Configuration Complexes and Tangential and Infinitesimal versions of Polylogarithmic Complexes

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Dedicated to

my father, mother and wife

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Abstract

In this thesis we consider the Grassmannian complex of projective configurations in weight 2 and 3, and Cathelineau's infinitesimal polylogarithmic complexes as well as a tangential complex to the famous Bloch-Suslin complex (in weight 2) and to Goncharov's "motivic" complex (in weight 3), respectively, as proposed by Cathelineau [5].

Our main result is a morphism of complexes between the Grassmannian complexes and the associated infinitesimal polylogarithmic complexes as well as the tangential complexes.

In order to establish this connection we introduce an F-vector space $\beta_2^D(F)$, which is an intermediate structure between a $\mathbb{Z}$-module $\mathcal{B}_2(F)$ (scissors congruence group for F) and Cathelineau's F-vector space $\beta_2(F)$ which is an infinitesimal version of it. The structure of $\beta_2^D(F)$ is also infinitesimal but it has the advantage of satisfying similar functional equations as the group $\mathcal{B}_2(F)$. We put this in a complex to form a variant of Cathelineau's infinitesimal complex for weight 2. Furthermore, we define $\beta_3^D(F)$ for the corresponding infinitesimal complex in weight 3. One of the important ingredients of the proof of our main results is the rewriting of Goncharov's triple-ratios as the product of two projected cross-ratios. Furthermore, we extend Siegel's cross-ratio identity ([21]) for 2×2 determinants over the truncated polynomial ring $F[\varepsilon]_\nu := F[\varepsilon]/\varepsilon^\nu$. We compute cross-ratios and Goncharov's triple-ratios in $F[\varepsilon]_2$ and $F[\varepsilon]_3$ and use them extensively in our compu-

tations for the tangential complexes. We also verify a ”projected five-term” relation in the group $T\mathcal{B}_2(F)$ which is crucial to prove one of our central statements Theorem 4.3.3.

Declaration

The work in this thesis is based on research carried out at the Pure Maths group, Department of Mathematical Sciences, University of Durham, England, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

In his seminal papers ([9],[10],[11],[12]), Goncharov uses the Grassmannian complex (first introduced by Suslin (see [23])) associated to points in $\mathbb{P}^2$ in order to prove Zagier's conjecture on polylogarithms and special L-values (see [26]) for weight $n = 3$. This conjecture in particular asserts that the values of Dedekind zeta function $\zeta_F(s)$ for some number field F at an integer point $s = n \geq 2$ can be expressed as a determinant of n-logarithms evaluated at points in F. It was known for $n = 2$ by work of Suslin, Borel and Bloch and also proved in a slightly weaker form by Zagier himself. Goncharov forms an ingenious proof for weight $n = 3$.

In the process, he introduces complexes $\Gamma(n)$ (which he called "motivic"). Cathelineau investigates variants of these complexes in the additive (both infinitesimal and tangential) setting (see [3],[4],[5]).

One of the most important ingredients of Goncharov's work is the triple-ratio (Goncharov called it generalized cross-ratio) which is first introduced by Goncharov (see [10]). In his earlier paper Goncharov had a formula (which is not visibly antisymmetric) for the morphism $f_2^{(3)} : C_6(3) \to \mathcal{B}_3(F)$, (see §4 in [9]), for any field F, where $C_6(3)$ is the free abelian group generated by the configurations of 6 points in 3 dimensional F-vector space modulo the action of $GL_3(F)$. The triple-ratio was discovered by Goncharov together with Zagier by anti-symmetrization of formula for $f_2^{(3)}$. Having defined the triple-ratio he described an antisymmetric formula for the morphism $f_6(3) : C_6(3) \to \mathcal{B}_3(F)$, but with the restriction that it applies to generic configuration only, where points are in generic position (see Formula 3.9 in [10]) (unfortunately, in [10] his proof of commutativity of left square of diagram (3.2) in [10] was incorrect(see Theorem 3.10 in [10]); a missing factor of $\frac{15}{2}$ was pointed out by Gangl and Goncharov provided a correct proof in the appendix of [13]). By using algebraic K-theory he constructed a map of complexes from the Grassmannian complex to his own complex and then he proved Zagier's conjecture for weight $n = 3$.

Our point of view is to bring the geometry of configuration spaces into infinitesimal and tangential settings. We tried to find suitable morphisms between the Grassmannian subcomplex $(C_*(n), d)$ (see diagram (2.1a) in 2.1) and Cathelineau's analogues of Goncharov's complexes $\Gamma(n)$. For weight $n = 2$, we have not only shown that the corresponding diagrams in both cases are commutative but also that they extend to morphisms of complexes involving both the Grassmannian and Cathelineau's complex (see §3.1 and §4.2). For weight $n = 3$, we also proved that the corresponding diagrams in the infinitesimal and the tangential setting connecting the Grassmannian subcomplex $(C_*(n), d)$ (see diagram (2.1a) in §2.1) are commutative (see §3.2 and §4.3).

Goncharov outlined the proof for commutativity of the left square of diagram (3.2c) at the end of Chapter 3 (see §3 in [10] for the actual diagram and appendix of [13] for the proof). For this he worked in $\bigwedge^2 F^\times \otimes F^\times$, using the factorisation of $1 - \frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}$, where $\Delta(l_i, l_j, l_k)$ denotes some 3×3-determinant, into a 3×3-determinant and a 6×6-determinant and also had to appeal to a deeper result in algebraic K-theory (see Lemma 5.1 and Proposition 5.2 in [13]).

We observe that each term in the triple-ratio can be rewritten as product of two "projected" cross-ratios in $\mathbb{P}^2$, which enables us to give an elementary proof (which does not use algebraic K-theory) of one of our main results (Theorem 3.2.5).

Furthermore, we define infinitesimal group $\beta_2^D(F)$ for any derivative $D \in Der_{\mathbb{Z}}F$ over a field F which has more or less similar functional equations as the group $\mathcal{B}_2(F)$ and use it to our advantage for the proof which works almost same for the two direct summand involving $\beta_2^D(F) \otimes F^\times$ and $F \otimes \mathcal{B}_2(F)$. In summary, the proof of Theorem 3.2.5 consists of rewriting the triple-ratio as the product of two cross-ratios, combinatorial techniques and the use of functional equations in $\beta_2^D(F)$ and $\mathcal{B}_2(F)$. Since $\mathcal{B}_2(F)$ has similar functional equations, we can also apply this technique in Goncharov's setting. For the convenience of the reader, we also included the proof of commutativity of the left square of Goncharov's diagram in the appendix A.

For weight $n > 3$, Goncharov generalized a version of the infinitesimal analogue(see [13]), involving the groups $\beta_n(F), \beta_{n-1}(F) \otimes F^\times, F \otimes \mathcal{B}_{n-1}(F), \dots, F \otimes \bigwedge^{n-1} F^\times$. We suggest a slight modification in the maps to guarantee its being a complex (see Lemma 2.4.2) and relate a variant of it to the Grassmannian complex in the top degree, using a natural generalization of maps defined in weight $n = 2$ and $n = 3$(see Proposition 4.3.5 and Proposition 3.2.7).

For given $(l_0, \dots, l_3) \in C_4(2)$, a well-known Siegel cross-ratio identity (See [21]) for associated 2×2-determinants $\Delta(l_i, l_j)$ for $0 \le i < j \le 3$ becomes a very important tool for the factorization of $1 - r(l_0, \dots, l_3)$, where $r(l_0, \dots, l_3) = \frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}$ is a cross-ratio of four points in the version used by Goncharov(see identity (4.1) or see §3 in [9]).

We elaborate on similar cross-ratio constructions in the tangential case, where instead of F we are working over the ring of dual numbers $F[\varepsilon]_\nu := F[\varepsilon]/\varepsilon^\nu$. At first, for $(l_0^*, \dots, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_\nu})$, we present an analogue to Siegel cross-ratio identity for 2×2-determinants $\Delta(l_i^*, l_j^*), 0 \le i < j \le 3$ for vectors in $(l_0^*, \dots, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_\nu})$ (see Lemma 4.1.1 and equations (4.3) and (4.2)) which is the analogue of (4.1), and consider their cross-ratios as an element over the truncated polynomial ring $F[\varepsilon]_\nu$, i.e., $\boldsymbol{r}(l_0^*, \dots, l_3^*) = \left(r_{\varepsilon^0}\varepsilon^0 + r_{\varepsilon^1}\varepsilon^1 + \cdots + r_{\varepsilon^{\nu-1}}\varepsilon^{\nu-1}\right)(l_0^*, \dots, l_3^*)$, where r_{ε^0} is the usual cross-ratio of four points in $\mathbb{A}^2_F$, while the other elements of $\boldsymbol{r}$ are computed in §4.1.1. We introduce a similar construction for the triple-ratio as well (see §4.1.2).

Due to this analogue of cross-ratios, we are able to find morphisms between the Grassmannian subcomplex $C_*(\mathbb{A}^n_{F[\varepsilon]_2}, d)$ for $n = 2, 3$ and the tangent complexes to the Bloch-Suslin and the Goncharov complexes (see §4.2 and §4.3). We also produce results for the projected five-term relation in $\beta_2^D(F)$ and $T\mathcal{B}_2(F)$ (see Lemma 3.1.5 and Lemma 4.2.4) which are analogous to Goncharov's projected five-term relation in $\mathcal{B}_2(F)$ (see Lemma 2.18 of [9]) and very helpful for the proof of our main results (Theorem 3.2.5 and 4.3.3). In appendix B, we provide a different way to look the tangential complexes to Bloch-Suslin and Goncharov's complexes, especially when one wants to look elements in $F[\varepsilon]_3$.

In §4.3, we provide a possible definition of a group $T\mathcal{B}_3(F)$ which was first defined hypothetically in §9 of [5]. On the basis of our definition, we mimic this construction with the F-vector space $\beta_3^D(F)$ and reproduce Cathelineau's 22-term functional equation for $T\mathcal{B}_3(F)$. At the end of chapter 4, we present a suitable definition of group $T\mathcal{B}_n(F)$ for any n and try to put in the complex.

Chapter 2

Preliminaries and Background

As we mentioned in the introduction, we are relating the Grassmannian complex to a variant of Cathelineau's complex and tangent complex to Bloch-Suslin and Goncharov's complexes. For this it is important to recall them in this chapter. We will also present the variant of Cathelineau's (infinitesimal) complex in §2.4.1 and will try to form a generalized complex for $\beta_n^D(F)$ as Goncharov's work in [12].

2.1 Grassmannian complex

In this section, we recall concepts from (see [9], [11]). Consider $\tilde{C}_m(X)$, which is the free abelian group generated by elements $(x_1, \ldots, x_m) \in X^m$ for some set X with $x_i \in X$. Then we have a simplicial complex $(\tilde{C}_*(X), d)$ generated by simplices whose vertices are the elements of X, where the differential in degree -1 is given on generators by

$$d : C_m(X) \to C_{m-1}(X)$$

$$d\colon (x_1, \ldots, x_m) \mapsto \sum_{i=0}^{m} (-1)^i (x_1, \ldots, \hat{x}_i, \ldots, x_m) \tag{2.1}$$

Let G be a group acting on X. The elements of $G \setminus X^m$ are called configurations of X, where G is acting diagonally on X^m. Further assume that $C_m(X)$ is the free abelian group generated by the configurations of m elements of X then there is a complex $(C_*(X), d)$, and $\tilde{C}_*(X)_G$ be the group of coinvariants of the natural action of G on $C_*(X) = \tilde{C}_*(X)$. For $m > n$, let us define $C_m(n)$ (or $C_m(\mathbb{P}_F^{n-1})$) which is the free abelian group, generated by the configurations of m vectors in an n-dimensional vector space $V_n = \mathbb{A}^n_F$ over a field F (any n vectors arising by using $X = V_n$) (or m points in $\mathbb{P}^{n-1}_F$) in generic position (an m-tuple of vectors in an n-dimensional vector space V_n is in generic position if n or fewer number of vectors are linearly independent). Apart from the above differential d, we have another differential map:

$$d' : C_{m+1}(n+1) \to C_m(n)$$
$$d' \colon (l_0, \ldots, l_m) \mapsto \sum_{i=0}^{m} (-1)^i (l_i | l_0, \ldots, \hat{l}_i, \ldots, l_m),$$

where $(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_m)$ is the configuration of vectors in $V_{n+1}/\langle l_i \rangle$ defined as the n-dimensional quotient space, obtained by the projection of vectors $l_j \in V_{n+1},\ j \neq i$, projected from $C_{m+1}(n+1)$ to $C_m(n)$ from which we have the following bicomplex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & C_{n+5}(n+2) \arrow[r, "d"] \arrow[d, "d'"] & C_{n+4}(n+2) \arrow[r, "d"] \arrow[d, "d'"] & C_{n+3}(n+2) \arrow[d, "d'"] \\
\cdots \arrow[r] & C_{n+4}(n+1) \arrow[r, "d"] \arrow[d, "d'"] & C_{n+3}(n+1) \arrow[r, "d"] \arrow[d, "d'"] & C_{n+2}(n+1) \arrow[d, "d'"] \\
\cdots \arrow[r] & C_{n+3}(n) \arrow[r, "d"] & C_{n+2}(n) \arrow[r, "d"] & C_{n+1}(n)
\end{tikzcd} \tag{2.1a}$$

which is called the Grassmannian bicomplex. We will verify here commutativity of the above diagram for this we just need to show that $d' \circ d = d \circ d'$ for the group $C_{n+k+m}(n+k)$.

$$d' \circ d(l_0, \ldots, l_{n+k+m-1}) = \sum_{i=0}^{n+k+m-1} (-1)^i \left\{ \sum_{\substack{j=0 \\ j\neq i}}^{n+k+m-1} (-1)^j (l_j | l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_{n+k+k+m-1}) \right\}$$

and

$$d \circ d'(l_0, \ldots, l_{n+k+m-1}) = \sum_{i=0}^{n+k+m-1} (-1)^i \left\{ \sum_{\substack{j=0 \\ j\neq i}}^{n+k+m-1} (-1)^j (l_i | l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_{n+k+m-1}) \right\}$$

for the following we will use a subcomplex $(C_*(n), d)$ called the Grassmannian complex, of the above

$$\cdots \xrightarrow{d} C_{n+3}(n) \xrightarrow{d} C_{n+2}(n) \xrightarrow{d} C_{n+1}(n)$$

We concentrate our studies to the subcomplex $(C_*(n), d)$, but in some cases we will also use the following subcomplex $(C_*(*), d')$ of the Grassmannian complex

$$\cdots \xrightarrow{d'} C_{n+3}(n+2) \xrightarrow{d'} C_{n+2}(n+1) \xrightarrow{d'} C_{n+1}(n)$$

2.2 Polylogarithmic Groups

From now on we will denote our field by F and $F - \{0, 1\}$ will be abbreviated as $F^{\bullet\bullet}$. In some texts $F^{\bullet\bullet}$ is also referred as doubly punctured affine line over F in ([7]). We will also denote $\mathbb{Z}[\mathbb{P}^1_F]$ as the free abelian group generated by $[x]$ where $x \in \mathbb{P}^1_F$.

Scissors congruence group:([22])The Scissors congruence group $\mathcal{B}(F)$ of F is defined as the quotient of the free abelian group $\mathbb{Z}[F^{\bullet\bullet}]$ by the subgroup generated by the elements of the form

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-y}{1-x}\right] + \left[\frac{1-y^{-1}}{1-x^{-1}}\right] \text{ where } x \neq y,\ x, y \neq 0, 1$$

The above relation is the famous Abel's five-term relation for the dilogarithm. It can also be interpreted geometrically (in terms of scissors congruences) whence its name: Consider a an ideal polyhedron hyperbolic 3-space with five vertices $x_1, \ldots, x_5$. Divide this polyhedron into five tetrahedra by leaving out one vertex at a time i.e $\{x_2, x_3, x_4, x_5\}$ and $\{x_1, x_3, x_4, x_5\}$ with common face $\{x_3, x_4, x_5\}$ and three other tetrahedra $\{x_1, x_2, x_4, x_5\}$, $\{x_1, x_2, x_3, x_5\}$ and $\{x_1, x_2, x_3, x_4\}$ so that the sum of first two volumes is same as the sum of last three volumes(when taken with the right orientation). This volumes identity is mimicked in the following relation (where $r(a, b, c, d)$ denotes the cross-ratio of four points)

$$[r(x_2, x_3, x_4, x_5)]+[r(x_1, x_3, x_4, x_5)] = [r(x_1, x_2, x_4, x_5)]+[r(x_1, x_2, x_3, x_5)]+[r(x_1, x_2, x_3, x_4)]$$

This relation is a version of the above five-term relation.

2.3 Bloch-Suslin and Goncharov's polylog complexes

In this section we will closely follow [9] and [10].

2.3.1 Weight 1:

We define subgroup $R_1(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ by

$$R_1(F) = \langle [xy] - [x] - [y], x, y \in F^\times - \{1\} \rangle$$

The map $\delta_1 : \mathcal{B}_1(F) \to F^\times, [a] \mapsto a$ is defined as an isomorphism (see §1 of [9]), so we have $\mathcal{B}_1(F) = F^\times$.

2.3.2 Weight 2:

First we define the subgroup $R_2(F) \subset \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]$

$$R_2(F) := \left\langle \sum_{i=0}^{4} (-1)^i [r(x_0, \dots, \hat{x}_i, \dots, x_4)], \quad x_i \in \mathbb{P}^1_F \right\rangle$$

where $r(x_0, x_1, x_2, x_3) = \frac{(x_0 - x_3)(x_1 - x_2)}{(x_0 - x_2)(x_1 - x_3)}$ is the cross-ratio of four points and δ_2 is defined as

$$\delta_2 : \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}] \to \bigwedge{}^2 F^\times$$

$$[x] \mapsto (1 - x) \wedge x$$

where $\bigwedge{}^2 F^\times = F^\times \otimes_{\mathbb{Z}} F^\times / \langle x \otimes_{\mathbb{Z}} x | x \in F^\times \rangle$. One has $\delta_2 (R_2(F)) = 0$. Now we can define the free abelian group $\mathcal{B}_2(F)$ which is generated by $[x] \in \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]$ and quotient by the subgroup $R_2(F) \subset \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]$, i.e.

$$\mathcal{B}_2(F) = \frac{\mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]}{R_2(F)}$$

and we get a complex $B_F(2)$ called the Bloch-Suslin complex of F

$$B_F(2) : \qquad \mathcal{B}_2(F) \xrightarrow{\delta} \bigwedge{}^2 F^\times$$

where first term is in degree 1 and second term in degree 2 and δ is induced from δ_2 due to fact $\delta_2 (R_2(F)) = 0$.

2.3.3 Weight 3:

Consider the triple-ratio of six points $r_3 \in \mathbb{Z}[\mathbb{P}^1_F]$ which is defined as $r_3 : C_6(\mathbb{P}^2_F) \to \mathbb{Z}[\mathbb{P}^1_F]$, where $C_6(\mathbb{P}^2_F)$ is a free abelian group generated by the configurations of 6 points in generic position over $\mathbb{P}^1_F$

$$r_3(l_0,\dots,l_5) = \mathrm{Alt}_6\frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)},$$

where l_i is the point in $\mathbb{P}^2_F$, $\Delta(l_i,l_j,l_k) = \langle\omega, l_i\wedge l_j\wedge l_k\rangle$ and $\omega\in\det V^*$. Now define the relation $R_3(F)\in\mathbb{Z}[\mathbb{P}^1_F]$

$$R_3(F) := \left\langle \sum_{i=0}^{6}(-)^i r_3(l_0,\dots,\hat{l}_i,\dots,l_6)\right| \quad (l_0,\dots,\hat{l}_i,\dots,l_6)\in C_6(\mathbb{P}^2_F)\left.\vphantom{\sum_{i=0}^{6}}\right\rangle$$

One can define $\mathcal{B}_3(F)$ as the free abelian group generated by $[x]\in\mathbb{Z}[\mathbb{P}^1_F]$ and quotient by $R_3(F)$, $[0]$ and $[\infty]$. Thus we get the complex $B_F(3)$

$$B_F(3): \qquad \mathcal{B}_3(F)\xrightarrow{\delta}\mathcal{B}_2(F)\otimes_{\mathbb{Z}}F^\times\xrightarrow{\delta}\bigwedge{}^3F^\times$$

2.3.4 Weight ≥ 3:

Here we will define group $\mathcal{B}_n(F)$. Suppose $\mathcal{R}_n(F)$ is defined already, we set

$$\mathcal{B}_n(F) = \frac{\mathbb{Z}[\mathbb{P}^1_F]}{\mathcal{R}_n(F)}$$

and the morphism

$$\delta_n : \mathbb{Z}[\mathbb{P}^1_F]\to\mathcal{B}_{n-1}(F)\otimes F^\times$$
$$[a]\mapsto\begin{cases}0 & \text{if } x = 0,1,\infty\\ [x]_{n-1}\wedge x & \text{otherwise}\end{cases}$$

where $[x]_n$ is class of $[x]$ in $\mathcal{B}_n(F)$. We find more important is the case for $n\geq 2$, where we define

$$\mathcal{A}_n(F) = \ker\delta_n$$

and $\mathcal{R}_n(F)\subset\mathbb{Z}[\mathbb{P}^1_F]$ is generated by the elements $\alpha(0)-\alpha(1)$,$[\infty]$ and $[0]$, where $\alpha(t)$ runs through all the elements of $\mathcal{A}_n(F(t))$, for an indeterminate t.

Lemma 2.3.1. *(Goncharov) For $n\geq 2$, $\mathcal{R}_n(F)\subset\ker\delta_n$*

Proof: See lemma 1.16 of [9]. □

Goncharov defines the following complex ([9],[10]) for the group $\mathcal{B}_n(F)$.

$$\mathcal{B}_n(F)\xrightarrow{\delta}\mathcal{B}_{n-1}\otimes F^\times\xrightarrow{\delta}\mathcal{B}_{n-2}\otimes\bigwedge{}^2F^\times\xrightarrow{\delta}\cdots\xrightarrow{\delta}\mathcal{B}_2(F)\bigwedge{}^{n-2}F^\times\xrightarrow{\delta}\frac{\bigwedge^nF^\times}{2-\text{torsion}} \qquad (2.2)$$

2.4 Infinitesimal Complexes (Cathelineau's Complexes)

There are two versions of infinitesimal complex or infinitesimal groups. In the literature the first one was introduced by Cathelineau [4] while the other version was introduced by Bloch-Esnault [1] also called "additive". The latter version is beyond the scope of this text we will discuss here only the former one.

Cathelineau ([4],[3]) has defined the group (in fact an F-vector spaces) as an infinitesimal analogue of Goncharov's groups $\mathcal{B}_n(F)$ as follows

1. We define $\beta_1(F) = F$

2. One can define $\beta_2(F)$ as

$$\beta_2(F) = \frac{F[F^{\bullet\bullet}]}{r_2(F)}$$

where $r_2(F)$ is the kernel of the map

$$\partial_2 : F[F^{\bullet\bullet}] \to F \otimes_F F^{\times}$$

$$[a] \mapsto a \otimes_F a + (1-a) \otimes_F (1-a)$$

Cathelineau [4] has shown that $r_2(F)$ is given as the subvector space of $F[F^{\bullet\bullet}]$ spanned by the elements

$$[a] - [b] + a\left[\frac{b}{a}\right] + (1-a)\left[\frac{1-b}{1-a}\right], a, b \in F^{\bullet\bullet}, a \neq b,$$

hence passing to the quotient by $r_2(F)$ we obtain the complex

$$\beta_2(F) \xrightarrow{\partial} F \otimes_F F^{\times} \tag{2.3}$$

$$\partial : \langle a \rangle_2 \mapsto a \otimes a + (1-a) \otimes (1-a)$$

3. For $n \geq 3$, the F-vector space $\beta_n(F)$ is defined as

$$\beta_n(F) = \frac{F[F^{\bullet\bullet}]}{r_n(F)}$$

where $r_n(F)$ is kernel of the map

$$\partial_n : F[F^{\bullet\bullet}] \to (\beta_{n-1}(F) \otimes F^{\times}) \oplus (F \otimes \mathcal{B}_{n-1}(F))$$

$$[a] \mapsto \langle a \rangle_{n-1} \otimes a + (1-a) \otimes [a]_{n-1}$$

where $\langle a\rangle_k$ is the class of $[a]$ in $\beta_k(F)$ and $[a]_k$ is the class of $[a]$ in $\mathcal{B}_k(F)$. For $n = 2$, we have the following complex of F-vector spaces.

$$\beta_3(F) \xrightarrow{\partial} (\beta_2(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) \xrightarrow{\partial} F \otimes \bigwedge{}^2 F^\times \tag{2.4}$$

where

$$\partial : \langle a\rangle_3 \mapsto \langle a\rangle_2 \otimes a + (1-a) \otimes [a]_2$$

$$\partial : \langle a\rangle_2 \otimes b + x \otimes [y]_2 \mapsto -(a \otimes a \wedge b + (1-a) \otimes (1-a) \wedge b) + x \otimes (1-y) \wedge y$$

Before the following lemma we shall introduce Kähler differentials (see §25 in [17] and §26 in [18]). First, recall the definition of a derivation map $D \in Der(A, M)$ for a ring A and an A-module M is $D : A \to M$ and this map satisfies $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$. Now an A-module $\Omega_{A/F}$ is generated by $\{da | a \in A\}$ so that the uniqueness of a linear map $f : \Omega_{A/F} \to M$ satisfying $D = f \circ d$ is obvious (see p192 of [17]). If $a \in A$ then the element $da \in \Omega_{A/F}$ and called the differential of a and the A-module $\Omega_{A/F}$ is called the module of Kähler differentials.

Lemma 2.4.1. *(Cathelineau [3],[4]) The complexes 2.3 and 2.4 are quasi-isomorphic to Ω^i_F through the maps $d\log : \bigwedge^i F^\times \to \Omega^i_F$ so that the following sequences*

$$0 \to \beta_2(F) \xrightarrow{\partial} F \otimes F^\times \xrightarrow{d\log} \Omega^1_F \to 0$$

$$0 \to \beta_3(F) \xrightarrow{\partial} (\beta_2(F) \otimes F^\times) \oplus (F \otimes B_2(F)) \xrightarrow{\partial} F \otimes \wedge^2 F^\times \xrightarrow{d\log} \Omega^2_F \to 0$$

are exact. Here Ω^i_F is the vector space of Kähler differential with the respective definitions of $d\log$ as $d\log(a \otimes b) = a\frac{db}{b}$ and $d\log(a \otimes b \wedge c) = a\frac{db}{b} \wedge \frac{dc}{c}$.

Functional equations in $\beta_2(F)$

Here we will mainly focus on the work in ([7])

1. The two-term relation

$$\langle a\rangle_2 = \langle 1-a\rangle_2$$

2. The inversion relation.

$$\langle a\rangle_2 = -a\left\langle \frac{1}{a}\right\rangle_2$$

3. The distribution relation

$$\langle a^m \rangle_2 = \sum_{\zeta^m=1} \frac{1-a^m}{1-\zeta a} \langle \zeta a \rangle_2$$

4. The four-term relation in $F[F^{\bullet\bullet}]$.

$$\langle a \rangle_2 - \langle b \rangle_2 + a \left\langle \frac{b}{a} \right\rangle_2 + (1-a) \left\langle \frac{1-b}{1-a} \right\rangle_2 = 0, \quad a \neq b \tag{2.5}$$

The above equation is an infinitesimal version of the famous five-term relation and it can be deduced directly from the following form of five term relation [22].

$$[a]_2 - [b]_2 + \left[\frac{b}{a}\right]_2 - \left[\frac{1-b}{1-a}\right]_2 + \left[\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right]_2 = 0$$

Functional equation in $\beta_3(F)$

Here as well we will mainly focus on the work of ([7])

1. The three-term relation.

$$\langle 1-a \rangle_3 - \langle a \rangle_3 + a \left\langle 1 - \frac{1}{a} \right\rangle_3 = 0 \tag{2.6}$$

2. The inversion relation.

$$\langle a \rangle_3 = -a \left\langle \frac{1}{a} \right\rangle_3 \tag{2.7}$$

The inversion relation is a consequence of the three-term relation (2.6) (see lemma 3.11 of [7]).

3. The distribution relation

$$\langle a^m \rangle_3 = m \sum_{\zeta^m=1} \frac{1-a^m}{1-\zeta a} \langle \zeta a \rangle_3$$

4. The 22-term relation.([7])

There are number of ways to write it and one of them is the following.

$$
\begin{aligned}
& c\langle a\rangle_3 - c\langle b\rangle_3 + (a-b+1)\langle c\rangle_3 \\
& +(1-c)\langle 1-a\rangle_3 - (1-c)\langle 1-b\rangle_3 + (b-a)\langle 1-c\rangle_3 \\
& -a\left\langle \frac{c}{a}\right\rangle_3 + b\left\langle \frac{c}{b}\right\rangle_3 + ca\left\langle \frac{b}{a}\right\rangle_3 \\
& -(1-a)\left\langle \frac{1-c}{1-a}\right\rangle_3 + (1-b)\left\langle \frac{1-c}{1-b}\right\rangle_3 + c(1-a)\left\langle \frac{1-b}{1-a}\right\rangle_3 \\
& +c(1-a)\left\langle \frac{a(1-c)}{c(1-a)}\right\rangle_3 - c(1-b)\left\langle \frac{b(1-c)}{c(1-b)}\right\rangle_3 - b\left\langle \frac{ca}{b}\right\rangle_3 \\
& +(1-c)a\left\langle \frac{a-b}{a}\right\rangle_3 + (1-c)(1-a)\left\langle \frac{b-a}{1-a}\right\rangle_3 \\
& -(a-b)\left\langle \frac{(1-c)a}{a-b}\right\rangle_3 - (1-b)\left\langle \frac{c(1-a)}{1-b}\right\rangle_3 \\
& -(b-a)\left\langle \frac{(1-c)(1-a)}{b-a}\right\rangle_3 + c(a-b)\left\langle \frac{(1-c)b}{c(a-b)}\right\rangle_3 \\
& +c(b-a)\left\langle \frac{(1-c)(1-b)}{c(b-a)}\right\rangle_3 = 0
\end{aligned}
\tag{2.8}
$$

For $n > 7$ inversion and distribution existing relations are the only known elements in $\beta_n(F)$, while for $n \leq 7$ one can derive non-trivial elements from functional equations for Li_n [7]. Cathelineau's complex for $\beta_n(F)$ and for the higher Bloch groups $\mathcal{B}_k(F)$ $(2 \leq k \leq n-1)$ is the following(see §2 of [12]):

$$
\beta_n(F) \xrightarrow{\partial} \begin{smallmatrix} \beta_{n-1}(F)\otimes F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F) \end{smallmatrix} \xrightarrow{\partial} \begin{smallmatrix} \beta_{n-2}(F)\otimes\wedge^2 F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-2}(F)\otimes F^\times \end{smallmatrix} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \begin{smallmatrix} \beta_2(F)\otimes\wedge^{n-2} F^\times \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3} F^\times \end{smallmatrix} \xrightarrow{\partial} F\otimes\wedge^{n-1}F^\times \tag{2.9}
$$

We correct here a misprint in the map defined in §2 of [12] for the above complex. The above becomes a complex if we use a slightly different formula with the difference to put alternative signs for ∂: when $n = 2$ then we put

$$\partial : \langle a\rangle_2 \mapsto -(a\otimes a + (1-a)\otimes(1-a))$$

and when $n \geq 3$ then we propose to use

$$\partial : \langle a\rangle_n \mapsto \langle a\rangle_{n-1}\otimes a + (-1)^{n-1}(1-a)\otimes[a]_{n-1}$$

Note: By definition above $\beta_1(F) = F$ and when ∂ is applied to the group $\mathcal{B}_n(F)$ then it agrees with δ_n defined by ([7])

$$
\delta : [a]_n \mapsto \begin{cases} [a]_{n-1}\otimes a, & \text{for } n \geq 3 \\ (1-a)\wedge a, & \text{for } n = 2 \end{cases}
$$

Lemma 2.4.2. *The sequence* (2.9) *is a complex under the map ∂ defined above.*

Proof: We can show that the above sequence is a complex by considering the part for $2 \le k \le n-2$

$$\cdots \xrightarrow{\partial} \begin{matrix} \beta_{n-k+1}(F)\otimes\wedge^{k-1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-2}F^\times \end{matrix} \xrightarrow{\partial} \begin{matrix} \beta_{n-k}(F)\otimes\wedge^{k}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k}(F)\otimes\wedge^{k-1}F^\times \end{matrix} \xrightarrow{\partial} \begin{matrix} \beta_{n-k-1}(F)\otimes\wedge^{k+1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k-1}(F)\otimes\wedge^{k}F^\times \end{matrix} \xrightarrow{\partial} \cdots$$

Let $\langle x\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} y_i + a \otimes [b]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} c_j \in \begin{matrix} \beta_{n-k+1}(F)\otimes\wedge^{k-1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-2}F^\times \end{matrix}$

Now compute $\partial\left(\partial\left(\langle x\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} y_i + a \otimes [b]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} c_j\right)\right)$.

To make calculation easy we calculate first

$$\begin{aligned}
&\partial\left(\partial\left(\langle x\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} y_i\right)\right)\\
=&\partial\left(\langle x\rangle_{n-k} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i + (-1)^{n-k}(1-x) \otimes [x]_{n-k} \otimes \bigwedge_{i=1}^{k-1} y_i\right)\\
=&\langle x\rangle_{n-k-1} \otimes \underbrace{x \wedge x}_{0} \wedge \bigwedge_{i=1}^{k-1} y_i + (-1)^{n-k-1}(1-x) \otimes [x]_{n-k-1} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i\\
&+ (-1)^{n-k}(1-x) \otimes [x]_{n-k-1} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i\\
=&0
\end{aligned}$$

then find

$$\begin{aligned}
\partial\left(\partial\left(a \otimes [b]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} c_j\right)\right) =&\partial\left(a \otimes [b]_{n-k} \otimes b \wedge \bigwedge_{j=1}^{k-2} c_j\right)\\
=&a \otimes [b]_{n-k-1} \otimes \underbrace{b \wedge b}_{0} \wedge \bigwedge_{j=1}^{k-2} c_j\\
=&0
\end{aligned}$$

There is only one case left for $k = 1$ with the correction $\bigwedge_{i=0}^{0} y_i = 1 \in \mathbb{Z}$ and using $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ for any ring R. □

2.4.1 Derivation in F-vector space

Let F be a field and $D \in Der_{\mathbb{Z}}(F, F)$ be an absolute derivation, (see §25 of [17] and §6 of [7]) we will also write simply as $D \in Der_{\mathbb{Z}}(F)$. For example if $x \in F$ then its derivative over $\mathbb{Z}$ will be represented by $D(x)$ and will be an element of F as well.

According to §6.1 in [7] we have $\tilde{f}_D : \mathbb{Z}[F] \to F[F^{\bullet\bullet}], [a] \mapsto \frac{D(a)}{a(1-a)}[a]$ induces a map

$$\tau_{2,D} : \mathcal{B}_2(F) \to \beta_2(F), [a]_2 \mapsto \frac{D(a)}{a(1-a)}\langle a\rangle_2$$

We define an F-vector space $\beta_2^D(F)$ generated by $[\![a]\!]^D$ for $a \in F^{\bullet\bullet}$ and subject to the five-term relation

$$[\![a]\!]^D - [\![b]\!]^D + \left[\!\!\left[\frac{b}{a}\right]\!\!\right]^D - \left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]^D + \left[\!\!\left[\frac{1-b^{-1}}{1-a^{-1}}\right]\!\!\right]^D \text{ where } a \neq b, \quad 1-a \neq 0,$$

where $[\![a]\!]^D := \frac{D(a)}{a(1-a)}[a]$ and $[a] \in F[F^{\bullet\bullet}]$. Furthermore, we have

$$\partial_2^D : F[F^{\bullet\bullet}] \to F \otimes F^\times$$

with

$$\partial_2^D : [\![a]\!]^D \mapsto -D\log(1-a) \otimes a + D\log(a) \otimes (1-a),$$

where $D\log a = \frac{D(a)}{a}$. We identify $\mathrm{Im}(\tau_{2,D})\,(\subset \beta_2(F))$ with $\beta_2^D(F)$. We can also write a variant of Cathelineau's complex by using the F-vector space

$$\beta_2^D(F) \subset F[F^{\bullet\bullet}]/(\text{five-term relation}),$$

as

$$\beta_2^D(F) \xrightarrow{\partial^D} F \otimes F^\times$$

with

$$\partial^D : [\![a]\!]_2^D \mapsto -D\log(1-a) \otimes a + D\log(a) \otimes (1-a)$$

where $[\![a]\!]_2^D = \frac{D(a)}{a(1-a)}\langle a\rangle_2$.

We also want to define F-vector spaces $\beta_n^D(F)$ for $n \geq 3$. For this we use a slightly different construction by Cathelineau which in the case $n = 2$ gives his $\mathbf{b}_2(F)$ (see [4]). For this he divides $F[F^{\bullet\bullet}]$ by the kernel of the map ∂_2, of which an important element is the Cathelineau's four-term relation. By Remark 2.4.3 below the differential of the five-term relation in $\mathcal{B}_2(F)$ leads to Cathelineau's four-term relation. For later purpose we note that the differential of Goncharov's 22-term relation in $\mathcal{B}_3(F)$ vanishes in $\beta_3(F)$ for any $D \in Der_{\mathbb{Z}}(F)$ (see Proposition 6.10 of [7]). We define

$$\beta_3^D(F) = \frac{F[F^{\bullet\bullet}]}{\rho_3^D(F)}$$

where $\rho_3^D(F)$ is the kernel of the map

$$\partial_3^D : [\![a]\!]^D \mapsto [\![a]\!]_2^D \otimes a + D\log(a) \otimes [a]_2$$

and $\beta_n^D(F) = \frac{F[F^{\bullet\bullet}]}{\rho_n^D(F)}$ for $n > 3$, where $\rho_n^D(F)$ is the kernel of the map

$$\partial_n^D : [\![a]\!]^D \mapsto [\![a]\!]_{n-1}^D \otimes a + (-1)^{n-1} D\log(a) \otimes [a]_{n-1}$$

The following is a complex which can be proved in a completely analogous way as Lemma 2.4.2 (except that for given F, a is replaced by $\frac{D(a)}{a}$):

$$\beta_n^D(F) \xrightarrow{\partial^D} \begin{matrix} \beta_{n-1}^D(F)\otimes F^\times \\ \oplus \\ F\otimes B_{n-1}(F) \end{matrix} \xrightarrow{\partial^D} \cdots \xrightarrow{\partial^D} \begin{matrix} \beta_2^D(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ F\otimes B_2(F)\otimes\wedge^{n-3}F^\times \end{matrix} \xrightarrow{\partial^D} F\otimes\wedge^{n-1}F^\times$$

Now we have an F-vector space $\beta_2^D(F)$ which is an intermediate stage between a $\mathbb{Z}$-module $\mathcal{B}_2(F)$ and an F-vector space $\beta_2(F)$ and has two-term and inversion relations same as $\mathcal{B}_2(F)$.

2.4.2 Functional Equations in $\beta_2^D(F)$

The inversion and two-term relations in $\beta_2^D(F)$ are quite similar to group $\mathcal{B}_2(F)$.

1. Two-term relation:

$$[\![a]\!]_2^D = -[\![1-a]\!]_2^D$$

We know from Cathelineau's F-vector space $\beta_2(F)$.

$$\begin{aligned} \langle a\rangle_2 &= \langle 1-a\rangle_2 \\ \frac{D(a)}{a(1-a)}\langle a\rangle_2 &= \frac{D(a)}{a(1-a)}\langle 1-a\rangle_2 \\ \frac{D(a)}{a(1-a)}\langle a\rangle_2 &= -\frac{D(1-a)}{(1-a)\{1-(1-a)\}}\langle 1-a\rangle_2 \\ [\![a]\!]_2^D &= -[\![1-a]\!]_2^D \end{aligned}$$

2. Inversion relation:

$$[\![a]\!]_2^D = -\left[\!\!\left[\frac{1}{a}\right]\!\!\right]_2^D$$

The inversion relation in $\beta_2(F)$ is

$$\begin{aligned}
\langle a\rangle_2 &= -\,a\left\langle\frac{1}{a}\right\rangle_2\\
\frac{D(a)}{a(1-a)}\,\langle a\rangle_2 &= \frac{D(a)}{a(1-a)}\cdot -a\left\langle\frac{1}{a}\right\rangle_2\\
\frac{D(a)}{a(1-a)}\,\langle a\rangle_2 &= \frac{\frac{1}{a^2}D(a)}{\frac{1}{a}\left(1-\frac{1}{a}\right)}\left\langle\frac{1}{a}\right\rangle_2\\
\frac{D(a)}{a(1-a)}\,\langle a\rangle_2 &= -\,\frac{D\left(\frac{1}{a}\right)}{\frac{1}{a}\left(1-\frac{1}{a}\right)}\left\langle\frac{1}{a}\right\rangle_2\\
[\![a]\!]_2^D &= -\left[\!\!\left[\frac{1}{a}\right]\!\!\right]_2^D
\end{aligned}$$

3. The five-term relation:

$$[\![a]\!]_2^D - [\![b]\!]_2^D + \left[\!\!\left[\frac{b}{a}\right]\!\!\right]_2^D - \left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]_2^D + \left[\!\!\left[\frac{1-b^{-1}}{1-a^{-1}}\right]\!\!\right]_2^D = 0$$

Remark 2.4.3. *If we use the definition of $[\![a]\!]_2^D$ for certain $D \in Der_{\mathbb{Z}}(F)$,i.e., $D = a(1-a)\frac{\partial}{\partial a} + b(1-b)\frac{\partial}{\partial b} \in Der_{\mathbb{Z}}(F,F)$ where $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ are the usual partial derivatives then we see that $\left[\!\!\left[\frac{1-b^{-1}}{1-a^{-1}}\right]\!\!\right]_2^D = 0$. This is how Cathelineau arrived at his four-term relation (see example 3.1.6 later in the next chapter).*

2.5 The Tangent Complex to the Bloch-Suslin Complex

In this section mainly we will discuss text from [5]. Let $F[\varepsilon]_2 = F[\varepsilon]/\varepsilon^2$ be the ring of dual numbers for an arbitrary field F. We can define an $F^\times$-action in $F[\varepsilon]_2$ as follows. For $\lambda \in F^\times$,

$$\lambda : F[\varepsilon]_2 \to F[\varepsilon]_2, a + a'\varepsilon \mapsto a + \lambda a'\varepsilon$$

we denote this act by $\star$, so we use $\lambda \star (a + a'\varepsilon) = a + \lambda a'\varepsilon$.

Definition:

The *tangent group* $T\mathcal{B}_2(F)$ is defined as a $\mathbb{Z}$-module generated by the combinations $[a + a'\varepsilon]-[a] \in \mathbb{Z}[F[\varepsilon]_2]$, $\quad (a, a' \in F)$: for which we put the shorthand $\langle a; a'] := [a+a'\varepsilon]-[a]$

and quotient by the following relation

$$\begin{aligned}\langle a; a'] - \langle b; b'] + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)' \right] - \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)' \right] \\ + \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)' \right], \quad a, b \neq 0, 1, a \neq b \end{aligned} \tag{2.10}$$

where

$$\left(\frac{b}{a}\right)' = \frac{ab' - a'b}{a^2},$$

$$\left(\frac{1-b}{1-a}\right)' = \frac{(1-b)a' - (1-a)b'}{(1-a)^2}$$

and

$$\left(\frac{a(1-b)}{b(1-a)}\right)' = \frac{b(1-b)a' - a(1-a)b'}{(b(1-a))^2}$$

Remark 2.5.1. *See [5] for a discussion of $T\mathcal{B}_2(F)$, where the definition of $T\mathcal{B}_2(F)$ was justified using Lemma 3.1 of [5])*

We give a list of relations in $T\mathcal{B}_2(F)$ from [5]. These relations use the $\star$-action in $T\mathcal{B}_2(F)$. By specialization of the five-term relation (2.10), we find

1. Two-term relation:

$$\langle a; b]_2 = -\langle 1-a; -b]_2$$

2. Inversion relation:

$$\langle a; b]_2 = \left\langle \frac{1}{a}; -\frac{b}{a^2} \right]_2$$

3. Four-term relation:

If we use $a' = a(1-a)$ and $b' = b(1-b)$ then (2.10) becomes four-term relation (see remark 2.4.3).

$$\langle a; a(1-a)]_2 - \langle b; b(1-b)]_2 + a \star \left\langle \frac{b}{a}; \frac{b}{a}\left(1 - \frac{b}{a}\right) \right]_2 + (1-a) \star \left\langle \frac{1-b}{1-a}; \frac{1-b}{1-a}\left(1 - \frac{1-b}{1-a}\right) \right]_2 = 0,$$

where $a, b \neq 0, 1, a \neq b$.

The following map is an infinitesimal analogue of δ (defined in §2.3) and ∂ (defined in §2.4) above and Cathelineau called it *tangential map*.

$$T\mathcal{B}_2(F) \xrightarrow{\partial_\varepsilon} (F \otimes F^\times) \oplus \left(\bigwedge{}^2 F\right)$$

with

$$\partial_\varepsilon(\langle a;b]_2) = \left(\frac{b}{a}\otimes(1-a) + \frac{b}{1-a}\otimes a\right) + \left(\frac{b}{1-a}\wedge\frac{b}{a}\right)$$

The first term of the complex is in degree one and ∂_ε has degree +1.

Note that we get the direct sum of two spaces on the right side.

We would like to see the comparison of various complexes discussed above in the tubular form.

	complex	group	defining functional equation
Bloch-Suslin	$\mathcal{B}_2(F)\xrightarrow{\delta}\bigwedge^2 F^\times$	$\mathcal{B}_2(F)$	5-term
infinitesimal	$\beta_2(F)\xrightarrow{\partial} F\otimes F^\times$	$\beta_2(F)$	4-term
tangent	$T\mathcal{B}_2(F)\xrightarrow{\partial_\varepsilon}\begin{smallmatrix}F\otimes F^\times\\ \oplus\\ \bigwedge^2 F\end{smallmatrix}$	$T\mathcal{B}_2(F)$	5-term

	complex	group	defining functional equation
Goncharov	$\mathcal{B}_3(F)\xrightarrow{\delta}\mathcal{B}_2(F)\otimes F^\times\xrightarrow{\delta}\bigwedge^3 F^\times$	$\mathcal{B}_3(F)$	Goncharov’s 840-term(*)
infinitesimal	$\beta_3(F)\xrightarrow{\partial}\begin{smallmatrix}\beta_2^D(F)\otimes F^\times\\ \oplus\\ F\otimes\mathcal{B}_2(F)\end{smallmatrix}\xrightarrow{\partial} F\otimes\bigwedge^2 F^\times$	$\beta_3(F)$	Cathelineau’s 22-term
tangent	$T\mathcal{B}_3(F)\xrightarrow{\partial_\varepsilon}\begin{smallmatrix}T\mathcal{B}_2(F)\otimes F^\times\\ \oplus\\ F\otimes\mathcal{B}_2(F)\end{smallmatrix}\xrightarrow{\partial_\varepsilon}\begin{smallmatrix}F\otimes\bigwedge^2 F^\times\\ \oplus\\ \bigwedge^3 F\end{smallmatrix}$	$T\mathcal{B}_3(F)$	Not known (**)

(*): Goncharov’s 840-term relation is known defining functional equation of $\mathcal{B}_3(F)$ and $\ker\delta$ might be large.

(**): The defining functional equation is not known for the group $T\mathcal{B}_3(F)$ but we give a tangential version of Cathelineau’s 22-term relation which lies in $\ker\partial_\varepsilon$ (see §4.3)

Remark 2.5.2.

In some text (see [8]), groups $\beta_n(F)$ (by Cathelineau in [4]) and $\mathrm{TB}_n(F)$ (by Bloch-Esnault in [1]) are called two infinitesimal versions of the K-theory of a field F. For clarity we mention here that group $T\mathcal{B}_n(F)$ is different from $\mathrm{TB}_n(F)$. Cathelineau named $T\mathcal{B}(F)$ tangent group so we also call $T\mathcal{B}_n(F)$ the tangent group of $\mathcal{B}_n(F)$ and $\beta_n(F)$ will be called as the infinitesimal n-logarithmic group throughout this text.

Chapter 3

Infinitesimal complexes

There are some homomorphisms which relate Bloch-Suslin and Goncharov's complexes to Grassmannian complex([9],[10],[11]). This chapter will relate variant of Cathelineau's infinitesimal complex to the geometric configurations of Grassmannian complex. We will suggest here some suitable maps for this relation and then will verify the commutativity of the underlying diagrams. Goncharov used K-theory to prove the commutativity of his diagram in which he related his complex with the Grassmannian complex (see appendix of [13]) but here we are giving proof of the commutativity of diagram (3.2a)(see §3.2 below) without using K-theory we shall use combinatorial techniques with the rewriting of triple ratio into a product of two cross-ratios. The same technique can also be used in Goncharov's case (see appendix A).

Throughout this chapter we will work with modulo 2-torsion and use $D \in Der_{\mathbb{Z}}F$ as an absolute derivation for a field F. For §3.1 determinant Δ is defined as $\Delta(l_i, l_j) := \langle \omega, l_i \wedge l_j \rangle$, for $l_i, l_j \in V_2$, where $\omega \in \det V_2^*$ is volume form in V_2. For §3.2 determinant Δ is defined as $\Delta(l_i, l_j, l_k) := \langle \omega, l_i \wedge l_j \wedge l_k \rangle$ for $l_i, l_j, l_k \in V_3$, where $\omega \in \det V_3^*$ is volume form in V_3.

3.1 Infinitesimal Dilogarithm

Let $C_m(2)$ (or $C_m(\mathbb{P}^1_F)$) be the free abelian group generated by configurations $(l_0, \dots, l_{m-1})$ of m vectors in a two dimensional vector space V_2 over a field F (or m points in $\mathbb{P}^1_F$) in generic position. Configurations of m vectors in vector space V_2 are 2-tuples of vectors modulo $\mathrm{GL}_2(V_2)$-equivalence. Grassmannian subcomplex (see diagram 2.1a in §2.1) for this case is the following.

$$\cdots \xrightarrow{d} C_5(2) \xrightarrow{d} C_4(2) \xrightarrow{d} C_3(2)$$

$$d\colon (l_0, \ldots, l_{m-1}) \mapsto \sum_{i=0}^{m} (-1)^i (l_0, \ldots, \hat{l}_i, \ldots, l_{m-1})$$

We will outline the procedure initially for V_2 and then will proceed further for V_3. We will also use the process of derivation (see §2.4.1) in combination with cross-ratio to define our maps.

Consider the following diagram

$$\begin{tikzcd} C_5(2) \arrow[r, "d"] & C_4(2) \arrow[r, "d"] \arrow[d, "\tau_1^2"] & C_3(2) \arrow[d, "\tau_0^2"] \\ & \beta_2^D(F) \arrow[r, "\partial^D"] & F \otimes F^\times \end{tikzcd} \tag{3.1a}$$

where $\beta_2^D(F)$ and ∂^D are defined in §2.4.1, we define

$$\tau_0^2\colon (l_0, l_1, l_2) \mapsto \sum_{i=0}^{2} \frac{D\{\Delta(l_i, l_{i+2})\}}{\Delta(l_i, l_{i+2})} \otimes \Delta(l_i, l_{i+1}) \\ - \frac{D\{\Delta(l_{i+1}, l_i)\}}{\Delta(l_{i+1}, l_i)} \otimes \Delta(l_i, l_{i+2})\} \ i \bmod 3 \tag{3.1}$$

Note: The above can also be written as:

$$\sum_{i=0}^{2} \frac{D\{\Delta(l_i, l_{i+2})\}}{\Delta(l_i, l_{i+2})} \otimes \frac{\Delta(l_i, l_{i+1})}{\Delta(l_{i-1}, l_{i+1})}, \ i \bmod 3.$$

Furthermore, we put

$$\tau_1^2\colon (l_0, \ldots, l_3) \mapsto [\![r(l_0, \ldots, l_3)]\!]_2^D \tag{3.2}$$

where $[\![a]\!]_2^D = \frac{D(a)}{a(1-a)}\langle a\rangle$ (defined in §2.4.1) and $r(l_0, \ldots, l_3) = \frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}$ is the cross ratio of the points $(l_0, \ldots, l_3) \in C_4(\mathbb{P}_F^1)$(defined in §2.3).

To ensure well-definedness of our homomorphisms τ_0^2 and τ_1^2 above, we first show that the definition is independent of length of the vectors and volume form ω. Here are some results for verification.

Lemma 3.1.1. *τ_0^2 is independent of the volume form ω by the vectors in V_2.*

Proof: According to (3.1), τ_0^2 can be written for the vectors (l_0, l_1, l_2) as

$$\begin{aligned}\tau_0^2(l_0,l_1,l_2) = & \frac{D\{\Delta(l_0,l_2)\}}{\Delta(l_0,l_2)} \otimes \Delta(l_0,l_1) - \frac{D\{\Delta(l_0,l_1)\}}{\Delta(l_0,l_1)} \otimes \Delta(l_0,l_2) \\ & + \frac{D\{\Delta(l_1,l_0)\}}{\Delta(l_1,l_0)} \otimes \Delta(l_1,l_2) - \frac{D\{\Delta(l_1,l_2)\}}{\Delta(l_1,l_2)} \otimes \Delta(l_1,l_0) \\ & + \frac{D\{\Delta(l_2,l_1)\}}{\Delta(l_2,l_1)} \otimes \Delta(l_2,l_0) - \frac{D\{\Delta(l_2,l_0)\}}{\Delta(l_2,l_0)} \otimes \Delta(l_2,l_1)\end{aligned}$$

further we can also write as

$$\tau_0^2(l_0,l_1,l_2) = \frac{D\{\Delta(l_0,l_2)\}}{\Delta(l_0,l_2)} \otimes \frac{\Delta(l_0,l_1)}{\Delta(l_2,l_1)} - \frac{D\{\Delta(l_0,l_1)\}}{\Delta(l_0,l_1)} \otimes \frac{\Delta(l_0,l_2)}{\Delta(l_1,l_2)} + \frac{D\{\Delta(l_1,l_2)\}}{\Delta(l_1,l_2)} \otimes \frac{\Delta(l_2,l_0)}{\Delta(l_1,l_0)}$$

Changing the volume form $\omega \mapsto \lambda\omega$ does not change the expression on RHS, due to homogeneity of the terms of the RH factors. □

Next lemma will show independence of the length of the vectors.

Lemma 3.1.2. $\tau_0^2 \circ d(l_0, \ldots, l_3)$ *does not depend on the length of the vectors* l_i *in* V_2.

Proof: By using a simple calculation we can write

$$\begin{aligned}\tau_0^2 \circ d(l_0,\ldots,l_3) = & \frac{D\{\Delta(l_0,l_1)\Delta(l_2,l_3)\}}{\Delta(l_0,l_1)\Delta(l_2,l_3)} \otimes \frac{\Delta(l_0,l_2)\Delta(l_1,l_3)}{\Delta(l_0,l_3)\Delta(l_1,l_2)} \\ & - \frac{D\{\Delta(l_1,l_2)\Delta(l_0,l_3)\}}{\Delta(l_1,l_2)\Delta(l_0,l_3)} \otimes \frac{\Delta(l_1,l_3)\Delta(l_0,l_2)}{\Delta(l_0,l_1)\Delta(l_2,l_3)} \\ & + \frac{D\{\Delta(l_0,l_2)\Delta(l_1,l_3)\}}{\Delta(l_0,l_2)\Delta(l_1,l_3)} \otimes \frac{\Delta(l_0,l_3)\Delta(l_2,l_1)}{\Delta(l_0,l_1)\Delta(l_2,l_3)}\end{aligned} \tag{3.3}$$

now consider $\lambda \in F^\times$ and we know that $\frac{D(\lambda x)}{\lambda x} = \frac{D(x)}{x}$ for $\lambda \in F^\times$ and the other part of the right hand side is a cross-ratio. □

Note: Since τ_1^2 is defined via cross-ratio and $d\log$ so there is no need to check things that are mandatory for τ_0^2.

Proposition 3.1.3. *The diagram below is commutative.*

$$\begin{tikzcd} C_4(2) \arrow[r, "d"] \arrow[d, "\tau_1^2"] & C_3(2) \arrow[d, "\tau_0^2"] \\ \beta_2^D(F) \arrow[r, "\partial^D"] & F \otimes F^\times \end{tikzcd} \tag{3.1b}$$

Proof: The first thing is to calculate $\partial^D \circ \tau_1^2(l_0, \ldots, l_3)$ because we have already computed $\tau_0^2 \circ d(l_0, \ldots, l_3)$ in (3.3) then by (3.2)

$$\tau_1^2(l_0, \cdots, l_3) = \left[\!\!\left[\frac{\Delta(l_0, l_3)\Delta(l_1, l_2)}{\Delta(l_0, l_2)\Delta(l_1, l_3)} \right]\!\!\right]_2^D$$

According to this we can identify $l_0, \ldots, l_3$ with points in $\mathbb{P}_F^1$, then by the 3-fold transitivity of $\mathrm{PGL}_2(F)$ any $(l_0, \ldots, l_3) \in (\mathbb{P}_F^1)^4$ in generic position is $\mathrm{PGL}_2(F)$-equivalent to $(0, \infty, 1, a)$ for some $a \in F$

$$\tau_1^2\,(0, \infty, 1, a) = \frac{D(a)}{a(1-a)}\langle a \rangle_2 = [\![a]\!]_2^D \text{ for any } a \in \mathbb{P}_F^1 - \{0, 1, \infty\}$$

where $D\log(a) = \frac{D(a)}{a}$. Calculate $\partial^D([\![a]\!]_2^D)$

$$= -\frac{D(1-a)}{(1-a)} \otimes a + \frac{D(a)}{a} \otimes (1-a)$$

For the vectors in $C_4(2)$ and by using the identity (4.1) we can write

$$\begin{aligned}\partial^D \circ \tau_1^2(l_0, \ldots, l_3) = &- \frac{D\left\{\frac{\Delta(l_0,l_1)\Delta(l_2,l_3)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}\right\}}{\frac{\Delta(l_0,l_1)\Delta(l_2,l_3)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}} \otimes \frac{\Delta(l_0, l_3)\Delta(l_1, l_2)}{\Delta(l_0, l_2)\Delta(l_1, l_3)} \\ &+ \frac{D\left\{\frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}\right\}}{\frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}} \otimes \frac{\Delta(l_0, l_1)\Delta(l_2, l_3)}{\Delta(l_0, l_2)\Delta(l_1, l_3)}\end{aligned}$$

by using $\frac{D(\frac{a}{b})}{(\frac{a}{b})} = \frac{D(a)}{a} - \frac{D(b)}{b}$ and then cancelling two terms we can convert the above into (3.3) and the diagram (3.1b) is commutative. □

Further consider the diagram (3.1a) and note that $\tau_1^2 \circ d$ becomes

$$\tau_1^2 \circ d(l_0, \ldots, l_4) = \sum_{i=0}^{4} (-1)^i [\![r(l_0, \ldots, \hat{l}_i, \ldots, l_4)]\!]_2^D \tag{3.4}$$

Now we can further check that $\tau_1^2 \circ d(l_0, \ldots, l_4) \in \ker(\partial^D)$

$$\begin{aligned}&\partial^D \circ (\tau_1^2 \circ d(l_0, \ldots, l_4)) \\ &= \sum_{i=0}^{4} \Big(- \frac{D\left\{1 - r(l_0, \ldots, \hat{l}_i, \ldots, l_4)\right\}}{1 - r(l_0, \ldots, \hat{l}_i, \ldots, l_4)} \otimes r(l_0, \ldots, \hat{l}_i, \ldots, l_4) \\ &\qquad + \frac{D\left\{r(l_0, \ldots, \hat{l}_i, \ldots, l_4)\right\}}{r(l_0, \ldots, \hat{l}_i, \ldots, l_4)} \otimes \left\{1 - r(l_0, \ldots, \hat{l}_i, \ldots, l_4)\right\} \Big)\end{aligned}$$

From now on we will write (ij) for $\Delta(l_i, l_j)$ in short. The above expression can also be written for each value of i's, e.g.

$$\text{for } i=0 \text{ we have } -\frac{D\left\{\frac{(12)(43)}{(13)(42)}\right\}}{\frac{(12)(43)}{(13)(42)}} \otimes \frac{(14)(23)}{(13)(24)} + \frac{D\left\{\frac{(14)(23)}{(13)(24)}\right\}}{\frac{(14)(23)}{(13)(42)}} \otimes \frac{(12)(43)}{(13)(42)}$$

and similarly for others as well.

If we multiply out, using $\frac{D(ab)}{ab} = \frac{D(a)}{a} + \frac{D(b)}{b}$ and start to collect each term of the form $\frac{D(ij)}{(ij)} \otimes \cdots$ from the above i.e. fix i and j, calculate the sum of all, then we will be able to see that every individual term of $\frac{D(ij)}{(ij)} \otimes \cdots$ is 0. For example $\frac{D(01)}{(01)} \otimes \frac{(04)(13)}{(03)(14)} \frac{(02)(14)}{(04)(12)} \frac{(03)(12)}{(02)(13)} = 0$ since the RHS is 2-torsion in $F^{\times}$ so we can easily say that the above is zero and $\tau_1^2 \circ d \in \ker(\partial^D)$.

Projeced cross-ratio: For $l_0, \ldots, l_4 \in \mathbb{P}^2_F$, $r(l_0|l_1, l_2, l_3, l_4)$ is the projected cross-ratio of four points $l_0, \ldots, l_4$ projeced from l_0 and is defined as

$$r(l_0|l_1, l_2, l_3, l_4) = \frac{\Delta(l_0, l_1, l_4)\Delta(l_0, l_2, l_3)}{\Delta(l_0, l_1, l_3)\Delta(l_0, l_2, l_4)},$$

where $\Delta(l_i, l_j, l_k)$ is a 3×3 determinant for $l_i, l_j, l_k \in \mathbb{P}^2_F$

Lemma 3.1.4. *(Goncharov, A. B., [9]) Let $x_0, \ldots, x_4$ be five points in generic position in $\mathbb{P}^2_F$. Then*

$$\sum_{i=0}^{4} (-1)^i [r(x_i|x_0, \ldots, \hat{x}_i, \ldots, x_4)] = 0 \in \mathcal{B}_2(F),$$

where $r(x_0|x_1, x_2, x_3, x_4)$ is the projected cross-ratio of four points $x_1, \ldots, x_4$ projected from x_0

See Lemma 2.18 in [9] for the proof. □

In continuation of the above lemma we have a similar result here which shows that the projected five-term (or four-term in special condition) relation can also be presented for $\beta_2^D(F)$ in the same way using geometric configurations of five points in $\mathbb{P}^2_F$.

Lemma 3.1.5. *Let $x_0, \ldots, x_4$ be 5 points in generic position in $\mathbb{P}^2_F$ then, for any $D \in Der_{\mathbb{Z}}F$*

$$\sum_{i=0}^{4} (-1)^i [\![r(x_i|x_0, \ldots, \hat{x}_i, \ldots, x_4)]\!]_2^D = 0 \in \beta_2^D(F) \tag{3.5}$$

Proof: If $x_0, \ldots, x_4$ in $\mathbb{P}^2_F$ then Lemma 3.1.4 gives projected five-term relation

$$\sum_{i=0}^{4} (-1)^i [r(x_i|x_0, \ldots, \hat{x}_i, \ldots, x_4)] = 0 \in B_2(F).$$

According to definition of $D \in Der_{\mathbb{Z}}F$ in §2.4.1 above (3.5) is the five-term relation in $\beta_2^D(F)$. □

Example 3.1.6.

By appropriate specialization of the configuration in $C_5(2)$, we can use (*) to produce Cathelineau's four-term relation from the geometric configurations by using the operator $D = a(1-a)\frac{\partial}{\partial a} + b(1-b)\frac{\partial}{\partial b}$ for $F = K(a,b)$ where a and b are indeterminates over the field K and $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ are the usual partial derivatives (see §6 of [7]). Let $(0,\infty,1,a,b) \in (\mathbb{P}_F^1)^5$ in generic position be the five-tuple corresponding to $(l_0,\dots,l_4) = \left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}a\\1\end{pmatrix},\begin{pmatrix}b\\1\end{pmatrix}\right) \in C_5(2)$. Calculate all possible determinants formed by $(l_0,\dots,l_4) \in C_5(2)$, i.e. $\Delta(l_i,l_j)$ for $0 \le i < j \le 4$, put all of them in (3.4), we get

$$[\![a]\!]_2^D - [\![b]\!]_2^D + \left[\!\!\left[\frac{b}{a}\right]\!\!\right]_2^D - \left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]_2^D + \left[\!\!\left[\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right]\!\!\right]_2^D = 0$$

since $\tau_1^2 \circ d \in \ker(\partial^D)$, then use D defined above, calculate each term of the above:

$$\begin{aligned}[\![a]\!]_2^D &= \frac{D(a)}{a(1-a)}\langle a\rangle_2\\ &= \frac{a(1-a)\frac{\partial}{\partial a}(a) + b(1-b)\frac{\partial}{\partial b}(0)}{a(1-a)}\langle a\rangle_2\\ &= 1\langle a\rangle_2\end{aligned}$$

for the second term

$$\begin{aligned}[\![b]\!]_2^D &= \frac{D(b)}{b(1-b)}\langle b\rangle_2\\ &= \frac{a(1-a)\frac{\partial}{\partial a}(0) + b(1-b)\frac{\partial}{\partial b}(b)}{b(1-b)}\langle b\rangle_2\\ &= 1\langle b\rangle_2\end{aligned}$$

for the third term

$$\begin{aligned}\left[\!\!\left[\frac{b}{a}\right]\!\!\right]_2^D &= \frac{D\left(\frac{b}{a}\right)}{\frac{b}{a}\left(1-\frac{b}{a}\right)}\left\langle\frac{b}{a}\right\rangle_2\\ &= \frac{a(1-a)\frac{\partial}{\partial a}\left(\frac{b}{a}\right) + b(1-b)\frac{\partial}{\partial b}\left(\frac{b}{a}\right)}{\frac{b}{a}\left(1-\frac{b}{a}\right)}\left\langle\frac{b}{a}\right\rangle_2\\ &= a\left\langle\frac{b}{a}\right\rangle_2\end{aligned}$$

for fourth term

$$\begin{aligned}\left[\!\!\left[\frac{1-b}{1-a}\right]\!\!\right]_2^D =& \frac{D\left(\frac{1-b}{1-a}\right)}{\frac{1-b}{1-a}\left(1-\frac{1-b}{1-a}\right)}\left\langle\frac{1-b}{1-a}\right\rangle_2\\ =&\frac{a(1-a)\frac{\partial}{\partial a}\left(\frac{1-b}{1-a}\right)+b(1-b)\frac{\partial}{\partial b}\left(\frac{1-b}{1-a}\right)}{\frac{1-b}{1-a}\left(1-\frac{1-b}{1-a}\right)}\left\langle\frac{1-b}{1-a}\right\rangle_2\\ =&-(1-a)\left\langle\frac{1-b}{1-a}\right\rangle_2\end{aligned}$$

for the last term

$$\begin{aligned}\left[\!\!\left[\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right]\!\!\right]_2^D =& \frac{D\left(\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right)}{\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\left(1-\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right)}\left\langle\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right\rangle_2\\ =&\frac{a(1-a)\frac{\partial}{\partial a}\left(\frac{(1-b)a}{(1-a)b}\right)+b(1-b)\frac{\partial}{\partial b}\left(\frac{(1-b)a}{(1-a)b}\right)}{\frac{(1-b)a}{(1-a)}b\left(1-\frac{(1-b)a}{(1-a)b}\right)}\left\langle\frac{1-\frac{1}{b}}{1-\frac{1}{a}}\right\rangle_2\\ =&0\end{aligned}$$

From this we retrieve Cathelineau's four-term relation.

$$\langle a\rangle-\langle b\rangle+a\left\langle\frac{b}{a}\right\rangle+(1-a)\left\langle\frac{1-b}{1-a}\right\rangle=0 \tag{3.6}$$

3.2 Infinitesimal Trilogarithm

Let $C_m(3)$ (or $C_m(\mathbb{P}^2_F)$) be the free abelian group generated by the configurations of m vectors in a three dimensional vector space V_3 over a field F (or m points in $\mathbb{P}^2_F$) in generic position. Consider the following diagram

$$\begin{tikzcd} C_6(3) \arrow[r, "d"] \arrow[d, "\tau_2^3"] & C_5(3) \arrow[r, "d"] \arrow[d, "\tau_1^3"] & C_4(3) \arrow[d, "\tau_0^3"] \\ \beta_3^D(F) \arrow[r, "\partial"] & (\beta_2^D(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F)) \arrow[r, "\partial"] & F\otimes\bigwedge^2 F^\times \end{tikzcd} \tag{3.2a}$$

where

$$\begin{aligned}\tau_0^3:(l_0,\dots,l_3)\mapsto\sum_{i=0}^{3}(-1)^i\frac{D\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)}&\otimes\frac{\Delta(l_0,\dots,\hat{l}_{i+1},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\\ &\wedge\frac{\Delta(l_0,\dots,\hat{l}_{i+3},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\end{aligned} \tag{3.7}$$

$$\tau_1^3 : (l_0, \ldots, l_4) \mapsto -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\{[\![r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)]\!]_2^D \otimes \prod_{j\neq i} \Delta(\hat{l}_i, \hat{l}_j)$$
$$+ \frac{D\left(\prod_{j\neq i} \Delta(\hat{l}_i, \hat{l}_j)\right)}{\prod_{j\neq i} \Delta(\hat{l}_i, \hat{l}_j)} \otimes [r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)]_2\}$$

$$\tau_2^3 : (l_0, \ldots, l_5) \mapsto \frac{2}{45}\text{Alt}_6 \left[\!\!\left[\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)} \right]\!\!\right]_3^D$$

where

$$[\![a]\!]_3^D = \frac{D(a)}{a(1-a)}\langle a\rangle_3 \text{ and } \Delta(\hat{l}_i, \hat{l}_j) = \Delta(l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_4)$$

$$\partial^D\left([\![a]\!]_3^D\right) = [\![a]\!]_2^D \otimes a + \frac{D(a)}{a} \otimes [a]_2$$

$$\partial^D\left([\![a]\!]_2^D \otimes b + x \otimes [y]_2\right) = \frac{D(1-a)}{1-a} \otimes a \wedge b - \frac{D(a)}{a} \otimes (1-a) \wedge b + x \otimes (1-y) \wedge y$$

First we need to show that our maps τ_0^3 and τ_1^3 are independent of the chosen volume form ω. There is no need to show that same thing for the map τ_2^3. The proofs of the following three lemmas are similar to those in §3 of [9].

Lemma 3.2.1. *τ_0^3 is independent of the volume element $\omega \in \det V_3^*$.*

Proof: We can write equation (3.7) in the form

$$\tau_0^3(l_0, \ldots, l_3) = \sum_{i=0}^{3}(-1)^{i+1}\frac{D\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3)}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3)} \otimes \frac{\Delta(l_0, \ldots, \hat{l}_{i+1}, \ldots, l_3)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3)} \wedge \frac{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3)}{\Delta(l_0, \ldots, \hat{l}_{i+3}, \ldots, l_3)} \tag{3.8}$$

If we apply the definition of Δ in terms of ω in the above then the last two factors will remain unchanged and we know that $\frac{D(\lambda a)}{\lambda a} = \frac{D(a)}{a}$ for all $\lambda \in F^\times$. □

Lemma 3.2.2. *τ_1^3 is independent of the volume element $\omega \in \det V_3^*$.*

Proof: To prove the above we will take the difference of the elements $\tau_1^3(l_0, \ldots, l_4)$ by using the volume forms $\lambda \cdot \omega$ and $\omega(\lambda \in F^\times)$, term of type $F \otimes \mathcal{B}_2(F)$ will be zero while the term of type $\beta_2^D(F) \otimes F^\times$ will be

$$= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Big([\![r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)]\!]_2^D \otimes \lambda^4 \prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)$$

$$- [\![r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)]\!]_2^D \otimes \prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)$$

$$= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i[\![r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)]\!]_2^D \otimes \lambda^4$$

We use lemma 3.1.5 which shows that left factor of the above is simply the projected five-term relation in $\beta_2^D(F)$. □

Now we need to show here that the composition map $\tau_1^3 \circ d$ is independent of the length of the vectors in V_3.

Lemma 3.2.3. *$\tau_1^3 \circ d$ does not depend on the length of the vectors l_i in V_3.*

Proof: The proof of this lemma is quite similar to the proof of proposition 3.9 of [9], but we will out line here main steps because this proof involves more calculations. It is enough to prove that the following

$$\tau_1^{(3)} \circ d\{(l_0,\ldots,l_5) - (\lambda_0 l_0,\ldots,\lambda_5 l_5)\} = 0 \quad (\lambda_i \in F^\times)$$

We will consider the case $\lambda_1 = \cdots = \lambda_5 = 1$ and $\lambda_0 = \lambda$

The first summand $(l_1,\ldots,l_5)$ will not give any contribution to the difference

$$\tau_1^3 \circ d\{(l_0,\ldots,l_5) - (\lambda_0 l_0,\ldots,l_5)\} \tag{3.9}$$

Now consider the second summand $-(l_0, l_2, l_3, l_4, l_5)$

$$\frac{1}{3}\Bigg(-[\![r(l_0|l_2, l_3, l_4, l_5)]\!]_2^D \otimes \lambda^3 \prod_{j=2}^{5} \Delta(\hat{l}_0, \hat{l}_j) + [\![r(l_2|l_0, l_3, l_4, l_5)]\!]_2^D \otimes \lambda^3 \prod_{j=0,3,4,5} \Delta(\hat{l}_2, \hat{l}_j)$$
$$-[\![r(l_3|l_0, l_2, l_4, l_5)]\!]_2^D \otimes \lambda^3 \prod_{j=0,2,4,5} \Delta(\hat{l}_3, \hat{l}_j) + [\![r(l_4|l_0, l_2, l_3, l_5)]\!]_2^D \otimes \lambda^3 \prod_{j=0,2,3,5} \Delta(\hat{l}_4, \hat{l}_j)$$
$$-[\![r(l_5|l_0, l_2, l_3, l_4)]\!]_2^D \otimes \lambda^3 \prod_{j=0,2,3,4} \Delta(\hat{l}_5, \hat{l}_j)$$
$$+\sum_{\substack{i=0\\ i\neq 1}}^{5} \frac{D\left(\prod_{j\neq 1,i} \Delta(\hat{l}_i, \hat{l}_j)\right)}{\Delta(\hat{l}_i, \hat{l}_j)} \otimes \left[r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)\right]\Bigg)$$
$$-\frac{1}{3}\Bigg(-[\![r(l_0|l_2, l_3, l_4, l_5)]\!]_2^D \otimes \prod_{j=2}^{5} \Delta(\hat{l}_0, \hat{l}_j) + [\![r(l_2|l_0, l_3, l_4, l_5)]\!]_2^D \otimes \prod_{j=0,3,4,5} \Delta(\hat{l}_2, \hat{l}_j)$$
$$-[\![r(l_3|l_0, l_2, l_4, l_5)]\!]_2^D \otimes \prod_{j=0,2,4,5} \Delta(\hat{l}_3, \hat{l}_j) + [\![r(l_4|l_0, l_2, l_3, l_5)]\!]_2^D \otimes \prod_{j=0,2,3,5} \Delta(\hat{l}_4, \hat{l}_j)$$
$$-[\![r(l_5|l_0, l_2, l_3, l_4)]\!]_2^D \otimes \prod_{j=0,2,3,4} \Delta(\hat{l}_5, \hat{l}_j)$$
$$+\sum_{\substack{i=0\\ i\neq 1}}^{5} \frac{D\left(\prod_{j\neq 1,i} \Delta(\hat{l}_i, \hat{l}_j)\right)}{\Delta(\hat{l}_i, \hat{l}_j)} \otimes \left[r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)\right]\Bigg)$$

This difference gives us

$$\frac{1}{3}\Big([\![r(l_2|l_0, l_3, l_4, l_5)]\!]_2^D - [\![r(l_3|l_0, l_2, l_4, l_5)]\!]_2^D + [\![r(l_4|l_0, l_2, l_3, l_5)]\!]_2^D - [\![r(l_5|l_0, l_2, l_3, l_4)]\!]_2^D\Big) \otimes \lambda^3 \tag{3.10}$$

If we apply lemma 3.1.5 to the 5-tuple $(l_0, l_2, l_3, l_4, l_5)$ of points in $\mathbb{P}_F^2$ then we see that

$$[\![r(l_0|l_2, l_3, l_4, l_5)]\!]_2^D = [\![r(l_2|l_0, l_3, l_4, l_5)]\!]_2^D - [\![r(l_3|l_0, l_2, l_4, l_5)]\!]_2^D + [\![r(l_4|l_0, l_2, l_3, l_5)]\!]_2^D - [\![r(l_5|l_0, l_2, l_3, l_4)]\!]_2^D$$

Then equation 3.10 can be written as

$$\frac{1}{3}[\![r(l_0|l_2, l_3, l_4, l_5)]\!]_2^D \otimes \lambda^3 \tag{3.11}$$

The contribution of the summand $(-1)^i(l_0, \ldots, \hat{l}_i, \ldots, l_5)$ in equation 3.11 is

$$\frac{1}{3}(-1)^{i-1}[\![r(l_0|l_1, \ldots, \hat{l}_i, \ldots, l_5)]\!]_2^D \otimes \lambda^3$$

Now for all summands

$$\frac{1}{3}\sum_{i=1}^{5}(-1)^{i-1}[\![r(l_0|l_1,\ldots,\hat{l}_i,\ldots,l_5)]\!]_2^D\otimes\lambda^3$$

According to lemma 3.1.5 left factor of the above is projected five-term relation in $\beta_2^D(F)$ and is zero. □

Theorem 3.2.4. *The following diagram*

$$\begin{array}{ccc} C_5(3) & \xrightarrow{\quad d \quad} & C_4(3) \\ \downarrow{\scriptstyle \tau_1^3} & & \downarrow{\scriptstyle \tau_0^3} \\ \left(\beta_2^D(F)\otimes F^\times\right)\oplus(F\otimes\mathcal{B}_2(F)) & \xrightarrow{\quad \partial \quad} & F\otimes\bigwedge^2 F^\times \end{array}$$

is commutative i.e. $\tau_0^3\circ d=\partial^D\circ\tau_1^3$

Proof: From now on we will denote $\Delta(l_0,l_1,l_2)=(l_0,l_1,l_2)$

$$\begin{aligned} &\tau_0^3\circ d(l_0,\ldots,l_4)\\ &=\tau_0^3\left(\sum_{i=0}^{4}(-1)^i(l_0,\ldots,\hat{l}_i,\ldots,l_4)\right)\\ &=\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\frac{D(l_0,\ldots,\hat{l}_i,\ldots,\hat{l}_3)}{(l_0,\ldots,\hat{l}_i,\ldots,\hat{l}_3)}\otimes\frac{(l_0,\ldots,\hat{l}_{i+1},\ldots,\hat{l}_3)}{(l_0,\ldots,\hat{l}_{i+2},\ldots,\hat{l}_3)}\right.\\ &\qquad\qquad\left.\wedge\frac{(l_0,\ldots,\hat{l}_{i+3},\ldots,\hat{l}_3)}{(l_0,\ldots,\hat{l}_{i+2},\ldots,\hat{l}_3)}\right),\qquad i \mod 4 \end{aligned}\tag{3.12}$$

where $\widetilde{\mathrm{Alt}}$ differs from usual alternation sum in the sense that we do not divide by the order of the group for $\widetilde{\mathrm{Alt}}$. If we expand the inner sum first then we will get 4 terms which can be simplified in 12 terms, i.e., we will have terms of the following shape:

$$\frac{D(l_1,l_2,l_3)}{(l_1,l_2,l_3)}\otimes(l_0,l_2,l_3)\wedge(l_0,l_1,l_3)\quad\text{and so on}$$

Then we pass to the alternation which gives us 60 terms so we keep together those terms which have same first factor e.g.,

$$\begin{aligned} +\frac{D(l_0,l_1,l_2)}{(l_0,l_1,l_2)}\otimes\{&(l_0,l_1,l_3)\wedge(l_1,l_2,l_3)-(l_0,l_1,l_4)\wedge(l_1,l_2,l_4)-(l_0,l_2,l_3)\wedge(l_1,l_2,l_3)\\ &+(l_0,l_2,l_4)\wedge(l_1,l_2,l_4)-(l_0,l_1,l_3)\wedge(l_0,l_2,l_3)+(l_0,l_1,l_4)\wedge(l_0,l_2,l_4)\}\\ \vdots \end{aligned}$$

and so on

The other part of the calculation is very long and tedious but we will try to include some steps here.

Going to the other side of the diagram, we find

$$\begin{aligned}\partial^D \circ \tau_1^3(l_0,\ldots,l_4) = -\frac{1}{3}\partial^D\Bigg(&\sum_{i=0}^{4}(-1)^i[\![r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)]\!]_2^D \otimes \prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)\\ &+\frac{D\left(\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)\right)}{\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)}\otimes[r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)]\}\Bigg)\end{aligned}$$

$$\begin{aligned}= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Big(&\frac{D\left(1-r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\right)}{1-r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)}\otimes r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\wedge\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)\\ &-\frac{D\left(r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\right)}{r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)}\otimes\{1-r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\}\wedge\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)\\ &+\frac{D\left(\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)\right)}{\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)}\otimes\left(1-r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\right)\wedge r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\Big)\end{aligned}\tag{3.13}$$

From now on we will use (ijk) instead of $\Delta(l_i,l_j,l_k)$ as a shorthand. If we expand the above sum with respect to i, then we will get a long expression. For example when $i=0$, we have

$$\begin{aligned}&+\frac{D\left(\frac{(012)(034)}{(013)(024)}\right)}{\frac{(012)(034)}{(013)(024)}}\otimes\frac{(014)(023)}{(013)(024)}\wedge(234)(134)(124)(123)\\ &-\frac{D\left(\frac{(014)(023)}{(013)(024)}\right)}{\frac{(014)(023)}{(013)(024)}}\otimes\frac{(012)(034)}{(013)(024)}\wedge(234)(134)(124)(123)\\ &+\frac{D\left((234)(134)(124)(123)\right)}{(234)(134)(124)(123)}\otimes\frac{(012)(034)}{(013)(024)}\wedge\frac{(014)(023)}{(013)(024)}\end{aligned}$$

and we can get four more similar expressions for the other values of i as well. If we collect terms of type $\frac{D(ijk)}{(ijk)}\otimes\cdots\wedge\cdots$ i.e., fix i, j and k in all five expressions (one of them is given above), then we will see a huge amount of terms but we cancel terms pairwise and collect terms of the same kind, we get each remaining term with the coefficient "3". So we can write in the following form.

$$
\begin{aligned}
&-3\frac{D(012)}{(012)} \otimes \{(013) \wedge (123) - (014) \wedge (124) - (023) \wedge (123) \\
&\qquad + (024) \wedge (124) - (013) \wedge (023) + (014) \wedge (024)\} \\
&-3\frac{D(013)}{(013)} \otimes \{(014) \wedge (134) + (023) \wedge (123) - (012) \wedge (123) \\
&\qquad - (034) \wedge (134) - (014) \wedge (034) + (012) \wedge (023)\} \\
&\vdots \\
&\text{and so on}
\end{aligned}
$$

It turns out that every term has "−3" as a coefficient that cancels the factor $-\frac{1}{3}$ in the definition of τ_1^3 then comparing the expression above with (3.12), we find after a long calculation that both agree (term-wise) □

Here we have another result which will then complete the commutativity of diagram (3.2a)

Theorem 3.2.5. *The following diagram*

$$
\begin{tikzcd}
C_6(3) \arrow[r, "d"] \arrow[d, "\tau_2^3"] & C_5(3) \arrow[d, "\tau_1^3"] \\
\beta_3^D(F) \arrow[r, "\partial"] & \left(\beta_2^D(F) \otimes F^\times\right) \oplus (F \otimes \mathcal{B}_2(F))
\end{tikzcd}
$$

is commutative i.e. $\tau_2^3 \circ \partial^D = d \circ \tau_1^3$.

Proof: The map τ_2^3 is based on generalized cross-ratios of 3×3 determinants. The total number of terms due to map τ_2^3 will be 720 which can further be reduced to 120 due to symmetry (cyclic and inverse). The direct procedure which was used in the previous proof will be very lengthy and tedious so we will use techniques of combinatorics and will rewrite the triple-ratio in to the product of two cross-ratios to prove this result.

We first compute $\partial \circ \tau_2^3(l_0, \ldots, l_5)$ and we already have

$$
\tau_2^3(l_0, \ldots, l_5) = \frac{2}{45}\mathrm{Alt}\left[\!\!\left[\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)} \right]\!\!\right]_3^D,
$$

from now, in this proof we will use (ijk) for $\Delta(l_i, l_j, l_k)$ and $(0.\ldots,5)$ for $(l_0, \ldots, l_5)$ as a short hand.

The above becomes

$$\tau_2^3(l_0,\dots,l_5) = \frac{2}{45}\mathrm{Alt}_6\left[\!\!\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]\!\!\right]_3^D$$

$$\begin{aligned}\partial^D \circ \tau_2^3(l_0,\dots,l_5) =& \frac{2}{45}\mathrm{Alt}_6\left(\left[\!\!\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]\!\!\right]_2^D \otimes \frac{(013)(124)(205)}{(014)(125)(203)}\right)\\ &+\frac{2}{45}\mathrm{Alt}_6\left(D\log\frac{(013)(124)(205)}{(014)(125)(203)} \otimes \left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]_2\right)\end{aligned} \tag{3.14}$$

First we will consider first term of the above

$$\begin{aligned}=\frac{2}{45}\Big(&\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (013)\right\} + \mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (124)\right\}\\ &+\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (205)\right\} - \mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (014)\right\}\\ &-\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (125)\right\} - \mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (203)\right\}\Big)\end{aligned}$$

where

$$r_3(0,\dots,5) = \frac{(013)(124)(205)}{(014)(125)(203)}$$

Use the even cycle (012)(345)

$$\mathrm{Alt}_6\left\{[\![r_3(012345)]\!]_2^D \otimes (013)\right\} = \mathrm{Alt}_6\left\{[\![r_3(120453)]\!]_2^D \otimes (124)\right\}$$

Now we use $[\![r_3(012345)]\!]_2^D = [\![r_3(120453)]\!]_2^D$ and similar for the others, then the above can be written as

$$= \frac{2}{45}\left(3\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (013)\right\} - 3\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (014)\right\}\right)$$

Use the odd cycle (34)

$$= \frac{2}{45}\left(6\mathrm{Alt}_6\left\{[\![r_3(0\dots5)]\!]_2^D \otimes (013)\right\}\right)$$

If we apply the odd permutation (03), then

$$= \frac{2}{45}\left(3\mathrm{Alt}_6\left\{[\![r_3(012345)]\!]_2^D \otimes (013)\right\} - 3\mathrm{Alt}_6\left\{[\![r_3(312045)]\!]_2^D \otimes (310)\right\}\right)$$

but (013)=(310) so up to 2-torsion

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left([\![r_3(012345)]\!]_2^D - [\![r_3(312045)]\!]_2^D\right) \otimes (013)\right\}$$

Now we will use here the crucial idea of this proof in which we will divide the triple-ratio into the product of two projected cross-ratios of four points each. There are exactly 3 ways to divide this ratio into such a product. i.e., if $r_3(a, b, c, d, e, f)$ then it can be divided by projection from a and b, projection from a and c or projection from b and c. In our case we will divide by projection from 1 and 2.

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left(\left[\!\left[\frac{r(2|1053)}{r(1|0234)}\right]\!\right]_2^D - \left[\!\left[\frac{r(2|1350)}{r(1|3204)}\right]\!\right]_2^D\right)\otimes(013)\right\}$$

Apply lemma 3.1.5 (five-term relation in $\beta_2^D(F)$) then we will have

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left(-\left[\!\left[\frac{r(2|1530)}{r(1|0342)}\right]\!\right]_2^D + [\![r(2|1053)]\!]_2^D - [\![r(1|0234)]\!]_2^D\right)\otimes(013)\right\} \tag{3.15}$$

We will treat the above three terms individually. We consider first term now,

$$\mathrm{Alt}_6\left\{\left[\!\left[\frac{r(2|1530)}{r(1|0342)}\right]\!\right]_2^D\otimes(013)\right\}$$

For each individual determinant, e.g. (013), we have the following terms.

$$\mathrm{Alt}_6\left\{\left[\!\left[\frac{r(2|1530)}{r(1|0342)}\right]\!\right]_2^D\otimes(013)\right\} = \mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(013)(245)}\left(\left[\!\left[\frac{r(2|1530)}{r(1|0342)}\right]\!\right]_2^D\otimes(013)\right)\right\}$$

We need a subgroup in S_6 which fixes (013) as a determinant i.e. $(013) \sim (310) \sim (301)\cdots$.

Here S_3 permuting $\{0, 1, 3\}$ and another one permuting $\{2, 4, 5\}$ i.e. $S_3 \times S_3$. Now consider

$$\begin{aligned}&\mathrm{Alt}_{(013)(245)}\left\{\left[\!\left[\frac{r(2|1530)}{r(1|0342)}\right]\!\right]_2^D\otimes(013)\right\}\\ =&\mathrm{Alt}_{(013)(245)}\left\{\left[\!\left[\frac{(210)(235)}{(213)(250)}\cdot\frac{(104)(132)}{(102)(135)}\right]\!\right]_2^D\otimes(013)\right\}\\ =&\mathrm{Alt}_{(013)(245)}\left\{\left[\!\left[\frac{(253)(104)}{(250)(134)}\right]\!\right]_2^D\otimes(013)\right\}\end{aligned}$$

By using odd permutation (25) the above becomes

$=0$

The new shape of (3.15) is

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left([\![r(2|1053)]\!]_2^D - [\![r(1|0234)]\!]_2^D\right)\otimes(013)\right\} \tag{3.16}$$

Now we will consider the first terms

$$\begin{aligned}
&\frac{2}{15}\mathrm{Alt}_6\left\{[\![r(2|1053)]\!]_2^D \otimes (013)\right\}\\
=&\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{6}\mathrm{Alt}_{(245)}[\![r(2|1053)]\!]_2^D \otimes (013)\right\}\\
=&\frac{1}{45}\mathrm{Alt}_6\{\Big([\![r(4|1023)]\!]_2^D - [\![r(2|1043)]\!]_2^D\\
&\qquad + [\![r(5|1043)]\!]_2^D - [\![r(4|1053)]\!]_2^D\\
&\qquad + [\![r(2|1053)]\!]_2^D - [\![r(5|1023)]\!]_2^D\Big) \otimes (013)\}
\end{aligned}$$

We are able to use lemma 3.1.5 (projected five-term relation in $\beta_2^D(F)$) here.

$$\begin{aligned}
=&\frac{1}{45}\mathrm{Alt}_6\{\Big([\![r(0|1234)]\!]_2^D - [\![r(1|0234)]\!]_2^D - [\![r(3|0124)]\!]_2^D\\
&+[\![r(0|1435)]\!]_2^D - [\![r(1|0435)]\!]_2^D + [\![r(3|0145)]\!]_2^D\\
&+[\![r(0|1532)]\!]_2^D - [\![r(1|0532)]\!]_2^D + [\![r(3|0152)]\!]_2^D\Big) \otimes (013)\}
\end{aligned}$$

Use the cycle (013)(245) then we get

$$= \frac{1}{45} \cdot 9\mathrm{Alt}_6\left\{[\![r(0|1234)]\!]_2^D \otimes (013)\right\} \tag{3.17}$$

We also have $-\frac{2}{15}\mathrm{Alt}_6\left\{[\![r(1|0234)]\!]_2^D \otimes (013)\right\}$ from (3.16) which can be written as

$$\frac{1}{45} \cdot -6\mathrm{Alt}_6\left\{[\![r(1|0234)]\!]_2^D \otimes (013)\right\}$$

then (3.16) can be written as

$$= \frac{1}{45}\mathrm{Alt}_6\{\Big(9[\![r(0|1234)]\!]_2^D - 6\,[\![r(1|0234)]\!]_2^D\Big) \otimes (013)\}$$

Use the cycle (01). We will get $\frac{1}{3}\mathrm{Alt}_6\left\{[\![r(0|1234)]\!]_2^D \otimes (013)\right\}$ as a result of (3.16).

This gives the first term in (3.14). For the second one, consider the second part of (3.14) which has a $D\log$ factor in F and we know that $\frac{D(ab)}{ab} = \frac{D(a)}{a} + \frac{D(b)}{b}$ and $\frac{D(\frac{a}{b})}{\frac{a}{b}} = \frac{D(a)}{a} - \frac{D(b)}{b}$, while the right factor of second term is in $\mathcal{B}_2(F)$ which is equipped with five-term relation so same procedure can be adopted for the second term as we did for first term. So, after passing through above procedure for second term, we get from the second term of (3.14) $\frac{1}{3}\mathrm{Alt}_6\{D\log(013) \otimes [r(0|1234)]_2\}$, at the end of the computation we have from the LHS of the diagram (simpler form of the diagram)

$$= \frac{1}{3}\mathrm{Alt}_6\left\{[\![r(0|1234)]\!]_2^D \otimes (013) + D\log(013) \otimes [r(0|1234)]_2\right\} \tag{3.18}$$

The above allows us to rewrite τ_1^3 using alternation sums. In fact, we have

$$\tau_1^3(l_0,\dots,l_4) = \frac{1}{3}\mathrm{Alt}_5\{[\![r(l_0|l_1,l_2,l_3,l_4)]\!]_2^D \otimes \Delta(l_0,l_1,l_2) + D\log(\Delta(l_0,l_1,l_2)) \otimes [r(l_0|l_1,l_2,l_3,l_4)]_2\}$$

In reduced notation, the above can also be written as

$$\tau_1^3(0\dots 4) = \frac{1}{3}\mathrm{Alt}_5\left\{[\![r(0|1234)]\!]_2^D \otimes (012) + D\log(012) \otimes [r(0|1234)]_2\right\}$$

It remains to compare $\partial \circ \tau_2^3(0,\dots,5)$ with $\tau_1^3 \circ d(0\dots5)$. For the latter, apply cycle (012345) for d and then expand Alt_5 from the definition of τ_1^3 so we get

$$\tau_1^3 \circ d(0\dots 5) = \frac{1}{3}\mathrm{Alt}_6\left\{[\![r(0|1234)]\!]_2^D \otimes (012) + D\log(012) \otimes [r(0|1234)]_2\right\}$$

Now use the odd permutation (23) then the above becomes

$$= -\frac{1}{3}\mathrm{Alt}_6\{[\![r(0|1324)]\!]_2^D \otimes (013) + D\log(013) \otimes [r(0|1324)]_2\}$$

Finally use the two-term relation to get the correct sign and it will be same as (3.18). This proves the theorem. □

Corollary 3.2.6. *The diagram (3.2a) is commutative, i.e. there is a morphism of complexes between the Grassmannian complex and a variant of Cathelineau's complex which involves the F-vector spaces $\beta_3^D(F)$ and $\beta_2^D(F)$ and the groups $\mathcal{B}_2(F)$ and $F \times \bigwedge^2 F^\times$.*

Proof: The proof follows from combining Theorem 3.2.4 and Theorem 3.2.5. □

Now consider the diagram (3.2a) and note that $\tau_1^3 \circ d \in \ker \partial^D$. It is clear from the commutativity of the diagram that $\partial^D\left(\tau_1^3(d(l_0,\dots,l_5))\right) = 0$.

Goncharov has given a morphism from the Grassmannian bicomplex to $\Gamma(n)$, here we try to establish a result in the following proposition for the infinitesimal case.

Proposition 3.2.7. *The following maps*

1. $C_4(3) \xrightarrow{d'} C_3(2) \xrightarrow{\tau_0^2} F \otimes F^\times$
2. $C_5(3) \xrightarrow{d'} C_4(2) \xrightarrow{\tau_1^2} \beta_2^D(F)$
3. $C_5(4) \xrightarrow{d'} C_4(3) \xrightarrow{\tau_0^3} F \otimes \wedge^2 F^\times$

4. $C_{n+1}(n+1) \xrightarrow{d'} C_{n+1}(n) \xrightarrow{\tau_0^n} F \otimes \bigwedge^{n-1} F^\times$

are zero, where

$$\begin{aligned}
&\tau_0^n(l_0, \ldots, l_n) \\
&= \sum_{i=0}^{n} (-1)^i \Bigg(\frac{D\Big(\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_n)\Big)}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_n)} \otimes \frac{\Delta(l_0, \ldots, \hat{l}_{i+1}, \ldots, l_n)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_n)} \\
&\qquad\qquad \wedge \cdots \wedge \frac{\Delta(l_0, \ldots, \hat{l}_{i+(n-1)}, \ldots, l_n)}{\Delta(l_0, \ldots, \hat{l}_{i+n}, \ldots, l_n)} \Bigg), \quad i \mod (n+1)
\end{aligned}$$

Proof: See the proof of the lemmas 4.2.3, 4.2.5, 4.3.4 and 4.3.5. □

Now we can relate the above with the work of Goncharov ([9] and [10]) to see the bigger picture.

Lemma 3.2.8. *(Elbaz-Vincent–Gangl)(see Lemma 6.1 and Proposition 6.2 of [7]) Let $D \in Der_{\mathbb{Z}}(F)$ be an absolute derivation for the field F. Then the following diagram is commutative.*

$$\begin{tikzcd}
\mathbb{Z}[F^{\bullet\bullet}] \arrow[r, "f_D"] \arrow[d, "\delta_n"] & F[F^{\bullet\bullet}] \arrow[d, "\partial_n"] \\
\mathcal{B}_{n-1}(F) \otimes F^\times \arrow[r, "g_D^n"] & \left(\beta_{n-1}^D(F) \otimes F^\times\right) \oplus (F \otimes \mathcal{B}_{n-1}(F))
\end{tikzcd}$$

Where $f_D : [a] \mapsto \frac{D(a)}{a(1-a)}[a]$

$$\delta_n : [a] \mapsto \begin{cases} [a]_{n-1} \otimes a & \textit{for } n > 2 \\ (1-a) \wedge a & \textit{for } n = 2 \end{cases}$$

$$\partial_n : [a] \mapsto \begin{cases} [\![a]\!]_{n-1} \otimes a + a \otimes [a]_{n-1} & \textit{for } n > 2 \\ -\frac{D(a)}{1-a} \otimes a + \frac{D(a)}{a} \otimes (1-a) & \textit{for } n = 2 \end{cases}$$

$$g_D^n : [a]_{n-1} \otimes b \mapsto [\![a]\!]_{n-1} \otimes b + \frac{D(b)}{b} \otimes [a]_{n-1}$$

Proof requires direct calculation(see the proof of Lemma 6.1 and Proposition 6.2 in [7]).

□

Recall the Diagram (2.9) of §2.3 in [9]. Goncharov proved that the following diagram is commutative.

$$\begin{tikzcd} C_4(2) \arrow[r, "d"] \arrow[d, "f_1^2"] & C_3(2) \arrow[d, "f_0^2"] \\ \mathcal{B}_2(F) \arrow[r, "\delta"] & \bigwedge^2 F^\times \end{tikzcd}$$

for the following maps

$$f_0^2(l_0, l_1, l_2) = \Delta(l_0, l_1) \wedge \Delta(l_0, l_2) - \Delta(l_0, l_1) \wedge \Delta(l_1, l_2) + \Delta(l_0, l_2) \wedge \Delta(l_1, l_2)$$

and

$$f_1^2(l_0, \ldots, l_3) = \left[\frac{\Delta(l_0, l_3)\Delta(l_1, l_2)}{\Delta(l_0, l_2)\Delta(l_1, l_3)}\right]_2$$

where d is defined in §2.1 and δ is defined in §2.3.

On the basis of this diagram, Lemma 3.2.8 and diagram (3.1a) in §3.1, we can construct a prism which has six faces and above discussion in this chapter shows that all square faces of the following diagram are commutative.

$$\begin{tikzcd} & C_4(2) \arrow[rr, "d"] \arrow[dl, "f_1^2"'] \arrow[dd, "\tau_1^2"] & & C_3(2) \arrow[dl, "f_0^2"'] \arrow[dd, "\tau_0^2"] \\ \mathcal{B}_2(F) \arrow[rr] \arrow[dr, "\tau_D^2"] & & \bigwedge^2 F^\times \arrow[dr, "g^1_{2,D}"] & \\ & \beta_2^D(F) \arrow[rr, "\partial"] & & F \otimes F^\times \end{tikzcd} \tag{3.2b}$$

where (see §6.1 in [7])

$$\tau_D^2([x]_2) = [\![x]\!]_2 \quad \text{and} \quad g^1_{2,D}(x \wedge y) = \frac{D(x)}{x} \otimes y - \frac{D(y)}{y} \otimes x$$

Corollary 3.2.9. *The diagram* (3.2b) *above is commutative, i.e. there is a morphism of complexes between all three complexes used in diagram* (3.2b).

Proof: We only need to show that $g^1_{2,D} \circ f_0^2(l_0, l_1, l_3) = \tau_0^2(l_0, l_1, l_3)$ and $\tau_D^2 \circ f_1^2(l_0, \ldots, l_3) =$

$\tau_1^2(l_0,\dots,l_3)$.

$$
\begin{aligned}
g^1_{2,D}\circ f^2_0(l_0,l_1,l_3) =& g^1_{2,D}\left(\Delta(l_0,l_1)\wedge\Delta(l_0,l_2)-\Delta(l_0,l_1)\wedge\Delta(l_1,l_2)+\Delta(l_0,l_2)\wedge\Delta(l_1,l_2)\right)\\
=&\frac{D\left(\Delta(l_0,l_1)\right)}{\Delta(l_0,l_1)}\otimes\Delta(l_0,l_2)-\frac{D\left(\Delta(l_0,l_2)\right)}{\Delta(l_0,l_2)}\otimes\Delta(l_0,l_1)\\
&-\frac{D\left(\Delta(l_0,l_1)\right)}{\Delta(l_0,l_1)}\otimes\Delta(l_1,l_2)+\frac{D\left(\Delta(l_1,l_2)\right)}{\Delta(l_1,l_2)}\otimes\Delta(l_0,l_1)\\
&+\frac{D\left(\Delta(l_0,l_2)\right)}{\Delta(l_0,l_2)}\otimes\Delta(l_1,l_2)-\frac{D\left(\Delta(l_1,l_2)\right)}{\Delta(l_1,l_2)}\otimes\Delta(l_0,l_2)\\
=&\tau^2_0(l_0,l_1,l_3)
\end{aligned}
$$

and

$$
\tau^2_D\circ f^2_1(l_0,\dots,l_3) =\tau^2_D\left(\left[\frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}\right]_2\right)=\left[\!\left[\frac{\Delta(l_0,l_3)\Delta(l_1,l_2)}{\Delta(l_0,l_2)\Delta(l_1,l_3)}\right]\!\right]_2
$$

We can construct the similar diagram for weight 3 case. We recall diagram (3.2) in §3 of [10]

$$
\begin{tikzcd}
C_6(3) \arrow[r, "d"] \arrow[d, "f^3_2"] & C_5(3) \arrow[r, "d"] \arrow[d, "f^3_1"] & C_4(3) \arrow[d, "f^3_0"] \\
\mathcal{B}_3(F) \arrow[r, "\delta"] & \mathcal{B}_2(F)\otimes F^\times \arrow[r, "\delta"] & \bigwedge^3 F^\times
\end{tikzcd}
\tag{3.2c}
$$

is commutative for the following maps

$$
f^3_0(l_0,\dots,l_3)=\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\Delta(l_0,\dots,\hat{l}_j,\dots,l_3),
$$

$$
f^3_1(l_0,\dots,l_4)=-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\left[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\right]_2\otimes\prod_{j\neq i}\Delta(\hat{l}_i,\hat{l}_j)
$$

and f^3_2 is defined via alternation sum for generic points.

$$
f^3_2(l_0,\dots,l_5)=\frac{2}{45}\mathrm{Alt}_6\left[\frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}\right]_3
$$

where $\delta\left([x]_3\right)=[x]_2\otimes x$ for all $x\neq 0,1\in F^\times$ and $\delta([x]_2)=(1-x)\wedge x$.

So, we can combine this diagram, diagram (3.2a) in §3.2 by using Lemma 3.2.8 to get the following diagram where all square faces are commutative.

$$
\begin{tikzcd}
& C_6(3) \arrow[rr, "d"] \arrow[dl, "f^3_2"'] \arrow[dd, "\tau^3_2"] & & C_5(3) \arrow[rr, "d"] \arrow[dl, "f^3_1"'] \arrow[dd, "\tau^3_1"] & & C_4(3) \arrow[dl, "f^3_0"'] \arrow[dd, "\tau^3_0"] \\
\mathcal{B}_3(F) \arrow[rr, "\delta"] \arrow[dr, "\tau^3_D"'] & & \mathcal{B}_2(F)\otimes F^\times \arrow[rr, "\delta"] \arrow[dr, "g^2_{3,D}"] & & \bigwedge^3 F^\times \arrow[dr, "g^1_{3,D}"] & \\
& \beta^D_3(F) \arrow[rr] & & \left(\beta^D_2(F)\otimes F^\times\right)\oplus\left(F\otimes\mathcal{B}_2(F)\right) \arrow[rr] & & F\otimes\bigwedge^2 F^\times
\end{tikzcd}
\tag{3.2d}
$$

where (see §6.1 in [7])

$$\tau^3_D([x]_3) = [\![x]\!]_3, \quad g^2_{3,D}([x]_2 \otimes y) = [\![x]\!]_2 \otimes y + \frac{D(y)}{y} \otimes [x]_2$$

and

$$g^1_{3,D}(x \wedge y \wedge z) = \frac{D(x)}{x} \otimes y \wedge z - \frac{D(y)}{y} \otimes x \wedge z + \frac{D(z)}{z} \otimes x \wedge y$$

Corollary 3.2.10. *The diagram* (3.2d) *above is commutative, i.e. there is a morphism of complexes between all three complexes used in diagram* (3.2d).

Proof: We only need to show that $g^1_{3,D} \circ f^3_0(l_0,\dots,l_3) = \tau^3_0(l_0,\dots,l_3)$, $g^2_{3,D} \circ f^3_1(l_0,\dots,l_4) = \tau^3_1(l_0,\dots,l_4)$ and $\tau^3_D \circ f^3_2(l_0,\dots,l_5) = \tau^3_2(l_0,\dots,l_5)$

$$\begin{aligned}
&g^1_{3,D} \circ f^3_0(l_0,\dots,l_3)\\
&\quad= g^1_{3,D}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0\\ j\neq i}}^{3} \Delta(l_0,\dots,\hat{l}_j,\dots,l_3)\right)\\
&\quad= \sum_{i=0}^{3}(-1)^i\left(\frac{D\left(\Delta(l_0,\dots,\hat{l}_i,\dots,l_4)\right)}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_4)} \otimes \frac{\Delta(l_0,\dots,\hat{l}_{i+1},\dots,l_4)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_4)} \wedge \frac{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_4)}{\Delta(l_0,\dots,\hat{l}_{i+3},\dots,l_4)}\right),\\
&\qquad\qquad i \mod 4
\end{aligned}$$

$$\begin{aligned}
&g^2_{3,D} \circ f^3_1(l_0,\dots,l_4)\\
&\quad= g^2_{3,D}\left(\sum_{i=0}^{4}(-1)^i \left[r\left(l_i|l_0,\dots,\hat{l}_i,\dots,l_4\right)\right]_2 \otimes \prod_{i\neq j}(\hat{l}_i,\hat{l}_j)\right)\\
&\quad= \sum_{i=0}^{4}(-1)^i \left[\!\left[r\left(l_i|l_0,\dots,\hat{l}_i,\dots,l_4\right)\right]\!\right]_2 \otimes \prod_{i\neq j}(\hat{l}_i,\hat{l}_j)\\
&\quad+ \sum_{\substack{j=0\\ j\neq i}}^{4} \frac{D(\hat{l}_i,\hat{l}_j)}{(\hat{l}_i,\hat{l}_j)} \otimes \left[r\left(l_i|l_0,\dots,\hat{l}_i,\dots,l_4\right)\right]_2
\end{aligned}$$

$$\begin{aligned}
\tau^3_D \circ f^3_2(l_0,\dots,l_5) &= \tau^3_D\left(\frac{2}{45}\mathrm{Alt}_6\left[\frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}\right]_3\right)\\
&= \frac{2}{45}\mathrm{Alt}_6\left[\!\left[\frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}\right]\!\right]_3
\end{aligned}$$

Chapter 4

Tangent Complexes

In the previous chapter, we have described a morphism between the Grassmannian complex and a variant of Cathelineau's infinitesimal complex. In this chapter we will discuss and try to write geometric configurations for the tangent complex to Bloch-Suslin complex and to Goncharov's complex (see §3 of [10]). By specializing the derivation of that tangent complex, we relate it with the variant of the Cathelineau's infinitesimal complex and with Goncharov's complexes (see examples 4.2.6 and 4.3.2 below). This chapter will also introduce cross-ratios and identities of determinants for the configurations of vectors in $C_m(\mathbb{A}^n_{F[\varepsilon]_\nu})$ for $n = 2, 3$, $m = 3, \ldots, 7$ and $\nu \geq 1$ ($\nu = 1$ is the usual case and we have used this previously) (see §4.1 below).

One of our main results is Theorem 4.3.3. In its proof we shall use combinatorial techniques and will rewrite the triple-ratio as the product of two projected cross-ratios in $F[\varepsilon]_2$.

4.1 Configurations of points in $C_m(\mathbb{A}^n_{F[\varepsilon]_\nu})$

Let F be a field of characteristic 0. For $\nu \geq 1$, we denote the νth truncated polynomial ring over F by $F[\varepsilon]_\nu := F[\varepsilon]/\varepsilon^\nu$. Further define the abelian group $C_m(\mathbb{A}^n_{F[\varepsilon]_\nu})$ generated by m points in $\mathbb{A}^n_{F[\varepsilon]_\nu}$ in generic position, where $\mathbb{A}^n_{F[\varepsilon]_\nu}$ is the n-dimensional affine space over the truncated polynomial ring $F[\varepsilon]_\nu$. We will not consider here degenerate points and we are assuming that no two points coinciding and no three points are lying on a line. Now for the case $n = 2$ and $\nu = 2$, any $l_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \mathbb{A}^2_F \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and $l_{i,\varepsilon} := \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix} \in \mathbb{A}^2_F$, we put $l_i^* = \begin{pmatrix} a_i + a_{i,\varepsilon}\varepsilon \\ b_i + b_{i,\varepsilon}\varepsilon \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix}\varepsilon = l_i + l_{i,\varepsilon}\varepsilon$ and define a differential

$$d : C_{m+1}(\mathbb{A}^2_{F[\varepsilon]_2}) \to C_m(\mathbb{A}^2_{F[\varepsilon]_2})$$

$$d : (l_0^*, \ldots, l_m^*) \mapsto \sum_{i=0}^{m} (-1)^i (l_0^*, \ldots, \hat{l}_i^*, \ldots, l_m^*).$$

Let $\omega \in V_2^*$ be a volume element formed in $V_2 := \mathbb{A}^2_F$ and $\Delta(l_i, l_j) = \langle \omega, l_i \wedge l_j \rangle$, where $l_i, l_j \in \mathbb{A}^2_F$. Here we define

$$\Delta(l_i^*, l_j^*) = \Delta(l_i^*, l_j^*)_{\varepsilon^0} + \Delta(l_i^*, l_j^*)_{\varepsilon^1}\varepsilon$$

where

$$\Delta(l_i^*, l_j^*)_{\varepsilon^0} = \Delta(l_i, l_j) \quad \text{and} \quad \Delta(l_i^*, l_j^*)_{\varepsilon^1} = \Delta(l_i, l_{j,\varepsilon}) + \Delta(l_{i,\varepsilon}, l_j);$$

more generally for $\nu = n + 1$, we have

$$l_i^* = l_i + l_{i,\varepsilon}\varepsilon + l_{i,\varepsilon^2}\varepsilon^2 + \cdots + l_{i,\varepsilon^n}\varepsilon^n \quad \text{and} \quad l_{i,\varepsilon^0} = l_i$$

and we get

$$\Delta(l_i^*, l_j^*) = \Delta(l_i, l_j) + \Delta(l_i^*, l_j^*)_{\varepsilon}\varepsilon + \Delta(l_i^*, l_j^*)_{\varepsilon^2}\varepsilon^2 + \cdots + \Delta(l_i^*, l_j^*)_{\varepsilon^n}\varepsilon^n,$$

where

$$\Delta(l_i^*, l_j^*)_{\varepsilon^n} = \Delta(l_i, l_{j,\varepsilon^n}) + \Delta(l_{i,\varepsilon}, l_{j,\varepsilon^{n-1}}) + \cdots + \Delta(l_{i,\varepsilon^n}, l_j)$$

Consider the Siegel cross-ratio identity for the 2×2 determinants of four vectors in $C_4(\mathbb{A}^2_F)$ (see [21], [9] or Remark 2 on p155 of [16])

$$\Delta(l_0, l_1)\Delta(l_2, l_3) = \Delta(l_0, l_2)\Delta(l_1, l_3) - \Delta(l_0, l_3)\Delta(l_1, l_2) \tag{4.1}$$

With the above notation, an analogous to Siegel cross-ratio identity turns out to be true for $\mathbb{A}^2_{F[\varepsilon]_{n+1}}$, and we can extract further results which are essential for the proof of our main results. Throughout this section we will assume that $\Delta(l_i, l_j) \neq 0$ for $i \neq j$.

Lemma 4.1.1. *For $(l_0^*, l_1^*, l_2^*, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_{n+1}})$, we have*

$$\Delta(l_0^*, l_1^*)\Delta(l_2^*, l_3^*) = \Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*) - \Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*) \tag{4.2}$$

where

$$l_i^* = l_i + l_{i,\varepsilon}\varepsilon + l_{i,\varepsilon^2}\varepsilon^2 + \cdots + l_{i,\varepsilon^n}\varepsilon^n \quad \textit{and} \quad l_{i,\varepsilon^0} = l_i$$

$$\Delta(l_i^*, l_j^*) = \Delta(l_i, l_j) + \Delta(l_i^*, l_j^*)_\varepsilon \varepsilon + \Delta(l_i^*, l_j^*)_{\varepsilon^2}\varepsilon^2 + \cdots + \Delta(l_i^*, l_j^*)_{\varepsilon^n}\varepsilon^n$$

where

$$\Delta(l_i^*, l_j^*)_{\varepsilon^n} = \Delta(l_i, l_{j,\varepsilon^n}) + \Delta(l_{i,\varepsilon}, l_{j,\varepsilon^{n-1}}) + \cdots + \Delta(l_{i,\varepsilon^n}, l_j)$$

Proof: For $r = 0, \dots, n$, we can write $l^* = \begin{pmatrix} \sum_{r\geq 0} l_r\varepsilon^r \\ \sum_{r\geq 0} l'_r\varepsilon^r \end{pmatrix}$ and $m^* = \begin{pmatrix} \sum_{r\geq 0} m_r\varepsilon^r \\ \sum_{r\geq 0} m'_r\varepsilon^r \end{pmatrix}$.

Now we have

$$\begin{aligned}\Delta(l^*, m^*) = \begin{vmatrix} \sum_{r\geq 0} l_r\varepsilon^r & \sum_{r\geq 0} m_r\varepsilon^r \\ \sum_{r\geq 0} l'_r\varepsilon^r & \sum_{r\geq 0} m'_r\varepsilon^r \end{vmatrix} &= \sum_{r\geq 0}\left(\sum_{k=0}^{r} l_k m'_{r-k} - \sum_{k=0}^{r} l'_k m_{r-k}\right)\varepsilon^r \\ &= \sum_{r\geq 0}\left(\sum_{k=0}^{r} \Delta\left(l_k, m_{r-k}\right)\right)\varepsilon^r\end{aligned}$$

Hence

$$\begin{aligned}\Delta(l_0^*, l_1^*)\Delta(l_2^*, l_3^*) &= \sum_{r\geq 0}\left(\sum_{k=0}^{r}\Delta(l_{0,k}, l_{1,r-k})\right)\varepsilon^r \cdot \sum_{s\geq 0}\left(\sum_{j=0}^{r}\Delta(l_{0,j}, l_{1,r-j})\right)\varepsilon^s \\ &= \sum_{t\geq 0}\varepsilon^t\left(\sum_{r=0}^{t}\left(\sum_{k=0}^{r}\Delta(l_{0,k}, l_{1,r-k})\sum_{j=0}^{t-r}\Delta(l_{2,j}, l_{3,t-r-j})\right)\right) \\ &= \sum_{t\geq 0}\varepsilon^t\left(\sum_{r=0}^{t}\left(\sum_{k=0}^{r}\sum_{j=0}^{t-r}\Delta(l_{0,k}, l_{1,r-k})\Delta(l_{2,j}, l_{3,t-r-j})\right)\right),\end{aligned}$$

and similarly for $\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)$ and $\Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*)$. Hence we use validity of (4.1) to deduce the analogue for $\Delta(l_i^*, l_j^*)$'s in place of $\Delta(l_i, l_j)$ passing from the ring $F[\![\varepsilon]\!]$ of power series to a truncated polynomial ring, say to $F[\varepsilon]_{n+1}$. □

As special cases we find for $n = 0$ the identity (4.1) while for $n = 1$ we have the following identity which will be used extensively below:

$$\begin{aligned}&\Delta(l_0, l_1)\Delta(l_2^*, l_3^*)_\varepsilon + \Delta(l_2, l_3)\Delta(l_0^*, l_1^*)_\varepsilon \\ &= \{\Delta(l_0, l_2)\Delta(l_1^*, l_3^*)_\varepsilon + \Delta(l_1, l_3)\Delta(l_0^*, l_2^*)_\varepsilon\} - \{\Delta(l_0, l_3)\Delta(l_1^*, l_2^*)_\varepsilon + \Delta(l_1, l_2)\Delta(l_0^*, l_3^*)_\varepsilon\}. \quad (4.3)\end{aligned}$$

if we write

$$(ab)_{\varepsilon^n} := a_{\varepsilon^n}b_{\varepsilon^0} + a_{\varepsilon^{n-1}}b_\varepsilon + \cdots + a_{\varepsilon^0}b_{\varepsilon^n}$$

then (4.3) can be more concisely written as

$$\{\Delta(l_0^*, l_1^*)\Delta(l_2^*, l_3^*)\}_\varepsilon = \{\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)\}_\varepsilon - \{\Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*)\}_\varepsilon.$$

4.1.1 Cross-ratio in $F[\varepsilon]_\nu$:

In this section we will try to find the cross-ratio of four points in $F[\varepsilon]_\nu$ for $\nu = n+1$. We will use the same technique here as we did for the identity (4.2) but the procedure here involves lengthy calculations. First we define the cross-ratio of four points $(l_0^*, \ldots, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_{n+1}})$ as

$$\mathbf{r}(l_0^*, \ldots, l_3^*) = \frac{\Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*)}{\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)}$$

If we expand $\mathbf{r}(l_0^*, \ldots, l_3^*)$ as a truncated polynomial over $F[\varepsilon]_{n+1}$, then

$$\mathbf{r}(l_0^*, \ldots, l_3^*) = \left(r_{\varepsilon^0} + r_\varepsilon \varepsilon + r_{\varepsilon^2}\varepsilon^2 + \cdots + r_{\varepsilon^n}\varepsilon^n\right)(l_0^*, \ldots, l_3^*) \tag{4.4}$$

If we truncate this for $n = 0$, then

$$\mathbf{r}(l_0^*, \ldots, l_3^*) = r_{\varepsilon^0}(l_0^*, \ldots, l_3^*) = r(l_0, \ldots, l_3) = \frac{\Delta(l_0, l_3)\Delta(l_1, l_2)}{\Delta(l_0, l_2)\Delta(l_1, l_3)} \tag{4.5}$$

If we truncate (4.4) for $n = 1$ then the coefficient of ε^0 will remain the same as for $n = 0$ and we compute the coefficient of ε in the following way:

Consider $(l_0^*, \ldots, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_2})$ in generic position, we get

$$\mathbf{r}(l_0^*, \ldots, l_3^*) = \frac{\Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*)}{\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)} = \frac{\{\Delta(l_0, l_3) + \Delta(l_0^*, l_3^*)_\varepsilon \varepsilon\}\{\Delta(l_1, l_2) + \Delta(l_1^*, l_2^*)_\varepsilon \varepsilon\}}{\{\Delta(l_0, l_2) + \Delta(l_0^*, l_2^*)_\varepsilon \varepsilon\}\{\Delta(l_1, l_3) + \Delta(l_1^*, l_3^*)_\varepsilon \varepsilon\}}$$

If $a \neq 0 \in F$ then the inverse of $(a + b\varepsilon) \in F[\varepsilon]_2$ is $\frac{1}{a} - \frac{b}{a^2}\varepsilon \in F[\varepsilon]_2$ (this is the same as the inversion relation in $T\mathcal{B}_2(F)$ discussed later in §2.5).

Simplify the above by multiplying the inverses of denominators and separate the coefficients of ε^0 and ε. The coefficient of ε is the following

$$r_\varepsilon(l_0^*, \ldots, l_3^*) = \frac{\{\Delta(l_0^*, l_3^*)\Delta(l_1^*, l_2^*)\}_\varepsilon}{\Delta(l_0, l_2)\Delta(l_1, l_3)} - r(l_0, \ldots, l_3)\frac{\{\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)\}_\varepsilon}{\Delta(l_0, l_2)\Delta(l_1, l_3)} \tag{4.6}$$

Now for $n = 2$, i.e. $(l_0^*, \ldots, l_3^*) \in C_4(\mathbb{A}^2_{F[\varepsilon]_3})$, we will use (l_i, l_j) instead of $\Delta(l_i, l_j)$ to get

$$\mathbf{r}(l_0^*, \ldots, l_3^*) = \frac{\{(l_0, l_3) + (l_0^*, l_3^*)_\varepsilon \varepsilon + (l_0^*, l_3^*)_{\varepsilon^2}\varepsilon^2\}\{(l_1, l_2) + (l_1^*, l_2^*)_\varepsilon \varepsilon + (l_1^*, l_2^*)_{\varepsilon^2}\varepsilon^2\}}{\{(l_0, l_2) + (l_0^*, l_2^*)_\varepsilon \varepsilon + (l_0^*, l_3^*)_{\varepsilon^2}\varepsilon^2\}\{(l_1, l_3) + (l_1^*, l_3^*)_\varepsilon \varepsilon + (l_1^*, l_3^*)_{\varepsilon^2}\varepsilon^2\}}$$

simplify and separate the coefficient of ε^0, ε^1 and ε^2. Coefficients of ε^0 and ε^1 are same as we computed in (4.5) and (4.6) respectively, and the coefficient of ε^2 is

$$r_{\varepsilon^2}(l_0^*, \ldots, l_3^*) = \frac{\{(l_0^*, l_3^*)(l_1^*, l_2^*)\}_{\varepsilon^2}}{(l_0, l_2)(l_1, l_3)} - r_\varepsilon(l_0^*, \ldots, l_3^*)\frac{\{(l_0^*, l_2^*)(l_1^*, l_3^*)\}_\varepsilon}{(l_0, l_2)(l_1, l_3)} - r(_0, \ldots, l_3)\frac{\{(l_0^*, l_2^*)(l_1^*, l_3^*)\}_{\varepsilon^2}}{(l_0, l_2)(l_1, l_3)} \tag{4.7}$$

Remark 4.1.2. *The computation of coefficient of ε^n which is $r_{\varepsilon^n}(l_0^*,\ldots,l_3^*)$ in the truncated polynomial* (4.4) *will give us the following:*

$$\sum_{k=0}^{n}\left(\{\Delta(l_0^*,l_2^*)\Delta(l_1^*,l_3^*)\}_{\varepsilon^k}\, r_{\varepsilon^{n-k}}(l_0^*,\ldots,l_3^*)\right)=\{\Delta(l_0^*,l_3^*)\Delta(_1^*,l_2^*)\}_{\varepsilon^n}\,, \tag{4.8}$$

where $\Delta(l_i,l_j)\neq 0$ for $i\neq j$ and $(l_0^*,\ldots,l_3^*)\in C_4\left(\mathbb{A}^2_{F[\varepsilon]_{n+1}}\right)$

4.1.2 Triple-ratio in $F[\varepsilon]_\nu$:

In this subsection we will discuss triple-ratio (generalized cross-ratio) of 6 points, i.e., $(l_0^*,\ldots,l_5^*)\in C_6(\mathbb{A}^3_{F[\varepsilon]_\nu})$ for $\nu=n+1$. We are pleased to see that the calculations in triple-ratio are similar as the cross-ratio of 4 points $(l_0^*,\ldots,l_3^*)\in C_4(\mathbb{A}^2_{F[\varepsilon]_\nu})$.

Case $\nu=2$:

First we take $(l_0^*,\ldots,l_5^*)\in C_6(\mathbb{A}^3_{F[\varepsilon]_2})$, for any $l_i^*\in(l_0^*,\ldots,l_5^*)$

$$l_i^*=\begin{pmatrix} a_i+a_{i,\varepsilon}\varepsilon\\ b_i+b_{i,\varepsilon}\varepsilon\\ c_i+c_{i,\varepsilon}\varepsilon\end{pmatrix}=\begin{pmatrix} a_i\\ b_i\\ c_i\end{pmatrix}+\begin{pmatrix} a_{i,\varepsilon}\\ b_{i,\varepsilon}\\ c_{i,\varepsilon}\end{pmatrix}\varepsilon=l_i+l_{i,\varepsilon}\varepsilon$$

$$\Delta(l_i^*,l_j^*,l_k^*)=\Delta(l_i,l_j,l_k)+\Delta(l_i^*,l_j^*,l_k^*)_\varepsilon\varepsilon$$

where $\Delta(l_i,l_j,l_k)$ is a 3×3-determinant,

$$\Delta(l_i^*,l_j^*,l_k^*)_\varepsilon=\Delta(l_{i,\varepsilon},l_j,l_k)+\Delta(l_i,l_{j,\varepsilon},l_k)+\Delta(l_i,l_j,l_{k,\varepsilon})$$

and

$$\Delta(l_i^*,l_j^*,l_k^*)_{\varepsilon^0}=\Delta(l_i,l_j,l_k)$$

As we can expand

$$\mathbf{r}_3(l_0^*,\ldots,l_5^*)=r_3(l_0,\ldots,l_5)+r_{3,\varepsilon}(l_0^*,\ldots,l_5^*)\varepsilon$$

From [10] we have

$$r_3(l_0,\ldots,l_5)=\ \mathrm{Alt}\ \frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}$$

for $\Delta(l_i, l_j, l_j) \neq 0$ multiplicative inverse of $\Delta(l_i^*, l_j^*, l_k^*)$ is $-\frac{1}{\Delta(l_i,l_j,l_j)} - \frac{\Delta(l_i^*,l_j^*,l_k^*)_\varepsilon}{\Delta(l_i,l_j,l_j)^2}\varepsilon$ and from now on we will use $(l_i^* l_j^* l_k^*)$ instead of $\Delta(l_i^*, l_j^*, l_k^*)$ unless specify.

$$\begin{aligned}\mathbf{r}(l_0^*, \ldots, l_5^*) &= \text{Alt}\,\frac{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)}{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)}\\ &= \text{Alt}\left\{\frac{\{(l_0 l_1 l_3) + (l_0^* l_1^* l_3^*)_\varepsilon \varepsilon\}\{(l_1 l_2 l_4) + (l_1^* l_2^* l_4^*)_\varepsilon \varepsilon\}\{(l_2 l_0 l_5) + (l_2^* l_0^* l_5^*)_\varepsilon \varepsilon\}}{\{(l_0 l_1 l_4) + (l_0^* l_1^* l_4^*)_\varepsilon \varepsilon\}\{(l_1 l_2 l_5) + (l_1^* l_2^* l_5^*)_\varepsilon \varepsilon\}\{(l_2 l_0 l_3) + (l_2^* l_0^* l_3^*)_\varepsilon \varepsilon\}}\right\}\end{aligned}$$

Simplify the above and separate coefficients of ε^0 and ε^1, we will see that the coefficient of ε^1 is the triple-ratio of six points $(l_0^*, \ldots, l_5^*) \in C_6(\mathbb{A}_F^3)$ and the coefficient of ε is the following:

$$\begin{aligned}&r_{3,\varepsilon}(l_0^*, \ldots, l_5^*)\\ &= \text{Alt}\left\{\frac{\{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)\}_\varepsilon}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)} - \frac{(l_0 l_1 l_3)(l_1 l_2 l_4)(l_2 l_0 l_5)}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)}\frac{\{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)\}_\varepsilon}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)}\right\} \qquad (4.9)\end{aligned}$$

we can write the explicit formula. For 6 points in $C_6(\mathbb{A}^3_{F[\varepsilon]_2})$

$$r_{3,\varepsilon}(l_0^*, \ldots, l_5^*) = \text{Alt}\left\{\frac{\{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)\}_\varepsilon}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)} - r_3(l_0, \ldots, l_5)\frac{\{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)\}_\varepsilon}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)}\right\}$$

Case $v = 3$:

For 6 points in $C_6(\mathbb{A}^3_{F[\varepsilon]_3})$ we will the following through same procedure as we did above, and for $a \neq 0$ we have $\frac{1}{a + a_\varepsilon \varepsilon + a_{\varepsilon^2}\varepsilon^2} = b + b_\varepsilon \varepsilon + b_{\varepsilon^2}\varepsilon^2$ where $b = a^{-1} \in F^\times$, $b_\varepsilon = (a, a_\varepsilon)$ and $b_{\varepsilon^2} = (a, a_\varepsilon, a_{\varepsilon^2})$

$$\begin{aligned}r_{3,\varepsilon^2} = \text{Alt}\Bigg\{&\frac{\{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)\}_{\varepsilon^2}}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)} - r_{3,\varepsilon}(l_0^*, \ldots, l_5^*)\frac{\{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)\}_\varepsilon}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)}\\ &- r_3(l_0, \ldots, l_5)\frac{\{(l_1^* l_2^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)\}_{\varepsilon^2}}{(l_0 l_1 l_4)(l_1 l_2 l_5)(l_2 l_0 l_3)}\Bigg\}\end{aligned}$$

Here we used the following notation just for simplification.

$$(abc)_\varepsilon := a_\varepsilon b_{\varepsilon^0} c_{\varepsilon^0} + a_{\varepsilon^0} b_\varepsilon c_{\varepsilon^0} + a_{\varepsilon^0} b_{\varepsilon^0} c_\varepsilon$$

4.2 Dilogarithmic Bicomplexes

In this subsection we will connect the Grassmannian bicomplex to the Cathelineau's tangential complex in weight 2.

We will use the following notations throughout this section

$$\Delta(l_i^*, l_j^*)_\varepsilon = \Delta(l_{i,\varepsilon}, l_j) + \Delta(l_i, l_{j,\varepsilon}) \qquad \text{and} \qquad \Delta(l_i^*, l_j^*)_{\varepsilon^0} = \Delta(l_i, l_j)$$

and we will assume that $\Delta(l_i, l_j) \neq 0$ (as we often want to divide by such determinants).

Let $C_m(\mathbb{A}^2_{F[\varepsilon]_2})$ be the free abelian group generated by the configuration $(l_0^*, \ldots, l_{m-1}^*)$ of m points in $\mathbb{A}^2_{F[\varepsilon]_2}$, where $\mathbb{A}^2_{F[\varepsilon]_2}$ is defined as an affine plane over $F[\varepsilon]_2$. Configurations of m points in $\mathbb{A}^2_{F[\varepsilon]_2}$ are 2-tuples of vectors over $F[\varepsilon]_2$ modulo $\mathrm{GL}_2(F[\varepsilon])$. In this case the Grassmannian complex will be in the following shape

$$\cdots \xrightarrow{d} C_5(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_3(\mathbb{A}^2_{F[\varepsilon]_2})$$

$$d\colon (l_0^*, \ldots, l_{m-1}^*) \mapsto \sum_{i=0}^{m} (-1)^i (l_0^*, \ldots, \hat{l}_i^*, \ldots, l_{m-1}^*)$$

where $l_i^* = \begin{pmatrix} a_i + a_{i,\varepsilon}\varepsilon \\ b_i + b_{i,\varepsilon}\varepsilon \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix}\varepsilon = l_i + l_{i,\varepsilon}\varepsilon$ and $a_i, b_i, a_{i,\varepsilon}, b_{i,\varepsilon} \in F$, $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Here we recall the $\mathbb{Z}$-module $T\mathcal{B}_2(F)$ generated by $\langle a; b]_2 := [a + b\varepsilon] - [a] \in \mathbb{Z}[F[\varepsilon]_2]$ (have discussed earlier in §2.5).

Consider the following diagram

$$\begin{array}{ccccc} C_5(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d} & C_4(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \\ & & \downarrow{\scriptstyle \tau^2_{1,\varepsilon}} & & \downarrow{\scriptstyle \tau^2_{0,\varepsilon}} \\ & & T\mathcal{B}_2(F) & \xrightarrow{\partial_\varepsilon} & F \otimes F^\times \oplus \bigwedge^2 F \end{array} \tag{4.2a}$$

where

$$\partial_\varepsilon : \langle a; b]_2 \mapsto \left(\frac{b}{a} \otimes (1-a) + \frac{b}{1-a} \otimes a\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a}\right)$$

We write the map $\tau^2_{0,\varepsilon}$ as a sum of two maps

$$\tau^{(1)} : C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \to F \otimes F^\times$$

and

$$\tau^{(2)} : C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \to \bigwedge{}^2 F$$

where

$$\begin{aligned}&\tau^{(1)}(l_0^*,l_1^*,l_2^*)\\=&\frac{\Delta(l_1^*,l_2^*)_\varepsilon}{\Delta(l_1,l_2)}\otimes\frac{\Delta(l_0,l_2)}{\Delta(l_0,l_1)}-\frac{\Delta(l_0^*,l_2^*)_\varepsilon}{\Delta(l_0,l_2)}\otimes\frac{\Delta(l_1,l_2)}{\Delta(l_1,l_0)}+\frac{\Delta(l_0^*,l_1^*)_\varepsilon}{\Delta(l_0,l_1)}\otimes\frac{\Delta(l_2,l_1)}{\Delta(l_2,l_0)}\end{aligned}$$

and

$$\begin{aligned}&\tau^{(2)}(l_0^*,l_1^*,l_2^*)\\=&\frac{\Delta(l_0^*,l_1^*)_\varepsilon}{\Delta(l_0,l_1)}\wedge\frac{\Delta(l_1^*,l_2^*)_\varepsilon}{\Delta(l_1,l_2)}-\frac{\Delta(l_0^*,l_1^*)_\varepsilon}{\Delta(l_0,l_1)}\wedge\frac{\Delta(l_0^*,l_2^*)_\varepsilon}{\Delta(l_0,l_2)}+\frac{\Delta(l_1^*,l_2^*)_\varepsilon}{\Delta(l_1,l_2)}\wedge\frac{\Delta(l_0^*,l_2^*)_\varepsilon}{\Delta(l_0,l_2)}\end{aligned}$$

Furthermore, we put

$$\tau^2_{1,\varepsilon}(l_0^*,\ldots,l_3^*)=\langle r(l_0,\ldots,l_3);r_\varepsilon(l_0^*,\ldots,l_3^*)]$$

where $r(l_0,\ldots,l_3)$, $r_\varepsilon(l_0^*,\ldots,l_3^*)$ are the coefficient of ε^0 and ε^1 respectively, in $\mathbf{r}(l_0^*,\ldots,l_3^*)$ as defined in 4.1.1. and Δ is defined in 4.1

Our maps $\tau^2_{0,\varepsilon}$ and $\tau^2_{1,\varepsilon}$ are based on ratios of determinants and cross-ratios respectively, so there is enough evidence that these are independent of the length of the vectors and the volume formed by these vectors. This independence can be seen directly through the definition of the maps.

We will also use the shorthand (l_il_j) instead of $\Delta(l_i,l_j)$ wherever we find less space to accommodate long expressions.

Now calculate the cross-ratio of the points in $\mathbb{A}^2_F[\varepsilon]_2$.

$$\begin{aligned}\mathbf{r}(l_0^*,\ldots,l_3^*)&=\frac{\Delta(l_0^*,l_3^*)\Delta(l_1^*,l_2^*)}{\Delta(l_0^*,l_2^*)\Delta(l_1^*,l_3^*)}\quad\text{we have already assumed that }\Delta(l_i,l_j)\neq 0\\&=\frac{(\Delta(l_0,l_3)+\{\Delta(l_0,l_{3,\varepsilon})+\Delta(l_{0,\varepsilon},l_3)\}\varepsilon)(\Delta(l_1,l_2)+\{\Delta(l_1,l_{2,\varepsilon})+\Delta(l_{1,\varepsilon},l_2)\}\varepsilon)}{(\Delta(l_0,l_2)+\{\Delta(l_0,l_{2,\varepsilon})+\Delta(l_{0,\varepsilon},l_2)\}\varepsilon)(\Delta(l_1,l_3)+\{\Delta(l_1,l_{3,\varepsilon})+\Delta(l_{1,\varepsilon},l_3)\}\varepsilon)}\\&=\frac{(l_0l_3)(l_1l_2)+\{(l_0l_3)((l_1l_{2,\varepsilon})+(l_{1,\varepsilon}l_2))+(l_1l_2)((l_0l_{3,\varepsilon})+(l_{0,\varepsilon}l_3))\}\varepsilon}{(l_0l_2)(l_1l_3)+\{(l_0l_2)((l_1l_{3,\varepsilon})+(l_{1,\varepsilon}l_3))+(l_1l_3)((l_0l_{2,\varepsilon})+(l_{0,\varepsilon}l_2))\}\varepsilon}\\&=\frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}+\frac{x}{(l_0l_2)^2(l_1l_3)^2}\varepsilon\end{aligned}$$

where

$$\begin{aligned}x=&+(l_0l_2)(l_1l_3)(l_0l_3)(l_1l_{2,\varepsilon})+(l_0l_2)(l_1l_3)(l_0l_3)(l_{1,\varepsilon}l_2)\\&+(l_0l_2)(l_1l_3)(l_1l_2)(l_0l_{3,\varepsilon})+(l_0l_2)(l_1l_3)(l_1l_2)(l_{0,\varepsilon}l_3)\\&-(l_0l_3)(l_1l_2)(l_0l_2)(l_1l_{3,\varepsilon})-(l_0l_3)(l_1l_2)(l_0l_2)(l_{1,\varepsilon}l_3)\\&-(l_0l_3)(l_1l_2)(l_1l_3)(l_0l_{2,\varepsilon})-(l_0l_3)(l_1l_2)(l_1l_3)(l_{0,\varepsilon}l_2)\end{aligned}$$

Similarly we calculate

$$
\begin{aligned}
1 - \mathbf{r}(l_0^*, \ldots, l_3^*) &= \frac{\Delta(l_0^*, l_1^*)\Delta(l_2^*, l_3^*)}{\Delta(l_0^*, l_2^*)\Delta(l_1^*, l_3^*)} \\
&= \frac{(l_0 l_1)(l_2 l_3)}{(l_0 l_2)(l_1 l_3)} + \frac{y}{(l_0 l_2)^2 (l_1 l_3)^2}\varepsilon
\end{aligned}
$$

where

$$
\begin{aligned}
y = &+ (l_0 l_2)(l_1 l_3)(l_0 l_1)(l_2 l_{3,\varepsilon}) + (l_0 l_2)(l_1 l_3)(l_0 l_1)(l_{2,\varepsilon} l_3) \\
&+ (l_0 l_2)(l_1 l_3)(l_2 l_3)(l_0 l_{1,\varepsilon}) + (l_0 l_2)(l_1 l_3)(l_2 l_3)(l_{0,\varepsilon} l_1) \\
&- (l_0 l_1)(l_2 l_3)(l_0 l_2)(l_1 l_{3,\varepsilon}) - (l_0 l_1)(l_2 l_3)(l_0 l_2)(l_{1,\varepsilon} l_3) \\
&- (l_0 l_1)(l_2 l_3)(l_1 l_3)(l_0 l_{2,\varepsilon}) - (l_0 l_1)(l_2 l_3)(l_1 l_3)(l_{0,\varepsilon} l_2)
\end{aligned}
$$

Remark 4.2.1. *The $F^\times$-action of $T\mathcal{B}_2(F)$ lifts to an $F^\times$-action on $C_4(\mathbb{A}^2_{F[\varepsilon]_2})$ in the obvious way:*

The $F^\times$-action is defined above for $F[\varepsilon]_2$ induces an $F^\times$-action in $\mathbb{A}^2_{F[\varepsilon]_2}$ diagonally as

$$
\lambda \star \begin{pmatrix} a + a_\varepsilon \varepsilon \\ b + b_\varepsilon \varepsilon \end{pmatrix} = \begin{pmatrix} a + \lambda a_\varepsilon \varepsilon \\ b + \lambda b_\varepsilon \varepsilon \end{pmatrix} \in \mathbb{A}^2_{F[\varepsilon]_2}, \lambda \in F^\times
$$

Lemma 4.2.2. *The diagram (4.2a) is commutative*

Proof: First we need to calculate $\tau^2_{1,\varepsilon}$.

$$
\tau^2_{1,\varepsilon}(l_0^*, \ldots, l_3^*) = \left\langle \frac{(l_0 l_3)(l_1 l_2)}{(l_0 l_2)(l_1 l_3)}; \frac{x}{(l_0 l_2)^2 (l_1 l_3)^2} \right]
$$

$$
\begin{aligned}
&\partial_\varepsilon \circ \tau^2_{1,\varepsilon}(l_0^*, \ldots, l_3^*) \\
=&\left(\frac{x}{(l_0 l_2)(l_1 l_3)(l_0 l_3)(l_1 l_2)} \otimes \frac{(l_0 l_1)(l_2 l_3)}{(l_0 l_2)(l_1 l_3)} + \frac{x}{(l_0 l_2)(l_1 l_3)(l_0 l_1)(l_2 l_3)} \otimes \frac{(l_0 l_3)(l_1 l_2)}{(l_0 l_2)(l_1 l_3)}, \right. \\
&\left. - \frac{x}{(l_0 l_2)(l_1 l_3)(l_0 l_3)(l_1 l_2)} \wedge \frac{x}{(l_0 l_2)(l_1 l_3)(l_0 l_1)(l_2 l_3)} \right)
\end{aligned}
$$

For the other side we first calculate d and then apply $\tau^2_{0,\varepsilon}$ on $d(l_0^*, \ldots, l_3^*)$.

$$
\tau^2_{0,\varepsilon} \circ d(l_0^*, \ldots, l_3^*) = \tau^2_{0,\varepsilon}\left(-(l_0^*, l_1^*, l_2^*) + (l_0^*, l_1^*, l_3^*) - (l_0^*, l_2^*, l_3^*) + (l_1^*, l_2^*, l_3^*)\right)
$$

This calculation can be done in two steps. In a first step we find $\tau^{(1)} \circ d(l_0^*, \ldots, l_3^*)$ and in a second step we calculate $\tau^{(2)} \circ d(l_0^*, \ldots, l_3^*)$

In the first step, we have

$$\begin{aligned}
&(\tau^{(1)} \circ d)(l_0^*, \dots, l_3^*) \\
&= \frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} \otimes \frac{(l_0l_1)}{(l_0l_2)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} \otimes \frac{(l_0l_1)}{(l_1l_2)} + \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} \otimes \frac{(l_0l_2)}{(l_1l_2)} \\
&- \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} \otimes \frac{(l_0l_1)}{(l_0l_3)} + \frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} \otimes \frac{(l_0l_1)}{(l_1l_3)} - \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} \otimes \frac{(l_0l_3)}{(l_1l_3)} \\
&+ \frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} \otimes \frac{(l_0l_2)}{(l_0l_3)} - \frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} \otimes \frac{(l_0l_2)}{(l_2l_3)} + \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} \otimes \frac{(l_0l_3)}{(l_2l_3)} \\
&- \frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} \otimes \frac{(l_1l_2)}{(l_1l_3)} + \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} \otimes \frac{(l_1l_2)}{(l_2l_3)} - \frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} \otimes \frac{(l_1l_3)}{(l_2l_3)} \\
&= \left(\frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} + \frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} - \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} \right) \otimes \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \\
&+ \left(\frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} + \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} - \frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} - \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} \right) \otimes \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)} \in F \otimes F^\times
\end{aligned} \tag{4.10}$$

By using identities (4.3) and (4.1) the above can be written as

$$= \frac{x}{(l_0l_2)(l_1l_3)(l_0l_3)(l_1l_2)} \otimes \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} + \frac{x}{(l_0l_2)(l_1l_3)(l_0l_1)(l_2l_3)} \otimes \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}$$

Now go to the second step, and by using $a \wedge a = 0$, modulo 2 torsion and $a \wedge b = -b \wedge a$ and identities (4.3),(4.1) we can get here

$$\begin{aligned}
&\tau^{(2)} \circ d(l_0^*, \dots, l_3^*) \\
&= \left(\frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} + \frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} - \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} \right) \wedge \left(\frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} + \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} - \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} \right) \\
&= -\frac{x}{(l_0l_2)(l_1l_3)(l_0l_3)(l_1l_2)} \wedge \frac{x}{(l_0l_2)(l_1l_3)(l_0l_1)(l_2l_3)}
\end{aligned} \tag{4.11}$$

□

In the remainder of this section we prove that the following diagram following is a bicomplex.

$$\begin{tikzcd}
C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "d'"] & C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[d, "d'"] \\
C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "\tau^2_{1,\varepsilon}"] & C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \arrow[d, "\tau^2_{0,\varepsilon}"] \\
T\mathcal{B}_2(F) \arrow[r, "\partial_\varepsilon"] & F \otimes F^\times \oplus \bigwedge^2 F
\end{tikzcd} \tag{4.2b}$$

To prove the above is a bicomplex, we are giving the following results.

Proposition 4.2.3. *The map* $C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{d'} C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{\tau^2_{0,\varepsilon}} (F \otimes F^\times) \oplus \left(\bigwedge^2 F\right)$ *is zero.*

Proof: Let $\omega \in \det V_3^*$ be the volume form in three-dimensional vector space V_3, i.e., $\Delta(l_i, l_j, l_k) = \langle \omega, l_i \wedge l_j \wedge l_k \rangle$ then $\Delta(l_i, \cdot, \cdot)$ is a volume form in $V_3/\langle l_i \rangle$. Use

$$\Delta(l_i^*, l_j^*, l_k^*) = \Delta(l_i, l_j, l_k) + \left\{\Delta(l_i^*, l_j^*, l_k^*)_\varepsilon\right\} \varepsilon$$

where

$$\Delta(l_i^*, l_j^*, l_k^*)_\varepsilon = \Delta(l_{i,\varepsilon}, l_j, l_k) + \Delta(l_i, l_{j,\varepsilon}, l_k) + \Delta(l_i, l_j, l_{k,\varepsilon})$$

We can directly compute $\tau^2_{0,\varepsilon} \circ d'$

$$\tau^2_{0,\varepsilon} \circ d'(l_0^*, \ldots, l_3^*) = \tau^2_{0,\varepsilon}\left((l_0^*|l_1^*, l_2^*, l_3^*) - (l_1^*|l_0^*, l_2^*, l_3^*) + (l_2^*|l_0^*, l_1^*, l_3^*) - (l_3^*|l_0^*, l_1^*, l_2^*)\right)$$

First we calculate the first part of the map $\tau^2_{0,\varepsilon}$.

$$\begin{aligned}
= &-\frac{(l_0^*l_2^*l_3^*)_\varepsilon}{(l_0l_2l_3)} \otimes \frac{(l_0l_1l_2)}{(l_0l_1l_3)} + \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} \otimes \frac{(l_0l_2l_1)}{(l_0l_2l_3)} - \frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \otimes \frac{(l_0l_3l_1)}{(l_0l_3l_2)} \\
&+ \frac{(l_1^*l_2^*l_3^*)_\varepsilon}{(l_1l_2l_3)} \otimes \frac{(l_1l_0l_2)}{(l_1l_0l_3)} - \frac{(l_1^*l_0^*l_3^*)_\varepsilon}{(l_1l_0l_3)} \otimes \frac{(l_1l_2l_0)}{(l_1l_2l_3)} + \frac{(l_1^*l_0^*l_2^*)_\varepsilon}{(l_1l_0l_2)} \otimes \frac{(l_1l_3l_0)}{(l_1l_3l_2)} \\
&- \frac{(l_2^*l_1^*l_3^*)_\varepsilon}{(l_2l_1l_3)} \otimes \frac{(l_2l_0l_1)}{(l_2l_0l_3)} + \frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)} \otimes \frac{(l_2l_1l_0)}{(l_2l_1l_3)} - \frac{(l_2^*l_0^*l_1^*)_\varepsilon}{(l_2l_0l_1)} \otimes \frac{(l_2l_3l_0)}{(l_2l_3l_1)} \\
&+ \frac{(l_3^*l_1^*l_2^*)_\varepsilon}{(l_3l_1l_2)} \otimes \frac{(l_3l_0l_1)}{(l_3l_0l_2)} - \frac{(l_3^*l_0^*l_2^*)_\varepsilon}{(l_3l_0l_2)} \otimes \frac{(l_3l_1l_0)}{(l_3l_1l_2)} + \frac{(l_3^*l_0^*l_1^*)_\varepsilon}{(l_3l_0l_1)} \otimes \frac{(l_3l_2l_0)}{(l_3l_2l_1)}
\end{aligned}$$

Clearly the above gives zero. Similarly calculate the second part of the map.

$$\begin{aligned}
= &\frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} \wedge \frac{(l_0^*l_2^*l_3^*)_\varepsilon}{(l_0l_2l_3)} + \frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \wedge \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} + \frac{(l_0^*l_2^*l_3^*)_\varepsilon}{(l_0l_2l_3)} \wedge \frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \\
&- \frac{(l_1^*l_0^*l_3^*)_\varepsilon}{(l_1l_0l_3)} \wedge \frac{(l_1^*l_2^*l_3^*)_\varepsilon}{(l_1l_2l_3)} - \frac{(l_1^*l_0^*l_2^*)_\varepsilon}{(l_1l_0l_2)} \wedge \frac{(l_1^*l_0^*l_3^*)_\varepsilon}{(l_1l_0l_3)} - \frac{(l_1^*l_2^*l_3^*)_\varepsilon}{(l_1l_2l_3)} \wedge \frac{(l_1^*l_0^*l_2^*)_\varepsilon}{(l_1l_0l_2)} \\
&+ \frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)} \wedge \frac{(l_2^*l_1^*l_3^*)_\varepsilon}{(l_2l_1l_3)} + \frac{(l_2^*l_0^*l_1^*)_\varepsilon}{(l_2l_0l_1)} \wedge \frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)} + \frac{(l_2^*l_1^*l_3^*)_\varepsilon}{(l_2l_1l_3)} \wedge \frac{(l_2^*l_0^*l_1^*)_\varepsilon}{(l_2l_0l_1)} \\
&- \frac{(l_3^*l_0^*l_2^*)_\varepsilon}{(l_3l_0l_2)} \wedge \frac{(l_3^*l_1^*l_2^*)_\varepsilon}{(l_3l_1l_2)} + \frac{(l_3^*l_0^*l_1^*)_\varepsilon}{(l_3l_0l_1)} \wedge \frac{(l_3^*l_0^*l_2^*)_\varepsilon}{(l_3l_0l_2)} + \frac{(l_3^*l_1^*l_2^*)_\varepsilon}{(l_3l_1l_2)} \wedge \frac{(l_3^*l_0^*l_1^*)_\varepsilon}{(l_3l_0l_1)} \\
= &0
\end{aligned}$$

□

The following result is very important for proving Theorem 4.3.3. Through this result we are able to see the projected-five term relation in $T\mathcal{B}_2(F)$.

Lemma 4.2.4. *Let $x_0^*, \ldots, x_4^* \in \mathbb{P}^2_{F[\varepsilon]_2}$ be 5 points in generic position, then*

$$\sum_{i=0}^{4} (-1)^i \left\langle r(x_i|x_0, \ldots, \hat{x}_i, \ldots, x_4); r_\varepsilon(x_i^*|x_0^*, \ldots, \hat{x}_i^*, \ldots, x_4^*)\right] = 0 \in T\mathcal{B}_2(F), \tag{4.12}$$

where $x_i^* = x_i + x_i'\varepsilon$ *and* $x_i, x_i' \in \mathbb{P}_F^2$

$$\boldsymbol{r}(x_i^*|x_0^*, \dots, \hat{x}_i^*, \dots, x_4^*) = r(x_i|x_0, \dots, \hat{x}_i, \dots, x_4) + r_\varepsilon(x_i^*|x_0^*, \dots, \hat{x}_i^*, \dots, x_4^*)\varepsilon,$$

where the LHS denotes the projected cross-ratio of any four points projected from the fifth from $x_0^*, \dots, x_4^* \in \mathbb{P}^2_{F[\varepsilon]_2}$.

Proof: Consider five points $y_0, \dots, y_4 \in \mathbb{P}_F^1$ in generic position. We can write the five-term relation in terms of cross-ratios in $\mathcal{B}_2(F)$ as (see Proposition 4.5 (2)b in [7]):

$$\sum_{i=0}^{4}(-1)^i[r(y_0, \dots, \hat{y}_i, \dots, y_4)]_2 = 0$$

These five points depend on 2 parameters modulo the action of $PGL_2(F)$, whose action on $\mathbb{P}_F^1$ is 3-fold transitive, so we can express these five points with two variables modulo this action, we can put

$$(y_0, \dots, y_4) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{b} \\ 1 \end{pmatrix}\right),$$

then we get one of the form of five-term relation in two variables (needs to use inversion in the last two terms).

$$[a]_2 - [b]_2 + \left[\frac{b}{a}\right]_2 + \left[\frac{1-a}{1-b}\right]_2 - \left[\frac{1-\frac{1}{a}}{1-\frac{1}{b}}\right]_2 = 0.$$

Now we consider five points $y_0^*, \dots, y_4^* \in \mathbb{P}^1_{F[\varepsilon]_2}$, in generic position, where $y_i^* = y_i + y_i'\varepsilon$ for $y_i, y_i' \in \mathbb{P}_F^1$. A generic 2×2 matrix in $PGL_2(F[\varepsilon]_2)$ depends on $6 = 2(2 \times 2) - 2(1)$ parameters, while each point in $\mathbb{P}^1_{F[\varepsilon]_2}$ depends on 2 parameters, so these five points in $\mathbb{P}^1_{F[\varepsilon]_2}$ modulo the action of $PGL_2(F[\varepsilon]_2)$ have 4 parameters. Now we can express them by using four variables we choose:

$$(y_0^*, \dots, y_4^*) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a} - \frac{a'}{a^2}\varepsilon \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{b} - \frac{b'}{b^2}\varepsilon \\ 1 \end{pmatrix}\right).$$

We calculate all possible determinants which are the following:

$$\Delta(y_0, y_1) = \Delta(y_0, y_2) = \Delta(y_0, y_3) = \Delta(y_0, y_4) = 1, \Delta(y_1, y_2) = -1,$$
$$\Delta(y_1, y_3) = -\frac{1}{a}, \Delta(y_1, y_4) = -\frac{1}{b}, \Delta(y_2, y_3) = 1 - \frac{1}{a}, \Delta(y_2, y_4) = 1 - \frac{1}{b}$$
$$\Delta(y_0^*, y_1^*)_\varepsilon = \Delta(y_0^*, y_2^*)_\varepsilon = \Delta(y_0^*, y_3^*)_\varepsilon = \Delta(y_0^*, y_4^*)_\varepsilon = \Delta(y_1^*, y_2^*)_\varepsilon = 0$$
$$\Delta(y_1^*, y_3^*)_\varepsilon = \Delta(y_2^*, y_3^*)_\varepsilon = \frac{a'}{a^2}, \Delta(y_1^*, y_4^*)_\varepsilon = \Delta(y_2^*, y_4^*)_\varepsilon = \frac{b'}{b^2}$$

For $y_0^*, \dots, y_4^* \in \mathbb{P}^1_{F[\varepsilon]_2}$, we can write the following expression in $T\mathcal{B}_2(F)$

$$\sum_{i=0}^{4}(-1)^i \left\langle r(y_0,\dots,\hat{y}_i,\dots,y_4); r_\varepsilon(y_0^*,\dots,\hat{y}_i^*,\dots,y_4^*)\right]_2$$

If we expand the above expression and we put all determinants in it we will get the following expression in two variables.

$$\langle a; a'\rangle_2 - \langle b; b'\rangle_2 + \left\langle \frac{b}{a}; \frac{ab'-a'b}{a^2}\right]_2 - \left\langle \frac{1-b}{1-a}; \frac{(1-b)a'-(1-a)b'}{(1-a)^2}\right]_2$$
$$+\left\langle \frac{a(1-b)}{b(1-a)}; \frac{b(1-b)a'-a(1-a)b'}{(b(1-a))^2}\right]_2$$

From (4.12) it is clear that the above is the LHS of the five-term relation in $T\mathcal{B}_2(F)$. We will reduce the claim to this latter form of five-term relation.

Consider $x_0, \dots, x_4 \in \mathbb{P}^2_F$ in generic position. These five points also depend on 2 parameters modulo the action of $PGL_2(F)$, so we can express these five points in terms of two variables by the following choice:

$$(x_0,\dots,x_4) = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{b}\\ \frac{1}{a}\\ 1\end{pmatrix}\right)$$

We compute all possible 3×3 determinants of the above and put them in the expansion of the following:

$$\sum_{i=0}^{4}(-1)^i\, [r(x_i|x_0,\dots,\hat{x}_i,\dots,x_4)]_2 \in \mathcal{B}_2(F),$$

we get the following expression in two variables

$$[a]_2 - b]_2 + \left[\frac{b}{a}\right]_2 + \left[\frac{1-a}{1-b}\right]_2 - \left[\frac{1-\frac{1}{a}}{1-\frac{1}{b}}\right]_2,$$

clearly the above is the LHS of one version of five-term relation in $\mathcal{B}_2(F)$.

Since by assumption $x_0^*, \dots, x_4^* \in \mathbb{P}^2_{F[\varepsilon]_2}$ are 5 points in generic position, we can express them modulo the action of $PGL_3(F[\varepsilon]_2)$ into 4 parameters then we can choose these points in terms of four variables in the following way:

$$(x_0^*,\dots,x_4^*) = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{b}-\frac{b'}{b^2}\varepsilon\\ \frac{1}{a}-\frac{a'}{a^2}\varepsilon\\ 1\end{pmatrix}\right)$$

We compute all possible 3×3 determinants and substitute them in an expansion of the following:

$$\sum_{i=0}^{4}(-1)^i \left\langle r(x_i|x_0,\ldots,\hat{x}_i,\ldots,x_4); r_\varepsilon(x_i^*|x_0^*,\ldots,\hat{x}_i^*,\ldots,x_4^*)\right]_2 \in T\mathcal{B}_2(F),$$

we get

$$\langle a; a']_2 - \langle b; b']_2 + \left\langle \frac{b}{a}; \frac{ab'-a'b}{a^2}\right]_2 - \left\langle \frac{1-b}{1-a}; \frac{(1-b)a'-(1-a)b'}{(1-a)^2}\right]_2$$
$$+ \left\langle \frac{a(1-b)}{b(1-a)}; \frac{b(1-b)a'-a(1-a)b'}{(b(1-a))^2}\right]_2$$

which is the five-term expression in $T\mathcal{B}_2(F)$ up to invoking the inversion relation for the last two terms, which also holds in $T\mathcal{B}_2(F)$ □

Lemma 4.2.4 indicates that we now have the projected five-term relation in $T\mathcal{B}_2(F)$ and this relation will help us to prove the commutative diagram for weight $n = 3$ in the tangential case.

Proposition 4.2.5. *The map $C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{d'} C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{\tau^2_{1,\varepsilon}} T\mathcal{B}(F)$ is zero.*

Proof: We can directly calculate $\tau^2_{1\varepsilon} \circ d'$.

$$\tau^2_{1,\varepsilon} \circ d'(l_0^*,\ldots,l_4^*) = \tau^2_{1,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i \left(l_i^*|l_0^*,\ldots,\hat{l}_i^*,\ldots,l_4^*\right)\right)$$
$$= \sum_{i=0}^{4}(-1)^i \left\langle r\left(l_i|l_0,\ldots,\hat{l}_i\ldots,l_4\right); r_\varepsilon\left(l_i^*|l_0^*,\ldots,\hat{l}_i^*,\ldots,l_4^*\right)\right] \tag{4.13}$$

The above is the projected five term relation in $T\mathcal{B}_2(F)$ by Lemma 4.2.4. □

Theorem 4.2.2 shows that the diagram (4.2a) is commutative and Propositions 4.2.3 and 4.2.5 shows that we have formed a bicomplex between the Grassmannian complex and Cathelineau's tangential complex.

4.2.1 Special Case of Derivation

Let us look at our results for a very special derivation $D \in Der_{\mathbb{Z}}(F[\varepsilon]_2, F)$ to make them more explicit. Let $D \in Der_{\mathbb{Z}}(F[\varepsilon]_2, F)$ be the derivation map in the following way (see [17] for the discussion related to derivation in F).

$$D(a + b\varepsilon) = b$$

If $u \in F[\varepsilon]_2$ then $D(u) \in F$ and holds the following rules:

$$D(u \pm v) = D(u) \pm D(v)$$

$$D(uv) = uD(v) + vD(u)$$

for all $u, v \in F[\varepsilon]_2$

We can use this D in our case where $\Delta(l_i^*, l_j^*) \in F[\varepsilon]_2$

$D\left(\Delta(l_i^*, l_j^*)\right) = D\left(\Delta(l_i, l_j) + \Delta(l_i^*, l_j^*)_\varepsilon \varepsilon\right) = \Delta(l_i^*, l_j^*)_\varepsilon$

Now (4.10) and (4.11) can be written here combine by using D.

$$\begin{aligned}
&\Bigg(\left(\frac{D(l_1^*l_2^*)}{(l_1l_2)} + \frac{D(l_0^*l_3^*)}{(l_0l_3)} - \frac{D(l_1^*l_3^*)}{(l_1l_3)} - \frac{D(l_0^*l_2^*)}{(l_0l_2)}\right) \otimes \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \\
&+\left(\frac{D(l_1^*l_3^*)}{(l_1l_3)} + \frac{D_\varepsilon(l_0^*l_2^*)}{(l_0l_2)} - \frac{D(l_2^*l_3^*)}{(l_2l_3)} - \frac{D(l_0^*l_1^*)}{(l_0l_1)}\right) \otimes \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\Bigg) \\
&+\Bigg(\left(\frac{D(l_1^*l_2^*)}{(l_1l_2)} + \frac{D(l_0^*l_3^*)}{(l_0l_3)} - \frac{D(l_1^*l_3^*)}{(l_1l_3)} - \frac{D(l_0^*l_2^*)}{(l_0l_2)}\right) \\
&\wedge\left(\frac{D(l_2^*l_3^*)}{(l_2l_3)} + \frac{D(l_0^*l_1^*)}{(l_0l_1)} - \frac{D(l_1^*l_3^*)}{(l_1l_3)} - \frac{D(l_0^*l_2^*)}{(l_0l_2)}\right)\Bigg) \in (F \otimes F^\times) \oplus \left(\bigwedge{}^2 F\right)
\end{aligned}$$

Example 4.2.6. *If we specialise $\varepsilon \mapsto 0$ then $F[\varepsilon]_2 \to F$ and replace $D \in Der_{\mathbb{Z}}(F[\varepsilon]_2, F)$ with $D \in Der_{\mathbb{Z}} F$ as defined in (2.4.1) then the above becomes*

$$\begin{aligned}
&\left(D\log\frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)} \otimes \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} + D\log\frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \otimes \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\right) \\
&+\left(D\log\frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \wedge D\log\frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\right) \in (F \otimes F^\times) \oplus \left(\bigwedge{}^2 F\right)
\end{aligned}$$

From the above expression it is clear that the first part of the sum is same as $\partial^D \circ \tau_1^2(l_0, \ldots, l_3) = \tau_0^2 \circ d(l_0, \ldots, l_3)$ which we have shown in Lemma 3.1.3. This indicates that (3.1a) in §3.1 is a special case of (4.2a). Further, when Goncharov found the morphisms between Bloch-Suslin complex and Grassmannian complex [9], he got

$$\delta \circ f_1^2(l_0, \ldots, l_3) = f_0^2 \circ d(l_0, \ldots, l_3) = \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \wedge \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)} \in \bigwedge{}^2 F^\times$$

from both sides of the commutative diagram, where δ, f_0^2 and f_1^2 are defined at the end of §3.2, but we have here through this example

$$D\log\frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)} \wedge D\log\frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)} \in \bigwedge{}^2 F$$

with the comparison of above two expressions one can say that the second part of the sum is $D\log$ of the Goncharov's case.

4.3 Trilogarithmic Complexes

We have already discussed the tangent group (or $\mathbb{Z}$-module) $T\mathcal{B}_2(F)$ over $F[\varepsilon]_2$ in §4.2. In this section we will discuss group $T\mathcal{B}_3(F)$ and its functional equations and will connect Grassmannian complex and tangential complex to Goncharov complex.

4.3.1 Definition and functional equations of $T\mathcal{B}_3(F)$:

The $\mathbb{Z}$-module $T\mathcal{B}_3(F)$ over $F[\varepsilon]_2$ is defined as the group generated by:

$$\langle a; b] = [a + b\varepsilon] - [a] \in \mathbb{Z}\left[F[\varepsilon]_2\right], \quad a, b \in F, \quad a \neq 0, 1$$

and quotiented by the kernel of the following map

$$\partial_{\varepsilon,3} : \mathbb{Z}\left[F[\varepsilon]_2\right] \to T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F), \langle a; b] \mapsto \langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2$$

Now we can say that $\langle a; b]_3 \in T\mathcal{B}_3(F) \subset \mathbb{Z}[F[\varepsilon]_2]/\ker \partial_{\varepsilon,3}$

We have the following relations which are satisfied in $T\mathcal{B}_3(F)$.

1. The three-term relation.

$$\langle 1 - a; (1 - a)_\varepsilon]_3 - \langle a; a_\varepsilon]_3 + \left\langle 1 - \frac{1}{a}; \left(1 - \frac{1}{a}\right)_\varepsilon\right]_3 = 0 \in T\mathcal{B}_3(F)$$

We can verify that the three-term relation lies in the kernel of ∂_ε, where ∂_ε is induced by $\partial_{\varepsilon,3}$ defined above.

$$\partial_\varepsilon : T\mathcal{B}_3(F) \to T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F), \langle a; b]_3 \mapsto \langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2$$

$$\partial_\varepsilon\left(\langle 1 - a; (1 - a)_\varepsilon]_3 - \langle a; a_\varepsilon]_3 + \left\langle 1 - \frac{1}{a}; \left(1 - \frac{1}{a}\right)_\varepsilon\right]_3\right)$$

$$\begin{aligned}
=&\langle 1 - a; (1 - a)_\varepsilon]_2 \otimes (1 - a) + \frac{(1 - a)_\varepsilon}{1 - a} \otimes [1 - a]_2 - \langle a; a_\varepsilon]_2 \otimes a \\
&- \frac{a_\varepsilon}{a} \otimes [a]_2 + \left\langle 1 - \frac{1}{a}; \left(1 - \frac{1}{a}\right)_\varepsilon\right]_2 \otimes \left(1 - \frac{1}{a}\right) + \frac{\left(1 - \frac{1}{a}\right)_\varepsilon}{1 - \frac{1}{a}} \otimes \left[1 - \frac{1}{a}\right]_2
\end{aligned} \tag{4.14}$$

For simplification separate the parts in $T\mathcal{B}_2(F) \otimes F^\times$ and $F \otimes \mathcal{B}_2(F)$.

The part $T\mathcal{B}_2(F) \otimes F^\times$ is

$$
\begin{aligned}
=&\langle 1-a;(1-a)_\varepsilon]_2 \otimes (1-a) - \langle a;a_\varepsilon]_2 \otimes a \\
&+ \left\langle 1-\frac{1}{a};\left(1-\frac{1}{a}\right)_\varepsilon\right]_2 \otimes \left(1-\frac{1}{a}\right)
\end{aligned}
$$

Use the two-term relation in $T\mathcal{B}_2(F)$ for the first term then combine it with second term. Similarly use two term relation in third term, then we will have

$$\langle a;a_\varepsilon]_2 \otimes \left(1-\frac{1}{a}\right) + \left\langle \frac{1}{a};\left(\frac{1}{a}\right)_\varepsilon\right]_2 \otimes \left(1-\frac{1}{a}\right);$$

we know that

$$\left(\frac{1}{a}\right)_\varepsilon = -\frac{a_\varepsilon}{a^2}$$

so the above becomes

$$\left(\langle a;a_\varepsilon]_2 + \left\langle \frac{1}{a};-\frac{a_\varepsilon}{a^2}\right]_2\right) \otimes \left(1-\frac{1}{a}\right)$$

the left hand factor vanishes due to the inversion relation in $T\mathcal{B}_2(F)$.

The $F \otimes \mathcal{B}_2(F)$ part of (4.14) will be

$$
\begin{aligned}
=&\frac{(1-a)_\varepsilon}{1-a} \otimes [1-a]_2 - \frac{a_\varepsilon}{a} \otimes [a]_2 + \frac{\left(1-\frac{1}{a}\right)_\varepsilon}{1-\frac{1}{a}} \otimes \left[1-\frac{1}{a}\right]_2 \\
=&\frac{-a_\varepsilon}{1-a} \otimes [1-a]_2 - \frac{a_\varepsilon}{a} \otimes [a]_2 + \frac{\frac{a_\varepsilon}{a^2}}{-\frac{1-a}{a}} \otimes \left[1-\frac{1}{a}\right]_2
\end{aligned}
$$

use two term relation in first and last terms and

then inversion relation in last term in $\mathcal{B}_2(F)$

$$=\left(\frac{a_\varepsilon}{1-a} - \frac{a_\varepsilon}{a} - \frac{a_\varepsilon}{a(1-a)}\right) \otimes [a]_2$$

This gives zero since we are working modulo 2 torsion.

2. The inversion relation

$$\langle a;a_\varepsilon]_3 = \left\langle \frac{1}{a};\left(\frac{1}{a}\right)_\varepsilon\right]_3$$

3. The Cathelineau 22-term relation ([7])

This relation $J(a,b,c)$ for the indeterminates a, b, c can be written in this way:

$$J(a,b,c) = [[a,c]] - [[b,c]] + a\left[\left[\frac{b}{a},c\right]\right] + (1-a)\left[\left[\frac{1-b}{1-a},c\right]\right], \tag{4.15}$$

where

$$[[a,b]] = (b-a)\tau(a,b) + \frac{1-b}{1-a}\sigma(a) + \frac{1-a}{1-b}\sigma(b),$$

while $\tau(a,b)$ is defined via five term relation and $\star$-action. We take $\langle x_i; x_{i,\varepsilon}]_3$ with coefficient $\frac{1}{1-x_i}$ which is handled by $\star$-action.

$$\begin{aligned}\tau(a,b) = &\left\langle a; a_\varepsilon \cdot \frac{1}{1-a}\right]_3 - \left\langle b; b_\varepsilon \cdot \frac{1}{1-b}\right]_3 + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)_\varepsilon \cdot \frac{1}{a-b}\right]_3 \\ &- \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)_\varepsilon \cdot \frac{1}{b-a}\right]_3 - \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)_\varepsilon \cdot \frac{1}{b-a}\right]_3\end{aligned}$$

and

$$\sigma(a) = \langle a; a_\varepsilon \cdot a]_3 + \langle 1-a; (1-a)_\varepsilon \cdot (1-a)]_3.$$

Then we can calculate Cathelineau's 22-term expression by substituting all values in (4.15).

$$\begin{aligned}J(a,b,c) =& \langle a; a_\varepsilon c]_3 - \langle b; b_\varepsilon c]_3 + \langle c; c_\varepsilon(a-b+1)]_3 \\ &+ \langle 1-a; (1-a)_\varepsilon(1-c)]_3 - \langle 1-b; (1-b)_\varepsilon(1-c)]_3 + \langle 1-c; (1-c)_\varepsilon(b-a)]_3 \\ &- \left\langle \frac{c}{a}; \left(\frac{c}{a}\right)_\varepsilon\right]_3 + \left\langle \frac{c}{b}; \left(\frac{c}{b}\right)_\varepsilon\right]_3 + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)_\varepsilon c\right]_3 \\ &- \left\langle \frac{1-c}{1-a}; \left(\frac{1-c}{1-a}\right)_\varepsilon\right]_3 + \left\langle \frac{1-c}{1-b}; \left(\frac{1-c}{1-b}\right)_\varepsilon\right]_3 + \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)_\varepsilon c\right]_3 \\ &+ \left\langle \frac{a(1-c)}{c(1-a)}; \left(\frac{a(1-c)}{c(1-a)}\right)_\varepsilon\right]_3 - \left\langle \frac{ca}{b}; \left(\frac{ca}{b}\right)_\varepsilon\right]_3 - \left\langle \frac{b(1-c)}{c(1-b)}; \left(\frac{b(1-c)}{c(1-b)}\right)_\varepsilon\right]_3 \\ &+ \left\langle \frac{a-b}{a}; \left(\frac{a-b}{a}\right)_\varepsilon (1-c)\right]_3 + \left\langle \frac{b-a}{1-a}; \left(\frac{b-a}{1-a}\right)_\varepsilon (1-c)\right]_3 \\ &+ \left\langle \frac{c(1-a)}{1-b}; \left(\frac{c(1-a)}{1-b}\right)_\varepsilon\right]_3 \\ &- \left\langle \frac{(1-c)a}{a-b}; \left(\frac{(1-c)a}{a-b}\right)_\varepsilon\right]_3 - \left\langle \frac{(1-c)(1-a)}{b-a}; \left(\frac{(1-c)(1-a)}{b-a}\right)_\varepsilon\right]_3 \\ &+ \left\langle \frac{(1-c)b}{c(a-b)}; \left(\frac{(1-c)b}{c(a-b)}\right)_\varepsilon\right]_3 + \left\langle \frac{(1-c)(1-b)}{c(b-a)}; \left(\frac{(1-c)(1-b)}{c(b-a)}\right)_\varepsilon\right]_3 \end{aligned} \tag{4.16}$$

For the special condition $a_\varepsilon = a(1-a)$,$b_\varepsilon = b(1-b)$ and $c_\varepsilon = c(1-c)$, this 22-term expression becomes zero in $T\mathcal{B}_3(F)$. The proof uses the four-term relation in $T\mathcal{B}_2(F)$ which is described with the help of $\star$-action in $T\mathcal{B}_2(F)$. We calculate $\partial_\varepsilon\,(J(a,b,c))$ in steps. First apply ∂_ε on each term of (4.16).

First, second and third terms will give us

$$\langle a; a_\varepsilon c]_2 \otimes a + \frac{a_\varepsilon c}{a} \otimes [a]_2 - \langle b; b_\varepsilon c]_2 \otimes b - \frac{b_\varepsilon c}{b} \otimes [b]_2 + \langle c; c_\varepsilon(a-b+1)]_2 \otimes c + \frac{c_\varepsilon(a-b+1)}{c} \otimes [c]_2,$$

and next three terms are the following:

$$
\begin{aligned}
&+\langle 1-a;(1-a)_\varepsilon(1-c)]_2\otimes(1-a)+\frac{(1-a)_\varepsilon(1-c)}{1-a}\otimes[1-a]_2\\
&-\langle 1-b;(1-b)_\varepsilon(1-c)]_2\otimes(1-b)-\frac{(1-b)_\varepsilon(1-c)}{1-b}\otimes[1-b]_2\\
&+\langle 1-c;(1-c)_\varepsilon(b-a)]_2\otimes(1-c)+\frac{(1-c)_\varepsilon(b-a)}{1-c}\otimes[1-c]_2
\end{aligned}
$$

and similar for others. There are two parts $T\mathcal{B}_2(F)\otimes F^\times$ and $F\otimes\mathcal{B}_2(F)$ so, first compute $T\mathcal{B}_2(F)\otimes F^\times$. We first collect all the terms of type $\cdots\otimes a$

$$
\begin{aligned}
=&\Bigg(\langle a;a(1-a)c]_2+\left\langle\frac{c}{a};\frac{c}{a}\left(1-\frac{c}{a}\right)a\right]_2-\left\langle\frac{b}{a};\frac{b}{a}\left(1-\frac{b}{a}\right)a\right]_2\\
&+\left\langle\frac{a(1-c)}{c(1-a)};\frac{a(1-c)}{c(1-a)}\left(1-\frac{a(1-c)}{c(1-a)}\right)c(1-a)\right]_2-\left\langle\frac{ca}{b};\frac{ca}{b}\left(1-\frac{ca}{b}\right)b\right]_2\\
&-\left\langle\frac{a(1-c)}{a-b};\frac{a(1-c)}{a-b}\left(1-\frac{a(1-c)}{a-b}\right)(a-b)\right]_2\Bigg)\otimes a
\end{aligned}
$$

$$
\begin{aligned}
=&\Bigg(\langle c;c(1-c)a]_2-\left\langle\frac{b}{a};\frac{b}{a}\left(1-\frac{b}{a}\right)a\right]_2\\
&-\left\langle\frac{ca}{b};\frac{ca}{b}\left(1-\frac{ca}{b}\right)b\right]_2-\left\langle\frac{a(1-c)}{a-b};\frac{a(1-c)}{a-b}\left(1-\frac{a(1-c)}{a-b}\right)(a-b)\right]_2\\
&+\langle a;a(1-a)c]_2-\langle c;c(1-c)a]_2+\left\langle\frac{c}{a};\frac{c}{a}\left(1-\frac{c}{a}\right)a\right]_2\\
&+\left\langle\frac{a(1-c)}{c(1-a)};\frac{a(1-c)}{c(1-a)}\left(1-\frac{a(1-c)}{c(1-a)}\right)c(1-a)\right]_2\Bigg)\otimes a
\end{aligned}
$$

$$
\begin{aligned}
=\Bigg(-a\star\Bigg\{&\left\langle\frac{b}{a};\frac{b}{a}\left(1-\frac{b}{a}\right)\right]_2-\langle c;c(1-c)]_2+\left\langle\frac{c}{\frac{b}{a}};\frac{c}{\frac{b}{a}}\left(1-\frac{c}{\frac{b}{a}}\right)\frac{b}{a}\right]_2\\
&-\left\langle\frac{1-c}{1-\frac{b}{a}};\frac{1-c}{1-\frac{b}{a}}\left(1-\frac{1-c}{1-\frac{b}{a}}\right)\left(1-\frac{b}{a}\right)\right]_2\Bigg\}\\
&-\left\langle\frac{1}{a};\frac{1}{a}\left(1-\frac{1}{a}\right)ac\right]_2+\left\langle\frac{1}{c};\frac{1}{c}\left(1-\frac{1}{c}\right)ac\right]_2\\
&-\left\langle\frac{a}{c};\frac{a}{c}\left(1-\frac{a}{c}\right)c\right]_2+\left\langle\frac{1-\frac{1}{c}}{1-\frac{1}{a}};\frac{1-\frac{1}{c}}{1-\frac{1}{a}}\left(1-\frac{1-\frac{1}{c}}{1-\frac{1}{a}}\right)c(1-a)\right]_2\Bigg)\otimes a
\end{aligned}
$$

$$
\begin{aligned}
=\Bigg(-a\star\{0\}-ac\star\Bigg\{&\left\langle\frac{1}{a};\frac{1}{a}\left(1-\frac{1}{a}\right)\right]_2-\left\langle\frac{1}{c};\frac{1}{c}\left(1-\frac{1}{c}\right)\right]_2+\left\langle\frac{a}{c};\frac{a}{c}\left(1-\frac{a}{c}\right)\frac{1}{a}\right]_2\\
&+\left\langle\frac{1-\frac{1}{c}}{1-\frac{1}{a}};\frac{1-\frac{1}{c}}{1-\frac{1}{a}}\left(1-\frac{1-\frac{1}{c}}{1-\frac{1}{a}}\right)\left(1-\frac{1}{a}\right)\right]_2\Bigg\}\Bigg)\otimes a
\end{aligned}
$$

$$
=0
$$

Similarly, it can be proved for all of them, i.e., $\cdots \otimes b$, $\cdots \otimes c$, $\cdots \otimes (1-a)$ and so on. The other part $F \otimes \mathcal{B}_2(F)$ directly gives two-terms and five-terms relations. so it is clear that $\partial_\varepsilon (J(a,b,c)) = 0$.

One can write the following complex for $T\mathcal{B}_3(F)$.

$$T\mathcal{B}_3(F) \xrightarrow{\partial_\varepsilon} \substack{T\mathcal{B}_2(F)\otimes F^\times \\ \oplus \\ F\otimes \mathcal{B}_2(F)} \xrightarrow{\partial_\varepsilon} \left(F \otimes \bigwedge{}^2 F^\times\right) \oplus \left(\bigwedge{}^3 F\right)$$

4.3.2 Mapping Grassmannian complexes to Tangential complexes in weight 3:

In this subsection, we will try to find morphisms between this complex and the Grassmannian complex and after a long computation we see that each square of the following diagram is commutative. Consider the following diagram

$$\begin{tikzcd}
C_6(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "\tau^3_{2,\varepsilon}"] & C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "\tau^3_{1,\varepsilon}"] & C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[d, "\tau^3_{0,\varepsilon}"] \\
T\mathcal{B}_3(F) \arrow[r, "\partial_\varepsilon"] & (T\mathcal{B}_2(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F)) \arrow[r, "\partial_\varepsilon"] & \left(F\otimes \wedge^2 F^\times\right)\oplus\left(\wedge^3 F\right)
\end{tikzcd} \tag{4.3a}$$

Here we define

$$\mathbf{r}(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*) = \frac{\Delta(l_0^*, l_1^*, l_4^*)\Delta(l_0^*, l_2^*, l_3^*)}{\Delta(l_0^*, l_1^*, l_3^*)\Delta(l_0^*, l_2^*, l_4^*)}$$

The projected cross-ratio is defined here

$$\mathbf{r}(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*) = r(l_0|l_1, l_2, l_3, l_4) + r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*)\varepsilon$$

where

$$r(l_0|l_1, l_2, l_3, l_4) = \frac{\Delta(l_0, l_1, l_4)\Delta(l_0, l_2, l_3)}{\Delta(l_0, l_1, l_3)\Delta(l_0, l_2, l_4)}$$

$$r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*) = \frac{u}{\Delta(l_0, l_1, l_3)^2\Delta(l_0, l_2, l_4)^2}$$

$$\begin{aligned}
u = &-\Delta(l_0, l_1, l_4)\Delta(l_0, l_2, l_3)\{\Delta(l_0, l_1, l_3)\Delta(l_0^*, l_2^*, l_4^*)_\varepsilon + \Delta(l_0, l_2, l_4)\Delta(l_0^*, l_1^*, l_3^*)_\varepsilon\} \\
&+ \Delta(l_0, l_1, l_3)\Delta(l_0, l_2, l_4)\{\Delta(l_0, l_1, l_4)\Delta(l_0^*, l_2^*, l_3^*)_\varepsilon + \Delta(l_0, l_2, l_3)\Delta(l_0^*, l_1^*, l_4^*)_\varepsilon\}
\end{aligned}$$

where the morphisms between the two complexes are defined as follows:

$$\begin{aligned}
&\tau^3_{0,\varepsilon}(l_0^*,\dots,l_3^*)\\
&=\sum_{i=0}^{3}(-1)^i\Bigg(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,l_3^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)}\otimes\frac{\Delta(l_0,\dots,\hat{l}_{i+1},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\\
&\qquad\wedge\frac{\Delta(l_0,\dots,\hat{l}_{i+3},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}+\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\frac{\Delta(l_0^*,\dots,\hat{l}_j^*,\dots,l_3^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_j,\dots,l_3)}\Bigg),\quad i \mod 4,
\end{aligned}$$

$$\begin{aligned}
&\tau^3_{1,\varepsilon}(l_0^*,\dots,l_4^*)\\
&=-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(\Big\langle r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4);r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l^*}_i,\dots,l_4^*)\Big]_2\otimes\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)\\
&\qquad+\sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_j^*,\dots,l_4^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\Bigg)
\end{aligned}$$

and

$$\tau^3_{2\varepsilon}(l_0^*,\dots,l_5^*)=\frac{2}{45}\mathrm{Alt}_6\,\langle r_3(l_0,\dots,l_5);r_{3,\varepsilon}(l_0^*,\dots,l_5^*)]_3$$

where

$$r_3(l_0,\dots,l_5)=\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$$

and

$$\begin{aligned}
&r_{3,\varepsilon}(l_0^*,\dots,l_5^*)\\
&=\frac{\{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}-\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}
\end{aligned}\tag{4.17}$$

the map ∂_ε is defined as

$$\begin{aligned}
&\partial_\varepsilon\left(\langle a;b]_2\otimes c+x\otimes[y]_2\right)\\
&=\left(-\frac{b}{1-a}\otimes a\wedge c-\frac{b}{a}\otimes(1-a)\wedge c+x\otimes(1-y)\wedge y\right)+\left(\frac{b}{1-a}\wedge\frac{b}{a}\wedge x\right)
\end{aligned}$$

and

$$\partial_\varepsilon(\langle a;b]_3)=\langle a;b]_2\otimes a+\frac{b}{a}\otimes[a]_2$$

Theorem 4.3.1. *The right square of the diagram (4.3a), i.e.*

$$\begin{tikzcd} C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "\tau^3_{1,\varepsilon}"] & C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[d, "\tau^3_{0,\varepsilon}"] \\ (T\mathcal{B}_2(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F)) \arrow[r, "\partial_\varepsilon"] & \left(F\otimes\bigwedge^2 F^\times\right)\oplus\left(\bigwedge^3 F\right) \end{tikzcd}$$

is commutative, i.e. $\tau^3_{0,\varepsilon}\circ d=\partial_\varepsilon\circ\tau^3_{1,\varepsilon}$

Proof: First we divide the map $\tau^3_{0,\varepsilon}=\tau^{(1)}+\tau^{(2)}$ then calculate $\tau^{(1)}\circ d(l_0^*,\dots,l_4^*)$

$$\tau^{(1)}\circ d(l_0^*,\dots,l_4^*)=\tau^3_{0,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i(l_0^*,\dots,\hat{l}_i^*,\dots,l_4^*)\right)$$

$$=\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\Big(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,l_3^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,l_3)}\otimes\frac{\Delta(l_0,\dots,\hat{l}_{i+1},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\wedge\frac{\Delta(l_0,\dots,\hat{l}_{i+3},\dots,l_3)}{\Delta(l_0,\dots,\hat{l}_{i+2},\dots,l_3)}\Big),\quad i\mod 4\right)\tag{4.18}$$

We expand the inner sum first that contains 12 terms and passing alternation to the sum, gives us 60 different terms overall. We collect terms involving same $\frac{\Delta(l_i^*,l_j^*,l_k^*)}{\Delta(l_i,l_j,l_k)}\otimes\cdots$ together for calculation purpose. On the other hand second part of the map is the following:

$$\begin{aligned}&\tau^{(1)}\circ d(l_0^*,\dots,l_4^*)\\ =&\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\frac{\Delta(l_0^*,\dots,\hat{l}_j^*,\dots,l_3^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_j,\dots,l_3)}\right)\end{aligned}\tag{4.19}$$

The other side of the proof requires very long computations. For the calculation of $\partial_\varepsilon\circ\tau^3_{1,\varepsilon}$ we will use short hand $(l_i^*l_j^*l_k^*)_\varepsilon$ for $\Delta(l_i^*,l_j^*,l_k^*)_\varepsilon$ and $(l_il_jl_k)$ for $\Delta((l_i,l_j,l_k)$. First we write $\partial_\varepsilon\circ\tau^3_{1,\varepsilon}(l_0^*,\dots,l_4^*)$ by using the definitions above.

$$\begin{aligned}=\partial_\varepsilon\Bigg(-\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Big(\Big\langle r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4);r_\varepsilon(l_i^*|l_0^*,\dots,\hat{l}^*{}_i,\dots,l_4^*)\Big]_2\otimes\prod_{i\neq j}\Delta(\hat{l}_i,\hat{l}_j)\\ +\sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(l_0^*,\dots,\hat{l}_i^*,\dots,\hat{l}_j^*,\dots,l_4^*)_\varepsilon}{\Delta(l_0,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_4)}\right)\otimes\Big[r(l_i|l_0,\dots,\hat{l}_i,\dots,l_4)\Big]_2\Big)\Bigg)\end{aligned}$$

then we divide $\partial_\varepsilon = \partial^{(1)} + \partial^{(2)}$. The first part $\partial^{(1)} \circ \tau^3_{1,\varepsilon}(l_0^*, \ldots, l_4^*)$ is

$$
\begin{aligned}
= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(& -\frac{r_\varepsilon(l_i^*|l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)}{1 - r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)} \otimes r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4) \wedge \prod_{i\neq j}(\hat{l}_i, \hat{l}_j) \\
& -\frac{r_\varepsilon(l_i^*|l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)}{r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)} \otimes (1 - r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)) \wedge \prod_{i\neq j}(\hat{l}_i, \hat{l}_j) \\
& + \sum_{\substack{j=0 \\ j\neq i}}^{4}\left(\frac{\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, \hat{l}_j^*, \ldots, l_4^*)_\varepsilon}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_4)}\right) \otimes (1 - r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)) \\
& \wedge r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)\Bigg)
\end{aligned} \tag{4.20}
$$

The second part $\partial^{(2)} \circ \tau^3_{1,\varepsilon}(l_0^*, \ldots, l_4^*)$ is

$$
\begin{aligned}
= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(& -\frac{r_\varepsilon(l_i^*|l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)}{r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)} \wedge \frac{r_\varepsilon(l_i^*|l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)}{1 - r(l_i|l_0, \ldots, \hat{l}_i, \ldots, l_4)} \\
& \wedge \sum_{\substack{j=0 \\ j\neq i}}^{4}\left(\frac{\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, \hat{l}_j^*, \ldots, l_4^*)_\varepsilon}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_4)}\right)\Bigg)
\end{aligned} \tag{4.21}
$$

then we calculate $\frac{b_\varepsilon}{a}$ and $\frac{b_\varepsilon}{1-a}$. i.e. all the values of the form $\frac{r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*)}{r(l_0|l_1, l_2, l_3, l_4)}$ and $\frac{r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*)}{1-r(l_0|l_1, l_2, l_3, l_4)}$. By using formula (4.6) we have

$$\frac{r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*)}{r(l_0|l_1, l_2, l_3, l_4)} = \frac{(l_0^*l_1^*l_4^*)_\varepsilon}{(l_0l_1l_4)} + \frac{(l_0^*l_2^*l_3^*)_\varepsilon}{(l_0l_2l_3)} - \frac{(l_0^*l_2^*l_4^*)_\varepsilon}{(l_0l_2l_4)} - \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)}$$

Similarly we can find this ratio for each value of $i = 0, \ldots, 4$. Now use formula (4.6) as well as identities (4.1) and 4.3, we have

$$\frac{r_\varepsilon(l_0^*|l_1^*, l_2^*, l_3^*, l_4^*)}{1 - r(l_0|l_1, l_2, l_3, l_4)} = \frac{(l_0^*l_2^*l_4^*)_\varepsilon}{(l_0l_2l_4)} + \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} - \frac{(l_0^*l_3^*l_4^*)_\varepsilon}{(l_0l_3l_4)} - \frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)}$$

After calculating all these values. Expand the sums (4.20) and (4.21) and put all values what we have calculated above. Let us talk about (4.20). In this sum we have huge amount of terms, so we group them in a suitable way. First collect all the terms involving $\frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \otimes \cdots$, we find that there are 6 different terms with coefficient -3 involving $\frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \otimes \cdots$

$$
\begin{aligned}
-3\frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \otimes \Big(& (l_0l_1l_3) \wedge (l_1l_2l_3) + (l_0l_2l_4) \wedge (l_1l_2l_3) + (l_0l_1l_4) \wedge (l_0l_2l_4) \\
& -(l_0l_1l_3) \wedge (l_0l_2l_3) - (l_0l_1l_4) \wedge (l_1l_2l_4) - (l_0l_2l_3) \wedge (l_1l_2l_3)\Big)
\end{aligned}
$$

There are exactly 10 possible terms of $\frac{(l_i^* l_j^* l_k^*)_\varepsilon}{(l_i l_j l_k)}$. Compute all of them individually. We will see that each will have the coefficient -3 that will be cancelled by $-\frac{1}{3}$ in (4.20) and then combine 60 different terms with 6 in a group of same $\frac{(l_i^* l_j^* l_k^*)_\varepsilon}{(l_i l_j l_k)}$, write in the sum form then we will note that it will be the same as (4.18).

Computation for the second part is relatively easy and direct. We need to put all values of the form $\frac{r_\varepsilon(l_0^*|l_1^*,l_2^*,l_3^*,l_4^*)}{r(l_0|l_1,l_2,l_3,l_4)}$ and $\frac{r_\varepsilon(l_0^*|l_1^*,l_2^*,l_3^*,l_4^*)}{1-r(l_0|l_1,l_2,l_3,l_4)}$ in (4.21), expand the sums, use $a \wedge a = 0$ modulo 2 torsion. Here we will have simplified result which can be recombined in the sum notation which will be same as (4.19). □

Example 4.3.2.

In this example we will discuss the part of the commutative diagrams which we have discussed in previous chapter i.e. $F \otimes \bigwedge^2 F^\times$, Goncharov has discussed in [9] i.e. $\bigwedge^3 F^\times$ and the last part of diagram (4.3a), i.e. $\left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right)$. We will also try to find some relations with the continuation of example 4.2.6. Let us use $D \in Der_{\mathbb{Z}}(F[\varepsilon]_2, F)$ defined in §4.2.1, here in this case, we have

$$D\left(\Delta(l_i^*, l_j^*, l_k^*)\right) = D\left(\Delta(l_i, l_j, l_k) + \Delta(l_i^*, l_j^*, l_k^*)_\varepsilon \varepsilon\right) = \Delta(l_i^*, l_j^*, l_k^*)_\varepsilon$$

For the comparison, it is enough to see the images of f_0^3 (defined in [9]), τ_0^3 and $\tau_{0,\varepsilon}^3$ in their respective qualifying complex. First find $\tau_{0,\varepsilon}^3(l_0^*, \ldots, l_3^*)$, which shall be the sum of

$$\widetilde{\mathrm{Alt}}_{(01234)}\Bigg(\sum_{i=0}^{3}(-1)^i\Bigg(\frac{D\left(\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_3^*, l_4^*)\right)}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3, l_4)} \otimes \frac{\Delta(l_0, \ldots, \hat{l}_{i+1}, \ldots, l_3, l_4)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3, l_4)} \wedge \frac{\Delta(l_0, \ldots, \hat{l}_{i+3}, \ldots, l_3, l_4)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3, l_4)}\Bigg) \quad i \mod 4\Bigg) \in F \otimes \bigwedge{}^2 F^\times$$

and

$$\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0\\ j\neq i}}^{3} \frac{D\left(\Delta(l_0^*, \ldots, \hat{l}_j^*, \ldots, l_3^*, l_4^*)\right)}{\Delta(l_0, \ldots, \hat{l}_j, \ldots, l_3, l_4)}\right) \in \bigwedge{}^3 F$$

As we did in example 4.2.6. For $\varepsilon \to 0$ we have $F[\varepsilon]_2 \to F$ and $(l_i^*, l_j^*, l_k^*) \to (l_i, l_j, l_k)$. In this situation one can replace $D_\varepsilon \in Der_{\mathbb{Z}}(F[\varepsilon]_2, F)$ by $D \in Der_{\mathbb{Z}}F$. Then above becomes the sum of

$$\widetilde{\mathrm{Alt}}_{(01234)}\Bigg(\sum_{i=0}^{3}(-1)^i\Big(\frac{D\Big(\Delta(l_0,\ldots,\hat{l}_i,\ldots,l_3,l_4)\Big)}{\Delta(l_0,\ldots,\hat{l}_i,\ldots,l_3,l_4)}\otimes\frac{\Delta(l_0,\ldots,\hat{l}_{i+1},\ldots,l_3,l_4)}{\Delta(l_0,\ldots,\hat{l}_{i+2},\ldots,l_3,l_4)}$$
$$\wedge\frac{\Delta(l_0,\ldots,\hat{l}_{i+3},\ldots,l_3,l_4)}{\Delta(l_0,\ldots,\hat{l}_{i+2},\ldots,l_3,l_4)}\Big)\quad i \mod 4\Bigg)\in F\otimes\bigwedge{}^2F^\times$$

and

$$\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\frac{D\Big(\Delta(l_0,\ldots,\hat{l}_j,\ldots,l_3,l_4)\Big)}{\Delta(l_0,\ldots,\hat{l}_j,\ldots,l_3,l_4)}\quad i\mod 4\right)\in\bigwedge{}^3F$$

First expression of the above is just $\tau_0^3\circ d(l_0,\ldots,l_4)$ and second expression of the above is $D\log$ of $f_0^3\circ d(l_0,\ldots,l_4)$. In other words one can make a remark that in diagram (3.2a), $F\otimes\bigwedge^2F^\times$ is a special case of $\tau_{0,\varepsilon}^3\circ d(l_0,\ldots,l_4)=\partial_\varepsilon\circ\tau_{1,\varepsilon}^3(l_0,\ldots,l_4)$ in diagram (4.3a) when $\varepsilon\to 0$ and under these special condition $D\log$ of $\bigwedge^3F^\times$ in diagram (6.10) in [9] is $\bigwedge^3F$ in diagram (4.3a).

Theorem 4.3.3. *The left square of the diagram (4.3a), i.e.*

$$\begin{tikzcd}
C_6(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[r, "d"] \arrow[d, "\tau^3_{2,\varepsilon}"] & C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \arrow[d, "\tau^3_{1,\varepsilon}"] \\
T\mathcal{B}_3(F) \arrow[r, "\partial_\varepsilon"] & (T\mathcal{B}_2(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F))
\end{tikzcd}$$

is commutative i.e. $\tau^3_{2,\varepsilon}\circ\partial_\varepsilon=d\circ\tau^3_{1,\varepsilon}$

Proof: The map $\tau^3_{2,\varepsilon}$ gives 720 terms and due to symmetry (cyclic and inverse) we find 120 different ones (up to inverse). We will use the same technique here which we have used in the proof of theorem 4.3.1. By definition, we have

$$\tau^3_{2,\varepsilon}(l_0^*,\ldots,l_5^*)=\frac{2}{45}\mathrm{Alt}_6\left\langle r_3(l_0,\ldots,l_5);r_{3,\varepsilon}(l_0^*,\ldots,l_5^*)\right]_3$$

For convenience and similar to our previous conventions, we will abbreviate our notation by dropping Δ and commas.

$$\begin{aligned}&\partial_\varepsilon\circ\tau^3_{2\varepsilon}(l_0^*\ldots l_5^*)\\ =&\frac{2}{45}\mathrm{Alt}_6\left\{\left\langle r_3(l_0\ldots l_5);r_{3,\varepsilon}(l_0^*\ldots l_5^*)\right]_2\otimes r_3(l_0\ldots l_5)+\frac{r_{3,\varepsilon}(l_0^*\ldots l_5^*)}{r_3(l_0\ldots l_5)}\otimes[r_3(l_0\ldots l_5)]_2\right\}\end{aligned}\tag{4.22}$$

We need to compute the value of $\frac{r_{3,\varepsilon}(l_0^*\ldots l_5^*)}{r_3(l_0\ldots l_5)}$ which is

$$= \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} + \frac{(l_1^*l_2^*l_4^*)_\varepsilon}{(l_1l_2l_4)} + \frac{(l_2^*l_0^*l_5^*)_\varepsilon}{(l_2l_0l_5)} - \frac{(l_0^*l_1^*l_4^*)_\varepsilon}{(l_0l_1l_4)} - \frac{(l_1^*l_2^*l_5^*)_\varepsilon}{(l_1l_2l_5)} - \frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)}$$

(4.22) can also be written as

$$\begin{aligned}=&\frac{2}{45}\mathrm{Alt}_6\Big\{ \langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\\ &+\left(\frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} + \frac{(l_1^*l_2^*l_4^*)_\varepsilon}{(l_1l_2l_4)} + \frac{(l_2^*l_0^*l_5^*)_\varepsilon}{(l_2l_0l_5)} - \frac{(l_0^*l_1^*l_4^*)_\varepsilon}{(l_0l_1l_4)} - \frac{(l_1^*l_2^*l_5^*)_\varepsilon}{(l_1l_2l_5)} - \frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)}\right) \otimes [r_3(l_0\ldots l_5)]_2\Big\}\end{aligned}$$

We will consider here only first part of the above relation.

$$\frac{2}{45}\mathrm{Alt}_6\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$$

Further,

$$\begin{aligned}=&\mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes (l_0l_1l_3)\right\} + \mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}n(l_0^*\ldots l_5^*)]_2 \otimes (l_1l_2l_4)\right\}\\ +&\mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes (l_2l_0l_5)\right\} - \mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes (l_0l_1l_4)\right\}\\ -&\mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes (l_1l_2l_5)\right\} - \mathrm{Alt}_6\left\{\langle r_3(l_0\ldots l_5); r_{3,\varepsilon}(l_0^*\ldots l_5^*)]_2 \otimes (l_2l_0l_3)\right\}\end{aligned} \tag{4.23}$$

We use the even cycle $(l_0l_1l_2)(l_3l_4l_5)$ (or $(l_0^*l_1^*l_2^*)(l_3^*l_4^*l_5^*)$) to obtain

$$\begin{aligned}&\mathrm{Alt}_6\left\{\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 \otimes (l_0l_1l_3)\right\}\\ =&\mathrm{Alt}_6\left\{\langle r_3(l_1l_2l_0l_4l_5l_3); r_{3,\varepsilon}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*)]_2 \otimes (l_1l_2l_4)\right\}\end{aligned}$$

We can also use here the symmetry

$$\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 = \langle r_3(l_1l_2l_0l_4l_5l_3); r_{3,\varepsilon}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*)]_2$$

since

$$r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = r_{3,\varepsilon}(l_1^*l_2^*l_0^*l_4^*l_5^*l_3^*) \quad \text{precisely both have the same factors}$$

and similar for the others as well so that (4.23) can be written as

$$\begin{aligned}=\frac{2}{15}\mathrm{Alt}_6\Big\{&\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 \otimes (l_0l_1l_3)\\ &-\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 \otimes (l_0l_1l_4)\Big\}\end{aligned}$$

If we apply the odd permutation (l_3l_4) (or $(l_3^*l_4^*)$), then we have

$$= \frac{2}{15}\cdot 2\mathrm{Alt}_6\left\{\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 \otimes (l_0l_1l_3)\right\}$$

Again apply an odd permutation (l_0l_3) (or $(l_0^*l_3^*)$)

$$\begin{aligned}= \frac{2}{15}\mathrm{Alt}_6\Big\{&\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2 \otimes (l_0l_1l_3)\\ &- \langle r_3(l_3l_1l_2l_0l_4l_5); r_{3,\varepsilon}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*)]_2 \otimes (l_3l_1l_0)\Big\}\end{aligned}$$

but up to 2-torsion, which we ignore here, we have $(l_0l_1l_3) = (l_3l_1l_0)$ and then the above will become

$$\begin{aligned}= \frac{2}{15}\mathrm{Alt}_6\Big\{\Big(&\langle r_3(l_0l_1l_2l_3l_4l_5); r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)]_2\\ &- \langle r_3(l_3l_1l_2l_0l_4l_5); r_{3,\varepsilon}(l_3^*l_1^*l_2^*l_0^*l_4^*l_5^*)]_2\Big) \otimes (l_0l_1l_3)\Big\}\end{aligned} \tag{4.24}$$

Recall from (3.2) that the triple-ratio $r_3(l_0l_1l_2l_3l_4l_5) = \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$ can be written as the ratio of two projected cross-ratios.

We will see here that $r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)$ can also be converted into the ratio of two first order cross-ratios.

Let a and b be two projected cross-ratios whose ratio is the triple-ratio $r_3(l_0l_1l_2l_3l_4l_5) = \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}$ then $r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)$ will be written as $\left(\frac{a^*}{b^*}\right)_\varepsilon$. Since we can also write as

$$\mathbf{r}_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = r_3(l_0l_1l_2l_3l_4l_5) + r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\varepsilon$$

or

$$\mathbf{r}_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = r_3(l_0l_1l_2l_3l_4l_5) + \left(\mathbf{r}_3(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)\right)_\varepsilon \varepsilon$$

we get

$$r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = \left(\frac{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)}{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)}\right)_\varepsilon$$

Now it is clear that $r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*)$ can also be written as the ratio or product of two projected cross-ratios. There are exactly three ways to write it (projected by (l_0^* and l_1^*), (l_1^* and l_2^*) and (l_0^* and l_2^*)) but we will use here l_1^* and l_2^*. The last expression can be written as

$$r_{3,\varepsilon}(l_0^*l_1^*l_2^*l_3^*l_4^*l_5^*) = \left(\frac{\mathbf{r}(l_2^*|l_1^*l_0^*l_5^*l_3^*)}{\mathbf{r}(l_1^*|l_0^*l_2^*l_3^*l_4^*)}\right)_\varepsilon$$

and 4.24 can be written as

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left(\left\langle \frac{r(l_2|l_1l_0l_5l_3)}{r(l_1|l_0l_2l_3l_4)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_0^*l_5^*l_3^*)}{\mathbf{r}(l_1^*|l_0^*l_2^*l_3^*l_4^*)}\right)_{\varepsilon}\right]_2 - \left\langle \frac{r(l_2|l_1l_3l_5l_0)}{r(l_1|l_3l_2l_0l_4)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_3^*l_5^*l_0^*)}{\mathbf{r}(l_1^*|l_3^*l_2^*l_0^*l_4^*)}\right)_{\varepsilon}\right]_2\right)\otimes(l_0l_1l_3)\right\}$$

Applying five-term relations in $T\mathcal{B}_2(F)$ which are analogous to the one in (2.10).

$$=\frac{2}{15}\mathrm{Alt}_6\{\Big(\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2 - \langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*)]_2$$
$$-\left\langle \frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{\mathbf{r}(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon}\right]_2\Big)\otimes(l_0l_1l_3)\} \tag{4.25}$$

For each individual determinant, e.g. $(l_0l_1l_3)$ will have three terms. First consider the third term of (4.25)

$$\frac{2}{15}\mathrm{Alt}_6\left\{\left\langle \frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{\mathbf{r}(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon}\right]_2\otimes(l_0l_1l_3)\right\}$$
$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left(\left\langle \frac{\mathbf{r}(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{\mathbf{r}(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon}\right]_2\otimes(l_0l_1l_3)\right)\right\}$$

We need a subgroup in S_6 which fixes $(l_0l_1l_3)$ as a determinant i.e. $(l_0l_1l_3)\sim(l_3l_1l_0)\sim(l_3l_0l_1)\cdots$

Here in this case S_3 permuting $\{l_0,l_1,l_3\}$ and another one permuting $\{l_2,l_4,l_5\}$ i.e. $S_3\times S_3$. Now consider

$$\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\left\langle \frac{r(l_2|l_1l_5l_3l_0)}{r(l_1|l_0l_3l_4l_2)};\left(\frac{\mathbf{r}(l_2^*|l_1^*l_5^*l_3^*l_0^*)}{\mathbf{r}(l_1^*|l_0^*l_3^*l_4^*l_2^*)}\right)_{\varepsilon}\right]_2\otimes(l_0l_1l_3)\right\}$$
$$=\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\left\langle \frac{(l_2l_5l_3)(l_1l_0l_4)}{(l_2l_5l_0)(l_1l_3l_4)};\left(\frac{(l_2^*l_5^*l_3^*)(l_1^*l_0^*l_4^*)}{(l_2^*l_5^*l_0^*)(l_1^*l_3^*l_4^*)}\right)_{\varepsilon}\right]_2\otimes(l_0l_1l_3)\right\}$$

By using odd permutation (l_2l_5) the above becomes zero.

then (4.25) becomes

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\Big(\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2 - \langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*)]_2\Big)\otimes(l_0l_1l_3)\right\} \tag{4.26}$$

Consider the first term now,

$$\frac{2}{15}\mathrm{Alt}_6\left\{\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2\otimes(l_0l_1l_3)\right\}$$
$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(l_0l_1l_3)(l_2l_4l_5)}\left\{\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2\otimes(l_0l_1l_3)\right\}\right\}$$

The permutation $(l_0l_2l_3)$ does not have any role because the ratio is projected by 2. So, it will be reduced to S_3.

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{6}\mathrm{Alt}_{(l_2l_4l_5)}\left\{\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2\otimes(l_0l_1l_3)\right\}\right\}$$

Write all possible inner alternation, then

$$= \frac{1}{45}\mathrm{Alt}_6\Big\{\Big(\langle r(l_4|l_1l_0l_2l_3); r_\varepsilon(l_4^*|l_1^*l_0^*l_2^*l_3^*)]_2 - \langle r(l_2|l_1l_0l_4l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_4^*l_3^*)]_2$$
$$+\langle r(l_5|l_1l_0l_4l_3); r_\varepsilon(l_5^*|l_1^*l_0^*l_4^*l_3^*)]_2 - \langle r(l_4|l_1l_0l_5l_3); r_\varepsilon(l_4^*|l_1^*l_0^*l_5^*l_3^*)]_2$$
$$+\langle r(l_2|l_1l_0l_5l_3); r_\varepsilon(l_2^*|l_1^*l_0^*l_5^*l_3^*)]_2 - \langle r(l_5|l_1l_0l_2l_3); r_\varepsilon(l_5^*|l_1^*l_0^*l_2^*l_3^*)]_2\Big)\otimes(l_0l_1l_3)\Big\}$$

Now we can use projected five-term relation for $T\mathcal{B}_2(F)$ here,

$$= \frac{1}{45}\mathrm{Alt}_6\Big\{\Big(\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2 - \langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*)]_2$$
$$-\langle r(l_3|l_0l_1l_2l_4); r_\varepsilon(l_3^*|l_0^*l_1^*l_2^*l_4^*)]_2 + \langle r(l_0|l_1l_4l_3l_5); r_\varepsilon(l_0^*|l_1^*l_4^*l_3^*l_5^*)]_2$$
$$-\langle r(l_1|l_0l_4l_3l_5); r_\varepsilon(l_1^*|l_0^*l_4^*l_3^*l_5^*)]_2 + \langle r(l_3|l_0l_1l_4l_5); r_\varepsilon(l_3^*|l_0^*l_1^*l_4^*l_5^*)]_2$$
$$+\langle r(l_0|l_1l_5l_3l_2); r_\varepsilon(l_0^*|l_1^*l_5^*l_3^*l_2^*)]_2 - \langle r(l_1|l_0l_5l_3l_2); r_\varepsilon(l_1^*|l_0^*l_5^*l_3^*l_2^*)]_2$$
$$+\langle r(l_3|l_0l_1l_5l_2); r_\varepsilon(l_3^*|l_0^*l_1^*l_5^*l_2^*)]_2\Big)\otimes(l_0l_1l_3)\Big\}$$

Use the cycle $(l_0l_1l_3)(l_2l_4l_5)$ then we get

$$= \frac{1}{45}\cdot 9\mathrm{Alt}_6\left\{\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2\otimes(l_0l_1l_3)\right\} \tag{4.27}$$

The second term of (4.26) can be written as

$$\frac{1}{45}\cdot -6\mathrm{Alt}_6\left\{\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*)]_2\otimes(l_0l_1l_3)\right\}$$

(4.27) can be combined with the above so we get

$$= \frac{1}{45}\mathrm{Alt}_6\left\{\left(9\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2 - 6\langle r(l_1|l_0l_2l_3l_4); r_\varepsilon(l_1^*|l_0^*l_2^*l_3^*l_4^*)]_2\right)\otimes(l_0l_1l_3)\right\} \tag{4.28}$$

Use the permutation $(l_0l_1l_3)(l_2l_4l_5)$ to get

$$= \frac{1}{3}\mathrm{Alt}_6\left\{\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2\otimes(l_0l_1l_3)\right\}$$

Since $\mathcal{B}_2(F)$ satisfies five-term relation then we can write the following.

$$= \frac{1}{3}\mathrm{Alt}_6\left\{\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2\otimes(l_0l_1l_3) + \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)}\otimes[r(l_0|l_1l_2l_3l_4)]_2\right\} \tag{4.29}$$

Now go to the other side. Map $\tau_{1,\varepsilon}^3$ can also be written in the alternation sum form

$$\tau_{1,\varepsilon}^3(l_0^*\dots l_4^*) = \frac{1}{3}\mathrm{Alt}\{\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2\otimes(l_0l_1l_2)$$
$$+\frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)}\otimes[r(l_0|l_1l_2l_3l_4)]_2\}$$

Compute $\tau^3_{1,\varepsilon} \circ d(l_0^* \ldots l_5^*)$ and apply cycle $(l_0l_1l_2l_3l_4l_5)$ for d and then expand Alt_5 from the definition of $\tau^3_{1,\varepsilon}$ so we get

$$\tau^3_{1\varepsilon} \circ d(l_0^* \ldots l_5^*) = \frac{1}{3}\mathrm{Alt}_6\{ \left\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*)\right]_2 \otimes (l_0l_1l_2) + \frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)} \otimes [r(l_0|l_1l_2l_3l_4)]_2\}$$

Use the odd permutation (l_2l_3), then

$$= -\frac{1}{3}\mathrm{Alt}_6\{ \left\langle r(l_0|l_1l_3l_2l_4); r_\varepsilon(l_0^*|l_1^*l_3^*l_2^*l_4^*)\right]_2 \otimes (l_0l_1l_3) + \frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)} \otimes [r(l_0|l_1l_3l_2l_4)]_2\}$$

Finally use two-term relation in $T\mathcal{B}_2(F)$ and $\mathcal{B}_2(F)$ to get the correct sign. The final answer will be the same as (4.29). □

If we combine Theorem 4.3.1 and 4.3.3, then we see that the diagram (B.2a) is commutative and have maps of morphisms between the Grassmannian complex and the tangential complex for weight 3. Here we have some results

Proposition 4.3.4. *The map* $C_5(\mathbb{A}^4_{F[\varepsilon]_2}) \xrightarrow{d'} C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{\tau^3_{0,\varepsilon}} \left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right)$ *is zero.*

Proof: The proof of this lemma is direct by calculation. Let $(l_0^*, \ldots, l_4^*) \in C_5(\mathbb{A}^4_{F[\varepsilon]_2})$ Where

$$l_i^* = \begin{pmatrix} a + a_\varepsilon\varepsilon \\ b + b_\varepsilon\varepsilon \\ c + c_\varepsilon\varepsilon \\ d + d_\varepsilon\varepsilon \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} a_\varepsilon \\ b_\varepsilon \\ c_\varepsilon \\ d_\varepsilon\varepsilon \end{pmatrix} = l_i + l_{i\varepsilon}\varepsilon$$

Let ω be the volume formed in four-dimensional vector space, and $\Delta(l_i, \cdot, \cdot, \cdot)$ be the volume form in $V_4/\langle l_i\rangle$.

$$\begin{aligned}&\tau^3_{0,\varepsilon} \circ d'(l_0^*, \ldots, l_4^*)\\ =&\tau^3_{0,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i(l_i^*|l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)\right)\end{aligned}$$

Consider the first coordinate of the map first

$$=\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\Big(\frac{\Delta(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_3^*, l_4^*)_\varepsilon}{\Delta(l_0, \ldots, \hat{l}_i, \ldots, l_3, l_4)} \otimes \frac{\Delta(l_0, \ldots, \hat{l}_{i+1}, \ldots, l_3, l_4)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3, l_4)} \wedge \frac{\Delta(l_0, \ldots, \hat{l}_{i+3}, \ldots, l_3, l_4)}{\Delta(l_0, \ldots, \hat{l}_{i+2}, \ldots, l_3, l_4)}\Big) \quad i \mod 4\right) \tag{4.30}$$

First, we expand inner sum which gives us 12 different terms after simplification. When we apply alternation sum then we get 60 terms and there is direct cancellation which leads to zero. Now consider the second coordinate , which gives us

$$\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0\\ j\neq i}}^{3} \frac{\Delta(l_0^*,\ldots,\hat{l}_j^*,\ldots,l_3^*,l_4^*)_\varepsilon}{\Delta(l_0,\ldots,\hat{l}_j,\ldots,l_3,l_4)}\right)$$

Again if we expand inner sum first, then we get only four different terms but after the application of alternation we get zero. □

As an analogy of Proposition 4.3.4 in higher weight, we present the following result

Proposition 4.3.5. *The map* $C_{n+2}(\mathbb{A}^{n+1}_{F[\varepsilon]_2}) \xrightarrow{d'} C_{n+1}(\mathbb{A}^{n}_{F[\varepsilon]_2}) \xrightarrow{\tau^n_{0,\varepsilon}} \left(F \otimes \bigwedge^{n-1} F^\times\right) \oplus (\bigwedge^n F)$ *is zero, where*

$$\begin{aligned}
&\tau^n_{0,\varepsilon}(l_0^*,\ldots,l_n^*)\\
&= \sum_{i=0}^{n}(-1)^i\Bigg(\frac{\Delta(l_0^*,\ldots,\hat{l}_i^*,\ldots,l_n^*)_\varepsilon}{\Delta(l_0,\ldots,\hat{l}_i,\ldots,l_n)} \otimes \frac{\Delta(l_0,\ldots,\hat{l}_{i+1},\ldots,l_n)}{\Delta(l_0,\ldots,\hat{l}_{i+2},\ldots,l_n)}\\
&\qquad \wedge\cdots\wedge \frac{\Delta(l_0,\ldots,\hat{l}_{i+(n-1)},\ldots,l_n)}{\Delta(l_0,\ldots,\hat{l}_{i+n},\ldots,l_n)}\Bigg) + \Bigg(\bigwedge_{\substack{j=0\\ j\neq i}}^{n} \frac{\Delta(l_0^*,\ldots,\hat{l}_j^*,\ldots,l_n^*)_\varepsilon}{\Delta(l_0,\ldots,\hat{l}_j,\ldots,l_n)}\Bigg),\\
&\qquad\qquad i \mod (n+1)
\end{aligned}$$

Proof: Let $(l_0^*,\ldots,l_{n+1}^*) \in C_{n+2}(\mathbb{A}^{n+1}_{F[\varepsilon]_2})$. We have

$$\tau^n_{0,\varepsilon}\circ d'(l_0^*,\ldots,l_{n+1}^*) = \tau^n_{0,\varepsilon}\left(\sum_{i=0}^{n}(-1)^i(l_i^*|l_0^*,\ldots,\hat{l}_i^*,\ldots,l_{n+1}^*)\right)$$

Now use definition of alternation to represent this sum then we have

$$\begin{aligned}
&\tau^n_{0,\varepsilon}\circ d'(l_0^*,\ldots,l_{n+1}^*)\\
&=\widetilde{\mathrm{Alt}}_{(0\cdots n+1)}\Bigg\{\sum_{i=0}^{n}(-1)^i\Bigg(\Bigg(\frac{\Delta(l_0^*,\ldots,\hat{l}_i^*,\ldots,l_n^*,l_{n+1}^*)_\varepsilon}{\Delta(l_0,\ldots,\hat{l}_i,\ldots,l_n,l_{n+1})} \otimes \frac{\Delta(l_0,\ldots,\hat{l}_{i+1},\ldots,l_n,l_{n+1})}{\Delta(l_0,\ldots,\hat{l}_{i+2},\ldots,l_n,l_{n+1})}\\
&\wedge\cdots\wedge \frac{\Delta(l_0,\ldots,\hat{l}_{i+(n-1)},\ldots,l_n,l_{n+1})}{\Delta(l_0,\ldots,\hat{l}_{i+n},\ldots,l_n,l_{n+1})}\Bigg) + \Bigg(\bigwedge_{\substack{j=0\\ j\neq i}}^{n} \frac{\Delta(l_0^*,\ldots,\hat{l}_j^*,\ldots,l_n^*,l_{n+1}^*)_\varepsilon}{\Delta(l_0,\ldots,\hat{l}_j,\ldots,l_n,l_{n+1})}\Bigg)\Bigg),\\
&\qquad\qquad i \mod n+1\Bigg\}
\end{aligned} \tag{4.31}$$

First expand the inner sum on first term that gives $n+1$ number of terms. Expand again by using the properties of wedge that gives $n(n+1)$ terms. Apply alternation sum on that gives us $n(n+1)(n+2)$ terms, so there are $n+2$ sets each consisting $n(n+1)$ terms and each term in $n(n+1)$ term has $n+1$ sets of n terms and good thing is that they cancelled set by set to give zero.

Now expand the inner sum in the second term of (4.31) that gives $n+1$ terms and then apply alternation sum which gives $n+2$ sets of $n+1$ terms, we find cancellation in the expansion of sum accordingly which gives zero as well. □

4.4 Tangent Complex for any n

In this subsection, we give suggestions how to define a tangent group $T\mathcal{B}_n(F)$ for any n in a similar spirit as in ([12]) and give technique for its appropriateness by relating them in a suitable complex.

We can write the tangent group $T\mathcal{B}_n(F)$ for any n by defining the map $\partial : \mathbb{Z}[F[\varepsilon]_2] \to T\mathcal{B}_{n-1}(F) \otimes F^\times \oplus F \otimes \mathcal{B}_{n-1}(F)$.

Define $T\mathcal{B}_n(F)$ as a $\mathbb{Z}$-module over $F[\varepsilon]_2$ which is generated by $\langle a; b] = [a+b\varepsilon] - [a] \in \mathbb{Z}[F[\varepsilon]_2]$ and quotiented by kernel of the following map

$$\partial : \mathbb{Z}[F[\varepsilon]_2] \to T\mathcal{B}_{n-1}(F) \otimes F^\times \oplus F \otimes \mathcal{B}_{n-1}(F)$$

$$\partial : \langle a; b] \mapsto \langle a; b]_{n-1} \otimes a + (-1)^{n-1}\frac{b}{a} \otimes [a]_{n-1}$$

Then the following becomes a complex

$$T\mathcal{B}_n(F) \xrightarrow{\partial_\varepsilon} \begin{smallmatrix} T\mathcal{B}_{n-1}(F)\otimes F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F) \end{smallmatrix} \xrightarrow{\partial_\varepsilon} \cdots \xrightarrow{\partial_\varepsilon} \begin{smallmatrix} T\mathcal{B}_2(F)\otimes\wedge^{n-2}F \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times \end{smallmatrix} \xrightarrow{\partial_\varepsilon} \left(F \otimes \bigwedge^{n-1} F^\times\right) \oplus \left(\bigwedge^n F\right) \quad (4.32)$$

where ∂_ε is induced by ∂ and defined by

$$\partial_\varepsilon : \langle a; b]_n \mapsto \langle a; b]_{n-1} \otimes a + (-1)^{n-1}\frac{b}{a} \otimes [a]_{n-1}$$

Note: when ∂_ε is applied to the group $\mathcal{B}_n(F)$ then it agrees with δ_n defined by ([7])

$$\delta_n : [a]_n \mapsto \begin{cases} [a]_{n-1} \otimes a, & \text{for } n \geq 3 \\ (1-a) \wedge a, & \text{for } n = 2 \end{cases}$$

We can show that the equation (4.32) is a complex by considering the part for $2 \leq k \leq n-2$

$$\cdots \xrightarrow{\partial_\varepsilon} \begin{array}{c} T\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-2}F^\times \end{array} \xrightarrow{\partial_\varepsilon} \begin{array}{c} T\mathcal{B}_{n-k}(F)\otimes\wedge^{k}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k}(F)\otimes\wedge^{k-1}F^\times \end{array} \xrightarrow{\partial_\varepsilon} \begin{array}{c} T\mathcal{B}_{n-k-1}(F)\otimes\wedge^{k+1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k-1}(F)\otimes\wedge^{k}F^\times \end{array} \xrightarrow{\partial} \cdots$$

Let $\langle x; x_1]_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} y_i + \frac{a_1}{a} \otimes [b]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} c_j \in \begin{array}{c} T\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-1}F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-k+1}(F)\otimes\wedge^{k-2}F^\times \end{array}$ We can do this in two steps. First calculate $\partial_\varepsilon\left(\partial_\varepsilon\left(\langle x; x_1]_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} y_i\right)\right)$

$$\begin{aligned}
=&\partial_\varepsilon\left(\langle x; x_1]_{n-k} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i + (-1)^{n-k}\frac{x_1}{x} \otimes [x]_{n-k} \otimes \bigwedge_{i=1}^{k-1} y_i\right) \\
=&\langle x; x_1]_{n-k-1} \otimes \underbrace{x \wedge x}_{0} \wedge \bigwedge_{i=1}^{k-1} y_i + (-1)^{n-k-1}\frac{x_1}{x} \otimes [x]_{n-k-1} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i \\
&+ (-1)^{n-k}\frac{x_1}{x} \otimes [x]_{n-k-1} \otimes x \wedge \bigwedge_{i=1}^{k-1} y_i \\
=&0
\end{aligned}$$

In the next step we calculate $\partial_\varepsilon\left(\partial_\varepsilon\left(\frac{a_1}{a} \otimes [b]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} c_j\right)\right)$

$$\partial_\varepsilon\left(\frac{a_1}{a} \otimes [b]_{n-k} \otimes b \wedge \bigwedge_{j=1}^{k-2} c_j\right) = \frac{a_1}{a} \otimes [b]_{n-k-1} \otimes \underbrace{b \wedge b}_{0} \wedge \bigwedge_{j=1}^{k-2} c_j = 0$$

We don't know the homology of Complex (4.32), because we don't have kernels of the maps ∂_ε but it is expected to be in a similar way as the homology of the complex (2.2).

Appendix A

A.1 Mapping Grassmannian to Goncharov's complex in weight 3

To show that the left hand square of the following diagram is commutative, we will re-prove Theorem 3.10 from [10] (see proof in the appendix of [13]) without using K-theory, where he construct a morphism from Grassmannian complex to his motivic complex in weight 3.

$$\begin{tikzcd} C_6(3) \arrow[r, "d"] \arrow[d, "f_2^3"] & C_5(3) \arrow[r, "d"] \arrow[d, "f_1^3"] & C_4(3) \arrow[d, "f_0^3"] \\ \mathcal{B}_3(F) \arrow[r, "\delta"] & \mathcal{B}_2(F) \otimes F^\times \arrow[r, "\delta"] & \bigwedge^3 F^\times \end{tikzcd} \tag{A.1a}$$

where the maps (as described in [10] and [13]) are the following:

$$f_2^3(l_0, \dots, l_5) = \frac{1}{15}\mathrm{Alt}_6 \left[\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)}\right]_3,$$

$$f_1^3(l_0, \dots, l_4) = \frac{1}{2}\mathrm{Alt}_5 \left\{[r(l_0|l_1, l_2, l_3, l_4)]_2 \otimes \Delta(l_0, l_1, l_2)\right\}$$

and

$$f_0^3(l_0, \dots, l_3) = \mathrm{Alt}_4 \left\{\Delta(l_0, l_1, l_2) \wedge \Delta(l_0, l_1, l_3) \wedge \Delta(l_0, l_2, l_3)\right\}$$

while

$$\delta([x]_3) = [x]_2 \otimes x \quad \text{and} \quad \delta([x]_2 \otimes y) = (1 - x) \wedge x \wedge y$$

First, we will compute $\delta \circ f_2^3(l_0, \ldots, l_5)$ and for short hand we will write (ijk) instead of $\Delta(l_i, l_j, l_k)$

$$\begin{aligned}
&\delta\circ f_2^3(l_0,\ldots,l_5)\\
=&\frac{1}{15}\mathrm{Alt}_6\left\{\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]_2 \otimes \frac{(013)(124)(205)}{(014)(125)(203)}\right\}\\
&\text{Use the even cycle (012)(345), then we have}\\
=&\frac{1}{5}\mathrm{Alt}_6\left\{\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]_2 \otimes \frac{(013)}{(014)}\right\}\\
&\text{Use the odd cycle (34)}\\
=&\frac{2}{5}\mathrm{Alt}_6\left\{\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]_2 \otimes \Delta(013)\right\}
\end{aligned}$$

If we apply the odd permutation (03)

$$= \frac{1}{5}\left(\mathrm{Alt}_6\left\{\left[\frac{(013)(124)(205)}{(014)(125)(203)}\right]_2 \otimes (013)\right\} - \mathrm{Alt}_6\left\{\left[\frac{(310)(124)(235)}{(314)(125)(230)}\right]_2 \otimes (013)\right\}\right)$$

Now we use the crucial step here in which we rewrite this triple-ratio in the product of two projected cross-ratios.

$$= \frac{1}{5}\mathrm{Alt}_6\left\{\left(\left[\frac{r(2|1053)}{r(1|0234)}\right]_2 - \left[\frac{r(2|1350)}{r(1|3204)}\right]_2\right)\otimes (013)\right\}$$

Apply five-term relation in $\mathcal{B}_2(F)$) then we will have

$$= \frac{1}{5}\mathrm{Alt}_6\left\{\left(-\left[\frac{r(2|1530)}{r(1|0342)}\right]_2 + [r(2|1053)]_2 - [r(1|0234)]_2\right)\otimes (013)\right\} \tag{A.1}$$

We will treat the above three terms individually. We consider first term now,

$$\mathrm{Alt}_6\left\{\left[\frac{r(2|1530)}{r(1|0342)}\right]_2 \otimes (013)\right\}$$

For each individual determinant, e.g. (013), will have the following terms.

$$\mathrm{Alt}_6\left\{\left[\frac{r(2|1530)}{r(1|0342)}\right]_2 \otimes (013)\right\} = \mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(013)(245)}\left(\left[\frac{r(2|1530)}{r(1|0342)}\right]_2 \otimes (013)\right)\right\}$$

We need a subgroup in S_6 which fixes (013) as a determinant i.e. $(013) \sim (310) \sim (301)\cdots$.

Here S_3 permuting $\{0,1,3\}$ and another one permuting $\{2,4,5\}$ i.e. $S_3 \times S_3$. Now consider

$$\begin{aligned}
&\mathrm{Alt}_{(013)(245)}\left\{\left[\frac{r(2|1530)}{r(1|0342)}\right]_2 \otimes (013)\right\}\\
=&\mathrm{Alt}_{(013)(245)}\left\{\left[\frac{(210)(235)}{(213)(250)}\cdot\frac{(104)(132)}{(102)(135)}\right]_2 \otimes (013)\right\}\\
=&\mathrm{Alt}_{(013)(245)}\left\{\left[\frac{(253)(104)}{(250)(134)}\right]_2 \otimes (013)\right\}
\end{aligned}$$

By using odd permutation (25) the above becomes

$$=0$$

The new shape of (A.1) is

$$= \frac{1}{5}\mathrm{Alt}_6\left\{([r(2|1053)]_2 - [r(1|0234)]_2) \otimes (013)\right\} \tag{A.2}$$

Now we will consider the first terms

$$\begin{aligned}
&\frac{1}{5}\mathrm{Alt}_6\left\{[r(2|1053)]_2 \otimes (013)\right\}\\
=&\frac{1}{5}\mathrm{Alt}_6\left\{\frac{1}{6}\mathrm{Alt}_{(245)}[r(2|1053)]_2 \otimes (013)\right\}\\
=&\frac{1}{30}\mathrm{Alt}_6\{\big([r(4|1023)]_2 - [r(2|1043)]_2\\
&\qquad + [r(5|1043)]_2 - [r(4|1053)]_2\\
&\qquad + [r(2|1053)]_2 - [r(5|1023)]_2\big) \otimes (013)\}
\end{aligned}$$

We are able to use projected five-term relation in $\mathcal{B}_2(F)$ here.

$$\begin{aligned}
=&\frac{1}{30}\mathrm{Alt}_6\{\big([r(0|1234)]_2 - [r(1|0234)]_2 - [r(3|0124)]_2\\
&+[r(0|1435)]_2 - [r(1|0435)]_2 + [r(3|0145)]_2\\
&+[r(0|1532)]_2 - [r(1|0532)]_2 + [r(3|0152)]_2\big) \otimes (013)\}
\end{aligned}$$

Use the cycle (013)(245) then we get

$$= \frac{1}{30}\cdot 9\mathrm{Alt}_6\left\{[r(0|1234)]_2 \otimes (013)\right\} \tag{A.3}$$

We also have $-\frac{1}{5}\mathrm{Alt}_6\left\{[\![r(1|0234)]\!]_2^D \otimes (013)\right\}$ from (A.2) which can be written as

$$\frac{1}{30}\cdot -6\mathrm{Alt}_6\left\{[r(1|0234)]_2 \otimes (013)\right\}$$

the above expression can be combined with (A.3) and gives

$$= \frac{1}{30}\mathrm{Alt}_6\{\Big(9[\![r(0|1234)]\!]_2^D - 6\,[\![r(1|0234)]\!]_2^D\Big) \otimes (013)\}$$

By using cycle (01), we get $\frac{1}{2}\mathrm{Alt}_6\left\{[\![r(0|1234)]\!]_2^D \otimes (013)\right\}$ as a result of (A.2). While the computation of $f_1^3 \circ d(l_0, \ldots, l_4)$ has no changes and can be performed in the usual way to get the commutativity.

Appendix B

Second order tangent complex

Here we will define second order tangent complex and will try to relate it with geometric configurations. This chapter will also describe the application of second component r_{ε^2} of the cross-ratio of vectors in $C_4(\mathbb{A}^2_{F[\varepsilon]_3})$ or triple-ratio of vectors in $C_6(\mathbb{A}^3_{F[\varepsilon]_3})$. Construction of groups and calculations in this chapter are similar to what we have in chapter 4. That's why we excluded it from the main text and presented here in the appendix.

B.1 Dilogarithmic complex

We remember from the first chapter that $F[\varepsilon]_3 := F[\varepsilon]/\varepsilon^3$ and $F^\times$-action in $F[\varepsilon]_3$ is defined as $\lambda : a + b_1\varepsilon + b_2\varepsilon^2 \mapsto a + \lambda b_1\varepsilon + \lambda b_2\varepsilon^2$, where $\lambda \in F^\times$ and will be denoted by $\star$-action. Now we define second order tangent group $T\mathcal{B}_2^2(F)$ as a $\mathbb{Z}$-module generated by the following

$$\langle a; b_1, b_2], \quad a, b_1, b_2 \in F, a \neq 0, 1$$

where $\langle a; b_1, b_2] = [a + b_1\varepsilon + b_2\varepsilon^2] - [a] \in \mathbb{Z}[F[\varepsilon]_3]$, quotient with the five term relation.

$$\begin{aligned}\langle a; a_\varepsilon, a_{\varepsilon^2}] - \langle b; b_\varepsilon, b_{\varepsilon^2}] + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)_\varepsilon, \left(\frac{b}{a}\right)_{\varepsilon^2} \right] - \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)_\varepsilon, \left(\frac{1-b}{1-a}\right)_{\varepsilon^2} \right] \\ + \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)_\varepsilon, \left(\frac{a(1-b)}{b(1-a)}\right)_{\varepsilon^2} \right] \qquad \text{(B.1)}\end{aligned}$$

is denoted by $\langle a; b_1, b_2]_2^2$, for $a, b \neq 0, 1, \quad a \neq b$ where

$$\begin{aligned}
\left(\frac{b}{a}\right)_\varepsilon =& \frac{ab_\varepsilon - a_\varepsilon b}{a^2}\\
\left(\frac{1-b}{1-a}\right)_\varepsilon =& \frac{(1-b)a_\varepsilon - (1-a)b_\varepsilon}{(1-a)^2}\\
\left(\frac{a(1-b)}{b(1-a)}\right)_\varepsilon =& \frac{b(1-b)a_\varepsilon - a(1-a)b_\varepsilon}{(b(1-a))^2}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{b}{a}\right)_{\varepsilon^2} =& \frac{a^2 b_{\varepsilon^2} - ab a_{\varepsilon^2} - a a_\varepsilon b_\varepsilon + b\,(a_\varepsilon)^2}{a^3}\\
\left(\frac{1-b}{1-a}\right)_{\varepsilon^2} =& \frac{(1-a)(1-b)a_{\varepsilon^2} - (1-a)^2 b_{\varepsilon^2} - (1-a)a_\varepsilon b_\varepsilon + (1-b)\,(a_\varepsilon)^2}{(1-a)^3}\\
\left(\frac{a(1-b)}{b(1-a)}\right)_{\varepsilon^2} =& \frac{A}{(b(1-a))^3},
\end{aligned}$$

where

$$\begin{aligned}
A =& (1-a)\Big(b(1-b)a_{\varepsilon^2} - a(1-b)b_{\varepsilon^2} - b(1-a)a_\varepsilon b_\varepsilon + a\,(b_\varepsilon)^2\\
&+ b\,(a_\varepsilon)^2\, b_\varepsilon\Big) - a(1-b)a_\varepsilon\,(b_\varepsilon)^2
\end{aligned}$$

We found some more relations in $T\mathcal{B}_2^2(F)$ through five term relation.

1. Two-term relation:

$$\langle a; b_1, b_2]_2^2 = -\langle 1-a; -b_1, -b_2]_2^2$$

2. Inversion relation:

$$\langle a; b_1, b_2]_2^2 = \left\langle \frac{1}{a}; -\frac{b_1}{a^2}, -\frac{ab_2 - b_1^2}{a^3} \right]_2^2$$

Let $C_m(\mathbb{A}^2_{F[\varepsilon]_3})$ be the free abelian group generated by the configuration $(l_0^*, \ldots, l_{m-1}^*)$ of m points in $\mathbb{A}^2_{F[\varepsilon]_3}$, where $\mathbb{A}^2_{F[\varepsilon]_3}$ is defined as an affine plane over $F[\varepsilon]_3$ (here we assume that all points are in generic position). In this case the Grassmanian complex will be in the following shape

$$\cdots \xrightarrow{d} C_5(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{d} C_4(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{d} C_3(\mathbb{A}^2_{F[\varepsilon]_3})$$

$$d\colon (l_0^*, \ldots, l_{m-1}^*) \mapsto \sum_{i=0}^{m} (-1)^i (l_0^*, \ldots, \hat{l}_i^*, \ldots, l_{m-1}^*)$$

where $l_i^* = \begin{pmatrix} a_i + a_{i,\varepsilon}\varepsilon + a_{i,\varepsilon^2}\varepsilon^2 \\ b_i + b_{i,\varepsilon}\varepsilon + b_{i,\varepsilon^2}\varepsilon^2 \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix}\varepsilon + \begin{pmatrix} a_{i,\varepsilon^2} \\ b_{i,\varepsilon^2} \end{pmatrix}\varepsilon^2 = l_i + l_{i,\varepsilon}\varepsilon + l_{i,\varepsilon^2}\varepsilon^2.$

Here we will use second order $\mathbb{Z}$-module $T\mathcal{B}_2^2(F)$.

Consider the following diagram

$$\begin{tikzcd} C_4(\mathbb{A}^2_{F[\varepsilon]_3}) \arrow[r, "d"] \arrow[d, "\tau^2_{1,\varepsilon^2}"] & C_3(\mathbb{A}^2_{F[\varepsilon]_3}) \arrow[d, "\tau^2_{0,\varepsilon^2}"] \\ T\mathcal{B}_2^2(F) \arrow[r, "\partial_{\varepsilon^2}"] & F \otimes F^\times \oplus \bigwedge{}^2 F \end{tikzcd} \tag{B.1a}$$

the maps of which we defined as follows:

1.

$$\partial_{\varepsilon^2} : \langle a; b_1, b_2] \mapsto \left(\frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{a} \otimes (1-a) + \frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{1-a} \otimes a \right) + \left(\frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{1-a} \wedge \frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{a} \right)$$

$$b_1^\varepsilon = \text{pure } b_\varepsilon \text{ parts from } b_1$$

$$b_2^{\varepsilon^2} = \text{pure } b_{\varepsilon^2} \text{ parts from } b_2$$

Definition:

The coefficient of ε^2 which comes through the product of coefficient of ε^2 and coefficient of ε^0 or itself will be called as pure part of the coefficient of ε^2., e.g., if

$$\frac{b + b_\varepsilon\varepsilon + b_{\varepsilon^2}\varepsilon^2}{a + a_\varepsilon\varepsilon + a_{\varepsilon^2}\varepsilon^2} = \frac{b}{a} + \frac{ab_\varepsilon - a_\varepsilon b}{a^2}\varepsilon + \frac{ab_{\varepsilon^2} - aba_{\varepsilon^2} - aa_\varepsilon b_\varepsilon + b(a_\varepsilon)^2}{a^3}\varepsilon^2$$

then ab_{ε^2} and $-aba_{\varepsilon^2}$ are pure in the coefficient of ε^2, while $-aa_\varepsilon b_\varepsilon$ and $b(a_\varepsilon)^2$ are non-pure.

Note: If we consider the above in the context of cross-ratios then we see that r_ε (as in §4.1) has only terms which are pure in b_ε while r_{ε^2} has some terms which are non-pure in b_{ε^2} so we eliminate those terms through the definition of the map ∂_{ε^2}. One major reason for eliminating non-pure terms is that we would like to recombine terms using a d log-like rule, but this is only guaranteed to work for pure terms.

2.

$$\tau^2_{1,\varepsilon^2}(l_0^*, \dots, l_3^*) = \langle r(l_0, \dots, l_3); r_\varepsilon(l_0^*, \dots, l_3^*), r_{\varepsilon^2}(l_0^*, \dots, l_3^*)]_2^2$$

3. For defining τ^2_{0,ε^2}, first we divide is $\tau^2_{0,\varepsilon^2} = \tau^1 + \tau^2$

$$\tau^1(l_0^*, l_1^*, l_2^*) = \sum_{i=0}^{2}(-1)^i\left(\frac{(l_i^* l_{i+2}^*)_\varepsilon + (l_i^* l_{i+2}^*)_{\varepsilon^2}}{(l_i l_{i+2})} \otimes \frac{(l_i l_{i+1})}{(l_{i+2} l_{i+1})}\right), \quad i \mod 3$$

and

$$\tau^2(l_0^*, l_1^*, l_2^*) = \sum_{i=0}^{2}(-1)^i\left(\frac{(l_i^* l_{i+1}^*)_\varepsilon + (l_i^* l_{i+1}^*)_{\varepsilon^2}}{(l_i l_{i+1})} \wedge \frac{(l_i^* l_{i+2}^*)_\varepsilon + (l_i^* l_{i+2}^*)_{\varepsilon^2}}{(l_i l_{i+2})}\right), \quad i \mod 3$$

where $r, r_\varepsilon, r_{\varepsilon^2}$ are coefficients of ε^0, ε^1 and ε^2 respectively in the following:

$$\begin{aligned}
\mathbf{r}(l_0,\dots,l_3) =& r(l_0,\dots,l_3) + r_\varepsilon(l_0^*,\dots,l_3^*)\varepsilon + r_{\varepsilon^2}(l_0^*,\dots,l_3^*)\varepsilon^2\\
r(l_0,\dots,l_3) =& \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\\
r_\varepsilon(l_0^*,\dots,l_3^*) =& \frac{\left\{(l_0^*l_3^*)(l_1^*l_2^*)\right\}_\varepsilon}{(l_0l_2)(l_1l_3)} - \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\cdot\frac{\left\{(l_0^*l_2^*)(l_1^*l_3^*)\right\}_\varepsilon}{(l_0l_2)(l_1l_3)}\\
r_{\varepsilon^2}(l_0^*,\dots,l_3^*) =& \frac{\left\{(l_0^*l_3^*)(l_1^*l_2^*)\right\}_{\varepsilon^2}}{(l_0l_2)(l_1l_3)} - r_\varepsilon(l_0^*,\dots,l_3^*)\cdot\frac{\left\{(l_0^*l_2^*)(l_1^*l_3^*)\right\}_\varepsilon}{(l_0l_2)(l_1l_3)} - \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}\cdot\frac{\left\{(l_0^*l_2^*)(l_1^*l_3^*)\right\}_{\varepsilon^2}}{(l_0l_2)(l_1l_3)}
\end{aligned}$$

for the notation purpose only we used in the upper definition of cross-ratios $(ab)_\varepsilon :=$ $a_\varepsilon b + ab_\varepsilon$ and $(ab)_{\varepsilon^2} := a_{\varepsilon^2}b + a_\varepsilon b_\varepsilon + ab_{\varepsilon^2}$.

Proposition B.1.1. *The diagram (B.1a) is commutative.*

Proof: It requires direct calculation. We will outline some steps here because the methodology of the calculation is quite similar to the previous calculation. The composition of $\partial_{\varepsilon^2} \circ \tau^2_{1,\varepsilon^2}(l_0^*,\dots,l_3^*)$ will be

$$\begin{aligned}
&\partial_{\varepsilon^2} \circ \tau^2_{1,\varepsilon^2}(l_0^*,\dots,l_3^*)\\
&= \partial_{\varepsilon^2}\left(\langle r(l_0,\dots,l_3); r_\varepsilon(l_0^*,\dots,l_3^*), r_{\varepsilon^2}(l_0^*,\dots,l_3^*)]_2^2\right)
\end{aligned}$$

We already have the values of $\frac{r_\varepsilon(l_0^*,\dots,l_3^*)}{r(l_0,\dots,l_3)}$ and $\frac{r_\varepsilon(l_0^*,\dots,l_3^*)}{1-r(l_0,\dots,l_3)}$ but here we will also need $\frac{r_{\varepsilon^2}(l_0^*,\dots,l_3^*)^{\varepsilon^2}}{r(l_0,\dots,l_3)}$ and $\frac{r_{\varepsilon^2}(l_0^*,\dots,l_3^*)^{\varepsilon^2}}{1-r(l_0,\dots,l_3)}$. We find that

$$\frac{r_{\varepsilon^2}(l_0^*,\dots,l_3^*)^{\varepsilon^2}}{r(l_0,\dots,l_3)} = \frac{(l_0^*l_3^*)_{\varepsilon^2}}{(l_0l_3)} + \frac{(l_1^*l_2^*)_{\varepsilon^2}}{(l_1l_2)} - \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} - \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)}$$

and

$$\frac{r_{\varepsilon^2}(l_0^*,\dots,l_3^*)^{\varepsilon^2}}{1-r(l_0,\dots,l_3)} = \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} + \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)} - \frac{(l_0^*l_1^*)_{\varepsilon^2}}{(l_0l_1)} - \frac{(l_2^*l_3^*)_{\varepsilon^2}}{(l_2l_3)}$$

We divide ∂_{ε^2} as $\partial_{\varepsilon^2} = \partial^1 + \partial^2$ such that

$$\partial^1 (\langle a; b_1, b_2]) = \frac{b_1^{\varepsilon} + b_2^{\varepsilon^2}}{a} \otimes (1-a) + \frac{b_1^{\varepsilon} + b_2^{\varepsilon^2}}{1-a} \otimes a$$

and

$$\partial^2 (\langle a; b_1, b_2]) = \frac{b_1^{\varepsilon} + b_2^{\varepsilon^2}}{1-a} \wedge \frac{b_1^{\varepsilon} + b_2^{\varepsilon^2}}{a}$$

Now $\partial^1 \circ \tau^2_{1,\varepsilon^2}(l_0^*, \ldots, l_3^*)$ is

$$\begin{aligned}
=&\left(\frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} + \frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} - \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} + \frac{(l_0^*l_3^*)_{\varepsilon^2}}{(l_0l_3)} + \frac{(l_1^*l_2^*)_{\varepsilon^2}}{(l_1l_2)} - \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} - \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)}\right)\\
&\otimes \frac{(l_0l_1)(l_2l_3)}{(l_0l_2)(l_1l_3)}\\
+&\left(\frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} + \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} - \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} - \frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} + \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} + \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)} - \frac{(l_0^*l_1^*)_{\varepsilon^2}}{(l_0l_1)} - \frac{(l_2^*l_3^*)_{\varepsilon^2}}{(l_2l_3)}\right)\\
&\otimes \frac{(l_0l_3)(l_1l_2)}{(l_0l_2)(l_1l_3)}
\end{aligned}$$

and $\partial^2 \circ \tau^2_{1,\varepsilon^2}(l_0^*, \ldots, l_3^*)$ is

$$\begin{aligned}
=&\left(\frac{(l_0^*l_3^*)_\varepsilon}{(l_0l_3)} + \frac{(l_1^*l_2^*)_\varepsilon}{(l_1l_2)} - \frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} - \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} + \frac{(l_0^*l_3^*)_{\varepsilon^2}}{(l_0l_3)} + \frac{(l_1^*l_2^*)_{\varepsilon^2}}{(l_1l_2)} - \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} - \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)}\right)\\
\wedge&\left(\frac{(l_0^*l_2^*)_\varepsilon}{(l_0l_2)} + \frac{(l_1^*l_3^*)_\varepsilon}{(l_1l_3)} - \frac{(l_0^*l_1^*)_\varepsilon}{(l_0l_1)} - \frac{(l_2^*l_3^*)_\varepsilon}{(l_2l_3)} + \frac{(l_0^*l_2^*)_{\varepsilon^2}}{(l_0l_2)} + \frac{(l_1^*l_3^*)_{\varepsilon^2}}{(l_1l_3)} - \frac{(l_0^*l_1^*)_{\varepsilon^2}}{(l_0l_1)} - \frac{(l_2^*l_3^*)_{\varepsilon^2}}{(l_2l_3)}\right)
\end{aligned}$$

for the other side we compute first $\tau^1 \circ d(l_0^*, \ldots, l_3^*)$.

$$\tau^1 \circ d(l_0^*, \ldots, l_3^*) = \tau^2_{0,\varepsilon^2}\left(\sum_{i=0}^{3}(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_3^*)\right)$$

By using alternation sum first part of this composition will be

$$\tau^1 \circ d(l_0^*, \ldots, l_3^*) = \widetilde{\mathrm{Alt}}_{(0123)}\left\{\sum_{i=0}^{2}(-1)^i\left(\frac{(l_i^*l_{i+2}^*)_\varepsilon + (l_i^*l_{i+2}^*)_{\varepsilon^2}}{(l_il_{i+2})} \otimes \frac{(l_il_{i+1})}{(l_{i+2}l_{i+1})}\right), \quad i \mod 3\right\}$$

By first expanding the inner sum we obtain three terms then pass the alternation through that sum, will give us 12 terms and combining them will give us an expression equal to $\partial^1 \circ \tau^2_{1,\varepsilon^2}(l_0^*, \ldots, l_3^*)$. A similar technique can be used for the second part giving

$$\begin{aligned}
&\tau^2 \circ d(l_0^*, \ldots, l_3^*)\\
=&\widetilde{\mathrm{Alt}}_{(0123)}\left\{\sum_{i=0}^{2}(-1)^i\left(\frac{(l_i^*l_{i+1}^*)_\varepsilon + (l_i^*l_{i+1}^*)_{\varepsilon^2}}{(l_il_{i+1})} \wedge \frac{(l_i^*l_{i+2}^*)_\varepsilon + (l_i^*l_{i+2}^*)_{\varepsilon^2}}{(l_il_{i+2})}\right), \quad i \mod 3\right\}
\end{aligned}$$

The inner sum will give us three terms which can be further distributed into 12 terms using the wedge product. The procedure will be same as in theorem 4.2.2 for wedge factors are of the same type i.e., both $()_\varepsilon$ or both $()_{\varepsilon^2}$. Then we combine mixed terms together for which one wedge factor is of type $()_\varepsilon$, while the other is of type $()_{\varepsilon^2}$, and the other with ε^2 term, combining those gives us completely $\partial^2 \circ \tau^2_{1,\varepsilon^2}(l_0^*, \ldots, l_3^*)$. □

B.2 Trilogarithmic Complex

In this section we will define a second order tangent group $T\mathcal{B}_3^2(F)$ which is a $\mathbb{Z}$-module over $F[\varepsilon]_3$ and generated by $\langle a; b_1, b_2]$, where

$$\langle a; b_1, b_2] = [a + b_1\varepsilon + b_2\varepsilon^2] - [a] \in \mathbb{Z}[F[\varepsilon]_3]$$

and quotient by the kernel of the following map.

$$\partial : \mathbb{Z}[F[\varepsilon]_3] \to \left(T\mathcal{B}_2^2(F) \otimes F^\times\right) \oplus (F \otimes B_2(F)),$$

$$\partial : \langle a; b_1, b_2] \mapsto \langle a; b_1, b_2]_2^2 \otimes a + \frac{b_1^\varepsilon}{a} \otimes [a]_2 + \frac{b_2^\varepsilon}{a} \otimes [a]_2$$

The following is a complex figuring $T\mathcal{B}_3^2(F)$.

$$T\mathcal{B}_3^2(F) \xrightarrow{\partial_{\varepsilon^2}} \begin{array}{c} T\mathcal{B}_2^2(F)\otimes F^\times \\ \oplus \\ F\otimes B_2(F) \end{array} \xrightarrow{\partial_{\varepsilon^2}} \left(F \otimes \bigwedge{}^2 F^\times\right) \oplus \left(\bigwedge{}^3 F\right)$$

Considerations above (§4.3) show that a suitable definition of $T\mathcal{B}_3^2(F)$ should preferably extend the above for $T\mathcal{B}_3^2(F)$ such that the following diagram is commutative.

$$\begin{array}{ccccc} C_6(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & C_5(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \\ \downarrow{\scriptstyle \tau^3_{2,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{1,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{0,\varepsilon^2}} \\ T\mathcal{B}_3^2(F) & \xrightarrow{\partial_{\varepsilon^2}} & \left(T\mathcal{B}_2^2(F) \otimes F^\times\right) \oplus (F \otimes \mathcal{B}_2(F)) & \xrightarrow{\partial_{\varepsilon^2}} & \left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right) \end{array} \quad \text{(B.2a)}$$

where (in the continuation of the previous subsection)

$$\begin{aligned} &\partial_{\varepsilon^2}\left(\langle a; b_1, b_2]_2^2 \otimes b + x \otimes [y]_2\right) \\ &= \left(-\frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{a} \otimes (1-a) \wedge b - \frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{1-a} \otimes a \wedge b + x \otimes (1-y) \wedge y\right) \\ &+ \left(\frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{a} \wedge \frac{b_1^\varepsilon + b_2^{\varepsilon^2}}{1-a} \wedge x\right), \end{aligned}$$

$$\tau^3_{2,\varepsilon^2}(l^*_0,\ldots,l^*_5) = \frac{2}{45}\mathrm{Alt}\left\langle r_3(l_0,\ldots,l_5); r_{3,\varepsilon}(l^*_0,\ldots,l^*_5), r_{3,\varepsilon^2}(l^*_0,\ldots,l^*_5)\right]^2_3,$$

and

$$\begin{aligned}
&\tau^3_{1,\varepsilon^2}(l^*_0,\ldots,l^*_4)\\
&= -\frac{1}{3}\sum_{i=0}^{4}\Bigg(\left\langle r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4); r_\varepsilon(l^*_i|l^*_0,\ldots,\hat{l}^*_i,\ldots,l^*_4), r_{\varepsilon^2}(l^*_i|l^*_0,\ldots,\hat{l}^*_i,\ldots,l^*_4)\right]^2_2 \otimes \prod_{i\neq j}(\hat{l}_i\hat{l}_j)\\
&\qquad + \sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{(l^*_0\ldots\hat{l}^*_i\ldots\hat{l}^*_j\ldots l^*_4)_\varepsilon}{(l_0\ldots\hat{l}_i\ldots\hat{l}_j\ldots l_4)}\right)\otimes\left[r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\right]_2\\
&\qquad + \sum_{\substack{k=0\\ k\neq i}}^{4}\left(\frac{(l^*_0\ldots\hat{l}^*_i\ldots\hat{l}^*_k\ldots l^*_4)_{\varepsilon^2}}{(l_0\ldots\hat{l}_i\ldots\hat{l}_k\ldots l_4)}\right)\otimes\left[r(l_i|l_0,\ldots,\hat{l}_i,\ldots,l_4)\right]_2\Bigg)
\end{aligned}$$

for the definition of τ^3_{0,ε^2}, we divide $\tau^3_{0,\varepsilon^2} = \tau^1 + \tau^2$ such that, $\tau^1(l^*_0,\ldots,l^*_3)$ is

$$\sum_{i=0}^{3}(-1)^i\left(\frac{(l^*_0\ldots\hat{l}^*_i\ldots l^*_3)_\varepsilon + (l^*_0\ldots\hat{l}^*_i\ldots l^*_3)_{\varepsilon^2}}{(l_0\ldots\hat{l}_i\ldots l_3)}\otimes\frac{(l_0\ldots\hat{l}_{i+1}\ldots l_3)}{(l_0\ldots\hat{l}_{i+2}\ldots l_3)}\wedge\frac{(l_0\ldots\hat{l}_{i+3}\ldots l_3)}{(l_0\ldots\hat{l}_{i+2}\ldots l_3)}\right)$$

$$i \mod 4$$

and $\tau^2(l^*_0,\ldots,l^*_3)$ is

$$\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\left(\frac{(l^*_0\ldots\hat{l}^*_j\ldots l^*_3)_\varepsilon + (l^*_0\ldots\hat{l}^*_j\ldots l^*_3)_{\varepsilon^2}}{(l_0\ldots\hat{l}_j\ldots l_3)}\right)$$

where $r_3, r_{3,\varepsilon}, r_{3,\varepsilon^2}$ are coefficients of ε^0, ε^1 and ε^2 respectively in the following:

$$\begin{aligned}
\mathbf{r}_3(l_0,\ldots,l_5) =& r_3(l_0,\ldots,l_5) + r_{3,\varepsilon}(l^*_0,\ldots,l^*_5)\varepsilon + r_{3,\varepsilon^2}(l^*_0,\ldots,l^*_5)\varepsilon^2\\
r_3(l_0,\ldots,l_5) =& \mathrm{Alt}_6\frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\\
r_{3,\varepsilon}(l^*_0,\ldots,l^*_5) =& \mathrm{Alt}_6\left\{\frac{\left\{(l^*_0l^*_1l^*_3)(l^*_1l^*_2l^*_4)(l^*_2l^*_0l^*_5)\right\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} - r_3(l_0,\ldots,l_5)\cdot\frac{\left\{(l^*_0l^*_1l^*_4)(l^*_1l^*_2l^*_5)(l^*_2l^*_0l^*_3)\right\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\right\}\\
r_{3,\varepsilon^2}(l^*_0,\ldots,l^*_5) =& \mathrm{Alt}_6\{\frac{\left\{(l^*_0l^*_1l^*_3)(l^*_1l^*_2l^*_4)(l^*_2l^*_0l^*_5)\right\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} - r_{3,\varepsilon}(l^*_0,\ldots,l^*_5)\cdot\frac{\left\{(l^*_0l^*_1l^*_4)(l^*_1l^*_2l^*_5)(l^*_2l^*_0l^*_3)\right\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\\
&- \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\cdot\frac{\left\{(l^*_0l^*_1l^*_4)(l^*_1l^*_2l^*_5)(l^*_2l^*_0l^*_3)\right\}_{\varepsilon^2}}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}\}
\end{aligned}$$

Theorem B.2.1. *The right hand square of the diagram (B.2a) is commutative ,i.e.,*

$$\begin{array}{ccc}
C_5(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{\quad d \quad} & C_4(\mathbb{A}^3_{F[\varepsilon]_3})\\
\downarrow{\scriptstyle \tau^3_{1,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{0,\varepsilon^2}}\\
\left(T\mathcal{B}^2_2(F)\otimes F^\times\right)\oplus\left(F\otimes\mathcal{B}_2(F)\right) & \xrightarrow{\partial_{\varepsilon^2}} & \left(F\otimes\bigwedge^2 F^\times\right)\oplus\left(\bigwedge^3 F\right)
\end{array}$$

Proof: We will do here direct calculation. First calculate $\tau^1 \circ d(l_0^*, \ldots, l_4^*)$ is

$$\begin{aligned}\tau^1 \circ d(l_0^*, \ldots, l_4^*) =&\tau^1\left(\sum_{i=0}^{4}(l_0^*, \ldots, \hat{l}_i^*, \ldots, l_4^*)\right)\\ =&\widetilde{\mathrm{Alt}}_{(01234)}\Bigg\{\sum_{i=0}^{3}(-1)^i\Bigg(\frac{(l_0^*\ldots\hat{l}_i^*\ldots l_3^*)_\varepsilon + (l_0^*\ldots\hat{l}_i^*\ldots l_3^*)_{\varepsilon^2}}{(l_0\ldots\hat{l}_i\ldots l_3)}\\ &\otimes\frac{(l_0\ldots\hat{l}_{i+1}\ldots l_3)}{(l_0\ldots\hat{l}_{i+2}\ldots l_3)}\wedge\frac{(l_0\ldots\hat{l}_{i+3}\ldots l_3)}{(l_0\ldots\hat{l}_{i+2}\ldots l_3)}\Bigg),\quad i \mod 4\Bigg\}\end{aligned}$$

First evaluate inner sum of the above then we will find 12 terms of type $\frac{a_\varepsilon}{a}\otimes b\wedge c$ and 12 terms of type $\frac{a_{\varepsilon^2}}{a}\otimes b\wedge c$. Pass each sum through the $\widetilde{\mathrm{Alt}}_{(01234)}$ which gives 60 terms of each type then regroup them on the basis of similar $\frac{(l_i^*l_j^*l_k^*)_\varepsilon}{(l_il_jl_k)}\otimes\ldots$ and $\frac{(l_i^*l_j^*l_k^*)_{\varepsilon^2}}{(l_il_jl_k)}\otimes\ldots$. We will see that there are 10 possible determinants of each type and each has 6 terms, we are listing here first few.

$$\begin{aligned}+\frac{(l_0^*l_1^*l_2^*)_\varepsilon}{(l_0l_1l_2)}\otimes\Big((l_0l_1l_3)\wedge(l_1l_2l_3)+(l_0l_2l_4)\wedge(l_1l_2l_3)+(l_0l_1l_4)\wedge(l_0l_2l_4)\\ -(l_0l_1l_3)\wedge(l_0l_2l_3)-(l_0l_1l_4)\wedge(l_1l_2l_4)-(l_0l_2l_3)\wedge(l_1l_2l_3)\Big)\end{aligned}$$

and

$$\begin{aligned}+\frac{(l_0^*l_1^*l_2^*)_{\varepsilon^2}}{(l_0l_1l_2)}\otimes\Big((l_0l_1l_3)\wedge(l_1l_2l_3)+(l_0l_2l_4)\wedge(l_1l_2l_3)+(l_0l_1l_4)\wedge(l_0l_2l_4)\\ -(l_0l_1l_3)\wedge(l_0l_2l_3)-(l_0l_1l_4)\wedge(l_1l_2l_4)-(l_0l_2l_3)\wedge(l_1l_2l_3)\Big)\end{aligned}$$

and $\tau^2 \circ d(l_0^*, \ldots, l_4^*)$ can also be evaluated in similar fashion that will be

$$= \widetilde{\mathrm{Alt}}_{(01234)}\left\{\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{j=0\\ j\neq i}}^{3}\left(\frac{(l_0^*\ldots\hat{l}_j^*\ldots l_3^*)_\varepsilon+(l_0^*\ldots\hat{l}_j^*\ldots l_3^*)_{\varepsilon^2}}{(l_0\ldots\hat{l}_j\ldots l_3)}\right)\right\}$$

Now we go the other side of the diagram. We first divide $\partial_{\varepsilon^2} = \partial^1 + \partial^2$ such that

$$\partial^1\left(\langle a; b_1, b_2]_2^2\otimes b + x\otimes[y]_2\right) = -\frac{b_1^\varepsilon+b_2^{\varepsilon^2}}{a}\otimes(1-a)\wedge b - \frac{b_1^\varepsilon+b_2^{\varepsilon^2}}{1-a}\otimes a\wedge b + x\otimes(1-y)\wedge y$$

and

$$\partial^1\left(\langle a; b_1, b_2]_2^2\otimes b + x\otimes[y]_2\right) = \frac{b_1^\varepsilon+b_2^{\varepsilon^2}}{a}\wedge\frac{b_1^\varepsilon+b_2^{\varepsilon^2}}{1-a}\wedge x$$

We first calculate $\partial^1\circ\tau^3_{1,\varepsilon^2}(l_0^*, \ldots, l_4^*)$, which is quite similar with the calculation in the proof of theorem 4.3.1. The only difference here is that, instead of collecting terms of type $\frac{(l_i^* l_j^* l_k^*)_\varepsilon}{(l_i l_j l_k)} \otimes \cdots$, only we also collect the terms of type $\frac{(l_i^* l_j^* l_k^*)_{\varepsilon^2}}{(l_i l_j l_k)} \otimes \cdots$ and we find that terms match with result of $\tau^1 \circ d(l_0^*, \ldots, l_4^*)$ with coefficient -3 that is killed by $-\frac{1}{3}$ which is already the coefficient of the map τ^3_{1,ε^2}. We have similar situation with $\partial^2 \circ \tau^3_{1,\varepsilon^2}(l_0^*, \ldots, l_4^*)$. Direct calculation gives us the same result as we have for $\tau^2 \circ d(l_0^*, \ldots, l_4^*)$ □

Theorem B.2.2. *The left hand square of the diagram (B.2a) is commutative, i.e.,*

$$\begin{array}{ccc} C_6(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{\quad d \quad} & C_5(\mathbb{A}^3_{F[\varepsilon]_3}) \\ \downarrow{\scriptstyle \tau^3_{2,\varepsilon^2}} & & \downarrow{\scriptstyle \tau^3_{1,\varepsilon^2}} \\ T\mathcal{B}_3^2(F) & \xrightarrow{\partial_{\varepsilon^2}} & \left(T\mathcal{B}_2^2(F) \otimes F^\times\right) \oplus (F \otimes B_2(F)) \end{array}$$

is commutative

Proof: First consider

$$\begin{aligned} &\partial_{\varepsilon^2} \circ \tau^3_{2,\varepsilon^2}(l_0^*, \ldots, l_5^*) \\ &= \frac{2}{45}\mathrm{Alt}_6\Bigg\{ \langle r_3(l_0, \ldots, l_5); r_{3,\varepsilon}(l_0^*, \ldots, l_5^*), r_{3,\varepsilon^2}(l_0^*, \ldots, l_5^*)]_2^2 \otimes r_3(l_0, \ldots, l_5) \\ &\qquad + \frac{r_{3,\varepsilon}(l_0^*, \ldots, l_5^*)^\varepsilon}{r_3(l_0, \ldots, l_5)} \otimes [r_3(l_0, \ldots, l_5)]_2 + \frac{r_{3,\varepsilon^2}(l_0^*, \ldots, l_5^*)^{\varepsilon^2}}{r_3(l_0, \ldots, l_5)} \otimes [r_3(l_0, \ldots, l_5)]_2 \Bigg\} \end{aligned} \tag{B.2}$$

The simplification of (B.2) is quite similar to (4.22) in the proof of theorem 4.3.3. We can use same steps here if we show that $r_{3,\varepsilon^2}(l_0^*, \ldots, l_5^*)$ can be written in terms of two projected cross-ratios whose ratio corresponds to $r_3(l_0, \ldots, l_5)$. We know that

$$\mathbf{r}_3(l_0^*, \ldots, l_5^*) = \frac{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)}{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)}$$

and is expanded as

$$\mathbf{r}_3(l_0^*, \ldots, l_5^*) = r_3(l_0, \ldots, l_5) + r_{3,\varepsilon}(l_0^*, \ldots, l_5^*)\varepsilon + r_{3,\varepsilon^2}(l_0^*, \ldots, l_5^*)\varepsilon^2$$

where $l_i^* \in \mathbb{A}^3_{F[\varepsilon]_3}$ and $(l_i^* l_j^* l_k^*) \in F[\varepsilon]_3$. so we can write

$$r_{3,\varepsilon^2}(l_0^*, \ldots, l_5^*) = \left(\frac{(l_0^* l_1^* l_3^*)(l_1^* l_2^* l_4^*)(l_2^* l_0^* l_5^*)}{(l_0^* l_1^* l_4^*)(l_1^* l_2^* l_5^*)(l_2^* l_0^* l_3^*)}\right)_{\varepsilon^2} = (\mathbf{r}_3(l_0^*, \ldots, l_5^*))_{\varepsilon^2}$$

As it is the coefficient of ε^2 from $\mathbf{r}_3(l_0^*, \ldots, l_5^*)$ which is a ratio of determinants in $F[\varepsilon]_3$ so it is clear that it can be written as the ratio of two projected cross-ratios. Now calculate

$\partial_{\varepsilon^2} \circ \tau^3_{2,\varepsilon^2}(l_0^*, \dots, l_5^*)$.

$$
\begin{aligned}
=&\frac{2}{45}\mathrm{Alt}_6\Bigg\{ \langle r_3(l_0,\dots,l_5); r_{3,\varepsilon}(l_0^*,\dots,l_5^*), r_{3,\varepsilon^2}(l_0^*,\dots,l_5^*)]_2^2 \otimes \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} \\
&+\left(\frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)}+\frac{(l_1^*l_2^*l_4^*)_\varepsilon}{(l_1l_2l_4)}+\frac{(l_2^*l_0^*l_5^*)_\varepsilon}{(l_2l_0l_5)}-\frac{(l_0^*l_1^*l_4^*)_\varepsilon}{(l_0l_1l_4)}-\frac{(l_1^*l_2^*l_5^*)_\varepsilon}{(l_1l_2l_5)}-\frac{(l_2^*l_0^*l_3^*)_\varepsilon}{(l_2l_0l_3)}\right)\otimes[r_3(l_0,\dots,l_5)]_2 \\
&+\left(\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)}+\frac{(l_1^*l_2^*l_4^*)_{\varepsilon^2}}{(l_1l_2l_4)}+\frac{(l_2^*l_0^*l_5^*)_{\varepsilon^2}}{(l_2l_0l_5)}-\frac{(l_0^*l_1^*l_4^*)_{\varepsilon^2}}{(l_0l_1l_4)}-\frac{(l_1^*l_2^*l_5^*)_{\varepsilon^2}}{(l_1l_2l_5)}-\frac{(l_2^*l_0^*l_3^*)_{\varepsilon^2}}{(l_2l_0l_3)}\right) \\
&\otimes[r_3(l_0,\dots,l_5)]_2\Bigg\}
\end{aligned}
\tag{B.3}
$$

By using similar technique as in the proof of theorem 4.3.3, equation (B.3) can be written as:

$$
\begin{aligned}
=\frac{1}{3}\mathrm{Alt}_6\Bigg\{ &\langle r(l_0|l_1l_2l_3l_4); r_\varepsilon(l_0^*|l_1^*l_2^*l_3^*l_4^*); r_{\varepsilon^2}(l_0^*|l_1^*l_2^*l_3^*l_4^*)]_2^2 \otimes (l_0l_1l_3) \\
&+\frac{(l_0^*l_1^*l_3^*)_\varepsilon}{(l_0l_1l_3)}\otimes[r(l_0|l_1l_2l_3l_4)]_2+\frac{(l_0^*l_1^*l_3^*)_{\varepsilon^2}}{(l_0l_1l_3)}\otimes[r(l_0|l_1l_2l_3l_4)]_2\Bigg\}
\end{aligned}
\tag{B.4}
$$

Calculation of $\tau^3_{1,\varepsilon^2} \circ d(l_0^*, \dots, l_5^*)$ will give us same result as we have in (B.4). □

We can further relate this second order second with first order by constructing the following result:

Proposition B.2.3. *The following maps*

1. $C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{d'} C_3(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{\tau^2_{0,\varepsilon^2}} F \otimes F^\times \oplus \bigwedge^2 F$
2. $C_5(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{d'} C_4(\mathbb{A}^2_{F[\varepsilon]_3}) \xrightarrow{\tau^2_{1,\varepsilon^2}} T\mathcal{B}_2^2(F)$
3. $C_5(\mathbb{A}^4_{F[\varepsilon]_3}) \xrightarrow{d'} C_4(\mathbb{A}^3_{F[\varepsilon]_3}) \xrightarrow{\tau^3_{0,\varepsilon^2}} F \otimes \bigwedge^2 F^\times \oplus \bigwedge^3 F$
4. $C_{n+1}(\mathbb{A}^{n+1}_{F[\varepsilon]_3}) \xrightarrow{d'} C_{n+1}(\mathbb{A}^{n}_{F[\varepsilon]_3}) \xrightarrow{\tau^n_0} F \otimes \bigwedge^{n-1} F^\times \oplus \bigwedge^n F$

are zero.

Proof: See the proof of the lemmas 4.2.3, 4.2.5, 4.3.4 and 4.3.5. □

Second Order Tangent Group for any n

we define $T\mathcal{B}_n^2(F)$ as a $\mathbb{Z}$-module over $F[\varepsilon]_3$ is generated by $\langle a; b_1, b_2] = [a+b_1\varepsilon+b_2\varepsilon^2]-[a] \in \mathbb{Z}[F[\varepsilon]_3]$ and quotient by the kernel of

$$\partial : \mathbb{Z}[F[\varepsilon]_2] \to T\mathcal{B}^2_{n-1}(F)\otimes F^\times \oplus F \otimes \mathcal{B}_{n-1}(F)$$

$$\partial : \langle a; b_1, b_2] \mapsto \langle a; b_1, b_2]^2_{n-1} \otimes a + (-1)^{n-1} \left(\frac{b_1^{\varepsilon}}{a} \otimes [a]_{n-1} + \frac{b_2^{\varepsilon^2}}{a} \otimes [a]_{n-1} \right)$$

then the following is a complex

$$T\mathcal{B}^2_n(F) \xrightarrow{\partial_{\varepsilon^2}} \begin{matrix} T\mathcal{B}^2_{n-1}(F)\otimes F^{\times} \\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F) \end{matrix} \xrightarrow{\partial_{\varepsilon^2}} \cdots \xrightarrow{\partial_{\varepsilon^2}} \begin{matrix} T\mathcal{B}^2_2(F)\otimes\bigwedge^{n-2} F \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\bigwedge^{n-3} F^{\times} \end{matrix} \xrightarrow{\partial_{\varepsilon^2}} \left(F \otimes \bigwedge{}^{n-1} F^{\times}\right) \oplus \left(\bigwedge{}^{n} F\right)$$

The correctness of the this complex can be shown in a similar way as we did in §4.4.

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