Affine Toda solitons and fusing rules

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by

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Abstract

This thesis is concerned with various soliton solutions to some of the affine Toda field theories. These are field theories in 1+1 dimensions that possess a rich underlying Lie algebraic structure and they are known to be integrable. The soliton solutions occur as a result of the multi-vacua that appear in the field theory when the coupling constant is taken to be purely imaginary.

In chapter one a review of the affine Toda field theories is undertaken. This is meant to be a relatively complete and exhaustive survey of the literature that has appeared on the subject in recent years. A brief introduction to the theory of solitons and the methods of obtaining such solutions in field theory is given in chapter two, resulting in the construction of the relevant machinery for the Toda theories.

In chapter three, Hirota's method is used to construct single and double soliton solutions to these theories. As a consequence of these explicit formulae the fusing structure of the solitons may be investigated and shown to be equivalent to that found in the classical particle regîme, supplemented by further 'annihilations' of 'soliton-antisoliton'. The calculations of the double soliton solutions are claimed to be original in this context. The fusing has also been examined by Olive, Turok and Underwood[16] through an abstract group-theoretical approach to the affine Toda field theories, however very few explicit formulae are given by them, and hence all the solutions given here are important in their own right.

An algebra-independent analysis of such phenomena is undertaken in chapter four where a vertex operator construction is given for the relevant interaction functions. Some properties of these functions are noted; (some of these facts correspond with those in [16] concerning the fusing structure of the solitons).

Preface

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1990 and January 1994, under the supervision of Professor E. Corrigan. This material has not been submitted previously for any degree, either in this or any other university.

No claim of originality is made for the review in chapter one, nor for the review of the aspects of soliton theory in chapter two, whether directly relevant or not. Most of the remainder of the thesis is claimed as original unless otherwise stated explicitly in the appropriate sections of the text. The motivation for the majority of this work were the preprints and papers by Hollowood[14], however the suggestion to study the leading order behaviour of the soliton tau functions was solely due to my supervisor E.Corrigan[64]. This suggestion was given to me in September 1992 while visiting the Isaac Newton Institute for Mathematical Sciences.

Some of the results found here with regards to double soliton solutions may also be found in Caldi and Zhu[63], though computed in a different manner.

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1. A REVIEW OF AFFINE TODA FIELD THEORY

1.1 Introduction

In recent years one of the most dominant problems in the study of quantum field theories has been that of the renormalisation group flow on the space of quantum field theories. This has primarily manifested itself in the study of critical phenomena in two-dimensional Euclidean quantum field theory. This is unique in the sense that the fixed points of the renormalisation group have an 'infinite dimensional' conformal group associated with them, and hence the symmetry may well prove to be powerful enough to solve the theory exactly.

There is a large class of two-dimensional quantum field theories associated with each of these points and each of these theories can be associated with a representation of the conformal algebra, more precisely that of the Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{cn(n^2-1)}{12} \cdot \delta_{n+m,0} \tag{1.01}$$

which is a central extension of the Lie algebra of conformal transformations and the L_n are the generators of the algebra. A complete classification of conformal field theories, however does not exist at present.

When these systems are shifted away from criticality the situation is by no means so well understood, but the field theories that correspond to the renormalisation group flow away from a fixed point, can be considered as a conformal field theory 'perturbed' by some operator.

Pioneering work into this area was undertaken by A.B. Zamolodchikov[1], who argued that certain relevant deformations away from criticality would preserve integrability and hence, the massive theory that resulted from the perturbation would have an infinite number of conserved quantities associated with it. By calculating and conjecturing the spins of these 'integrals of motion' he was able to conjecture the form of the relevant S-matrices for these perturbed field theories once an initial mass-ratio and coupling had been introduced. The 'natural' description of the field theories in terms of their S-matrices resulted in a renewal of interest in S-matrices for integrable two-dimensional systems in general. Because of the connection between the original work on perturbed conformal field theories and Toda field theories a flurry of activity soon developed around the construction of S-matrices for affine Toda theories based on various Lie algebras. This connection resulted from the fact that the integrals of motion in the P.C.F.T. had spin values that were labelled by the exponents of the Lie algebra e_8 modulo its Coxeter number and that the subsequent mass-ratios were equivalent. The Toda theories are a series of relativistic quantum field theories connected to the set of simple Lie algebras through their potential terms. If the root system of the Lie algebra is simply-laced, the quantum field theory is uniquely associated to that root system for all values of the coupling constant. Whereas for the non-simply-laced algebras a more complicated situation arises, where a single quantum field theory is associated with a dual pair of the algebras through the coupling. The field theories based on the finite-dimensional Lie algebras exhibit conformal invariance in the sense that their Lagrangians are scale invariant in light cone co-ordinates and hence, are massless. Those associated with the infinite-dimensional affine algebras have a well defined ground state and hence, are massive field theories.

It has been conjectured that Toda field theories at specific values of the coupling constant correspond to conformal field theories in the minimal series of the Virasoro algebra, extended to a W-algebra based on a Lie algebra[2]. In particular, the statistical system the Ising model, which corresponds to a $c = 1/2$ conformal field theory can be described by both an e_8 and a_1 Toda field theory. The affine versions of these theories then represent the perturbations of the statistical system away from criticality. The magnetic perturbation is described by the affine e_8 Toda field theory, whereas the affine a_1 theory is a representation of the thermal perturbation. Braaten, Curtright, Ghandour and Thorn[3] calculated the value of the central charge that may be associated with the Toda field theory:

$$c = r\left[1 + h(h+1)\left(\frac{1}{\beta} + \frac{\beta}{4\pi}\right)^2\right] \tag{1.02}$$

where r, h are the rank and Coxeter numbers of the Lie algebra under question, and β is a coupling constant. So it can be seen that taking β to be purely imaginary allows the possibility of values of c less than one, and hence representations of the discrete unitary series of conformal field theories. The construction of S-matrices for affine Toda theories based on the simply-laced Lie algebras $a_n^{(1)}$, $d_n^{(1)}$ and $e_6^{(1)}$, $e_7^{(1)}$, $e_8^{(1)}$ was successfully carried out a couple of years ago by a series of authors[4],[5],[6],[7],[8]. Only recently, however, has the answer been discovered for the non-simply-laced algebras, since a more generalised method of attack was required[9]. Furthermore, the construction of the scattering matrices for Toda theories based on Lie super-algebras has been initiated[10]. It appears as though only the minimal part of the S-matrix formulae that have been conjectured for the simply-laced A.T.F.T's are relevant to the discussion of a connection with the perturbed conformal field theory. This is because it is possible to compute the central charge of the ultraviolet limit of a theory solely from its scattering matrix, and only the minimal parts give rise to the charges of the relevant coset constructions of conformal field theories[11]. Moreover, the question of unitarity, in the field theory sense, is still not fully understood, since the potential terms in the Toda theories are in general not hermitian. This fact also is discussed in Klassen and Melzer[11].

Many of the puzzling features that were inherent in the initial studies of classical and quantum affine Toda field theory have now been given a much clearer understanding by the gradual application of detailed group theory to the quantum field theories. Specifically such questions as those of, the universal coupling formulae for simply-laced algebras, why the conserved quantities appeared as eigenvectors for the relevant Cartan matrix and the Clebsch-Gordon property of couplings have all been answered by utilisation of Lie algebra theory. Moreover, for example in the last case, the relevant algebraic theory had only been proved as recently ago as 1988 but had been conjectured to be true twenty years earlier. An important aspect in gaining this knowledge has been the Coxeter element of the Lie algebra since it has turned up in connections with a general formula for the S-matrices, the coupling rule conjectured by Dorey[12],[13] and the conserved charge spectrum, all which will be expanded upon later in this review chapter.

Following recent work by T.J. Hollowood[14], the A.T.F.T's with imaginary coupling constant have been shown to possess soliton solutions due to the existence of multi-vacua. Roughly speaking, solitons are complex 'lump-like' solutions of non-linear equations which are localised in space and move at constant speed with little or no change in shape. They exist because the dispersion effects are exactly balanced by the non-linearity of the equations of motion and they interpolate between the vacuum states of the theory. This is quite unlike the case of real coupling affine Toda field theories where oscillations about the ground state only give rise to particle states. Therefore, the extension of the coupling into the complex regîme, must complicate the S-matrices even more - Hollowood has conjectured S-matrices for the 'soliton states' in the $a_n^{(1)}$ series of A.T.F.T's and these involve representations of quantum groups.

Using techniques developed by Hirota[15], Hollowood managed to construct single soliton solutions for the $a_n^{(1)}$ theories and wrote down a generalisation for n-solitons. He therefore extended the case of $g = a_1^{(1)}$, that is the sine-Gordon equation, which has been well studied in the past, and was manifestly the rôle model for study in this field.

This is the area this thesis will be concerned with - the subject of solitons in A.T.F.T.. Specifically, single soliton solutions have been constructed by the author for some of the remaining simply-laced Lie algebras - namely $e_6^{(1)}$, $e_7^{(1)}$ and moreover, double solitons for all the simply-laced Lie algebras. As a consequence, their fusing relationships have been analysed which have produced a coupling rule similar to that found in the real coupling regîme, but augmented slightly by further 'annihilation' couplings. An initial study of the interaction terms has also been undertaken and it is shown how the leading order terms may be constructed in terms of the underlying algebraic theory. An abstract approach to the solutions has been constructed by Olive, Turok and Underwood[16], in a manifestly group - theoretical way, but they give very few explicit results. Other approaches to this problem have also included the construction of Bäcklund transformations[17], but here there appears to be difficulties with regard to generalising it beyond the $a_n^{(1)}$ series of algebras.

1.2 Classical Affine Toda Field Theory.

The affine Toda theories associated with the affine untwisted Kač-Moody algebras $\hat{g}$ are massive two-dimensional bosonic field theories represented by the Lagrangians of the form:

$$L=\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{m^2}{\beta^2}\sum_{i=0}^{r}n_i e^{\beta\alpha_i\cdot\Phi}. \tag{1.03}$$

Here, r denotes the rank of the respective finite Lie algebra g and Φ is a vector of the real scalar fields, which takes values in the Cartan subalgebra and describes r massive particles. β is a coupling constant and the $\{\alpha_i\}$ are the simple roots of g augmented by the 'negative' of the highest root in the adjoint representation for the untwisted algebra. The inner products amongst these vectors are described by the affine Dynkin diagram. The Kač labels for this algebra (or 'marks') n_i are such that,

$$\sum_{i=0}^{r} n_i\alpha_i = 0$$

and the mark corresponding to α_0 is normalised to unity; m sets the mass - scale for the field theory. (For the case of Lie super algebras - which will not be mentioned again - the set of roots is divided into bosonic and fermionic ones, and the Lagrangians contain fermions as well).

The exponential term involving α_0 can be regarded as a perturbing term to the potential of the ordinary Toda field theory. It has the effect of breaking the conformal invariance that is manifest in the original Toda Lagrangian when the light-cone co-ordinates are transformed by

$$x^\pm \to x'^\pm = f^{(\pm)}(x^\pm)$$

and the fields by

$$\phi(x)\to\phi'(x')=\phi(x)-\frac{\delta}{\beta}\ln(\partial_+ f^{(+)}\partial_- f^{(-)}).$$

Here $\delta=\sum_{i=1}^{r}\lambda_i$ and λ_i are the fundamental weights, defined by $\lambda_i\cdot\frac{2\alpha_j}{|\alpha_j|^2}=\delta_{ij}$. (So, $\delta\cdot\alpha_i=1$ for i = 1,...,r and the Lagrangian is scaled by the product $\partial_+ f^{(+)}\partial_- f^{(-)}$). Hence, the A.T.F.T. can be regarded in this sense as a 'perturbed conformal field theory' and as such the extra potential term has the effect of stabilising the vacuum of the conformal field theory so that massive particle excitations exist around the minimum of the potential. The perturbing piece in the Lagrangian also has the property of maintaining the integrability of the equations, which results from the existence of a Lax pair, infinitely many conserved quantities and exact solvability[18]. The infinite set of currents $J_{\pm}^{(s+1)}$ of increasing spin s+1 are labelled by the exponents 's' of the Lie algebra g modulo its Coxeter number (the first of which is the stress-tensor). These are conserved by virtue of the Toda field equations derived from the Lagrangian:

$$\partial^{\mu}\partial_{\mu}\Phi = -\frac{m^2}{\beta}\sum_{i=0}^{r} n_i\alpha_i e^{\beta\alpha_i\cdot\Phi}. \tag{1.04}$$

Hence, the Lagrangian may admit an infinite number of symmetries described by the corresponding conserved currents. Quantum mechanically, the existence of these symmetries has a profound effect on the structure of the scattering matrices of these theories, implying that the n-particle matrix must factorise into a product of elastic two-particle ones. This will be described later in this review.

The infinitely many conserved charges Q_s (that are in involution) are similarly labelled by their spins since they transform under two-dimensional Lorentz transformations ($x^{\pm} \rightarrow \lambda^{\pm}x^{\pm}$) as

$$Q_s \rightarrow \lambda^{-s}Q_s.$$

The Lagrangian for A.T.F.T. can be expanded about the minimum at $\Phi = 0$, in order to perturbatively extract such data as the mass matrix and three point couplings. For the potential part it is found that,

$$V(\Phi) = \frac{m^2}{\beta}\sum_{i=0}^{r} n_i + \frac{m^2}{2}\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b\Phi^a\Phi^b + \frac{m^2\beta}{6}\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b\alpha_i^c\Phi^a\Phi^b\Phi^c + \ldots\ldots$$

and so the mass matrix and couplings are given by (neglecting combinatorial factors):

$$M^2 = m^2\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b \quad , \quad c^{abc} = \beta m^2\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b\alpha_i^c \tag{1.05}$$

The calculations of the mass eigenstates were initiated by Arinshtein et al.[4] when they obtained those for the $a_n^{(1)}$ series of Toda models. Contributions by a multitude of authors have resulted in all such quantities (together with the couplings between them) for the simply-laced cases, and a complete list of them may be found, for example, in[19]. It turns out that these two pieces of data contain the key to the quantum S-matrix for the simply-laced Lie algebras.

A point worth noting is that the data for the non-simply-laced cases (untwisted and twisted theories) can be obtained, as it were 'free', from the simply-laced cases. This follows from the fact that the Dynkin diagrams for each of these theories may be obtained by "folding" one and only one of the untwisted simply-laced diagrams as explained by Olive and Turok[20]. A reduction using a symmetry of the finite algebra's diagram results in an untwisted non-simply-laced case, whereas using any additional symmetry of the extended Dynkin diagram yields affine Toda theories based on the twisted affine Dynkin diagrams.

However, when analysed carefully enough there appears to be an inherent difference between the field theories derived from the two classes of foldings. In the case of foldings leading to untwisted theories, degeneracies in the mass spectrum are removed, resulting in non-mass-degenerate fields that are linear combinations of the original degenerate fields from the parent theory. Attempting to derive S-matrices in this manner, however, reveals that they are not diagonal in this new basis of states which is invariant under the automorphism of the Dynkin diagram. Hence the non-simply-laced quantum theory will violate unitarity when restricted to these states. Therefore, already there is a definite distinction between the simply-laced and non-simply-laced based Toda theories. By comparison, foldings which lead to twisted theories leave some particles unchanged whilst removing others outright. The particles left merely form a subset of the three-point couplings.

Once all the masses were known, it soon became evident[11],[21] that (apart from in the twisted cases) they constituted the components of the Perron-Frobenius eigenvector of the Cartan matrix, associated with the finite dimensional Lie algebra g. Setting

$$\underset{\sim}{m} = (m_1, \ldots\ldots, m_r)$$

it was noticed that,

$$C\underset{\sim}{m} = \lambda_{min}\underset{\sim}{m} = (4\sin^2\frac{\pi}{2h})\underset{\sim}{m},$$

where

$$C_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} \qquad i, j = 1, \ldots\ldots, r.$$

(A generalisation of this result occurs in the quantum theory, where it is found that the values of the conserved charges of the theory are proportional to the entries of the other eigenvectors of the Cartan matrix)[11]. The particles can then be unambiguously assigned to the spots of the Dynkin diagram, so a particle of mass m_a could be assigned to a fundamental representation with the highest weight λ_a. A list of the diagrams with particle labels attached to them may be found in [6] or [12]..

Any mass degeneracy corresponds to a symmetry of the Dynkin diagram. However the mass-degenerate particles are different, not only because they have different representations of the Lie algebra associated with them, but because the other conserved quantities distinguish them.

A proof however, of this 'Perron-Frobenius' fact was not immediately forthcoming, and it was not until the application of more algebraic theory[22],[23] that the veil shrouding these observations was lifted. This will be explained later in this chapter (section 1.3.3).

Once the masses were known, other curiosities were noticed when the couplings between the mass eigenstates were calculated. For example, it transpired that the magnitude of the coupling satisfied a universal rule. Namely that the non-zero, three-point couplings obey the 'area rule':

$$c_{ijk} = \varepsilon_{ijk} \frac{4\beta}{\sqrt{h}} \Delta_{ijk}$$

where $\varepsilon_{ijk} = \pm 1$, h is the Coxeter number of the Lie algebra and Δ_{ijk} is the area of the triangle with sides m_i, m_j, and m_k. (Modified slightly in the case of the non-simply laced algebras; if the length^2 of the highest root is denoted by Ψ^2 then $\sqrt{\frac{2}{\Psi^2}}\, \varepsilon_{ijk}$ always has the value ±1 unless all three particles correspond to short roots, in which case it is $\pm\frac{1}{\sqrt{2}}$ for $b_n^{(1)}$, $c_n^{(1)}$, $f_4^{(1)}$ and $\pm\frac{2}{\sqrt{3}}$ for $g_2^{(1)}$). It did not appear sufficient that the masses had to make a triangle for there to be a non-zero coupling, since the angles in the triangle were also required to be in integer multiples of $\frac{\pi}{h}$. Moreover, a curious fact was noted[11],[21], that the assignment of representations to the particles was also reflected in the couplings, but not in the most obvious manner. Namely that a necessary (although not always sufficient) condition for c_{ijk} to be non-zero was that the irreducible decomposition of the tensor product of representations associated with a, b, c, should contain the trivial representation. That is:

$$c^{abc} \neq 0 \Rightarrow (a)\otimes(b)\otimes(c) \supset (1) \qquad (1.06)$$

which became known as the Clebsch-Gordon property of A.T.F.T., for obvious reasons.

It was not until the conjecture (and case-by-case proof) by Dorey[12] and its subsequent Lie algebraic proof by Olive, Liao and Fring[23], that these problems were finally put to rest and established firmly on a more group theoretical foothold. The rule that Dorey proposed specified precisely which couplings were non-vanishing and took the form:

"A non-zero three point coupling c_{abc} exists if there are integers r, s such that,

$$\gamma_a + \omega^r \gamma_b + \omega^s \gamma_c = 0\text{"} \qquad (1.07)$$

where ω is a Coxeter element for the Lie algebra, that is, a product of the Weyl reflections in the simple roots of g. The simple roots have been pre-multiplied by the colour index $c(i) = \pm 1$, which comes from any bicolouration of the Dynkin diagram (i.e. adjoining spots possess opposite colour), to give $\gamma_i = c(i)\alpha_i$.

Alternative forms for the rule which have proved to be just as useful may be found in[24], where the simple roots α_a, α_b, α_c are replaced by their respective fundamental weights, or the 'alternative roots', $\phi_i = (1 - \omega^{-1})\,\lambda_i$. Using one of the alternative forms of the rule, i.e.

$$\lambda_i + \omega^r \lambda_j + \omega^s \lambda_k = 0 \Leftrightarrow c_{ijk} \neq 0$$

the Clebsch-Gordon property is proved by showing that $V(\lambda_{\bar{i}})$ is an irreducible component of $V(\lambda_j)\otimes V(\lambda_k)$, where $V(\lambda_i)$ denotes the irreducible representation with highest weight λ_i. The PRV conjecture[26], proved in [27], specifies that for any element σ in the Weyl group of g and highest weights λ, μ; $V([\lambda + \sigma\mu])$ must occur in the irreducible decomposition of $V(\mu)\otimes V(\lambda)$, where the brackets denote the dominant weight conjugate to the argument.

Hence, since $[-\omega^{h-s}\lambda_i] = [-\lambda_i] = \lambda_{\bar{i}}$, the Clebsch-Gordon property appears as an immediate consequence of the Dorey-rule and the A.T.F.T's have been shown to possess this at the classical level. The apparent 'holes' in the Clebsch-Gordon correspondence, where a classically vanishing coupling is allowed, occur due to the fact that the PRV result is less restrictive, in a sense, than that required by the Dorey-rule.

For example, in the algebra D_5; $V(\lambda_2) \subset V(\lambda_2) \otimes V(\lambda_2)$ where $V(\lambda_2)$ is the adjoint rep, but $C_{222} = 0$. Here $\lambda_2 + \omega^3\lambda_2 = (\omega_{\alpha_4}\omega_{\alpha_5}\omega_{\alpha_3}\omega_{\alpha_2})\lambda_2$ which is not of the relevant form for a coupling since the product of Weyl reflections is not a power of the Coxeter element.

It is also possible to show[12] for the simply-laced (and even the twisted non-simply-laced) field theories, that each particle has h-2 non-vanishing couplings, that is $C_{ijk} \neq 0$ for a fixed i and (j, k) an ordered pair. This fact is slightly altered for the untwisted non-simply-laced cases[24].

Expanding the potential term in (1.03) further, questions may be asked concerning the nature of the n-point couplings with $n \geq 4$. However, nothing new 'in a sense' occurs, since Fring[28] has shown that, by expanding upon the work in[22],[23], all these couplings can be completely determined in terms of the masses and the three-point couplings. Furthermore, he has produced a general fusing rule, formulated in the root space of the Lie algebra for the couplings which, therefore generalises the three-point couplings rule of Dorey.

Specifically, a generating function, for all of the couplings, is given by the compact form:

$$C_{l_1}.....C_{l_n} = (-)^{n+1} \sum_{t=1}^{n-2} \sum_{\overleftrightarrow{x}} \aleph_t^{-1} x_1.....x_t \, m_{l_{n-1}}^2 \delta_{\bar{l}_{n-1} l_n}$$

where the x_i for each particular t are given either in terms of the three-point couplings, or the masses, and

$$x_1, \ldots x_{2(t+1)-n} = \sum_{\bar{k}} \frac{C_{\mu\nu\bar{k}}}{m_k^2} \qquad , \qquad x_{2t+3-n}, \ldots, x_t = m_{l_\nu}^2 \delta_{\bar{l}_\nu l_\mu}$$

Here $\sum_{\overleftrightarrow{x}}$ denotes the sum over all possible permutations of the x_i and the factor $\aleph_t$ takes care of the overcounting of permutations of any symmetric terms. Explicitly,

$$\aleph_t = (2t+2-n)!(n-t-2)!$$

A proper understanding of this n-point coupling is necessary when the knowledge of the field theory is extended off-shell in order to compute form factors[29].

The generalised fusing rule is as follows: $C_{l_1 \cdots l_n}$ for $n \geq 4$ is non-zero if, and only if, there exist n roots in Ω_i for $i = 1, \ldots, n$ which sum to zero, that is,

$$\sum_{i=1}^{n} \omega^{\xi(i)} \gamma_i = 0$$

where γ_i are again the pre-multiplied simple roots and Ω_i denotes the orbit of γ_i under the Coxeter element. This occurs together with the extra following constraints:

$$\begin{aligned}
&\omega^{\xi(1)}\gamma_1 + \omega^{\xi(2)}\gamma_2 = \omega^{\xi(\bar{1}_1)}\gamma_{\bar{1}_1} \\
&\omega^{\xi(\bar{1}_1)}\gamma_{\bar{1}_1} + \omega^{\xi(3)}\gamma_3 = \omega^{\xi(\bar{1}_2)}\gamma_{\bar{1}_2} \\
&\quad\vdots \\
&\omega^{\xi(\bar{1}_{n-3})}\gamma_{\bar{1}_{n-3}} + \omega^{\xi(n-1)}\gamma_{n-1} = \omega^{\xi(n)}\gamma_n,
\end{aligned}$$

such that

$$\gamma_{\bar{1}_i} \in \Omega_i \text{ for some } i = 0, \ldots, r.$$

That is, another n-2 triangles exist, which can be used to triangulate the resulting n-gon when the generalised fusing equation is projected onto any 'Coxeter invariant' subspace in the root space. Hence, the fusing rules which give a 'non-vanishing n-point coupling rule', are again based on the fundamental three-point coupling rule of Dorey.

Many of the results of classical affine Toda field theory are carried through into the quantum regîme. Specifically, those associated with the simply-laced algebras appear to follow through with little or no change. It is the classical data in these cases that enables construction of the scattering matrices through meticulous use of a 'bootstrap principle' and the fact that the renormalised masses appear to be in the same ratio as the classical ones, as suggested by one loop calculations[5]. The classical couplings then appear as fusing relations at imaginary rapidity values and no such fusings occur which are not present as three-point couplings in the classical theory when all particles are on their mass-shell. The account for theories based on the non-simply-laced algebras is much more subtle and will be expanded upon in section 1.3.2. It will suffice to say, at the moment, that the classical masses do not renormalise in the same way and the pole residues in the S-matrices display a different pattern. In fact a single quantum field theory is not associated with each of these theories, but with 'pairs' in the sense that algebras that are dual give rise to one quantum field theory. (The exception is $a_{2n}^{(2)}$ which is self-dual). This quantum field theory then interpolates between the 'dual pair' as the coupling constant varies, the masses lying between the masses of the two non-simply-laced theories.

1.3 Quantum Affine Toda Field Theory.

The basic objects in the quantum field theory are the fields and the multi-particle states, the latter being labelled by the momenta and species of the particles. In two dimensions the momentum can be written in terms of a rapidity angle θ_a, and hence there exist states

$$|p^{(a_1)}, p^{(a_2)}, \ldots\rangle$$

where $p^{(a)} = m_a(ch(\theta_a), sh(\theta_a))$. These states are seen to be on-shell and are well defined when each particle is spatially well separated.

The over-riding aim in the quantum theory is to calculate the scattering matrices and hence, the probability of a particular outcome of an event given some initial conditions, or states. Of course, the outcome is certain in all such integrable theories. Nevertheless S-matrices contain a great deal of information about the field theory, for example the fusing rules.

Classically, the integrability of affine Toda field theory implies the existence of an infinite number of conserved charges associated with the theory. By assuming these remain so when considered in the quantum theory, then the scattering picture is considerably simpler. As it is possible to show[30] that there can be no annihilation or production of particles in any physical final state and that the momenta must be preserved individually.

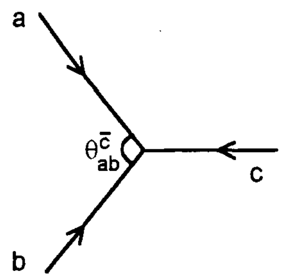
For the Toda theories with a real coupling constant and associated with the simply-laced Lie algebras, the particles that appear in the quantum theory are just those that are manifest in the Lagrangian and nothing new appears - quite unlike the case of imaginary coupling where solitonic states also exist. The conservation of charges also enables a factorisation of any n-particle S-matrix into a product of $\frac{1}{2}n(n-1)$ two-particle S-matrices, elegantly portrayed in [19] by consideration of their effect on the wave functions of the states. The fact that the momenta are individually preserved implies that the interaction must be a phase:

$$\left|p^{(a)}, p^{(b)}\right\rangle_{out} = S_{ab}(\Theta)\left|p^{(a)}, p^{(b)}\right\rangle_{in} \qquad : \Theta = \theta_a - \theta_b \qquad (1.08)$$

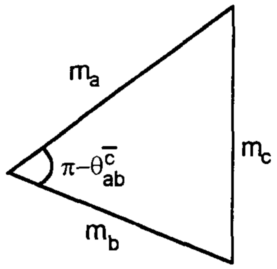
Because of Lorentz invariance, the scattering phase is only a function of the rapidity difference of the two interacting particles. Each S-matrix element is not trivial, since it can be, and generally is, a meromorphic function of Θ. The two-particle S-matrix also has to satisfy the conditions of unitarity and crossing, after analytic continuations have been made. As a consequence of these two facts, all S-matrix elements are invariant under the shift $\Theta \to \Theta + 2\pi i$ and hence, may be expressed in terms of products of hyperbolic functions. The other key factor, and arguably most important concept in the generation of the S-matrices, is that of fusing and the 'bootstrap'. Simply put, the S-matrices may have bound state poles at purely imaginary values of Θ in the range $0 \le \mathrm{Im}(\Theta) \le \pi$, which has been given the name 'the physical strip'. Two particles a,b can 'fuse' to a third c when their S-matrix contains this pole; odd poles only appear to correspond to bound states. Hence, in considering a forward channel process, the bound state particle $\bar{c}$ will have momentum^2 equal to the Mandelstam quantity 's' and the pole occurs in the amplitude when $\bar{c}$ is on shell. The rapidity difference $\theta_a - \theta_b = i\theta_{ab}^{\bar{c}}$ where $\theta_{ab}^{\bar{c}}$ is given the name 'the fusing angle' and the bootstrap hypothesis is that any bound state is itself one of the possible asymptotic states of the field theory.

Hence, $$m_{\bar{c}}^2 = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a m_b \cos\theta_{ab}^{\bar{c}}$$

for,

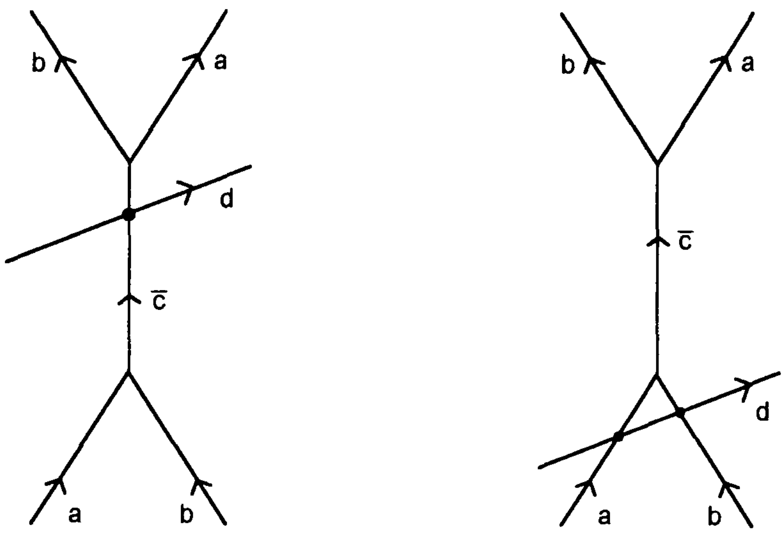


which therefore, has the dual diagram,



When the Lagrangian is considered perturbatively, the pole can be traced to a Feynman diagram containing this vertex, and for the bound state to occur there has to be a non-zero three-point coupling c_{abc}.

Poles in Θ on the physical strip correspond to s-channel or t-channel bound states according to the sign of the residue (+1,-1 respectively). The idea behind the bootstrap equations is that near the rapidity difference $\Theta = i\theta_{ab}^{\bar{c}}$, the two particle state $|p^{(a)}(\theta_a), p^{(b)}(\theta_b)\rangle$ should be dominated by the one particle state $|p^{(\bar{c})}(\theta_{\bar{c}})\rangle$, where $\theta_{\bar{c}} = \theta_a + i(\pi - \theta_{a\bar{c}}^{b})$, fixed by conservation of momentum. The immediate consequence of this is that the following 'scattering' pictures may be equated,



to yield the bootstrap equations:

$$S_{d\bar{c}}(\Theta) = S_{da}(\Theta - i\bar{\theta}_{ac}^{b})S_{db}(\Theta + i\bar{\theta}_{bc}^{a}), \qquad \bar{\theta} = \pi - \theta \tag{1.09}$$

These allow the generation of some candidates for the S-matrices in the quantum field theory, given some suitable starting point, normally associated with the lightest particle(s).

These equations are an over-determined set of equations, but the solution to them is not unique.

Before progressing, it would be worth while noting briefly the difference in the bootstrap equations which can be used for the simply-laced and non-simply-laced algebras. For the simply-laced cases these equations are applicable to any odd order pole with positive residue in the physical strip, and these poles do not move since the mass ratios are renormalisation invariant. In the non-simply-laced cases (other than $a_{2n}^{(2)}$), the poles depend on a parameter whose range lies between the relevant pairs of Coxeter numbers of dual pairs of algebras. It has been proposed[33] that only those poles which 'float' within the physical strip, with positive coefficients throughout this range, should participate in the bootstrap. Therefore, a more 'generalised' bootstrap has been suggested for the non-simply-laced algebras.

The Yang-Baxter equation which appears due to a permutation of three two-particle S-matrix elements is trivially satisfied in A.T.F.T. because of the facts that there is no particle production in the theory and the particles are uniquely distinguished by the conserved quantities (that is, the S-matrices are diagonal).

The conditions on the S-matrices that have discussed above are not the only pieces of information that the bootstrap yields, since another by-product is the fact that it places conditions on the conserved quantities in the quantum theory as well.

Classically the existence of a Lax-pair implies the existence of an infinite set of conserved quantities as has already been mentioned. If they persist quantum mechanically then they will represent a set of mutually commuting operators whose eigenstates are the particle states and may be sufficient to define a form of quantum integrability.

It is conjectured

$$Q_s \left| p^{(a)} \right\rangle = q_s^a \cdot e^{s\theta_a} \left| p^{(a)} \right\rangle$$

where the rapidity dependence is a consequence of the correct Lorentz transformation property for Q_s. It is also easily seen that the charge q_1^a corresponding to spin 1 is purely the mass, m_a of the particle state. The charge acts additively on multi-particle states for obvious asymptotic reasons. Again, the one-particle domination effect near $\Theta = i\theta_{ab}^{\bar{c}}$ places restrictions on the conserved quantity eigenvalues since $Q_s|p^{(a)}, p^{(b)}\rangle \approx Q_s|p^{(\bar{c})}\rangle$ gives a set of relations for the quantum numbers of any three particles involved in a 'fusing':

$$q_s^a e^{-is\bar{\theta}_{ac}^{\bar{b}}} + q_s^b \cdot e^{is\bar{\theta}_{bc}^{\bar{a}}} = q_s^{\bar{c}}. \tag{1.10}$$

These are known as the charge bootstrap equations.

As has already been mentioned in the classical theory, the masses calculated from the mass matrix, form an eigenvector of the Cartan matrix, associated with the lowest eigenvalue. There is in fact a generalisation of this in the quantum theory, for the simply-laced cases (and $a_{2n}^{(2)}$). Using the classically allowed angles derived from the non-zero, three-point couplings, Klassen and Melzer verified originally on a case-by-case basis, that the eigenvalues of the conserved charges for a given spin s solved the charge bootstrap equations and could be assembled into a single vector,

$$\underset{\sim}{q}_s = (q_s^1, \ldots, q_s^r)$$

such that this was an eigenvector of the Cartan matrix:

$$C\underset{\sim}{q}_s = \mu_s \underset{\sim}{q}_s \qquad\qquad \mu_s = 4\sin^2\frac{S\pi}{2h} \tag{1.11}$$

This should not be surprising and in essence may be considered a classical fact for the simply-laced cases, because here the mass ratios do not renormalise (at least to one loop) and so the fusing angles do not move away from their respective classical values.

It should be noted that equation (1.11) is only a statement about the ratios of the eigenvalues of the conserved charges and the actual eigenvalues will be scaled by an unknown function of s and the coupling constant β.

Details with regard to the conserved charges associated with the non-simply-laced cases may be found in [19], where it is described how the folding of the Dynkin diagram gives rise to the relevant reduced set of spins, for the untwisted and twisted cases, again these are the exponents of the respective Lie algebras.

1.3.1 The S-matrices for the Simply-Laced Affine Toda Field Theories

For these cases the classical data appears to be sufficient to generate a solution to the S-matrix bootstrap which satisfies the additional constraints of unitarity and crossing. What is not so obvious however, is how these S-matrix elements could possibly be created from first principles, starting with the Lagrangian (1.03) and this still remains an open question. Perturbation theory is manifestly inadequate for such a task and so a new approach is required, maybe in the direction of quantum inverse scattering (also referred to as the Quantum Inverse Spectral method).

Firstly, the conjectured S-matrix elements will merely be listed and then a brief description of their immediate properties will be given. Their explicit construction can be found in many papers[4],[5],[7],[8],[11]. They are,

$$a_n^{(1)}: \qquad S_{ab}(\Theta) = \prod_{\substack{|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{x\}$$

$$d_n^{(1)}: \qquad S_{ab}(\Theta) = \prod_{\substack{|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{x\}\{h-x\}$$

$$S_{ss}(\Theta) = S_{s's'}(\Theta) = \prod_{\substack{1 \\ \text{step } 4}}^{h-1} \{x\}$$

$$S_{ss'}(\Theta) = \prod_{\substack{3 \\ \text{step } 4}}^{h-3} \{x\}$$

$$S_{sa}(\Theta) = S_{s'a}(\Theta) = \prod_{\substack{0 \\ \text{step } 2}}^{2a-2} \{n-a+x\} \qquad\qquad a = 1, 2, \ldots, n-2$$

and for $e_n^{(1)}$; consult [19].

The S-matrices for the ADE algebras are, as can be seen, constructed from a universal building block,

$$\{x\} = \frac{(x+1)(x-1)}{(x+1-B)(x-1+B)} \tag{1.12}$$

where $$(x) = \frac{\sinh(\frac{\Theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\Theta}{2} - \frac{i\pi x}{2h})} \tag{1.13}$$

and $B = B(\beta)$; $\{x\}$ has the following properties: $\{0\} = \{h\} = 1$, $\{x \pm 2h\} = \{x\}$, $\{-x\} = \{x\}^{-1}$. The object (x) is a unitary block that contains no implicit dependence on the coupling constant β.

Deletion of all the 'unitary' blocks that contain dependence on β through B, results in another solution to the bootstrap and S-matrix constraints, known as the 'minimal' matrix. In fact this minimal S-matrix was what was initially constructed by certain groups, for example [5], using the classical data and the dependence on β added later, since classically the coupling plays no rôle at all. The dependence is added in such a way, that taking the limit $\beta \to 0$ a free field theory is obtained and so the S-matrix elements will tend to unity. It must also be such that no further poles are added in the physical strip and the residues of the odd order poles of the scattering matrices are consistent with perturbation theory. The postulate

$$B(\beta) = 2\beta^2(\beta^2 + 4\pi)^{-1} \tag{1.14}$$

due to Arinshtein, Fateev and Zamolodchikov[4] for the $a_n^{(1)}$-series over a decade ago, appears to be true for $d_n^{(1)}$ and the simply-laced exceptional algebras also, as suggested by low order perturbation theory[6],[31].

Clearly $\{x\}_B = \{x\}_{2-B}$ and, coupled with the fact that $B(\frac{4\pi}{\beta}) = 2 - B(\beta)$, this implies the property,

$$S(\Theta; \beta) = S(\Theta; \frac{4\pi}{\beta}) \tag{1.15}$$

and hence, there is a clear symmetry between weak and strong coupling, which is exactly what would be expected if A.T.F.T's were to describe perturbed conformal field theories, since the central charge in (1.02) exhibits this exact same symmetry as well.

For the $d_n^{(1)}$-series, it is manifest that for n even, all of the S-matrices are crossing symmetric representing the fact that all the particles are self conjugate. Whereas, in the case of n odd, $S_{ss'}$ and S_{ss} interchange under the crossing relation, since (s', s) are regarded as the anti-particles of one another, that is, as a conjugate pair. Similar properties hold for the other algebras; in the cases of $a_n^{(1)}$, the particles labelled $(a, n+1-a)$ are a conjugate pair; only $e_6^{(1)}$ from the exceptional algebras has any such pairs. Another major fact which becomes apparent upon examination of these scattering matrices, is that upon closing the bootstrap, poles of order higher than one have been produced in many of the formulae, in addition to those higher order poles corresponding to fusings. However, it is proposed that all these may be explained away purely on the basis of perturbation theory and a detailed example of this concerning $d_6^{(1)}$ may be found in [5], where the double and fourth order poles occur at values of rapidity not corresponding to a particle mass. The mechanism for all double and triple poles of the simply-laced cases may be found in [6]. Examination of all simply-laced cases shows that this is true for all double order poles in the conjectured S-matrices, which gives some credence towards the fact that the bootstrap principle is consistent when only the odd order poles are considered.

The Clebsch-Gordon property also appears to hold when Feynman diagrams at higher order are considered, since they cancel when all the particles are taken on-shell, in all the cases that have been examined.

1.3.2 The S-matrices for the Non-Simply-Laced Affine Toda Field Theories

Initial investigations into the structure of the S-matrices in these cases portrayed a distinct difference between those associated with the simply-laced algebras. For a start, the fact that the mass ratios were not renormalization invariant meant that the field theory connection was problematical. A purely formal solution to the bootstrap was presented in [5], but in all cases it appeared as though little, if in fact any, of the multipole structure could be explained within perturbation theory based on the affine Toda Lagrangian. A distinct difference was noted between the twisted and the untwisted cases, since in the cases arising from folding using the extra symmetry of the affine diagram, "the S-matrix is a subset of the parent theory, closing under the bootstrap to form a sort of subalgebra", but there again the pole structure appeared inexplicable on the reduced set of particles. It could have been suggested that the classical integrability broke down at the quantum level in these models due to anomalies and hence, factorizable, elastic S-matrices did not exist for these Toda theories. However, this was not the case as Delius, Grisaru and Zanon[32] were able to construct higher spin conserved currents at the quantum level, for both the simply-laced and non-simply-laced Toda theories, and so the difficulties in constructing S-matrices for the non-simply-laced cases were not because of such anomalies.

The insightful jump that was required to solve this problem was provided again by this group[9], based upon the speculation that one merely had to give up the idea that the blocks (x) which determined the pole positions satisfied the bootstrap independently of $(x \pm B)$ and inherited instead an implicit β dependence. Therefore, there should be no 'minimal' β - independent S-matrix associated with these models. A new 'hybrid' block was the key which could take care of such things as the mass distortion and multipole structure. In fact, the existence of coupling constant dependent quantum corrections particular to each mass ratio, should have pointed the way to coupling constant dependent poles in the numerator blocks. A consistent construction for the S-matrices for $b_n^{(1)}$, $c_n^{(1)}$, $a_{2n-1}^{(2)}$, $d_{n+1}^{(2)}$ and $g_2^{(1)}$ was given in [9], with the remaining algebras $d_4^{(3)}$, $e_6^{(2)}$, and $f_4^{(1)}$ dealt with later by different groups[33]. The conclusion Delius et al. came to was that the construction could be achieved with blocks very similar to the (x) used previously, but with one important difference; the Coxeter number which appeared in the mass formulae and blocks, was to be substituted by a coupling constant dependent Coxeter number $H(\beta)$. In fact, what was really happening was that coupled with a renormalisation of the mass scale (to take account of the divergent tadpole diagrams which also occur in the simply-laced cases), in order to have a finite quantum theory, there is also a shift in the vacuum expectation values of the fields, which leads to a renormalisation of the Kač labels. The bare forms of the mass scale and Kač labels are taken in order that the quantum Lagrangian coincide with the classical Lagrangian (and with normal ordered exponentials of course). Grisaru et al. constructed the S-matrices by similar routes as those used in the simply-laced cases, but with two manifest differences: the S-matrix had simple particle poles at positions shifted away from the classical mass values by radiative corrections and the bootstrap principle was altered since a few of the simple poles were 'shifted' away from their respective single particle positions due to anomalous Landau singularities. (Feynman diagrams in two-dimensions that have all internal propagators on-shell).

In [9] a very explicit example is given, the case of $a_{2n-1}^{(2)}$ and the authors show how after having satisfied crossing symmetry, $S(\Theta, \beta = 0) = I$, tree level amplitudes and no additional poles on the physical strip, the large number of extra poles that have been produced (in the sense that they can not be explained in perturbation theory) may be avoided by cancelling the relevant blocks to obtain a minimal number for the S-matrix. This is achieved by choosing constraints for the functions of β involved in the bootstrap generated matrices. After the cancellations, the simple poles correspond to the correct fusings at the β - dependent angles (obtained by consideration of the Mandelstam quantities) in the s, u channels; moreover, there are several double poles which can all be explained by the Landau diagrams.

The manipulation of notation allows the S-matrices for the $a_{2n-1}^{(2)}$ case to be written in terms of:

$$\{x\}_H = \frac{(x-1)_H(x+1)_H}{(x-1+B)_H(x+1-B)_H} \quad \& \quad (x)_H = \frac{\mathrm{sh}(\frac{\Theta}{2}+\frac{i\pi x}{2H})}{\mathrm{sh}(\frac{\Theta}{2}-\frac{i\pi x}{2H})}$$

where $H = h + \frac{B}{2}$, such that $B = -2h\frac{\varepsilon}{1+\varepsilon}$ and the renormalised mass conjecture has been taken as,

$$\hat{m}_a = 2\hat{m}\sin(\frac{a\pi}{h}(1+\varepsilon(\beta))) \equiv 2\hat{m}\sin(\frac{a\pi}{H})$$

So, it can be seen that the renormalised masses have the equivalent classical mass form, but $h \to H(\beta)$, that is, the Coxeter number of the Lie algebra has been replaced by the 'coupling constant dependent Coxeter number'.

Subsequent perturbation theory on the respective classical Lagrangian was able to check agreement with such considerations as coupling constant dependence, pole coefficients, tree level-agreements and that for corrections to the masses at loop-level. The conjecture $B(\beta) = 2\beta^2(\beta^2 + 4\pi)^{-1}$ was again consistent with Delius's results, achieved by comparing a specific residue from the exact S-matrix to the respective one loop calculation of the residue of a single particle pole and hence, the 'universal' simply-laced coupling constant dependence was manifestly evident here as well.

Similar results were found in all the cases that were examined by Grisaru et al., apart from the appearance of the S-matrices for $g_2^{(1)}$, which could not be written in terms of the block {x}, but (x) only.

However, this was not quite the end of the story for the field theories based on non-simply-laced algebras. As was well known, the S-matrices for the simply-laced theories exhibit an invariance under the so-called 'weak-coupling, strong-coupling' exchange $\beta \to \frac{4\pi}{\beta}$, since $(x-1+B)(x+1-B)$ is invariant, if $B(\beta)$ is as conjectured. The transformation in this context should produce something altogether different though, since (x) has an implicit β dependence. Examination of the $a_{2n-1}^{(2)}$ 'renormalised' Coxeter number showed that it transformed

$$H_{a_{2n-1}^{(2)}} = 2n-1+\frac{B}{2} \to 2n-1+\frac{2-B}{2} = 2n-\frac{B}{2} = H_{b_n^{(1)}}$$

to that of the 'renormalised' Coxeter number associated with the algebra $b_n^{(1)}$. Moreover, $S_{nn}^{a_{2n-1}^{(2)}} \to S_{nn}^{b_n^{(1)}}$ and hence, because of the fact that all the other elements are generated through the bootstrap, this showed that the $a_{2n-1}^{(2)}$ and $b_n^{(1)}$ S-matrices switched under $\beta \leftrightarrow \frac{4\pi}{\beta}$. Similarly, $c_n^{(1)}$ and $d_{n+1}^{(2)}$ also switched and at $\beta^2 = 4\pi$ are equivalent. So what in effect was occuring was that the low β coupling aspect of one non-simply-laced theory became that corresponding to the high β coupling for another 'dual' theory and vice versa.

Following the work of Watts and Weston[34], Cho, Koh and Kim[35], it became clear that this 'duality' property was very fundamental. The word 'duality' refers to the fact that the non-simply-laced affine root systems fall into dual pairs under the transformation $\alpha_i \to \hat{\alpha}_i = \frac{2\alpha_i}{|\alpha_i|^2}$, namely, $(b_n^{(1)}, a_{2n-1}^{(2)})$, $(c_n^{(1)}, d_{n+1}^{(2)})$, $(g_2^{(1)}, d_4^{(3)})$and $(f_4^{(1)}, e_6^{(2)})$, excepting the case of $a_{2n}^{(2)}$ which is self dual, as are all the simply-laced systems.

Hence generalising, what in effect existed was that for each dual pair of root lattices there existed a parameterized set of quantum field theories (the parameter lying between the Coxeter numbers of the dual pair, since $B(0)=0$, $B(\infty)=2$). The masses of the Q.F.T's 'float' between the classical masses in each dual pair, as β changes and the 'two conjectured S-matrices' are just manifestations of the S-matrix for this parameterised field theory. $[S(\beta = 0) \equiv S(H = h^{(1)}) = S(H = h^{(2)}) \equiv S(\beta = \infty) = I]$.

Following on this, Corrigan et al.[33] provided the missing ($f_4^{(1)}$, $e_6^{(2)}$) case and proposed a 'generalised bootstrap principle'. This principle gave a restriction on the poles of the S-matrix which could be used in the bootstrap. The conjecture was that only odd order poles in the physical strip with positive coefficients throughout the parameter range could participate. This was consistent with all the S-matrices and, furthermore, all other singularities were explained by them, 'at least in principle', to be variations upon the theme of a Coleman-Thun mechanism[36]. The latter first appeared in the literature in order to explain the double poles in the breather scattering matrix elements of the sine-Gordon model, ($a_1^{(1)}$ - Toda field theory with imaginary coupling constant). The introduction of a new, more generalised form of block:

$$\{x\}_v = \frac{(x - vB - 1)(x + vB + 1)}{(x + vB + B - 1)(x - vB - B + 1)} \tag{1.16}$$

allowed the 'dual-pair' S-matrices to be written in a form more akin to that of the self-dual cases. However, the drawback was that the new block did not possess such an easy manipulation with respect to the bootstrap. Furthermore, the new notation helped with the problem that many 'zero / pole' cancellations were hidden in the older way of writing the matrices. It can be briefly noted that the case of $a_{2n}^{(2)}$ fits in exactly with all that has gone before, even though it is a non-simply-laced algebra, the classical and quantum mass ratios are equivalent[7] and 'floating' due to uneven renormalisation does not occur[5].

The Coleman-Thun mechanism and the generalisations which have been found, can as already stated, at least provide examples of how all the semi-positive simple, (double) and cubic poles, arise in terms of the field theory and are given at the end of [33]. It will suffice to say that it is the 'floating' zeros of the S-matrix elements that provide the crux of an explanation for the eventual 'many-signed' poles in these matrices. As noted, the fact that these zeros have a rôle is tantamount to exaggerating the non-perturbative nature of all these mechanisms. The scattering matrices for the dual pairs are in some respects similar to those of the sine-Gordon S-matrices for breathers. Perturbation theory would have to be extrapolated to a very high order in order to see the full effect of the 'floating' over its Coxeter bounded range, quite unlike the case of the self-dual theories. There, the existence of fixed position multi-poles is explicable in standard perturbation theory and the coefficients of the poles calculated to any finite order in β.

Nevertheless, it should be pointed out that despite the differences, the common ground for all the Toda theories is that the bootstrap can be performed on those odd order poles that possess a unique sign in the physical strip and leads in a consistent way to candidate S-matrices.

The question may well be posed as to whether it is possible to find any other 'related features' between these two superficially very different classes of scattering matrix elements. In fact, a recent publication by Dorey[37] has made some headway into resolving this problem. A 'naturally dual' notation is introduced, which enables both types of S-matrix to be written in terms of a new 'interpolating' block. Simultaneously a generalisation of the 'simply-laced' $B(\beta)$ is noted to be appropriate in the context of these blocks and for theories based on a general semi-simple Lie algebra as a whole.

The way the new function of β has been introduced, has been to define a function specific to each Lie algebra that satisfies the same two boundary conditions as B, that is, $B(0)=0$, $B(\infty)=2$, but supplemented by the fact that all the poles and zeros in the S-matrices corresponding to that Lie algebra, must depend linearly on this new function. Since the poles 'float' with β, then it is sufficient to check a single pole to fix this new $B^{[g]}$ and it turns out that for all cases the function is given by:

$$B^{[g]}(\beta) = 2\beta^2 \cdot (\beta^2 + 4\pi\frac{h}{h^{\vee}})^{-1} \tag{1.17}$$

where $h^{\vee}$ denotes the dual Coxeter number. This function satisfies the new dual property

$$B^{[g]}(\beta) = 2 - B^{[g^{\vee}]}(\frac{4\pi}{\beta}).$$

The interpolating block is defined to be:

$$<x, y> \ = \ <(2-B)\frac{x}{2h} + (B)\frac{y}{2h^{\vee}}> \tag{1.18}$$

where $$<x>=\frac{\mathrm{sh}(\frac{\Theta}{2}+\frac{i\pi x}{2})}{\mathrm{sh}(\frac{\Theta}{2}-\frac{i\pi x}{2})}$$

so that as $\beta\to 0$, $<x,y>\sim(x)_h$ and as $\beta\to\infty$, $<x,y>\sim(y)_{h^\vee}$. In order for the correspondence with the classical data to hold, the two indices x and y will label integer values since the pole positions should tend to integer multiples of $\frac{i\pi}{h}$ and $\frac{i\pi}{h^\vee}$ respectively. A new building block for the S-matrix elements can then be defined (which is akin to the block {x} in the simply-laced cases),

$$\{x,y\}=\frac{<x-1,y-1><x+1,y+1>}{<x-1,y+1><x+1,y-1>} \tag{1.19}$$

For the self-dual cases it merely reduces to $\{x,x\}=\{x\}$. As in the case of the old building block, $\{x,y\}\to I$ as either $\beta\to 0$ or $\beta\to\infty$.

Dorey succeeds in showing how the cancellations between superfluous physical-strip zeros and poles may then occur in terms of these blocks, resulting in a generalised object with the same number of poles and zeros as the new building block,

$$_a\{x,y\}_b=\frac{<x-a,y-b><x+a,y+b>}{<x-a,y+b><x+a,y-b>}. \tag{1.20}$$

As has already been stated, but repeated here to emphasise the point; for the non-simply-laced cases (bar $a_{2n}^{(2)}$) the zeros and poles are not independent, but go through the bootstrap as it were, 'hand-in-hand' and hence, a minimal 'β-independent' S-matrix cannot be defined here. In the simply-laced cases they are independent and therefore, there is a minimal S-matrix. All of the S-matrix elements for all 'simply-laced and non-simply-laced' affine Toda field theories can then be written either as a product of these new building blocks {x,y} on their own, or, after the cancellations, in terms of the more sophisticated blocks $_a\{x,y\}_b$. So there exists a unified way of looking at the scattering matrices for all affine Toda field theories and hence it has become evident that the non-simply-laced S-matrices have a great deal more in common with those of the simply-laced cases than was realised when these objects were first analysed.

The duality of these blocks works very simply and comes as a direct consequence of the dual property noted before given to B[9]. From this it can be seen that the elements for the dual theory are transcribed from those of the original blocks by simply interchanging the labels x and y and any other suffices, hence

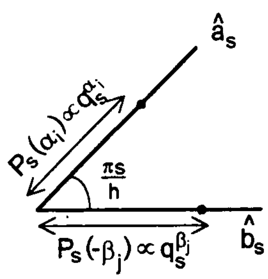
$$_a\{x, y\}_b \longleftrightarrow {_b\{y, x\}_a}$$

Since there now exists some common ground for the S-matrices, it would be an interesting investigation to see if the new formulae could be 'recast' in a more group theoretical or algebraic way, as has already been done for those of the simply-laced cases[12], and which will be covered in the following section. As was hinted at by Dorey, the way forward may lie in a geometrical interpretation.

1.3.3 Further Developments in the Quantum Regîme

It has already been mentioned in the section on classical theory (section 1.2) that the masses in one of the 'untwisted' A.T.F.T's can be ordered such that they form an eigenvector of the corresponding Cartan matrix. It turns out that in the quantum field theory, a generalisation of this occurs for the simply-laced algebras, and the higher conserved charges q_s^a become components of vectors which mimic the remaining eigenvectors of this matrix. That is $C_{ab}q_s^b = \lambda_s q_s^a$, where $\lambda_s = 2 - 2\cos\frac{\pi s}{h}$ [11],[19],[21].

At least for the simply-laced cases, Dorey's rule for the three point couplings becomes quantum mechanically equivalent to the bootstrap equations and for fusings obeying this rule, it can be shown that these bootstrap equations are automatically satisfied by the eigenvectors of the Cartan matrix[12]. Hence, the components of these vectors become candidates for the set of conserved charges; both the eigenvectors and the spins of the charges being labelled by the exponents of the relevant Lie algebra modulo the Coxeter number. This is achieved by defining vectors $a_s = \sum_{\circ} q_s^{\alpha_i}\hat{\alpha}_i$, $b_s = \sum_{\bullet} q_s^{\beta_i}\hat{\beta}_i$, for $s \in \{$ exponents of g$\}$ where again the bicolouration is used, (such that the q_s^i are the components



$|\hat{a}_s| = |\hat{b}_s|$

of the eigenvector corresponding to λ_s and $\{\hat{\alpha}_i, \hat{\beta}_j\}$ is the dual basis to the simple roots) which span a 2-plane in $\Re^r$ that the dihedral group $<\prod_{\circ}\omega_{\alpha_i}, \prod_{\bullet}\omega_{\beta_i}>$ acts 'naturally' on, in the sense that ω acts as a rotation through an angle $\frac{2\pi s}{h}$ in each subspace, from a_s to b_s. It is clear that the period of ω is the Coxeter number (using $s=1$). The aftermath of this is that the projections of the simple roots into this 's^{th}' subspace have components that are proportional to the $q_s^{\,i}$, that is the components of the eigenvectors of the Cartan Matrix,

$$P_s(\alpha_i) = q_s^{\alpha_i} \cdot \hat{a}_s \quad \& \quad P_s(-\beta_j) = q_s^{\beta_j} \cdot \hat{b}_s$$

noting that it is the 'signed' simple roots that are important in this context and that $\{\hat{a}_s, -\hat{b}_s\}$ are vectors dual to $\{a_s, b_s\}$ in the 2-plane.

Hence, the overall result is that the projections $P_s(\omega^p\alpha_i)$ and $P_s(\omega^p(-\beta_j))$ lie on the rotation of $\hat{a}_s$, $\hat{b}_s$ respectively, by $\frac{2\pi ps}{h}$ for each s. The case $s=1$ where h-gons are obtained fixed in circles of radii given by the masses of the particles, allows the conclusion that $\{\omega^p\gamma_i\,;\ p=0,\ldots,h-1\,;\ i=1,\ldots,r\}$ gives the set of all roots Φ of g. Generally, there exists a complete description of the projections of all the roots into the ω-invariant subspaces and the fact that $\omega^{h/2}(-\alpha)$ and $\bar{\alpha}$ have a common projection onto every invariant subspace is sufficient information to define charge conjugation for the particles.

This rule which corresponds with the couplings[5],[7] in the field theory, then gives a solution to the bootstrap equations since the closing of a 'root-triangle' $(\omega^i\gamma_i + \omega^j\gamma_j + \omega^k\gamma_k = 0)$ in $\Re^r$ projects to a closed triangle in each subspace. Noting that the charge bootstrap equations (1.10) can be rewritten as:

$$q_s^a + q_s^b\, e^{isU_{ab}^c} + q_s^c\, e^{is(U_{ab}^c + U_{bc}^a)} = 0$$

where U are the fusing angles, they can be equated with the projections (since true for $s=1$), to give the required solution to the bootstrap in terms of the relevant eigenvector of the Cartan Matrix. Moreover, this bootstrap is solved for all non-vanishing couplings simultaneously since the charge attributed to each particle is independent of the others.

The coupling rule answers many more questions however, that have been raised following a purely calculational approach to the models. For example, it allows the fusing angles for each coupling to be calculated in terms of the indices that the Coxeter element is raised to in each coupling. Moreover, as a by-product of the fact that roots corresponding to different colours are projected onto different axes in the 2-planes (and the intervening angle being an odd multiple of $\frac{\pi s}{h}$), it is easily seen that, irrespective of the bound state particle produced, two particles of the same colour must fuse at an even multiple of $\frac{\pi}{h}$ and those of opposite colour at an odd multiple of $\frac{\pi}{h}$. Hence, this verifies the so-called 'brick wall' observation[19] that in all S-matrix elements $S_{ij}(\Theta)$, all the simple poles, which correspond to bound state formation, are spaced by even multiples of the basic angle $\frac{\pi}{h}$.

Dorey's fusing rule has also been analysed in a thorough manner by Fring and Olive[38], who give a more detailed analysis of the structure of the rule, alternative solutions and formulations, and naturally the relation to the quantum conservation laws. Emphasis is placed upon the fact that given one solution to the fusing, there always exists another inequivalent solution, namely,

$$\sum_{t=a,b,c} \omega^{\left(-\xi(t)+\frac{c(t)-c(i)}{2}\right)} \cdot \gamma_t \equiv \sum_{t=a,b,c} \omega^{\xi'(t)} \cdot \gamma_t = 0 \tag{1.21}$$

where, c(i) is arbitrary, and where $(\xi(a), \xi(b), \xi(c))$ is the triplet of Coxeter element integer powers for the first equivalence class. This can then be shown to lead to a contradiction if the two classes are equated and hence, they must be different. (The properties used in deriving this are simply the identity $\omega_{c(i)}\omega_{c(j)} = \omega^{\frac{c(j)-c(i)}{2}}$, $\omega_{c(j)}\gamma_j = -\gamma_j$ and $\omega_{c(i)}\omega^{P} = \omega^{-P}\omega_{c(i)}$). These two versions of the fusing rule, together with the ones used in proving the Clebsch-Gordon property of the coupling, $0 = \sum_{t=a,b,c} \omega^{-\xi'(t)} \cdot \lambda_t$ and $0 = \sum_{t=a,b,c} \omega^{-\xi(t)} \cdot \lambda_t$ where the λ's are the fundamental weights of the Lie algebra associated with the particles, all play a crucial rôle in establishing properties of a hypothetical 'general' S-matrix to be reviewed shortly.

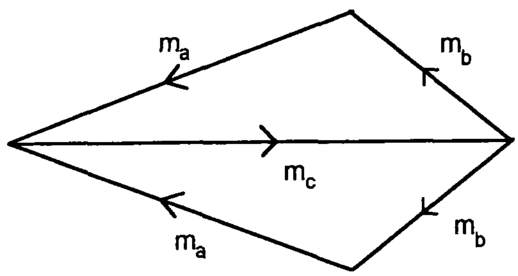
Fring and Olive's analysis of the connection with the conservation laws, though much less geometrically orientated, results in exactly the statement that Dorey made. This appears by expanding the eigenvector of the Coxeter element in terms of those of the Cartan matrix:

$$\omega \underset{\sim}{x}_s = e^{\frac{2\pi i s}{h}} \underset{\sim}{x}_s \tag{1.22}$$

where $$\underset{\sim}{x}_s = \frac{1}{\sqrt{2}\,\sin\theta_s}\left((\sum_{\bullet} q_s^{\alpha_i}\alpha_i) + e^{\frac{i\pi s}{h}} (\sum_{\circ} q_s^{\alpha_i}\alpha_i) \right) \tag{1.23}$$

and again s is an exponent of the algebra. Taking the scalar product of this with the fusing equation then yields the equations which can be regarded as conservation laws for the three particle process allowed by the coupling c_{abc}. Where their discussion then differs is in pointing out that only r (the rank) linearly independent conservation laws appear as a result of the three particle process. They are therefore, manifestly inadequate for representing the integrability of the theory. The conservation laws for the negative exponents are linearly related to the ones for positive exponents. Those of any spin equal to an exponent mod h are proportional to one of the 'r' fundamental equations when considering a three particle process; so there exists a curtailed system of conservation laws to that probably expected. The conservation laws are re-interpreted as components of a generalised energy momentum conservation for the process $a+b+c \to 0$ allowed by the fusing rule. Viewing this geometrically (as Olive[38] did) it is then obvious that there are two, and only two, equivalence class solutions when the coupling is non-zero, since there are only two triangles which may be constructed from three lengths of a given magnitude in an orientated manner.

(s=1)



Another by-product of the fusing rule is, given the fact that the three masses of the coupling particles must form a closed triangle, then by the triangle inequality it is evident that any one of these particle will be stable with respect to the decay into the other two antiparticles permitted by the coupling. This is simply because its mass is less than the sum of the other two, and hence, will be disallowed on the grounds of the kinematics involved. Until [12] appeared, the S-matrices for the simply-laced A.T.F.T's had been constructed on a case-by-case basis, and even though certain notation had been introduced which gave them all a common ground, there had been no breakthrough in writing them in an algebra-independent way. That is, in such a manner that only depended on the much more abstract root systems and representation theory of the underlying Lie algebra. The first attempt (as far as is known) that resulted in general expressions for the S-matrix elements for any simply-laced A.T.F.T. was established in a 'colour' dependent way. It was then demonstrated that they satisfied the bootstrap equations, the colour dependence being inherent because of the form the fusing equations took. The proposed expressions for the general S-matrix element in the ADE scattering theories again had a manifest dependence upon the action of the Coxeter element in the space of roots of a Lie algebra, which had already occurred in describing the coupling rule in a more group theoretical way. The properties that the S-matrix elements had to satisfy, unitarity, crossing etc., are then reduced to identities satisfied by the inner products associated with the root and weight lattices, such as,

$$\left.\begin{aligned}
(\hat{\alpha}_i, \omega^p \alpha_j) &= (\hat{\alpha}_j, \omega^p \alpha_i) = -(\hat{\alpha}_i, \omega^{-p-1} \alpha_j) \\
(\hat{\alpha}_i, -\omega^p \beta_j) &= (\hat{\beta}_j, \omega^p \alpha_i) = -(\hat{\alpha}_i, -\omega^{-p} \beta_j) \\
(\hat{\beta}_i, -\omega^p \beta_j) &= (\hat{\beta}_j, -\omega^p \beta_i) = -(\hat{\beta}_i, -\omega^{-p+1} \beta_j)
\end{aligned}\right\} \quad (1.24)$$

where α, β denotes the bicolouration; α_i, β_i, the simple roots; $\hat{\alpha}_i$, $\hat{\beta}_i$ the dual roots (they are fundamental weights for the simply-laced algebras) and ω the Coxeter element. Merely the fact that the conjectures satisfied the bootstrap however, gave no evidence towards establishing that the correct physical pole structure was reproduced. The checks that were made with the previously calculated ADE S-matrix formulae centred on showing that the expansion of the supposed Coxeter element in terms of inner products in a basis of the simple roots was consistent with the properties which were supposed to hold. That is $\omega^h = 1$, and that the characteristic polynomials of the matrices were in agreement with those of the known Coxeter element[39]. It was not until the conjectured formulae were written in one encompassing way that the physical pole structure was eventually elucidated. In [13],[38] general formulae for the purely elastic S-matrices were proposed which resulted in a derivation of many of the properties of the ADE matrices in a universal manner. Curiously, it was noted that the crossing and bootstrap conditions could be verified in a general way which were valid for any Lie algebra, whether simply-laced, untwisted non-simply-laced or twisted, but the property of analyticity required the extra condition that restricted analysis solely to the simply-laced algebras. More importantly, attention was paid to the structure and positions of the poles in the proposed formulae and it was shown that the poles of odd order were the relevant ones to indicate bound states in the quantum theory. Furthermore, they possessed residues with fixed signature. Various universally observed features of the S-matrices were in effect, simple consequences of the properties of the root systems.

Moreover[13], a new labelling of the orbits of the Coxeter element based on the work of Kostant[40] was utilized by Dorey. The main result was that if Γ_i labels the orbit of 'the alternative roots' ϕ_i under $<\omega>$, then the Γ_i's are disjoint, each having h elements and thus their totality is the set of all roots Φ, of magnitude $rh = (\dim g) - r$. (The fundamental weight λ_i corresponding to the simple root α_i can then be seen to be related by $\phi_i = (1-\omega^{-1})\lambda_i$).

Using these facts together with some convenient manipulation of the fundamental building block, the expression for the two-particle S-matrix that had been introduced before (albeit in a colour dependent way), can be rewritten compactly as

$$S_{ab} = \prod_{p=0}^{h-1} \{2p+1+u_{ij}\}_{-}^{(\lambda_a,\omega^{-p}\phi_b)} \quad . \tag{1.25}$$

Here the block,

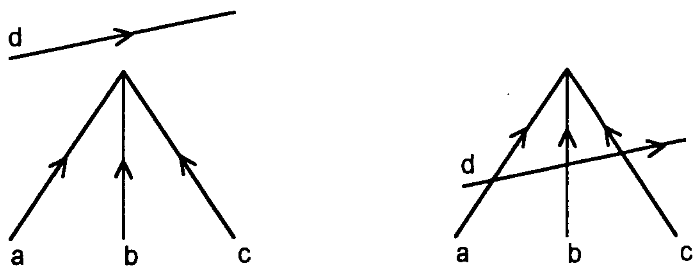
$$\{x\}_{-} \equiv \{-x\}_{+}^{-1} \equiv \frac{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(x-B+1))\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(x+B-1))}{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(x+1))\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(x-1))}$$

is used for convenience in the discussion of the pole structure in the S-matrix formulae; $u_{ij} \equiv u(\phi_i, \phi_j)$ is defined such that $\frac{\pi s}{h}.u_{ij}$ gives the (signed) angle between the projections of the roots ϕ_i ,ϕ_j into the ω^s eigenspace of the Coxeter element. Unitarity is then checked using $(\lambda_i, \omega^{-p}\phi_j) = -(\lambda_i, \omega^{p+1+u_{ij}}\phi_j)$ and symmetry via $(\lambda_i, \omega^{-p}\phi_j) = (\lambda_j, \omega^{-p-u_{ij}}\phi_i)$. (The coupling dependent part plays no rôle in the analysis of the pole structure).

The fact that the formulae satisfy the S-matrix bootstrap is most easily seen by rewriting the bootstrap in a more succinct way. The crossing and unitarity identities allow it to be transformed to

$$S_{da}(\Theta)S_{db}(\Theta+i\theta^{c}_{ab})S_{dc}(\Theta-i\theta^{b}_{ac})=1 \tag{1.26}$$

which has a pictorial representation:



(and is analogous to the rewritten conserved charge bootstrap equations).

By defining $(\tau_y f)(\Theta)=f(\Theta+\frac{i\pi y}{h})$, the bootstrap may be written as

$$(\tau_{u(\phi_d,\alpha_{(a)})}S_{da})(\tau_{u(\phi_d,\alpha_{(b)})}S_{db})(\tau_{u(\phi_d,\alpha_{(c)})}S_{dc})=1$$

where the $\alpha_{(i)}\in\Gamma_i$ such that $\alpha_{(a)}+\alpha_{(b)}+\alpha_{(c)}=0$. (The fusing relation). The bootstrap equations in this form are then seen to be immediately satisfied, from the properties of the block $\{x\}_-$, the fact that the scattering matrix element can again be rewritten as,

$$S_{ab}=\prod_{\alpha_{(b)}\in\Gamma_b}\{u(\phi_a,\alpha_{(b)})+1\}_-^{(\lambda_a,\alpha_{(b)})} \tag{1.27}$$

and given that the fusing relation holds.

The S-matrix formula (1.27) is readily open to a detailed examination of the physical pole structure and has been forthcoming in providing algebraic answers to questions that were left 'in situ' after study of the case-by-case calculations. For example, when the S-matrix elements were pictorially represented by 'blocks'[19], it became apparent that the positions of expected forward or crossed channel poles precisely corresponded to the downhill or uphill sections of the wall, which portrayed a very-well-established hypothesis between the fusing structure and value of pole residue. This was that the poles of odd order always have an interpretation in terms of the production of a bound state, the downhill poles with a '+i' residue being forward channel, whereas uphill with '–i' residue being crossed channel. (The nomenclature 'residue' referring to the coefficient of the most singular part). Moreover, examination of the ADE elements in the case-by-case style made it apparent that the height of the wall never changed by more that one unit; excepting the points 0, $i\pi$ of the physical strip, where the S-matrix is analytic and change is from $\pm 1 \to \mp 1$.

All these facts possess an explanation within the context of the conjectured general S-matrix formula (1.27) and the properties of the simply-laced root systems. Furthermore, a total examination of the analytic structure is possible.

As $\alpha_{(b)}$ runs through Γ_b, the quantity $u(\phi_a, \alpha_{(b)})$ lies between 0 and h, and given the definition of the block, S_{ab} may have poles such that $\frac{-i\pi}{h} \le \frac{i\pi}{h} \cdot u(\phi_a, \alpha_{(b)}) \le i\pi + \frac{i\pi}{h}$. However, algebraic analysis has shown that any possible pole must lie inside the physical strip $(0 \le \mathrm{Im}(\Theta) \le \pi)$, since all the blocks contributing poles to locations outside this strip have zero exponents. The relevant parts of the S-matrix that contribute to a pole at $\frac{i\pi}{h}(\phi_a, \alpha_{(b)}) \equiv \theta_{ab}$ are,

$$S_{ab} = \cdots\cdots\{u(\phi_a, \omega\alpha_{(b)}) + 1\}^{(\lambda_a, \omega\alpha_{(b)})} . \{u(\phi_a, \alpha_{(b)}) + 1\}^{(\lambda_a, \alpha_{(b)})}_{\cdots\cdots\cdots} \qquad \text{(using the full block)}$$

$$\sim \cdots\cdots\cdots \left(\frac{+i}{\Theta - \theta_{ab}}\right)^{(\lambda_a, \omega\alpha_{(b)})} \cdot \left(\frac{-i}{\Theta - \theta_{ab}}\right)^{(\lambda_a, \alpha_{(b)})}_{\cdots\cdots\cdots} \qquad \text{(to the highest order of pole).}$$

Hence, given that $(\lambda_a, \alpha_{(b)}) - (\lambda_a, \omega\alpha_{(b)}) = ((1 - \omega^{-1})\lambda_a, \alpha_{(b)}) = (\phi_a, \alpha_{(b)})$, the change in wall height at this relative rapidity is $\delta h = (\phi_a, \alpha_{(b)})$, and the use of simple algebraic arguments gives the result required. That is $\delta h = -1 \Leftrightarrow$ a bound state in the forward channel, similarly $\delta h = +1 \Leftrightarrow$ a bound state in the crossed channel and $\delta h = \pm 2$ implies $\phi_a = \pm\alpha_{(b)}$ hence, a relative rapidity of 0 or $i\pi$. Similarly, the residue is a positive real multiple of $i^{-(\phi_a, \alpha_{(b)})}$ and accordingly gives residues that are proportional to $+i$ or $-i$ for forward or crossed channel poles.

The fact that the wall height is always positive in the physical strip can also be explained in terms of this new 'language' and refers to the fact that traversing each of the orbits Γ_a, it is the positive roots which correspond to the region bounded by 0 and $i\pi$ rapidity values, hence any inner product with a fundamental weight will obviously be positive. Similar results have been found by others[38], whose starting point was notationally slightly different. In that case also, very detailed checks were made to ensure symmetry in the indices, the crossing property held and that the bootstrap was satisfied. Again this was all based on properties of the block and identities satisfied by the Coxeter element. This group also noted the interesting fact, already briefly mentioned, that all the properties - apart from meromorphy - held equally well when the algebra under concern is non-simply-laced. The breakdown appearing from the fact that in the non-simply-laced cases, the powers to which the blocks are raised are not all integers, simply because of the variation in root length. Furthermore, they prove the equivalence between their conjectured formulae and that which appears in a vertex operator construction of the scattering matrix, due to Corrigan and Dorey[25].

$$S_{ab}(\Theta) = \prod_{q=1}^{h} \left\{2q + 1 + \frac{c(a) - c(b)}{2}\right\}^{\lambda_a \cdot \omega^{-q} \lambda_b} \tag{1.28}$$

As already briefly mentioned, at the end of section (1.2) the vertex operators are constructed to provide a representation of the exchange relation which contains the S-matrix of a real-coupling simply-laced affine Toda theory. The S-matrix is found to be the ratio of the factors that are left over after normal ordering of the pairs of vertex operators has taken place in the usual manner. Here again, the Coxeter element is found to play a very important rôle in the sense that it 'twists' the annihilation part of one of the vertex operators, which results in the destruction of all conformal character and hence, the eventual outcome of the non-trivial braiding relation.

For the Toda theories a suitable operator has been found to take the form:

$$V^a(\theta_a, \bar{\theta}_a) = V^{\lambda_a}(\theta_a)\bar{V}^{\lambda_a}(\bar{\theta}_a)W^{\lambda_a}(\theta_a)\overline{W}^{\lambda_a}(\bar{\theta}_a)$$

where the exchange of the V's provides a representation of one of the analytic continuations of the minimal S-matrix (off the $\bar{\Theta} = -\Theta$ submanifold) and the exchange of the W's is required in order to produce the extra coupling constant dependent part inherent in A.T.F.T..

$$V^{\lambda}(\theta) = e^{X_-^{\lambda}(\theta)} e^{X_+^{\lambda}(\theta)}$$

$$\bar{V}^{\lambda}(\bar{\theta}) = e^{\bar{X}_-^{\lambda}(\bar{\theta})} e^{\bar{X}_+^{\omega\lambda}(\bar{\theta})}$$

The V's have the above form, illustrating the placement of ω, which acts on the weight in the 'stringy' field X to shift the rapidity by $\frac{2\pi i}{h}$; similarly $\overline{W}$.

Another feature borne out of the use of the vertex operators in this manner was that the bootstrap relations for the S-matrices followed as a direct consequence of those for the conserved quantities, the 'delocalisation' having no effect in the sense that it was independent of particle type.

Interestingly, it was noted that the construction of classical soliton solutions for other field theories in terms of τ-functions involved the use of vertex operators with a similar structure[41],[42]. It does turn out that a basic 'building block' in the interaction terms for solitons associated with the Toda theories can be constructed using operators that are slightly different to the ones mentioned. This will be shown in section (4.3) and is claimed to be original in such a context. The 'full' answer to the problem of constructing such operators that create soliton solutions to the affine Toda field theory equations may be found in [16] and [56].

2. SOLITON THEORY

2.1 Solitons and Methods of Solution in Classical Field Theory.

Classical soliton theory deals with the study of localised 'lump-like' solutions of 'special' non-linear partial differential equations. Just to obtain a specific solution of a non-linear system is in general very difficult to do and so these soliton solutions are indeed special. There are equations that have become popular nowadays that exhibit chaotic properties in the sense that infinitesimal changes in any initial data propagate over a short period of time into wildly divergent evolutions.

Very few non-linear wave equations exhibit soliton solutions since the inherent dispersion effects would dissipate the lump solution eventually, but in the few equations where the dispersion effects are exactly balanced by the non-linearity there is the possibility of soliton solutions that do not collapse.

If, for the moment, the linear wave equation in two dimensions is considered,

$$\partial^\mu \partial_\mu \phi = 0$$

then this has the solution

$$\phi = \phi_1(x - ct) + \phi_2(x + ct)$$

and, imposing boundary conditions of $|\phi| \to 0$ as $|x| \to \infty$, any localised wave packet travelling with uniform velocity will satisfy these conditions. The wave packet will not disperse since all plane wave solutions to the wave equation have the same phase velocity $\frac{\omega}{k} = c$ and of course, the packet can be Fourier expanded in such a complete set.

Solitons are in a sense the equivalent of these wave packets for non-linear equations and, as such, also possess interaction properties (in the linear case due to the principle of linear superposition). Specifically, in considering classical relativistic field theories, the solutions in Minkowski space-time will possess finite energy with a localised energy density that does not disperse with time, and moreover, propagate with uniform velocity with little or no change in shape. In this sense the solitons are very much akin to particles, but are in fact solutions to non-linear wave equations. It was this fact that led Skyrme to use relativistic solitons in a model of the nucleus of an atom[59].

The vast majority of the soliton solutions are non-perturbative, in the sense that they are not obtained by considering the exact solutions to the associated linear problem and then the non-linear parts in some perturbative scenario, but are usually constructed by some totally different method. The last twenty-five years has seen a rapid growth in the study of solitary wave-like phenomena and as such there are now many very powerful methods and techniques available for the analysis of these equations. Most notably 'the inverse scattering transform'[43], Hirota's method[15] and the use of the Bäcklund transformations[44].

The inverse scattering transform was developed after study of the Korteweg-de Vries (KdV) equation and its related conservation laws, and is in essence a form of non-linear analogue of the Fourier transform. (The conservation laws are of course derived from the result that there is an associated current in the theory

$$\partial_\mu j^\mu \Rightarrow \partial_0 Q = 0 \quad : \quad Q = \int_{-\infty}^{+\infty} j^0 dx$$

where for simplification the example of 1+1 dimensions is given and a suitable boundary condition on the spatial part of the current is assumed).

The KdV equation has an infinite number of conserved quantities[45]. This is implied by the crux of the transform's mechanism, i.e. the fact that the equation can be written as the compatibility condition for an overdetermined set of linear equations - the Lax pair:

$$\left.\begin{aligned} B\Psi &= \partial_0 \Psi \\ L\Psi &= \lambda \cdot \Psi \end{aligned}\right\} \Rightarrow \partial_o L - [B, L]_- = \partial_o \lambda$$

where B, L are differential operators, λ the so-called spectral parameter and $[,]_+$ a commutator bracket; $\partial_0 \lambda$ vanishes if and only if the equations of motion are satisfied. The Lax system controls the time development of some associated initial scattering data which may then be transformed back into a solution of the original equation at time t, (at least in the cases were there is a discrete set of data that appears in the spectrum of L). Hence, the inverse method solves the initial value problem in principle. Among the set of these solutions are the solitons.

The fact that a field equation can be written as the compatibility condition for a Lax pair defines, essentially, what it means for the theory to be 'integrable' classically - as it goes hand-in-hand, that this theory then possesses an infinite number of conserved quantities and has exact solubility.

However, the point that a field equation may be integrable does not immediately imply the constructibility of soliton solutions, even though it should be possible in principle. This is portrayed by the fact that there exist many integrable hierarchies with equations associated with any number of dimensions and generalised Lax systems, but very few exact soliton solutions have been constructed explicitly in more than 1+1 dimensions because of many technical mathematical details.

The subject of Bäcklund transformations constitutes a transformation theory that in essence relates soliton solutions at some fixed time in their evolution through the use of recursion relationships. For the sine-Gordon equation (which will be analysed in detail in section (2.3)) the actual transformations were worked out a long time ago[46] and arose in the context of differential geometry rather than soliton theory[44]. The transformations have been an active subject of research in the theory of solitons since the discovery by Estabrook and Wahlquist[47] of the relevant transformations for the KdV equation, resulting recently in the construction of Bäcklund transformations for some of the affine Toda field theories[17].

One of the most interesting results of this transformation theory is that it culminates in a simple superposition formula - the theorem of permutability - which allows the construction of multi-solitons from single (or even no) solitons by purely algebraic methods, given some appropriate boundary conditions. This is illustrated quite simply using the following diagram:

$$\begin{tikzcd}
 & & U_1 \arrow[dr, "a_2"] & \\
\text{Initial Solution} \arrow[r] & U_0 \arrow[ur, "a_1"] \arrow[dr, "a_2"'] & & U_3 = U_4 \\
 & & U_2 \arrow[ur, "a_1"'] &
\end{tikzcd}$$

where a_1, a_2 are the parameters of the Bäcklund transformations. Denoting $B(a_i)$ to be the transformation associated to a_i then $B(a_2)(B(a_1)u_0)$ is equated with $B(a_1)(B(a_2)u_0)$ to obtain an expression for u_3 in terms of u_0, u_1, u_2 by algebraic means, without having to solve any of the initial differential equations. The action of $B(a_i)$ adds a soliton to an already known solution.

Having mentioned the sine-Gordon equation it may be appropriate, at this point, to note that it turns out in some field theories that the solitons may also possess a 'topological number', which is a consequence of the asymptotic behaviour of these solutions. For solitons this number turns out to be a conserved quantity, which later, hopefully, can be used to label the quantum state of the soliton if a way to quantize the classical field theory has been found. Referred to as 'topological charge' in the context of A.T.F.T. with imaginary coupling and taking values in the weight lattice of the associated Lie algebra. The way to calculate these quantities is far less understood in these field theories. Only recently has progress been made on a general understanding of these topological quantities for the solitons associated with the simplest of the series, that is those related to $a_n^{(1)}$ [48]. The topological quantity is defined in (1+1) dimensions as:

$$Q \propto \Phi(x=+\infty) - \Phi(x=-\infty) \tag{2.01}$$

with an associated conserved current,

$$j^{\mu} \propto \varepsilon^{\mu\nu}\partial_{\nu}\Phi \tag{2.02}$$

Hence, it is just proportional to the difference in asymptotic values of the field. It can be shown, for example in Rajaraman[49], that for a single scalar field in two dimensions, non-trivial static solutions are necessarily topological - in the sense that $Q \neq 0$ for all such solutions. Moreover, these static solutions must interpolate between neighbouring minima of the potentials for such systems with multi-vacua where the potential is real.

Before giving a detailed examination of the sine-Gordon model however, a review of the Hirota method will be undertaken since it is a method central to constructing explicit soliton solutions for the A.T.F.T.'s.

2.2 Hirota's Method

Initially, Hirota constructed his 'direct method' in 1972 as a means of obtaining the N-soliton solution to the KdV equation and, after manipulation, it gave exactly the same determinantal solution as calculated from the inverse scattering method by Gardner et al.. The method is direct in the sense that no reference is made to an associated 'simpler' system such as those that occur in the scattering problem and the Bäcklund transformations.

Although primarily ad-hoc, the method has been abstracted to a deep algebraic setting by the Kyoto school, that is Sato, Date, Jimbo, Miwa and Kashiwara[42],[50]. This has involved the use of many areas of mathematics such as infinite dimensional Graßmannian manifolds, vertex operators and Kač-Moody algebras.

The crux of the Hirota method is the ability to find a substitution for the field variable such that the equations of motion can be written in the 'Hirota bilinear form'.

That is, a substitution

$$\Phi(x,t) = \Phi(\tau_1(x,t), \ldots, \tau_n(x,t) \tag{2.03}$$

is made where the τ_i's are the tau functions for the transformation, one for each of the independent fields that constitute the system. The non-linear evolution equation takes the form:

$$F(D_t, D_x)\tau \cdot \tau = 0, \tag{2.04}$$

where F is a polynomial or exponential function of the operators D_t and D_x, known as the Hirota derivatives. These Hirota derivatives are defined by:

$$D_t^i D_x^j \tau \cdot \tau \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^i \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^j \tau(t,x)\tau(t',x')|_{\substack{t=t' \\ x=x'}} = 0 \tag{2.05}$$

For example,

$$D_x^2\, f \cdot g \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 f(x)g(x')|_{x=x'}$$

$$= f''(x)g(x) - 2(f'(x)g'(x)) + f(x)g''(x)$$

A comprehensive list of properties of such operators may be found, for example, in [51].

If this is possible, then by expressing the τ_i in a power series in say ε and recursively obtaining the terms in the expansion, it may also be possible to consistently truncate the expansion after a finite number of terms. Therefore the problem of summing an infinite series to resurrect τ_i in a closed form does not arise. It has been shown by Hirota that if F satisfies the conditions:

$$F(0,0) = 0$$

and
$$F(D_t, D_x) = F(-D_t, -D_x)$$

then the equation will possess at least a two-soliton solution (and usually N-soliton solutions). This is also true in more than one spatial dimension. In the original solution to the KdV equation, for example, it is possible to truncate at order one to obtain a one-soliton solution which takes the form,

$$\Phi = 2(\log \tau)_{xx}$$

where
$$\tau = 1 + \lambda \exp(kx - k^3 t) \qquad \lambda,\ k \text{ constants.}$$

Addition of more single solitons to give an N-soliton solution can then be achieved consistently by truncating at order N:

$$\tau = \sum_{\mu=0,1} \exp\Big(\sum_{i>j}^{N} A^{(ij)}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\xi_i\Big)$$

where
$$\exp \xi_i = \lambda_i \exp(k_i x - k_i^3 t)$$

and
$$\exp A^{(ij)} = (k_i - k_j)^2/(k_i + k_j)^2$$

($\sum\limits_{\mu=0,1}$ is shorthand for $\sum\limits_{\mu_1=0}^{1} \sum\limits_{\mu_2=0}^{1} \dots \sum\limits_{\mu_N=0}^{1}$, and $\sum\limits_{i>j}^{N}$ denotes the sum over all ordered pairs).

In the sense that if it is possible to write an evolution equation in a bilinear way, then the truncation is possible to obtain a compact form for soliton solutions, there is a certain amount of mystery surrounding the Hirota method. However, its use as a computational tool was immediately obvious and exploited to obtain exact N-soliton solutions to an ever increasing number of non-linear P.D.E's and difference equations, including - as will be seen - the affine Toda field theories. However, in all the cases (bar $a_n^{(1)}$) a subtle massage of the method is required in order to obtain soliton solutions since the equations of motion do not strictly take the bilinear form of Hirota after the proposed substitution has been made.

2.3 A specific example of a soliton system - the sine-Gordon equation

The sine-Gordon equation (or $a_1^{(1)}$ affine Toda field theory with imaginary coupling) arises in several areas of physics. For example it arises in the Josephson junction in superconductivity where itdescribes the motion of magnetic flux in the system. It also appears in two-dimensional models of elementary particles such as the Skyrme model of the nucleon. It even arises in a very simple mechanical model. The system consists of a single scalar field $\Phi(x,t)$ in (1+1) dimensions with an action:

$$S(\Phi,\partial^\mu\Phi)=\int_{-\infty}^{\infty} d^2x\frac{\mu}{8\lambda}\left[\frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi)+\lambda^2(\cos\Phi-1)\right] \tag{2.06}$$

where λ,μ are constants.

Hence, considering the usual Euler Lagrange equations for a stationary point of the action

$$\partial^\mu\left(\frac{\partial L}{\partial^\mu\Phi}\right)-\frac{\partial L}{\partial\Phi}=0$$

the equation of motion is obtained:

$$\partial^\mu\partial_\mu\Phi=-\lambda^2\sin\Phi \tag{2.07}$$

that is, the sine-Gordon equation.

By Taylor expansion it is obvious that the equation is a non-linear continuation of the Klein-Gordon equation, which is presumably how the name was obtained[49].

Examination of the energy,

$$E=\int_{-\infty}^{\infty} dx'\frac{\mu}{8\lambda}\left[\frac{1}{2}(\partial_0\Phi)^2+\frac{1}{2}(\partial_1\Phi)^2+\lambda^2(1-\cos\Phi)\right] \tag{2.08}$$

reveals that the sine-Gordon system has a discrete infinity of degenerate minima, given by,

$$\Phi=2\pi N \qquad :N\in\mathbf{Z}$$

which also satisfy the equation of motion These are the trivial solutions Φ =constant. As already stated, in section (2.1), any non-trivial static solution (in the sense of a single soliton) must interpolate between neighbouring minima and so (working mod 2π) will have to interpolate between 0 and $\pm 2\pi$, or vice versa. Being a relativistic equation the moving soliton can then be obtained from the static solution by a Lorentz boost.

The Lagrangian and field equation also possess the discrete symmetries $\Phi \leftrightarrow -\Phi$ and $\Phi \leftrightarrow \Phi + 2N\pi$ and as before the topological index can be defined for the sine-Gordon equation to be given by

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \frac{\partial \Phi}{\partial x} \tag{2.09}$$

The static single soliton solutions to equation (2.07) are given by:

$$\Phi(x) = \pm 4 \tan^{-1}(\exp \lambda(x - x_0))$$

and have index ± 1 respectively and hence, are denoted the soliton and antisoliton. The moving soliton is then just the Lorentz boost of this:

$$\Phi(x) = 4 \tan^{-1}\left(\exp \lambda \cdot \left(\frac{x - x_0 - vt}{\sqrt{1 - v^2}} \right) \right) \tag{2.10}$$

the antisoliton is related to this by the symmetry $\Phi \leftrightarrow -\Phi$. Calculations of the single soliton energy (2.08) and momentum (both of which are conserved)

$$E = \frac{\mu}{\sqrt{1 - v^2}}, \quad P = \frac{\mu}{8\pi}\int_{-\infty}^{\infty} dx' \partial_0 \Phi \partial_1 \Phi = \frac{\mu v}{\sqrt{1 - v^2}}$$

give

$$E^2 - p^2 = \mu^2$$

which is in total accordance with the interpretation of the soliton as a relativistic particle of rest mass μ and, gives weight to the idea of modelling elementary particles by solitons.

There is in fact a third type of soliton solution to the equation of motion commonly denoted 'the breather'. It occurs in the analysis of a specific soliton - antisoliton pair, whose scattering solution is given by:

$$\Phi_{SA}(x) = 4 \tan^{-1}\left[\frac{\sinh(vt/\sqrt{1 - v^2})}{v \cosh(x/\sqrt{1 - v^2})} \right]. \tag{2.11}$$

Asymptotically (i.e. as $t \to \pm\infty$) it can be shown, using $\left\{\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\frac{x \pm y}{1 \mp xy}\right\}$ that this solution is equivalent to that of the superposition of a single soliton and antisoliton that are approaching or leaving one another with relative velocity $\frac{2v}{1+v^2}$. However, the final configuration $t \to +\infty$ has the second soliton shifted by an extra amount $\delta x = \pm(v^2 - 1)\log v$ relative to the first. This is a by-product of the collision that has taken place at finite t and related to the non-linearity of the equation. This fact may be interpreted as an attractive force acting between the soliton and antisoliton that manifests itself when they are in local vicinity of one another. This system is an example of a solution that has an overall topological index of zero, but of course it does not correspond to one that can be brought to rest in any frame.

The breather solution which also satisfies the equations of motion, is obtained from this by setting $v = iu$ (u real) and again results in a real solution,

$$\Phi_{Br}(x) = 4\tan^{-1}\left[\frac{\sin(ut/\sqrt{1+u^2})}{u\cosh(x/\sqrt{1+u^2})}\right]. \tag{2.12}$$

However, this corresponds to a 'bound' pair in the sense that the solution is periodic in t with period $\frac{2\pi\sqrt{1+u^2}}{u}$, and moreover the energy of the breather is less than 2μ. The soliton and antisoliton oscillate about some centre of mass with this period and hence, are totally consistent with the notion of an attractive force at work, since they cannot separate past some finite distance.

Similarly, there exist soliton-soliton, antisoliton-antisoliton solutions Φ_{ss}, $\Phi_{AA} = -\Phi_{ss}$ which belong to the class of solutions with $Q = 2, -2$ respectively, but having no associated bound states.

The general N-soliton solutions have been constructed using the inverse-scattering method by Ablowitz et al.[52] and hence, all solutions of the time dependent equations are at least in practice known, even if difficult to write down. They may be generated recursively using the Bäcklund transformations which are the following first order coupled (partial) differential equations

$$\frac{1}{2}(u+v)_\xi = \lambda \cdot a^{-1} \cdot \sin(\frac{u-v}{2})$$
$$\frac{1}{2}(u-v)_\eta = \lambda \cdot a \cdot \sin(\frac{u+v}{2})$$

where 'a' is the parameter; $\xi, \eta = \frac{1}{2}(x \pm t)$ are light-cone co-ordinates and u, v solutions of the sine-Gordon equation. The elegance of this method lies in the fact that only first order equations need to be solved rather than the original second order equation. It provides a simple way of obtaining the double soliton solutions above.

The single soliton solution (2.10) can be rewritten in several ways; one of these (using a slightly different normalisation of the fields)

$$\Phi = \frac{2}{i}\left[\log(1 + i\exp\frac{\lambda(x - x_0 - vt)}{\sqrt{1-v^2}}) - \log(1 - i\exp\frac{\lambda(x - x_0 - vt)}{\sqrt{1-v^2}})\right] \tag{2.13}$$

gives one clue to the correct substitution in order to obtain soliton solutions to affine Toda field theory by Hirota's method. However, it is not immediately obvious that there should be such solutions since the potential terms for general A.T.F.T. with imaginary coupling - unlike the case of $a_1^{(1)}$ Toda here - are complex. Therefore they are inherently difficult to interpret, but this will be discussed further later.

2.4 Hirota's Method for Solitons in Affine Toda Field Theory

As has been shown, the sine-Gordon equation (which after rescaling the fields corresponds to $a_1^{(1)}$ Toda field theory) has a vacuum configuration that is manifestly degenerate. This is ultimately responsible for the topological soliton solutions that can be found in the field theory, given that there also exists an infinite number of conserved quantities associated with the system.

Similarly, if the coupling constant β is taken to be purely imaginary in the classical Toda Lagrangian (1.03), then the potential term takes the form:

$$V(\Phi) = \frac{m^2}{(i\gamma)^2}\sum_{j=0}^{r} n_j e^{i\gamma\alpha_j\cdot\Phi} \tag{2.14}$$

where β has been replaced by $i\gamma$; where $\gamma \in \Re$.

It is immediately apparent that with such a coupling the potential term is left invariant under the shift in fields $\Phi \to \Phi + \frac{2\pi}{\gamma}\lambda$ where $\lambda \in \Lambda_W^*$, the co-weight lattice of g. Moreover, unlike the case of real coupling where the potential is minimized by a unique value $\Phi = 0$, the potential is now zero whenever $\Phi = \frac{2\pi}{\gamma}\lambda$ and hence, there is a countable infinity of degenerate vacua in the field theory.

Again the topological charge for the field Φ can be defined to be:

$$t = \frac{\gamma}{2\pi}\int_{-\infty}^{+\infty} \partial_x \Phi \, dx \equiv (\Phi|_{x=+\infty} - \Phi|_{x=-\infty}) \cdot \frac{\gamma}{2\pi} \tag{2.15}$$

and hence, if the field interpolates between two of the degenerate vacuum states then the topological charge is the difference of two weights, so from the underlying construction of a lattice, is also an element of Λ_W^*. If the weights happen to be in the *same representation* of the Lie algebra then the charge $t \in \Lambda_R$ - the root lattice of g; a sublattice of Λ_W^*. It is now easily seen that the infinity of vacua in the sine-Gordon equation just corresponds to the weight lattice of su(2).

The fact that there *exist* such degenerate vacua in a general A.T.F.T. (putting aside for the moment the questions concerning the complex nature of the potential terms and non-unitarity of the theory), prompted Hollowood[14] to investigate the existence of solitonic solutions that interpolate these vacua. Guided by the Hirota substitution for the Toda lattice equations and sine-Gordon model the following substitution was conjectured:

$$\Phi = -\frac{1}{i\gamma}\sum_{j=0}^{r} \alpha_j \log \tau_j \equiv -\frac{1}{i\gamma}\sum_{j=1}^{r} \alpha_j \log\left(\frac{\tau_j}{\tau_0^{n_j}}\right) \tag{2.16}$$

(This has been generalised slightly in[53], but the improvement is irrelevant for the case of the simply-laced Lie algebras, which is all that will be required here, since it reduces to the above formula. Moreover, some solutions for the non-simply-laced cases may be obtained from folding the single soliton solutions of the simply-laced algebras).

The question may be asked as to why there are r+1 tau functions in the substitution compared with the fact that the number of fields present is r. Aratyn et al.[54] note that it is a remnant of the fact that the affine Toda field theory can be embedded in a larger field theory (through the introduction of two extra fields η, ν) that possesses a form of conformal invariance. The extra tau function can then be traced back to the field ν in this "conformal affine Toda system" (C.A.T.).

Substitution of (2.16) into the equations of motion (2.14), where β has been replaced by $i\gamma$ then gives

$$\sum_{j=0}^{r} \alpha_j(\partial^\mu \partial_\mu \log \tau_j) = \sum_{j=0}^{r} \alpha_j (m^2 n_j \prod_{p=0}^{r} \tau_p^{-\alpha_p \cdot \alpha_j}). \tag{2.17}$$

This may be decoupled to give the set of equations

$$(\partial_0^2 \tau_j - \partial_1^2 \tau_j)\tau_j + (\partial_1 \tau_j)^2 - (\partial_0 \tau_j)^2 = m^2 n_j (\prod_{p=0}^{r} \tau^{A_{jp}} - \tau_j^2) \tag{2.18}$$

for $j = 0, \ldots, r$ (where A_{jp} is the adjacency matrix for g defined to be $2\delta_{jp}$-$\alpha_j.\alpha_p$). These then take the Hirota form:

$$(D_t^2 - D_x^2)\tau_j \cdot \tau_j = 2m^2 n_j (\prod_{p=0}^{r} \tau_p^{A_{jp}} - \tau_j^2). \tag{2.19}$$

Strictly, if $\Pi \tau_p^{A_{jp}} \equiv \tau_x \tau_y$ then the bilinear form of Hirota is obtained, however this only occurs for all j, when the affine algebras $a_n^{(1)}$ are considered. Fortunately, all is not lost and consistent series expansions of the τ_j's can be found that truncate after a finite number of terms.

Each tau function is expanded in a power series of a dummy variable 'ε' which is later set to unity, or similarly rescaled into the exponential terms:

$$\tau_j = 1 + \tau_j^{(1)} \cdot \varepsilon^1 + \tau_j^{(2)} \cdot \varepsilon^2 + \ldots\ldots$$

and the terms are equated order by order after substitution into the equations of motion.

Using the minimal ansatz that $\tau_0 = 1 + e^{\Psi} \cdot \varepsilon^1$, where $\Psi = \sigma(x - vt) + \xi$, the system of equations has a closed solution which takes the form

$$\tau_j = 1 + \tau_j^{(1)} \cdot \varepsilon^1 + \cdots + \tau_j^{(n_j)} \cdot \varepsilon^{n_j} \tag{2.20}$$

where n_j is the Kač label of the Dynkin spot j and

$$\tau_j^{(k)} = \delta_j^{(k)} \cdot e^{k \cdot \Psi} \qquad : \delta_j^{(k)} \in \mathbf{C};\ \forall_j \in \{0, \ldots, r\},\ k \in \{1, \ldots, r\}. \tag{2.21}$$

It is also required that $\sigma^2(1 - v^2) = \lambda m^2$, where λ is an eigenvalue of the matrix $K_{ij} = n_i \alpha_i \cdot \alpha_j$. As has been noticed previously[21], the eigenvalues of this matrix are the same as those of $M^{ab} = \sum_{i=1}^{r} n_i \alpha_i^a \alpha_i^b$ and hence, correspond to the classical particle masses² of the relevant affine Toda field theory with real coupling constant. The fact that the expansion of each tau function terminates at the exponent corresponding to the Kač label can also be seen by examining the second form of the Hirota substitution for Φ, (2.16). The fields are required to be finite as $x \to \infty$, and as has been discussed, must correspond to an element of the coweight lattice. If σ is taken to be positive then $\Phi|_{x=-\infty} = 0$ trivially and only the $+\infty$ limit becomes important. Hence the numerator in each τ_j must terminate at the n_j^{th} power of e^{Ψ} for this finite property to hold.

Expansion of the equations of motion for the τ's to order e^{Ψ}, then gives:

$$\sigma^2(v^2 - 1)\delta_j^{(1)} \equiv m^2 \cdot n_j \cdot \left[\prod_{k=0}^{r} (1 + \delta_k^{(1)} \cdot e^{\Psi} + \cdots)^{2\delta_{jk} - \alpha_j \cdot \alpha_k} - \tau_j^2 \right]$$

$$\equiv m^2 \cdot n_j \cdot \left(-\sum_{k=0}^{r} \alpha_j \cdot \alpha_k \delta_k^{(1)} \right)$$

and hence, $\delta_j^{(1)}$ is an eigenvector of the matrix $K_{ij} (i, j \in \{0, \ldots, r\})$ corresponding to eigenvalue $\sigma^2(1 - v^2)m^{-2} \equiv \lambda$, that is the classical particle mass². So there are r non-trivial distinct classes of soliton which can be associated to the relevant spots of the Dynkin diagram in just the same way as the classical particles can be attached via the Perron-Frobenius eigenvector. The trivial solution is just a manifestation of the fact that the extended Cartan matrix $\hat{C}_{ij}$ has a right eigenvector (corresponding to zero eigenvalue) of a vector comprising the Kač labels,

$$\hat{C}_{ij} n_j \equiv \left(\frac{2\underset{\sim}{\alpha}_i \cdot \underset{\sim}{\alpha}_j}{|\alpha_j|^2} \right) n_j = 0 \qquad : \quad i, j \in \{0, \ldots, r\}$$

implying tau functions of the form $\tau_j = (1 + e^{\Psi})^{n_j}$ and hence, a zero field.

When $\lambda \neq 0$ it can be seen that $\sum_{i=0}^{r} \delta_i^{(1)} = 0$ from the sum over 'i' in: $n_i \alpha_i^a \alpha_j^a \delta_j^{(1)} = \lambda \delta_i^{(1)}$.

The association of the soliton solutions to the spots on the Dynkin diagram also goes hand-in-hand with the fact that the topological charges of these solutions lie amongst the set of weights corresponding to those fundamental representations. (Although not all the weights are found for the single solitons, e.g. see [48]) . Moreover, the solitons having such a charge attached to a specific representation are degenerate in mass, this mass is proportional to the classical particle mass.

Firstly, however, an important distinction must be noted between the solutions to the field theory based on a general Lie algebra and that of the sine-Gordon model. This is that the soliton solutions are real in the latter case, but manifestly complex in the former. The question then immediately arises concerning the nature of the masses of such objects since the Hamiltonian and momentum densities :

$$\mathbf{H} = \frac{1}{2}((\partial_0\Phi)^2 + (\partial_1\Phi)^2) - \frac{m^2}{\gamma^2}\sum_{j=0}^{r} n_j \cdot e^{i\gamma\alpha_j\cdot\Phi} \tag{2.22}$$

$$\mathbf{P} = -(\partial_0\Phi)(\partial_1\Phi). \tag{2.23}$$

are complex-valued.

Initial computations of the masses on a case-by-case basis[14],[53] suggested that they were real and moreover, proportional to those of the classical particles, the constant of proportionality being $\frac{2h}{\beta^2}$. Hollowood[14] has argued that this is because it is possible to construct classical conservation equations for the solitons analogous to the quantum charge bootstrap equations and which possess precisely the same form. Since the solutions are unique given the spectrum of conserved charges for the algebra, then the integrals of motion are proportional up to an overall scale factor independent of particle type, as portrayed by the calculations of masses for the simply-laced cases.

Subsequent rigorous proofs of such facts have been given from a variety of directions of attack. Namely, via the use of Bäcklund transformations[17]; from consideration of the conformal affine Toda field theory[54],[55], and by utilization of a generalised Leznov-Savaliev solution for the affine Toda models[16],[56]. The latter is ultimately linked to a construction of a vertex operator formalism for such theories and the representation theory of the Kač Moody algebras in general. The relevant Bäcklund transformations were found only for $a_n^{(1)}$ and $d_4^{(1)}$; those corresponding to the $a_n^{(1)}$ series were, in fact, essentially those discovered over a decade ago by Fordy and Gibbons[57],

$$\partial_+(\phi_n - \tilde{\phi}_n) = \frac{m}{\sqrt{2}\,\beta} A\left[e^{i\beta(\tilde{\phi}_n - \phi_{n+1})} - e^{i\beta(\tilde{\phi}_{n-1} - \phi_n)}\right] \qquad : \ \phi_n \equiv (\Phi)_n$$

$$\partial_-(\phi_n - \tilde{\phi}_{n-1}) = \frac{m}{\sqrt{2}\,\beta} A^{-1}\left[e^{i\beta(\phi_n - \tilde{\phi}_n)} - e^{i\beta(\phi_{n-1} - \tilde{\phi}_{n-1})}\right]$$

where $x^{\pm}$ denote the light-cone co-ordinates $\frac{(t \pm x)}{\sqrt{2}}$ and A is the Bäcklund parameter. When these are satisfied, the energy and momentum densities become total divergences leading to 'real' topological surface terms[17] and hence, the reality of the stress energy tensor. It is also conjectured to be true for all affine Toda theories but the relevant Bäcklund transformations have been hard to uncover; notably for $d_4^{(1)}$, the equations corresponding to the outer spots of the Dynkin diagram are algebraic, whereas the equation corresponding to the central point has a square root associated with it. As in the $a_n^{(1)}$ cases, the transformations can be integrated to give single soliton solutions from the vacuum. Moreover, the exponential of the single soliton field takes the form of a ratio of functions akin to the tau functions, in total analogy with the Hirota substitution (2.16), after a specific representation of the simple roots has been taken, that is $\underset{\sim}{\alpha}_i = \underset{\sim}{e}_i - \underset{\sim}{e}_{i+1}$, such that the $\underset{\sim}{e}_i$ are orthonormal vectors, $i = 0, \ldots, r$ in $\Re^{r+1}$, and the physical fields Φ lie in an 'r' dimensional subspace.

The 'N' soliton solution then takes a particularly neat form for the $a_n^{(1)}$ series in this language. Using the Bäcklund transformations, it can be shown that,

$$e^{i\gamma\phi_j} = A_1 A_2 \cdots A_N \frac{(\det T^{1,2,\ldots,N}_{j-1,\ldots,j-N})}{(\det T^{1,2,\ldots,N}_{j,\ldots,j-N+1})}$$

where for example,

$$\det T^{1,2}_{j-1\,j-2} = T^1_{j-1} T^2_{j-2} - T^2_{j-1} T^1_{j-2}$$

and the T^i are exactly equivalent to the i^{th} single soliton tau functions. In such a compact form, this can be compared favourably to the series solution found by Hollowood[14], which is similar in structure to that of the KdV solution.

The method of Aratyn et al.[54] again involves the observation that all contributions to the soliton masses in the affine Toda models come from terms that can be written as a total divergence, but differing from the previous method in that they involve the Conformal affine Toda field ν. Specifically the momentum equations gives

$$M_{sol}\frac{v}{(1-v^2)^{1/2}} = \gamma\int_{-\infty}^{+\infty} dx\partial_x\partial_t(\sum_{j=1}^{r}\phi_j + h\cdot\nu)$$
$$= \gamma\cdot\partial_t(\sum_{j=1}^{r}\phi_j + h\cdot\nu)|_{-\infty}^{+\infty}$$

and hence, the masses of the solitons can be calculated given the asymptotic nature of the ν field[54]. This is not so surprising given the fact that the affine Toda equations appear in some 'gauge-fixed' sense from the conformally invariant C.A.T. ones, a residual appearance of such a field in the mass term may be expected.

So, despite the non-unitary form of the Lagrangian, what has appeared is that the classical solitons possess real energy and momenta. However, it is also possible to show more than this, namely that the solitons are stable with respect to small perturbations around their solutions[58].

Denoting χ to be the perturbation around any static solution $\hat{\Phi}$, then for small deviations around $\hat{\Phi}$ equation (1.04) gives

$$\partial_\mu\partial^\mu\chi + m^2\sum_{j=0}^{r} n_j\alpha_j(\alpha_j\cdot\chi)e^{i\gamma\alpha_j\cdot\hat{\Phi}} = 0. \tag{2.24}$$

Therefore, only considering the asymptotic behaviour of χ i.e as $x\to\pm\infty$, the above gives

$$\partial_\mu\partial^\mu\chi + m^2\sum_{j=0}^{r} n_j\alpha_j(\alpha_j\cdot\chi) = 0 \tag{2.25}$$

which is simply equivalent to the linear classical equation of motion around the vacuum configuration.

There are r linearly independent solutions to (2.25) since χ and $\hat{\Phi}$ lie in a vector space with dimension given by the rank of the Lie algebra and possess the asymptotic behaviour

$$\chi_a(x^\mu) \xrightarrow[\pm\infty]{x} e^{i(kx+vt+\xi_\pm)}\sum_{j=0}^{r}\omega_a^j\alpha_j \qquad a = 1,\ldots,r$$

such that the frequency is given by:

$$\nu^2 = k^2 + m_a^2$$

in accordance with the Toda fundamental particle modes. Consideration of double-soliton solutions then gives the exact form of the solution. However, the fundamental point to note is that the solitons are classically stable since there are no modes with imaginary frequency. As has been discussed, this is expected in the sine-Gordon model since the potential terms are real for $a_1^{(1)}$ Toda, however, it is not at all obvious for the more general complex valued potential terms for $a_n^{(1)}$ (n>1) and the other affine algebras. Coupled with the fact that the 2-momenta of the solitons are real it may ultimately be explained by the discovery that there is a unitary theory somehow embedded in a non-unitary field theory.

The proof by Olive et al.[16] of the energy-momentum reality and positivity again involves consideration of the fact that affine Toda can be embedded in the higher conformal system and not only do they show such properties for the single soliton, but for the n-soliton solution in general. Arguably, their approach is the most systematic and powerful way of constructing such solutions for the Toda models since it involves the application of the representation theory of the simply-laced Kač-Moody algebras incorporated into a general solution for the affine models. This is akin to the Leznov-Saveliev solution to the Toda equations of motion and it takes the form,

$$e^{-\beta\gamma_i\cdot\Phi} = \frac{\langle\Lambda_i|A_{(x^+)}B_{(x^-)}|\Lambda_i\rangle}{\langle\Lambda_0|A_{(x^+)}B_{(x^-)}|\Lambda_0\rangle}\cdot e^{-\beta\gamma_i\cdot\Phi_0}. \qquad i = 1,\ldots,r.$$

Here A, B are chiral fields; $\Lambda_0,\ldots,\Lambda_r$ the highest weights of the fundamental representations of the Kač-Moody algebra; Φ_0 is a free field (set to zero or one of the degenerate vacua for the case of imaginary coupling soliton solutions) and the m_i are the Kač labels. However, the theory is complicated, and for the sake of constructing single and double solitons 'explicitly', the Hirota method may be considered preferable, though the inherent lack of any deep Lie algebraic theory associated with the method curtails explanation of many of the phenomena that appear in the multi-soliton solutions. Nevertheless, a certain amount of headway can be made after case-by-case consideration and formulae conjectured that go part of the way to describing the interaction pieces that appear in the simply-laced cases.

Ultimately, a full explanation must lie in the domain of the underlying group theory, as elucidated in [16],[56].

3. THE SINGLE / DOUBLE SOLITONS AND FUSING RULES FOR SIMPLY-LACED AFFINE TODA FIELD THEORY.

3.1 Introduction

The Hirota method is used to construct explicit expressions for the single and double soliton solutions for all the simply-laced cases of affine Toda field theory. The solutions portray well the subtle form of non-linear superposition principle that is inherent in the method. However, the fact that only the $a_n^{(1)}$ series of affine Toda models can be written in the strict 'Hirota - bilinear' form disturbs the principle somewhat, in a way that is only understood in this method via explicit calculation. (A group-theoretical understanding may be necessary to answer such questions fully).

The double soliton solutions possess a natural physical interpretation in which two objects scatter as time evolves (when both velocities are taken to be real). This interpretation generalises to the n-soliton case, for all n>1.

Slightly more rigorously, it can be supposed that $\mathrm{Re}(\xi_1) < \mathrm{Re}(\xi_2)$, $v_1 > v_2$ and as usual, the size parameters $\sigma_i > 0$, $i = 1,2$. Then in the limit $t \to -\infty$, examination of the solution in the proximity of the first soliton such that $x \approx v_1 t - \sigma^{-1} \cdot \mathrm{Re}(\xi_1)$ leads to the conclusion that the solution is given approximately by the single soliton tau functions:

$$\tau_j^{(-\infty)} \approx 1 + \delta_{j_1}^{(1)} e^{\Psi^{(1)}} + \delta_{j_1}^{(2)} e^{2\Psi^{(1)}} + \cdots + \delta_{j_1}^{(n_j)} \cdot e^{n_j \cdot \Psi^{(1)}}$$

where

$$\Psi^{(1)} = \sigma_1(x - v_1 t) + \xi_1 ,$$

which is what would, be naïvely expected in comparing this system to two 'non-interacting' isolated solitons. However, as will be seen, in taking the limit $t \to +\infty$ and again focusing on the region of x above, the solution in all simply-laced cases takes the approximate form,

$$\begin{aligned}
\tau_j^{(+\infty)} &\approx \delta_{j_2}^{(n_j)} \cdot e^{n_j \Psi^{(2)}} + \{\text{interaction parts involving } \delta_{j_2}^{(n_j)} \cdot e^{n_j \Psi^{(2)}}\} \\
&\equiv \delta_{j_2}^{(n_j)} \cdot e^{n_j \Psi^{(2)}} \cdot [1 + \delta_{j_1}^{(1)} \cdot e^{\Psi^{(1)}} \cdot A^{(12)} + \cdots + \delta_{j_1}^{(n_j)} \cdot e^{n_j \Psi^{(1)}} \cdot (A^{(12)})^{n_j}] \\
&\equiv \delta_{j_2}^{(n_j)} \cdot e^{n_j \Psi^{(2)}} \cdot \hat{\tau}_j^{(-\infty)} .
\end{aligned}$$

Here, $A^{(12)}$ is a 'basic' interaction block associated with the extra spot on the affine Dynkin diagram in such a way that the ansatz

$$\tau_0 = 1 + e^{\Psi^{(1)}} + e^{\Psi^{(2)}} + A^{(12)} \cdot e^{\Psi^{(1)} + \Psi^{(2)}} \tag{3.01}$$

is given for the double soliton tau function attached to this point. This then gives the full set of tau functions from the appropriate equations of motion.

(This interaction piece implicitly depends on the rapidity difference Θ of the two constituent solitons, however, for brevity in this chapter, this will not be explicitly portrayed, and only the other 'less fundamental' interaction terms will be written with such explicit dependence).

Hence, comparing the limits $t \to \pm\infty$ for one of the solitons, it can be seen that the interaction causes the soliton to be displaced in such a way that $\tau_j \to \hat{\tau}_j$, which is equivalent to $\mathrm{Re}(\xi_1) \to \mathrm{Re}(\xi_1) + \log|A^{(12)}|$. The size parameter, σ, and velocity, v, of the soliton remain unchanged after the interaction, a reflection of the fact that there exists an infinite number of conservation laws associated with the soliton. So, in effect the result of the interaction is merely to shift the soliton along the x-axis by the amount $\log|A^{(12)}|$.

The single soliton solutions for the both the $a_n^{(1)}$ and $d_4^{(1)}$ (and n-solitons for $a_n^{(1)}$) were first written down by Hollowood[14], following the discovery of the relevant Hirota substitution. His methods were generalised slightly, resulting in all the single solitons for the simply-laced Lie algebras, by the author (see following reference), MacKay and McGhee[53] and Aratyn et al.[54],[55].

The calculations of the double solitons are claimed to be original and will be used to show that the classical fusing rule of Dorey, in the case of particle excitations, also holds for the solitons but supplemented by some extra 'annihilation' couplings for the soliton and antisoliton. The fusing has been examined by Olive et al.[16] in the context of their group-theorectical approach and gives exactly the same results as found here in the process of explicit calculation of the double soliton solutions.

The fusing structure may be revealed by shifting the arguments of the exponential terms in the tau functions (i.e. a rescaling) and by considering the poles in the interaction quantities $A^{(12)}$. The shift is taken to be

$$\Psi^i \to \Psi^i - \frac{1}{2}\log\left|A^{(12)}\right| \tag{3.02}$$

for each i, and it transpires that the solitons possess a fusing structure which occurs at the classical level, that is, in most part, identical to that of the real coupling particle fusing. The fusing that occurs in the real coupling regîme, however, is manifestly different in the sense that it occurs at tree level in the quantum field theory. The values of rapidity difference at which all objects fuse are of course complex for on-shell physics, which is a reflection of the integrable structure of the field equations. The fact that the fusing angles are equivalent for the particles and solitons goes hand-in-hand with the proposal by Hollowood that there exist classical conservation equations for the solitons exactly the same as the quantum charge bootstrap equations (1.10). However, there is also another piece of information that is associated with the solitons, that is their topological charge. It turns out that the fusing of the solitons' charges reflects the Clebsch-Gordon property of the classical particles, that is taking $\lambda_{a_1} \in \Lambda_{a_1}$, $\lambda_{a_2} \in \Lambda_{a_2}$, $\lambda_{a_3} \in \Lambda_{a_3}$ where λ_{a_1,a_2,a_3} are the topological charges of particles a_1, a_2, a_3 respectively and Λ_{a_1,a_2,a_3} the fundamental representations, then:

$$c_{a_1a_2a_3} \neq 0 \Rightarrow \Lambda_{a_3} \subset \Lambda_{a_1} \otimes \Lambda_{a_2}$$

The equivalence of the $a_n^{(1)}$ 'fusings' for the solitons and the fundamental particles was first discovered by Hollowood[14] through explicit calculation. The check of all the remaining simply-laced cases is also carried out here in an explicit manner and claimed as original in such a context.

An investigation into the terms that appear in the double soliton solutions which are interpretable as the interaction terms (since they depend upon the rapidity difference of the two constituent solitons) will begin in the following chapter. However, the introduction of some relevant notation is advantageous here, for writing the leading terms of the tau functions in a compact manner. It is this notation that will lead to the association of these particular terms to the orbits of the simple roots, associated with the solitons, under the

Coxeter element, that will play a prominent part in chapter four, where objects similar to the vertex operators that appear in the construction of the Toda S-matrix[25] will be shown to produce these highest order terms upon normal ordering.

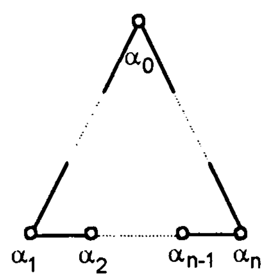
3.2 Solitons for the $a_n^{(1)}$ Series.

The soliton solutions associated to this series of Kač-Moody algebras are by far the simplest in the sense that the Hirota form of the equations of motion takes its true bilinear nature. The 'n' soliton tau functions then take the standard form (c.f. section (2.2) in the case of the KdV equation) such that the interaction parameter becomes a covariant function of the respective rapidity difference.

For these algebras all the Kač labels n_i are unity, hence $K_{ij} \equiv \hat{c}_{ij}$, the extended Cartan matrix. The eigenvalues are:

$$\lambda_a = (2\sin\frac{\pi a}{h})^2 \qquad\qquad a = 0, \dots, n$$

where the Coxeter number $h = n+1$ for the a_n series. The Dynkin diagram for the affine algebra $a_n^{(1)}$ is given by:



$n \geq 2$

Given that all labels are understood to be taken mod h, due to the cyclic symmetry of the Dynkin diagram, the equations of motion are:

$$f(\tau_j) = m^2(\tau_{j+1}\tau_{j-1} - \tau_j^2) \qquad\qquad j = 0, \dots, n \tag{3.03}$$

where the operator f is defined by,

$$(\ddot{\tau}_j - \tau_j'')\tau_j - \dot{\tau}_j^2 + \tau_j'^2 \equiv f(\tau_j).$$

These can be solved to give the single soliton solutions[14] (using the specific choice $\tau_0 = 1 + e^{\Psi}$):

$$\Phi_{(a)} = -\frac{1}{i\gamma}\sum_{k=1}^{r} \underset{\sim}{\alpha}_k \log\left(\frac{1+\omega^{ak}e^{\Psi}}{1+e^{\Psi}}\right) \tag{3.04}$$

where ω is an 'r+1' th root of unity and the dispersion relation $\sigma^2(1-v^2)=\lambda_a \cdot m^2$ holds. There is obviously a non-trivial solution for each $a \in \{1,\ldots,n\}$, which may be associated to the respective spot on the Dynkin diagram. As has already been mentioned in section (2.4) each of the $\Phi_{(a)}$ may also be connected to the a^{th} fundamental representation corresponding to the spot through their topological charge. The topological charge manifests itself as a weight in the representation, the actual weight being dependent on the imaginary part of ξ and only recently has a thorough understanding of the charges emerged[48]. The single soliton charges do not fill all the fundamental representations, but only through excessive combinations of the single solitons (i.e. multi-soliton solutions) can all the weights be found.

The double soliton solutions result from terminating the tau function expansion at a higher power of the expansion parameter. As has been stated, a simple form of non-linear superposition principle pervades in these cases since the tau functions obey proper bilinear equations and in fact, the whole argument easily generalises to the N-soliton case.

By defining,

$$\Psi_j^{(q)} = \sigma_q(x - v_q t) + \xi_q + \frac{2\pi i}{n+1} a_q \cdot j \qquad q = 1,2$$

where σ_q, v_q, $Re(\xi_q)$, $Im(\xi_q)$, denote the size parameter, velocity, position, offset and topological parameter of the q^{th} soliton respectively. Then after the expansion parameter has been absorbed into the exponential terms, the double soliton tau functions take the following form:

$$\tau_j = 1 + e^{\Psi_j^{(1)}} + e^{\Psi_j^{(2)}} + A^{(12)} \cdot e^{\Psi_j^{(1)} + \Psi_j^{(2)}} \qquad j = 0, \ldots, n. \tag{3.05}$$

The function $A^{(12)}$ may be written in several equivalent ways:

$$A^{(12)} = -\frac{(p_1 - p_2)^2 - m^2_{a_1 - a_2}}{(p_1 + p_2)^2 - m^2_{a_1 + a_2}} \tag{3.06}$$

where

$$p_i = (\sigma_i, -\sigma_i v_i), \quad m_j = 2m \sin\left(\frac{j\pi}{n+1}\right)$$

which allows the relativistic invariance of the interaction parameter to be self-evident. Alternatively it can be re-written as,

$$A^{(12)} = -\frac{F(\sigma_1 - \sigma_2, \sigma_1 v_1 - \sigma_2 v_2, a_1 - a_2)}{F(\sigma_1 + \sigma_2, \sigma_1 v_1 + \sigma_2 v_2, a_1 + a_2)} \tag{3.07}$$

where

$$F(a, b, c) = a^2 - b^2 - m_c^2$$

such that each single soliton's set of data $\{\sigma_i, v_i, \xi_i, a_i\}$ satisfies $F(\sigma_i, \sigma_i v_i, a_i) = 0$.

Introduction of the rapidity variable, θ, such that the soliton's associated two-momentum can be written as, $m_a(\cosh\theta, \sinh\theta)$ allows the interaction to take the form,

$$A^{(12)} = \frac{\cosh\Theta - \cos(\frac{\pi}{n+1}(a_1 - a_2))}{\cosh\Theta - \cos(\frac{\pi}{n+1}(a_1 + a_2))} \tag{3.08}$$

$$\equiv \frac{\sinh(\frac{\Theta}{2} + \frac{i\pi(a_1-a_2)}{2(n+1)})\sinh(\frac{\Theta}{2} - \frac{i\pi(a_1-a_2)}{2(n+1)})}{\sinh(\frac{\Theta}{2} + \frac{i\pi(a_1+a_2)}{2(n+1)})\sinh(\frac{\Theta}{2} - \frac{i\pi(a_1+a_2)}{2(n+1)})} \tag{3.09}$$

where $\Theta = \theta_1 - \theta_2$. This is equivalent to,

$$\frac{(1 - \omega^{(\frac{a_1-a_2}{2})}.e^{\Theta})(1 - \omega^{-(\frac{a_1-a_2}{2})}.e^{\Theta})}{(1 - \omega^{(\frac{a_1+a_2}{2})}.e^{\Theta})(1 - \omega^{-(\frac{a_1+a_2}{2})}.e^{\Theta})}. \tag{3.10}$$

The interaction function as a product of sinh functions has been known for a long time to possess a vertex operator construction, especially in the case of the sine-Gordon equation, see, for example, Skyrme[59].

It turns out that not only the $a_n^{(1)}$ series, but all simply-laced series possess this character, that is, that the leading terms of the 'interaction' (multi-soliton) tau functions can be expressed as quotients of sinh functions. With such a point in mind it would be hoped that a vertex operator construction of such general objects would be possible within the frame work of the Lie algebra structure. As will be seen this can be done and involves the use of the simple roots and Coxeter element in a way very akin to that of the construction of the minimal S-matrix[25]. It must be borne in mind, however, that this construction is only for the leading terms and not for the soliton solution as a whole.

More generally, the N-soliton solution is given by the following, where the sums are equivalent to those in the KdV example:

$$\tau_j = \sum_{\mu=0,1} \exp(\sum_{p=1}^{N} \mu_p \Psi_j^{(p)} + \sum_{1\le p<q\le N} \mu_p\mu_q \log A^{(pq)}).$$

In addition to these scattering double solitons there also exist 'breathing solutions', i.e generalisations of the breather of $a_1^{(1)}$ affine Toda theory and moreover, there are static configurations of double solitons which only exist for pairs associated to different spots on the affine Dynkin diagram. Such static double solitons are another artifact borrowed from the case of a real potential in two dimensional physics. It is possible to show [for example, Rajaraman[49]] that it is impossible to have identically static solitons which interpolate non-adjacent minima of the potential. In A.T.F.T. the situation is much more subtle since the potential terms are complex, and moreover, any of the solitons that possess topological charge in the same fundamental representation are taken to be 'identical'. From (3.08) it can be seen that this definition of 'identical' for a static solution would result in a zero value for $A^{(12)}$.

Taking $v_1 = v_2$ in (3.05), so that with respect to some frame both solitons are stationary, the tau functions take the form:

$$\tau_j = 1 + \omega^{a_1 j}\cdot\exp(\Psi) + \omega^{a_2 j}\cdot\exp\left(\frac{\sigma_2}{\sigma_1}(\Psi-\xi_1)+\xi_2\right) + A^{(12)}\cdot\omega^{(a_1+a_2)j}\exp\left(\left(\frac{\sigma_1+\sigma_2}{\sigma_1}\right)\Psi + \left(\frac{\sigma_1-\sigma_2}{\sigma_1}\right)\xi_1+\xi_2\right)$$

where,

$$A^{(12)} = -\left\{\frac{(\sigma_1-\sigma_2)^2-(2m\sin\frac{\pi(a_1-a_2)}{n+1})^2}{(\sigma_1+\sigma_2)^2-(2m\sin\frac{\pi(a_1+a_2)}{n+1})^2}\right\} \quad \text{and} \quad \Psi = \sigma_1 x + \xi_1 .$$

As has been noted by Aratyn et al. (but from a different point of view, involving the non-trivial kernel of an operator), when $\frac{\sigma_2}{\sigma_1} \equiv \frac{m_{a_2}}{m_{a_1}} \in \mathbf{Z}^+$ (using the dispersion relation) then such double soliton solutions appear naturally in the Hirota formalism. Apart from the trivial case when the $\mathbf{Z}_2$ symmetry of the Dynkin diagram for a_n is considered, such a condition only arises for $a_{6p-1}^{(1)} \equiv \hat{su}(6p)$, when $\{\sin(\frac{a_1\pi}{h}), \sin(\frac{a_2\pi}{h})\} \subset \{1,-1,\frac{1}{2},-\frac{1}{2}\}$.

For example: $a_1 + a_2 = n+1 \equiv h$, gives

$$A^{(12)} = \left\{\frac{2m\sin(\frac{\pi}{n+1}(n+1-2a_1))}{2\sigma_1}\right\}^2 = 1-\left(\sin\frac{\pi a_1}{n+1}\right)^2 = 1-\frac{1}{4}m_{a_1}^2$$

and the static double soliton takes the form: (after a shift in Ψ)

$$\tau_j = 1+\omega^{a_1 j}\cdot e^{\xi_1}\cdot e^{\Psi} + \omega^{-a_1 j}e^{\xi_2}e^{\Psi} + e^{\xi_1+\xi_2}(1-\frac{1}{4}m_{a_1}^2)e^{2\Psi}.$$

It is easily seen that if either ξ_1 or $\xi_2 \to -\infty$ then these tau functions revert to those corresponding to the single soliton since the other soliton has been forced away to infinity.

3.2.1 Fusing for $a_n^{(1)}$

For the case $n = 1$, that is, the sine-Gordon model the phenomenum known as fusing does not occur. This is a reflection of the fact that the solitons (antisolitons) have a topological index ± 1 (in units of 2π). For a complete discussion of the cases $n \geq 2$, see [14], however, a brief outline will be given below.

The fusing occurs when a two soliton solution reduces to that of a single soliton at a rapidity difference taking a non-physical, complex value. Consider the double soliton solution (3.05), a simple rescaling $\xi_i \to \xi_i - \frac{1}{2}\log A^{(12)}$ gives the new tau functions

$$\tau_j = 1 + A^{(12)^{-1/2}} \cdot (e^{\Psi_j^{(1)}} + e^{\Psi_j^{(2)}}) + e^{\Psi_j^{(1)} + \Psi_j^{(2)}} \tag{3.11}$$

It can be seen that a pole in $A^{(12)}$ removes the middle pair of terms, leaving a tau function appropriate to a single soliton. From the equivalent forms of $A^{(12)}$, the pole is seen to exist at rapidity difference Θ when

$$\mathrm{ch}(\Theta) = \cos((a_1 + a_2)\tfrac{\pi}{h}).$$

Hence, restricting analysis to the physical strip, that is the region of the complex plane $0 \leq \mathrm{Im}(\Theta) \leq \pi$, the fusing 'angles' $\theta_{a_1 a_2}$ are given by:

$$\theta_{a_1 a_2} = \begin{cases} (a_1 + a_2)\frac{\pi}{h} & \text{if } a_1 + a_2 < h \\ (2h - a_1 - a_2)\frac{\pi}{h} & \text{if } a_1 + a_2 \geq h \end{cases}$$

These are exactly equivalent to the three-point particle couplings for the classical Toda particles but here correspond to the closing of the soliton mass triangle since the masses are proportional to those in the real coupling case.

At these fusing values the solution (3.11) reduces to

$$\tau_j = 1 + e^{\Psi_j^{(3)}} \qquad : \Psi_j^{(3)} = \Psi_j^{(1)} + \Psi_j^{(2)}$$

a single soliton tau function. The solution possesses mass, $m_{a_1+a_2}$, momentum, $p_{a_1} + p_{a_2}$, and topological charge $\lambda_{a_3} = \lambda_{a_1} + \lambda_{a_2}$, such that $\lambda_{a_3} \in \Lambda^{(a_1+a_2 \bmod n+1)}$.

3.3 Solitons for the $d_n^{(1)}$ series.

Because of the extra symmetry involved in the Dynkin diagram for $d_4^{(1)}$ when compared with the rest of the $d_n^{(1)}$-series for values of $n \geq 5$, these cases will be treated separately. The extra symmetry which gives rise to a triple degeneracy in the mass spectrum for the case $d_4^{(1)}$ also results in an equation of motion of a type not found in the higher 'n' cases. As will be seen, the equations of motion for all these affine algebras do not fit into the category of 'Hirota bilinear type', but it is nevertheless consistently possible to obtain a solution to the equations, as mentioned in the previous chapter (section 2.4). The fact that such a 'bilinearization' does not appear also complicates the structure of the multi-soliton solutions in comparison to the $a_n^{(1)}$ cases.

In a similar fashion to the method used for obtaining double solitons for the $a_n^{(1)}$ series, the tau function associated with the extra spot on the affine diagram is conjectured to take the form

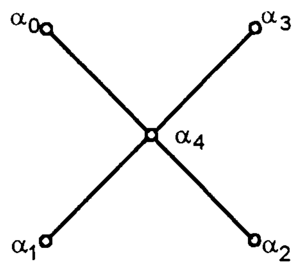
$$\tau_0 = 1 + e^{\Psi^{(1)}} + e^{\Psi^{(2)}} + A^{(12)} \cdot e^{\Psi^{(1)} + \Psi^{(2)}} \tag{3.12}$$

and the remaining tau functions to have exponential terms possessing orders up to $n_j \cdot (\Psi^{(1)} + \Psi^{(2)})$, where n_j is the associated Kač label to spot j. All interaction terms then manifest themselves as polynominals in $A^{(12)}$ with coefficients that are functions of the rapidity difference of the two constituent solitons.

Again, it is found that in all cases the quantity $A^{(12)}$, which will be refered to as the basic block, can be written as a product of hyperbolic functions, moreover a new notation is introduced in order to write this quantity in a more compact manner.

3.3.1 $d_4^{(1)}$:

With respect to the rank of the affine Lie algebra this is the smallest of the $d_n^{(1)}$ series, but is unique in the sense that it possesses a form of symmetry that the other members of the series do not possess. All the Kač labels corresponding to the outer spots of the Dynkin diagram (the spinor or antispinor and vector representations of the algebra) are unity, whereas that of the inner spot (the adjoint representation) takes the value two. The Dynkin diagram can be labelled:



This is in fact a different labelling to the standard one used in the literature, for example Goddard and Olive[60] and is used for convenience when considering the equations of motion and the matrix K_{ij}.

The eigenvalues of the matrix K_{ij} are $\lambda = 0, 2, 2, 2, 6$ which reflects the triple degeneracy of the diagram corresponding to the finite dimensional Lie algebra.

With such a labelling of the extended root system, the equations of motion for this algebra take the form,

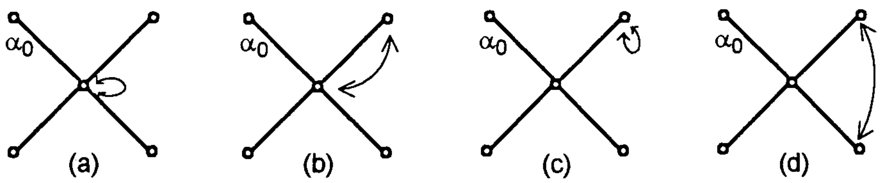
$$\left.\begin{array}{ll} f(\tau_j) = 1 \cdot m^2(\tau_4 - \tau_j^2) & j \in \{0, \dots, 3\} \\ f(\tau_4) = 2 \cdot m^2(\prod_{j=0}^{3} \tau_j - \tau_4^2). & \end{array}\right\} \quad (3.13)$$

These may be solved using the minimal ansatz that was mentioned in chapter two, to give single soliton solutions which take the form

$$\left.\begin{array}{llll} \lambda = 2: & \tau_0 = 1 + e^{\Psi} & \lambda = 6: & \tau_{0,1,2,3} = 1 + e^{\Psi} \\ & \tau_{1,2,3} = 1 \pm e^{\Psi} & & \tau_4 = 1 - 4e^{\Psi} + e^{2\Psi}, \\ & \tau_4 = 1 + e^{2\Psi}, & & \end{array}\right\} \quad (3.14)$$

with the restriction that in the degenerate case $\sum_0^3 \delta_i^{(1)} = 4$ and hence, that there are three distinct solutions - one for each of the degenerate λ's. These non-trivial solutions can again be associated with the Dynkin diagram in the manner which by now must be familiar.

In considering the double soliton solutions it was found advantageous to separate the solutions into four distinct classes (a),...,(d) as follows:



Here the pictorial representation is quite readily explained, since an arrow connecting any two points of the Dynkin diagram denotes a double soliton whose constituents are associated with the two spots - one from either spot.

It is also convenient to use the notation l, l', l''; h with reference to the points 1,...;4 respectively, to denote the light and heavy mass structure associated with them.

A basic point to note is that in calculating the solutions it is also required that each solution should appear as the two single solitons when these objects are widely separated. Therefore, part of the structure of the double soliton is immediately laid down 'a priori'.

The substitution of the general ansatz

$$\tau_j = 1 + \delta^{(1)}_{j(1)} \cdot f_1 + \delta^{(1)}_{j(2)} \cdot f_2 + W_j(\Theta) f_1 f_2 \qquad j \in \{1,2,3\}$$

$$\tau_4 = 1 + \delta^{(1)}_{4(1)} \cdot f_1 + f_1^2 + \delta^{(1)}_{4(2)} \cdot f_2 + f_2^2 + X(\Theta) \cdot f_1 f_2 + Y^{(1)}(\Theta) \cdot f_1^2 f_2 + Y^{(2)}(\Theta) \cdot f_1 f_2^2 + Z(\Theta) \cdot f_1^2 f_2^2$$

into the equations of motion (3.13), where the τ_0 substitution (3.12) has been used and the $\delta_j^{(i)}$'s are given by coefficients from (3.14), reveals a certain amount of symmetry in the double soliton tau functions. Here f_i denotes $e^{\sigma_i(x - v_i t) + \xi_i}$ and the comment above has been kept in mind, that is, $x \sim v_i t - \xi_i$ gives the single soliton approximation.

What is found after the substitution (keeping in mind $m_i^2 = \lambda_i$ and the fact that parameterization with respect to the rapidity variable Θ can be utilized) is that in all cases the interaction quantities W, Y, Z are given simply by the products of the $\delta_j^{(i)}$'s and powers of $A^{(12)}$. Indeed, they take the form that might be guessed naïvely, if such a solution was to be attempted. The tau functions take the form:

$$\left.\begin{aligned}
\tau_0 &= 1+f_1+f_2+A^{(12)}\cdot f_1f_2 \\
\tau_j &= 1+\delta^{(1)}_{j(1)}\cdot f_1+\delta^{(1)}_{j(2)}\cdot f_2+\delta^{(1)}_{j(1)}\delta^{(1)}_{j(2)}\cdot A^{(12)}\cdot f_1f_2 \\
\tau_4 &= 1+\delta^{(1)}_{4(1)}\cdot f_1+f_1^2+\delta^{(1)}_{4(2)}\cdot f_2^2+f_2^2+X(\Theta)\cdot f_1f_2+ \\
&\quad +\delta^{(1)}_{4(1)}\cdot A^{(12)}\cdot f_1f_2^2+\delta^{(1)}_{4(2)}\cdot A^{(12)}\cdot f_1^2f_2+A^{(12)^2}\cdot f_1^2f_2^2.
\end{aligned}\right\} \tag{3.15}$$

The remaining interaction term $X(\Theta)$ which is the coefficient of the f_1f_2 piece of the tau function associated with the centre spot of the Dynkin diagram is in all cases given by the following formula

$$\begin{aligned}
X(\Theta) &= (2-(p_1+p_2)^2)\cdot A^{(12)}+(2-(p_1-p_2)^2) \\
&\equiv f_-(\Theta)\cdot A^{(12)}+f_+(\Theta)
\end{aligned} \tag{3.16}$$

where

$$f_\pm(\Theta) = (2-\lambda-\lambda'\pm 2\sqrt{\lambda\lambda'}\,ch(\Theta)\ . \tag{3.17}$$

[For the case of I, I', that is (d), it was found more convenient to use the ansatz that the tau functions were given by $\tau_i = 1+(-)^{p_i}\cdot f_1+(-)^{q_i}\cdot f_2+(-)^{r_i}\cdot A^{(12)}f_1f_2$ for the outer spots, with the obvious constraints on the sums of the quantities p_i and q_i respectively. The equations of motion then give $(-)^{p_i+q_i+r_i}=1$, as above].

The basic blocks $A^{(12)}$ and the quantities $X(\Theta)$ derived from them may now be listed using (3.16),(3.17) for each of the separate cases (a),...,(d):

$$\begin{aligned}
A^{(12)} &\overset{(a)}{=} \frac{(ch\Theta-\frac{1}{2})(ch\Theta-1)}{(ch\Theta+\frac{1}{2})(ch\Theta+1)} & X(\Theta) &= \frac{16(ch\Theta-\frac{\sqrt{5}}{2\sqrt{2}})(ch\Theta+\frac{\sqrt{5}}{2\sqrt{2}})}{(ch\Theta+\frac{1}{2})(ch\Theta+1)} \\
&\overset{(b)}{=} \frac{(ch\Theta-\frac{\sqrt{3}}{2})}{(ch\Theta+\frac{\sqrt{3}}{2})} & &= 0 \\
&\overset{(c)}{=} \frac{(ch\Theta+\frac{1}{2})(ch\Theta-1)}{(ch\Theta-\frac{1}{2})(ch\Theta+1)} & &= \frac{2}{(ch\Theta-\frac{1}{2})(ch\Theta+1)} \\
&\overset{(d)}{=} \frac{(ch\Theta-\frac{1}{2})}{(ch\Theta+\frac{1}{2})}\ , & &= 0\ .
\end{aligned}$$

As in the $a_n^{(1)}$ series, the interaction pieces $A^{(12)}$ that appear as the leading terms in the multisoliton solutions can now be written in terms of products of hyperbolic trigonometric functions. One way of achieving this is to use the fact that

$$sh(\frac{\Theta}{2}+x)sh(\frac{\Theta}{2}-x) = sh^2(\frac{\Theta}{2})ch^2(x) - ch^2(\frac{\Theta}{2})sh^2(x)$$

and hence that,

$$sh(\frac{\Theta}{2}+ix)sh(\frac{\Theta}{2}-ix) = sh^2(\frac{\Theta}{2})\cos^2(x) + ch^2(\frac{\Theta}{2})\sin^2(x). \tag{3.18}$$

Then the basic interaction blocks may be written in the following way,

$$\left.\begin{aligned}
A^{(12)} &\overset{(a)}{=} \frac{sh(\frac{\Theta}{2}+\frac{i2\pi}{2h})sh(\frac{\Theta}{2}-\frac{i2\pi}{2h})sh(\frac{\Theta}{2})sh(\frac{\Theta}{2})}{sh(\frac{\Theta}{2}+\frac{i4\pi}{2h})sh(\frac{\Theta}{2}-\frac{i4\pi}{2h})sh(\frac{\Theta}{2}+\frac{i6\pi}{2h})sh(\frac{\Theta}{2}-\frac{i6\pi}{2h})} \\
&\overset{(b)}{=} \frac{sh(\frac{\Theta}{2}+\frac{i\pi}{2h})sh(\frac{\Theta}{2}-\frac{i\pi}{2h})}{sh(\frac{\Theta}{2}+\frac{i5\pi}{2h})sh(\frac{\Theta}{2}-\frac{i5\pi}{2h})} \\
&\overset{(c)}{=} \frac{sh(\frac{\Theta}{2}+\frac{i4\pi}{2h})sh(\frac{\Theta}{2}-\frac{i4\pi}{2h})sh(\frac{\Theta}{2})sh(\frac{\Theta}{2})}{sh(\frac{\Theta}{2}+\frac{i2\pi}{2h})sh(\frac{\Theta}{2}-\frac{i2\pi}{2h})sh(\frac{\Theta}{2}+\frac{i6\pi}{2h})sh(\frac{\Theta}{2}-\frac{i6\pi}{2h})} \\
&\overset{(d)}{=} \frac{sh(\frac{\Theta}{2}+\frac{i2\pi}{2h})sh(\frac{\Theta}{2}-\frac{i2\pi}{2h})}{sh(\frac{\Theta}{2}+\frac{i4\pi}{2h})sh(\frac{\Theta}{2}-\frac{i4\pi}{2h})}
\end{aligned}\right\} \tag{3.19}$$

One immediate fact to notice from these formulae is that on bicolouration of the Dynkin diagram then the $A^{(12)}$ interaction term corresponding to a white - black configuration is constructed from sinh functions with an odd argument x in $sh(\frac{\Theta}{2}\pm\frac{i\pi x}{2h})$, whereas, those corresponding to the same colour all have even values for x; a similar fact has been noted and shown to be true in the scattering matrices and was discussed in the review chapter (section 1.3.3).

Another fact to note is that in $A^{(12)}_{hh}$ and $A^{(12)}_{ll}$ there are double poles in Θ corresponding to a rapidity value of '$i\pi$'. In the formulae given above this is simply a consequence of the fact that sinh is anti-periodic in $i\pi$.

A new notation could be further introduced at this stage that will compactify the expressions for these blocks and prove to be very useful later when the simply-laced exceptional algebras are considered.

Denoting $\omega = e^{\frac{2\pi i}{h}}$, then on taking

$$(1-\omega^x e^{\Theta})^y \equiv (x)^y_{\Theta} \equiv (x)^y, \tag{3.20}$$

the quantities $A^{(12)}$ simply become the following in this new notation:

$$A^{(12)} \overset{(a)}{=} (0)^2(1)^1(2)^{-1}(3)^{-2}(4)^{-1}(5)^1$$

$$
\left.
\begin{aligned}
&\overset{(b)}{=} (\tfrac{1}{2})^{1}(\tfrac{5}{2})^{-1}(\tfrac{7}{2})^{-1}(\tfrac{11}{2})^{1} \\
&\overset{(c)}{=} (0)^{2}(1)^{-1}(2)^{1}(3)^{-2}(4)^{1}(5)^{-1} \\
&\overset{(d)}{=} (1)^{1}(2)^{-1}(4)^{-1}(5)^{1}.
\end{aligned}
\right\} \quad (3.21)
$$

3.3.2 The Fusing Rules for $d_4^{(1)}$

Again the fusing rules mimic those for the real coupling particles that may be found, for example, in Braden et al.[5], but here they are supplement by some 'annihilation' couplings that always occur at rapidity difference $\Theta = i\pi$. The shift $\Psi \to \frac{1}{2}\log A^{(12)}$ again reveals that the fusing occurs at the poles of the interaction block $A^{(12)}$, as is true in all the simply-laced cases. As the Dynkin diagram has been labelled slightly differently from the aforementioned paper, the notation will not be borrowed, but a re-labelling gives exactly the same results as were found in that paper.

3.3.2*(i)* **h, h**

This case will be covered in slightly more detail and then the others should be self explanatory. From (3.19) it can be seen that the poles of $A^{(12)}$ occur at rapidity differences which lie in the physical strip at values $\frac{4\pi i}{h}$ and $i\pi$. After the shift in the arguments of the exponential terms and 'evaluation at a pole' then the coefficient of $f_1 f_2$ in the tau function associated with the inner spot of the Dynkin diagram becomes

$$
X(\Theta)A^{(12)^{-1}}\Bigg|_{\text{pole}} \equiv \frac{16(\mathrm{ch}\Theta - \frac{\sqrt{5}}{2\sqrt{2}})(\mathrm{ch}\Theta + \frac{\sqrt{5}}{2\sqrt{2}})}{(\mathrm{ch}\Theta - \frac{1}{2})(\mathrm{ch}\Theta - 1)}\Bigg|_{\Theta = \frac{4\pi i}{h}, i\pi} = -4, 2
$$

respectively. Hence, the following fusings $hh \to h$ and $hh \to$trivial, occur at rapidity values $\Theta = \frac{4\pi i}{h}$, $i\pi$ repectively, since the following tau functions are obtained:

$$
\begin{aligned}
&\tau_i = 1 + f_1 f_2 && i \in \{0, \ldots, 3\} \\
&\tau_4 = 1 + (-4,\ 2)\cdot f_1 f_2 + f_1^2 f_2^2\ .
\end{aligned}
$$

This is in complete accordance with[5], but in addition there is an 'annihilation' coupling where presumably all quantum numbers cancel in the fusion to give the zero field. In this situation the topological charges must correspond to $+\beta$ and $-\beta$ where $\beta \in \Lambda_R$, since this spot of the Dynkin diagram corresponds to the adjoint representation.

3.3.2*(ii)* **l, h**

From (3.19) it can be seen that the pole of $A^{(12)}$ occurs in the physical strip at $\Theta = \frac{5\pi i}{h}$. Moreover, the coefficient of f_1f_2 in τ_4 is found to be zero:

$$X \cdot A^{(12)^{-1}} = 0,$$

hence, the fusing $lh \to l$ is obtained at rapidity difference $\Theta = \frac{5\pi i}{h}$, as is found in the classical particle regîme.

3.3.2*(iii)* **l, l**

Again, from (3.19) the poles of the interaction block $A^{(12)}$ can be seen to occur at rapidity differences $\Theta = \frac{2\pi i}{h}, \pi i$ respectively, in the physical strip. Following the shift in the exponential terms of the tau functions the new coefficients of the fused soliton can then be obtained:

$$X(\Theta) \cdot A^{(12)^{-1}} \Big|_{\text{pole}} = \frac{2}{(\text{ch}\Theta + \frac{1}{2})(\text{ch}\Theta - 1)} \Bigg|_{\Theta = \frac{2\pi i}{h}, i\pi} = -4, +2$$

respectively, which gives the fusings $l, l \to h$ and $l, l \to$trivial solution, at rapidity values $\Theta = \frac{2\pi i}{h}, i\pi$. So, in the similar manner to that of case *(i)* it is again found that an '$i\pi$' pole has resulted in an 'annihilation' of the two constituent solitons after fusing.

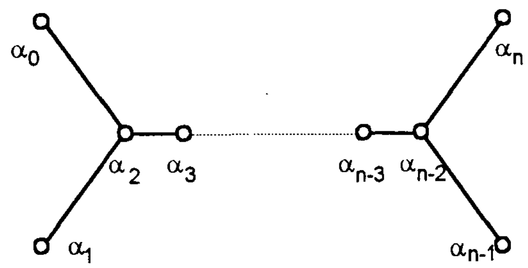
3.3.2*(iv)* **l', l**

Here, the fusing angle is found to be at $\Theta = \frac{4\pi i}{h}$, and again trivially $X \cdot A^{(12)^{-1}} = 0$. Hence, as a direct consequence of the more general ansatz that was commented on above, it can easily be seen that $l, l' \to l''$ at the rapidity difference stated above. That is, the fact that $(-)^{p_i+q_i+r_i} = +1$ where, for example, $(-)^{p_i}$ was the coefficient of the leading term for the i^{th} tau function ($i \in \{0,1,2,3\}$) of one of the single soliton solutions). Here, l, l', l'' all lie in different fundamental representations of the Lie algebra d_4; corresponding to the outer spots of the d_4 Dynkin diagram. The topological charge of the fused soliton is a representation of the fact that $\Lambda_{l''} \subset \Lambda_l \otimes \Lambda_{l'}$.

3.3.3 $d_n^{(1)}$ ($n \geq 5$)

As in the $d_4^{(1)}$ case, none of these algebras give rise to equations of motion for the tau functions which take the true bilinear form, but again it is possible to obtain solutions using the conjecture (2.20) which builds upon the minimal ansatz of $\tau_0 = 1 + e^{\Psi}$.

Again in all these cases the Kač labels associated with the outer spots of the affine Dynkin diagram are unity, whereas all the labels corresponding to the inner arm of the diagram are two and the affine diagram is labelled as follows:



The non-zero eigenvalues of the matrix $K_{ij} = n_i \underset{\sim}{\alpha}_i \cdot \underset{\sim}{\alpha}_j$ are given by

$$\lambda_a = 8\sin^2\theta_a \quad (1 \leq a \leq n-2), \quad \lambda_{n-1} = \lambda_n = 2 \tag{3.22}$$

where the angle is defined as $\theta_a = \frac{a\pi}{h} \equiv \frac{a\pi}{2(n-1)}$ for this series of affine Lie algebras. The fact that λ_{n-1} and λ_n are equal is again a reflection of the degeneracy in the classical particle masses associated with the spinor and anti-spinor representations of these algebras. With such a labelling of the extended root system (on this occasion, corresponding with[60]) the equations of motion (2.18) take the form:

$$\left.\begin{aligned}
&f(\tau_i) = 1m^2(\tau_2 - \tau_i^2) && i = 0, 1 \\
&f(\tau_2) = 2m^2(\tau_0\tau_1\tau_3 - \tau_2^2) && \\
&f(\tau_j) = 2m^2(\tau_{j-1}\tau_{j+1} - \tau_j^2) && j = 3, \ldots\ldots, n-3 \\
&f(\tau_{n-2}) = 2m^2(\tau_n\tau_{n-1}\tau_{n-3} - \tau_{n-2}^2) && \\
&f(\tau_k) = 1m^2(\tau_{n-2} - \tau_k^2) && k = n-1, n.
\end{aligned}\right\} \tag{3.23}$$

This set of equations may be consistently solved to all orders, using the technique mentioned briefly above, to give the following set of single soliton solutions[53].

$\lambda = \lambda_a$:

$$\delta_0^{(1)} = \delta_1^{(1)} = 1, \quad \delta_{n-1}^{(1)} = \delta_n^{(1)} = (-)^n,$$

$$\delta_j^{(1)} = \frac{2\cos((2_j-1)\theta_a)}{\cos\theta_a}, \quad \delta_j^{(2)} = +1 \qquad \text{for } j \in \{2, \dots, n-2\}$$

where the angle θ_a is defined as above, and true for all values of n whether odd or even.

$$\lambda = 2: \qquad (3.24)$$

$$\delta_0^{(1)} = -\delta_1^{(1)} = 1$$

$$\delta_j^{(1)} = 0, \quad \delta_j^{(2)} = (-)^j \qquad \text{for } j \in \{2, \dots, n-2\}$$

$$\delta_{n-1}^{(1)} = -\delta_n^{(1)} = \pm 1, \text{ (n even)}; \quad \delta_{n-1}^{(1)} = -\delta_n^{(1)} = \pm i, \text{ (n odd)}.$$

(These will be used in constructing part of the double soliton tau functions, and may be attached to the Dynkin diagram in the foresaid manner:)

Again, a general tau function can be conjectured for the double soliton solution such that the ansatz (3.01) holds and that the other tau functions possess interaction terms with orders of the exponential terms up to $n_j(\Psi^{(1)}+\Psi^{(2)})$. Similarly, it will be required that such functions approximate the single soliton tau functions when the constituent solitons are widely separated and the local proximity of one of them examined. The substitution of such a general solution into the equations of motion (2.18) again gives rise to tau functions of a similar type to those that appeared in $d_4^{(1)}$:

$$\tau_0 = 1 + f_1 + f_2 + A^{(12)} f_1 f_2$$

$$\tau_1 = 1 + \delta_{1_{(1)}}^{(1)} f_1 + \delta_{1_{(2)}}^{(1)} f_2 + \delta_{1_{(1)}}^{(1)} \delta_{1_{(2)}}^{(1)} A^{(12)} f_1 f_2$$

$$\tau_j = 1 + \delta_{j_{(1)}}^{(1)} f_1 + \delta_{j_{(1)}}^{(2)} f_1^2 + \delta_{j_{(2)}}^{(1)} f_2 + \delta_{j_{(2)}}^{(2)} f_2^2 + X_j(\theta) \cdot f_1 f_2 +$$
$$+ \delta_{j_{(1)}}^{(1)} \cdot \delta_{j_{(2)}}^{(2)} A^{(12)} f_1 f_2^2 + \delta_{j_{(2)}}^{(1)} \cdot \delta_{j_{(1)}}^{(2)} A^{(12)} f_1^2 f_2 + \delta_{j_{(1)}}^{(2)} \delta_{j_{(2)}}^{(2)} A^{(12)^2} \cdot f_1^2 f_2^2$$

$$\tau_{n-1} = 1 + \delta_{n-1_{(1)}}^{(1)} f_1 + \delta_{n-1_{(2)}}^{(1)} f_2 + \delta_{n-1_{(1)}}^{(1)} \delta_{n-1_{(2)}}^{(1)} A^{(12)} f_1 f_2$$

$$\tau_n = 1 + \delta_{n_{(1)}}^{(1)} f_1 + \delta_{n_{(2)}}^{(1)} f_2 + \delta_{n_{(1)}}^{(1)} \delta_{n_{(2)}}^{(1)} A^{(12)} f_1 f_2$$

Here, all the $\delta_j^{(i)}$'s are those that appear in the relevant single soliton solutions (two paragraphs above) and again, $X_2(\theta)$ is given by the formula (3.16). So it can be seen that all the members of the $d_n^{(1)}$-Toda field theories have common features when considering the form of their tau functions.

The double solitons can be separated into distinct classes, in particular they can be taken to be $(\lambda, \lambda') = (\lambda_a, \lambda_b)$, $(\lambda_a, 2)$ and $(2, 2)$. Parameterization with respect to Θ is again utilized and also the quantity $\delta_j^{(1)}$ $(2 \le j \le n-2)$ for $\lambda = \lambda_a$ is given the label $\eta_j^{(a)}$ in order to shorten the expressions in the interaction quantities. The calculations to obtain the interaction pieces in each of the three cases will be given in elaborate detail since they require the solution of difference equations which have particularly interesting properties. It will suffice to say that such properties allow relatively simple expressions for the remaining interaction pieces in the double soliton solutions to be obtained as will now be seen.

3.3.3*(i)* (λ_a, λ_b)

After the substitution of the double soliton ansatz into the equations of motion, consistency requires that the basic block $A^{(12)}$ must take the form:

$$A^{(12)} = \frac{(p_1 \cdot p_2)^2 + (p_1 \cdot p_2)A + B}{(p_1 \cdot p_2)^2 - (p_1 \cdot p_2)A + B}$$

where $$A = -\frac{1}{8}(-8 + 4p_1^2 + 4p_2^2 + 2\eta_2^{(a)}\eta_2^{(b)})$$

and $$B = -\frac{1}{8}(\eta_2^{(a)}\eta_2^{(b)} \cdot (4 - p_1^2 - p_2^2) - 4(\eta_3^{(a)}\eta_3^{(b)}))$$

which, after a somewhat lengthy trigonometric manipulation and parameterization, with respect to rapidity difference, gives the interaction piece as:

$$A^{(12)} = \frac{\mathrm{ch}^2\Theta + \mathrm{ch}\Theta(-2\sin\theta_a \sin\theta_b) + (1 - \sin^2\theta_a - \sin^2\theta_b)}{\mathrm{ch}^2\Theta + \mathrm{ch}\Theta(+2\sin\theta_a \sin\theta_b) + (1 - \sin^2\theta_a - \sin^2\theta_b)}$$

$$\equiv \frac{(\mathrm{ch}\Theta + \cos(\theta_a + \theta_b))(\mathrm{ch}\Theta - \cos(\theta_a - \theta_b))}{(\mathrm{ch}\Theta - \cos(\theta_a + \theta_b))(\mathrm{ch}\Theta + \cos(\theta_a - \theta_b))}. \tag{3.25}$$

Hence, it is easily seen that the poles of $A^{(12)}$ occur at rapidity values of $\Theta = \frac{i\pi(a+b)}{h}$ and $\frac{i\pi(h+a-b)}{h}$, which, before going any further, may be noted to correspond to the fusing values of rapidity difference for the real coupling particles.

The calculation of the remaining interaction pieces, that is, the $X_k(\Theta)$'s involves the solution of a difference equation which can then be matched with the boundary conditions X_1 and X_2; (X_1 is obtained, as it were, backwards through the difference equation involving X_2, X_3, and is utilised because of its simpler form when compared with the form of X_3).

The difference equation takes the following form and at first sight appears formidable:

$$2X_{n+1}+2X_{n-1}-f_{(-)}(\theta_a,\theta_b,\Theta)\cdot X_n = \eta_n^{(a)}\eta_n^{(b)}\cdot f_{(+)}(\theta_a,\theta_b,\Theta)-2(\eta_{n+1}^{(a)}\eta_{n-1}^{(b)}+\eta_{n-1}^{(a)}\eta_{n+1}^{(b)}) \qquad (3.26)$$

where

$$f_{(\pm)}=(4-8\sin^2\theta_a-8\sin^2\theta_b\pm 16\sin\theta_a\sin\theta_b\mathrm{ch}\Theta). \qquad (3.27)$$

Fortunately, this inhomogeneous equation can be solved relatively easily by rewriting the inhomogeneous part using several trigonometric identities. After much reshuffling it can be written in the form:

$$2\cos((2n-1)(\theta_{a+b}))\cdot\{16\tan\theta_a\tan\theta_b(\mathrm{ch}\Theta-\cos(\theta_{a-b}))\}+ 2\cos((2n-1)(\theta_{a-b}))\cdot\{16\tan\theta_a\tan\theta_b(\mathrm{ch}\Theta+\cos(\theta_{a+b}))\}. \qquad (3.28)$$

Hence, a particular solution to the difference equation which has the form

$$a_n=a_+(\theta_a,\theta_b,\Theta)\cdot\cos((2n-1)(\theta_{a+b}))+a_-(\theta_a,\theta_b,\Theta)\cdot\cos((2n-1)(\theta_{a-b})) \qquad (3.29)$$

may be sought, which gives

$$a_\pm=2\sec\theta_a\sec\theta_b\frac{(\mathrm{ch}\Theta\mp\cos(\theta_a\mp\theta_b)}{(\mathrm{ch}\Theta\mp\cos(\theta_a\pm\theta_b)}. \qquad (3.30)$$

However, in the full solution to the inhomogeneous difference equation (3.26), it is found that the homogeneous part can not be written in a closed form purely involving a function of the form $\mathrm{ch}(k\Theta)$. Fortunately, however, an extremely lucky fact occurs, since a comparison with the boundary conditions $X_1(\Theta)=2A^{(12)}+2$ and $X_2(\Theta)$ (equation (3.16)), reveals that the solution to the homogeneous part plays no rôle in the final solution of the difference equation and that the functions $X_k(\Theta)=a_k(\theta_a,\theta_b,\Theta)$ for all $k\in\{2,\dots,n-2\}$.

Re-examining the basic block $A^{(12)}$, it can again be seen that the function can be written as a quotient of sinh functions:

$$A^{(12)}=\frac{\mathrm{sh}(\frac{\Theta}{2}+\frac{i(a-b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}-\frac{i(a-b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}+\frac{i(h+a+b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}-\frac{i(h+a+b)\pi}{2h})}{\mathrm{sh}(\frac{\Theta}{2}+\frac{i(a+b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}-\frac{i(a+b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}+\frac{i(h+a-b)\pi}{2h})\mathrm{sh}(\frac{\Theta}{2}-\frac{i(h+a-b)\pi}{2h})} \qquad (3.31)$$

where the Coxeter number $h=2(n-1)$ for $d_n^{(1)}$. Again this may be written in terms of the new notation with $\omega=e^{\frac{2\pi i}{2(n-1)}}\equiv e^{\frac{2\pi i}{h}}$:-

$$A^{(12)}=\left(\tfrac{a-b}{2}\right)\left(\tfrac{2h-a+b}{2}\right)\left(\tfrac{a+b}{2}\right)^{-1}\left(\tfrac{2h-a-b}{2}\right)^{-1}\left(\tfrac{h+a+b}{2}\right)\left(\tfrac{h-a-b}{2}\right)\left(\tfrac{h+a-b}{2}\right)^{-1}\left(\tfrac{h-a+b}{2}\right)^{-1} \qquad (3.32)$$

Similarly, it can easily be seen that the arguments of such functions are integers if a, b are both even or both odd (or equivalently coloured the same), or they are half integers.

3.3.3*(ii)* **(λ_a, 2)**

This case is by far the simplest of the three, since substitution of the general ansatz into the equations of motion (2.18) merely results in consistency at all orders if $X_k(\Theta)=0$ for $k \in \{2,\ldots,n-2\}$ and the basic interaction block $A^{(12)}$ is given by the following:

$$A^{(12)} = \frac{\mathrm{ch}\Theta - \sin\theta_a}{\mathrm{ch}\Theta + \sin\theta_a}$$

$$= \frac{\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{2h}(3(n-1)+a)) \cdot \mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(3(n-1)+a))}{\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{2h}(n-1+a))\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2h}(n-1+a))}$$

$$= \left(\frac{n-1-a}{2}\right)\left(\frac{n-1+a}{2}\right)^{-1}\left(\frac{3(n-1)+a}{2}\right)\left(\frac{3(n-1)-a}{2}\right)^{-1} \tag{3.33}$$

Hence, the question as to whether the arguments of the (x) blocks are integers or half-integers depends on the algebra and also which point of the Dynkin diagram is being considered.

3.3.3*(iii)* **($\lambda = 2$, $\lambda' = 2$)**

Again, in this case the determination of the $X_k(\Theta)$'s requires the solution of a difference equation. However, here the equation is homogeneous and the solution to it may also be written in a neat closed form. The difference equation under consideration is given by:

$$X_{k+1} + (p_1 \cdot p_2)X_k + X_{k-1} = 0$$

which may be solved in the standard manner. Rewriting in terms of the rapidity difference and using the hyperbolic version of De Moivre's theorem,

$$(\mathrm{ch}\Theta \pm \mathrm{sh}\Theta)^k = \mathrm{ch}(k\Theta) \pm \mathrm{sh}(k\Theta)$$

results (after comparison with the boundary conditions $X_1(\Theta)$, $X_2(\Theta)$ as above) in the interaction pieces,

$$X_k(\Theta) = 2(-)^{k-1} \cdot \left\{ A^{(12)} \cdot \left(\frac{\mathrm{sh}((\frac{2k-1}{2})\Theta)}{\mathrm{sh}(\frac{\Theta}{2})} \right) - \left(\frac{\mathrm{ch}((\frac{2k-1}{2})\Theta)}{\mathrm{ch}(\frac{\Theta}{2})} \right) \right\} \tag{3.34}$$

a formula which is true whether n is odd or even.

Finally, the calculation of the basic interaction piece $A^{(12)}$ reveals that it is dependent not only on the value of n, but also on $\delta^{(1)}_{n-1}$ and $\delta_n^{(1)}$. This may then be substituted into (3.34). What is found is that the functions are given by:

$$A^{(12)} = \frac{ch((\frac{n-1}{2})\Theta)sh(\frac{\Theta}{2})}{sh((\frac{n-1}{2})\Theta)ch(\frac{\Theta}{2})} \qquad (3.35a)$$

for (n odd, $\delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = +1$; n even, $\delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = -1$)

$$A^{(12)} = \frac{sh((\frac{n-1}{2})\Theta)sh(\frac{\Theta}{2})}{ch((\frac{n-1}{2})\Theta)ch(\frac{\Theta}{2})} \qquad (3.35b)$$

for (n odd, $\delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = -1$; n even, $\delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = +1$).

Hence, they give rise to the following interaction pieces:

$$X_{k+1}(\Theta) = 2(-)^k\left\{\frac{sh((\frac{2k+2-n}{2})\Theta)}{sh((\frac{n-1}{2})\Theta)ch(\frac{\Theta}{2})}\right\} \qquad (\text{n odd}, \delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = +1; \ldots\text{etc}\ldots) \qquad (3.36a)$$

$$X_{k+1}(\Theta) = 2(-)^{k+1}\left\{\frac{ch((\frac{2k+2-n}{2})\Theta)}{ch((\frac{n-1}{2})\Theta)ch(\frac{\Theta}{2})}\right\} \qquad (\text{n odd}, \delta^{(1)}_{n_1}\delta^{(1)}_{n_2} = -1; \ldots\text{etc}\ldots) \qquad (3.36b)$$

Therefore, all the rapidity dependent interaction pieces have been found in order to form a consistent double soliton solution to the equations of motion.

Before examining the fusings it may be noted that the pole structure of the $A^{(12)}$'s in this case is a little more delicate than may first appear, due to a couple of cancellations in the numerator and denominator of these functions. In case (a) the $\Theta = 0$ pole and zero cancel for all values of n and is supplemented in the 'n even' case by the cancellation of the '$i\pi$' pole. For n odd, an '$i\pi$' double pole remains. All the poles at their respective rapidity differences may be listed as follows:

$$\Theta = \frac{4\pi i}{h}, \frac{8\pi i}{h}, \ldots\ldots, (i\pi)\times 2, \qquad \text{:for n odd}$$

$$\equiv 2(n-1-a)\frac{\pi i}{h} \qquad \text{where } a = 0, 2, 4, \ldots, n-3 \quad (\text{n odd, a even});$$

$$\Theta = \frac{4\pi i}{h}, \frac{8\pi i}{h}, \ldots\ldots, \frac{2(n-2)}{2(n-1)}\cdot i\pi \qquad \text{:for n even}$$

$$\equiv 2(n-1-a)\frac{\pi i}{h} \qquad \text{where } a = 1, 3, 5, \ldots, n-3 \quad (\text{n even, a odd}).$$

In case (b), the reverse of the above occurs in the sense that the '$i\pi$' pole cancels with an equivalent zero for n odd, whereas in the case n even a double '$i\pi$' pole persists. The poles exist at rapidity difference:

$$\Theta = \frac{2\pi i}{h}, \frac{6\pi i}{h}, \ldots\ldots, \frac{2(n-2)}{2(n-1)} \cdot i\pi \qquad \text{:for n odd}$$

$$\equiv 2(n-1-a)\frac{\pi i}{h} \qquad \text{where } a = 1, 3, 5, \ldots, n-2 \qquad \text{(n odd, a odd);}$$

$$\Theta = \frac{2\pi i}{h}, \frac{6\pi i}{h}, \ldots\ldots, (i\pi) \times 2 \qquad \text{:for n even}$$

$$\equiv 2(n-1-a)\frac{\pi i}{h} \qquad \text{where } a = 0, 2, \ldots, n-2 \qquad \text{(n even, a even).}$$

Hence, (3.35a) may be written in the following manner (with similar expressions holding for (3.35b)) the proof of the equivalence of the formulae being completed by induction:

$$A^{(12)} = \prod_{\substack{a=1 \\ \text{step } 2}}^{n-3} \frac{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{h}(n-2-a))\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{h}(n-2-a))}{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{h}(n-1-a))\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{h}(n-1-a))} \qquad \text{for n even} \qquad (3.37a)$$

$$A^{(12)} = \prod_{\substack{a=0 \\ \text{step } 2}}^{n-3} \frac{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{h}(n-2-a))\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{h}(n-2-a))}{\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{h}(n-1-a))\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{h}(n-1-a))} \qquad \text{for n odd} \qquad (3.37b)$$

Hence, as in all previous cases, the interaction piece can be written as the product of sinh functions. The double pole in '$i\pi$' in the second expression has been disguised again using the periodicity property of the sinh function.

For the sake of clarity at this point, it may be prudent to give a simple example - specifically that of the affine algebra $d_5^{(1)}$ with $\delta^{(1)}_{5_{(1)}}\delta^{(1)}_{5_{(2)}} = +1$. From (3.37b) the interaction block $A^{(12)}$ may be seen to be written as,

$$A^{(12)} = \frac{\mathrm{ch}(2\Theta)\mathrm{sh}(\frac{\Theta}{2})}{\mathrm{sh}(2\Theta)\mathrm{ch}(\frac{\Theta}{2})}.$$

This may then be rearranged systematically (since n is comparatively small in this context) using the facts that $\mathrm{ch}x = \pm\frac{1}{i}\mathrm{sh}(x \pm \frac{i\pi}{2})$ and $\mathrm{sh}2x = 2\,\mathrm{sh}x\,\mathrm{ch}x$. Alternatively, the above formula for general n (with $n = 5$, $h = 8$) gives,

$$A^{(12)} = (-)\frac{\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{8})\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{8})\mathrm{sh}(\frac{\Theta}{2} + \frac{3i\pi}{8})\mathrm{sh}(\frac{\Theta}{2} - \frac{3i\pi}{8})}{\mathrm{sh}(\frac{\Theta}{2} + \frac{i\pi}{4})\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{4})\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2})\mathrm{sh}(\frac{\Theta}{2} - \frac{i\pi}{2})}.$$

3.3.4 The Fusing Rules for $d_n^{(1)}$, ($n \geq 5$)

The fusings of the solitons again correspond with those of the real coupling particles and may be treated on a case-by-case basis. Additional 'iπ' poles always appear in the soliton / antisoliton $A^{(12)}$ interaction functions.

3.3.4*(i)* (λ_a, λ_b)

From (3.31) it can be seen that the poles occur at rapidity differences $\Theta = i(a+b)\frac{\pi}{h}$ and $i(h+a-b)\frac{\pi}{h}$ in the physical strip. Following the shift in the arguments of the exponential terms the coefficient of the $f_1 f_2$ term in τ_k for $2 \leq k \leq n-2$ becomes,

$$X_k(\Theta)A^{(12)^{-1}} = \frac{2\cos((2k-1)(\theta_{a+b}))}{\cos\theta_a \cos\theta_b} \cdot \left(\frac{ch\Theta + \cos(\theta_{a-b})}{ch\Theta + \cos(\theta_{a+b})}\right)$$

$$+\frac{2\cos((2k-1)(\theta_{a-b}))}{\cos\theta_a \cos\theta_b} \cdot \left(\frac{ch\Theta - \cos(\theta_{a+b})}{ch\Theta - \cos(\theta_{a-b})}\right)$$

Hence, evaluation of this quantity at the rapidity values $i(a+b)\frac{\pi}{h}$, $i(h+a-b)\frac{\pi}{h}$ gives $\eta_k^{(a+b)}$ and $\eta_k^{(h+a-b)}$ respectively. Therefore, the fusing of the double soliton solution corresponding to (λ_a, λ_b) results in the single solitons corresponding to $(\lambda_{a+b}, \lambda_{h+a-b})$ at the respective imaginary rapidity differences above. If $a = b$ there is an 'iπ' pole in $A^{(12)}$, this corresponds to an annihilation of the soliton and antisoliton (since 'a' is self-conjugate) resulting in a trivial solution.

3.3.4*(ii)* $(\lambda_a, 2)$

This case is by far the simplest because of the nature of the interaction terms. The interaction block $A^{(12)}$ possesses poles at $ch\Theta = \sin(\theta_a + \pi) = \cos(\frac{\pi}{2} + \theta_a)$, that is at the rapidity difference, $\Theta = \frac{i\pi}{h}(n-1+a)$.

Since $X_k(\Theta) = 0$ for $2 \leq k \leq n-2$ it can be seen that the fusing results in tau functions of the form,

$$\tau_0 = 1 + f_3, \quad \tau_1 = 1 - f_3$$

$$\tau_k = 1 + (-)^k \cdot f_3^2 \qquad 2 \leq k \leq n-2$$

$$\tau_{n-1} = 1 + \delta_{n-1}^{(1)} \cdot (-)^a \cdot f_3$$

$$\tau_n = 1 + \delta_n^{(1)} \cdot (-)^a \cdot f_3$$

where

$$f_3 \equiv e^{\Psi^{(1)} + \Psi^{(2)}}.$$

Examination of the cases n even, n odd, then reveals exactly the same fusing structure that is found, for example, in Braden et al.[5] for the real coupling particles.

3.3.4*(iii)* $(\lambda = 2, \lambda' = 2)$

The pole structure of the two cases of $A^{(12)}$ that appear through the calculations has already been dealt with, but summarizing, the poles occur at $\Theta^{(a)} = 2(n-1-a)\frac{\pi i}{h}$ where,

$a = 0, 2, 4, \ldots, n-3$ (n odd, $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = +1$); $a = 1, 3, 5, \ldots, n-3$ (n even, $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = -1$);

$a = 0, 2, 4, \ldots, n-2$ (n even, $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = +1$); $a = 1, 3, 5, \ldots, n-2$ (n odd, $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = -1$).

From (3.34) it may be seen that for all n and values of $\delta_n^{(1)}$;

$$X_k(\Theta)\, A^{(12)^{-1}}\big|_{pole} = 2(-)^{k-1} \frac{sh\left((\frac{2k-1}{2})2(n-1-a)(\frac{\pi i}{h})\right)}{sh\left(\frac{2(n-1-a)}{2}(\frac{\pi i}{h})\right)}$$

which again may be re-written using some trigonometric identities as

$$\frac{2(-)^{k-1}(-)^{k+1} \cdot i \cdot ch((2k-1)\hat{\theta}_a)}{i \cdot ch(\hat{\theta}_a)} \equiv \frac{2\cos((2k-1)\theta_a)}{\cos\theta_a} \equiv \eta_k^{(a)}$$

where the notation $\hat{\theta}_a = i\theta_a$ has been adopted.

From consideration of the individual cases that arise in this example, it is apparent that $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = (-)^a$ and hence, the following fusing rules are obtained using the notation of [5]. For n odd "$s' = \bar{s}$", and for n even "$s = \bar{s}, s' = \overline{s'}$" (that is, they are both self-conjugate) it is again seen that the fusings of the real coupling case are supplemented by 'soliton and antisoliton' annihilations.

	<u>n odd</u> ($s' = \bar{s}$),		<u>n even</u> ($s = \bar{s}$)	
	ss'	ss, s's'	ss, s's'	ss'
$\Theta^{(a)}$	↓	↓	↓	↓
	a even	a odd	a even	a odd
	+		+	
	{0}		{0}	

where, for n odd: $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = +1 \equiv ss'$

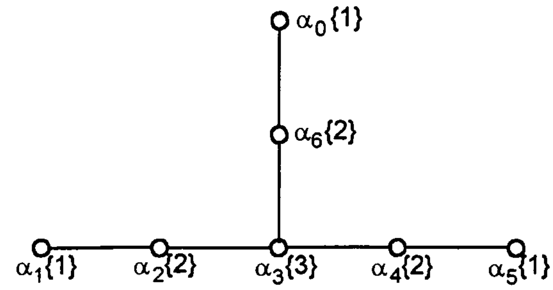
$\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = -1 \equiv ss, s's'$

n even: $\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = +1 \equiv ss, s's'$

$\delta_{n_1}^{(1)}\delta_{n_2}^{(1)} = -1 \equiv ss'$

3.4 Solitons for $e_6^{(1)}$

For this algebra, the affine Dynkin diagram has been labelled with respect to the extended root system in the following manner,



and correspondingly, the Kač labels associated with these roots have been bracketed next to the root labels. The non-zero eigenvalues of K_{ij} for this exceptional Lie algebra again reflect the symmetry embedded in the unextended diagram corresponding to the finite algebra and are given by $\lambda_1 = \lambda_5 = 3 - \sqrt{3}$, $\lambda_2 = \lambda_4 = 3 + \sqrt{3}$, $\lambda_3 = 2(3 + \sqrt{3})$ and finally, $\lambda_6 = 2(3 - \sqrt{3})$. With such a labelling of the root system, the equations of motion for this algebra are,

$$\left.\begin{array}{lll} f(\tau_0) = m^2(\tau_6 - \tau_0^2), & f(\tau_6) = 2m^2(\tau_0\tau_3 - \tau_6^2), & \\ f(\tau_1) = m^2(\tau_2 - \tau_1^2), & f(\tau_2) = 2m^2(\tau_1\tau_3 - \tau_2^2), & \\ f(\tau_5) = m^2(\tau_4 - \tau_5^2), & f(\tau_4) = 2m^2(\tau_5\tau_3 - \tau_4^2), & \& \quad f(\tau_3) = 3m^2(\tau_2\tau_4\tau_6 - \tau_3^2), \end{array}\right\} \tag{3.38}$$

and display the threefold symmetry of the extended diagram. The Coxeter number $h = 12$ for e_6. The equations of motion may be solved consistently to all orders to give the single soliton solutions of the form (2.20) which, as usual, can be associated with the spots on the Dynkin diagram through $\delta_j^{(1)}$.

$\lambda = 3 \pm \sqrt{3}$:

$$\begin{aligned} \tau_0 &= 1 + e^{\Psi} \\ \tau_1 &= 1 + \omega \cdot e^{\Psi} \\ \tau_2 &= 1 + (2-\lambda)\omega \cdot e^{\Psi} + \omega^2 \cdot e^{2\Psi} \\ \tau_3 &= 1 + e^{3\Psi} \\ \tau_4 &= 1 + (2-\lambda)\omega^2 \cdot e^{\Psi} + \omega \cdot e^{2\Psi} \\ \tau_5 &= 1 + \omega^2 \cdot e^{\Psi} \\ \tau_6 &= 1 + (2-\lambda)e^{\Psi} + e^{2\Psi}, \end{aligned}$$

$\lambda = 6 \pm 2\sqrt{3}$:

$$\begin{aligned} \tau_0 &= 1 + e^{\Psi} \\ \tau_1 &= 1 + e^{\Psi} \\ \tau_2 &= 1 + (2-\lambda)e^{\Psi} + e^{2\Psi} \\ \tau_3 &= 1 + 3(\lambda-3)e^{\Psi} + 3(\lambda-3)e^{2\Psi} + e^{3\Psi} \\ \tau_4 &= 1 + (2-\lambda)e^{\Psi} + e^{2\Psi} \\ \tau_5 &= 1 + e^{\Psi} \\ \tau_6 &= 1 + (2-\lambda)e^{\Psi} + e^{2\Psi}. \end{aligned}$$

The other two solutions corresponding to the eigenvalues $\lambda = 3 \pm \sqrt{3}$ are obtained through the exchange symmetry, $\omega \leftrightarrow \omega^2$, where ω is defined to be the primitive third root of unity.

As in the previous algebras a general tau function can be conjectured for the double soliton solution with the premiss that the ansatz (3.01) holds and that the tau functions are required to take the approximate form of the single soliton tau functions when the widely separated double solitons are observed locally. The substitution of the general form into (2.18) similarly gives rise to tau functions with a form of symmetry when the $\delta_j^{(i)}$'s are taken to be the coefficients that occur in the single soliton solutions. Specifically, double soliton tau functions of the following form are obtained,

$$\tau_1 = 1 + \delta^{(1)}_{1_{(1)}} \cdot f_1 + \delta^{(1)}_{1_{(2)}} \cdot f_2 + \delta^{(1)}_{1_{(1)}} \delta^{(1)}_{1_{(2)}} \cdot A^{(12)} \cdot f_1 f_2$$

$$\tau_2 = 1 + \delta^{(1)}_{2_{(1)}} \cdot f_1 + \delta^{(2)}_{2_{(1)}} \cdot f_1^2 + \delta^{(1)}_{2_{(2)}} \cdot f_2 + \delta^{(2)}_{2_{(2)}} \cdot f_2^2 + X_2(\Theta) \cdot f_1 f_2 +$$
$$+ \delta^{(1)}_{2_{(1)}} \delta^{(2)}_{2_{(2)}} \cdot A^{(12)} \cdot f_1 f_2^2 + \delta^{(2)}_{2_{(1)}} \delta^{(1)}_{2_{(2)}} \cdot A^{(12)} \cdot f_1^2 f_2 + \delta^{(2)}_{2_{(1)}} \delta^{(2)}_{2_{(2)}} \cdot A^{(12)^2} \cdot f_1^2 f_2^2$$

$$\tau_3 = 1 + \delta^{(1)}_{3_{(1)}} \cdot f_1 + \delta^{(2)}_{3_{(1)}} \cdot f_1^2 + f_1^3 + \delta^{(1)}_{3_{(2)}} \cdot f_2 + \delta^{(2)}_{3_{(2)}} \cdot f_2^2 + f_2^3 + X_3(\Theta) \cdot f_1 f_2 +$$
$$+ Y_3(\Theta)\left(f_1^2 f_2 + f_1 f_2^2\right) + X_3(\Theta) \cdot A^{(12)} \cdot f_1^2 f_2^2 + \delta^{(1)}_{3_{(1)}} \cdot A^{(12)} \cdot f_1 f_2^3 + \delta^{(1)}_{3_{(2)}} \cdot A^{(12)} \cdot f_1^3 f_2 +$$
$$+ \delta^{(2)}_{3_{(1)}} \cdot A^{(12)^2} \cdot f_1^2 f_2^3 + \delta^{(2)}_{3_{(2)}} \cdot A^{(12)^2} \cdot f_1^3 f_2^2 + A^{(12)^3} \cdot f_1^3 f_2^3$$

τ_4, τ_6: similar form to τ_2 with $\delta^{(i)}_{2_{(j)}} \to \delta^{(i)}_{4_{(j)}}, \delta^{(i)}_{6_{(j)}}$ and $X_2 \to X_4, X_6$ respectively.

τ_5, τ_0: similar form to τ_1 with $\delta^{(i)}_{1_{(j)}} \to \delta^{(i)}_{5_{(j)}}, \delta^{(i)}_{0_{(j)}}$, such that $\delta^{(i)}_{0_{(j)}} = 1$ is understood.

Hence, the tau functions display the characteristic three-fold symmetry of the extended Dynkin diagram.

Only one interaction term is required for the $f_i^1 f_j^2$ terms in τ_3. This fact follows via the equations of motion, as a direct consequence of the fact that $\delta_3^{(1)} = \delta_3^{(2)}$ for each single soliton tau function.

Again, the double solitons can be separated into classes in order to list the interaction functions, in fact these may be taken to be, $(\lambda, \lambda') = (6 \pm 2\sqrt{3}, 6 \pm 2\sqrt{3})$, $(6 \pm 2\sqrt{3}, 3 \pm \sqrt{3})$ and $(3 \pm \sqrt{3}, 3 \pm \sqrt{3})$ respectively. The interaction terms $X_3(\Theta), Y_3(\Theta)$ are expressible in terms of $A^{(12)}$, but these expressions are of no apparent importance and will be omitted; only the explicit form of these functions will be given in each case. Of course, all the expressions could have been collectively left in terms of chΘ, λ, λ' in each case, but because of the fact that the number of subcases is comparatively small for this algebra when compared with the other two simply-laced exceptional algebras, the explicit expressions for all the interaction terms for all cases will be given.

3.4.0*(i)* $(\lambda = 6 \pm 2\sqrt{3}, \lambda' = 6 \pm 2\sqrt{3})$

The interaction terms for consistency at all orders of the equations of motion were found to be given by the following (where the fact that $\frac{1}{2}(\lambda^2 - 6\lambda + 6) = 3(\lambda - 3)$ proved to be of great use):

(a) $\lambda = \lambda' = 6 \pm 2\sqrt{3}$:

$$A^{(12)}_{(\pm)} = \frac{(\mathrm{ch}\Theta - 1)(2\mathrm{ch}\Theta - 1)(2\mathrm{ch}\Theta \mp \sqrt{3})}{(\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta \pm \sqrt{3})},$$

$$X_{2_{(\pm)}}(\Theta) = \frac{(28 \pm 16\sqrt{3})\mathrm{ch}\Theta(4\mathrm{ch}^2\Theta + (2 \mp 3\sqrt{3}))}{(\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta \pm \sqrt{3})},$$

$$= X_{4_{(\pm)}}(\Theta), X_{6_{(\pm)}}(\Theta)$$

and

$$X_{3_{(\pm)}}(\Theta) = \pm 9\left(\frac{A_{(\pm)}\mathrm{ch}^3\Theta + B_{(\pm)}\mathrm{ch}^2\Theta + C_{(\pm)}\mathrm{ch}\Theta + D_{(\pm)}}{(\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta \pm \sqrt{3})}\right),$$

$$Y_{3_{(\pm)}}(\Theta) = \pm 9\left(\frac{A_{(\pm)}\mathrm{ch}^3\Theta - B_{(\pm)}\mathrm{ch}^2\Theta + C_{(\pm)}\mathrm{ch}\Theta - D_{(\pm)}}{(\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta \pm \sqrt{3})}\right),$$

where

$$A_{(\pm)} = 12(4\sqrt{3} \pm 7),\ B_{(\pm)} = 2(19\sqrt{3} \pm 33),\ C_{(\pm)} = -33\sqrt{3} \mp 58,\ D_{(\pm)} = -25\sqrt{3} \mp 44$$

It may be noted that there exists the symmetry $3 \leftrightarrow -\sqrt{3}$ in all the above expressions when $\lambda = 6 + 2\sqrt{3} \leftrightarrow \lambda = 6 - 2\sqrt{3}$.

(b) $\lambda = 6 + 2\sqrt{3}$, $\lambda' = 6 - 2\sqrt{3}$:

$$A^{(12)} = \frac{(2\sqrt{2}\,\mathrm{ch}\Theta - (1 + \sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta + (1 - \sqrt{3}))}{(2\sqrt{2}\,\mathrm{ch}\Theta + (1 + \sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta - (1 - \sqrt{3}))},$$

$$X_2(\Theta) = \frac{32\mathrm{ch}^2\Theta - 40}{(2\sqrt{2}\,\mathrm{ch}\Theta + (1 + \sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta - (1 - \sqrt{3}))},$$

and
$$X_3(\Theta) = 18\left(\frac{A\,\mathrm{ch}^2\Theta + B\,\mathrm{ch}\Theta + C}{(2\sqrt{2}\,\mathrm{ch}\Theta + (1+\sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta - (1-\sqrt{3}))}\right),$$

$$Y_3(\Theta) = 18\left(\frac{A\,\mathrm{ch}^2\Theta - B\,\mathrm{ch}\Theta + C}{(2\sqrt{2}\,\mathrm{ch}\Theta + (1+\sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta - (1-\sqrt{3}))}\right),$$

where $A = -12$, $B = -2\sqrt{6}$, $C = 5$.

Again $X_2(\Theta) = X_4(\Theta) = X_6(\Theta)$ in this case.

3.4.0*(ii)* $(\lambda = 6 \pm 2\sqrt{3},\ \lambda' = 3 \pm \sqrt{3})$.

Here it is found that many of the interaction terms disappear when consistency at all orders is required, specifically that both functions $X_3(\Theta) = Y_3(\Theta) = 0$. It was also found advantageous to introduce the notation used in[5], with respect to the masses of the particles and utilized in the classification of the double soliton interaction terms. This was not necessarily required but is used to emphasise the degeneracy of the $\lambda = 3 \pm \sqrt{3}$ solitons. The equations of motion for the algebra also reveal that $X_2(\Theta) = \delta^{(1)}_{2_{(1)}}\delta^{(1)}_{2_{(2)}} \cdot X_6(\Theta)$ and $X_4(\Theta) = \delta^{(1)}_{4_{(1)}}\delta^{(1)}_{4_{(1)}} \cdot X_6(\Theta)$. Hence, these quantities are merely, $(\omega/\omega^2)X_6(\Theta)$, where the coefficient is subcase dependent.

(a) $\lambda = 6 + 2\sqrt{3}$, $\lambda' = 3 + \sqrt{3}$:

$$A^{(12)}_{Hh} = A^{(12)}_{H\bar{h}} = \frac{(\sqrt{2}\,\mathrm{ch}\Theta - 1)(2\sqrt{2}\,\mathrm{ch}\Theta - (1+\sqrt{3}))}{(\sqrt{2}\,\mathrm{ch}\Theta + 1)(2\sqrt{2}\,\mathrm{ch}\Theta + (1+\sqrt{3}))},$$

$$X_6(\Theta) = \frac{4(10+6\sqrt{3})(\mathrm{ch}^2\Theta - (5-\sqrt{3}))}{(\sqrt{2}\,\mathrm{ch}\Theta + 1)(2\sqrt{2}\,\mathrm{ch}\Theta + (1+\sqrt{3}))},$$

(b) $\lambda = 6 - 2\sqrt{3}$, $\lambda' = 3 + \sqrt{3}$:

$$A^{(12)}_{Lh} = A^{(12)}_{L\bar{h}} = \frac{2\mathrm{ch}\Theta - \sqrt{3}}{2\mathrm{ch}\Theta + \sqrt{3}}, \qquad X_6(\Theta) = \frac{2(2\sqrt{3} - 2)\mathrm{ch}\Theta}{2\mathrm{ch}\Theta + \sqrt{3}},$$

(c) $\lambda = 6 + 2\sqrt{3}$, $\lambda' = 3 - \sqrt{3}$:

$$A^{(12)}_{Hl} = A^{(12)}_{H\bar{l}} = \frac{2\mathrm{ch}\Theta - \sqrt{3}}{2\mathrm{ch}\Theta + \sqrt{3}}, \qquad X_6(\Theta) = \frac{2(-2-2\sqrt{3})\mathrm{ch}\Theta}{2\mathrm{ch}\Theta + \sqrt{3}},$$

(d) $\lambda = 6 - 2\sqrt{3}$, $\lambda' = 3 - \sqrt{3}$:

$$A^{(12)}_{Ll} = A^{(12)}_{L\bar{l}} = \frac{(\sqrt{2}\,\mathrm{ch}\Theta - 1)(2\sqrt{2}\,\mathrm{ch}\Theta - (1-\sqrt{3}))}{(\sqrt{2}\,\mathrm{ch}\Theta + 1)(2\sqrt{2}\,\mathrm{ch}\Theta + (1-\sqrt{3}))},$$

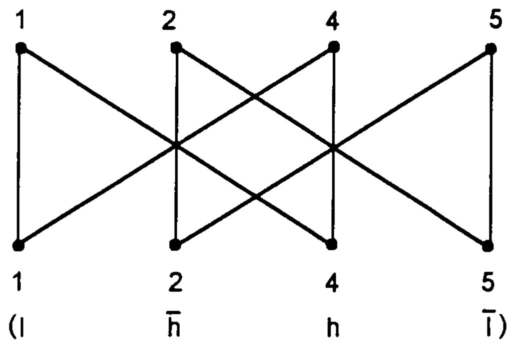
$$X_6(\Theta) = \frac{4(10-6\sqrt{3})(\mathrm{ch}^2\Theta - (5+\sqrt{3}))}{(\sqrt{2}\,\mathrm{ch}\Theta + 1)(2\sqrt{2}\,\mathrm{ch}\Theta + (1-\sqrt{3}))}.$$

(That is, these are obtained by $\sqrt{3} \to -\sqrt{3}$ in the case, $\lambda = 6 + 2\sqrt{3}$, $\lambda' = 3 + \sqrt{3}$).

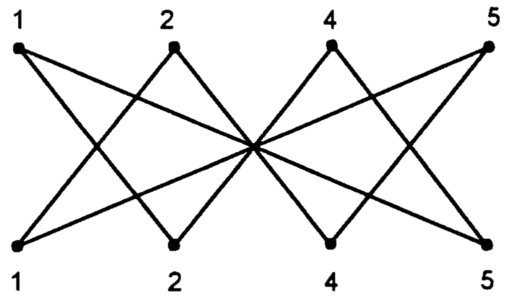
3.4.0*(iii)* $(\lambda = 3 \pm \sqrt{3}\ ,\ \lambda' = 3 \pm \sqrt{3}\ )$

This set of calculations may be split into two sub-sections, those that have $\delta^{(1)}_{1_{(1)}} = \delta^{(1)}_{1_{(2)}}$ and those with $\delta^{(1)}_{1_{(1)}} = \overline{\delta^{(1)}_{1_{(2)}}}$, where $\bar{A}$ denotes the complex conjugate of A. These double soliton solutions may be pictorially represented by the following, where the number label represents the spot of the Dynkin diagram with which the soliton is associated and the 'mass notation' is given below. Any line joining two such points represents a possible double soliton in each case:

(a) $\delta^{(1)}_{1_{(1)}} = \delta^{(1)}_{1_{(2)}}$



(b) $\delta^{(1)}_{1_{(1)}} = \overline{\delta^{(1)}_{1_{(2)}}}$



Again, $X_2(\Theta) = \delta^{(1)}_{2_{(1)}} \cdot \delta^{(1)}_{2_{(2)}} X_6(\Theta)$ and $X_4(\Theta) = \delta^{(1)}_{4_{(1)}} \cdot \delta^{(1)}_{4_{(2)}} X_6(\Theta)$ in all of these cases where $X_6(\Theta)$ is given by the expression (?). The interaction terms can now be merely listed using the mass notation. All these terms again appear as polynomials in the basic block $A^{(12)}$ with coefficients that are functions of the rapidity variable, that is, more precisely, 'chΘ'.

(a)
$$A^{(12)}_{hh} = A^{(12)}_{\bar{h}\bar{h}} = \frac{4\mathrm{ch}\Theta(\mathrm{ch}\Theta - 1)}{(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + \sqrt{3})},$$

$$A^{(12)}_{ll} = A^{(12)}_{\bar{l}\bar{l}} = \frac{4\mathrm{ch}\Theta(\mathrm{ch}\Theta - 1)}{(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta - \sqrt{3})},$$

(that is $\sqrt{3} \to -\sqrt{3}$ in the above expression),

and
$$A^{(12)}_{lh} = A^{(12)}_{\bar{l}\bar{h}} = \frac{(\mathrm{ch}\Theta - \frac{1}{\sqrt{2}})(\mathrm{ch}\Theta + \frac{1}{\sqrt{2}})}{(\mathrm{ch}\Theta + (\frac{1+\sqrt{3}}{2\sqrt{2}}))(\mathrm{ch}\Theta + (\frac{\sqrt{3}-1}{2\sqrt{2}}))}.$$

From these are obtained,

$$X_6(\Theta)_{hh} = X_6(\Theta)_{\bar{h}\bar{h}} = \frac{4(4+2\sqrt{3})\mathrm{ch}^2\Theta + 2(1+\sqrt{3})\mathrm{ch}\Theta - 2(3+2\sqrt{3})}{(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + \sqrt{3})}$$

$$X_6(\Theta)_{ll} = X_6(\Theta)_{\bar{l}\bar{l}} = [\sqrt{3} \to -\sqrt{3} \text{ in the above}],$$

$$X_6(\Theta)_{lh} = X_6(\Theta)_{\bar{l}\bar{h}} = \frac{-2(\mathrm{ch}\Theta + (\frac{\sqrt{3}-\sqrt{19}}{4\sqrt{2}}))(\mathrm{ch}\Theta + (\frac{\sqrt{3}+\sqrt{19}}{4\sqrt{2}}))}{(\mathrm{ch}\Theta + (\frac{1+\sqrt{3}}{2\sqrt{2}}))(\mathrm{ch}\Theta + (\frac{\sqrt{3}-1}{2\sqrt{2}}))};$$

finally,

$$X_3(\Theta) = 0 \text{ for all pairs } (h,h),\ (\bar{h},\bar{h}),\ (l,l),\ (\bar{l},\bar{l}),\ (l,h),\ (\bar{l},\bar{h}).$$

and

$$Y_3(\Theta)_{hh} = Y_3(\Theta)_{\bar{h}\bar{h}} = \frac{18 + 9\sqrt{3}}{(2\mathrm{ch}\Theta + 1)(2\mathrm{ch}\Theta + \sqrt{3})}$$

$$Y_3(\Theta)_{ll} = Y_3(\Theta)_{\bar{l}\bar{l}} = [\sqrt{3} \to -\sqrt{3} \text{ in the above}]$$

$$Y_3(\Theta)_{lh} = Y_3(\Theta)_{\bar{l}\bar{h}} = \frac{-18}{(2\sqrt{2}\,\mathrm{ch}\Theta + (1+\sqrt{3}))(2\sqrt{2}\,\mathrm{ch}\Theta + (\sqrt{3}-1))}.$$

(b) The interaction terms are given by the following:

$$A^{(12)}_{h\bar{h}} = \frac{(2\mathrm{ch}\Theta - 1)(2\mathrm{ch}\Theta - \sqrt{3})}{4\mathrm{ch}\Theta(\mathrm{ch}\Theta + 1)},$$

$$A^{(12)}_{l\bar{l}} = [\sqrt{3} \to -\sqrt{3} \text{ in the above}],$$

$$A^{(12)}_{l\bar{h}} = A^{(12)}_{\bar{l}h} = \frac{(\mathrm{ch}\Theta - (\frac{1+\sqrt{3}}{2\sqrt{2}}))(\mathrm{ch}\Theta - (\frac{\sqrt{3}-1}{2\sqrt{2}}))}{(\mathrm{ch}\Theta - \frac{1}{\sqrt{2}})(\mathrm{ch}\Theta + \frac{1}{\sqrt{2}})}.$$

and from these are obtained

$$X_6(\Theta)_{h\bar{h}} = \frac{(4+2\sqrt{3})\mathrm{ch}^2\Theta + (-\frac{1}{2} - \frac{\sqrt{3}}{2})\mathrm{ch}\Theta + (-\frac{3}{2} - \sqrt{3})}{\mathrm{ch}\Theta(\mathrm{ch}\Theta + 1)}$$

$$X_6(\Theta)_{\bar{l}l} = [\sqrt{3} \rightarrow -\sqrt{3} \text{ in the above}]$$

$$X_6(\Theta)_{l\bar{h}} = X_6(\Theta)_{\bar{l}h} = \frac{-2(\mathrm{ch}\Theta - (\frac{\sqrt{3}-\sqrt{19}}{4\sqrt{2}}))(\mathrm{ch}\Theta - (\frac{\sqrt{3}+\sqrt{19}}{4\sqrt{2}}))}{(\mathrm{ch}\Theta - \frac{1}{\sqrt{2}})(\mathrm{ch}\Theta + \frac{1}{\sqrt{2}})}$$

and finally,

$$Y_3(\Theta) = 0 \text{ for all pairs } (h,\bar{h}),\ (l,\bar{l}),\ (l,\bar{h}), \text{ and } (\bar{l},h)$$

$$X_3(\Theta)_{h\bar{h}} = \frac{18+9\sqrt{3}}{4\mathrm{ch}\Theta(\mathrm{ch}\Theta+1)},$$

$$X_3(\Theta)_{l\bar{l}} = [\sqrt{3} \rightarrow -\sqrt{3} \text{ in the above}]$$

$$X_3(\Theta)_{l\bar{h}} = Y_3(\Theta)_{\bar{l}h} = \frac{-9}{2(\sqrt{2}\,\mathrm{ch}\Theta - 1)(\sqrt{2}\,\mathrm{ch}\Theta + 1)}.$$

Hence, it can be seen that there is a substantial amount of symmetry displayed between the interaction quantities of the two subcases. Namely, there are (-) sign shifts in the odd powers of (chΘ) in the numerators of the interaction pieces $X_6(\Theta)$ and slightly different to the previous cases, the numerators of the two sets of interaction terms $X_3(\Theta)$, $Y_3(\Theta)$ are swapped between subcases (this is similar to the Coxeter rotated simple roots using the symmetry of the Dynkin diagram - see Appendix A).

All the interaction terms $A^{(12)}$ that appear in cases (i),(ii) and (iii) may be written as products of sinh functions, as has occurred in all the other cases so far. Since these expressions are quite lengthy and not of direct importance (other than their existence) the quantities $A^{(12)}$ shall only be listed in terms of the notation (x) where ω is now taken to be $e^{\frac{2\pi i}{h}} \equiv e^{\frac{\pi i}{6}}$ and not to be confused with the third root of unity used in the single soliton solutions. They will be listed in an appendix to this chapter.

3.4.1 The Fusing Rules for $e_6^{(1)}$

All the fusions possible for the double solitons either correspond to those appearing as three point couplings in the real coupling classical particle theory or, those that do not, can be explained away as 'annihilation' couplings of soliton - antisoliton. The latter always occur at rapidity difference 'iπ'. Namely there are HH, LL, $l\bar{l}$, $h\bar{h}$ annihilations, the first two of these due to the fact that the particles/solitons are self conjugate in the classical theory. The fusmerely result in the production of the trivial field associated with the zeroth spot of the extended Dynkin diagram.

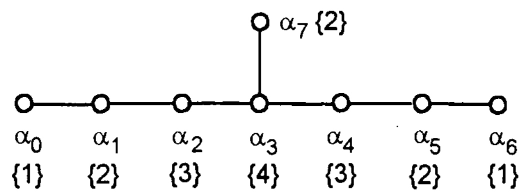
3.4.2 Appendix

Here is the list of interaction quantities $A^{(12)}$ for $e_6^{(1)}$, written in terms of the notation (x):

$$
\left.\begin{aligned}
&\text{(i)} && A_{\pm}^{(12)} = (1)^{\pm1}(2)^{+1}(4)^{-1}(5)^{\mp1}(6)^{-2}(7)^{\mp1}(8)^{-1}(10)^{+1}(11)^{\pm1}(12)^{+2} \\
& && A_{+-}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{5}{2})^{+1}(\tfrac{7}{2})^{-1}(\tfrac{11}{2})^{-1}(\tfrac{13}{2})^{-1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{+1}(\tfrac{23}{2})^{+1} \\
&\text{(ii)} && A_{Hh,H\bar{h}}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{+1}(\tfrac{9}{2})^{-1}(\tfrac{11}{2})^{-1}(\tfrac{13}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{+1}(\tfrac{23}{2})^{+1} \\
& && A_{Lh,L\bar{h}}^{(12)} = A_{Hl,H\bar{l}}^{(12)} = (1)^{+1}(5)^{-1}(7)^{-1}(11)^{+1} \\
& && A_{Ll,L\bar{l}}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{5}{2})^{-1}(\tfrac{7}{2})^{+1}(\tfrac{9}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{17}{2})^{+1}(\tfrac{19}{2})^{-1}(\tfrac{21}{2})^{+1} \\
&\text{(iii)(a)} && A_{hh,\overline{hh}}^{(12)} = (0)^{+2}(3)^{+1}(4)^{-1}(5)^{-1}(7)^{-1}(8)^{-1}(9)^{+1} \\
& && A_{ll,\overline{ll}}^{(12)} = (0)^{+2}(1)^{-1}(3)^{+1}(4)^{-1}(8)^{-1}(9)^{+1}(11)^{-1} \\
& && A_{Hh,H\bar{h}}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{7}{2})^{-1}(\tfrac{9}{2})^{+1}(\tfrac{11}{2})^{-1}(\tfrac{13}{2})^{-1}(\tfrac{15}{2})^{+1}(\tfrac{17}{2})^{-1}(\tfrac{21}{2})^{+1} \\
&\text{(iii)(b)} && A_{h\bar{h}}^{(12)} = (1)^{+1}(2)^{+1}(3)^{-1}(6)^{-2}(9)^{-1}(10)^{+1}(11)^{+1} \\
& && A_{l\bar{l}}^{(12)} = (2)^{+1}(3)^{-1}(5)^{+1}(6)^{-2}(7)^{+1}(9)^{-1}(10)^{+1} \\
& && A_{l\bar{h},\bar{l}h}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{-1}(\tfrac{5}{2})^{+1}(\tfrac{9}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{19}{2})^{+1}(\tfrac{21}{2})^{-1}(\tfrac{23}{2})^{+1}
\end{aligned}\right\} \tag{3.39}
$$

3.5 Solitons for $e_7^{(1)}$

The extended Dynkin diagram corresponding to this affine algebra is given by the following (where again the Kač labels associated with the extended root system have been bracketed after the root labels):



The non-zero eigenvalues of the matrix K_{ij} are given in this case by:

$$\lambda_1 = 8\sqrt{3}\,\sin\left(\frac{\pi}{18}\right)\sin\left(\frac{2\pi}{9}\right), \qquad \lambda_4 = 8\sin^2\left(\frac{4\pi}{9}\right) \quad \& \quad \lambda_2 = 8\sin^2\left(\frac{\pi}{3}\right) = 6$$

$$\lambda_3 = 8\sqrt{3}\,\sin\left(\frac{7\pi}{18}\right)\sin\left(\frac{4\pi}{9}\right), \qquad \lambda_6 = 8\sin^2\left(\frac{\pi}{9}\right)$$

$$\lambda_5 = 8\sqrt{3}\,\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{\pi}{9}\right), \qquad \lambda_7 = 8\sin^2\left(\frac{2\pi}{9}\right)$$

With such a labelling of the extended root system, the equations of motion take the form:

$$\left.\begin{array}{ll} f(\tau_0) = 1\cdot m^2(\tau_1-\tau_0^2), & f(\tau_6) = 1\cdot m^2(\tau_5-\tau_6^2) \\ f(\tau_1) = 2\cdot m^2(\tau_0\tau_2-\tau_1^2), & f(\tau_5) = 2\cdot m^2(\tau_4\tau_6-\tau_5^2) \\ f(\tau_2) = 3\cdot m^2(\tau_1\tau_3-\tau_2^2), & f(\tau_4) = 3\cdot m^2(\tau_3\tau_5-\tau_4^2) \\ \text{and} & \\ f(\tau_3) = 4\cdot m^2(\tau_2\tau_4\tau_7-\tau_3^2), & f(\tau_7) = 2\cdot m^2(\tau_3-\tau_7^2) \end{array}\right\} \quad (3.40)$$

The symmetry of the affine diagram is manifestly portrayed in the first six of these equations under the substitution, $0\leftrightarrow 6$, $1\leftrightarrow 5$, $2\leftrightarrow 4$. The equations may be solved to all orders to give single soliton solutions for each eigenvalue. These take the following form:

λ_2:

$$\tau_0 = \tau_6 = 1+f, \qquad \tau_1 = \tau_5 = \tau_7 = 1-4f+f^2$$

$$\tau_2 = \tau_4 = (1+f)^3, \qquad \tau_3 = (1+f)^4$$

$\lambda_{1,3,5}$:

$$\tau_0 = \tau_6 = 1+f, \qquad \tau_1 = \tau_5 = 1+(2-\lambda)f+f^2, \qquad \tau_7 = 1+2(\lambda-2)f+f^2$$

$$\tau_2 = \tau_4 = 1+\frac{1}{2}(\lambda^2-6\lambda+6)f+\frac{1}{2}(\lambda^2-6\lambda+6)f^2+f^3$$

$$\tau_3 = 1-(\lambda^2-6\lambda+8)f+2(2\lambda^2-9\lambda+9)f^2-(\lambda^2-6\lambda+8)f^3+f^4$$

$\lambda_{4,6,7}$:

$$\tau_0 = 1+f, \qquad \tau_6 = 1-f$$

$$\tau_1 = 1+(2-\lambda)f+f^2, \qquad \tau_5 = 1-(2-\lambda)f+f^2$$

$$\tau_2 = 1+\frac{1}{2}(\lambda^2-6\lambda+6)f+\frac{1}{2}(\lambda^2-6\lambda+6)f^2+f^3$$

$$\tau_4 = 1-\frac{1}{2}(\lambda^2-6\lambda+6)f+\frac{1}{2}(\lambda^2-6\lambda+6)f^2-f^3$$

$$\tau_3 = 1+2(\lambda-1)f^2+f^4$$

$$\tau_7 = 1-f^2$$

and display the characteristic symmetry of the soliton solutions which appears due to a rescaling of the fields. That is, if $n_k = 3$ then $\delta_k^{(1)} = \pm\delta_k^{(2)}$, $\delta_k^{(0)} = \pm\delta_k^{(3)}$ and if $n_k = 4$ then $\delta_k^{(1)} = \delta_k^{(3)}$, $\delta_k^{(0)} = \delta_k^{(4)}$.

All the non-trivial eigenvalues satisfy cubic polynomial constraints which are given by:

$$\lambda^3 - 12\cdot\lambda^2 + 36\cdot\lambda - 24 = 0 \qquad \text{for} \qquad \lambda \in \{\lambda_4, \lambda_6, \lambda_7\} \tag{3.41a}$$

$$\lambda^3 - 18\cdot\lambda^2 + 72\cdot\lambda - 72 = 0 \qquad \text{for} \qquad \lambda \in \{\lambda_1, \lambda_3, \lambda_5\} \tag{3.41b}$$

These characteristic polynomials are essential in simplifying the otherwise unwieldy expressions that appear for the $\delta_j^{(i)}$'s which result from the order-by-order check.

For this affine algebra the double soliton tau functions take the following form; again this results from the substitution of a general ansatz into the equations of motion. (The $\delta_j^{(i)}$'s are similarly the coefficients that appear in the single soliton solutions.)

$$\tau_0 = 1 + \delta_{0_{(1)}}^{(1)}\cdot f_1 + \delta_{0_{(2)}}^{(1)}\cdot f_2 + \delta_{0_{(1)}}^{(1)}\delta_{0_{(2)}}^{(1)}\cdot A^{(12)}\cdot f_1 f_2$$

$$\begin{aligned}\tau_1 = 1 &+ \delta_{1_{(1)}}^{(1)}\cdot f_1 + \delta_{1_{(1)}}^{(2)}\cdot f_1^2 + \delta_{1_{(2)}}^{(1)}\cdot f_2 + \delta_{1_{(2)}}^{(2)}\cdot f_2^2 + A_1(\Theta) f_1 f_2 + \delta_{1_{(1)}}^{(1)}\delta_{1_{(2)}}^{(2)}\cdot A^{(12)}\cdot f_1 f_2^2 + \\ &\delta_{1_{(2)}}^{(1)}\delta_{1_{(1)}}^{(2)}\cdot A^{(12)}\cdot f_1^2 f_2 + \delta_{1_{(1)}}^{(2)}\delta_{1_{(2)}}^{(2)}\cdot A^{(12)^2}\cdot f_1^2 f_2^2\end{aligned}$$

$$\begin{aligned}\tau_2 = 1 &+ \delta_{2_{(1)}}^{(1)}\cdot f_1 + \delta_{2_{(1)}}^{(2)}\cdot f_1^2 + \delta_{2_{(1)}}^{(3)}\cdot f_1^3 + \delta_{2_{(2)}}^{(1)}\cdot f_2 + \delta_{2_{(2)}}^{(2)}\cdot f_2^2 + \delta_{2_{(2)}}^{(3)}\cdot f_2^3 + A_2(\Theta) f_1 f_2 + \\ &B_2^{(1)}(\Theta) f_1^2 f_2 + B_2^{(2)}(\Theta) f_1 f_2^2 + C_2(\Theta)\cdot A^{(12)}\cdot f_1^2 f_2^2 + \\ &\quad \delta_{2_{(1)}}^{(1)}\delta_{2_{(2)}}^{(3)}.A^{(12)}.f_1 f_2^3 + \delta_{2_{(1)}}^{(2)}\delta_{2_{(2)}}^{(3)}.A^{(12)^2}.f_1^2 f_2^3 + \delta_{2_{(1)}}^{(3)}\delta_{2_{(2)}}^{(1)}.A^{(12)}.f_1^3 f_2 + \\ &\quad\quad \delta_{2_{(1)}}^{(3)}\delta_{2_{(2)}}^{(2)}.A^{(12)^2}.f_1^3 f_2^2 + \delta_{2_{(1)}}^{(3)}\delta_{2_{(2)}}^{(3)}.A^{(12)^3}.f_1^3 f_2^3.\end{aligned}$$

$$\begin{aligned}\tau_3 = 1 &+ \delta_{3_{(1)}}^{(1)} f_1 + \delta_{3_{(1)}}^{(2)} f_1^2 + \delta_{3_{(1)}}^{(3)} f_1^3 + f_1^4 + \delta_{3_{(2)}}^{(1)} f_2 + \delta_{3_{(2)}}^{(2)} f_2^2 + \delta_{3_{(2)}}^{(3)} f_2^3 + f_2^4 + A_3(\Theta) f_1 f_2 + \\ &B_3^{(1)}(\Theta) f_1^2 f_2 + B_3^{(2)}(\Theta) f_1 f_2^2 + C_3(\Theta) f_1^2 f_2^2 + D_3(\Theta).(f_1^3 f_2 + f_1 f_2^3) + \\ &\quad B_3^{(2)} A^{(12)} f_1^3 f_2^2 + B_3^{(1)}(\Theta) A^{(12)} f_1^2 f_2^3 + A_3(\Theta) A^{(12)^2} f_1^3 f_2^3 + \delta_{3_{(1)}}^{(1)} A^{(12)} f_1 f_2^4 + \\ &\quad\quad \delta_{3_{(1)}}^{(2)} A^{(12)^2} f_1^2 f_2^4 + \delta_{3_{(1)}}^{(3)} A^{(12)^3} f_1^3 f_2^4 + \delta_{3_{(2)}}^{(1)} A^{(12)} f_1^4 f_2 + \delta_{3_{(2)}}^{(2)} A^{(12)^2} f_1^4 f_2^2 + \\ &\quad\quad\quad \delta_{3_{(2)}}^{(3)} A^{(12)^3} f_1^4 f_2^3 + A^{(12)^4}\cdot f_1^4 f_2^4.\end{aligned}$$

τ_4: similar form to τ_2 with $\delta_{2_{(j)}}^{(i)} \to \delta_{4_{(j)}}^{(i)}$ and $A_2(\Theta)$, $B_2^{(i)}(\Theta)$, $C_2(\Theta) \to A_4(\Theta)$, $B_4^{(i)}(\Theta)$, $C_4(\Theta)$.

τ_5, τ_7: similar form to τ_1 with $\delta_{1_{(j)}}^{(i)} \to \delta_{5_{(j)}}^{(i)}, \delta_{7_{(j)}}^{(i)}$ and $A_1(\Theta) \to A_5(\Theta)$, $A_7(\Theta)$ respectively.

τ_6: similar form to τ_0 with $\delta_{0_{(j)}}^{(i)} \to \delta_{6_{(j)}}^{(i)}$,

where the following constraints also hold:

$$A_5(\Theta)=\frac{\delta^{(1)}_{5_{(1)}}\delta^{(1)}_{5_{(2)}}}{\delta^{(1)}_{1_{(1)}}\delta^{(1)}_{1_{(2)}}}\cdot A_1(\Theta)\equiv\delta^{(1)}_{6_{(1)}}\delta^{(1)}_{6_{(2)}}\cdot A_1(\Theta),\qquad A_4(\Theta)=\frac{\delta^{(1)}_{4_{(1)}}\delta^{(1)}_{4_{(2)}}}{\delta^{(1)}_{2_{(1)}}\delta^{(1)}_{2_{(2)}}}\cdot A_2(\Theta)\equiv\delta^{(1)}_{6_{(1)}}\delta^{(1)}_{6_{(2)}}\cdot A_2(\Theta),$$

and similarly,

$$B^{(1)}_4(\Theta)=\frac{\delta^{(2)}_{4_{(1)}}\delta^{(1)}_{4_{(2)}}}{\delta^{(2)}_{2_{(1)}}\delta^{(1)}_{2_{(2)}}}\cdot B^{(1)}_2(\Theta)\equiv\delta^{(1)}_{6_{(2)}}\cdot B^{(1)}_2(\Theta),\qquad B^{(2)}_4(\Theta)=\frac{\delta^{(1)}_{4_{(1)}}\delta^{(2)}_{4_{(2)}}}{\delta^{(1)}_{2_{(1)}}\delta^{(2)}_{2_{(2)}}}\cdot B^{(2)}_2(\Theta)\equiv\delta^{(1)}_{6_{(1)}}\cdot B^{(2)}_2(\Theta),$$

also,

$$C_2(\Theta)=\frac{\delta^{(2)}_{2_{(1)}}\delta^{(2)}_{2_{(2)}}}{\delta^{(1)}_{2_{(1)}}\delta^{(1)}_{2_{(2)}}}\cdot A_2(\Theta)\equiv A_2(\Theta)\qquad C_4(\Theta)=\frac{\delta^{(2)}_{4_{(1)}}\delta^{(2)}_{4_{(2)}}}{\delta^{(1)}_{4_{(1)}}\delta^{(1)}_{4_{(2)}}}\cdot A_4(\Theta)\equiv\delta^{(1)}_{6_{(1)}}\delta^{(1)}_{6_{(2)}}A_4(\Theta)=A_2(\Theta).$$

Hence, the set of tau functions display the two-fold symmetry associated with the affine Dynkin diagram. (As they should do, being solutions to the series of equations (3.40)).

The remaining interaction functions may now be listed in a case-by-case manner, where the separate cases refer to the 'groupings' of the eigenvalues λ_a which were utilised in the single soliton solutions.

An interesting pure-mathematical problem concerning the subject of finite field extensions arose in the derivation of the quantity $A^{(12)}$ for the case of $\lambda,\lambda'\in\{\lambda_1,\lambda_3,\lambda_5\}$. This has been commented upon in the following listings.

In all the cases where $\lambda=\lambda'$ any degeneracy reflected by the presence of an upper index disappears and hence, accordingly, such functions will only be noted once. For example, $B^{(1)}_3(\Theta)\equiv B^{(2)}_3(\Theta)$ will be written as $B_3(\Theta)$.

Moreover, a specific pattern occurs throughout particular interaction terms in the soliton solutions, and this is true in all cases. (This is also true in the case of $e_6^{(1)}$ but is restricted to the solutions involving one or more of the self conjugate spots of the Dynkin diagram). This will be commented on further in chapter four, when its connection to the Lie algebra structure underlying part of the tau functions solutions is illuminated.

If n_k denotes the Kač label corresponding to the k^{th} tau function then as has been mentioned already, there is a symmetry of $\delta_k^{(a)} = \pm\delta_k^{(n_k-a)}$ in the single soliton solutions. Taking the numerator of the interacton coefficient for the term $f_1^a f_2^a$ $\left(0 < a < \frac{n_k}{2}\right)$ to have the form:

$$a_p \cdot ch^p\Theta + a_{p-1} \cdot ch^{p-1}\Theta + a_{p-2} \cdot ch^{p-2}\Theta + a_{p-3} \cdot ch^{p-3}\Theta + \cdots\cdots.$$

then the numerator of the $f_1^{n_k-a} f_2^a$, $f_1^a f_2^{n_k-a}$ coefficients can be seen to be the function

$$a_p \cdot ch^p\Theta - a_{p-1} \cdot ch^{p-1}\Theta + a_{p-2} \cdot ch^{p-2}\Theta - a_{p-3} \cdot ch^{p-3}\Theta + \cdots\cdots..$$

Hence, when n_k is even (that is $n_k = 2t$ where $t \in \mathbf{Z}^+$), the numerator of the coefficient for the $f_1^t f_2^t$ term is either a function of purely odd or purely even powers of $ch\Theta$. This is found to be true in all explicit calculations.

In all cases, the term $A^{(12)}$ will be given first, followed by the numerators of all the remaining interaction pieces. These all possess the same denominator as the fundamental interaction block, apart from the $C_3(\Theta)$ term where the denominator is squared. Again all the fundamental interaction pieces $A^{(12)}$ will be given in terms of the notation (x) in an appendix, after the full list of interaction terms.

3.5.0*(i)* $\lambda = \lambda' = \lambda_2$

This is manifestly the simplest case of the double soliton solutions. The fundamental interaction term is given by

$$A^{(12)} = \frac{2ch^2\Theta - 3ch\Theta + 1}{2ch^2\Theta + 3ch\Theta + 1} \equiv \frac{(ch\Theta - \frac{1}{2})(ch\Theta - 1)}{(ch\Theta + \frac{1}{2})(ch\Theta + 1)}$$

and the numerators of the remaining interaction terms are as follows, (the denominators are as already stated, the same as that of $A^{(12)}$ - apart from the function $C_3(\Theta)$ where it is squared):

$$A_1(\Theta) = A_7(\Theta) = 4(8ch^2\Theta - 5)$$

$$A_2(\Theta) = 9(2ch^2\Theta + ch\Theta + 1)$$
$$B_2(\Theta) = 9(2ch^2\Theta - ch\Theta + 1)$$

$$A_3(\Theta) = 8(4ch^2\Theta + 3ch\Theta + 2)$$

$$D_3(\Theta) = 8(4\mathrm{ch}^2\Theta - 3\mathrm{ch}\Theta + 2)$$
$$B_3(\Theta) = 24(2\mathrm{ch}^2\Theta + 1)$$
$$C_3(\Theta) = 36(4\mathrm{ch}^4\Theta + \mathrm{ch}^2\Theta + 1).$$

3.5.0*(ii)* $\lambda = \lambda_{4,6,7}$, $\lambda' = \lambda_2$

$$A^{(12)} = \frac{8\mathrm{ch}^2\Theta - 2\sqrt{6\lambda}\,\mathrm{ch}\Theta + (\lambda - 2)}{8\mathrm{ch}^2\Theta + 2\sqrt{6\lambda}\,\mathrm{ch}\Theta + (\lambda - 2)},$$

and the other interaction numerators are given by the functions:

$$A_1(\Theta) = 32(\lambda - 2)\mathrm{ch}^2\Theta + 2(\lambda + 4)(2 - \lambda)$$
$$A_2(\Theta) = \alpha\,\mathrm{ch}^2\Theta + \beta\,\mathrm{ch}\Theta + \gamma$$
$$B_2(\Theta) = \alpha\,\mathrm{ch}^2\Theta - \beta\,\mathrm{ch}\Theta + \gamma$$

where $\alpha = 12(\lambda^2 - 6\lambda + 6)$, $\beta = (\lambda^2 - 6\lambda + 6)\sqrt{6\lambda}$, $\gamma = 3(-2\lambda^2 + 15\lambda - 18)$

$$A_3(\Theta) = B_3^{(2)}(\Theta) = D_3(\Theta) = A_7(\Theta) = 0$$

and finally,

$$B_3^{(1)}(\Theta) = 64(\lambda - 1)\mathrm{ch}^2\Theta + 8(2 - \lambda)(1 + 2\lambda)$$
$$C_3(\Theta) = 768(\lambda - 1)\mathrm{ch}^4\Theta + 96(4 + 3\lambda - 3\lambda^2)\mathrm{ch}^2\Theta + 12(68 - 92\lambda + 23\lambda^2).$$

3.5.0*(iii)* $\lambda = \lambda_{1,3,5}$, $\lambda' = \lambda_2$

$$A^{(12)} = \frac{4C^3 - 6\lambda \cdot C^2 + 3(\lambda^2 - 12)C + 9\lambda(6 - \lambda)}{4C^3 + 6\lambda \cdot C^2 + 3(\lambda^2 - 12)C - 9\lambda(6 - \lambda)}$$

where C has been defined to be $\sqrt{6\lambda}\,\mathrm{ch}\Theta$. The remaining numerators are given by:

$$A_1(\Theta) = -\frac{1}{2} \cdot A_7(\Theta) = 16(\lambda - 2)C^3 + 48(-3 + 6\lambda - 2\lambda^2)C$$

$$A_2(\Theta) = \alpha \cdot C^3 + \beta \cdot C^2 + \gamma \cdot C + \delta$$
$$B_2(\Theta) = \alpha \cdot C^3 - \beta \cdot C^2 + \gamma \cdot C - \delta$$

where

$$\alpha = 6(\lambda^2 - 6\lambda + 6),\quad \beta = 18(2\lambda^2 - 11\lambda + 12),$$
$$\gamma = 27(-17\lambda^2 + 96\lambda - 108),\quad \delta = 27(-107\lambda^2 + 582\lambda - 648).$$

$$A_3(\Theta) = \mu \cdot C^3 + \nu \cdot C^2 + \sigma \cdot C + \rho$$
$$D_3(\Theta) = \mu \cdot C^3 - \nu \cdot C^2 + \sigma \cdot C - \rho$$

where

$$\mu = -16(\lambda^2 - 6\lambda + 8), \quad \nu = 48(-3\lambda^2 + 16\lambda - 18),$$
$$\sigma = 48(10\lambda^2 - 63\lambda + 78), \quad \rho = 144(35\lambda^2 - 192\lambda + 216).$$

and

$$B_3^{(1)}(\Theta) = 32(2\lambda^2 - 9\lambda + 9)C^3 + 72(-59\lambda^2 + 300\lambda - 324)C$$
$$B_3^{(2)}(\Theta) = -24(\lambda^2 - 6\lambda + 8)C^3 + 72(25\lambda^2 - 144\lambda + 168)C$$
$$C_3(\Theta) = 192(2\lambda^2 - 9\lambda + 9)C^6 + 1296(-41\lambda^2 + 208\lambda - 224)C^4 +$$
$$3888(337\lambda^2 - 1746\lambda + 1896)C^2 + 209954(142\lambda^2 - 733\lambda + 794).$$

3.5.0*(iv)* $\lambda = \lambda_{1,3,5},\ \lambda' = \lambda_{4,6,7}$

This case is found to divide into three particular subcases. Namely that the fundamental interaction block takes the form of a quotient of cubic polynomials in 'chΘ' for the double solitons associated with the pairs (1, 7), (3, 4) & (5, 6); whereas this function factorizes further in the remaining cases, to give a quadratic function for (1, 6), (3, 7) & (5, 4) and ultimately a linear function for (1, 4), (3, 6) & (5, 7). This information may be neatly summarized by the following table, where the entry refers to the degree of the polynomial associated with the fundamental interaction piece.

Second Soliton

		4	6	7
	1	1	2	3
First Soliton	3	3	1	2
	5	2	3	1

Another feature which can be noted is that all of the interaction functions without an upper index are invariant under $\lambda \leftrightarrow \lambda'$. This is not apparent at first sight and has occured due to the simplification of such quantities through meticulous use of the constraints involving λ. Obviously, $B_3^{(1)}(\Theta) \leftrightarrow B_3^{(2)}(\Theta)$ under such a transformation.

In all of the subcases

$$A_3(\Theta) = B_3^{(1)}(\Theta) = D_3(\Theta) = A_7(\Theta) = 0.$$

(a) (1, 4), (3, 6), (5, 7)

$$A^{(12)} = \frac{4C - \lambda\lambda'}{4C + \lambda\lambda'}$$

$$A_1(\Theta) = 4(2-\lambda)(2-\lambda')C,$$

$$A_2(\Theta) = \alpha C + \beta$$

$$B_2(\Theta) = \alpha C - \beta$$

where,

$$\alpha = (\lambda^2 - 6\lambda + 6)(\lambda'^2 - 6\lambda' + 6)$$

$$\beta = (12\lambda + 36\lambda' - 12\lambda^2 - 36\lambda'^2 - 51\lambda\lambda' + 24\lambda^2\lambda' + 39\lambda\lambda'^2 - 11\lambda^2\lambda'^2)$$

and

$$B_3^{(2)}(\Theta) = -8(\lambda' - 1)(\lambda^2 - 6\lambda + 8)C$$

$$C_3(\Theta) = 64(\lambda' - 1)(2\lambda^2 - 9\lambda + 9)C^2 + 4(25920 + 25056\lambda + 44928\lambda' - 5352\lambda^2 - 18216\lambda'^2 - 43056\lambda\lambda' + 9036\lambda^2\lambda' + 17208\lambda\lambda'^2 - 3503\lambda^2\lambda'^2)$$

(b) (1, 6), (3, 7), (5, 4)

The fundamental interaction term is given by:

$$A^{(12)} = \frac{\alpha C^2 + \beta C + \gamma}{\alpha C^2 - \beta C + \gamma}$$

where

$$\alpha = 16,\ \beta = -4\lambda\lambda',$$

$$\gamma = (144 - 96(\lambda + \lambda') + 12(\lambda^2 + \lambda'^2) + 48\lambda\lambda' - 6(\lambda^2\lambda' + \lambda\lambda'^2) + \lambda^2\lambda'^2).$$

The remaining numerators are,

$$A_1(\Theta) = 16(2-\lambda)(2-\lambda')C^2 + 16(-108 + 84\lambda + 66\lambda' - 15\lambda^2 - 12\lambda'^2 - 45\lambda\lambda' + 9\lambda^2\lambda' + 9\lambda\lambda'^2 - 2\lambda^2\lambda'^2)$$

$$A_2(\Theta) = \alpha C^2 + \beta C + \gamma,$$

$$B_2(\Theta) = \alpha C^2 - \beta C + \gamma,$$

where

$$\alpha = 4(\lambda^2 - 6\lambda + 6)(\lambda'^2 - 6\lambda' + 6),$$

$$\beta = 4(432 - 348\lambda - 396\lambda' + 60\lambda^2 + 72\lambda'^2 + 309\lambda\lambda' - 54\lambda^2\lambda' - 57\lambda\lambda'^2 + 10\lambda^2\lambda'^2)$$

$$\gamma = 3(144 - 48\lambda + 1584\lambda' + 12\lambda^2 - 468\lambda'^2 - 1560\lambda\lambda' + 294\lambda^2\lambda'$$

$$+450\lambda\lambda'^2 - 85\lambda\lambda'^2)$$

and

$$B_3^{(2)}(\Theta) = -32(\lambda'-1)(\lambda^2-6\lambda+8)C^2 +$$
$$8(4032-3648\lambda-8976\lambda'+708\lambda^2+2484\lambda'^2+8172\lambda\lambda'-1584\lambda^2\lambda'$$
$$-2256\lambda\lambda'^2+437\lambda^2\lambda'^2)$$

$$C_3(\Theta) = 1024(\lambda'-1)(2\lambda^2-9\lambda+9)C^4 +$$
$$64(1054944-972576\lambda-1436832\lambda'+187824\lambda^2+335808\lambda'^2$$
$$+1324656\lambda\lambda'-255816\lambda^2\lambda'-309588\lambda\lambda'^2+59779\lambda^2\lambda'^2)C\hat{}2 +$$
$$576(25453728-23468376\lambda-34596576\lambda'+4533180\lambda^2$$
$$+8140476\lambda'^2+31898100\lambda\lambda'-6161480\lambda^2\lambda'-7505532\lambda\lambda'^2$$
$$+1449779\lambda^2\lambda'^2).$$

Finally,

(c) <u>(1, 7), (3, 4), (5, 6)</u>

In this example the coefficients f(λ, λ') of the polynomials in 'C' may be further reduced through use of the constraint:

$$\lambda\lambda' - 6\lambda - 6\lambda' + 24 = 0$$

This then gives the following set of interaction functions [numerators]. The leading coefficients however, have been left in their original state in order to emphasise their explicit nature.

$$A^{(12)} = \frac{\alpha C^3 + \beta C^2 + \gamma C + \delta}{\alpha C^3 - \beta C^2 + \gamma C - \delta}$$
$$\alpha = 4$$
$$\beta = 6(4-\lambda-\lambda')$$
$$\gamma = 3(-36+4\lambda+4\lambda'+\lambda^2+\lambda'^2)$$
$$\delta = 3(108-18\lambda-12\lambda'-2\lambda^2-3\lambda'^2).$$

$$A_1(\Theta) = 4(2-\lambda)(2-\lambda')C^3 + 12(156-8\lambda-14\lambda'+8\lambda^2-7\lambda'^2)C,$$

$$A_2(\Theta) = \mu C^3 + \nu C^2 + \sigma C + \rho$$
$$B_2(\Theta) = \mu C^3 - \nu C^2 + \sigma C - \rho$$

where

$$\mu = (\lambda^2-6\lambda+6)(\lambda'^2-6\lambda'+6),$$
$$\nu = 6(-348+59\lambda+33\lambda'+7\lambda^2+12\lambda'^2)$$

$$\sigma = 3(4644 - 204\lambda - 672\lambda' - 239\lambda^2 - 141\lambda'^2)$$
$$\delta = 9(13932 - 1178\lambda - 1680\lambda' - 582\lambda^2 - 473\lambda'^2)$$

and

$$B_3^{(2)}(\Theta) = -8(\lambda' - 1)(\lambda^2 - 6\lambda + 8)C^3 + 48(-864 + 7\lambda + 121\lambda' + 53\lambda^2 + 30\lambda'^2)C$$
$$\begin{aligned} C_3(\Theta) = {} & 64(\lambda' - 1)(2\lambda^2 - 9\lambda + 9)C^6 \\ & + 48(44040 - 3372\lambda - 1937\lambda^2 - 5256\lambda' - 1539\lambda'^2)C^4 \\ & + 144(-1115496 + 120930\lambda + 40283\lambda^2 + 117522\lambda' + 40605\lambda'^2)C^2 \\ & + 432(7250688 - 754230\lambda - 269603\lambda^2 - 779856\lambda' - 261906\lambda'^2). \end{aligned}$$

3.5.0*(v)* $\lambda = \lambda_{4,6,7}$, $\lambda' = \lambda_{4;6,7}$

In a sense nothing 'abnormal' occurs here with respect to the subject of finite field extensions - see following case. Both fundamental blocks $A^{(12)}$ for $\lambda = \lambda'$ and $\lambda \neq \lambda'$ are quotients of cubic polynomials in chΘ and moreover, the restriction of $\lambda = \lambda'$ applied to that for $\lambda \neq \lambda'$ gives the respective interaction terms. This was noted from direct calculation of both cases and, hence, it is only necessary to give the case $\lambda \neq \lambda'$ with the proviso that $C = \lambda.\text{ch}\Theta$ when the restriction is made, and that (3.41a) may be utilised to reduce the degree of the coefficients.

$$A^{(12)} = \frac{\alpha C^3 + \beta C^2 + \gamma C + \delta}{\alpha C^3 - \beta C^2 + \gamma C - \delta}$$

where

$$\alpha = 16$$
$$\beta = -4\lambda\lambda'$$
$$\gamma = (-96 + 96(\lambda + \lambda') - 12(\lambda^2 + \lambda'^2) - 108\lambda\lambda' + 12(\lambda^2\lambda' + \lambda\lambda'^2) - \lambda^2\lambda'^2)$$
$$\begin{aligned} \delta = {} & (192 - 216(\lambda + \lambda') + 24(\lambda^2 + \lambda'^2) + 240\lambda\lambda' \\ & -24(\lambda^2\lambda' + \lambda\lambda'^2) + 2\lambda^2\lambda'^2). \end{aligned}$$

The numerators of the other interaction terms are:

$$A_1(\Theta) = 16(2 - \lambda)(2 - \lambda')C^3 + 12\lambda\lambda'(-4 + 2(\lambda + \lambda') - \lambda\lambda')C,$$

$$A_2(\Theta) = \mu C^3 + \nu C^2 + \sigma C + \rho$$
$$B_2(\Theta) = \mu C^3 - \nu C^2 + \sigma C - \rho$$

where

$$\mu = 4(\lambda^2 - 6\lambda + 6)(\lambda'^2 - 6\lambda' + 6)$$
$$\begin{aligned} \nu = {} & 12(16 - 20(\lambda + \lambda') + 4(\lambda^2 + \lambda'^2) + 25\lambda\lambda' \\ & -5(\lambda^2\lambda' + \lambda\lambda'^2) + \lambda^2\lambda'^2) \end{aligned}$$

$$\sigma = 3(-480 + 592(\lambda + \lambda') - 116(\lambda^2 + \lambda'^2) - 724\lambda\lambda'$$
$$+140(\lambda^2\lambda' + \lambda\lambda'^2) - 27\lambda^2\lambda'^2)$$
$$\rho = 6(-576 + 756(\lambda + \lambda') - 160(\lambda^2 + \lambda'^2) - 992\lambda\lambda'$$
$$+210(\lambda^2\lambda' + \lambda\lambda'^2) - 45\lambda^2\lambda'^2),$$

$$A_3(\Theta) = \kappa C + \varepsilon$$
$$B_3(\Theta) = \kappa C - \varepsilon,$$
$$A_7(\Theta) = -\kappa$$

where

$$\kappa = 96(16 - 20(\lambda + \lambda') + 4(\lambda^2 + \lambda'^2) + 25\lambda\lambda'$$
$$-5(\lambda^2\lambda' + \lambda\lambda'^2) + \lambda^2\lambda'^2)$$
$$\varepsilon = 48(128 - 168(\lambda + \lambda') + 36(\lambda^2 + \lambda'^2) + 220\lambda\lambda'$$
$$-47(\lambda^2\lambda' + \lambda\lambda'^2) + 10\lambda^2\lambda'^2)$$

$$B_3^{(1)}(\Theta) = B_3^{(2)}(\Theta) = 0$$

and finally,

$$C_3 = 1024(\lambda - 1)(\lambda' - 1)C^6 + 192(-128 + 128(\lambda + \lambda') - 120\lambda\lambda' - 4(\lambda^2\lambda' + \lambda\lambda'^2)$$
$$+\lambda^2\lambda'^2)C^4 + 384(-528 + 1044(\lambda + \lambda') - 462(\lambda^2 + \lambda'^2) - 1822\lambda\lambda'$$
$$+685(\lambda^2\lambda' + \lambda\lambda'^2) - 188\lambda^2\lambda'^2)C^2 + 768(46032$$
$$-63780(\lambda + \lambda') + 15567(\lambda^2 + \lambda'^2) + 88290\lambda\lambda'$$
$$-21483(\lambda^2\lambda' + \lambda\lambda'^2) + 5173\lambda^2\lambda'^2)$$

3.5.0*(vi)* $\boldsymbol{\lambda = \lambda_{1,3,5}, \lambda' = \lambda_{1,3,5}}$,

This is quite unlike the previous case, where the 'naïve' restriction of $\lambda = \lambda'$ in the fundamental interaction block for $\lambda \neq \lambda'$ gave the required block for $\lambda = \lambda'$. In this case the fact that the coefficients of the polynomials involved are elements of finite field extensions comes into play. Ultimately, it results in a quotient of quartics for $\lambda = \lambda'$ and a quotient of quadratics for $\lambda \neq \lambda'$. Naturally, this is highly counter-intuitive and just does not occur when fields of characteristic zero are being considered. To see how such a factorisation occurs it may be deemed prudent to give a simple example, since it is only the essence of how such an event occurs that is important.

Therefore, let λ take the values a, b, c , then it is required to show that

$$\frac{\mathrm{ch}\Theta + f(\lambda, \lambda')}{\mathrm{ch}\Theta + g(\lambda, \lambda')}$$

is non-trivial when $\lambda = \lambda'$, but trivial when $\lambda \neq \lambda'$. That is equivalent to finding $f(\lambda, \lambda')$ & $g(\lambda, \lambda')$ such that $f \neq g$ when $\lambda = \lambda'$, and $f = g$ when $\lambda \neq \lambda'$.

Hence, taking f = g + h, it is required that

$$h(\lambda, \lambda') \neq 0 \qquad \text{when } \lambda = \lambda'$$
$$= 0 \qquad \text{when } \lambda \neq \lambda'.$$

It can easily be seen that[61]

$$h(\lambda, \lambda') = ((\lambda + \lambda') - (a + b)) \cdot ((\lambda + \lambda') - (a + c)) \cdot ((\lambda + \lambda') - (b + c))$$

satisfies such conditions when a, b, c are all unequal.

Having seen - at least in principle - how it is possible to obtain a factorisation of the required design, the interaction terms may now be given for both cases.

(a) $\lambda = \lambda'$

$$A^{12} = \frac{\alpha \cdot C^4 + \beta C^3 + \gamma C^2 + \delta C + \varepsilon}{\alpha \cdot C^4 - \beta C^3 + \gamma C^2 - \delta C + \varepsilon}$$

where

$$\alpha = 4,$$
$$\beta = -\lambda^2,$$
$$\gamma = 144 - 138\lambda + 25\lambda^2,$$
$$\delta = -972 + 882\lambda - 162\lambda^2,$$
$$\varepsilon = 2160 - 1980\lambda + 378\lambda^2.$$

$$A_1(\Theta) = 4(2-\lambda)(2-\lambda)C^4 + 4(-684 + 636\lambda - 125\lambda^2)C^2$$
$$+72(-1392 + 1282\lambda - 247\lambda^2),$$

$$A_2(\Theta) = \kappa C^4 + \mu C^3 + \nu C^2 + \sigma C + \rho$$
$$B_2(\Theta) = \kappa C^4 - \mu C^3 + \nu C^2 - \sigma C + \rho$$

where

$$\kappa = (\lambda^2 - 6\lambda + 6)^2 \equiv 12(39 - 36\lambda + 7\lambda^2),$$
$$\mu = 3(2160 - 1992\lambda + 385\lambda^2),$$
$$\nu = 27(-1824 + 1682\lambda - 325\lambda^2),$$
$$\sigma = 54(-15810 + 14579\lambda - 2817\lambda^2),$$
$$\rho = 162(-11064 + 10202\lambda - 1971\lambda^2).$$

$$A_3(\Theta) = \xi C^4 + \phi C^3 + \psi C^2 + \eta C + \pi$$
$$D_3(\Theta) = \xi C^4 - \phi C^3 + \psi C^2 - \eta C + \pi$$

where

$$\xi = 4(\lambda^2 - 6\lambda + 8)^2 \equiv 32(62 - 57\lambda + 11\lambda^2),$$
$$\phi = 32(1269 - 1170\lambda + 226\lambda^2),$$
$$\psi = 64(-369 + 339\lambda - 65\lambda^2),$$
$$\eta = 72(-63864 + 58882\lambda - 11373\lambda^2),$$
$$\pi = 144(-157548 + 145262\lambda - 28059\lambda^2)$$

$$B_3(\Theta) = -8(\lambda^2 - 6\lambda + 8)(2\lambda^2 - 9\lambda + 9)C^4 + 144(14370 - 13249\lambda + 2559\lambda^2)C^2 + 864(-125670 + 115869\lambda - 22381\lambda^2),$$

$$A_7(\Theta) = 16(\lambda - 2)(\lambda - 2)C^4 + 16(-1386 + 1284\lambda - 251\lambda^2)C^2 + 72(15384 - 14186\lambda + 2741\lambda^2)$$

and

$$C_3(\Theta) = 64(2\lambda^2 - 9\lambda + 9)^2 C^8 + 144(-530076 + 488754\lambda - 94415\lambda^2)C^6 + 32(384137802 - 354181329\lambda + 68413977\lambda^2)C^4 + 11664(-72374812 + 66730652\lambda - 12889703\lambda^2)C^2 + 93312(223443942 - 206018625\lambda + 39794585\lambda^2).$$

(b) $\lambda \neq \lambda'$

The fundamental block is given by a quotient of quadratics:

$$A^{(12)} = \frac{48C^2 - (12\lambda\lambda')C + \lambda\lambda'(\lambda\lambda' - 12)}{48C^2 + (12\lambda\lambda')C + \lambda\lambda'(\lambda\lambda' - 12)}$$

and the numerator terms by,

$$A_1(\Theta) = 48(2-\lambda)(2-\lambda')C^2 + 4(-36(\lambda^2 + \lambda'^2) - 12\lambda\lambda' + 42(\lambda^2\lambda' + \lambda\lambda'^2) - 17\lambda^2\lambda'^2),$$

$$A_2(\Theta) = \alpha C^2 + \beta C + \gamma$$
$$B_2(\Theta) = \alpha C^2 - \beta C + \gamma$$

where

$$\alpha = 12(\lambda^2 - 6\lambda + 6)(\lambda'^2 - 6\lambda' + 6)$$
$$\beta = 36(144 - 132(\lambda + \lambda') + 24(\lambda^2 + \lambda') + 121\lambda\lambda' - 22(\lambda^2\lambda' + \lambda\lambda'^2) + 4\lambda^2\lambda'^2)$$
$$\gamma = 9(-6912 + 6480(\lambda + \lambda') - 1248(\lambda^2 + \lambda'^2) - 6060\lambda\lambda' + 1164(\lambda^2\lambda' + \lambda\lambda'^2) - 223\lambda^2\lambda'^2)$$

$$A_3(\Theta) = \delta C^2 + \varepsilon C + \kappa$$
$$D_3(\Theta) = \delta C^2 - \varepsilon C + \kappa$$

$$\delta = 48(\lambda^2 - 6\lambda + 8)(\lambda'^2 - 6\lambda' + 8),$$

$$\varepsilon = 96(324 - 288(\lambda + \lambda') + 54(\lambda^2 + \lambda'^2) + 256\lambda\lambda' - 48(\lambda^2\lambda' + \lambda\lambda'^2) + 9\lambda^2\lambda'^2),$$

$$\kappa = 16(-49896 + 45900(\lambda + \lambda') - 8856(\lambda^2 + \lambda'^2) - 42222\lambda\lambda' + 8145(\lambda^2\lambda' + \lambda\lambda'^2) - 1571\lambda^2\lambda'^2).$$

$$B_3^{(1)}(\Theta) = -96(2\lambda^2 - 9\lambda + 9)(\lambda'^2 - 6\lambda' + 8)C^2 + 144(-26208 + 24312\lambda + 24084\lambda' - 4728\lambda^2 - 4620\lambda'^2 - 22344\lambda\lambda' + 4346\lambda^2\lambda' + 4287\lambda\lambda'^2 - 834\lambda^2\lambda'^2)$$

$B_3^{(2)}(\Theta) : (\lambda \leftrightarrow \lambda'$ as above)

$$A_7(\Theta) = 192(\lambda - 2)(\lambda' - 2)C^2 + 4(12960 - 11664(\lambda + \lambda') + 2124(\lambda^2 + \lambda'^2) + 10428\lambda\lambda' - 1884(\lambda^2\lambda' + \lambda\lambda'^2) + 337\lambda^2\lambda'^2)$$

and finally,

$$C_3(\Theta) = 9216(2\lambda^2 - 9\lambda + 9)(2\lambda'^2 - 9\lambda' + 9)C^4 + 5184(-625728 + 577728(\lambda + \lambda') - 111944(\lambda^2 + \lambda'^2) - 533416\lambda\lambda' + 103360(\lambda^2\lambda' + \lambda\lambda'^2) - 20029\lambda^2\lambda'^2)C^2 + 31104(15032520 - 13892316(\lambda + \lambda') + 2678646(\lambda^2 + \lambda'^2) + 12783206\lambda\lambda' - 2470127(\lambda^2\lambda' + \lambda\lambda'^2) + 477308\lambda^2\lambda'^2).$$

3.5.1 The Fusing Rules for $e_7^{(1)}$

All the (non-' $i\pi$') fusings for this affine algebra were calculated numerically to arbitrary precision and found to agree with [5]; the ' $i\pi$' self-conjugate fusings corresponded to an annihilation of the constituent solitons. The numerical computations were carried out using Mathematica.

3.5.2 Appendix

The interaction terms $A^{(12)}$ for $e_7^{(1)}$ are again listed in terms of the notation (x) :

$$A_{11}^{(12)} = (1)^{-1}(3)^{+1}(4)^{-1}(5)^{+1}(6)^{-1}(8)^{+1}(9)^{-2}(10)^{+1}(12)^{-1}(13)^{+1}(14)^{-1}(15)^{+1}(17)^{-1}(18)^{+2}$$

$$A_{21}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{-1}(\tfrac{5}{2})^{+1}(\tfrac{13}{2})^{-1}(\tfrac{15}{2})^{+1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{-1}(\tfrac{21}{2})^{+1}(\tfrac{23}{2})^{-1}(\tfrac{31}{2})^{+1}(\tfrac{33}{2})^{-1}(\tfrac{35}{2})^{+1}$$

$$A_{31}^{(12)} = (1)^{+1}(4)^{+1}(5)^{-1}(8)^{-1}(10)^{-1}(13)^{-1}(14)^{+1}(17)^{+1}$$

$$A_{41}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{33}{2})^{+1}$$

$$A_{51}^{(12)} = (2)^{+1}(4)^{-1}(5)^{+1}(7)^{-1}(11)^{-1}(13)^{+1}(14)^{-1}(16)^{+1}$$

$$A_{61}^{(12)} = (\tfrac{5}{2})^{+1}(\tfrac{7}{2})^{-1}(\tfrac{11}{2})^{+1}(\tfrac{13}{2})^{-1}(\tfrac{23}{2})^{-1}(\tfrac{25}{2})^{+1}(\tfrac{29}{2})^{-1}(\tfrac{31}{2})^{+1}$$

$$A_{71}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{5}{2})^{-1}(\tfrac{7}{2})^{+1}(\tfrac{11}{2})^{-1}(\tfrac{13}{2})^{+1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{23}{2})^{+1}(\tfrac{25}{2})^{-1}(\tfrac{29}{2})^{+1}(\tfrac{31}{2})^{-1}(\tfrac{33}{2})^{+1}$$

$$A_{22}^{(12)} = (3)^{+1}(6)^{-1}(9)^{-2}(12)^{-1}(15)^{+1}(18)^{+2}$$

$$A_{32}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{+1}(\tfrac{7}{2})^{+1}(\tfrac{11}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{17}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{25}{2})^{-1}(\tfrac{29}{2})^{+1}(\tfrac{33}{2})^{+1}(\tfrac{35}{2})^{+1}$$

$$A_{42}^{(12)} = (1)^{+1}(2)^{+1}(7)^{-1}(8)^{-1}(10)^{-1}(11)^{-1}(16)^{+1}(17)^{+1}$$

$$A_{52}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{5}{2})^{+1}(\tfrac{7}{2})^{-1}(\tfrac{11}{2})^{+1}(\tfrac{13}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{23}{2})^{-1}(\tfrac{25}{2})^{+1}(\tfrac{29}{2})^{-1}(\tfrac{31}{2})^{+1}(\tfrac{33}{2})^{+1}$$

$$A_{62}^{(12)} = (2)^{+1}(4)^{-1}(5)^{+1}(7)^{-1}(11)^{-1}(13)^{+1}(14)^{-1}(16)^{+1}$$

$$A_{72}^{(12)} = (1)^{+1}(4)^{+1}(5)^{-1}(8)^{-1}(10)^{-1}(13)^{-1}(14)^{+1}(17)^{+1}$$

$$A_{33}^{(12)} = (1)^{+1}(2)^{+1}(3)^{+1}(6)^{-1}(7)^{-1}(8)^{-1}(9)^{-2}(10)^{-1}(11)^{-1}(12)^{-1}(15)^{+1}(16)^{+1}(17)^{+1}(18)^{+2}$$

$$A_{43}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{+1}(\tfrac{5}{2})^{+1}(\tfrac{13}{2})^{-1}(\tfrac{15}{2})^{-1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{23}{2})^{-1}(\tfrac{31}{2})^{+1}(\tfrac{33}{2})^{+1}(\tfrac{35}{2})^{+1}$$

$$A_{53}^{(12)} = (1)^{+1}(2)^{+1}(7)^{-1}(8)^{-1}(10)^{-1}(11)^{-1}(16)^{+1}(17)^{+1}$$

$$A_{63}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{33}{2})^{+1}$$

$$A_{73}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{5}{2})^{+1}(\tfrac{13}{2})^{-1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{-1}(\tfrac{23}{2})^{-1}(\tfrac{31}{2})^{+1}(\tfrac{35}{2})^{+1}$$

$$A_{44}^{(12)} = (2)^{+1}(4)^{+1}(5)^{-1}(7)^{-1}(9)^{-2}(11)^{-1}(13)^{-1}(14)^{+1}(16)^{+1}(18)^{+2}$$

$$A_{54}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{7}{2})^{+1}(\tfrac{11}{2})^{-1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{-1}(\tfrac{25}{2})^{-1}(\tfrac{29}{2})^{+1}(\tfrac{35}{2})^{+1}$$

$$A_{64}^{(12)} = (1)^{+1}(2)^{-1}(3)^{+1}(6)^{-1}(7)^{+1}(8)^{-1}(10)^{-1}(11)^{+1}(12)^{-1}(15)^{+1}(16)^{-1}(17)^{+1}$$

$$A_{74}^{(12)} = (1)^{+1}(3)^{+1}(4)^{-1}(5)^{+1}(6)^{-1}(8)^{-1}(10)^{-1}(12)^{-1}(13)^{+1}(14)^{-1}(15)^{+1}(17)^{+1}$$

$$A_{55}^{(12)} = (2)^{-1}(3)^{+1}(4)^{+1}(5)^{-1}(6)^{-1}(7)^{+1}(9)^{-2}(11)^{+1}(12)^{-1}(13)^{-1}(14)^{+1}(15)^{+1}(16)^{-1}(18)^{+2}$$

$$A_{65}^{(12)} = (\tfrac{1}{2})^{+1}(\tfrac{3}{2})^{-1}(\tfrac{7}{2})^{+1}(\tfrac{11}{2})^{-1}(\tfrac{15}{2})^{+1}(\tfrac{17}{2})^{-1}(\tfrac{19}{2})^{-1}(\tfrac{21}{2})^{+1}(\tfrac{25}{2})^{-1}(\tfrac{29}{2})^{+1}(\tfrac{33}{2})^{-1}(\tfrac{35}{2})^{+1}$$

$$A_{75}^{(12)} = (\tfrac{3}{2})^{+1}(\tfrac{15}{2})^{-1}(\tfrac{21}{2})^{-1}(\tfrac{33}{2})^{+1}$$

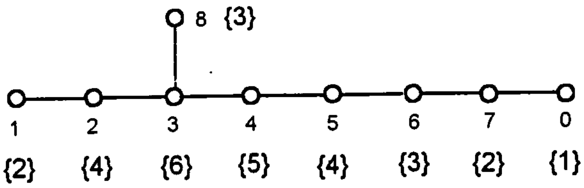
$$A_{66}^{(12)} = (1)^{-1}(4)^{+1}(5)^{-1}(8)^{+1}(9)^{-2}(10)^{+1}(13)^{-1}(14)^{+1}(17)^{-1}(18)^{+2}$$

$$A_{76}^{(12)} = (2)^{+1}(3)^{-1}(4)^{+1}(5)^{-1}(6)^{+1}(7)^{-1}(11)^{-1}(12)^{+1}(13)^{-1}(14)^{+1}(15)^{-1}(16)^{+1}$$

$$A_{77}^{(12)} = (1)^{-1}(2)^{+1}(7)^{-1}(8)^{+1}(9)^{-2}(10)^{+1}(11)^{-1}(16)^{+1}(17)^{-1}(18)^{+2}$$

3.6 Solitons for $e_8^{(1)}$

The affine Dynkin diagram for $e_8^{(1)}$ (including the Kač labels) is given by the following:



The eigenvalues of the matrix K_{ij} (using the above labelling of the extended root system) may be listed as:

$$\lambda_1 = 32\sqrt{3}\,\sin(\tfrac{\pi}{30})\sin(\tfrac{\pi}{5})\cos^2(\tfrac{\pi}{5})$$

$$\lambda_2 = 8\sqrt{3}\,\sin(\tfrac{7\pi}{30})\sin(\tfrac{2\pi}{5})$$

$$\lambda_3 = 512\sqrt{3}\,\sin(\tfrac{\pi}{30})\sin(\tfrac{\pi}{5})\cos^2(\tfrac{2\pi}{15})\cos^4(\tfrac{\pi}{5})$$

$$\lambda_4 = 8\sqrt{3}\,\sin(\tfrac{13\pi}{30})\sin(\tfrac{2\pi}{5})$$

$$\lambda_5 = 8\sqrt{3}\,\sin(\tfrac{11\pi}{30})\sin(\tfrac{\pi}{5})$$

$$\lambda_6 = 32\sqrt{3}\,\sin(\tfrac{\pi}{30})\sin(\tfrac{\pi}{5})\cos^2(\tfrac{\pi}{30})$$

$$\lambda_7 = 8\sqrt{3}\,\sin(\tfrac{\pi}{30})\sin(\tfrac{\pi}{5})$$

$$\lambda_8 = 128\sqrt{3}\,\sin(\tfrac{\pi}{30})\sin(\tfrac{\pi}{5})\cos^2(\tfrac{\pi}{5})\cos^2(\tfrac{7\pi}{30})$$

where a slight correction regarding λ_8 has been made, see [53]. These eigenvalues may further be split into two sets and shown to satisfy the following crucial constraints:

$$\lambda \in \{\lambda_1, \lambda_3, \lambda_6, \lambda_8\} :- \qquad \lambda^4 = 30\lambda^3 - 240\lambda^2 + 720\lambda - 720$$

$$\lambda \in \{\lambda_2, \lambda_4, \lambda_5, \lambda_7\} :- \qquad \lambda^4 = 30\lambda^3 - 300\lambda^2 + 1080\lambda - 720.$$

Thses constraints appear from the factorization of the characteristic polynomial of the matrix $K_{ij}^{[5]}$.

The equations of motion reduce to the set of equations for the tau functions of:

$$\left.\begin{aligned}
f(\tau_1) &= 2\,m^2(\tau_2 - \tau_1^2),\\
f(\tau_2) &= 4\,m^2(\tau_1\tau_3 - \tau_2^2),\\
f(\tau_3) &= 6\,m^2(\tau_2\tau_4\tau_8 - \tau_3^2),\\
f(\tau_4) &= 5\,m^2(\tau_3\tau_5 - \tau_4^2),\\
f(\tau_5) &= 4\,m^2(\tau_4\tau_6 - \tau_5^2),\\
f(\tau_6) &= 3\,m^2(\tau_5\tau_7 - \tau_6^2),\\
f(\tau_7) &= 2\,m^2(\tau_6\tau_0 - \tau_7^2),\\
f(\tau_0) &= 1\,m^2(\tau_7 - \tau_0^2),\\
f(\tau_8) &= 3\,m^2(\tau_3 - \tau_8^2)
\end{aligned}\right\} \qquad (3.42)$$

This set of equations exhibits the fact that there is no symmetry in the affine diagram.

The single soliton solutions to the field theory were first written down by Mc Ghee and appear in [53], although they will be given here for completeness.

The minimal ansatz $\tau_0 = 1 + f$ (where f denotes $e^{\Psi} \equiv \exp(\sigma(x - vt) + \xi)$) gives the following set of tau functions which satisfy (3.42) at all orders of 'f'. The structure of the tau functions is apparent under rescaling and shifts in Ψ.

$$\begin{aligned}
\tau_0 &= 1 + f \\
\tau_1 &= 1 + Af + f^2 \\
\tau_2 &= 1 + Bf + Cf^2 + Bf^3 + f^4 \\
\tau_3 &= 1 + Df + Ef^2 + Ff^3 + Ef^4 + Df^5 + f^6 \\
\tau_4 &= 1 + Gf + Hf^2 + Hf^3 + Gf^4 + f^5 \\
\tau_5 &= 1 + If + Jf^2 + If^3 + f^4 \\
\tau_6 &= 1 + Kf + Kf^2 + f^3 \\
\tau_7 &= 1 + Lf + f^2 \\
\tau_8 &= 1 + Mf + Mf^2 + f^3
\end{aligned}$$

where for $\lambda \in \{\lambda_1, \lambda_3, \lambda_6, \lambda_8\}$

$$\begin{aligned}
A &= -\frac{1}{6}(\lambda^3 - 24\lambda^2 + 132\lambda - 192), \\
B &= \frac{1}{6}(\lambda^3 - 6\lambda^2 + 24), \\
C &= \frac{2}{3}(5\lambda^3 - 60\lambda^2 + 225\lambda - 261), \\
D &= -\frac{1}{2}(\lambda - 2)(\lambda^2 - 6\lambda + 6), \\
E &= (64\lambda^3 - 668\lambda^2 + 2214\lambda - 2325), \\
F &= -(303\lambda^3 - 3186\lambda^2 + 10614\lambda - 11180), \\
G &= \frac{5}{12}(\lambda^3 - 12\lambda^2 + 48\lambda - 60), \\
H &= \frac{5}{4}(11\lambda^3 - 116\lambda^2 + 384\lambda - 400), \\
I &= -\frac{1}{6}(\lambda^3 - 12\lambda^2 + 36\lambda - 24), \\
J &= \frac{1}{6}(7\lambda^3 - 78\lambda^2 + 288\lambda - 324), \\
K &= \frac{1}{2}(\lambda^2 - 6\lambda + 6), \\
L &= (2 - \lambda), \\
M &= \frac{1}{4}(\lambda^3 - 18\lambda^2 + 84\lambda - 108).
\end{aligned}$$

Similarly for $\lambda \in \{\lambda_2, \lambda_4, \lambda_5, \lambda_7\}$

$$\begin{aligned}
A &= \frac{1}{3}(\lambda^3 - 21\lambda^2 + 114\lambda - 84), \\
B &= -\frac{1}{6}(5\lambda^3 - 102\lambda^2 + 540\lambda - 384),
\end{aligned}$$

$$C=-\frac{2}{3}(\lambda^3-24\lambda^2+135\lambda-99),$$

$$D=(\lambda^2-9\lambda+6),$$

$$E=(3\lambda^3-50\lambda^2+234\lambda-165),$$

$$F=-2(3\lambda^3-54\lambda^2+267\lambda-190),$$

$$G=\frac{5}{12}(\lambda^3-18\lambda^2+84\lambda-60),$$

$$H=\frac{5}{4}(\lambda-8)(3\lambda^2-26\lambda+20),$$

$$I=-\frac{1}{6}(\lambda^3-12\lambda^2+36\lambda-24),$$

$$J=\frac{1}{6}(7\lambda^3-108\lambda^2+468\lambda-324),$$

$$K=\frac{1}{2}(\lambda^2-6\lambda+6),$$

$$L=(2-\lambda),$$

$$M=\frac{1}{4}(\lambda^3-24\lambda^2+144\lambda-108).$$

As with the case of the algebra $e_7^{(1)}$, the spots of the Dynkin diagram are self-conjugate in the sense that the fundamental representations associated with them and hence, particles/ solitons attached to them, are self-conjugate. In the previous case this led to tau functions that had an intrinsic pattern connected with them (also notable for the self-conjugate cases for the other affine algebras). What is found here is that this pattern is repeated for the tau functions and expanded upon in an obvious manner for those functions connected with the spots with Kač labels five and six. However, before a discussion of this is undertaken the problems associated with constructing the explicit double soliton solutions will be mentioned.

The interaction function $A^{(12)}(\Theta)$ which appears in the ansatz (3.01) may be found by equating the nine simultaneous equations appearing at order f_1f_2 in the nine tau function equations (3.42). The remaining interaction terms can then be systematically constructed and all equations can be shown to be consistent at all orders $f_1^i f_2^j$ for $0 \le i,\ j \le n_k$: the Kač label of spot 'k'.

However, it was found that the factorization of the object $A^{(12)}(\Theta)$ (which appears as a quotient of polynomials of degree nine in 'chΘ' with coefficients that are elements of the finite field extension $Q[\lambda, \lambda']$) proved to require too much memory space when using "Mathematica" or "Maple V" on the Sparc systems at Durham.

Construction of this 'factorised quantity' backwards (using the data from Appendix 'A' and the fusing information in [5]) again proved to require too much memory for algebraic (that is exact) calculation, and moreover, was only possible in the simplest set of self-interacting cases, using a floating-point numerical calculation from "Maple V". This involved the inversion of a 12×12 matrix whose elements were products of up to four dozen trigonometric quantities.

The numerical algorithm in "Maple V" appears to breakdown for the other set of the self- interactions where the matrix involved is of the order 20×20. [The numerical routines in "Mathematica" ran out of memory in all cases.]

Nevertheless, this set of double soliton solutions is sufficient to display the afore mentioned pattern in the tau functions and this will be discussed now. The explicit interaction terms will be relegated to an appendix since they are very lengthy and only their existence is of any importance. However, it will be noted that the term $A^{(12)}(\Theta)$ for $\lambda \in \{\lambda_2, \lambda_4, \lambda_5, \lambda_7\}$ may be written as a quotient of sinh functions in each case.

$$A^{(12)}(\Theta) = \frac{\alpha C^4 + \beta C^3 + \gamma C^2 + \delta C + \varepsilon}{\alpha C^4 - \beta C^3 + \gamma C^2 - \delta C + \varepsilon}$$

where

$$\begin{aligned}
\alpha &= 4 \\
\beta &= -\lambda^2 \\
\gamma &= 3(-20 + 30\lambda - 10\lambda^2 + \lambda^3) \\
\delta &= -15(-132 + 192\lambda - 46\lambda^2 + 3\lambda^3) \\
\varepsilon &= 30(-264 + 378\lambda - 84\lambda^2 + 5\lambda^3)
\end{aligned}$$

and $C = \lambda \operatorname{ch}\Theta$. Using the notation (x) they may be written as:

$$A^{(12)}_{\lambda=\lambda_2} = (3)^{+1}(5)^{+1}(7)^{-1}(8)^{+1}(10)^{-1}(12)^{-1}(15)^{-2}(18)^{-1}(20)^{-1}(22)^{+1}(23)^{-1}(25)^{+1}(27)^{+1}(30)^{+2}$$

$$A^{(12)}_{\lambda=\lambda_4} = (2)^{+1}(3)^{+1}(5)^{+1}(10)^{-1}(12)^{-1}(13)^{-1}(15)^{-2}(17)^{-1}(18)^{-1}(20)^{-1}(25)^{+1}(27)^{+1}(28)^{+1}(30)^{+2}$$

$$A^{(12)}_{\lambda=\lambda_5} = (4)^{+1}(5)^{+1}(6)^{-1}(9)^{+1}(10)^{-1}(11)^{-1}(15)^{-2}(19)^{-1}(20)^{-1}(21)^{+1}(24)^{-1}(25)^{+1}(26)^{+1}(30)^{+2}$$

$$A^{(12)}_{\lambda=\lambda_7} = (1)^{-1}(5)^{+1}(6)^{-1}(9)^{+1}(10)^{-1}(14)^{+1}(15)^{-2}(16)^{+1}(20)^{-1}(21)^{+1}(24)^{-1}(25)^{+1}(29)^{-1}(30)^{+2} \tag{3.43}$$

3.6.1 The general tau function pattern for self-conjugate solitons.

The internal tau function structure for the case $e_8^{(1)}$ generalizes all the self-conjugate double soliton tau function structure that has appeared before. The functions corresponding to the Dynkin spots with labels 2, 3, and 6 may be graphically given as follows, where the interpolation from 3 to 6 should be immediate from the figures and accompanying information. Also, for simplicity, the leading order δ -term of the single soliton tau function has been taken to be one:

(a) τ_j : Kač label '2'

$$\begin{array}{c}
1 \\[1em]
\delta^{(1)}_{j(1)} f_1 \qquad\qquad \delta^{(1)}_{j(2)} f_2 \\[1em]
f_1^2 \qquad\qquad A_j(\Theta) \qquad\qquad f_2^2 \\[1em]
A^{(12)}\,x \qquad\qquad \delta^{(1)}_{j(2)} f_1^2 f_2 \qquad\qquad \delta^{(1)}_{j(1)} f_1 f_2^2 \\[1em]
A^{(12)^2}x \qquad\qquad f_1^2 f_2^2
\end{array}$$

(b) τ_j : Kač label '3'

$$\begin{array}{c}
1 \\[1em]
\delta^{(1)}_{j(1)} f_1 \qquad\qquad \delta^{(1)}_{j(2)} f_2 \\[1em]
\delta^{(2)}_{j(1)} f_1^2 \qquad\qquad A_j(\Theta) f_1 f_2 \qquad\qquad \delta^{(2)}_{j(2)} f_2^2 \\[1em]
f_1^3 \qquad\qquad B_j(\Theta) f_1^2 f_2 \qquad\qquad B_j(\Theta) f_1 f_1^2 \qquad\qquad f_2^3 \\[1em]
A^{(12)}x \qquad\qquad \delta^{(1)}_{j(2)} f_1^3 f_2 \qquad\qquad A_j(\Theta) f_1^2 f_2^2 \qquad\qquad \delta^{(1)}_{j(1)} f_1 f_2^3 \\[1em]
A^{(12)^2}x \qquad\qquad \delta^{(2)}_{j(2)} f_1^3 f_2^2 \qquad\qquad \delta^{(2)}_{j(1)} f_1^2 f_2^3 \\[1em]
A^{(12)^3}x \qquad\qquad f_1^3 f_2^3
\end{array}$$

[where in (b), if the numerator of the interaction term $A_j(\Theta)$ is given by

$$A_j(\Theta) = a_p(\lambda, \lambda')ch^p\Theta + a_{p-1}(\lambda, \lambda')ch^{p-1}\Theta + a_{p-2}(\lambda, \lambda')ch^{p-2}\Theta + \ldots\ldots$$

then

$$B_j(\Theta) = a_p(\lambda, \lambda')ch^p\Theta - a_{p-1}(\lambda, \lambda')ch^{p-1}\Theta + a_{p-2}(\lambda, \lambda')ch^{p-2}\Theta - \ldots\ldots \qquad].$$

(c) τ_j : Kač label '6'

The structure of the tau function is shown on the following page.

The relationships between the numerator of the interaction terms on the diagonal lines in this graphic description of the tau function, generalises that found for τ_j ($n_j = 3$) in a straight forward way.

[Interchange of the superscript (1) $\leftrightarrow$ (2) corresponds to the replacement of $\lambda \leftrightarrow \lambda'$ in the relevant formulae.]

It was found that,

i) $A_j(\Theta)$, $B_j^i(\Theta)$, $C_j^i(\Theta)$, $E_j^i(\Theta)$ and $G_j(\Theta)$ are linear polynomials in $A^{(12)}(\Theta)$. Moreover, the numerators of the interaction pairs ($A_j(\Theta)$, $G_j(\Theta)$) and similarily ($B_j^i(\Theta)$, $E_j^i(\Theta)$) are related by the polynomial coefficient signature pattern (+ + + + ...,+ - + - ....) that occured in the previous case. This goes hand-in-hand with the fact that the numerator of $C_j^i(\Theta)$ was found to be a function of only odd or even powers of $ch\Theta$.

ii) $D_j(\Theta)$, $F_j^i(\Theta)$ and $H_j(\Theta)$ are quadratic polynomials in $A^{(12)}(\Theta)$. The pair ($D_j(\Theta)$, $H_j(\Theta)$) has numerators related by the above sign pattern extended to 2p+1 terms (rather than p+1 in the previous case). Again $F_j^i(\Theta)$ is a function of only odd or even powers of $ch\Theta$.

iii) $I_j(\Theta)$ is a cubic polynomial in $A^{(12)}(\Theta)$ and its numerator is again a function of only odd or even powers of $ch\Theta$.

Hence the interaction coefficient of $f_1^i f_2^j$ may be expressed as a polynomial in $A^{(12)}(\Theta)$ of degree min(i,j) with coefficients that are functions of the rapidity difference.

$$
\begin{array}{c}
1 \\[1ex]
\delta^{(1)}_{j(1)} f_1 \qquad \delta^{(1)}_{j(2)} f_2 \\[1ex]
\delta^{(2)}_{j(1)} f_1^2 \qquad A_j(\Theta) f_1 f_2 \qquad \delta^{(2)}_{j(2)} f_2^2 \\[1ex]
\delta^{(3)}_{j(1)} f_1^3 \qquad B^{(1)}_j(\Theta) f_1^2 f_2 \qquad B^{(2)}_j(\Theta) f_1 f_2^2 \qquad \delta^{(3)}_{j(2)} f_2^3 \\[1ex]
\delta^{(4)}_{j(1)} f_1^4 \qquad C^{(1)}_j(\Theta) f_1^3 f_2 \qquad D_j(\Theta) f_1^2 f_2^2 \qquad C^{(2)}_j(\Theta) f_1 f_2^3 \qquad \delta^{(4)}_{j(2)} f_2^4 \\[1ex]
\delta^{(5)}_{j(1)} f_1^5 \qquad E^{(1)}_j(\Theta) f_1^4 f_2 \qquad F^{(1)}_j(\Theta) f_1^3 f_2^2 \qquad F^{(2)}_j(\Theta) f_1^2 f_2^3 \qquad E^{(2)}_j(\Theta) f_1 f_2^3 \qquad \delta^{(5)}_{j(2)} f_2^5 \\[1ex]
f_1^6 \qquad G_j(\Theta) f_1^5 f_2 \qquad H_j(\Theta) f_1^4 f_2^2 \qquad I_j(\Theta) f_1^3 f_2^3 \qquad H_j(\Theta) f_1^2 f_2^4 \qquad G_j(\Theta) f_1 f_2^5 \qquad f_2^6 \\[1ex]
\mathbf{A^{(12)^1} x} \qquad \delta^{(5)}_{j(2)} f_1^6 f_2 \qquad E^{(2)}_j(\Theta) f_1^5 f_2^2 \qquad F^{(2)}_j(\Theta) f_1^4 f_2^3 \qquad F^{(1)}_j(\Theta) f_1^3 f_2^4 \qquad E^{(1)}_j(\Theta) f_1^2 f_2^5 \qquad \delta^{(5)}_{j(1)} f_1 f_2^6 \\[1ex]
\mathbf{A^{(12)^2} x} \qquad \delta^{(4)}_{j(2)} f_1^6 f_2^2 \qquad C^{(2)}_j(\Theta) f_1^5 f_2^3 \qquad D_j(\Theta) f_1^4 f_2^4 \qquad C^{(1)}_j(\Theta) f_1^3 f_2^5 \qquad \delta^{(4)}_{j(1)} f_1^2 f_2^6 \\[1ex]
\mathbf{A^{(12)^3} x} \qquad \delta^{(3)}_{j(2)} f_1^6 f_2^3 \qquad B^{(2)}_j(\Theta) f_1^5 f_2^4 \qquad B^{(1)}_j(\Theta) f_1^4 f_2^5 \qquad \delta^{(3)}_{j(1)} f_1^3 f_2^6 \\[1ex]
\mathbf{A^{(12)^4} x} \qquad \delta^{(2)}_{j(2)} f_1^6 f_2^4 \qquad A_j(\Theta) f_1^5 f_2^5 \qquad \delta^{(2)}_{j(1)} f_1^4 f_2^6 \\[1ex]
\mathbf{A^{(12)^5} x} \qquad \delta^{(1)}_{j(2)} f_1^6 f_2^5 \qquad \delta^{(1)}_{j(1)} f_1^5 f_2^6 \\[1ex]
\mathbf{A^{(12)^6} x} \qquad f_1^6 f_2^6
\end{array}
$$

3.6.2 The Fusing Rules for $e_8^{(1)}$

The fusings for the few solutions that were explicitly found were again in accordance with [5] supplemented by an 'iπ' annihilation. As with the case of $e_7^{(1)}$, these were carried out numerically (to high precision) using Mathematica.

4. AN ALGEBRA-INDEPENDENT APPROACH

4.1 Introduction

Following the explicit construction of the double soliton solutions for the simply-laced Lie algebras via Hirota's method, an immediate question to be addressed is that of what the interaction terms have in common. Moreover how these features are related to the underlying Lie algebraic structure.

As has been seen, the ansatz (3.01) leads to interaction functions (in all cases) that are polynomials in the basic interaction piece '$A^{(12)}$' with coefficients that are functions of the rapidity difference Θ. Much more importantly, the function $A^{(12)}$ can always be written as a product of hyperbolic trigonometric functions. This fact in turn hints at the possibility of an all-encompassing construction of such quantities in terms of vertex operators.

The purpose of this chapter is to show how this can be achieved for the case of the quantities $A^{(12)}$, fundamental in the fact that these are the entities upon which all the rest of the interaction functions are based. Moreover, this construction is achieved as a consequence of noting that these 'fundamental' quantities may be written in an algebra-independent manner. Geometry can again be said to play a major role here in the sense that the orbits of the simple roots under the Coxeter element are crucial to the description of these objects. A few obvious properties of the term $A^{(12)}$ will be noted given such a description.

The vertex operator construction may in effect only be regarded as a piece of mathematics since it only goes partway to achieving the representation of the soliton phenomena in terms of such entities. However it is interesting solely in it's own right and resembles very closely that of the operator construction for the minimal part of the scattering matrix that has appeared in [25].

Before proceeding it must also be noted that such operators have since appeared independently elsewhere in the literature (though written slightly diffently in [62]). In Olive, Turok & Underwood[56] they appear as part of a 'thorough' analysis of the soliton solutions for imaginary coupling affine Toda field theory, through the application of the representation theory of the Kač-Moody algebras to a generalised Leznov-Saveliev solution.

4.2 The Fundamental Function $A^{(12)}(\Theta)$

Central to the construction of the double soliton solutions was the ansatz for the tau function associated with the 'affine' spot of the extended Dynkin diagram. This took the form (3.01) and hence gave rise to leading coefficients for the remaining tau functions that were powers of the fundamental function, up to a trivial factor. These powers are exactly the marks of the Dynkin diagram. In turn the quantity $A^{(12)}$ was found to be expressible - in all cases - in terms of a quotient of sinh functions, and hence the quantities (x). A comparison of the equations (3.19a,b,c,d) for $d_4^{(1)}$, (3.43) for $e_8^{(1)}$, and the appendices 3.4.2, 3.5.2 for $e_6^{(1)}$, $e_7^{(1)}$, with the tables of inner products in appendix A, gave rise to the conjecture that

$$A^{\alpha_i\alpha_j}(\Theta) = \prod_{p=1}^{h} \left(1 - \omega^p e^{\Theta + (c(i) - c(j))\frac{\pi i}{2h}}\right)^{c(i)c(j)(\alpha_i \bullet \underline{\omega}^p \alpha_j)} . \tag{4.01}$$

(This was first pointed out to me for the case $d_4^{(1)}$ by E.Corrigan[64]). Here ω ,$\underline{\omega}$ are taken to be the p^{th} root of unity and the Coxeter element, respectively. The bracketed superscripts 1,2 (denoting the first and second soliton) have been replaced by their respective associated simple roots.

To show that this is also true for the infinite sets of simply-laced Lie algebras, i.e. $a_n^{(1)}$ and $d_n^{(1)}$ is manifestly more involved, and the route is to expand the inner product in another basis and then directly calculate through known results.

The more rigorous proof is achieved by expanding the simple roots in terms of one of the complex bases for the eigenvectors of the Coxeter element, which has the properties:

$$\underline{\omega}(\mathbf{e}_s) = e^{\frac{2\pi i s}{h}} \mathbf{e}_s, \quad \mathbf{e}_s \bullet \mathbf{e}_{s'} = \delta_{s+s',h} \tag{4.02}$$

where the label s is one of the (r rank) exponents of the Lie algebra. The following representation of the eigenvectors may be taken:

$$\mathbf{e}_s = \sqrt{2}\,(\hat{a}^{(s)}_{\bullet} - e^{\frac{i\pi s}{h}}\hat{a}^{(s)}_{\circ}) \tag{4.03}$$

where for example $\hat{a}^{(s)}_{\bullet} = \sum_{i\in\bullet} q_i^{(s)}\hat{\alpha}_i = \sum_{i\in\bullet} q_i^{(s)}\lambda_i$ for the simply-laced algebras. These can then be seen to satisfy (4.02) provided that the eigenvectors of the Cartan matrix have unit length and satisfy the conditions:

$$q_i^{(s)} = c(i)q_i^{(h-s)}, \quad q_j^{(s)} \bullet q_j^{(s')} = \delta_{s,s'}. \tag{4.04}$$

Given this, the inner product of an eigenvector with a simple root gives

$$\alpha_l \bullet \mathbf{e}_k = c(l)\sqrt{2}\,e^{\left(\frac{1-c(l)}{2}\right)\frac{i\pi k}{h}} q_l^{(k)} \tag{4.05}$$

and hence the inner product $(\alpha_i \bullet \underline{\omega}^p\alpha_j)$ may be written in terms of the eigenvectors of the Cartan matrix via (4.05).

$$(\alpha_l \bullet \underline{\omega}^p\alpha_j) = \sum_n (\alpha_l \bullet \mathbf{e}_n)(\alpha_j \bullet \mathbf{e}_{h-n})e^{\frac{2\pi ipn}{h}}$$

$$= 2c(l)c(j)\sum_n e^{\left(2p+\frac{c(j)-c(l)}{2}\right)\frac{i\pi n}{h}} q_l^{(n)}q_j^{(n)} \tag{4.06}$$

Here the sum is taken over the exponents of the relevant algebra. Explicit values for such eigenvectors may be found for all the simply-laced Lie algebras in [12] and hence the inner product can be explicitly evaluated.

For example, taking the $a_n^{(1)}$ series, the normalised eigenvectors are given by $q_l^{(s)} = \sqrt{\frac{2}{h}}\sin(\frac{l\pi s}{h})$ and the exponents s are 1,2,3........,h-1. Therefore the term above becomes:

$$\sum_{n=1}^{h-1} \frac{(-)}{h} c(l)c(j)e^{\left(2p+\frac{c(j)-c(l)}{2}\right)\frac{i\pi n}{h}}(e^{\frac{i\pi nl}{h}} - e^{\frac{-i\pi nl}{h}})(e^{\frac{i\pi nj}{h}} - e^{\frac{-i\pi nj}{h}})$$

which gives zero unless $(2p+\frac{c(j)-c(l)}{2} \pm l \pm j) \propto h$. Explicit calculation determines that

$$A^{\alpha_i\alpha_j}(\Theta) = \frac{(1-\omega^{(\frac{i-j}{2})}e^{\Theta})(1-\omega^{-(\frac{i-j}{2})}e^{\Theta})}{(1-\omega^{(\frac{i+j}{2})}e^{\Theta})(1-\omega^{-(\frac{i+j}{2})}e^{\Theta})} \tag{4.07}$$

in accordance with (3.10), irrespective of colour and ordering. Similar results give the interaction pieces for $d_n^{(1)}$.

4.3 The Operator Construction

Given (4.01) and its equivalence to a quotient of sinh functions, the question begs as to whether it is possible to obtain such a quantity through the usual normal ordering procedure of vertex operators.

In order to achieve this, a variation on Corrigan & Dorey[25] is used, and a string-like rapidity dependent field is defined for each simple root via the formula:

$$X^{\alpha_i}(\theta_i)=\sum_{r=s+kh}\frac{h}{r}\,e^{r\left(\theta_i+c(i)\frac{i\pi}{2h}\right)}\gamma_i^{(h-s)}.d_r \tag{4.08}$$

The summation is taken over the exponents s of the algebra and all integers k; h denotes the relevant Coxeter number. Here, the quantity γ_i denotes the i^{th} signed simple root $c(i)\alpha_i$ resulting from a bicolouration of the Dynkin diagram, and c(i) can be taken to be '+1' if 'i' is black, '-1' if 'i' is white, as in [12]. The superscript denotes the $(h\text{-}s)^{th}$ component of an object expanded in a complex basis of the eigenvectors of the Coxeter element. That is

$$\alpha_i=\sum_s \alpha_i^{(s)}\mathbf{e}_s=\sum_s\ (\alpha_i\bullet\mathbf{e}_{h-s}) \tag{4.09}$$

The operators d_r satisfy the commutation relations

$$\left[d_r\,,d_{r'}\right]=\frac{r}{h}\,\delta_{r+r',0} \tag{4.10}$$

and it is implicitly assumed that there exists a 'ground state' annihilated by all d_r, $r>0$. It is easily seen that the $X^{\alpha}(\theta)$ are $2\pi i$-periodic in θ and moreover that the field is 'twisted' in the sense that a Coxeter rotation of the root shifts the rapidity by the value $2\pi i/h$:

$$\alpha_i'=\sum_s\ (\alpha_i^{(s)})'.\mathbf{e}_s=\underline{\omega}\ (\alpha_i)\quad:\ (\alpha_i^{(s)})'=e^{\frac{2\pi is}{h}}.\alpha_i^{(s)}$$

Then the commututation relation involving the creation and annihilation parts of such operators $X^{\alpha}(\theta)$ may be calculated,

$$\left[X_+^{\alpha_i}(\theta_i),X_-^{\alpha_j}(\theta_j)\right]=\left[\sum_{r>0}\tfrac{h}{r}e^{r\left(\theta_i+c(i)\frac{i\pi}{2h}\right)}\gamma_i^{(h-s)}d_r\,,\sum_{r'>0}\tfrac{h}{r'}e^{r'\left(\theta_j+c(j)\frac{i\pi}{2h}\right)}\gamma_j^{(h-s')}d_{r'}\right]$$

$$=\sum_{r>0}\ (-)\tfrac{h}{r}\,e^{r\left(\Theta+(c(i)-c(j))\frac{i\pi}{2h}\right)}\gamma_i^{(h-s)}\gamma_j^{(h-s)} \tag{4.11}$$

where (4.10) has been utilised and Θ has been defined to be θ_i-θ_j, that is the rapidity difference. This expression is equivalent to

$$-h\sum_s \left(\sum_{r\equiv s(\mathrm{mod}\, h)} \frac{1}{r} X^r \right) \gamma_i^{(h-s)} \gamma_j^{(h-s)} \tag{4.12}$$

where X is taken to be $e^{r\left(\Theta+(c(i)-c(j))\frac{i\pi}{2h}\right)}$. This is further simplified using the summation formula

$$\sum_{n\equiv s(\mathrm{mod}\, h)} \frac{1}{n} t^n = (-)\frac{1}{h} \sum_{l=1}^{h} \omega^{-ls}.\ln(1-\omega^l t) \tag{4.13}$$

to give,

$$\sum_{l=1}^{h} \left(\sum_s \gamma_i^{(h-s)} \omega^{-ls} \gamma_j^{(h-s)} \right) \ln(1-\omega^l X) \qquad : \mathrm{Re}(\Theta) < 0$$

Noting that $\omega^{-ls}\gamma_j^{(h-s)} = c(j)(\underline{\omega} \bullet \alpha_j)^{(h-s)}$, the commutator is easily seen to be equivalent to

$$\sum_{p=1}^{h} \left(c(i)c(j)(\alpha_i \bullet \underline{\omega}^p \alpha_j) \right) \ln(1-\omega^p X). \tag{4.14}$$

Hence defining a vertex operator to be the normal ordered exponential of such a field:

$$Y^{\alpha_i}(\theta_i) = \ :\exp X^{\alpha_i}(\theta_i): \ \equiv \ \exp X_-^{\alpha_i}(\theta_i) \exp X_+^{\alpha_i}(\theta_i)$$

then the product of two such operators can be normal ordered using the Baker-Campbell-Hausdorf formula to give an extra rapidity dependent factor.

$$Y^{\alpha_i}(\theta_i) Y^{\alpha_j}(\theta_j) = A^{\alpha_i \alpha_j}(\Theta) : Y^{\alpha_i}(\theta_i) Y^{\alpha_j}(\theta_j) :$$

where

$$A^{\alpha_i \alpha_j}(\Theta) = \exp\left[X_+^{\alpha_i}(\theta_i), X_-^{\alpha_j}(\theta_j) \right]$$

$$= \prod_{p=1}^{h} \left(1 - \omega^{(p+\frac{c(i)-c(j)}{4})} e^{\Theta} \right)^{c(i)c(j)(\alpha_i \bullet \underline{\omega}^p \alpha_j)} \tag{4.15}$$

which is the required interaction function.

Before proceeding into a discussion of the aspects of such a function, it could be noted that the vertex operators that appear here are 'conformal' in the sense that they give rise to a trivial braiding relation. There is no need to perturb them in any way (compare [25]) since this is all that is required. It will be seen however, that the interaction function satisfies some similar properties as the scattering matrices.

4.4 Properties of the function $A^{(12)}(\Theta)$

As has already been mentioned in section 4.1, the leading order coefficient of a double soliton tau function must be trivially proportional to $\left(A^{\alpha_1\alpha_2}(\Theta)\right)^{n_j}$. Hence this function governs the fusing proceedure for the classical solitons. A thorough analysis of this phenomenon may be undertaken through the use of (4.01), where the function is written in an algebra-independent form.

The fusings occur when the leading term has a pole in Θ and hence when $c(i)c(j)(\alpha_i,\underline{\omega}^p\alpha_j)$ takes a negative value. Since $\underline{\omega}^p\alpha_j$ is a root of a simply-laced algebra, this value can only either be -1 or -2. Utilising the fact that the orbits of γ_i are distinct, then the value of -2 is obtained only when $\gamma_i = -\underline{\omega}^p\gamma_j$ i.e. when γ_i lies in the Coxeter orbit of $-\gamma_j$. Comparison with the fact that

$$\gamma_{\bar{i}} = -\underline{\omega}^{\left(-\frac{h}{2}+\frac{c(i)-c(\bar{i})}{4}\right)}.\gamma_i \tag{4.16}$$

i.e the conjugate 'signed' root lies in the same Coxeter orbit as the negative of the same 'signed' root, requests i and j to be a conjugate pair, and p to take the value corresponding to the power of ω, since all the orbits are disjoint. [This exponent is easily seen to lie in the integers since $c(i)c(\bar{i})=(-)^h$]. Hence the pole occurs at the value of Θ when

$$0 = 1-\underline{\omega}^{\left(p+\frac{c(i)-c(j)}{4}\right)}.e^{\Theta} = 1-\underline{\omega}^{\left(-\frac{h}{2}\right)}.e^{\Theta} = 1-e^{\Theta-i\pi}$$

that is, at $\Theta= i\pi$. This fact was borne out in all cases of conjugate pairs.

The remaining case, when $(\gamma_i \bullet \underline{\omega}^p\gamma_j) = -1$, implies that $\gamma_i + \underline{\omega}^p\gamma_j = \underline{\omega}^q\gamma_k$ since the Weyl reflection of any root in a simple root must be another root, and then this is also expressible as the Coxeter rotation of one of the γ_i. Therefore

$$\gamma_i + \underline{\omega}^p\gamma_j + \underline{\omega}^r\gamma_{\bar{k}} = 0$$

where $r = q + \frac{h}{2} - \frac{c(k)-c(\bar{k})}{4}$, which is Dorey's rule for the three-point couplings in the real -coupling regime of affine Toda field theory.

Hence it has been shown that in the imaginary coupling regime (specifically with soliton solutions in mind) Dorey's rule is augmented further by soliton-antisoliton fusings (annihilations) at rapidity difference $i\pi$, purely using the algebraic construction of the leading order term.

It is also possible to note via (4.16) that when a root is self-conjugate, $\underline{\omega}^{\frac{h}{2}}\alpha_i = -\alpha_i$, and hence the interaction function takes the form:

$$\frac{\mathrm{sh}\left(\frac{\Theta}{2}+\frac{i\pi y_1}{2h}\right)\mathrm{sh}\left(\frac{\Theta}{2}-\frac{i\pi y_1}{2h}\right)\ldots\ldots\ldots\ldots\mathrm{sh}\left(\frac{\Theta}{2}+\frac{i\pi y_n}{2h}\right)\mathrm{sh}\left(\frac{\Theta}{2}-\frac{i\pi y_n}{2h}\right)}{\mathrm{sh}\left(\frac{\Theta}{2}+\frac{i\pi x_1}{2h}\right)\mathrm{sh}\left(\frac{\Theta}{2}-\frac{i\pi x_1}{2h}\right)\ldots\ldots\ldots\ldots\mathrm{sh}\left(\frac{\Theta}{2}+\frac{i\pi x_n}{2h}\right)\mathrm{sh}\left(\frac{\Theta}{2}-\frac{i\pi x_n}{2h}\right)}$$

where $y_i = x_i + h \pmod{2h}$ and the poles of $A^{(12)}$ occur at $\Theta_j = \frac{i\pi x_j}{h}$. This is equivalent to the form

$$\frac{(\mathrm{ch}\Theta+\cos\Theta_1)\ldots\ldots\ldots\ldots(\mathrm{ch}\Theta+\cos\Theta_n)}{(\mathrm{ch}\Theta-\cos\Theta_1)\ldots\ldots\ldots\ldots(\mathrm{ch}\Theta-\cos\Theta_n)} \tag{4.17}$$

for the function $A^{(12)}$, which is ultimately responsible for the phenomena noted in section 3.6.1 through expressions derived from the equations of motion for the relevant interaction terms.

Given that the interaction piece appears naturally through a similar vertex operator construction as the minimal S-matrix, it might be considered that such a function would possess some properties akin to those held by the scattering matrix. This proposition is in fact true since it is easily shown that if particles (solitons) a,b fuse to a third, c say, then the function $A^{(12)}$ satisfies the same 'bootstrap' property as that of the scattering matrices, i.e. (1.09).

$$A^{da}(\Theta - i(\pi-\theta^b_{ac}))\, A^{db}(\Theta + i(\pi-\theta^a_{bc}))$$

$$=\prod_{p=1}^{h}\left(1-\omega^{\left(p+\frac{c(d)-c(a)}{4}-\frac{U^b_{ac}}{2}\right)}.e^{\Theta}\right)^{(\gamma_d\bullet\underline{\omega}^p\gamma_a)}\cdot\prod_{q=1}^{h}\left(1-\omega^{\left(q+\frac{c(d)-c(b)}{4}+\frac{U^a_{bc}}{2}\right)}.e^{\Theta}\right)^{(\gamma_d\bullet\underline{\omega}^p\gamma_b)}$$

$$=\prod_{p'=1}^{h}\left(1-\omega^{\left(p'+\frac{c(d)-c(c)}{4}\right)}.e^{\Theta}\right)^{\left(\gamma_d\bullet\underline{\omega}^{p'+\frac{U^b_{ac}}{2}-\frac{c(c)-c(a)}{4}}.\gamma_a+\underline{\omega}^{p'-\frac{U^a_{bc}}{2}-\frac{c(c)-c(b)}{4}}.\gamma_b\right)}$$

$$\equiv \prod_{p=1}^{h} \left(1-\omega^{\left(p+\frac{c(d)-c(c)}{4}\right)}.e^{\Theta}\right)^{(\gamma_d \bullet \underline{\omega}^p \gamma_c)}$$

$$\equiv A^{dc}(\Theta) \tag{4.18}$$

where,

$$\gamma_c = \underline{\omega}^{\frac{U^b_{ac}}{2}-\frac{c(c)-c(a)}{4}}.\gamma_a + \underline{\omega}^{-\frac{U^a_{bc}}{2}-\frac{c(c)-c(b)}{4}}.\gamma_b \tag{4.19}$$

and hence is equivalent to Dorey's fusing rule.

In the process of the calculation, the U's were defined by $\overline{\theta^c_{ab}} \equiv \frac{i\pi}{h}.U^c_{ab}$ and the fact that $\omega,\underline{\omega}$ were periodic in h allowed the appropriate relabelling to be carried out:

$$\begin{aligned} p' &= \frac{c(c)-c(a)}{4} + p - \frac{U^b_{ac}}{2} \quad \in Z \\ &\equiv \frac{c(c)-c(b)}{4} + q + \frac{U^a_{bc}}{2} \quad \in Z. \end{aligned}$$

Another property of this 'leading order' function which is quite readily proved is that

$$\left(A^{i\bar{i}}(\Theta - \tfrac{i\pi}{2}).A^{i\bar{i}}(\Theta + \tfrac{i\pi}{2})\right)_{\Theta=i\pi} = 1 \tag{4.20}$$

when 'i' is taken to correspond to any self-conjugate spot on the Dynkin diagram. From (4.01) it is easily seen that the L.H.S. of (4.20) is equivalent to

$$\prod_{p=1}^{h} \left(1+\omega^{2\left(p+\frac{c(i)-c(\bar{i})}{4}\right)}\right)^{(\gamma_i \bullet \underline{\omega}^p \gamma_{\bar{i}})}$$

and hence by taking $q = p - \frac{h}{2} + \frac{c(i)-c(\bar{i})}{4} \in Z$ after substituting the expression (4.16) for $\gamma_{\bar{i}}$, the above becomes:

$$\prod_{q=1}^{h} \left(1+\omega^{2\left(q+\frac{h}{2}\right)}\right)^{-(\gamma_i \bullet \underline{\omega}^q \gamma_i)} \equiv \prod_{q=1}^{h} \left(1+\omega^{2q}\right)^{-(\gamma_i \bullet \underline{\omega}^q \gamma_i)} .$$

Comparing the two formulae it can be seen that (4.20) must hold when 'i' is self-conjugate. For example, taking the case of the Lie algebras $a_n^{(1)}$, the fundemental interaction function is given by the expression (4.07), and hence

$$A^{i\bar{i}}(\Theta - \tfrac{i\pi}{2}).A^{i\bar{i}}(\Theta + \tfrac{i\pi}{2})$$

$$= \frac{\left(1+i\omega^{\left(\frac{2i-h}{2}\right)}.e^{\Theta}\right)\left(1+i\omega^{-\left(\frac{2i-h}{2}\right)}.e^{\Theta}\right)\left(1-i\,\omega^{\left(\frac{2i-h}{2}\right)}.e^{\Theta}\right)\left(1-i\omega^{\left(\frac{2i-h}{2}\right)}.e^{\Theta}\right)}{\left(1+i\omega^{\left(\frac{h}{2}\right)}.e^{\Theta}\right)\left(1+i\omega^{-\left(\frac{h}{2}\right)}.e^{\Theta}\right)\left(1-i\omega^{\left(\frac{h}{2}\right)}.e^{\Theta}\right)\left(1-i\omega^{-\left(\frac{h}{2}\right)}.e^{\Theta}\right)}$$

$$\equiv \frac{\left(1+\omega^{2i}.e^{\Theta}\right)\left(1+\omega^{-2i}.e^{\Theta}\right)}{\left(1+e^{\Theta}\right)\left(1+e^{\Theta}\right)}.$$

Therefore $\left(A^{i\bar{i}}(\Theta-\frac{i\pi}{2}).A^{i\bar{i}}(\Theta+\frac{i\pi}{2})\right)_{\Theta=i\pi}$ is undefined unless $i=\bar{i}=\frac{h}{2}$ when it is equivalent to unity. This property holds for all the relevant simply-laced double soliton solutions.

5 CONCLUSION AND OUTLOOK

This thesis has been concerned with the explicit construction of single and double soliton solutions to the set of field theories known as the simply-laced affine Toda field theories. Given these solutions, a thorough analysis of the fusing structure of such classical objects has been undertaken and has also been abstracted into the setting of the underlying Lie algebraic structure. A few basic properties of the functions involved have been noted and were commented upon.

A decade ago[2],[3], it was found that taking the coupling constant to be purely imaginary led to representations of the unitary minimal series of conformal field theories in the form of quantized Toda field theories. In the context of their associated affine models, it leads to the equally important phenomena of topological soliton solutions interpolating the weight lattice of the finite Lie algebra. This appears through the infinite number of degenerate minima that occur in the potential terms.

Even though the Hamiltonian is non-Hermitian, the explicit single soliton solutions possess real energy and momenta. It is, therefore, hoped that a unitary field theory is somehow embedded in this larger non-unitary system. These single solitons can be taken pairwise and scattered classically and their resulting interactions can be analysed. To this end, double soliton solutions were constructed explicitly using Hirota's method for all the simply-laced theories, although it was found that the tau function substitutional ansatz involved, only led to 'proper' bilinear equations for the simpler $a_n^{(1)}$ - series of algebras. The finite expansion of the tau functions found is considered to be a manifestation of the integrability of the underlying field theory.

The double soliton solutions reveal the existence of the 'hidden' Lie algebraic framework through their leading order tau function behaviour. As also discovered in the real-coupling regîme of affine Toda theory, the Coxeter orbits of the simple roots are found to be a crucial element in the analysis of the relevant interaction terms. These solutions are in essence fundamental, since the N-soliton scattering matrices should factorize into products of two-soliton scatterings because of the presence of the infinite number of conserved charges associated with the system. Only those corresponding to the $a_n^{(1)}$ series of algebras have, however, been conjectured and these have involved aspects of the theory of quantum groups[14]. Hence, this is an obvious area for immediate further research, since an extension to the other simply-laced and non-simply-laced theories is required.

The leading order tau function behaviour reveals a great deal more information. Namely, the topological nature of such solutions via the single solitons (not discussed here), and their fusing relationships when double soliton solutions are considered. It was found that these classically allowed fusings occur at precisely the same rapidity differences that appear for the three point couplings [in the real-coupling affine Toda field theory], but in all simply-laced cases were supplemented by soliton-antisoliton annihilations.

Given that the topological charges of the single solitons classically do not appear to fill out the whole of the weight systems of the fundamental representations (except for $a_1^{(1)}$, $a_2^{(1)}$), it would be an interesting investigation to determine whether a 'breathing' solution could be constructed with such a property. More exotic breathers may be expected in the higher rank algebras when compared with the sine-Gordon model, given the extra dimensions of the weight lattices associated with such theories.

Moreover, given the difficulties with constructing the $e_8^{(1)}$ double soliton solutions , it would be instructive to determine the level of complexity involved in producing these from the generalised Leznov-Saveliev solution (or vertex operator formalism) introduced by Olive, Turok and Underwood[56]. Certainly the Hirota method is tedious for all the exceptional algebras however it may be deemed preferential for generating explicit solutions.

APPENDIX A

The inner products for the orbits of the simple roots under the Coxeter element are given here for the algebras d_4, e_6, e_7 and e_8.

d_4:

Taking l, l', l'' & h to correspond with the outer and inner spots of the Dynkin diagram, then the following values for the scalar products are obtained:

$$(\alpha_h, \underline{\omega}^p \alpha_h) = (+1, -1, -2, -1, +1, +2)$$

$$(\alpha_l, \underline{\omega}^p \alpha_h) = (\ 0, +1, +1,\ \ 0, -1, -1\ )$$

$$(\alpha_l, \underline{\omega}^p \alpha_l) = (-1, +1, -2, +1, -1, +2\ )$$

$$(\alpha_l, \underline{\omega}^p \alpha_{l'}) = (+1, -1,\ \ 0, -1, +1,\ \ 0)$$

where p takes the values from 1,..,6 respectively. The label 'l' corresponds to any of the degenerate 'light' spots, and the labels l,l' refer to any non-equal pair from the set of such spots.

Invariance of the inner products under the symmetry $\underline{\omega}^p \to \underline{\omega}^{-p}$, $\underline{\omega}^p \to \underline{\omega}^{-p-1}$ may be noted for equally-coloured, unequally-coloured simple roots respectively, after the Dynkin diagram has been bicoloured as mentioned in chapter four. More precisely there is an invariance under $p \to -p + \frac{c(1) - c(2)}{2}$, where c(1), c(2) refer to the colour indices of the first, second simple roots in the inner product. This is true for all the algebras.

e_6:

With the labelling of the simple roots taken to be that corresponding to the subdiagram of the affine Dynkin diagram on page '85', they may be represented by the vectors:

$$\alpha_i = e_i - e_{i+1}, \qquad \alpha_6^T = (\alpha\ , \alpha\ , \alpha\ , \alpha + 1, \alpha + 1, \alpha + 1)$$

where i =1,...,5 and the $\{e_j\}$ denotes the standard orthonormal basis for R^6. The constant α also satisfies the equation

$$6\alpha^2 + 6\alpha + 1 = 0$$

if all the simple roots possess a normalisation of two. Then the Coxeter element in this basis can be given by:

$$\underline{\omega} = \underline{\omega}_\bullet \underline{\omega}_\circ$$

$$\equiv (\underline{\omega}_{\alpha_6}\underline{\omega}_{\alpha_4}\underline{\omega}_{\alpha_2})(\underline{\omega}_{\alpha_5}\underline{\omega}_{\alpha_3}\underline{\omega}_{\alpha_1})$$

$$\equiv \begin{pmatrix} 1-\alpha^2 & -\alpha^2 & -\alpha^2 & -\alpha\beta & -\alpha\beta & -\alpha\beta \\ -\alpha^2 & 1-\alpha^2 & -\alpha^2 & -\alpha\beta & -\alpha\beta & -\alpha\beta \\ -\alpha^2 & -\alpha^2 & 1-\alpha^2 & -\alpha\beta & -\alpha\beta & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & 1-\beta^2 & -\beta^2 & -\beta^2 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & -\beta^2 & 1-\beta^2 & -\beta^2 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & -\beta^2 & -\beta^2 & 1-\beta^2 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} -\alpha^2 & 1-\alpha^2 & -\alpha\beta & -\alpha^2 & -\alpha\beta & -\alpha\beta \\ -\alpha^2 & -\alpha^2 & -\alpha\beta & 1-\alpha^2 & -\alpha\beta & -\alpha\beta \\ 1-\alpha^2 & -\alpha^2 & -\alpha\beta & -\alpha^2 & -\alpha\beta & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & -\beta^2 & -\alpha\beta & -\beta^2 & 1-\beta^2 \\ -\alpha\beta & -\alpha\beta & 1-\beta^2 & -\alpha\beta & -\beta^2 & -\beta^2 \\ -\alpha\beta & -\alpha\beta & -\beta^2 & -\alpha\beta & 1-\beta^2 & -\beta^2 \end{pmatrix}$$

where $\beta = \alpha + 1$. [This construction easily generalises for e_7 and e_8.]

Using (4.16) and the fact that the symmetry of the e_6 Dynkin diagram implies $c(i) = c(\bar{i})$, then $\underline{\omega}^{\frac{h}{2}}\alpha_i = -\alpha_{\bar{i}}$. Therefore $\underline{\omega}^{\frac{h}{2}}\alpha_1 = -\alpha_5$, $\underline{\omega}^{\frac{h}{2}}\alpha_2 = -\alpha_4$ and for the self-conjugate simple roots α_3, α_6 : $\underline{\omega}^{\frac{h}{2}}\alpha_i = -\alpha_i$.

Hence, as a result of these facts, it is merely sufficient to calculate each $(\alpha_i, \underline{\omega}^p\alpha_j)$ from '1' to '$\frac{h}{2}$' (= 6) in order to possess the relevant information to construct the inner product tables for the whole of the Coxeter orbits. These scalars (for p=1,...,6) are now listed:

$$(\alpha_1, \underline{\omega}^p\alpha_1) = (-1,\ 0, +1, -1,\ 0,\ 0)$$

$$(\alpha_2, \underline{\omega}^p\alpha_1) = (-1, +1, -1,\ 0, +1,\ 0)$$

$$(\alpha_3, \underline{\omega}^p\alpha_1) = (+1,\ 0,\ 0,\ 0, +1,\ 0)$$

$$(\alpha_4, \underline{\omega}^p\alpha_1) = (\ 0, -1,\ 0, +1, -1, +1)$$

$$(\alpha_5, \underline{\omega}^p\alpha_1) = (\ 0, +1, -1,\ 0, +1, -2)$$

$$(\alpha_6, \underline{\omega}^p\alpha_1) = (\ 0, -1, +1, -1, +1,\ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_2)=(+1,-1,\ 0,+1,\ 0,\ 0)$$
$$(\alpha_2,\underline{\omega}^P\alpha_2)=(\ 0,\ 0,+1,-1,-1,\ 0)$$
$$(\alpha_3,\underline{\omega}^P\alpha_2)=(-1,\ 0,\ 0,+1,+1,+1)$$
$$(\alpha_4,\underline{\omega}^P\alpha_2)=(+1,+1,-1,\ 0,\ 0,-2)$$
$$(\alpha_5,\underline{\omega}^P\alpha_2)=(-1,\ 0,+1,-1,+1,+1)$$
$$(\alpha_6,\underline{\omega}^P\alpha_2)=(+1,\ 0,\ 0,\ 0,-1,\ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_3)=(+1,\ 0,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_2,\underline{\omega}^P\alpha_3)=(-1,-1,\ 0,\ 0,+1,+1)$$
$$(\alpha_3,\underline{\omega}^P\alpha_3)=(+1,+1,\ 0,-1,-1,-2)$$
$$(\alpha_4,\underline{\omega}^P\alpha_3)=(-1,-1,\ 0,\ 0,+1,+1)$$
$$(\alpha_5,\underline{\omega}^P\alpha_3)=(+1,\ 0,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_6,\underline{\omega}^P\alpha_3)=(-1,\ 0,-1,+1,\ 0,+1);$$

$$(\alpha_1,\underline{\omega}^P\alpha_4)=(-1,\ 0,+1,-1,+1,+1)$$
$$(\alpha_2,\underline{\omega}^P\alpha_4)=(+1,+1,-1,\ 0,\ 0,-2)$$
$$(\alpha_3,\underline{\omega}^P\alpha_4)=(-1,\ 0,\ 0,+1,+1,+1)$$
$$(\alpha_4,\underline{\omega}^P\alpha_4)=(\ 0,\ 0,+1,-1,-1,\ 0)$$
$$(\alpha_5,\underline{\omega}^P\alpha_4)=(+1,-1,\ 0,+1,\ 0,\ 0)$$
$$(\alpha_6,\underline{\omega}^P\alpha_4)=(+1,\ 0,\ 0,\ 0,-1,\ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_5)=(\ 0,+1,-1,\ 0,+1,-2)$$
$$(\alpha_2,\underline{\omega}^P\alpha_5)=(\ 0,-1,\ 0,+1,-1,+1)$$
$$(\alpha_3,\underline{\omega}^P\alpha_5)=(+1,\ 0,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_4,\underline{\omega}^P\alpha_5)=(-1,+1,-1,\ 0,+1,\ 0)$$
$$(\alpha_5,\underline{\omega}^P\alpha_5)=(-1,\ 0,+1,-1,\ 0,\ 0)$$
$$(\alpha_6,\underline{\omega}^P\alpha_5)=(\ 0,-1,+1,-1,+1,\ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_6)=(-1,+1,-1,+1,\ 0,\ 0)$$
$$(\alpha_2,\underline{\omega}^P\alpha_6)=(+1,\ 0,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_3,\underline{\omega}^P\alpha_6)=(\ 0,-1,+1,\ 0,+1,+1)$$
$$(\alpha_4,\underline{\omega}^P\alpha_6)=(+1,\ 0,\ 0,\ 0,-1,\ 0)$$

$$(\alpha_5, \underline{\omega}^p\alpha_6) = (-1,+1,-1,+1,\ 0,\ 0)$$

$$(\alpha_6, \underline{\omega}^p\alpha_6) = (-1,+1,\ 0,-1,+1,-2)\,.$$

e_7:

Unlike the case of the algebra e_6, the Dynkin diagram for e_7 is devoid of symmetry and hence all the simple roots are self-conjugate and satisfy $\underline{\omega}^{\frac{h}{2}}\alpha_i = -\alpha_i$. (This is also true for the case e_8). Therefore, only the values for p=1,....,$\frac{h}{2}$ = 9 need to be calculated, the others coming as-it-were 'gratis'.

$$(\alpha_1, \underline{\omega}^p\alpha_1) = (-1,\ 0,+1,-1,+1,-1,\ 0,+1,-2)$$

$$(\alpha_2, \underline{\omega}^p\alpha_1) = (-1,+1,-1,\ 0,\ 0,\ 0,+1,-1,+1)$$

$$(\alpha_3, \underline{\omega}^p\alpha_1) = (+1,\ 0,\ 0,+1,-1,\ 0,\ 0,-1,\ 0)$$

$$(\alpha_4, \underline{\omega}^p\alpha_1) = (\ 0,-1,\ 0,\ 0,\ 0,\ 0,\ 0,+1,\ 0)$$

$$(\alpha_5, \underline{\omega}^p\alpha_1) = (\ 0,+1,\ 0,-1,+1,\ 0,-1,\ 0,\ 0)$$

$$(\alpha_6, \underline{\omega}^p\alpha_1) = (\ 0,\ 0,-1,+1,\ 0,-1,+1,\ 0,\ 0)$$

$$(\alpha_7, \underline{\omega}^p\alpha_1) = (\ 0,-1,+1,-1,\ 0,+1,-1,+1,\ 0)\,;$$

$$(\alpha_1, \underline{\omega}^p\alpha_2) = (+1,-1,\ 0,\ 0,\ 0,+1,-1,+1,+1)$$

$$(\alpha_2, \underline{\omega}^p\alpha_2) = (\ 0,\ 0,+1,\ 0,\ 0,-1,\ 0,\ 0,-2)$$

$$(\alpha_3, \underline{\omega}^p\alpha_2) = (-1,\ 0,-1,\ 0,+1,\ 0,+1,+1,+1)$$

$$(\alpha_4, \underline{\omega}^p\alpha_2) = (+1,+1,\ 0,\ 0,\ 0,\ 0,-1,-1,\ 0)$$

$$(\alpha_5, \underline{\omega}^p\alpha_2) = (-1,-1,+1,\ 0,-1,+1,+1,\ 0,\ 0)$$

$$(\alpha_6, \underline{\omega}^p\alpha_2) = (\ 0,+1,\ 0,-1,+1,\ 0,-1,\ 0,\ 0)$$

$$(\alpha_7, \underline{\omega}^p\alpha_2) = (+1,\ 0,\ 0,+1,-1,\ 0,\ 0,-1,\ 0)\,;$$

$$(\alpha_1, \underline{\omega}^p\alpha_3) = (+1,\ 0,\ 0,+1,-1,\ 0,\ 0,-1,\ 0)$$

$$(\alpha_2, \underline{\omega}^p\alpha_3) = (-1,-1,\ 0,-1,\ 0,+1,\ 0,+1,+1)$$

$$(\alpha_3, \underline{\omega}^p\alpha_3) = (+1,+1,+1,\ 0,\ 0,-1,-1-1,-2)$$

$$(\alpha_4, \underline{\omega}^p\alpha_3) = (-1,-1,-1,\ 0,\ 0,\ 0,+1,+1,+1)$$

$$(\alpha_5, \underline{\omega}^p\alpha_3) = (+1,+1,\ 0,\ 0,\ 0,\ 0,-1,-1,\ 0)$$

$$(\alpha_6, \underline{\omega}^p\alpha_3) = (\ 0,-1,\ 0,\ 0,\ 0,\ 0,\ 0,+1,\ 0)$$

$$(\alpha_7, \underline{\omega}^p\alpha_3) = (-1,\ 0,-1,\ 0,\ 0,\ 0,+1,\ 0,+1)\,;$$

$$(\alpha_1,\underline{\omega}^p\alpha_4)=(-1,0,0,0,0,0,+1,0,0)$$

$$(\alpha_2,\underline{\omega}^p\alpha_4)=(+1,+1,0,0,0,0,-1,-1,0)$$

$$(\alpha_3,\underline{\omega}^p\alpha_4)=(-1,-1,0,0,0,+1,+1,+1,+1)$$

$$(\alpha_4,\underline{\omega}^p\alpha_4)=(0,+1,0,+1,-1,0,-1,0,-2)$$

$$(\alpha_5,\underline{\omega}^p\alpha_4)=(0,0,-1,0,+1,0,0,+1,+1)$$

$$(\alpha_6,\underline{\omega}^p\alpha_4)=(+1,-1,+1,0,0,-1,+1,-1,0)$$

$$(\alpha_7,\underline{\omega}^p\alpha_4)=(+1,0,+1,-1,+1,-1,0,-1,0);$$

$$(\alpha_1,\underline{\omega}^p\alpha_5)=(0,+1,0,-1,+1,0,-1,0,0)$$

$$(\alpha_2,\underline{\omega}^p\alpha_5)=(0,-1,-1,+1,0,-1,+1,+1,0)$$

$$(\alpha_3,\underline{\omega}^p\alpha_5)=(+1,+1,0,0,0,0,-1,-1,0)$$

$$(\alpha_4,\underline{\omega}^p\alpha_5)=(-1,0,0,-1,0,+1,0,0,+1)$$

$$(\alpha_5,\underline{\omega}^p\alpha_5)=(0,-1,+1,+1,-1,-1,+1,0,-2)$$

$$(\alpha_6,\underline{\omega}^p\alpha_5)=(-1,+1,0,-1,0,+1,0,-1,+1)$$

$$(\alpha_7,\underline{\omega}^p\alpha_5)=(0,-1,0,0,0,0,0,+1,0);$$

$$(\alpha_1,\underline{\omega}^p\alpha_6)=(0,-1,+1,0,-1,+1,0,0,0)$$

$$(\alpha_2,\underline{\omega}^p\alpha_6)=(0,+1,0,-1,+1,0,-1,0,0)$$

$$(\alpha_3,\underline{\omega}^p\alpha_6)=(-1,0,0,0,0,0,+1,0,0)$$

$$(\alpha_4,\underline{\omega}^p\alpha_6)=(+1,-1,+1,0,0,-1,+1,-1,0)$$

$$(\alpha_5,\underline{\omega}^p\alpha_6)=(+1,0,-1,0,+1,0,-1,+1,+1)$$

$$(\alpha_6,\underline{\omega}^p\alpha_6)=(-1,0,0,+1,-1,0,0,+1,-2)$$

$$(\alpha_7,\underline{\omega}^p\alpha_6)=(0,+1,-1,+1,-1,+1,-1,0,0);$$

$$(\alpha_1,\underline{\omega}^p\alpha_7)=(-1,+1,-1,0,+1,-1,+1,0,0)$$

$$(\alpha_2,\underline{\omega}^p\alpha_7)=(+1,0,0,+1,-1,0,0,-1,0)$$

$$(\alpha_3,\underline{\omega}^p\alpha_7)=(0,-1,0,0,0,+1,0,+1,+1)$$

$$(\alpha_4,\underline{\omega}^p\alpha_7)=(+1,0,+1,-1,+1,-1,0,-1,0)$$

$$(\alpha_5,\underline{\omega}^p\alpha_7)=(-1,0,0,0,0,0,+1,0,0)$$

$$(\alpha_6,\underline{\omega}^p\alpha_7)=(0,+1,-1,+1,-1,+1,-1,0,0)$$

$$(\alpha_7,\underline{\omega}^p\alpha_7)=(-1,+1,0,0,0,0,-1,+1,-2).$$

$\mathbf{e_8}$:

Again, all the simple roots are 'self-conjugate.' The Coxeter number h = 30 for this finite algebra and therefore only the values for p = 1,.....,15 may be given:.

$$(\alpha_1,\underline{\omega}^p\alpha_1)=(-1,\ 0,+1,-1,+1,\ 0,-1,+1,\ 0,-1,+1,-1,\ 0,+1,-2)$$
$$(\alpha_2,\underline{\omega}^p\alpha_1)=(-1,+1,-1,\ 0,\ 0,-1,+1,\ 0,-1,+1,\ 0,\ 0,+1,-1,+1)$$
$$(\alpha_3,\underline{\omega}^p\alpha_1)=(+1,\ 0,\ 0,+1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_4,\underline{\omega}^p\alpha_1)=(\ 0,-1,\ 0,\ 0,-1,+1,-1,\ 0,+1,-1,+1,\ 0,\ 0,+1,\ 0)$$
$$(\alpha_5,\underline{\omega}^p\alpha_1)=(\ 0,+1,\ 0,\ 0,\ 0,\ 0,+1,-1,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0)$$
$$(\alpha_6,\underline{\omega}^p\alpha_1)=(\ 0,\ 0,-1,\ 0,+1,-1,\ 0,\ 0,\ 0,+1,-1,\ 0,+1,\ 0,\ 0)$$
$$(\alpha_7,\underline{\omega}^p\alpha_1)=(\ 0,\ 0,+1,-1,\ 0,+1,-1,+1,-1,\ 0,+1,-1,\ 0,\ 0,\ 0)$$
$$(\alpha_8,\underline{\omega}^p\alpha_1)=(\ 0,-1,+1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,+1,-1,+1,\ 0);$$

$$(\alpha_1,\underline{\omega}^p\alpha_2)=(+1,-1,\ 0,\ 0,-1,+1,\ 0,-1,+1,\ 0,\ 0,+1,-1,+1,+1)$$
$$(\alpha_2,\underline{\omega}^p\alpha_2)=(\ 0,\ 0,+1,\ 0,+1,\ 0,-1,+1,\ 0,-1,\ 0,-1,\ 0,\ 0,-2)$$
$$(\alpha_3,\underline{\omega}^p\alpha_2)=(-1,\ 0,-1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,+1,+1,\ 0,+1,+1,+1)$$
$$(\alpha_4,\underline{\omega}^p\alpha_2)=(+1,+1,\ 0,+1,\ 0,\ 0,+1,-1,\ 0,\ 0,-1,\ 0,-1,-1,\ 0)$$
$$(\alpha_5,\underline{\omega}^p\alpha_2)=(-1,-1,\ 0,\ 0,\ 0,-1,\ 0,+1,\ 0,\ 0,\ 0,+1,+1,\ 0,\ 0)$$
$$(\alpha_6,\underline{\omega}^p\alpha_2)=(\ 0,+1,+1,-1,\ 0,+1,\ 0,\ 0,-1,\ 0,+1,-1,-1,\ 0,\ 0)$$
$$(\alpha_7,\underline{\omega}^p\alpha_2)=(\ 0,-1,\ 0,+1,-1,\ 0,\ 0,\ 0,+1,-1,\ 0,+1,\ 0,\ 0,\ 0)$$
$$(\alpha_8,\underline{\omega}^p\alpha_2)=(+1,\ 0,\ 0,+1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0,-1,\ 0);$$

$$(\alpha_1,\underline{\omega}^p\alpha_3)=(+1,\ 0,\ 0,+1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0,-1,\ 0)$$
$$(\alpha_2,\underline{\omega}^p\alpha_3)=(-1,-1,\ 0,-1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,+1,+1,\ 0,+1,+1)$$
$$(\alpha_3,\underline{\omega}^p\alpha_3)=(+1,+1,+1,+1,+1,\ 0,\ 0,\ 0,\ 0,-1,-1,-1,-1,-1,-2)$$
$$(\alpha_4,\underline{\omega}^p\alpha_3)=(-1,-1,-1,-1,\ 0,-1,\ 0,\ 0,\ 0,+1,\ 0,+1,+1,+1,+1)$$
$$(\alpha_5,\underline{\omega}^p\alpha_3)=(+1,+1,+1,\ 0,\ 0,+1,\ 0,\ 0,-1,\ 0,\ 0,-1,-1,-1,\ 0)$$
$$(\alpha_6,\underline{\omega}^p\alpha_3)=(\ 0,-1,-1,\ 0,\ 0,\ 0,-1,\ 0,+1,\ 0,\ 0,\ 0,+1,+1,\ 0)$$
$$(\alpha_7,\underline{\omega}^p\alpha_3)=(\ 0,+1,\ 0,\ 0,\ 0,\ 0,+1,-1,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 0)$$
$$(\alpha_8,\underline{\omega}^p\alpha_3)=(-1,\ 0,-1,\ 0,-1,\ 0,\ 0,\ 0,\ 0,\ 0,+1,\ 0,+1,\ 0,+1);$$

$$(\alpha_1, \underline{\omega}^p\alpha_4) = (-1,\ 0,\ 0, -1, +1, -1,\ 0, +1, -1, +1,\ 0,\ 0, +1,\ 0,\ 0)$$
$$(\alpha_2, \underline{\omega}^p\alpha_4) = (+1, +1,\ 0, +1,\ 0,\ 0, +1, -1,\ 0,\ 0, -1,\ 0, -1, -1,\ 0)$$
$$(\alpha_3, \underline{\omega}^p\alpha_4) = (-1, -1, -1,\ 0, -1,\ 0,\ 0,\ 0, +1,\ 0, +1, +1, +1, +1, +1)$$
$$(\alpha_4, \underline{\omega}^p\alpha_4) = (\ 0, +1, +1,\ 0, +1,\ 0,\ 0,\ 0,\ 0, -1,\ 0, -1, -1,\ 0, -2)$$
$$(\alpha_5, \underline{\omega}^p\alpha_4) = (\ 0, -1,\ 0, -1,\ 0,\ 0,\ 0,\ 0,\ 0, +1,\ 0, +1,\ 0, +1, +1)$$
$$(\alpha_6, \underline{\omega}^p\alpha_4) = (+1,\ 0,\ 0, +1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, -1,\ 0,\ 0, -1,\ 0)$$
$$(\alpha_7, \underline{\omega}^p\alpha_4) = (-1, +1, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, +1, -1, +1,\ 0,\ 0)$$
$$(\alpha_8, \underline{\omega}^p\alpha_4) = (+1,\ 0, +1,\ 0,\ 0, +1, -1, +1, -1,\ 0,\ 0, -1,\ 0, -1,\ 0);$$

$$(\alpha_1, \underline{\omega}^p\alpha_5) = (\ 0, +1,\ 0,\ 0,\ 0,\ 0, +1, -1,\ 0,\ 0,\ 0,\ 0, -1,\ 0,\ 0)$$
$$(\alpha_2, \underline{\omega}^p\alpha_5) = (\ 0, -1, -1,\ 0,\ 0,\ 0, -1,\ 0, +1,\ 0,\ 0,\ 0, +1, +1,\ 0)$$
$$(\alpha_3, \underline{\omega}^p\alpha_5) = (+1, +1, +1,\ 0,\ 0, +1,\ 0,\ 0, -1,\ 0,\ 0, -1, -1, -1,\ 0)$$
$$(\alpha_4, \underline{\omega}^p\alpha_5) = (-1,\ 0, -1,\ 0, -1,\ 0,\ 0,\ 0,\ 0,\ 0, +1,\ 0, +1,\ 0, +1)$$
$$(\alpha_5, \underline{\omega}^p\alpha_5) = (\ 0,\ 0,\ 0, +1, +1, -1,\ 0,\ 0, +1, -1, -1,\ 0,\ 0,\ 0, -2)$$
$$(\alpha_6, \underline{\omega}^p\alpha_5) = (-1,\ 0, +1, -1, -1,\ 0, +1,\ 0, -1,\ 0, +1, +1, -1,\ 0, +1)$$
$$(\alpha_7, \underline{\omega}^p\alpha_5) = (+1, -1,\ 0, +1,\ 0,\ 0, -1, +1,\ 0,\ 0, -1,\ 0, +1, -1,\ 0)$$
$$(\alpha_8, \underline{\omega}^p\alpha_5) = (\ 0, +1,\ 0, -1, +1, -1,\ 0,\ 0,\ 0, +1, -1, +1,\ 0, +1,\ 0);$$

$$(\alpha_1, \underline{\omega}^p\alpha_6) = (\ 0, -1,\ 0, +1, -1,\ 0,\ 0,\ 0, +1, -1,\ 0, +1,\ 0,\ 0,\ 0)$$
$$(\alpha_2, \underline{\omega}^p\alpha_6) = (\ 0, +1, +1, -1,\ 0, +1,\ 0,\ 0, -1,\ 0, +1, -1, -1,\ 0,\ 0)$$
$$(\alpha_3, \underline{\omega}^p\alpha_6) = (-1, -1,\ 0,\ 0,\ 0, -1,\ 0, +1,\ 0,\ 0,\ 0, +1, +1,\ 0,\ 0)$$
$$(\alpha_4, \underline{\omega}^p\alpha_6) = (+1,\ 0,\ 0, +1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, -1,\ 0,\ 0, -1,\ 0)$$
$$(\alpha_5, \underline{\omega}^p\alpha_6) = (\ 0, +1, -1, -1,\ 0, +1,\ 0, -1,\ 0, +1, +1, -1,\ 0, +1, +1)$$
$$(\alpha_6, \underline{\omega}^p\alpha_6) = (\ 0, -1,\ 0, +1, +1, -1, -1, +1, +1, -1, -1,\ 0, +1,\ 0, -2)$$
$$(\alpha_7, \underline{\omega}^p\alpha_6) = (+1,\ 0,\ 0, -1,\ 0, +1,\ 0, -1,\ 0, +1,\ 0,\ 0, -1, +1, +1)$$
$$(\alpha_8, \underline{\omega}^p\alpha_6) = (\ 0, +1,\ 0,\ 0,\ 0,\ 0, +1, -1,\ 0,\ 0,\ 0,\ 0, -1,\ 0,\ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_7)=(\ 0,\ \ 0,+1,-1,\ \ 0,+1,-1,+1,-1,\ \ 0,+1,-1,\ \ 0,\ \ 0,\ \ 0)$$

$$(\alpha_2,\underline{\omega}^P\alpha_7)=(\ 0,\ \ 0,-1,\ \ 0,+1,-1,\ \ 0,\ \ 0,\ \ 0,+1,-1,\ \ 0,+1,\ \ 0,\ \ 0)$$

$$(\alpha_3,\underline{\omega}^P\alpha_7)=(\ 0,+1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0)$$

$$(\alpha_4,\underline{\omega}^P\alpha_7)=(\ 0,-1,+1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-1,+1,\ \ 0)$$

$$(\alpha_5,\underline{\omega}^P\alpha_7)=(+1,-1,\ \ 0,+1,\ \ 0,\ \ 0,-1,+1,\ \ 0,\ \ 0,-1,\ \ 0,+1,-1,\ \ 0)$$

$$(\alpha_6,\underline{\omega}^P\alpha_7)=(-1,+1,\ \ 0,\ \ 0,-1,\ \ 0,+1,\ \ 0,-1,\ \ 0,+1,\ \ 0,\ \ 0,-1,+1)$$

$$(\alpha_7,\underline{\omega}^P\alpha_7)=(-1,\ \ 0,\ \ 0,\ \ 0,+1,-1,\ \ 0,\ \ 0,+1,-1,\ \ 0,\ \ 0,\ \ 0,+1,-2)$$

$$(\alpha_8,\underline{\omega}^P\alpha_7)=(\ 0,\ \ 0,-1,+1,-1,+1,-1,\ \ 0,+1,-1,+1,-1,+1,\ \ 0,\ \ 0);$$

$$(\alpha_1,\underline{\omega}^P\alpha_8)=(-1,+1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-1,+1,\ \ 0,\ \ 0)$$

$$(\alpha_2,\underline{\omega}^P\alpha_8)=(+1,\ \ 0,\ \ 0,+1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0,-1,\ \ 0)$$

$$(\alpha_3,\underline{\omega}^P\alpha_8)=(\ 0,-1,\ \ 0,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,\ \ 0,+1,\ \ 0,+1,+1)$$

$$(\alpha_4,\underline{\omega}^P\alpha_8)=(+1,\ \ 0,+1,\ \ 0,\ \ 0,+1,-1,+1,-1,\ \ 0,\ \ 0,-1,\ \ 0,-1,\ \ 0)$$

$$(\alpha_5,\underline{\omega}^P\alpha_8)=(-1,\ \ 0,-1,+1,-1,\ \ 0,\ \ 0,\ \ 0,+1,-1,+1,\ \ 0,+1,\ \ 0,\ \ 0)$$

$$(\alpha_6,\underline{\omega}^P\alpha_8)=(\ 0,+1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0)$$

$$(\alpha_7,\underline{\omega}^P\alpha_8)=(\ 0,-1,+1,-1,+1,-1,\ \ 0,+1,-1,+1,-1,+1,\ \ 0,\ \ 0,\ \ 0)$$

$$(\alpha_8,\underline{\omega}^P\alpha_8)=(-1,-1,\ \ 0,\ \ 0,+1,-1,+1,-1,+1,-1,\ \ 0,\ \ 0,-1,-1,-2).$$

APPENDIX B

The interaction functions for the $\{\lambda_2, \lambda_4, \lambda_5, \lambda_7\}$ soliton self-interactions in $e_8^{(1)}$ affine Toda field theory are as follows: (superscripts have been omitted since $\lambda = \lambda'$)

$$A^{(12)} = \frac{\alpha C^4 + \beta C^3 + \gamma C^2 + \delta C + \varepsilon}{\alpha C^4 + \beta C^3 + \gamma C^2 + \delta C + \varepsilon}$$

where,

$$\alpha = 4,\ \beta = -\lambda^2,\ \gamma = 3(-20 + 30\lambda - 10\lambda^2 + \lambda^3),$$
$$\delta = -15(-132 + 192\lambda - 46\lambda^2 + 3\lambda^3),$$
$$\varepsilon = 30(-264 + 378\lambda - 84\lambda^2 + 5\lambda^3),$$

and the remaining numerators: (the denominator being the obvious power of the fundemental function denominator)

$$A_1(\Theta) \equiv 4.\delta_1^{(1)^2}.C^4 + 8(-300 + 300.\lambda + 75.\lambda^2 - 16.\lambda^3)C^2$$
$$+240(-6972 + 9933.\lambda - 2157.\lambda^2 + 128.\lambda^3)$$

$$A_2(\Theta) \equiv 4.\delta_2^{(1)^2}.C^4 + 8(-3420 + 4980.\lambda - 1199.\lambda^2 + 80.\lambda^3)C^3$$
$$+16(-1860 + 2520.\lambda - 405.\lambda^2 + 13.\lambda^3)C^2$$
$$+240(23964 - 34209.\lambda + 7501.\lambda^2 - 449.\lambda^3)C$$
$$+240(143112 - 203634.\lambda + 43929.\lambda^2 - 2575.\lambda^3)$$

$$B_2(\Theta) \equiv 4.\delta_2^{(1)}\delta_2^{(2)}.C^4 + 4(-522360 + 737640.\lambda - 152970.\lambda^2 + 8503.\lambda^3)C^2$$

$$+480(242514-344493.\lambda+73680.\lambda^2-4270.\lambda^3)$$

$$\begin{aligned}D_2(\Theta) \equiv\ & 16.\delta_2^{(2)^2}.C^8+24(515640-755220.\lambda+186430.\lambda^2-12659.\lambda^3)C^6\\ &+720(-5906748+8432076.\lambda-1848857.\lambda^2+110512.\lambda^3)C^4\\ &+8640(48226995-68626800.\lambda+14809055.\lambda^2-867838.\lambda^3)C^2\\ &+129600(-104911776+149238024.\lambda-32148463\lambda^2+1879756\lambda^3)\end{aligned}$$

$$\begin{aligned}A_3(\Theta) \equiv\ & 4\delta_3^{(1)^2}.C^4+12(-6120+8700.\lambda-1868.\lambda^2+109.\lambda^3)C^3\\ &+36(-20760+29590\lambda-6440.\lambda^2+383.\lambda^3)C^2\\ &+360(-13152+18798.\lambda-4148.\lambda^2+251.\lambda^3)C\\ &+360(-50472+72090.\lambda-15852.\lambda^2+953.\lambda^3)\end{aligned}$$

$$\begin{aligned}B_3(\Theta) \equiv\ & 4\delta_3^{(1)}\delta_3^{(2)}.C^4+3(-202320+288120.\lambda-62410.\lambda^2+3671.\lambda^3)C^3\\ &+90(85116-121160.\lambda+26191.\lambda^2-1540.\lambda^3)C^2\\ &+90(1604100-2282406.\lambda+492290.\lambda^2-28841.\lambda^3)C\\ &+180(2295360-3263706.\lambda+701445.\lambda^2-40883.\lambda^3)\end{aligned}$$

$$\begin{aligned}C_3(\Theta) \equiv\ & 4\delta_3^{(1)}\delta_3^{(3)}.C^4+120(-263112+374562.\lambda-80998.\lambda^2+4761.\lambda^3)C^2\\ &+720(3680760-5236806.\lambda+1129082.\lambda^2-66103.\lambda^3)\end{aligned}$$

$$\begin{aligned}D_3(\Theta) \equiv\ & 16\delta_3^{(2)^2}.C^8+120(-714816+1017024.\lambda-219299.\lambda^2+12844.\lambda^3)C^7\\ &+90(960072-1367564.\lambda+296618.\lambda^2-17489.\lambda^3)C^6\end{aligned}$$

$$+180(123956760 - 176341080.\lambda + 37999366.\lambda^2 - 2222777.\lambda^3)C^5$$

$$+180(884592180 - 1258220820.\lambda + 270911247.\lambda^2 - 15830762.\lambda^3)C^4$$

$$+16200(-78983660 + 112359554.\lambda - 24209074.\lambda^2 + 1415813.\lambda^3)C^3$$

$$+10800(-1743680673 + 2480297838.\lambda - 534183825\lambda^2. + 31225022.\lambda^3)C^2$$

$$+32400(-2117539680 + 3012174390.\lambda - 648823582.\lambda^2 + 37933297.\lambda^3)C$$

$$+32400(-2619730080 + 3726966600.\lambda - 803263563.\lambda^2 + 46998298.\lambda^3)$$

$$F_3(\Theta) \equiv 16.\delta_3^{(2)}\delta_3^{(3)}.C^8 + 120(-16823424 + 23928156.\lambda - 5150942.\lambda^2 + 300957.\lambda^3)C^6$$

$$+360(890988360 - 1267306140.\lambda + 272853034.\lambda^2 - 15942331.\lambda^3)C^4$$

$$+21600(-1206004926 + 1715452707.\lambda - 369426992.\lambda^2 + 21591682.\lambda^3)C^2$$

$$+64800(16972221120 - 24141891360.\lambda + 5199198514.\lambda^2 - 303891721.\lambda^3)$$

$$I_3(\Theta) \equiv 64.\delta_3^{(3)^2}.C^{12} + 5760(640792 - 909828.\lambda + 194146.\lambda^2 - 11233.\lambda^3)C^{10}$$

$$+5760(561516120 - 798889860.\lambda + 172233822.\lambda^2 - 10080427.\lambda^3)C^8$$

$$+86400(-13129975768 + 18677132220.\lambda - 4022951658.\lambda^2 + 235189817.\lambda^3)C^6$$

$$+1036800(142646342940 - 202905283095.\lambda + 43697747970.\lambda^2 - 2554132966\lambda^3.)C^4$$

$$+31104000(-276144609828 + 392794583643.\lambda - 84588612246.\lambda^2 + 4943918203.\lambda^3)C^2$$

$$+93312000(1944421460080 - 2765786282160.\lambda + 595609246932.\lambda^2 - 34810973603.\lambda^3)$$

$$A_4(\Theta) \equiv 4\delta_4^{(1)^2}.C^4 + 75(-1296 + 1848.\lambda - 403\lambda^2. + 24.\lambda^3)C^3$$
$$+25(-33324 + 47418.\lambda - 10230.\lambda^2 + 599.\lambda^3)C^2$$
$$+225(-14100 + 19992.\lambda - 4234.\lambda^2 + 241.\lambda^3)C$$
$$+450(-23720 + 33590.\lambda - 7068.\lambda^2 + 399.\lambda^3)$$

$$B_4(\Theta) \equiv 4\delta_4^{(1)}\delta_{4.}^{(2)}C^4 + 50(-11448 + 16278.\lambda - 3499.\lambda^2 + 204.\lambda^3)C^3$$
$$+25(375288 - 533568.\lambda + 114630.\lambda^2 - 6679.\lambda^3)C^2$$
$$+75(894480 - 1271460.\lambda + 272854.\lambda^2 - 15871.\lambda^3)C$$
$$+600(-152130 + 216555.\lambda - 46812.\lambda^2 + 2749.\lambda^3)$$

$$D_4(\Theta) \equiv 16.\delta_4^{(2)^2}.C^8 + 800(-87084 + 123876.\lambda - 26683.\lambda^2 + 1560.\lambda^3)C^7$$
$$+2400(501262 - 713037.\lambda + 153586.\lambda^2 - 8979.\lambda^3)C^6$$
$$+600(39351000 - 55976352.\lambda + 12057364.\lambda^2 - 704931.\lambda^3)C^5$$
$$+600(-131696880 + 187319340.\lambda + 40329036.\lambda^2 + 2356277.\lambda^3)C^4$$
$$+9000(-279676632 + 397822908.\lambda - 85676154.\lambda^2 + 5007823.\lambda^3)C^3$$
$$+36000(-73156074 + 104064570.\lambda - 22416549.\lambda^2 + 1310614.\lambda^3)C^2$$
$$+432000(187453290 - 266635782.\lambda + 57417824.\lambda^2 - 3355693.\lambda^3)C$$
$$+216000(1057151400 - 1503696960.\lambda + 323800293.\lambda^2 - 18923318.\lambda^3)$$

$$A_5(\Theta) \equiv 4\delta_5^{(1)^2}.C^4 + 8(-20340 + 28950.\lambda - 6254\lambda^2. + 367.\lambda^3)C^3$$
$$+4(33600 - 47520.\lambda + 9930.\lambda^2 - 553.\lambda^3)C^2$$
$$+60(326256 - 463884.\lambda + 99688.\lambda^2 - 5809.\lambda^3)C$$
$$+360(266888 - 379560.\lambda + 81664.\lambda^2 - 4767.\lambda^3)$$

$$B_5(\Theta) \equiv 4.\delta_5^{(1)}\delta_5^{(2)}.C^4 + 8(-1066680 + 1517490.\lambda - 327030.\lambda^2 + 19129.\lambda^3)C^2$$
$$+120(3829896 - 5446728.\lambda + 1171848.\lambda^2 - 68399.\lambda^3)$$

$$D_5(\Theta) \equiv 16.\delta_5^{(2)^2}.C^8 + 24(13204440 - 18787860.\lambda + 4052070.\lambda^2 - 237299.\lambda^3)C^6$$
$$+2160(-24366436 + 34661556.\lambda - 7466771.\lambda^2 + 436596.\lambda^3)C^4$$
$$+4320(855277590 - 1216572480.\lambda + 261995970.\lambda^2 - 15313351.\lambda^3)C^2$$
$$+129600(-712676976 + 1013730168.\lambda - 218309431.\lambda^2 + 12759612.\lambda^3)$$

$$A_6(\Theta) \equiv 4.\delta_6^{(1)^2}.C^4 + 3(-5760 + 8280.\lambda - 1879.\lambda^2 + 118.\lambda^3)C^3$$
$$+9(13620 - 19630.\lambda + 4510.\lambda^2 - 287.\lambda^3)C^2$$
$$+45(66588 - 95064.\lambda + 20854.\lambda^2 - 1249.\lambda^3)C$$
$$+90(84168 - 119682.\lambda + 25728\lambda^2 - 1499.\lambda^3)$$

$$A_7(\Theta) \equiv 4.\delta_7^{(1)^2}.C^4 + 24(20 - 35.\lambda + 15.\lambda^2 - 2.\lambda^3)C^2$$
$$+120(3336 - 4758.\lambda + 1038.\lambda^2 - 61.\lambda^3)$$

$$
\begin{aligned}
A_8(\Theta) \equiv 4.\delta_8^{(1)^2}.C^4 &+ 27(-400 + 560.\lambda - 111.\lambda^2 + 6.\lambda^3)C^3 \\
&+9(10980 - 15390.\lambda + 3070.\lambda^2 - 167.\lambda^3)C^2 \\
&+45(34668 - 49296.\lambda + 10602.\lambda^2 - 623.\lambda^3)C \\
&+270(26856 - 38358.\lambda + 8432.\lambda^2 - 505.\lambda^3)
\end{aligned}
$$

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