The Schrodinger representation for (ˆ4) theory and the O(N) σ-model

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by

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ABSTRACT

The Schrödinger Representation for ϕ^4 theory and the $O(N)$ σ-model

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In this work we apply the field theoretical Schrödinger representation to the massive ϕ^4 theory and the $O(N)$ σ model in $1+1$ dimensions. The Schrödinger equation for the ϕ^4 theory is reviewed and then solved classically and semiclassically, to obtain the vacuum functional as an expansion of local functionals. These results are compared with equivalent ones derived from the path integral formulation to prove their agreement with the conventional field theoretical methods.

For the $O(N)$ σ model we construct the functional Laplacian, which is the principal ingredient of the corresponding Schrödinger equation. This result is used to construct the generalised Virasoro operators for this model and study their algebra.

DECLARATION

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1993 and August 1996. This material has not been submitted previously for any degree in this or any other university.

No claim of originality is made for the second and third chapters; the work in Chapters 4, 5, 6, 7 and 8 is claimed as original, except where the authors have been specifically acknowledged in the text.

Part of Chapter 6 has been done in collaboration with Dr. P. Mansfield and published as [26].

Parts of the diagram calculations in Chapter 5 and the program construction in Appendix E have been done in collaboration with Marcos Sampaio.

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Dedicated to the memory of Euan Squires

Contents

Chapter 1

Introduction

In this work we study the Schrödinger Representation in Quantum Field Theory [1]-[8], applied to the ϕ^4 interacting theory and the $O(N)$ σ model. Our interest in the Schrödinger representation is due to its usefulness for working out non-perturbative features of the field theories. While the high energy sector of quantum chromodynamics (QCD) is well set in terms of the path integrals and semiclassical expansion, the Schrödinger equation can provide a natural way to study low energy behaviour. Ultimately this may help us understand major problems in particle physics like confinement, chiral symmetry breaking and quantisation of gravity ([9], [10], [11]).

For fields that vary slowly on the scale of the lightest mass it was shown by Mansfield [12] that the logarithm of the vacuum functional can be expanded as a sum of local functionals. However, this expansion does not satisfy the obvious form of the Schrödinger equation. We construct the appropriate equation for the ϕ^4 theory, where the basic features of the local expansion are revealed. We are also interested in the $O(N)$ σ model, which apart from its usefulness as a simplified gravitational model, has many features in common with Yang-Mills theory.

In Chapter 2 we present the field theoretical Schrödinger representation as a generalisation of Quantum mechanics. The Schrödinger equation is constructed for the free massive scalar field and for the ϕ^4 interactive case.

In Chapter 3 we study the renormalisation of the Schrödinger representation for the ϕ^4

interactive theory in $3+1$ and $1+1$ dimensions. In $3+1$ dimensions, apart from the usual renormalisation procedure needed for the infinitely extended space-time, we face the problem of the "boundary" singularities, connected with the quantisation surface. For the ϕ^4 theory in $1+1$ dimensions no similar singularities appear as this model is super-renormalisable [13].

In Chapter 4 we review the construction of the Schrödinger equation, which the local expansion of the vacuum functional has to satisfy, for the ϕ^4 theory in $1+1$ dimensions. Then we proceed in solving it classically and semiclassically.

In Chapter 5 we construct and calculate the connected Feynman diagrams with propagators on the boundary, for which the logarithm of the vacuum functional can be viewed as their generating functional. This provides a test of comparison of the results we get by solving the Schrödinger equation with the ones we get by using path integrals.

In Chapter 6 we study the $O(N)$ σ model's Schrödinger equation, the principal ingredient of which is the regulated functional Laplacian. We construct the Laplacian to the leading and next to the leading order, acting on local functionals. It is determined by imposing rotational invariance in the internal space together with closure of the Poincaré algebra.

In Chapter 7 it is shown that it is not possible to construct a Laplacian for a general curved manifold with the requirements set in the previous chapter.

In Chapter 8 is presented a way to construct the modified Virasoro algebra for the $O(N)$ σ model. As this model is not conformal invariant after its quantisation, the algebra is expected to have operator-like terms in place of the usual central charge. The functional formalism provides a natural way to perform such a calculation.

In Chapter 9 we present the conclusions of the thesis and some directions for further work.

Finally, in one of the Appendices is presented a computer program performing certain functional calculations in Maple programming language.

Chapter 2

The Field Theoretical Schrödinger Representation

2.1 Introduction

Quantum Field Theory, is a generalisation of Quantum Mechanics to a theory with an infinite number of degrees of freedom. There are two ways to approach the problem of quantising of a field theory. The usual one is to calculate cross-sections using the path integral formalism. This has the advantage of a well established mathematical framework, which includes the perturbation expansion. Another way of quantising field theories is by the Hamiltonian formalism. Familiar in Quantum Mechanics, this method is not much studied in field theory. It has the advantage of being able to describe operators and their eigenvalues without the need for perturbation. Hence, we can calculate physical quantities which would be difficult or impossible to consider under a perturbative approach. The main disadvantage is the difficulty in manipulating such a theory in some situations, which are more easily resolved by path integral formulation.

We will try to construct the Hamiltonian formalism for the Field Theory. Attempting to apply the Schrödinger representation from Quantum Mechanics, the basic idea is to have a "diagonalisation" of the field analogous to the diagonalisation of the position operator in Quantum Mechanics. Additionally, the canonical commutation relations with its conjugate momentum have to be satisfied. In the following we will see how it is possible

to build up such a consistent theory, with the above requirements.

2.2 Quantum Mechanics and Quantum Field Theory

Quantum Mechanics describes the small scale world with a particle-wave character. To study a particle of mass m in a $D+1$ space-time with a time independent potential $V(x)$, one can find its Schrödinger wave function $\Psi(x,t)$ which contains all the information about the particle. Let us build up the equation Ψ satisfies. The non-relativistic energy conservation of the particle is

$$E = \frac{p^2}{2m} + V(x). \tag{2.1}$$

The wave character can be given by replacing the variables x, p with operators $\hat{x}$, $\hat{p}$ which satisfy the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar. \tag{2.2}$$

This gives the basic physical observation of *the uncertainty principle.* Moreover, the Schrödinger representation is declared by asking the position operator to be diagonalised; that is to act multiplicatively on the wave function $\hat{x}\Psi(x,t) = x\Psi(x,t)$. From (2.2) the momentum operator can be chosen to be

$$\hat{p}\Psi(x,t) = -i\hbar\frac{\partial}{\partial x}\Psi(x,t). \tag{2.3}$$

Ψ can be found by solving the Schrödinger equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = H\Psi(x,t) \equiv -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) \tag{2.4}$$

which is a combination of the energy conservation (2.1) and the operator representation of x and p so that we get a wave equation. It can be solved by separation of variables and finding the eigenstates and eigenvalues of the Hamiltonian.

In Quantum Field Theory we describe the particle with a wave functional Ψ, where the variables we are interested in are not the position and time of the particle but rather the configuration of the function-field $\phi(x,t)$ through space-time. The wave functionals give rise to a Hilbert space H on which the field $\phi(x,t)$ and its conjugate momentum $\pi(x,t)$ act. We need to bear in mind the implications of the diagonalisation of the fields ϕ given by

$$\phi(x,0)\Psi[\varphi] = \varphi(x)\Psi[\varphi] \tag{2.5}$$

where $t = 0$ is the quantisation surface on which the field has a specific configuration. We have conventionally chosen the time $t = 0$ for the diagonalisation. Commutation relations read

$$[\phi(x,t), \pi(x',t')]|_{t=t'=0} = i\hbar\delta(x-x'),$$
$$[\pi(x,t), \pi(x',t')]|_{t=t'=0} = [\phi(x,t), \phi(x',t')]|_{t=t'=0} = 0 \tag{2.6}$$

From (2.5) and (2.6) we see that we can choose, as in (2.3), the momentum operator to be

$$\pi(x,0)\Psi[\varphi] = -i\hbar\frac{\delta}{\delta\varphi(x)}\Psi[\varphi] \tag{2.7}$$

The Hamiltonian can be constructed using the relations (2.5) and (2.7), and the dynamics of the system is given by the Schrödinger equation

$$H\Psi_t[\varphi] = i\hbar\frac{\partial}{\partial t}\Psi_t[\varphi] \tag{2.8}$$

We have chosen for the diagonalisation of the field operator $\phi(x,t)$ the time $t = 0$ so that the equal time commutation relations (2.6) is defined also on the surface $t = 0$. This leads the momentum operator, π, as well as the Hamiltonian, H, to be defined on the same surface.

In addition to the operator formalism already discussed, we can use functional integrals. By using the basic definitions of path integrals in field theory, the Schrödinger wave functional can be interpreted as the generating functional of certain Feynman Diagrams on the half plane $t \geq 0$. This is due to the definition of the Schrödinger functional as the matrix element of the Euclidean time evolution operator between the eigenkets of the field

$$\langle\varphi'|e^{-tH}|\varphi\rangle = \int \mathcal{D}\tilde{\phi}e^{-S_E[\tilde{\phi}]} \tag{2.9}$$

where S_E is the Euclidean action for the $1+1$ dimensional volume bounded by space-like surfaces a time t apart and $\tilde{\phi}(x,0) = \varphi(x)$, $\tilde{\phi}(x,t) = \varphi'(x)$.

In the limiting case of $t \to \infty$ the above definition gives the vacuum functional for a field ϕ. This can be seen in the following way: from (2.9) we have

$$\langle\varphi'|e^{-tH}|\varphi\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\varphi'|n\rangle\langle n|e^{-tH}|m\rangle\langle m|\varphi\rangle \tag{2.10}$$

where $|n\rangle$ and $|m\rangle$ are elements of the orthonormal eigenstates of the energy, i.e. of the Hamiltonian H. Thus (2.10) becomes

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\varphi'|n\rangle\langle n|e^{-tE_m}|m\rangle\langle m|\varphi\rangle = \sum_{n=0}^{\infty}\langle\varphi'|n\rangle e^{-tE_n}\langle n|\varphi\rangle \tag{2.11}$$

The inner product $\langle\varphi|n\rangle$ is equal to the wave functional $\Psi_n[\varphi]$, of the field φ with energy E_n. We can extract the vacuum wave functional $\Psi_0[\varphi']$ from (2.11)

$$\langle\varphi'|e^{-tH}|\varphi\rangle \underset{t\to\infty}{\sim} \Psi_0[\varphi']\Psi_0^*[\varphi]e^{-tE_0} \tag{2.12}$$

If we normalise the vacuum energy to zero we are left with only the ground state $\Psi_0[\varphi']$ up to a normalisation factor.

2.3 Scalar Field

If we study the case of a real scalar free field with mass m in $1+1$ dimensions, the classical Hamiltonian will read

$$H = \int dx \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right\}$$

where $\nabla \equiv \partial/\partial x$. By using the relations (2.5) and (2.7) and setting $\hbar = 1$, we get the quantised equation

$$H\Psi_0[\varphi] \equiv -\frac{1}{2}\int dx \frac{\delta^2}{\delta\varphi\delta\varphi}\Psi_0[\varphi] + \frac{1}{2}\int dx((\nabla\varphi)^2 + m^2\varphi^2)\Psi_0[\varphi] = E_0\Psi_0[\varphi] \tag{2.13}$$

where $\Psi_0[\varphi]$ is the vacuum functional. We can guess a form for $\Psi_0[\phi]$

$$\Psi_0[\varphi] = \exp\left(-\frac{1}{2}\int dx\varphi\Gamma\varphi\right) \tag{2.14}$$

which when substituted into (2.13) gives

$$H\Psi_0[\varphi] = \left(\frac{1}{2}\int dx(-\varphi\Gamma^2\varphi + (\nabla\varphi)^2 + m^2\varphi^2) + \frac{1}{2}\mathrm{Tr}\ (\Gamma)\right)\Psi_0[\varphi] \tag{2.15}$$

In order for Ψ_0 to be an eigenstate of the Hamiltonian, the right hand side of (2.15) has to be constant. This happens for $\Gamma = \pm\sqrt{-\nabla^2 + m^2}$. In order for the functional to be normalisable we choose the positive sign, so that finally $\Psi_0[\varphi] = \exp\left(-\frac{1}{2}\int dx\varphi\sqrt{-\nabla^2 + m^2}\varphi\right)$, which is an eigenstate of the Hamiltonian belonging to the eigenvalue, E_0, proportional to the functional trace of Γ. The latter is a divergent quantity and its renormalisation will be given in the next Chapter, as the $g = 0$ limit of the interacting theory.

Turning to the interacting theory, the Hamiltonian H for the ϕ^4 theory is given by

$$H = \int dx \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \right\}. \tag{2.16}$$

so that the Hamiltonian operator reads

$$\widehat{H} = -\frac{1}{2}\Delta + \int dx \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}g\varphi^4 \right\} \tag{2.17}$$

where Δ stands for the two functional differentiations inserted from the conjugate momentum. An interesting characteristic of the Hamiltonian is that it depends only on the space coordinate, while the time one does not appear for a time independent potential. In other words the Hamiltonian exist on the $t = 0$ surface where its eigen-value problem is solved. The Schrödinger equation is

$$i\frac{\partial}{\partial t}\Psi_t[\varphi] = \widehat{H}\Psi_t[\varphi]$$

where $\widehat{H}$ is given by (2.17).

In Quantum Mechanics $|\Psi(x,t)|^2$ is the probability of finding a particle at time t at the point x (i.e. $\widehat{x}(t)$ is diagonal), while in Quantum Field Theory $|\Psi[\varphi]|^2$ is the probability that the field $\phi(x,t)$ takes the value $\varphi(x)$ at time $t = 0$ (i.e. the field ϕ is diagonal). $\varphi(x)$ has three interpretations. First, as the boundary value of the field $\phi(x,t)$ for $t = 0$. Second, it is the variable for the Hamiltonian in the Schrödinger equation. Thirdly, it plays the role of the source in the Feynman diagram expansion of Ψ. We are going to study the last characteristic after considering the regularisation and renormalisation of the field theory Schrödinger Representation. Finally we see that the Schrödinger equation is the relation the wave functional has to satisfy when the boundary sources are varied, that is, it describes the dynamics of the field in relation to the boundary. However, after finding its solution the boundary takes the form of the quantisation surface $t = 0$ from where the solution can be uniquely defined on the whole plane with the use of the evolution operator $e^{-i\widehat{H}t}$.

Chapter 3

Renormalisation of the Schrödinger Representation

3.1 Introduction

In this chapter we study the renormalisation of the Schrödinger representation through the path integral formulation where the Green functions are the constructing elements. In this way another perception of the theory will appear.

The renormalisation of the theory is essential to prove its existence and finiteness. Symanzik studied the Schrödinger representation, [13], for the ϕ^4 theory in $3+1$ dimensions using perturbation theory. We will outline the basic features of his arguments on renormalisation based on [14], [15] and [16].

Apart from the usual renormalisation involving the mass, coupling and field renormalisation constants, the ϕ^4_{3+1} interacting theory needs additional counter-terms to renormalise new ultraviolet divergences. This is the basic point Symanzik made in order to prove the existence of the Schrödinger representation in renormalisable field theories. He showed that field operators which are diagonalisable in the sense of the Schrödinger representation differ from the usual renormalised field operators by (in perturbation theory, logarithmically) divergent factors, in a similar manner to the way renormalised field operators themselves differ by such factors from the "bare" ones, in the usual sense canonical, field

theory. For example in some renormalisation scheme the Green's functions will appear to diverge as one of their arguments approach the boundary where the diagonalisation is defined.

Also from (2.13) we see two functional derivatives acting at the same point which eventually give a delta function evaluated at zero when, for example, they act on local functionals. Therefore, we need to split the two arguments of the functional derivatives, as is necessary also in the free case. Super-renormalisable theories, such as the ϕ^4 in $1+1$ dimensions, only need, in addition to the usual renormalisation, a procedure to regularise the Laplacian appearing in (2.13).

In field theory the Lagrangian is constructed so that the corresponding generating functional will give the desirable dynamics of the system under consideration. In the Schrödinger Representation we are seeking to construct the wave functional which will satisfy the Schrödinger equation. This quantity can be interpreted as the generating functional for Feynman Diagrams in space-time with boundaries.

3.2 Free Field

Let us consider the free scalar field in $3+1$ dimensions. The Euclidean Lagrangian is given by

$$L = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2. \tag{3.1}$$

The diagonalisation of the field ϕ given in relation (2.5), is effected by a boundary term added to L. We can define Γ to be the half space $t > 0$, so that its boundary $\partial\Gamma$ is the plane $t = 0$, while its complement, Γ', is given by $t < 0$. Consider now the additional term in the Lagrangian (3.1):

$$L_{\partial\Gamma\varphi} - L = \delta(t)\phi\partial_t\phi - \delta(t)\varphi\partial_t\phi. \tag{3.2}$$

We can see the effect of $L_{\partial\Gamma\varphi} - L$ on the Green functions, by taking the functional integral of the Lagrangian $L_{\partial\Gamma\varphi}$ with source J [13]. It will give

$$\int \mathcal{D}\phi \exp\left[-\int L_{\partial\Gamma} + \int J\phi\right] =$$

$$const.\ \exp\left[\frac{1}{2}\int_\Gamma\int_\Gamma JG_DJ + \frac{1}{2}\int_{\Gamma'}\int_{\Gamma'} JG_NJ +\right.$$

$$\frac{1}{2}\int_{\partial\Gamma}\int_{\partial\Gamma}\varphi\partial_t G_D \overleftarrow{\partial'_t}\varphi - \int_{\partial\Gamma}\int_{\Gamma}\varphi\partial_t G_D J\Big]. \tag{3.3}$$

In (3.3) G_D are the Dirichlet Green functions, which are constrained in Γ, and they satisfy the relations

$$(m^2-\partial^2)G_D(x,x') = \delta(x-x'),\ x,\ x'\in\Gamma,$$

$$G_D(x,x') = 0,\ x\in\partial\Gamma,\text{ and } x'\in\Gamma \tag{3.4}$$

and G_N are the Neumann ones in Γ', satisfying

$$(m^2-\partial^2)G_N(x,x') = \delta(x-x'),\ x,x'\in\Gamma'$$

$$\partial_t G_N(x,x') = 0,\ x\in\partial\Gamma,\text{ and } x'\in\Gamma' \tag{3.5}$$

There does not appear any correlation between the two regions Γ and Γ', as any Green function with one argument in Γ and another in Γ' will be zero.

We see that relation (3.3) factorises into Dirichlet and Neumann parts. This is because the functional integration in ϕ takes on average the specific value $\varphi(x)$ on the $t=0$ plane, as is required by the term (3.2) in the Lagrangian. This does not allow any fluctuations of the field for $t=0$, by which a propagator could cross the boundary. This property also holds in the interacting theory.

3.3 Interacting Theory

Adding the interaction term $-\frac{1}{4!}g\phi^4$ in the Lagrangian produces unrenormalised Feynman diagrams. Adopting dimensional regularisation, we can work in 1 time and $3-\epsilon$ space dimensions and give the appropriate counter-terms.

Away from the boundary the required counter-terms are the usual ones

$$\Delta L = \frac{1}{2}(Z_3-1)\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}(Z_2-1)m^2\phi^2 +$$

$$+\frac{1}{2}m_{B_0}^2 Z_3\phi^2 + \frac{1}{4!}(Z_1-1)g\phi^4$$

for the coupling, mass and field regularisation. We choose for the constants Z_1, Z_2 and Z_3 the "minimal" form

$$Z_i(g,\epsilon) = 1+\epsilon^{-1}f_{i1}(g)+\epsilon^{-2}f_{i2}(g)+... \tag{3.6}$$

so that the Green's functions exist for $\epsilon \to 0$.

The boundary terms (3.2), added to the interacting Lagrangian, need additional regularisation as there are new divergences in the Green's functions when one or more legs are attached to the boundary. As seen in [14] the necessary counter-terms are

$$\Delta L_{\partial\Gamma\varphi} = -(Z_4 - 1)\delta(t)\phi\partial_t\phi + (Z_5 - 1)\delta(t)\varphi\partial_t\phi - c_1\Lambda\delta(t)\phi^2 - c_3\Lambda\delta(t)\varphi\phi - c_5\Lambda\delta(t)\varphi^2. \tag{3.7}$$

Λ is a new cut-off as the terms with c-numbers cannot be written in dimensional regularisation. Z_4 and Z_5 can be expressed in the same way as in (3.6) and one finds

$$Z_4 - 1 = (16\pi^2\epsilon)^{-1}g + O(g^2)$$

$$Z_5 - 1 = -(32\pi^2\epsilon)^{-1}g + O(g^2)$$

up to the first order in g. Using the Lagrangian with the inhomogeneous boundary condition and the equivalent counter-terms we find the functional integral of its exponential to be $\Psi(\varphi|J) = \Psi_{Dir}(\varphi|J) \cdot \Psi_{Neu}(|J)$. The first factor depends only on J restricted to $t > 0$ and the second only on J restricted to $t < 0$, as in the free case. We will restrict our interest to Ψ_{Dir}, as Ψ_{Neu} fails to retain the Neumann property after renormalisation even at the first order in g, because of the presence of the c_1 term in (3.7), as can been seen with a variation of the action with the boundary and counter terms included.

3.4 Green Functions Attached on the Boundary

Under this renormalisation the inhomogeneous Dirichlet condition takes the modified form[1]

$$\lim_{t\to 0}\{a(g,\mu t, mt)\phi(xt)|\varphi\rangle\} = |\varphi\rangle\varphi(x)$$

where $|\varphi\rangle$ is the eigenstate of the renormalised field $\phi(xt)|_{t=0}$ with eigenvalue $\varphi(x)$, $a(g,\mu t,mt) = 1 - \frac{g}{32\pi^2}\ln(\mu t) + O(g^2)$ and μ is the normalised mass entering with the coupling constant g dimensionless in $1+3-\epsilon$ dimensions in the combination $g\mu^\epsilon$.

[1] see [13].

Figure 3.1: The free propagator between the sources $J(x)$ and $J(y)$.

The renormalisation of $\partial_t\phi(xt)$ as t approaches the boundary, makes it necessary to use the (now normalised) functional derivative:

$$\lim_{t\to 0}\{c(g,\mu t,mt)\partial_t\phi(xt)|\varphi\rangle\} = \frac{\delta}{\delta\varphi(x)}|\varphi\rangle \tag{3.8}$$

where

$$c(g,\mu t,mt) = 1 + \frac{g}{32\pi^2}\left[\ln(\mu t) + 1\right] + O(g^2) \tag{3.9}$$

Relations (3.8) and (3.9) reflect the way the Green functions approach the boundary. In the free theory the diagram (3.1) has an amplitude given by

$$-\frac{i}{2}\int J(x)G(x-y)J(y)d^4xd^4y$$

where $J(x)$ is a smooth source. The Green's function $G(x-y)$ is a finite quantity. When we insert a boundary at $t = 0$, then an additional type of free theory diagram results, with legs attached on the boundary.

As seen from (3.3) we may consider $\varphi(x)$ to be a source on $\partial\Gamma$, and $\partial_t G(x-y)|_{t=0}$ to be a Green's function with one leg on the boundary. In Symanzik's words "an external leg of G is upon normal differentiation bent to the boundary", which also holds for the normalised interacting ϕ^4 theory [13]. A Green's function with both legs on the boundary has the form $\partial_t\partial_{t'}G(x-y)|_{t=t'=0}$. The amplitude for this diagram is

$$-\frac{1}{2}\int \varphi(x)\left.\frac{\partial^2 G_D(x,y)}{\partial t\partial t'}\right|_{t=t'=0}\varphi(y)d^3xd^3y$$

The relation

$$\frac{\partial}{\partial t}\frac{\partial}{\partial t'}(T\phi(x')\phi(x)) = T\frac{\partial}{\partial t'}\phi(x')\frac{\partial}{\partial t}\phi(x) - \delta(t'-t)\delta(x'-x) \tag{3.10}$$

shows this procedure, as the free two point function is $\langle 0|T\phi(x)\phi(x')|0\rangle$, while the two point function with both legs on the boundary $t = 0$ is $\langle 0|T\ \partial_t\phi(x)|_{t=0}\ \partial_{t'}\phi(x')|_{t'=0}|0\rangle$. However, the two point amplitude produced (first term on the r.h.s.) is accompanied by a divergence as the arguments are approaching the boundary $t = t' = 0$.

In the interacting case viewing $\ln\Psi(\varphi|J)$ as the generating functional for both kinds of connected diagrams mentioned, then

$$\ln \Psi_{Dir}(\varphi|J) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{l} \int dz_j \varphi(z_j) \prod_{i=1}^{n} \int dx_i \int_0^{\infty} dt_i J(x_i t_i) \times$$
$$G(z_1 ... z_l | x_1 t_1 ... x_n t_n; \mu, g, m)$$

for $z_i \in \partial\Gamma$ and we can extract them by functional differentiation

$$G(z_1 ... z_l | x_1 t_1 ... x_n t_n) = \prod_{j=1}^{l} \frac{\delta}{\delta\varphi(z_j)} \prod_{i=0}^{n} \frac{\delta}{\delta J(x_i t_i)} \ln \Psi(\varphi|J) \Big|_{\varphi=J=0}$$

where the connected Green functions $G(z_1 ... z_l | x_1 t_1 ... x_n t_n)$ are non-zero only if $n + l$ is an even number as the sources J and φ come in pairs[2]. As

$$\Psi_{Dir}(\varphi|J) = \langle \varphi | T \exp \int_0^{\infty} dt \int dx J(xt) \phi(xt) | 0 \rangle$$

we can calculate from (3.8) and (3.9) that [3]

$$\lim_{t \to 0} \{ a(t) G(z_1 ... z_l | x t x_1 t_1 ... x_n t_n) \} = \delta_{l1} \delta_{n0} \delta(x - z_1)$$

and

$$\lim_{t \to 0} \Big\{ c(t) \left[\partial_t G(z_1 ... z_l | x t x_1 t_1 ... x_n t_n) - \delta_{n0} \delta_{l1} \delta(x - z_1) \partial_t \tilde{G}(0|0t) \right] \Big\} =$$
$$G(z_1 ... z_l x | x_1 t_1 ... x_n t_n) \tag{3.11}$$

Symanzik's statement can now be expressed with formulas (3.11), where the arguments standing on the left of the bar are on the boundary and the ones on the right are in the half plane $t > 0$. $\tilde{G}(0|0t)$ is the Fourier transform of the corresponding Green's function and its purpose is to subtract the divergences arising from $\partial_t G(z_1 ... z_l | x t x_1 t_1 ... x_n t_n)$ as $t \to 0$. In the free case, $\tilde{G}$ is substituted by the delta function of time as seen in (3.10).

3.5 Renormalisation of ϕ^4 in $1+1$ Dimensions

The ϕ^4 theory in $1+1$ dimensions is a super-renormalisable theory. This means that the divergent diagrams are finite in number. In particular, the only divergence comes from a self-contraction of the field at the same space-time point. Therefore renormalisation can be done, at least within perturbation theory, via normal ordering, using Wick's theorem and subtracting the infinite terms appearing. This involves expanding powers of φ as a

[2] See relation (5.7) in Chapter 5.

[3] See [13].

sum of normal-ordered terms with more and more self-contractions. In this way we can separate the convergent terms (with no contraction) from the divergent ones (with at least one contraction). Renormalisation then, is applied by subtracting the divergences by equal and opposite counter-terms.

Since we want to use a momentum cut-off, s, we are going to compare the normal-ordered Hamiltonian

$$:\widehat{H}:=\int dx:\left(\frac{1}{2}(\hat{\pi}(x)^2+\hat{\varphi}'(x)^2-M_r^2\hat{\varphi}(x)^2)+\frac{g}{4!}\hat{\varphi}^4\right): \tag{3.12}$$

with the operator $\widehat{H}_s$, constructed from momentum cut off fields

$$\widehat{H}_s=\int dx\int\left(\frac{1}{2}(\hat{\pi}_s+\hat{\varphi}_s'^2+M^2(s)\hat{\varphi}_s^2)+\frac{g}{4!}\hat{\varphi}_s^4-\mathcal{E}(s)\right) \tag{3.13}$$

where

$$\hat{\varphi}_s(x)=\int dy\mathcal{G}_s(x,y)\hat{\varphi}(y),\qquad \hat{\pi}_s(x)=\int dy\mathcal{G}_s(x,y)\hat{\pi}(y)$$

and the kernel

$$\mathcal{G}_s(x,x')=\int_{p^2<1/s}\frac{dp}{2\pi}e^{ip(x-x')} \tag{3.14}$$

implements the momentum cut-off. The Hamiltonian $\widehat{H}$ has been normal ordered with respect to oscillators with finite mass M_r. The coupling constant is also finite. There are no divergences appearing due to the boundaries which would need further field renormalisation as our theory is super-renormalisable [14]. Our aim is to define the divergent quantities $M(s)$ and $\mathcal{E}(s)$ so that $\lim_{s\to 0}\widehat{H}_s\Psi=\widehat{H}\Psi$. We start with (3.13), trying to express the various terms in a normal ordered form. Rewriting it as

$$\widehat{H}_s=\int dx\left(\frac{1}{2}(\hat{\pi}_s+\hat{\varphi}_s'^2+M_r^2\hat{\varphi}_s^2)+\frac{1}{2}(M^2(s)-M_r^2)\hat{\varphi}_s^2+\frac{g}{4!}\hat{\varphi}_s^4-\mathcal{E}(s)\right)$$

we can normal-order the first bracket as we do for the free field in terms of the creation and annihilation operators $a_s(k)=\frac{1}{\sqrt{\omega_k}}[\omega_k\hat{\varphi}_s(k)+i\hat{\pi}_s(k)]$ and $a_s^+(k)=\frac{1}{\sqrt{\omega_k}}[\omega_k\hat{\varphi}_s(k)-i\hat{\pi}_s(k)]$ where $\omega_k=\sqrt{k^2+m^2}$. The commutation relation $[a_s(k),a_s^+(k')]=\overline{\mathcal{G}}_s(k-k')$, gives for $\widehat{H}_{s0}$, the free field Hamiltonian

$$\widehat{H}_s=\int dx\left(:\widehat{H}_{s0}:+\frac{1}{2}\int_{p^2<1/s}\frac{dp}{2\pi}\omega_p+\frac{1}{2}(M^2(s)-M_r^2)\hat{\varphi}_s^2+\frac{g}{4!}\hat{\varphi}_s^4-\mathcal{E}(s)\right) \tag{3.15}$$

Now we can use Wick's theorem to normal order $\hat{\varphi}_s^2$ and $\hat{\varphi}_s^4$ as their vacuum to vacuum amplitude is evaluated at the same space-time point and will diverge. Labelling T_s as the vacuum to vacuum amplitude $\langle 0|\hat{\varphi}_s(x)\hat{\varphi}_s(x)|0\rangle$ we can write

$$\hat{\varphi}_s^2=:\hat{\varphi}_s^2:+T_s$$

and

$$\widehat{\varphi}_s^4 =: \widehat{\varphi}_s^4 : +6T_s : \widehat{\varphi}_s^2 : +3T_s^2$$

Expression (3.15) becomes

$$\widehat{H}_s = \int dx \left(: \widehat{H}_{s0} : + \frac{1}{2} \int_{p^2<1/s} \frac{dp}{2\pi} \omega_p + \frac{1}{2}(M^2(s) - M_r^2)(: \widehat{\varphi}_s^2 : + T_s) + \right.$$

$$\left. \frac{g}{4!}(: \widehat{\varphi}_s^4 : + 6T_s : \widehat{\varphi}_s^2 : + 3T_s^2) - \mathcal{E}(s) \right) =$$

$$\int dx \left(: \widehat{H}_s : + \frac{1}{2} \int_{p^2<1/s} \frac{dp}{2\pi} \omega_p + \frac{1}{2}(M^2(s) - M_r^2)(: \widehat{\varphi}_s^2 : + T_s) + \right.$$

$$\left. \frac{g}{4!}(6T_s : \widehat{\varphi}_s^2 : + 3T_s^2) - \mathcal{E}(s) \right) \tag{3.16}$$

where $: \widehat{H}_s :=: \widehat{H}_{s0} : + g/4! : \widehat{\varphi}_s^4 :$. In order that all except the first term on the r.h.s. vanish we require

$$M_r^2 - M^2(s) = \frac{g}{2} T_s \tag{3.17}$$

and

$$\mathcal{E}(s) = \frac{1}{2} \int_{p^2<1/s} \frac{dp}{2\pi} \omega_p + \frac{1}{2}(M^2(s) - M_r^2) T_s + \frac{g}{8} T_s^2. \tag{3.18}$$

T_s from its definition as the propagator with coinciding points has the analytic form

$$T_s = \frac{1}{2} \int_{p^2<1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + m^2}}$$

so that (3.17) and (3.18) become

$$M^2(s) = M_r^2 - \frac{g}{4} \int_{p^2<1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + m^2}} \tag{3.19}$$

and

$$\mathcal{E}(s) = \frac{1}{2} \int_{p^2<1/s} \frac{dp}{2\pi} \sqrt{p^2 + m^2} + \frac{g}{32} \int_{p^2<1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + m^2}} \tag{3.20}$$

These divergences refer to the free space-time diagrams. They come from a loop expansion in the propagator, with both ends coinciding. As we know, each loop is accompanied by a factor of $\hbar$ which in our case makes the mass correction in (3.19) proportional to $\hbar$ as well as in (3.20) the first term being of order $\hbar$ and the second of $\hbar^2$.

3.6 Time and Energy

When we want to interpret the creation of a non-charged particle, described by the real scalar field ϕ at the point (x, t), and its annihilation at the point (x', t'), we can write the

corresponding amplitude as the "two point function"

$$\langle 0|\theta(t'-t)\phi(x',t')\phi(x,t)|0\rangle \tag{3.21}$$

for $t' > t$. For $t' < t$ it takes the form

$$\langle 0|\theta(t-t')\phi(x,t)\phi(x',t')|0\rangle \tag{3.22}$$

The sum of the two amplitudes is the *Dyson's time ordered product*

$$\langle 0|T\phi(x',t')\phi(x,t)|0\rangle = \langle 0|\theta(t'-t)\phi(x',t')\phi(x,t) + \theta(t-t')\phi(x,t)\phi(x',t')|0\rangle. \tag{3.23}$$

The operators which occur under the T symbol are arranged from right to left with increasing times. This leads to the relation

$$\frac{\partial^2}{\partial t'^2}T\phi(x',t')\phi(x,t) = T\frac{\partial^2}{\partial t'^2}\phi(x',t')\phi(x,t) + \delta(t'-t)\delta(x'-x) \tag{3.24}$$

where the delta function with respect to time comes from the derivative of the theta function through the relation

$$\frac{\partial}{\partial t}\theta(t) = \delta(t).$$

(3.24) is the relation we used to attach the two legs of the free propagator on the boundary $t = 0$ (see (3.10)). The sources are restricted on the $t = 0$ surface, generating propagators with both legs on the boundary for which the delta function of (3.24) becomes infinite, whereas the conventional field theory spreads its sources over the whole plane without a similar divergent ambiguity appearing. Alternatively, as can been seen from (2.12), the vacuum functional is

$$\Psi = \langle \phi|E_0\rangle = \langle D|e^{i\int_{t=0}\hat{\pi}\phi dx}|E_0\rangle$$

where $\hat{\pi} = \dot{\hat{\phi}}$, so $\hat{\pi}$ is represented in the functional integral by $\dot{\phi}$ plus terms coming from $\partial/\partial t$ acting on the T-ordering, because the functional integral interpretation of Ψ represents T-ordered products. This demands the use of the relation (3.24) which at the points $t = t' = 0$ gives an infinity.

With boundary conditions which are not sharp in space-time, but are spread with a probability function, this infinity gets a finite value (see [17]). This phenomenon comes directly from the *uncertainty principle* for time and energy

$$\Delta t \Delta E \geq \frac{1}{2}\hbar$$

where for $\Delta t = 0$ (for sharp time boundaries) the energy becomes infinite.

Chapter 4

The Wave Functional for Slowly Varying Fields

4.1 Introduction

In this chapter we study the behaviour of the vacuum functional. As the method used aims at the non-perturbative construction of this functional a different kind of expansion has to be used. We will restrict our interest to slowly varying fields on the scale of the lightest mass. This enables the vacuum functional to be expanded in terms of the fields derivatives. In particular, we are going to see that its logarithm can be expanded as a sum of local functionals and satisfies a modified form of the Schrödinger Equation [12]. This procedure which can be generalised to any kind of massive fields will be performed here for the free and the ϕ^4 theory.

4.2 Expansion of the Wave Functional

As we have seen, the vacuum wave functional $\Psi[\varphi]$ can be derived from the Schrödinger wave functional

$$\langle\varphi'|e^{-tH}|\varphi\rangle \underset{t\to\infty}{\sim} \Psi_{vac}[\varphi']\Psi^*_{vac}[\varphi]$$

for large time t, which by the relation

$$\langle\varphi'|e^{-tH}|\varphi\rangle = \int \mathcal{D}\tilde{\phi}e^{-S_E[\tilde{\phi}]}$$

for $\tilde{\phi}(x,0) = \varphi(x)$ and $\tilde{\phi}(x,t) = \varphi'(x)$, suggests Ψ could be interpreted as a generating functional for certain Feynman Diagrams on the half plane $t \geq 0$. In particular, the expansion of its logarithm can be represented as a sum of connected Feynman diagrams. Additionally, the existence of a non-zero mass generates an exponentially damped factor in the propagators at large distances. In two Euclidean dimensions (x_1, x_2), this is expressed by the asymptotic expansion of the Green function, G, of the operator

$$-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + m^2$$

which is for $r = |x - y|$

$$G(x,y) = -\frac{e^{-mr}}{2\sqrt{2\pi mr}}\left(1 - \frac{1}{8mr} + \frac{9}{128(mr)^2} - ...\right).$$

The mass does not let the field propagate very far. This enables as to conclude that these diagrams reduce to local functionals for slowly varying fields. The arguments hold for the free, as well as for the interacting, theory.

In order to treat the scaling of the distance in a uniform way we introduce a scale factor s. This allows us to perform scaling transformations on the field $\varphi(x)$ as

$$\varphi^s(x) = \varphi(\frac{x}{\sqrt{s}})$$

For small s the argument of the right hand side is large, forcing the scaled field to take the value φ at infinity except for $x = 0$ where $\varphi^s(x) = \varphi(0)$. For large s the field varies slowly in space as it gets the value $\varphi^s(x) \approx \varphi(0)$. This will help us to study the behaviour of $\Psi[\varphi^s]$ for small and large s and to find a connection between the two regions.

For the free massive case we found that for $W[\varphi] \equiv \ln \Psi[\varphi]$ we have

$$W[\varphi] = -\frac{1}{2}\int dx\varphi\sqrt{-\nabla^2 + m^2}\varphi.$$

For a slowly varying field φ (i.e. with Fourier transform which vanishes for momentum, k_0, greater than the mass, m), this can be expanded as

$$W[\varphi] = -\int dx\left(\frac{m}{2}\varphi^2 + \frac{1}{4m}(\nabla\varphi)^2 - \frac{1}{16m^3}(\nabla^2\varphi)^2 + ...\right). \tag{4.1}$$

With the argument as φ^s the functional becomes $W[\varphi^s] = -\frac{1}{2}\int dx\varphi\sqrt{-\nabla^2 + sm^2}\varphi$, having the following expansion

$$W[\varphi^s] = -\int dx \left(s^{1/2}\frac{m}{2}\varphi^2 + s^{-1/2}\frac{1}{4m}(\nabla\varphi)^2 - s^{-3/2}\frac{1}{16m^3}(\nabla^2\varphi)^2 + ... \right) \tag{4.2}$$

For $s = 1$ we get $W[\varphi]$ from $W[\varphi^s]$. We will demonstrate how from a series like (4.2) we can obtain the value of $W[\varphi]$ from large s. A truncated expansion of (4.2), is a better approximation to the full functional when s is large. This generalised to the interacting theory will enable us to calculate the wave functional up to a certain order in its local expansion (valid for large s) and hence to get the desirable small s behaviour.

We will try to establish this method by studying first the vacuum energy and then the vacuum functional itself. A regularisation of the Laplacian could be achieved with a momentum cut-off like

$$\Delta_s = \int_{p^2<1/s} \frac{dp}{2\pi} \frac{\delta^2}{\delta\overline{\varphi}(p)\delta\overline{\varphi}(-p)},$$

where $\overline{\varphi}(p) = (1/2\pi)\int dx\varphi(x)\exp(-ipx)$. This will be a regularisation of the eigenvalue of the energy, given as the trace of the Γ operator. That is the vacuum energy density $\mathcal{E} = E/V$ is given by the cut-off dependent relation

$$\mathcal{E} = -\frac{1}{2V}\Delta_s W[\varphi] = \frac{1}{2}\int_{p^2<1/s} \frac{dp}{2\pi}\sqrt{p^2 + m^2} \sim \frac{1}{4\pi}\frac{1}{s} \text{ as } s \to 0. \tag{4.3}$$

On the other hand, doing the same calculation by using the local expansion of W we get

$$-\frac{1}{2V}\Delta_s W[\varphi] = \int_{p^2<1/s} \frac{dp}{2\pi}\left(\frac{m}{2} - \frac{p^2}{4m} - \frac{(p^2)^2}{16m^3}...\right) = \sum_{n=0}^{\infty} \frac{\alpha_n}{(m^2 s)^{n+1/2}} \tag{4.4}$$

where $\alpha_n = 1/(4\pi)m^2\Gamma(3/2)/(\Gamma(3/2-n)\Gamma(n+1)(1+2n))$. As we can see this is a large s expansion of the original quantity $-1/(2V)\Delta_s W$ whereas to get its proper behaviour for the cut-off dependence we are interested in its small s expansion to let $s \to 0$. There are ways to get the small s behaviour from the large s expansion. First we need to define the continuation of the vacuum energy[1] to complex s plane by

$$\mathcal{E}(s) = \frac{1}{2s^{1/2}}\int_{p^2<1} \frac{dp}{2\pi}\sqrt{m^2 + p^2/s}$$

This is analytic throughout the complex s-plane with the negative real axis removed (due to a cut of the square root).

[1] As we will see, this procedure can be applied to several well behaved functions with physical interest.

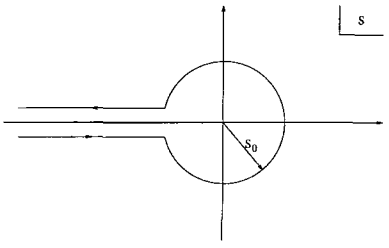

Figure 4.1: The contour of integration

For $|s|m^2 > 1$ this has the large s expansion of (4.4). Let C be the key-hole shaped contour of figure (4.1). The integral

$$I(\lambda) = \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}\mathcal{E}(s) \tag{4.5}$$

which could be defined as the action of the re-summation operator $R(s) \equiv 1/(2\pi i)$ $\int ds/s\ e^{\lambda s}$ on $\mathcal{E}(s)$, may be evaluated using the expansion (4.4). We can find in the mathematical literature (e.g. [19]) that

$$\int_C e^z z^n dz = -2i \sin n\pi\ n!$$

for n non-negative-integer, where C is the contour of (4.1). If n is negative, for $n = -m$, where $m > 0$, the integral takes the form

$$\int_C \frac{e^z}{z^m} dz = -2i \sin(-m\pi)\ (-m)! =$$

$$2i \sin m\pi\ \frac{\pi m}{\sin m\pi\ m!}$$

as

$$m!(-m)! = \frac{\pi m}{\sin m\pi}$$

so that finally we have

$$\int_C \frac{e^z}{z^m} dz = \frac{2\pi i}{(m-1)!} \Rightarrow \frac{1}{2\pi i}\int_C \frac{e^z}{z^m} dz = \frac{1}{\Gamma(m)}$$

For $m = n + 1 + 1/2$ as is needed to calculate (4.5) we evaluate

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(n+3/2)} \left(\frac{\lambda}{m^2}\right)^{n+1/2} \tag{4.6}$$

The value of the integral will not change if we collapse the contour C to a small circle centered on $s = 0$ and a contour that just surrounds the real negative axis. By taking λ to be positive and very large, the contribution from the negative real axis becomes exponentially suppressed away from the neighbourhood of the origin and hence the integral

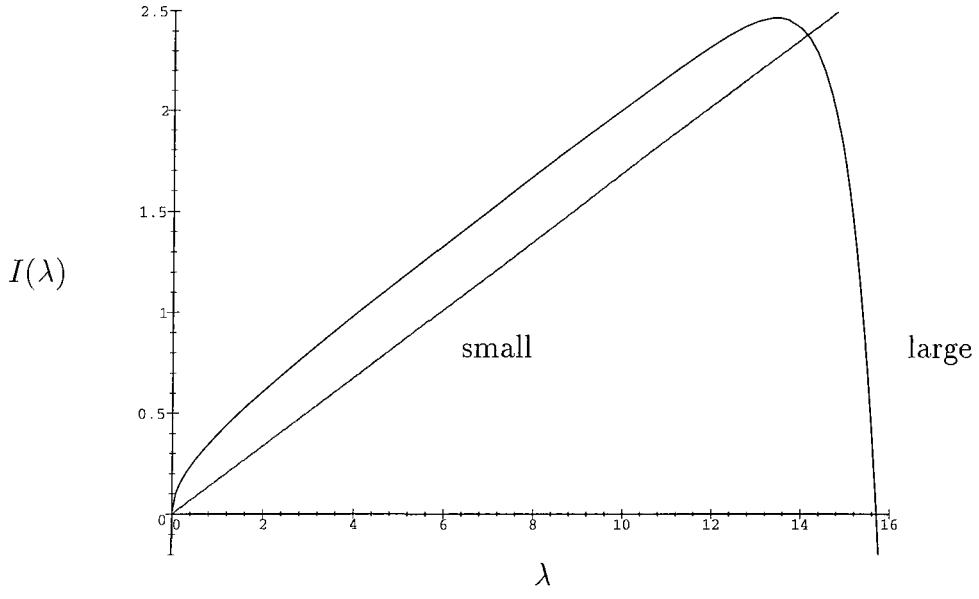

Figure 4.2: $I(\lambda)$ with the large and small s expansion for the energy density.

is determined by $\mathcal{E}(s)$ for small s. We can see that from (4.5), where for negative real s the exponential is suppressed for large λ while the only point which survives is for $s = 0$. This is a way to get the small s behaviour out of a large s expansion. In the diagram (4.2) we see that the energy density, given by (4.5) where $\mathcal{E}(s)$ is taken from (4.3) and from (4.4) respectively, has the same pattern in the sense that in the large-s expansion there is still the information of the small s.

We can follow similar steps to extract the small s behaviour of $W[\varphi]$ out of its local expansion (see [18]). Since we are interested for the value $s = 1$, let us make the expansion $W[\varphi^s]$ around infinity expressed in terms of $s - 1$. This will have the form

$$W[\varphi^s] = -\frac{m}{2}\sum_{n=0}^{\infty}\frac{\Gamma(3/2)}{\Gamma(n+1)\Gamma(3/2-n)}(s-1)^{1/2-n}\int dx\varphi\left(1-\frac{\nabla^2}{m^2}\right)^n\varphi$$

Using the re-summation operator $R(s)$, with the circle of the contour now centered at $s = 1$ as seen in Figure (4.3) we get

$$\frac{1}{2\pi i}\int_C\frac{ds}{s-1}e^{\lambda(s-1)}W[\varphi^s] =$$

$$-\frac{m}{2}\sum_0^{\infty}\frac{\Gamma(3/2)}{\Gamma(n+1)\Gamma(3/2-n)}\int dx\varphi\left(1-\frac{\nabla^2}{m^2}\right)^n\varphi\int_C\frac{e^{\lambda(s-1)}}{(s-1)^{n+1/2}}ds =$$

$$\frac{m}{4\sqrt{\pi}}\sum_0^{\infty}\frac{(-1)^n\lambda^{n-1/2}}{n!(n-1/2)}\int dx\varphi\left(1-\frac{\nabla^2}{m^2}\right)^n\varphi \qquad (4.7)$$

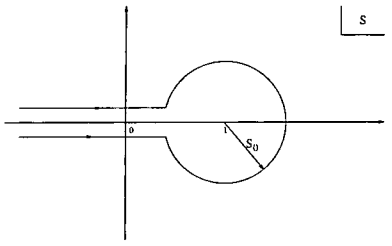

Figure 4.3: The contour of integration with the circle centered at 1

Because of the factor $n!$ the sum converges for any value of λ and of the integral which depends upon the momentum cut-off, k_0, of the field φ. If we expand the exponentials in the expression

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\left(\frac{1}{\sqrt{\lambda}}e^{-\lambda(1-\nabla^2/m^2)}+\int_0^\lambda d\lambda'\frac{1}{\sqrt{\lambda'}}e^{-\lambda'(1-\nabla^2/m^2)}\left(1-\frac{\nabla^2}{m^2}\right)\right)\varphi= \tag{4.8}$$

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{\lambda}}\frac{(-1)^n}{n!}\lambda^n\left(1-\frac{\nabla^2}{m^2}\right)^n+\int_0^\lambda d\lambda'\frac{1}{\sqrt{\lambda'}}\frac{(-1)^n}{n!}\lambda'^n\left(1-\frac{\nabla^2}{m^2}\right)^{n+1}\right)=$$

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{\lambda}}\frac{(-1)^n}{n!}\lambda^n\left(1-\frac{\nabla^2}{m^2}\right)^n+\frac{1}{\sqrt{\lambda}}\frac{(-1)^n}{n!(n+1/2)}\lambda^{n+1}\left(1-\frac{\nabla^2}{m^2}\right)^{n+1}\right)=$$

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\frac{1}{\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\lambda^n\left(1-\frac{\nabla^2}{m^2}\right)^n\left(1-\frac{n}{n-1/2}\right)=$$

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\frac{1}{\sqrt{\lambda}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\lambda^n\left(1-\frac{\nabla^2}{m^2}\right)^n\frac{-1/2}{n-1/2}$$

which is equal to (4.7), we can rewrite (4.8) as

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\left(\frac{1}{\sqrt{\lambda}}e^{-\lambda(1-\nabla^2/m^2)}+\int_0^\lambda d\lambda'\frac{1}{\sqrt{\lambda'}}e^{-\lambda'(1-\nabla^2/m^2)}\left(1-\frac{\nabla^2}{m^2}\right)\right)\varphi=$$

$$-\frac{m}{2\sqrt{\pi}}\int dx\varphi\left(\frac{1}{\sqrt{\lambda}}e^{-\lambda(1-\nabla^2/m^2)}+\int_0^\infty d\lambda'\frac{1}{\sqrt{\lambda'}}e^{-\lambda'(1-\nabla^2/m^2)}\left(1-\frac{\nabla^2}{m^2}\right)-\right.$$

$$\left.-\int_\lambda^\infty d\lambda'\frac{1}{\sqrt{\lambda'}}e^{-\lambda'(1-\nabla^2/m^2)}\left(1-\frac{\nabla^2}{m^2}\right)\right)\varphi=$$

$$-\frac{1}{2}\int dx\varphi\sqrt{-\nabla^2+m^2}\varphi+\frac{m}{4\sqrt{\pi}}\int dx\varphi\left(\int_\lambda^\infty d\lambda'\frac{1}{\sqrt{\lambda'^3}}e^{-\lambda'(1-\nabla^2/m^2)}\right)\varphi \tag{4.9}$$

since for $a>0$

$$\int_0^\infty dx\frac{e^{-ax}}{\sqrt{x}}=\frac{\sqrt{\pi}}{\sqrt{a}}$$

As λ grows large, (4.9) approaches $W[\varphi]$ with its form easily read off and with an error in approximating $W[\varphi]$ with the series (4.7) exponentially suppressed. As this series is alternating, truncating it at some order λ^n will result an error smaller than the first term

neglected. We see then, that we can have an acceptable knowledge of $W[\varphi]$ from a local truncated series of the form (4.7) or (4.2) with an error which depends on how large is n, the truncating order, and the size of λ relative to n.

The previous arguments apply to the case of a free field where we know the analytic form of W. In the interacting case we want to calculate an expansion of the wave functional as the full solution seems to be a too difficult if not impossible problem. As the functional dependence of W on φ can be very well defined by the linear combination of local functionals of φ, we are left with the task of evaluating the coefficients of these combinations.

A local expansion for W for the interacting theory could be

$$W[\varphi] = \int dx \left(a_1\varphi^2 + a_2(\varphi')^2 + a_3\varphi^2(\varphi')^2 + ...\right)$$

or in terms of φ^s

$$W[\varphi^s] = \int dx \left(a_1\varphi^{s2} + a_2(\varphi^{s\prime})^2 + a_3\varphi^2(\varphi^{s\prime})^2 + ...\right) =$$

$$\int dx \left(a_1\sqrt{s}\varphi^2 + a_2\frac{1}{\sqrt{s}}(\varphi')^2 + \frac{1}{\sqrt{s}}a_3\varphi^2(\varphi')^2 + ...\right) \tag{4.10}$$

Using (4.10) we can expand the wave functional $\Psi[\varphi^s] = e^{W[\varphi^s]}$ in inverse powers of $s-1$, with coefficients that depend on the original configuration φ, $\Psi[\varphi^s] \sim \sum(s-1)^{-n}\psi_n[\varphi]$. As it will be shown later if analytically continued to the complex plane $\Psi[\varphi^s]$ will have cuts and poles along the negative real axis. Now we can use the re-summation operator $R(s)$ to get

$$I(\lambda) = R(s)\Psi[\varphi^s] =$$

$$\frac{1}{2\pi i}\int_C \frac{ds}{s-1}e^{\lambda(s-1)}\Psi[\varphi^s]$$

where the contour C is taken to be key-hole shaped as shown in the Figure (4.3). If we take s_0 to be large then we can compute the integral using the local expansion for W shown in (4.10) from which we can compute the coefficients $\psi_n[\varphi]$. Finally we have

$$I(\lambda) = \sum_n \frac{\lambda^n\psi_n[\varphi]}{\Gamma(n+1)}$$

We can also evaluate the integral $I(\lambda)$ by collapsing the contour C until it breaks into two disconnected pieces: a small circle centered on $s = 1$ and a contour that surrounds the negative real axis. The integral over the circle gives $\Psi[\varphi]$. By taking λ to be real,

positive and very large the contribution from the negative real axis will be exponentially suppressed, (provided it is not singular, as we check using perturbation theory in following sections), so that for large λ,

$$\Psi[\varphi] \approx \sum_n \frac{\lambda^n \psi_n[\varphi]}{\Gamma(n+1)} \tag{4.11}$$

which provides a re-summation of the local expansion. As ψ_n depends only on terms of order up to s^{-n} in (4.10) we believe that we can obtain an approximation by truncating (4.11) at some order in λ.

In order to understand the kind of poles and cuts of the functions we will use have, we need to see in more detail their s dependence. Our main interest lies in the s behaviour of the action of the Laplacian Δ_s on the wave functional $\Psi[\varphi]$. As the representations $\hat{\pi}(x) = -i\delta/\delta\varphi(x)$ and $\Psi[\varphi] = \langle\varphi|0\rangle$ hold (at time $t = 0$) we can use bra and ket formalism

$$\langle\varphi|\hat{\pi}(x,t)\hat{\pi}(x',t')|0\rangle \tag{4.12}$$

with $t = t' = 0$ and then we integrate against $\mathcal{G}_s(x,x')$ in order to get $\Delta_s\Psi$. Expressing this in terms of $\dot{\varphi}$ under the formalism of functional integrals will exhibit a linear divergence, the meaning and renormalisation of which we have seen in section (3.5). So (4.12) up to additional delta functions is equal to

$$-\int \mathcal{D}\widetilde{\varphi} e^{-S_E[\widetilde{\varphi}]+\int dx\, \varphi\dot{\widetilde{\varphi}}}\, \dot{\widetilde{\varphi}}\,(x,t)\, \dot{\widetilde{\varphi}}\,(x',t') \tag{4.13}$$

We now set $t = t' = 0$. In order to be able to treat this we rotate co-ordinates so that the distance of the points $(x,0)$ and $(x',0)$ lies along the time axis with difference $\tau = |x - x'|$. The new form that (4.13) takes is

$$\int \mathcal{D}\widetilde{\varphi}_r e^{S_{E_r}[\widetilde{\varphi}]+\int dt\varphi\widetilde{\varphi}'_r} \widetilde{\varphi}'_r(0,\tau)\widetilde{\varphi}'_r(0,0) \tag{4.14}$$

where the variable of the functional integral $\widetilde{\varphi}_r$ is defined on the rotated half plane $x > 0$. This can be interpreted as the time-ordered vacuum expectation value of fields that evolve in Euclidean time with a Hamiltonian $\widetilde{H}$ defined on the half line $x > 0$. Turning back to the bra-ket formalism we have

$$T_E\langle 0_r|e^{\int dt\varphi(t)\widetilde{\varphi}'(0,t)}\widetilde{\varphi}'(0,\tau)\widetilde{\varphi}'(0,0)|0_r\rangle \tag{4.15}$$

which is written in terms of the vacuum $|0_r\rangle$ corresponding to the rotated Hamiltonian. One way to study this is to look at the terms appearing in the expansion of the exponential. We have to take care of the time-ordering which puts the quantities with smaller time

variable on the right. As we have the freedom to break the t integration which goes from $-\infty$ to $+\infty$, into smaller intervals, we make sure that the terms in the expansion have the integration variable varying in such a way that the integrand is time ordered for the whole range of its corresponding t integration variable. Thus we get

$$\int_\tau^\infty dt_n \int_\tau^{t_n} dt_{n-1} ... \int_\tau^{t_{p+1}} dt_p \int_0^\tau dt_{p-1} ... \int_0^{t_{q+1}} dt_q \int_{-\infty}^0 dt_{q-1} ... \int_{-\infty}^{t_2} dt_1 \varphi(t_n) ... \varphi(t_1)$$

$$\langle 0_r | \widehat{\varphi}' e^{-\widetilde{H}(t_n - t_{n-1})} \widehat{\varphi}' ... e^{-\widetilde{H}(t_p - \tau)} \widehat{\varphi}' e^{-\widetilde{H}(\tau - t_{p-1})} \widehat{\varphi}' ... e^{-\widetilde{H} t_q} \widehat{\varphi}' e^{\widetilde{H} t_{q-1}} \widehat{\varphi}' ... e^{-\widetilde{H}(t_2 - t_1)} \widehat{\varphi}' | 0_r \rangle \quad (4.16)$$

where we have also ordered the integral variables dt_i. The vacuum state $|0\rangle_r$ is time-invariant. Note that the time interval $(-\infty, \infty)$ has been broken in three pieces $(-\infty, 0)$, $(0, \tau)$ and (τ, ∞). This is due to the points where the $\widehat{\varphi}'$ operators appearing in (4.15) are defined. The Fourier decomposition of the functions $\varphi(t_i)$ is

$$\varphi(t_i) = \frac{1}{2\pi} \int dk_i \overline{\varphi}(k_i) \exp(i k_i t_i)$$

We can picture the result. By inserting a basis of eigenstates of $\widetilde{H}$ between each operator we do the time integrations. After the performing the intermediate t_i integrations we are left with a last t integration which runs from minus infinity to plus infinity, appearing only in the exponentials of the k_i Fourier variable and as a result it will give a delta function for the conservation of the total momentum $\sum k_i$. From the other t_i integrations we have, for $i \geq p$, the insertions of $(E_i - i \sum \overline{k_i})^{-1}$ whereas for $i < q$ it gives $(E_i + i \sum \overline{k_i})^{-1}$. Here $\sum \overline{k_i}$ includes only some of the k_i's whereas $\sum k_i$ is the sum over all momentum. From the remaining t-integrations we get a sum of products of energy denominators of the form $(E_j - i \sum \overline{k_j})^{-1}$ multiplied by exponentials of τ of the form $\exp -(E_j - i \sum \overline{k_j})\tau$, so that we have the τ dependence explicit as a sum of integrals over the spectrum of $\widetilde{H}$

$$\int dE dk_1 ... dk_n \overline{\varphi}(k_1) ... \overline{\varphi}(k_n) \delta(\sum k_i)(\rho_0 + \rho_1 e^{i k_1 \tau} + \rho_2 e^{i(k_1 + k_2)\tau} + ...) e^{-E\tau}$$

To obtain $\Delta_s \Psi$ we integrate these terms against $\mathcal{G}_s$ to get a sum of terms of the form

$$\int_{p^2 < s^{-1}} \frac{dp}{\pi} \int dE dk_1 ... dk_n \overline{\varphi}(k_1) ... \overline{\varphi}(k_n) \delta(\sum k_i) \sum \frac{\rho_j(E, k_1, ..., k_n)}{E - i(p + \sum k_i)}$$

As our purpose is to see the s dependence of $\Delta_s \Psi$ explicitly, we set $p = q/\sqrt{s}$ and substitute φ with $\varphi^s(x) = \varphi(x/\sqrt{s})$, or in momentum space $\overline{\varphi}(k)$ with $\overline{\varphi}^s(k\sqrt{s})\sqrt{s}$, and take $\overline{\varphi}^s(k)$ to vanish outside $|k| < \kappa << 1$. This enables us to scale the k_i integrals to obtain

$$\int_{q^2 < 1} \frac{dq}{\pi} \int dE dk_1 ... dk_n \overline{\varphi}^s(k_1) ... \overline{\varphi}^s(k_n) \sqrt{s} \delta(\sum k_i) \sum \frac{\rho_j(E, k_1, ..., k_n)}{\sqrt{s} E - i(q + \sum k_i)}$$

The functions ρ_j acquire a dependence on s via the energy denominators which can be written as $1/(E - i(\sum k)/\sqrt{s})$ after scaling of k_i.

We can conclude that our initial expression, $(\Delta_s \Psi)[\varphi^s]$, originally a function of real positive s, can be continued as an analytic function on the whole complex s-plane, excluding the negative real axis because of the half integer power of s. This is a result which we proved order by order in powers of φ^s and this is the way we are going to work in the following.

In the following we will need to find what kind of expansion in large s we get when we expand the coefficients of connected Feynman diagrams. These diagrams appear in W as a sum of certain local functionals of the form $\int dx \prod_i a_i$ where a_i is φ with i derivatives taken with respect to x and raised to a power u_i, i.e. $a_i = \varphi^{(i)\,u_i}$. Substituting $\varphi = \varphi^s = \varphi(x/\sqrt{s})$ in W we get terms of the form

$$\int dx \prod_i a_i^s = \int dx \sqrt{s} \prod_i \left(\frac{1}{\sqrt{s}}\right)^{i\,u_i} a_i = \frac{1}{\sqrt{s}^{\sum_i i\,u_i - 1}} \int dx \prod_i a_i$$

As we are dealing with the scalar ϕ^4 theory we may assume that it is invariant under parity transformations $x \to -x$ which reflects to the restriction on W to be built up from local functionals which have an even overall number of x derivatives. That is $\sum_i i\,u_i - 1$ is odd. So finally $W[\varphi^s]$ is constructed from terms which have a coefficient $1/s$ in half odd integer power. By applying the Laplacian Δ_s to $W[\varphi^s]$ only integer powers of $1/s$ will multiply $1/\sqrt{s}^{\sum_i i\,u_i - 1}$ because the half integer ones will be cancelled by symmetry [2]. So the Feynman diagrams constructed this way will include a large s series with terms of the form $1/s^{n/2}$ where n is odd. This function of s is the one we need to treat with re-summation in order to extract its value at $s = 0$ to calculate the various quantum corrections.

[2]The Kernel $\mathcal{G}$ in the Laplacian is symmetric with respect to its two arguments x and x' and actually depends on their defference. An odd number of derivatives of the one argument acting on it will result an odd function. By the symmetry our expression has with respect to x and x' it will make the odd derivative terms to pair, so that they cancel each other. These are the ones which could contribute a half integer power of $1/s$ coming from the Kernel.

4.3 Re-summation Procedures

It will be very important to be able to discover small s behaviour from the large with as much accuracy as possible. What we have is a function $f(s)$ with cuts and poles on the negative real axis. Expanding it for small s will give[3]

$$f(s) \approx f(0) + a_1\sqrt{s} + a_2 s + a_3\sqrt{s}^3 + a_4 s^2 + ... \tag{4.17}$$

where a_n's are some numbers which can be calculated for specific $f(s)$. The large s expansion of $f(s)$ will be[4]

$$f(s) \approx \frac{b_0}{\sqrt{s}} + \frac{b_1}{\sqrt{s^3}} + \frac{b_2}{\sqrt{s^5}} + ... \tag{4.18}$$

Relation (4.18) has, for certain b_n's, finite limit for $s \to \infty$. We can define the operator $R(s)$ to be

$$R(s) = \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s} \tag{4.19}$$

while the integral is taken after the integrand is multiplied with a desired function of s. $R(s)$ applied to (4.17) gives $f(0)$ up to the half integer powers of s, which now becomes $1/\lambda$, with modified coefficients (the integer powers will give zero). $R(s)$ applied to (4.18) will give another series of half integer powers of λ. It will have similar behaviour up to exponentially suppressed terms due to the integration contour around the negative real axis. But still we need $f(0)$ which we get from the small s series up to the terms added to $f(0)$ (see (4.17)). To get around to this error of approximation we can use one of the following methods.

First we can linearly combine terms of the form

$$\lambda^n R(s) s^n f(s)$$

so that we cancel out some of the additional terms appearing in (4.17) next to $f(0)$. The action of $R(s)$ on s^k is

$$R(s)s^k = \frac{1}{2\pi i}\int_C \frac{e^{\lambda s}}{s} s^k ds = \frac{1}{\lambda^k}\frac{\sin k\pi}{\pi}(k-1)!$$

which is zero for integer k.

[3] This is the general way the functions, we are going to use, expand as it will be seen in Chapter 5.

[4] As shown at the end of the previous section.

We can then calculate, for example, the linear combination $R(s)f(s)+2\lambda R(s)sf(s)$ to be

$$f(0)+0+0+a_3\frac{2}{3\pi^{3/2}}\frac{1}{\lambda^{3/2}}+...$$

as well as the combination $R(s)f(s)+4\lambda R(s)sf(s)+4/3\lambda^2R(s)s^2f(s)$ gives

$$f(0)+0+0+0+0+a_5\frac{2}{\sqrt{\pi}}\frac{1}{\lambda^{5/2}}+...$$

In this way we eliminate the most influential corrections to $f(0)$. The disadvantage of this method is that the coefficient of the first non-zero term might get larger than its corresponding one in the $R(s)f(s)$ series, increasing the error.

Another way could be to substitute t^2 for s in $f(s)$. This will move the cuts of the complex s-plane from the negative real axis to the imaginary axis of the t-plane. The expansion (4.18) will become

$$f(t^2)\approx\frac{b_0}{t}+\frac{b_1}{t^3}+\frac{b_2}{t^5}+... \tag{4.20}$$

We can observe that putting t^2 instead of s moves the cuts on the imaginary t axis. To understand better the procedure let us take an example. The function

$$f(s)=\sqrt{s+1}-\sqrt{s}$$

which is well behaved for $s\to 0$ and $s\to\infty$ takes the form

$$\sqrt{t^2+1}-t$$

If we want to make the small t expansion then we expand the square root. Expanding it as

$$\sqrt{t^2+1}=\sqrt{(t+i)}\sqrt{(t-i)}$$

we observe that the cuts are for the first square root in the interval $(-\infty,-i)$ and for the second in the interval $(-\infty,i)$ as is seen in figure (4.4). Asking the behaviour of this function under $t\to -t$ transformations in the t plane we should calculate the argument the function has for the two points A and A'. The first square root has an argument $-\phi_1/2$ while the second $\phi_2/2$ and their product $(\phi_2-\phi_1)/2$. On the other hand, taking into account the position of the cuts, the argument of A' is $(\pi+\phi_2)/2$ and $(\pi-\phi_1)/2$ respectively for the two square roots. Eventually for $\sqrt{t^2+1}$ we have

$$\arg A=\frac{\phi_2-\phi_1}{2}\qquad\text{and}\qquad\arg A'=\pi+\frac{\phi_2-\phi_1}{2}$$

where the two arguments differ by π. This reveals the odd character of $\sqrt{t^2+1}$ with the specific cuts we chose in Figure (4.4). In addition computing the arguments for the points B and B' we get that $\sqrt{t^2+1}$ can be continued in the region $(-i,\infty)$ leaving the cuts in the finite limits interval $(-i,i)$.

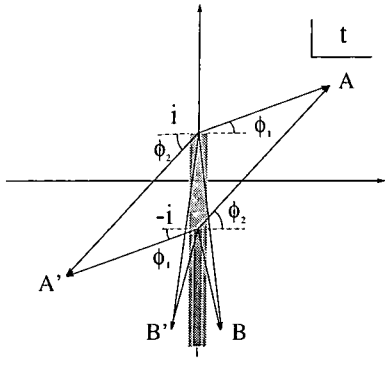

Figure 4.4: The cuts are overlapping.

To make the large t expansion then we can expand

$$f(t^2) = t\sqrt{1+\frac{1}{t^2}} - t \tag{4.21}$$

in powers of t, which is odd in t. This function has cuts for $t \in (-i,i)$ a region which

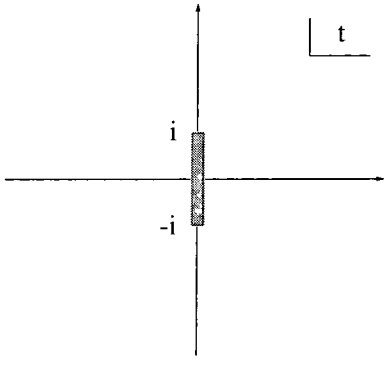

Figure 4.5: The cuts are defined in the interval $(-i,i)$.

allows the large t expansion to be well defined (see Figure (4.5)). These characteristics are similar to the functions we are going to use.

The result from the action of $R(t)$ upon a function $f(t)$ can be represented in a general way. The expansion (4.20) is odd in t. This odd behaviour can be extended in the small t region as seen in the previous example. The contour C in (4.19) is a circle centered on the origin with large radius. Because we are able to make a large t expansion, we can assume that our function, in a form similar to (4.21), has cuts and poles in a closed interval of

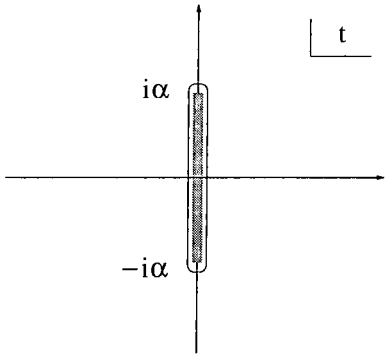

Figure 4.6: The cuts are in the interval $(-i\alpha, i\alpha)$.

the imaginary axis, say $t \in (-i\alpha, i\alpha)$. The existence of such a finite α relies upon the fact that our theory is massive, which allows us to make a large s expansion and get well behaved series like (4.18). C can be collapsed around this interval as seen in Figure (4.6). The integral takes the form

$$\frac{1}{2\pi i}\oint \frac{dt}{t} e^{\lambda t} f(t) =$$

$$\int_{-\alpha}^{\alpha} \frac{dy}{\epsilon + iy} e^{\lambda(\epsilon+iy)} f(\epsilon + iy) + \int_{\alpha}^{-\alpha} \frac{dy}{-\epsilon + iy} e^{\lambda(-\epsilon+iy)} f(-\epsilon + iy) =$$

$$\int_{-\alpha}^{\alpha} \frac{dy}{\epsilon + iy} e^{\lambda(\epsilon+iy)} f(\epsilon + iy) - \int_{-\alpha}^{\alpha} \frac{dy}{-(\epsilon + iy)} e^{\lambda(-\epsilon-iy)} f(-\epsilon - iy) =$$

$$\int_{-\alpha}^{\alpha} \frac{dy}{\epsilon + iy} \left\{ e^{\lambda(\epsilon+iy)} - e^{-\lambda(\epsilon+iy)} \right\} =$$

$$\int_{-\alpha}^{\alpha} \frac{dy}{y} \sin(\lambda y) f(\epsilon + iy) \tag{4.22}$$

The ratio $\sin(\lambda y)/y$ eventually will extract the value $f(0)$ up to an error depending on how large we can make λ. This error is equivalent to the exponentially suppressed term we get with the previous method with the additional error from the sin function, which will appear in our diagrams as an undulation (oscillation) upon the value we want to determine[5].

In the interacting theory we are only able to compute a finite number of terms for the expansions (4.17) or (4.18). As they are alternating series we can take λ up to certain value, above which the last term of the series becomes dominant causing the series to jump to infinity. The more the terms we take the larger the value of λ we can use and

[5] Phenomenon [20], where the oscillations there, can be suppressed drastically by the use of Lanczos convergence factors.

the less the error of approximation of the original function with its truncated expansion. In what follows we will use the re-summation operator given by (4.19).

In Appendix A we see a straightforward way, in how the two series (4.17) and (4.18) are connected.

4.4 ϕ^4 Theory

Here we study the re-summation techniques for the ϕ^4 theory in $1+1$ dimensions (which is super-renormalisable). There are no extra divergences in the Schrödinger functional due to the boundaries. The only divergence comes from those Feynman diagrams which have both ends of the propagator contracted at the same point as in the usual treatment without boundaries. As we have seen, the Hamiltonian in terms of the momentum cut-off reads

$$H_s = \int dx \left(\frac{1}{2} \left(\hat{\pi}_s^2 + \hat{\varphi}_s'^2 + M^2(s) \hat{\varphi}_s^2 \right) + \frac{g}{4!} \hat{\varphi}_s^4 - \mathcal{E}(s) \right)$$

where

$$\hat{\varphi}_s(x) = \int dy \, \mathcal{G}_s(x,y) \hat{\varphi}(y), \quad \hat{\pi}_s(x) = \int dy \, \mathcal{G}_s(x,y) \hat{\pi}(y)$$

($\mathcal{G}_s(x,y)$ given in (3.14)). We have shown previously that $\lim_{s \to 0} \widehat{H}_s \Psi = \widehat{H} \Psi$ if

$$\mathcal{E}(s) = \frac{1}{2} \int_{p^2 < 1/s} \frac{dp}{2\pi} \sqrt{p^2 + M_r^2} - \frac{g}{32} \left(\int \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M_r^2}} \right)^2 \tag{4.23}$$

and

$$M_r^2 = M^2(s) + \frac{g}{4} \int_{p^2 < 1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M_r^2}}, \tag{4.24}$$

so that the Schrödinger equation for the vacuum is just $\lim_{s \to o} \langle \varphi | \widehat{H}_s | 0 \rangle = E \langle \varphi | 0 \rangle$ or

$$\lim_{s \downarrow 0} \left(-\frac{1}{2} \Delta_s + \int dx \left(\frac{1}{2} \left(\varphi_s'(x)^2 + M^2(s) \varphi_s(x)^2 \right) + \frac{g}{4!} \varphi_s(x)^4 - \mathcal{E}(s) \right) \right) \Psi$$

$$\equiv \lim_{s \to 0} \left(-\frac{1}{2} \Delta_s + V_s \right) \Psi \equiv \lim_{s \to 0} \; H_s \Psi = E \Psi$$

where

$$V_s = \int dx \left(\frac{1}{2} \left(\varphi_s'(x)^2 + M^2(s) \varphi_s(x)^2 \right) + \frac{g}{4!} \varphi_s(x)^4 - \mathcal{E}(s) \right). \tag{4.25}$$

If we evaluate this expression for a φ that has no Fourier modes with momentum greater than $1/s$ we can replace φ_s in this expression by φ itself.

4.5 Solving the Schrödinger Equation

We have seen that the action of the Laplacian on the vacuum functional, $(\Delta_s\Psi)[\varphi_s]$, with regulator s, can be continued as a function of s from real and positive to the whole complex s-plane excluding the negative real axis. In the same way the Schrödinger equation of the ϕ^4 theory for the ground state wave functional, $((H_s - E)\Psi)[\varphi_s]$, extends to an analytic function of s with singularities over the negative real axis. Eventually its value for $s = 0$ can be obtained from a contour integral with radius arbitrarily large (see Figure (4.1)). Using the operator $R(s)$ we obtain

$$\lim_{\lambda\to\infty} R(s)((H_s - E)\Psi)[\varphi_s] = 0 \Rightarrow \tag{4.26}$$

$$\lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}((H_s - E)\Psi)[\varphi_s] = 0 \tag{4.27}$$

where

$$R(s) = \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}, \tag{4.28}$$

V_s is given by (4.25) and E is the energy eigenvalue. Taking $\Psi = e^W$, (4.27) becomes

$$\lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}\left(-\frac{1}{2}\left(\Delta_s W + \int dxdy\mathcal{G}_s(x,y)\frac{\delta W}{\delta\varphi(x)}\frac{\delta W}{\delta\varphi(y)}\right) + V_s - E\right)[\varphi_s] = 0$$

where for $\kappa << 1$ (see page 29) we can replace $\mathcal{G}_s(x,y)$ by a delta function, so that

$$\lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}\left(-\frac{1}{2}\left(\Delta_s W + \int dx\left(\frac{\delta W}{\delta\varphi(x)}\right)^2\right) + V_s - E\right)[\varphi_s] = 0 \tag{4.29}$$

Treating W as the generating functional for the connected Feynman diagrams we assume that an expansion in powers of φ is possible, that is

$$W[\varphi] = \sum_{n=1}^{\infty}\int \frac{dk_1}{2\pi}...\frac{dk_{2n}}{2\pi}\overline{\varphi}(k_1)...\overline{\varphi}(k_{2n})\delta(\sum k_i)\Gamma_n(k_1,...k_{2n}).$$

The functions $\Gamma_n(k_1,...k_{2n})$ can be expanded in positive powers of k_i's for small momenta, so W can be written in terms of local functionals. In a similar way, if we apply Δ_s to $W[\varphi]$ we get

$$\Delta_s W[\varphi] = \int_{p^2<\frac{1}{s}} \frac{dp}{2\pi}\sum_{n=1}^{\infty} n(n-1)\int \frac{dk_3}{2\pi}...\frac{dk_{2n}}{2\pi}\overline{\varphi}(k_3)...\overline{\varphi}(k_{2n})\delta(\sum k_i)\Gamma_n(p,-p,k_3,...k_{2n})$$

Substituting $\varphi = \varphi_s$ into W we can expand for large s in positive powers of p as well as in the rest of the momenta k_i's. Again a local expansion is obtained which actually coincides with the action of Δ_s on the local expansion of W followed by the substitution

$\varphi = \varphi_s$. Taking advantage of this we can take a linear combination of local functionals for W and reduce the eigenvalue problem of the Hamiltonian to the problem of solving an algebraic set of equations for the coefficients of the linear combinations.

The most general ansatz for the local functionals of W is $\int dx\ \varphi^{v_0}(\varphi')^{v_1}\dots(\varphi^{(n)})^{v_n}$ where v_n and n are non-negative integers. Since integration by parts gives linear relations between terms in the local expansion, it can be verified that one should take v_n (the power of the highest derivative) to be greater than or equal to two (see Appendix C). This ensures linear independence. Moreover, an even number of φ's assures that $\varphi \to -\varphi$ is an unbroken symmetry of the lagrangian, while parity invariance leads us to take only an even number of derivatives. Consequently, the local expansion for W_λ can be written as

$$\begin{aligned} W_\lambda = \int dx \Big(& \varphi^2(b_0 + c_0\varphi^2) + \varphi'^2(b_1 + c_1\varphi^2 + d_1\varphi'^2) \\ & + \sum_{n=2}^{\infty} \varphi^{(n)2}(b_n + c_n\varphi^2 + d_n\varphi'^2 + e_n\varphi\varphi'' + ..) \\ & + \varphi^4(f_0\varphi^2 + f_1\varphi'^2 + f_2\varphi''^2 + ..) + .. \Big) \end{aligned} \tag{4.30}$$

where the coefficients $b_i, c_i, ..$ are determined via the Schrödinger equation and, since Ψ is finite, they ought to be finite as $\lambda \to \infty$.

Inserting (4.30) back into (4.29) we can show that

$$\begin{aligned} \int dx \ \Bigg(& \left(2\mathcal{E}(\lambda) + 2\mathcal{E} + \frac{\sqrt{\lambda}}{\sqrt{\pi}^3}\left(4b_0 + \sqrt{\pi}\sum_{n=1}^{\infty}\rho(n)\ b_n\ \lambda^n \right) \right) \\ & + \varphi^2 \left(-\overline{M}^2(\lambda) + \frac{4}{\sqrt{\pi}}b_0^2 + \frac{\sqrt{\lambda}}{\pi}\left(12c_0 + \sum_{n=1}^{\infty}\rho'(n)\ c_n\ \lambda^n \right) \right) \\ & + \varphi^4 \left(-\frac{2}{4!\sqrt{\pi}}g + \frac{16}{\sqrt{\pi}}b_0c_0 + \frac{\sqrt{\lambda}}{\pi}\left(30f_0 + \sum_{n=1}^{\infty}\rho'(n)\ f_n\ \lambda^n \right) \right) \\ & + \varphi'^2 \left(-\frac{2}{\sqrt{\pi}} + \frac{16}{\sqrt{\pi}}b_0b_1 + \frac{\sqrt{\lambda}}{\pi}\left(2c_1 + 2d_1\lambda + \frac{4c_2\lambda}{3} + \cdots \right) \right) \\ & + \varphi^2\varphi'^2 \left(\frac{32}{\sqrt{\pi}}b_0c_1 + \frac{96}{\sqrt{\pi}}b_1c_0 + \frac{\sqrt{\lambda}}{\pi}12f_1 + \cdots \right) \\ & + \varphi''^2 \left(\frac{32b_0b_2 + 16b_1^2}{3\sqrt{\pi}} + \frac{\sqrt{\lambda}}{\pi}\left(c_2 + \frac{d_1\lambda}{9} - \frac{e_2\lambda}{3} + \cdots \right) \right) \\ & + \cdots \end{aligned}$$

$$+(\varphi^{(n)})^2\left(\sum_{m\geq 3}\frac{4b_m b_{n-m}}{\Gamma(n+1/2)}+\frac{\sqrt{\lambda}}{\pi}\left(\frac{2c_n}{n!}+\frac{2(d_n-e_n)\lambda^{\frac{3}{2}}}{3(n+1)!}+\cdots\right)\right)$$
$$+\cdots$$
$$+\varphi^{2k}\left(\frac{4}{\sqrt{\pi}}\sum_{n=0}^{k-1}a_0^n a_0^{k-1-n}(k-n)(n+1)+\frac{\sqrt{\lambda}}{\pi}\left(2(k+1)(2k+1)a_0^k+\sum_{n=1}\rho'(n)\ a_n^k\ \lambda^n\right)\right)$$
$$+\cdots=\mathbf{0}. \tag{4.31}$$

where $\rho(n)=2/(\Gamma(n+\frac{3}{2})(2n+1))$, $\rho'(n)=2/(\Gamma(n+1)(2n+1))$, $\overline{M}^2(\lambda)=\sqrt{\lambda}/\sqrt{\pi}\hat{R}(s)$ $1/\sqrt{s}M^2(s)$ and $\mathcal{E}(\lambda)=R(s)\mathcal{E}(s)$. Also we have substituted φ for φ_s for $s=1/\lambda$. The coefficient of $\varphi^{2k}(k\geq 3)$ above is such that a_j^i is the coefficient of the subset

$$W'=\int dx\sum_{i=0}^{\infty}a_j^i\varphi^{2j}\varphi^{(i)^2} \tag{4.32}$$

of W_λ, i.e., $a_i^0=b_i$, $a_i^1=c_i$, $a_i^2=f_i$, etc. Setting the coefficient of each element of the basis to vanish yields an infinite set of algebraic equations.

4.6 Classical Equations

In order to study an expansion in $\hbar$ we are initially interested in the classical description of our model. This is given by the Hamilton-Jacobi classical equation of motion. In $1+1$ dimensions the classical field theory has a Euclidean Action S_E given in terms of the Euclidean Lagrangian density $\mathcal{L}_E$ by

$$S_E=\int d^2x\mathcal{L}_E$$

where we assume that $\mathcal{L}_E$ is a function of the scalar field, ϕ, and its first derivative with respect to time, $\dot{\phi}$. Its conjugate momentum is defined by

$$\pi(x)=\frac{\partial\mathcal{L}_E}{\partial\dot{\phi}(x)}=\frac{\delta S_E}{\delta\phi(x)}$$

The *Hamilton-Jacobi equation* for a time-independent potential is

$$H(\pi,\phi)=0\Rightarrow H(\frac{\delta S_E}{\delta\phi},\phi)=0 \tag{4.33}$$

For the ϕ^4 theory the Lagrangian $\mathcal{L}_E$ has the form

$$\mathcal{L}_E=\frac{1}{2}(\dot{\phi}^2+\phi'^2)+\frac{1}{2}m^2\phi^2+\frac{g}{4!}\phi^4$$

so that the Hamiltonian is

$$H_E = \pi\dot{\phi} - \mathcal{L}_E = \frac{1}{2}\pi^2 - \frac{1}{2}\phi'^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 = \frac{1}{2}\pi^2 - V_{class}(\phi)$$

so that (4.33) becomes

$$\frac{1}{2}\left(\frac{\delta S_E}{\delta\phi}\right)^2 - \frac{1}{2}\phi'^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 = 0 \Rightarrow$$

$$\frac{1}{2}\left(\frac{\delta S_E}{\delta\phi}\right)^2 - V_{class}(\phi) = 0 \tag{4.34}$$

We can obtain the set of the classical equations from the full set of algebraic equations, as the zeroth order approximation of equation (4.31) in $\hbar$. It is necessary to recover $\hbar$ factors in (4.26) and expand W in $\Psi_0 = e^W$ as $W = \sum_{n=0}\hbar^{n-1}W_n$. Also, $E_\lambda = E_{class} + \hbar E_1 + \cdots$ so that the classical equation becomes

$$R(s)\left\{\frac{1}{2}\int dx\ \left(\frac{\delta W_0}{\delta\phi}\right)^2 - V_s^{class}\right\} = 0. \tag{4.35}$$

In the equation above

$$V_s^{class} = \int dx\ \frac{1}{2}\left(\varphi_s'^{\,2} + M_r^2\varphi_s^2\right) + \frac{g}{4!}\varphi_s^4 + \mathcal{E}_{class}$$

$\mathcal{E}_{class}$ is zero because the Hamiltonian has been constructed to match the normal ordered Hamiltonian $:H:$ with zero eigenvalue. There are no contributions from the mass or the energy renormalisation as both counter-terms are of order $\hbar$ or higher[6]. Comparing (4.34) and (4.35) we deduce that $-W_{class}/\hbar$ can be represented by S_E. We see, that the on-shell Euclidean action, S_E, is the classical approximation to W, the logarithm of the vacuum functional, satisfying the Minkowski Hamiltonian. $R(s)$ leads to an equivalent equation and in this case redefines the field φ_s accordion to its scaling properties. We can also arrive at the classical equations by setting $\lambda = 0$ in (4.31).

It is possible to calculate the term $\int dx \left(\frac{\delta W_{class}}{\delta\phi}\right)^2$ for the specific subset of local vectors of the form $\int \phi^{2m}\phi^{(n)2}$ quite easily. In this way we can evaluate the coefficients of their linear combination in W_{class}. This will be done in the next section.

[6]The mass and energy divergencies are quantum effects and they do not appear in the classical level.

4.6.1 Analytic Computation of the Classical Equations

We can compute the classical equations of the coefficients a_n^m of $\int \phi^{2m}\phi^{(n)2}$ in a general way. In Appendix C we see that the classical equations produced by this set are complete to determine a_n^m, without insertions of coefficients of other types of local functionals of W. If we say that the term $\int \phi^{2m}\phi^{(n)^2}$ comes from the product $M_{k_1\lambda_1 k_2\lambda_2} = \int \frac{\delta(\phi^{2k_1}\phi^{(\lambda_2)^2})}{\delta\phi}$ $\frac{\delta(\phi^{2k_2}\phi^{(\lambda_2)^2})}{\delta\phi}$. then the number of ϕ's and the number of σ derivatives should be the same. That is:

$$2m + 2 = 2k_1 + 2 - 1 + 2k_2 + 2 - 1 \Rightarrow m = k_1 + k_2 \tag{4.36}$$

and

$$2\lambda_1 + 2\lambda_2 = 2n \Rightarrow \lambda_1 + \lambda_2 = n \tag{4.37}$$

But we have

$$\int \frac{\delta(\phi^{2k}\phi^{(\lambda)^2})}{\delta\phi} = 2k\phi^{2k-1}\phi^{(\lambda)^2} + (-1)^\lambda 2(\phi^{2k}\phi^{(\lambda)})^{(\lambda)} \tag{4.38}$$

so that for M having the variables k_1, λ_1, k_2, λ_2 it will be

$$M_{k_1\lambda_1 k_2\lambda_2} =$$

$$4\Big[k_1k_2\phi^{2m-2}\phi^{(\lambda_1)^2}\phi^{(\lambda_2)^2} +$$

$$k_1(-1)^{\lambda_2}\phi^{2k_1-1}\phi^{(\lambda_1)^2}\sum_{r=0}^{\lambda_2}\binom{\lambda_2}{r}\left(\phi^{2k_2}\right)^{(\lambda_2-r)}\phi^{(\lambda_2+r)} +$$

$$k_2(-1)^{\lambda_1}\phi^{2k_2-1}\phi^{(\lambda_2)^2}\sum_{r=0}^{\lambda_1}\binom{\lambda_1}{r}\left(\phi^{2k_1}\right)^{(\lambda_1-1)}\phi^{(\lambda_1+r)} +$$

$$(-1)^n\sum_{r=0}^{\lambda_1}\binom{\lambda_1}{r}\left(\phi^{2k_1}\right)^{(\lambda_1-r)}\phi^{(\lambda_1+r)}\sum_{w=0}^{\lambda_2}\binom{\lambda_2}{w}\left(\phi^{2k_2}\right)^{(\lambda_2-w)}\phi^{(\lambda_2+w)}\Big]$$

Finally the classical equations will be

$$\sum_{\lambda_1,k_1} M_{k_1\lambda_1(m-k_1)(n-\lambda_1)}a_{k_1}^{\lambda_1}a_{m-k_1}^{n-\lambda_1} - V_{class} = 0 \tag{4.39}$$

as the relations (4.36) and (4.37) are imposed. V_{class} is composed of functionals of the form $\int \phi^{2m}\phi^{(n)^2}$. We can calculate $M_{k_1\lambda_1 k_2\lambda_2}$ by distinguishing the following cases:

1. The first term is $k_1k_2\phi^{2m-2}\phi^{(\lambda_1)^2}\phi^{(\lambda_2)^2}$ and it will contribute only if $\lambda_1 = 0$ or $\lambda_1 = n \Rightarrow \lambda_2 = 0$. Then, it will become

$$k_1k_2\phi^{2m}\phi^{(n)^2} \tag{4.40}$$

2. The second and third terms are similar:

$$k_2(-1)^{\lambda_1}\phi^{2k_2-1}\phi^{(\lambda_2)^2}\sum_{r=0}^{\lambda_1}\binom{\lambda_1}{r}\left(\phi^{2k_1}\right)^{(\lambda_1-r)}\phi^{(\lambda_1+r)} \tag{4.41}$$

If we take $\lambda_2 \neq 0$ the only contribution can come from $\lambda_2 = n$ where (4.41) becomes

$$k_2\phi^{2m}\phi^{(n)^2}. \tag{4.42}$$

On the other hand if $\lambda_2 = 0$ then (4.41) becomes

$$k_2(-1)^n\phi^{2k_2+1}\sum_{r=0}^{n}\binom{n}{r}\left(\phi^{2k_1}\right)^{(n-r)}\phi^{(n+r)} \tag{4.43}$$

which if $r \neq n$ then

$$k_2(-1)^n\phi^{2k_2+1}\sum_{r=0}^{n-1}\binom{n}{r}\left(\phi^{2k_1}\right)^{(n-r)}\phi^{(n+r)} =$$

$$k_2(-1)^n\sum_{r=0}^{n-1}\binom{n}{r}(-1)^r 2k_1\phi^{2m}\left(\phi^{(n)}\right)^2$$

while when $r = n$

$$k_2(-1)^n\phi^{2k_2+1}\phi^{2k_1}\phi^{(2n)} =$$

$$k_2(2m+1)\phi^{2m}\left(\phi^{(n)}\right)^2$$

The overall contributions from (4.41) are going to be

$$k_2(-1)^n\sum_{r=0}^{n-1}\binom{n}{r}(-1)^r 2k_1\phi^{2m}\left(\phi^{(n)}\right)^2 + k_2(2m+1)\phi^{2m}\left(\phi^{(n)}\right)^2 \tag{4.44}$$

for $\lambda_2 = 0$, and for $\lambda_2 = n$ it is

$$k_2\phi^{2m}\phi^{(n)2}. \tag{4.45}$$

3. The last term in (4.6.1) will make contributions according to the various values of λ_1 as:

a) for $\lambda_1 = 0$ it is

$$.. = 2k_2(-1)^n\sum_{w=0}^{n-1}\binom{n}{w}(-1)^w\phi^{2m}\left(\phi^{(n)}\right)^2 + (2m+1)\phi^{2m}\left(\phi^{(n)}\right)^2 \tag{4.46}$$

b) for $\lambda_1 = n$ it is

$$.. = 2k_1(-1)^n\sum_{r=0}^{n-1}\binom{n}{r}(-1)^r\phi^{2m}\left(\phi^{(n)}\right)^2 + (2m+1)\phi^{2m}\left(\phi^{(n)}\right)^2 \tag{4.47}$$

c) for $0 < \lambda_1 < n$ the only contribution is of the form $\phi^{2m}\phi^{(n)2}$

Because it is hard to keep track of all these cases it is useful to create a program which can generate the classical equations for the various values of m and n. This can be done easily in *Maple programing language* (see also Appendix E) with the following program:

```
clser:=proc(m,n)
local M,s,NN,k,NNN,k1,MM,l,l1,N,MMM,ss;
 if m=0 and n=1 then ss:=1 else ss:=0 fi;
 if n=1 then s:=0 else s:=1 fi;
  if n<>0 then
   NN[0]:=0;
   NNN[0]:=0;
   k:=0;
    for k1 from 0 to m do
     k:=k+1;
     MM[n-1]:=s*sum(a[k1][l1]*a[m-k1][n-l1],l1=1..n-1);
     N[k1]:=a[k1][0]*a[m-k1][n]*(k1*(m-k1)+(k1+1)*(2*k1+1)+m-k1)+
           a[k1][n]*a[m-k1][0]*(k1*(m-k1)+(m-k1+1)*(2*m+1-2*k1)+k1);
     NN[k]:=NN[k-1]+N[k1]+MM[n-1];
     NNN[k]:=NNN[k-1]+NN[k] -ss*1/4;
    od;
   MMM[m,n]:=NULL;
  elif n=0 then
   if m=0 then
    MMM[m,n]:=-1+4*sum(a[k1][0]*a[m-k1][0]*(k1*(m-k1)+m+1),k1=0..m);
   elif m=1 then
    MMM[m,n]:=-1/12+4*sum(a[k1][0]*a[m-k1][0]*(k1*(m-k1)+m+1),k1=0..m);
   elif m=2 then
    MMM[m,n]:=4*sum(a[k1][0]*a[m-k1][0]*(k1*(m-k1)+m+1),k1=0..m);
   else
    MMM[m,n]:=4*sum(a[k1][0]*a[m-k1][0]*(k1*(m-k1)+m+1),k1=0..m);
   fi;
  NNN[k]:=NULL;
fi;
```

```
RETURN(NNN[k],MMM[m,n]);
end;
```

This program provides the defining classical equation for the coefficient $a[m][n] \equiv a_n^m$ of the basis vector $\int \phi^{2m}\phi^{(n)2}$. After "reading" the program in a Maple worksheet one can execute it by the command *clser(m,n);* where m and n denote the corresponding indices in $\phi^{2m}\phi^{(n)2}$, for which we want to find its coefficient equation. For computational ease we have set $M_r = 1$ and $g = 1$.

In Appendix D we see that a closed set of equations for the coefficients b_i, c_i, f_i[7], etc, can be determined owing to the symmetry properties of the subset W' of W (see equation (4.32)): the equations which classically determine the coefficients in W' are such that in order to calculate a_n^m we need to know only a_j^i with $i < m$ and $j < n$. It can be shown[8] that

$$b_0 = -1/2,\ b_1 = -1/4,\ b_n = -\frac{1}{2}\binom{1/2}{n}\ for\ n = 2,3,4,\dots\ ,$$

$$c_0 = -1/96,\ c_n = b_0 c_0 \frac{12}{2^n}\frac{(-1)^n}{b_0^{n+1}} det\ \mathcal{B}_n\ for\ n = 1,2,\dots$$

where

$$\mathcal{B}_n = \begin{pmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ 2b_0 & b_1 & \dots & b_{n-2} & b_{n-1} \\ 0 & 2b_0 & \dots & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2b_0 & b_1 \end{pmatrix}. \tag{4.48}$$

Similar and more complicated formulae for the other a_i^j's can be computed. We can solve these equations by using the following computer program:

```
cleqn:=proc(m,n)
local inp1,inp2,inp3,inp4,inp33,inp44,inp11,i,inp5,inp6,j,inp7;
inp5[-1]:=NULL;
inp6[-1]:=NULL;
 for j from 0 to m do
  inp1[-1]:=NULL;
```

[7]The letters b, c and f stand for the first terms of a[i][j] for i equal to zero, one and two equivalently.

[8]See Appendix D.

```
   inp11[-1]:=NULL;
   inp44[-1]:=NULL;
    for i from 0 to n do
     inp1[i]:=inp1[i-1],clser(j,i);
    od;
   inp2:={inp1[n]};
    for i from 0 to n do
     inp11[i]:=inp11[i-1],a[j][i];
    od;
   inp5[j]:=inp5[j-1],inp1[n];
   inp6[j]:=inp6[j-1],inp11[n];
 od;
inp7:=solve({inp5[m]},{inp6[m]});
RETURN(op(2,[inp7]));
end;
```

Executing this program with the command *cleqn(m,n);*, we get all the coefficients a^i_j with $i \leq m$ and $j \leq n$.

4.7 Semiclassical Calculations

Now let us turn our attention to solving the equations (4.31) by the semi-classical procedure, that is, up to order $\hbar$ in Minkowski space. In order $\hbar$, (4.26) becomes

$$\hbar \left(R(s) \left(\frac{1}{2} \Delta_s W_0 + \int dx \, \frac{\delta W_0}{\delta \phi} \frac{\delta W_1}{\delta \phi} - V_s^{\hbar} \right) \right) [\varphi] = 0 \tag{4.49}$$

In the equation above

$$V_s^{\hbar} = \int dx \, \frac{1}{2} \delta M^2(s) \varphi_s^2 - \mathcal{E}^{\hbar}(s) - \mathcal{E}^{\hbar},$$

where

$$\delta M^2(s) = M^2(s) - M_r^2 = -\frac{g}{4} \int_{p^2 < 1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M_r^2}}. \tag{4.50}$$

and

$$\mathcal{E}^{\hbar}(s) = \frac{1}{2} \int_{p^2 < 1/s} \frac{dp}{2\pi} \sqrt{p^2 + M_r^2} \tag{4.51}$$

are of order $\hbar$ [9]. If we write $W_1 = \int(b_0^\hbar \phi^2 + c_0^\hbar \phi^4 + \cdots)$, we can iteratively compute corrections to the classical coefficients b_0, c_0, etc. of $W_0 = \int(b_0\phi^2 + c_0\phi^4 + ...)$ by substituting the leading order solutions into $\Delta_s W_0$ and solving the $O(\hbar)$ equation for the corrections $b_0^\hbar$, $c_0^\hbar$,... . These will be given as functions of λ. But from what we have said above, W_λ as a solution of equation (4.31) approximates the exact vacuum state's logarithm. This approximation is refined when more terms of the Laplacian expansion are included, such that λ can be taken to greater value. Of course an alternating series, such as the expansion of the Laplacian acting on W, which has been truncated at a certain order, will have the same behaviour as the full series up to a certain value of λ. Beyond this value the last term will dominate causing the truncated series to jump to infinity. A clearer understanding of this is achieved by plotting the alternating series and reading from it the value it tends to, which should be a finite constant.

From the Maple programs of the previous section we can construct the classical equations, solve them and then substitute the values in the Laplacian part of the equation. Let us see how the different re-summation techniques influence the outcome of the Laplacian.

In Chapter 5 we will calculate the Feynman diagrams corresponding to the first order quantum corrections of the constants b's, c's and so on. They can give us the exact values of the $\hbar$ corrections. For those few cases we can compare the results with what we get by applying the Laplacian on the truncated classical W_0.

Some terms of the semiclassical equation are

$$\int dx \left(\left(2\mathcal{E}^\hbar(\lambda) + 2\mathcal{E}^\hbar + \frac{\sqrt{\lambda}}{\sqrt{\pi}^3} \left(4b_0 + \sqrt{\pi} \sum_{n=1}^{\infty} \rho(n)\ b_n\ \ \lambda^n \right) \right) + \right.$$

$$\varphi^2 \left(-\delta\overline{M}^2(\lambda)\ + \frac{4}{\sqrt{\pi}} b_0 b_0^\hbar + \frac{\sqrt{\lambda}}{\pi} \left(12c_0 + \sum_{n=1}^{\infty} \rho'(n)\ c_n\ \lambda^n \right) \right) +$$

$$\varphi^4 \left(-\frac{2}{4!\sqrt{\pi}} g + \frac{8}{\sqrt{\pi}} (b_0 c_0^\hbar + b_0^\hbar c_0) + \frac{\sqrt{\lambda}}{\pi} \left(30 f_0 + \sum_{n=1}^{\infty} \rho'(n)\ f_n\ \lambda^n \right) \right) +$$

$$\varphi'^{\,2} \left(-\frac{2}{\sqrt{\pi}} + \frac{8}{\sqrt{\pi}} (b_0 b_1^\hbar + b_0^\hbar b_1) + \frac{\sqrt{\lambda}}{\pi} \left(2c_1 + 2d_1\lambda + \frac{4c_2\lambda}{3} + \cdots \right) \right) +$$

$$\left. \varphi^2 \varphi'^2 \left(\frac{16}{\sqrt{\pi}} (b_0 c_1^\hbar + b_0^\hbar c_1) + \frac{48}{\sqrt{\pi}} (b_1 c_0^\hbar + b_1^\hbar c_0) + \frac{\sqrt{\lambda}}{\pi} 12 f_1 + \cdots \right) + \right.$$

[9] See Chapter 3.

$$\varphi''^2\left(\frac{16(b_0b_2^\hbar+b_0^\hbar b_2)+16b_1^\hbar b_1}{3\sqrt{\pi}}+\frac{\sqrt{\lambda}}{\pi}\left(c_2+\frac{d_1\lambda}{9}-\frac{e_2\lambda}{3}+\cdots\right)\right)+$$
$$\cdots+$$
$$(\varphi^{(n)})^2\left(\sum_{m\geq 3}\frac{2(b_m b_{n-m}^\hbar+b_m^\hbar b_{n-m})}{\Gamma(n+1/2)}+\frac{\sqrt{\lambda}}{\pi}\left(\frac{2c_n}{n!}+\frac{2(d_n-e_n)\lambda^{\frac{3}{2}}}{3(n+1)!}+\cdots\right)\right)+$$
$$\cdots+$$
$$\varphi^{2k}\left(\frac{2}{\sqrt{\pi}}\sum_{n=0}^{k-1}(a_0^n a^\hbar{}_0^{k-1-n}+a^\hbar{}_0^n a_0^{k-1-n})(k-n)(n+1)+\right.$$
$$\left.\frac{\sqrt{\lambda}}{\pi}\left(2(k+1)(2k+1)a_0^k+\sum_{n=1}\rho'(n)\ a_n^k\ \lambda^n\right)\right)+$$
$$\cdots=\mathbf{0}. \tag{4.52}$$

where $k\geq 3$, $\delta\overline{M}^2(\lambda)=\sqrt{\lambda}/\sqrt{\pi}R(s)1/\sqrt{s}\delta M^2(s)$, $\mathcal{E}^\hbar(\lambda)=R(s)\mathcal{E}^\hbar(s)$ and the other symbols are the same as in equation (4.31). For simplicity we will refer to the Laplacian term with b_n's, as the b-series, with c_n's, as the c-series and with f_n's, as the f-series, which also apply to the truncated parts of them, up to a certain order. Since we know the value of the classical coefficients from the previous section, we can calculate one by one their first order corrections starting from the top of equation (4.52). The first equation

$$\left(2\mathcal{E}^\hbar(\lambda)+2\mathcal{E}^\hbar+\frac{\sqrt{\lambda}}{\sqrt{\pi}^3}\left(4b_0+\sqrt{\pi}\sum_{n=1}^{\infty}\rho(n)\ b_n\ \ \lambda^n\right)\right)=0$$

gives, after substituting[10] $b_n=-1/(2M_r^{2n-1})\binom{1/2}{n}$ and the expansion of $\mathcal{E}^\hbar(\lambda)=1/2R(s)$ $\int_{p^2<1/s}dp/(2\pi)\sqrt{p^2+M_r^2}$ given by

$$\mathcal{E}^\hbar(\lambda)=\frac{1}{\sqrt{\pi}^3}\sum_{m=0}^{\infty}P(m,M_r)\lambda^{m+1/2}, \tag{4.53}$$

where

$$P(m,M_r)\equiv\frac{(-1)^{m+1}}{(2m+1)^2(2m-1)\Gamma(m+1)M_r^{2m-1}},$$

that $\mathcal{E}^\hbar$ is equal to zero. This holds for any power of λ as the b-series cancels order by order in λ the $\mathcal{E}$-series, reflecting the normal ordering condition of the Hamiltonian. This demonstrates a way to find the renormalisation subtractions; something also true for the mass counter-term as shown below.

In graph[11] (4.7) we have taken the expansion (4.53) with $m=11$ and 21. We see that by taking more terms, the alternating series describes the exact function for larger values

[10] See Appendix D.

[11] In the following diagrams we set $M_r=g=1$ for simplicity.

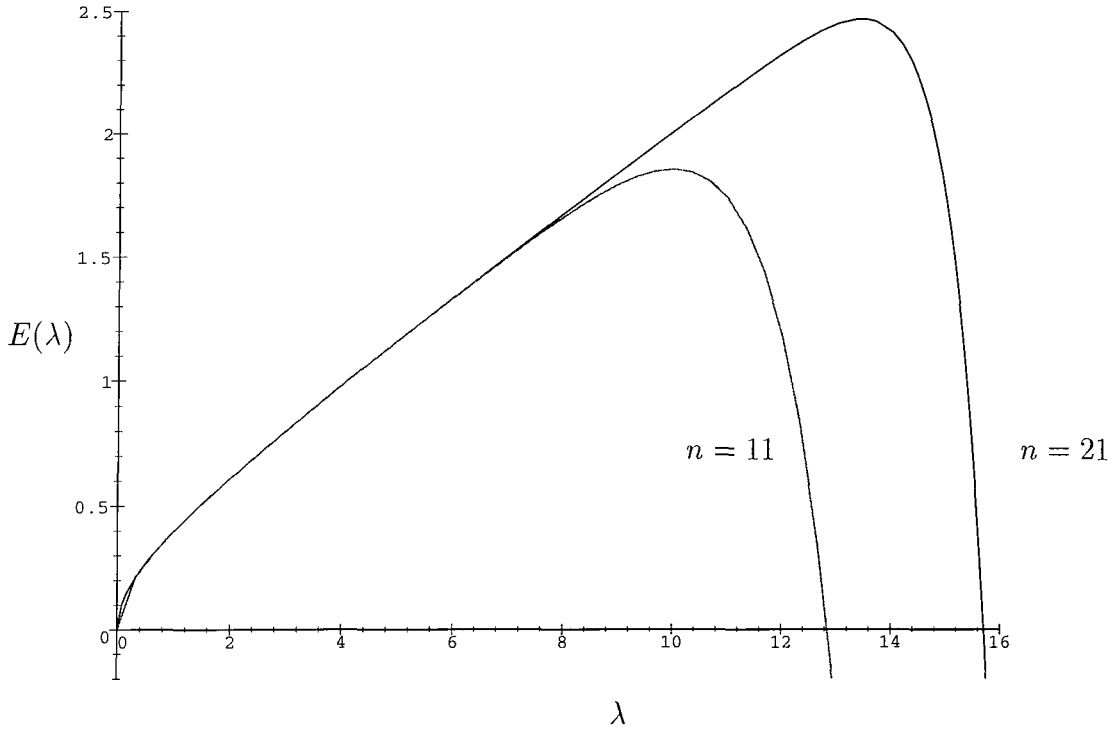

Figure 4.7: The Energy graphs, for 11 and 21 terms.

of λ. We have seen at the beginning of this chapter that the energy is proportional to the inverse of s. In the diagram we see that the energy tends to be proportional to the regulator λ which agrees with the previous result[12] for $\lambda = 1/s$.

The second equation reads

$$-\delta\overline{M}^2(\lambda) + \frac{4}{\sqrt{\pi}} b_0 b_0^{\hbar} + \frac{\sqrt{\lambda}}{\pi}\left(12c_0 + \sum_{n=1}^{\infty} \rho'(n)\ c_n\ \lambda^n\right) = 0 \tag{4.54}$$

The expansion of $\delta\overline{M}^2(\lambda)$, as given in (4.50), is

$$\delta\overline{M}^2(\lambda) = -\frac{g}{4\pi}\left(\frac{\sqrt{\lambda}}{M_r} + \sum_{m=0}^{\infty} H(m, M_r)\lambda^{m+3/2}\right)$$

where

$$H(m, M_r) = \frac{\Gamma(2m+2)(-1)^{m+1}}{2^{2m+1}(2m+3)\Gamma(m+1)\Gamma(m+2)^2 M_r^{2m+2}}$$

while its diagram is given in Figure (4.8)

The c series diagram is given in (4.9) and their subtraction in (4.10), which has a finite value for λ in an appropriate region. Diagrams (4.8) and (4.9) have been taken with 22

[12]See Section (4.2).

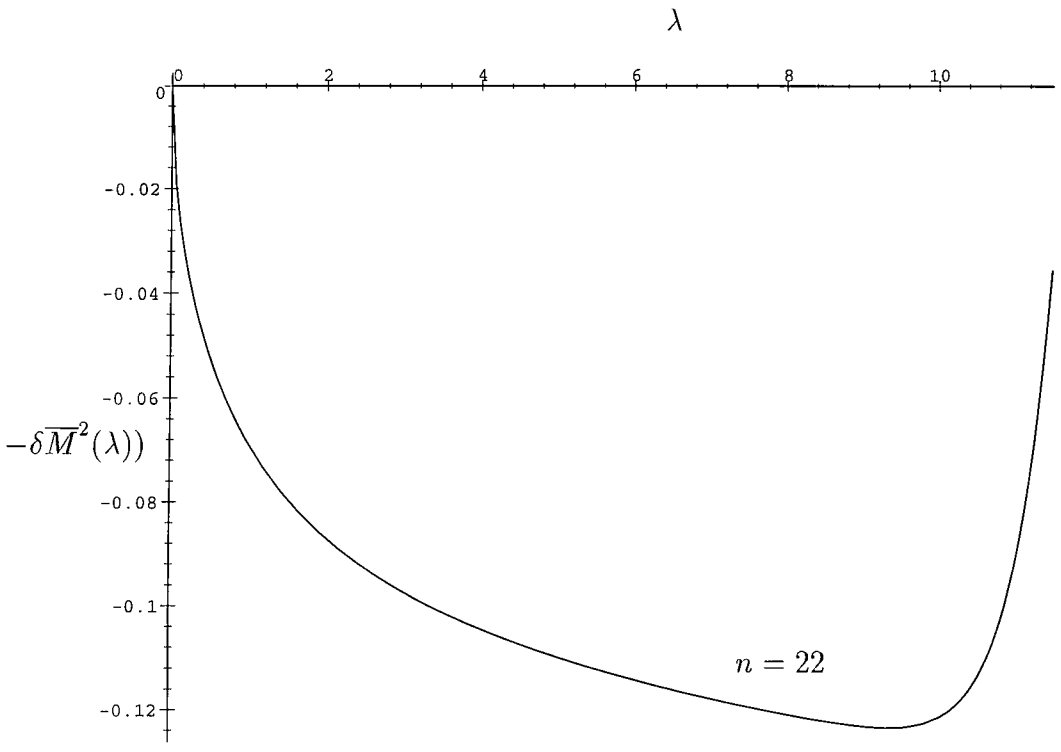

Figure 4.8: The Mass graph for 22 terms. It is logarithmically divergent.

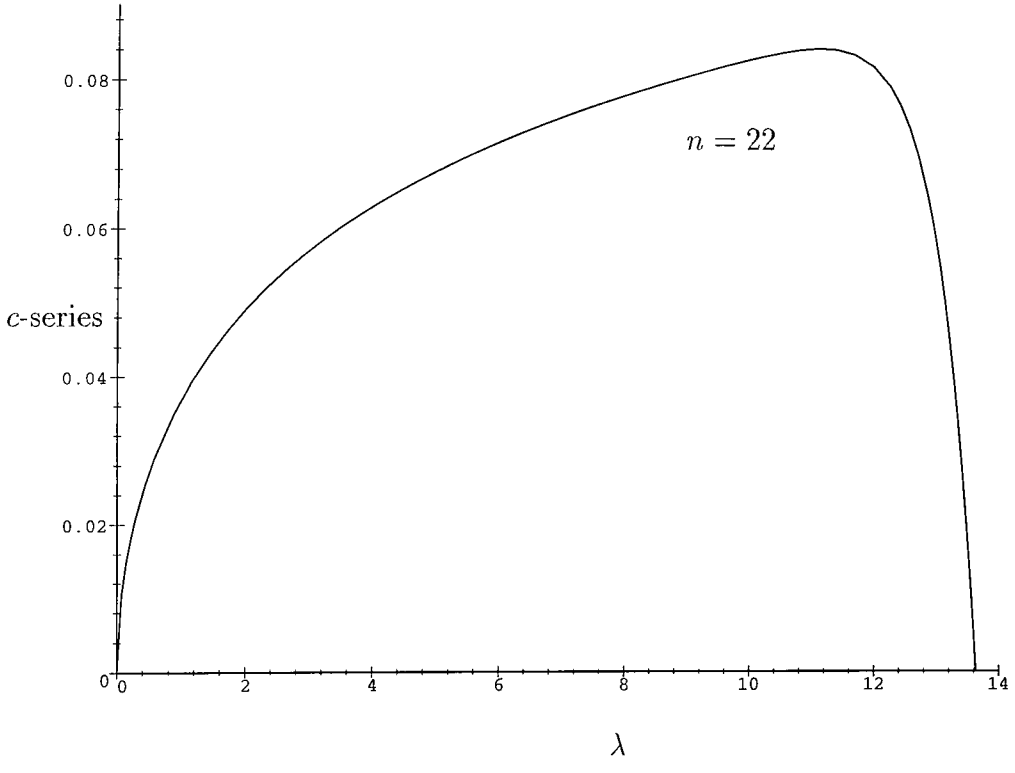

Figure 4.9: The c series graph for 22 terms.

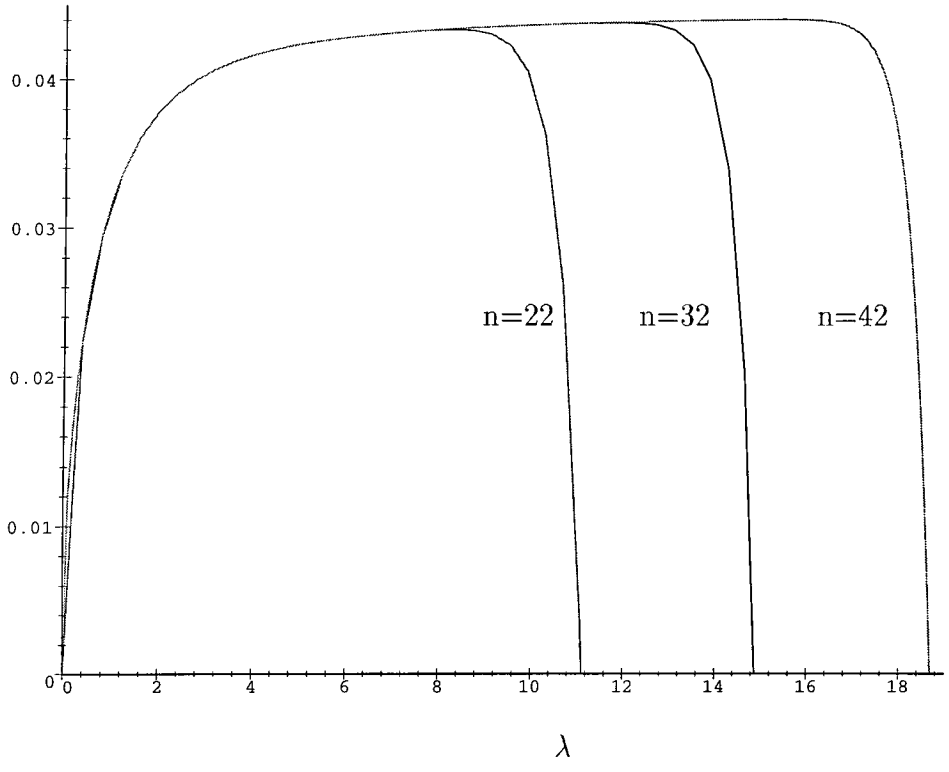

Figure 4.10: The c minus the mass series graphs for 22, 32 and 42 terms.

terms of the expansion while (4.10) has been taken with 22, 32 and 42 terms. From these we can read within a margin of error the value of the Laplacian acting on the c series and calculate the first order correction, $b_0^{\hbar}$, from equation (4.54).

To deduce this equation, as well as the other equations in (4.52), we have used the scaled field φ_s, which contributes additional $\sqrt{s}$ factors. With the application of the operator $R(s)$ the equations change from the form with the unscaled field, but they lead to an equivalent equation which results in the same values for the quantum corrections. In the following we give values for the Laplacian term with the s terms from the scaled field, φ_s, included, as well as the values from the Laplacian acting on the ansatz, W, constructed out of the unscaled field, φ [13]. The second kind of series are of the form (4.18) and the techniques of re-summation mentioned in section (4.3) are applied here. The error depends on where we truncate the series. The following diagrams show how this value changes as we include more terms, as well as using different re-summation procedures.

For the f-series we have the diagram (4.14), constructed for 12, 22, 32 and 42 terms,

[13]The two values are connected as can been checked from the small s expansion (4.17)

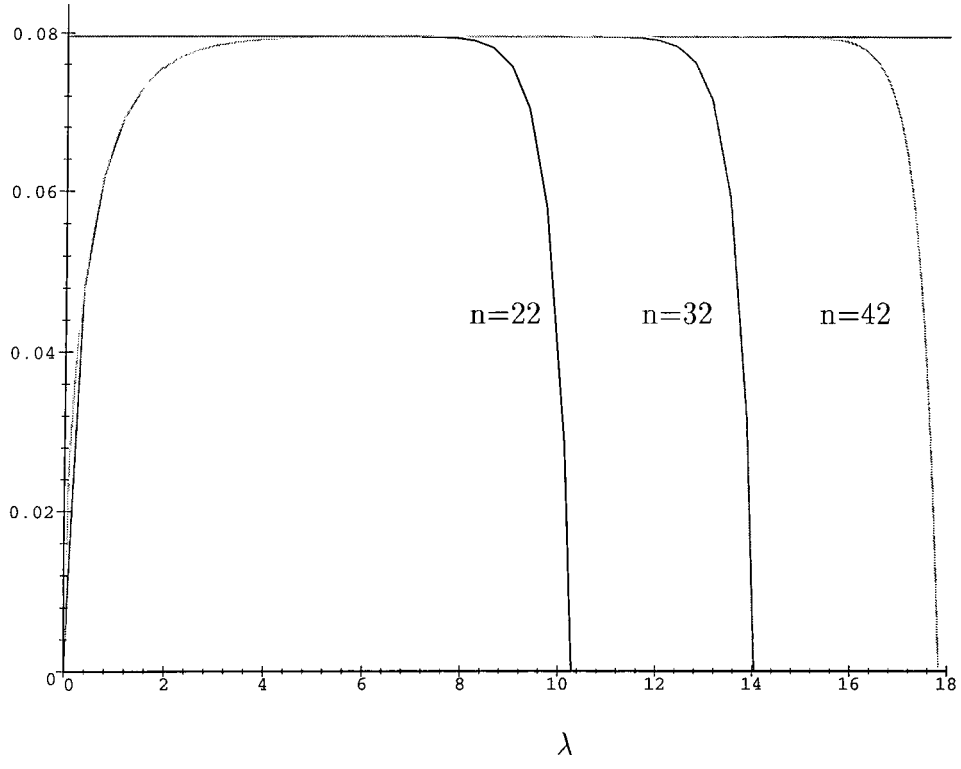

Figure 4.11: The c minus the mass series graphs with $R(s) + 2\lambda R(s)s$ re-summation operator for 22, 32 and 42 terms.

which eventually gives the value 0.001793.

We can plot the result and read off the value to which the series tends. From the diagrams (4.10) and (4.14), we can get the suggested values for the "plateau" in each one. We will distinguish the cases where there are even numbers of terms in the series, where the desired value is the maximum of the diagram, and an odd number of terms, where the value can be read of as its turning point (see Figure (4.13)). As we can see from the following tables the percentage error for the odd number of terms of the series approximating the actual value is smaller than the corresponding one for neighbouring even number of terms. This is because the series approaches the *actual* value from "below" and the turning point of a graph, like in (4.13), has a larger value than the maximum of the neighboring series with even number of terms.

For various number of terms we obtain the following Table for the Laplacian term as it is

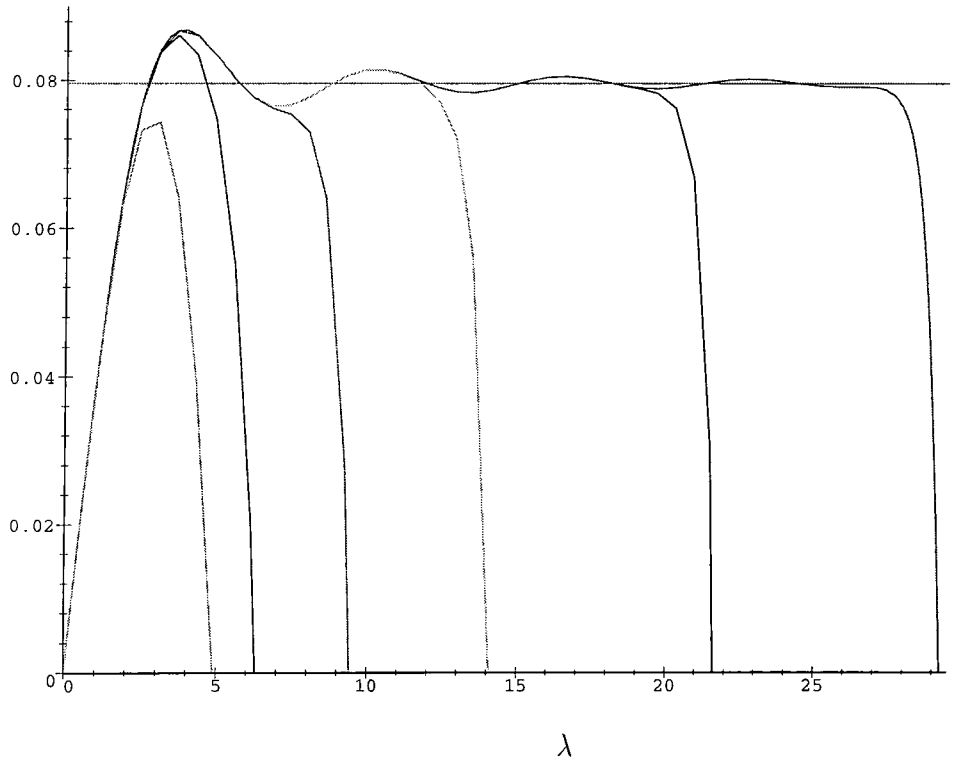

Figure 4.12: The c minus the mass series graphs with the $s \to t^2$ re-summation technique for 2, 4, 8, 14, 24 and 34 terms.

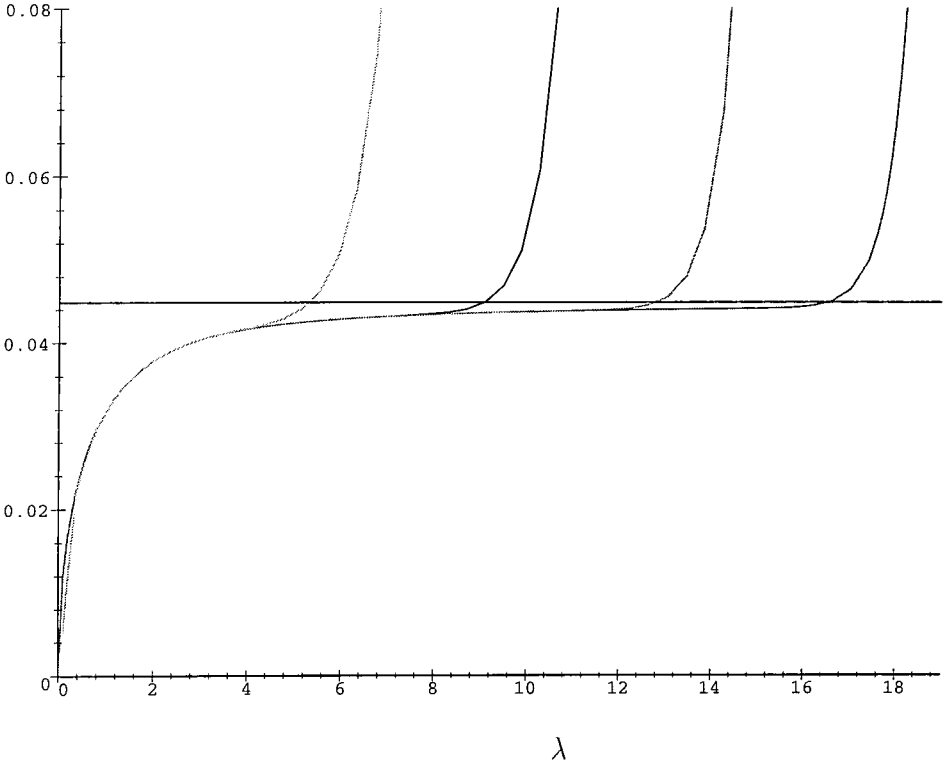

Figure 4.13: The c minus the mass series graphs for 11, 21, 31 and 41 terms.

given in relation (4.54).

Table 1

#	Even	%	#	Odd	%
2	0.030629	31.7782	3	0.037381	16.7410
4	0.036346	19.0455	5	0.038613	13.9962
6	0.038993	13.1500	7	0.039999	10.9093
8	0.040483	9.8302	9	0.040984	8.7143
10	0.041413	7.7596	11	0.041677	7.1722
12	0.042035	6.3737	13	0.042177	6.0588
14	0.042475	5.3939	15	0.042549	5.2284
16	0.042800	4.6700	17	0.042836	4.5900
18	0.043049	4.1156	19	0.043062	4.0857
20	0.043245	3.6781	21	0.043245	3.6783
22	0.043404	3.3244	23	0.043396	3.3426
24	0.043049	3.0326	25	0.043522	3.0616
26	0.043645	2.7878	27	0.043629	2.8230
28	0.043739	2.5794	29	0.043721	2.6180
30	0.043819	2.3999	31	0.043801	2.4401
32	0.043889	2.2438	33	0.043871	2.2844

The first column denotes the number of terms of the series, the second the actual value of the Laplacian acting on the c terms of W_0 and the third is the percentage error of the truncated series with respect to the correct value, $1/(4\pi^{3/2}) = 0.044897$, read off from Chapter 5. The percentages have been given by the computer allowing more digits to be taken into account. From equation (4.54) and the above value which is inserted with a negative sign we get the first order correction $b_0^{\hbar} = -1/8\pi$. For the linear combination $R(s) + 2\lambda R(s)s$ of the re-summation operator acting on the c series we get Table 2. This combination has been taken to act on the Laplacian term constructed from the unscaled field φ. The scaled field, φ_s, provides $\sqrt{s}$ terms combined in the expression, on which $R(s)$ acts. From this procedure we get Tables 1, 2 and 3. Tables 2, 4 and 6 are produced from the unscaled field expressions. In Table 2 the value we want to approach is $1/(4\pi) = 0.079577$; it is connected with the one from the previous Table with the relation $R(s)\sqrt{s}A = A/\sqrt{\pi}$. Here A represents the value we get from the combination $R(s)+2\lambda R(s)s$ acting on an equivalent equation to (4.54) with the unscaled field, φ. This

differs from the scaled field case, which contributes the factor $\sqrt{s}$ as

$$\int dx\varphi_s^2(x) = \int dx\varphi^2(x/\sqrt{s}) = \sqrt{s}\int dx\varphi^2(x)$$

Equivalent relations can be deduced for φ_s^4, $\varphi_s'^2$ etc.

Table 2

#	Even	%	#	Odd	%
2	0.059862	24.7747	3	0.073508	7.6275
4	0.071004	10.7732	5	0.075483	5.1450
6	0.075459	5.1757	7	0.077330	2.8245
8	0.077509	2.5994	9	0.078371	1.5159
10	0.078511	1.3396	11	0.078930	0.8131
12	0.079019	0.7021	13	0.079230	0.4378
14	0.079281	0.3725	15	0.079389	0.2369
16	0.079418	0.1995	17	0.079475	0.1287
18	0.079492	0.1076	19	0.079522	0.0702
20	0.079531	0.0583	21	0.079547	0.0384
22	0.079552	0.0318	23	0.079561	0.0211
24	0.079564	0.0174	25	0.079568	0.0116
26	0.079570	0.0095	27	0.079572	0.0064
28	0.079573	0.0053	29	0.079575	0.0035
30	0.079575	0.0029	31	0.079576	0.0020
32	0.079576	0.0016	33	0.079566	0.0011

Table 3 gives a similar result for the f series as we obtain from (4.52) with the value of convergence 0.001793. Taking account this value, equation (4.52) gives the quantum correction $c_0^{\hbar} = -0.020868$.

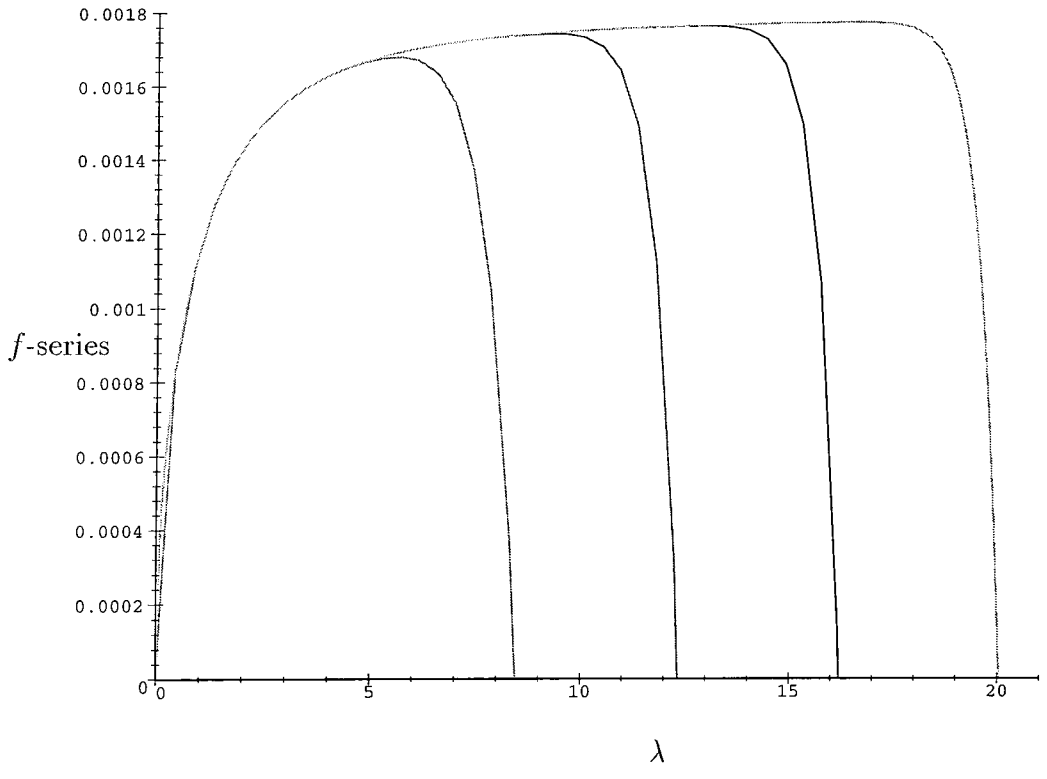

Figure 4.14: The f series graphs for 12, 22, 32 and 42 terms.

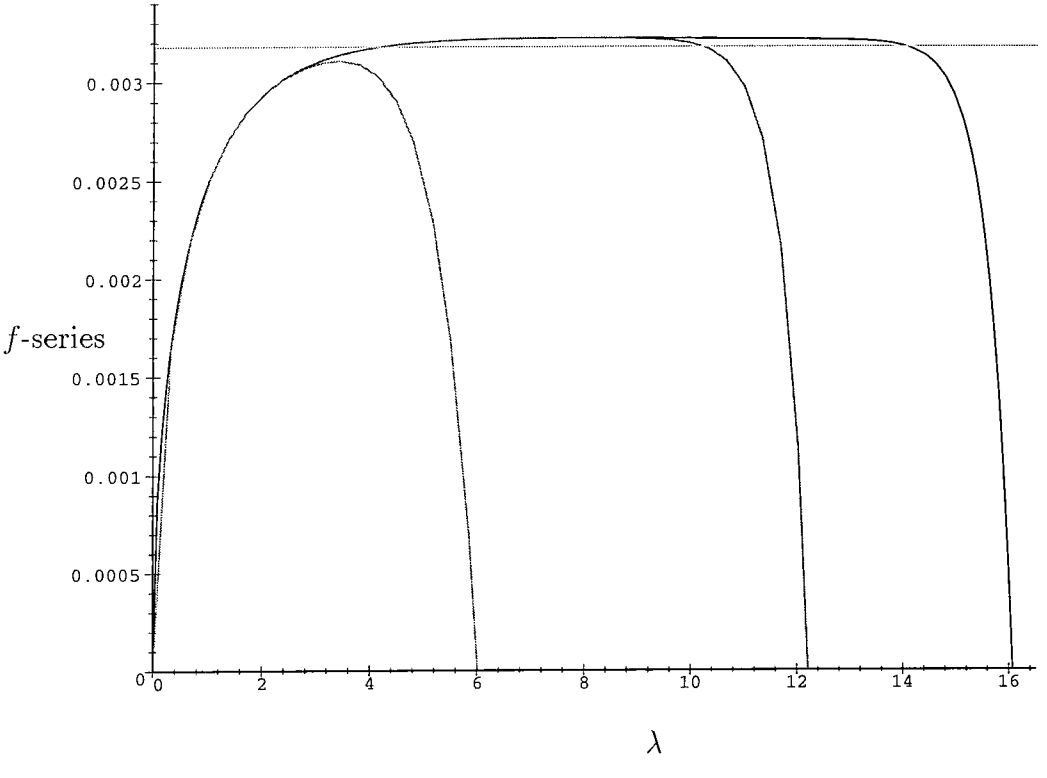

Figure 4.15: The f series graphs for 8, 24 and 34 terms, by using the re-summation operator $R(s) + 2\lambda R(s)s$.

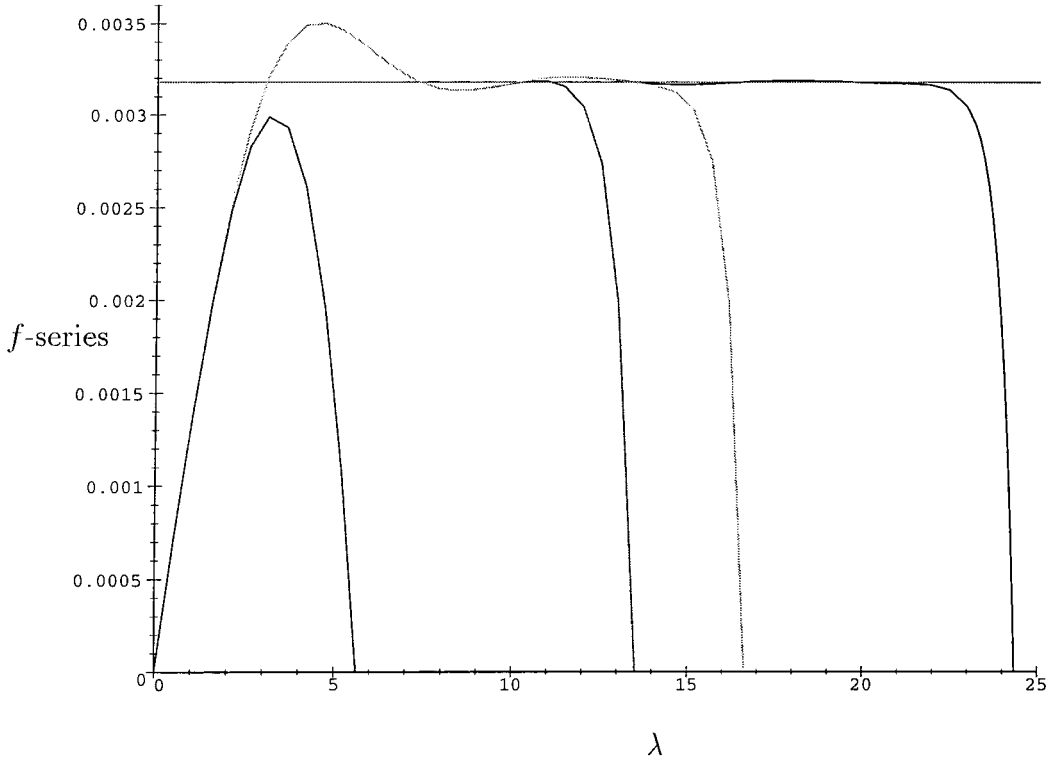

Figure 4.16: The f series graphs for 2, 12, 16 and 26 terms with the $s \to t^2$ re-summation technique.

Table 3

#	Even	%	#	Odd	%
2	0.001224	31.7718	3	0.001488	17.0305
4	0.001445	19.4046	5	0.001533	14.5415
6	0.001551	13.5380	7	0.001589	11.3784
8	0.001612	10.091	9	0.001632	9.0054
10	0.001653	7.8492	11	0.001663	7.2688
12	0.001680	6.2952	13	0.001686	5.9778
14	0.001701	5.1692	15	0.001704	4.9975
16	0.001716	4.3253	17	0.001717	4.2376
18	0.001727	3.6754	19	0.001728	3.6375
20	0.001737	3.1637	21	0.001737	3.1558
22	0.001737	2.7533	23	0.001744	2.7635
24	0.001750	2.4189	25	0.001750	2.4398
26	0.001755	2.1426	27	0.001754	2.1697
28	0.001759	1.9116	29	0.001758	1.9420
30	0.001763	1.7164	31	0.001762	1.7483
32	0.001765	1.5500	33	0.001765	1.5821

while the linear combination $R(s) + 2\lambda R(s)s$ gives

Table 4

#	Even	%	#	Odd	%
2	0.002391	24.7676	3	0.002930	7.8330
4	0.002830	10.9659	5	0.003009	5.3387
6	0.003015	5.1444	7	0.003094	2.6743
8	0.003108	2.2148	9	0.003147	0.9856
10	0.003159	0.6207	11	0.003180	−0.0385
12	0.003187	−0.2824	13	0.003199	−0.6547
14	0.003204	−0.8032	15	0.003211	−1.0217
16	0.003214	−1.1031	17	0.003218	−1.2349
18	0.003219	−1.2715	19	0.003221	−1.3519
20	0.003222	−1.3606	21	0.003223	−1.4083
22	0.003223	−1.4020	23	0.003224	−1.4264
24	0.003224	−1.4166	25	0.003224	−1.4203
26	0.003224	−1.4166	27	0.003223	−1.3994
28	0.003224	−1.4166	29	0.003222	−1.3697
30	0.003224	−1.4166	31	0.003222	−1.3357
32	0.003224	−1.4166	33	0.003221	−1.3007

Table 4 gives negative valued percentages corresponding to Figure (4.15). This is due to the fact that the re-summation operator $R(s) + 2\lambda R(s)s$ eliminates the next significant term to the actual value in the small s expansion, but changes the coefficients of the following terms. This causes the series to jump above the actual value.

In Figures (4.12), (4.16), (4.18) we adopted the second re-summation technique studied in section (4.3), based on the substitution $s \to t^2$. In these figures the oscillation character we deduced theoretically is revealed (see relation (4.22)).

For the next series, which is the Laplacian-part of the coefficient of the φ'^2 term, the con-

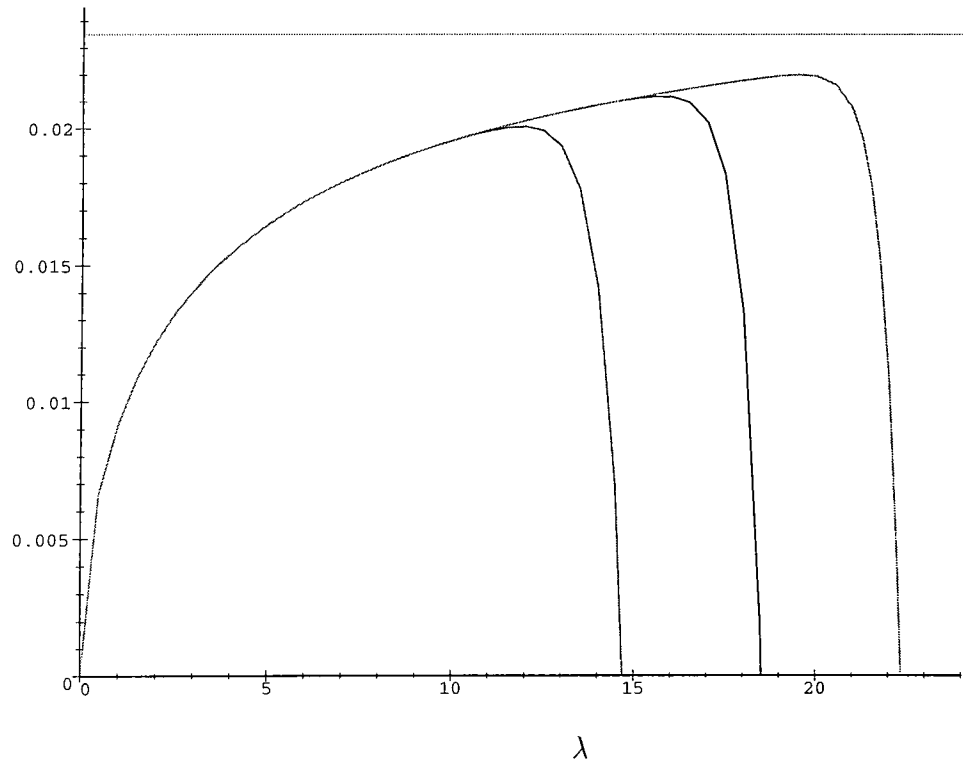

Figure 4.17: The ϕ'^2 series graphs for 22, 32 and 42 terms.

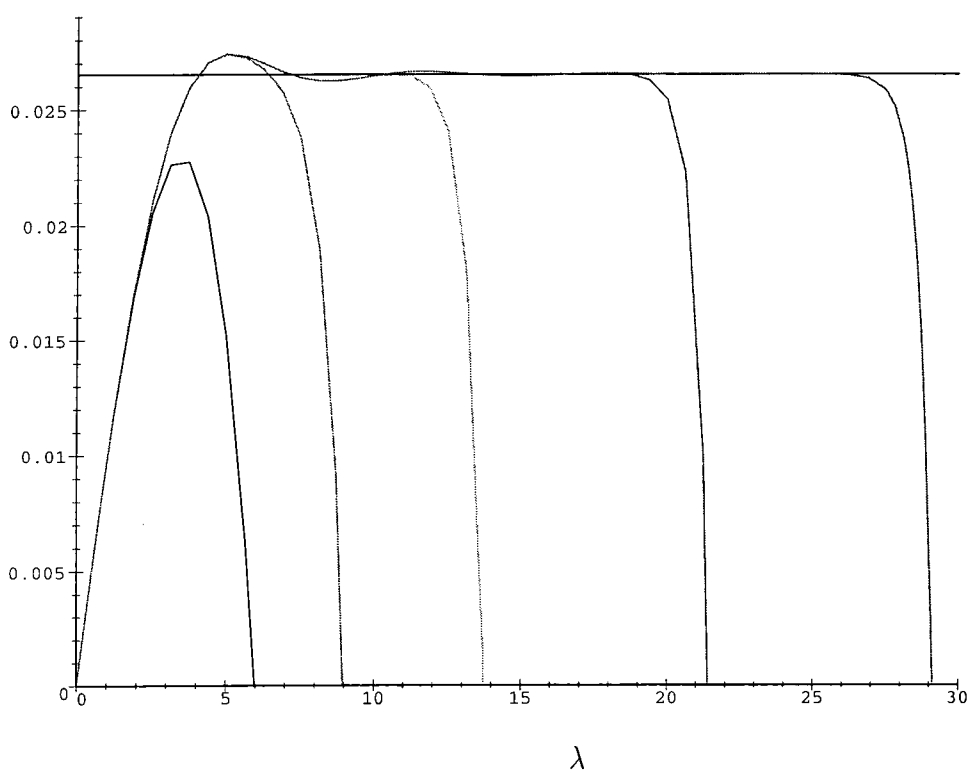

Figure 4.18: The ϕ'^2 series graphs for $s \to t^2$ resummation technique for 2, 6, 12, 22 and 32 terms.

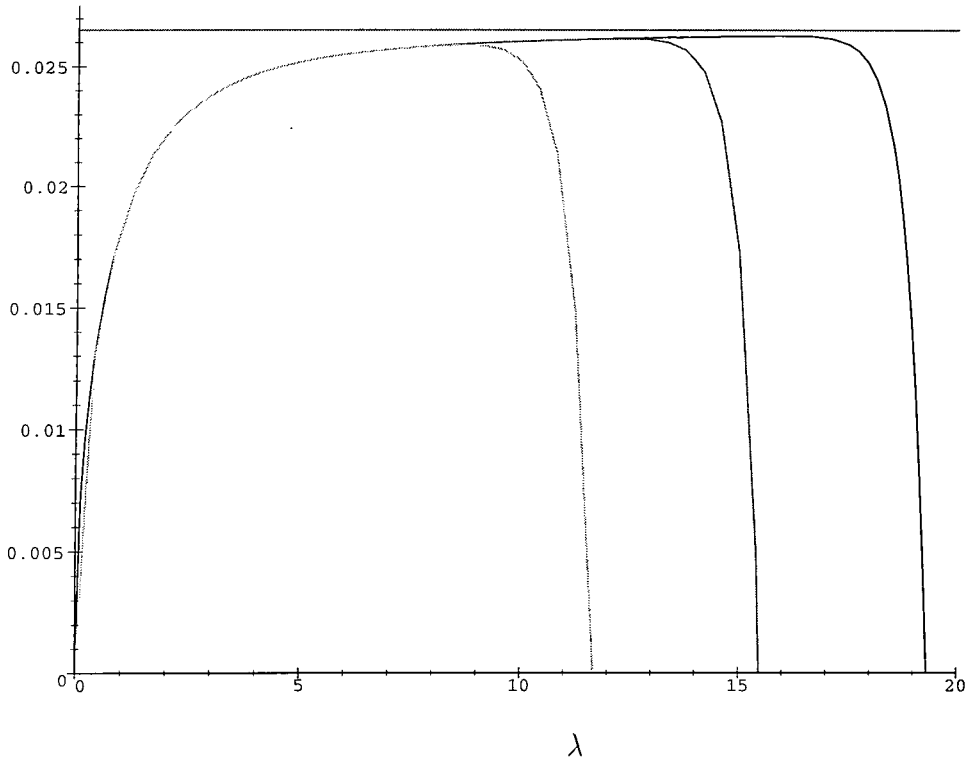

Figure 4.19: The ϕ'^2 series graphs with the $R(s) + 2\lambda R(s)s$ resummation technique for 12, 22 and 32 terms.

vergence value is 0.014956. This gives from (4.52) the quantum correction $b_1^{\hbar} = -0.473478$.

Table 5

#	Even	%	#	Odd	%
2	0.013263	43.5810	3	0.015462	34.2279
4	0.015168	35.4759	5	0.015655	33.4070
6	0.016268	30.7977	7	0.016344	30.4723
8	0.017074	27.3697	9	0.017002	27.6770
10	0.017717	24.6321	11	0.017582	25.2093
12	0.018254	22.3479	13	0.018090	23.0459
14	0.018715	20.3895	15	0.018539	21.1385
16	0.019117	18.6785	17	0.018937	19.4431
18	0.019473	17.1625	19	0.019294	17.9238
20	0.019793	15.8044	21	0.019617	16.5523
22	0.020081	14.5767	23	0.019910	15.3057
24	0.020344	13.4585	25	0.020178	14.1660
26	0.020585	12.4337	27	0.020424	13.1184
28	0.020807	11.4893	29	0.020651	12.1510
30	0.021013	10.6148	31	0.020862	11.2538
32	0.021204	9.8016	33	0.021059	10.4186

while the linear combination $R(s) + \lambda R(s)s$ gives Table 6.

Table 6

#	Even	%	#	Odd	%
2	0.018329	30.9011	3	0.022349	15.7477
4	0.021707	18.1663	5	0.023049	13.1059
6	0.023274	12.2574	7	0.023873	10.0026
8	0.024165	8.9013	9	0.024466	7.7656
10	0.024724	6,7934	11	0.024885	6.1839
12	0.025099	5.3795	13	0.025188	5.0424
14	0.025363	4.3839	15	0.025413	4.1967
16	0.025556	3.6552	17	0.025583	3.5542
18	0.025702	3.1049	19	0.025715	3.0550
20	0.025815	2.6783	21	0.025820	2.6591
22	0.025905	2.3401	23	0.025905	2.3397
24	0.025978	2.0670	25	0.025975	2.0779
26	0.026037	1.8427	27	0.026032	1.8605
28	0.026087	1.6560	29	0.026081	1.6778
30	0.026128	1.4986	31	0.026122	1.5225
32	0.026164	1.3646	33	0.026157	1.3893

In the same way we can find the first order correction for all the coefficients. We can see from the diagrams that the classical value of the coefficients, as well as their first order quantum correction decreases as the number of derivatives or the number of ϕ's increases. Practically, we need only to calculate the first terms of the expansion of the vacuum functional's logarithm up to the point of sufficient approximation for our purpose.

4.8 Conclusions

In this Chapter we studied the different ways in which we can extract the small s behaviour from the large one, by using different techniques in order to minimise the approximation error.

In general we wish to extract the value that the series tends to, with the least number of terms. Here we were able to calculate as many classical values for the c's and find the first

order correction for b_0 with as great a degree of accuracy as we determine. However, this will not be generally possible. When we want to consider the equations as a set, without considering any sort of perturbation expansion in the Planck constant $\hbar$ or the coupling g, we need the fewest possible terms and combined equations to solve, since its complexity and difficulty increases rapidly when more terms are incorporated. By studying the tables above we conclude that the operator $R(s) + 2\lambda R(s)s$ gives better values than the simple application of $R(s)$ on the scaled fields. However, if we have many terms in the series we can use the second re-summation technique, which apart from the strong oscillation for small λ, gives the correct values to a higher degree of accuracy (see Figures (4.17), (4.18)).

By using these results we can choose the appropriate resummation method when we want to solve the Schrödinger equation for various models and in various ways (semiclassically or non-perturbatively).

Chapter 5

Feynman Diagrams

5.1 Introduction

Field theory has been studied through the path integral formalism, with which, one can construct the generating functional, the propagators, and then be able to treat interactions with the help of perturbation. In this way you can calculate the desired cross sections without the need of solving a dynamical equation. A great simplification to these steps has been achieved by the Feynman diagrams and rules. These provide a pictorial representation of the physical process with a symbolic calculational meaning.

As we have calculated the first order corrections of the ϕ^4 theory with the use of the Schrödinger representation, we proceed to verify these results with equivalent calculations of Feynman Diagrams.

5.2 Feynman Rules on the Plane

Let us take the case of a free scalar field $\phi(x)$ on the whole 2-dimensional plane x, with its source $J(x)$. The vacuum to vacuum transition amplitude of the field ϕ in the presence of the source J is

$$Z_0[J] = \int \mathcal{D}\phi \exp\left\{\int d^2x \left[-\mathcal{L}_E(\phi) + J(x)\phi(x)\right]\right\} \propto < 0, \infty | 0, -\infty >^J$$

where $\mathcal{L}_E$ is the Euclidean Klein-Gordon Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) + \frac{m^2}{2}\phi^2$$

After using functional integral techniques we find

$$Z_0[J] = N \exp\left[-\frac{1}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right] \tag{5.1}$$

where N is a constant which will be absorbed in the normalisation, $\Delta_F(x-y)$ satisfies

$$(\Box + m^2)\Delta_F(x) = -\delta^2(x)$$

which is called the Feynman propagator. Its analytic form is

$$\Delta_F(x) = \frac{1}{(2\pi)^2}\int d^2k \frac{e^{-ikx}}{k^2+m^2}$$

written in Euclidean space, where $k^2 = k_1^2 + k_2^2$. We can expand expression (5.1) as

$$\begin{aligned} Z_0[J] = N\Big\{1 &+ \frac{-1}{2}\int J(x)\Delta_F(x-y)J(y)dxdy \\ &+ \frac{1}{2!}\left(\frac{-1}{2}\right)^2\left[\int J(x)\Delta_F(x-y)J(y)dxdy\right]^2 \\ &+ \frac{1}{3!}\left(\frac{-1}{2}\right)^3\left[\int J(x)\Delta_F(x-y)J(y)dxdy\right]^3 + ...\Big\} \end{aligned} \tag{5.2}$$

This expansion has a suggestive pictorial interpretation. The second term in the curly brackets can be illustrated by the diagram of a particle generated by the source J at the point x, then propagated from x to y and then absorbed by another source at the point y. The other terms in the expansion, which are the n-th powers of the same integral, can be represented by n particles propagating between these sources as the one described. The factors in front of the integrals should be inserted as symmetry factors of the diagrams. From (5.2) a further important characteristic arises. By asking for a normalised Z as $Z[J=0] = 1$, $Z_0[J]$ becomes

$$Z_0[J] = \exp\left[-\frac{1}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right]$$

and can be interpreted as the generating functional of the n point Green functions

$$\tau(x_1,...,x_n) = \left.\frac{\delta^n Z_0[J]}{\delta J(x_1)...\delta J(x_n)}\right|_{J=0}.$$

For example we get

$$\tau(x) = 0$$

$$\tau(x,y) = -\Delta_F(x-y).$$

$$\tau(x,y,z) = 0$$

$$\tau(x_1,x_2,x_3,x_4) = [\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)$$

$$+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)]$$

By construction, Z is the vacuum-to-vacuum transition amplitude in the presence of a source J. That is

$$\langle 0|T(\phi(x_1)..\phi(x_n))|0\rangle = \left.\frac{\delta^n Z_0[J]}{\delta J(x_1)...\delta J(x_n)}\right|_{J=0}$$

which suggests

$$\tau(x_1,...,x_n) = \langle 0|T(\phi(x_1)...\phi(x_n))|0\rangle$$

Interactions can be included in a similar way. For example we can use the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4 = \mathcal{L}_0+\mathcal{L}_{int} \tag{5.3}$$

where the last term is the interaction. The generating functional for a general interacting Lagrangian is

$$Z[J] = N\int \mathcal{D}\phi \exp\left(-\int(\mathcal{L}_0+\mathcal{L}_{int} - J\phi)dx\right)$$

where N is a factor which normalises $Z[0]$ to be one. After some algebra it can take the more tractable form

$$Z[J] = N\exp\left[-\int \mathcal{L}_{int}\left(\frac{\delta}{\delta J(z)}\right)dz\right]\exp\left[-\frac{1}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right]$$

which by expanding the exponentials can give perturbations in g and in $\hbar$. For example $\mathcal{L}_{int}$ is proportional to g, so by expanding the first exponential we get a perturbation series with respect to the coupling. The expansion with respect to $\hbar$ will be revealed as we act with the interaction term upon the second exponential, and is equivalent to a loop expansion of the constructed diagrams [21]. For example the 2-point function for (5.3) is given by

$$\tau(x_1,x_2) = \left.\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} =$$

$$-\Delta_F(x_1-x_2) - \frac{g}{2}\Delta_F(0)\int dz\Delta_F(z-x_1)\Delta_F(z-x_2) + O(g^2)$$

while the 4-point function is

$$\tau(x_1,x_2,x_3,x_4) = \left.\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} =$$

$$3\Delta_F(x_1-x_2)\Delta_F(x_3-x_4) - 3g\Delta_F(0)\int dz\Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(x_3-x_4)$$
$$-g\int dz\Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z)$$

Here the Green's function Δ_F is given again by

$$\Delta_F(x-y) = \frac{1}{(2\pi)^2}\int \frac{e^{ik\cdot(x-y)}}{k^2+m^2}d^2k$$

and it represents the free propagator of the theory.

5.3 Feynman Rules on the Half Plane

In this case the Feynman Diagrams can be separated into two categories. If we call surface propagator the one which has at least one leg on the boundary we can distinguish the Feynmam Diagrams into the free space ones and the remainder, which involves the surface propagator at least once. Thus, the connected graphs are either free-space ones, if they involve free-space propagators only, or surface ones.

Normalising the generating functional of Feynman Diagrams, is equivalent to subtracting all the vacuum diagrams with no legs on the boundary. This is a true statement to all orders in perturbation theory, and is a general property of normalised generating functionals [22]. Hence, we have only to worry about the Feynman Diagrams with legs on the boundary, which we are going to study below.

To construct the generating functional for the half plane it will be helpful to use the vector bra-ket formalism. We will use the $t > 0$ region, which will call Γ with boundary $\partial\Gamma$, defined by $t = 0$. Let us define the state $\langle\varphi|$ so that the field $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$ have the following action on it

$$\langle\varphi|\hat{\phi}(x,0) = \langle\varphi|\varphi(x) \tag{5.4}$$

and

$$\langle\varphi|\hat{\pi}(x,0) = -i\frac{\delta}{\delta\varphi(x)}\langle\varphi|$$

The bra $\langle\varphi|$ which satisfies these relations can be analytically given by

$$\langle\varphi| = \langle D|e^{\int dx\varphi(x)\hat{\pi}(x)}$$

where $\langle D|$ is a state which is annihilated by $\widehat{\phi}$, that is it satisfies the homogeneous Dirichlet boundary condition $\langle D|\widehat{\phi}(x,0)=0$. We will be interested in the inner product

$$\langle\varphi|e^{-HT}|\varphi'\rangle=\langle D|e^{\int\widehat{\pi}\varphi}e^{-HT}e^{-\int\widehat{\pi}\varphi'}|D\rangle \tag{5.5}$$

from which we will get the generating functional. Turning to the path integral picture, relation (5.5) can be written as

$$\int\mathcal{D}\phi\ e^{\int_{t=0}dx\dot{\phi}\varphi-\int_{t=T}dx\dot{\phi}\varphi'-S_E}$$

for which the boundary conditions $\phi(x,0)=\phi(x,T)=0$ hold, as the functional integral has been constructed with respect to the $\langle D|$ states. Taking T to infinity and assuming that φ' vanishes there, we get the generating functional for the boundary diagrams

$$\langle\varphi|0,\infty\rangle=\int\mathcal{D}\phi\ e^{\int dx\dot{\phi}\varphi-S_E}$$

where $|0,\infty\rangle$ stands for the vacuum state at infinity and S_E is the Euclidean action

$$S_E=\int_{t>0}dxdt\left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+\frac{1}{2}m^2\phi^2+\mathcal{L}_{int}\right) \tag{5.6}$$

In (5.6) $\partial_\mu\phi\partial_\mu\phi$ denotes $\partial_t^2\phi+\partial_x^2\phi$ and $\mathcal{L}_{int}$ stands for the interacting term. In an equivalent way to the treatment of the "whole plane" we can find that the generating functional, $Z[J,\varphi]$, for the interacting theory on the half plane will be [13]

$$Z[J,\varphi]=N\exp\left[-\int\mathcal{L}_{int}(\frac{\delta}{\delta J(x,t)})d^2x\right]\times$$

$$\exp\left[\frac{1}{2}\int_\Gamma\int_\Gamma JG_DJ+\frac{1}{2}\int_{\partial\Gamma}\int_{\partial\Gamma}\varphi\partial_tG_D\overleftarrow{\partial}_{t'}\varphi-\int_{\partial\Gamma}\int_\Gamma\varphi\partial_tG_DJ\right] \tag{5.7}$$

where $\overleftarrow{\partial}_{t'}$ means that the time derivative acts on the left, $\partial_tG_D\overleftarrow{\partial}_{t'}$ is equal to $\partial_tG_D(x,x')\overleftarrow{\partial}_{t'}$ for $x(\text{or } x')\in\partial\Gamma$ and $x'(\text{or } x)\to\partial\Gamma$ from Γ. The Dirichlet propagator G_D is given by

$$G_D(x,y)=\Delta_F(x,y)-\Delta_F(\widehat{x},y)$$

where $x=(x^0,x^1)$, $y=(y^0,y^1)$ and $\widehat{x}=(-x^0,x^1)$, and $\Delta_F(x,y)\equiv G(x,y)$ is the Feynman propagator we used in the previous section. Now the logarithm of $Z[J,\varphi]$, is the generating functional of the connected Feynman Diagrams. $Z[J,\varphi]|_{J=0}$, as defined in (5.5), is also, the vacuum functional, which satisfies the Schrodinger equation. In equation (4.31), what we want to calculate, is the coefficients of the expansion of the logarithm of $Z[J,\varphi]|_{J=0}$, in local functionals. They can be determined from the connected diagrams calculated from $W[J,\varphi]\equiv\ln Z[J,\varphi]$. We can choose these diagrams from the general ones generated from

(5.7). Calculating their amplitude without setting $\varphi = 0$ will provide us with terms of the functional $W[0, \varphi]$.

Some of the Feynman diagrams with legs on the boundary are presented in Figures (5.1), (5.2), (5.3).

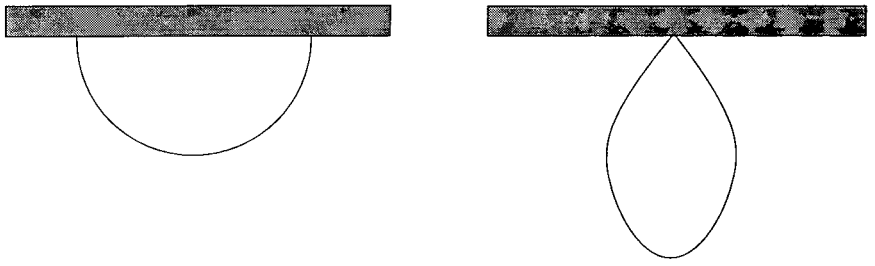

Figure 5.1: Boundary propagators with both legs on the boundary.

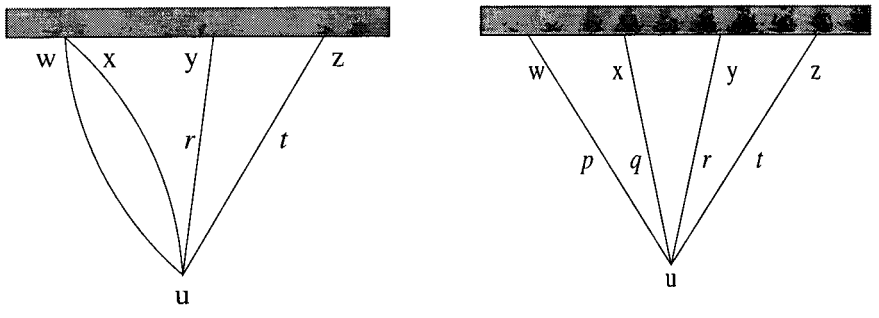

Figure 5.2: Four legged boundary diagrams.

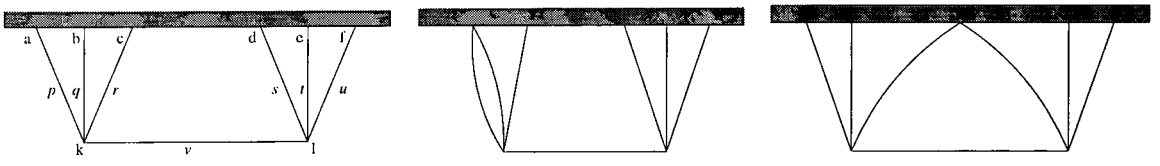

Figure 5.3: Six legged boundary diagrams.

We can calculate the energy density from the diagram (5.1). Its amplitude is

$$\int_{x_2 y_2} \varphi(x_2)\varphi(y_2)\frac{\partial^2 G_D(\hat{x},\hat{y})}{\partial x_1 \partial y_1} \tag{5.8}$$

We can see that the Dirichlet Green function in (5.8) can be written in terms of the usual Green function as it has legs on the boundary. This gives

$$\frac{\partial G_D(\hat{x},\hat{y})}{\partial x_1} = \frac{\partial(G(\hat{x},\hat{y}) - G(\hat{x},\underline{\hat{y}}))}{\partial x_1} = \frac{\partial G(\hat{x},\hat{y})}{\partial x_1} - \frac{\partial G((x_1,x_2),(-y_1,y_2))}{\partial x_1} =$$

$$\frac{\partial G(\hat{x},\hat{y})}{\partial x_1} - \frac{\partial G((-x_1,x_2),(y_1,y_2))}{\partial x_1} =$$

$$\frac{\partial G(\widehat{x},\widehat{y})}{\partial x_1}+\frac{\partial G((-x_1,x_2),(y_1,y_2))}{\partial(-x_1)}= \tag{5.9}$$

$$\frac{\partial G(\widehat{x},\widehat{y})}{\partial x_1}+\frac{\partial G((x_1,x_2),(y_1,y_2))}{\partial x_1}=$$

$$2\frac{\partial G(\widehat{x},\widehat{y})}{\partial x_1}. \tag{5.10}$$

Hence

$$\int_{x_2y_2}\varphi(x_2)\varphi(y_2)\frac{\partial^2 G(\widehat{x},\widehat{y})}{\partial x_1\partial y_1}=\int_{x_2y_2}\frac{d^2p}{(2\pi)^2}\varphi(x_2)\varphi(y_2)i^2p_0(-p_0)\frac{e^{i(\widehat{x}-\widehat{y})\cdot\widehat{p}}}{p^2+m^2}=$$

$$\int_{x_2y_2}\frac{d^2p}{(2\pi)^2}\varphi(x_2)\varphi(y_2)p_0^2\frac{e^{i(\widehat{x}-\widehat{y})\cdot\widehat{p}}}{p^2+m^2} \tag{5.11}$$

As both legs of the propagator are on the boundary it is $x_1=y_1=0$. Acting on it with the Laplacian

$$\Delta_s=\int\limits_{x'}\int\limits_{y'}\int\limits_{q^2<\frac{1}{s}}\frac{dq}{2\pi}e^{iq(x'-y')}\frac{\delta}{\delta\phi(x')}\frac{\delta}{\delta\phi(y')} \tag{5.12}$$

we get

$$\Delta_s\int_{x_2y_2}\varphi(x_2)\varphi(y_2)\frac{\partial^2 G_D(\widehat{x},\widehat{y})}{\partial x_1\partial y_1}=$$

$$\frac{2}{2\pi}\int_{q^2<1/s}\int\frac{e^{i(x'-y')(q+p_2)}p_1^2}{p^2+m^2}dx'dy'=\frac{2}{2\pi}\int_{q^2<1/s}\frac{p_1^2}{q^2+p_1^2+m^2}dp_1dq=$$

$$\frac{2}{2\pi}\int_{q^2<1/s}\left(1-\frac{q^2+m^2}{p_1^2+q^2+m^2}\right)dp_1dq$$

The first term in the brackets is going to give infinity after the p_1 integration. We can cancel it with a counter-term in the Lagrangian of the form $-\varphi^2\delta(0)$ (see section (3.6)). The second term, after p_1 integration gives

$$\int_{q^2<1/s}dq\sqrt{q^2+m^2} \tag{5.13}$$

which is the energy density[1] we got in relation (4.3). The expansion of (5.11) will reveal the b series included in W.

We can calculate also, the amplitude of the diagram (5.2). It is

$$A=g\int_{w_2,x_2,y_2,z_2,\widehat{u}}\varphi(\widehat{w})\varphi(\widehat{x})\varphi(\widehat{y})\varphi(\widehat{z})\frac{\partial G_D(\widehat{w},\widehat{u})}{\partial w_1}\frac{\partial G_D(\widehat{x},\widehat{u})}{\partial x_1}\frac{\partial G_D(\widehat{y},\widehat{u})}{\partial y_1}\frac{\partial G_D(\widehat{z},\widehat{u})}{\partial z_1} \tag{5.14}$$

[1] We see that from this diagram we get two different kinds of infinities. The one due to the boundary and the other due to the contraction of the two legs of the boundary propagator, which gives the divergent energy density, similarly to the theory without boundaries.

As all the propagators have one leg on the boundary we can substitute instead of the Dirichlet Green functions, the usual ones as (5.10) suggests. This is

$$A = 16g \int_{w_2,x_2,y_2,z_2,\widehat{u}} \varphi(\widehat{w})\varphi(\widehat{x})\varphi(\widehat{y})\varphi(\widehat{z}) \frac{\partial G(\widehat{w},\widehat{u})}{\partial w_1} \frac{\partial G(\widehat{x},\widehat{u})}{\partial x_1} \frac{\partial G(\widehat{y},\widehat{u})}{\partial y_1} \frac{\partial G(\widehat{z},\widehat{u})}{\partial z_1} =$$

$$16g \int_{w_2,x_2,y_2,z_2,\widehat{u}} \varphi(\widehat{w})\varphi(\widehat{x})\varphi(\widehat{y})\varphi(\widehat{z}) \int_{\widehat{p},\widehat{q},\widehat{r},\widehat{t}} p_1 q_1 r_1 t_1 \times$$

$$\frac{e^{i(\widehat{w}-\widehat{u})\cdot\widehat{p}}}{(p^2+m^2)} \frac{e^{i(\widehat{x}-\widehat{u})\cdot\widehat{q}}}{(q^2+m^2)} \frac{e^{i(\widehat{y}-\widehat{u})\cdot\widehat{r}}}{(r^2+m^2)} \frac{e^{i(\widehat{z}-\widehat{u})\cdot\widehat{t}}}{(t^2+m^2)} \tag{5.15}$$

where $\int_{\widehat{p}} \equiv \int \frac{d\widehat{p}}{(2\pi)^2}$. Because one leg of each propagator is on the boundary $w_1 = x_1 = y_1 = z_1 = 0$. The $\widehat{u}$ integration goes for u_1 from $-\infty$ to 0 and for u_2 from $-\infty$ to ∞. But the integrations of the momentum have limits $-\infty$ and ∞ so that with a change of variables we can substitute in the above integral $\int_0^\infty du_1 ... = \frac{1}{2}\int_{-\infty}^{\infty} du_1 ....$ Now we can perform the u_1 and u_2 integrations which will give us the delta functions $\delta(p_1+q_1+r_1+t_1)$ and $\delta(p_2+q_2+r_2+t_2)$. Putting these back in (5.15) we get

$$g \int_{w_2,x_2,y_2,z_2,\widehat{u}} \varphi(\widehat{w})\,\varphi(\widehat{x})\,\varphi(\widehat{y})\,\varphi(\widehat{z}) \int_{\widehat{p},\widehat{q},\widehat{r}} \{$$

$$\frac{(-1)\,p_1 q_1 r_1 (p_1+q_1+r_1)\; e^{i\,(w_2 p_2 + x_2 q_2 + y_2 r_2 + z_2(-p_2-q_2-r_2))}}{(p^2+m^2)\,(q^2+m^2)\,(r^2+m^2)\,((p_1+q_1+r_1)^2+(p_2+q_2+r_2)^2+m^2)}\}$$

Now we can make p_1, q_1 and r_1 integrations to get

$$g \int_{w_2,x_2,y_2,z_2,\widehat{u}} \varphi(w_2)\,\varphi(x_2)\,\varphi(y_2)\,\varphi(z_2) \int_{p_2,q_2,r_2} \{$$

$$\frac{(-1)\,\pi^3\, e^{i\,(w_2 p_2 + x_2 q_2 + y_2 r_2 + z_2(-p_2-q_2-r_2))}}{\sqrt{p_2^2+m^2}+\sqrt{q_2^2+m^2}+\sqrt{r_2^2+m^2}+\sqrt{(p_2+q_2+r_2)^2+m^2}}\} \tag{5.16}$$

Acting on (5.16) with the Laplacian given in (5.12) we get

$$6g \int\limits_{p^2<\frac{1}{s}} \int_{w_2} \int_{x_2} \varphi(w_2)\,\varphi(x_2) \int_{q_2,r_2} \frac{e^{i(x_2-w_2)q_2}}{\sqrt{q_2^2+m^2}+\sqrt{p^2+m^2}} \tag{5.17}$$

From this amplitude we can calculate the coefficients of the terms $\varphi^{(n)}(x)^2$. What we have to do is to expand the $\varphi(x_2)$ function in (5.17), in its variable x_2, so that it gives

$$6g \int\limits_{p^2<\frac{1}{s}} \int_{w_2} \int_{x_2} \varphi(w_2) \left(\varphi(w_2) + ... + \frac{(w_2-x_2)^{2n}}{(2n)!}\varphi^{(2n)}(w_2) + ...\right) \times$$

$$\int_{q_2,r_2} \frac{e^{i(x_2-w_2)q_2}}{\sqrt{q_2^2+m^2}+\sqrt{p^2+m^2}} \tag{5.18}$$

where the odd terms become a total deferential and vanish under the integral. The first term we get is the coefficient of $\varphi^2(x)$. It has the form

$$\int\limits_{p^2<\frac{1}{s}} \frac{dp}{2\pi} \frac{1}{m^2+\sqrt{p^2+m^2}}$$

For $s \to 0$ the integral diverges, as it needs the mass counter-term (see [12]). In short, the finite term is going to be

$$\int\limits_{p^2<\frac{1}{s}} \frac{dp}{2\pi} \left(\frac{1}{m^2+\sqrt{p^2+m^2}} - \frac{1}{\sqrt{p^2+m^2}} \right)$$

The general term includes derivatives of a delta function

$$\int (x_2-w_2)^{2n} e^{i(x_2-w_2)q_2} dx_2 = \int \frac{1}{i^{2n}} \frac{\partial^{2n}}{\partial^{2n} q_2} e^{i(x_2-w_2)q_2} dx_2 =$$

$$\frac{1}{i^{2n}} \frac{\partial^{2n}}{\partial^{2n} q_2} \delta(q_2) = \frac{1}{i^{2n}} \delta^{(2n)}(q_2)$$

Also in (5.18) the fields term becomes

$$\int \varphi(w_2)\, \varphi^{(2n)}(w_2) dw_2 = (-1)^n \int {\varphi^{(n)}}^2 dw_2$$

so finally by substituting back to (5.18) we get

$$6g \sum_{n=1}^{\infty} \int_x {\varphi^{(n)}(x)}^2 \int_{p^2<1/s} \frac{dp}{2\pi} \int dq \frac{\delta^{(2n)}(q)}{\sqrt{p^2+m^2}+\sqrt{q^2+m^2}} \tag{5.19}$$

We can calculate all the coefficients of the corrections of the semiclassical approach for the terms ${\varphi^{(n)}}^2$ in the $W[\varphi]$ local expansion. This is done by expanding (5.19) for each n in the large s region. For example, for $n=0$ we get from (5.19)

$$6g \int_x \varphi^2(x) \int_{p^2<1/s} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2+m^2}+m}. \tag{5.20}$$

Expanding the integrand for small p and performing the p integration we get a similar expression with the one obtained from the action of the Laplacian on the $\sum_n c_n \int \varphi^2 {\varphi^{(n)}}^2$ terms by substituting the values of c_n's and collecting the coefficients of $\int \varphi^2$ as in (5.20). As we have seen in section (4.7) this is divergent for $s \to 0$ and it needs the mass counter-term subtraction to give the desired finite result. The form of the mass counter-term is given in (4.50) and combined with (5.20) gives

$$6g \int_x \varphi^2(x) \int_{p^2<1/s} \frac{dp}{2\pi} \left(\frac{1}{\sqrt{p^2+m^2}+m} - \frac{1}{\sqrt{p^2+m^2}} \right) \tag{5.21}$$

Expression (5.21) gives us the possibility to calculate the exact value of the limiting case $s \to 0$. By expanding (5.21) for small p we can perform the p integration with integration

limits $-\infty$ to ∞, i.e. for $s = 0$. By fixing the appropriate factor in order the expansion of (5.21) is identical to the c-series minus the mass counter-term we find the value $1/(4\pi)$. Similar treatment for the coefficient of $\int \varphi'^2$ results the value 0.014956.

Let us turn our attention to the diagram of Figure (5.3). To calculate the Laplacian terms of the Schrodinger equation with four φ's we need to calculate the sum of the amplitude of the second and third topologically distinct diagrams presented in (5.3), which are produced from the action of the Laplacian on the first, bearing in mind their symmetry factors. The amplitude of the first diagram in Figure (5.3) is

$$g^2 \int_{a_2,b_2,c_2,d_2,e_2,f_2,\widehat{k},\widehat{l}} \varphi(a_2)\varphi(b_2)\varphi(c_2)\varphi(d_2)\varphi(e_2)\varphi(f_2)\times$$
$$\frac{\partial G_D(\widehat{a},\widehat{k})}{\partial a_1}\bigg|_{a_1=0} \frac{\partial G_D(\widehat{b},\widehat{k})}{\partial b_1}\bigg|_{b_1=0} \frac{\partial G_D(\widehat{c},\widehat{k})}{\partial c_1}\bigg|_{c_1=0} G_D(\widehat{k},\widehat{l})\times$$
$$\frac{\partial G_D(\widehat{d},\widehat{l})}{\partial d_1}\bigg|_{d_1=0} \frac{\partial G_D(\widehat{e},\widehat{l})}{\partial e_1}\bigg|_{e_1=0} \frac{\partial G_D(\widehat{f},\widehat{l})}{\partial f_1}\bigg|_{f_1=0} \tag{5.22}$$

The application of the Laplacian on (5.22) will produce the other two diagrams and finally will result for the φ^4 term

$$g^2 \int_x \varphi^4(x) \int_{p^2<1/s} \frac{dp}{2\pi} \frac{\sqrt{p^2+1}+4}{(\sqrt{p^2+1}+1)(\sqrt{p^2+1}+2)} \tag{5.23}$$

whereas for $\int \varphi^2 \varphi'^2$ it reads

$$-\frac{g^2}{16} \int_x \varphi^2(x)\varphi'^2(x) \int_{p^2<1/s} \frac{6p^2+28+22\sqrt{p^2+1}+p^2\sqrt{p^2+1}}{(\sqrt{p^2+1}+2)^2(\sqrt{p^2+1}+1)^3} \tag{5.24}$$

In both relations m is set equal to 1. After adjusting an appropriate factor in (5.23) to match the corresponding f series we obtain the value for the limit $s \to 0$ to be 0.003178.

By developing these analytic expansions of the diagrams we achieve to show the validity of the results obtained from the modified Schrödinger equation (4.31). Additionally we are able to see for these few cases the s dependence of the functions we have to re-sum. They are analytic for any complex valued s apart from the real negative axis. It will be interesting to study the s dependent term of expression (5.20). It reads for $m = 1$

$$\int_{p^2<1/s} dp \frac{1}{\sqrt{p^2+1}+1} = 2\left(\sqrt{s} - \sqrt{1-s} + \operatorname{arcsin h}\left(\frac{1}{\sqrt{s}}\right)\right)$$

with the small s expansion

$$-2 + 2\ln 2 + \ln\frac{1}{s} + 2\sqrt{s} - \frac{1}{2}s + \frac{1}{16}s^2 - \frac{1}{48}s^3 + \ldots$$

From there we see the logarithmic divergence which is subtracted by the mass counter-term. Also, in this expansion apart from $\ln 1/s$ and $\sqrt{s}$ terms the rest are integer powers of s, which will be eliminated if we act with the re-summation operator $R(s)$. Acting with the operator $R(s) + 2\lambda R(s)s$ on the renormalised series (with the logarithmic divergence subtracted) we eliminate additionally the square root term and we are left with the constant value -2. However, this will not be true for the coefficients of $\int \varphi^{(n)\,2}$ with $n \geq 1$ as the delta function derivatives in (5.19) will produce higher half integer powers of s.

Chapter 6

The $O(N)$ σ Model

6.1 Introduction

In the following chapters we are going to study the $1+1$ dimensional $O(N)$ σ model. Our interest in the $O(N)$ σ model relies upon the fact that apart from its usefulness as a simplified gravitational model, shares many features with Yang-Mills theory. They are both conformally invariant in the classical level, but they generate mass quantum mechanically which breaks their conformal invariance. Also, they are renormalisable [23], asymptotically free, and have large-N expansions [24]. However, the σ model is much simpler to work with so that, features like the mass generation can be explicitly demonstrated within the large-N expansion, as we will see in the following.

In [12], a new method is proposed for solving the eigenvalue problem for the Hamiltonian of massive quantum field theories that are classically massless. It is based on the version of the Schrödinger equation we studied in Chapter 4. As we have seen, the equation acts directly on a local expansion of the vacuum functional. In the following we will see the construction of the $O(N)$ σ model Laplacian, which is the principal ingredient in the Schrödinger representation approach, acting on local functionals.

6.2 Geometrical View

Through studying the Harmonic Maps of C. Misner [25] a better understanding of the geometrical concepts of the σ model as well as other nonlinear theories can be obtained. The most important objects in these theories are the two manifolds where the mapping is taking place. Let us name them M and M'. The first structure we demand for them is that they be pseudo-Riemannian manifolds; that is, they admit a pseudo-Riemannian metric g that satisfies the following axioms at each point p of the manifold

$$(i) \quad g_p(U,V) = g_p(V,U)$$

$$(ii) \quad \text{if } g_p(U,V) = 0 \text{ for every } U \in T_pM \text{ then } V = 0$$

We can construct related coordinate systems for each one. If x^μ parametrises the manifold M and ϕ^A parametrises the manifold M' and we *define* M' to be generated from the set of all possible values of some field ϕ, then the map is

$$\phi : M \to M', \; x^\mu \to \phi^A(x^\mu) \equiv \phi^A.$$

We require the field to be smooth and infinitely differentiable. The metric of each manifold will give rise to the invariant lengths

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \text{ for } M \tag{6.1}$$

$$dL^2 = G_{AB}(\phi)d\phi^A d\phi^B \text{ for } M'. \tag{6.2}$$

Usually M is given the flat Minkowski or Euclidean space with the metric g having the appropriate form. However, if we ask manifold M' to be curved then the metric G has a nontrivial form which corresponds to a nonlinearity for the field ϕ. In terms of relation (6.2) this is expressed through the fact that there is no reparametrised metric G for which the infinitesimal element dL^2 could be reduced in linear parts, for a linear combination of fields $c_1\phi_1 + c_2\phi_2$ taking the place of the field ϕ in (6.2). These maps will be called *Harmonic Maps* providing they satisfy the Euler-Lagrange equations of the variational principle $\delta I = 0$ for the action

$$I = \frac{1}{2}\int \sqrt{|g|}d^n x g^{\mu\nu}(x)\frac{\partial\phi^A}{\partial x^\mu}\frac{\partial\phi^B}{\partial x^\nu}G_{AB}(\phi). \tag{6.3}$$

I is the action we are going to use with g, a $1+1$ dimensional Minkowski metric.

In the following section we will see how this framework will help us to understand some physical consequences of the short and large wavelength cases.

6.3 Short-Large Wavelength

The model described above has a very interesting structure when it is quantised. Because of its non-linearity it gives characteristics similar to the ones we have in Yang-Mills or the Einstein cases. What we ask for, is that the quantum theory of a field ϕ be defined on a two dimensional Minkowski space, with values on a compact Riemannian symmetric space of positive sectional curvature which for simplicity we will choose to be an N dimensional sphere.

To see how this model behaves for short and large wavelength let us make a lattice in the spatial Minkowski direction with sites x_n, where its nearest neighbours will be called $x_{n'}$ and are set at distance $\Delta \equiv \Delta x$ apart. We can write the action (6.3) in the form

$$I = \int dt \left\{ \frac{1}{2} \frac{\hbar c}{\mathrm{L}^2} \int dx \left(\frac{1}{c^2} \left\| \frac{\partial \phi}{\partial t} \right\|^2 - \|\nabla \phi\|^2 \right) \right\}$$

where

$$\|\partial \phi\|^2 = G_{AB}(\phi) \partial \phi^A \partial \phi^B.$$

When this action is approximated on a lattice this becomes

$$I = \int dt \left\{ \frac{1}{2} \frac{\hbar c}{\mathrm{L}^2} \sum_n \Delta \left(\frac{1}{c^2} \left\| \frac{\partial \phi_n}{\partial t} \right\|^2 - \frac{1}{2\Delta^2} \sum_{n'} \|\phi_{n'} - \phi_n\|^2 \right) \right\}$$

We have to give an interpretation for $\|\phi_1 - \phi_2\|$. An obvious one is the least geodesic distance between these points on the Riemannian space, i.e.

$$\|\phi_1 - \phi_2\| = d(\phi_1, \phi_2)$$

However, we have to have in mind that the target manifold is a compact one, so that it has a finite volume or finite diameter. This makes the lattice interpretation of the spatial derivative of the field ϕ finite as can be seen in figure (6.1). The maximum distance that the field ϕ can map to the lattice distance Δ is the diameter of the target manifold, let us name it d. As the mapping is continuous, taking smaller distances on the lattice will

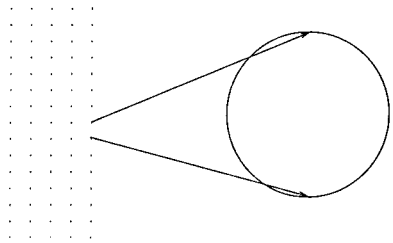

Figure 6.1: The mapping from the lattice to the sphere.

result in mapping fractions of the distance d. Hence

$$\|\phi_1 - \phi_2\| \leq d \tag{6.4}$$

But the uncertainty principle transforms (6.4) to a lower bound for the conjugate momentum of the field ϕ, which is defined by

$$p_n = \frac{1}{\hbar}\frac{\partial \mathcal{L}}{\partial \phi_n} ,$$

where $\mathcal{L}$ is the Lagrangian of the action I, and satisfies the commutation relation $[\phi_n, p_m] = i\delta_{nm}$. The Hamiltonian corresponding to (6.3) is

$$H = \left(\frac{\hbar c}{\mathrm{L}^2}\Delta\right)\sum_n \left[\left(\frac{\mathrm{L}^2}{\Delta^2}\right)^2 \|p_n\|^2 + \frac{1}{2}\sum_{n'} \|\phi_{n'} - \phi_n\|^2\right].$$

Approximating even further, by considering the nearest neighbours interactions significant (as the others behave like free field modes) it becomes

$$H_{12} = \left(\frac{\hbar c}{\mathrm{L}^2}\Delta\right)\left[\left(\frac{\mathrm{L}^2}{\Delta^2}\right)^2 \|p\|^2 + \|\phi - \phi_0\|^2\right]$$

where ϕ_0 is fixed (close neighbours in the Minkowski space does not mean necessarily close mappings in the Riemannian space). Since $\|\phi - \phi_0\|^2$ is bounded, the behaviour of the quantum system (i.e. the pattern of the eigenvalues) depends upon the dimensionless number $J^2 = \Delta^2/\mathrm{L}^2$. Two limiting cases have great interest. The $J^2 >> 1$ and $J^2 << 1$ or equivalently $\Delta^2/\mathrm{L}^2 >> 1$ and $\Delta^2/\mathrm{L}^2 << 1$. For all modes with $\Delta^2/\mathrm{L}^2 << 1$ the interaction term can be neglected, since the space derivative terms in the action are negligible and our system has strong quantum effects. But for $\Delta^2/\mathrm{L}^2 >> 1$ the free field behaviour results (large momentum). The case $\Delta^2/\mathrm{L}^2 << 1$ can be interpreted as a slowly moving field due to its large wavelength. This makes, as mentioned, the spatial derivatives small or negligible and our theory behaves as an ultralocal in first approximation (that is the eigenstates are only functions of the co-ordinate x and the field ϕ); no spatial

derivatives appear, only time derivatives. In a better approximation one can construct a local theory which takes into account eigenstates which are a local expansion of spatial derivatives of the field ϕ up to a certain order. The following chapters are involved with local expansions of the field ϕ.

6.4 Mass Generation

The $O(N)$ σ model is conformal invariant at the classical level (see Chapter 8). However, it generates mass when it is quantised. This shared feature between $O(N)$ σ model and Yang-Mills theory is one of their similarities. We are going to construct this procedure for the simple case of the two dimensional nonlinear $O(N)$ σ model.

The action of this model is

$$S = -\frac{1}{2\alpha}\int d\sigma d\tau\, g_{\mu\nu}\left(\dot{z}^{\mu}\dot{z}^{\nu} - z'^{\mu}z'^{\nu}\right) \tag{6.5}$$

z is a field with N components, which depends on σ and τ, the space and time coordinates respectively. It is constrained by $z\cdot z = a^2$ which retains the $O(N)$ symmetry in the target manifold. The summation convention $z\cdot z = z^{\mu}z_{\mu}$, where μ runs from 1 to N, is used also in the following. The generating functional Z can be written as

$$Z = \int Dz(x)\left(\prod_x \delta(z(x)^2 - a^2)\right) exp\left(-\frac{1}{2\alpha}\int d^2x\, \partial_i z^{\mu}\partial^i z_{\mu}\right) \tag{6.6}$$

where the constraint has been inserted as a delta function, which with the application of the Lagrange multiplier λ, introduced as:

$$\prod_x \delta(z(x)^2 - a^2) = \prod_x \int_{C-i\infty}^{C+i\infty} d\lambda\, exp\left\{\lambda(z(x)^2 - a^2)\right\} = \int \mathcal{D}\lambda\, exp\left\{\int d^2x \lambda(z(x)^2 - a^2)\right\} \tag{6.7}$$

makes Z to be

$$Z = \int_{C-i\infty}^{C+i\infty} \mathcal{D}\lambda \int \mathcal{D}z(x) exp\left(-\frac{1}{2\alpha}\int d^2x\, \partial_i z^{\mu}\partial^i z_{\mu} + \lambda(z(x)^2 - a^2)\right) \tag{6.8}$$

where λ is a function of space and time coordinates. This is the elementary scalar field which will break our symmetry, that is the conformal symmetry of the σ model, by acquiring a non-zero expectation value.

We can perform the z functional integration, which is a standard Gaussian integral, and will result

$$Z = \int_{C-i\infty}^{C+i\infty} \mathcal{D}\lambda \, exp\left\{ -\frac{1}{2\alpha} \int d^2x \, \lambda a^2 - \frac{N}{2} log \, det\| - \partial^2 + \lambda a^2 \| \right\} \tag{6.9}$$

The logarithm of the determinant can be represented by the Feynman graphs of Figure (6.2) where the dashed lines correspond to the λ field (like interaction) and the z field is

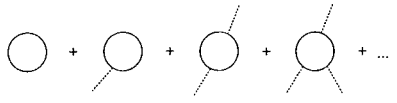

Figure 6.2: Diagrammatical expansion of $log \, det\| - \partial^2 + \lambda a^2 \|$.

represented by the solid line. The propagator of z is $1 \, / \, p^2$. While the infrared divergence appearing in, is taken care by the theory itself, we have to insert a cut off for the ultraviolet one.

Passing to the momentum representation we expand the λ field around zero momentum or in other words, we approximate the above integral around the saddle point of the λ field. At this point we will demand our theory to be well behaved for large N. This reflects, as we will see, in making the λ field to have a finite expectation value.

From the graph (6.2) it is obvious that the derivative of the logarithm of the determinant with respect to λ is the Green function

$$\frac{N}{2} G(x, x; \lambda) = \frac{N}{2} \frac{\delta}{\delta \lambda(x)} log \, det\| - \partial^2 + \lambda a^2 \| = \frac{a^2}{2\alpha} \tag{6.10}$$

where the second equation defines the saddle point of relation (6.9). As can be seen by the mathematical identity [1]

$$\delta \, log \, det \, A = \delta \, \mathrm{Tr} \, log \, A = Tr \, A^{-1} \, \delta A \tag{6.11}$$

the Green function evaluated at two different points x and x' is

$$G(x, x'; \lambda) = \langle x | (-\partial^2 + \lambda a^2)^{-1} | x' \rangle \tag{6.12}$$

and in momentum space

$$G(x, x'; \lambda) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip(x-x')}}{p^2 + \lambda a^2} \tag{6.13}$$

[1] or from Figure (6.2)

so that equation (6.13) becomes for $x = x'$

$$\frac{N\alpha}{a^2}G(x,x;\lambda) = \frac{N\alpha}{a^2}\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+\lambda a^2} \tag{6.14}$$

which after the insertion of a cut off Λ we finally have

$$\tilde{\lambda} = \frac{\Lambda^2}{a^2}exp\left(-\frac{4\pi a^2}{N\ \alpha}\right) \tag{6.15}$$

where $\tilde{\lambda}$ is the saddle point value of the field $\lambda(x)$. The square root of $\tilde{\lambda}$ can be interpreted as the mass which is generated via the λ field. The $1\ /\ \alpha$ factor in front of the action is the coupling and a is the radius. They are connected as a ratio (i.e. for large radius -small coupling and via-versa) a connection which can be also read from the action (6.5). But the product α times N is constant as N goes to infinity so that expression (6.9) is totally proportional to N. That is, the saddle point equation demands the part of the action proportional to N to be zero so that we have a good behaviour of the expansion for large N.

6.5 $O(N)$ σ Model and Schrödinger Representation

We can construct the Schrödinger representation for the $O(N)$ σ model as a generalisation of its quantum mechanical analog (see [26]). Consider a non-relativistic particle of mass, m, moving on the N-dimensional sphere with co-ordinates $z(\tau)$ at time τ. For $g_{\mu\nu}$ the metric of the sphere with radius a we can define the particle's action as $S = m/2\int d\tau g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu$ which is invariant under the rotation transformations on the sphere. The corresponding Hamiltonian in the Schrödinger representation is $H = -1/(2m)\Delta$, where Δ is the Laplacian on the sphere with eigenfunctions the spherical harmonics.

Considering the generalisation of this quantum mechanical case in field theory, we insert another spatial dimension σ, which can be thought as parametrising a curve on the sphere. This curve takes the place of the particle in quantum mechanics. The $O(N)$ σ model is then defined as the infinite dimensional theory of a particle on an N dimensional sphere parametrised by the function $z^\mu(\sigma,\tau)$ as the variable τ varies. We ask the (σ, τ) space to be Minkowskian. The action of the theory has to be a scalar with respect to the Lorentz transformations and also to the reparametrisations on the sphere. So we can choose it to

be

$$S = \frac{1}{2\alpha}\int d\sigma d\tau g_{\mu\nu}(\dot{z}^\mu \dot{z}^\nu - z'^\mu z'^\nu) \tag{6.16}$$

where $'$ and $\cdot$ denote differentiations with respect σ and τ respectively, and α is a coupling constant.

Turning to the Schrödinger Representation we take $z(\sigma,\tau)$ to be diagonalised at $\tau = 0$ satisfying the relation

$$z^\mu(\sigma,0)\Psi[z] = z^\mu(\sigma)\Psi$$

for Ψ the Schrödinger wave functional and its conjugate momentum $\pi(\sigma,\tau)$ to be at $\tau = 0$

$$\pi_\nu(\sigma)\Psi = i\alpha \mathbf{D}_\nu(\sigma)\Psi \tag{6.17}$$

so that the equal time commutation relation

$$[z^\mu(\sigma), \pi_\nu(\sigma')] = i\alpha\delta^\mu_\nu\delta(\sigma,\sigma') \tag{6.18}$$

is satisfied. In (6.17) the differential operator is defined with respect to a covariant differentiation whose meaning and structure will be given later on. Though, as z^μ is a scalar $\mathbf{D}_\nu(\sigma)$ takes the usual functional derivative form, $\delta/\delta z^\nu(\sigma)$. From (6.16) we can read the Hamiltonian

$$H = \frac{\alpha}{2}\Delta + \frac{1}{2\alpha}\int d\sigma g_{\mu\nu} z'^\mu z'^\nu \tag{6.19}$$

where Δ is now the Laplacian constructed from an inner product on variations of the co-ordinates

$$(\delta z, \delta z) = \int d\sigma g_{\mu\nu}\delta z^\mu \delta z^\nu$$

The type of tensor we deal with are the ones with indices the finite valued μ, ν and the infinite valued σ. Because of the different σ variables appearing in our expression, as in (6.18), we need to make the distinction between the infinite component tensors (denoted in the following as bold-faced), which are used for the description of the σ model, and the ordinary tensors on S^N. Then $\mathbf{g}_{\mu_1\mu_2}(\sigma_1,\sigma_2) = g_{\mu_1\mu_2}(z(\sigma_1))\delta(\sigma_1 - \sigma_2)$ where we can treat the pair (μ_1,σ_1) as a single index, as well as (μ_2,σ_2). Hence, $\mathbf{g}$ can be considered as a two-indexed tensor. Its inverse is $\mathbf{g}^{\mu_1\mu_2}(\sigma_1,\sigma_2) = g^{\mu_1\mu_2}(z(\sigma_1))\delta(\sigma_1 - \sigma_2)$ so their contraction gives

$$\int d\sigma \mathbf{g}_{\mu_1\mu}(\sigma_1,\sigma)\mathbf{g}^{\mu\mu_2}(\sigma,\sigma_2) = \delta^{\mu_2}_{\mu_1}\delta(\sigma_1 - \sigma_2) \equiv \mathbf{I}^{\mu_2}_{\mu_1}(\sigma_1,\sigma_2)$$

where $\mathbf{I}^{\mu_2}_{\mu_1}(\sigma_1, \sigma_2)$ is the infinite dimensional Kronecker delta and it is also equal to the functional derivative of $z^{\mu_2}(\sigma_2)$ with respect to $z^{\mu_1}(\sigma_1)$

$$[z^\mu(\sigma), \pi_\nu(\sigma')] = i\alpha \mathbf{I}^\mu_\nu(\sigma, \sigma') \tag{6.20}$$

A total Riemannian geometrical interpretation of this infinite dimensional model would ask for a covariant transformation of $\mathbf{g}$ and $\mathbf{I}$ under the general co-ordinate transformations $z^\mu \to \tilde{z}^\mu(\sigma)$ where $\tilde{z}$ is a functional of z, i.e. it depends on the entire curve $z = z(\sigma)$. However, the utility of this for quantum field theory is not clear. But we can restrict ourselves to the rotation transformations in the internal space. Invariance of the theory with respect to them underpins its renormalisability [23]. These are rigid co-ordinate transformations, for which the shape of the curve is not important. So we can take $\tilde{z}^\mu(\sigma)$ to be a function of $z^\nu(\sigma)$ i.e. depending on a specific point of σ on the curve. Under this restricted class of transformations a finite dimensional vector $V^\mu(z)$ on S^N may be thought of as an infinite dimensional vector $\mathbf{V}^\mu(\sigma) = V^\mu(z(\sigma))$.

6.6 Operators on S^N

The momentum operator we used in the commutation relation (6.20) has to be covariant. It is also needed for the Laplacian appearing in (6.19). Given the infinite dimensional metric we can follow the usual construction of the *Levi-Civita* connection, $\mathbf{D}$, which will transform covariantly under general co-ordinate transformations and therefore under their restricted class of rotations. Thus, if it acts on a scalar will reduce to the usual functional derivative. For an infinite dimensional vector $\mathbf{V}^{\mu_1}(\sigma_1)$ we get

$$\mathbf{D}_{\mu_2}(\sigma_2)\mathbf{V}^{\mu_1}(\sigma_1) = \frac{\delta \mathbf{V}^{\mu_1}(\sigma_1)}{\delta z^{\mu_2}(\sigma_2)} + \int d\sigma_3 \, \mathbf{\Gamma}^{\mu_1}_{\mu_2\mu_3}(\sigma_1, \sigma_2, \sigma_3)\mathbf{V}^{\mu_3}(\sigma_3) \tag{6.21}$$

where the infinite dimensional *Christoffel symbol* is related to that on S^N by

$$\mathbf{\Gamma}^{\mu_1}_{\mu_2\mu_3}(\sigma_1, \sigma_2, \sigma_3) = \delta(\sigma_1 - \sigma_2)\,\delta(\sigma_2 - \sigma_3)\;\Gamma^{\mu_1}_{\mu_2\mu_3}(z(\sigma_1)) \tag{6.22}$$

If we apply this to a vector that depends on σ and $z(\sigma)$ but not on its derivatives (i.e. it is ultra local), then it is straightforward to compute

$$\mathbf{D}_{\mu_2}(\sigma_2)\mathbf{V}^{\mu_1}(\sigma_1) = (D_{\mu_2}V^{\mu_1})|_{z(\sigma_1)}\,\delta(\sigma_1 - \sigma_2), \tag{6.23}$$

where D is the covariant derivative on S^N. Similarly we can compute the covariant derivative of z'^{μ} as

$$\mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1) = \delta^{\mu_1}_{\mu_2}\,\delta'(\sigma_1-\sigma_2) + \left(\Gamma^{\mu_1}_{\mu_2\rho} z'^{\rho}\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) \tag{6.24}$$

with Γ the finite dimensional Christoffel symbol. We can define a finite dimensional intrinsic derivative $\mathcal{D} = \partial/\partial\sigma + z'^{\mu}D_{\mu}$ which maps finite dimensional vectors to finite dimensional vectors. Hence we can use it to define new infinite component vectors as

$$\mathcal{D}|_{\sigma}\mathbf{V}^{\mu}(\sigma) \equiv (\mathcal{D}V)^{\mu}|_{z(\sigma)} \tag{6.25}$$

The tensor $\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2)$ is an element of the product of the tangent space at $z(\sigma_1)$ and the co-tangent space at $z(\sigma_2)$. The intrinsic derivative with respect to σ_1 acts only on the (μ_1,σ_1) index, so it gives

$$\mathcal{D}|_{\sigma_1}\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2) = \delta^{\mu_1}_{\mu_2}\,\delta'(\sigma_1-\sigma_2) + \left(\Gamma^{\mu_1}_{\mu_2\rho} z'^{\rho}\right)|_{z(\sigma_2)}\,\delta(\sigma_1-\sigma_2) = \mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1). \tag{6.26}$$

Similarly the intrinsic derivative with respect to σ_2 acts only on the (μ_2,σ_2) index to give

$$\mathcal{D}|_{\sigma_2}\mathbf{I}^{\mu_1}_{\mu_2}(\sigma_1,\sigma_2) = -\mathbf{D}_{\mu_2}(\sigma_2)\,\mathbf{z}'^{\mu_1}(\sigma_1) \tag{6.27}$$

Since $z'^{\mu} = \mathcal{D}z^{\mu}(\sigma)$ this implies that $[\mathbf{D},\mathcal{D}]z^{\mu} = 0$, so that this commutator also annihilates any ultra-local scalar. For the following it will be useful to evaluate this commutator when it acts on vectors. If V^{μ} is ultra-local then the commutator $[\mathbf{D}_{\mu_1}(\sigma_1),\,\mathcal{D}|_{\sigma_2}]\,\mathbf{V}^{\mu_2}(\sigma_2)$ becomes

$$[\mathbf{D}_{\mu_1}(\sigma_1),\,\mathcal{D}|_{\sigma_2}]\,\mathbf{V}^{\mu_2}(\sigma_2) =$$

$$\mathbf{D}_{\mu_1}\left(\frac{\partial}{\partial\sigma_2}\mathbf{V}^{\mu_2}(\sigma_2) + z'^{\mu}D_{\mu}\mathbf{V}_{\mu_2}(\sigma_2)\right) - \frac{\partial}{\partial\sigma_2}\left(\mathbf{D}_{\mu_1}\mathbf{V}^{\mu_2}(\sigma_2)\right) - z'^{\mu}D_{\mu}\left(\mathbf{D}_{\mu_1}\mathbf{V}^{\mu_2}(\sigma_2)\right) =$$

$$\mathbf{D}_{\mu_1}\frac{\partial}{\partial\sigma_2}\mathbf{V}^{\mu_2}(\sigma_2) + \delta'(\sigma_2-\sigma_1)D_{\mu_1}\mathbf{V}^{\mu_2}(\sigma_2) + z'^{\mu}\mathbf{D}_{\mu_1}D_{\mu}\mathbf{V}^{\mu_1}(\sigma_2) - \mathbf{D}_{\mu_1}\frac{\partial}{\partial\sigma_2}\mathbf{V}^{\mu_2}(\sigma_2) -$$

$$\delta'(\sigma_2-\sigma_1)D_{\mu_1}\mathbf{V}^{\mu_2}(\sigma_2) - z'^{\mu}D_{\mu}\mathbf{D}_{\mu_1}\mathbf{V}^{\mu_2}(\sigma_2) =$$

$$z'^{\mu}\left(\mathbf{D}_{\mu_1}(\sigma_1)D_{\mu}\mathbf{V}^{\mu_2}(\sigma_2) - D_{\mu}\mathbf{D}_{\mu_1}(\sigma_1)\mathbf{V}^{\mu_2}(\sigma_2)\right) =$$

$$z'^{\mu}\left(D_{\mu_1}D_{\mu}V^{\mu_2} - D_{\mu}D_{\mu_1}V^{\mu_2}\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) =$$

$$z'^{\mu}\left([D_{\mu_1},D_{\mu}]\,V^{\mu_2}\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) = z'^{\mu}\left(R_{\mu\mu_1\rho}{}^{\mu_2}V^{\rho}\right)\Big|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) \Rightarrow$$

$$[\mathbf{D}_{\mu_1}(\sigma_1),\,\mathcal{D}|_{\sigma_2}]\,\mathbf{V}^{\mu_2}(\sigma_2) = z'^{\mu}(\sigma_1)\left(R_{\mu\mu_1\rho}{}^{\mu_2}V^{\rho}\right)|_{z(\sigma_1)}\,\delta(\sigma_1-\sigma_2) \tag{6.28}$$

where R is the finite dimensional *Riemann tensor* given by $[D_{\mu},D_{\lambda}]\,V^{\beta} = R_{\lambda\mu\rho}{}^{\beta}V^{\rho}$. On the sphere S^N it takes the form $R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta})/a^2$ while the *Ricci tensor* is

$R_{\beta\delta} \equiv g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} = \frac{N-1}{a^2}g_{\beta\delta} \equiv kg_{\beta\delta}$. For the vector $V^\mu = z'^\mu$ the previous expression becomes

$$[\mathbf{D}_{\mu_1}(\sigma_1), \mathcal{D}|_{\sigma_2}]\, \mathbf{z}'^{\mu_2}(\sigma_2) = z'^{\rho}(\sigma_1)z'^{\lambda}(\sigma_1)\, R_{\mu_1\rho\lambda}{}^{\mu_2}|_{z(\sigma_1)}\, \delta(\sigma_1 - \sigma_2). \tag{6.29}$$

The Laplacian given in (6.19) as

$$\Delta = \int d\sigma_1 d\sigma_2\, \mathbf{g}^{\mu_1\mu_2}(\sigma_1, \sigma_2)\, \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2) = \int d\sigma\, g^{\mu_1\mu_2}\, \mathbf{D}_{\mu_1}(\sigma)\mathbf{D}_{\mu_2}(\sigma) \tag{6.30}$$

is not well defined because the two functional derivatives act at the same point σ. Also the determinant of the infinite dimensional metric $\mathbf{g}$ is ill-defined as the integral on its diagonal $\sigma_1 = \sigma_2$ gives infinity. We can get around this problem by defining the Laplacian to have the regulated expression

$$\Delta_s = \int d\sigma_1 d\sigma_2\, \mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2)\, \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2). \tag{6.31}$$

The Kernel, $\mathbf{G}$, can be determined by a number of physical requirements. We will see that this is possible at least to leading and next to leading order, when the Laplacian acts on local functionals. Firstly we require that it is a regularisation of the inverse metric, so we will assume that it depends on a cut-off parameter, s, with the dimensions of squared length, and takes the form

$$\mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2) = \mathcal{G}_s(\sigma_1 - \sigma_2)\;\; K^{\mu_1\mu_2}(\sigma_1, \sigma_2; s) \tag{6.32}$$

where $\mathcal{G}_s(\sigma_1 - \sigma_2) \to \delta(\sigma_1 - \sigma_2)$ as $s \to 0$, and K is expandable as a power series in positive integer powers of s so that it has a finite limit as s goes to zero. Thus $K = \sum_{n=0}^{\infty} K_n s^n$ and $K_0{}^{\mu_1\mu_2}(\sigma, \sigma) = g^{\mu_1\mu_2}(z(\sigma))$ so that

$$\lim_{s\to 0} \mathbf{G}^{\mu_1\mu_2}(\sigma_1, \sigma_2) = \mathbf{g}^{\mu_1\mu_2}(\sigma_1)\delta(\sigma_1 - \sigma_2).$$

To preserve the invariance of the theory under internal rotational symmetry the kernel, $\mathbf{G}$, must be a second rank tensor under the restricted class of co-ordinate transformations. Finally, since we work in the Hamiltonian formalism, Poincaré invariance must be imposed, by demanding that the generators of these transformations satisfy the Poincaré algebra.

6.6.1 Poincaré Algebra

Let us study the Poincaré algebra by ignoring the problem of regularisation for the time being. The Poincaré generators are the Hamiltonian, given in (6.19) which generates time,

τ, translations, the momentum $P = \int d\sigma z'^{\mu} \mathbf{D}_{\lambda}(\sigma)$ which generates space, σ, translation and the Lorentz generator $L = -M + N$, where

$$M = \frac{1}{2} \int d\sigma \, \sigma g^{\mu_1 \mu_2} \, \mathbf{D}_{\mu_1}(\sigma) \mathbf{D}_{\mu_2}(\sigma), \quad N = \frac{1}{2} \int d\sigma \, \sigma g_{\mu_1 \mu_2} z'^{\mu_1} z'^{\mu_2} \tag{6.33}$$

which generates Lorentz transformations in the (σ, τ) Minkowski space-time. Formally, operators L, H and P satisfy the Poincaré algebra

$$[P, H] = 0, \quad [L, P] = H, \quad [L, H] = P. \tag{6.34}$$

We require that this algebra holds for the regularised operators. The momentum operator does not need to be regulated. From the regularised Laplacian (6.31) the Hamiltonian acquires a cut-off dependence, H_s. We have seen in Chapter 4 that for the scalar φ^4 theory (see also Symanzik [13]) the Schrödinger representation wave-functionals have a finite limit as the regulator is removed, and since the Hamiltonian generates displacements in τ it has a finite action on these wave-functionals. Thus the limit as $s \to 0$ of $H_s \Psi$ exists and is what we mean by the Hamiltonian applied to Ψ. We assume that this property of the Hamiltonian also holds for the $O(N)$ σ model once we have made the radius, a, depend appropriately on the cut-off, s. Similarly the cut-off dependent Lorentz operator, L_s, should have a finite limit when applied to the physical states. The commutator $[L, P] = H$ implies that L should be regulated with the same kernel as H, so we replace in (6.33) the operator M by

$$\int d\sigma_1 d\sigma_2 \, \frac{\sigma_1 + \sigma_2}{2} \, \mathbf{G}^{\mu_1 \mu_2}(\sigma_1, \sigma_2) \, \mathbf{D}_{\mu_1}(\sigma_1) \mathbf{D}_{\mu_2}(\sigma_2) \equiv M_s \tag{6.35}$$

The regularised versions of the relations (6.34) impose conditions on the Kernel when acting on local functionals F. For example the regularised version of $([L, H] - P)F = 0$ is

$$\frac{1}{4} \left[-a M_{s_1} + a^{-1} N, -a \Delta_{s_2} + a^{-1} V \right] F = PF$$

as $s_1, s_2 \to 0$. We demand that this equation holds order by order in $1/s$ up to order zero. These are the terms that, in the absence of a regulator, involve two functional derivatives at the same point on a single local functional. The terms with positive powers of s will disappear at the limit $s \to 0$. They are equivalent to the $O(s^n)$ terms with $n > 0$ in expansion (4.17), which we treated with the various re-summation procedures in order to extract the desired zeroth order term. Thus, by requiring $[M_s, \Delta_s]F = 0$, as a restriction for the Kernel, and by ignoring the positive powers of s in the intermediate steps of the calculation we obtain a better approximation for the first term of the expansion (4.17). Also we demand $M_s V = 0$ and $\Delta_s N = 0$

6.6.2 Local Expansion

As we have seen in Chapter 4 we are interested in constructing the Schrödinger equation for slowly varying fields. This allows us to expand the vacuum functional in local functionals. For the $O(N)$ σ model they are integrals of functions of σ, $z(\sigma)$ and a finite number of its derivatives at the point σ. In order to construct the kernel $\mathbf{G}$, we will consider the conditions that arise from applying the regularised form of (6.34) to such test functionals. It will be convenient to order them according to the powers of $\mathcal{D}$. If we consider the result of the action of the two functional derivatives from Δ (or M) on a local test functional as a differential operator of σ acting on a delta function, then the order of the operator, that is the biggest number of covariant σ derivatives acting on the delta function, depends on the highest number of differentiations on the z's used to construct the local functional. This operator acting on one of the σ arguments of the Kernel, via integration by parts and setting its two arguments equal to each other, with the application of the delta function, will demand the use of more terms of the Kernel expansion with respect to s, depending on the order of the operator[2].

In section (4.2) we showed that for a scalar field theory a special form of the Schrödinger equation is required to be applied to the expansion of the vacuum functional, in terms of local functionals, as the local expansion does not have the proper cut-off dependence for the subtraction of the appearing singularities. However, by studying the analyticity properties of the Laplacian it was shown how to re-sum the cut-off dependence of $\Delta_s\Psi$ so as to be able to get the correct small-s behaviour. We assume that such a re-summation may be performed here.

6.6.3 First Order Calculations

The first kind of local functionals we are going to use are of the form $F_n \equiv \int d\sigma\ f(z(\sigma),\sigma)_{\mu_1\ldots\mu_n}\ z'^{\mu_1}\ldots z'^{\mu_n}$ where f is ultra-local. When Δ_s is acted upon F_n its two functional derivatives will act on the $z'^{\mu_1}\ldots z'^{\mu_n}$ to generate a second order differential operator acting on $\delta(\sigma_1-\sigma_2)$. We can express then $\Delta_s F_n$, in terms of the second derivatives

[2] and in conclusion on the order of the test functional.

of the Kernel evaluated at co-inciding points. We treat F_n as a scalar so that

$$\mathbf{D}_\mu(\sigma)F_n = \frac{\delta F_n}{\delta z^\mu(\sigma)} =$$

$$\int d\tilde{\sigma}\,\{D_\mu f_{\rho_1\ldots\rho_n} z'^{\rho_1}\ldots z'^{\rho_n}\delta(\tilde{\sigma}-\sigma) +$$

$$n f_{\rho_1\ldots\rho_n}\,\mathcal{D}|_{\tilde{\sigma}}\, I_\mu^{\rho_1}(\tilde{\sigma}-\sigma) z'^{\rho_2}\ldots z'^{\rho_n}\Big\} =$$

$$(D_\mu f_{\mu_1\ldots\mu_n} z'^{\mu_1}\ldots z'^{\mu_n} - n\mathcal{D}(f_{\mu\mu_2\ldots\mu_n} z'^{\mu_2}\ldots z'^{\mu_n}))|_\sigma$$

which is an infinite component co-vector. Using the commutator (6.28) we get

$$\Delta_s F_n = \int d\sigma\,\left(\mathbf{G}^{\mu\nu}(\sigma,\sigma)(D_\mu D_\nu f_{\rho_1\ldots\rho_n} - nR_{\mu\rho_1\nu}{}^{\lambda} f_{\lambda\rho_2\ldots\rho_n})\, z'^{\rho_1}\ldots z'^{\rho_n} +\right.$$

$$n\int d\sigma\,((\mathcal{D}|_\sigma + \mathcal{D}|_{\sigma'})\mathbf{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'}\, D_\mu f_{\nu\rho_2\ldots\rho_n} z'^{\rho_2}\ldots z'^{\rho_n} +$$

$$\left. n(n-1)\int d\sigma\,(\mathcal{D}|_\sigma\,\mathcal{D}|_{\sigma'}\,\mathcal{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'}\, f_{\mu\nu\rho_3\ldots\rho_n}\, z'^{\rho_3}\ldots z'^{\rho_n}\right) \tag{6.36}$$

Having in mind the transformation properties (rotation invariance) of $\mathbf{G}$ and its dimension (inverse length) we can set

$$\mathbf{G}^{\mu\nu}(\sigma,\sigma) = \frac{1}{\sqrt{s}} b_0^0 g^{\mu\nu} \tag{6.37}$$

so that

$$((\mathcal{D}|_\sigma + \mathcal{D}|_{\sigma'})\mathbf{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'} = \mathcal{D}\left(\frac{1}{\sqrt{s}} b_0^0 g^{\mu\nu}\right) = 0 \tag{6.38}$$

up to zeroth order in s and

$$(\mathcal{D}|_\sigma\,\mathcal{D}|_{\sigma'}\,\mathcal{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'} = -\frac{1}{\sqrt{s}^3}(b_0^1 g^{\mu\nu} + s b_1^1 g_{\lambda\rho} z'^{\lambda} z'^{\rho} g^{\mu\nu} + s b_2^1 z'^{\mu} z'^{\nu}) \tag{6.39}$$

where b_1^0, b_0^1, b_1^1, b_2^1, ... are dimensionless constants. b_0^0 and b_0^1 are determined by our choice of regularisation of the delta-function, $\mathcal{G}_s$

$$b_0^0 = \sqrt{s}\mathcal{G}_s(0), \quad b_0^1 = \sqrt{s}^3\mathcal{G}''(0)$$

What we want is to relate them to the remaining coefficients by imposing the closure of the Poincaré algebra. Using these expressions we can write $\Delta_s F_n$ as

$$\Delta_s F_n = -\frac{n(n-1)b_0^1}{\sqrt{s}^3}\int d\sigma g^{\mu\nu} f_{\mu\nu\rho_3\ldots\rho_n} z'^{\rho_3}\ldots z'^{\rho_n} + \frac{1}{\sqrt{s}}\int d\sigma (J_n f)_{\rho_1\ldots\rho_n} z'^{\rho_1}\ldots z'^{\rho_n} \tag{6.40}$$

where

$$(J_n f)_{\rho_1\ldots\rho_n} = b_0^0 g^{\mu\nu}(D_\mu D_\nu f_{\rho_1\ldots\rho_n} + n f_{\lambda(\rho_2\ldots\rho_n} R_{\rho_1)\mu\nu}{}^{\lambda}) -$$

$$n(n-1)(b_1^1 g^{\mu\nu} f_{\mu\nu(\rho_3\ldots\rho_n} g_{\rho_1\rho_2)} + b_2^1 f_{\rho_1\ldots\rho_n})$$

The bracket means symmetrisation of the enclosed indices.

In a similar way we can calculate $M_s F_n$, in which instead of $\mathbf{G}$ we have $\frac{\sigma_1+\sigma_2}{2}\mathbf{G}$, so that there is an additional piece coming from the second integral on the r.h.s. of (6.36). That is

$$M_s F_n = -\frac{n(n-1)b_0^1}{\sqrt{s}^3}\int d\sigma\,\sigma\, g^{\mu\nu} f_{\mu\nu\rho_3\ldots\rho_n} z'^{\rho_3}\ldots z'^{\rho_n} + \frac{1}{\sqrt{s}}\int d\sigma\,\sigma\,(J_n f)_{\rho_1\ldots\rho_n} z'^{\rho_1}\ldots z'^{\rho_n} + \\ + \frac{nb_0^0}{\sqrt{s}}\int d\sigma D^\mu f_{\mu\rho_2\ldots\rho_n} z'^{\rho_2}\ldots z'^{\rho_n}. \tag{6.41}$$

As we have already seen, by imposing the closure of the Poincaré algebra relations $[M_{s_1}, \Delta_{s_2}]F_n = 0$, $M_{s_1} V = 0$ and $\Delta_{s_2} N = 0$ have to hold. We can use the results above repeatedly to compute

$$[M_{s_1}, \Delta_{s_2}]F_n = (s_2^{-1/2} s_1^{-3/2} - s_2^{-3/2} s_1^{-1/2}) 2n(n-1)b_0^1 \times \\ (kb_0^0 + (N-1)b_1^1 + (4n-6)(b_1^1 + b_2^1)) - \\ 2n(n-1)b_0^0 (s_1 s_2)^{-1/2}\int d\sigma \left(\left(b_1^1 + \frac{kb_0^0}{N-1}\right) D_{(\rho_2}(\mathrm{tr}\, f)_{\rho_3\ldots\rho_n)} + \right. \\ \left. \left(b_2^1 - \frac{kb_0^0}{N-1}\right) D^\mu f_{\mu\rho_2\ldots\rho_n}\right) z'^{\rho_2}\ldots z'^{\rho_n} \tag{6.42}$$

where $(\mathrm{tr}\, f)_{\rho_3\ldots\rho_n} = g^{\mu\nu} f_{\mu\nu\rho_3\ldots\rho_n}$. Relation (6.42) will vanish, as demanded, for any n by taking

$$b_1^1 = -b_2^1 = -\frac{kb_0^0}{N-1} = \frac{b_0^0}{a^2} \tag{6.43}$$

Also from (6.40) and (6.41) the conditions $M_s V = 0$ and $\Delta_s N = 0$ take the form

$$M_s V = \Delta_s N = -\frac{2Nb_0^1}{\sqrt{s}^3}\int d\sigma\,\sigma - \frac{2}{\sqrt{s}}(kb_0^0 + Nb_1^1 + b_2^1)\int d\sigma\,\sigma\, g_{\mu\nu} z'^\mu z'^\nu = 0$$

which give identical relations as in (6.43), while the ill-defined integral $\int d\sigma\sigma$ vanishes as the integrand is odd.

Having these information we can built $\mathbf{G}$ up to first order. Substituting (6.43) into (6.37) and (6.39) we get

$$\mathbf{G}^{\mu\nu}(\sigma,\sigma) = \mathcal{G}_s(0) g^{\mu\nu}$$

and

$$(\mathcal{D}|_\sigma\, \mathcal{D}|_{\sigma'}\, \mathbf{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'} = -\mathcal{G}_s''(0) g^{\mu\nu} + \mathcal{G}_s(0) R^\mu{}_\lambda{}^\nu{}_\rho z'^\lambda z'^\rho$$

To this we can add the condition $(\mathcal{D}|_\sigma\, \mathbf{G}^{\mu\nu}(\sigma,\sigma'))|_{\sigma=\sigma'} = 0$, which follows from dimensional analysis and rotational invariance.

6.6.4 Second Order Calculations

In the same way we may assume that for the next order the constraint equations resulting from the closure of the Poincaré algebra, will be independent of the general form of the test functional, as soon as the differential operator is of order four. We can apply the Laplacian on $\int d\sigma\, f(z(\sigma),\sigma)_{\mu\nu}\mathcal{D}z'^{\mu}\mathcal{D}z'^{\nu} \equiv F$, where f is ultra-local. Then, the two functional derivatives in the Laplacian will generate a fourth order differential operator acting on $\delta(\sigma_1 - \sigma_2)$. Integrating by parts allows this operator to act on one of the σ arguments of the kernel, whilst the delta-function sets both arguments equal. The consequence of this is that $\Delta_s F$ now depends on the fourth derivative of the Kernel evaluated at co-incident points. Demanding the closure of the Poincaré algebra acting on F will constrain this quantity. F is the lowest order functional of the general form $\int d\sigma f_{\rho_1\ldots\rho_n}\mathcal{D}z'^{\rho_1}\ldots\mathcal{D}z'^{\rho_n}$ that gives a constraint to this order.

To simplify the calculations we notice that the part of the result of the operation of the Lorentz operator on a general functional, which will contribute to the commutation relations of the Poincaré algebra, is the non-homogenous one in σ. To prove it let us consider a test functional h and the action of Δ_s and M_s on it. They can be generally written as

$$\Delta_s h = \int d\sigma \bar{\Delta}_s h \tag{6.44}$$

and

$$M_s h = \int d\sigma \sigma \bar{\Delta}_s h + \int d\sigma \tilde{h} \tag{6.45}$$

where $\tilde{h}$ is a new functional not linearly dependent on σ and $\bar{\Delta}_s$ is the density of the operator Δ_s with respect to the variable σ. Thus

$$M_s\Delta_s h = \int d\sigma \sigma \bar{\Delta}_s(\bar{\Delta}_s h) + \int d\sigma \widetilde{\bar{\Delta}_s h} \tag{6.46}$$

and

$$\Delta_s M_s h = \int d\sigma \sigma \bar{\Delta}_s(\bar{\Delta}_s h) + \int d\sigma \bar{\Delta}_s \tilde{h} \tag{6.47}$$

so that

$$(\Delta_s M_s - M_s\Delta_s)h = \int d\sigma \bar{\Delta}_s \tilde{h} - \int d\sigma \widetilde{\bar{\Delta}_s h} \tag{6.48}$$

which means that in our calculations, only the inhomogeneous terms of (6.46) and (6.47) will be needed.

We treat F as a scalar so that

$$\mathbf{D}_\mu(\sigma)F = \frac{\delta F}{\delta z^\mu(\sigma)} = D_\mu f_{\kappa\lambda}\mathcal{D}z'^\kappa\mathcal{D}z'^\lambda + 2\mathcal{D}^2(f_{\mu\lambda}\mathcal{D}z'^\lambda) + 2f_{\kappa\lambda}z'^i z'^j R_{\mu ij}{}^\kappa \mathcal{D}z'^\lambda \tag{6.49}$$

which is an infinite component co-vector. Using again the commutators of $\mathbf{D}$ and $\mathcal{D}$ worked out above we can show that

$$\begin{aligned}
M_s F = \int d\sigma\sigma \mathbf{G}^{\mu\nu}(\sigma,\sigma)\Big((D_\mu D_\nu f_{\rho_1\rho_2} + 2R_{\mu\rho_1}{}^\lambda{}_\nu\, f_{\lambda\rho_2})\mathcal{D}z'^{\rho_1}\mathcal{D}z'^{\rho_2} \\
2f_{\rho_1\rho_2}R_{\nu i_1 i_2}{}^{\rho_1}R_{\mu j_1 j_2}{}^{\rho_2} z'^{i_1}z'^{i_2}z'^{j_1}z'^{j_2} + 4D_\mu f_{\rho_1\rho_2}R_{\nu i_1 i_2}{}^{\rho_1} z'^{i_1}z'^{i_2}\mathcal{D}z'^{\rho_2}\Big) + \\
4\int d\sigma\,\sigma\,\mathcal{D}_\sigma^2\,\mathbf{G}^{\mu\nu}(\sigma,\sigma')\Big|_{\sigma=\sigma'}(D_\nu f_{\mu\rho_2}\mathcal{D}z'^{\rho_2} + f_{\rho\mu}R^\rho_{\nu ij}z'^i z'^j) + \\
2\int d\sigma\sigma\,\mathcal{D}_\sigma^4\mathbf{G}^{\mu\nu}(\sigma,\sigma')\Big|_{\sigma=\sigma'} f_{\mu\nu} + \\
4\int d\sigma\,\mathbf{G}^{\mu\nu}(\sigma,\sigma) f_{\rho_1\rho_2}R^{\rho_1}_{\nu i\mu}z'^i\mathcal{D}z'^{\rho_2}
\end{aligned} \tag{6.50}$$

Using relations (6.37), (6.38), (6.39) and the additional

$$\begin{aligned}
\Big(\mathcal{D}_\sigma^4\mathbf{G}^{\mu\nu}(\sigma,\sigma')\Big)_{\sigma=\sigma'} = \frac{1}{\sqrt{s}^5}\big(\, b_0^2\, g^{\mu\nu} + s b_1^2\, g_{\lambda\rho}z'^\lambda z'^\rho\, g^{\mu\nu} + s b_2^2\, z'^\mu z'^\nu \\
+ s^2 b_3^2\,(g_{\lambda\rho}z'^\lambda z'^\rho)^2\, g^{\mu\nu} + s^2 b_4^2\, g_{\lambda\rho}z'^\lambda z'^\rho\, z'^\mu z'^\nu \\
+ s^2 b_5^2\,\mathcal{D}z'^\mu\mathcal{D}z'^\nu + s^2 b_6^2\, g_{\lambda\rho}\mathcal{D}z'^\lambda\mathcal{D}z'^\rho\, g^{\mu\nu} \\
+ s^2 b_7^2\, z'^{(\mu}\mathcal{D}^2 z'^{\nu)} + s^2 b_8^2\, g_{\lambda\rho}z'^{(\lambda}\mathcal{D}^2 z'^{\rho)} g^{\mu\nu}\ \big) \\
+ O(\sqrt{s})
\end{aligned} \tag{6.51}$$

we can write $M_s F$ in terms of the constants b_j^i. Because of the property

$$\mathcal{D}^k|_\sigma \mathbf{G}^{\mu\nu}(\sigma,\sigma')\Big|_{\sigma=\sigma'} = -\,\mathcal{D}^k|'_\sigma \mathbf{G}^{\mu\nu}(\sigma,\sigma')\Big|_{\sigma=\sigma'} + O(\sqrt{s}) \tag{6.52}$$

for k an odd integer, and the symmetry of Δ_s and M_s in σ,σ' there will not be any odd number of intrinsic derivatives acting on $\mathbf{G}$ in our final expressions so we will not need their expansions in terms of the b's.

After substituting into (6.50) we have

$$\begin{aligned}
M_s\int d\tilde{\sigma} f_{\mu\nu}\mathcal{D}z'^\mu\mathcal{D}z'^\nu = \int d\sigma\,\sigma\Bigg(\frac{2}{\sqrt{s}^5}b_0^2 f_\nu^\nu + \\
\frac{1}{\sqrt{s}^3}(4b_0^1 D^\nu f_{\nu\rho_2}\mathcal{D}z'^{\rho_2} + 4b_0^1 f_{\mu\nu}R^\mu_{ij}{}^\nu z'^i z'^j + 2b_1^2 f_\nu^\nu g_{ij}z'^i z'^j + 2b_2^2 f_\mu\nu z'^\mu z'^\nu) + \\
\frac{1}{\sqrt{s}}((\bar{J}_2 f)_{\mu\nu}\mathcal{D}z'^\mu z'^\nu + (\bar{J}_4 f)_{\mu\nu\kappa\lambda}z'^\mu z'^\nu z'^\kappa z'^\lambda + (\bar{J}_3 f)_{\mu\nu}z'^{(\mu}\mathcal{D}^2 z'^{\nu)})\Bigg) +
\end{aligned}$$

$$\int d\sigma \frac{4}{\sqrt{s}} b_0^0 \frac{1-N}{\alpha^2} f_{\mu\nu} z'^{\mu} \mathcal{D} z'^{\nu}$$

where

$$(\bar{J}_2 f)_{\mu\nu} = b_0^0 \tilde{\Delta} f_{\mu\nu} + 2b_0^0 R^{\rho}_{(\mu} f_{\nu)\rho} + 2b_5^2 f_{\mu\nu} + 2b_6^2 f^{\kappa}_{\kappa} g_{\mu\nu}$$

$$(\bar{J}_4 f)_{\mu\nu\kappa\lambda} = 2(b_3^2 - \frac{b_0^0}{\alpha^4}) f^{\rho}_{\rho} g_{(\mu\nu} g_{\kappa\lambda)} + 2(b_4^2 + \frac{b_0^0}{\alpha^4}) f_{(\mu\nu} g_{\kappa\lambda)}$$

$$(\bar{J}_3 f)_{\mu\nu} = 2b_7^2 f_{\mu\nu} + 2b_8^2 f^{\kappa}_{\kappa} g_{\mu\nu}$$

We can easily read of the action of Δ_s from above (terms proportional to σ). In order to compute the commutator of M_s and Δ_s on $\int f_{\mu\nu} \mathcal{D} z'^{\mu} \mathcal{D} z'^{\nu}$, we need the following relations:

$$M_s \int f_{\kappa} \mathcal{D} z'^{\kappa} = .. + \int d\sigma 2 \frac{1}{\sqrt{s}} b_0^0 R^{\rho}_{\gamma} f_{\rho} z'^{\gamma} \tag{6.53}$$

$$M_s \int f_{\kappa\lambda\rho} z'^{\kappa} z'^{\lambda} \mathcal{D} z'^{\rho} = .. + \int d\sigma (-\frac{1}{\sqrt{s^3}} b_0^1 f^{\nu}_{\lambda\nu} z'^{\lambda}) \tag{6.54}$$

$$M_s \int f_{\kappa\lambda} z'^{(\kappa} \mathcal{D}^2 z'^{\lambda)} = .. + \int d\sigma 3 \frac{1}{\sqrt{s^3}} b_0^1 D^{\nu} f_{\kappa\nu} z'^{\kappa} \tag{6.55}$$

$$M_s \int f_{\mu\nu} z'^{\mu} z'^{\nu} = .. + \int d\sigma \frac{2}{\sqrt{s}} b_0^0 D^{\mu} f_{\mu\rho} z'^{\rho} \tag{6.56}$$

$$\Delta_s \int f_{\kappa\lambda} z'^{\kappa} \mathcal{D} z'^{\lambda} = \int d\sigma \left(\frac{1}{\sqrt{s^3}} 2b_0^1 D^{\nu} f_{\kappa\nu} z'^{\kappa} + \frac{1}{\sqrt{s}} b_0^0 (\tilde{\Delta} f_{\mu\nu} + 2\frac{1-N}{\alpha^2} f_{\mu\nu}) z'^{\mu} \mathcal{D} z'^{\nu} \right) \tag{6.57}$$

where .., represents the homogenous part of the action of M_s on the specific functional. Using relations (6.53) to (6.57) we can derive the action of the operators $M_s \Delta_s$ and $\Delta_s M_s$ on $\int f_{\mu\nu} \mathcal{D} z'^{\mu} \mathcal{D} z'^{\nu}$. They are given by

$$M_s \Delta_s \int f_{\mu\nu} \mathcal{D} z'^{\mu} \mathcal{D} z'^{\nu} =$$

$$.. + \int d\sigma \left(\frac{1}{s} (4b_0^0 \frac{1-N}{\alpha^2} b_0^0 \tilde{\Delta} f_{\mu\nu} + 2b_0^0 R^{\nu}_{\mu} f_{\nu\rho} + 2b_5^2 f_{\mu\nu} + 2b_6^2 f^{\kappa}_{\kappa} g_{\mu\nu}) z'^{\mu} \mathcal{D} z'^{\nu} + \right.$$

$$4b_0^0 D^{\mu} (2(b_3^2 - \frac{b_0^0}{\alpha^4}) f^{\rho}_{\rho} g_{(\mu\nu} g_{\kappa\lambda)} + 2(b_4^2 + \frac{b_0^0}{\alpha^4}) f_{(\mu\nu} g_{\kappa\lambda)}) z'^{\nu} z'^{\kappa} z'^{\lambda} + \mathcal{M}_s (J_3 f)_{\mu\nu} z'^{(\mu} D^2 z'^{\nu)})$$

$$\frac{1}{s^2} (4b_0^1 b_0^0 2 \frac{1-N}{\alpha^2} D^{\nu} f_{\nu\rho} + 4b_0^0 b_1^2 D_{\rho} f^{\kappa}_{\kappa} + 4b_0^0 b_2^2 D^{\mu} f_{\mu\rho} +$$

$$\left. 6b_0^1 b_7^2 D^{\nu} f_{\nu\rho} + 6b_0^1 b_8^2 D_{\rho} f^{\kappa}_{\kappa} + 4\frac{b_0^1 b_0^0}{\alpha^2} (2D_{\rho} f^{\kappa}_{\kappa} - 2D^{\nu} f_{\nu\rho})) z'^{\rho} \right)$$

and

$$\Delta_s M_s \int f_{\mu\nu} \mathcal{D} z'^{\mu} \mathcal{D} z'^{\nu} =$$

$$.. + \int d\sigma \left(\frac{4}{s} b_0^{0\,2} \frac{1-N}{\alpha^2} (\tilde{\Delta} f_{\mu\nu} + 2\frac{1-N}{\alpha^2} f_{\mu\nu}) z'^{\mu} \mathcal{D} z'^{\nu} \right.$$

$$+\frac{1}{s^2}4b_0^0\frac{1-N}{\alpha^2}2b_0^1 D^\nu f_{\rho\nu}z'^\rho\Big)$$

$\mathcal{M}_s$ is the density of M_s with respect to σ integration. The $O(1/s^3)$ order does not contribute to the commutator. The $O(1/s^2)$ order can be factorised with coefficients the various z combinations. The resulting equations for the b's are

$$b_3^2 = -b_4^2 = \frac{b_0^0}{\alpha^4} \quad \text{and} \quad b_5^2 = b_6^2 = b_7^2 = b_8^2 = 0 \tag{6.58}$$

The $O(1/s)$ order contributes the following relations

$$-4b_0^1 b_0^0\frac{1}{\alpha^2} + 2b_0^0 b_2^2 + 3b_0^1 b_7^2 = 0$$

$$4\, b_0^1 b_0^0\frac{1}{\alpha^2} + 2b_0^0 b_1^2 + 3b_0^1 b_8^2 = 0$$

Which with the use of (6.43) they become

$$-\,2b_0^1\frac{1}{\alpha^2} + b_2^2 = 0 \tag{6.59}$$

$$2\, b_0^1\frac{1}{\alpha^2} + b_1^2 = 0 \tag{6.60}$$

We can see that from these relations the b's needed up to this order in the Kernel are completely determined. We can assume that if the theory is consistent these b's are the same for every test functional we use with the same number of σ derivatives, acting on z, in F.

Chapter 7

The Laplacian on a General Curved Manifold

7.1 Introduction

We have seen how the Laplacian can be constructed on an N-dimensional sphere up to the second order. However, when we attempt to construct the Laplacian for a general manifold with metric $g_{\mu\nu}(X(\sigma))$, difficulties arise. We will see that such a construction is not possible because we used a Kernel to split the functional derivatives of the Laplacian. A Kernel is able to approximate the physical object we constructed within a small region around a definite point σ. Trying to describe a general manifold with a tool like this will fail. Though, it is possible to approximate a positive curvature manifold in a small region to first order by a sphere for which we have already constructed the Laplacian. In the following section we see how we can construct an ansatz for the Laplacian on a general manifold by using geometrical means and thereafter proceed to how this object fails to satisfy the Poincaré algebra by using a first order test functional.

7.2 Construction of the Kernel

In Chapter 6 we saw that for a curved manifold (sphere), the Kernel $\mathbf{G}^{\mu\nu}(\sigma,\sigma')$, which we used to split the action of the two functional derivatives, can be defined by deter-

mining its various derivatives at the point $\sigma' = \sigma$. We can discover a general ansatz for them, from dimensional demands and symmetry properties. The Kernel has dimensions of inverse length of σ and zero in length of z, and it is a second rank covariant tensor. On the manifold, the only "intrinsic" geometrical objects we have are the metric $g_{\mu\nu}$, the Riemannian Curvature $R^m{}_{nkl}$, with its contractions, and the vector $z'^{\mu}(\sigma)$ with its higher derivatives. Using these properties we have for

$$\mathbf{G}^{\mu\nu}(\sigma,\sigma') = \mathcal{G}_s(\sigma-\sigma')\, K^{\mu\nu}(\sigma,\sigma';s) \tag{7.1}$$

the following decomposition

$$G^{\mu\nu}(\sigma,\sigma) = \frac{1}{\sqrt{s}} c_0^0 g_{\mu\nu} \tag{7.2}$$

$$\mathcal{D}^2 G^{\mu\nu}(\sigma,\sigma) = \frac{1}{\sqrt{s}^3}(c_0^1 g_{\mu\nu} + s(c_1^1 R\ z'^{\mu}z'^{\nu} + c_1^2 R^{\mu\nu} g_{\kappa\lambda} z'^{\kappa} z'^{\lambda} + c_1^3 R\ g_{\mu\nu}\ g_{\kappa\lambda} z'^{\kappa} z'^{\lambda} +$$
$$c_1^4 g^{\mu\nu} R_{\kappa\lambda}\ z'^{\kappa} z'^{\lambda} + c_1^5 R^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda} z'^{\kappa} z'^{\lambda} + c_1^6 R^{(\mu}{}_{|\kappa|} z'^{\nu)} z'^{\kappa})) \tag{7.3}$$

where c_0^0 and c_0^1 are determined by our choice of regularisation of the delta function, $\mathcal{G}_s$, as

$$c_0^0 = \sqrt{s}\mathcal{G}_s(0),\ c_0^1 = \sqrt{s}^3 \mathcal{G}''(0) \tag{7.4}$$

and the constants c_j^i are functions of s. We see that for the curvature of a sphere the Kernel reduces to the same expansion we used in (6.37), (6.38) and (6.39).

7.3 Poincaré Algebra

We can proceed, as before, to calculate the action of the Laplacian and the Lorentz operator on a first order test functional $\int d\sigma f_{\mu_1\ldots\mu_n} z'^{\mu_1}..z'^{\mu_n}$. This is a generalisation of the formulae we had for the sphere

$$M_s \int d\sigma f_{\rho_1\ldots\rho_n} z'^{\rho_1}..z'^{\rho_n} =$$
$$\int d\sigma\sigma\{G^{\mu\nu} D_\mu D_\nu f_{\rho_1..\rho_n}\ z'^{\rho_1}..z'^{\rho_n} + nG^{\mu\nu} z'^{\gamma} R_{\mu\gamma\nu}{}^{\rho} f_{\rho\rho_2..p_n} z'^{\rho_2}..z'^{\rho_n} -$$
$$n(n-1)\mathcal{D}^2 G^{\mu\nu} f_{\mu\nu\rho_3..\rho_n} z'^{\rho_3}..z'^{\rho_n}\} +$$
$$\int d\sigma n G^{\mu\nu} D_\nu f_{\mu\rho_2..\rho_n} z'^{\rho_2..\rho_n} \tag{7.5}$$

and substituting (7.3) and (7.2) in (7.5) we have

$$M_s \int d\sigma f_{\rho_1..\rho_n} z'^{\rho_1}..z'^{\rho_n} =$$

$$\int d\sigma \frac{\sigma}{\sqrt{4\pi s}} \left\{ \frac{n\,(n-1)}{2s} c_0^0 f^{\nu}{}_{\nu\rho_3..\rho_n} z'^{\rho_3}..z'^{\rho_n} + (J_n f)_{(\rho_1..\rho_n)}\, z'^{\rho_1}..z'^{\rho_n} \right\} +$$

$$\int d\sigma \frac{n}{\sqrt{4\pi s}} c_0^0 D^{\nu} f_{\nu\rho_2..\rho_n} z'^{\rho_2}..z'^{\rho_n}$$

where

$$(J_n f)_{(\rho_1..\rho_n)} = c_0^0 \tilde{\Delta} f_{\rho_1..\rho_n} + n c_0^0 R^{\rho}_{(\rho_1} f_{|\rho|\rho_2..\rho_n)} -$$

$$-n(n-1)(c_1^1 R f_{\rho_1..\rho_n} + R^{\mu\nu} f_{\mu\nu(\rho_3..\rho_n} g_{\rho_1\rho_2)} +$$

$$c_0^3 R g^{\mu\nu} f_{\mu\nu(\rho_3..\rho_n} g_{\rho_1\rho_2)} + g^{\mu\nu} f_{\mu\nu(\rho_3..\rho_n} R_{\rho_1\rho_2)} +$$

$$c_0^5 R^{\mu}{}_{(\rho_1}{}^{\nu}{}_{\rho_2} f_{|\mu\nu|\rho_3..\rho_n)} + c_0^6 R^{\mu}_{(\rho_1} f_{|\mu|\rho_2..\rho_n)})$$

Now we can calculate the inhomogeneous part, arising from the action of the successive applications of M_s and Δ_s or Δ_s and M_s on the test functional. This will give

$$\Delta_s M_s \int d\sigma f_{\rho_1...\rho_n} z'^{\rho_1}..z'^{\rho_n} = .. + \int d\sigma \frac{n}{4\pi s} c_0^0 \{ \frac{(n-1)(n-2)}{2s} c_0^0 D^{\nu} f_{\nu}{}^{\kappa}{}_{\kappa\rho_4..\rho_n} z'^{\rho_4}..z'^{\rho_n} +$$

$$(J_{n-1} D^{\nu} f_{\nu\rho_2..\rho_n})_{(\rho_2..\rho_n)} z'^{\rho_2}..z'^{\rho_n} \}$$

and

$$M_s \Delta_s \int d\sigma f_{\rho_1...\rho_n} z'^{\rho_1}..z'^{\rho_n} = .. + \int d\sigma \frac{n}{4\pi s} c_0^0 \{ \frac{n(n-1)}{2s} c_0^0 (n-2) D^{\nu} f^{\kappa}{}_{\kappa\nu\rho_4..\rho_n} z'^{\rho_4}..z'^{\rho_n} +$$

$$n D^{\nu} (J_n f)_{(\nu\rho_2..\rho_n)} z'^{\rho_2}..z'^{\rho_n} \}$$

so that

$$(\Delta_s M_s - M_s \Delta_s) \int d\sigma f_{\rho_1...\rho_n} z'^{\rho_1}..z'^{\rho_n} =$$

$$\int d\sigma \frac{n}{4\pi s} \left\{ (J_{n-1} D^{\nu} f_{\nu\rho_2..\rho_n})_{(\rho_2..\rho_n)} - D^{\nu} (J_n f)_{(\nu\rho_2..\rho_n)} \right\} z'^{\rho_2}..z'^{\rho_n}$$

In order to simplify the calculations we will take $n = 2$. The results derived from this special case will enforce the same restrictions as for a general n. So we have

$$(\Delta_s M_s - M_s \Delta_s) \int d\sigma f_{\rho_1\rho_2} z'^{\rho_1} z'^{\rho_2} =$$

$$\int d\sigma \frac{2c_0^0}{4\pi s} \{ c_0^0 [-2R^{\mu\beta\lambda}{}_{\rho_2} D_{\beta} f_{\mu\lambda} + R^{\beta\lambda} D_{\beta} f_{\lambda\rho_2} -$$

$$D_{\beta} f_{\lambda\rho_2} - D_{\beta} R^{\mu\beta\lambda}{}_{\mu} f_{\lambda\rho_2} - D_{\beta} R^{\mu\beta\lambda}{}_{\rho_2} f_{\mu\lambda}] +$$

$$c_0^0 R^{\rho}_{\rho_2} D^{\nu} f_{\nu\rho} - 2c_0^0 D^{\nu} (R_{(\nu|}{}^{\rho} f_{\rho|\rho_2)}) + 2c_0^1 (R f_{\nu\rho_2})_{;}{}^{\nu} + 2c_0^2 (R^{\mu\nu} f_{\mu\nu})_{;\rho_2} +$$

$$2c_0^3 (R f^{\kappa}_{\kappa})_{;\rho_2} + 2c_0^4 (R_{\nu\rho_2} f^{\kappa}_{\kappa})_{;}{}^{\nu} +$$

$$2c_0^5 D^{\nu} (R^{\mu}{}_{(\nu|}{}^{\kappa}{}_{|\rho_2)} f_{\mu\kappa}) + 2c_0^6 (R^{\mu}{}_{(\nu|} f_{\mu|\rho_2)})_{;}{}^{\nu} \} z'^{\rho_2} \qquad (7.6)$$

We can use the *Bianchi identity* which the Riemannian Curvature $R^{\lambda}{}_{\mu\nu\rho}$ satisfies

$$R^{\lambda}{}_{\mu[\nu\rho;\sigma]} = 0$$

which for the $\lambda = \sigma$ contraction gives

$$R^{\lambda}_{\mu\nu\rho;\lambda} = R_{\mu\nu;\rho} - R_{\mu\rho;\nu}$$

By contracting further the μ and ρ indices, we get

$$R^{\lambda\nu}{}_{;\lambda} = \frac{1}{2} g^{\lambda\nu} R_{;\lambda} = \frac{1}{2} R_{;}{}^{\nu}$$

With the help of these identities we can simplify expression (7.6) by grouping terms, like

$$(\Delta_s M_s - M_s \Delta_s) \int d\sigma f_{\rho_1\rho_2} z'^{\rho_1} z'^{\rho_2} =$$

$$\int d\sigma \frac{2c_0^0}{4\pi s} \{ f_{\mu\lambda} [-c_0^0 R^{\mu\lambda}{}_{;\rho_2} + 2c_0^2 R^{\mu\lambda}{}_{;\rho_2} + 2c_0^3 g^{\mu\lambda} R_{;\rho_2} +$$

$$c_0^6 R^{\mu}{}_{\rho_2;}{}^{\lambda} + c_0^4 g^{\mu\lambda} R_{;\rho_2} + 2c_0^5 (-R^{\mu}{}_{\rho_2;}{}^{\lambda} + R^{\mu\lambda}{}_{;\rho_2})] +$$

$$f_{\nu\rho_2} R_{;}{}^{\nu} (2c_0^1 + \frac{1}{2} c_0^6) -$$

$$2c_0^0 R^{\mu\beta\lambda}{}_{\rho_2} f_{\mu\lambda;\beta} + 2c_0^1 R f^{\nu}{}_{\rho_2;\nu} + 2c_0^2 R^{\mu\nu} f_{\mu\nu;\rho_2} +$$

$$2c_0^3 R f^{\kappa}{}_{\kappa;\rho_2} + 2c_0^4 R^{\nu}{}_{\rho_2} f^{\kappa}{}_{\kappa;\nu} + 2c_0^5 R^{\mu\nu\kappa}{}_{\rho_2} f_{\mu\kappa;\nu} + c_0^6 (R^{\mu\nu} f_{\mu\rho_2;\nu} + R^{\mu}{}_{\rho_2} f_{\mu\nu;}{}^{\nu}) \} z'^{\rho_2}$$

We can decompose the Riemannian Curvature in terms of trace-free parts

$$R^{\mu\beta\lambda}{}_{\rho_2} = C^{\mu\beta\lambda}{}_{\rho_2} + \frac{1}{N-2} \left(g^{\mu\lambda} R^{\beta}{}_{\rho_2} - \delta^{\mu}{}_{\rho_2} R^{\beta\lambda} - g^{\beta\lambda} R^{\mu}{}_{\rho_2} + \delta^{\beta}{}_{\rho_2} R^{\mu\lambda} \right) -$$

$$\frac{R}{(N-1)(N-2)} \left(g^{\mu\lambda} \delta^{\beta}{}_{\rho_2} - g^{\beta\lambda} \delta^{\mu}{}_{\rho_2} \right)$$

where $C^{\mu\beta\lambda}{}_{\rho_2}$ is the *Weyl tensor*. Finally the commutator becomes

$$(\Delta_s M_s - M_s \Delta_s) \int d\sigma \, f_{\rho_1\rho_2} z'^{\rho_1} z'^{\rho_2} =$$

$$\int d\sigma \frac{2c_0^0}{4\pi s} \{ \, f_{\mu\lambda} [(-c_0^0 + 2c_0^2 + 2c_0^5) R^{\mu\lambda}{}_{;\rho_2} + (2c_0^3 + c_0^4) g^{\mu\lambda} R_{;\rho_2} + (c_0^6 - 2c_0^5) R^{\mu}{}_{\rho_2;}{}^{\lambda}] +$$

$$f_{\nu\rho_2} R_{;}{}^{\nu} [2c_0^1 + \frac{1}{2} c_0^6] +$$

$$2C^{\mu\beta\lambda}{}_{\rho_2} f_{\mu\lambda;\beta} [-c_0^0 + c_0^5] +$$

$$f_{\rho_2\lambda;\beta} [(-2 \frac{-c_0^0 + c_0^5}{N-2} + c_0^6) R^{\beta\lambda} + (2 \frac{-c_0^0 + c_0^5}{(N-1)(N-2)} + 2c_0^1) R g^{\lambda\beta}] +$$

$$f_{\kappa\lambda;\beta} [(2 \frac{-c_0^0 + c_0^5}{N-2} + 2c_0^4) g^{\kappa\lambda} R^{\beta}{}_{\rho_2} + (-2 \frac{-c_0^0 + c_0^5}{N-2} + c_0^6) g^{\lambda\beta} R^{\kappa}{}_{\rho_2}] +$$

$$f_{\mu\lambda;\rho_2}[(2\frac{-c_0^0+c_0^5}{N-2}+2c_0^2)R^{\mu\lambda}+(-2\frac{-c_0^0+c_0^5}{(N-1)(N-2)}+2c_0^3)Rg^{\mu\lambda}]\}z'^{\rho_2}=0 \tag{7.7}$$

In order for this to be zero for general f and R, each set of c combinations has to be zero. For the third term of the r.h.s. we have two options. Either $C^{\mu\beta\lambda}{}_{\rho_2}=0$ or $-c_0^0+c_0^5=0$. If we chose $c_0^5=c_0^0$ then we have the following set of equations

$$(c_0^0+2c_0^2)R^{\mu\lambda}{}_{;\rho_2}+(2c_0^3+c_0^4)g^{\mu\lambda}R_{;\rho_2}+(c_0^6-2c_0^0)R^{\mu}{}_{\rho_2;}{}^{\lambda}=0 \tag{7.8}$$

$$2c_0^1+\frac{1}{2}c_0^6=0 \tag{7.9}$$

$$c_0^6R^{\beta\lambda}+2c_0^1Rg^{\lambda\beta}=0 \tag{7.10}$$

$$2c_0^4g^{\kappa\lambda}R^{\beta}{}_{\rho_2}+c_0^6g^{\lambda\beta}R^{\kappa}{}_{\rho_2}=0 \tag{7.11}$$

$$2c_0^2R^{\mu\lambda}+2c_0^3Rg^{\mu\lambda}=0 \tag{7.12}$$

With (7.9), equation (7.10) becomes

$$R^{\beta\lambda}=\frac{1}{2}Rg^{\beta\lambda} \tag{7.13}$$

Substituting (7.13) into (7.8) we obtain

$$[\frac{1}{2}(c_0^0+2c_0^2)+(2c_0^3+c_0^4)]R_{;\rho_2}+(c_0^6-2c_0^0)R_{;}{}^{\lambda}\delta^{\mu}{}_{\rho_2}=0$$

For $\mu\neq\rho_2$ we get $g^{\mu\lambda}R_{;\rho_2}=0\Rightarrow R_{;\rho_2}=0$ for general μ and λ. This implies only the metric of a sphere with a positive or negative sign. We have studied this case above. So we are left with the option of the coefficients being zero:

$$\frac{1}{2}(c_0^0+2c_0^2)+(2c_0^3+c_0^4)=0 \tag{7.14}$$

$$c_0^6-2c_0^0=0 \tag{7.15}$$

Now substituting (7.13) into (7.11) we have

$$2c_0^4g^{\kappa\lambda}\delta^{\beta}{}_{\rho_2}+c_0^6g^{\lambda\beta}\delta^{\kappa}{}_{\rho_2}=0 \tag{7.16}$$

which means for $\beta\neq\rho_2\Rightarrow g^{\lambda\beta}\delta^{\kappa}{}_{\rho_2}=0$ or equivalently, $g^{\lambda\beta}=0$, for $\kappa=\rho_2$ and for any choice of λ and β (for any value of β). This of course is not an option at all, so we have to consider the case of having their coefficients equal to zero. That is $c_0^6=0$. But this again is in contrast with (7.15), where c_0^0 is different from zero as demanded from relation (7.4). We can conclude that the set of equations (7.8)-(7.12) does not allow any manifold other than the sphere.

The second case, $C^{\mu\beta\lambda}{}_{\rho_2} = 0$, makes expression (7.7) produce the following equations

$$\frac{k}{N-2} + 2c_0^4 = 0 \tag{7.17}$$

$$-\frac{k}{N-2} + c_0^6 = 0 \tag{7.18}$$

$$\frac{k}{(N-1)(N-2)} + 2c_0^1 = 0 \tag{7.19}$$

$$(\frac{k}{N-2} + 2c_0^2)R^{\mu\lambda} + (-\frac{k}{(N-1)(N-2)} + 2c_0^3)Rg^{\mu\lambda} = 0 \tag{7.20}$$

$$2c_0^1 + \frac{1}{2}c_0^6 = 0 \tag{7.21}$$

$$(-c_0^0 + 2c_0^2 + c_0^5)R^{\mu\lambda}{}_{;\rho_2} + (2c_0^3 + c_0^4)g^{\mu\lambda}R_{;\rho_2} + (c_0^6 - 2c_0^5)R^{\mu}{}_{\rho_2;}{}^{\lambda} = 0 \tag{7.22}$$

where $k \equiv 2(-c_0^0 + c_0^5) \neq 0$. The first two equations follow from a parallel argument to equation (7.16). Now from (7.18), (7.19) and (7.21) we get

$$-\frac{k}{(N-1)(N-2)} + \frac{1}{2}\frac{k}{N-2} = 0 \Rightarrow N = 3$$

This fixes the dimension of the manifold to three. From equation (7.20), assuming that $\frac{k}{N-2} + 2c_0^2 \neq 0$, we can get a simplified expression $R^{\mu\lambda} = \Lambda R g^{\mu\lambda}$ where $\Lambda \equiv (-\frac{k}{(N-1)(N-2)} + 2c_0^3)/(\frac{k}{N-2} + 2c_0^2)$ is assumed not to be zero. Substituting into (7.22) we get

$$(-c_0^0 + 2c_0^2 + c_0^5)\Lambda g^{\mu\lambda}R_{;\rho_2} + (2c_0^3 + c_0^4)g^{\mu\lambda}R_{;\rho_2} + (c_0^6 - 2c_0^5)\Lambda\delta^{\mu}{}_{\rho_2}R_{;}{}^{\lambda} = 0$$

which implies $c_0^6 - 2c_0^5 = 0$. This together with (7.18) gives

$$-(-2c_0^0 + 2c_0^5) + 2c_0^5 = 0 \Rightarrow c_0^0 = 0$$

which is contrary to with the normalisation of the Kernel (see relation (7.4)).

7.4 Conclusions

From this treatment we see that for the proposed Kernel we cannot construct a Laplacian for a general manifold other than the sphere. This could be a product of the strong constraint of demanding the vanishing of the commutator when acting on a general test function. What we really need is that the commutator vanishes when it acts on eigenstates of the Hamiltonian as the cut-off is removed. However, we cannot construct the Hamiltonian with this method without calculating the Kernel first. This weakens the theory. Still, as has been mentioned in the introduction that the Kernel for the sphere can be used as a local approximation for a general manifold with positive curvature.

Chapter 8

Virasoro Algebra and $O(N)$ σ Model

8.1 Introduction

During the last twenty years a great deal of work has been done on conformally symmetric theories. Importantly, they can be exactly solved to give critical exponents of two dimensional theories, aiding their classification. Moreover, their conformal symmetry enables the association of strongly interacting fields with weakly coupled ones, which are easy to elaborate. To make physically interesting theories out of them it is necessary to incorporate interactions (e.g. curvature) in the free case. The cases of interacting fields which preserve the conformal symmetry in a stronger or a weaker sense have been studied in the literature [27], as well as theories where their interactions destroy this symmetry. One of the latter is the non-linear $O(N)$ σ model[1].

A considerable amount of interest is concentrated on the cylindrical space-time $\mathbf{R}^1 \times S^1$, which shares many features with string theory [28]. We will face one of them, the Virasoro algebra, which has been previously studied through different quantisation procedures, by using the functional formalism.

Our aim is to set up a general formalism for the study of a modified form of the Virasoro algebra for the $O(N)$ σ model. Instead of the usual central charge term we expect operator-like terms as a quantum anomaly extension of this algebra.

[1] In Chapter 6 we have seen how the $O(N)$ symmetry generates a mass term in the quantum level, which destroys the classical conformal symmetry of the model.

8.2 Conformal Symmetry

Firstly, a brief review of the conformal transformations. They are defined on an N dimensional Riemannian manifold with coordinates x^μ, $\mu = 1, .., N$ and metric g with

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu.$$

For *general coordinate transformations*

$$x^\mu \to \widetilde{x}^\mu(x)$$

which change the metric

$$\widetilde{g}_{\lambda\rho}(\widetilde{x}) = g_{\mu\nu}(x(\widetilde{x}))\frac{\partial x^\mu}{\partial \widetilde{x}^\lambda}\frac{\partial x^\nu}{\partial \widetilde{x}^\rho}$$

and for *Weyl transformations*

$$g_{\mu\nu}(x) = e^{\rho(x)}g_{\mu\nu}(x)$$

that is, a scaling of the metric with the coordinates unchanged, we can find in some theories coordinate transformations which compensate a Weyl transformation. In other words

$$\widetilde{g}_{\lambda\rho}(\widetilde{x}) = g_{\mu\nu}(x(\widetilde{x}))\frac{\partial x^\mu}{\partial \widetilde{x}^\lambda}\frac{\partial x^\nu}{\partial \widetilde{x}^\rho} = \widetilde{g}_{\lambda\rho}(\widetilde{x}) = e^{\varphi(\widetilde{x})}g_{\mu\nu}(\widetilde{x})$$

holds for an appropriate $\varphi(\widetilde{x})$. The conformal transformations are the ones which are a combination of these coordinate transformations followed by a compensating Weyl transformation. Their basic characteristic is that they lead to a representation of coordinates and metric for which to the point $\widetilde{x}$ is attached a metric similar to the one existing in x coordinates

$$g_{\mu\nu}(x) \to g_{\mu\nu}(\widetilde{x}).$$

Usually we construct theories which are classically general coordinate invariant. So to guaranty that they are conformally invariant we only need to check their Weyl invariance. For example we can consider the two dimensional σ model action

$$S = \frac{1}{2}\int d^2x\sqrt{g(x)}g^{\mu\nu}(x)\partial_\mu X(x)\partial_\nu X(x) \tag{8.1}$$

for $X(x)$ being a scalar field. Clearly S is conformal invariant. Though, these transformations change $X(x)$ to $\widetilde{X}(x)$. Infinitesimally we have

$$X(x) \to \widetilde{X}(x) = X(x - \epsilon(x)) = X(x) - \epsilon^i(x)\frac{\partial X(x)}{\partial x^i} \Rightarrow$$

$$\widetilde{X}(x) - X(x) = -\epsilon^i(x)\frac{\partial X(x)}{\partial x^i} \equiv \delta_\epsilon X$$

For a flat target space of X, the quantisation leads to an action which still preserves conformal invariance. The generators of the conformal transformations are

$$L[\epsilon] = \frac{1}{4}\int d\sigma\ \epsilon(\sigma) : (P + X')^2 : \tag{8.2}$$

which are called Virasoro operators. They give infinitesimally

$$\delta_\epsilon X \equiv -\epsilon\ \partial X = [iL[\epsilon], X]$$

However, quantisation of (8.1) with g, a non-flat metric, generates a mass term which destroys the Weyl invariance. This is shown in section (6.4), where for the $O(N)$ σ model a mass has been generated quantum mechanically. So even if conformal symmetry exists at the classical level, it is broken by the quantisation procedure.

8.3 Virasoro Algebra

We can have a string on a manifold parametrised by X^μ while the string spans a $1+1$ dimensional Minkowski world sheet with coordinates σ and τ. The position of the string on the manifold is given by the functions $X^\mu(\sigma, \tau)$. The string can be open or closed. In the second case we demand $X(\sigma, \tau)$ to be periodic in σ, which we assume runs around the string in the interval $[-\pi, \pi]$. The action of the string can be written

$$S = -\frac{1}{2\pi}\int d\sigma d\tau\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X \tag{8.3}$$

which is the action for a σ model, now chosen to be in a flat background manifold. From it we get the wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0 \tag{8.4}$$

The solutions of (8.4) for the closed string are

$$X^\mu_R(\sigma, \tau) = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau - \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\sigma-\tau)}$$

and

$$X^\mu_L(\sigma, \tau) = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau + \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\sigma+\tau)}$$

while for the open string we have the standing waves

$$X^\mu = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\tau} \cos n\sigma$$

where α and $\tilde{\alpha}$ are arbitrary constants.

At the quantum level the action (8.3) is accompanied by the commutation relations

$$[X^\mu(\sigma), P_\nu(\sigma')] = i\delta_\nu^\mu \delta(\sigma - \sigma') \tag{8.5}$$

where $P = \dot{X}$ ($\cdot$ stands for a derivative with respect to time).

For the case of the closed string there are two oscillating modes going "left" and "right". These modes can be generated by the Virasoro operators

$$L_n = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma : (P^\mu - X'^\mu)(P_\mu - X'_\mu) : e^{-in\sigma}$$

$$\tilde{L}_m = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma : (P^\mu + X'^\mu)(P_\mu + X'_\mu) : e^{im\sigma}$$

for m and n integers different in general, that is the two modes are independent from each other. The operators

$$\alpha_n^\mu \equiv \frac{1}{\sqrt{\pi}} \int d\sigma e^{-in\sigma} \left(P^\mu(\sigma) - X'^\mu(\sigma)\right)$$

and

$$\tilde{\alpha}_n^\mu \equiv \frac{1}{\sqrt{\pi}} \int d\sigma e^{in\sigma} \left(P^\mu(\sigma) + X'^\mu(\sigma)\right)$$

play the role of the creation operators for $n < 0$ and the annihilation ones for $n > 0$ with respect to the vacuum state $|0\rangle$. α and $\tilde{\alpha}$ satisfy the commutation relations

$$[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu} n \delta_{m+n,0} = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]$$

$$[\alpha_n^\mu, \tilde{\alpha}_m^\nu] = 0$$

following from (8.5). We can re-write the Virasoro operators as

$$L_n = \frac{1}{2} \sum : \alpha_{n+p}^\mu \alpha_{-p}^\nu : \eta_{\mu\nu} \tag{8.6}$$

and

$$\tilde{L}_n = \frac{1}{2} \sum : \tilde{\alpha}_{n+p}^\mu \tilde{\alpha}_{-p}^\nu : \eta_{\mu\nu} \tag{8.7}$$

where now the meaning of the normal ordering symbol :: is clearly defined with respect to the operators α and $\tilde{\alpha}$. It is actually needed only for the case $n = 0$, as $L_0 + \tilde{L}_0 + 2$ represents the Hamiltonian of the closed string and needs renormalisation of the vacuum energy (performed here by normal ordering). We can check that the renormalised Virasoro operators satisfy the well-known Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{N}{12}\delta_{n,-m}(n^3 - n). \tag{8.8}$$

The same algebra is also satisfied by the $\tilde{L}$ operators.

8.4 The Vacuum State of the String

We can construct the wave functional $\langle X|0\rangle$, that represents the vacuum state $|0\rangle$, explicitly by using the annihilation relations

$$\alpha_n^\mu|0\rangle = 0$$

for $n > 0$. The equation it has to satisfy is

$$\alpha_{n\mu}\langle X|0\rangle = 0 \Rightarrow$$

$$\frac{1}{\sqrt{\pi}}\int d\sigma e^{-in\sigma}\left(-i\frac{\delta}{\delta X^\mu(\sigma)} - X'_\mu(\sigma)\right)\langle X|0\rangle = 0 \Rightarrow$$

$$\int d\sigma e^{-in\sigma}\left(-i\frac{\delta}{\delta X^\mu(\sigma)} - inX_\mu(\sigma)\right)\langle X|0\rangle = 0 \tag{8.9}$$

From (8.9) we see that the vacuum state will have a Gaussian localised form. Let us take

$$\langle X|0\rangle = \exp\left(\int d\sigma d\sigma' X^\mu(\sigma)H(\sigma,\sigma')X_\mu(\sigma')\right)$$

where $H(\sigma,\sigma')$ is a symmetric function of σ and σ', to be calculated. As all the points on the closed string are equivalent, H should be a function of the difference $\sigma - \sigma'$. So its decomposition in modes will be

$$H(\sigma,\sigma') = H(\sigma-\sigma') = \sum_{m=-\infty}^{\infty} H^m e^{im(\sigma-\sigma')}$$

where H^m are the Fourier components of $H(\sigma,\sigma')$. After substituting into (8.9) we obtain

$$H^m = -\frac{m}{4\pi}$$

for $m \geq 0$. As $H(\sigma, \sigma')$ is symmetric in its arguments we finally have

$$H^m = -\frac{|m|}{4\pi}, \qquad \text{for every } m.$$

So the vacuum functional has the form

$$\langle X|0\rangle = \exp\left(\int d\sigma d\sigma' X^\mu(\sigma) H(\sigma,\sigma') X_\mu(\sigma')\right) \tag{8.10}$$

with

$$H(\sigma,\sigma') = -\sum_{m=-\infty}^{\infty} \frac{|m|}{4\pi} e^{im(\sigma-\sigma')}$$

8.5 Virasoro Algebra

To calculate the central charge term of the Virasoro algebra, as is given in (8.8), we can apply the algebra to the vacuum state. Let us assume it has the form

$$[L_n, L_m] = (n-m)L_{n+m} + A(n)\delta_{m+n,0}$$

that is, being the classical algebra, extended by the term $A(n)$ existing only when $m+n = 0$, as only then the operator L_0 appears on the r.h.s. with a quantum ambiguity. Taking $m = -n$ we get

$$L_n L_{-n}\langle X|0\rangle = A(n)\langle X|0\rangle$$

as $L_{-n}L_n\langle X|0\rangle = 0$ and $L_0\langle X|0\rangle = 0$ (the operators have been normal ordered). Successive application of the operators L_{-n} and L_n on the vacuum state will give us the value of $A(n)$, i.e.

$$L_{-n}\langle X|0\rangle = \frac{1}{4}\int (P^\mu(\sigma) - X'^\mu(\sigma))(P_\mu(\sigma) - X'_\mu(\sigma))e^{in\sigma} d\sigma \langle X|0\rangle =$$

$$\frac{1}{4}\int (P^\mu(\sigma) - X'^\mu(\sigma))(P_\mu(\sigma') - X'_\mu(\sigma'))e^{in\sigma}\frac{1}{2\pi}e^{im(\sigma-\sigma')} d\sigma d\sigma' \langle X|0\rangle$$

But

$$\int (P_\mu(\sigma') - X'_\mu(\sigma'))e^{-im\sigma'} d\sigma' \langle X|0\rangle = -2im\int X^\mu(\sigma')e^{-im\sigma'}\langle X|0\rangle \quad \text{for } m<0$$

$$= 0 \qquad \text{for } m>0 \tag{8.11}$$

as it can be easily verified, so that

$$L_{-n}\langle X|0\rangle = \frac{1}{8\pi}\sum_{m<0}\int (P_\mu(\sigma) - X'_\mu(\sigma))e^{in\sigma+im\sigma} d\sigma(-2im)\int X^\mu(\sigma')e^{-im\sigma'} d\sigma' \langle X|0\rangle =$$

$$\frac{1}{2\pi}\sum_{m=-1}^{-n} m(m+n)\int X^{\mu}(\sigma')e^{-im\sigma'}d\sigma'\int X^{\mu}(\sigma)e^{-i(-n-m)\sigma}d\sigma|X\rangle \qquad (8.12)$$

If we apply L_n on (8.12) and take into account relation (8.11) we get zero for its action on the vacuum functional, while when the two functional derivatives act on the other multiplicative terms they give

$$L_nL_{-n}\langle X|0\rangle = -\frac{N}{2}\sum_{m=1}^{n-1} m(m-n)\langle X|0\rangle = \frac{N}{2}\frac{n(n^2-1)}{6}\langle X|0\rangle \qquad (8.13)$$

which is the central charge term for the open or closed string.

In [8] we can see a way to extract the central charge term with the use of the commutation relations of the creation and annihilation operators α_n for $n > 0$ or $n < 0$ respectively. We have proceeded through the functional calculus in order to get an insight into the structure of the algebra, aiming to calculate the commutator of the Virasoro operators with the use of another renormalisation procedure.

In Appendix F we see how the Virasoro algebra constructed with the Virasoro operators of the form

$$L[u] = -\frac{1}{4}\int d\sigma d\sigma' u(\sigma,\sigma')K^{\mu\nu}(\sigma,\sigma')(P_{\mu}(\sigma) - X'_{\mu}(\sigma))(P_{\nu}(\sigma') - X'_{\nu}(\sigma')) \qquad (8.14)$$

holds applied on the vacuum state. In (8.14) $K_s^{\mu\nu}(\sigma,\sigma')$ is a Kernel to point split the double action of the functional differentiations, satisfying the condition $\lim_{s\to 0} K_s^{\mu\nu}(\sigma,\sigma') = \eta^{\mu\nu}\delta(\sigma,\sigma')$, $u(\sigma,\sigma')$ is the component of a vector field on the circle, S^1, on which $X(\sigma)$ is defined and is symmetric in σ and σ'. For $u(\sigma,\sigma') = 1$ we get the divergent quantity $L[1]$, which is equivalent to the divergency present in the unrenormalised Hamiltonian of the string.

The Virasoro algebra has the form

$$[L[u], L[v]] = -iL[[u,v]] + C(u,v)$$

where

$$C(u,v) = \frac{iN}{2\sqrt{2}\pi}\int d\sigma \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{1}{s} -$$
$$-\frac{iN}{24\pi}\int d\sigma\frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma) - \frac{iN}{24\pi}\int d\sigma\frac{\partial^3}{\partial\sigma^3}u(\sigma,\sigma)v(\sigma,\sigma) \qquad (8.15)$$

The first term in C is linearly divergent as $s \to 0$, if the antisymmetrisation of the term $\int d\sigma \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)$ with respect to u and v is not zero. When we regularise the Virasoro

operators by the use of the Kernel in (F.5) we face a problem only for the $L[1]$, as we have already said. The divergence appearing there (see relation (F.3)) is the same as in relation (8.15). That is, the renormalisation of the operator $L[1]$ makes automatically the Virasoro algebra renormalised. The linear divergence coming from $L[1]$, is connected with the vacuum energy divergence, which is equivalent to the linear divergence we faced in relation (4.3) for the scalar field.

8.6 Applications to the $O(N)$ σ Model

A similar treatment can be applied to the $O(N)$ σ model, where we expect the corresponding algebra to differ from the Virasoro algebra, because the β function is nonzero. We will use the form (8.14) for our generalised Virasoro operators, where here the coordinates $X(\sigma) \equiv z(\sigma)$ represent a manifold with $O(N)$ symmetry. We can work with operators which have similar form to the Hamiltonian, Lorentz and momentum operator. By studying their algebra we can connect them with the generalised Virasoro operators and deduce the algebra the latter satisfy. Instead of a general vector $u(\sigma)$, we can use powers of the σ variable (i.e. σ^k, $k \geq 0$). Let us make the following definitions

$$P \equiv -i \int d\sigma d\sigma' K^{\mu\nu}(\sigma,\sigma') g_{\nu\kappa} z'^{\kappa} \frac{D}{Dz^{\mu}(\sigma)} \equiv -i \int d\sigma \mathcal{P}(\sigma)$$

$$H \equiv \frac{1}{2}\left[- \int d\sigma d\sigma' K^{\mu\nu}(\sigma,\sigma') \frac{D}{Dz^{\mu}(\sigma)} \frac{D}{Dz^{\nu}(\sigma')} + \int d\sigma g_{\mu\nu}(\sigma) z'^{\mu} z'^{\nu} \right] \equiv \frac{1}{2} \int d\sigma \mathcal{H}$$

$$L^k \equiv \frac{1}{2}\left[- \int d\sigma d\sigma' \left(\frac{\sigma+\sigma'}{2} \right)^k K^{\mu\nu}(\sigma,\sigma') \frac{D}{Dz^{\mu}(\sigma)} \frac{D}{Dz^{\nu}(\sigma')} + \right.$$

$$\left. \int d\sigma d\sigma' \left(\frac{\sigma+\sigma'}{2} \right)^k g_{\mu\nu}(\sigma) z'^{\mu} z'^{\nu} \right] \equiv \frac{1}{2} \int d\sigma \sigma^k \mathcal{H}$$

$$\mathcal{B} \equiv \int d\sigma g_{\mu\nu}(\sigma) z'^{\mu} z'^{\nu} \quad \text{and} \quad \tilde{\mathcal{B}} \equiv \int d\sigma \sigma^k g_{\mu\nu}(\sigma) z'^{\mu} z'^{\nu}$$

where $\tilde{\mathcal{B}}$ could be described with two σ integrations connected by the Kernel K instead of the metric g and the term σ^k written as $(\sigma+\sigma')^k/2$. However, this will only differ from the given expression by terms of order $O(s^n)$ with $n > 0$, which will vanish when the limit $s \to 0$ is taken. Now if we take the commutator of H and L^k we have

$$\left[\int d\sigma \mathcal{H}, \int d\sigma \sigma^k \mathcal{H} \right] \Psi =$$

$$\left[-\int d\sigma\bar{\Delta}(\sigma)+\mathcal{B},-\int d\sigma\sigma^k\bar{\Delta}(\sigma)+\tilde{\mathcal{B}}\right]\Psi=$$

$$\left[-\int d\sigma\bar{\Delta},-\int d\sigma\sigma^k\bar{\Delta}\right]\Psi+\left[\mathcal{B},-\int d\sigma\sigma^k\bar{\Delta}\right]\Psi+\left[-\int d\sigma\bar{\Delta},\tilde{\mathcal{B}}\right]\Psi \tag{8.16}$$

The second term becomes

$$\left[\mathcal{B},\int d\sigma\sigma^{k-1}\bar{\Delta}\right]\Psi=$$

$$-\int d\sigma\sigma^k\left((\bar{\Delta}\,\mathcal{B})\Psi+2K^{\mu\nu}\frac{D}{Dz^\mu}\mathcal{B}\frac{D}{Dz^\mu}\Psi\right)=$$

$$-k(k-1)\frac{b_0^0}{\sqrt{s}}N\int d\sigma\sigma^{k-2}-2N\frac{b_0^1}{\sqrt{s}^3}\int d\sigma\sigma^k-$$

$$-\int d\sigma d\sigma'\left(\frac{\sigma+\sigma'}{2}\right)^k 2K^{\mu\nu}(\sigma,\sigma')\,(-g_{\gamma\mu}2\mathcal{D}z'^\gamma)|_\sigma\,\frac{D\Psi}{Dz^\nu(\sigma')}$$

while the third

$$\left[-\int d\sigma\bar{\Delta}\,,\tilde{\mathcal{B}}\right]\Psi=$$

$$-\int d\sigma\left[(\bar{\Delta}\,\tilde{\mathcal{B}})\Psi+2K^{\mu\nu}\frac{D\tilde{\mathcal{B}}}{Dz^\mu}\frac{D\Psi}{Dz^\nu}\right]=$$

$$0-\int d\sigma d\sigma' 2K^{\mu\nu}(\sigma,\sigma')\left(-2g_{\gamma\mu}k\sigma^{k-1}z'^\gamma-2g_{\gamma\mu}\sigma^k\mathcal{D}z'^\gamma\right)\frac{D\Psi}{Dz^\nu(\sigma')}$$

The second and third terms together give

$$\frac{k(k-1)}{4}\frac{b_0^0}{\sqrt{s}}N\int d\sigma\sigma^{k-2}\Psi-\frac{2N}{4}\frac{b_0^1}{\sqrt{s}^3}\int d\sigma\sigma^k\Psi+\int d\sigma k\sigma^{k-1}\mathcal{P}(\sigma)\Psi= \tag{8.17}$$

$$\int d\sigma k\sigma^{k-1}\mathcal{P}(\sigma)\Psi$$

as the first and second terms of (8.17) vanish under the antisymmetry of the commutator between $(\frac{\sigma+\sigma'}{2})^k$ and 1. The previous results are independent of the choice of the functional Ψ. However, the first term depends on the specific form of Ψ. We can take Ψ, as in the previous section, to be the wave vacuum functional. Working in the lowest order of approximation for slowly varying fields the vacuum functional becomes $\Psi=e^{\int f_{\mu\nu}z'^\mu z'^\nu}$ where $f_{\mu\nu}$ is an ultra-local function, which can be determined from the Schrödinger equation. The following calculations will be done up to two derivatives with respect to σ. For example, the action of the two functional derivatives on Ψ gives

$$\frac{\delta^2}{\delta z(\sigma)\delta z(\sigma')}e^{\int f_{\mu\nu}z'^\mu z'^\nu}=\frac{\delta^2\int f_{\mu\nu}z'^\mu z'^\nu}{\delta z(\sigma)\delta z(\sigma')}e^{\int f_{\mu\nu}z'^\mu z'^\nu}+\frac{\delta\int f_{\mu\nu}z'^\mu z'^\nu}{\delta z(\sigma)}\frac{\delta\int f_{\mu\nu}z'^\mu z'^\nu}{\delta z(\sigma')}e^{\int f_{\mu\nu}z'^\mu z'^\nu}$$

from which we only consider the first term, as the second involves four derivatives with respect to σ. So we only need to consider the $\int f_{\mu\nu}z'^\mu z'^\nu$ functional in the expansion of the exponential Ψ. For this case we have

$$L^k\int f_{\mu\nu}\,z'^\mu z'^\nu=$$

$$\int d\sigma\sigma^k \left\{ (J_2 f)_{\mu\nu} z'^\mu z'^\nu + \frac{1}{s} b_0^0 f^\nu{}_\nu \right\} + \int d\sigma k \sigma^{k-1} 2 b_0^0 D^\nu f_{\nu\mu} z'^\mu + \int d\sigma \frac{k(k-1)}{4} \sigma^{k-2} b_0^0 2 f^\nu{}_\nu$$

also

$$\Delta L^k \int f_{\mu\nu} z'^\mu z'^\nu =$$

$$.. + \int d\sigma k \sigma^{k-1} 2 b_0^0 (J_1 D^\nu f_\nu)_\mu z'^\mu + \int d\sigma \frac{k(k-1)}{4} \sigma^{k-2} b_0^0 2 (J_0 f^\nu{}_\nu)$$

and

$$L^k \Delta \int f_{\mu\nu} z'^\mu z'^\nu =$$

$$.. + \int d\sigma k \sigma^{k-1} 2 b_0^0 D^\nu (J_2 f)_{\nu\mu} z'^\mu + \int d\sigma k(k-1) \sigma^{k-2} b 2 (J_2 f)^\nu{}_\nu$$

Taking the identities

$$(\text{tr } J_n f)_{\rho_3 .. \rho_n} - (J_{n-2} \text{tr } f)_{\rho_3 .. \rho_n} = 0$$

$$(D \cdot J_n f) - J_{n-1}(D \cdot f))_{\rho_2 .. \rho_n} = 0$$

from the previous, when we asked for the Poincaré algebra to be satisfied ($k \equiv 1$), we get

$$\left[\Delta L^k - L^k \Delta \right] \int f_{\mu\nu} z'^\mu z'^\nu = 0$$

So altogether for the vacuum functional $\Psi = e^{\int f_{\mu\nu} z'^\mu z'^\nu}$ the commutator becomes

$$\frac{1}{4} \left[\int d\sigma \mathcal{H}, \int d\sigma \sigma^k \mathcal{H} \right] \Psi = \int d\sigma k \sigma^{k-1} \mathcal{P} \Psi \tag{8.18}$$

No additional term results. This could be derived, more easily, by substituting $f_{\mu\nu} = a g_{\mu\nu}$ which is the most general form the first term could have, in the expansion of the logarithm of the vacuum functional, for an appropriate constant a. From (8.18) we deduce that in addition to the vacuum state, all the exited states will not contribute in the commutator (8.16) at the first order. But this will not be the case for higher order terms. The way an additional term in (8.18) would adjust in the algebra of the generalised Virasoro operators is shown in the next section.

8.7 Modification of the Central Charge Term

Let us define the generalised Virasoro operators as

$$L[u] = \frac{1}{4} \int_{-\pi}^{\pi} u(\sigma)(P^\mu P_\mu + X'^\mu X'_\mu - X'^\mu P_\mu - P^\mu X'_\mu) d\sigma$$

$$\tilde{L}[u] = \frac{1}{4}\int_{-\pi}^{\pi} u(-\sigma)(P^\mu P_\mu + X'^\mu X'_\mu + X'^\mu P_\mu + P^\mu X'_\mu)d\sigma$$

where the operator $\tilde{L}$ is the same with L but σ replaced with $-\sigma$. The fields $X(\sigma)$ satisfy the $O(N)$ symmetry. Using these definitions we can express H, L_0 and P in terms of L and $\tilde{L}$ as

$$L[1] + \tilde{L}[1] = \frac{1}{2}\int (P^\mu P_\mu + X'^\mu X'_\mu) = H$$

$$L[1] - \tilde{L}[1] = -\frac{1}{2}\int (X'^\mu P_\mu - \frac{1}{2}X'^\mu P_\mu) = -P$$

$$L[\sigma] + \tilde{L}[\sigma] = -\int \sigma X'^\mu P_\mu = -\int \sigma \mathcal{P}$$

$$L[\sigma] - \tilde{L}[\sigma] = \frac{1}{2}\int \sigma (P^\mu P_\mu + X'^\mu X'_\mu) = L_0$$

Using these identifications we can see how an extension to the algebra (8.18), produced from another functional than $\int f_{\mu\nu} z'^\mu z'^\nu$, will be placed in the modified Virasoro algebra. Let us extend the algebra with the additional term A, as

$$\left[L[1], L[\sigma^k]\right] = -L\left[[1, \sigma^k]\right] + A$$

where

$$[u, v] = uv' - u'v \Rightarrow [1, \sigma^k] = k\sigma^{k-1}$$

and

$$\left[\tilde{L}[1], \tilde{L}[\sigma^k]\right] = -\tilde{L}\left[[1, \sigma^k]\right] + \tilde{A}$$

We separate two cases: for $k = 2l$ and for $k = 2l + 1$. Then

$$L[\sigma^{2l}] = \frac{1}{4}\int \sigma^{2l}(P^2 + X'^2 - X'P - PX')$$

$$\tilde{L}[\sigma^{2l}] = -\frac{1}{4}\int \sigma^{2l}(P^2 + X'^2 + X'P + PX')$$

so that

$$L[\sigma^{2l}] + \tilde{L}[\sigma^{2l}] = \frac{1}{2}\int \sigma^{2l}(P^2 + X'^2) = \int \sigma^{2l}\mathcal{H}$$

and similarly

$$L[\sigma^{2l+1}] = \frac{1}{4}\int \sigma^{2l+1}(P^2 + X'^2 - X'P - PX')$$

$$\tilde{L}[\sigma^{2l+1}] = -\frac{1}{4}\int \sigma^{2l+1}(P^2 + X'^2 + X'P + PX')$$

so that

$$L[\sigma^{2l+1}] - \tilde{L}[\sigma^{2l+1}] = \frac{1}{2}\int \sigma^{2l+1}(P^2 + X'^2) = \int \sigma^{2l+1}\mathcal{H}$$

For the first case we have

$$\left[\int \mathcal{H}, \int \sigma^{2l}\mathcal{H}\right] = \left[L[1] + \tilde{L}[1], L[\sigma^{2l}] + \tilde{L}[\sigma^{2l}]\right] =$$

$$-L\left[[1, \sigma^{2l}]\right] + A - \tilde{L}\left[[1, \sigma^{2l}]\right] + \tilde{A} =$$

$$2l \int \sigma^{2l-1} X^{\mu} P_{\mu} + A + \tilde{A}$$

while for the second

$$\left[\int \mathcal{H}, \int \sigma^{2l+1}\mathcal{H}\right] = \left[L[1] + \tilde{L}[1], L[\sigma^{2l+1}] - \tilde{L}[\sigma^{2l+1}]\right] =$$

$$-L\left[[1, \sigma^{2l+1}]\right] + A + \tilde{L}\left[[1, \sigma^{2l+1}]\right] - \tilde{A} =$$

$$(2l+1) \int \sigma^{2l} X^{\mu} P_{\mu} + A - \tilde{A}$$

If we compare the outcome of the commutator $\left[\int \mathcal{H}, \int \sigma^{k}\mathcal{H}\right]$ acting on a general functional, then we can identify the quantities A and $\tilde{A}$. For the functional $\Psi = e^{\int f_{\mu\nu} z'^{\mu} z'^{\nu}}$ it gives $A = \tilde{A} = 0$.

Finally, we note that the central charge we arrived at in section (8.5) was a result of the non locality of the vacuum state. This is described by the function $H(\sigma, \sigma')$, which in contrast to the usual delta function (appearing when we use local functionals), brings in the non-local character of the vacuum. As we have seen in (8.13), the function $H(\sigma, \sigma')$ is the one which gives the specific form to the central charge term. However, we expect to get operator-like terms for the central charge of the algebra on S^N.

Chapter 9

Conclusions

In this thesis we developed techniques for applying the Schrödinger representation to the ϕ^4 theory and the $O(N)$ σ model. Having set this framework one can proceed in applying our techniques to more realistic theories like QCD as it shares many features with these models. Solving the Schrödinger equation in the low energy region (local expansion) provides a natural way for approaching problems of QCD like confinement and chiral symmetry breaking. The Schrödinger representation has the advantage of giving analytic results compared with the numerical ones from the lattice formulation, which in addition has an ambiguity in the transition from the discrete to the continous limit.

For the ϕ^4 theory we constructed the Schrödinger equation, which a local expansion of the logarithm of the vacuum functional has to satisfy, and then compared its semiclassical solution with similar results, derived from the standard path integral approach. As we have seen in Chapter 5, these two results are in complete agreement with each other, showing the correctness of the local expansion method. Also, the different resummation methods proposed in Section 4.7 provide a choice in the way we can extract the ultraviolet behaviour out of the infrared one, enabling us to select the best possible resummation method for the various ways with which we can solve the Schrödinger equation (e.g. non-perturbatively).

For the $O(N)$ σ model we constructed the functional Laplacian, which is the principal ingredient of the corresponding Schrödinger equation. The Poincaré algebra proposed to hold when acting on local functionals enabled us to determine the Laplacian up to the

second order. Though, for a general manifold rather than the sphere it is not possible to find a Kernel for the Laplacian satisfying this demand. The form of the Laplacian for the $O(N)$ σ model can be used to solve the equation, as a further work. Also, we took advantage of the functional calculus already used to study the modified form of the Virasoro algebra for the non-conformal $O(N)$ σ model.

Finally, a computer program is presented in the appendices, which helps in the construction of the vacuum functional for the ϕ^4 theory, in the Schrödinger representation. It has been used extensively for checking the validity of our results, as it provides an easy way to get the solution of the Schrödinger equation semiclassically, once the determining the order of truncation of the local expansion has been determined.

Appendix A

Re-summation procedure

Let us look at the problem of re-summation from a more mathematical point of view. We have a function $f(s)$ which can be expanded for small and large s like

$$f(s) = \sum_{n=0}^{\infty} a_n s^n \tag{A.1}$$

and

$$f(z) = \sum_{m=0}^{\infty} b_m \frac{1}{z^m} \tag{A.2}$$

for certain a_n and b_m. What we want is to find a way to express a_m's in terms of b_n's. Also, let us assume that the limits $s \to 0$ and $s \to \infty$ exist and that there is a path connecting these two points (e.g. the positive real axis) where the function f has no divergences or cuts. We will try with repetitive expansions and re-summations to go from (A.2) to (A.1). The point of expansion of (A.2) can be changed in a limiting process, from infinity to a large number N, which at the end is meant to be sent to infinity. Defining y and w as

$$y = \frac{N-z}{N} \text{ and } w = \frac{1}{N}\frac{y}{1-y}$$

so that

$$\frac{1}{z} = \frac{1}{z} - \frac{1}{N} + \frac{1}{N} = \frac{N-z}{N}\frac{1/N}{1+(z-N)/N} + \frac{1}{N} =$$

$$= \frac{1}{N}\frac{y}{1-y} + \frac{1}{N} = w + \frac{1}{N}$$

and substituting into (A.2) we get

$$\sum_{m=0}^{\infty} b_m \frac{1}{z^m} = \sum_{m=0}^{\infty} b_m \left(w + \frac{1}{N}\right)^m =$$

$$\sum_{m=0}^{\infty} b_m \sum_{k=0}^{m} \binom{m}{k} N^{-m+k} w^k =$$

$$\sum_{m=0}^{\infty} b_m \sum_{k=0}^{m} \binom{m}{k} N^{-m+k} \left(\frac{1}{N} y \frac{1}{1-y} \right)^k$$

if $|y| < 1$ and $k \neq 0$ then

$$\sum_{m=0}^{\infty} b_m \sum_{k=0}^{m} \binom{m}{k} N^{-m} y^k \sum_{q=0}^{\infty} \binom{q+k-1}{k-1} y^q$$

which by changing the variables of the sums becomes

$$\sum_{m=0}^{\infty} \left(\sum_{q=0}^{m} \sum_{n=q}^{\infty} b_n \binom{n}{q} N^{-n} \binom{m-1}{q-1} \right) y^m$$

which has the form of (A.1). So we can read the coefficients a_m, which eventually will be independent of N. But if we have only a finite number of b_n's then we get an approximated value for the a_m's which will be read for large N. N is similar to λ we got in the re-summation procedures already mentioned in the main text.

Appendix B

The "Tadpole" Diagram

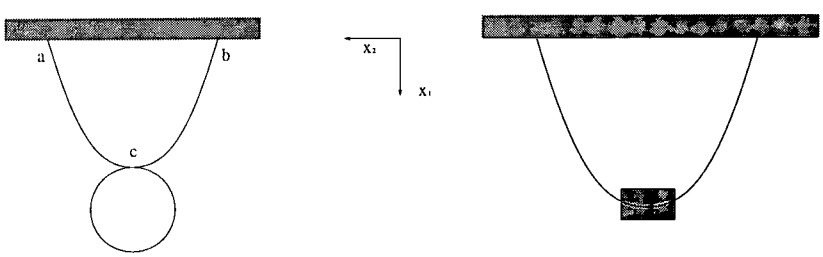

Figure B.1: Tadpole and counter-term diagrams.

Here we are going to calculate the value of $b_0^{\hbar}$ (i.e. the order $\hbar$ correction) in another way than the one we used in the main text. Rather than using the structure of the Schrödinger equation with the application of the Laplacian, we try to find the renormalised value of $b_0^{\hbar}$ as it can be given by the two diagrams in figure (B.1). The value of the counter-term diagram is given by the demand that it cancels exactly the divergences appearing in the "tadpole" diagram. The mass counter-term is the only one needed in the 1 + 1-dimensional ϕ^4 theory to make it renormalised. This is the natural way to calculate the quantum corrections from Feynman diagrams. The disadvantage of this method with the one we use in Chapter 5 is that here we cannot reduce the Dirichlet Green functions to ordinary ones, which makes the calculations more complicated.

Using the Dirichlet propagator we calculate the amplitude of the "mass correction"

$$\frac{1}{2}g\int da_2\int db_2\int d\widehat{c}\;\phi(a_2)\phi(b_2)\left.\frac{\partial G_D(\widehat{a},\widehat{c})}{\partial a_1}\right|_{a_1=0}\left.\frac{\partial G_D(\widehat{b},\widehat{c})}{\partial b_1}\right|_{b_1=0}G_D(\widehat{c},\widehat{c}) \qquad \text{(B.1)}$$

where $G_D(\widehat{a},\widehat{c}) = G(\widehat{a},\widehat{c}) - G(\widehat{\overline{a}},\widehat{c})$, $\widehat{\overline{a}} = (-a_1, a_2)$, $G(\widehat{a},\widehat{c}) = \int_p e^{i\widehat{p}\cdot(\widehat{a}-\widehat{c})}/(p^2+m^2)$ and $\int_p \equiv \int d^2\widehat{p}/(2\pi)^2$. After evaluating the integrals in (B.1) and subtracting the infinity

appearing in (mass counter-term), we find, for $g = 1$, the value $1/(8\pi)$, which is the first order quantum correction to b_0 and consists the 7.96% of its classical value. This result is in agreement with the one we find by solving the Schrödinger equation, or by calculating the equivalent correction diagrams in Chapter 5.

Appendix C

The Space of Local Functionals

C.1 Definition of the Functional Space

We can define the space of local functionals as the one with elements linear combinations of

$$A = \int d\sigma \prod_{i=0}^{n} \left(\phi^{(i)}\right)^{u_i} \tag{C.1}$$

where ϕ is an analytic function of σ which is zero at the limits of integration, u_i is a non-negative integer and (i) means differentiating i times with respect to σ. We also impose that the total number of ϕ's and the number of derivatives are even, that is

$$\sum_i u_i \text{ and } \sum_i i u_i \text{ are even numbers.} \tag{C.2}$$

In this space we have as the main 'procedure' the integration by parts. With it we can define a set of basis vectors (b.v.'s) for our space as the vectors of the form above for which u_n is greater than or equal to 2. That is we will prove that any vector of the form A can be reduced, using integrations by parts, to a linear combination of basis vectors, and no basis vector can be written as a linear combination of other b.v.'s

C.2 Reduction to Basis Vectors

A general vector like A will be a non b.v. if the last power u_n is equal to one. Then it will have the form

$$A = \int d\sigma \phi^{u_0} (\phi')^{u_1} \cdots \left(\phi^{(n-1)}\right)^{u_{n-1}} \phi^{(n)}$$

To reduce A to a basis vector we can use the following procedure which holds for u_{n-1} greater or equal to zero

$$\left(\phi^{(n-1)}\right)^{u_{n-1}} \phi^{(n)} = \left(\frac{\left(\phi^{(n-1)}\right)^{u_{n-1}+1}}{u_{n-1}+1} \right)' \tag{C.3}$$

so that after integration by parts the overall derivative will produce a vector with the last term to the power $u_{n-1} + 1 \geq 1$. If u_{n-1} is not zero then we have a basis vector. Otherwise, if it is equal to zero we continue this procedure until we get a term previous to the last one with a non-zero power. This is the only way to decompose a vector.

If you try to integrate by parts derivatives different than the ones from the last ϕ then terms with higher derivatives will be produced, which will lead to the necessity to integrate them by parts in order to produce a b.v., that is a vector with the final term in power two or higher. For a j with $j \leq n$, and $u_n \geq 2$, we have

$$\begin{aligned}
&\int d\sigma \phi^{u_0} (\phi')^{u1} (\cdots) \left(\phi^{(j)}\right)^{u_j} (\cdots) \left(\phi^{(n)}\right)^{u_n} = \\
&- \int d\sigma \left((\cdots)' \left(\phi^{(j)}\right)^{u_j-1} (\cdots) \left(\phi^{(n)}\right)^{u_n} \right. \\
&+(\cdots)(u_j - 1) \left(\phi^{(j)}\right)^{u_j-2} \phi^{(j+1)} (\cdots) \left(\phi^{(n)}\right)^{u_n} \\
&+(\cdots) \left(\phi^{(j)}\right)^{u_j-1} (\cdots)' \left(\phi^{(n)}\right)^{u_n} \\
&\left. + (\cdots) \left(\phi^{(j)}\right)^{u_j-1} (\cdots) u_n \left(\phi^{(n)}\right)^{u_n-1} \phi^{(n+1)}) \phi^{(j-1)} \right)
\end{aligned} \tag{C.4}$$

where $(\cdots)$ stands for whatever is on the left or on the right of $\left(\phi^{(j)}\right)^{u_j}$ respectively. Now we either can continue partial integrations of derivatives from terms like j, which will lead even further from the construction of a b.v., or we can take care of the last term of the equation, which is the only non-b.v., using the method described above. It will generate terms which cancel exactly the right hand side of (C.4) plus the vector we started with. This means that we cannot generate other b.v.'s from b.v.'s or reduce vectors to b.v. from using a middle term labeled by j. We can conclude then, that the b.v.'s we defined are

linearly independent under the procedure of partial integration, because of the previous argument for $u_n \geq 2$. Also, any vector can be reduced to a linear combination of b.v.'s using the methods described in the beginning. So our space of local functionals with its basis is well defined.

C.3 A Subspace of Functionals

Let us represent a functional vector with A. Then it could be $A = \int \prod_{i=0}^{n} a^i$ where a^i is ϕ, i times differentiated with respect to σ, to some power. If A is not a b.v. then you can decompose it to basis vectors by integrating by parts the ϕ with the highest number of derivatives and only that. This in general will produce vectors constructed by a^i's with none or more derivatives higher than the ones in the original vector A. Clearly you cannot construct a b.v. out of A, say A' (after a total decomposition) with an element a'^i $(0 \leq i < n)$ with less number of derivatives on it than a^i. In this sense if you want to determine from the classical equations a specific coefficient of a vector A, then you can choose the ones whose their product, after they have been functionally differentiated, have the same or greater number of ϕ's without derivative (a^0), as well as the next term, and so on. For example you cannot get rid of the derivatives on $\phi\phi'' \cdots$ or $\phi'^2 \cdots$ in order to get a basis vector with the first term of the form $\phi^2 \cdots$.

An interesting subspace of the full local functional space is the one which is produced by $B_n^m = \int \phi^{2n} \, \phi^{(m)\,2}$ b.v.'s. Let us claim and prove that in the classical definition equations of their coefficients, the only vectors which contribute are the ones of the same form B_n^m. That is we can construct the classical equations out of b.v.'s of this form without need to use all the b.v.'s that could contribute as long as the numbers of ϕ's and $'$'s are preserved. If W is a linear combination of all b.v.'s then the classical equations are defined to be (see *Hamilton-Jacobi Equation* (4.35)):

$$C.E. = \int d\sigma \left(\frac{\delta W}{\delta \phi(\sigma)} \right)^2 - V^{class} = 0$$

Let us name $\widetilde{W}$ the linear combination of vectors of the form B_n^m in W. Then our claim can be expressed in the following way: if we take the difference

$$\int d\sigma \left(\frac{\delta W}{\delta \phi(\sigma)} \right)^2 - \int d\sigma \left(\frac{\delta \widetilde{W}}{\delta \phi(\sigma)} \right)^2 =$$

$$\int d\sigma \left(\frac{\delta(W-\widetilde{W})}{\delta\phi(\sigma)}\right)\left(\frac{\delta(W+\widetilde{W})}{\delta\phi(\sigma)}\right) \tag{C.5}$$

then (C.5) cannot be reduced to b.v. of the form B_n^m. $W-\widetilde{W}$ can be written in terms of b.v.'s not of type B_n^m, while $W+\widetilde{W}$ contains all b.v.'s. For this let us take two general vectors in $W-\widetilde{W}$ and $W+\widetilde{W}$ and see the product of their functional derivatives if it can be reduced to B_n^m b.v.'s. We only need to check R_1 and R_2

$$R_1 = \int \prod_{i=0}^{k1} (\phi^{(i)})^{v_i}$$

$$R_2 = \int \prod_{i=0}^{k2} (\phi^{(i)})^{u_i}$$

where at least one of them is not of the form B_n^m. Their functional derivative (f.d.) is going to be

$$\frac{\delta R}{\delta\phi} = \sum_{j=0}^{k} \int \prod_{a=0}^{j=1} (\phi^{(a)})^{u_a} (\phi^{(j)})^{u_j-1} \delta^{(j)} \prod_{b=j+1}^{k1} (\phi^{(b)})^{u_b} \tag{C.6}$$

where the σ dependence as a variable has been omitted. The overall number of ϕ's and the overall number of σ derivatives of the product $(\frac{\delta R_1}{\delta\phi})(\frac{\delta R_2}{\delta\phi})$ have to be the same as in B_n^m. So we deduce the relations

$$\sum_i i v_i + \sum_j j u_j = 2m \tag{C.7}$$

$$\sum_i v_i + \sum_j u_j = 2n+4 \tag{C.8}$$

The various terms produced in relation (C.5) could be candidates for the form of B_n^m. We can distinguish four types. Writing j_1 and j_2 for the two vectors R_1 and R_2 in analogy to j in (C.6) for R, we have that the overall number of ϕ's with no derivatives on them, let us call it k, is going to be the most[1] equal to

$$v_0 + u_0 - 2 \text{ for } j_1 = 0 \text{ and } j_2 = 0 \tag{C.9}$$

$$v_0 + u_0 - 1 \text{ for } j_1 \neq 0 \text{ and } j_2 = 0 \tag{C.10}$$

$$v_0 + u_0 - 1 \text{ for } j_1 = 0 \text{ and } j_2 \neq 0 \tag{C.11}$$

$$v_0 + u_0 \text{ for } j_1 \neq 0 \text{ and } j_2 \neq 0 \tag{C.12}$$

We consider three cases. (a) $k < 2n$. Then it is impossible to reduce it to B_n^m, as we did before. [2] (b) $k = 2n$. Then there are four ϕ's left to carry derivatives. If they are

[1] After reduction to b.v. in (C.5) the number of ϕ's without derivatives on them will be actually smaller, but this is a subcase which falls in the cases we are going to study.

[2] See beginning of this chapter.

separated between the two vectors R_1 and R_2 like $2+2$ then each two for the same vector should have the same number of derivatives on them, i.e. $\int \phi^{u_0}\phi^{(k)^2}$, so both are going to be of the form of B_n^m, and it is not the case we want to consider. If the ϕ's are separated like $3+1$, then clearly one of them is not a b.v.. Finally the case $0+4$ cannot be reduced to a b.v. after taking the f.d. and then their product, as we have too many ϕ's with derivatives to get a vector with two only σ-differentiated ϕ's. (c) $k > 2n$ then because of (C.8) we can only have the following cases of the relations (C.9)-(C.12). If (C.9) then there are two possibilities, $v_0 + u_0 = 2n + 3$ or $v_0 + u_0 = 2n + 4$. If $v_0 + u_0 = 2n + 3$ then one of them has only one ϕ with derivatives and this is not a b.v.. If $v_0 + u_0 = 2n + 4$ it means that there are no ϕ's with derivatives at all and this gives again vectors of the form B_n^m. If (C.10) or (C.11) we can have $v_0 + u_0 = 2n + 2$ or $v_0 + u_0 = 2n + 3$ or $v_0 + u_0 = 2n + 4$, which do not contribute as we have seen. If (C.12) then $v_0 + u_0 = 2n + 2$ or $v_0 + u_0 = 2n + 1$. The second (the only one which could contribute) suggests the two following b.v.'s

$$R_1 = \int \phi^{v_0} \text{ and } R_2 = \int \phi^{u_0}\phi^{(k1)}\phi^{(k2)^2} \tag{C.13}$$

This is the only case we have to consider. In it $v_0 + u_0 = 2n + 1$ and $k1$ is even. After going through the f.d. and taking their product we get only one interesting part of the result

$$\cdots + \int v_0\phi^{v_0+u_0-1}\left((-1)^{k1}\left(\phi^{k2^2}\right)^{(k1)} + 2(-1)\left(\phi^{(k1)}\phi^{(k2)}\right)^{(k2)}\right) \tag{C.14}$$

We can use *Leibnitz rule*

$$D^n(uv) = \sum_{r=0}^{n}\binom{n}{r}D^{n-r}uD^r v \tag{C.15}$$

where here D stands for σ differentiation. If we take $u = \phi^{(k2)}$, $v = \phi^{(k2)}$ and $n = k1$ then

$$\sum_{r=0}^{k1}\binom{k1}{r}\phi^{(k2+k1-r)}\phi^{(k2+r)} = \phi^{(k2+\frac{k1}{2})^2}\sum_{r=0}^{k1}\binom{k1}{r}(-1)^{\frac{k1}{2}-r} \tag{C.16}$$

whereas if $u = \phi^{(k1)}$, $v = \phi^{(k2)}$ and $n = k2$ then

$$\sum_{r=0}^{k2}\binom{k2}{r}\phi^{(k2+k1-r)}\phi^{(k2+r)} = \phi^{(k2+\frac{k1}{2})^2}\sum_{r=0}^{k2}\binom{k2}{r}(-1)^{\frac{k1}{2}-r} \tag{C.17}$$

But $\sum_{r=0}^{k1}\binom{k1}{r}(-1)^{\frac{k1}{2}-r} = 0$ as well as $\sum_{r=0}^{k2}\binom{k2}{r}(-1)^{\frac{k1}{2}-r} = 0$. So we conclude that there are no contributions to the classical equations of B_n^m from b.v.'s with form different from B_n^m.

Appendix D

The b, c and f Coefficients

The equations for the b's read:

$$\sum_{n=0}^{m} b_n b_{m-n} = 0$$

for $m \geq 2$ as well as $b_0 = -1/2$ and $b_1 = -1/4$ (for the mass $m = 1$). We can solve this equation to find the value for a general b_n in the following way. These equations can be written in a power series form like

$$\left(\sum_{n=0}^{\infty} b_n z^n \right)^2 = b_0^2 + 2 b_0 b_1 z$$

for z a real variable. Solving with respect to b_n we have

$$b_n = b_0 \left(\begin{array}{c} 1/2 \\ n \end{array} \right) \left(\frac{2 b_1}{b_0} \right)^n \text{ for } n = 2, 3, 4, ..$$

or in other words

$$b_n = -\frac{1}{2} \left(\begin{array}{c} 1/2 \\ n \end{array} \right) \text{ for } n = 2, 3, 4, ..$$

Some b's are $b_2 = 1/16$, $b_3 = -1/32$, $b_4 = 5/256$, ...

For the c series we have the following equations (for $g = 1$)

$$c_0 = -\frac{1}{96}$$

$$2 b_0 c_1 + 6 b_1 c_0 = 0$$

$$2 b_0 c_2 + b_1 c_1 + 6 b_2 c_0 = 0$$

$$2 b_0 c_3 + b_1 c_2 + b_2 c_1 + 6 b_3 c_0 = 0$$

...

We can rename b_0 and c_0 as $b_0 \to \frac{1}{2}b_0$ and $c_0 \to \frac{1}{6}c_0$ so that the equations can be written as

$$b_0c_1 + b_1c_0 = 0$$

$$b_0c_2 + b_1c_1 + b_2c_0 = 0$$

$$b_0c_3 + b_1c_2 + b_2c_1 + b_3c_0 = 0$$

...

or

$$\left(\sum_{i=0}^{\infty} b_i z^i\right)\left(\sum_{j=0}^{\infty} c_j z^j\right) = b_0c_0 \Rightarrow$$

$$\left(\sum_{j=0}^{\infty} c_j z^j\right) = \frac{b_0c_0}{(\sum_{i=0}^{\infty} b_i z^i)} \Rightarrow$$

$$c_n = b_0c_0\frac{(-1)^n}{b_0^{n+1}}\begin{vmatrix} b_1 & b_2 & .. & & b_n \\ b_0 & b_1 & .. & & b_{n-1} \\ 0 & b_0 & .. & & b_{n-2} \\ .. & & & & \\ 0 & 0 & & b_0 & b_1 \end{vmatrix}$$

for $n = 1, 2, 3, ..$ where $||$ means determinant. Substituting back the original b_0 and c_0 we have

$$c_n = \frac{12}{2^n}b_0c_0\frac{(-1)^n}{b_0^{n+1}}\begin{vmatrix} b_1 & b_2 & .. & & b_n \\ 2b_0 & b_1 & .. & & b_{n-1} \\ 0 & 2b_0 & .. & & b_{n-2} \\ .. & & & & \\ 0 & 0 & & 2b_0 & b_1 \end{vmatrix}$$

Some of the first c's are $c_1 = 1/64$, $c_2 = -1/128$, $c_3 = 5/1024$.

In a similar way we can find the f series. The equations read

$$2c_0^2 + 3f_0b_0 = 0$$

$$16c_0c_1 + 6b_0f_1 + 30b_1f_0 = 0$$

$$16c_0c_2 + c_1^2 + 6b_0f_2 + b_1f_1 + 30b_2f_0 = 0$$

$$16c_0c_3 + 2c_1c_2 + 6b_0f_3 + 2b_1f_2 + 2b_2f_1 + 30b_3f_0 = 0$$

...

We can write these equations, but the first one, in terms of a power series after substituting $b_0 \to 1/3b_0$, $c_0 \to 1/8c_0$ and $f_0 \to 1/15f_0$. Then

$$\left(\sum_{i=0}^{\infty} c_i z^i\right)^2 + 2\left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{l=0}^{\infty} f_l z^l\right) = c_0^2 + 2b_0 f_0$$

where $f_0 = -2c_0^2/(3b_0)$. From this equation we deduce

$$\left(\sum_{l=0}^{\infty} f_l z^l\right) = \frac{c_0^2 + 2b_0 f_0 - \left(\sum_{i=0}^{\infty} c_i z^i\right)^2}{2\left(\sum_{k=0}^{\infty} b_k z^k\right)} =$$

$$= \frac{1}{2}\left(c_0^2 + 2b_0 f_0\right)\left(\sum_{k=0}^{\infty} (b^{-1})_k z^k\right) - \frac{1}{2}\left(\sum_{i=0}^{\infty} (c^2 b^{-1})_i z^i\right) \Rightarrow$$

$$f_n = \frac{1}{2}\left(c_0^2 + 2b_0 f_0\right)(b^{-1})_n - \frac{1}{2}(c^2 b^{-1})_n \text{ for } n = 1, 2, 3, ..$$

and by substituting back the original b_0, c_0 and f_0 we have

$$f_n = \frac{1}{2}\left(64c_0^2 + 90b_0 f_0\right)(b^{-1})_n - \frac{1}{2}(c^2 b^{-1})_n \text{ for } n = 1, 2, 3, ..$$

where b^{-1} and $c^2 b^{-1}$ can be found easily using the formulas for the inverse and the product of power series (see e.g. [29]). Finally we can derive the first few f's: $f_0 = 1/6912$, $f_1 = -17/13824$, $f_2 = 269/331776$.

Similar treatment can give the other classical coefficients of the functional $\int \varphi^{2n}\left(\varphi^{(m)}\right)^2$.

Appendix E

Maple Programming

Maple is a powerful tool which can be used by everybody who deals with mathematical calculations [30]. It is especially useful for symbolic computations as that was the purpose of its construction at the University of Waterloo in Ontario, Canada [31]. Its basic features as elementary data structures, input/output, arithmetic with numbers and elementary simplification are coded in a systems programming language for describing algebraic algorithms. They consist of a relatively small core program, programed in C [32], and it is accompanied by a large number of libraries and routines which are mainly developed using Maple's own language. With this language the user can build additional libraries and routines to match his/hers calculational needs.

The main advantage of programing in Maple is that a large number of functions like the permutation function and "natural" simplifications like factorisation are already present in the main library. This makes the programing, which is in a high level programing language, much easier and faster. At the same time there are the basic disadvantages of memory inefficiency and lack of speed at some stages in comparison with ordinary programing languages. In addition Maple gives in a few cases "strange" results, that is, wrong answers in algebraic or other computations, which should prevent us from a "blind" trust in Maple. For our purpose we will see that Maple can be proved useful up to a certain point.

Let us have a look at some basic input/output operations [33]. Maple procedures, sessions

or variables can be read into Maple using the *read* command:

read

Definition :	read 'filename[.ms]'
Usage :	read variables, procedures or whole sessions.
Example :	read 'MyFile.ms';

This is used to read codes written in a text editor and executed in the Maple worksheet. These codes can be named with an extension **.ms*. These procedures can also been saved into the main library with the use of the command *save* in place of *read*.

Another useful procedure is *simplify*:

simplify

Definition :	simplify expression, (identity)
Usage :	simplification using standard or defined identities
Example :	simplify($\sin^2 x + \cos^2 x$); simplify($a + b + 2c, \{a + b + c = 0\}$);

With it you can do mathematical simplifications or to impose defined identities in certain expressions.

Two very important functions in Maple are the *op* and *nops*, which make it possible to pick out certain sub-terms or sub-expressions from structured expressions or indices of an indexed type variable:

op, nops

Definition :	op(integer,expression); nops(expression);
Usage :	pick out and determine the number of subexpressions in expression
Example :	op(1,a+b)=a nops(a+b)=2

To see in what elements of an expression A and in which order the *op* function puts them we can evaluate $op(A)$ which will return us ordered the primary components of A. Then we can extract them by using *op* in the initial way or we can decompose further composed components.

The command *subs* has the following properties:

subs

Definition :	subs(variable1=expression1,expression2);
Usage :	substitude expression1 for every occurance of variable1 in expression2
Example :	subs(a=c*d,a+b)=c*d+b

and proves to be very useful tool. Maple attributes on each object (expressions and variables) with a type. The function *type* offers the possibility to check if an expression

is of a specific type:

type

Definition :	type(expression,expression-type)
Usage :	returns true or false if the type of expression is given by expression-type or not
Example :	type(A[d],indexed)=true

These are a few examples of the functions in Maple library. We can define new procedures or executable programs by using the following format:

```
ProcedureName:=proc(Variable1,Variable2,...)
local LVariable1, LVariable2,...;
      <commands>
RETURN(RVariable);
end;
```

A more extended discussion of procedures in Maple can be found in [34].

One of the procedures we are going to use is the functional differentiation of an expression. Some basic features of it are the following

$$\frac{\delta}{\delta f(y)} f(x) = \delta(x-y)$$

$$\frac{\delta}{\delta f(y)} f'(x) = \delta'(x-y)$$

$$\frac{\delta}{\delta f(y)} f^2(y) = 2f(y)\delta(x-y)$$

$$\frac{\delta}{\delta f(y)} f'^2(x) = -2f'(x)\delta'(x-y)$$

These procedures can be done by Maple by using the modified operator D:

D

Definition :	functional differentiation
Usage :	gives the functionally differentiation of an expression with respect to a given function f with variable x
Example :	$D(f(x)) = Dirac(x-y)$, $D(diff(f(x),x)) = Dirac(1,x-y)$

The function $Dirac(n, x-y)$ represents the n-th x-derivative of the delta function $\delta(x-y)$. To define the symbol D we have to use the code

```
fdiff:=proc(F,X,n)
local i;
 D(X)=0;
```

```
 D(F(X)):=Dirac(X-y);
  for i from 1 to n do
   D(diff(F(X),X$i)):=Dirac(i,X-y);
   D(z[i]):=0;
  od;
RETURN(FDifferentiation_defined);
end;
```

This code will define D to be able to act upon given functions F with variable X giving the expected functional differentiation while acting upon X or $z[i]$ will give zero. We must be careful, though, because initially D has a specific structure of a differential operator in Maple libraries, obeying the relations

$$D(f+g) = D(f) + D(g)$$

$$D(f*g) = D(f)*g + f*D(g)$$

$$D(f@g) = D(f)@g * D(g)$$

and so forth, where @ is the composition of functions symbol. What we did is to extend its initial definition suitably for our purpose.

Another useful example is the production of the b series (see section (4.6)). If we set $g = 1$ and $M = 1$ in the classical equations of the b's then the following program gives the Laplacian part of the field independent term in equation (4.31).

```
bees:=proc(n)
local inp1,inp2,inp3,eqn,vars,sols,b,l,i,j,h,t,ser;
l:=1;
h[1]:=NULL;
inp2[1]:=NULL;
 for i from 2 to n do
  l:=l+1;
  inp1[i]:=sum(b[j]*b[i-j],j=0..i);
  inp2[l]:=inp2[l-1],inp1[i];
  h[l]:=h[l-1],b[i];
```

```
 od;
eqns:=inp2[n],b[0]^2-1/4, 2*b[0]*b[1]-1/4;
vars:=h[n],b[0],b[1];
sols:=[solve(eqns,vars)];
 for t from 0 to n do
  inp3[t]:=subs(sols[2],b[t]);
 od;
ser:=2*sum(x^(1/2)*inp3[r]*x^r/(GAMMA(r+1)*(2*r+1)*(Pi)),r=0..n);
RETURN(ser);
end;
```

The number n gives the order where the series is truncated. After executing this program we can obtain the behaviour of the divergence of the energy, which is renormalised with the use of the term $\mathcal{E}(\lambda)$, as seen in section (4.7).

A general way to produce the classical equations of *basic vectors* through Maple programing has been shown in section (4.6). Another code has been presented there which solves the equations so that finally we get the classical value of the coefficients of the *basic vectors*. The basic elements of these programs is to find, after a theoretical study,the general equations in way accepted by Maple taking care of all the subcases. Then using the command *solve* Maple is able to solve our system of equations for any number of them.

In the previous we used Maple to reproduce formulas we already know their general form, and then solve them. We can ask for something more ambitious. We can make a program which for given local functionals it can reproduce the corresponding terms in the Schrödinger equation. Then, by taking a big and "complete" set of local functionals we can have a closed set of algebraic equations to solve generated as in (4.31).

In general, what we are asking for is a program which initially generates this "complete" set of *basic vectors*. Then we want these vectors to be functionally differentiated to produce the Laplacian term and the squared term shown in the Schrödinger equation. These terms, to be used for the algebraic equations, should be decomposed to *basic vectors* (see Appendix C). We can do that, also, with another program, as we can see in the following.

Let us have a look at the code which gives the action of the Laplacian upon a given basis vector, reduced also to a basis vector. The main difficulty lies on how can we make the program recognise b.v.'s. For this purpose we use a way to translate the conventionally written vectors as

$$\int dx \phi^{u_0} ... \phi^{(n)\, u_n} \tag{E.1}$$

to a more approachable way for the program, like

$$[[0, u_0], [1, u_1], ..., [n, u_n]] \tag{E.2}$$

where in each square bracket appears the number of derivatives and the power for each ϕ.

The following program is a composition of small programs, each one serving a specific purpose. At first we define the functional differentiation with the appropriate variables. Then, the Laplacian is acted upon the test functionals and we get a linear combination of b.v.'s and non-basis vectors with coefficients depending on s. Each one of them is translated in the form (E.2). Then, the non-b.v.'s are picked and reduced to b.v.'s and the overall result is presented. Let us see the program with a few explanations inserted for the various steps.

THIS IS A PROCEDURE TO EVALUATE THE ACTION OF THE LAPLACIAN UPON THE ANSATZ W. THE FIRST STEP IS TO DEFINE A FUNCTIONAL DIFFERENTIATION PROCEDURE UPON $\int F(\phi(a), \phi'(a), ...)da$.

```
diffy:= proc(F,X,Z)
local i;
D(X):=0;
D(F(X)):= Dirac(X-Z);
 for i from 1 to 20 do
  D(diff(F(X),X$i)):= Dirac(i,X-Z);
  D(z1[i]):=0;
  D(z2[i]):=0;
  D(z3[i]):=0;
  D(z4[i]):=0;
```

```
  D(z5[i]):=0;
 od;
RETURN(FDifferentiation_defined);
end;
```

GE IS THE INPUT FUNCTIONAL. *LAPL* INITIATES THE PROCEDURE OF THE DIFFERENTIATIONS. G IS ITS FIRST FUNCTIONAL DERIVATIVE AND $D(G)$ THE SECOND. *PI* IS THE REGULARISED ACTION OF THE LAPLACIAN ON THE FUNCTIONAL WITH A MOMENTUM CUTOFF $1/s$. *FWTINI* IS A SUBROUTINE WHICH TAKES *PI* THROUGH THE NECESSARY ROUTINES OF DECOMPOSING THE OUTCOMING FUNCTIONALS IN BASIC VECTOR ONES COMBINED WITH COEFFICIENTS DEPENDING ON THE CUTOFF s OR RE-SUMMED, TO DEPEND ON λ.

```
lapl:= proc(GE)
local j,aux1,dfxim,result,final,k,PI,RPI,in2,
rrr,in3,rrrr,G,inp2,r,inp3,rr,FWT;
fdiffy(f,a,y);
in2[15]:=D(GE);
 for rrr from 15 by -1 to 1 do
  in2[rrr-1]:=subs(diff(f(a),arrr)=gg[rrr](a),in2[rrr]);
 od;
in3[0]:=int(in2[0],a=-infinity..infinity);
 for rrrr from 1 to 15 do
  in3[rrrr]:=subs(gg[rrrr](y)=diff(f(y),yrrrr),in3[rrrr-1]);
 od;

fdiffy(f,y,x);
G:=in3[15];
inp2[35]:=expand(D(G)*exp(I*p*(x-y)));
 for r from 35 by -1 to 1 do
  inp2[r-1]:=subs(diff(f(y),yr)=g[r](y),inp2[r]);
 od;
inp3[0]:=int(inp2[0],y=-infinity..infinity);
```

```
 for rr from 1 to 35 do
  inp3[rr]:=subs(g[rr](x)=diff(f(x),xrr),inp3[rr-1]);
 od;
PI:=expand(int(inp3[35],p=-s^(-1/2).s^(-1/2)));
FWT:=fwtini(PI);
RETURN(FWT);
end;
```

******* AUXILIARY ROUTINES *******

MASTER

MASTER IS A ROUTINE TO TRANSLATE A VECTORS FROM THE FORM (E.1) TO THE FORM (E.2). *FAFF* READS THE NUMBER OF DERIVATIVES ON EACH INDIVIDUAL ϕ WHILE *POW* READS ITS POWER.

```
master:=proc(inp1,n)
local inp2,i1;
inp2:=NULL;
 if type(inp1,'+') then
  for i1 to nops(inp1) do
   inp2:=inp2,[pow(op(i1,inp1),n)];
  od;
 elif type(inp1,'*') then inp2:=[pow(inp1,n)];
 elif type(inp1,'**') then
  inp2:=[faff(op(1,inp1),n),op(2,inp1)];
 elif type(inp1,'numeric') then inp2:=[[inp1,'number']];
 elif type(inp1,'function') then inp2:=[faff(inp1,n),1];
 else ERROR('Wrong format for us!');
 fi;
RETURN([inp2]);
end;

faff:=proc(q,n)
```

```
local inp1,inp2,inp3,l,i;
l:=0;
inp2[0]:=NULL;
 for i from 1 to n do
  l:=l+1;
   if q=diff(f(t),ti) then
    inp1[i]:=i;
   else inp1[i]:=NULL;
   fi;
  inp2[l]:=inp2[l-1],inp1[i];
 od;
 if inp2[n]=NULL and q=f(t) then
  inp3:=0 elif inp2[n]=NULL and type(q,indexed then
  inp3:=q;
 else inp3:=inp2[n]
 fi;
RETURN(inp3);
end;

pow:=proc(q,n)
local inp1,inp2,inp3,ind2,i,j,l,k;
l:=0;
k:=0;
inp2[0]:=NULL;
 if type(op(1,q),numeric) and nops(q)<>1 then
  if nops(q)=2 and type(op(2,q),'**') then
   inp2[nops(q)-1]:=[faff(op(1,op(2,q)),n),op(2,op(2,q))]
  elif type((q/(op(1,q))),'*') then
    for i from 2 to nops(q) do
     l:=l+1;
      if type(op(i,q),'**') then
       inp1[i]:=(faff(op(1,op(i,q)),n),op(2,op(i,q)));
      else inp1[i]:=(faff(op(i,q),n),1);
```

```
      fi;
     np2[l]:=inp2[l-1],[inp1[i]];
    od;
  else inp2[nops(q)-1]:=[faff(op(2,q),n),1];
  fi;
  inp3:=([op(1,q),'number'],inp2[nops(q)-1]);
 elif nops(q)=1 and type(q,numeric) then
  inp3:=([q,'number']); else if type(q,'^') then
  inp2[nops(q)]:=[faff(op(1,q),n),op(2,q)]
 elif type(q,'*') then
  for j from 1 to nops(q) do
   k:=k+1;
    if
     type(op(j,q),'**') then
     inp1[j]:=(faff(op(1,op(j,q)),n),op(2,op(j,q)));
    else inp1[j]:=(faff(op(j,q),n),1);
    fi;
   inp2[k]:=inp2[k-1],[inp1[j]];
  od;
 else inp2[nops(q)]:=[faff(q,n),1];
 fi;
inp3:=inp2[nops(q)]
fi;
RETURN(inp3);
end;
```

COMPARE

COMPARE SEPARATES THE BASIS VECTORS FROM THE NON B.V.'S IN A GIVEN EXPRESSION.

```
compare:=proc(inp1)
local var0,var1,var2,var3,var4,var5,var11,i1,l,sas,l1,k1,
var13,var14,var12,i11,var15,var16,var17;
```

```
l:=0;
var3[0]:=NULL;
var11[0]:=NULL;
 if nops(inp1)=1 and nops(op(1,inp1))=1 and
  nops(op(1,op(1,inp1)))=2
  and op(2,op(1,op(1,inp1)))=number then
  var4:='Number';
  var5:=op(1,op(1,op(1,inp1)));
 elif nops(inp1)=1 then
  if nops(op(1,inp1))=1 and
   type(op(2,op(1,op(1,inp1))),'numeric') then
    if op(2,op(1,op(1,inp1)))=1 and
     type(op(1,op(1,op(1,inp1))),'numeric') then
     var4:='It is 1'; var5:=op(1,inp1);
    else var4:='It is not 1';
     var5:=op(1,inp1);
    fi;
   elif nops(op(1,inp1))>1 and
    op(2,op(nops(op(1,inp1)),op(1,inp1)))=1 and
    type(op(1,op(nops(op(1,inp1)),op(1,inp1))),'numeric')
then var4:='It is 1';
    var5:=op(1,inp1);
   else var4:='It is not 1';
    var5:=op(1,inp1);
   fi;
  elif nops(inp1)>1 then
   for i1 from 1 to
    nops(inp1) do
    l:=l+1;
     if nops(op(i1,inp1))=1 and nops(op(1,op(i1,inp1)))=2
and op(2,op(1,op(i1,inp1)))=number then
      var2[i1]:='Number';
      var1[i1]:=op(i1,inp1);
```

```
     elif nops(op(i1,inp1))=1 and nops(op(1,op(i1,inp1)))=2
and type(op(2,op(1,op(i1,inp1))),'numeric') then
      if op(2,op(1,op(i1,inp1)))=1 and
type(op(1,op(1,op(i1,inp1))),'numeric') then
       var2[i1]:='It is 1';
       var1[i1]:=op(i1,inp1);
      else var2[i1]:='It is not 1';
       var1[i1]:=op(i1,inp1);
      fi;
     elif op(2,op(nops(op(i1,inp1)),op(i1,inp1)))=1 and
type(op(1,op(nops(op(i1,inp1)),op(i1,inp1))),'numeric') then
var2[i1]:='It is 1'; var1[i1]:=op(i1,inp1);
     else var2[i1]:='It is not 1';
      var1[i1]:=op(i1,inp1)
     fi;
    var3[l]:=var3[l-1],var2[i1];
    var11[l]:=var11[l-1],var1[i1];
   od;
  var4:=var3[l];
  var5:=var11[l];
 fi;
sas:=[[var4],[var5]];
l1:=0;
k1:=0;
var13[0]:=NULL;
var14[0]:=NULL;
var11:=op(1,sas);
var12:=op(2,sas);
 for i11 from 1 to nops(var11) do
  if op(i11,var11)='It is 1' then
   l1:=l1+1;
   var13[l1]:=var13[l1-1],op(i11,var12);
  elif op(i11,var11)='It is not 1' then
```

```
    k1:=k1+1;
    var14[k1]:=var14[k1-1],op(i11,var12);
   fi;
  od;
var15:=var13[l1];
var16:=var14[k1];
var17:=[[var15],[var16]];
RETURN(var17);
end;
```

MASA

MASA RE-EXPRESSES VECTORS OF THE FORM (E.2) TO THE FORM (E.1).

```
masa1:=proc(inp1)
local var0,var1,i0,l;
l:=0;
var1[0]:=0;
 if nops(inp1)<>1 then
  for i0 from 1 to nops(inp1) do
   l:=l+1;
   var1[l]:=var1[l-1]+masa(op(i0,inp1));
  od;
  var0:=NULL;
 elif nops(inp1)=1 then
  var0:=masa(op(1,inp1));
  var1[0]:=NULL;
 fi;
RETURN(var0,var1[l]);
end;

dif:=proc(inp1,n)
local var1;
 if n=0 then
```

```
  var1:=inp1;
 elif n<>0 and type(n,numeric) then
  var1:=diff(inp1,t$n);
 elif type(n,indexed) then
  var1:=n;
 fi;
RETURN(var1);
end;

masa:=proc(inp1)
local var1,l,var0,i;
l:=0;
var1[1]:=1;
 if nops(inp1)=1 then
  if op(2,op(1,inp1))=number then
   var0:=op(1,o(1,inp1));
  else var0:=dif(f(t),op(1,op(1,inp1)))^op(2,op(1,inp1));
  fi;
  var1[1]:=NULL;
 elif nops(inp1)<>1 and
  op(2,op(1,inp1))=number then
   for i from 2 to nops(inp1) do
    l:=l+1;
    var1[1]:=op(1,op(1,inp1));
    var1[l+1]:=var1[l]*dif(f(t),op(1,op(i,inp1)))^op(2,op(i,inp1));
   od;
  var0:=NULL;
 elif nops(inp1)<>1 and
  type(op(2,op(1,inp1)),numeric) then
   for i from 1 to nops(inp1) do
    l:=l+1;
    var1[l+1]:=var1[l]*dif(f(t),op(1,op(i,inp1)))^op(2,op(i,inp1));
   od;
```

```
  var0:=NULL;
 fi;
RETURN(var0,var1[l+1]);
end;
```

MEGA

MEGA REDUCES THE NON-B.V.'S TO B.V.'S FROM A GIVEN EXPRESSION.

```
mega:=proc(A)
local q,p,up,uq,var1,var2,l1,var3,var00,T2,l,var11,T3,T,B,TM,T4,T5;
B:=A;
with(student);
l1:=0;
var3[0]:=0;
while B<>0 do
 if type(B,'+') then
  T:=op(1,B);
 else T:=B;
 fi;
TM:=op(1,master(T,20));
l:=0;
var2[0]:=NULL;
q:=TM[nops(TM)][1];
p:=TM[nops(TM)-1][1];
uq:=TM[nops(TM)][2];
up:=TM[nops(TM)-1][2];
 if q=p+1 then
  var00:=-diff(product('op(j,T)','j'=1..(nops(T)-2)),t)*
                    diff(f(t),tp)}^(up+1)/(up+1);
  var11:=sorte(expand(var00),15);
  l1:=l1+1;
  var3[l1]:=var3[l1-1]+var11;
  B:=B-T;
```

```
 else
  T2:=intparts(Int(T,t),product(op(j,T),j=1..(nops(T)-1)))
  -op(1,intparts(Int(T,t),product(op(j,T),j=1..(nops(T)-1))));
  T3:=-sorte(expand(sorte(op(1,-T2),15)),15);
  T4:=compare(master(T3,20));
   if op(2,T4)<>NULL then
    l1:=l1+1;
    var3[l1]:=var3[l1-1]+masa1(op(2,T4));
   fi;
  T5:=masa1(op(1,T4));
  B:=B-T+T5;
 fi;
od;
RETURN(var3[l1]);
end;
```

SORTE

SORTE IS A USEFUL ROUTINE WHICH PUTS THE ϕ IN INCREASING ORDER OF THEIR DERIVATIVES.

```
sorte:=proc(inp1,n)
loca li,var2,var1;
var2[0]:=f(t);
 for i from 1 to n do
  var2[i]:=var2[i-1],diff(f(t),t$i);
 od;
var1:=sort(inp1,[var2[n]]);
RETURN(var1);
end;
```

*****END OF AUXILIARY ROUTINES*****

ELE

ELE CONNECTS THE PREVIOUS SUBROUTINES TOGETHER SO THAT WE GET THE DESIRED RESULT.

```
ele:=proc(QQ)
local QQ1,EE,RR,WW,AA,FINAL;
WW:=master(expand(QQ),15);
EE:=compare(WW);
RR:=masa1((op(1,EE)));
AA:=mega(RR);
FINAL:=AA+masa1(op(2,EE));
RETURN(FINAL);
RETURN(WW);
end;
```

FWTINI

FWTINI IS A ROUTINE TO PERFORM THE RE-SUMMATION PROCEDURE AND INTRODUCES THE CUTOFF PARAMETER λ.

```
fwtini:=proc(inp1)
local l,var1,var2,var3,var4,var5,var6,var7,var8,i,inp11,var22;
l:=0;
var7[0]:=0;
elen(0):=0;
inp11:=sorte(inp1,15);
var1:=subs(s=1/LL^2,inp1);
var2:=subs(1/sqrt(Pi)=1,var1);
var22:=simplify(subs(x=t,var2));
var3:=subs(csgn(conjugate(LL))=1,sorte(collect(var22,LL),15));
 for i from 0 to nops(inp11)-1 do
  l:=l+1;
  var4[i]:=coeff(var3,LL^(2*i+1));
  var5[i]:=elen(sorte(var4[i],15));
  var6[i]:=var5[i]*1/s^((2*i+1)/2);
```

```
  var7[l]:=var7[l-1]+var6[i];
  var8[l]:=var8[l-1],var5[i];
 od;
RETURN(var7[l]);
end;
```

This program generates the Laplacian term of equation (4.31). After reading the program in a Maple worksheet we can produce the outcome of the laplacian acting on an expression G, by using the command *ele(G);*. The $(\delta W/\delta\varphi)^2$ term is generated by a similar procedure, which uses the same basic elements of the previous one.

Appendix F

Functional Method for the Central Charge Term

We can define the regularised Virasoro operators as

$$L[u] = \frac{1}{4}\int d\sigma d\sigma' u(\sigma,\sigma')K^{\mu\nu}(\sigma,\sigma')(P_\mu(\sigma) - X'_\mu(\sigma))(P_\nu(\sigma') - X'_\nu(\sigma')) \equiv$$

$$\frac{1}{4}\int d\sigma d\sigma' u(\sigma,\sigma')R_\mu(\sigma)R_\nu(\sigma')$$

where $K_s^{\mu\nu}(\sigma,\sigma')$ is a Kernel to point split the double action of the functional differentiations, satisfying the condition $\lim_{s\to 0} K_s^{\mu\nu}(\sigma,\sigma') = \eta^{\mu\nu}\delta(\sigma,\sigma')$, $R_\mu(\sigma) \equiv (P_\mu(\sigma) - X'_\mu(\sigma))$, $u(\sigma,\sigma')$ is the component of a vector field on the circle, S^1, on which $X(\sigma)$ is defined and is symmetric in σ and σ'.

We will calculate the commutator $[L[u], L[v]]$ acting on the vacuum state $\langle X|0\rangle$ given in (8.10). We have

$$[L[u], L[v]]\,\langle X|0\rangle =$$

$$\frac{1}{16}\int d\sigma d\sigma' d\bar\sigma d\bar\sigma' u(\sigma,\sigma')v(\bar\sigma,\bar\sigma')K^{\mu\nu}(\sigma,\sigma')K^{\kappa\lambda}(\bar\sigma,\bar\sigma')\times$$

$$[R_\mu(\sigma)R_\nu(\sigma'), R_\kappa(\bar\sigma)R_\lambda(\bar\sigma')]\,\langle X|0\rangle =$$

$$-\frac{1}{4}\int d\sigma d\sigma' d\bar\sigma d\bar\sigma' u(\sigma,\sigma')v(\bar\sigma,\bar\sigma')K^{\mu\nu}(\sigma,\sigma')K^{\kappa\lambda}(\bar\sigma,\bar\sigma')\times$$

$$i\eta_{\mu\kappa}\delta'(\sigma,\bar\sigma)\left\{R_\nu(\sigma')R_\lambda(\bar\sigma') + R_\lambda(\bar\sigma')R_\nu(\sigma')\right\}\langle X|0\rangle. \qquad \text{(F.1)}$$

(F.1) results after applying the relation $[R_\mu(\sigma), R_\kappa(\bar\sigma)] = -2i\delta'(\sigma,\bar\sigma)\eta_{\mu\kappa}$ and re-arranging the σ variables as well as the indices. By commuting $R_\lambda(\bar\sigma')$ and $R_\nu(\sigma')$ and integrating

by parts, we get

$$[L[u], L[v]]\,\langle X|0\rangle =$$

$$\frac{i}{2}\int d\sigma d\sigma' \Big\{ u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial}{\partial\sigma}K^{\lambda\nu}(\sigma,\sigma')+$$

$$\frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma)K^{\mu\nu}(\sigma,\sigma')\Big\} R_\nu(\sigma')R_\lambda(\sigma)\langle X|0\rangle \qquad \text{(F.2)}$$

where the rest of the terms give zero, as they are symmetric in u and v or include the quantity $\frac{\partial}{\partial\sigma}K^{\mu\nu}(\sigma,\sigma')\Big|_{\sigma'=\sigma}$ which is zero[1]. Applying the combination $R_\nu(\sigma')R_\lambda(\sigma)$ on the vacuum state $\langle X|0\rangle$, the only term with singularity coming out will be from the two functional derivatives. That is

$$\frac{\delta}{\delta X^\mu(\sigma')}\frac{\delta}{\delta X^\nu(\sigma)}\langle X|0\rangle =$$

$$\left[4\int H(\sigma',\sigma'')X_\mu(\sigma'')d\sigma''\int H(\sigma,\sigma'')X_\nu(\sigma'')d\sigma'' + 2H(\sigma,\sigma')\eta_{\mu\nu}\right]\langle X|0\rangle \qquad \text{(F.3)}$$

As the Kernel acts on (F.3) the only divergency will come from the last term in the curly brackets. The other terms will be combined as in the normal ordered case to give the normalised Virasoro operator sitting on the r.h.s. of the algebra. Let us study the term

$$T \equiv -i\int d\sigma d\sigma' \Big\{ u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial}{\partial\sigma}K^{\lambda\nu}(\sigma,\sigma')+$$

$$\frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma)K^{\mu\nu}(\sigma,\sigma')\Big\}\eta_{\lambda\nu}H(\sigma,\sigma'). \qquad \text{(F.4)}$$

We can take the Kernel to be of the form

$$K^{\mu\nu}(\sigma,\sigma') = \eta^{\mu\nu}\mathcal{G}_s(\sigma,\sigma') = \eta^{\mu\nu}\frac{e^{-(\sigma-\sigma')^2/4s}}{\sqrt{2\pi s}}. \qquad \text{(F.5)}$$

where

$$\lim_{s\to 0}\frac{e^{-(\sigma-\sigma')^2/4s}}{\sqrt{2\pi s}} = \delta(\sigma-\sigma').$$

Expression (F.4) can be symmetrised with respect to σ and σ' so that the summation in H can be re-written as

$$H(\sigma,\sigma') = -\frac{1}{4\pi}\sum_{m=-\infty}^{\infty}|m|e^{im(\sigma-\sigma')} \to$$

$$\to -\frac{2}{4\pi}\sum_{m=1}^{\infty}me^{im(\sigma-\sigma')} = -\frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{\partial}{\partial\sigma}e^{im(\sigma-\sigma')} = -\frac{1}{2\pi i}\frac{\partial}{\partial\sigma}\left(\frac{1}{1-e^{i(\sigma-\sigma')}}\right)$$

[1] See Chapter 6.

Substituting this into (F.4) we get

$$T = -iN \int d\sigma d\sigma' \left\{ -u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial^2}{\partial\sigma^2}\mathcal{G}_s(\sigma,\sigma') - \right.$$

$$\left. \frac{\partial^2}{\partial\sigma^2}u(\sigma,\sigma')v(\sigma,\sigma)\mathcal{G}_s(\sigma,\sigma') \right\} \frac{-1}{2\pi i}\frac{1}{1-e^{i(\sigma-\sigma')}} \qquad \text{(F.6)}$$

By expanding

$$\frac{1}{1-e^{i(\sigma-\sigma')}} = \frac{i}{\sigma-\sigma'} + \frac{1}{2} - \frac{i}{12}(\sigma-\sigma') + ...$$

the first term of (F.6) gives

$$-iN \int d\sigma d\sigma' u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial^2}{\partial\sigma^2}\mathcal{G}_s(\sigma,\sigma')\frac{1}{2\pi i}\frac{1}{1-e^{i(\sigma-\sigma')}} =$$

$$-iN \int d\sigma d\sigma' \left\{ \frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial}{\partial\sigma}\mathcal{G}_s(\sigma,\sigma')\frac{1}{2\pi i}\left(\frac{i}{\sigma-\sigma'} + \frac{1}{2} - \frac{i}{12}(\sigma-\sigma') + ...\right) - \right.$$

$$\left. -u(\sigma,\sigma')v(\sigma,\sigma)\frac{\partial}{\partial\sigma}\mathcal{G}_s(\sigma,\sigma')\frac{1}{2\pi i}\left(-\frac{i}{(\sigma-\sigma')^2} - \frac{i}{12} + ...\right) \right\}. \qquad \text{(F.7)}$$

We can set $x = \sigma - \sigma'$. As the exponential damps all contributions for large x we can extend the integration of x from $-\infty$ to $+\infty$. Hence (F.7) becomes

$$-iN \int d\sigma dx \left\{ \frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma)\left(-\frac{2x}{4s\sqrt{s}}\right)\frac{e^{-x^2/4s}}{\sqrt{2\pi}}\frac{1}{2\pi i}\left(\frac{i}{x} + \frac{1}{2} - \frac{i}{12}x + ...\right) - \right.$$

$$\left. -u(\sigma,\sigma)v(\sigma,\sigma)\left(-\frac{2x}{4s\sqrt{s}}\right)\frac{e^{-x^2/4s}}{\sqrt{2\pi}}\frac{1}{2\pi i}\left(-\frac{i}{x^2} - i + ...\right) \right\} =$$

$$-iN \int d\sigma dx \left\{ \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{(-1)}{4\pi i}\frac{e^{-x^2/4s}}{\sqrt{2\pi}}\left(\frac{i}{s\sqrt{s}} + \frac{1}{2s\sqrt{s}}x - i\frac{x^2}{12s\sqrt{s}} + ...\right) + \right.$$

$$\left. u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{\sqrt{2\pi}}\frac{1}{s\sqrt{s}}\frac{1}{2\pi i}\left(-\frac{i}{x} - ix + ...\right) \right\}$$

We can use the relations

$$\int_{-\infty}^{\infty} dx \frac{e^{-x^2/4s}}{x^2} = \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{e^{-x^2/4s}}{s}, \quad \int_{-\infty}^{\infty} dx \frac{e^{-x^2/4s}}{x} = 0, \quad \int_{-\infty}^{\infty} dx\, e^{-x^2/4s} = 2\sqrt{s}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} dx\, x e^{-x^2/4s} = 0, \quad \int_{-\infty}^{\infty} dx\, x^2 e^{-x^2/4s} = 2\int_{-\infty}^{\infty} dx\, s e^{-x^2/4s}, \quad ...$$

to get

$$-iN \int d\sigma dx \left\{ \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{(-1)}{4\pi i}\frac{e^{-x^2/4s}}{\sqrt{2\pi}}\left(\frac{i}{s\sqrt{s}} - \frac{2i}{12\sqrt{s}} + ...\right) + 0 \right\} \qquad \text{(F.8)}$$

where the second term in the curly brackets in (F.8) is zero because of its symmetry in u and v. Up to zeroth order in s we obtain

$$\frac{iN}{4\pi} \int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{s\sqrt{2\pi s}} -$$

$$-\frac{iN}{2\pi}\int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{12\sqrt{2\pi s}} =$$

$$\frac{iN}{4\pi\sqrt{2\pi}}\int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{s\sqrt{s}} - \frac{iN}{24\pi}\int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{\sqrt{2\pi s}} \rightarrow$$

$$\rightarrow \frac{iN}{4\pi\sqrt{2\pi}}\int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma)\frac{e^{-x^2/4s}}{s\sqrt{s}} - \frac{iN}{24\pi}\int d\sigma \frac{\partial}{\partial\sigma}u(\sigma,\sigma)v(\sigma,\sigma) \qquad \text{(F.9)}$$

for $s \rightarrow 0$.

Treating in the same way the second term in T, we get altogether

$$T = \frac{iN}{4\pi}\int d\sigma dx \frac{\partial}{\partial\sigma}u(\sigma)v(\sigma)\frac{e^{-x^2/4s}}{s\sqrt{2\pi s}} -$$

$$-\frac{iN}{24\pi}\int d\sigma \frac{\partial}{\partial\sigma}u(\sigma)v(\sigma) - \frac{iN}{24\pi}\int d\sigma \frac{\partial^3}{\partial\sigma^3}u(\sigma)v(\sigma) \qquad \text{(F.10)}$$

We can see that the rest of the terms of (F.2), apart from T, construct the renormalised part of

$$\frac{i}{2}\int d\sigma d\sigma' \frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma)K^{\mu\nu}(\sigma,\sigma')R_\nu(\sigma')R_\lambda(\sigma)\langle X|0\rangle$$

and as we can antisymmetrise in u and v we have

$$\frac{i}{4}\int d\sigma d\sigma' \left\{\frac{\partial}{\partial\sigma}u(\sigma,\sigma')v(\sigma,\sigma) - u(\sigma,\sigma)\frac{\partial}{\partial\sigma}v(\sigma,\sigma')\right\} \times$$

$$K^{\mu\nu}(\sigma,\sigma')R_\nu(\sigma')R_\lambda(\sigma)\langle X|0\rangle =$$

$$-iL[[u,v]]\langle X|0\rangle$$

for $[u,u] = uv' - u'v$. This is the desired result as it is calculated in [36] with another method. For $u = e^{-in\sigma}$ and $v = e^{-im\sigma}$ it gives the result we obtained in (8.8).

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