GROUP-THEORETIC QUANTISATION AND CENTRAL EXTENSIONS

by

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A thesis presented for the degree
of Doctor of Philosophy at the
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September 1990

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Ph.D. Thesis, 1990

Abstract

This work is concerned with the applications of Isham's group-theoretic quantisation programme to simple systems which involve central extensions of some symmetry group. Of particular interest are those systems with a 'Wess-Zumino'-like term in their actions where other nontrivial modifications are necessary.

In Chapters 1 and 2, a review of the necessary tools used in this work as well as outlines of the group-theoretic quantisation programme are given to facilitate a smooth discussion in the latter chapters. The programme is first exemplified by the normal quantum mechanics on $\mathbb{R}^n$. This example also involves central extensions but of a slightly different nature from those which arise from systems with a 'Wess-Zumino'-like term.

Chapter 3 forms the core of the whole work. The discussions there provide the basis for further examples. It is concerned with the group-theoretic quantisation of the system of a particle moving on the two-torus in a constant magnetic field with quantised flux. The case without the magnetic field is also given for comparison. The canonical group for the case with the magnetic field is required to be the central extension of the universal cover of the canonical group for the case without the magnetic field. These results are then generalised to the corresponding systems on the n-torus.

Chapter 4 is a digression from the main topic of quantisation and central extensions to the discussions of σ-models with Wess-Zumino term. The main purpose of this chapter is to provide a parallel between these σ-models and the systems of a particle moving in a magnetic field (as in Chapter 3). The general construction of a Wess-Zumino term is given along with the discussion of an Abelian gauge symmetry that the term provides for the σ-models. The σ-models can be interpreted as systems of a 'particle' moving on an infinite-dimensional configuration space in a background 'functional magnetic field'. This interpretation is further reinforced by the discussions of Noether's theorem and topological effects.

Finally in Chapter 5, a review of Isham's work on the group-theoretic quantisation of strings on the tori is given. The inclusion of an antisymmetric tensor field into the system arising from the Wess-Zumino term is then considered. This results in a similar effect to the inclusion of the magnetic field considered in Chapter 3 namely the canonical group acquires a central extension. Other effects particular to strings are also discussed.

Preface

This thesis is the result of work carried out in the Department of Mathematical Sciences at the University of Durham, between October 1987 and September 1990, under the supervision of Dr. R. S. Ward. No part of it has been previously submitted for any degree, either in this or any other university.

No originality is claimed for the material reviewed in Chapters 1 and 2. The material in Chapter 3 is an expansion of the author's work published in *Phys. Rev.* **D40** (1989) 636-641. Chapter 4 is based on the author's work in the Durham preprint, *Noether's Theorem in Nonlinear σ-Models with Wess-Zumino Term,* (DTP-89/55) which is to appear in *Journal of Mathematical Physics.* Finally, the material in Chapter 5 is part review and part original work. The discussions concerning central extensions are due to the author.

I am most grateful to Richard Ward for his valuable suggestions and discussions and his supervision and patience. I am also grateful to the Malaysian Public Services Department and the University of Agriculture, Malaysia for their support throughout this work. I would like to thank all my colleagues in particular P. Dorey, P. Fletcher and R. Leese for their help in various ways. Many thanks are also given to Maher Rashid for his keen interest in my work and reading the manuscripts. Finally, my dearest thanks to my parents and my fiancée for their support, care and patience without which this work would never have been finished.

Contents

Chapter 1

Introduction

1.1 General Introduction

Both mathematicians and physicists have kept a continual interest in quantum theory ever since its birth. Much of the interest is probably due to its (initial) imprecise formulation and the significance it carries. A simple example would be the idea of 'quantising' a classical system. Even at this level, one finds intriguing questions and the implications that they can bring may be far reaching and physically relevant in further developments of theoretical physics (*e.g.* see [1]). Various quantisation schemes have been set up to make the procedure of quantising a classical system more 'well-defined' *e.g.* Feynman path integrals,[2] quantisation by $*$-products,[3] geometric quantisation,[4,5] stochastic quantisation,[6] C*-algebra quantisation[7] *etc.* These procedures mostly differ in their starting points. To many physicists, the starting point would seem to be the imposition of the canonical commutation relations (CCR)

$$[q^a, p_b] = i\hbar\delta^a_b \tag{1.1}$$

on the position variables q^a of the configuration space of the system studied and their corresponding conjugate momenta p_a. This is normally known as the canonical quantisation procedure. For the other schemes, they usually have the CCR built in as an outcome at a later stage of the procedure. However, the CCR may be inappropriate as a basis for quantising classical systems on say, nonlinear configuration spaces. Thus one requires some other guiding principles to serve as a basis for quantisation. A natural ingredient would be the consideration of symmetries of the system to be quantised. One particular scheme that uses symmetry is Isham's group-theoretic quantisation programme.[8]

The group-theoretic quantisation programme grew out of Isham's attempt to quantise gravity.[9] It has also recently been applied to strings on tori.[10] The basic idea of the programme is first to identify the symmetry group of the classical phase space of the system studied and to modify it, if necessary. Quantisations of the system are then given by irreducible unitary representations of the group. The main advantage of this programme is that it addresses symmetry at a fundamental level of the quantisation procedure. Such a consideration is useful for understanding the quantum mechanical symmetries of a system in comparison to their classical counterparts and in particular, it might lead to possible insights on the subject of anomalies. The other main advantage of the programme is that it uses geometrical notions, and deals specifically with the question of whether the objects defined on the system studied are globally well-defined. These geometrical aspects in fact allow one to study the well-known topological effects of quantum mechanics (*e.g.* Aharonov-Bohm effect and charge quantisation) in a group-theoretical context. These are but a few of the intriguing aspects of the programme, intertwining global structure of the system studied with the properties of the group describing its symmetries. The geometrical tools are mainly symplectic geometry[11,12] and fibre bundle techniques,[13,14] used in a similar way as in the geometric quantisation scheme; while the group-theoretical aspects rely very heavily on Mackey's techniques of induced representations.[15,16]

In discussions of symmetries in a quantum mechanical system, one often finds the idea of central extensions,[17] in which the group describing the classical symmetries of a system can be centrally extended to describe the corresponding quantum mechanical systems. In some cases, such simple extensions cannot be achieved; nontrivial modifications of the group are necessary.[18] Central extensions find a natural place in quantum theory since it is known that the axioms that build up a quantum mechanical system form a projective geometry.[19] A classical theorem of Wigner[20] states that the symmetry group of a quantum system must be realised by either unitary or anti-unitary projective representations. These projective representations may then be lifted to representations of an extension of the original group. These extensions can also be thought of as a consequence of an (Abelian) gauge symmetry in the system.[21] The main concern of this thesis is to investigate simple examples of central extensions that arise from applying the group-theoretical programme to systems in an external background field. Of particular interest will be those with a Wess-Zumino-like term in the action. These terms will induce a line bundle structure over the (possibly infinite-dimensional) configuration space of the system studied. This line bundle structure is very much connected to the Abelian gauge structure mentioned earlier.

Organisation

The organisation of the thesis will be as follows.

Chapter 1: The rest of this chapter will review topics from symplectic geometry, fibre bundles and induced representations. The reviews will only be brief introductions to the essential tools that are needed for the rest of the chapters. They also serve the purpose of setting up the appropriate notation contained in this thesis. Most proofs of theorems *etc.* will not be given, they can be found elsewhere in the literature.

Chapter 2: The second chapter will introduce the basic outlines of Isham's group-theoretic quantisation programme, with some elaborations. The standard example of quantum mechanics on $\mathbb{R}^n$ will be given according to this scheme. This example will also be the first example that mentions central extension, even though it is of a slightly different nature from the examples of prime concern in this work.

Chapter 3: The third chapter initially discusses the application of the programme to the system of a particle on the two-torus. This example leads naturally to the study of the simplest example of quantisation of a particle moving on a nonlinear configuration space (T^2) in a constant background magnetic field. Here is the first encounter of a 'Wess-Zumino'-like term *i.e.* the field strength of the magnetic field with quantised flux. This term significantly changes the symmetry group of the case without the magnetic field to one which also includes some form of central extension. Finally, the results for the two-torus are then generalised to the case of the n-torus in a latticised form.

Chapter 4: Here, the discussion digresses into a slightly different topic, namely nonlinear σ-models with Wess-Zumino term. The Wess-Zumino term provides an analogue of an external background 'magnetic field' in the configuration space of fields of the model and hence furnishes these theories with an Abelian gauge symmetry. This allows one to treat them as systems of a particle moving on an infinite-dimensional configuration space with a magnetic field which provides a parallel with the example studied in Chapter 3. This analogy is brought closer by looking into Noether's theorem which will be modified in the presence of an external field. Some examples of such theories are given along with a discussion of possible topological effects and global problems.

Chapter 5: The final chapter make use of the results of the previous two chapters to discuss the quantisation of strings on the tori. First, a review of Isham's results of quantisation of a string on circle/torus for the case without the Wess-Zumino term is given. This is followed by the consideration of the case with Wess-Zumino term.

1.2 Symplectic Geometry

Quantisation schemes have always been built around familiar concepts of classical mechanics. This is in particular true for Isham's group-theoretic quantisation programme. It makes use of the modern geometrical approach to classical mechanics based on symplectic manifolds.[11,12] A symplectic manifold is a manifold $\mathcal{S}$ whose space is modelled on $\mathbb{R}^{2n}$. The coordinates of $\mathcal{S}$ corresponds to some state space of a particle moving on an n-dimensional configuration space Q. A structure defined along with $\mathcal{S}$ is a two-form ω on $\mathcal{S}$ such that

(a) ω is closed; $d\omega = 0$, and

(b) for each $x \in \mathcal{S}$, $\omega_x : T_x\mathcal{S} \times T_x\mathcal{S} \rightarrow \mathbb{R}$ is nondegenerate;

ω_x and $T_x\mathcal{S}$ is the two-form ω and the tangent space respectively at x. The two-form ω is called the *symplectic form (structure)* and the pair $(\mathcal{S}, \omega)$ is a *symplectic manifold.* To make the connection with classical mechanics more transparent, we work with a concrete example of $\mathcal{S}$. The state space in classical mechanics is determined by the positions and velocities (or momenta) of the particle. An appropriate model space for the state space would be the tangent bundle TQ of the configuration space of the particle. Alternatively one can use the dual to TQ *i.e.* the cotangent bundle T^*Q. Let the coordinates of Q be denoted by $\{q^i\}$ (i runs from 1 to $n = \dim Q$). The tangent bundle is then coordinatised by $\{q^i, dq^i\} := \{q^i, \dot{q}^i\}$ while the cotangent bundle is coordinatised by $\{q^i, \partial/\partial q^i\} := \{q^i, p_i\}$.[22] Some explanatory notes are necessary for this notation. The symbols dq^i and $\partial/\partial q^i$ are really coordinate functions on TQ and T^*Q respectively such that when they act on $\dot{q}^j\partial/\partial q^j \in T_qQ$ and $p_j dq^j \in T^*_qQ$ ($q \in Q$) (implicitly summed),

$$\begin{aligned} dq^i(\dot{q}^j \frac{\partial}{\partial q^j}) &= \dot{q}^i \\ \frac{\partial}{\partial q^i}(p_j dq^j) &= p_i \quad , \end{aligned} \tag{1.2}$$

they reproduce the corresponding fibre coordinates. The coordinate $\dot{q}^i$ is considered to be the velocity coordinate defining the state space and by abuse of notation, it is identified with the coordinate function dq^i. Similarly for p_i and $\partial/\partial q^i$ where p_i is the momentum coordinate of the phase space.[The notation is useful when discussing systems on a lattice *e.g.* see Section 3.4.] We will now only use (the phase space) T^*Q as the state space since it is more convenient to deal with forms than vector fields, and also T^*Q is endowed with a natural symplectic form which is constructed as follows.

Let $T(T^*Q)$ be the tangent bundle to T^*Q and consider the commutative diagram

$$\begin{array}{ccc} T(T^*Q) & \xrightarrow{\lambda_Q^T} & TQ \\ \pi_{T^*Q}\downarrow & & \downarrow \pi_Q \\ T^*Q & \xrightarrow{\lambda_Q} & Q \end{array} \quad , \tag{1.3}$$

where the maps are defined as

$$\begin{aligned} \pi_Q &: (q^i, \dot{q}^i) \in TQ \mapsto q^i \in Q \\ \lambda_Q &: (q^i, p_i) \in T^*Q \mapsto q^i \in Q \\ \pi_{T^*Q} &: (q^i, p_i; dq^i, dp_i) \in T(T^*Q) \mapsto (q^i, p_i) \in T^*Q \\ \lambda_Q^T &: (q^i, p_i; dq^i, dp_i) \in T(T^*Q) \mapsto (q^i, dq^i) \in TQ \quad . \end{aligned} \tag{1.4}$$

There is a dual to the upper line in (1.3) *i.e.*

$$\begin{array}{ccc} T(T^*Q) & \xrightarrow{\lambda_Q^T} & TQ \\ \text{dual}\downarrow & & \downarrow \text{dual} \\ T^*(T^*Q) & \xleftarrow{(\lambda_Q^T)^*} & T^*Q \end{array} \quad . \tag{1.5}$$

Consider now the following

$$\begin{aligned} \alpha_q &:= \alpha_i \; dq^i \in T_q^*Q \\ \xi_{\alpha_q} &:= \xi^i \frac{\partial}{\partial q^i} + \xi_i^* \frac{\partial}{\partial p_i} \in T_{\alpha_q}(T^*Q) \quad . \end{aligned} \tag{1.6}$$

[The subscript q will be dropped to denote the one-form and the vector field respectively in contrast to the tangent vector and covector above.] One has then the identity

$$\lambda_Q^T(\xi_\alpha) \lrcorner \alpha = \xi_\alpha \lrcorner (\lambda_Q^T)^* \alpha = \xi^i \alpha_i \quad , \tag{1.7}$$

where $\lrcorner$ denotes the contraction between appropriate vectors and one-forms. Using the dual map $(\lambda_Q^T)^*$, one can now construct the following one-form on T^*Q:

$$\theta := (\lambda_Q^T)^*\alpha \in T^*(T^*Q) \quad . \tag{1.8}$$

From (1.7), θ can be written as

$$\theta = p_i \; dq^i \tag{1.9}$$

since $p_i(\alpha) = \alpha_i$. Thus, the symplectic form on T^*Q can now be defined as

$$\omega := -d\theta = dq^i \wedge dp_i \quad . \tag{1.10}$$

For this reason, θ is sometimes called the *symplectic potential* (or *canonical one-form*). Note that ω is obviously closed and nondegenerate. Hence, the pair $(\mathcal{S}, \omega)$, where $\mathcal{S} = T^*Q$, is a symplectic manifold.

One property of the symplectic structure is that it behaves like a skew-symmetric metric. An isomorphism between the vector fields and one-forms on T^*Q can be established using the symplectic form via the contraction operation. For example, if ξ_α is given by (1.6) then the one-form that corresponds to ξ_α is

$$\xi_\alpha \lrcorner\, \omega = \xi^i dp_i - \xi_i^* dq^i \quad . \tag{1.11}$$

In a similar way, if β is a one-form on T^*Q given by

$$\beta := \beta_i dq^i + \beta^{*i} dp_i \quad , \tag{1.12}$$

then a vector field that corresponds to β is

$$\xi_\beta = \beta^{*i} \frac{\partial}{\partial q^i} - \beta_i \frac{\partial}{\partial p_i} \quad . \tag{1.13}$$

A special case is when β is the exact form df where $f \in C^\infty(\mathcal{S}, \mathbb{R})$; its corresponding vector field is

$$\xi_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \quad , \tag{1.14}$$

obtained from the relation

$$\xi_f \lrcorner\, \omega = df \quad . \tag{1.15}$$

This vector field is called *Hamiltonian.* If f is the Hamiltonian function H on $\mathcal{S}$, the

Hamiltonian vector field is then

$$\xi_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \quad . \tag{1.16}$$

The integral curves of ξ_H are given by solutions to the Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \ ; \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad , \tag{1.17}$$

where t is the time evolution parameter parametrising the curves. This gives the usual connection to Hamiltonian mechanics. Another important relation which can be obtained by further contractions with the symplectic form is the following. If ξ_f, ξ_g are two Hamiltonian vector fields corresponding to functions f and g respectively, then

$$\begin{aligned} \xi_g \lrcorner\, \xi_f \lrcorner\, \omega &= \omega(\xi_f, \xi_g) \\ &= \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} = \{f, g\} \quad , \end{aligned} \tag{1.18}$$

which is the well-known Poisson bracket. If $[\cdot,\cdot]$ is the Lie bracket for two vector fields, it is not difficult to check that

$$[\xi_f, \xi_g] = -\xi_{\{f,g\}} \quad . \tag{1.19}$$

This establishes an antihomomorphism of the commutator algebra of vector fields with the Poisson bracket algebra of functions, both on $\mathcal{S}$. It is one of the key relations in the quantisation procedure. [Note that the observables that are to be quantised are given by functions on $\mathcal{S}$.] It provides the first step of representing the Poisson bracket algebra as an operator algebra.

As mentioned in the introduction, one of the first major steps in the quantisation programme is to identify the symmetry group $\mathcal{G}$ of the classical phase space. These group transformations should preserve the equations of motion and the Poisson bracket algebra. This amounts to finding transformations that preserve the symplectic structure. Consider then a local flow $\phi_t^\xi : I \times Q \longrightarrow Q$ to a vector field ξ where I is some interval in $\mathbb{R}$. This satisfies

$$\phi_0^\xi(q) = q \qquad \text{for all } q \in Q \tag{1.20}$$

$$\phi_t^\xi(\phi_s^\xi(q)) = \phi_{t+s}^\xi \qquad \text{for all } t, s, (t+s) \in I \quad . \tag{1.21}$$

This generates a (local) one-parameter group of diffeomorphisms of Q. For a symmetry transformation, the local flow is given by

$$\phi_t^{\xi}(q) := q\exp(-tA) \tag{1.22}$$

where A is a generator of the Lie algebra $\mathcal{L}(\mathcal{G})$ of some symmetry group $\mathcal{G}$ and the vector field ξ^A is given by

$$\xi_q^A(f) = \frac{df}{dt}(q\exp(-tA)\ |_{t=0} \quad . \tag{1.23}$$

Note that the right hand side of (1.22) is a right translation of the point $q \in Q$ by an element of $\mathcal{G}$. This group of diffeomorphisms is said to be generating *symplectomorphisms* or *canonical transformations* if

$$\phi_t^{\xi^A *}\omega = \omega \tag{1.24}$$

which is required of the symmetry group $\mathcal{G}$. The condition (1.24) is in fact equivalent to the condition

$$\mathcal{L}_{\xi^A}\omega = 0 \tag{1.25}$$

on ξ^A where $\mathcal{L}_{\xi^A}$ denotes the Lie derivative in the direction of ξ^A. For a Hamiltonian vector field ξ_f, this is always true since

$$\begin{aligned} \mathcal{L}_{\xi_f}\omega &= d(\xi_x \lrcorner\, \omega) + \xi_f \lrcorner\, d\omega \\ &= d(df) = 0 \quad , \end{aligned} \tag{1.26}$$

where we have used the homotopy formula $\mathcal{L}_{\xi}(\cdot) = d(\xi \lrcorner \cdot) + \xi \lrcorner\, d(\cdot)$ and (1.15). It is also possible to have a vector field ξ^A such that (1.25) holds without the property of (1.15). Such vector fields are said to be only *locally Hamiltonian.* However, they will not be of much use in the quantisation programme for reasons to be discussed later.

Finally, it is important to note that the discussions above have their generalisations or modifications to the case of infinite-dimensional symplectic manifolds.[23] We shall not discuss them here, but a mention will be made when any such necessary modifications or related technical points arise.

1.3 Fibre Bundles

Before going into ideas of group theory and induced representations which form the core of Isham's quantisation programme, it is helpful to delve into some general notions of fibre bundles.[13,14] In particular, fibre bundles have been very useful in providing the framework of induced representations.[24,25] They have also been used in the formulation of gauge theories and σ-models in physics.[26,27] A fibre bundle, intuitively, looks like a product $\mathbb{R}^n \times \mathbb{R}^m$ but maybe glued together in a nontrivial way globally. Tangent (cotangent) bundles encountered in the last section are familiar examples of fibre bundles, given by the product of the given configuration space with its tangent (cotangent) spaces. Formal definitions of fibre bundles and related concepts will now be given below.

A *fibre bundle* $F \longrightarrow E \longrightarrow X$ is a triple of manifolds E, X and F with a smooth *projection* map π from E onto X. At each point $x \in X$, the set $\pi^{-1}(x) =: F_x$ is diffeomorphic to F and is called the *fibre* at x. The space E is called the *total space* and for simplicity, it will be interchangeably referred to as the bundle itself. A *section* of the bundle E is a map $s : X \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X$, the identity map on X. A *trivial bundle* is a bundle whose total space is the product $X \times F$ (globally) with a natural projection map onto X. A bundle is therefore always trivialisable locally. Consider the cover $\{U_a\}$ of X. On each U_a, there is the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U_a) & \xleftarrow{s_a} & U_a \times F \\ \downarrow & & \downarrow \mathrm{pr}_{U_a} \\ U_a & \xleftarrow{\mathrm{id}_X} & U_a \end{array} \qquad (1.27)$$

where pr_{U_a} is the natural projection onto U_a and s_a is the local trivialisation of E given by the diffeomorphism $U_a \times F \longrightarrow \pi^{-1}(U_a)$. This trivialisation serves as a local chart for the bundle. Thus, a section $s(x)$ of E may now be given a pair of coordinates (namely that of the base space X and that of the fibre F) using the local trivialisation map s_a :

$$s(x) = s_a(x, f) =: (x, f) \quad , \qquad (1.28)$$

where f is the fibre coordinate and $x \in U_a$. Note that such a coordinatisation depends on the local chart U_a. In order to see the coordinatisation in other (overlapping) charts and how the bundle is glued together from such local charts, consider two neighbourhoods U_a

and U_b of x. On the overlap $U_a \cap U_b$, one can define the transition function

$$\begin{aligned} \Omega_{ab} &: U_a \cap U_b \longrightarrow \text{Diff } F \quad , \\ \Omega_{ab}(x) &:= s_a^{-1} \circ s_b(x) \quad , \end{aligned} \tag{1.29}$$

where the parenthesised x on the right hand side of (1.29) denotes the evaluation of $s_a^{-1} \circ s_b$ at x. In most physical examples, Diff F is realised by a much smaller symmetry group G through the monomorphism $\rho : G \longrightarrow \text{Diff } F$,

$$\Omega_{ab} = \rho \circ g_{ab} \quad , \tag{1.30}$$

where $g_{ab} : U_a \cap U_b \longrightarrow G$ obeying relations

$$\begin{aligned} g_{aa}(x) &= e \qquad \forall x \in U_a \\ g_{ab}(x) g_{ba}(x) &= e \qquad \forall x \in U_a \cap U_b \\ g_{ab}(x) g_{bc}(x) g_{ca}(x) &= e \qquad \forall x \in U_a \cap U_b \cap U_c \end{aligned} \tag{1.31}$$

where e is the identity in G. The bundle E is then called a *G-bundle* and the group G is said to be the *structure group* of E. Note that under change of local charts from that determined by U_a to that of U_b, the coordinatisation of the section changes as

$$\begin{aligned} s_b(x, f) &= s_a(x, f) \Omega_{ab}(x) \\ &= (x, g_{ab}^{-1}(x) f) \end{aligned} \tag{1.32}$$

[the symbol ρ has been dropped for simplicity; F now has an effective G-action.]

Equipped with the above definitions, one has the following theorem.

Theorem 1.1: Given the spaces X and F, a covering $\{U_a\}$, transition functions $\{g_{ab}\}$ satisfying (1.31), there exists a G-bundle $F \longrightarrow E \longrightarrow X$ over X determined up to an isomorphism. □

Isomorphic G-bundles over X form equivalence classes. The following proposition gives one condition when an isomorphism between G-bundles can be established.

Proposition 1.2: Let $\{g_{ab}\}$ and $\{g'_{ab}\}$ be two sets of transition functions defined on the covering $\{U_a\}$ on X. They define isomorphic G-bundles over X if and only if there exist functions $\lambda_a : U_a \longrightarrow G$ such that

$$g'_{ab} = \lambda_a g_{ab} \lambda_b \quad . \qquad \square \tag{1.33}$$

Note that from Theorem 1.1 and Proposition 1.2, the isomorphic G-bundles over X fall into the same set of equivalence classes irrespective of the fibre F. It will be enough to consider one representative of the fibre spaces F to demonstrate the necessary properties of the G-bundles. One fibre space that has a natural G-action on it, both from the left and the right, is the group G itself. The G-bundle that has G as its fibre is called a *principal G-bundle* $(G \longrightarrow P \longrightarrow X)$ where its total space is now denoted by P. The group G is often called the *gauge group* (as in the context of gauge theories). The (local) section σ of P is trivialised by

$$\sigma(x) := (x, g_a) \qquad x \in U_a \quad . \tag{1.34}$$

There is a natural right G-action on the bundle P defined by

$$r_g \sigma(x) := (x, g_a g) \qquad x \in U_a, \ g \in G \quad . \tag{1.35}$$

This gives an automorphism of P which maps each fibre to itself :

$$\begin{tikzcd} G \arrow[r] & P \arrow[rr, "r_g"] \arrow[dr] & & P \arrow[dl] \\ & & X & \end{tikzcd} \quad . \tag{1.36}$$

Such automorphisms are called *gauge transformations* of P.

Associated to the bundle P are various other G-bundles built out of different fibre spaces, depending on the structure that is required of F. The construction of such bundle is given by first defining a right G-action on $P \times F$ by

$$\tau_g(p, \psi) := (r_g p, \mathcal{U}(g^{-1})\psi(x)) \qquad g \in G, \ p \in P \quad , \tag{1.37}$$

where ψ is a function of $x = \pi(p)$ taking values in F and $\mathcal{U}(g^{-1})$ is the representation of g^{-1} on F. The *associated bundle* E to P with fibre F then has the total space taken to be the quotient of $P \times F$ with respect to the G-action (1.37) *i.e.*

$$E = (P \times F)/G =: P \times_G F \quad . \tag{1.38}$$

The projection map π_E from E to X is obtained by the following commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{\mathrm{pr}_P} & P \\ \chi \downarrow & & \downarrow \pi \\ E & \xrightarrow{\pi_E} & X \end{array} \tag{1.39}$$

where χ is the projection map of $P \times F$ to the set of equivalence classes under (1.37) and thus

$$\pi_E(\chi(p, \psi)) = \pi(p) \quad . \tag{1.40}$$

A section Ψ of E is given by

$$\Psi(x) := [\sigma(x), \psi(x)] \tag{1.41}$$

where $[\cdot, \cdot]$ denotes the equivalence class of $P \times F$ under the equivalence relation

$$[\sigma(x), \psi(x)] \equiv [r_g \sigma(x), \mathcal{U}(g^{-1})\psi(x)] \quad . \tag{1.42}$$

Note under the change of (local) sections $\sigma_b(x) = \sigma_a(x)\Omega_{ab}(x)$ for $x \in U_a \cap U_b$, one finds

$$\begin{aligned} \Psi(x) &= [\sigma_a(x), \psi_a(x)] = [\sigma_b(x), \psi_b(x)] \\ &= [\sigma_a(x)\Omega_{ab}(x), \psi_b(x)] \\ &\equiv [\sigma_a(x), \Omega_{ab}(x)\psi_b(x)] \end{aligned} \tag{1.43}$$

where $\psi_a(x)$ and $\psi_b(x)$ are the functions ψ on U_a and U_b respectively, and ρ in (1.30) is now given by $\mathcal{U}$. Hence the function ψ obeys the relation

$$\psi_a(x) = \Omega_{ab}(x)\psi_b(x) \quad . \tag{1.44}$$

To close this section, we shall briefly look into the notion of liftings associated to the structure of fibre bundles. A lifting is said to be considered if the objects or specific properties defined on the base space are extended or generalised to corresponding objects or properties defined on the whole bundle. One important consequence of liftings considered in this work is central extensions of a symmetry group. In view of this main aim of studying central extensions, the relevant topic of lifting to be discussed is that of a group action.[28-31] Consider the bundle $F \longrightarrow E \longrightarrow X$ where X has a G-action τ_G defined on it. The action τ_G is said to be lifted to E if there exists a G-action $\tau_G^\dagger$ on E such that the following diagram commutes

$$\begin{array}{ccc} G \times E & \xrightarrow{\tau_G^\dagger} & E \\ (\mathrm{id}_G, \pi) \downarrow & & \downarrow \pi_E \\ G \times X & \xrightarrow{\tau_G} & X \end{array} \tag{1.45}$$

The G-action $\tau_G^\dagger$ satisfies the right action identity

$$\tau_{g_2}^\dagger \circ \tau_{g_1}^\dagger = \tau_{g_1 g_2}^\dagger \qquad g_1,\ g_2 \in G \quad . \tag{1.46}$$

Below are some useful results on lifting group actions.

Proposition 1.3: Let P be a principal fibre bundle with base space G, a topological group. Let $\tau : G \times G \longrightarrow G$ be the group action on G *i.e.* $\tau_{g_2} g_1 = g_1 g_2$, then the action τ can be lifted if and only if P is a trivial bundle. $\square$

An advantage of constructing associated bundles in the way described above is that one can immediately see that the following proposition is true.

Proposition 1.4: If a G-action τ_G on X has a lifting in a principal bundle P over X, then τ_G has also a lift in every bundle associated to P. $\square$

Proposition 1.5: If the G-action τ_G on X has a lifting in a principal K-bundle P over X, then there is a naturally defined action of $G \times K$ on P [This action could be that of a semidirect product between G and K]. If τ_G is transitive (free) then the newly defined action of $G \times K$ on P will also be transitive (free). $\square$

This proposition hints towards the result that central extensions of a symmetry group can be established in the presence of a line bundle structure as indicated in the general introduction. Finally, one can prove via isomorphism of fibre bundles over a given space X the following theorem.

Theorem 1.6: A lifted G-action on a bundle E over X is only unique up to a bundle equivalence. $\square$

1.4 Group Representations

The problem of quantisation to some in the end boils down to finding unitary representations of some symmetry group. Group representation theory[32-34] has always occupy an important position in mathematics with its wide ranging connections with other areas, and it has been developed at a very technical level. It will be very useful to give a brief exposition of the relevant tools needed in the forecoming chapters. We begin this section by introducing the basic ideas forming representation theory and then continue with Mackey's induced representation techniques and other topics involving semidirect products and group extensions.

Basic Ideas

A *representation* ρ of a group G on a *representation space* V is a homomorphism of G into invertible (linear) maps of V into itself such that the resulting map (action)

$$\rho : G \times V \longrightarrow V \tag{1.47}$$

is continuous. An *invariant subspace* for ρ is a vector subspace U of V such that $\rho(g)(U) \subseteq U \; \forall g \in G$. The representation ρ is then said to be *irreducible* if it has no invariant subspace other than 0 and V. Otherwise ρ is *reducible*. The representation ρ is *unitary* if $\rho(g)$ is unitary *i.e.*

$$\rho^{\dagger}(g)\rho(g) = \rho(g)\rho^{\dagger}(g) = \mathbb{1} \qquad \forall g \in G \quad , \tag{1.48}$$

where $\dagger$ denotes the hermitian conjugation and $\mathbb{1}$ is the identity homomorphism. Two representations of G, ρ on V and ρ' on V' are (unitarily) *equivalent* if there is a unitary operator $\Omega : V \longrightarrow V'$ such that

$$\Omega\rho(g) = \rho'(g)\Omega \qquad \forall g \in G \quad . \tag{1.49}$$

The operator Ω is called an *intertwining operator.* Thus, among the representations of G, the main objects of interest are the equivalence classes of (unitary) irreducible representations. Later, these objects would correspond to different quantisations of a system with the symmetry group G. One important problem of representation theory is to classify representations of G according to these equivalence classes. To do so, the function that labels each equivalence class should be independent of the representations that constitute the equivalence class *i.e.* it should be an invariant of the equivalence class. For a finite-dimensional representation ρ, it is easily defined as

$$\chi_\rho(g) := \mathrm{Tr}\ \rho(g) \qquad \forall g \in G \quad . \tag{1.50}$$

These functions are called *characters*. For an infinite-dimensional representation, there are technical difficulties in defining (1.50) though they can be circumvented (*e.g.* see Atiyah in [33] and [32]). For our purposes, it is sufficient to consider characters as 'generalised functions' from G to $\mathbb{C}$. These characters are said to form a *dual space* to G, Char(G). One now has the following theorem.

Theorem 1.7: An irreducible representation ρ is defined by its character up to an equivalence. □

One distinguishing feature for the Abelian groups is that the dual space forms a topological (dual) group. First, note that by Schur's Lemma the irreducible representations ρ of an Abelian group G are always one-dimensional. The character of G is a continuous function $\chi :\ G \longrightarrow \mathbb{C}$ satisfying

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2) \qquad g_1,\ g_2 \in G \tag{1.51}$$

$$|\chi(g)| \equiv 1 \qquad \forall g \in G \quad . \tag{1.52}$$

Hence, from (1.51) and (1.52) we have

$$\chi(e) = 1 \quad , \tag{1.53}$$

$$\chi(g^{-1}) = \overline{\chi(g)} = \chi^{-1}(g) \quad . \tag{1.54}$$

Therefore the character is a one-dimensional unitary representation of G, and hence Char(G) is an Abelian group itself. Char(G) sometimes denoted $\hat{G}$ is called the *Pontryagin dual* of G. An important theorem due to Pontryagin follows.

Theorem 1.8: The dual space $\hat{\hat{G}}$ of $\hat{G}$ is topologically isomorphic to G *i.e.*

$$G \cong \hat{\hat{G}} \quad . \quad \square \tag{1.55}$$

Below are some useful results concerning dual groups:

$$\hat{\mathbb{R}} = \mathbb{R}\ , \quad \widehat{U(1)} = \mathbb{Z}\ , \quad \hat{\mathbb{Z}} = U(1)\ , \quad \hat{\mathbb{Z}}_n = \mathbb{Z}_n \quad . \tag{1.56}$$

Induced Representations

A standard technique of obtaining irreducible representations has always been the induction method. This is to to say that given a known irreducible representation of a subgroup H of G, an irreducible representation of G may be constructed out of this known representation of H. The machinery that underlies the technique involves a deep and technical theory of measures. This is ignored in this review for the sake of simplicity; their reviews may be found instead in Mackey's work in [15, 16, 33] and also of others in [19, 32, 34]. Here, we focus more on the surface of the useful techniques and related results involved.

Let H be a closed subgroup of G and $h \longrightarrow L(h)$ be a unitary representation of $h \in H$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^L$ be the set of functions $\psi : \ G \longrightarrow \mathcal{H}$ such that

(*i*) $\psi(gh) = L(h^{-1})\psi(g) \qquad \forall h \in H, \ g \in G;$

(*ii*) the inner product $(\psi(g), \psi(g))_L$ on $\mathcal{H}^L$ is 'measurable' for which

$$\int_{G/H} (\psi(g), \psi(g))_L \, d\mu < \infty$$

where $d\mu$ is a measure on the quotient space G/H (invariant under G-action),

then

$$(U^L(g_1)\psi)(g_2) := \psi(g_2 g_1) \qquad g_1, g_2 \in G \tag{1.57}$$

defines a unitary representation of G in $\mathcal{H}^L$ which is called the *induced representation* of G by L.

Lemma 1.9: The space $\mathcal{H}^L$ is isomorphic to the Hilbert space $L^2(G/H, \mu, \mathcal{H})$ of square integrable functions with domain in G/H and values in $\mathcal{H}$ via relation

$$\psi(g) = L(h_g)\psi'(gH) \qquad g \in G \tag{1.58}$$

where h_g is the factor of subgroup H in g and $\psi' \in L^2(G/H, \mu, \mathcal{H})$. $\square$

This lemma allows one to construct the above induced representation naturally on a bundle associated to the bundle $H \longrightarrow G \longrightarrow G/H$. The construction is as follows. Associated to the 'master' bundle G over G/H, is a principal $U(n)$-bundle

$$U(n) \longrightarrow G \times_L U(n) \longrightarrow G/H \tag{1.59}$$

The equivalence relation defining this bundle is given by

$$[g, U] \equiv [gh, L(h^{-1})U] \qquad h \in H,\ U \in U(n) \quad , \tag{1.60}$$

where $L(h^{-1})$ above is an irreducible unitary representation of H. The appropriate bundle to build the induced representations from is the associated vector bundle to the above principal $U(n)$-bundle,

$$\mathbb{C}^n \longrightarrow G \times_L \mathbb{C}^n \longrightarrow G/H \quad . \tag{1.61}$$

The cross section Ψ of this bundle is given by

$$\Psi(g) := [g, \psi(g)] \qquad g \in G \quad , \tag{1.62}$$

where ψ is a square-integrable $\mathbb{C}^n$-valued function of G which obeys

$$\psi(gh) = L(h^{-1})\psi(g) \tag{1.63}$$

from (1.60). The function ψ does in fact form the induced representation of G by L as in (1.57).

It is obvious that the equivalence class of irreducible representations of H will somehow determine that of the induced representations of G. To make the statement more precise, it is necessary to look into the G-actions on characters of H *i.e.* the G-*orbits* of $\hat{H}$. Let H_χ denote the stabiliser group of $\chi \in \hat{H}$ which includes H itself. [A *stabiliser group* of χ is a subgroup of G whose elements g obey $g\chi g^{-1} = \chi$.] For every $\chi \in \hat{H}$, there is a continuous one-to-one map

$$gH_\chi \mapsto g\chi \tag{1.64}$$

of G/H_χ onto the G-orbit $\mathcal{O}$ of χ.

Theorem 1.10: Let H be a closed normal Abelian subgroup of G and $\mathcal{O}$, the G-orbit of $\chi \in \hat{H}$. Let $\mathcal{V}(\chi)$ be the set of elements V in $\hat{H}_\chi$ such that $V(h) :=< \chi, h > \mathbb{1}$ where $h \in H$ and $< \cdot, \cdot >$ denotes the inner product between elements of H with their duals. Then the map

$$V \longrightarrow U^V \qquad (V \in \mathcal{V}(\chi)) \tag{1.65}$$

is a one-to-one map of $\mathcal{V}(\chi)$ onto $\hat{G}|_{\mathcal{O}}$ (the elements in $\hat{G}$ associated to $\mathcal{O}$). □

This theorem effectively gives a specific correspondence between characters of H with those of G and hence a correspondence between their equivalence classes of representations. A special case of the above theorem is when $\mathcal{O}$ itself is a one-element orbit $\{\mathbb{1}\}$. Then the elements of $\hat{G}$ correspond to the equivalence classes of irreducible unitary representations of G lifted from those of G/H *i.e.*those of the form

$$g \longrightarrow U(gH) \tag{1.66}$$

where U is an irreducible unitary representation of G/H.

Semi-direct Products

The theory of induced representations and their orbital analysis takes a more definite form when the group G is that of a semi-direct product. First, we define what a semi-direct product is. Consider two (Lie) groups K and H. K is said to act on H by automorphism if a smooth map $\tau : K \times H \longrightarrow H$ is specified such that $\tau(k, \cdot)$ $(k \in K)$ is a homomorphism of K into the automorphism group of H. The *semi-direct product* group of K and H denoted $G := K \underset{\tau}{\ltimes} H$ is constructed by the pair (K, H) whose group multiplication is

$$(k_1, h_1)(k_2, h_2) := (k_1 k_2, \tau(k_2^{-1}, h_1) h_2) \qquad (k_i \in K,\ h_i \in H) \tag{1.67}$$

and the inverse elements are given by

$$(k, h)^{-1} = (k^{-1}, \tau(k, h^{-1})) \qquad (k \in K,\ h \in H) \quad . \tag{1.68}$$

Note that one sometimes write G as $K \ltimes H$ where a specific action τ is already assumed. The notation $H \rtimes K$ may also be used for G. It is easy to check from (1.67) and (1.68) that H is a normal subgroup of G.

To study the irreducible unitary representations $\mathcal{U}$ of G, it is convenient to introduce the notation

$$L(h) := \mathcal{U}(e,h)\ ,\quad T(k) := \mathcal{U}(k,e) \qquad (k \in K,\ h \in H)\quad , \tag{1.69}$$

where e is the identity in the respective subgroups. L and T are the irreducible unitary representations of subgroups H and K respectively such that

$$\mathcal{U}(k,h) = T(k)L(h)\quad . \tag{1.70}$$

They satisfy the relation

$$T(k_1)L(h_1)T(k_2)L(h_2) = T(k_1)T(k_2)L(\tau(k_2^{-1},h_1))L(h_2) \qquad (k_i \in K,\ h_i \in H)\quad , \tag{1.71}$$

which reduces to

$$T^{-1}(k_2)L(h_1)T(k_2) = L(\tau(k_2^{-1},h_1))\quad . \tag{1.72}$$

The relation (1.72) induces a K-action on $\chi \in \hat{H}$ via

$$\chi \mapsto k.\chi\ ;\quad < h, k.\chi > = < k^{-1}hk, \chi > \qquad (k \in K,\ h \in H)\quad . \tag{1.73}$$

Let $\mathcal{O}$ be a K-orbit on $\hat{H}$ and $\chi \in \mathcal{O}$. Given an irreducible representation V of the stability subgroup K_χ of K, one can define the irreducible representation $V\chi$ of $K_\chi \ltimes H$ by

$$V\chi(k,h) := \chi(h)V(k)\quad . \tag{1.74}$$

Theorem 1.11: Given the irreducible representation $V\chi$ of $K_\chi \ltimes H$, one can construct the induced representation $U^{V\chi}$ of G which is irreducible. If $\mathcal{O}'$ is a K-orbit on $\hat{H}$ with $\chi' \in \mathcal{O}'$ and V' is an irreducible representation of $K_{\chi'}$, then $U^{V\chi}$ is equivalent to $U^{V'\chi'}$ if the orbits $\mathcal{O}$ and $\mathcal{O}'$ coincide. □

The corollary to Theorem 1.10 when G is the semi-direct product $K \ltimes H$ is given by the following theorem.

Theorem 1.12: Let H be a closed normal Abelian subgroup of $G = K \rtimes H$. Let $\mathcal{O}$ be a G-orbit in $\hat{H}$ with $\chi \in \mathcal{O}$. For each V in $(\widehat{K \cap H_\chi})$, let $V\chi \in \hat{H}_\chi$ be given by $(V\chi)(k,h) := < h, \chi > V(k) \quad (k \in K \cap H_\chi,\ h \in H)$ then the map

$$V \longrightarrow U^{V\chi} \tag{1.75}$$

is a one-to-one map of $(\widehat{K \cap H_\chi})$ onto $\hat{G}|_{\mathcal{O}}$. $\square$

Equipped with these theorems, one can now simply construct all the irreducible representations of G by first determining the characters of H and classifying their orbits under G (or K).

Group Extensions

To end, we direct our discussions to the 'central' theme of the thesis *i.e.* central extensions and how it connects with everything we have just mentioned. The connection with projective representations (and hence projective geometry) will also be briefly illustrated.

An *extension* of the group G by A is the short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \bar{G} \xrightarrow{\pi} G \longrightarrow 0 \tag{1.76}$$

of groups where ι is the inclusion map and π is the projection map. [An *exact sequence* is a sequence of maps between objects of which the kernel of one map is equal to the image of the preceding map.] The sequence (1.76) is said to *split* if there is a (local) section σ such that $\sigma \circ \pi = \mathrm{id}_G$. The sequence splits if it is isomorphic to the sequence

$$0 \longrightarrow A \longrightarrow A \rtimes G \longrightarrow G \longrightarrow 0 \tag{1.77}$$

i.e. the following diagram commutes

$$\begin{array}{ccccc} A & \longrightarrow & \bar{G} & \longrightarrow & G \\ \mathrm{id}_A \downarrow & & \downarrow \gamma & & \downarrow \mathrm{id}_G \\ A & \longrightarrow & A \rtimes G & \longrightarrow & G \end{array} \tag{1.78}$$

where γ is an isomorphism of $\bar{G}$ into $A \rtimes G$. The extension (1.76) is *central* if $\iota(A)$ lies in the centre of $\bar{G}$ (*i.e.* $\bar{g}a\bar{g}^{-1} = a\ \forall \bar{g} \in \bar{G},\ a \in A$).

Given the machinery of induced representations of semi-direct product groups, one can construct irreducible unitary representations of $\bar{G}$ induced by an irreducible unitary representation of A. Suppose that A is a central Abelian subgroup and $\chi \in \hat{A}$. Let $\bar{g} \in \bar{G}$ and $a_{\bar{g}}$ be the factor of A in $\bar{g}$, then an irreducible unitary representation of $\bar{G}$ on $L^2(\bar{G}/A, \mu, \mathcal{H})$ may be given by

$$(U^{\chi}(\bar{g}_2)\psi)(\bar{g}_1 A) := \chi(a_{\bar{g}_2})\psi(\bar{g}_1\bar{g}_2 A) \qquad \bar{g}_i \in \bar{G},\ \psi \in L^2(\bar{G}/A, \mu, \mathcal{H}) \quad . \tag{1.79}$$

Let g_i be the factor of G in $\bar{g}_i$ and writing the above equation (1.79) as

$$(U^{\chi}(g_2)\psi)(g_1) := \chi(g_1, g_2)\psi(g_1 g_2) \tag{1.80}$$

defines a *projective (multiplier) representation* where $\chi(g_1, g_2)$ is called a *multiplier*. The multiplier obeys the cocycle identity

$$\chi(g_1, g_2 g_3) = \chi(g_1, g_2)\chi(g_1 g_2, g_3) \tag{1.81}$$

and

$$\chi(g_1, e) = 1 \quad . \tag{1.82}$$

[This implies extensions may be classified by cohomology of groups (see *e.g.* [19,35]).] Thus this establishes the correspondence between central extensions and projective representations.

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Chapter 2

Group-Theoretic Quantisation

2.1 Quantum Theory

Quantum theory took shape in the early 1900's and went through several phases of development (see [1] for a pedagogical introduction to the subject developed along historical lines). Today, its conventional formulation rests more or less on the following postulates.

Postulate 1: The (pure) *states* of a physical system are represented by vectors in a complex (projective) Hilbert space $\mathcal{H}$ in which probabilistic information of the system are encoded.

Postulate 2: Physical quantities known as *observables* O are represented by self-adjoint operators $\hat{O}$ defined on $\mathcal{H}$.

Postulate 3: Result of a measurement of an observable O in a state represented by the vector ψ is given by the expectation value

$$< O >_{\psi} := \frac{< \psi, \hat{O}\psi >}{< \psi, \psi >} \tag{2.1}$$

where $< \cdot, \cdot >$ denotes the inner product on $\mathcal{H}$.

Postulate 4: In a physical system with no external influence, states ψ_t, $\psi_{t'}$ at corresponding different times t and t' are unitarily related by

$$\psi_{t'} = U(t' - t)\psi_t \tag{2.2}$$

where U is a unitary operator given by

$$U(t) := \exp(-it\hat{H}/\hbar) \quad . \tag{2.3}$$

The operator $\hat{H}$ is said to be the generator of time translations and is called the *Hamiltonian* of the system. It may be added here that the operator $\hat{p}$ corresponding to the momentum observable, generates space translations.

The future discussions of Isham's group-theoretic quantisation programme[2] will assume these four postulates to hold. Other schemes may want to add these further two postulates.

Postulate 5: Self-adjoint operators $\hat{q}$ and $\hat{p}$ of position and momentum observables respectively obey the canonical commutation relation (CCR)

$$[\hat{q},\hat{p}] = i\hbar\mathbb{1} \quad . \tag{2.4}$$

Postulate 6: A quantum mechanical state vector $\psi \in \mathcal{H}$ is symmetric under the permutation of identical bosons and antisymmetric under the permutation of identical fermions.

Earlier in Chapter 1, it has been mentioned that Postulate 5 might be undesirable for quantisation on nonlinear configuration spaces. In fact, the CCR only holds for systems with underlying linear configuration spaces in the group-theoretic quantisation scheme. The scheme being based on the correct symmetries of the phase space, disallows the CCR for configuration spaces other than the linear ones. Later in the chapter, the example of configuration space $Q = \mathbb{R}^n$ will be given showing how the CCR arises from the symmetry group of the system's corresponding phase space. Postulate 6, though concerns with a kind of symmetry related to spin and statistics, it is of a different level and will not be discussed here at all.

It is Postulate 2 that poses the main problem of quantisation. It was once hoped that the quantisation map

$$f \xrightarrow{\wedge} \tfrac{1}{i}\hat{f} \tag{2.5}$$

mapping the observable f (a function of the state space) to a self-adjoint operator $\frac{1}{i}\hat{f}$ on a Hilbert space $\mathcal{H}$ obeys

(*i*) $(f+g) \xrightarrow{\wedge} \frac{1}{i}(\hat{f}+\hat{g})$;

(*ii*) $\lambda f \xrightarrow{\wedge} \frac{1}{i}\lambda\hat{f}, \quad \lambda \in \mathbb{R}$;

(*iii*) $\{f,g\} \xrightarrow{\wedge} -[\hat{f},\hat{g}]$;

(*iv*) $1 \xrightarrow{\wedge} \frac{1}{i}\mathbb{1}$;

(*v*) $\hat{q}^i$ and $\hat{p}_j$ act irreducibly on $\mathcal{H}$.

This is known as the *Dirac problem*. It is now known that this is not possible by Groenwald-Van Hove Theorem (see [3] p. 434 and [4] for a more modern discussion).

It turned out that there is a quantisation which respects (i) - (iv) only when the 'multiplicity' of the representation of the p's and q's is infinite. Alternatively, one can select only a subclass of the observables for the quantisation process. A key observation to be made in the above problem is that not all observables generate flows globally on the state space studied.[3] Those which generate only local flows are not expected to have good quantum counterparts and should then be discounted. Such a consideration is one of the global aims of the group-theoretic quantisation programme.[2] The observables considered in the programme are those whose vector fields are strictly Hamiltonian, generating a global group of symplectomorphisms on the phase space. The outlines of the programme will now be given below in Section 2.2.

2.2 Group-Theoretic Quantisation Programme

The programme, as the name suggests, rests heavily on a group which describes the symmetry of the phase space $\mathcal{S} = T^*Q$; Q being the configuration space of the particle/string considered. This symmetry group is called the *canonical group.* Once it is identified, the quantisation of the system will then be given by finding its irreducible unitary representations. The programme may be summarised (following Isham[2,5]) in the four main steps below.

Step 1: Identify a Lie group $\mathcal{G}$ of symplectomorphisms of $\mathcal{S}$. Each element A of the Lie algebra $\mathcal{L}(\mathcal{G})$ of $\mathcal{G}$ will generate a one-parameter family of symplectomorphisms $s \mapsto s\exp(tA)$, $s \in \mathcal{S}$. The induced vector field (1.23)

$$\xi^A(f) := \frac{df}{dt}(s\exp(-tA)|_{t=0} \qquad f \in C^\infty(\mathcal{S}, \mathbb{R}),\ s \in \mathcal{S}$$

is Hamiltonian whose flow is globally defined on $\mathcal{S}$. Such vector fields are the homomorphic image of $\mathcal{L}(\mathcal{G})$. It is an isomorphic image if the $\mathcal{G}$-action on $\mathcal{S}$ is almost effective (*i.e.* if only a discrete subgroup of $\mathcal{G}$ acts trivially).

Step 2: To each ξ^A there will be an observable $f^A \in C^\infty(\mathcal{S}, \mathbb{R})$ such that $\xi^A \lrcorner\, \omega = df^A$ where ω is a given symplectic structure on $\mathcal{S}$. Thus, on forming the Poisson bracket algebra of these observables, it is hoped that this algebra will be isomorphic to $\mathcal{L}(\mathcal{G})$. Otherwise an extension of $\mathcal{G}$ is required to achieve an isomorphism of its Lie algebra with the Poisson bracket algebra of the observables.

Step 3: Once Step 2 is achieved, a canonical group $\mathcal{G}$ has then been established in which its Lie algebra is represented faithfully by the Poisson bracket algebra of a preferred set of observables $\{f^A \mid A \in \mathcal{L}(\mathcal{G})\} =: O$. The set O is called the *canonical observables*. They are then required to be large enough to generate every other observable in $C^\infty(\mathcal{S}, \mathbb{R})$. Given a basis $\{A_i\}$ of $\mathcal{L}(\mathcal{G})$, the continuity of the map $s \longrightarrow \{f^{A_i}\}$ may require the embedding of $\mathcal{S}$ into some vector space $\mathbb{R}^k$ where $k \geq \dim \mathcal{S}$.

Step 4: Given $\mathcal{G}$, the quantisation of the system is then given by a state space represented by the Hilbert space of an irreducible unitary representation of $\mathcal{G}$. Inequivalent representations provide different quantisations of the system. The set $\{\hat{f}^A \mid A \in \mathcal{L}(\mathcal{G})\} =: \hat{O}$ of self-adjoint generators of $\mathcal{L}(\mathcal{G})$ provide the quantisation of the corresponding observables in O and hence also give an operator representation of the Poisson bracket algebra.

It must be said at this point that while there is a relation between the choice of canonical observables and the canonical group, it is not clear what really determines the correct choice for each Q. A position of trial and error is needed. For this reason, the programme may be applied in a slightly different order than presented above to some cases depending on convenience. For example, one could equally well start with the canonical observables and work backwards to obtain the canonical group. Another point that is worth mentioning regarding the choice of the canonical group is the possibility of having different canonical groups equally acting on $\mathcal{S}$ as required in Step 1. They amount to giving significantly different quantisations of the system. An example of such a case will be given in Chapter 3. In this section, we shall continue to elaborate some aspects involved in each step of the programme.

Step 1

The task of finding $\mathcal{G}$ as mentioned above is far from obvious and hence it will be useful to know what type of symmetry group is needed. The first condition that one requires of $\mathcal{G}$ is that its action on $\mathcal{S}$ must be transitive. The reason for this requirement comes from its aforementioned connection with the canonical observables from which other observables of the system are to be generated. The details of the explanation shall be deferred to the discussion of Step 3. One group that acts transitively on Q is the diffeomorphism group Diff Q itself. To have a canonical group acting transitively on the whole of $\mathcal{S} = T^*Q$, one needs to supplement Diff Q with a group acting along the fibres of T^*Q. This group can be shown to be the Abelian group $C^\infty(Q, \mathbb{R})$ and the full canonical group will be Diff $Q \ltimes C^\infty(Q, \mathbb{R})$.[2] Note that this group is infinite-dimensional even when $\dim Q < \infty$.

While it is possible to quantise using this group, one encounters several difficulties in giving the resultant theory a physical interpretation, not to mention the technical ones. A more modest and sensible attempt would be to find finite-dimensional subgroups of both Diff Q and $C^\infty(Q, \mathbb{R})$ that would work as a canonical group.

Once a candidate for the canonical group $\mathcal{G}$ is found, the generators A of its Lie algebra $\mathcal{L}(\mathcal{G})$ generate one-parameter subgroups of symplectomorphisms

$$s \mapsto s \exp(-tA) \qquad s \in \mathcal{S},\ t \in \mathbb{R} \quad . \tag{2.6}$$

The tangent vector to the flow (2.6) then defines the vector field

$$\xi^A(f) := \frac{df}{dt}(s \exp(-tA))|_{t=0} \qquad f \in C^\infty(\mathcal{S}, \mathbb{R}) \quad . \tag{2.7}$$

A homomorphism $A \mapsto \xi^A$ between $\mathcal{L}(\mathcal{G})$ and the commutator algebra of these vector fields is obtained by the relation

$$[\xi^{A_1}, \xi^{A_2}] = \xi^{[A_1, A_2]} \qquad A_1, A_2 \in \mathcal{L}(\mathcal{G}) \quad . \tag{2.8}$$

An isomorphism can be achieved if and only if there is no element $A \in \mathcal{L}(\mathcal{G})$ such that

$$s = s \exp(tA) \qquad \forall s \in \mathcal{S} \tag{2.9}$$

near $t = 0$. Hence the additional condition on $\mathcal{G}$ is that its action is *almost effective* *i.e.*

$$sg = s \qquad \forall s \in \mathcal{S} \qquad \Longrightarrow g \in D \tag{2.10}$$

where D is a discrete subgroup of $\mathcal{G}$. By definition, the Lie derivative of the symplectic form ω on $\mathcal{S}$ with respect to vector fields (2.7) is

$$\mathcal{L}_{\xi^A}\omega = 0 \quad . \tag{2.11}$$

Equation (2.11) immediately implies that the ξ^A's are locally Hamiltonian vector fields. The programme however requires that these vector fields are strictly Hamiltonian in order to get globally well-defined observables via relation

$$\xi^A \lrcorner\, \omega = df^A \qquad f \in C^\infty(\mathcal{S}, \mathbb{R}) \quad . \tag{2.12}$$

This will be related to the question of embedding/immersion in Step 3.

Step 2

The relation (2.12) ($\xi_f \lrcorner\, \omega = df$) from Step 1 establishes the homomorphism

$$\begin{gathered} C^\infty(\mathcal{S}, \mathbb{R}) \xrightarrow{j} Ham\ VF(\mathcal{S}) \\ f \mapsto -\xi_f \end{gathered} \tag{2.13}$$

between observables and Hamiltonian vector fields on $\mathcal{S}$. The minus sign on ξ^A comes from the fact that

$$[\xi_f, \xi_g] = -\xi_{\{f,g\}} \qquad f, g \in C^\infty(\mathcal{S}, \mathbb{R}) \quad . \tag{2.14}$$

Thus with each $A \in \mathcal{L}(\mathcal{G})$ one has the map F,

$$A \xrightarrow{F} f^A \in C^\infty(\mathcal{S}, \mathbb{R}) \tag{2.15}$$

called the *momentum map.* From relations (2.12) - (2.15) together with (2.8), a homomorphism between $\mathcal{L}(\mathcal{G})$ and the Poisson bracket algebra may now be established. It is now important to note that this homomorphism may fail to be an isomorphism. This is because the homomorphism (2.13) is unique only up to the addition of constant functions to f. Thus if the constant functions are included as observables on $\mathcal{S}$, the homomorphism of $\mathcal{L}(\mathcal{G})$ with the Poisson bracket algebra is not an isomorphism. To summarise the whole situation, one has the following exact sequence/commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "\iota"] & C^\infty(\mathcal{S}, \mathbb{R}) \arrow[r, "j"] & Ham\ VF(\mathcal{S}) \arrow[r] & 0 \\ & & & \mathcal{L}(\mathcal{G}) \arrow[ul, "F"] \arrow[u] & \end{tikzcd} \tag{2.16}$$

For F to be a Lie algebra isomorphism, it is required that

$$\{f^A, f^B\} = f^{[A,B]} \qquad A, B \in \mathcal{L}(\mathcal{G}) \quad . \tag{2.17}$$

But so far only the relation

$$\{f^A, f^B\} = f^{[A,B]} + z(A, B) \quad , \tag{2.18}$$

where $z(A, B)$ is a constant, follows. The constant $z(A, B)$ can be shown to obey

$$z(A, B) = -z(B, A) \quad ; \tag{2.19}$$

$$z(A, [B, C]) + z(B, [C, A]) + z(C, [A, B]) = 0 \quad . \tag{2.20}$$

The functions $z : \mathcal{L}(\mathcal{G}) \times \mathcal{L}(\mathcal{G}) \longrightarrow \mathbb{R}$ is in fact a *two-cocycle* of $\mathcal{L}(\mathcal{G})$. If $z(A, B)$ is of the form

$$z(A,B) = <\alpha, [A,B]> \tag{2.21}$$

for some $\alpha \in \mathcal{L}(\mathcal{G})^*$ then it is called a *two-coboundary.* The space of two-cocycles modulo two-coboundaries of $\mathcal{L}(\mathcal{G})$ are said to form the second cohomology group $H^2(\mathcal{L}(\mathcal{G}); \mathbb{R})$ of $\mathcal{L}(\mathcal{G})$.[6] Thus to guarantee that there is an isomorphism between $\mathcal{L}(\mathcal{G})$ and the Poisson bracket algebra, $H^2(\mathcal{L}(\mathcal{G}); \mathbb{R})$ has to be trivial. Otherwise, an extension of the algebra and hence of $\mathcal{G}$ itself is required (see Section 1.4). The new extended Lie algebra $\mathcal{L}(\mathcal{G}) + \mathbb{R}$ is defined by the following Lie bracket relations

$$[(A,r),(B,s)] := ([A,B], z(A,B)) \tag{2.22}$$

where $A, B \in \mathcal{L}(\mathcal{G})$ and $r, s \in \mathbb{R}$. Its new isomorphic Poisson bracket algebra are given by

$$\{f^{(A,r)}, f^{(B,s)}\} = \{f^A, f^B\} = f^{[A,B]} + z(A,B) = f^{([A,B],z(A,B))} \quad . \tag{2.23}$$

Step 3

Once the Lie algebra $\mathcal{L}(\mathcal{G})$ is realised by the Poisson bracket of the observables, it remains to be seen that the canonical observables generate a 'sufficiently large' set of other observables. One way to ensure that they do generate these other observables is to employ the local generating principle.[2]

Local Generating Principle: Let $\{F_i,\ i = 1, \ldots, k\}$ be a basis of $\mathcal{L}(\mathcal{G})$. Consider an open neighbourhood U_s of $s \in \mathcal{S}$. Given any $f \in C^\infty(\mathcal{S}, \mathbb{R})$ whose support is contained in U_s, there exists a function $\mathcal{F}_f \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that for all $s \in U_s$,

$$f(s) = \mathcal{F}_f(F_1(s), \cdots, F_k(s)) \quad . \quad \square \tag{2.24}$$

This implies a local embedding $\jmath : \mathcal{S} \longrightarrow \mathbb{R}^k$ *i.e.*

$$\jmath(s) := (F_1(s), \cdots, F_k(s)) \tag{2.25}$$

whose rank is k. Thus, the map $\jmath$ is an *immersion* and hence the corresponding induced map $\jmath_* : T_s\mathcal{S} \longrightarrow T_{\jmath(s)}\mathbb{R}^k$ is injective. The transitive action of $\mathcal{G}$ on $\mathcal{S}$ then guarantees or overspan (if $k > 2n$) the tangent space $T_s\mathcal{S}$ for each point $s \in \mathcal{S}$. For if $\mathcal{G}$ is not acting transitively on $\mathcal{S}$, there would be at least one direction in which there is no one-parameter subgroup of $\mathcal{G}$ to translate points of $\mathcal{S}$ along that direction and hence k must be less than $2n$. Thus the local generating principle would not hold in that case. Given that $\mathcal{G}$ is transitive, consider an arbitrary vector field $\rho \in T\mathcal{S}$ and the relation

$$\rho \lrcorner \xi^A \lrcorner \omega = \omega(\xi^A, \rho) = < df^A, \rho > \quad . \tag{2.26}$$

The non-degenerate property of ω then implies that the one-forms $\{df^A,\ A \in \mathcal{L}(\mathcal{G})\}$ span or overspan every cotangent space $T^*_s\mathcal{S}$. This implies that the canonical observables do generate other observables as required.

In another perspective, the continuity of map $\jmath$ ensures that the observables are globally well-defined on $\mathcal{S}$. This is very desirable since it avoids any possible obstruction in carrying out the quantisation map of the observables. This question sheds light in the subject of anomalies for the quantum mechanical system[7] (see Section 3.3).

Step 4

Finally, the quantisation process involves finding irreducible unitary representations of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$, The irreducibility aspect of the representation is desirable since any self-adjoint operator on $\mathcal{H}$ is then always a function of the operator representation of the canonical observables, reflecting the classical picture of any observable being written in terms of the canonical ones. The use of the unitary operators help to ease the technical aspects of the programme since they are bounded. The representations considered in this work are those realisable on the cross-sections of the bundles over the configuration spaces Q. This leads to the idea of twisted wavefunctions on Q.[8] These representations are constructed using the induced representation techniques introduced in Section 1.4. In particular, the technique involving semi-direct products will be often used to find representations of the canonical group which is a semi-direct product, $G \rtimes K$; G and K being the subgroups of Diff Q and $C^\infty(\mathcal{S}, \mathbb{R})$. With regards to other possible representations on other spaces, it has been pointed out that they may be equally important[2,9] but they will not be considered here.

Once the representations are found, the quantisation map (2.5) can now be obtained by the correspondence between the self-adjoint generators $\{\hat{f}^A\}$ of $\mathcal{L}(\mathcal{G})$ and the observables

$\{f^A\}$ on $\mathcal{S}$ $(A \in \mathcal{L}(\mathcal{G}))$ *i.e.*

$$f^A \xrightarrow{\wedge} \hat{f}^A \qquad A \in \mathcal{L}(\mathcal{G}) \quad . \tag{2.27}$$

An important result that one seeks from the representation theory is an analogue of the Stone-Von Neumann uniqueness theorem in the case of quantum mechanics say on $\mathbb{R}^n$. This theorem tells us that after fixing some physical scale, there is only one possible quantisation of the system on $\mathbb{R}^n$. Since the possible quantisations of a system are given by inequivalent irreducible representations of $\mathcal{G}$, it is only necessary to classify the irreducible representations and hence obtain some general version of the Stone-Von Neumann theorem. This is done by using the orbital analysis discussed in Section 1.4.

2.3 Quantisation on $Q = \mathbb{R}^n$

There is no better example to demonstrate the application of group-theoretic quantisation programme than quantum mechanics on $\mathbb{R}^n$. The well-known results of quantum mechanics serve as a comparison and a guideline to the programme's performance. Also, from the point of view of this work, it is the first simple example that demonstrates the idea of central extensions. The steps mentioned in the previous section shall be followed closely for the group-theoretic quantisation of the system of a particle moving on the configuration space $Q = \mathbb{R}^n$.

The phase space of the system is given by the cotangent bundle $\mathcal{S} = T^*Q = \mathbb{R}^{n*} \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. The bundle being trivial and a product of two linear spaces, has globally well-defined coordinates; $\{q^i,\ i = 1, \ldots, n\}$ for Q and $\{p_i,\ i = 1, \ldots, n\}$ for the fibres. It can be said at this stage that these globally well-defined coordinate functions serve well as (part of) the set of canonical observables. The natural symplectic form on $\mathcal{S}$ is given by

$$\omega = dq^i \wedge dp_i \tag{2.28}$$

(implicit sum over repeated indices is assumed). An obvious candidate of the canonical group is the Abelian group $\mathcal{G} = \mathbb{R}^n \times \mathbb{R}^n$, acting as the group of translations on $\mathcal{S}$. The Lie algebra of $\mathcal{G}$ is $\mathbb{R}^n \oplus \mathbb{R}^n$ and its exponential map from the algebra to the group is given by

$$\exp(A^i, B_i) = (a^i, b_i) \tag{2.29}$$

where (A^i, B_i) is the corresponding Lie algebra element to $(a^i, b_i) \in \mathbb{R}^n \times \mathbb{R}^n$. [The use of capital letters for the elements of the algebra is adopted to differentiate them from the elements of the group.] Consider then the one-parameter subgroup generated by (A^i, B_i) from the map $t \mapsto (ta^i, tb_i)$. The induced vector field from this one-parameter subgroup is

$$\xi^{(A^i, B_i)} = A^i \frac{\partial}{\partial q^i} - B_i \frac{\partial}{\partial p_i} \quad . \tag{2.30}$$

The vector field (2.30) is Hamiltonian and its corresponding observable is

$$f^{(A^i, B_i)} = A^i p_i + B_i q^i \quad . \tag{2.31}$$

Equation (2.31) then gives the desired momentum map $(A^i, B_i) \mapsto f^{(A^i, B_i)}$. The Poisson bracket algebra of the observables (2.31) is given by

$$\{f^{(A^i, B_i)}, f^{(A'^i, B'_i)}\} = B_i A'^i - B'_i A^i \quad . \tag{2.32}$$

Note however that the group $\mathbb{R}^n \times \mathbb{R}^n$ is Abelian and hence

$$[(A^i, B_i), (A'^i, B'_i)] = 0 \quad . \tag{2.33}$$

This implies that the Poisson bracket algebra is not isomorphic to $\mathcal{L}(\mathcal{G})$. The disagreement lies in the nontrivial cocycle

$$z((A^i, B_i), (A'^i, B'_i)) := B_i A'^i - B'_i A^i \tag{2.34}$$

of $\mathcal{L}(\mathcal{G})$ in (2.32). A way out of this problem is to extend the Lie algebra $\mathcal{L}(\mathcal{G})$ to $\mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$ as described in Step 2 of the programme. The extended algebra has the Lie bracket relation

$$[(A^i, B_i, C), (A'^i, B'_i, C')] := (0, 0, B_i A'^i - B'_i A^i) \tag{2.35}$$

where C and C' are the elements of the central subalgebra $\mathbb{R}$. The new extended canonical group $\bar{\mathcal{G}}$ corresponding to algebra (2.35) is the Heisenberg-Weyl group,

$$\bar{\mathcal{G}} = \mathbb{R}^n \ltimes (\mathbb{R}^n \times \mathbb{R}) \quad . \tag{2.36}$$

Its group product law is given by

$$(a^i, b_i, c)(a'^i, b'_i, c') := (a^i + a'^i, b_i + b'_i, c + c' + \tfrac{1}{2}(b_i a'^i - b'_i a^i)) \quad . \tag{2.37}$$

The new momentum map is given by

$$\begin{aligned} (A^i, B_i, C) &\longrightarrow f^{(A^i,B_i,C)} \quad , \\ f^{(A^i,B_i,C)} &:= A^i p_i + B_i q^i + C \quad . \end{aligned} \tag{2.38}$$

Note that the local generating principle still holds in the extended case since the $\bar{\mathcal{G}}$-action on $\mathcal{S}$ is still transitive though it is not (almost) effective. [The non-effective action is not a problem since the prime interest here is more in achieving the isomorphism between the Poisson bracket algebra and $\mathcal{L}(\mathcal{G})$ rather than with the commutator algebra of the Hamiltonian vector fields. See also related remark at the end of this section.]

To study the irreducible unitary representations $\mathcal{U}$ of the Heisenberg-Weyl group, the following notation is introduced. Let U and V be the unitary representations of the two $\mathbb{R}^n$ subgroups which are defined by

$$U(a^i) := \mathcal{U}(a^i, 0, 0) \ , \quad V(b_i) := \mathcal{U}(0, b_i, 0) \quad . \tag{2.39}$$

The central subgroup $\mathbb{R}$ is unitarily represented by the one-dimensional representation

$$\mathcal{U}(0, 0, c) := e^{-i\mu C} \tag{2.40}$$

where μ is a real parameter. From (2.37), the operator (2.39) satisfy the relations

$$U(a^i)U(a'^i) = U(a^i + a'^i) \quad , \tag{2.41}$$

$$V(b_i)V(b'_i) = V(b_i + b'_i) \quad , \tag{2.42}$$

$$U(a^i)V(b_i) = V(b_i)U(a^i)e^{i\mu A^i B_i} \quad , \tag{2.43}$$

which are called the *Weyl commutation relations.* If the generators of U and V are written as $\hat{p}_i$ and $\hat{q}^i$ respectively *i.e.*

$$U(a^i) := e^{-iA^i\hat{p}_i} \quad , \tag{2.44}$$

$$V(b_i) := e^{-iB_i\hat{q}^i} \quad , \tag{2.45}$$

then the relation (2.43) produces the desired CCR

$$[\hat{q}^i, \hat{p}_j] = i\mu\delta^i_j \quad . \tag{2.46}$$

The representation of the full Heisenberg-Weyl group is probably best realised by what is now known as the *Schroedinger representation*[10] which is a representation induced by a unitary representation of the subgroup $\mathbb{R}^n \times \mathbb{R}$. The Hilbert space of this representation is given by $L^2(\mathbb{R}^n)$, the square-integrable functions of $\mathbb{R}^n$. The representation U is given by

$$U(a^i)\psi(q^i) := \psi(q^i - a^i) \quad , \tag{2.47}$$

and V is the character representation

$$V(b_i)\psi(q^i) := e^{-iB_i q^i}\psi(q^i) \quad ; \tag{2.48}$$

where $\psi \in L^2(\mathbb{R}^n)$. The central subgroup is still represented by the multiplicative operator (2.40). The full representation $\mathcal{U}$ of the Heisenberg-Weyl group is then given by the induced representation $\mathcal{U}^\chi$ by the characters χ of $\mathbb{R}^n \times \mathbb{R}$ from (2.48) and (2.40). With the above class of representation and the techniques of orbital analysis in Section 1.4, one can arrive to the following famous theorem.

Stone-Von Neumann Theorem: Let $\mathcal{U}(a^i, b_i, c)$ be an irreducible unitary representation of (a^i, b_i, c) of the Heisenberg-Weyl group on a Hilbert space $\mathcal{H}$ such that $\mathcal{U}(0,0,c) = e^{-i\mu C}$ for some $\mu \in \mathbb{R}$. Then $\mathcal{U}$ is unitarily equivalent to the induced representation $\mathcal{U}^\chi$ by character χ of $\mathbb{R}^n \times \mathbb{R}$ *i.e.* the Schroedinger representation given above. □

At this point, it is important to note that there is a whole family of CCR parametrised by μ. The contact with physics is when μ is identified with $\hbar$. This intuitively means fixing the physical scale of the theory. Note that the Stone-Von Neumann uniqueness theorem is only applied after such physical scale is determined.

Finally, a brief explanation of the connection of the central extension with a line bundle structure for this case is appropriate. The line bundle structure is actually implied when it is noted earlier that the $\bar{\mathcal{G}}$ fails to be (almost) effective. This failure also means the failure of the isomorphism of the commutator algebra of the Hamiltonian vector fields on $\mathcal{S}$ with $\mathcal{L}(\mathcal{G})$. The isomorphism however can be restored if there is another degree of freedom added to $\mathcal{S}$ in order to define a 'new Hamiltonian vector field' corresponding to the constant functions.[11] This degree of freedom is provided by the fibres of a line bundle over $\mathcal{S}$. This line bundle structure has been known to be the *quantisation bundle* in the geometric quantisation school[12] and the bundle is characterised by its curvature which is given by the symplectic form. This example is to be contrasted with the example that will be given in Section 3.3 where the central extension in that example arises from a specific line bundle (up to equivalence) over the configuration space itself rather than the phase space.

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Chapter 3

Quantisation of a Particle on the Torus

3.1 Introduction

In the previous chapter, the outlines of Isham's group-theoretic quantisation programme were discussed and then exemplified by quantisation of a system of a particle moving on the configuration space $Q = \mathbb{R}^n$. This example, however, does not fully demonstrate the advantages of this programme, owing to the linearity of the configuration space. Here, the coordinates of the phase space $\mathcal{S} = \mathbb{R}^n \times \mathbb{R}^n$, namely $\{q^a, p_a\}$ readily serve as part of the required set of globally well-defined canonical observables. In a nonlinear configuration space, this is not possible, as the coordinates are no longer globally well-defined. The choice of globally well-defined canonical observables necessitates the embedding of the phase space $\mathcal{S}$ in some vector space $\mathbb{R}^m$, mentioned in Step 3 of the programme, where $m > \dim \mathcal{S}$. Hence the nonlinearity of the configuration space is being taken into account from the beginning of the quantisation programme, by choosing some canonical variables (*cf.* $\{q,p\}$ in $Q = \mathbb{R}$) intrinsic to Q. For example, take the simplest case of $Q = S^1$ whose coordinate is the angle ϕ. Here one requires $m = 3$, as the canonical observables are $\cos\phi$, $\sin\phi$, and the momentum coordinate J. Note that $\cos\phi$ is a well-defined continuous function on S^1 while ϕ is not; the observable $\sin\phi$ is then needed to close the Poisson bracket algebra of the observables.

The above modification of the choice of observables then leads to an underlying canonical group for quantisation involving nonlinear spaces, different from the usual Heisenberg-Weyl group. This in turn disallows the use of the usual canonical commutation relations (CCR) $[q^a, p_b] = i\hbar\delta^a_b$. Such relations, as pointed out earlier, assume the linearity of the configuration space. The appropriate canonical group obtained by the above consideration should then very well describe the symmetries of the underlying nonlinear spaces. This is the basic advantage of this programme. The usual erroneous attempt using the CCR is now 'corrected' by taking into account the symmetries of the underlying phase space. Various such examples leading to different commutation relations have been discussed in Isham's Les Houches lectures.[1]

In this chapter, we shall be discussing another example: namely quantisation of a system of a particle moving on the two-torus ($Q = \mathrm{T}^2$). This example is simply a straightforward generalisation of the case of $Q = S^1$, which has been discussed in detail by Isham.[1] The reason why this example is discussed here is to enable us to discuss next the simplest example of quantisation of a system of a particle moving in an external background magnetic field on a nonlinear configuration space. Here, the presence of the background field modifies the natural symplectic form. This leads to a nontrivial modification of the canonical group itself. It is found that one has to use the central extension of the universal cover of the canonical group found for the case without the background field. This example extends Isham's programme to include nontrivial quantisation bundles on the configuration space, as opposed to only bundles with flat connections.[1] The different possible quantisations for the example are then discussed by looking at inequivalent representations of the new canonical group. Finally, all the above discussions will be generalised to the case of the general n-torus ($Q = \mathrm{T}^n$) but now formulated in the language of lattices. This will facilitate the discussions of quantisation of strings on the tori in the final chapter.

3.2 Quantisation on $Q = \mathrm{T}^2$

In this section, we shall proceed in a slightly different way from the example of $Q = \mathbb{R}^n$ in Chapter 2. Instead of guessing what the canonical group is, we first identify the canonical observables and then proceed to find the appropriate group from them. In this way, the methods used in this section can easily be carried over to the next section when discussing the case with the magnetic field. This suits our main purpose of finding the global structure of the canonical group $\mathcal{G}$. In this section and the next, the indices $a, b, c, \ldots$ will take values 1 and 2 and they are not summed over repeated indices unless there is an explicit summation sign.

Canonical Group $\mathcal{G} = E_2 \times E_2$

Consider a particle moving on the configuration space $Q = \mathrm{T}^2$ coordinatised by the angles ϕ^a. The phase space S is given by the cotangent bundle $T^*\mathrm{T}^2 = \mathrm{T}^2 \times \mathbb{R}^2$ which is endowed with the natural symplectic form

$$\omega = \sum_a d\phi^a \wedge dJ_a \quad , \tag{3.1}$$

where J_a are the coordinates of the fibres $\mathbb{R}^2$. As in the case of S^1 mentioned in the introduction, one chooses the following set of well-defined functions on $T^*\mathrm{T}^2$,

$$u^a = \cos\phi^a \ , \quad v^a = \sin\phi^a \ , \quad J_a \quad , \tag{3.2}$$

as the canonical observables. The Hamiltonian vector fields corresponding to these observables are obtained by the relation $\xi_f \lrcorner\, \omega = df$, where f is the observable and ξ_f, its vector field. They are given by

$$\xi_u^a = \sin\phi^a \frac{\partial}{\partial J_a} \quad , \tag{3.3}$$

$$\xi_v^a = -\cos\phi^a \frac{\partial}{\partial J_a} \quad , \tag{3.4}$$

$$\xi_{Ja} = \frac{\partial}{\partial \phi^a} \quad . \tag{3.5}$$

The Poisson bracket algebra of these observables is

$$\{J_b,\ u^a\} := \omega(\xi_{Jb},\ \xi_u^a) = v^a \delta_b^a \quad , \tag{3.6}$$

$$\{J_b,\ v^a\} := \omega(\xi_{Jb},\ \xi_v^a) = -u^a \delta_b^a \quad , \tag{3.7}$$

$$\{u^a,\ v^b\} \ = 0 = \{J_a,\ J_b\} \quad . \tag{3.8}$$

It is important to note that the commutator algebra of these Hamiltonian vector fields is isomorphic to the Poisson bracket algebra above. This implies that these vector fields can be identified with the generators of the canonical group of symplectomorphisms on $\mathcal{S}$. Denote the exponential mapping of the vector fields by

$$m_a := \exp(M_a \xi_u^a) \quad , \tag{3.9}$$

$$n_a := \exp(N_a \xi_v^a) \quad , \tag{3.10}$$

$$\eta^a := \exp(H^a \xi_{Ja}) \quad , \tag{3.11}$$

where M_a, N_a and H^a are parameters associated to the Lie algebra of the canonical group. The terms on the left hand side are the group elements whose action on the points of $\mathcal{S}$

(on the right) are given by evaluating the corresponding terms on the right hand side on those points. The group action on $\mathcal{S}$ may now be given by

$$\begin{aligned}&\tau_{(m_a,n_a,\eta^a)}(\phi^a,\ J_a)\\&\quad:=((\phi^a+\eta^a) \bmod 2\pi,\ J_a+m_a\sin(\phi^a+\eta^a)-n_a\cos(\phi^a+\eta^a))\end{aligned} \tag{3.12}$$

for each a. This is the action of the Euclidean group E_2. Thus, the canonical group is

$$\mathcal{G}=E_2\times E_2=(\mathbb{R}^2 \rtimes SO(2))\times(\mathbb{R}^2 \rtimes SO(2)) \quad . \tag{3.13}$$

The elements $(m_a,\ n_a)$ and η^a belong respectively to the $\mathbb{R}^2$ and $SO(2)$ subgroups of $(E_2)_a$ – the a-th Euclidean group. The group law is given by

$$(m_a,\ n_a,\ \eta^a)(m'_a,\ n'_a,\ \eta^{a\prime})=(m''_a,\ n''_a,\ (\eta^a+\eta^{a\prime}) \bmod 2\pi) \tag{3.14}$$

where

$$\begin{pmatrix} m''_a \\ n''_a \end{pmatrix}=\begin{pmatrix} m_a \\ n_a \end{pmatrix}+\begin{pmatrix} \cos\eta^a & -\sin\eta^a \\ \sin\eta^a & \cos\eta^a \end{pmatrix}\begin{pmatrix} m'_a \\ n'_a \end{pmatrix} \tag{3.15}$$

for each a.

It is also possible to use the universal cover of $\mathcal{G}$ as the canonical group. The group action on $\mathcal{S}$ is the same, and much of what follows can be easily generalised for the covering group. The only difference is in the discussion of its 'nontrivial' representations, which will be given later.

Representations of $\mathcal{G}$

Given that the canonical group is $\mathcal{G}=E_2\times E_2$, the next step of the programme involves finding irreducible unitary representations

$$(m_1,n_1,\eta^1;m_2,n_2,\eta^2)\mapsto\mathcal{U}(m_1,n_1,\eta^1;m_2,n_2,\eta^2)$$

of $\mathcal{G}$ on some Hilbert space $\mathcal{H}$. At this stage, it is convenient to define the operators

$$V(m_1,n_1):=\mathcal{U}(m_1,n_1,0;0,0,0) \quad , \tag{3.16}$$

$$U(\eta^1):=\mathcal{U}(0,0,\eta^1;0,0,0) \quad , \tag{3.17}$$

etc. which satisfy the relations

$$U(\eta^a)U(\eta^{a\prime}) = U((\eta^a + \eta^{a\prime}) \bmod 2\pi) \quad , \tag{3.18}$$

$$V(m_a,\ n_a)V(m'_a,\ n'_a) = V(m_a + m'_a,\ n_a + n'_a) \quad , \tag{3.19}$$

$$U(\eta^1)U(\eta^2) = \mathcal{U}(0,0,\eta^1;0,0,\eta^2) = U(\eta^2)U(\eta^1) \quad , \tag{3.20}$$

$$V(m_1,n_1)V(m_2,n_2) = \mathcal{U}(m_1,n_1,0;m_2,n_2,0) = V(m_2,n_2)V(m_1,n_1) \quad , \tag{3.21}$$

$$U(\eta^1)V(m_2,n_2) = V(m_2,n_2)U(\eta^1) \quad , \tag{3.22}$$

$$U(\eta^a)V(m_a,n_a) = V(m'_a,n'_a)U(\eta^a) \quad , \tag{3.23}$$

etc., where

$$\begin{pmatrix} m'_a \\ n'_a \end{pmatrix} = \begin{pmatrix} \cos\eta^a & -\sin\eta^a \\ \sin\eta^a & \cos\eta^a \end{pmatrix} \begin{pmatrix} m_a \\ n_a \end{pmatrix} \quad . \tag{3.24}$$

Suppose that the generators of $U(\eta^a)$ and $V(m_a,\ n_a)$ are given by $\hat{J}_a$, $\hat{u}^a$ and $\hat{v}^a$ *i.e.*

$$U(\eta^a) := e^{iH^a \hat{J}_a} \quad , \tag{3.25}$$

$$V(m_a,\ n_a) := e^{i(M_a \hat{u}^a + N_a \hat{v}^a)} \quad , \tag{3.26}$$

then from (3.18) - (3.23), we obtain the following quantum commutators

$$[\hat{u}^a,\ \hat{v}^b] = 0 = [\hat{J}_a,\ \hat{J}_b] \quad , \tag{3.27}$$

$$[\hat{J}_b,\ \hat{u}^a] = i\hat{v}^a \delta^a_b \quad , \tag{3.28}$$

$$[\hat{J}_b,\ \hat{v}^a] = -i\hat{u}^a \delta^a_b \quad . \tag{3.29}$$

An obvious representation of $\mathcal{G}$ is on the space of square-integrable functions on T^2 itself. This is given by

$$(U(\eta^a)\psi)(\phi^1,\phi^2) := \psi((\phi^1 + \delta^1_a\eta^a) \bmod 2\pi, (\phi^2 + \delta^2_a\eta^a) \bmod 2\pi) \quad , \tag{3.30}$$

$$(V(m_a,n_a)\psi)(\phi^1,\phi^2) := e^{i(M_a\cos\phi^a + N_a\sin\phi^a)}\psi(\phi^1,\phi^2) \quad , \tag{3.31}$$

and these correspond to

$$(\hat{J}_a\psi)(\phi^1,\phi^2) = -i\frac{\partial}{\partial\phi^a}\psi(\phi^1,\phi^2) \quad , \tag{3.32}$$

$$(\hat{u}^a\psi)(\phi^1,\phi^2) = \cos\phi^a\psi(\phi^1,\phi^2) \quad , \tag{3.33}$$

$$(\hat{v}^a\psi)(\phi^1,\phi^2) = \sin\phi^a\psi(\phi^1,\phi^2) \quad . \tag{3.34}$$

There are, however, other inequivalent representations and hence quantisations of the system on this representation space. They are characterised by using Mackey's orbital techniques in the theory of representations of semi-direct products[2,3] (see Theorem 1.12). Here, each of the E_2 subgroups of $\mathcal{G}$ may be treated separately for this analysis. With respect to the above representation space, this simply implies the restriction of wavefunctions ψ to functions of only one of the angular variables while keeping the other fixed.

In the group E_2, the $SO(2)$ subgroup acts on the $\mathbb{R}^2$ subgroup as in (3.24). This induces an action of $SO(2)$ on Char($\mathbb{R}^2$) by using the isomorphism between Char($\mathbb{R}^2$) with the dual space $\mathbb{R}^{2*}$ and the natural identification of $\mathbb{R}^{2*}$ with $\mathbb{R}^2$. So, if a character of $\mathbb{R}^2$, χ_w, is given by

$$\chi_w(v) := e^{-iw.v} \quad , \tag{3.35}$$

where $w \in \mathbb{R}^{2*}$ ($\sim \mathbb{R}^2$), $v \in \mathbb{R}^2$ and $w.v$ is the inner product between them, then the action τ_R of $R = \begin{pmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{pmatrix} \in SO(2)$ on $\chi_w (=: \chi_{(w_1,w_2)})$ is

$$\tau_R\chi_w = \chi_{(w_1\cos\eta - w_2\sin\eta,\, w_1\sin\eta + w_2\cos\eta)} = \chi_{Rw} \quad . \tag{3.36}$$

Thus the orbits of the $SO(2)$-action in Char($\mathbb{R}^2$) are circles of radius λ :

$$S^1_\lambda := \{(w_1,w_2) \in \mathbb{R}^{2*} \mid {w_1}^2 + {w_2}^2 = \lambda^2\} \quad . \tag{3.37}$$

Note that the isotropy group of any point on S^1_λ is trivial. Hence associated to each λ is only one irreducible representation of E_2. Returning to the representation of the whole of $\mathcal{G}$, there will then be two parameters λ_a, each associated to an E_2 subgroup, characterising the representations of $\mathcal{G}$. These representations are given by (3.30) and

$$\begin{aligned}&(V^{(\lambda)}(m_1,n_1;m_2,n_2)\psi)(\phi^1,\phi^2)\\ &\quad := e^{i\lambda_1(M_1\cos\phi^1+N_1\sin\phi^1)+i\lambda_2(M_2\cos\phi^2+N_2\sin\phi^2)}\psi(\phi^1,\phi^2) \quad .\end{aligned} \tag{3.38}$$

Lifting Actions and Universal Cover of G

From above, we have seen how representations of $\mathcal{G}$ are realisable on the space of square-integrable functions on T^2. This can be further generalised in a very geometrical manner using the space of sections of a line bundle over T^2.

Consider the trivial principal $U(1)$-bundle over T^2, $P = \mathrm{T}^2 \times U(1)$. On the base space, there is the transitive action r_G of $G = SO(2) \times SO(2)$. This action r_G has a trivial lift[4,5] $r_G^\dagger$ onto P making the following diagram commutative

$$\begin{array}{ccccc} U(1) & \longrightarrow & P & \xrightarrow{r_G^\dagger} & P \\ & & \downarrow & & \downarrow \\ & & \mathrm{T}^2 & \xrightarrow{r_G} & \mathrm{T}^2 \end{array} \quad . \tag{3.39}$$

Let $(\phi^1, \phi^2; U)$ be the (smooth) trivialisation of a section $\sigma(\phi^1, \phi^2)$ of P. Given that the action $r_{(\eta^1,\eta^2)}$ on T^2 ($(\eta^1, \eta^2) \in G$) is

$$r_{(\eta^1,\eta^2)}(\phi^1, \phi^2) = ((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi) \quad , \tag{3.40}$$

then the trivial lift $r^\dagger_{(\eta^1,\eta^2)}$ on P is simply

$$\begin{aligned} r^\dagger_{(\eta^1,\eta^2)}\sigma(\phi^1, \phi^2) &:= (r_{(\eta^1,\eta^2)}(\phi^1, \phi^2); U) \\ &= ((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi; U) \quad . \end{aligned} \tag{3.41}$$

This induces a (trivial) lifted G-action $\tau_G^\dagger$ on the associated line bundle $E = P \times_{U(1)} \mathbb{C}$. If the trivialisation of the section $\Psi(\phi^1, \phi^2)$ of E is given by

$$\Psi(\phi^1, \phi^2) := [\sigma(\phi^1, \phi^2); \psi(\phi^1, \phi^2)] \quad , \tag{3.42}$$

where ψ is a complex-valued function on T^2 and $[\cdot\ ;\cdot]$ denotes equivalence classes under $U(1)$-action, then $\tau^\dagger_{(\eta^1,\eta^2)}$ is defined by

$$\begin{aligned} \tau^\dagger_{(\eta^1,\eta^2)}\Psi(\phi^1, \phi^2) &:= [r^\dagger_{(\eta^1,\eta^2)}\sigma(\phi^1, \phi^2); \psi(r_{(\eta^1,\eta^2)}(\phi^1, \phi^2))] \\ &= [\sigma((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi; \\ &\qquad \psi((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi)] \quad . \end{aligned} \tag{3.43}$$

This reproduces the representation (3.30) when the above function ψ is identified with that of (3.30).

In a similar way, one can construct inequivalent representations analogous to (3.30) ((3.43)) but they are now representations of the covering group of G and they are generated by different liftings of the group action. Let us consider first an arbitrary lift of an action of an arbitrary group K (be that of G or its universal cover) on P. An action r_k $(k \in K)$ is lifted to $r_k^\dagger$ by

$$r_k^\dagger \sigma(\phi^1, \phi^2) := (r_k(\phi^1, \phi^2); f(k; (\phi^1, \phi^2))U) \quad , \tag{3.44}$$

where f is a (smooth) mapping from $K \times \mathrm{T}^2$ to $U(1)$. The induced lifted action on E will then be given by

$$\begin{aligned} \tau_k^\dagger \Psi(\phi^1, \phi^2) &= [r_k^\dagger \sigma(\phi^1, \phi^2);\ \psi(r_k(\phi^1, \phi^2))] \\ &= [\sigma(r_k(\phi^1, \phi^2));\ f(k; (\phi^1, \phi^2))\psi(r_k(\phi^1, \phi^2))] \quad , \end{aligned} \tag{3.45}$$

where the last equality in (3.45) follows from the property of the equivalence class. Strictly speaking one should use a unitary representation of $f(k; (\phi^1, \phi^2))$ in (3.45) on the representation space of functions $\psi(\phi^1, \phi^2)$ rather than f itself. However, here we will always use the identity representation. Note that f obeys the cocycle condition,

$$f(k_1 k_2; (\phi^1, \phi^2)) = f(k_2; (\phi^1, \phi^2))\ f(k_1; r_{k_2}(\phi^1, \phi^2)) \quad , \tag{3.46}$$

where $k_1, k_2 \in K$. One can in fact define equivalent lifts through f by noting that under change of sections $\sigma'(\phi^1, \phi^2) = \sigma(\phi^1, \phi^2)\Omega(\phi^1, \phi^2)$ where $\Omega : \mathrm{T}^2 \longrightarrow U(1)$, f is transformed to

$$f'(k; (\phi^1, \phi^2)) = \Omega(r_k(\phi^1, \phi^2))^{-1} f(k; (\phi^1, \phi^2))\Omega(\phi^1, \phi^2) \quad . \tag{3.47}$$

Such f and f' are said to be cohomologous and the liftings generated by them are equivalent. Note that if one equates $f(k; (\phi^1, \phi^2))$ with the identity (*i.e.* the trivial lift) in (3.47) we find

$$f'(k; (\phi^1, \phi^2)) = \Omega(r_k(\phi^1, \phi^2))^{-1}\Omega(\phi^1, \phi^2)) \quad . \tag{3.48}$$

Such functions f' are then called coboundaries and they generate lifts equivalent to the trivial one.

Returning to the group $G = SO(2) \times SO(2)$, one may define the following cocycle for this group

$$f^{(\alpha)}((\eta^1, \eta^2); (\phi^1, \phi^2)) := e^{i(\alpha_1\eta^1 + \alpha_2\eta^2)} \quad , \tag{3.49}$$

as an attempt to define a nontrivial lift of G-action on P, where $(\eta^1, \eta^2) \in G$ and the α_a's are parameters which are required to be integers to ensure periodicity of $f^{(\alpha)}$ under the transformation $\eta^a \mapsto \eta^a + 2\pi$. However, one can define

$$\Omega^{(\alpha)}(\phi^1, \phi^2) := e^{-i(\alpha_1\phi^1 + \alpha_2\phi^2)} \quad , \tag{3.50}$$

and find that

$$f^{(\alpha)}((\eta^1, \eta^2); (\phi^1, \phi^2)) = \Omega^{(\alpha)}(\phi^1 + \eta^1, \phi^2 + \eta^2)^{-1}\, \Omega^{(\alpha)}(\phi^1, \phi^2) \quad . \tag{3.51}$$

Hence (3.49) would only generate a lift equivalent to the trivial one. In order to get a nontrivial lift one should consider instead the universal cover of G, *i.e.* $\mathbb{R} \times \mathbb{R}$ in the canonical group. In this case the cocycle (3.49) is changed to

$$f^{(\theta)}((\eta^1, \eta^2); (\phi^1, \phi^2)) := e^{i(\theta_1\eta^1 + \theta_2\eta^2)} \quad , \tag{3.52}$$

where (η^1, η^2) now belongs to $\mathbb{R} \times \mathbb{R}$ and the θ_a's are real parameters. An attempt similar to (3.50) would no longer work as the corresponding function is no longer a continuous map from T^2 to $U(1)$ unless the θ_a's are integers. With this nontrivial lift, one can now obtain inequivalent representations to (3.30), now parametrised by θ_a, *i.e.*

$$\begin{aligned}(U(\eta^1, \eta^2)\psi)(\phi^1, \phi^2) \\ = e^{i(\theta_1\eta^1 + \theta_2\eta^2)}\psi((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi) \quad .\end{aligned} \tag{3.53}$$

For the complete representations of the new canonical group (*i.e.* the universal cover of $\mathcal{G}$), the representation (3.53) is supplemented by (3.38). It is worth pointing out that the phase factors involving the θ_a's are simply the phase factors that give the usual θ-structure in quantum theory[6] arising from $\mathrm{Hom}(\pi_1(Q), U(1))$.[7] One can in fact show that the representations (3.53) can be obtained by considering $\pi_1(\mathrm{T}^2) \longrightarrow \mathbb{R} \times \mathbb{R} \longrightarrow \mathrm{T}^2$ as a 'master bundle' which carries the desired lift of a group action naturally.[1] Also, the parameters θ_a can be considered as components of a flat connection on P. This can be seen from the generator of (3.53) (*cf.* (3.32)) which is

$$(\hat{J}_a\psi)(\phi^1,\phi^2) = -i\frac{\partial\psi}{\partial\phi^a}(\phi^1,\phi^2) + \theta_a\psi(\phi^1,\phi^2) \quad . \tag{3.54}$$

This is to be compared with the minimal coupling rule of electromagnetism. The significance of this interpretation will be much more evident in the next section.

Contraction of $\mathcal{L}(\mathcal{G})$

Finally in this section, we shall turn briefly to another interesting topic, namely that of contraction of the Lie algebra of the canonical group. Contraction[8] involves taking a limiting process for a particular parameter which parametrises the Lie (sub-) algebra in question. It is often discussed within a physical context[9] in which the parameter will be some 'external' physical parameter of the system studied. Here, contraction of $\mathcal{L}(G)$ relates quantisation of a particle on T^2 to the ordinary quantum mechanics on $\mathbb{R}^2$. Physically, this is welcomed, as the quantum system on T^2 should locally resemble that of $\mathbb{R}^2$. The relevant physical parameters that should be considered here are the radii ρ_a of the generating cycles of T^2 embedded in $\mathbb{R}^3$. Taking the limit of these radii to infinity would then in effect gives the local character of the quantum system on T^2. In order to see how these radii would parametrise $\mathcal{L}(G)$, we first observe that locally in a neighbourhood of $\phi^a = 0$, the canonical observables u^a, v^a behave like 1, ϕ^a respectively. Thus in this neighbourhood, the quantity $\rho_a\phi^a$ would serve as the 'canonical coordinate' x^a in $\mathbb{R}^2$. In a similar dimensional argument, the quantity $\rho_a^{-1}J_a$ would be the 'momentum' p_a conjugate to x^a. Given this local parametrisation of the observables, it is easily deduced that the required Lie algebra parametrisation is the following homomorphism of $\mathcal{L}(G)$ into itself

$$A_\rho : \{\hat{u}^a, \hat{v}^a, \hat{J}_a\} \mapsto \{\hat{u}^a_{(\rho)}, \hat{v}^a_{(\rho)}, \hat{J}_{a(\rho)}\} = \{\hat{u}^a, \rho_a\hat{v}^a, \rho_a^{-1}\hat{J}_a\} \ , \tag{3.55}$$

where the subscript ρ is just a label to denote the corresponding quantity depends on the parameters ρ_a. This replaces the commutator relations (3.27) - (3.29) by

$$[\hat{J}_b,\ \hat{u}^a]_{(\rho)} := A_\rho^{-1}[\hat{J}_{b(\rho)},\ \hat{u}^a_{(\rho)}] = i\rho_a^{-2}\hat{v}^a\delta^a_b \quad , \tag{3.56}$$

$$[\hat{J}_b,\ \hat{v}^a]_{(\rho)} := A_\rho^{-1}[\hat{J}_{b(\rho)},\ \hat{v}^a_{(\rho)}] = -i\hat{u}^a\delta^a_b \quad , \tag{3.57}$$

$$[\hat{u}^a,\ \hat{v}^b]_{(\rho)} = 0 = [\hat{J}_a,\ \hat{J}_b]_{(\rho)} \quad . \tag{3.58}$$

Taking the limit $\rho_a \longrightarrow \infty$ reduces the relations to

$$[\hat{J}_b,\ \hat{u}^a]_{(\infty)} = [\hat{u}^a,\ \hat{v}^b]_{(\infty)} = [\hat{J}_a,\ \hat{J}_b]_{(\infty)} = 0 \quad , \tag{3.59}$$

$$[\hat{J}_b,\ \hat{v}^a]_{(\infty)} = -i\hat{u}^a\delta^a_b \quad . \tag{3.60}$$

To illuminate these relations further, one should consider the unitary map of representation spaces,

$$W_\rho : L^2(\mathrm{T}^2, \frac{d\phi^1 d\phi^2}{4\pi^2}) \longrightarrow L^2(I_1 \times I_2, \frac{dx^1 dx^2}{4\pi^2\rho_1\rho_2}) \quad , \tag{3.61}$$

where I_a is the interval $[0,\ 2\pi\rho_a]$ and

$$(W_\rho\psi)(x^1, x^2) := \psi_\rho(\rho_1^{-1}x^1, \rho_2^{-1}x^2) \quad , \tag{3.62}$$

where $\psi \in L^2(\mathrm{T}^2)$ and $\psi_\rho \in L^2(I_1 \times I_2)$. Under such a map, the operators $\hat{J}_{a(\rho)}$, $\hat{u}^a_{(\rho)}$, $\hat{v}^a_{(\rho)}$ have the following form:

$$(W_\rho\hat{J}_{a(\rho)}W_\rho^{-1}\psi_\rho)(x^1, x^2) = -i\frac{\partial\psi_\rho}{\partial x^a}(x^1, x^2) \quad , \tag{3.63}$$

$$(W_\rho\hat{u}^a_{(\rho)}W_\rho^{-1}\psi_\rho)(x^1, x^2) = \lambda^a\cos(\rho_a^{-1}x^a)\psi_\rho(x^1, x^2) \quad , \tag{3.64}$$

$$(W_\rho\hat{v}^a_{(\rho)}W_\rho^{-1}\psi_\rho)(x^1, x^2) = \lambda^a\rho_a\sin(\rho_a^{-1}x^a)\psi_\rho(x^1, x^2) \quad , \tag{3.65}$$

where λ^a are the parameters mentioned in (3.38). Thus as $\rho \longrightarrow \infty$, $\hat{u}^a$ tends to λ^a and hence (3.60) is given by

$$[\hat{J}_b,\ \hat{v}^a]_{(\infty)} = -i\lambda^a\delta^a_b \quad , \tag{3.66}$$

which is precisely a multiple of the CCR. This justifies the earlier claim that the quantum system on T^2 locally resembles normal quantum mechanics on $\mathbb{R}^2$.

3.3 Quantisation on $Q = \mathrm{T}^2$ with a Constant Magnetic Field

The quantisation of a particle on $Q = \mathrm{T}^2$ in the previous section serves the purpose of setting up the necessary ideas needed to discuss the case of the same system but now with the inclusion of a background constant magnetic field. It also enables us to make comparisons between the two cases. Many of the points raised in the last section have either their analogues or their generalisations for the case with the magnetic field discussed in this section.[10]

Canonical Group $\tilde{\mathcal{G}} = \tilde{E}_2 \rtimes (\tilde{E}_2 \times U(1))$

The framework for the system of a particle moving on T^2 in a constant magnetic field is very much the same as the last section. The phase space $\mathcal{S}$ is the cotangent bundle $T^*\mathrm{T}^2$ but now the symplectic structure is changed from (3.1) to

$$\omega_F := \omega + F \quad , \tag{3.67}$$

where F is the field strength two-form

$$F := \sum_{a,b} \tfrac{1}{2} F_{ab}\, d\phi^a \wedge d\phi^b \quad . \tag{3.68}$$

The F_{ab}'s are the constant components of the magnetic field introduced into the system. With the same set of observables (3.2), one can use ω_F to find the corresponding new Hamiltonian vector fields to the observables. It is only (3.5) that is changed to

$$\xi_{J_a} = \frac{\partial}{\partial \phi^a} + \sum_b F_{ab} \frac{\partial}{\partial J_b} \quad ; \tag{3.69}$$

the vector fields (3.3) and (3.4) remain the same. This new set of vector fields along with ω_F form a new Poisson bracket algebra. It is found that it no longer closes because

$$\{J_a, J_b\} := \omega_F(\xi_{J_a}, \xi_{J_b}) = -F_{ab} \quad . \tag{3.70}$$

However, the vector fields still obey the same commutator algebra as in the last section, with $[\xi_{J_a}, \xi_{J_b}] = 0$. Thus, there is no isomorphism between the Poisson bracket algebra and the Lie algebra of $\mathcal{G}$. One possible way out might be to add constants to the observables (which does not affect the Hamiltonian vector fields), in the hope of restoring the isomorphism. But adding constants to J_a leaves $\{J_a, J_b\}$ unchanged, and so we cannot change (3.70).

What one therefore has to do is to replace the Lie algebra of $\mathcal{G}$ by its central extension. This is done by replacing the commutator $[\hat{J}_a, \hat{J}_b] = 0$ by the 'quantum commutator' corresponding to (3.70), namely

$$[\hat{J}_a, \hat{J}_b] = -iF_{ab}\hat{1} \quad ; \tag{3.71}$$

here F_{ab} are the central elements of the new Lie algebra and $\hat{1}$ is the identity operator. Thus one is required to replace (3.69) with some other operator, to generate the new algebra. At this stage, it is helpful to look at the problem in a different way. The insertion of a magnetic field suggests that one has a $U(1)$-gauge theory on T^2. Hence, a natural thing to do is to employ the minimal coupling rule, replacing $-i\partial/\partial\phi^a$ as the operator representation for J_a with the corresponding gauge covariant one, namely

$$\hat{J}_a = -iD_a := -i\frac{\partial}{\partial\phi^a} + A_a(\phi^1, \phi^2) \quad , \tag{3.72}$$

where $A_a(\phi^1, \phi^2)$ is the connection (gauge potential for F_{ab}) on the $U(1)$-bundle over T^2. This automatically gives the required commutator (3.71). It is useful to note that there is a Dirac quantisation condition on the total flux of the magnetic field (see Appendix); the field strength two-form is given by

$$F = \frac{m}{2\pi} d\phi^1 \wedge d\phi^2 \quad , \tag{3.73}$$

where the magnetic charge m is an integer. For calculational convenience, we will choose the gauge

$$A_1(\phi^1, \phi^2) := 0 \quad , \quad A_2(\phi^1, \phi^2) := \frac{m}{2\pi}\phi^1 \quad . \tag{3.74}$$

Given the explicit representation of $\hat{J}_a$ (3.72) with the above gauge choice, one can now construct the new extended canonical group by exponentiating the generators of the new algebra. For simplicity, let us just consider only the generators $\hat{J}_a$ of $(SO(2))_a$. These generators act on sections of a line bundle over T^2. These sections is in one-to-one correspondence with the complex-valued functions $\psi = \psi(\phi^1, \phi^2)$ which satisfy the appropriate boundary conditions given in the Appendix.

Exponentiating the action of $\hat{J}_a$ on the sections gives

$$\exp(iH^1\hat{J}_1)\psi(\phi^1, \phi^2) = \psi((\phi^1 + \eta^1) \bmod 2\pi, \phi^2) \quad , \tag{3.75}$$

$$\exp(iH^2\hat{J}_2)\psi(\phi^1, \phi^2) = e^{imH^2\phi^1/2\pi}\psi(\phi^1, (\phi^2 + \eta^2) \bmod 2\pi) \quad , \tag{3.76}$$

where $H^a \in \mathbb{R}$ is a coordinate for the Lie algebra of $(SO(2))_a$ while η^a is a coordinate on the corresponding group $(SO(2))_a$. Of particular interest are the following products of exponentiated actions of $\hat{J}_a$:

$$\begin{aligned}
&\exp(iH^1\hat{J}_1)\exp(iH^2\hat{J}_2)\psi(\phi^1,\phi^2) \\
&\quad = e^{imH^2\phi^1/2\pi}e^{imH^1H^2/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi) \quad ,
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
&\exp(iH^2\hat{J}_2)\exp(iH^1\hat{J}_1)\psi(\phi^1,\phi^2) \\
&\quad = e^{imH^2\phi^1/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi) \quad .
\end{aligned} \tag{3.78}$$

Parametrise the central extension of $SO(2) \times SO(2)$ by $(\eta^1 \bmod 2\pi, \eta^2 \bmod 2\pi, e^{imr})$, where e^{imr} is the central element. So the exponential mapping may be denoted by

$$\exp(H^1,0,0) := (\eta^1 \bmod 2\pi, 0, 1) \quad , \tag{3.79}$$

$$\exp(0,H^2,0) := (0, \eta^2 \bmod 2\pi, 1) \quad , \tag{3.80}$$

$$\exp(0,0,r) := (0,0,e^{imr/2\pi}) \quad . \tag{3.81}$$

Hence, (3.77) and (3.78) give

$$(\eta^1 \bmod 2\pi, 0, 1)(0, \eta^2 \bmod 2\pi, 1) = (\eta^1 \bmod 2\pi, \eta^2 \bmod 2\pi, 1) \quad , \tag{3.82}$$

$$(0, \eta^2 \bmod 2\pi, 1)(\eta^1 \bmod 2\pi, 0, 1) = (\eta^1 \bmod 2\pi, \eta^2 \bmod 2\pi, e^{imH^1H^2/2\pi}) \quad , \tag{3.83}$$

from which the following general product law is deduced,

$$\begin{aligned}
&(\eta^1 \bmod 2\pi, \eta^2 \bmod 2\pi, e^{imr/2\pi})(\zeta^1 \bmod 2\pi, \zeta^2 \bmod 2\pi, e^{ims/2\pi}) \\
&\quad ((\eta^1+\zeta^1) \bmod 2\pi, (\eta^2+\zeta^2) \bmod 2\pi, e^{im(H^1Z^2+r+s)/2\pi}) \quad ,
\end{aligned} \tag{3.84}$$

where Z^a are the Lie algebra parameters corresponding to group element ζ^a. However, as η^a and ζ^a are only defined mod 2π, the product H^1Z^2 in (3.84) is not well-defined. A solution to this problem is to use the covering group $\widetilde{SO(2)} = \mathbb{R}$ of $SO(2)$ as the group generated by $\hat{J}_a$, instead of $SO(2)$. Hence (3.79), (3.80) and (3.84) are defined without the 'mod 2π'. This will change the group product (3.84) to

$$\begin{aligned}
&(\eta^1, \eta^2, e^{imr/2\pi})(\zeta^1, \zeta^2, e^{ims/2\pi}) \\
&\quad = (\eta^1+\zeta^1, \eta^2+\zeta^2, e^{im(H^1Z^2+r+s)/2\pi})
\end{aligned} \tag{3.85}$$

which is well-defined. This is the product law of the group $\tilde{G} = \widetilde{SO(2)} \ltimes (\widetilde{SO(2)} \times U(1))$. One may check that the other properties of a group holds with (3.85); the identity is $(0,0,1)$ and the inverse of $(\eta^1, \eta^2, e^{imr/2\pi})$ is $(-\eta^1, -\eta^2, e^{im(H^1H^2-r)/2\pi})$.

Combining the results of section 3.2 for the other observables u^a and v^a (see (3.33) and (3.34)) with those above, one has now found the new canonical group $\tilde{\mathcal{G}}$ describing the symmetries of the new system $(\mathcal{S}, \omega_F)$, namely

$$\tilde{\mathcal{G}} = \tilde{E}_2 \ltimes (\tilde{E}_2 \times U(1)) \quad , \tag{3.86}$$

where $\tilde{E}_2$ denotes the covering group $\mathbb{R} \ltimes \mathbb{R}^2$ of E_2. Having found this, we must now find (other) unitary inequivalent representations of $\tilde{\mathcal{G}}$ and classify them. Before doing so, a useful observation to make is that the abstract group $\tilde{\mathcal{G}}$ is independent of m for $m \neq 0$, as one can rescale one of the $\widetilde{SO(2)}$ Lie algebra parameters by $\frac{1}{m}$ and still have the same group product law. The factor m will then be dropped for this reason. One can make further simplification of notation by identifying the central group element with the element of its algebra

$$(\eta^1, \eta^2, e^{imr/2\pi}) \sim (\eta^1, \eta^2, r) \quad . \tag{3.87}$$

Representations of $\tilde{\mathcal{G}}$

We will now reconstruct the representations (3.75) and (3.76) of $\tilde{\mathcal{G}}$ in a more geometrical and condensed form analogous to that given in the previous section. Consider a nontrivial $U(1)$-bundle over T^2, $U(1) \longrightarrow P_m \longrightarrow \mathrm{T}^2$, where the subscript m on P_m is just to differentiate it from the trivial $U(1)$-bundle P in Section 3.2, implying the existence of a magnetic field (nonvanishing curvature). Locally, its (measurable) section σ_m is given by $(\phi^1, \phi^2; U)$. A right $\tilde{G}$-action on T^2 is given by

$$r_{(\eta^1,\eta^2,r)}(\phi^1, \phi^2) := ((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi) \quad , \tag{3.88}$$

where the $U(1)$ subgroup has a trivial action on T^2. This action has a nontrivial lift to P_m given by

$$r^{\dagger}_{(\eta^1,\eta^2,r)}(\phi^1, \phi^2; U) := ((\phi^1 + \eta^1) \bmod 2\pi, (\phi^2 + \eta^2) \bmod 2\pi; e^{i(H^2\phi^1 + r)/2\pi} U) \quad . \tag{3.89}$$

This action can be seen to satisfy the required right-action identity

$$r^{\dagger}_{(\eta^1,\eta^2,r)} r^{\dagger}_{(\zeta^1,\zeta^2,s)} = r^{\dagger}_{(\zeta^1,\zeta^2,s)(\eta^1,\eta^2,r)} \quad . \tag{3.90}$$

This action induces a lifted $\tilde{G}$-action on the associated line bundle to P_m, $E_m = P_m \times_{U(1)} \mathbb{C}$

i.e.

$$\begin{aligned} & r^\dagger_{(\eta^1,\eta^2,r)}\Psi_m(\phi^1,\phi^2) \\ := & [r^\dagger_{(\eta^1,\eta^2,r)}\sigma_m(\phi^1,\phi^2); \psi(r_{(\eta^1,\eta^2,r)}(\phi^1,\phi^2))] \\ = & [\sigma_m((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi); \\ & e^{i(H^2\phi^1+r)/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi)] \quad , \end{aligned} \tag{3.91}$$

where $\Psi_m(\phi^1,\phi^2)$ is the local section of E_m trivialised by $[\sigma_m(\phi^1,\phi^2);\psi(\phi^1,\phi^2)]$ and ψ being a complex-valued function defined only locally on T^2 (or alternatively it obeys the boundary conditions given in the Appendix, but we will not need this information in this construction). Note that the $\tilde{G}$-action gives rise to an action of the $U(1)$-structure group of P_m and E_m on the fibres. It does leave the inner product on the fibres of E_m invariant:

$$< r^\dagger_{(\eta^1,\eta^2,r)}\psi, r^\dagger_{(\eta^1,\eta^2,r)}\psi' >_{r_{(\eta^1,\eta^2,r)}(\phi^1,\phi^2)} = < \psi,\psi' >_{(\phi^1,\phi^2)} \quad , \tag{3.92}$$

where the inner product is given by

$$< \psi,\psi' > := \int_{\mathrm{T}^2} \psi^*(\phi^1,\phi^2)\psi'(\phi^1,\phi^2)d\phi^1 d\phi^2/4\pi^2 \quad , \tag{3.93}$$

giving the required Hilbert space structure.

Equation (3.91) reproduces the representations (3.75) and (3.76) on the space of sections of E_m:

$$\begin{aligned} & (U(\eta^1,\eta^2,r)\psi)(\phi^1,\phi^2) \\ & \qquad = e^{i(H^2\phi^1+r)/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi) \quad . \end{aligned} \tag{3.94}$$

This is in fact a representation of $\tilde{G}$ induced by a unitary representation of its $U(1)$-subgroup manifested by the structure group of E_m. One now looks for other inequivalent representations of $\tilde{G}$ for quantisation on $Q = \mathrm{T}^2$, if they exist, and tries to classify them. This is done by looking at $\tilde{G}$-orbits on the space of characters $\mathrm{Char}(U(1))$ of $U(1)$.[11] A character of $U(1)$ is given by

$$\chi_k(w) := e^{-ikw/2\pi} \quad , \tag{3.95}$$

where $\chi_k \in \mathrm{Char}(U(1))$ and $w \in U(1)$ with the parameter k required to be integer. Using the isomorphism between $\mathrm{Char}(U(1))$ and the dual objects to elements of $U(1)$, the action of $\tilde{G}$ on χ_k is given by

$$\tau_{(\eta^1,\eta^2,r)}\chi_k = \chi_k \quad , \tag{3.96}$$

i.e. the $\tilde{G}$-orbits in Char($U(1)$) is just trivial. Hence, by a theorem of Mackey (see Theorem 1.12[11]), there is a one-to-one correspondence between Char($\tilde{G}$) and Char($U(1)$). This means that k characterises the inequivalent irreducible unitary representations of $\tilde{G}$. This is what one expects physically – interpreting k as the magnetic charge m which labels the different physical systems on (S, ω_F). In general, the representation of $\tilde{G}$ on sections of E_m will be

$$\begin{aligned}(U(\eta^1,\eta^2,r)\psi)(\phi^1,\phi^2)\\ = e^{im(H^2\phi^1+r)/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi) \quad ,\end{aligned} \tag{3.97}$$

where m is the magnetic charge. It is important to note that there exist other inequivalent representations of $\tilde{G}$, induced by representations of the subgroup $\mathbb{R} \times U(1)$. However, they seem physically irrelevant to our problem of quantisation on $Q = \mathrm{T}^2$ as they involve functions of only one variable.

To obtain the representation of the whole canonical group $\tilde{\mathcal{G}}$, one employs again Mackey's theory on semidirect products[11,2,3] and note that the central $U(1)$-subgroup of $\tilde{G}$ does not act on the $(\mathbb{R}^2)_a$ subgroups of $\tilde{\mathcal{G}}$. In fact, the analysis for the representations V in Section 3.2 follows here and one simply complements the representation (3.97) with the representations of $\mathbb{R}^2 \times \mathbb{R}^2$ subgroup of $\tilde{E}_2 \times \tilde{E}_2$ given by

$$\begin{aligned}(V^{(\lambda)}(m_1,n_1;m_2,n_2)\psi)(\phi^1,\phi^2)\\ = e^{i\lambda_1(M_1\cos\phi^1+N_1\sin\phi^1)+i\lambda(M_2\cos\phi^2+n_2\sin\phi^2)}\psi(\phi^1,\phi^2) \quad ,\end{aligned} \tag{3.98}$$

where (m_a, n_a) and (M_a, N_a) are respectively elements of the group $(\mathbb{R}^2)_a$ and its Lie algebra, while λ_a is the parameter characterising the $(\tilde{E}_2)_a$ orbit in Char$(\mathbb{R}^2)_a$. One easily checks that these representations do obey the required boundary conditions (see Appendix).

This completes the study of possible quantisations on (S, ω_F) (modulo θ-angles – see below).

Related Remarks on Lifting Actions

From above, we have seen the modification of the symplectic form to include an external magnetic field into the physical system significantly changes the canonical group of the case without the field to the central extension of its universal cover. Here, the central extension arises differently from the example of $Q = \mathbb{R}^n$. It arises from the nontrivial line bundle structure over the configuration space, rather than the phase space as in $Q = \mathbb{R}$.[12] The change to the universal cover is expected due to the nontrivial lifting of the G-action on T^2 (to that of $\tilde{G}$) as in Section 3.2. However, in this case the change is necessary since it is known that the $SO(2) \times SO(2)$-action on T^2 has only a lift on the product bundle.[5] Passing to the covering group is also desirable as it is related to the possibility of θ-states (see Section 3.2), given that there is now a magnetic field in the physical system. This arises when we add a constant potential to (3.74) *i.e.*

$$A_1 = m\theta_1/2\pi \quad , \quad A_2 = \tfrac{m}{2\pi}(\phi^1 + \theta_2) \quad , \tag{3.99}$$

for some constants θ_1 and θ_2. This will generalise the representation (3.97) to

$$\begin{aligned}(U(\eta^1,\eta^2,r)\psi)(\phi^1,\phi^2)\\ = e^{im(H^1\theta_1+H^2\theta_2+H^2\phi^1+r)/2\pi}\psi((\phi^1+\eta^1) \bmod 2\pi, (\phi^2+\eta^2) \bmod 2\pi) \quad .\end{aligned} \tag{3.100}$$

It is important to note that the semidirect product structure in $\tilde{G}$ seems to suggest that one of the generating cycles of T^2 is preferred over the other. However, this is only artificial since one can find another lift of $\tilde{G}$-action on P_m which would suggest a different preference of the cycles, but this lift is only cohomologous to the lift (3.89). Consider the cocycle $f((\eta^1,\eta^2,r);(\phi^1,\phi^2)) := e^{i(H^2\phi^1+r)/2\pi}$ given by the lift (3.89) where the cycle coordinatised by the angle ϕ^1 is being preferred. Define a map $\Omega : \mathrm{T}^2 \longrightarrow U(1)$ locally by

$$\Omega(\phi^1,\phi^2) := e^{i\phi^1\phi^2/2\pi} \quad . \tag{3.101}$$

Note that Ω being globally ill-defined is consistent with the fact that ψ can only be given locally by the trivialisation Ψ_m (alternatively, impose the same boundary conditions on Ω as for ψ). Using (3.47), one can define a new cocycle f' by

$$\begin{aligned}f'((\eta^1,\eta^2,r);(\phi^1,\phi^2)) :&= \Omega(r_{(\eta^1,\eta^2,r)}(\phi^1,\phi^2))^{-1}f((\eta^1,\eta^2,r);(\phi^1,\phi^2))\Omega(\phi^1,\phi^2)\\ &= e^{-i(H^1\phi^2+H^1H^2-r)/2\pi} \quad .\end{aligned} \tag{3.102}$$

The cocycle f' then defines an equivalent lifting of $\tilde{G}$-action to (3.89) but now the 'preferred' cycle is coordinatised by the angle ϕ^2. This actually follows from a theorem which states that the lifted action of $\tilde{G}$ is unique up to bundle equivalence (see Theorem 1.6[5]). One can also understand the above problem in another way by noting that the representations of $\tilde{G}$ given by f and f' are equivalent in the sense of locally operating (or vector bundle) representations.[13] This originates from the idea of gauge equivalence[14] where in the above problem, it will involve performing a singular gauge transformation to (3.74) in order to get the new representation given by cocyle f'.

Contraction of $\mathcal{L}(\tilde{\mathcal{G}})$

In the last section we have also looked into the contraction of $\mathcal{L}(\mathcal{G})$. It will be of interest to us how the new Lie algebra $\mathcal{L}(\tilde{\mathcal{G}})$ will contract and how it relates to a local picture of quantum mechanics. Consider the previous homomorphism A_ρ (3.55), but now the generators $\hat{u}^a, \hat{v}^a, \hat{J}_a$ will be of (subalgebra of) $\mathcal{L}(\tilde{\mathcal{G}})$. We only need to know how the generator $\hat{1}$ of the centre of $\mathcal{L}(\tilde{\mathcal{G}})$ is mapped under an extension $\tilde{A}_\rho$ of A_ρ. Again, one should consider the physical context of the problem. Earlier, from (3.71), one observes that the central element corresponds to the field strength tensor F. Hence its generator should carry the same dimension as F, which is $(\text{length})^{-2}$ in natural units. Thus we define $\tilde{A}_\rho$ as

$$\tilde{A}_\rho : \{\hat{u}^a, \hat{v}^a, \hat{J}_a, \hat{1}\} \mapsto \{\hat{u}^a_{(\rho)}, \hat{v}^a_{(\rho)}, \hat{J}_{a(\rho)}, \hat{1}_{(\rho)}\} := \{\hat{u}^a, \rho_a \hat{v}^a, \rho_a^{-1} \hat{J}_a, (\rho_1\rho_2)^{-1}\hat{1}\} \quad . \tag{3.103}$$

The new commutator algebra defined by $\tilde{A}_\rho$ is the same as (3.56) - (3.58) apart from the last relation, which is replaced by

$$[\hat{J}_a, \hat{J}_b]_{(\rho)} := \tilde{A}^{-1}_{(\rho)}[\hat{J}_{a(\rho)}, \hat{J}_{b(\rho)}] = F_{ab}\hat{1} \quad . \tag{3.104}$$

Taking $\rho_a \longrightarrow \infty$, will give the new contracted algebra which is:

$$[\hat{J}_a, \hat{J}_b]_{(\infty)} = F_{ab}\hat{1} \quad , \tag{3.105}$$

together with (3.59) and (3.60). These relations then give the correct commutator algebra for the system of a particle moving on $\mathbb{R}^2$ in a constant magnetic field. This is consistent from the intuitive picture that we had earlier in Section 3.2.

'Anomalous' Constants of Motion

Finally, it is interesting to note a particular feature of this example of quantisation on $(\mathcal{S}, \omega_F)$. The quantum mechanical system of a particle on T^2 in a magnetic field has been described[15] to exhibit a kind of anomaly in which the expectation value of $\hat{J}_a$ is not conserved although it does commute with the Hamiltonian (in the canonical quantisation scheme). It has been shown[16] that such phenomenon can be attributed to the fact that the momentum operator does not preserve the domain on which the Hamiltonian is hermitian. One way of understanding this problem is to realise that the canonical quantisation scheme does not respect the symmetries of the phase space. A similar kind of 'domain problem' has also been commented on by Isham[1] in one of his arguments for the group-theoretic quantisation programme (see also [17]). We shall now consider the problem in the context of this programme.

The Hamiltonian operator of this system is given by

$$\hat{H} := \tfrac{1}{2} \sum_a \hat{J}_a \hat{J}^a = -\tfrac{1}{2} \sum_a D_a D^a \quad , \tag{3.106}$$

where D_a is the covariant derivative operator representation of $\hat{J}_a$ (3.72). Its commutator with $\hat{J}_b$ is

$$[\hat{H}, \hat{J}_b] = -i \sum_a \hat{J}_a \ F^a_{\ b} \quad , \tag{3.107}$$

and hence

$$\frac{d}{dt}\hat{J}_b = i[\hat{H}, \hat{J}_b] = \sum_a \hat{J}_a \ F^a_{\ b} \neq 0 \quad . \tag{3.108}$$

Thus an anomalous situation such as in [16] does not arise. However, the fact that one cannot find a conserved momentum remains. Suppose that one attempts to redefine the operator $\hat{J}_b$ such that its commutator with $\hat{H}$ vanishes and hence obtains a new generalised momentum which is a constant of motion. This 'can' be done by defining

$$\hat{J}'_b := \hat{J}_b + \sum_c F_{bc} \hat{\phi}^c \quad , \tag{3.109}$$

where $\hat{\phi}^c$ is the multiplicative position operator on T^2. This will commute with $\hat{H}$,

$$[\hat{J}'_b, \hat{H}] = 0 \quad . \tag{3.110}$$

However, the operator $\hat{\phi}^c$ corresponding to observable ϕ^c is not well-defined and has already been ruled out by the quantisation programme. The fact that ϕ^a is not globally well-defined, has in fact been considered[18] to be the origin of the domain problem mentioned earlier. Hence, in this example, one cannot find a generalised momentum which is a constant of motion. This is to be contrasted with the case $Q = \mathbb{R}^2$ where the position operator $\hat{x}^a$ is a well-defined operator and hence $\hat{p}'_a$ defined in a similar way to (3.109) is a constant of motion. The idea of ill-defined (classical) constants of motion as in the above example leads to an interesting question whether it has any generalisation to field theory and thus possible field-theoretic 'anomalies'. This will be discussed in the next chapter on σ-models with Wess-Zumino term.

3.4 Quantisation on $Q = \mathrm{T}^n$

This section deals with the generalisation of the results of the previous sections for the n-torus ($n > 2$). The generalisation is straightforward, but the treatment given here will involve the language of lattices. The motivation of introducing this slightly different treatment is that it follows from the usual lattice construction of string in the string literature (see *e.g.* [19]). Thus the results given here will be of use in the discussion of quantisation of strings on tori[20] in Chapter 5.

The Case Without the Magnetic Field

Consider T^n, expressed as the quotient space $W/2\pi\Lambda$ where W is an n-dimensional real vector space and $\Lambda \subset W$ is the lattice

$$\Lambda := \{\sum_{i=1}^{n} n^i A_i \mid n^i \in \mathbb{Z}\} \quad ; \tag{3.111}$$

here $\{A_i\ ,\ i = 1,\ldots,n\} =: E$ is the set of basis vectors for W. The dual vector space W^* has the basis $E^* := \{B^i\ ,\ i = 1,\ldots,n\}$ such that $< B^i, A_j >= \delta^i_j$. Thus, given that the configuration space for a particle moving freely on T^n is being expressed as $Q = W/2\pi\Lambda$, the phase space is $\mathcal{S} = T^*Q = W^* \times W/2\pi\Lambda$. The set of fibre coordinates $\{J_A \mid A \in E\}$ serves as a part of the canonical observables for this system of a particle on T^n. Note that the basis of W labels the fibre coordinates on W^* (*cf.* the notation in Section 1.2). For other observables, the example of $Q = \mathrm{T}^2$ suggests the following construction adapted to the lattice structure.

As outlined in the introduction, the nonlinearity of Q necessitates its embedding in some vector space V to obtain globally well-defined $2n$ variables intrinsic to Q (*cf.* $Q =$

T^2). An element v of V shall be given by the components $\mathrm{v}(B) := (\mathrm{v}_x(B), \mathrm{v}_y(B)) \in \mathbb{R}^2$ in the direction of $B \in E^*$. The relevant observables are then given by functions of $\mathcal{S}$ taking values in V *i.e.*

$$\mathrm{v}_{(\phi)}(B) = (\mathrm{v}_{(\phi)x}(B), \mathrm{v}_{(\phi)y}(B)) := (u^B(\Phi,[\phi]), v^B(\Phi,[\phi])) \quad ; \tag{3.112}$$

$$u^B(\Phi,[\phi]) := \cos < B,\phi > \quad , \quad v^B(\Phi,[\phi]) := \sin < B,\phi > \quad , \tag{3.113}$$

where $\Phi \in W^*$ and $[\phi] \in W/2\pi\Lambda$ is the equivalence class of $\phi \in W$ modulo $2\pi\Lambda$ (*cf.* the 'angle' ϕ^a with the locally defined coordinate function ϕ^a on T^2). The natural symplectic structure on the phase space $\mathcal{S}$ is given by

$$\omega = \sum_i d\phi^{B^i} \wedge dJ_{A_i} \quad , \tag{3.114}$$

where ϕ^{B^i} is the coordinate of W in the A_i-direction (dual to the B^i-direction). Note that $d\phi^{B^i}$ is a well-defined one-form on $W/2\pi\Lambda$. Thus, the Poisson bracket algebra of observables (3.113) and J_A is

$$\{u^B, v^{B'}\} = 0 \ , \quad \{J_A, u^B\} =< B, A > v^B \ , \quad \{J_A, v^B\} = - < B, A > u^B \quad , \tag{3.115}$$

where $A \in E$ and $B, B' \in E^*$. From this algebra, the canonical group G that acts on Q is $W/2\pi\Lambda$ itself, and the full canonical group is the direct product

$$\mathcal{G} = \underbrace{E_2 \times E_2 \times \cdots \times E_2}_{n \text{ times}} \quad , \tag{3.116}$$

where E_2 is the Euclidean group in two dimensions, built out of V and $W/2\pi\Lambda$. The subgroup $W/2\pi\Lambda$ acts on V as an automorphism group in the following way:

$$\tau_{[\eta]}\mathrm{v}(B) := \begin{pmatrix} \cos < B,\eta > & -\sin < B,\eta > \\ \sin < B,\eta > & \cos < B,\eta > \end{pmatrix} \begin{pmatrix} \mathrm{v}_x(B) \\ \mathrm{v}_y(B) \end{pmatrix} \quad , \tag{3.117}$$

where $[\eta] \in W/2\pi\Lambda$ and $\mathrm{v} \in V$. The full canonical group $\mathcal{G}$ acts on $\mathcal{S}$ by

$$\tau_{([\eta],\mathrm{v})}([\phi], J) := ([\phi + \eta], J + \mathrm{v}_x(\tau_{[\eta]}\mathrm{v}_{(\phi)})_y - \mathrm{v}_y(\tau_{[\eta]}\mathrm{v}_{(\phi)})_x) \quad , \tag{3.118}$$

where $[\phi], [\eta] \in W/2\pi\Lambda$, $J \in W^*$ and $\mathrm{v}, \mathrm{v}_{(\phi)} \in V$.

The quantisation of the system involves finding irreducible unitary representations of $\mathcal{G}$ and hence the Hermitian representation of the algebra (3.115):

$$[\hat{u}^B, \hat{v}^{B\prime}] = 0 \ , \quad [\hat{J}_A, \hat{u}^B] = i < B, A > \hat{v}^B \ , \quad [\hat{J}_A, \hat{v}^B] = -i < B, A > \hat{u}^B \quad . \tag{3.119}$$

This is done in a similar way to that of Section 3.2. The classification of the representations also follows directly from that section, now involving n parameters λ_i $(i = 1, \ldots, n)$ parametrising the $W/2\pi\Lambda$-orbit on V. It is important to note that the group actions (3.117) and (3.118) also allows the use of the cover W of $W/2\pi\Lambda$ for the canonical group. Similar remarks on the universal cover also follow from Section 3.2, leading to the possibility of θ-states with n parameters θ_i.

The Case With the Magnetic Field

For the system of a particle moving on T^n in a constant magnetic field, the main difference from what is given above, lies in the usage of a modified symplectic form for $\mathcal{S}$:

$$\omega_F := \sum_{i,j=1}^{n} (d\phi^{B^i} \wedge dJ_{A_i} + F(A_i, A_j) d\phi^{B^i} \wedge d\phi^{B^j}) \quad , \tag{3.120}$$

where F can be thought as a skew-symmetric bilinear form on $W/2\pi\Lambda$ such that $F(A_i, A_j)$ are the components of the constant magnetic field in the direction of A_i and A_j of E. Using the same set of canonical observables, the Poisson bracket algebra then changes to

$$\{J_A, J_{A'}\} = -F(A, A') \qquad (A, A' \in E) \tag{3.121}$$

together with relations (3.115). The previous canonical group will now no longer work but a larger extended group is needed. Analogous to the construction in Section 3.3, one can find the global structure of the new canonical group by exponentiating the action of an operator representation of J_A on sections of a line bundle over T^n where these operators obey the 'quantum commutator' corresponding to (3.121) *i.e.*

$$[\hat{J}_A, \hat{J}_{A'}] = -iF(A, A')\hat{1} \quad . \tag{3.122}$$

The sections can be given by complex-valued wavefunctions $\psi_{(n)}$ obeying the boundary conditions given in the Appendix (*cf.* $Q = \mathrm{T}^2$). The appropriate canonical group acting on $W/2\pi\Lambda$ is then

$$\tilde{G} = (W_{\sigma(1)} \ltimes (W_{\sigma(2)} \ltimes (\cdots \ltimes (W_{\sigma(n)} \times U(1)) \cdots)) \quad , \tag{3.123}$$

where $\sigma(i)$ is the ith element in the permutation σ of $(12\ldots n)$ and W_j is the jth component of W. The order of the permutation in $\tilde{G}$ does not matter as they correspond to taking a particular representation within an equivalence class of bundle representations (*cf.* Section 3.3). The group product of $\tilde{G}$ is given by

$$\begin{aligned}&(\eta^1, \eta^2, \cdots, \eta^n, e^{ir})(\zeta^1, \zeta^2, \cdots, \zeta^n, e^{is})\\ =&(\eta^1 + \zeta^1, \eta^2 + \zeta^2, \cdots, \eta^n + \zeta^n, e^{i(r+s+\sum_{j,k=1,(j<k)}^{n} m_{jk} H^j Z^k)}) \quad ,\end{aligned} \tag{3.124}$$

where $\eta^j(\eta'^j) \in W_j$, and $H^j(H'^j)$ are the corresponding Lie algebra parameters while the exponential terms belong to the central $U(1)$-subgroup. The parameters m_{jk} are the $\frac{1}{2}n(n-1)$ integers that labels the bundle representation of $\tilde{G}$ over T^n. They correspond to the quantised flux of the constant magnetic field through closed two-dimensional submanifolds of T^n. The full canonical group is

$$\tilde{\mathcal{G}} = \tilde{E}_2 \ltimes (\tilde{E}_2 \ltimes (\cdots \ltimes (\tilde{E}_2 \times U(1)) \cdots)) \quad , \tag{3.125}$$

where $\tilde{E}_2$ is the universal cover of E_2. The representations of the subgroup V of $\tilde{\mathcal{G}}$ are still labelled by the aforementioned λ_i's.

3.5 Summary

This chapter has examined the application of group-theoretic quantisation programme to systems of a particle moving on a torus, for both cases of with and without an external background constant magnetic field. The example of T^2 is given with some elaborations. The inclusion of a magnetic field brings about significant changes to the quantisation of the system. In particular the canonical group that is used as the basis for the quantisation of the system without the magnetic field has to be changed to the central extension of its universal cover. This change can be largely understood in terms of lifting group actions from T^2 onto the line bundle over T^2 which is considered as the representation space. The inequivalent unitary representations of this new canonical group in fact correspond to the known physical systems characterised by the inequivalent bundles over T^2. Other results on how the Lie algebra of the canonical group contracts to give the algebra that describes ordinary quantum mechanics on $\mathbb{R}^2$ (for both cases), and the possibility of 'anomalous' constants of motion are also given. Some of these results are then generalised for the n-torus T^n adapted to a lattice structure.

Below is a summary of the main results of this chapter:

(*i*) The canonical group $\mathcal{G}$ for the system of a particle moving on T^2 without the magnetic field is the direct product of two Euclidean groups *i.e.* $\mathcal{G} = E_2 \times E_2$. The different quantisations of the system are given by the inequivalent irreducible unitary representations of $\mathcal{G}$ which are parametrised by positive real parameters λ_a $(a = 1, 2)$ associated to the $SO(2)$-orbits of Char($\mathbb{R}^2$) for each E_2 subgroup. The group $\mathcal{G}$ can also be replaced by its universal cover giving rise to two extra parameters θ_a known as the θ-angles.

(*ii*) For the case of the particle in a magnetic field, the canonical group has to be changed to the central extension of its universal cover *i.e.* $\tilde{\mathcal{G}} = \tilde{E}_2 \ltimes (\tilde{E}_2 \times U(1))$. The use of the universal cover is necessary in order to generate a nontrivial lift of the group action on T^2 to a nontrivial line bundle over T^2.

(*iii*) The abstract group $\tilde{\mathcal{G}}$ is independent of the magnetic charge m. The different quantisations of the system are however given by the magnetic charge as well as the other parameters λ_a and θ_a mentioned in (i) above.

(*iv*) The use of globally well-defined canonical observables $\cos \phi^a$ and $\sin \phi^a$ eliminates the possibility of anomalous constants of motion though the theory is still 'anomalous' in not possessing a conserved generalised momentum.

(*v*) For the general case of a particle moving freely on T^n, the canonical group is simply the direct product of n Euclidean groups *i.e.* $\mathcal{G} = E_2 \times E_2 \times \cdots \times E_2$. Associated to each irreducible unitary representations of $\mathcal{G}$ are n parameters λ_i $(i = 1, \ldots, n)$.

(*vi*) The canonical group $\tilde{\mathcal{G}}$ for the general case of a particle moving on T^n with a magnetic field is $\tilde{\mathcal{G}} = \tilde{E}_2 \ltimes (\tilde{E}_2 \cdots \ltimes (\tilde{E}_2 \times U(1)) \cdots))$. The representations of $\tilde{\mathcal{G}}$ are labelled by $\frac{1}{2}n(n-1)$ integers m_{ij} $(i, j = 2, \ldots, n)$ which denotes the quantised flux of the magnetic field over closed two-dimensional surfaces of T^n. Other parameters are the aforementioned λ_i and also the θ-angles θ_i.

Appendix: Dirac Quantisation on the Torus.

(i) Two-torus T^2

Consider the set of functions ψ of the angular variables of (ϕ^1, ϕ^2) of T^2. One imposes the following boundary conditions on these wavefunctions:

$$\psi(\phi^1, 2\pi) = \psi(\phi^1, 0) \quad ,$$
$$\psi(2\pi, \phi^2) = g(\phi^1, \phi^2)\psi(0, \phi^2) \quad ,$$

where $g(\phi^1, \phi^2) := \exp(-im\phi^2)$. One requires that the covariant derivative $D_a := \partial_a + iA_a$ acting on the wavefunctions obey the same conditions, in order to keep the theory gauge-invariant *i.e.*

$$(D_a\psi)(\phi^1, 2\pi) = (D_a\psi)(\phi^1, 0) \quad ,$$
$$(D_a\psi)(2\pi, \phi^2) = g(\phi^1, \phi^2)(D_a\psi)(0, \phi^2) \quad .$$

This implies that the connection must satisfy the boundary conditions

$$A_a(\phi^1, 2\pi) = A_a(\phi^1, 0) \quad ,$$
$$A_a(2\pi, \phi^2) = A_a(0, \phi^2) + i(\partial_a g)g^{-1} \quad .$$

A compatible gauge choice that corresponds to a constant gauge field strength is

$$A_1(\phi^1, \phi^2) := 0 \quad ,$$
$$A_2(\phi^1, \phi^2) := \tfrac{m}{2\pi}\phi^1 \quad .$$

The magnetic charge is

$$\frac{1}{2\pi}\int_{T^2} F = \frac{1}{2\pi}\int_{T^2} dA = m \quad ,$$

and this has to be an integer, in order that the gauge transformation function g should be well-defined.

(ii) n-torus T^n

The above quantisation condition is easily generalised to the general n-torus T^n. One uses instead the wavefunctions $\psi_{(n)}$ of angular variables ϕ^a $(a = 1, \ldots, n)$ of T^n whose boundary conditions are

$$\begin{aligned}
\psi_{(n)}(\phi^1,\cdots,\phi^{n-1},2\pi) &= \psi_{(n)}(\phi^1,\cdots,\phi^{n-1},0) \quad , \\
\psi_{(n)}(\phi^1,\cdots,2\pi,\phi^n) &= e^{-im_{2n}\phi^n}\psi_{(n)}(\phi^1,\cdots,0,\phi^n) \quad , \\
&\vdots \\
\psi_{(n)}(2\pi,\cdots,\phi^{n-1},\phi^n) &= e^{-i(m_{n2}\phi^2+\cdots+m_{nn}\phi^n)}\psi_{(n)}(0,\cdots,\phi^{n-1},\phi^n) \quad .
\end{aligned}$$

where m_{ij} $(i,j = 2,\ldots,n)$ are the $\frac{1}{2}n(n-1)$ integers generating the second homology group $H_2(\mathrm{T}^n)$ of T^n. The connection then obeys

$$\begin{aligned}
A_a(\phi^1,\cdots,\phi^{n-1},2\pi) &= A_a(\phi^1,\cdots,\phi^{n-1},0) \quad , \\
A_a(\phi^1,\cdots,2\pi,\phi^n) &= A_a(\phi^1,\cdots,0,\phi^n) + m_{2n}\delta_{an} \quad , \\
&\vdots \\
A_a(2\pi,\cdots,\phi^{n-1},\phi^n) &= A_a(0,\cdots,\phi^{n-1},\phi^n) + (m_{n2}\delta_{a2}+\cdots+m_{nn}\delta_{an}) \quad .
\end{aligned}$$

A suitable gauge choice would then be

$$\begin{aligned}
A_1(\phi^1,\cdots,\phi^n) &:= 0 \quad , \\
A_2(\phi^1,\cdots,\phi^n) &:= \tfrac{1}{2\pi}m_{n2}\phi^1 \quad , \\
&\vdots \\
A_n(\phi^1,\cdots,\phi^n) &:= \tfrac{1}{2\pi}(m_{nn}\phi^1+\cdots+m_{2n}\phi^n) \quad .
\end{aligned}$$

The quantisation condition is then

$$\frac{1}{2\pi}\int_\Sigma F = m_{ij} \ , \qquad m_{ij}\in\mathbb{Z}$$

where Σ is any generating 2-surface of T^n.

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Chapter 4

Abelian Gauge Symmetry in Sigma-Models with Wess-Zumino Term

4.1 Introduction

A relevant topic for future discussions is the Abelian gauge symmetry in nonlinear symmetry in nonlinear sigma- (σ-) models arising from the inclusion of a 'topological term' now known as the Wess-Zumino term.[1] This chapter will digress from the main topic of group-theoretic quantisation and central extensions to look into this Abelian gauge symmetry in more detail with the motivation of setting up a parallel between these σ-models and the system of a particle moving in a constant magnetic field. This will be given along with a preceding discussion of the general construction of a Wess-Zumino term in a two-dimensional Minkowskian space-time. An investigation of Noether's theorem for the σ-models with the Wess-Zumino term[2] is included to elaborate the close analogy with the case of a particle in a magnetic field. Associated global problems will also be discussed.

Sigma-models have been frequently studied in theoretical physics in various ways. It was originally studied as an effective theory of scalar mesons (see *e.g.* Chapter 5 of [3]). The σ-models however are much more frequently exploited as model field theories exhibiting rich geometrical structures.[4] In particular their supersymmetric versions have in fact been used as 'mathematical tools', for example in improving Morse inequalities[5] and rederiving index theorems.[5,7] Today, their uses are often directed to the study of conformal field theories and string theories.[8–13] In these theories, the inclusion of Wess-Zumino terms have found to be crucial. The term has been used to cure anomalies,[13] establish equivalence between bosonic theories and fermionic theories and restore conformal invariance,[9] among other things.

A *sigma-model* is defined as a set of fields $\{\Phi^i\}$ mapping a $(d+1)$-dimensional (Minkowskian) space-time into a Riemannian manifold M ($i = 1, \ldots, \dim M$). The normal kinetic energy Lagrangian density is given by

$$\mathcal{L}_0 := \tfrac{1}{2}\partial_\mu \Phi^i \partial^\mu \Phi^j g_{ij}(\Phi) \quad . \tag{4.1}$$

(Sum over repeated indices is assumed.) Note that the Greek indices are the space-time indices and the indices $i, j, \ldots$ belong to the manifold M. For the purpose of this work, the space-time is assumed to be flat and of $(1+1)$ dimensions with space coordinate x and time coordinate t. The nonlinearity of the theory comes from the Φ-dependence of the metric g that describes the geometry of M. It is important to observe that $\mathcal{L}_0$ is independent of the coordinate patches on M *i.e.*

$$\partial_\mu \Phi^i \partial^\mu \Phi^j g_{ij}(\Phi) = \partial_\mu \Phi'^i \partial^\mu \Phi'^j g'_{ij}(\Phi') \quad , \tag{4.2}$$

where

$$g'_{ij}(\Phi') = g_{kl}(\Phi) \frac{\partial \Phi^k}{\partial \Phi'^i} \frac{\partial \Phi^l}{\partial \Phi'^j} \quad . \tag{4.3}$$

In usual discussions of the σ-model, the following boundary condition is normally imposed on Φ:

$$\Phi \longrightarrow \Phi_0 \in M \qquad \text{as } |x| \longrightarrow \infty \tag{4.4}$$

for Φ_0, a point on M. This implies that Φ is basically the map

$$\Phi : S^1 \longrightarrow M \quad , \tag{4.5}$$

where S^1 is space with a distinguished point mapped to Φ_0. The configuration space of the fields is then the *loop space* ΩM[14] *i.e.* loops in M with basepoint Φ_0. Having defined the (normal) σ-model above, one can go on to construct its Wess-Zumino term in the next section.

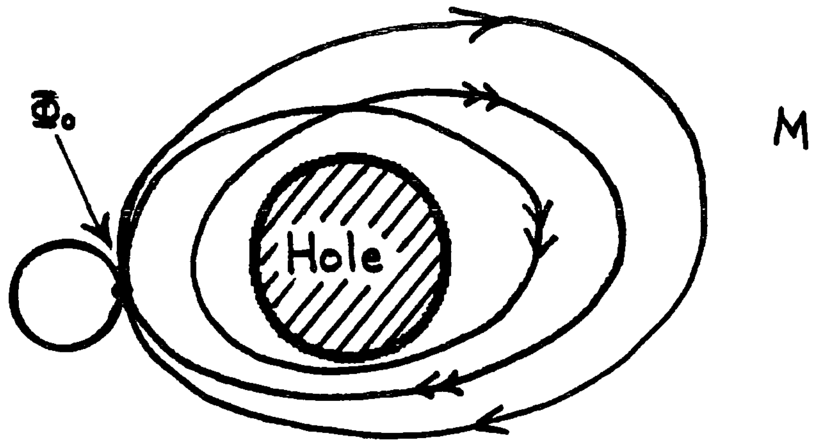


Fig.1 : Joining of loops in M.

4.2 Wess-Zumino Term

A *Wess-Zumino term* is a topological term added to the normal kinetic energy action (Lagrangian density) of the σ-model. It is topological in the sense that the physically relevant quantities derived from it are independent of deformations of the fields. In a general $(d+1)$ dimensions of space-time, the Wess-Zumino action consists of a $(d+2)$-form on M integrated over $(d+2)$-chains of M. One example would be the field strength two-form of a Dirac monopole on T^2 in the previous chapter, where $d = 0$ and $M = T^2$.

Before constructing the Wess-Zumino term, it is useful to look into the maps Φ (4.5) more closely. Its image, which shall also be called Φ, are said to be *one-cycles* of M and they can be decomposed in terms of *fundamental cycles* of M as

$$\Phi := \sum_a n_a C_a + \partial\tilde{\phi} \quad , \tag{4.6}$$

where C_a is a set of nontrivial loops generated by $\pi_1(M)$ (the first homotopy group of M) with nonzero winding numbers n_a, and the map $\tilde{\phi}$ is the extension

$$\tilde{\phi} \; : \; D^2 \longrightarrow M \quad , \tag{4.7}$$

of the map

$$\phi \; = \; \partial\tilde{\phi} \; : \; S^1 \longrightarrow M \quad , \tag{4.8}$$

where p is the boundary operator such that $\partial D^2 \; = \; S^1$ (with the distinguished point). Basically, ϕ is a map which is homotopic to the constant map. The '+' symbol in (4.6) means the joining of oriented loops at the basepoint as in the discussion of the group property of the fundamental group.[14–16] (See Fig. 1.)

In general, this (group) operation may be non-Abelian. In such a case one considers only the Abelianized version of the group. However the examples of M that shall be considered here will all have Abelian fundamental group and thus the discussion will not be pursued any further.

The time-dependence of the fields Φ is introduced by considering a family of maps $\Phi_t : S^1 \times \{t\} \longrightarrow M$, parametrised by $t \in I = [t_0, t_1] \subset \mathbb{R}$. The whole family of maps will be denoted by

$$\Phi_I : S^1 \times I \longrightarrow M \quad , \tag{4.9}$$

with

$$\Phi_I = \sum_a n_a C_{Ia} + (\partial\tilde{\phi})_I \quad , \tag{4.10}$$

where

$$\begin{aligned} \tilde{\phi}_I &: D^2 \times I \longrightarrow M \quad , \\ \phi_I = (\partial\tilde{\phi})_I &: S^1 \times I \longrightarrow M \quad . \end{aligned} \tag{4.11}$$

Note that one could take ϕ_I to be $(\partial\tilde{\phi}_I)$ where there will be an extra contribution from the endpoints of I. This amounts only to a total time derivative in the action and hence can be ignored. [Further explanations can be found in note 1 below and the discussion of gauge symmetry in the next section.]

The relevant differential form on M for the construction of the Wess-Zumino term is the three-form,

$$\Omega + d\Lambda \quad , \tag{4.12}$$

where Ω is a generator of the third cohomology group $H^3(M)$ of M, and Λ is some two form on M. The Wess-Zumino action can now be formed from (4.12) together with the decomposition (4.10) by the following functionals:

$$\Gamma[\Phi_I] := \sum_a n_a \Gamma[C_{Ia}] + \Gamma[(\partial\tilde{\phi})_I] + \Gamma'[\Phi_I] \quad , \tag{4.13}$$

where

$$\Gamma[(\partial\tilde{\phi})_I] := \int_{D^2\times I} \tilde{\phi}_I^* \Omega \quad , \tag{4.14}$$

and

$$\Gamma'[\Phi_I] := \int_{S^1 \times I} \Phi_I^* \Lambda \quad . \tag{4.15}$$

The asterisk in (4.14) and (4.15) denotes the pullback of the forms Ω and Λ by the maps $\tilde{\phi}_I$ and Φ_I respectively. The first term in equation (4.13) is an arbitrary fixed real number given by the following construction of Krichever *et al*.[17] This is done by first realising that C_a is a class of homologous one-cycles. Denote two cycles in the class by C_a and C'_a. Consider a pair (N^2, χ_a) where N^2 is a two-dimensional topological space whose boundary is $S^1 \dot{\cup} (-S^{1'})$. Here, $S^{1'}$ is a 'different' circle from S^1 and is mapped by C'_a to image of C'_a with the same basepoint as that of C_a. The map χ_a is the mapping

$$\chi_a \; : \; N^2 \longrightarrow M \quad , \tag{4.16}$$

such that

$$\chi_a \mid_{S^1} = C_a \quad , \tag{4.17}$$

$$\chi_a \mid_{S^{1'}} = C'_a \quad . \tag{4.18}$$

The pair (N^2, χ_a) can now be used to construct $\Gamma[C_{Ia}]$ globally by defining

$$\Gamma[C_{Ia}] := \int_{N^2 \times I} \chi_{Ia}^* \Omega \quad , \tag{4.19}$$

where χ_{Ia} is simply the mapping

$$\chi_{Ia} \; : \; N^2 \times I \longrightarrow M \quad . \tag{4.20}$$

It is now important to note the following:

1. The form Ω has been assumed to be closed. This is necessary to make the functional (4.14) a topological action *i.e.* to be independent of the way in which ϕ is extended to $\tilde{\phi}$. This can be seen as follows. (The same can be done for the functional (4.19) simply by replacing ϕ and $\tilde{\phi}$ by C_a and χ_a respectively.) Let $\tilde{\phi}_I$

and $\tilde{\phi}'_I$ be two different extensions of ϕ_I. If $\pi_2(M) = 0$, then a homotopy

$$h \; : \; D^3 \times I \longrightarrow M \tag{4.21}$$

from $\tilde{\phi}_I$ to $\tilde{\phi}'_I$ always exists where D^3 is a 3-disk (see Fig. 2) and

$$h \,|_{S\times I} = \tilde{\phi}_I \quad , \tag{4.22}$$

$$h \,|_{N\times I} = \tilde{\phi}'_I \quad , \tag{4.23}$$

$$h \,|_{E\times I} = \phi_I \quad . \tag{4.24}$$

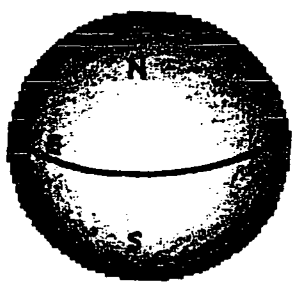


Fig. 2 : D^3 (a solid ball) divided by equator E giving two hemispheres N and S.

If Ω is closed, then

$$\begin{aligned} 0 &= \int_{D^3\times I} h^* d\Omega \\ &= \int_{\partial(D^3\times I)} h^*\Omega \\ &= \int_{N\times I} \tilde{\phi}'^*_I\Omega - \int_{S\times I} \tilde{\phi}^*_I\Omega + \int_{D^2} h^*\Omega \,|_{t=t_1} - \int_{D^2} h^*\Omega \,|_{t=t_0} \quad . \end{aligned} \tag{4.25}$$

Note that the last two terms combine to give a total time derivative under integral $\int dt$. Thus they may be ignored as they do not contribute to the dynamics. (Equivalently one may use the gauge freedom discussed in Section 4.3 to gauge them away.) Hence the Wess-Zumino action is independent of extensions $\tilde{\phi}_I$ (or χ_{Ia}) modulo endpoint contributions and thus is well-defined as a part of a physical action. For those manifolds M with $\pi_2(M) \neq 0$, there is an ambiguity in the possible extensions of ϕ_I (or C_{Ia}). The illustration above then implies the extensions $\tilde{\phi}_I$ and $\tilde{\phi}'_I$ are no longer homotopic to each other and hence are inequivalent. To resolve this problem one requires an extra datum to make the Wess-Zumino action well-defined. An example of such a case is given in Section 4.6 but the discussion concerning the ambiguity however shall be deferred to Section 4.7 for ease of explanation.

2. Most of the time the attention is drawn only to the functional (4.14) of the whole Wess-Zumino action. This functional in fact corresponds to the usual definition of the Wess-Zumino action which is independent of any deformations of $\tilde{\phi}_I$ in M. The functional (4.15) is uninteresting since it can be written consistently and globally as an integral over space-time without any difficulty. With this term in mind, one can always add further exact forms to Ω with their integrals contributing only to integral (4.15). Hence to define (4.14) one uses only the generators of the third cohomology group $H^3(M)$ of M. Thus there are $b_3(M) = \dim H^3(M)$ independent Wess-Zumino actions from this functional. The functional $\Gamma[C_{Ia}]$ will be treated as fixed numbers given by (4.19). The contributions from the fields C_a shall in fact be ignored later.

3. This construction of the Wess-Zumino action is different from its usual construction *e.g.* that of Braaten *et al.*[18,19] Here, the normal construction involving Euclideanised space-time is avoided by using a time-parametrised family of maps Φ. An important consequence is that the fields Φ_I need no longer be cycles of M but are general two-chains on M.

To write the functionals (4.14) and (4.19) in the usual fashion of the integral of the Lagrangian density, we use the Poincare Lemma[14] to write Ω as an exact form in some local patch of M,

$$\Omega = d\omega \quad . \tag{4.26}$$

The integral (4.14) may now be written as

$$\begin{aligned}\Gamma[(\partial\tilde{\phi})_I] &= \int\limits_{D\times I} \tilde{\phi}_I^* d\omega \\ &= \int\limits_{S^1\times I} \phi_I^*\omega + \{\text{total time derivatives}\} \\ &= \int\limits_{S^1\times I} dt\ dx\{\epsilon^{\mu\nu}\partial_\mu\phi_I^j\partial_\nu\phi_I^k\omega_{jk}(\phi)\} \quad ,\end{aligned} \tag{4.27}$$

where $\omega(\phi)$ has singularities in ϕ. For the integral (4.19) there seems to be no consistent way of writing it locally. This is due to the problem associated with the gauge transformations belonging only to trivial winding number sector of the fields Φ (see next section). To avoid a cumbersome notation, the subscript I will be dropped from now on.

4.3 Abelian Gauge Symmetry

It has long been noted that there are similarities between the system of a particle in the presence of Dirac monopole and the σ-model which includes the topological Wess-Zumino term.[20−22] The similarity in terms of the presence of a line bundle structure over the configuration space is also well-known.[23] The Abelian gauge symmetry in the σ-model with a Wess-Zumino term is first observed explicitly by Wu and Zee[1] through this analogy with the particle case. Here, the symmetry is made much more explicit by the interpretation of the σ-model as a system of a particle moving in a constant magnetic field on an infinite-dimensional configuration space ΩM.

First, consider the kinetic energy Lagrangian from $\mathcal{L}_0$ in (4.1),

$$L_0 = \int\limits_{S^1} dx\{\tfrac{1}{2}\partial_\mu\Phi^i\partial^\mu\Phi^j g_{ij}(\Phi)\} \quad . \tag{4.28}$$

One can make simplifications to L_0 when one realises that the fields C_a are linear functions of x (generating the nontrivial winding numbers). Furthermore it may be made independent of t owing to the topological nature of the fields C_a. Thus derivatives of the fields C_a in the kinetic term merely add constants to the kinetic term involving ϕ's (the cross terms in (4.28) may be integrated out) and therefore the contributions from the fields C_a will be ignored for the rest of this chapter. Equations (4.28) together with

(4.27) form the total Lagrangian density (modulo constants)

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi^j)(\partial^\mu \phi^k) g_{jk}(\phi) + \epsilon^{\mu\nu}(\partial_\mu \phi^j)(\partial_\nu \phi^k)\omega_{jk}(\phi) \quad . \tag{4.29}$$

The correspondence between the σ-model and the system of a particle in a magnetic field[1] can be observed by first writing down the conjugate momenta to $\phi^j(x)$ from equation (4.29) :

$$\pi_j(x) := \frac{\delta \mathcal{L}}{\delta \dot{\phi}^j(x)} = \dot{\phi}^k(x) g_{jk}(\phi) + 2\partial_x \phi^k(x)\omega_{jk}(\phi) \quad . \tag{4.30}$$

Note that the second term provides an analogue of the gauge potential A_j (*cf.* $p_j = \dot{q}_j + A_j(q)$) in the space of field configurations ΩM *i.e.*

$$\mathcal{A}_j := 2\partial_x \phi^k(x)\omega_{jk}(\phi) \quad . \tag{4.31}$$

To relate $\mathcal{A}_j$ to topological properties of the configuration space, one must write it as a differential form on ΩM (*cf.* $A = A_j dx^j$). It is then useful to digress briefly and describe the relevant ideas of differential geometry on ΩM.

A vector field on ΩM is intuitively given by an infinitesimal deformation of based-point loops in M. Thus the set of basis vectors at each point of ΩM may be denoted by $\{\delta\phi^j\}$. To describe a one-form on ΩM, it is convenient to observe that $\delta\phi^j$ evaluated at x gives a real number $\delta\phi^j(x)$. Thus $\delta\phi^j(x)$ is likened to be an object that sends $\delta\phi^j$ to the real numbers which is precisely the definition of a one-form. Hence denote the basis of one-forms on ΩM as $\{\delta\phi^j(x)\}$.[24,25] These objects anticommute (resembling the wedge product of ordinary one-forms), *i.e.*

$$\delta\phi^j(x)\delta\phi^k(x') = -\delta\phi^k(x')\delta\phi^j(x) \quad . \tag{4.32}$$

The exterior derivative is denoted by δ and it obeys

$$\delta^2 = 0 \quad . \tag{4.33}$$

The object $\phi^j(x)$ is considered to be the zero-form that sends the function ϕ^j to the number $\phi^j(x)$ for each x. This implies that $\phi^j(x)$ is a closed one-form, *i.e.* it obeys

$$\delta(\delta\phi^j(x)) = 0 \quad . \tag{4.34ϕ}$$

A general k-form $\mathcal{K}$ may be written as

$$\mathcal{K} := \int\limits_{S^1} dx^{(1)} \int\limits_{S^1} dx^{(2)} \cdots \int\limits_{S^1} dx^{(k)} \mathcal{K}_{i_1 i_2 \ldots i_k}\big(\phi(x^{(1)}), \ldots, \phi(x^{(k)})\big) \delta\phi^{i_1}(x^{(1)}) \ldots \delta\phi^{i_k}(x^{(k)}) \quad , \tag{4.35}$$

where the bracketed indices on the x's are just labels for different x's and $\mathcal{K}_{i_1 i_2 \ldots i_k}$ is a functional of ϕ at the various points. Note that apart from the usual summation convention over the indices i_1, i_2,...., i_k, there is also a 'summation' over the different x's denoted by the integral sign, showing the infinite dimension of ΩM. It is important to note that the forms that are to be considered here is only a subclass of such general forms namely those in which the functional $\mathcal{K}_{i_1 i_2 \ldots i_k}$ is a functional of ϕ at one point x. This is because such functionals will be appearing in the Lagrangian for the theory and hence required to be local. Lastly, given a vector $\delta\phi^j = v^j$ on ΩM, its contraction with $\mathcal{K}$ is given by

$$v \lrcorner \mathcal{K} = \sum_{i_j} \int\limits_{S^1} dx^{(1)} \int\limits_{S^1} dx^{(2)} \cdots \int\limits_{S^1} dx^{(k)} v^{i_j} \mathcal{K}_{i_1 i_2 \ldots i_k} \delta\phi^{i_1}(x^{(1)}) \ldots \widehat{\delta\phi^{i_j}}(x^{(j)}) \ldots \delta\phi^{i_k}(x^{(k)}) \quad , \tag{4.36}$$

where the careted symbol is missing from the given expression.

Returning to the discussion of $\mathcal{A}_j$, it can now be understood as the components of the gauge potential one-form

$$\begin{aligned} \mathcal{A}[\delta\phi^j] :&= \int\limits_{S^1} dx \big\{ 2\partial_x \phi^k(x) \omega_{jk}(\phi) \delta\phi^j(x) \big\} \\ &= \int\limits_{S^1} dx \big\{ \mathcal{A}_j(\phi(x)) \delta\phi^j(x) \big\} \quad . \end{aligned} \tag{4.37}$$

The field strength two-form on ΩM (*cf.* $F = dA$) can be obtained by applying the exterior derivative δ at point x, *i.e.*

$$\begin{aligned}\mathcal{F}[\delta\phi^l,\delta\phi^j] &:= \delta(\mathcal{A}[\delta\phi^j])[\delta\phi^l] \\ &= \int_{S^1} dx\{2(\partial_x\delta\phi^k(x))\omega_{jk}\delta\phi^j(x) + 2(\partial_x\phi^k)\omega_{jk,l}\delta\phi^l(x)\delta\phi^j(x)\} \\ &= \int_{S^1} dx\{3(\partial_x\phi^k)\omega_{jk,l}\delta\phi^l(x)\delta\phi^j(x)\} \quad ,\end{aligned} \tag{4.38}$$

where $\omega_{jk,l}$ denotes the derivative of ω_{jk} with respect to the field ϕ^l. In deriving equation (4.38), the periodicity of ϕ^j in x and the following symmetries of ω have been used:

$$\omega_{jk} = -\omega_{kj} \qquad ; \qquad \omega_{jk,l} = -\omega_{jl,k} \quad . \tag{4.39}$$

Note that $\mathcal{F}$ corresponds to a constant gauge field strength on ΩM where for each indices j,k,l as well as 'x' in (4.38) there is an assigned 'constant'. The Wess-Zumino action can now be written in terms of forms on ΩM:

$$\begin{aligned}\Gamma[(\partial\tilde{\phi})_I] &= \int dt \int_{S^1} dx\{2\dot{\phi}^j(x)(\partial_x\phi^k(x))\omega_{jk}(\phi)\} \\ &= \int dt \int_{S^1} dx\{\dot{\phi}^j(x)\mathcal{A}_j(\phi)\} \\ &= \int_{S^1} dx \int dt\{\frac{\partial\phi^j}{\partial t}\mathcal{A}_j(\phi)\} \\ &= \int_{S^1} dx \int_\gamma \delta\phi^j(x)\mathcal{A}_j(\phi) \\ &= \int_\gamma \mathcal{A} \quad ,\end{aligned} \tag{4.40}$$

where γ is the 'path traversed' in ΩM. A more useful form will be

$$\Gamma[(\partial\tilde{\phi})_I] = \int dt\mathcal{A}[\dot{\phi}] \quad , \tag{4.41}$$

treating $\dot{\phi}$ as a vector at each point of ΩM and contracting it with $\mathcal{A}$ (*cf.* $\int dtA_i\dot{x}^i$).

In order to describe the gauge symmetry, one must obtain the analogue of a gauge transformation of $\mathcal{A}$ *i.e.*

$$\mathcal{A} \longrightarrow \mathcal{A}' = \mathcal{A} + \delta \Lambda \quad , \tag{4.42}$$

for some zero-form Λ on ΩM, and check that the objects defined on ΩM have the correct gauge symmetry properties. From the form of the Wess-Zumino Lagrangian, $\delta\Lambda$ is required to be

$$\delta\Lambda := \int_{S^1} dx \{ 2\partial_x \phi^k \alpha_{k,j} \delta\phi^j(x) \} \quad , \tag{4.43}$$

for some functional one-form $\alpha(\phi)$ on M. This implies that

$$\Lambda = \int_{S^1} dx \{ \partial_x \phi^k(x) \alpha_k(\phi) \} \quad . \tag{4.44}$$

In deriving equation (4.44) from (4.43), it is necessary that Λ is a functional of fields only from the trivial sector, namely $\phi^j(x)$. This means that the gauge transformation for the whole action comes from the trivial sector. This is precisely the reason why (4.19) cannot be put in a Lagrangian form, as it depends on the choice of functional $\omega(C_a)$ (*i.e.* it is no longer gauge (quasi-) invariant). It is easy to check that $\mathcal{F}$ stays invariant under the transformation (4.42) :

$$\begin{aligned} \mathcal{F} \longrightarrow \mathcal{F}' &= \int_{S^1} dx \{ 3(\omega_{jk,l} + \alpha_{k,jl})(\partial_x \phi^k) \delta\phi^l(x) \delta\phi^j(x) \} \\ &= \int_{S^1} dx \{ 3\omega_{jk,l} (\partial_x \phi^k) \delta\phi^l(x) \delta\phi^j(x) \} = \mathcal{F} \quad . \end{aligned} \tag{4.45}$$

The Wess-Zumino action is also well-defined under transformation (4.42) as it changes by a total time derivative given by

$$\begin{aligned} \int dt \int_{S^1} dx \{ \epsilon^{\mu\nu} \partial_\mu \phi^j \partial_\nu \phi^k \alpha_{j,k} \} &= \int dt \int_{S^1} dx \{ \partial_\nu (\epsilon^{\mu\nu} \partial_\mu \phi^j \alpha_j) \} \\ &= \int dt \, \frac{\partial}{\partial t} (- \int_{S^1} dx \{ \partial_x \phi^j \alpha_j \}) \quad . \end{aligned} \tag{4.46}$$

(*cf.* $I_{\text{int}} := \int dt (A_i \dot{x}^i) \to I_{\text{int}} + \int dt (\partial_i \Lambda \, \dot{x}^i) = I_{\text{int}} + \int dt \, \frac{d\Lambda}{dt}$). This can be ignored as it does not contribute to the dynamics of the theory.

4.4 Noether's Theorem and Constants of Motion

With the results of the previous section, the Wess-Zumino action can be clearly interpreted as the interacting part of a total action for a 'particle' in a background 'magnetic field' on ΩM exhibiting the appropriate gauge symmetries. In this section, the space-time symmetries of the σ-model will be examined. Like any other system in an external background gauge field, one expects that the Noether's theorem gives constants of motion modified by a contribution from the 'background field'.[26] This further elaborates the particle analogy.

Consider the previous Lagrangian density (4.29) as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{WZ}} \quad ,$$

where

$$\mathcal{L}_0 := \tfrac{1}{2}(\partial_\mu \phi^j)(\partial^\mu \phi^k) g_{jk}(\phi) \quad , \tag{4.47}$$

$$\mathcal{L}_{\mathrm{WZ}} := \epsilon^{\mu\nu}(\partial_\mu \phi^j)(\partial_\nu \phi^k)\omega_{jk}(\phi) \quad . \tag{4.48}$$

Let ϕ^j transform as

$$\phi^j \longrightarrow \phi^j + \delta\phi^j \quad ,$$

$$\text{with} \qquad \delta\phi^j = v^j \quad . \tag{4.49}$$

First, let $\mathcal{L}_0$ be invariant under this transformation

$$\begin{aligned} \delta\mathcal{L}_0 &= \tfrac{1}{2}(\partial_\mu v^j \partial^\mu \phi^k g_{jk} + \partial_\mu \phi^j \partial^\mu v^k g_{jk} + \partial_\mu \phi^j \partial^\mu \phi^k g_{jk,l} v^l) \\ &= \tfrac{1}{2}(\partial_\mu \phi^l \partial^\mu \phi^k)(v^j{}_{,l} g_{jk} + v^j{}_{,k} g_{lj} + g_{lk,j} v^j) = 0 \quad . \end{aligned}$$

This implies

$$v^j{}_{,l} g_{jk} + v^j{}_{,k} g_{lj} + g_{lk,j} v^j = 0 \quad , \tag{4.50}$$

i.e. v must be a Killing vector on (M, g).

To see how $\mathcal{L}_{\rm WZ}$ responds to transformations (4.49) and in particular to see when they are symmetry transformations, it is important to recall that the Wess-Zumino action may be written in terms of a gauge potential one-form (see (4.41)). Interpreting $\delta\phi^j$ as a vector field on ΩM, the gauge potential one-form $\mathcal{A}$ transforms under (4.49) in a way given by its Lie derivative with respect to $\delta\phi = v$, *i.e.*

$$\mathcal{A} \longrightarrow \mathcal{A}' = \mathcal{A} + \pounds_v \mathcal{A} \quad . \tag{4.51}$$

Thus to make $\mathcal{L}_{\rm WZ}$ invariant one can impose $\pounds_v\mathcal{A} = 0$, but note that we can use the gauge freedom to modify this into

$$\pounds_v \mathcal{A} = \delta \mathcal{W}_v(\phi) \quad , \tag{4.52}$$

for some scalar $\mathcal{W}_v(\phi)$ on ΩM. From equation (4.46) this is equivalent to the condition that $\mathcal{L}_{\rm WZ}$ may change by a total time derivative. The change in $\mathcal{L}_{\rm WZ}$ under transformation (4.49) is explicitly given by

$$\begin{aligned} \delta\mathcal{L}_{\rm WZ} &= 2\epsilon^{\mu\nu}(\partial_\mu v^j)(\partial_\nu\phi^k)\omega_{jk} + \epsilon^{\mu\nu}(\partial_\mu\phi^j)(\partial_\nu\phi^k)\omega_{jk,l}v^l \\ &= \epsilon^{\mu\nu}(\partial_\mu\phi^j)(\partial_\nu\phi^k)(2\omega_{lk}v^l{}_{,j} + \omega_{jk,l}v^l) \quad . \end{aligned} \tag{4.53}$$

This can be set to equal the total derivative

$$\partial_\mu(\epsilon^{\mu\nu}\phi^j(\partial_\nu\phi^k)(2\omega_{lk}v^l{}_{,j} + \omega_{jk,l}v^l)) \quad , \tag{4.54}$$

provided that the following condition holds (*cf.* [27]):

$$2\partial_{[m}\omega_{k]l}v^l{}_{,j} + \partial_{[m}\omega_{k]j,l}v^l = 0 \quad . \tag{4.55}$$

By using symmetries in (4.39) and from equation (4.46), a sufficient condition for equation (4.52) to hold is then

$$\partial_m(2\omega_{kl}v^l{}_{,j} + \omega_{kj,l}v^l) = 0 \quad . \tag{4.56}$$

Note that in general, condition (4.56) is not true. For such cases it is necessary to treat equation (4.53) case by case for different M and ω (one such case is our second example in Section 4.6). For simplicity, we shall assume equation (4.56) in order to illustrate the point of the modified constants of motion.

Now the Lie derivative of $\mathcal{A}$ can also be expressed formally using the homotopy formula $\pounds_v(\cdot) := \delta(v \lrcorner \cdot) + v \lrcorner \delta(\cdot)$, in particular

$$\begin{aligned}\pounds_v \mathcal{A} &= \delta(v \lrcorner \mathcal{A}) + v \lrcorner (\delta \mathcal{A}) \\ &= \delta(\mathcal{A}[v]) + v \lrcorner \mathcal{F} \quad .\end{aligned} \tag{4.57}$$

Comparing equation (4.57) with equation (4.52) implies a new condition :

$$v \lrcorner \mathcal{F} = -\delta \psi_v \quad , \tag{4.58}$$

for some scalar ψ_v on ΩM with

$$\mathcal{W}_v = \mathcal{A}[v] - \psi_v \tag{4.59}$$

(*cf.* equation (1.8) in [26]). Of relevance to the discussions on constants of motion is the gauge invariant object ψ_v. From equation (4.58) note that ψ_v is globally well-defined only if $v \lrcorner \mathcal{F}$ is exact. Note that $v \lrcorner \mathcal{F}$ is necessarily closed since $\mathcal{F}$, being gauge invariant, must be invariant under the symmetry transformation (4.49) , *i.e.*

$$\pounds_v \mathcal{F} = \delta(v \lrcorner \mathcal{F}) = 0 \quad . \tag{4.60}$$

Thus $v \lrcorner \mathcal{F}$ belongs to the first cohomology class of ΩM. A sufficient condition for a globally well-defined ψ_v is then

$$H^1(\Omega M) = 0 \quad . \tag{4.61}$$

This is always the case for simply connected configuration spaces, *i.e.*

$$\pi_1(\Omega M) \cong \pi_2(M) = 0 \quad . \tag{4.62}$$

For other spaces, however, there is the possibility of $v \lrcorner \mathcal{F}$ being closed but nonexact and thus equation (4.58) is only true locally.

In addition to equation (4.58), one can also obtain another equation for ψ_v, involving further contraction of $\mathcal{F}$ with $v = [w, u]$ for some vector fields w, u on ΩM, namely

$$\psi_{[w,u]} = \mathcal{F}[w, u] \tag{4.63}$$

(*cf.* equation (1.14*b*) in [26]).

Proof : Consider the following identity

$$\pounds_w \pounds_u \mathcal{A} - \pounds_u \pounds_w \mathcal{A} = \pounds_{[w,u]} \mathcal{A} = \delta \mathcal{W}_{[w,u]} \quad . \tag{4.64}$$

The left-hand side of equation (4.64) gives

$$\begin{aligned} \pounds_w(\delta \mathcal{W}_u) - \pounds_u(\delta \mathcal{W}_w) &= \delta(w \lrcorner \delta \mathcal{W}_u) - \delta(u \lrcorner \delta \mathcal{W}_w) \\ &= \delta(\pounds_w \mathcal{W}_u - \pounds_u \mathcal{W}_w) \quad . \end{aligned} \tag{4.65}$$

The right-hand side of equation (4.64) with equation (4.65) gives the identity

$$\mathcal{W}_{[w,u]} = \pounds_w \mathcal{W}_u - \pounds_u \mathcal{W}_w \quad . \tag{4.66}$$

Using equation (4.59) in equation (4.66) one can obtain

$$\begin{aligned} \mathcal{A}\big[[w,u]\big] - \psi_{[w,u]} &= \pounds_w \mathcal{A}[u] - \pounds_w \psi_u - \pounds_u \mathcal{A}[w] + \pounds_u \psi_w \\ &= w(\mathcal{A}[u]) - (w \lrcorner \delta\psi_u) - u(\mathcal{A}[w]) + (u \lrcorner \delta\psi_w) \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \psi_{[w,u]} = u(\mathcal{A}[w]) - w(\mathcal{A}[u]) &+ \mathcal{A}\big[[w,u]\big] \\ &- u \lrcorner (\delta\psi_w) + w \lrcorner (\delta\psi_u) \quad . \end{aligned} \tag{4.67}$$

Using equation (4.59) with the identity

$$\mathcal{F}[u,w] = u(\mathcal{A}[w]) - w(\mathcal{A}[u]) + \mathcal{A}\big[[w,u]\big] \quad , \tag{4.68}$$

in equation (4.67) will now give the desired identity,

$$\psi_{[w,u]} = \mathcal{F}[u,w] + \mathcal{F}[w,u] - \mathcal{F}[u,w] = \mathcal{F}[w,u] \quad . \qquad \square$$

A useful computation is that of $\pounds_v \mathcal{A}$ using equations (4.57) and (4.37) :

$$\begin{aligned} \pounds_v \mathcal{A} &= \int_{S^1} dx \big\{ \delta(2(\partial_x \phi^k)\omega_{jk} v^j) + 3(\partial_x \phi^k)\omega_{jk,l} v^l \delta\phi^j(x) - 3(\partial_x \phi^k)\omega_{jk,l} v^j \delta\phi^l(x) \big\} \\ &= \int_{S^1} dx \big\{ 2\partial_x(\delta\phi^k(x)\omega_{jk} v^j) - 2\delta\phi^k(x)\omega_{jk,l} v^j(\partial_x \phi^l) - 2\delta\phi^k(x)\omega_{jk} v^j{}_{,l}(\partial_x \phi^l) \\ &\qquad + 2(\partial_x \phi^k)\omega_{jk} v^j{}_{,l}\delta\phi^l(x) + 4(\partial_x \phi^k)\omega_{jk,l} v^l \delta\phi^j(x) \big\} \\ &= \int_{S^1} dx \big\{ 2\omega_{jk} v^j{}_{,l}(\delta\phi^l(x)\partial_x \phi^k - \delta\phi^k(x)\partial_x \phi^l) + 2(\partial_x \phi^k)\omega_{jk,l} v^l \delta\phi^j(x) \big\} \\ &= \int_{S^1} dx \big\{ (\partial_x \phi^k \delta\phi^l(x) - \delta\phi^k(x)\partial_x \phi^l)(2\omega_{jk} v^j{}_{,l} + \omega_{lk,j} v^j) \big\} \quad . \end{aligned} \tag{4.69}$$

This is, in fact, consistent with the change in $\mathcal{L}_{WZ}$ when the action is written in terms of $\mathcal{A}$:

$$\begin{aligned}\int_{S^1} dx \;\; \delta\mathcal{L}_{WZ} &= \pounds_v \mathcal{A}[\dot{\phi}] \\ &= \int_{S^1} dx \big\{ (\dot{\phi}^l \partial_x \phi^k - \dot{\phi}^k \partial_x \phi^l)(2\omega_{jk} {v^j}_{,l} + \omega_{lk,j} v^j) \big\} \\ &= \int_{S^1} dx \big\{ \epsilon^{\mu\nu} \partial_\mu \phi^l \partial_\nu \phi^k (2\omega_{jk} {v^j}_{,l} + \omega_{lk,j} v^j) \big\} \quad .\end{aligned}$$

Given the above results one can now discuss conserved currents and hence the associated constants of motion with respect to the symmetry transformations (4.49). For completeness, the following standard discussion of Noether's theorem is included. A Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ under the transformation (4.49) changes (without using equations of motion) as

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu \quad , \tag{4.70}$$

for some K^μ. With the equations of motion, it transforms as

$$\begin{aligned}\mathcal{L} \longrightarrow \mathcal{L}' &\approx \mathcal{L} + \partial_\mu \delta\phi^j \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^j)} \right) + \delta\phi^j \left(\frac{\delta \mathcal{L}}{\delta \phi^j} \right) \\ &= \mathcal{L} + \partial_\mu \left(\delta\phi^j \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^j)} \right) \quad .\end{aligned} \tag{4.71}$$

Thus a conserved current can be constructed from the identity

$$\partial_\mu K^\mu = \partial_\mu \left(\delta\phi^j \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^j)} \right) \quad , \tag{4.72}$$

namely,

$$J^\mu = \delta\phi^j \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^j)} - K^\mu \quad . \tag{4.73}$$

From the total Lagrangian density, (4.47) with (4.48), K^μ is given by

$$K^\mu = \epsilon^{\mu\nu} \phi^j (\partial_\nu \phi^k)(2\omega_{lk} {v^l}_{,j} + \omega_{jk,l} v^l) \tag{4.74}$$

(see (4.53)). Hence the current J^μ is

$$\begin{aligned}J^\mu = \partial^\mu \phi^j g_{jk}(\phi) v^k &+ 2\epsilon^{\mu\nu} \partial_\nu \phi^k \omega_{lk} v^l \\ &- \epsilon^{\mu\nu} \phi^j (\partial_\nu \phi^k)(2{v^l}_{,j} \omega_{lk} + \omega_{jk,l} v^l) \quad .\end{aligned} \tag{4.75}$$

One can verify using the equations of motion that

$$\partial_\mu J^\mu = 0 \quad .$$

Hence one can construct constants of motion C_v, out of the time component of J^μ such that

$$\frac{\partial C_v}{\partial t} = \int_{S^1} dx\, \frac{\partial J^0}{\partial t} = -\int_{S^1} dx\, \frac{\partial J^1}{\partial x} = 0 \quad .$$

Computation of C_v from equation (4.75) gives

$$C_v = \int_{S^1} dx \{ \dot{\phi}^k g_{jk} v^j + 2\partial_x \phi^k \omega_{jk} v^j - \phi^j (\partial_x \phi^k)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) \} \quad . \tag{4.76}$$

The main point now is to understand what the terms in equation (4.76) mean. First, compute the exterior derivative of the last term:

$$\begin{aligned}
&\int_{S^1} dx \{ \delta(\phi^j (\partial_x \phi^k)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l)) \} \\
&= \int_{S^1} dx \{ \delta\phi^j(x)(\partial_x \phi^k)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) + \phi^j (\partial_x \delta\phi^k(x))(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) \\
&\qquad + \phi^j(\partial_x \phi^k)\partial_m(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l)\delta\phi^m(x) \} \\
&= \int_{S^1} dx \{ \delta\phi^j(x)(\partial_x \phi^k)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) + \partial_x (\phi^j \delta\phi^k(x)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l)) \\
&\qquad - (\partial_x \phi^j)\delta\phi^k(x)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) - \phi^j \delta\phi^k(x)\partial_m(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l)(\partial_x \phi^m) \} \\
&= \int_{S^1} dx \{ (2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l)(\delta\phi^j(x)\partial_x \phi^k - \delta\phi^k(x)\partial_x \phi^j) \} \\
&= \mathcal{L}_v \mathcal{A} \quad ,
\end{aligned} \tag{4.77}$$

where equations (4.56) and (4.69) and the fact that ϕ^j's are periodic functions of x have been used. Hence equations (4.77) and (4.56) imply that the third term in (4.76) is simply

$$\mathcal{W}_v = \int_{S^1} dx \{ \phi^j (\partial_x \phi^k)(2v^l{}_{,j}\omega_{lk} + \omega_{jk,l} v^l) \} \quad . \tag{4.78}$$

The second term in (4.76) is straightforwardly given by $\mathcal{A}[v]$, while the first term is just the normal contribution from the kinetic term. Writing the first term as C_{v_0}, one has

$$\begin{aligned} C_v &= C_{v_0} + \mathcal{A}[v] - \mathcal{W}_v \\ &= C_{v_0} + \psi_v \quad . \end{aligned} \tag{4.79}$$

Thus one finds that the normal constant of motion C_{v_0} is supplemented by ψ_v, the contraction of the field strength $\mathcal{F}$ with the Killing vector field v. This justifies the earlier claim that Noether's theorem gives a modified constant of motion which includes a contribution from the background field.

Having obtained these results, specific examples of σ-models with Wess-Zumino term will now be used to illustrate the above results on modified constants of motion.

4.5 σ-Model on $M = T^3$

The first example is the σ-model on target manifold $M = T^3$. Here, all the results derived in the previous two sections hold. They will be made more explicit for this particular model.

From the construction of the Wess-Zumino action (in the usual sense), there is only one independent action given by the (integral of the) generator of $H^3(T^3)$, which is the volume form

$$\Omega := d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \quad , \tag{4.80}$$

where ϕ^i $(i = 1, 2, 3)$ are the angular (field) variables of T^3. Ω can be represented locally as the exterior derivative (d) of the two-form

$$\omega := \tfrac{1}{6}\epsilon_{ijk}\phi^i d\phi^j \wedge d\phi^k \quad . \tag{4.81}$$

Given such an ω, the gauge potential one-form (4.37) is simply

$$\mathcal{A} = \int_{S^1} dx \left\{ \tfrac{1}{3}\epsilon_{ijk}\phi^i \partial_x \phi^k \delta\phi^j(x) \right\} \quad , \tag{4.82}$$

while the field strength two-form (4.38) is

$$\mathcal{F} = \int_{S^1} dx \left\{ \tfrac{1}{2}\epsilon_{ijk}\partial_x \phi^k \delta\phi^i(x) \delta\phi^j(x) \right\} \quad . \tag{4.83}$$

The Lagrangian density of this model may now be written as

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^j\partial^\mu\phi^k\eta_{jk} + \tfrac{1}{6}\epsilon^{\mu\nu}\epsilon_{ijk}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k \quad , \tag{4.84}$$

where η is the flat metric on T^3.

The Killing vectors on (T^3, η) that will generate the symmetry transformations are just the vectors v generating translations, so that

$$v := v^1\frac{\partial}{\partial\phi^1} + v^2\frac{\partial}{\partial\phi^2} + v^3\frac{\partial}{\partial\phi^3} \quad , \tag{4.85}$$

where v^i's are constants. The induced vector field on the configuration space ΩT^3 is obtained by Lie-dragging the coordinate functions by the vector field v, *i.e.*

$$\delta\phi^j = \pounds_v\phi^j = v^j \quad . \tag{4.86}$$

Under this symmetry transformation, the Lagrangian density (4.84) changes by a total derivative as in (4.53) (note that ω and v satisfy condition (4.56)) :

$$\delta\mathcal{L} = \partial_\mu K^\mu \quad ,$$

where

$$K^\mu = \tfrac{1}{6}\epsilon^{\mu\nu}\epsilon_{ijk}v^i\phi^j\partial_\nu\phi^k \quad . \tag{4.87}$$

Thus the conserved current J^μ (4.73) is simply given by

$$J^\mu = v^j\partial^\mu\phi^k\eta_{jk} + \tfrac{1}{2}\epsilon^{\mu\nu}\epsilon_{ijk}v^i\phi^j\partial_\nu\phi^k \quad . \tag{4.88}$$

The constant of motion associated with the above current is then

$$\begin{aligned} C_v &= \int_{S^1} dx J^0 \\ &= \int_{S^1} dx\left\{v^j\dot\phi^k\eta_{jk} + \tfrac{1}{2}\epsilon_{ijk}v^i\phi^j\partial_x\phi^k\right\} \quad . \end{aligned} \tag{4.89}$$

Note that the second term can be written as the contraction of the field strength two-form (4.83) with ϕ and v, *i.e.* $\mathcal{F}[\phi, v]$ (with an abuse of notation; ϕ is not a vector on ΩT^3). This can be compared with the case of a particle in magnetic field in which the analogous term is $F_{jk}x^jv^k$ (x^j is the coordinate function of the configuration space).[28]

Being preoccupied with global questions in this work, it is not inappropriate to address the possibility of the constants of motion being ill-defined. In [28], it is noted that for the case of a particle on T^n in a magnetic field, the term $F_{jk}x^j v^k$ is not globally defined, owing to the multiple-valued coordinate function x^j on the nonsimply-connected space T^n. In the present example, however, this problem does not occur. As the field variable $\phi^j(x)$ undergoes a translation of its period, 2π,

$$\phi^j(x) \longrightarrow \phi^j(x) + 2\pi \quad , \tag{4.90}$$

the change in C_v is trivial :

$$\Delta C_v = \int_{S^1} dx \{ \epsilon_{ijk} \pi v^j \partial_x \phi^k \} = 0 \quad , \tag{4.91}$$

as the function ϕ^k is periodic in x. This is consistent with the fact that the configuration space is now a loop space, ΩT^3, of T^3 and is simply connected, *i.e.*

$$\pi_1(\Omega T^3) = \pi_2(T^3) = 0 \quad . \tag{4.92}$$

In fact, $\psi_v = \mathcal{F}[\phi, v]$ must be globally defined as a consequence of this (see the remarks after equation (4.61)).

Thus to find any possible phenomena of 'anomalous' constants of motion, one must first require that $\pi_2(M)$ is nontrivial. Such an example will be discussed in the next section.

4.6 σ-Model on $M = S^2 \times S^1$

This is a more interesting example than the previous one as the target manifold $M = S^2 \times S^1$ has a nontrivial second homotopy group, which means that the space of field configurations is no longer simply connected. However this also means that one encounters an ambiguity in the construction of the Wess-Zumino action (see Note 1 in Section 4.2). We will nevertheless proceed as in Section 4.5. A comment regarding the ambiguity will be made in the next section.

The Wess-Zumino action is constructed from the one generator of $H^3(S^2 \times S^1)$ which is given by the volume form :

$$\Omega := \sin\phi^1 d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \quad , \tag{4.93}$$

where ϕ^1 and ϕ^2 are now the spherical coordinates on S^2 and ϕ^3 is the angular coordinate on S^1. Locally this is given by $\Omega = d\omega$ where

$$\omega := -(\cos\phi^1 \mp 1) d\phi^2 \wedge d\phi^3 \quad . \tag{4.94}$$

With the metric of $S^2 \times S^1$ given by

$$\begin{aligned} ds^2 :&= g_{jk} d\phi^j \otimes d\phi^k \\ &= (d\phi^1)^2 + \sin^2\phi^1 (d\phi^2)^2 + (d\phi^3)^2 \quad , \end{aligned} \tag{4.95}$$

the total Lagrangian density of the σ-model is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi^j \partial^\mu \phi^k g_{jk}(\phi) - \epsilon^{\mu\nu}\partial_\mu \phi^2 \partial_\nu \phi^3 (\cos\phi^1 \mp 1) \quad . \tag{4.96}$$

The gauge potentials computed from the above Lagrangian density are given by

$$\mathcal{A}_N := \int_{S^1} dx \left\{ -(\cos\phi^1 - 1)(\partial_x \phi^3 \delta\phi^2(x) - \partial_x \phi^2 \delta\phi^3(x)) \right\} \quad , \tag{4.97}$$

$$\mathcal{A}_S := \int_{S^1} dx \left\{ -(\cos\phi^1 + 1)(\partial_x \phi^3 \delta\phi^2(x) - \partial_x \phi^2 \delta\phi^3(x)) \right\} \quad . \tag{4.98}$$

Note that these are only well-defined locally in the regions

$$N_\epsilon := \left\{ (\phi^1, \phi^2) \mid 0 \le \phi^1 < \tfrac{\pi}{2} + \epsilon \ , \ 0 \le \phi^2 < 2\pi \right\} \quad , \tag{4.99}$$

$$S_\epsilon := \left\{ (\phi^1, \phi^2) \mid \tfrac{\pi}{2} - \epsilon < \phi^1 \le \pi \ , \ 0 \le \phi^2 < 2\pi \right\} \tag{4.100}$$

of S^2 respectively ($\epsilon > 0$). A gauge potential which is well-defined on the whole of S^2

(and hence of ΩM) can then be given by

$$\mathcal{A} = \begin{cases} \mathcal{A}_N & \text{on } N_\epsilon \\ \mathcal{A}_S & \text{on } S_\epsilon \end{cases} \tag{4.101}$$

with the observation that $\mathcal{A}_N$ and $\mathcal{A}_S$ are gauge related on $(N_\epsilon \cap S_\epsilon)$ by

$$\xi := \mathcal{A}_S - \mathcal{A}_N = -2 \int_{S^1} dx \{ \partial_x \phi^3 \delta\phi^2(x) - \partial_x \phi^2 \delta\phi^3(x) \} \quad . \tag{4.102}$$

Note that this is an exact one-form

$$\xi = \delta e \quad , \tag{4.103}$$

where e is the zero-form

$$e := -2 \int_{S^1} dx \{ (\partial_x \phi^3) \phi^2(x) \} \tag{4.104}$$

(e is well defined under translations $\phi^i \longrightarrow \phi^i + 2\pi$ for $i = 2, 3$). The field strength $\mathcal{F}$ is simply given by

$$\mathcal{F} = \int_{S^1} dx \{ \tfrac{1}{2} \sin\phi^1 \epsilon_{ijk} \partial_x \phi^k \delta\phi^i(x) \delta\phi^j(x) \} \quad . \tag{4.105}$$

For this model, the symmetry transformations are generated by the following Killing vectors on $S^2 \times S^1$:

$$v_{(1)} := \sin\phi^2 \frac{\partial}{\partial \phi^1} + \cot\phi^1 \cos\phi^2 \frac{\partial}{\partial \phi^2} \quad , \tag{4.106}$$

$$v_{(2)} := \frac{\partial}{\partial \phi^2} \quad , \tag{4.107}$$

$$v_{(3)} := \frac{\partial}{\partial \phi^3} \quad , \tag{4.108}$$

$$v_{(4)} := -(\cos\phi^2 \frac{\partial}{\partial \phi^1} - \cot\phi^1 \sin\phi^2 \frac{\partial}{\partial \phi^2}) \quad . \tag{4.109}$$

Here, the bracketed indices are just labels denoting different Killing vectors. One can now construct the associated constants of motion for each of the symmetry transformations given by $\delta\phi^j = v^j_{(i)}$ $(i = 1, \ldots, 4)$. It is important to note that the condition (4.56) does not hold for these cases and one has to repeat any necessary computations of Section 4.4 which assume this condition.

<u>$i=1$</u>

The vector field on ΩM is given by

$$\delta\phi^j = \sin\phi^2 \delta^{j1} + \cot\phi^1 \cos\phi^2 \delta^{j2} \tag{4.110}$$

where δ^{ij} on the right hand side is the Kronecker delta. The change in Lagrangian density by such a transformation is a total derivative, *i.e.* $\delta\mathcal{L} = \partial_\mu K^\mu_{(1)}$ where

$$K^\mu_{(1)} = -\epsilon^{\mu\nu}(\csc\phi^1 \mp \cot\phi^1)\cos\phi^2 \partial_\nu \phi^3 \quad . \tag{4.111}$$

Hence the associated constant of motion constructed from transformation (4.110) will be

$$C_{v_{(1)}} = \int_{S^1} dx \{\dot{\phi}^1 \sin\phi^2 + \dot{\phi}^2 \sin\phi^1 \cos\phi^1 \cos\phi^2 + (\partial_x\phi^3)\sin\phi^1 \cos\phi^2\} \quad . \tag{4.112}$$

Here the contribution from the field strength $\mathcal{F}$,

$$\psi_{v_{(1)}} = \int_{S^1} dx \{\sin\phi^1 \cos\phi^2 \partial_x \phi^3\} \quad , \tag{4.113}$$

is no longer as transparent as ψ_v in Section 3. However if we take the exterior derivative δ of $\psi_{v_{(1)}}$ we find

$$\begin{aligned} \delta\psi_{v_{(1)}} &= \int_{S^1} dx \{\cos\phi^1 \cos\phi^2 (\partial_x\phi^3 \delta\phi^1(x) - \partial_x\phi^1 \delta\phi^3(x)) \\ &\qquad + \sin\phi^1 \sin\phi^2 (\partial_x\phi^2 \delta\phi^3(x) - \partial_x\phi^3 \delta\phi^2(x))\} \\ &= -v_{(1)} \lrcorner \mathcal{F} \quad , \end{aligned} \tag{4.114}$$

thus confirming the previous results.

<u>$i=2$</u>

The vector field on ΩM induced by $v_{(2)}$ is simply given by

$$\delta\phi^j = \delta^{j2} \quad , \tag{4.115}$$

and hence the change in Lagrangian density, $\delta\mathcal{L}$, is trivial. The constant of motion is then,

$$\begin{aligned} C_{v_{(2)}} &= \int_{S^1} dx \{ \dot{\phi}^2 \sin^2 \phi^1 - (\partial_x \phi^3) \cos \phi^1 \} \\ &= C_{v_{(2)}0} + \psi_{v_{(2)}} \end{aligned} \tag{4.116}$$

($C_{v_{(2)}0}$ is the normal kinetic term contribution). As before, we find that

$$\begin{aligned} \delta\psi_{v_{(2)}} &= \int_{S^1} dx \{ \sin \phi^1 (\partial_x \phi^3 \delta\phi^1(x) - \partial_x \phi^1 \delta\phi^3(x)) \} \\ &= -v_{(2)} \lrcorner \mathcal{F} \quad . \end{aligned} \tag{4.117}$$

$\underline{i=3}$

Similar calculations for this case produce the following results :

$$\delta\phi^j = \delta^{j3} \quad , \tag{4.118}$$

$$\begin{aligned} C_{v_{(3)}} &= \int_{S^1} dx \{ \dot{\phi}^3 + (\partial_x \phi^2) \cos \phi^1 \} \\ &= C_{v_{(3)}0} + \psi_{v_{(3)}} \quad , \end{aligned} \tag{4.119}$$

$$\begin{aligned} \delta\psi_{v_{(3)}} &= \int_{S^1} dx \{ \sin \phi^1 (\partial_x \phi^1 \delta\phi^2(x) - \partial_x \phi^2 \delta\phi^1(x)) \} \\ &= -v_{(3)} \lrcorner \mathcal{F} \quad . \end{aligned} \tag{4.120}$$

$\underline{i=4}$

One could proceed for this case with similar calculations to the above; however we will instead make use of the observation that

$$v_{(4)} = [v_{(1)}, v_{(2)}] \quad . \tag{4.121}$$

Using equation (4.63), one finds the field strength contribution to the constant of motion associated with the symmetry transformation $\delta\phi^j = v^j_{(4)}$ to be

$$\begin{aligned}\psi_{v_{(4)}} &= \mathcal{F}[v_{(1)}, v_{(2)}] \\ &= \int_{S^1} dx \{\sin\phi^1 \sin\phi^2 \partial_x \phi^3\} \quad .\end{aligned} \tag{4.122}$$

One finds again that

$$\begin{aligned}\delta\psi_{v_{(4)}} = \int_{S^1} & dx \{\cos\phi^1 \sin\phi^2 (\partial_x\phi^3 \delta\phi^1(x) - \delta\phi^3(x)\partial_x\phi^1) \\ & + \sin\phi^1 \cos\phi^2 (\partial_x\phi^3\delta\phi^2(x) - \partial_x\phi^2\delta\phi^3(x))\} \\ = & -v_{(4)} \lrcorner \mathcal{F} \quad .\end{aligned} \tag{4.123}$$

Having constructed the constants of motion, it is interesting to check whether the $\psi_{v_{(i)}}$'s are globally well-defined or not. As discussed earlier, possible obstructions may occur when $\pi_1(\Omega M)$ is nontrivial. Thus it is natural to look at a noncontractible loop in the configuration space which is generated by this homotopy group. One such loop is shown in Fig. 3 below.

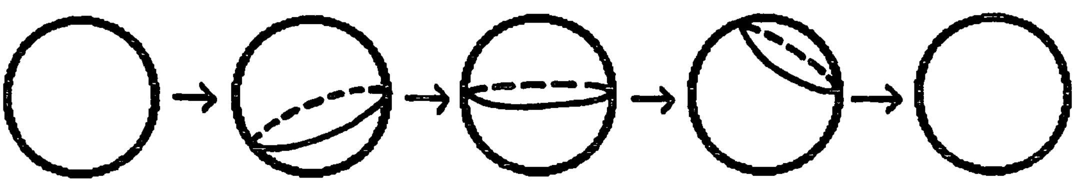


Fig. 3: Evolution of a loop around S^2 of M (S^1 not shown) giving a resultant noncontractible loop of ΩM.

The reason why such a loop can give a possible obstruction to a well-defined constant of motion, can be seen as follows. Consider the loop on S^2 in Fig. 3 as given by the map ϕ. In defining the Wess-Zumino action, the map ϕ has to be extended to $\tilde{\phi}$ (see Section 4.2). Replacing the loop given by the map ϕ in Fig. 3 by the image of (a particular) extension $\tilde{\phi}$ gives the corresponding Fig. 4. This figure shows that the Wess-Zumino

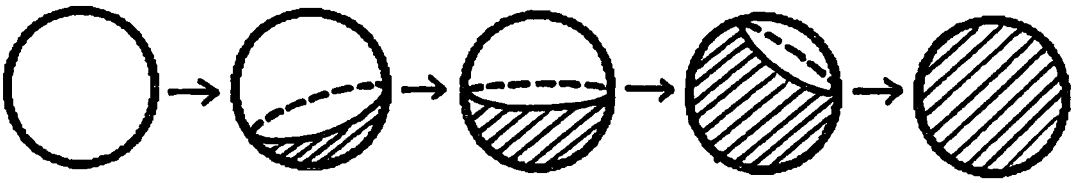


Fig. 4: The corresponding evolution of the image of the extension $\tilde{\phi}$ of ϕ.

action changes value as ϕ evolves under such noncontractible loop. Thus constants of motion derived from such action can also change values under such evolution of ϕ and hence are ill-defined. One has to check this explicitly.

A construction of one such noncontractible loop in ΩM is given as follows. First, the submanifold S^2 of M is embedded in $\mathbb{R}^3$ with a triplet of coordinate functions x_i's :

$$\begin{aligned} \hat{n} :&= (x_1,\ x_2,\ x_3) \\ &= (\sin\phi^1 \cos\phi^2,\ \sin\phi^1 \sin\phi^2,\ \cos\phi^1) \quad , \end{aligned} \tag{4.124}$$

which satisfies $\hat{n}\cdot\hat{n} = 1$ (ϕ^i's are coordinates on M). The loop may now be constructed via such a triplet of functions where they now map $[0,\pi]\times S^1$ to S^2, *i.e.*

$$\hat{n} := (\sin\lambda \sin x,\ \sin^2\lambda \cos x + \cos^2\lambda,\ \sin\lambda\cos\lambda(\cos x - 1)) \quad , \tag{4.125}$$

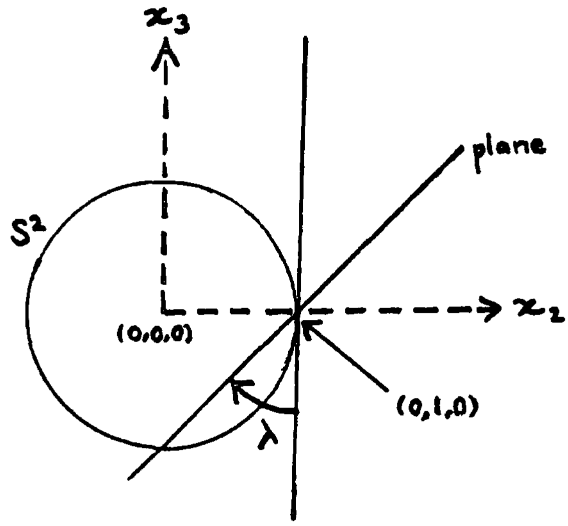
where $\lambda \in [0,\pi]$ is some parameter and $x \in S^1$ is the coordinate of space. The vector $\hat{n}$ has all the properties required of a noncontractible loop in ΩM :

(*i*). $\hat{n}\cdot\hat{n} = 1$.

(*ii*). For fixed x,

$$\hat{n}\ |_{\lambda=0} = \hat{n}\ |_{\lambda=1} = (0,\ 1,\ 0) \quad ,$$

is a fixed point through which the one-parameter family of loops (parametrised by λ) appear on the submanifold S^2. The map (4.125) actually describes the intersection of a plane with the two-sphere of unit radius as shown in the figure below.



$$\tan\lambda = \frac{x_2 - 1}{x_1}$$

Fig. 5: The intersection of a plane with S^2 in the x_2-x_3 plane.

This map possesses the required loop-like property in ΩS^2 as λ goes from 0 to π.

(*iii*). It has a topological winding number one. This can be seen by noting that with the coordinate functions (4.124), the volume form of S^2 is given by

$$\Omega := \epsilon^{ijk} x_i dx_j \wedge dx_k \quad . \tag{4.126}$$

For the map (4.125), the volume form is

$$\Omega = \sin\lambda(1 - \cos x) d\lambda \wedge dx \quad . \tag{4.127}$$

Integrating (4.127) gives the winding number multiplied by the volume :

$$\int_0^{\pi} d\lambda \int_0^{2\pi} dx \ \sin\lambda(1 - \cos x) = 4\pi \quad .$$

Hence the winding number is one.

Having obtained the map (4.125), it is now easily verified that the constants of motion are globally well-defined (with respect to function (4.125)),

$$\Delta\psi_{v_{(i)}} = \psi_{v_{(i)}} \mid_{(\lambda=\pi)} - \psi_{v_{(i)}} \mid_{(\lambda=0)} = 0 \quad (i = 1, \cdots, 4) \quad . \tag{4.128}$$

In fact, one finds that $v_{(i)} \lrcorner \mathcal{F}$ is an exact form for each $i = 1, \cdots, 4$ (using the triplet of coordinate functions), *i.e.*

$$v_{(1)} \lrcorner \mathcal{F} = \int_{S^1} dx\delta\{-x_1\partial_x\phi^3\} \quad , \tag{4.129}$$

$$v_{(2)} \lrcorner \mathcal{F} = \int_{S^1} dx\delta\{x_3\partial_x\phi^3\} \quad , \tag{4.130}$$

$$v_{(3)} \lrcorner \mathcal{F} = \int_{S^1} dx\delta\{-x_1\partial_x\phi^2\} \quad , \tag{4.131}$$

$$v_{(4)} \lrcorner \mathcal{F} = \int_{S^1} dx\delta\{-x_2\partial_x\phi^3\} \quad . \tag{4.132}$$

This brings us to the conclusion that while the requirement of ΩM is nonsimply-connected is necessary for the existence of 'anomalous' constants of motion, it is not sufficient.

4.7 Topological $\pi_1(\Omega M)$ Effects

In the previous sections, the correspondence between σ-models with a Wess-Zumino term and the system of a particle in a magnetic field has been made closer through discussions of both the gauge symmetries and the space-time symmetries. One other possible similarity between the two systems is that of a topological effect. In the second example of σ-model on $M = S^2 \times S^1$, the possibility of an anomalous constant of motion due to the nonsimply-connectedness of the configuration space was investigated, though such a constant of motion is not found. There is another topological effect that arises from nonsimply connected space, namely the Aharonov-Bohm effect from the background gauge field. It is possible to demonstrate this effect for the example of σ-model on $S^2 \times S^1$. But prior to this, some comments on the ambiguity in the construction of the Wess-Zumino action associated with the nontrivial $\pi_2(M)$ (see Note 1 of Section 4.2) are necessary.

Consider the σ-model on $M = S^2 \times S^1$ of the previous section with ϕ mapping space S^1 into the submanifold S^2 of M. This map has different extensions $\tilde{\phi}$ which are not deformable to each other due to the 'obstruction' from S^2 of M. For example, the map ϕ that sends S^1 to the equator of S^2 *e.g.*

$$\phi^1 := \tfrac{\pi}{2} \quad , \qquad \phi^2 := x \quad , \qquad \phi^3 := t \quad , \tag{4.133}$$

have extensions

$$\tilde{\phi}^1 := \tfrac{r\pi}{2} \quad , \qquad \tilde{\phi}^2 := x \quad , \qquad \tilde{\phi}^3 := t \quad , \tag{4.134}$$

and

$$\tilde{\phi}^1 := \tfrac{r\pi}{2} + n(1-r)\pi \quad , \qquad \tilde{\phi}^2 := x \quad , \qquad \tilde{\phi}^3 := t \quad , \tag{4.135}$$

where $r \in [0,1]$ is the radial coordinate of the two-dimensional disk D^2 ($\partial D^2 = S^1$) and n is a positive integer The integer n is actually the number of times the map (4.134) together with map (4.135) winds around S^2 of M.

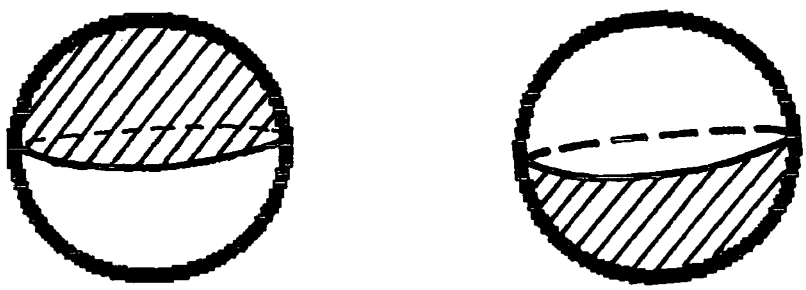


Fig. 6: The shaded regions are the image of different extensions of ϕ (4.133) given by (4.134) and (4.135) (with $n = 1$) respectively.

These extensions in fact give different values to the Wess-Zumino action *i.e.*

$$\begin{aligned}
&\int\limits_{D^2\times I} \tilde{\phi}^*\Omega \\
&= \int\limits_{D^2\times I} \epsilon^{\mu\nu\rho} \sin\tilde{\phi}^1 \partial_\mu\tilde{\phi}^1 \partial_\nu\tilde{\phi}^2 \partial_\rho\tilde{\phi}^3 \; r dr dx dt \\
&= \int\limits_I dt \int\limits_0^{2\pi} dx \int\limits_0^{1} dr \{ r\sin(\tfrac{r\pi}{2} + n(1-r)\pi)\} \\
&= \frac{(-1)^n 4I}{(2n-1)} \left\{ \cos\left(\frac{(2n-1)\pi}{2}\right) - \frac{2}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2}\right) \right\}
\end{aligned} \tag{4.136}$$

where I is the length of the time interval and n takes values from $0, 1, 2, \ldots$ ($n = 0$ corresponds to extension (4.134)). Thus to resolve the ambiguity of the extensions, one needs to specify this 'winding number' n which then gives a unique Wess-Zumino action.

Having done this, one can now discuss the 'Aharonov-Bohm effect' in ΩM. An essential ingredient in this topological effect is that one can obtain a different gauge $\mathcal{A}'$ by performing a singular gauge transformation on $\mathcal{A}$.[29] Consider then

$$\mathcal{A}_N = \int_{S^1} dx\{-(\cos\phi^1 - 1)(\partial_x\phi^3\delta\phi^2(x) - \partial_x\phi^2\delta\phi^3(x))\} \tag{4.137}$$

from the last section. One can perform a singular gauge transformation on $\mathcal{A}_N$ by the (nonexact) one-form

$$\xi' := \int_{S^1} dx\delta[-(\cos\phi^1 - 1)\phi^3(\partial_x\phi^2 - \partial_x\phi^1)] \tag{4.138}$$

to give the gauge potential

$$\begin{aligned}\mathcal{A}'_N = \int_{S^1} dx\{&\phi^3\sin\phi^1(\partial_x\phi^2\delta\phi^1(x) - \partial_x\phi^1\delta\phi^2(x)) \\ &+ (\cos\phi^1 - 1)(\partial_x\phi^1\delta\phi^3(x) - \partial_x\phi^3\delta\phi^1(x))\} \quad .\end{aligned} \tag{4.139}$$

Similarly one can do the same for $\mathcal{A}_S$ to get

$$\begin{aligned}\mathcal{A}'_S = \int_{S^1} dx\{&\phi^3\sin\phi^1(\partial_x\phi^2\delta\phi^1(x) - \partial_x\phi^1\delta\phi^2(x)) \\ &+ (\cos\phi^1 + 1)(\partial_x\phi^1\delta\phi^3(x) - \partial_x\phi^3\delta\phi^1(x))\} \quad .\end{aligned} \tag{4.140}$$

Both $\mathcal{A}'_N$ and $\mathcal{A}'_S$ can now be 'patched' up in the same way as $\mathcal{A}_N$ and $\mathcal{A}_S$ in the last section to obtain the desired new gauge potential $\mathcal{A}'$ on ΩM which gives the same $\mathcal{F}$ (4.105). In the Aharonov-Bohm effect however the relevant physical quantity to be determined is the phase factor[30]

$$\exp i(\oint \mathcal{A}) \quad , \tag{4.141}$$

where $\oint$ denotes an integral over a noncontractible closed path in ΩM.

Thus one only needs to obtain different holonomies $\oint \mathcal{A}$ giving different phase factors, to demonstrate the existence of Aharonov-Bohm effect in ΩM. Here, they are easily

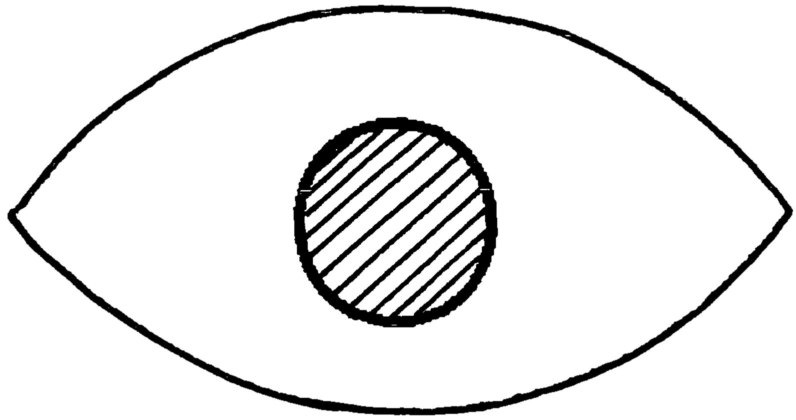


Fig. 7: A schematic diagram of a noncontractible loop in ΩM with the shaded region being Int$(S^2) \times S^1$.

given by the two gauges $\mathcal{A}$ and $\mathcal{A}'$. To show that this is the case we shall use a particular mapping ϕ namely, that given by (4.125) *i.e.*

$$\hat{n} := (\sin t \sin x,\ \sin^2 t \cos x + \cos^2 t,\ \sin t \cos t(\cos x - 1)) \quad , \tag{4.142}$$

and

$$\phi^3 := 1 \qquad \text{(a constant mapping)} \quad . \tag{4.143}$$

The parameter t in $\hat{n}$ now denotes the time which parametrizes the noncontractible closed path traversed in ΩM. Using this set of functions, one finds that the holonomy of $\mathcal{A}$ is just trivial :

$$\begin{aligned}\oint \mathcal{A} &= \int_0^{\frac{\pi}{2}} dt \int_0^{2\pi} dx \{-(\cos\phi^1 - 1)(\partial_x\phi^3\dot{\phi}^2 - \partial_x\phi^2\dot{\phi}^3)\} \\ &\qquad + \int_{\frac{\pi}{2}}^{\pi} dt \int_0^{2\pi} dx \{-(\cos\phi^1 + 1)(\partial_x\phi^3\dot{\phi}^2 - \partial_x\phi^2\dot{\phi}^3)\} \\ &= 0\end{aligned} \tag{4.144}$$

as $\dot{\phi}^3 = \partial_x\phi^3 = 0$. For $\mathcal{A}'$, the computation of its holonomy,

$$\oint \mathcal{A}' = \int_0^{\pi} dt \int_0^{2\pi} dx \{\phi^3 \sin\phi^1(\partial_x\phi^2\dot{\phi}^1 - \partial_x\phi^1\dot{\phi}^2)\} \quad , \tag{4.145}$$

is messy. The integral is done numerically and it gives

$$\oint \mathcal{A}' = -0.4159 \neq 0 \quad \text{(to 4 dec. pl.)} \quad . \tag{4.146}$$

These two results, (4.144) and (4.146), then give different phase factors and hence imply the significance of the gauge potentials themselves (as in normal Aharonov-Bohm effect). It is now important to note that the use of a different gauge potential $\mathcal{A}'$ implies the use of a different local expression of the Wess-Zumino Lagrangian density from that of (4.96) namely,

$$\mathcal{L}_{WZ} := \epsilon^{\mu\nu}(\phi^3 \sin\phi^1 \partial_\mu \phi^1 \partial_\nu \phi^2 + (\cos\phi^1 \pm 1)\partial_\mu \phi^3 \partial_\nu \phi^1) \quad . \tag{4.147}$$

Thus the Aharonov-Bohm effect in ΩM would then imply that the local expression for the Wess-Zumino Lagrangian density has a physical significance. At this point, it is interesting to see whether there is any relation between the ambiguity of the extensions for the construction of the Wess-Zumino action (which comes from $\pi_2(M) \neq 0$) and this Aharonov-Bohm effect (which comes from $\pi_1(\Omega M) \neq 0$). First, observe that the extensions are labelled by integral winding numbers while the Aharonov-Bohm effect is labelled by a continuous parameter, say $\lambda \in \mathbb{R}$, given by the holonomy of $\lambda\mathcal{A} + (1-\lambda)\mathcal{A}'$. A possible relation would be that λ, or more precisely the holonomy characterises the representation of $\pi_1(\Omega M)$ (the 'winding number') on $U(1)$ in the same way as in the normal Aharonov-Bohm effect from the representations of $\mathrm{Hom}(\pi_1(Q), U(1))$.[31,32] However, such a relation would be obscured by the functional character of the holonomy.

4.8 Summary

The main purpose of this chapter has been to show how close is the analogy between the σ-models with Wess-Zumino term and the system of a particle in a constant magnetic field in various aspects. The motivation underlying this purpose is that it is hoped that one can use the particle analogy in applying group-theoretic quantisation to strings/σ-models with Wess-Zumino term in the next chapter in the same way as in Chapter 3. We conclude this chapter by summarising the main results of this chapter in this respect:

(*i*) A σ-model with a Wess-Zumino term can be treated as a system of a particle in a background magnetic field on the configuration space ΩM. The Wess-Zumino term provides an analogue of the gauge potential on ΩM which then gives the Lagrangian of the σ-model a gauge symmetry.

(*ii*) Like other systems in a background gauge field, the constants of motion associated to the symmetry transformations of the system are modified by a contribution from the gauge field $\mathcal{F}$ on ΩM.

(*iii*) There is no 'anomalous' phenomenon of ill-defined constants of motion for σ-models on $M = T^3$ and $M = S^2 \times S^1$. The second example shows that $\pi_1(\Omega M) \neq 0$ is not a sufficient condition for such a phenomena.

(*iv*) For nonsimply connected configuration spaces ΩM, there is a functional analogue of the Aharonov-Bohm effect in ΩM. The holonomy of the gauge potential $\mathcal{A}$ has to be specified to obtain a unique theory. This implies that the local expression of the Wess-Zumino Lagrangian density has a physical significance. In addition to this, one has to specify the 'winding number' n to have a well-defined Wess-Zumino action.

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Chapter 5

Quantisation of Strings on Tori

5.1 Introduction

String theory[1,2] provides a suitable framework in which gravity can be incorporated successfully in quantum theory. It has been so far the most popular among the unification theories. However the theory has developed in a rather haphazard manner (see *e.g.* [3]). Together with all the sophisticated machinery associated with the theory, it is not an easy subject to follow. One way to put the theory at a more coherent level is to provide it a geometrical framework. Among the attempts to do so, the group-theoretic quantisation programme[4] has managed to address some of its basic issues in a geometrical and group-theoretical consistent way.[5,6] Below, a review of such a work by Isham and Linden[5] on the quantisation of strings on the tori will be given. At the end of this chapter, the quantisation of strings in the presence of a background antisymmetric tensor field arising from a Wess-Zumino term will also be considered.

The action of a string can be generally given as

$$S := \int dx dt \{ \tfrac{1}{2} \partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta g_{\alpha\beta}(\Phi) + \epsilon^{\mu\nu} \omega_{\alpha\beta}(\Phi) \partial_\mu \Phi^\alpha \partial_\nu \Phi^\beta \} \tag{5.1}$$

where $x \in S^1$ is the string parameter and t parametrises its evolution; the indices μ, ν refers to these two parameters. The fields $\{\Phi^\alpha\}$ denote the map from the world-sheet swept out by the string into space-time whose metric is given by $\{g_{\alpha\beta}\}$ (α, β are the space-time indices). The fields $\omega_{\alpha\beta}$ is a background antisymmetric tensor field (with possible singularities) introduced into the system which couples to oriented string surfaces[7] and $\epsilon^{\mu\nu}$ is the usual alternating tensor. The antisymmetric tensor field $\omega_{\alpha\beta}$ plays several important roles in string theory namely cancellation of anomalies,[8] string compactifications,[9] and other topological effects.[10] The space-time considered here is of the form $L_d \times M$ where L_d is the d-dimensional Minkowski space-time and M is a compact n-manifold. The fields $g_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are assumed to have nontrivial components only in the compactified directions. We will ignore the action that arises from L_d since the quantisation associated with this action can always be done using the normal field-theoretic Heisenberg-Weyl group. We will only be concerned with M. One particular M which is of prime interest to string theory is the n-torus T^n.[9,11] The part of the action (5.1) which is on T^n can be written as

$$S = \int dx dt \{ \tfrac{1}{2} \partial_\mu \Phi^i \partial^\mu \Phi^j \eta_{ij} + \epsilon^{\mu\nu} \omega_{jk}(\Phi) \partial_\mu \Phi^j \partial_\nu \Phi^k \} \tag{5.2}$$

where η is the flat metric on T^n and indices $i, j, k, \ldots$ belong to T^n. This is precisely the action of a σ-model with a Wess-Zumino term on T^n (*cf.* (4.29)). [There may be a difficulty in defining the second term of (5.2) for fields with nonzero winding number (see Section 4.2) but it can be overcome (see Section 5.4).] However note that the target manifold M is now the (compactified part of) space-time and the domain of the fields is the parameter space of the world-sheet. At this stage it is important to note that the configuration space is now the free loop space LT^n instead of the based-point loop space $\Omega\mathrm{T}^n$ considered in Chapter 4. One can however express LT^n as the semidirect product[12,13]

$$\mathrm{LT}^n \approx \mathrm{T}^n \ltimes \Omega \mathrm{T}^n \quad . \tag{5.3}$$

For other relations between σ-models and strings see *e.g.* [14,15]. The second term in (5.2) provides an analogue of a gauge potential one-form $\mathcal{A}$ (see Section 4.3) on LT^n,

$$\mathcal{A} = \int_{S^1} dx \{ 2(\partial_x \Phi^k) \omega_{jk}(\Phi) \delta\Phi^j(x) \} \tag{5.4}$$

where its corresponding field strength two-form $\mathcal{F}$ is

$$\mathcal{F} = \int_{S^1} dx \{ 3(\partial_x \Phi^k) \omega_{jk,l}(\Phi) \delta\Phi^l(x) \delta\Phi^j(x) \} \quad . \tag{5.5}$$

[Note that $\omega_{jk,l}$ denotes the derivative of ω_{jk} with respect to Φ^l and it forms the components of a well-defined three-form Ω on M.] However we will first ignore the 'interaction term' with the antisymmetric tensor field and reproduce the results of Isham and Linden[5] of quantisation of strings on a circle in Section 5.2 and their generalisation to T^n in Section 5.3. They will serve as a comparison for the case with the antisymmetric tensor field and facilitate the discussions in Section 5.4.

5.2 Quantisation of String on a Circle

The configuration space Q of a string moving on a circle T^1 is the infinite-dimensional loop space $Q = C^\infty(S^1, \mathrm{T}^1) =: \mathrm{LT}^1$. The phase space $\mathcal{S}$ is then given by $\mathcal{S} = T^*(\mathrm{LT}^1) \approx \mathrm{L}(T^*\mathrm{T}^1)$. There are various subtleties involved in considering infinite-dimensional manifolds and structures defined on them.[16–20] We will assume that whatever is given below will be well-defined in some form or another. Let the 'coordinate functions' of $\mathcal{S}$ be given by $\Phi(x)$ and $J(x)$ ($x \in S^1$). The natural symplectic form on $\mathrm{L}(T^*\mathrm{T}^1)$ is[21]

$$\sigma := \int_{S^1} dx\{\delta\Phi(x)\delta J(x)\} \quad . \tag{5.6}$$

Note that we have changed the notation for the symplectic form in the infinite-dimensional case from ω to σ to avoid the confusion with the antisymmetric tensor field $\omega_{ij}(\Phi)$. One can also decompose the 'fields' $\Phi(x)$ as in Chapter 4 into fields with nonzero winding number ($C(x)$) and fields with zero winding number ($\phi(x)$) but we will only do so when it is necessary. As in the case of particle on S^1,[4] the canonical group for the system can be taken as

$$\mathcal{G} = \mathrm{L}E_2 \approx \mathrm{L}\mathbb{R}^2 \rtimes \mathrm{L}SO(2) \quad . \tag{5.7}$$

The canonical observables on LT^1 can be obtained by embedding the target space T^1 in $\mathbb{R}^2$ *i.e.*

$$\mathrm{v}_\Phi(x) := (u(x), v(x)) = (\cos\Phi(x), \sin\Phi(x)) \qquad x \in S^1 \quad . \tag{5.8}$$

Together with $J(x)$, they form the set of canonical observables on $\mathcal{S}$. Note that LT^1 is disconnected *i.e.* $\pi_0(\mathrm{LT}^1) \approx \pi_1(\mathrm{T}^1) = \mathbb{Z}$ where the integer is given by the winding number

$$w(\Phi) := \frac{1}{2\pi}\int_0^{2\pi} dx\{u\frac{dv}{dx} - v\frac{du}{dx}\} \quad , \tag{5.9}$$

as indicated by the decomposition of the fields. The decomposition is also reflected in the transformation group $\mathrm{L}SO(2)$ of LT^1 *i.e.* $\pi_0(\mathrm{L}SO(2)) \approx \pi_1(U(1)) = \mathbb{Z}$. One can decompose the loop group $LU(1)$ as

$$LU(1) \approx (LU(1))_0 \rtimes \mathbb{Z} \tag{5.10}$$

and the pair $(\phi, n) \in (LU(1))_0 \rtimes \mathbb{Z}$ (where ϕ has winding number zero) corresponds to the loop with winding number n via the map

$$x \longrightarrow \phi(x)e^{inx} \quad . \tag{5.11}$$

Similarly $\mathrm{L}E_2$ is decomposed as

$$\mathrm{L}E_2 \approx (\mathrm{L}E_2)_0 \rtimes \mathbb{Z} \approx (\mathrm{L}\mathbb{R}^2 \rtimes \mathrm{L}SO(2))_0 \rtimes \mathbb{Z} \quad . \tag{5.12}$$

With the symplectic form (5.6), one can obtain the following Poisson bracket algebra which corresponds to the Lie algebra $(\mathrm{L}E_2)_0$:

$$\{J(x), u(x')\} = v(x)\delta(x - x') \quad , \tag{5.13}$$

$$\{J(x), v(x')\} = -u(x)\delta(x - x') \quad , \tag{5.14}$$

$$\{u(x), v(x')\} = 0 = \{J(x), J(x')\} \tag{5.15}$$

where $u(x)$ and $v(x)$ are now $\cos\phi(x)$ and $\sin\phi(x)$ respectively.

To quantise the system, one must find the irreducible unitary representations of $\mathcal{G}$ and hence the self-adjoint representations of $\mathcal{L}(\mathcal{G})$ on a Hilbert space $\mathcal{H}$. It is convenient to smear the $\hat{J}(x)$ operator with a test function $h \in C^\infty(S^1, \mathbb{R})$ *i.e.*

$$\hat{J}(h) := \frac{1}{2\pi}\int_0^{2\pi} dx\{\hat{J}(x)h(x)\} \tag{5.16}$$

in order to allow us to write the quantum commutator algebra corresponding to (5.13) - (5.15) (*i.e.* $\mathcal{L}(\mathrm{L}E_2)_0$) as

$$[\hat{J}(h), \hat{u}(x)] = ih(x)\hat{v}(x) \quad , \tag{5.17}$$

$$[\hat{J}(h), \hat{v}(x)] = -ih(x)\hat{u}(x) \quad , \tag{5.18}$$

$$[\hat{u}(x), \hat{v}(x)] = 0 = [\hat{J}(h), \hat{J}(h')] \tag{5.19}$$

wher h' is another test function. In addition to this algebra, one must find the representation of the discrete subgroup $\mathbb{Z}$ of $\mathcal{G}$. It will involve the family $\{U_n \mid n \in \mathbb{Z}\}$ of unitary operators satisfying

$$U_n U_m = U_{n+m} \qquad ; \qquad U_n^\dagger = U_{-n} \qquad (n, m \in \mathbb{Z}) \tag{5.20}$$

and they intertwine with the other generators $\hat{J}(h), \hat{u}(x)$ and $\hat{v}(x)$ as

$$U_n \hat{J}(h) U_n^{-1} = \hat{J}(h) \quad , \tag{5.21}$$

$$U_n \hat{u}(x) U_n^{-1} = e^{-inx} \hat{u}(x) \quad , \tag{5.22}$$

$$U_n \hat{v}(x) U_n^{-1} = e^{-inx} \hat{v}(x) \quad . \tag{5.23}$$

Note that

$$\hat{u}^2(x) + \hat{v}^2(x) = R^2 \tag{5.24}$$

is a Casimir operator for $\mathcal{G}$ where R is a constant (*cf.* particle on S^1 [4]).

Left-Right Split

An interesting problem of quantisation of strings is obtaining and maintaining the independence of the left and right moving modes of the strings. The usual approach in the string literature is to associate the winding number with one half of a canonical set of operators and use U_n to generate the other half.[22,23] A much more systematic and consistent approach is to use the symmetry group that respects the global structure of the phase space for the left-right split. First, consider the winding number w (5.9) with the operator status

$$\hat{\mathbf{w}} := \frac{1}{2\pi R^2} \int_0^{2\pi} dx \{\hat{u} \frac{d\hat{v}}{dx} - \hat{v} \frac{d\hat{u}}{dx}\} \tag{5.25}$$

where R^2 is given by the Casimir operator (5.24). Its commutation relations with the operators $\hat{u}(x)$, $\hat{v}(x)$, $\hat{J}(h)$ and U_n are

$$[\hat{\mathbf{w}}, \hat{u}(x)] = [\hat{\mathbf{w}}, \hat{v}(x)] = [\hat{\mathbf{w}}, \hat{J}(h)] = 0 \quad , \tag{5.26}$$

$$[\hat{\mathbf{w}}, U_n] = -n U_n \quad . \tag{5.27}$$

Note that the relations (5.26) imply that the Hilbert space $\mathcal{H}$ decomposes into the direct sum $\oplus_n \mathcal{H}_n$ where $\mathcal{H}_n$ is the eigenspace of $\hat{\mathbf{w}}$ with eigenvalue n. The irreducibility of the representation is restored by the intertwining relation (5.27). If the aim is to obtain two independent sets of operators that will correspond to the left and right moving modes of the string, then the relations (5.26) suggest that $\hat{\mathbf{w}}$ belongs to the set different from that containing $\hat{u}(x)$, $\hat{v}(x)$ and $\hat{J}(x)$. Also observe that $\hat{\mathbf{w}}$ has a discrete spectrum like the angular momentum operator and hence one expects that the set of operators that $\hat{\mathbf{w}}$ belongs to, form a representation of $\mathcal{L}(E_2)$. One can in fact construct the analogues of $\hat{u}(x)$ and $\hat{v}(x)$ for this new E_2 group by defining

$$\hat{\mathbf{u}} := \tfrac{1}{2}(U_{-1} + U_1) \quad , \tag{5.28}$$

$$\hat{\mathbf{v}} := \tfrac{1}{2i}(U_{-1} - U_1) \quad . \tag{5.29}$$

They satisfy the relations

$$[\hat{\mathbf{w}}, \hat{\mathbf{u}}] = i\hat{\mathbf{v}} \quad , \tag{5.30}$$

$$[\hat{\mathbf{w}}, \hat{\mathbf{v}}] = -i\hat{\mathbf{u}} \quad , \tag{5.31}$$

$$[\hat{\mathbf{u}}, \hat{\mathbf{v}}] = 0 \quad , \tag{5.32}$$

which are precisely the commutation relations of E_2. The corresponding Casimir operator $\hat{\mathbf{u}}^2 + \hat{\mathbf{v}}^2$ has a fixed value *i.e.*

$$\hat{\mathbf{u}}^2 + \hat{\mathbf{v}}^2 = \tfrac{1}{4}(U_{-2} + U_2 + 2) - \tfrac{1}{4}(U_{-2} + U_2 - 2) = \hat{\mathbb{1}} \quad . \tag{5.33}$$

One can construct the representations of this topological E_2 group on a 'configuration space' which is effectively a circle. First, note that U_m maps the eigenspace $\mathcal{H}_n$ of $\hat{\mathbf{w}}$ isomorphically onto $\mathcal{H}_{n-m}$ from the relation (5.27). Hence one can write any vector $\Psi \in \mathcal{H}$ in terms of vectors belonging to the set $\Psi = \{U_{-m}\Psi_m \mid m \in \mathbb{Z}\}$ where $\Psi_m \in \mathcal{H}_0$. One can write the action of the operators U_n on $\mathcal{H}_0$ by

$$(U_n\Psi)_m = \Psi_{m+n} \quad . \tag{5.34}$$

One can then construct the isomorphism $i\ :\ \mathcal{H} \longrightarrow L^2(\mathbf{S}^1, \mathcal{H}_0)$ defined by

$$(i(\Psi))(\vartheta) := \sum_{n=-\infty}^{\infty} \exp(in\vartheta)\Psi(\vartheta) \qquad \Psi \in \mathcal{H},\ \Psi_n \in \mathcal{H}_0,\ \vartheta \in \mathbf{S}^1 \tag{5.35}$$

where $\mathbf{S}^1$ is the effective 'topological' circle. The operators U_n acting on $L^2(\mathbf{S}^1, \mathcal{H}_0)$ have the effect

$$(U_n\Psi)(\vartheta) = \exp(-in\vartheta)\Psi(\vartheta) \qquad \Psi \in L^2(\mathbf{S}^1, \mathcal{H}_0) \quad . \tag{5.36}$$

Thus on this representation space we find the operators $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are represented by the operators

$$(\hat{\mathbf{u}}\Psi)(\vartheta) = \cos\vartheta\Psi(\vartheta) \quad , \tag{5.37}$$

$$(\hat{\mathbf{v}}\Psi)(\vartheta) = \sin\vartheta\Psi(\vartheta) \quad , \tag{5.38}$$

$$(\hat{\mathbf{w}}\Psi)(\vartheta) = -i\frac{d}{d\vartheta}\Psi(\vartheta) \quad . \tag{5.39}$$

Having constructed the topological E_2 group, one must now show that this new set of generators to this group is indeed 'independent' from the set $\{\hat{u}(x), \hat{v}(x), \hat{J}(h)\}$. It is important to note that $\hat{u}(x)$ and $\hat{v}(x)$ do not commute with $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ because of the relations (5.22) and (5.23). One can however use the relation (5.27) to remove the 'winding mode' from $\hat{u}(x)$ and $\hat{v}(x)$ by redefining them to be

$$\hat{u}'(x) := \exp(i\hat{\mathbf{w}}x)\hat{u}(x) \quad , \tag{5.40}$$

$$\hat{v}'(x) := \exp(i\hat{\mathbf{w}}x)\hat{v}(x) \tag{5.41}$$

respectively. These new operators with $\hat{J}(h)$ still form the Lie algebra of $(\mathrm{L}E_2)_0$ but now with the desired property of commuting with the generators of the topological E_2 group. To get the appropriate splitting of the phase space, one must first construct an E_2 group out of $(\mathrm{L}E_2)_0$ associated with the 'constant loops' to pair with the topological E_2. The momentum generator for the constant loops is given by

$$\hat{J} := \hat{J}(1) = \frac{1}{2\pi}\int_0^{2\pi} \hat{J}(x)dx \quad . \tag{5.42}$$

The operators $\hat{u}$ and $\hat{v}$ conjugate to $\hat{J}$ are to be picked out arbitrarily from the set of operators $\{\hat{u}'(x)\}$ and $\{\hat{v}'(x)\}$ respectively. The arbitrary choice corresponds to the arbitrariness of the embedding of the submanifold of constant loops T^1 in LT^1. Given that such a choice is made, the generators $(\hat{u}, \hat{v}, \hat{J})$ then satisfy the condensed relation

$$[\hat{J}, \hat{a}] = -\hat{a} \tag{5.43}$$

where

$$\hat{a} := \hat{u} - i\hat{v} \quad . \tag{5.44}$$

These are to be paired with

$$[\hat{\mathbf{w}}, \hat{\mathbf{a}}] = -\hat{\mathbf{a}} \quad ; \tag{5.45}$$

$$\hat{\mathbf{a}} := \hat{\mathbf{u}} - i\hat{\mathbf{v}} = U_1 \tag{5.46}$$

from the topological E_2.

A further redefinition of the generators of the two E_2 groups can now give the 'left' and 'right' set of generators which is consistent with the dynamical evolution of the left and right moving modes; the solutions of the string equations of motion are functions of $t - x$ and $t + x$ respectively.[23] They are given by

$$\hat{J}_L := \frac{1}{\sqrt{2}}(\hat{J} + \hat{\mathbf{w}}) \quad , \quad \hat{u}_L - i\hat{v}_L = \hat{a}_L := \frac{1}{\sqrt{2}}\hat{a}\hat{\mathbf{a}} \quad , \tag{5.47}$$

$$\hat{J}_R := \frac{1}{\sqrt{2}}(\hat{J} - \hat{\mathbf{w}}) \quad , \quad \hat{u}_R - i\hat{v}_R = \hat{a}_R := \frac{1}{\sqrt{2}}\hat{\mathbf{a}}^\dagger\hat{a} \quad . \tag{5.48}$$

It is easy to check that they reproduce the (condensed) commutation relations of the two E_2 groups *i.e.*

$$[\hat{J}_L, \hat{a}_L] = -\hat{a}_L \quad , \tag{5.49}$$

$$[\hat{J}_R, \hat{a}_R] = -\hat{a}_R \quad , \tag{5.50}$$

and the two sets are independent of each other:

$$[\hat{J}_L, \hat{a}_R] = 0 \quad , \tag{5.51}$$

$$[\hat{J}_R, \hat{a}_L] = 0 \quad . \tag{5.52}$$

Thus it is now established that the canonical left-right split of the phase space is built in within the canonical group describing the symmetries of the phase space.

5.3 Quantisation of String on T^n

One can now generalise the results obtained for T^1 in the previous section to the case of the n-torus T^n. But as in Section 3.4, it will be done by expressing T^n as the quotient space $W/2\pi\Lambda$ where $\Lambda := \{\sum_{i=1}^{n} n^i A_i \mid n^i \in \mathbb{Z}\}$ and $\{A_i\} =: E$ is the basis of W. [The dual lattice has the basis $E^* := \{B^i\}$.] Hence the configuration space of the system is $Q = \mathrm{L}(W/2\pi\Lambda)$ and the phase space is $\mathcal{S} \approx \mathrm{L}W^* \times \mathrm{L}(W/2\pi\Lambda)$. Before going further, it is useful to rephrase the description of Q being disconnected and the 'winding number' that classifies the disconnected classes in terms of the lattice structure. Given that W is contractible, the exact homotopy sequence of the bundle $\Lambda \longrightarrow W \longrightarrow \mathrm{T}^n$ implies the relation $\pi_0(\mathrm{LT}^n) \approx \pi_1(\mathrm{T}^n) \approx \pi_0(\Lambda) \approx \Lambda$.[24] Thus there is a preferred element ι of $H^1(\mathrm{T}^n;\Lambda)$ corresponding to the identity map from Λ to Λ since $H^1(\mathrm{T}^n;\Lambda) \approx \mathrm{Hom}(\pi_1(\mathrm{T}^n),\Lambda) \approx \mathrm{Hom}(\Lambda,\Lambda)$. One can now express the winding number of the fields $\{\Phi^B \mid B \in E^*\}$ as the element of $H^1(S^1;\Lambda) \approx \Lambda$ given by the pull-back $\Phi^{B*}(\iota)$ $(\iota \in H^1(\mathrm{T}^n;\Lambda))$. Given any string configuration $\Phi : S^1 \longrightarrow W/2\pi\Lambda$, one can always lift the map Φ to $\tilde{\Phi}$ on the space[25]

$$PW := \{\tilde{\Phi} \in C^\infty(\mathbb{R},W) \mid \Phi(y+2\pi) - \Phi(y) \in \Lambda \quad \forall y \in \mathbb{R}\} \tag{5.53}$$

where its 'winding number' is given by

$$w(\Phi) := (\tilde{\Phi}(2\pi) - \tilde{\Phi}(0)/2\pi \quad \in \Lambda \quad . \tag{5.54}$$

One can in fact express $\mathrm{L}(W/2\pi\Lambda)$ as $PW/2\pi\Lambda$.

The canonical observables on Q can be given by (*cf.* (3.113) and (5.8))

$$u^B(x)(\Phi) := \cos < B, \tilde{\Phi}(x) > \quad , \tag{5.55}$$

$$v^B(x)(\Phi) := \sin < B, \tilde{\Phi}(x) > \quad , \tag{5.56}$$

where $B \in E^*$. One can now form the basic set of generators for the connected component of the canonical group $\mathcal{G} = (\underbrace{\mathrm{L}E_2 \times \mathrm{L}E_2 \times \cdots \times \mathrm{L}E_2}_{n \text{ times}})$, namely $\{u^B(x), v^B(x), J(h)\}$ where $J(h)$ $(h \in \mathrm{L}W)$ is the smeared generator of $\mathcal{L}(\mathrm{LT}^n)_0$. With the natural symplectic form, the Poisson bracket of these observables are

$$\{J(h), u^B(x)\} =< B, h(x) > v^B(x) \quad , \tag{5.57}$$

$$\{J(h), v^B(x)\} = - < B, h(x) > u^B(x) \quad , \tag{5.58}$$

$$\{u^B(x), v^{B'}(x')\} = 0 = \{J(h), J(h')\} \tag{5.59}$$

where $B, B' \in E^*$ and $h, h' \in \mathrm{L}W$. The corresponding quantum commutators can be simply given by

$$[\hat{J}(h), \hat{a}^B(x)] = - < B, h(x) > \hat{a}^B(x) \quad , \tag{5.60}$$

$$[\hat{a}^B(x), \hat{a}^{B'}(x')] = 0 \tag{5.61}$$

where

$$\hat{a}^B(x) := \hat{u}^B(x) - i\hat{v}^B(x) \quad . \tag{5.62}$$

One must also include a unitary representation U_λ, $\lambda \in \Lambda$ of the disconnected part of $\mathcal{G}$; $\pi_0(\mathcal{G}) \approx \pi_0(\mathrm{LT}^n) \approx \pi_1(W/2\pi\Lambda) \approx \Lambda$. It intertwines with the other operators as

$$U_\lambda \hat{a}^B(x) U_\lambda^{-1} = \exp(-i < B, \lambda > x)\ \hat{a}^B(x) \quad , \tag{5.63}$$

$$U_\lambda \hat{J}(h) U_\lambda^{-1} = \hat{J}(h) \quad . \tag{5.64}$$

To discuss the left-right split for this system, we proceed in the same way as in Section 5.2 by first giving an operator status to the 'winding number' $w^B(\Phi) :=< B, w(\Phi) >$. The resultant set of Hermitian operators $\hat{\mathbf{w}}^B$ intertwine with the representation U_λ of Λ as

$$U_\lambda \hat{\mathbf{w}}^B U_\lambda^{-1} = \hat{\mathbf{w}}^B + < B, \lambda > \hat{\mathbb{1}} \quad . \tag{5.65}$$

The pair of operators conjugate to $\hat{\mathbf{w}}^B$ can be obtained from the operators $\{U_\lambda\}$ in a similar way to (5.28) and (5.29). They are

$$\hat{\mathbf{u}}_A := \tfrac{1}{2}(U_{-A} + U_A) \quad , \tag{5.66}$$

$$\hat{\mathbf{v}}_A := \tfrac{1}{2i}(U_{-A} - U_A) \quad , \tag{5.67}$$

where $A \in E$. They give the commutation relations

$$[\hat{\mathbf{w}}^B, \hat{\mathbf{u}}_A] = i < B, A > \hat{\mathbf{v}}_A \quad , \tag{5.68}$$

$$[\hat{\mathbf{w}}^B, \hat{\mathbf{v}}_A] = -i < B, A > \hat{\mathbf{u}}_A \quad , \tag{5.69}$$

($B \in E^*$, $A \in E$). Thus one has obtain the topological $E_2 \times E_2 \times \cdots E_2$ (n copies of E_2). The other $E_2 \times E_2 \times \cdots E_2$ group is constructed out of $(\mathrm{L}E_2 \times \cdots \times \mathrm{L}E_2)_0$ associated with the constant loops. Its generators are given by $\hat{u}^B$, $\hat{v}^B$ and $\hat{J}_A$ where $\hat{J}_A$ is given by $\hat{J}(h)$ with $h(x) := A \in E$.

To obtain the desired split, (5.47) and (5.48) suggest that the operators $\hat{J}_A$ and $\hat{\mathbf{w}}^B$ should be added/subtracted. This is no longer possible in this case since these operators are labelled by the different spaces *i.e.* E and E^* respectively. One needs to add a further structure in order to proceed to get the appropriate split. The metric $g_{ij} = g(A_i, A_j)$ on W is introduced to induce an isomorphism between W and W^* *i.e.*

$$\iota : W \longrightarrow W^* \quad ; \quad < \iota(A), A' > := g(A, A') \qquad A, A' \in W \quad . \tag{5.70}$$

The components of the metric are required to be integers so that $\iota(\Lambda) \subset \Lambda^*$. The analogues of (5.47) and (5.48) can now be given as

$$\hat{K}_L^B := \tfrac{1}{\sqrt{2}}(\hat{J}_{\kappa(B)} + \hat{\mathbf{w}}^B) \quad ; \quad \hat{a}_{LA} := \tfrac{1}{\sqrt{2}}\hat{a}^{\iota(A)}\hat{\mathbf{a}}_A \quad , \tag{5.71}$$

$$\hat{K}_R^B := \tfrac{1}{\sqrt{2}}(\hat{J}_{\kappa(B)} - \hat{\mathbf{w}}^B) \quad ; \quad \hat{a}_{RA} := \tfrac{1}{\sqrt{2}}\hat{\mathbf{a}}_A^\dagger \hat{a}^{\iota(A)} \quad , \tag{5.72}$$

where $A \in E$, $B \in E^*$ and κ is the inverse of ι ($\kappa : W^* \longrightarrow W$). Note that this construction is invertible only if $\kappa(\Lambda^*) \subset \Lambda$. This implies that the lattice Λ must be self-dual.

5.4 Quantisation of String in a Background Antisymmetric Tensor Field

Having done quantisation of strings moving freely on T^n in the previous section, one can now generalise the situation to include a background antisymmetric tensor field. This has the effect of modifying the natural symplectic form on T^n to

$$\sigma_{\mathcal{F}} := \sum_{i,j} \int_{S^1} dx \{\delta\Phi^{B^i}(x)\delta J_{A_i}(x) + \mathcal{F}(A_i, A_j)\delta\Phi^{B^i}(x)\delta\Phi^{B^j}(x)\} \tag{5.73}$$

(*cf.* Section 3.3). The bilinear form $\mathcal{F}(A_i, A_j)$ on $\mathrm{L}(W/2\pi\Lambda)$ gives the components of the field strength two-form in (5.5) (adapted to the lattice structure) *i.e.*

$$\mathcal{F}(A_i, A_j) := \sum_k \partial_x \Phi^{B^k} \Omega_{ijk}(\Phi) \tag{5.74}$$

where Ω is a closed nonexact three-form on T^n *i.e.* a generator of the third cohomology group of T^n (Ω_{ijk} denotes the components of Ω along the directions A_i, A_j and A_k).

Note that one does not have a difficulty in defining $\sigma_{\mathcal{F}}$ for fields with nontrivial winding numbers unlike the Lagrangian formulation in Chapter 4 since Ω is a globally well-defined three-form on T^n. Let us first concentrate on the trivial sector namely the fields ϕ that generate the constant loops. With (5.73), the Poisson bracket algebra of the observables $\{u^B(x), v^B(x), J(h)\}$ $(h \in \mathrm{L}W)$ is given by

$$\{J(h), u^B(x)\} = < B, h(x) > v^B(x) \quad , \tag{5.75}$$

$$\{J(h), v^B(x)\} = - < B, h(x) > u^B(x) \quad , \tag{5.76}$$

$$\{u^B(x), v^{B'}(x')\} = 0 \quad , \tag{5.77}$$

$$\{J(h), J(h')\} = -\mathcal{F}(h, h') \quad , \tag{5.78}$$

where $B, B' \in E^*$ and $h, h' \in \mathrm{L}W$ (*cf.* Section 3.4). They form a representation of the Lie algebra of the subgroup of the canonical group $\tilde{\mathcal{G}}$ of the system that corresponds to the constant loops. Their corresponding quantum commutators are

$$[\hat{J}(h), \hat{u}^B(x)] = i < B, h(x) > \hat{v}^B(x) \quad , \tag{5.79}$$

$$[\hat{J}(h), \hat{v}^B(x)] = -i < B, h(x) > \hat{u}^B(x) \quad , \tag{5.80}$$

$$[\hat{u}^B(x), \hat{v}^{B'}(x')] = 0 \quad , \tag{5.81}$$

$$[\hat{J}(h), \hat{J}(h')] = -i\mathcal{F}(h, h')\hat{\mathbb{1}} \quad . \tag{5.82}$$

The above relations suggest that the canonical group $\tilde{G}$ which acts on $\mathrm{L}(W/2\pi\Lambda)$ is given by

$$\tilde{G} := \mathrm{L}W_{p(1)} \ltimes (\mathrm{L}W_{p(2)} \ltimes (\cdots \ltimes (\mathrm{L}W_{p(n)} \times U(1)) \cdots)) \tag{5.83}$$

(*cf.* Section 3.4) where $\mathrm{L}W_j$ is the jth component of $\mathrm{L}W$ and $p(i)$ is the ith element in the permutation p of $(12\cdots n)$. Note that the use of the universal cover of $\mathrm{L}(W/2\pi\Lambda)$ is necessary for the canonical group to be centrally extended. It is important to note that at this stage that the resultant full canonical group

$$\tilde{\mathcal{G}} := \mathrm{L}\tilde{E}_2 \ltimes (\mathrm{L}\tilde{E}_2 \ltimes (\cdots \ltimes (\mathrm{L}\tilde{E}_2 \times U(1)) \cdots)) \tag{5.84}$$

is not disconnected. However one can still adopt the decomposition (5.11) to decompose $\tilde{G}$ (and hence $\tilde{\mathcal{G}}$) into $\tilde{G}_0 \rtimes W$ where $\tilde{G}_0$ is the subgroup that generates the constant loops and W generates the analogue of the operators that intertwine the constant loops with the other 'winding modes'. The generators of W can be given by the unitary operators

$$U_r := \sum_i r^i U_{A_i} \quad ; \quad r := \sum_i r^i A_i \in W \quad (A_i \in E) \quad . \tag{5.85}$$

They satisfy the relations

$$U_r U_s = U_{r+s} \quad ; \quad U_r^\dagger = U_{-r} \quad r, s \in W \tag{5.86}$$

and intertwine with the 'winding number' operators $\hat{\mathbf{w}}^B$ $(B \in E^*)$ in the same way as in (5.65) *i.e.*

$$U_r \hat{\mathbf{w}}^B U_r^{-1} = \hat{\mathbf{w}}^B + < B, r > \hat{\mathbb{1}} \quad . \tag{5.87}$$

Note that the 'winding numbers' are no longer separated by integers and hence the 'winding numbers' are no longer necessarily integers. This rather peculiar effect may however turn out to be desirable as pointed out in [5] as it leads to left-right asymmetry.

Before discussing the left-right split, it is interesting to note that there is a somewhat surprising new result for this case with the antisymmetric tensor field. The 'winding number' operators are required to satisfy the commutation relation

$$[\hat{\mathbf{w}}^B, \hat{\mathbf{w}}^{B'}] = -i\mathcal{F}(\kappa(B), \kappa(B'))\hat{\mathbb{1}} \qquad B, B' \in E^* \tag{5.88}$$

where κ is the map $\kappa : W^* \longrightarrow W$ mentioned in the previous section (*cf.* (5.82)). Later we will find that this relation is necessary to obtain the desired canonical left-right split. The relation (5.88) implies that the central subgroup of $\tilde{\mathcal{G}}$ also intertwines with the winding modes of the fields. Note that the peculiar feature of the (eigen-) values of the 'winding number' operator being nonintegral is probably due to the presence of the line bundle structure associated to this central subgroup in the sense that it 'opens up' the winding modes in the quantum theory. [Compare this feature with the effect of adding a gauge potential to the momentum operator $\hat{J}$.] However it is not clear what kind of physical situation such phenomenon corresponds to and why it is so.

Given these results, one can proceed in almost a similar way to Section 5.3 to obtain the canonical left-right split. First, we construct the operators $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{v}}_A$ as in (5.66) and

(5.67) and obtain the topological $\tilde{E}_2 \rtimes (\tilde{E}_2 \rtimes (\cdots \rtimes (\tilde{E}_2 \times U(1)) \cdots))$ group. The same is done for the constant loops to get the other $\tilde{E}_2 \rtimes (\tilde{E}_2 \rtimes (\cdots \rtimes (\tilde{E}_2 \times U(1)) \cdots))$ group to pair with the topological one. The desired canonical let-right split is then obtained in the same way as in the previous section by defining the operators $\hat{K}_L^B$, $\hat{K}_R^B$, $\hat{a}_{LA}$ and $\hat{a}_{RA}$ (see (5.71) and (5.72)). But now they have the new relations

$$[\hat{K}_L^B, \hat{K}_L^{B'}] = -i\mathcal{F}(\kappa(B), \kappa(B')) \quad , \tag{5.89}$$

$$[\hat{K}_R^B, \hat{K}_R^{B'}] = -i\mathcal{F}(\kappa(B), \kappa(B')) \quad . \tag{5.90}$$

Note also that

$$[\hat{K}_L^B, \hat{K}_R^{B'}] = 0 \quad , \tag{5.91}$$

ensuring the independence of the left and right moving modes of the string. Finally, we note that the use of the universal cover of E_2 in the paired groups above leads to two sets of θ-angles in the quantum sector (one for each group). These sets could be different from each other which then implies a left-right asymmetry associated to the definitions of $\hat{K}_L^B$ and $\hat{K}_R^B$ (as mentioned earlier on). Such a situation is of relevance to the construction of the heterotic strings.[23]

5.5 Summary and Outlook

We have now observed how Isham's group-theoretic quantisation programme is able to describe bosonic strings in a framework that respects the global structure of the phase space. The usual splitting of the phase space into sectors corresponding to the left and right moving modes of the string is intrinsic in the description of the canonical group in both cases of with and without the background antisymmetric tensor field. However in the case of string in the background field, a new feature arises from the centrally extended canonical group. It is found that the 'winding number' operators have noninteger (eigen) values and these operators do not commute. Here is a specific example of how this quantisation programme's global concerns are found to be advantageous in finding new (intrinsic) features of the theory. It will be interesting to see what kind of impact this would bring to the understanding of string theory.

To close this work, it must be said that the aims of the study here have been rather modest despite the potential of the group-theoretic quantisation programme. Nevertheless it is useful to bring out the simple aspects of the programme in relation to central extensions as described in this work. One hopes that the results found here are of significance in any future work. There are many possibilities for future areas of study in the context of this programme. Of particular interest will be the study of the Wess-Zumino-Witten models[26–28] and also theories with Chern-Simons term[29] – they share many of the properties of the examples discussed in this work in relation to central extensions.

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