} = \frac{i\,e^3 m(1-\xi)}{16\pi^2} \int_0^1 dz\; 2(1-z)\frac{k^\mu}{k^2}\left[\log\frac{k^2}{p^2}\right]$$

$$= \frac{i\,e^3 m(1-\xi)}{16\pi^2}\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}$$

and

$$I_{4,D} = 0.$$

Thus giving us:–

$$I_4 = \frac{i\, e^3 m(1 - \xi)}{16\pi^2}\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}.$$

Putting this all together we have:–

$$\begin{aligned}
I_1 + I_2 + I_3 + I_4 = \frac{i\, e^3}{16\pi^2}\Bigg[&\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{\not p\gamma^\mu \not k}{k^2}\log\frac{k^2}{p^2} \\
&+ (1 - \xi)\left(\gamma^\mu \log\frac{k^2}{\Lambda^2} + k^\mu \not p \log\frac{k^2}{p^2}\right) \\
&+ 4m\frac{k^\mu}{k^2}\log\frac{k^2}{p^2} + m(1 - \xi)\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}\Bigg] \\
= \frac{i\, e\alpha}{4\pi}\Bigg[&\xi\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{1}{k^2}\left(\not p\gamma^\mu \not k + (1 - \xi)k^\mu \not p\right)\log\frac{k^2}{p^2} \\
&+ m(3 + \xi)\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \Gamma^\mu = \gamma^\mu - \frac{\alpha}{4\pi}\Bigg[&\xi\gamma^\mu \log\frac{k^2}{\Lambda^2} + \frac{1}{k^2}\left(\not p\gamma^\mu \not k + (1 - \xi)k^\mu \not p\right)\log\frac{k^2}{p^2} \\
&+ m(3 + \xi)\frac{k^\mu}{k^2}\log\frac{k^2}{p^2}\Bigg] \qquad (2.19)
\end{aligned}$$

in Euclidean space-time.

From section 2.5c (iii) we know that for the real vertex $F(p^2)$ and $\Sigma(p^2)$ are, up to $O(\alpha)$ in leading logs:–

$$\begin{aligned}
F(p^2) &= 1 + \frac{\alpha\xi}{4\pi}\log\frac{p^2}{\Lambda^2} \\
\Sigma(p^2) &= m\left[1 - \frac{3\alpha}{4\pi}\log\frac{p^2}{\Lambda^2}\right]
\end{aligned} \qquad (2.20)$$

we also know that $\Gamma_T = \Gamma - \Gamma_L = \Gamma - \Gamma_{BC}$, so substituting (2.20) into the identity for Γ_{BC}, (2.6), and using (2.19) we can find Γ_T for $O(\alpha)$ up to $O(k^0, k^{-1})$. Doing this we have:–

$$\Gamma^{\mu}_{BC} = \gamma^{\mu} - \gamma^{\mu}\frac{\alpha\xi}{4\pi}\log\frac{pk}{\Lambda^2} - \frac{\alpha\xi}{8\pi}\frac{(k^{\mu}\not{k} + k^{\mu}\not{p} + p^{\mu}\not{k})}{k^2}\log\frac{p^2}{k^2} + \frac{m\alpha}{4\pi}(3+\xi)\frac{k^{\mu}}{k^2}\log\frac{k^2}{p^2}.$$

This is in Minkowski space-time, as that is where we defined the W-TI and our Feynman rules. To undertake our comparison we need to perform a Wick rotation on (2.19) to bring it into Minkowski space-time as well. We then have:–

$$\Gamma^{\mu}_{T} = -\frac{\alpha\xi}{8\pi}\gamma^{\mu}\log\frac{k^2}{p^2} - \frac{\alpha}{4\pi}\frac{1}{k^2}\left(\not{p}\gamma^{\mu}\not{k} + (1-\xi)k^{\mu}\not{p}\right)\log\frac{k^2}{p^2}$$
$$+ \frac{\alpha\xi}{8\pi}\frac{1}{k^2}(k^{\mu}\not{k} + k^{\mu}\not{p} + p^{\mu}\not{k})\log\frac{p^2}{k^2}$$

to $O(\alpha, k^{-1})$. When we substitute the gauge independent part of this into the fermion S-DE we find as expected it does not contribute to the leading order contribution of the mass. Its contribution is, from (2.1):–

$$\int\frac{d^4k}{k^2}tr\left[\gamma^{\mu}(\not{k} + \Sigma(k^2))(\not{p}\gamma^{\nu}\not{k} - k^{\nu}\not{p})\right]\frac{1}{k^2}\log\frac{k^2}{p^2}\frac{\delta_{\mu\nu}}{q^2} \qquad (\Gamma^{\mu}_{T}q_{\mu} = 0)$$
$$= \int\frac{d^4k}{k^4q^2}tr\left[\gamma^{\mu}\Sigma(k^2)(\not{p}\gamma^{\mu}\not{k} - k^{\mu}\not{p})\right]\log\frac{k^2}{p^2}$$

$\neq$leading logs.

Surprisingly however the gauge independent part also does not contribute to the leading log contribution of the wavefunction:–

$$\int\frac{d^4k}{k^4q^2}tr\left[\not{p}\gamma^{\mu}(\not{k} + \Sigma(k^2))(\not{p}\gamma^{\mu}\not{k} - k^{\mu}\not{p})\right]\log\frac{k^2}{p^2}$$
$$= \int\frac{d^4k}{k^4q^2}tr\left[\not{p}\gamma^{\mu}\not{k}\not{p}\gamma^{\mu}\not{k} - \not{p}\not{k}\not{k}\not{p}\right]\log\frac{k^2}{p^2}$$
$$= \int\frac{d^4k}{k^4q^2}tr\left[\,4(k.p)^2 - p^2k^2\right]\log\frac{k^2}{p^2}$$
$$\sim \int\frac{dk^2}{k^2}\left[\,\frac{p^4}{k^2} + p^2 - p^2\right]\log\frac{k^2}{p^2} \sim \int\frac{dk^2}{k^4}\log\frac{k^2}{p^2}$$

$\neq$leading logs.

Now as stated before we are only interested in the parts of Γ_T that give leading log contributions in the fermion S-DE. We are therefore at liberty to drop the above gauge independent parts. We are then left with:–

$$\Gamma_T^\mu = \frac{\alpha\xi}{8\pi}\left[-\gamma^\mu + \frac{1}{k^2}\left(k^\mu \not{k} - k^\mu \not{p} + p^\mu \not{k}\right)\right] \log\frac{k^2}{p^2}.$$

The transverse vertex lies in the vector space transverse to q_μ and so we shall write Γ_T in terms of a basis set of this transverse vector space, which is eight dimensional. The basis set we shall choose is the one introduced by Ball and Chiu[42], viz:–

$$T_1^\mu = p^\mu(k.q) - k^\mu(p.q)$$

$$T_2^\mu = T_1^\mu(\not{k} + \not{p})$$

$$T_3^\mu = q^2\gamma^\mu - q^\mu \not{q}$$

$$T_4^\mu = T_1^\mu p^\nu k^\rho \sigma_{\nu\rho}$$

$$T_5^\mu = \sigma^{\mu\nu} q_\nu$$

$$T_6^\mu = \gamma^\mu(k^2 - p^2) - (k+p)^\mu(\not{k} - \not{p})$$

$$T_7^\mu = \frac{1}{2}(k^2 - p^2)\left[\gamma^\mu(\not{k} + \not{p}) - p^\mu - k^\mu\right] + (k+p)^\mu p^\nu k^\rho \sigma_{\nu\rho}$$

$$T_8^\mu = -\gamma^\mu p^\nu k^\rho \sigma_{\nu\rho} + p^\mu \not{k} - k^\mu \not{p},$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

Now T_1, T_4, T_5, T_7 have the wrong number of γ-matrices to write Γ_T in terms of them, they would have been associated with anything non-zero coming from the massive terms $I_{3,4}$ that couldn't be accounted for by Γ_{BC}. If we look at the $k^2 \gg p^2$ limit of the remaining tensors we see that:–

$$T_6^\mu \overset{k^2 \gg p^2}{=} k^2 \left[\gamma^\mu - \frac{1}{k^2} \left(k^\mu \not{k} - k^\mu \not{p} + p^\mu \not{k} \right) \right]$$

just the form we need. As terms not contributing to leading logs in the fermion can be neglected we can write:–

$$\Gamma_T^\mu = -\frac{\alpha\xi}{8\pi} \frac{T_6^\mu}{k^2} \log \frac{k^2}{p^2}. \tag{2.21}$$

We could have obtained an indication of this result by solely using M.R. to constrain our result, as from (2.15 – 2.18) in section 2.5c we know that the Ball–Chiu ansatz gives the following leading log results for the S-DE's in the perturbative region:–

$$\begin{aligned}
\frac{1}{F(p^2)} &= 1 - \frac{\alpha\xi}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{3\xi}{8} + \frac{\xi^2}{2} \right) \left(\frac{\alpha}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} \\
\frac{\Sigma(p^2)}{F(p^2)} &= m \left[1 - (3 + \xi) \frac{\alpha}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{9}{2} + \frac{15}{4}\xi + \frac{\xi^2}{2} \right) \left(\frac{\alpha}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} \right]
\end{aligned}$$

and the real vertex gives:–

$$\begin{aligned}
\frac{1}{F(p^2)} &= 1 - \frac{\alpha\xi}{4\pi} \log \frac{p^2}{\Lambda^2} + \frac{\xi^2}{2} \left(\frac{\alpha}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} \\
\frac{\Sigma(p^2)}{F(p^2)} &= m \left[1 - (3 + \xi) \frac{\alpha}{4\pi} \log \frac{p^2}{\Lambda^2} + \left(\frac{9}{2} + 3\xi + \frac{\xi^2}{2} \right) \left(\frac{\alpha}{4\pi} \right)^2 \log^2 \frac{p^2}{\Lambda^2} \right].
\end{aligned}$$

So the extra transverse part must contribute $-3\xi/8(\alpha/4\pi)^2 \log^2(p^2/\Lambda^2)$ to the $1/F$ equation and $-m3\xi/4(\alpha/4\pi)^2 \log^2(p^2/\Lambda^2)$ to the Σ/F equation. The extra leading log piece that they contribute to the $1/F$ equation is:–

$$\frac{-\alpha}{4p^2 4\pi^3} \int_E \frac{d^4k F(k^2)}{k^2 q^2} tr\left[\not{p}\gamma^\mu \not{k}\Gamma_T^\nu\right]\delta_{\mu\nu} \tag{2.22}$$

and to the Σ/F equation:–

$$\frac{\alpha}{4\pi^3 4} \int_E \frac{d^4k F(k^2)}{k^2 q^2} tr\left[\gamma^\mu(\not{k} + \Sigma(k^2))\Gamma_T^\nu\right]\delta_{\mu\nu}. \tag{2.23}$$

As $\Gamma_T^\mu = \sum_{i=1}^8 a_i T_i^\mu$ we can substitute in each of the basis vectors in turn into (2.22 and 2.23) and generate some relations. We in fact substitute in $a_i(k^2, p^2, \alpha)T_i^\mu$, where, because the extra pieces needed from Γ_T are $\sim \alpha^2 \log^2(p^2/\Lambda^2)$, we know that $a_i \sim \alpha/(4\pi)\log^2(k^2/p^2)$, as the only way to get $\log^2(p^2/\Lambda^2)$ is from $\int \frac{dk^2}{k^2}\log k^2$. Γ_T^μ has dimensions $(\text{momentum})^0$ and so in this leading log calculation we also include in a_i enough powers of k^2 to ensure this. As pointed out before $T_{1,4,5,7}$ are associated with the massive terms, $I_{3,4}$, and $T_{2,3,6,8}$ (being odd in γ-matrices) with the massless terms $I_{1,2}$. In order to obtain the correct dimensions we need to multiply $T_{1,4,5,7}$ by m^1 and $T_{2,3,6,8}$ by m^0. As can be seen from the Ball-Chiu ansatz the natural way to obtain $(\alpha/4\pi)\log(k^2/p^2)$ using $F(p^2)$ and $\Sigma(p^2)$, (which is where this structure has to come from) is from $1/F(p^2) - 1/F(k^2)$ terms which give $\xi(\alpha/4\pi)\log(k^2/p^2)$, whilst to obtain $m(\alpha/4\pi)\log(k^2/p^2)$ we should use $\Sigma(p^2)/F(p^2) - \Sigma(k^2)/F(k^2)$ terms which give $m(3+\xi)(\alpha/4\pi)\log(k^2/p^2)$. So we are lead naturally to write

$$a_{1,4,5,7} = -m(3+\xi)\frac{\alpha}{4\pi}\frac{t_i \log k^2/p^2}{X_i(k^2)}$$

$$\text{and } a_{2,3,6,8} = -\frac{\alpha\xi}{4\pi}\frac{t_i \log k^2/p^2}{X_i(k^2)}$$

where t_i are constants and $X_i(k^2)$ ensure the correct dimensions. $X_{1,2,7} = k^4$, $X_{3,5,6,8} = k^2$ and $X_4 = k^6$.

Putting these into (2.22 and 2.23) and working to leading logs $O(\alpha^2)$ (where as we have $a_i \sim \alpha \log(k^2/p^2)$ we take $F(k^2) = 1$ and $\Sigma(k^2) = m$ in (2.22 and 2.23)) we obtain the following results:–

Tensor i	**l.l. cont. to** $1/\mathcal{F}(p^2)$ $\times - \frac{3\alpha^2}{128\pi^2}\ln^2\frac{p^2}{\Lambda^2}$	**l.l. cont. to** $\Sigma(p^2)/\mathcal{F}(p^2)$ $\times - \frac{3\alpha^2 m}{64\pi^2}\ln^2\frac{p^2}{\Lambda^2}$
1	–	–
2	$-\xi$	–
3	-2ξ	$+2\xi$
4	–	–
5	–	$+2(3+\xi)$
6	$+2\xi$	$+2\xi$
7	–	$-(3+\xi)$
8	$+2\xi$	–
Pert. Answer	$+\xi$	$+\xi$

Table 2.2. Leading log. ontributions to the O(α^2) part of the fermion S-DE's from the O(α) component of the traverse basis set.

Thus we have the identities:–

$$(-t_2 - 2t_3 + 2t_6 + 2t_8) = 1$$

$$2\xi t_3 + 2(3+\xi)t_5 + 2\xi t_6 - (3+\xi)t_7 = \xi$$

equating powers of ξ in the second equation we then have:–

$$0 = t_7 - t_5, \quad 2t_3 + 2t_6 = 1, \quad t_2 + 4t_3 - 2t_8 = 0.$$

Attempting to introduce the minimum number of parameters we see that we can in fact set $t_7 = 0 = t_5 = t_2 = t_3 = t_8$, $t_6 = 1/2$ and still satisfy the equations.

Indeed the equations are most suggestive of this kind of simplification, this gives us:–

$$\Gamma_T^{\mu} = -\frac{\alpha\xi}{8\pi}\frac{T_6^{\mu}}{k^2}\log\frac{k^2}{p^2}$$

which is nothing but (2.21).

There is of course no 'ipso facto' reason to set $t_{2,3,5,7,8} = 0$, but as this let's us write Γ_T^{μ} in the most minimalist form it is aesthetically very satisfying to see that it gives us the same form that we had from explicitly calculating Γ_T^{μ}.

2.5e *The Vertex (non-perturbative)* :

We have a form for the leading log form of Γ_T up to $O(\alpha)$. How does this help us to obtain this non-perturbative ansatz for Γ_T that was hinted to be the result of this chapter? The answer is that we use the perturbative form as a shadow of 'the' non-perturbative answer and try to construct the best ansatz we can for the non-perturbative answer using such aids as the k, p symmetry of the vertex, analiticity of physical objects, etc.... We then see whether our ansatz satisfies the M.R. constraints on F and Σ to next-to-leading order to check its validity (at least to next-to-leading order). So let us go ansatz building:–

We have to $O(\alpha)$ in leading logs:–

$$\Gamma_T^{\mu} = -\frac{\alpha\xi}{8\pi}\frac{T_6^{\mu}}{k^2}\log\frac{k^2}{p^2}. \tag{2.24}$$

Now first of all we'd like to write this in a slightly less obviously perturbative way. To do this we remember that the structure of the ansatz can only depend upon F and Σ as they are the only functions left in qQED. To $O(\alpha)$ in leading logs we know that

$$F(p^2) = 1 + \frac{\xi\alpha}{4\pi}\log\frac{p^2}{\Lambda^2}$$
$$\Sigma(p^2) = m\left[1 + \frac{3\alpha}{4\pi}\log\frac{p^2}{\Lambda^2}\right]$$

(2.24) has no term m in it and as we expect, from W-TI and M.R. (ie. $Z_1 = Z_2$), that these functions come into Γ in the form $1/F$ and Σ/F then it is easy to see that we should write

$$-\frac{\xi\alpha}{4\pi}\log\frac{p^2}{k^2} = \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)$$

when (2.24) becomes

$$\Gamma_T^\mu(k,p) = \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{T_6^\mu}{k^2}. \tag{2.25}$$

Which certainly looks a lot more non-perturbative than before. The denominator k^2 is a perturbative object and so we wish to write it as $d(k,p)$, say, which is non-perturbative. Then:–

$$\Gamma_T^\mu(k,p) = \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{T_6^\mu}{d(k,p)}. \tag{2.26}$$

We want to find a sensible form for $d(k,p)$ and then we shall be content to call (2.26) our ansatz for Γ_T.

As $1/F(k^2)\ -1/F(p^2)$ and T_6 are both k,p anti-symmetric and the overall form for Γ_T must be k,p symmetric we know that:–

(i) $d(k,p)$ must be k,p symmetric.

From dimensionality of t_6 and that Γ_T must be dimensionless we know that:–

(ii) $d(k,p)$ must have dimensions of (momentum)^2 and $\rightarrow k^2$ as $k^2 \rightarrow \infty$.

From the lack of kinematic singularities in Γ and Γ_{BC} we know that:–

(iii) $d(k,p)$ must be such that Γ_T is free of kinematic singularities.

For Γ_T to be a sensible physical object we know that:–

(iv) $d(k,p)$ must be an analytic function of k,p.

and finally as we believe that Minkowski and Euclidean space-times should be related via a Wick rotation then:–

(v) $d(k,p)$ should be such that Γ_T does not change sign under a Wick rotation.

An ansatz for $d(k,p)$ which obeys all these requirements is given by:–

$$d(k,p) = \frac{(k^2 - p^2)^2 + \left(\Sigma^2(k^2) + \Sigma^2(p^2)\right)^2}{k^2 + p^2}.$$

In the massive theory this then satisfies (i), (ii), (iii) (at least for $k^2 \in \Re$ which is what we want),(iv) and (v).

What though about the massless, $\Sigma = 0$, theory? We can hand wave at that, after all is there truely such a thing as a massless theory? (If you believe in α_c alright, but if not then $\Sigma(k^2,\alpha) \neq 0$ and no problem). In a theoretical sense there is a big difference between a massless theory, defined as one in which mass $= 0$ before calculations, and one in which mass $\rightarrow 0$ after calculations, the former often leading to theories that are too divergent to work with. This question crops up frequently in such things as quark models of hadrons[51]. In a strict theoretic sense we define here the massless theory to be the limit mass $\rightarrow 0$ of the massive theory. Numerically it makes no odds as the extra pole, when $\Sigma = 0$, from Γ in the fermion equation is finite when we integrate over it, but it is an important point to understand theoretically and had to be discussed. So we end up with:–

$$\Gamma^\mu = \Gamma^\mu_{BC} + \Gamma^\mu_T$$

where

$$\Gamma^\mu_T(k,p) = \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{T^\mu_6}{d(k,p)}$$

$$d(k,p) = \frac{(k^2 - p^2)^2 + \left(\Sigma^2(k^2) + \Sigma^2(p^2)\right)^2}{k^2 + p^2} \qquad (2.27)$$

as our ansatz.

2.6 *Checking against Multiplicative Renormalisation* :

In order to see whether our ansatz stands any chance of being useful we check to see whether it satisfies M.R. to next-to-leading logs for a perturbative expansion of F and Σ. As the ansatz is constructed to satisfy M.R. to $O(\alpha)$ in leading logs checking next-to-leading logs for all orders in α will be a reasonable initial test of the ansatz, at least in the perturbative region. From (B.15 and B.16), for a perturbative leading log calculation (where $k^2, p^2 \gg \Sigma^2 \sim m^2$) we have that:–

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha\xi}{4\pi\ p^2} \int_0^{\Lambda^2} dk^2 [\ \theta(p^2-k^2)\frac{k^2}{p^2} + \ \theta(k^2-p^2)\frac{F(k^2)}{F(p^2)}\frac{p^2}{k^2}]$$

$$\text{ie. } 1 = F(p^2) + \frac{\alpha\xi}{4\pi\ p^2} \int_0^{\Lambda^2} dk^2 [\ \theta(p^2-k^2)\ F(p^2)\frac{k^2}{p^2} + \ \theta(k^2-p^2)\ F(k^2)\frac{p^2}{k^2}] \tag{2.28}$$

and

$$\begin{aligned}
\frac{\Sigma(p^2)}{F(p^2)} = m + \frac{2\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 F(k^2) \Bigg[\\
& \frac{3}{4}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right)\Sigma(p^2)\left(\ \theta(p^2-k^2)\frac{1}{p^2} + \ \theta(k^2-p^2)\frac{1}{k^2}\right) \\
& + \frac{\xi}{4}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right)\Sigma(k^2)\left(\ \theta(p^2-k^2)\frac{1}{p^2} + \ \theta(k^2-p^2)\frac{1}{k^2}\right) \\
& + \frac{3}{4}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{\Sigma(k^2)}{k^2-p^2}\left(\ \theta(p^2-k^2)\frac{k^2}{p^2} + \ \theta(k^2-p^2)\frac{p^2}{k^2}\right) \\
& - \frac{\xi}{4}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\Sigma(k^2)\left(\ \theta(p^2-k^2)\frac{1}{p^2} - \ \theta(k^2-p^2)\frac{1}{k^2}\right) \\
& - \frac{1}{k^2-p^2}\left(\frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}\right)\left(\ \theta(p^2-k^2)\frac{k^2}{p^2} + \ \theta(k^2-p^2)\frac{p^2}{k^2}\right) \\
& - \frac{\xi}{2}\left(\frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}\right)\left(\ \theta(k^2-p^2)\frac{1}{k^2}\right)
\end{aligned}$$

$$+ \frac{3}{4}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\Sigma(k^2)\frac{k^2+p^2}{k^2-p^2}$$
$$\times\left(\theta(p^2-k^2)\frac{1}{p^2} + \theta(k^2-p^2)\frac{1}{k^2}\right)\Bigg]$$

$$\text{ie. } \Sigma(p^2) = mF(p^2) + \frac{2\alpha}{4\pi}\int_0^{\Lambda^2} dk^2 \Bigg[$$
$$\theta(p^2-k^2)\frac{1}{p^2}\Sigma(k^2)\left(\xi F(k^2) + \frac{3}{2}\left(F(p^2) + F(k^2)\right)\right)$$
$$+ \theta(k^2-p^2)\frac{1}{k^2}\left(\xi F(k^2)\Sigma(p^2) + \frac{3}{2}\left(F(p^2) + F(k^2)\right)\Sigma(k^2)\right)$$
$$+ \theta(p^2-k^2)\frac{3}{2}\frac{1}{k^2-p^2}\left[\Sigma(k^2)F(p^2)\left(\frac{k^2}{p^2}+1\right)\right.$$
$$\left. - \Sigma(k^2)F(k^2)\left(2\frac{k^2}{p^2}+1\right) + \Sigma(p^2)F(k^2)\frac{k^2}{p^2}\right]$$
$$+ \theta(k^2-p^2)\frac{3}{2}\frac{1}{k^2-p^2}\left[\Sigma(k^2)F(p^2)\left(\frac{p^2}{k^2}+1\right)\right.$$
$$\left. - \Sigma(k^2)F(k^2)\left(2\frac{p^2}{k^2}+1\right) + \Sigma(p^2)F(k^2)\frac{p^2}{k^2}\right]\Bigg]. \qquad (2.29)$$

To check whether M.R. is satisfied we use:–

$$F(p^2) = \sum_{n=0}^{\infty}\alpha^n A_n\left(\log^n\frac{p^2}{\Lambda^2} + B_n\log^{n-1}\frac{p^2}{\Lambda^2}\right)$$
$$\text{and}\quad \Sigma(p^2) = \sum_{n=0}^{\infty}\alpha^n C_n\left(\log^n\frac{p^2}{\Lambda^2} + D_n\log^{n-1}\frac{p^2}{\Lambda^2}\right)$$

where $B_0 = 0 = D_0$. Substituting these in to (2.28) we get:–

$$
\begin{aligned}
1 = & \sum_{n=0}^{\infty} \alpha^n A_n \left(\log^n \frac{p^2}{\Lambda^2} + B_n \log^{n-1} \frac{p^2}{\Lambda^2} \right) \\
& + \sum_{n=0}^{\infty} \frac{A_n \xi \alpha^{n+1}}{4\pi \; p^2} \int_0^{\Lambda^2} dk^2 \Bigg[\theta(p^2 - k^2) \left(\frac{k^2}{p^2} \log^n \frac{p^2}{\Lambda^2} + B_n \frac{k^2}{p^2} \log^{n-1} \frac{p^2}{\Lambda^2} \right) \\
& + \theta(k^2 - p^2) \left(\frac{p^2}{k^2} \log^n \frac{p^2}{\Lambda^2} + B_n \frac{p^2}{k^2} \log^{n-1} \frac{p^2}{\Lambda^2} \right) \Bigg] \\
= & \sum_{n=0}^{\infty} \alpha^n A_n \left(\log^n \frac{p^2}{\Lambda^2} + B_n \log^{n-1} \frac{p^2}{\Lambda^2} \right) \\
& + \sum_{n=0}^{\infty} \frac{A_n \xi \alpha^{n+1}}{4\pi} \left[\frac{1}{2} \log^n \frac{p^2}{\Lambda^2} - \frac{1}{n+1} \log^{n+1} \frac{p^2}{\Lambda^2} - B_n \frac{1}{n} \log^n \frac{p^2}{\Lambda^2} \right]
\end{aligned}
$$

to next-to-leading logs. Then equating coefficients of α^0 we have:–

$$
A_0 = 1.
$$

Equating coefficients of α^1 we have:–

$$
0 = A_1 \log \frac{p^2}{\Lambda^2} + A_1 B_1 + \frac{A_0 \xi}{4\pi} \left(\frac{1}{2} - \log \frac{p^2}{\Lambda^2} \right),
$$

equating logs we get:–

$$
A_1 = \frac{\xi}{4\pi} \text{and } B_1 = -\frac{1}{2}.
$$

Equating coefficients of α^N we have:–

$$
\begin{aligned}
& 0 = A_N \log^N \frac{p^2}{\Lambda^2} + A_N B_N \log^{N-1} \frac{p^2}{\Lambda^2} + A_{N-1} \frac{\xi}{4\pi} \Big[\frac{1}{2} \log^{N-1} \frac{p^2}{\Lambda^2} \\
& \quad - \frac{1}{N} \log^N \frac{p^2}{\Lambda^2} - B_{N-1} \frac{1}{N-1} \log^{N-1} \frac{p^2}{\Lambda^2} \Big] \qquad (2.30) \\
\Rightarrow \quad & 0 = A_N + A_{N-1} \frac{\xi}{4\pi} \left(\frac{-1}{N} \right) \\
& A_N = A_{N-1} \frac{1}{N} \left(\frac{\xi}{4\pi} \right) = A_0 \frac{1}{N!} \left(\frac{\xi}{4\pi} \right)^N \\
& \quad = \frac{1}{N!} (A_1)^N .
\end{aligned}
$$

So leading logs are satisfied to all orders in α. (2.30) also gives us:–

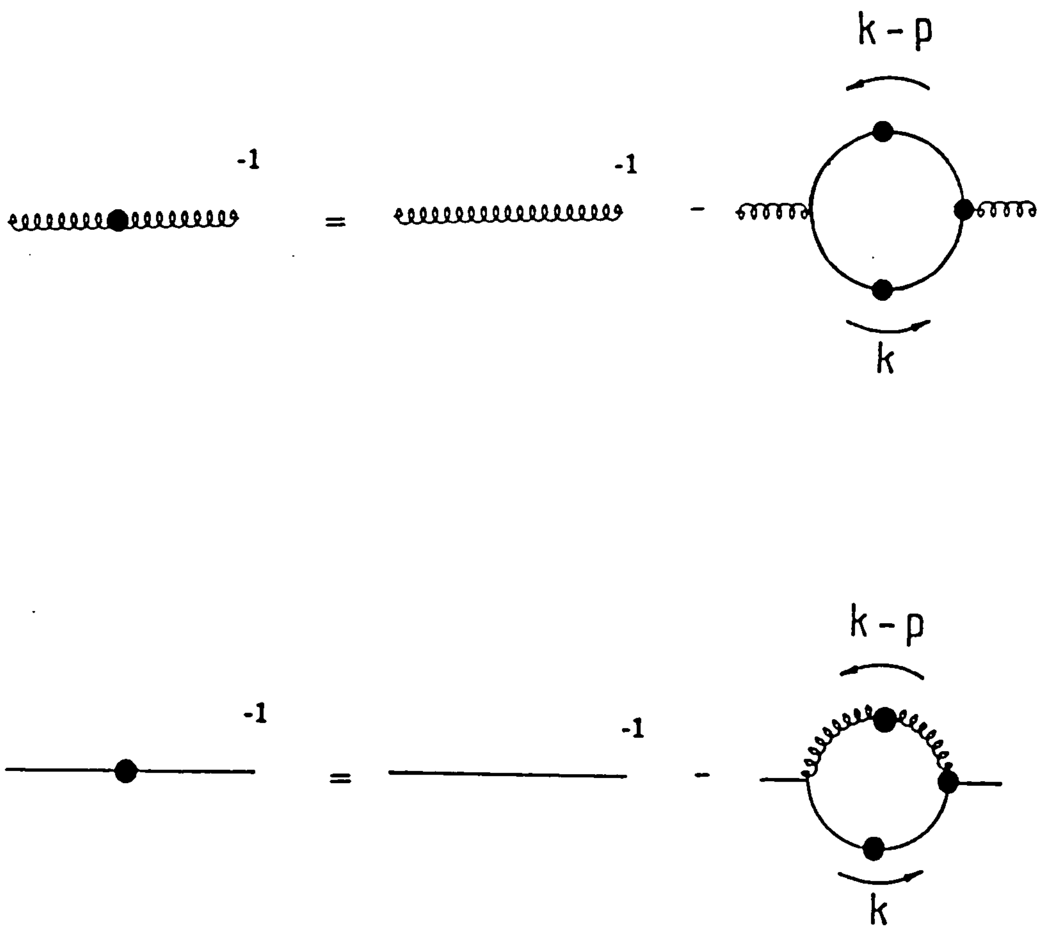
$$
\begin{aligned}
0 &= A_N B_N + A_{N-1}\frac{\xi}{4\pi}\frac{1}{2} - A_{N-1}B_{N-1}\frac{\xi}{4\pi}\frac{1}{N-1} \\
\Rightarrow \quad 0 &= \frac{1}{N}B_N + \frac{1}{2} - B_{N-1}\frac{1}{N-1} \\
B_N &= B_{N-1}\frac{N}{N-1} - \frac{N}{2} \quad \rightarrow \quad B_2 = -2 \\
\Rightarrow \quad B_N &= B_2\frac{N}{2} - \frac{N}{2}(N-2) \\
\Rightarrow \quad B_{N+2} &= \frac{1}{2}B_2(N+2) - \frac{1}{2}(N+2)N \\
&= \frac{1}{2}B_2(N+2)(N+1) + \frac{1}{2}(N+2)N \\
&= \frac{1}{2}B_2(N+2)(N+1) - B_1 N(N+2)
\end{aligned}
$$

ie. equation (2.12), the M.R. condition on next-to-leading logs. Thus our ansatz satisfies M.R. constraints on $F(p^2)$ for next-to-leading logs for all orders in α.

Repeating this process for $\Sigma(p^2)$, (2.29), is longwinded in the extreme and of no use except for the result, which is that the ansatz satisfies the M.R. constraints on $\Sigma(p^2)$ for next-to-leading logs for all orders in α, with:–

$$C_1 = -\frac{3}{4\pi}, \; D_1 = -\frac{\xi}{6} - 1 \text{ and } D_2 = -\frac{\xi^2}{9} - 3\xi.$$

Our ansatz appears to be reasonable in the perturbative regime, where we constructed it, which isn't really too surprising. This does not, however, mean it is necessarily useful in the full non-perturbative space. What we need to do is to investigate the non-perturbative behaviour of the propagator, using our ansatz. The nature of the integrals in the non-perturbative region necessitates the use of a numerical approach. We look at the propagator for the massless case (in the next chapter) and for the massive case (in subsequent chapters), where by massless we mean that we hold $\Sigma(p^2) = 0 \;\; \forall p^2$ throughout. In the massive case we set the current/bare mass $(m) = 0$ and study only the dynamical mass.



ig. 2.1 The S-D.E's for the boson and fermion propagators in QED. Curly lines represent the bosons and non-curly lines represent the fermions. The black dots represent full (as opposed to bare) Green's functions.

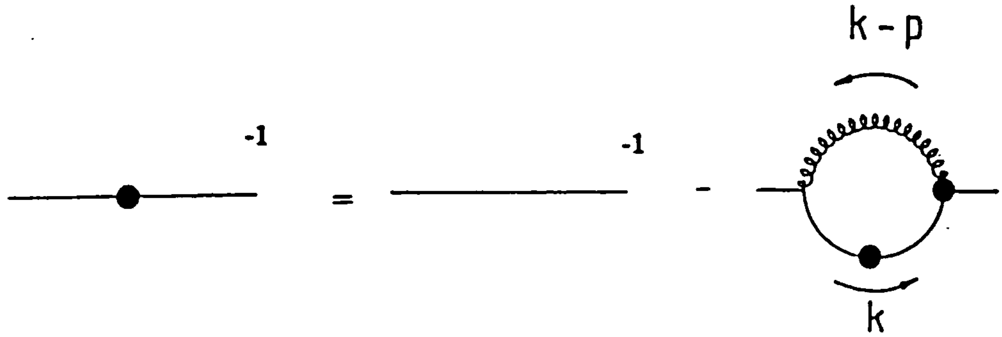


Fig. 2.2 The S-D.E's for the fermion propagator in qQED.
Lines and dots are as in fig. 2.1.

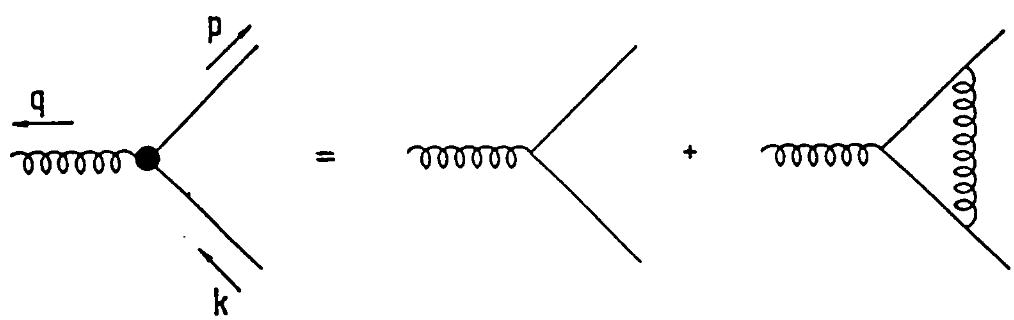


Fig. 2.4 The perturbative expansion of Γ to $O(\alpha)$. Lines and dots are as in fig. 2.1.

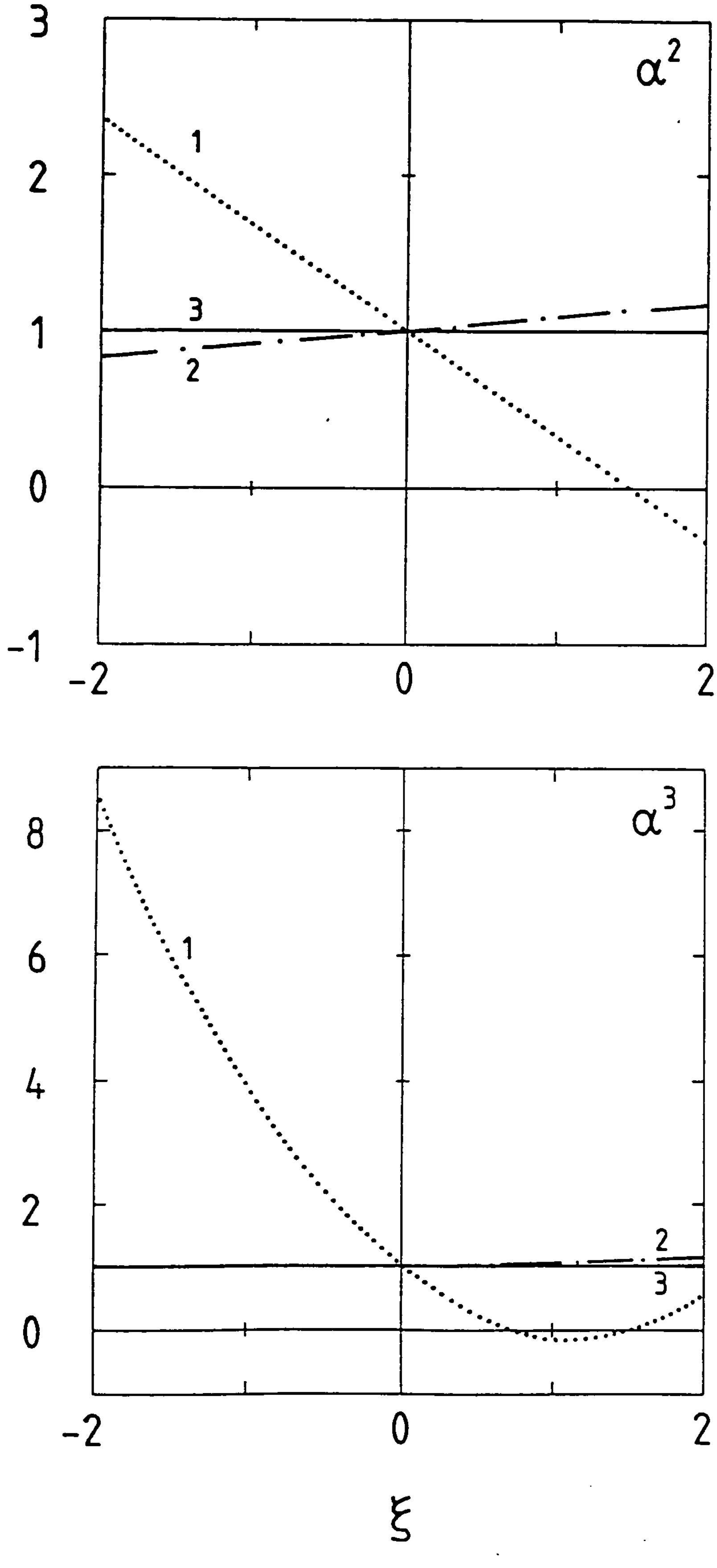


Fig. 2.3 The gauge dependance of α^2 and α^3 coefficients in the leading log. expansion of the mass, $\Sigma(p^2)$, for bare (line 1), Ball-Chiu (line 2) and full (line 3) verticies.

CHAPTER THREE

MASSLESS qQED4

3.1 *Introduction* :

We start off our non-perturbative studies by taking a comparative look at the bare, minimal Ball-Chiu and full ansatze in massless qQED[52]. We do this for two main reasons. Firstly in order to look numerically at the massive theory it is easier to use the massless case as a stepping stone (we get a handle on $F(p^2)$ before looking at $F(p^2)$ and $\Sigma(p^2)$). Secondly in massless qQED the RG equation is very simple and so we are able to check whether our ansatz, the full ansatz, satisfies M.R. (ie. the RG equation) non-perturbatively. From (2.7) the RG equation for $S_{F_R}(p^2)$ in massless qQED is:–

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} - \gamma_F(e) + \delta(e)\frac{\partial}{\partial\xi}\right) S_{F_R}^{-1}\left(\frac{p^2}{\mu^2}\right) = 0. \tag{3.1}$$

In qQED the photon propagator is equal to the bare photon propagator and hence $Z_A = 1 = \text{const.}$, $\alpha = e^2/4\pi = \text{const.}$ and $\xi = \text{const.}$ therefore:–

$$\beta(e) = \mu\frac{\partial e}{\partial\mu} = 0$$
$$\delta(e) = \mu\frac{\partial\xi}{\partial\mu} = 0.$$

Then (3.1) becomes:–

$$\left(\mu\frac{\partial}{\partial\mu} - \gamma_F(e)\right) S_{F_R}^{-1}(p^2) = 0. \tag{3.2}$$

In massless qQED

$$S_{F_R}^{-1}(p^2) = \frac{\not p}{F_R(p^2)}$$

$$\text{then } (3.2) \Rightarrow \qquad F_R\left(p^2/\mu^2\right) = A\left(\frac{p^2}{\mu^2}\right)^{\gamma} \tag{3.3}$$

$$\text{and} \qquad F\left(p^2/\Lambda^2\right) = B\left(\frac{p^2}{\Lambda^2}\right)^{\gamma} \qquad \text{where} \qquad \gamma = \frac{\gamma_F}{2}$$

but of course we don't know what γ is! (A is fixed by the normalisation of the renormalised wavefunction, eg $A = 1$ if $F_R(\mu^2) = 1$), but we do know that if the full vertex is valid then the answer it gives for $F(p^2)$ should be of this form. When we undertake our numerical procedure to find the solution to the S-DE if we use a logarithmic mesh for our integral momentum, k, and exterior momentum, p, then the mesh points are related by $x_{i+1} = a\ x_i \ \ \forall\ i \in$ mesh, $x = k^2$ or p^2, where $a =$ const.. Thus if a given ansatz is M.R. then from (3.3) we know that:-

$$\frac{F(p_{i+1}^2)}{F(p_i^2)} = a^{\gamma} = \text{const.} \tag{3.4}$$

Thus if we use a logarithmic mesh we can plot this ratio from our solution and so obtain a graphical guide as to how close to M.R. the solution to a given ansatz is.

Using a logarithmic mesh is useful in another respect and that is that much of the dynamics of the systems occur evenly spaced out in logarithmic momentum space rather than in linear momentum space (the dynamics occur in the low momentum region). In order to carry out the integrations accurately it is important to use a log. mesh. An easy way to view this argument is to remember that the expected solution (3.3) is momentum to a power, and in such a case it is always numerically more accurate to use a log mesh rather than a linear mesh, because while linear curves are trivial to handle numerically more structured curves, like $1/(1-x)$, can often be very tricky.

3.2 *Numerical methods* :

There are a number of possible methods for numerically solving integral equations (cf. chapter 2 of ref. [23]), but they are all of the following basic form:–

(1) Take some input data for the function you are trying to solve for.

(2) Put the input data into the integrals and use the S-DE's to calculate new 'output' data for the functions.

(3) Compare the output data to the input data, obtain a χ^2 and generate an updated set of input data.

This iterative process is then repeated until the χ^2 goes to zero in a suitable fashion. (The term "in a suitable fashion" will be explained later). There is of course a problem with this technique and that is that the χ^2 only measures the percentage difference between input and output data sets and not the output to the real solution. This is because we obviously don't know the real solution, if we did we wouldn't need to try to solve the equations numerically! This problem may not seem to be very important but let us use an example to show the potential problem. Consider the following example: define $S_N = \sum_{n=1}^{N} 1/n$. This is well known not to converge, $S_\infty = \infty$, however if we define $\chi^2_N = (S_N - S_{N-1})/S_N$ then $\chi^2_N \to 0$ as $N \to \infty$, which shows the problem of the χ^2 as described above. However $\chi^2_N \to 0$ in a slow fashion, it tends to crawl slowly towards 0 and S_N drifts across its target space ($\Re$) both of which indicate that there is a problem with finding a solution. If we now look at $T_N = \sum_{n=1}^{N} 1/n^2$ then we know that this will converge (to $\zeta(2)$)and then $\chi^2_N = (T_N - T_{N-1})/T_N \to 0$ as $N \to \infty$ very rapidly and T_N exhibits the typical behaviour of a series tending towards a finite asymptotic point, where the points squash up against some asymptotic wall. This is what was meant by "in a suitable fashion" in the above.

In the work we undertake we shall be looking at more complicated 1-D functions with a potentially complex target space. It is quite possible that there will be local minima and other target space topologies that might make finding 'the' solution a very complex process. One way to check to see whether we are in a local, rather than global, minimum is to nudge the solution and see whether it is returned to the same solution or not. Starting the iterative process $1 \rightarrow 3$ from a significantly different initial data set and checking that the final solution is the same as that obtained from the first initial data set is also another good method for checking the solution. However the process is numerical in nature and hence there is no sure–fire way of saying that we have the correct solution. All reasonable checks are made.

There is another problem that is inherent with numerical methods and that is due to numerical accuracy (this has been discussed in great detail in chapter 2 of ref. [23] and we shall only sketch out the problem here). We can divide this problem into two subsets. Firstly all computers have only a finite amount of accuracy and so trivial analytic cancellations that may occur in some regions of the integral will not be able to be reproduced numerically. This will be a major problem if the cancellations involve poles. In qQED4 we do not have this problem as our angular integrals give simple results, but in QED3[23], for example, it is a big problem. Secondly we get noise because of the form of our integrals and the fact that we do not initially know what the solution is. The way that this comes into our specific case is through so called GRADIENT terms present in the Ball-Chiu and full ansatze. These terms are of the form:-

$$\frac{1}{k^2 - p^2}\left(X(k^2) - X(p^2)\right)$$

where $X = 1/F$ or $X = \Sigma/F$ (in the massless case only the former is present). As we do not know the true solution to begin with we get the following problem. For close values of k^2 and p^2 any difference we have in X from the true solution will be highly enhanced, leading to large errors either as noise (which is easy to deal with) or leading to large deformations from the true solution (which is not so easy to handle). For example, if the true solutions for X at $k^2 = 1.0001p^2$ and p^2 are

1.0002 and 1.0000 respectfully then the gradient term, $\left(X(p^2) - X(k^2)\right)/(k^2 - p^2)$, is 2, but if our input solution has an error of 1% on it, eg 1.0102 and 0.9900 when the gradient term is 202, it can lead to very large errors in the value of the gradient term– in our example upto a factor of about 100.

In our present work it is the gradient terms that are our biggest problem and we have to use a variety of techniques to solve this problem, such as smoothing subroutines. The method that we shall use to obtain our solutions is the following: we shall use an $n \times n$ grid of points $(k_n^2,\ p_n^2)$ where $k_i^2 = p_i^2$. For each point p_i^2 integrate over the k^2's, using our input data for F (in the massive case Σ data as well), and then use the S-DE for F to compare the output for F at each point p_i^2 with the input. We write

$$\chi^2 = \frac{1}{n}\sum_{p_1^2}^{p_n^2}\left[\left(F_{IN}(p_i^2) - F_{OUT}(p_i^2)\right)/F_{IN}(p_i^2)\right]^2$$

and create our new input by putting

$$F_{NEW}(p_i^2) = \left(F_{IN}(p_i^2) + F_{OUT}(p_i^2)\right)/2.$$

We take the average of F_{IN} and F_{OUT} in order to increase the stability of our process. Our process is similar to Newton's iterative method for finding the roots of an equation.

3.3 *The equations* :

We now move on to the section where we shall state the equations that we are to solve numerically. We do this for completeness and to make absolutely clear what we are actually computing. We work in the Feynman gauge, $\xi = 1$, as from our work in section 2.5c we know that the effects of the transverse part of the vertex show up more clearly the larger $|\xi|$ becomes. There being no perturbative difference in the results in the Landau gauge, $\xi = 0$, which in this respect is special (more trivial) compared to other gauges. The integral in the S-DE for the bare vertex can not be rendered finite by the normal means. So in order to be able to compare our different ansatze we do not renormalise any of our S-DE's. This does not affect the identity (3.4) as in a massless theory if we have a M.R. object then its dependence on p^2 will be the same for the renormalised and unrenormalised versions of it, as all we have done is replace μ^2 by Λ^2 in the ratio p^2/μ^2 and may be change an overall normalisation factor (see (3.3)). As we are not renormalising our equations the integrals will run over the fixed interval $[0,\ \Lambda^2]$ and we rescale our variables by $1/\Lambda^2$ in order to have the convenience of integrating over the region $[0,\ 1]$. From Appendix B we then have the following S-DE's:–

3.3a *The bare vertex* :

where $\Gamma^\mu = \gamma^\mu$, we have (B.10):–

$$F^{-1}(y) = 1 + \frac{\alpha_0 \xi}{4\pi y} \int_0^1 dx\ F(x) \left[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x} \right]. \tag{3.5}$$

3.3b *The central kernel* [23] :

where $\Gamma^\mu = \frac{1}{2}\left(\frac{1}{F(p^2)} + \frac{1}{F(k^2)}\right)\gamma^\mu$, we have (B.11):–

$$F^{-1}(y) = 1 + \frac{\alpha_0 \xi}{4\pi y} \int_0^1 dx\ \frac{1}{2}\left(1 + \frac{F(x)}{F(y)}\right) \left[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x} \right]. \tag{3.6}$$

The central kernel is often used as a numerical stepping stone to the minimal Ball-Chiu vertex as it is simple, has non-perturbative functions, F, in it and the massless case is renormalisable. To obtain the central kernel we have dropped all of the gradient terms in the Ball-Chiu vertex and hence we do not have any noise problems from them, which makes things alot easier. The solution of the central kernel is then used as the initial input for the Ball-Chiu case.

3.3c *The minimal Ball-Chiu ansatz* :

where $\Gamma^\mu = \Gamma^\mu_{BC}$, we have (B.12):–

$$F^{-1}(y) = 1 + \frac{\alpha_0}{4\pi y}\int_0^1 dx\left[\theta(y-x)\xi\frac{x}{y} + \theta(x-y)\frac{F(x)}{F(y)}\xi\frac{y}{x}\right.$$
$$\left. - \frac{3}{4}\left(1-\frac{F(x)}{F(y)}\right)\frac{x+y}{x-y}\left(\theta(y-x)\frac{x}{y} + \theta(x-y)\frac{y}{x}\right)\right]. \tag{3.7}$$

3.3d *The full ansatz* :

where $\Gamma^\mu = \Gamma^\mu_{BC+T}$, where the denominator $d(x,y) = (x-y)^2/(x+y)$ in the massless theory, we have (B.13):–

$$F^{-1}(y) = 1 + \frac{\alpha_0\xi}{4\pi y}\int_0^1 dx\left[\theta(y-x)\frac{x}{y} + \theta(x-y)\frac{F(x)}{F(y)}\frac{y}{x}\right]. \tag{3.8}$$

We then solve (3.5-3.8) for a range of couplings $\alpha_0 = 2.0,\ 1.5,\ 1.0,\ 0.5$. When we undertake our numerical calculations we use a log mesh, as described above. When using such a mesh we obviously cannot actually integrate from 0, we in practice integrate over $[10^{-A},\ 1]$ where A is sufficiently large so that the region $[0,\ 10^{-A}]$ can be safely neglected. In our non-perturbative studies how big should we make A ? Well in our perturbative studies of chapter 2 we found that for all the above ansatze

$$F(y) = 1 + \frac{\alpha_0\xi}{4\pi}\log y + O(\alpha_0^2)$$

in leading logs and we also expect

$$F(y) = \sum_{n=0}^{\infty}\frac{1}{n\,!}\left(\frac{\alpha_0\xi}{4\pi}\log y\right)^n$$

in leading logs from M.R.. In some sense then we can view $\frac{\alpha_0\xi}{4\pi}\log y$ as our perturbative expansion parameter and in order to make $\left|\frac{\alpha_0\xi}{4\pi}\log y\right|$ of non-perturbative size, ie $\left|\frac{\alpha_0\xi}{4\pi}\log y\right| > 1$, we need to extend our range of integration down to at least $y = 10^{-11}$ (for $\alpha_0 = 0.5$, $\xi = 1$). In order to have the region in which $\left|\frac{\alpha_0\xi}{4\pi}\log y\right| > 1$ of a reasonably large enough size in order to be able to introduce non-perturbative behaviour we extend the region down to 10^{-15}. (In practice we integrate over $[10^{-16}, 1]$ and drop the lowest decade in order to remove edge effects).

We plot the solutions for all the ansatze, for a given α_0, on the same graph in order to be able to compare them, fig. 3.1-3.4. Curves I, II, III and IV correspond, respectively, to the bare, central, Ball-Chiu and full ansatze. We define our χ^2 equal to the average percentage error per point between input and output values of $F(y)$. Typical values for χ^2 are then:–

ansatz	χ^2
Bare	$O(1 \times 10^{-14})$
Central	$O(1 \times 10^{-14})$
Ball–Chiu	$O(1 \times 10^{-3})$
Full	$O(1 \times 10^{-14})$

Table 3.1. Typical values for the χ^2 error on the minimisation of solutions for various ansatze in massless qQED.

$\chi^2 = 1 \times 10^{-14}$ corresponds pretty much with the machine accuracy (calculations being done in real*8, Fortran). The reason why the χ^2_{BC} is so much larger than the others is because, as seen from (3.7), it is the only ansatz which has gradient terms in the integral of its S-DE and is therefore the only one beset by the problems of noise. The smoothing subroutines used cope well with the noise, but getting χ^2_{BC} down to machine accuracy is not possible as smoothing subroutines are not strictly mathematically defineable, they depend nontrivially on the persona of the programmer.

We then plot the ratio $F(y_{i+1})/F(y_i)$ for each ansatz, at a given α_0, on the same graph and compare them fig. 3.5.

3.4 *Comparing the results* :

We start by comparing the solution curves in fig. 3.1-3.4. We can see that as expected, from the form of the S-DE's, the solutions all decrease monotonically as the momentum decreases. All the solutions, for a given α, go to the same point as $y \to 1$, $p^2 \to \Lambda^2$, this is not surprising as all the vertices give $1 + \frac{\alpha\xi}{4\pi}\log(p^2/\Lambda^2) + O(\alpha^2)$ in the perturbative, high p^2 region. The bare solution (curve I) decreases only very slowly. The central solution (II) decreases only slightly more rapidly than the bare one (being about a factor of 2 smaller than (I) at $y = 10^{-15}\ \forall\ \alpha_0$). This is not surprising due to the similarity of the central kernal and the bare ansatz. What is a surprise though is just how close the full solution, (IV), is to the central one being only a factor of between 5 and 10 smaller than the central solution at $y = 10^{-15}\ \forall\ \alpha_0$. On these $\log F/\log y$ plots a M.R. solution (3.3) will give a straight line. All of the curves (I), (II) and (III) appear to be fairly straight, which is a good start towards M.R. solutions. We know that (I) cannot be M.R. (as the S-DE for the bare vertex is not renormalisable). The Ball-Chiu solution, (III), does not look too good, being so many orders of magnitude smaller than the rest and obviously not being a straight line. Things may not be as bad as they seem though: For high energies ($y \sim 1$) the curve is pretty straight, it then undergoes a region of high curvature and at low energy it is again pretty straight. The reason why it gets so small is that during the region of high curvature it gets pointed sharply downwards.

We have commented on the general features of the solution curves, it is now time to look at the more important graphs of $F(y_{i+1})/F(y_i)$, which tell us about the M.R. of the solutions (See fig. 3.5). The curve for the bare ansatz is clearly not constant and indeed is never flat in any region of momentum space. The curve for the central kernel does not vary quite so much, but still is never flat. Clearly these two ansatze are not M.R.. The Ball-Chiu curve however is really quite interesting.

It starts off by being no worse than the above two curves. This is not surprising as in the perturbative region $\Gamma^{\mu}_{BC} \rightarrow \Gamma^{\mu}_{\text{bare/central}}$. It then undergoes a period of rapid change, corresponding to the highly curved region of its $F(p^2)$ solution curve, before becoming flat at low energies (to 1 part in 10^7). This tells us that for high energies the minimal Ball-Chiu ansatz gives us a solution as good as any perturbative type ansatz, whilst for low energies it gives us M.R. solutions. However, in the intermediate region it has some problems. We can show formally that for low momentum, y, M.R. solutions, Ay^B, are supported by the Ball-Chiu S-DE (by carrying out a consistency check on it). This involves substituting the form, Ay^B, for $F(x \text{ or } y)$ into the RHS of equation (3.7) and checking that the result is consistent with the LHS, y^{-B}/A. This method, however, does not give us the values of A or B, just whether such solutions can be supported. It is perhaps interesting to note that the value of the ratio $F(y_{i+1})/F(y_i)$ at which it becomes constant is the same for a wide range of α_0 (constant to 3 parts in 10^7). Indeed for $\alpha_0 < 0.9$, where a constant ratio is not reached before $y = 10^{-15}$, we find that the ratio curves are undergoing behaviour which suggests very strongly that the constant value of the ratio will be the same as for those curves for $\alpha_0 > 0.9$. (In the Ball-Chiu case we have a lot of intermediate solution curves for α_0 between 0.5, 1.0, 1.5, 2.0 , in fact every 0.1 step plus $\alpha_0 = 0.55$, this is because when dealing with an equation which suffers from noise it was found necessary to use as initial input a solution for a nearby solution in parameter space, α_0 here. In this case varying α_0 by steps of 0.1 was as large as we could do and still use the previous solution as a useful initial input). We then have that in the flat region:–

$$\log\left(\frac{F(y_{i+1})}{F(y_i)}\right) = B\log\left(\frac{y_{i+1}}{y_i}\right).$$

Then with our mesh size of

$$\left(\frac{y_{i+1}}{y_i}\right) = 10^{1/50}$$

and the observation that

$$\left(\frac{F(y_{i+1})}{F(y_i)}\right) = 0.9659659(3)$$

in this region gives us:–

$$B = 0.751910(7).$$

This is independent of α_0, but not necessarily independent of ξ. We do not need to examine the effect on B of varying ξ because:-

The curve for the **full** ansatz is flat to machine accuracy. We have a M.R. solution–even non-perturbatively! Indeed looking at (3.8) we can see that it is a very simple looking equation. One might even be tempted to substitute in a M.R. form, (3.3), to see what happens. If we substitute in $F(w) = Bw^{\gamma_F/2}$ into equation (3.8) we get:–

$$B^{-1}y^{\gamma} = 1 + \frac{\alpha_0\xi}{4\pi}\left[\frac{1}{2} + \int_y^1 dx \frac{x^{-\gamma-1}}{y^{-\gamma}}\right]$$

if $\gamma = 0 \quad \Rightarrow \quad B \neq$ constant $\Rightarrow\Leftarrow$ setup,

if $\gamma \neq 0 \quad \Rightarrow$

$$B^{-1}y^{-\gamma} = 1 + \frac{\alpha_0\xi}{4\pi}\left[\frac{1}{2} + \frac{1}{\gamma}y^{-\gamma} - \frac{1}{\gamma}\right]$$

$$\Rightarrow \quad \gamma = \frac{K}{1+\frac{1}{2}K}, \quad B = \frac{1}{1+\frac{1}{2}K}, \quad \text{where } K = \frac{\alpha_0\xi}{4\pi}.$$

For the renormalised solution, F_R, we normalise such that $F_R(1) = 1$, in which case we have that:–

$$F_R\left(p^2/\mu^2\right) = \left(\frac{p^2}{\mu^2}\right)^{\gamma}, \quad \gamma = \frac{\left(\frac{\alpha_0\xi}{4\pi}\right)}{1+\frac{1}{2}\left(\frac{\alpha_0\xi}{4\pi}\right)}. \tag{3.9}$$

Now while the full ansatz has a complicated tensor structure, (2.27), in the massless case under the integral a lot of cancellation occurs, giving us the simple form:–

$$F^{-1}(y) = 1 + \frac{\alpha_0 \xi}{4\pi y} \int_0^1 dx \left[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{F(x)}{F(y)}\frac{y}{x} \right]. \tag{3.10}$$

Now by looking at the S-DE for the bare ansatz, (3.5), we can easily see that we can get equation (3.10) by using instead the simple effective ansatz:–

$$\Gamma^{\mu}_{\text{effective}}(k,\ p) = \gamma^{\mu}\ \frac{1}{F[\max(k^2,\ p^2)]}. \tag{3.11}$$

3.5 *Conclusions* :

In massless qQED the bare and central ansatze do not give M.R. solutions. The minimal Ball-Chiu ansatz gives solutions that are M.R. for low energies, but no where else. The full ansatz gives solutions which are M.R. everywhere:–

$$F_R\left(p^2/\mu^2\right) = \left(\frac{p^2}{\mu^2}\right)^{\gamma},\ \ \gamma = \frac{\left(\frac{\alpha_0 \xi}{4\pi}\right)}{1 + \frac{1}{2}\left(\frac{\alpha_0 \xi}{4\pi}\right)}.$$

If we assume this solution to be the 'real' one (rather than just a solution which happens to be M.R.) then the Ball-Chiu solutions are many orders of magnitude too small in the low ($y \sim< 10^{-5}$) energy region.

We have checked our ansatz $\Gamma_{BC+\ T}$ in massless qQED, where it is good. We now have to proceed to check it in massive qQED, which is what we shall do in the next chapter.

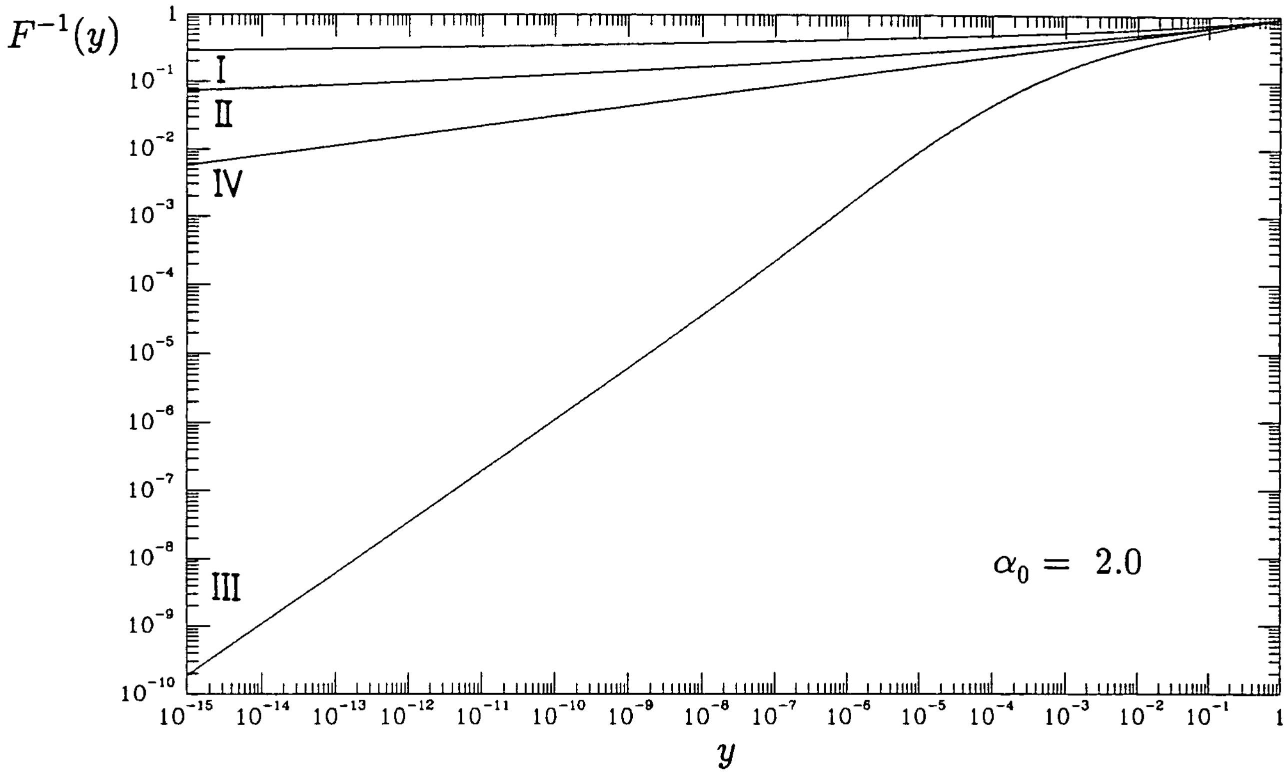


Fig. 3.1 The solution curves for $F(y)$ using the bare(I), central (II), Ball-Chiu (III) and full (IV) ansatie, in the Feynman gauge ($\xi = 1$) with $\alpha_0 = 2.0$.

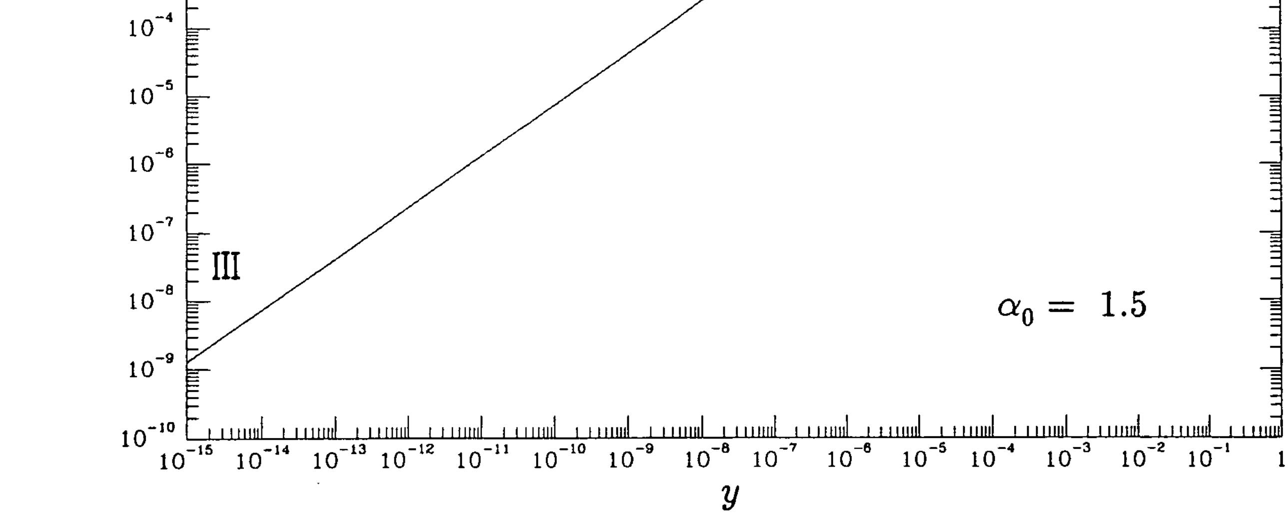


Fig. 3.2 As in fig. 3.1 but with $\alpha_0 = 1.5$.

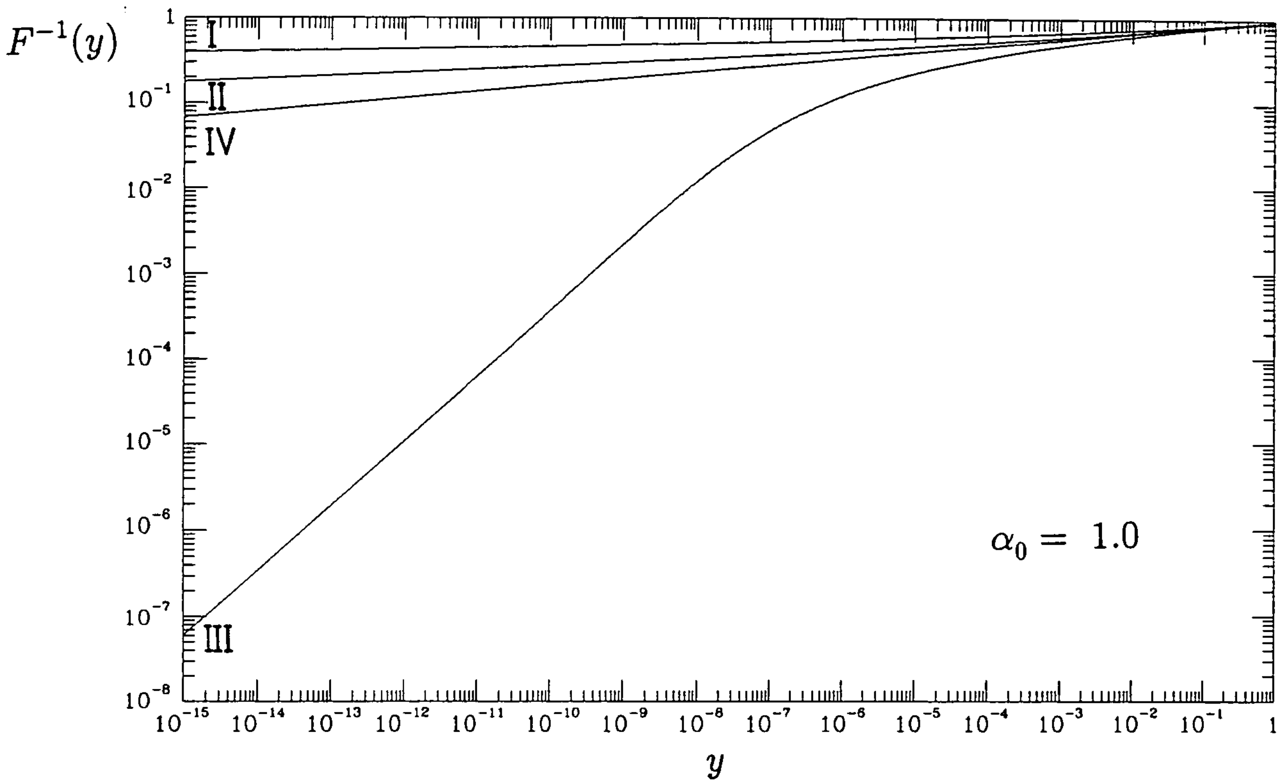


Fig. 3.3 As in fig. 3.1 but with $\alpha_0 = 1.0$.

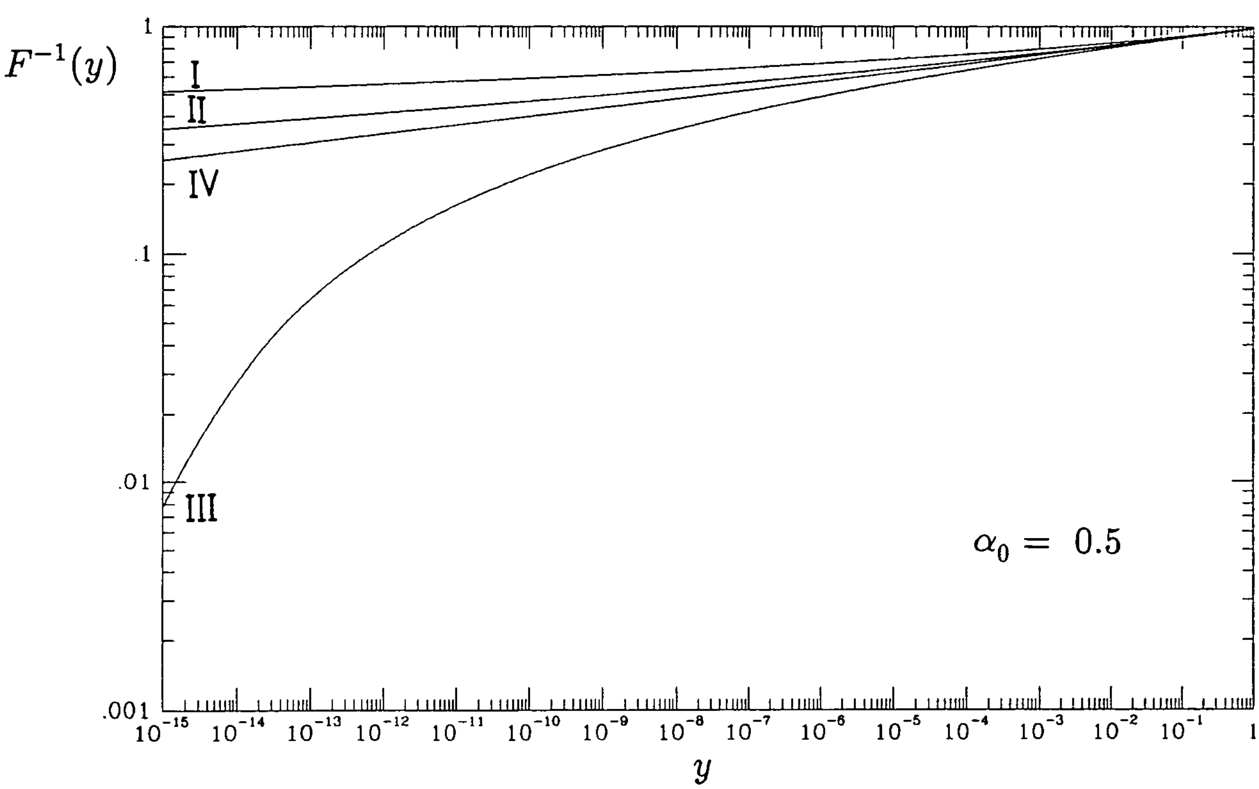


Fig. 3.4 As in fig. 3.1 but with $\alpha_0 = 0.5$.

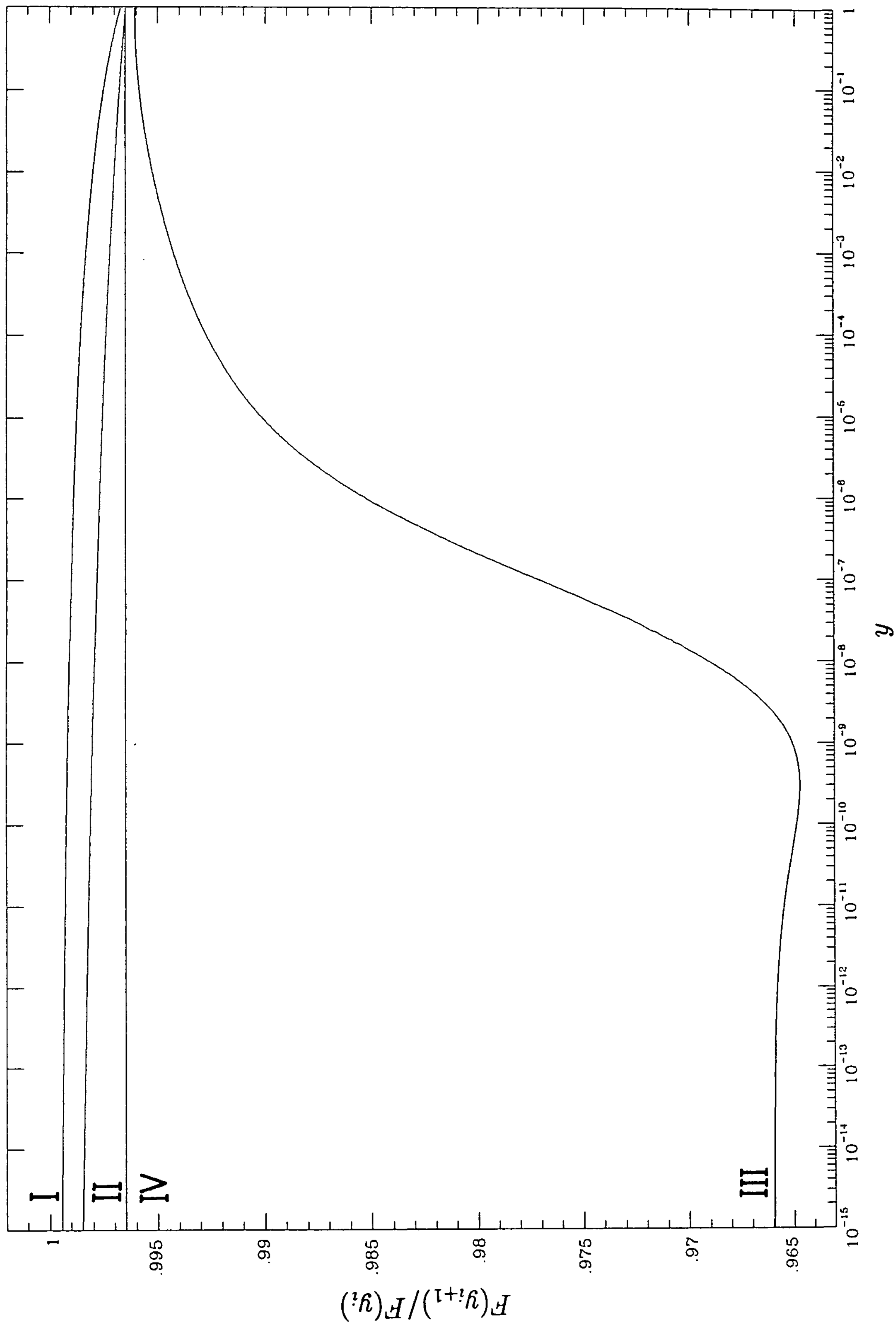


Fig. 3.5 The ratio of nextdoor points, $F(y_{i+1})/F(y_i)$, for the solution $F(y)$ in the Feynman gauge with $\alpha_0 = 1.0$ (ie. using data from fig. 3.3).

CHAPTER FOUR

MASSIVE qQED4

4.1 *Introduction* :

We now extend our non-perturbative studies by looking at the full ansatz in massive qQED4. However before actually looking at the case in hand we first of all undertake some useful training exercises.

4.2 *The bare vertex* :

We look at the S-DE's for the bare vertex. Now we know that the bare vertex ansatz only obeys the W-TI, and M.R., in the Landau ($\xi = 0$) gauge, when $F(p^2) = 1 \ \forall p^2$. In the massless case of chapter 3 we didn't renormalise the S-DE for $F(p^2)$ for the various ansatze and so mathematically we were formally able to look at the equations with $\xi = 1$ (Feynman gauge). We did this because the form expected for a M.R. solution was simple and only one function was being studied (the renormalised and unrenormalised functions being related by $p^2/\mu^2 \rightarrow p^2/\Lambda^2$). Setting $\xi = 0$ would have lead to $F_{bare}(p^2) = 1 = F_{central}(p^2) = F_{full}(p^2) \ \forall p^2$, by inspection of (3.5-3.7) (a trivial and unenlightening study!). However in the massive case we cannot solve the R-GEs to give functional forms for $F(p^2)$ and $\Sigma(p^2)$. This is because the R-GEs are perturbative equations whilst the $\Sigma(p^2)$ function we are studying is purely due to non-perturbative effects. Hence we do not know how to relate the non-renormalised functions to the renormalised ones. Thus studying the unrenormalised equations is, in this case, pointless. As the bare ansatz is only renormalisable in the Landau gauge we must study it there, where $F(p^2) = 1 \ \forall \ p^2$.

The physical mass, m, occurs when $m^2 = \Sigma^2(m^2)$. This, however, is in Minkowski space-time. When we Wick rotate into Euclidean space-time (where we solve our equations) the physical mass obviously occurs at $m^2 = -\Sigma^2(m^2)$. The evaluation of this physical mass would require an analytic continuation of the result of the S-DE to $p^2 < 0$. To avoid this, we instead define a Euclidean mass, p_0, which is given by $p_0^2 = \Sigma^2(p_0^2)$, which is standard. It is this Euclidean mass that we shall refer to as the mass throughout the rest of this chapter. It is expected to be of the same order of magnitude as the physical mass.

4.2a *The bare ansatz with finite cutoff*:

At present a common way of renormalising the S-DE for the bare ansatz is the standard cut-off method used for instance by Miransky et. al. [36,37] in which the cutoff in the S-DE is left finite:–

$$\Sigma(p^2) = \frac{3\alpha}{4\pi}\int_0^{\Lambda^2} \frac{dk^2\Sigma(k^2)}{k^2 + \Sigma^2(k^2)}\left[\theta(p^2 - k^2)\frac{k^2}{p^2} + \theta(k^2 - p^2)\right]. \tag{4.1}$$

This introduces a scale, Λ^2, into the theory and so the dynamical mass must go as $\Sigma(p^2) \sim \Lambda$. In order to renormalise the theory Miransky et. al. make the bare coupling α_0 depend on Λ^2 in a specific way, as described in section 2.3 and then

$$m_{dyn} = 4\Lambda \exp\left[\frac{-\Theta}{\sqrt{\frac{\alpha_0}{\alpha_c} - 1}}\right].$$

In our numerical calculations we change to dimensionless variables y and $S(y)$, where $p^2 = \Lambda^2 y$ and $\Sigma(p^2) = \Lambda S(y)$. The mass is defined to be the point p_0 at which $p_0 = \Sigma(p_0^2)$ and thus we get:–

$$y_0^{1/2} = S(y_0)$$

the 'dimensionless' mass. The S-DE with these dimensionless variables is:–

$$S(y) = \frac{3\alpha}{4\pi}\int_0^1 \frac{dx\ S(x)}{x + S^2(x)}\left[\theta(y - x)\frac{x}{y} + \theta(x - y)\right] \tag{4.2}$$

and from Miransky et. al. we expect the dimensionless mass to vary with α as:–

$$S(y_0) = 4\exp\left[\frac{-\Theta}{\sqrt{\frac{\alpha_0}{\alpha_c} - 1}}\right]. \tag{4.3}$$

We then undertake a numerical study of (4.2) in order to see how well we can cope with working with a theory with a critical point, α_c, in it and to see what numerical behaviour signals the onset of it. We do this as a training exercise in case the full ansatz has a critical point in its mass solutions (We work on a log. mesh as before). The results for a variety of α are shown in fig. 4.1..

By looking at the $S(y)$ curves for different values of α we can see that the momentum at which the function ceases to be flat and starts to fall off decreases as $(\alpha - \alpha_c\ )$ decreases, indeed for $\alpha =$ 1.065 and 1.060 we had to increase the range of integration from $[10^{-A},\ 1]$ with $A =$ 20 to $A =$ 26 and $A =$ 28 respectively. This preludes the collapse of the solution function at, and below, $\alpha = \alpha_c$.

As we took $\alpha \rightarrow \alpha_c$ the machine time needed to obtain convergence increased rapidly. Also the difference between doing the same calculations with different numbers of points per decade (ppd) on the mesh got to such a high degree that for the last few points it was necessary to extrapolate the answer by calculating it with three successive degrees of courseness for the mesh which had the number of ppd in each successive mesh of a constant ratio. It was then assumed that the change in $S(y_0)$ was wholly due to the errors in the Simpson's Rule integration technique used and not to any great extent by further wanderings of the $S(y)$ function over the solution space. Hence we were able to extrapolate to the answer for an infinite number of ppd, $S_\infty(y_0)$, by taking the answers for the three successive mesh sizes as the beginning of a geometric progression (fig. 4.2. is a plot of $S_\infty(y_0)$ vs. α_0). We tabulate the values for $\alpha =$ 1.080, 1.075, 1.070, 1.065 and 1.060 in table 4.1..

α	$S_\infty(y_0)$
1.080	0.9737×10^{-7}
1.075	0.2091×10^{-7}
1.070	0.2769×10^{-8}
1.065	0.1618×10^{-9}
1.060	0.1912×10^{-11}

Table 4.1. The value for the dimensionless mass (extrapolated to an infinite number of points per decade) for a variety of values of α.

We then carried out a numerical fit to the $S(y_0)$ solutions using a function with the same form as (4.3) viz:–

$$S(y_0) = A \exp\left[\frac{-B}{\sqrt{\alpha_0 C - 1}}\right]$$

remembering that (4.3) is valid only for $\sqrt{\alpha_0/\alpha_c - 1} \ll 1$. We use the CERN library minimising package MINUIT and minimised over the solutions for the four values $\alpha_0 =$ 1.080, 1.075, 1.070, 1.070 and 1.065 . We did not minimise to $\alpha_0 = 1.060$ as the change between its three mesh solutions were large and so potentially there could be a non-negligble error in the extrapolation to $S_\infty(p_0)$. For $\alpha_0 = 1.080$ $\sqrt{\alpha_0/\alpha_c - 1} = 0.1770$. Of course there is the problem of whether this is $\ll 1$? We are limited somewhat by the solutions we have to hand but its not unreasonable to say that $0.1770 \ll 1$.

The solution to the minimisation that we get is:–

$$A = 4.4295$$

$$B = 3.1057 = 0.98858\pi$$

$$C = 0.95465 = 0.99971\frac{3}{\pi}$$

with $\chi^2 = 0.2612 \times 10^{-5}$ percent per point.

The values of B and C agree pretty well with those in refs. [36,37], $B = \pi$ and $C = 3/\pi$. The value for A does not agree so well, $A = 4$ in refs. [36,37].

This does not mean that the results are incompatible just that any errors in the extrapolation tend to show up more in A than in B or C. This is because being in the exponential, the errors on B and C are naturally damped as, for example, a ten percent error on $S(y_0)$ is equivalent to an error of log(1.1) on the exponent. When $S(y_0) \approx 1 \times 10^{-7}$ the central value of the exponent is ≈ -16 and the error on the exponent is only about 0.6 percent. In this work we cover a change of $S(y_0)$ of 11 orders of magnitude. This is a lot better than the lattice calculations of refs. [19-22] which can only cover a change of $S(y_0)$ of at best $2-3$ orders of magnitude. Thus we are able to follow $S(y_0)$ a lot closer to $\alpha_0 = \alpha_c$ than usual and hence get a value for α_c that is only 0.03 percent away from the analytic value.

4.2b *The bare ansatz with infinite cutoff:*

The S-DE is now:–

$$\Sigma(p^2) = \frac{3\alpha}{4\pi} \int_0^\infty \frac{dk^2 \Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \left[\theta(p^2 - k^2)\frac{k^2}{p^2} + \theta(k^2 - p^2) \right]. \tag{4.4}$$

One might naively expect that in the limit $\Lambda^2 \to \infty$ equation (4.1) gives the same solution as (4.4). However, this is NOT the case! System (4.1) and (4.4) have COMPLETELY DIFFERENT PHYSICS [53]. (4.1) has a critical point, (4.4) does not have a critical point. The reason for this can be seen in two ways:–

Firstly (4.4) has scale invariance, ie. we can take the transformation:–

$$\Sigma(p^2) \to \{\tilde{\Sigma}(p^2) : \tilde{\Sigma}(\lambda^2 p^2) = \lambda \Sigma(p^2) \ , \ \lambda \in \Re\}$$

in a consistent fashion. Whereas in (4.1) Λ^2 sets the scale and there is NO scale invariance. Thus in (4.1) we have a critical point where for $\alpha_0 < \alpha_c$ there is different physics than for $\alpha_0 > \alpha_c$, a phase change. In (4.4) , due to the scale invariance, if $\Sigma(p^2)$ has a certain behaviour at a given α it must have the same behaviour $\forall \alpha$ and so we can't have a critical point α_c

Secondly the U.V. boundary condition of (4.1) is:–

$$\left(p^2 \frac{\partial \Sigma(p^2)}{\partial p^2} - \Sigma(p^2) \right) \Bigg|_{p^2 = \Lambda^2} = 0 \tag{4.5}$$

and of (4.4) is:–

$$\lim_{\Lambda^2 \to \infty} \left(p^2 \frac{\partial \Sigma(p^2)}{\partial p^2} - \Sigma(p^2) \right) \Bigg|_{p^2 = \Lambda^2} = 0. \tag{4.6}$$

In (4.5) we want $\Sigma(p^2)$ to cross the zero axis at a finite Λ^2, a similar set up to a damped oscillator, which leads to *sine* type solutions having zero solutions for other $p^2 > \Lambda^2$. In (4.6) we want $\Sigma(p^2)$ to meet the zero axis only at infinity, like for an over-damped oscillator, which leads to *sinh* type solutions that have no further zeros. The *sine* -like solutions of (4.5) lead to a Λ^2 independent critical point $\alpha_c = \pi/3$ and for $\alpha_0 > \alpha_c$ the further zeros lead to bound Cooper pairs of electrons. Whereas the *sinh*-like solutions of (4.6) lead to a continuous set of solutions.

Because it is a simple set up with $F(p^2) = 1 \ \forall p^2$, $\Sigma(p^2)$ unknown (and so a good training exercise for the full vertex where both $F(p^2)$ and $\Sigma(p^2)$ are unknown) and as a matter of pure interest we set about solving (4.4) numerically.

As (4.4) is scale invariant there is an infinite degeneracy in the solution for $\Sigma(p^2)$ (via the relationship $\tilde{\Sigma}(\lambda^2 p^2) = \lambda\Sigma(p^2), \ \ \forall \lambda \in \Re$). Quite obviously a computer will not be able to cope with this sort of problem. To tackle this, after each iteration we rescale our output for $\Sigma(p^2)$ to some predetermined value, $\Sigma_{\text{const}}(X^2)$ say. This then means that we only ask the computer to solve for **one** of the infinitely degenerate solutions. Now by putting in forms for Σ into the RHS of (4.4) and seeing if the results are consistent with the LHS we can tell that the solution for $\Sigma(p^2)$ should be constant for small p^2 and die off for large p^2. The fact that $\Sigma(p^2) \simeq$ const., for small p^2, suggests that a sensible value to rescale to is $\Sigma_{\text{const}}(p^2 = 0)$. What we do is to normalise $\Sigma(p^2) = 1 \times 10^{-3}$ for the lowest n_{flat} points in our region of intergration (we vary n_{flat} in order to make sure that our solution is independent of it). We then take the output, $\tilde{\Sigma}$, from our integration and by comparing this to our input over the first n_{flat}/N points we can calculate λ, as:–

$$\frac{\tilde{\Sigma}(p^2)}{\Sigma(p^2)} = \frac{1}{\lambda}\frac{\Sigma(p^2/\lambda^2)}{\Sigma(p^2)} = \frac{1}{\lambda}.$$

N is chosen so that we are comfortably in the momentum region where we have that $\Sigma(p^2/\lambda^2) = \Sigma(p^2) = 1 \times 10^{-3}$. We then use the calculated value of λ to rescale the whole of the output values, $\tilde{\Sigma}$, back to the solution with $\Sigma(0) = 1 \times 10^{-3}$. Since $\Sigma(p^2)$ is flat for many decades of p^2 this simplifies the rescaling and means that we can take the average of λ over a number of points, hence reducing any numerical instabilities. This is the new analytic difficulty that we learn to overcome by using (4.4) as a training exercise. It would seem a little unfair if we didn't also run into a new numerical problem that we had to overcome, and indeed we do run into one! The new numerical problem is another aspect of the age old computer problem of finite memory space. This time we have the problem that we integrate up to infinity in (4.4), a feet that we can not emulate on the computer, which leads to the following problem:

If we write:-

$$\Sigma(p^2) = \frac{3\alpha}{4\pi}\int_0^{X^2} \frac{dk^2\Sigma(k^2)}{k^2 + \Sigma^2(k^2)}\left[\theta(p^2 - k^2)\frac{k^2}{p^2} + \theta(k^2 - p^2)\right] + \Delta R$$

where X^2 is large enough so that $p^2 \gg \Sigma^2(p^2)$ and $\Sigma(p^2) = A(p^2/\mu^2)^{-\beta}$ (for ease we absorb $(\mu^2)^{-\beta}$ into A). Then:–

$$\Delta R = \frac{3\alpha}{4\pi}\int_{X^2}^{\infty} \frac{dk^2\; A\;(k^2)^{-\beta}}{k^2} = \frac{3\alpha}{4\pi}\frac{A}{\beta}\,(X^2)^{-\beta}$$

A and β are calculated a few decades away from the upper limit of the integration region, to avoid edge effects, and then used to calculate ΔR which is added to all points. The problem is then that it is easier to minimise to a flat line and thus the minimisation process makes $\beta \rightarrow 0$. ΔR then swamps out the rest of the integral and due to our rescaling mentioned previously we end up with the flat line $\Sigma(p^2) = 1 \times 10^{-3}\ \forall p^2$. Why is this due to the finite size of the machine? Well the problem is that our region of integration is a finite window on the infinite region we should really be integrating over and what $\beta \rightarrow 0$ has the effect of doing is just to shift this window down to cover only a low momentum region where $\Sigma(p^2) = 1 \times 10^{-3}$.

So what can we do to solve this? Well by looking at (4.4) and doing a consistency check for large p^2 we know that solutions of the form $\Sigma(p^2) \propto (p^2)^{-\beta}$ are supported only when $0 < \beta \leq 1$. From condensate theory we expect $\beta = 1$. Thus what we do is to hold $\beta = 1$ fixed and then minimise. This might well give largish edge effects near $p^2 = X^2$, but as $\beta = 1$ is most certainly the correct order of magnitude for β any errors will be negligible more than two decades away from the top edge of our integration region and after we have calculated the solution we discard the top two decades, which have these edge effects in them.

We plot the graphs of $\Sigma(p^2)$ for a range of values of α_0 in fig. 4.3.. As can be seen they all have similar behaviour and there is no evidence for a critical point (this is as expected). We then calculated the value of $p_0 = \Sigma(p_0^2)$ and a list of these appear in table 4.2. We also plot $\Sigma(p_0^2)$ vs. α_0 in fig.4.4., which clearly shows that there is no critical coupling point,α_c (compare to fig. 4.2. where there is a critical coupling point).

α	$\Sigma(p_0)$
2.00	0.8327×10^{-3}
1.75	0.8534×10^{-3}
1.50	0.8751×10^{-3}
1.30	0.8931×10^{-3}
1.10	0.9115×10^{-3}
1.05	0.9161×10^{-3}
1.00	0.9208×10^{-3}
0.90	0.9302×10^{-3}
0.70	0.9490×10^{-3}
0.50	0.9674×10^{-3}

Table 4.2. The mass for a range of values of α for the bare vertex and infinite cut-off.

4.3 *The full ansatz* :

4.3a *The equations* :

Finally we arrive at the main point of this chapter –the numerical study of the S-DE's for the full ansatz, (2.27), for the vertex in a massive theory. Now we are in qQED, so as we have stated before the relationship between the renormalised and unrenormalised Green's functions are:–

$$\begin{aligned}
\varphi_R(\frac{p^2}{\mu^2}) &= Z_2^{-1/2}(\frac{\mu^2}{\Lambda^2})\ \varphi(\frac{p^2}{\Lambda^2}) \\
A_R^\mu(\frac{p^2}{\mu^2}) &= A^\mu(\frac{p^2}{\Lambda^2}) \\
\Gamma_R^\mu(p,k,\mu) &= Z_1(\mu,\Lambda)\ \Gamma^\mu(p,k,\Lambda) \\
e_R(\mu) &= \frac{Z_2}{Z_1}\ e = \ e
\end{aligned}$$

as $Z_1 = Z_2$ from W-TI. So we can see that $F_R(p^2/\mu^2) = Z_2^{-1}(\mu^2/\Lambda^2)F(p^2/\Lambda^2)$ as F is linearly dependent on the propagator $\langle 0 \mid \overline{\varphi}\varphi \mid 0\rangle$. $Z_2(\mu^2/\Lambda^2)$ is regulated by the cut-off Λ^2 because it is divergent –in the perturbative region logarithmically so. The next question then is how do we renormalise the mass function $\Sigma(p^2/\Lambda^2)$? As the bare mass is zero the Z factor for Σ is a bit different in that it is a finite constant and so doesn't need to be regulated. (This stems from there being no mass fields in the Lagrangian):–

$$\Sigma_R(\frac{p^2}{\mu^2}) = Z_{finite}(\frac{\mu^2}{\Lambda^2})\ \Sigma(\frac{p^2}{\Lambda^2}).$$

Hence during the regularisation phase we can normalise $Z_{finite}(\mu^2/\Lambda^2) = 1$ giving us

$$\Sigma_R(\frac{p^2}{\mu^2}) = \Sigma(\frac{p^2}{\Lambda^2}).$$

The renormalisation of the S-DE's can then occur giving us the finite equations (B.16 and B.17 appendix B):–

$$\frac{F_R(\mu^2)}{F_R(p^2)} = 1 - \frac{\alpha_0}{4\pi}\int\limits_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)} I(k^2,p^2) + \frac{\alpha_0}{4\pi}\int\limits_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)} I(k^2,\mu^2)\frac{F_R(\mu^2)}{F_R(p^2)}$$
$$+ \frac{\alpha_0}{4\pi}\int\limits_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)}\frac{F_R(k^2)}{F_R(p^2)} \tag{4.7}$$

and

$$\Sigma(p^2) = +\frac{\alpha_0}{4\pi}\int\limits_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)}\left[J(k^2,p^2) + \Sigma(p^2)I(k^2,p^2)\right] \tag{4.8}$$

where I and J are defined in appendix B, (B.14 and B.15). In them F_R only appears in ratio (eg. $F_R(k^2)/F_R(p^2)$).

4.3b *Gauge invariance* :

Before we embark upon explaining how we solved the above equations numerically, its important for us to be clear in our minds what are the gauge invariant objects for our theory ie. for the fermion propagator. This is important as it makes our tactics understandable and maybe even logical. The main benefit is that it illuminates a sensible path to a gauge invariant renormalisation of the theory, after all it would be silly to solve the equations using a non-gauge invariant renormalisation scheme and then compare the objects we believe to be gauge invariant! Anyway, to achieve a gauge invariant renormalisation scheme we must start off by studying the theory to understand its gauge invariant objects.

In full QED the gauge invariant objects of the fermion are:–

1) the fermion mass: $p_0 = \Sigma(p_0^2)$ (strictly it is the physical mass rather than the Euclidean one).

2) the fermion coupling: $e_R = Z_2\, Z_3^{1/2} e/Z_1 = Z_3^{1/2} e$

(ie. the mass and charge on shell are physical objects therefore gauge independent). In qQED 2) becomes trivial and so we are left with the gauge invariant object:–

1) the fermion mass: $p_0 = \Sigma(p_0^2)$

What about F? How does that behave? Well, from 1) in qQED (and QED) the natural gauge invariant point to renormalise F_R at is $p_0 = \Sigma(p_0^2)$, at which point we define $F_R(p_0^2, \xi) = F_0$, some constant. We point out an important fact here and that is that in (4.7 and 4.8) F_R only appears in ratio and therefore what we are really determining is NOT $F_R(p^2, \xi)$ but $F_R(p^2, \xi)/F_0$ which means that when we find a solution for $F_R(p^2, \xi)$ from (4.7) then $\{\tilde{F}_R(p^2, \xi) : \tilde{F}_R(p^2, \xi) = AF_R(p^2, \xi)\ , A \in \Re\}$ is also a solution as $F_R/F_0 \to \tilde{F}_R/\tilde{F}_0 = AF_R/AF_0 = F_R/F_0$, we shall return to this later. So what we want to do is to renormalise the theories at $p_0 = \Sigma(p_0^2)$ at which point we shall choose $F_R(p_0^2, \xi) = 1$, as is conventional.

4.3c *Numerical tactics* :

Now the renormalisation just put forward is what we should use. However, it doesn't lend itself to easy minimisation. This is because even though we can, in principle, fix the scale of the Σ function (as (4.8) is scale invariant) by choosing a mesh point p_0^2 at which we make $p_0 = \Sigma(p_0^2)$ we don't know whether this is a good point, given the region $[10^{-A}, 10^{B}]$ that we are integrating over. We could, for instance, accidently choose p_0^2 such that the solution doesn't start to die off before 10^B, making the ΔR (section 4.2) calculation impossible, or such that the solution doesn't flatten off by the time we get down to 10^{-A}, making a sensible solution impossible to find (This assumes that $\Sigma(p^2)$ has the same sort of form as it does for Γ^{μ}_{Bare} with infinite cut off). We can of course just change our region of integration, but there are other problems as well. An important one is that if we define p_0^2 we don't know what value, Σ_0, $\Sigma(p^2)$ will have when it becomes constant for low p^2. It is important to know Σ_0 in order to damp the noise in the minimisation routine. This is because when a complicated integrand gives a solution that is constant for a large number of decades, mesh (lattice) waves tend to build up along this constant region which quickly wreck the minimisation. It is better to minimise by fixing Σ_0 so that we can put in damping algorithms. This may seem to preclude any hope of using a gauge invariant renormalisation scheme, because how do we know how to vary Σ_0 from gauge to gauge, but in fact it is possible to incorparate the two by minimising in two steps and this is what we shall now outline.

So for a given coupling α, say, we start off in a gauge $\xi = \xi_1$ (say $\xi = 1$), we fix $\Sigma_0 = 1\times 10^{-3}$ and integrate over a suitable range (found to be $[10^{-15}, 10^4]$). As we don't yet know what p_0^2 is we renormalise such that $F(p^2 = 1) = 1$. We solve the equations iteratively then calculate what $p_0^2(\xi_1)$ is, add this in as a new mesh point and rescale $F(p^2)$ such that $F(p_0^2) = 1$ (we do this by using the transformation $F(p^2) \to \tilde{F}(p^2) = F(p^2)/F(p_0^2)$ which we can do as we are really solving (4.7) for $F(p^2)/F_0$, see section 4.3*b*). We then move to a different gauge $\xi = \xi_2$ (say $\xi = 0$), fix $\Sigma_0 = 1 \times 10^{-3}$ and $F(p^2 = 1) = 1$ and solve. We then found that $p_0^2(\xi_2)$ was very close to $p_0^2(\xi_1)$ and so it was easy to add $p_0^2(\xi_1)$ to the mesh and resolve with renormalisations $\Sigma(p_0^2(\xi_1)) = p_0^2(\xi_1)$ and $F(p^2 = 1) = 1$ (the difference between $p_0^2(\xi_1)$ and $p_0^2(\xi_2)$ renormalised solutions, for $\xi = \xi_2$, were so close that noise didn't have a chance to build up). We then rescaled $F(p^2)$ so that $F(p_0^2(\xi_1)) = 1$.

Hence we have solved the equations, for a given coupling α, using the required renormalisation:–

$$p_0 = \Sigma(p_0^2) \quad \text{and} \quad F(p_0^2) = 1$$

where p_0^2 is the gauge independent, as required. We fix the scale for $\Sigma(p^2)$ by saying that p_0^2 is such that $\Sigma_0(\xi = 1) = 1 \times 10^{-3}$ for arbitary coupling α_0. We then solve the equations (4.7 and 4.8) (with $\xi = 1$ and 0) for a range of α_0.

When we solve we shall use all the normal smoothing algorithms that we have been using and discussing throughout this thesis. We also have the same problem as in section 4.2*b* in calculating ΔR with $\Sigma \simeq A(p^2)^{-\beta}$ where $\beta \to 0$ in the minimisation. We again get around this by fixing $\beta = 1$, however this time the errors introduced by this are even smaller as $(X^2)^{-\beta} A/\beta$ is not the only term present. We now choose X^2 big enough so that not only does $\Sigma \simeq A(p^2)^{-\beta}$ but $F \simeq C(p^2)^{\delta}$ as well (we know that F will have this form for large p^2 because as $p^2 \to \infty$, $\Sigma \to 0$ quickly and so F has the form that it would have in a massless theory, which we have calculated in section 3.4). With these forms it is found that:–

$$\Delta R = \frac{\alpha_0}{4\pi}\left[\frac{3A(p^2)^{1-\delta}(X^2)^{\delta-\beta-1}}{2(\delta-\beta-1)} - \frac{3A(p^2)^{1-\delta-\beta}(X^2)^{\delta-1}}{2(\delta-1)} + \frac{3A}{\beta}(X^2)^{-\beta}\right].$$

In table 4.3 we list $p_0(\xi_1)$ and $p_0(\xi_2)$ for a range of α_0. The reason that we list $p_0(\xi_2)$ as well is to show just how close it is to $p_0(\xi_1)$ for the whole range of α_0. This then gives us confidence that our theory has a high degree of intrinsic gauge invariance. We plot solution graphs for some values of α_0 in fig. 4.5.-4.8. for ξ_1 and ξ_2 renormalised at $p_0^2(\xi_1)$ (it is noticable just how close the entire solution $\Sigma_{\xi_1}(p^2)$ is to $\Sigma_{\xi_2}(p^2)$ for each given α_0). Also plotted is $p_0(\xi_1)$ vs. α_0 in fig. 4.9.. We note that there is no critical point α_c.

α	$p_0(\xi_1)$	$\chi^2_{p_0(\xi_1)}$	$p_0(\xi_2)$	$\chi^2_{p_0(\xi_2)}$
2.00	0.8957×10^{-3}	0.21×10^{-1}	0.8887×10^{-3}	0.15×10^{-1}
1.75	0.9046×10^{-3}	0.10×10^{-1}	0.8991×10^{-3}	0.21×10^{-1}
1.50	0.9132×10^{-3}	0.16×10^{-1}	0.9104×10^{-3}	0.10×10^{-1}
1.30	0.9230×10^{-3}	0.76×10^{-2}	0.9202×10^{-3}	0.12×10^{-1}
1.10	0.9333×10^{-3}	0.26×10^{-2}	0.9310×10^{-3}	0.28×10^{-2}
1.00	0.9389×10^{-3}	0.35×10^{-4}	0.9368×10^{-3}	0.48×10^{-5}
0.90	0.9448×10^{-3}	0.74×10^{-3}	0.9429×10^{-3}	0.43×10^{-3}
0.70	0.9576×10^{-3}	0.33×10^{-3}	0.9560×10^{-3}	0.80×10^{-3}
0.50	0.9715×10^{-3}	0.92×10^{-5}	0.9702×10^{-3}	0.36×10^{-6}

Table 4.3. The mass for $\xi_1 = 1$ and $\xi_2 = 0$ for a range of α,

with normalisation $\Sigma_0 = 1 \times 10^{-3}$ in all cases.

4.3d *Fitting functional forms to the solutions :*

Our solutions for F and Σ are obtained by using an iterative minimisation process over a number of points. This gives us good solutions, but it would be nice if we could write them in some simple functional form and this is what we shall attempt in this section.

4.3d(i) *The $F(p^2)$ function :*

What we do here is to look at the solution graphs for $F(p^2)$ for $\xi = 1$ and $\xi = 0$ for a given α_0 and see if we can come up with a good functional form for $F(p^2)$. Looking at $F(p^2)$ for $\xi = 1$ we note that:–

1) for large p^2 $F(p^2) \propto (p^2)^\gamma$, indeed by comparing to solutions of F in the massless case we can see that:–

$$\gamma = \frac{K}{1 + \frac{1}{2}K} \quad \text{where } K = \frac{\alpha_0 \xi}{4\pi} \tag{4.9}$$

just as in the massless case. This is not surprising as for large p^2, $\Sigma(p^2)$ has died off and so has very little effect in the $F(p^2)$ S-DE.

2) for small p^2, $F(p^2) = \text{const.}$.

3) from our renormalisation $F(p_0^2) = 1$.

In order to combine 1) and 2) we could use some form like $F(p^2) \propto (D + p^2)^\gamma$ where D (with dimensions of momentum^2) is a constant (like $\Sigma^2(p_0^2)$) or is some function that is constant when $p^2 < D(p^2)$ (like $\Sigma^2(p^2)$ for example). To satisfy 3) we can write F as:–

$$F(p^2) \propto \left(\frac{D + p^2}{E}\right)^\gamma \tag{4.10}$$

where E has dimensions of momentum^2, is such that at p_0^2 $E = D + p_0^2$ and does not cause the function to violate 1) or 2). So two possible forms are (4.10) with $D = \Sigma^2(p_0^2)$, $E = 2\Sigma^2(p_0^2)$ and $D = \Sigma^2(p^2)$, $E = 2\Sigma^2(p_0^2)$.

Looking at $F(p^2)$ for $\xi = 0$ (when $\gamma = 0$) in figs.4.5-4.8, we note that the function only varies by about 2 percent over the whole range of p^2. This is not surprising as when we set $\xi = 0$ all we have left in the integrand of the F S-DE are objects that are premultiplied by Σ^2, ie small totally non-perturbative objects. The solution looks like a one dimensional soliton and so a function of the form $F(p^2) = (1 + Xp^2)/(Y + Zp^2)$ is needed. We want $F(p_0^2) = 1$ and so we can then write:–

$$F(p^2) = \frac{1 + A(\frac{p^2}{p_0^2} - 1)}{1 + B(\frac{p^2}{p_0^2} - 1)}. \tag{4.11}$$

By using a fitting routine, such as MINUIT, we find that (4.11) is a good function to fit to the solution.

Now we want to use a general $F(p^2)$ that will work for all values of ξ, in particular $\xi = 1$ and $\xi = 0$. As $F(p^2)$ for $\xi = 0$ only varies by ~ 2 percent over the whole range of p^2 and is centered about 1, multiplying (4.10) by (4.11) will have very little effect on the solution for $\xi = 1$. So we guess a general form for F of:–

$$F(p^2) = \frac{1 + A(\frac{p^2}{p_0^2} - 1)}{1 + B(\frac{p^2}{p_0^2} - 1)} \left(\frac{D + p^2}{2\Sigma^2(p_0^2)} \right)^\gamma$$

where we fit A and B using the $\xi = 0$ data, as its more sensitive there, and see what D should be from the $\xi = 1$ data. It turns out that the best form for D is $\Sigma^2(p^2)$ when we get a very good fit. Indeed it is encouraging to note that if we drop the prefactor for $\xi = 1$ and use only $((\Sigma^2(p^2) + p^2)/2\Sigma^2(p_0^2))^\gamma$ we get a worse fit than if we use the prefactor fitted to $\xi = 0$ data! So a simple general functional form for $F(p^2)$ is:–

$$F(p^2) = \frac{1 + A(\frac{p^2}{p_0^2} - 1)}{1 + B(\frac{p^2}{p_0^2} - 1)} \left(\frac{\Sigma^2(p^2) + p^2}{2\Sigma^2(p_0^2)} \right)^\gamma \quad \forall \xi$$

where

$$\gamma = \frac{K}{1 + \frac{1}{2}K} \quad \text{with} \quad K = \frac{\alpha\xi}{4\pi}. \tag{4.12}$$

In table 4.4 we give a list of A and B for given couplings as well as the χ^2 for these functional forms compared to the solutions for $\xi = 0$ and $\xi = 1$.

α	A	B	$\chi^2_{\xi=0}$	$\chi^2_{\xi=1}$
2.00	0.35682	0.33988	0.79×10^{-3}	0.15×10^{-1}
1.75	0.36302	0.34948	0.37×10^{-3}	0.43×10^{-2}
1.50	0.36451	0.35426	0.13×10^{-3}	0.46×10^{-2}
1.30	0.36249	0.35467	0.54×10^{-4}	0.36×10^{-2}
1.10	0.35243	0.34682	0.19×10^{-4}	0.77×10^{-2}
1.00	0.34543	0.34080	0.10×10^{-4}	0.80×10^{-2}
0.90	0.33868	0.33494	0.51×10^{-5}	0.96×10^{-3}
0.70	0.32296	0.32074	0.76×10^{-6}	0.11×10^{-1}
0.50	0.27806	0.27703	0.65×10^{-7}	0.10×10^{-1}

Table 4.4. The values for A and B in (4.12) after minimisation to $\xi = 0$ data and the χ^2's for the fits to $\xi = 0$ and $\xi = 1$ data.

4.3d(ii) *The* $\Sigma(p^2)$ *function* :

By looking at the graphs for the solutions of $\Sigma(p^2)$ we see that Σ has pretty much the same form regardless of α or ξ. The question then is can we find a function that has this form? We shall try the generic form:–

$$\begin{aligned} I &= Tanh\Big[\ -\log(p^2 +\ \Sigma^2(p^2))\Big] +\ 1 \\ &= \frac{[p^2 +\ \Sigma^2(p^2)]^{-1} -\ [p^2 +\ \Sigma^2(p^2)]}{[p^2 +\ \Sigma^2(p^2)]^{-1} +\ [p^2 +\ \Sigma^2(p^2)]} +\ 1 \\ &= \frac{2}{1 +\ [p^2 +\ \Sigma^2(p^2)]^2}. \end{aligned}$$

where p^2 is taken to be scaled by a momentum large compared to p_0^2. Why this form? Well for $p^2 <\ p_0^2$, $I \sim\ 2$, due to $\Sigma^2(p_0^2)$ being small (ie. we have that below p_0^2 $\Sigma =$ constant). For $p^2 \rightarrow \infty$ $I \sim 2/p^4$ (ie. inverse power behaviour). Now obviously $I =\ 2$ for $p^2 <\ p_0^2$ is too large and $I \sim\ 1/p^4$ $p^2 \rightarrow \infty$ is probably to steep a decay, but we have the general feel of what $\Sigma(p^2)$ is. The next step is to make it more sophisticated, so let's try:–

$$I =\ C\left[Tanh\left(\frac{1}{B}\Big[\ -\log(p^2 +\ \Sigma^2(p^2)) +\ \log A^2\Big]\right) +\ 1\right]$$

where A^2 helps set the point at which the constant value region ends, $1/B$ sets the power law behaviour as $p^2 \rightarrow \infty$ and C sets the constant value of $\Sigma(p^2)$ as $p^2 \rightarrow\ 0$. When we minimise this we get a pretty good result. However, it is apparent that for $p^2\ \rightarrow \infty$ the Σ solution has a slight convex curve to it, rather than being a pure power law. To rectify this we premultiply the $\log(p^2 +\ \Sigma^2(p^2))$ piece by Y which varies slowly with p^2 in such a way as only to affect the power law decay region. After looking at how ratios of next door points behaved we decided to use $Y =\ 1 +\ D\log\,[\,(p^2 +\ \Sigma^2(p^2))/2p_0^2\,]$. Thus the final form we used was:–

$$\Sigma(p^2) =\ C\left[Tanh\left(\frac{1}{B}\Big[\ -Y\log(p^2 +\ \Sigma^2(p^2)) +\ \log A^2\Big]\right) +\ 1\right]$$

where

$$Y = 1 + D\log\left(\frac{p^2 + \Sigma^2(p^2)}{2p_0^2}\right). \tag{4.13}$$

This gave us good minimisation results which are presented in table 4.5. The factor Y should not be regarded as physically significant.

α	ξ	A	B	C	D	χ^2
2.00	1	0.7816×10^{-3}	3.698	1.186×10^{-3}	1.352×10^{-2}	0.62×10^{-2}
1.75	1	0.9342×10^{-3}	3.876	1.065×10^{-3}	1.353×10^{-2}	0.95×10^{-2}
1.50	1	1.133×10^{-3}	4.087	0.9654×10^{-3}	1.347×10^{-2}	1.6×10^{-2}
1.30	1	1.358×10^{-3}	4.293	0.8959×10^{-3}	1.349×10^{-2}	2.7×10^{-2}
1.10	1	1.723×10^{-3}	4.533	0.8252×10^{-3}	1.297×10^{-2}	3.9×10^{-2}
1.00	1	1.968×10^{-3}	4.670	0.7950×10^{-3}	1.266×10^{-2}	4.4×10^{-2}
0.90	1	2.328×10^{-3}	4.822	0.7611×10^{-3}	1.194×10^{-2}	4.9×10^{-2}
0.70	1	3.490×10^{-3}	5.169	0.6998×10^{-3}	1.040×10^{-2}	6.4×10^{-2}
0.50	1	6.365×10^{-3}	5.577	0.6391×10^{-3}	0.7489×10^{-2}	7.8×10^{-2}
2.00	0	0.6552×10^{-3}	3.881	1.326×10^{-3}	1.365×10^{-2}	0.45×10^{-2}
1.75	0	0.8352×10^{-3}	4.005	1.138×10^{-3}	1.349×10^{-2}	0.87×10^{-2}
1.50	0	1.069×10^{-3}	4.159	0.9970×10^{-3}	1.329×10^{-2}	1.7×10^{-2}
1.30	0	1.326×10^{-3}	4.316	0.9081×10^{-3}	1.302×10^{-2}	2.6×10^{-2}
1.10	0	1.694×10^{-3}	4.511	0.8323×10^{-3}	1.258×10^{-2}	3.7×10^{-2}
1.00	0	1.948×10^{-3}	4.628	0.7979×10^{-3}	1.226×10^{-2}	4.3×10^{-2}
0.90	0	2.279×10^{-3}	4.759	0.7652×10^{-3}	1.182×10^{-2}	4.9×10^{-2}
0.70	0	3.363×10^{-3}	5.073	0.7035×10^{-3}	1.045×10^{-2}	6.4×10^{-2}
0.50	0	5.938×10^{-3}	5.465	0.6436×10^{-3}	0.7816×10^{-2}	7.9×10^{-2}

Table 4.5. The values for A, B, C and D in (4.13), for various α and ξ, and their χ^2.

4.4 *Conclusions* :

We showed in chapter 2 how it was possible, using the lowest order corrections to the bare vertex, to generate a non-perturbative ansatz for the three point function $\Gamma^\mu(k,p)$. This involved knowing the form of the transverse basis set for Γ^μ, rewriting Γ^μ_{pert} in terms of the functions $F(p^2)$ and $\Sigma(p^2)$ (which we needed to know to O(α^2)), using the constraints imposed upon the theory by M.R. and the constraints imposed upon Γ^μ by physics (eg symmetry, W-TI,...). We could check our ansatz to next-to- next-to-leading logs in perturbation theory after which it became too time consuming to continue. So we had an ansatz which we thought might be all right, it is :–

$$\Gamma^\mu(k,p) = \Gamma^\mu_{B.C.}(k,p) + \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{\gamma^\mu(k^2 - p^2) - (k+p)^\mu(\not k - \not p)}{d\,(k,p)}$$

where

$$d\,(k,p) = \frac{(k^2 - p^2)^2 + \left(\Sigma^2(k^2) + \Sigma^2(p^2)\right)^2}{k^2 + p^2} \tag{4.14}$$

and $\Gamma^\mu_{B.C.}(k,p)$ is the Ball-Chiu vertex[42].

In chapter 3 we showed that in the massless theory, in which $\Sigma(p^2) \to 0$, our ansatz gives a fully non-perturbative M.R. solution:–

$$\begin{gathered} F_R(p^2) = \left(\frac{p^2}{\mu^2}\right)^\gamma \\ \gamma = \frac{K}{1 + \frac{1}{2}K} \quad , \quad K = \frac{\alpha_0 \xi}{4\pi} \end{gathered} \tag{4.15}$$

which is a good sign.

Then in chapter 4, after using Γ^μ_{Bare} with finite and infinite cutoffs as training exercises, we solved the S-DE's for $F(p^2)$ and $\Sigma(p^2)$ using a gauge invariant renormalisation procedure. We obtained masses $p_0 = \Sigma(p_0^2)$ for a range of α_0, which were all gauge independent and we did not find a critical coupling α_c. We were also able to obtain good functional fits to the solutions for $F(p^2)$ and $\Sigma(p^2)$, even though the S-DE's are far too complicated to be ever checked analytically, we have that:–

$$
\begin{aligned}
F(p^2) &= \frac{1+\ A(\frac{p^2}{p_0^2}-1)}{1+\ B(\frac{p^2}{p_0^2}-1)}\left(\frac{p^2+\ \Sigma^2(p^2)}{2\Sigma^2(p_0^2)}\right)^{\gamma} \\
\gamma &= \frac{K}{1+\ \frac{1}{2}K} \quad , \quad K = \frac{\alpha_0\xi}{4\pi}
\end{aligned}
$$

and (4.16)

$$
\begin{aligned}
\Sigma(p^2) &= C\left[Tanh\left(\frac{1}{B}\Big[-Y\log(p^2+\ \Sigma^2(p^2))+\ \log A^2\Big]\right)+\ 1\right] \\
Y &= 1+D\log\left(\frac{p^2+\ \Sigma^2(p^2)}{2p_0^2}\right).
\end{aligned}
$$

We are thus lead to conclude that our ansatz is probably a good one.

Our ansatz does not give results that have a critical coupling and this suggests that full QED4 might not have a critical coupling point either. This lack of critical coupling is backed up by results in QED3[23]. A question then is how can we explain the GSI results? After all a phase change at some critical coupling was postulated as a way to explain these results. One theoretical idea that has been suggested by Arbuzov et al.[55] is that a critical coupling is not needed. In their work they looked at the Bethe-Salpeter equation for the full four-point Green's function in the quasi-potential approach (where x_0 for the initial states are kept equal as are those for the final states, of course $x_{0\ initial} \neq x_{0\ final}$ in general). What they found was that as well as the usual Coulomb levels (for negative binding energies) they were able to obtain narrow width resonances (for positive binding energies) that they could associate with the GSI results. All this was done without the need for a critical coupling. A further point to note is the disagreement over the GSI results that has occured in the experimental community. Some groups have seen back-to-back narrow peaks and some groups have not[56]. The understanding and resolution of these experimental differences lies far out side the scope of this thesis, we leave it to the experimentalists!

4.5 *Further Study* :

We conclude this thesis by briefly outlining some possible future studies.

The first and easiest thing that can be done is to solve the S-DE's for Γ^{μ}_{full} with finite cut-off, Λ^2, when we can see whether we obtain different physics (eg a critical point α_c, as for Γ^{μ}_{bare}) or if we get related physics. This would be of interest to check whether the results of Miransky et al. are at all physically valid or if they are just a quirk of Γ^{μ}_{bare}.

Secondly to extend this work on qQED to full QED. In order to do this it will be necessary to calculate lowest order corrections to Γ^{μ} that come from the full photon propagator. These corrections will stand out because they will be premultiplied by N_f, from fermion loops. Thus the new pieces in Γ^{μ}, that come from the photon, will be solely due to the two $O(\alpha^2)$ diagrams with fermion loops. Using M.R. to help us construct an ansatz will be more complicated, as now $Z_3 \neq 1$ for A^{μ}, but we have $\Gamma^{\mu}_{Full}(Z_3 = 1)$ as a guide.

Thirdly we could try to expand this to QCD. This will be much more complicated as we have triple gluon vertex functions, which will come into the lowest order calculations of $\Gamma^{\mu\nu\rho}$ and also give extra C_{2A} factors in the lowest order corrections to the gluon propagator (from gluon loops). This will be extremely hard to do (as we have gluon and fermion loops[57]), but if successful will possibly make the calculations of $\Sigma(p^2)$ tractable. Without Γ^{μ}_T, the calculation of $\Sigma(p^2)$ in QCD is not really a well posed question as there are great problems getting a consistent in/out numerical fit to $\Sigma(p^2)$ in the intermediate energy region. There Γ^{μ}_T has its greatest effect since in the perturbative region, where $p^2 \gg 1$, $\Gamma^{\mu}_L + \Gamma^{\mu}_T \rightarrow \Gamma^{\mu}_L$ and at low momenta $\Gamma^{\mu}_T \rightarrow 0$ as $k^{\mu} - p^{\mu} \rightarrow 0$.

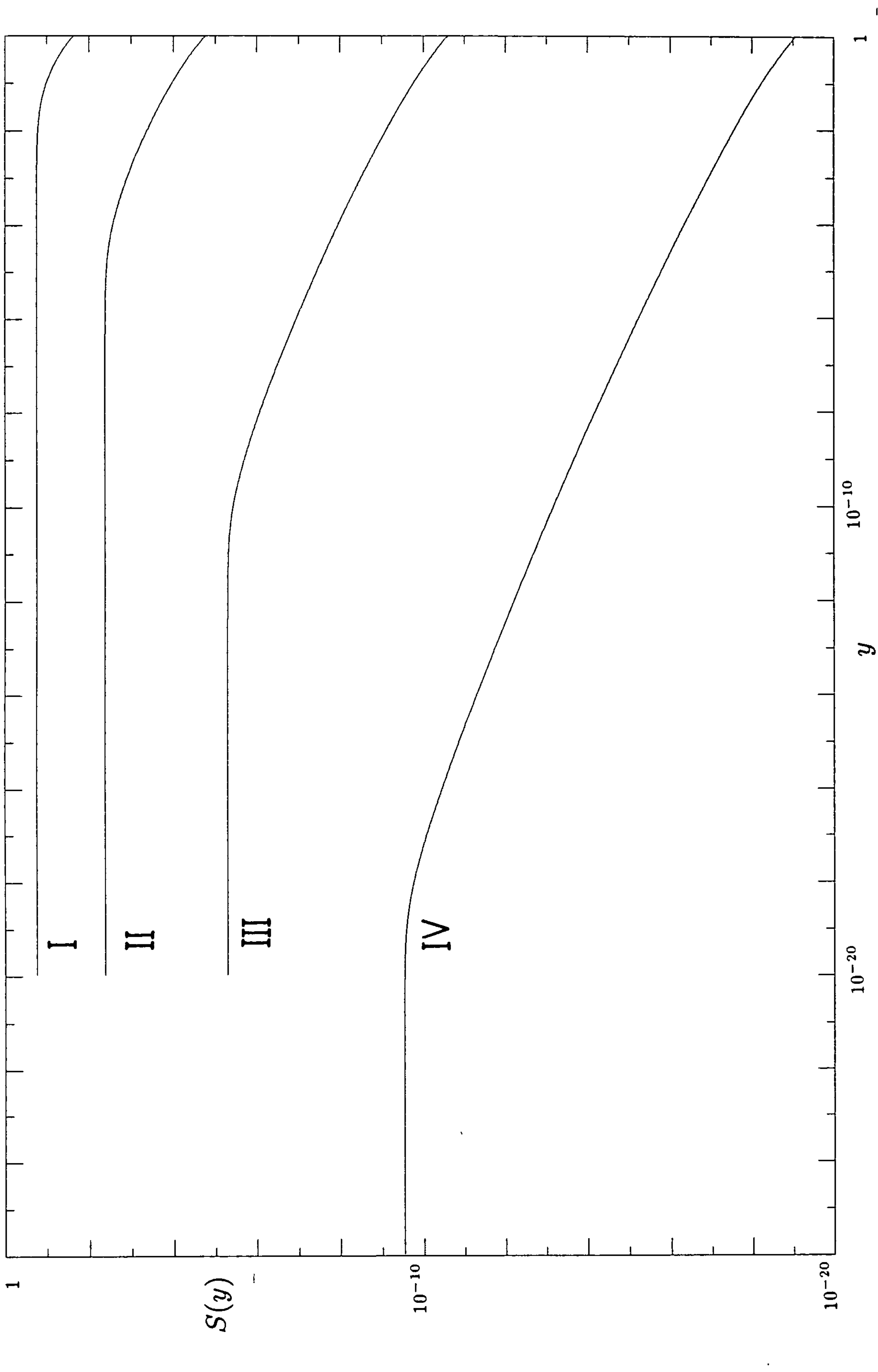


Fig. 4.1 The solution curves for $S(y)$ with bare ansatz and **finite** cutoff, Λ^2, for $\alpha_0 =$ 2.0 (I), $\alpha_0 =$ 1.25 (II), $\alpha_0 =$ 1.10 (III) and $\alpha_0 =$ 1.065 (IV).

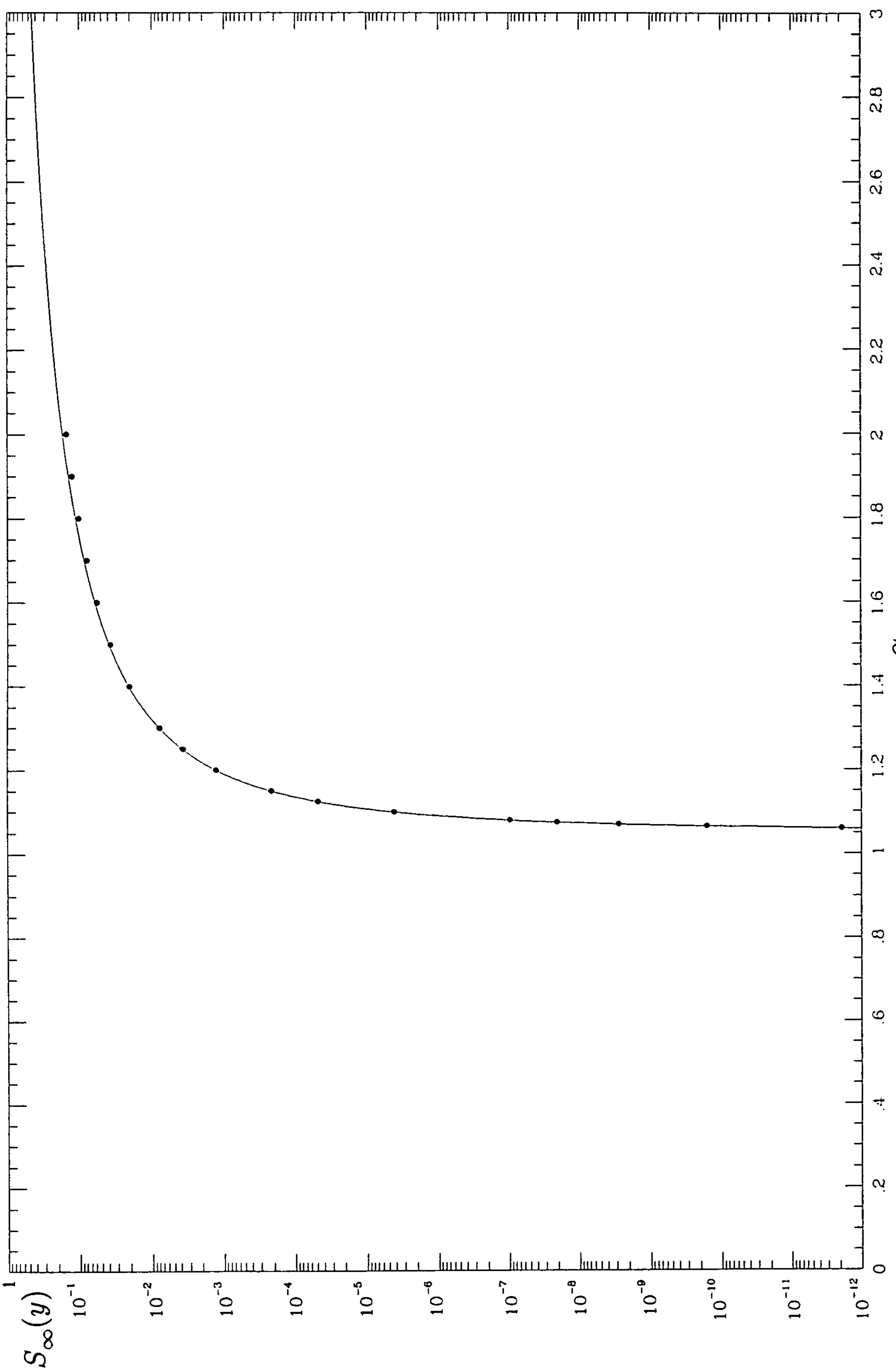


Fig. 4.2 The points represent $S_\infty(y_0)$ at various values of α_0, whilst the line represents the fitted functional form. It clearly shows the critical point, α_c.

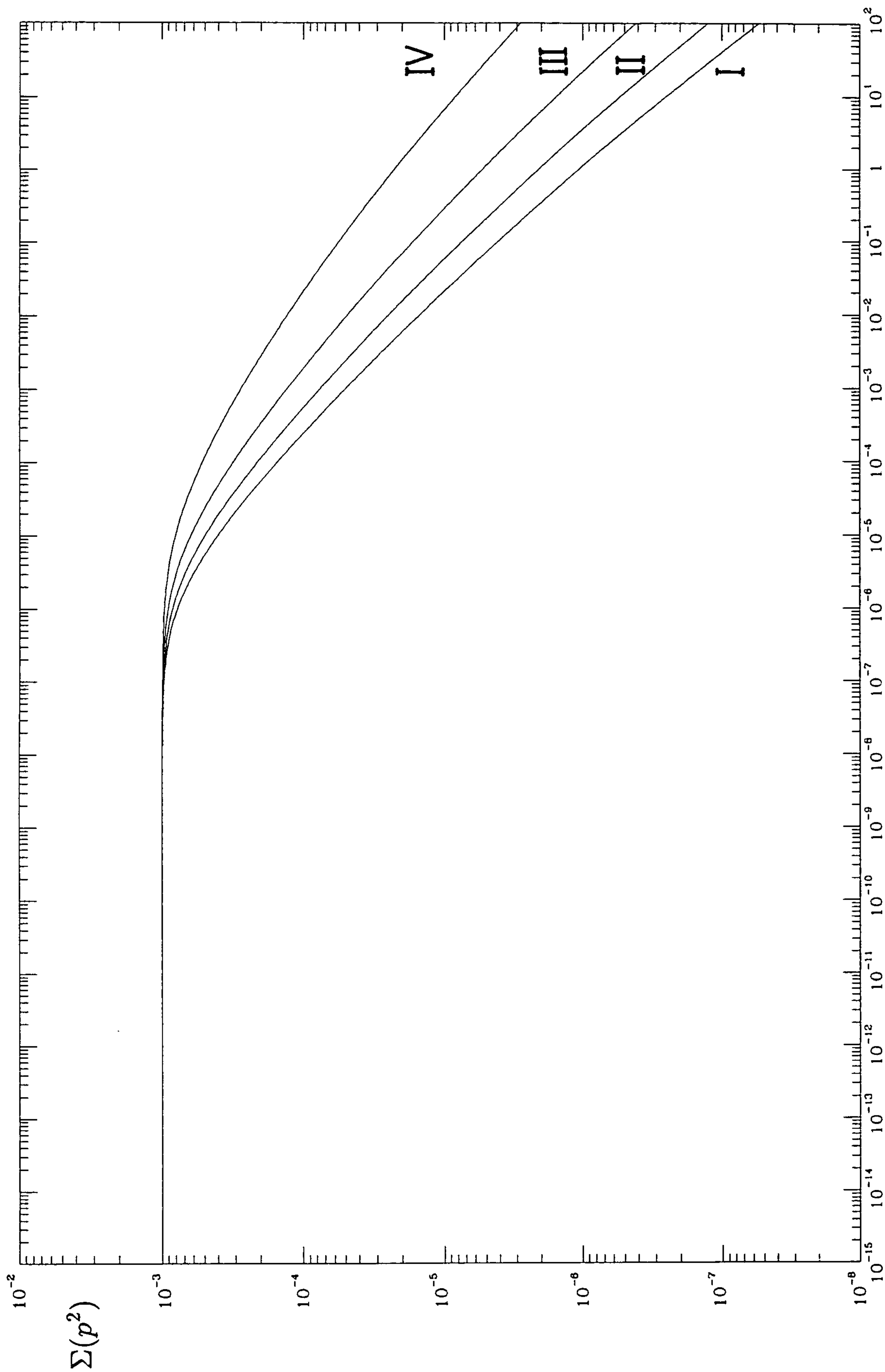


Fig. 4.3 The solution curves for $\Sigma(p^2)$ with bare ansatz and **infinite** cutoff for $\alpha_0 = 2.0$ (I), $\alpha_0 = 1.5$ (II), $\alpha_0 = 1.0$ (III) and $\alpha_0 = 0.5$ (IV).

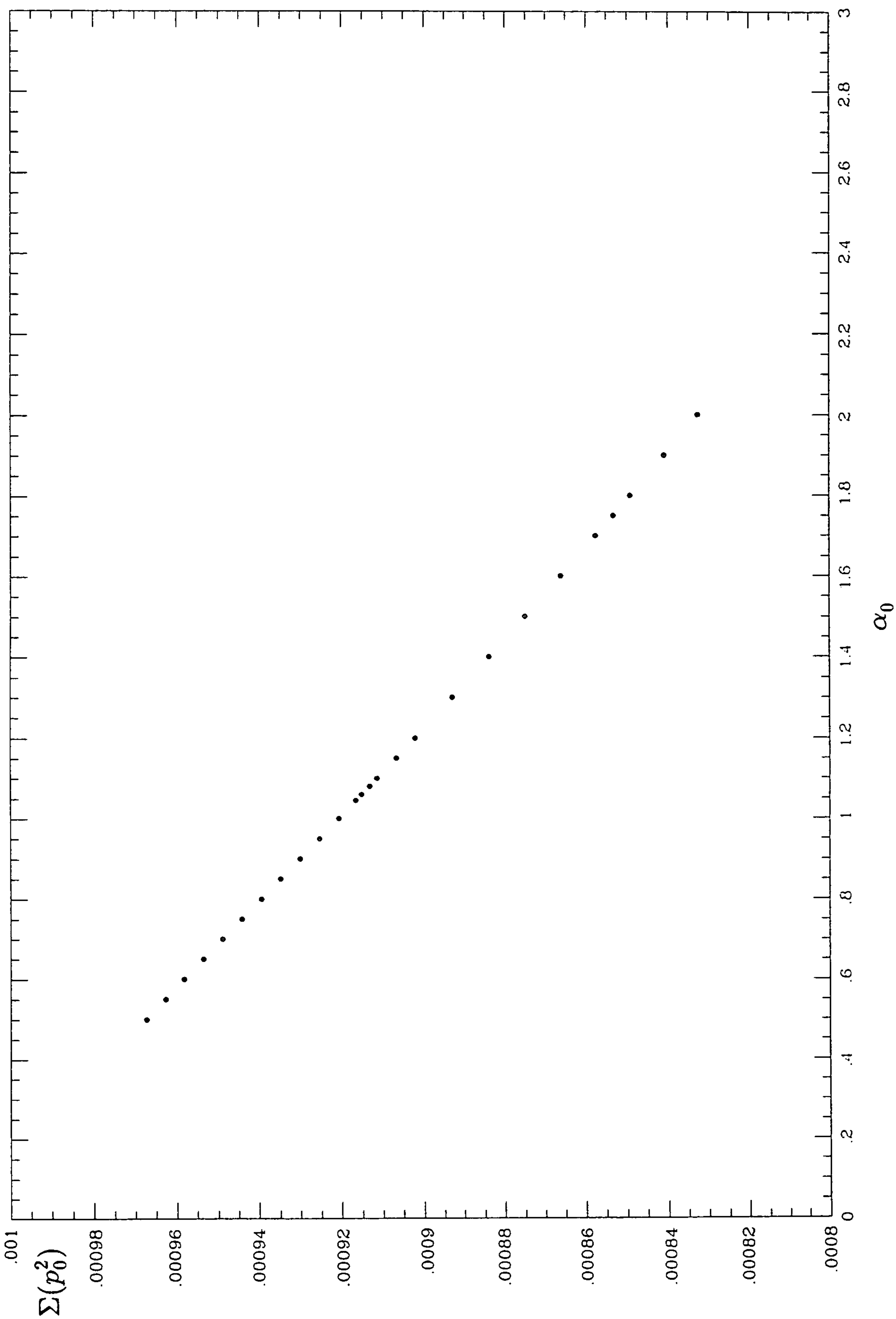


Fig. 4.4 $\Sigma(p_0^2)$ at various values of α_0 for the **bare** ansatz.
Clearly there is no critical point, α_c.

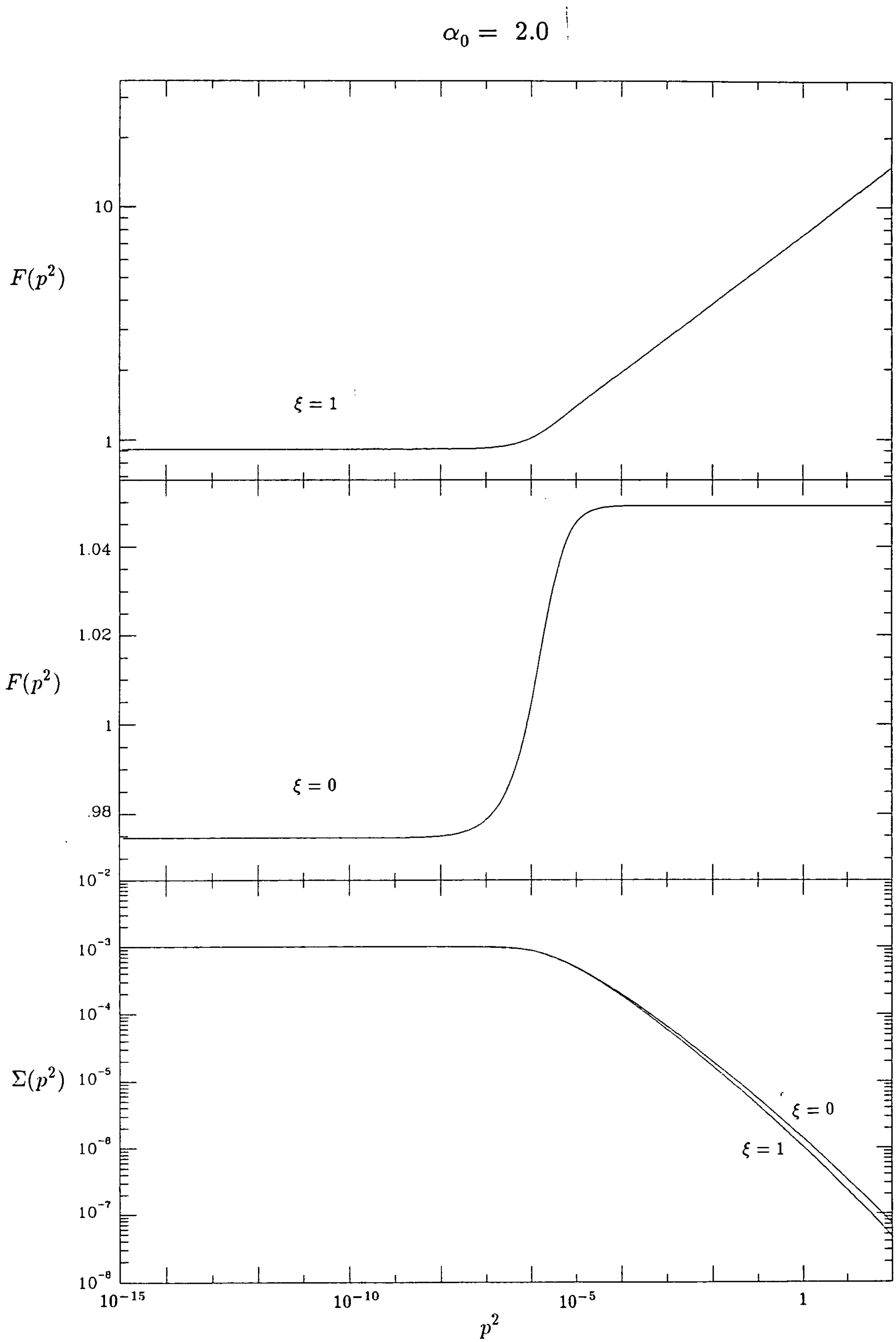


Fig. 4.5 $\alpha_0 =$ 2.0. Solutions for the **full** vertex (infinite cut off). Top graph is $F(p^2)$ for $\xi = 1$, middle graph is $F(p^2)$ for $\xi = 0$ and bottom graph is $\Sigma(p^2)$ for both $\xi = 1$ and $\xi = 0$.

$\alpha_0 = 1.5$

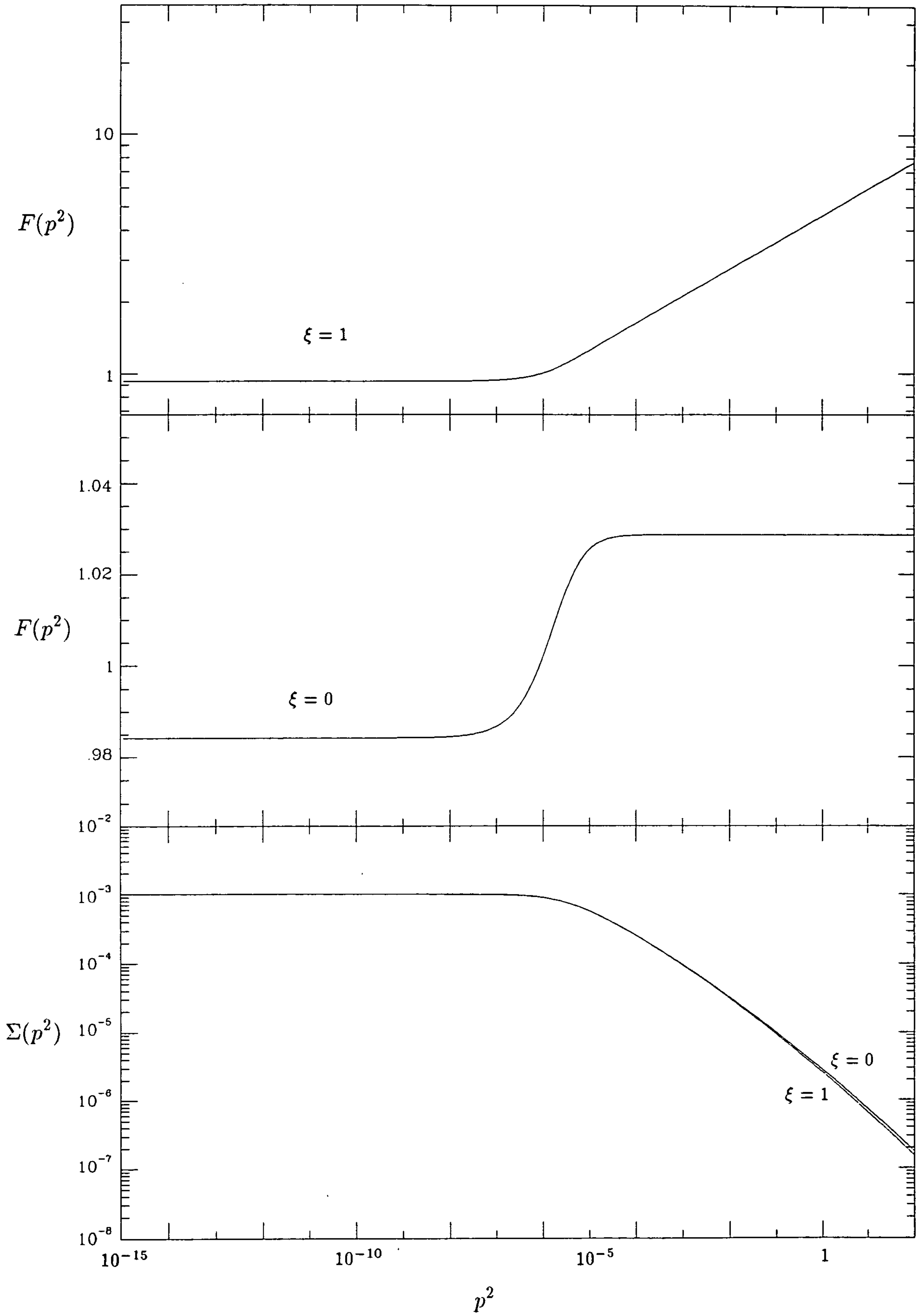


Fig. 4.6 $\alpha_0 = 1.5$. Otherwise as for fig. 4.5..

$\alpha_0 = 1.0$

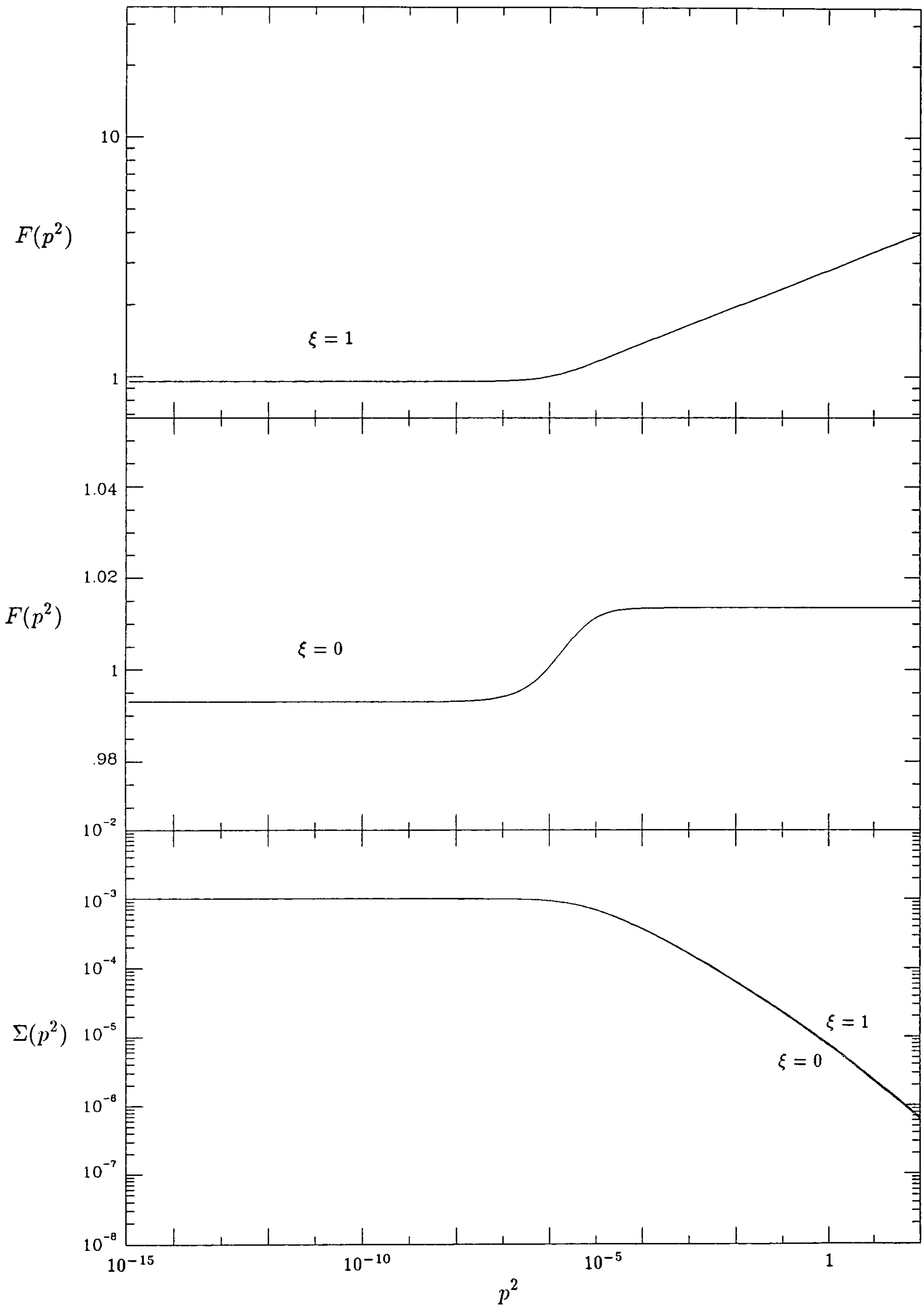


Fig. 4.7. $\alpha_0 =$ 1.0. Otherwise as for fig. 4.5..

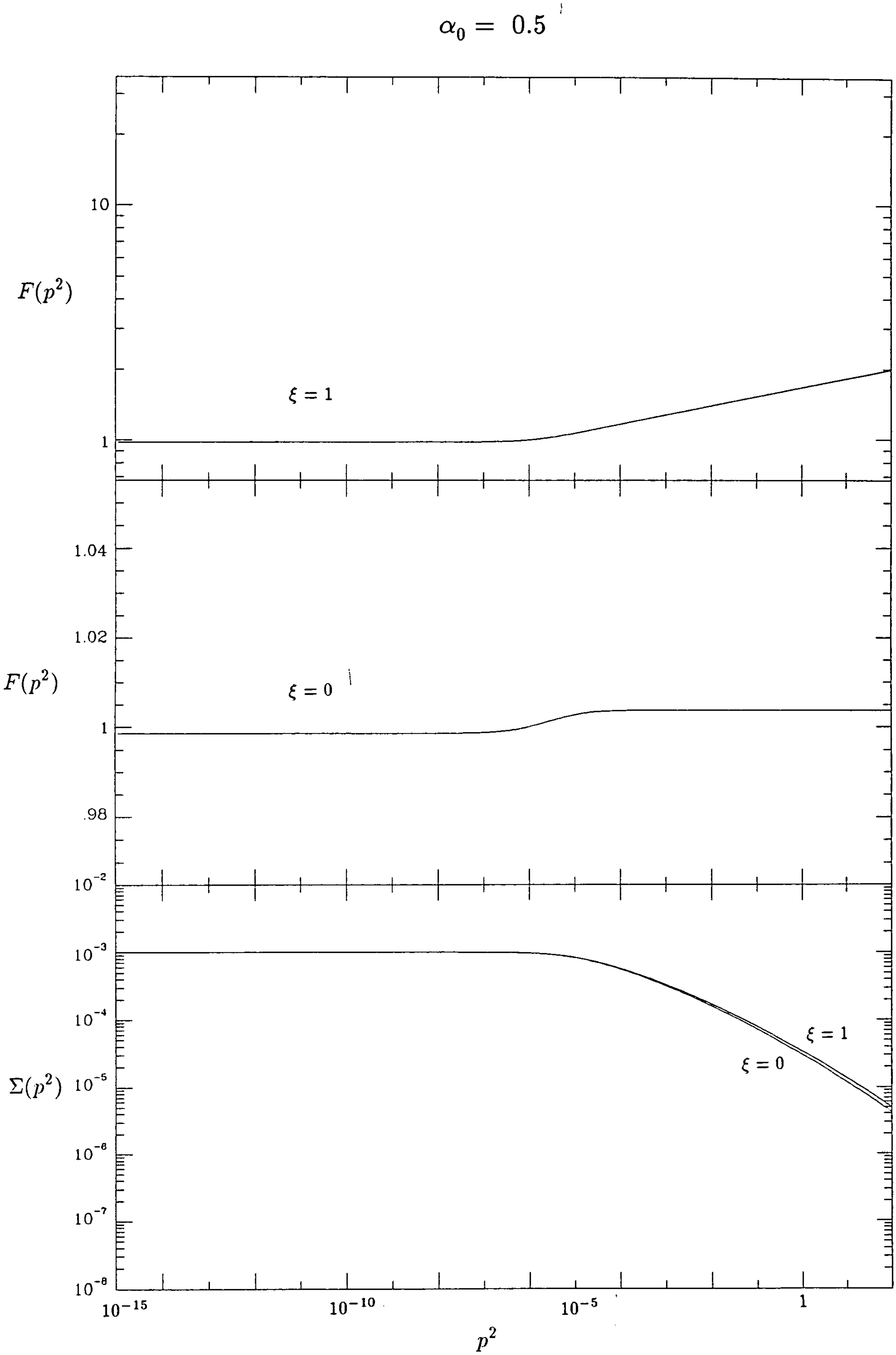


Fig. 4.8 $\alpha_0 =$ 0.5. Otherwise as for fig. 4.5..

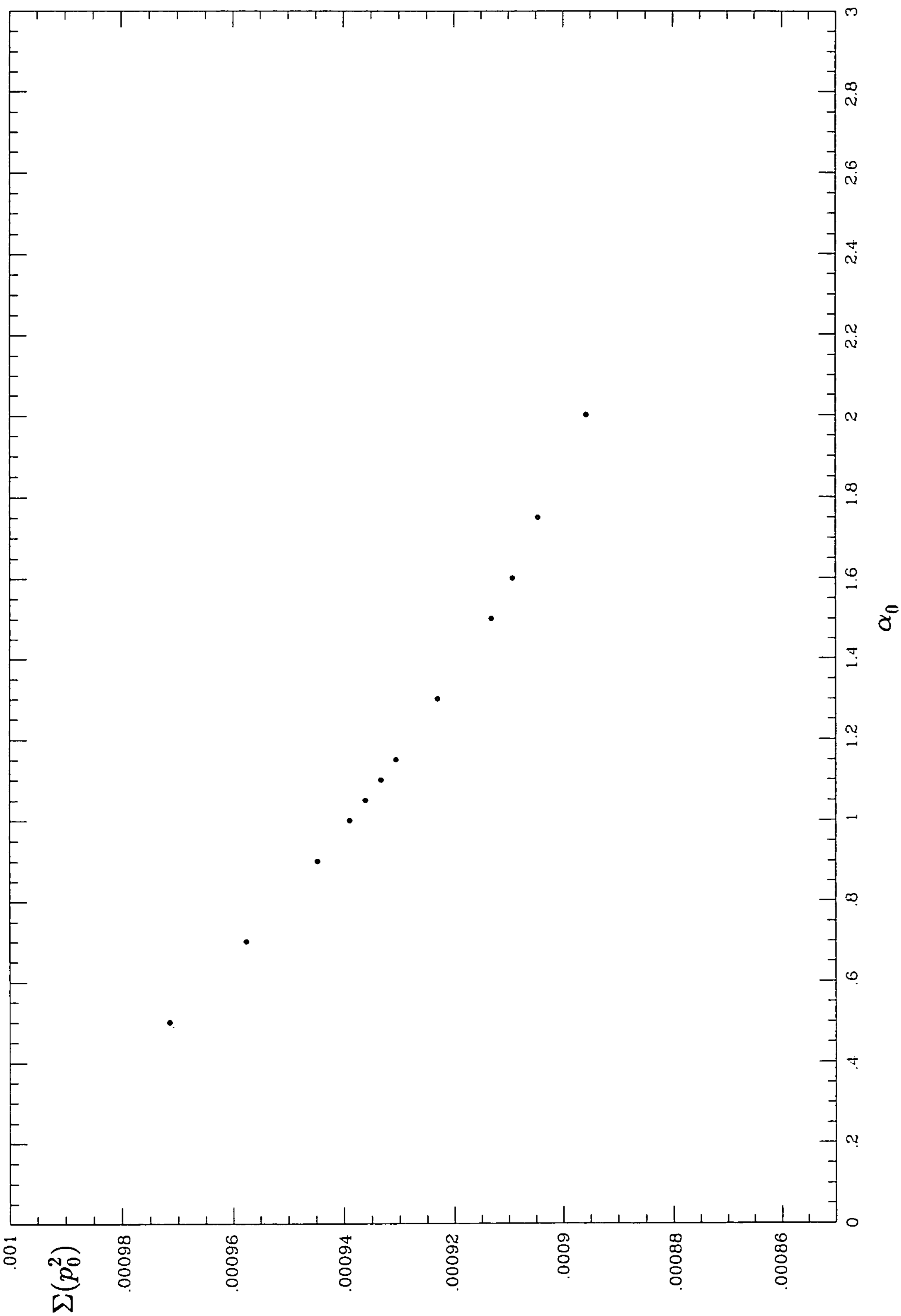


Fig. 4.9 $\Sigma(p_0^2)$ at various values of α_0 for the **full** vertex. Clearly there is no critical point, α_c.

APPENDIX A

ANGULAR INTEGRALS

In this appendix we shall cover the angular integrations that are used in other parts of this thesis. It will take the form of a recap of appendix A of the thesis of N. Brown[43]. We are at liberty to choose our coordinate system in such a fashon that the external momentum is:–

$$p^{\mu} = (p,\ 0,\ 0,\ 0)\,. \tag{A.1}$$

Our integrals are over Euclidean space-time and we write our integration variable, k^{μ}, in terms of 4 -dimensional spherical polars:–

$$k^{\mu} = (k\cos\psi,\ k\sin\psi\sin\theta\cos\phi,\ k\sin\psi\sin\theta\sin\phi,\ k\sin\psi\cos\theta)\,. \tag{A.2}$$

where $k \in [0,\ \infty)$, $\psi \in [0,\ \pi]$, $\theta \in [0,\ \pi]$ and $\phi \in [0,\ 2\pi]$. We can easily calculate the Jacobian and then:–

$$d^4k = \frac{k^2}{2}\ dk^2 \sin^2\psi\ d\psi \sin\theta\ d\theta\ d\phi. \tag{A.3}$$

All of the integrands depend only upon k^2, p^2 and $k{\cdot}p$, thus from (A.1) the integrands are independant of θ and ϕ. We are then able to perform the integrals over θ and ϕ which give an overall factor of 4π, hence (A.3) becomes:–

$$d^4k = 2\pi k^2\ dk^2 \sin^2\psi\ d\psi. \tag{A.4}$$

The general form of the integrals involving ψ are:–

$$\int_0^{\pi} \sin^2\psi\ d\psi\ \frac{(k\cdot p)^n}{(q^2)^m}$$

from (A.2) and (A.3) $k \cdot p = |k||p| \cos\psi$ and we shall define $I_{n,m}$ to be given by:–

$$I_{n,m} = |k|^n |p|^n \int_0^\pi \sin^2\psi \, d\psi \, \frac{\cos^n \psi}{(q^2)^m}.$$

Now we can write $q^2 = k^2 + p^2 - 2|p||k|\cos\psi \equiv a - b\cos\psi$, where $a = k^2 + p^2$ and $b = 2|p||k|$. In order to calculate these integrals we start off by looking at $I_{0,1}$, we make a change of variable to $z = \cos\psi$ then another, $y = a - bz$, to simplify the denominator. We can then express $I_{0,1}$ in terms of integrals of the form

$$\int_{a-b}^{a+b} \frac{y^i}{\sqrt{R}} \, dy$$

which we can look up[A1].

We then calculate $I_{r,0}$, which is easy. For $r =$ odd $I_{r,0} = 0$, by symmetry, and for $r =$ even we change the variable to $w = \cos^2\psi$ to give us a Beta function integral. So we know $I_{0,1}$ and $I_{r,0}$. To calculate the other integrals, up to $m = 2$, we make use of the iterative relations:–

$$I_{n,1} = -\frac{1}{2} I_{n-1,0} + \frac{a}{2} I_{n-1,1}$$

and:–

$$\frac{\partial}{\partial a} I_{n,1} = -\frac{b}{2} I_{n,2} \, .$$

Now if we define:–

$$h(x) = \frac{1}{2}(1 + x - |1 - x|) = \begin{cases} \mathrm{x} & x < 1; \\ 1 & x \geq 1. \end{cases}$$

Then we can write:–

$$
\begin{aligned}
I_{0,1} &= \frac{\pi}{2k^2}\, h\left(\frac{k^2}{p^2}\right) \\
I_{1,1} &= \frac{\pi p^2}{4k^2}\, h\left(\frac{k^4}{p^4}\right) \\
I_{2,1} &= \frac{\pi p^2}{8k^2}\left(p^2 + k^2\right) h\left(\frac{k^4}{p^4}\right) \\
I_{3,1} &= \frac{\pi p^6}{16k^2}\, h\left(\frac{k^8}{p^8}\right) + \frac{\pi p^4}{8}\, h\left(\frac{k^4}{p^4}\right) \\
I_{0,2} &= \frac{\pi}{2k^2}\frac{1}{|p^2 - k^2|}\, h\left(\frac{k^2}{p^2}\right) \\
I_{1,2} &= \frac{\pi p^2}{2k^2}\frac{1}{|p^2 - k^2|}\, h\left(\frac{k^4}{p^4}\right) \\
I_{2,2} &= \frac{3\pi p^4}{8k^2}\frac{1}{|p^2 - k^2|}\, h\left(\frac{k^6}{p^6}\right) + \frac{\pi p^2}{8}\frac{1}{|p^2 - k^2|}\, h\left(\frac{k^2}{p^2}\right).
\end{aligned}
\tag{A.5}
$$

APPENDIX B

SCHWINGER–DYSON EQUATIONS

In this appendix we shall generate the S-DE's that are used in the bulk of the thesis. We shall use the results (A.5) of appendix A in order to do the angular integrals. We start off by writting the vertex Γ^μ in the following way:–

$$\begin{aligned}\Gamma^\mu(k,\, p) = A\ \gamma^\mu + \ B\ \left(\not k + \not p\right)(k+\,p)^\mu -\ C\ (k+\,p)^\mu \\ +\ D\,\left[\gamma^\mu\left(k^2 -\, p^2\right) - \ (k+\,p)^\mu\left(\not k - \not p\right)\right].\end{aligned} \tag{B.1}$$

If we define

$$\begin{aligned}A' &= \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right)\\ B' &= \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{k^2 -\, p^2}\\ C' &= \left(\frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}\right)\frac{1}{k^2 -\, p^2}\\ D' &= \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{d(k,\,p)}\end{aligned} \tag{B.2}$$

$$d(k,\,p) = \frac{\left(k^2 -\, p^2\right)^2 + \ \left(\Sigma^2(k^2) + \ \Sigma^2(p^2)\right)^2}{k^2 + \ p^2}.$$

Then we can use the generic form for Γ^μ, (B.1), to calculate a general S-DE once and then use the following table to convert the result into the S-DE for the required ansatz.

ansatz	A	B	C	D
Bare	1	0	0	0
Central	A'	0	0	0
Ball–Chiu	A'	B'	C'	0
Full	A'	B'	C'	D'

(B.3)

to work in the massless theory we just throw away the $\Sigma(p^2)/F(p^2)$ S-DE and put $\Sigma(p^2 \text{ or } k^2) = 0$ in the $F^{-1}(p^2)$ S-DE. From fig. 2.2 the S-DE for qQED is:–

$$F^{-1}(p^2)[\not p - \Sigma(p^2)] = \not p + i\frac{e^2}{16\pi^4}\int_M \frac{d^4k\ F(k^2)}{k^2 - \Sigma^2(k^2)}\Gamma^\nu(k,\ p)\left(\not k + \Sigma(k^2)\right)\gamma^\mu\Delta^{\mu\nu}(q)$$

where the photon propagator

$$\Delta^{\mu\nu}(q) = \frac{1}{q^2}\left(g^{\mu\nu} + (\xi - 1)\frac{q^\mu q^\nu}{q^2}\right), \quad q^\mu = (k - p)^\mu.$$

In order to separate out the equation into those parts that are proportional to $\not p$ (1) we multiply through by $\not p$ (1) and trace, giving:–

$$F^{-1}(p^2) = 1 + i\frac{\alpha}{4\pi^3 p^2 4}\int_M \frac{d^4k\ F(k^2)}{k^2 - \Sigma^2(k^2)} tr\left[\not p\ \Gamma^\nu(k,\ p)\left(\not k + \Sigma(k^2)\right)\gamma^\mu\right]\Delta^{\mu\nu}(q) \tag{B.4}$$

and

$$\frac{\Sigma(p^2)}{F(p^2)} = -i\frac{\alpha}{4\pi^3 4}\int_M \frac{d^4k\ F(k^2)}{k^2 - \Sigma^2(k^2)} tr\left[\Gamma^\nu(k,\ p)\left(\not k + \Sigma(k^2)\right)\gamma^\mu\right]\Delta^{\mu\nu}(q) \tag{B.5}$$

respectively. Writing $\Gamma^\nu(k,\ p)\left(\not k + \Sigma(k^2)\right)$ in terms of odd and even powers of gamma matrices we have:–

$$
\begin{aligned}
\Gamma^{\nu}(k,\ p)\left(\not k+\ \Sigma(k^2)\right) = A\gamma^{\nu}\not k +\ B\left(k^2+\ \not p\not k\right)(k+\ p)^{\nu} -\ C\Sigma(k^2)\,(k+\ p)^{\nu} \\
+\ D[\left(k^2-\ p^2\right)\gamma^{\nu} -\ (k+\ p)^{\nu}\not q]\not k \qquad \text{(even)} \\
+\ A\Sigma(k^2)\gamma^{\nu}+\ B\Sigma(k^2)\left(\not k+\ \not p\right) -\ C\not k\,(k+\ p)^{\nu} \\
+\ D\Sigma(k^2)[\left(k^2-\ p^2\right)\gamma^{\nu} -\ (k+\ p)^{\nu}\not q] \qquad \text{(odd).}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
& tr\left[\not p\ \Gamma^{\nu}(k,\ p)\left(\not k+\ \Sigma(k^2)\right)\gamma^{\mu}\right]\Delta^{\mu\nu}(q) = \\
&\quad \frac{4}{q^2}\Bigg[A\{\ -2k\cdot p+\ (\xi-\ 1)\frac{1}{q^2}\left((k^2+\ p^2)k\cdot p-\ 2k^2p^2\right)\} \\
&\quad +\ B\{(k^2+\ p^2)k\cdot p+\ 2k^2p^2+\ (\xi-\ 1)\frac{1}{q^2}(k^2-\ p^2)^2k\cdot p\} \\
&\quad -C\Sigma(k^2)\{k\cdot p+\ p^2+\ (\xi-\ 1)\frac{1}{q^2}(k^2-\ p^2)(k\cdot p-\ p^2)\} \\
&\quad -\ 3D(k^2-\ p^2)k\cdot p\Bigg]
\end{aligned}
\tag{B.6}
$$

and

$$
\begin{aligned}
& tr\left[\Gamma^{\nu}(k,\ p)\left(\not k+\ \Sigma(k^2)\right)\gamma^{\mu}\right]\Delta^{\mu\nu}(q) = \\
&\quad \frac{4}{q^2}\Bigg[(3+\ \xi)A\Sigma(k^2)+\ B\Sigma(k^2)\{k^2+\ p^2+\ 2k\cdot p+\ (\xi-\ 1)\frac{1}{q^2}(k^2-\ p^2)^2\} \\
&\quad -C\{k^2+\ k\cdot p+\ (\xi-\ 1)\frac{1}{q^2}(k^2-\ k\cdot p)(k^2-\ p^2)+\ 3D\Sigma(k^2)(k^2-\ p^2)\Bigg]
\end{aligned}
\tag{B.7}
$$

Substituting (B.6) into (B.4) and (B.7) into (B.5) we have, after Wick rotation:–

$$F^{-1}(p^2) = 1 - \frac{\alpha}{4\pi^3 p^2} \int_E \frac{d^4k\ F(k^2)}{k^2 + \Sigma^2(k^2)} \frac{1}{q^2} \Bigg[A\{-2k\cdot p + (\xi - 1)\frac{1}{q^2}\left((k^2 + p^2)k\cdot p - 2k^2p^2\right)\}$$
$$+ B\{(k^2 + p^2)k\cdot p + 2k^2p^2 + (\xi - 1)\frac{1}{q^2}(k^2 - p^2)^2 k\cdot p\}$$
$$+ C\Sigma(k^2)\{k\cdot p + p^2 + (\xi - 1)\frac{1}{q^2}(k^2 - p^2)(k\cdot p - p^2)\}$$
$$- 3D(k^2 - p^2)\ k\cdot p \Bigg]$$

and

$$\frac{\Sigma(p^2)}{F(p^2)} = \frac{\alpha}{4\pi^3} \int_E \frac{d^4k\ F(k^2)}{k^2 + \Sigma^2(k^2)} \frac{1}{q^2} \Bigg[(3 + \xi)A\Sigma(k^2)$$
$$+ B\Sigma(k^2)\{k^2 + p^2 + 2k\cdot p + (\xi - 1)\frac{1}{q^2}(k^2 - p^2)^2\}$$
$$- C\{k^2 + k\cdot p + (\xi - 1)\frac{1}{q^2}(k^2 - k\cdot p)(k^2 - p^2)$$
$$+ 3D\Sigma(k^2)(k^2 - p^2) \Bigg].$$

Using the angular integrals (A.5) we can simplify the above into 1 -dimensional integrals over k^2:–

$$F^{-1}(p^2) = 1 + \frac{\alpha}{4\pi p^2} \int_0^{\Lambda^2} \frac{dk^2\ k^2\ F(k^2)}{k^2 + \Sigma^2(k^2)} \Bigg[\xi A\{\theta_0 \frac{k^2}{p^2} + \theta_1 \frac{p^2}{k^2}\}$$
$$- (k^2 + p^2)\frac{3B}{2}\{\theta_0\frac{k^2}{p^2} + \theta_1\frac{p^2}{k^2}\} + \xi B(k^2 - p^2)\{\theta_0\frac{k^2}{p^2} - \theta_1\frac{p^2}{k^2}\}$$
$$- \frac{3C}{2}\Sigma(k^2)\{\theta_0\frac{k^2}{p^2} + \theta_1\frac{p^2}{k^2}\} + \xi C(k^2 - p^2)\Sigma(k^2)\{\theta_0\frac{1}{p^2}\}$$
$$+ \frac{3D}{2}(k^2 - p^2)\{\theta_0\frac{k^2}{p^2} + \theta_1\frac{p^2}{k^2}\} \Bigg] \tag{B.8}$$

and

$$\frac{\Sigma(p^2)}{F(p^2)} = \frac{\alpha}{4\pi}\int_0^{\Lambda^2} \frac{dk^2\ k^2\ F(k^2)}{k^2 + \Sigma^2(k^2)}\Bigg[3A\Sigma(k^2)\{\theta_0\frac{1}{p^2} + \theta_1\frac{1}{k^2}\} + \xi A\Sigma(k^2)\{\theta_0\frac{1}{p^2} + \theta_1\frac{1}{k^2}\}$$
$$+ 3B\Sigma(k^2)\{\theta_0\frac{k^2}{p^2} + \theta_1\frac{p^2}{k^2}\} - \xi B\Sigma(k^2)(k^2 - p^2)\{\theta_0\frac{1}{p^2} - \theta_1\frac{1}{k^2}\}$$
$$- \frac{3C}{2}\{\theta_0\frac{k^2}{p^2} + \theta_1\frac{p^2}{k^2}\} - \xi C(k^2 - p^2)\{\theta_1\frac{1}{k^2}\}$$
$$+ 3D\Sigma(k^2)(k^2 - p^2)\{\theta_0\frac{1}{p^2} + \theta_1\frac{1}{k^2}\}\Bigg] \tag{B.9}$$

where we have explicitly put in the integration limits 0 and Λ (as this is the unrenormalised S-DE's) and $\theta_0 \equiv \theta(p^2 - k^2)$ and $\theta_1 \equiv \theta(k^2 - p^2)$. We can now use equations (B.8) and (B.9), in conjunction with table (B.3) and the identities (B.2) to generate the S-DE's for all the various cases we are interested in.

MASSLESS qQED :

(The equations for chapter 3). We drop equation (B.9), the mass equation, and set $\Sigma(k^2 \mathrm{or} p^2) = 0$ identically ($C' = 0$ and $d(k, p) = (k^2 - p^2)^2/(k^2 + p^2)$). We also make the transformation: $p^2 = y\Lambda^2$, $k^2 = x\Lambda^2$, $F(p^2) = F(y)$, $\Sigma(p^2) = \Lambda S(y), \ldots$ Thence from (B.8), (B.3) and (B.2) we have:–

Bare ansatz :

Here we set $A = 1$ and $B = C = 0 = D$ when we get:–

$$F^{-1}(y) = 1 + \frac{\alpha\xi}{4\pi y}\int_0^1 dx\ F(x)\Bigg[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\Bigg]. \tag{B.10}$$

Central Kernal ansatz :

In this case we have $A = A'$ and $B = C = 0 = D$ and we have:–

$$F^{-1}(y) = 1 + \frac{\alpha\xi}{4\pi y}\int_0^1 dx\ \frac{1}{2}\left(1 + \frac{F(x)}{F(y)}\right)\Bigg[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\Bigg]. \tag{B.11}$$

Minimal Ball–Chiu ansatz :

Setting $A = A'$, $B = B'$, $C = C' = 0$ and $D = 0$ gives us:–

$$
\begin{aligned}
F^{-1}(y) = 1 + \frac{\alpha}{4\pi y}\int_0^1 dx F(x)\Bigg[& \\
& \frac{\xi}{2}\left(\frac{1}{F(x)} + \frac{1}{F(y)}\right)\{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\} \\
& -\frac{3}{4}\left(\frac{1}{F(x)} - \frac{1}{F(y)}\right)\frac{x+y}{x-y}\{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\} \\
& +\frac{\xi}{2}\left(\frac{1}{F(x)} - \frac{1}{F(y)}\right)\{\, \theta(y - x)\frac{x}{y} - \theta(x - y)\frac{y}{x}\}\Bigg]
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
\Rightarrow \quad F^{-1}(y) = 1 + \frac{\alpha}{4\pi y}\int_0^1 dx \Bigg[& \xi\theta(y - x)\frac{x}{y} + \xi\frac{F(x)}{F(y)}\theta(x - y)\frac{y}{x} \\
& -\frac{3}{4}\left(1 - \frac{F(x)}{F(y)}\right)\frac{x+y}{x-y}\{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\}\Bigg].
\end{aligned}
$$

Full ansatz :

Now we set $A = A'$, $B = B'$, $C = C' = 0$ and $D = D'$, thus we obtain:–

$$
\begin{aligned}
F^{-1}(y) = 1 + \frac{\alpha}{4\pi y}\int_0^1 dx\ F(x)\Bigg[& \\
& \frac{\xi}{2}\left(\frac{1}{F(x)} + \frac{1}{F(y)}\right)\{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\} \\
& -\frac{3}{4}\left(\frac{1}{F(x)} - \frac{1}{F(y)}\right)\frac{x+y}{x-y}\{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\} \\
& +\frac{\xi}{2}\left(\frac{1}{F(x)} - \frac{1}{F(y)}\right)\{\, \theta(y - x)\frac{x}{y} - \theta(x - y)\frac{y}{x}\} \\
& -\frac{3}{4}\left(\frac{1}{F(x)} - \frac{1}{F(y)}\right)\frac{(x+y)(x-y)}{(x-y)^2} \\
& \quad \times \{\, \theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\}\Bigg]
\end{aligned}
$$

$$
\Rightarrow \quad F^{-1}(y) = 1 + \frac{\alpha\xi}{4\pi y}\int_0^1 dx\left[\theta(y - x)\frac{x}{y} + \frac{F(x)}{F(y)}\theta(x - y)\frac{y}{x}\right] \tag{B.13}
$$

MASSIVE qQED :

(The equations for chapter 4). We start from eqs. (B.9) and (B.10). For the bare ansatz we transform to dimensionless variables: $p^2 = y\ \Lambda^2$, $k^2 = y\ \Lambda^2$, $F(p^2) = F(y)$ and $\Sigma(p^2) = \Lambda S(y)$. For the full ansatz we do not so transform.

Bare ansatz :

In this case we have $A = 1$ and $B = C = 0 = D$, when we get:–

$$
\begin{aligned}
F^{-1}(y) &= 1 + \frac{\alpha\xi}{4\pi y}\int_0^1 \frac{dx\ x\ F(x)}{x + S^2(x)}\left[\theta(y - x)\frac{x}{y} + \theta(x - y)\frac{y}{x}\right] \\
\frac{S(y)}{F(y)} &= (3 + \xi)\frac{\alpha}{4\pi}\int_0^1 \frac{dx\ x\ F(x)S(x)}{x + S^2(x)}\left[\theta(y - x)\frac{1}{y} + \theta(x - y)\frac{1}{x}\right].
\end{aligned}
\tag{B.14}
$$

Full ansatz :

For this ansatz we have $A = A'$, $B = B'$, $C = C'$ and $D = D'$ which gives us:–

$$
\begin{aligned}
F^{-1}(p^2) = 1 + \frac{\alpha}{4\pi}\int_0^{\Lambda^2} \frac{dk^2}{k^2 + \Sigma^2(k^2)} \Bigg[& -\xi\{\ \theta(p^2 - k^2)\frac{k^4}{p^4} + \theta(k^2 - p^2)\frac{F(k^2)}{F(p^2)}\} \\
& - \xi\frac{\Sigma^2(k^2)}{k^2}\left(1 - \frac{F(k^2)\Sigma(p^2)}{F(p^2)\Sigma(k^2)}\right)\{\ \theta(p^2 - k^2)\frac{k^4}{p^4}\} \\
& + \frac{3(k^2 + p^2)}{4(k^2 - p^2)}\frac{\left(\Sigma^2(k^2) + \Sigma^2(p^2)\right)^2}{(k^2 - p^2)^2 + \left(\Sigma^2(k^2) + \Sigma^2(p^2)\right)^2} \\
& \quad \times \left(1 - \frac{F(k^2)}{F(p^2)}\right)\{\ \theta(p^2 - k^2)\frac{k^4}{p^4} + \theta(k^2 - p^2)\} \\
& + \frac{3\Sigma^2(k^2)}{2(k^2 - p^2)}\left(1 - \frac{F(k^2)\Sigma(p^2)}{F(p^2)\Sigma(k^2)}\right)\{\ \theta(p^2 - k^2)\frac{k^4}{p^4} + \theta(k^2 - p^2)\}\Bigg]
\end{aligned}
$$

and:–

$$\frac{\Sigma(p^2)}{F(p^2)} = \frac{\alpha}{4\pi}\int_0^{\Lambda^2} \frac{dk^2}{k^2+\Sigma^2(k^2)}\Bigg[\xi\frac{F(k^2)}{F(p^2)}\{\Sigma(k^2)\theta(p^2-k^2)\frac{k^2}{p^2}+\Sigma(p^2)\theta(k^2-p^2)\}$$
$$-\frac{3p^2}{2(k^2-p^2)}\frac{F(k^2)}{F(p^2)}\left(\Sigma(k^2)-\Sigma(p^2)\right)\{\theta(p^2-k^2)\frac{k^4}{p^4}+\theta(k^2-p^2)\}$$
$$+\frac{3\Sigma(k^2)}{2}\Bigg[\left(1+\frac{F(k^2)}{F(p^2)}\right)$$
$$+\frac{k^4-p^4}{(k^2-p^2)^2+(\Sigma^2(k^2)+\Sigma^2(p^2))^2}\left(1-\frac{F(k^2)}{F(p^2)}\right)\Bigg]$$
$$\times\{\theta(p^2-k^2)\frac{k^2}{p^2}+\theta(k^2-p^2)\}\Bigg]$$

$$\Rightarrow \quad F^{-1}(p^2) = 1-\frac{\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2+\Sigma^2(k^2)}I(k^2,p^2)+\frac{\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2+\Sigma^2(k^2)}\frac{F(k^2)}{F(p^2)} \tag{B.15}$$

where:–

$$I(k^2,p^2) =$$
$$\frac{3\,(k^2+p^2)}{4\,(k^2-p^2)}\frac{\left(\Sigma^2(k^2)+\Sigma^2(p^2)\right)^2}{(k^2-p^2)^2+(\Sigma^2(k^2)+\Sigma^2(p^2))^2}\left(1-\frac{F(k^2)}{F(p^2)}\right)$$
$$\times\{\theta(p^2-k^2)\frac{k^4}{p^4}+\theta(k^2-p^2)\}$$
$$+\frac{3\,\Sigma^2(k^2)}{2\,(k^2-p^2)}\left(1-\frac{F(k^2)\Sigma(p^2)}{F(p^2)\Sigma(k^2)}\right)\{\theta(p^2-k^2)\frac{k^4}{p^4}+\theta(k^2-p^2)\}$$
$$-\xi\frac{\Sigma^2(k^2)}{k^2}\left(1-\frac{F(k^2)\Sigma(p^2)}{F(p^2)\Sigma(k^2)}\right)\{\theta(p^2-k^2)\frac{k^4}{p^4}\}-\xi\{\theta(p^2-k^2)\frac{k^4}{p^4}\}$$

and

$$\frac{\Sigma(p^2)}{F(p^2)} = \frac{\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2+\Sigma^2(k^2)}J(k^2,p^2)+\frac{\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2+\Sigma^2(k^2)}\frac{\Sigma(p^2)F(k^2)}{F(p^2)} \tag{B.16}$$

where:–

$$J(k^2,p^2) =$$

$$\frac{3\ \Sigma(k^2)}{2}\Bigg[\left(1+\frac{F(k^2)}{F(p^2)}\right)$$

$$+\ \frac{k^4-\ p^4}{(k^2-\ p^2)^2+\ (\Sigma^2(k^2)+\ \Sigma^2(p^2))^2}\left(1-\ \frac{F(k^2)}{F(p^2)}\right)\Bigg]$$

$$\times\ \{\theta(p^2-\ k^2)\frac{k^2}{p^2}+\ \theta(k^2-\ p^2)\}$$

$$-\ \frac{3p^2}{2(k^2-\ p^2)}\frac{F(k^2)}{F(p^2)}(\Sigma(k^2)-\ \Sigma(p^2))\{\ \theta(p^2-\ k^2)\frac{k^4}{p^4}+\ \theta(k^2-\ p^2)\}$$

$$+\ \xi\frac{F(k^2)}{F(p^2)}\Sigma(k^2)\theta(p^2-\ k^2)\frac{k^2}{p^2}$$

We renormalised by setting $F_R(p^2/\mu^2)\ =\ Z_2^{-1}(\mu^2/\Lambda^2)F(p^2/\Lambda^2)$ and $\Sigma_R(p^2/\mu^2)\ =\ \Sigma(p^2/\Lambda^2)$. The reason why the Z factor for Σ is 1 is discussed in section 4.3. The ratio $F(k^2)/F(p^2)$ in I and J become $F_R(k^2)/F_R(p^2)$. We renormalise at $p^2\ =\ \mu^2$ when $F_R(\mu^2)$ = some constant to be defined (see section 4.3). Then we have:–

$$\frac{F_R(\mu^2)}{F_R(p^2)}\left(1-\ \frac{\alpha}{4\pi}\int_0^\infty\frac{dk^2}{k^2+\ \Sigma^2(k^2)}I(k^2,\mu^2)+\ \frac{\xi\alpha}{4\pi}\int_{\mu^2}^{X^2\to\infty}\frac{dk^2}{k^2+\ \Sigma^2(k^2)}\frac{F_R(k^2)}{F_R(\mu^2)}\right)$$

$$=1-\ \frac{\alpha}{4\pi}\int_0^\infty\frac{dk^2}{k^2+\ \Sigma^2(k^2)}I(k^2,p^2)+\ \frac{\xi\alpha}{4\pi}\int_{p^2}^{X^2\to\infty}\frac{dk^2}{k^2+\ \Sigma^2(k^2)}\frac{F_R(k^2)}{F_R(p^2)}$$

and

$$\Sigma(p^2)\left(1-\ \frac{\alpha}{4\pi}\int_0^\infty\frac{dk^2}{k^2+\ \Sigma^2(k^2)}I(k^2,p^2)+\ \frac{\xi\alpha}{4\pi}\int_{p^2}^{X^2\to\infty}\frac{dk^2}{k^2+\ \Sigma^2(k^2)}\frac{F_R(k^2)}{F_R(p^2)}\right)$$

$$=\frac{\alpha}{4\pi}\int_0^\infty\frac{dk^2}{k^2+\ \Sigma^2(k^2)}J(k^2,p^2)+\ \frac{\xi\alpha}{4\pi}\int_{p^2}^{X^2\to\infty}\frac{dk^2}{k^2+\ \Sigma^2(k^2)}\frac{\Sigma(p^2)F_R(k^2)}{F_R(p^2)}$$

$$\Rightarrow \frac{F_R(\mu^2)}{F_R(p^2)} = 1 - \frac{\alpha}{4\pi}\int_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)} I(k^2,p^2)$$

$$+ \frac{\alpha}{4\pi}\int_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)} I(k^2,\mu^2)\frac{F_R(\mu^2)}{F_R(p^2)} + \frac{\xi\alpha}{4\pi}\int_{p^2}^{\mu^2} \frac{dk^2}{k^2 + \Sigma^2(k^2)}\frac{F_R(k^2)}{F_R(p^2)} \tag{B.17}$$

and

$$\Sigma(p^2) = \frac{\alpha}{4\pi}\int_0^\infty \frac{dk^2}{k^2 + \Sigma^2(k^2)}[J(k^2,p^2) + \Sigma(p^2)I(k^2,p^2)] \tag{B.18}$$

APPENDIX C

FEYNMAN INTEGRALS

In this appendix we give the integral rules and expansions in k^2 that we used in the perturbative calculation of Γ^μ in chapter 2.5d

Feynman integral rules :

The Feynman integral rules tell us how to relate integrals in which the denominator is a complicated product to integrals in which the denominator is a simple sum to a power:–

$$\int d^4w \frac{1}{a_1 a_2 \ldots a_n} = (n-1)! \int d^4w \int_0^1 dx_1 \ldots \int_0^1 dx_n \frac{\delta(1-x_1 \ldots - x_n)}{[x_1 a_1 \ldots + x_n a_n]^n}$$
$$\int d^4w \frac{1}{a_1^2 a_2 \ldots a_n} = n! \int d^4w \int_0^1 dx_1 \ldots \int_0^1 dx_n \frac{x_1\, \delta(1-x_1 \ldots - x_n)}{[x_1 a_1 \ldots + x_n a_n]^{(n+1)}} \tag{C.1}$$

Then we can easily get:–

$$\begin{aligned}
&\int d^4w \frac{f(w)}{w^2[(k-w)^2+m^2][(p-w)^2+m^2]} \\
&= 2 \int d^4w \int_0^1 dz \int_0^1 dy \int_0^1 dx \frac{f(w)\, \delta(1-x-y-z)}{[zw^2 + x\,(k-w)^2 + xm^2 + y\,(p-w)^2 + ym^2]^3} \\
&= 2 \int d^4w' \int_0^1 dx \int_0^{1-x} dy \frac{f(w' + xk + yp)}{[w'^2 + L]^3} \\
&\equiv 2 \int d^4w \int_0^1 dx \int_0^{1-x} dy \frac{f(w + xk + yp)}{[w^2 + L]^3}
\end{aligned}$$

Where $w' = w - xk - yp$, $L = k^2x\,(1-x) + p^2y\,(1-y) - 2xyk\cdot p + xm^2 + ym^2$.

In an obvious way we also have that:–

$$\int d^4w \frac{f(w)}{w^4[(k-w)^2+m^2][(p-w)^2+m^2]}$$
$$= 3! \int d^4w \int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)f(w+xk+yp)}{[w^2+L]^4}$$

The $\int^{\Lambda} d^4w$ integrals :

After we have used the Feynman integral rules we are able to accomplish the $\int^{\Lambda} d^4w$ integrals using the following results:–

$$\int^{\Lambda} \frac{d^4w}{[w^2+L]^3} = \pi^2 \int_0^{\Lambda^2} \frac{dw^2 w^2}{[w^2+L]^3} = -\frac{\pi^2}{2}\left[\frac{w^2}{[w^2+L]^2}\right]_0^{\Lambda^2} + \frac{\pi^2}{2}\int_0^{\Lambda^2} \frac{dw^2}{[w^2+L]^2}$$
$$= -\frac{\Lambda^2\pi^2}{[\Lambda^2+L]^2} + \frac{\pi^2}{2}\left[\frac{-1}{[w^2+L]}\right]_0^{\Lambda^2}$$
$$\overset{\Lambda^2\to\infty}{=} \frac{\pi^2}{2L} \tag{C.2}$$

$$\int^{\Lambda} \frac{d^4w\, w^2}{[w^2+L]^3} = \pi^2 \int_0^{\Lambda^2} \frac{dw^2 w^4}{[w^2+L]^3}$$
$$= \pi^2 \int_0^{\Lambda^2} dw^2 \left(\frac{1}{[w^2+L]} - \frac{2L}{[w^2+L]^2} + \frac{L^2}{[w^2+L]^3}\right)$$
$$= \pi^2 \left(\log\left(\frac{\Lambda^2+L}{L}\right) + 2L\left[\frac{1}{[w^2+L]}\right]_0^{\Lambda^2} - \frac{L^2}{2}\left[\frac{1}{[w^2+L]^2}\right]_0^{\Lambda^2}\right)$$
$$\overset{\Lambda^2\to\infty}{=} \pi^2 \log\frac{\Lambda^2}{L} - \frac{3\pi^2}{2} \tag{C.3}$$

$$\int^{\Lambda} \frac{d^4w}{[w^2+L]^4} = \frac{-1}{3}\frac{\partial}{\partial L}\int^{\Lambda} \frac{d^4w}{[w^2+L]^3} \overset{\Lambda^2\to\infty}{=} \frac{\pi^2}{6L^2} \tag{C.4}$$

$$\int^{\Lambda} \frac{d^4w\ w^2}{[w^2 +\ L]^4} = \frac{-1}{3}\frac{\partial}{\partial L}\int^{\Lambda} \frac{d^4w\ w^2}{[w^2 +\ L]^3} \overset{\Lambda^2 \to \infty}{=} \frac{\pi^2}{3L} \tag{C.5}$$

$$\int^{\Lambda} \frac{d^4w\ w^4}{[w^2 +\ L]^4} = \int^{\Lambda} \frac{d^4w\ w^2}{[w^2 +\ L]^3} - L\int^{\Lambda} \frac{d^4w\ w^2}{[w^2 +\ L]^4} \overset{\Lambda^2 \to \infty}{=} \pi^2 \log\frac{\Lambda^2}{L} - \frac{11\pi^2}{6}. \tag{C.6}$$

The $\int_0^1 dx \int_0^1 dz$ *integrals :*

We go from the integral of y over the region $[0, 1 - x]$ to the integral of z over the region $[0, 1]$ by making the transformation $y = z\,(1 - x)$. The Jacobian of the transformation is of course $(1 - x)$. Then:–

$$\int_0^1 dx \int_0^{1-x} dy \log L = \int_0^1 dx \int_0^1 dz\ (1 - x) \log L$$

where now $L = a + bx + cx^2$ with $a = p^2z - p^2z^2 + m^2z$, $b = k^2 - 2k \cdot pz - p^2z + 2p^2z^2 + m^2 - m^2z$ and $c = -k^2 + 2k \cdot pz - p^2z^2$. Then, as $k^2 \gg (p^2,\ m^2)$, it is clear that:–

$$\int_0^1 dx \int_0^1 dz\ (1 - x) \log L \equiv \int_0^1 dx \int_0^1 dz\ (1 - x) \log k^2$$

to first two orders, and hence:–

$$\int_0^1 dx \log\frac{\Lambda^2}{L} \equiv \int_0^1 dx \log\frac{\Lambda^2}{k^2}. \tag{C.7}$$

In chapter 2.5d we have a number of integrands of the form x^n/L^m which we need to look at. To do this we make extensive use of the identities in Ref.[A1] and expand in terms of k^2. We now state our conventions and give some preliminary results:–

$$L = a + bx + cx^2$$

where $a = p^2 z - p^2 z^2 + m^2 z$, $b = k^2 - 2k\cdot pz - p^2 z + 2p^2 z^2 + m^2 - m^2 z$ and $c = -k^2 + 2k\cdot pz - p^2 z^2$.

$$\Delta = 4ac - b^2 = -k^4 + 4k^2\, k\cdot pz + O(k^2)$$

as we can take $m^2 \to 0$ with respect to k^2 and p^2. Some preliminary results that we need are:–

$$\log\left(\frac{b + 2cx - \sqrt{b^2 - 4ac}}{b + 2cx + \sqrt{b^2 - 4ac}}\right)\Bigg|_0^1 = \log\left[\left(\frac{-k^2}{m^2}\right)\left(\frac{k^2}{p^2z^2 - p^2 z - m^2 z}\right)\right]$$
$$= \log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2}z + z - z^2\right)$$

$$\log L\Bigg|_0^1 = \log\frac{m^2}{p^2} - \log\left(\frac{m^2}{p^2}z + z - z^2\right)$$

Then from Ref.[A1] we get the following:–

$$\int_0^1 dx\frac{1}{L} = \frac{1}{\sqrt{b^2 - 4ac}}\log\left(\frac{b + 2cx - \sqrt{b^2 - 4ac}}{b + 2cx + \sqrt{b^2 - 4ac}}\right)\Bigg|_0^1$$

$$= \frac{1}{k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2}z + z - z^2\right)\right] + O(k^{-4})$$

$$\int_0^1 dx\frac{x}{L} = \frac{1}{2c}\log L\Bigg|_0^1 - \frac{b}{2c}\int_0^1 dx\frac{1}{L}$$

$$= \frac{1}{2k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right] + O(k^{-4})$$

$$\int_0^1 dx \frac{x^2}{L} = \frac{x}{c}\Big|_0^1 - \frac{b}{2c^2}\log L\Big|_0^1 + \frac{b^2 - 2ac}{2c^2}\int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= -\frac{1}{k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right) \\ &\quad + \frac{1}{2k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^3}{L} = \frac{x^2}{2c}\Big|_0^1 - \frac{bx}{c^2}\Big|_0^1 + \frac{b^2 - 2ac}{2c^3}\log L\Big|_0^1 - \frac{b\,(b^2 - 3ac)}{2c^3}\int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= -\frac{3}{2k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right) \\ &\quad + \frac{1}{2k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^4}{L} = \frac{x^3}{3c}\Big|_0^1 - \frac{b}{c}\int_0^1 dx \frac{x^3}{L} - \frac{a}{c}\int_0^1 dx \frac{x^2}{L}$$

$$\begin{aligned} &= -\frac{11}{6k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right) \\ &\quad + \frac{1}{2k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{1}{L^2} = \left.\frac{b + 2cx}{\Delta L}\right|_0^1 + \frac{2c}{\Delta} \int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= \frac{1}{k^2}\left(\frac{1}{m^2} + \frac{1}{a}\right)\left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad + \frac{2}{k^4}\left(1 + \frac{4k \cdot p}{k^2} z\right) \\ &\qquad \times \left[\log \frac{k^2}{m^2} + \log \frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2} z + z - z^2\right)\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x}{L^2} = -\left.\frac{2a + bx}{\Delta L}\right|_0^1 - \frac{b}{\Delta} \int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= \frac{1}{k^2}\frac{1}{m^2}\left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad + \frac{1}{k^4}\left(1 + \frac{4k \cdot p}{k^2} z\right) \\ &\qquad \times \left[\log \frac{k^2}{m^2} + \log \frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2} z + z - z^2\right)\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^2}{L^2} = \left.\frac{ab + (b^2 - 2ac)\, x}{c\Delta L}\right|_0^1 + \frac{2a}{\Delta} \int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= \frac{1}{k^2}\frac{1}{m^2}\left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^3}{L^2} = \frac{1}{2c^2} \log L \Big|_0^1 + \frac{a\,(2ac - b^2) + b\,(3ac - b^2)\,x}{c^2 \Delta L} \Big|_0^1 - \frac{b\,(6ac - b^2)}{2c^2 \Delta} \int_0^1 dx \frac{1}{L}$$

$$\begin{aligned} &= \frac{1}{k^2} \frac{1}{m^2} \left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad - \frac{1}{2k^4} \left(1 + \frac{4k \cdot p}{k^2} z\right) \left[\log \frac{k^2}{m^2} + \log \frac{k^2}{p^2} + \log \frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^4}{L^2} = \frac{x^3}{cL} \Big|_0^1 - \frac{2b}{c} \int_0^1 dx \frac{x^3}{L^2} - \frac{3a}{c} \int_0^1 dx \frac{x^2}{L^2}$$

$$\begin{aligned} &= \frac{1}{k^2} \frac{1}{m^2} \left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad - \frac{1}{k^4} \left(1 + \frac{4k \cdot p}{k^2} z\right) \left[\log \frac{k^2}{m^2} + \log \frac{k^2}{p^2} + \log \frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^5}{L^2} = \frac{x^4}{2cL} \Big|_0^1 - \frac{3b}{2c} \int_0^1 dx \frac{x^4}{L^2} - \frac{2a}{c} \int_0^1 dx \frac{x^3}{L^2}$$

$$\begin{aligned} &= \frac{1}{k^2} \frac{1}{m^2} \left(1 + \frac{2k \cdot p}{k^2} z\right) \\ &\quad - \frac{3}{2k^4} \left(1 + \frac{4k \cdot p}{k^2} z\right) \left[\log \frac{k^2}{m^2} + \log \frac{k^2}{p^2} + \log \frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

$$\int_0^1 dx \frac{x^6}{L^2} = \left.\frac{x^5}{3cL}\right|_0^1 - \frac{4b}{3c}\int_0^1 dx \frac{x^5}{L^2} - \frac{5a}{3c}\int_0^1 dx \frac{x^4}{L^2}$$

$$\begin{aligned} &= \frac{1}{k^2}\frac{1}{m^2}\left(1 + \frac{2k\cdot p}{k^2}z\right) \\ &\quad - \frac{2}{k^4}\left(1 + \frac{4k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right] \\ &\quad + O(k^{-4}) \end{aligned}$$

By inspection of the above results we can easily see that:–

$$\int_0^1 dx \frac{x^n(1-x)}{L} = 0 + O(k^{-2},\ k^{-4}\log k^2) \qquad n = 1,\ 2,\ 3,\ \ldots \tag{C.8}$$

$$\begin{aligned} \int_0^1 dx \frac{(1-x)}{L} &= \frac{1}{k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\Bigg[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2}z + z - z^2\right) \\ &\qquad - \frac{1}{2}\left(\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right)\Bigg] + O(k^{-4}) \\ &= \frac{1}{k^2}\left(1 + \frac{2k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{p^2}\right] + O(k^{-4}) \end{aligned} \tag{C.9}$$

$$\int_0^1 dx \frac{x^n(1-x)^2}{L^2} = 0 + O(k^{-4},\ k^{-6}\log k^2) \qquad n = 2,\ 3,\ 4,\ \ldots \tag{C.10}$$

$$\begin{aligned} \int_0^1 dx \frac{x(1-x)^2}{L^2} &= \frac{1}{k^4}\left(1 + \frac{4k\cdot p}{k^2}z\right)\Bigg[\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} - \log\left(\frac{m^2}{p^2}z + z - z^2\right) \\ &\qquad - \frac{1}{2}\left(\log\frac{k^2}{m^2} + \log\frac{k^2}{p^2} + \log\frac{p^2}{m^2}\right)\Bigg] + O(k^{-4}) \\ &= \frac{1}{k^4}\left(1 + \frac{4k\cdot p}{k^2}z\right)\left[\log\frac{k^2}{p^2}\right] + O(k^{-6}) \end{aligned} \tag{C.11}$$

$$\int_0^1 dx \frac{(1-x)^2}{L^2} = \frac{1}{k^2}\frac{1}{a}\left(1 + \frac{2k\cdot p}{k^2}z\right) + O(k^{-4},\ k^{-6}\log k^2) \tag{C.12}$$

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