Resumming QCD perturbation series

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Resumming QCD Perturbation Series

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Abstract

Since the advent of Quantum Field Theory (QFT) in the late 1940's, perturbation theory has become one of the most developed and successful means of extracting phenomenologically useful information from a QFT. In the ever-increasing enthusiasm for new phenomenological predictions, the mechanics of perturbation theory itself have often taken a back seat. It is in this light that this thesis aims to investigate some of the more fundamental properties of perturbation theory.

The benefits of resumming perturbative series are highlighted by the explicit calculation of the three-jet rate in e^+e^- annihilation, resummed to all orders in leading and next-to-leading large logarithms. It is found that the result can be expressed simply in terms of exponentials and error functions.

In general it is found that perturbative expansions in QED and QCD diverge at large orders. The nature of these divergences has been explored and found to come from two sources. The first are instanton singularities, which correspond to the combinatoric factors involved in counting Feynman diagrams at large orders. The second are renormalon singularities, which are closely linked to non-perturbative effects through the operator product expansion (OPE).

By using Borel transform techniques, the singularity structure in the Borel plane for the QCD vacuum polarization is studied in detail. The renormalon singularity structure is as expected from OPE considerations. These results and existing exact large-N_f results for the QCD Adler D-function and Deep Inelastic Scattering sum rules are used to resum to all orders the portion of the QCD perturbative coefficients which is leading in b, the first coefficient of the QCD beta-function. This part is expected asymptotically to dominate the coefficients in a large-N_f expansion.

Resummed results are also obtained for the e^+e^- R-ratio and the τ-lepton decay ratio. The renormalization scheme dependence of these resummed results is discussed in some detail.

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Declaration

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

All the research in this thesis has been carried out in collaboration with Dr C.J.Maxwell. Part of Chapter 3 has previously appeared in

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Chapters 5 and 6 are partly summarised in

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- C.N.Lovett-Turner and C.J.Maxwell, University of Durham preprint DTP/95/36 [hep–ph/9505224] (to be published in Nucl. Phys. B).

To Mum, Dad and Jo

Contents

Chapter 1

Introduction

1.1 A Brief History

The history of particle physics is as long as the history of rational thought itself. The desire of man to categorise and rationalise the world around him has led to many and varied ideas in the quest to discover what composes the matter which we observe. The diversity of the physical world appeared to defy any straightforward description of the nature of matter. One of the earliest attempts at simplification was Anaximenes of Miletus's division of physical phenomena into the four 'elements': earth, air, fire and water. Despite its obvious drawbacks, this theory was a significant philosophical step, in that it was founded on the premise that it is possible to categorise physical phenomena. After all, it might not be unreasonable to assume that, for example, rocks obey the laws of rocks, trees obey the laws of trees, mice obey the laws of mice and so on. With Anaximenes' theory came the seeds of the idea that rocks, trees, mice and all physical phenomena obey the laws of physics.

Equally crucially, the theory of the four elements hinged on another assumption of physics, namely that the laws of physics are simple. It is because of these two assumptions that physics can justifiably claim to be the most fundamental of all the sciences. Particle physicists today still work around the premise that the world is governed at heart by simple principles and that the phenomena we observe are just exceptionally diverse manifestations of 'special cases' of those principles.

While the description of the nature of matter is a fundamental goal of particle physics, it is only half the story. The way in which particles interact with each

other is of equally crucial importance. It is only through the interactions of matter that we are able to observe anything at all. The nature of the forces we observe is arguably more difficult to decipher than the constitution of matter. It is for a good reason that in the study of quantum field theory one invariably begins with the case of a non-interacting theory!

When the Greek philosophers observed the physical world, they saw a vast diversity of matter; but they catalogued only three forces: gravity, electricity and magnetism. The last two have since been found to be manifestations of the same force and two more, the weak and the strong nuclear forces, have been discovered. The history of the study of forces is to some extent more simple than the history of the study of the nature of matter in that there was no necessity to categorise forces in the same way as there was to categorise matter.

Rather than give a detailed account of how modern particle physics was born from the study of atoms, nuclei and radiation, we refer the reader to Figure 1.1 in which a diagrammatic modern history of the subject is presented. The shaded box at the right-hand end of the central table represents the current state of knowledge of the fundamental particles found in nature. The idea, born from Heisenberg's uncertainty principle, that the transmission of forces between matter is due to the exchange of particles explains the transition from forces to particles in the lower row of the central table. The important result obtained from the uncertainty principle is that the range of a force is inversely proportional to the mass of the exchanged particle. That is,

$$\Delta E \Delta t \sim \hbar \tag{1.1}$$

$$\Rightarrow mc^2 t \sim \hbar \quad \text{(from relativity)}\,. \tag{1.2}$$

Yet ct is just the maximum distance travelled in time t, hence

$$\begin{aligned} \text{mass} \times \text{max.range} &= \text{constant} \\ \Rightarrow \text{max.range} &\sim \frac{1}{\text{mass}} \end{aligned} \tag{1.3}$$

Thus the photon, which carries the electromagnetic force, is massless, since the electromagnetic force has infinite range. The application of this idea to the strong nuclear force led Yukawa to his famous prediction of the mass of the pion [1]. The

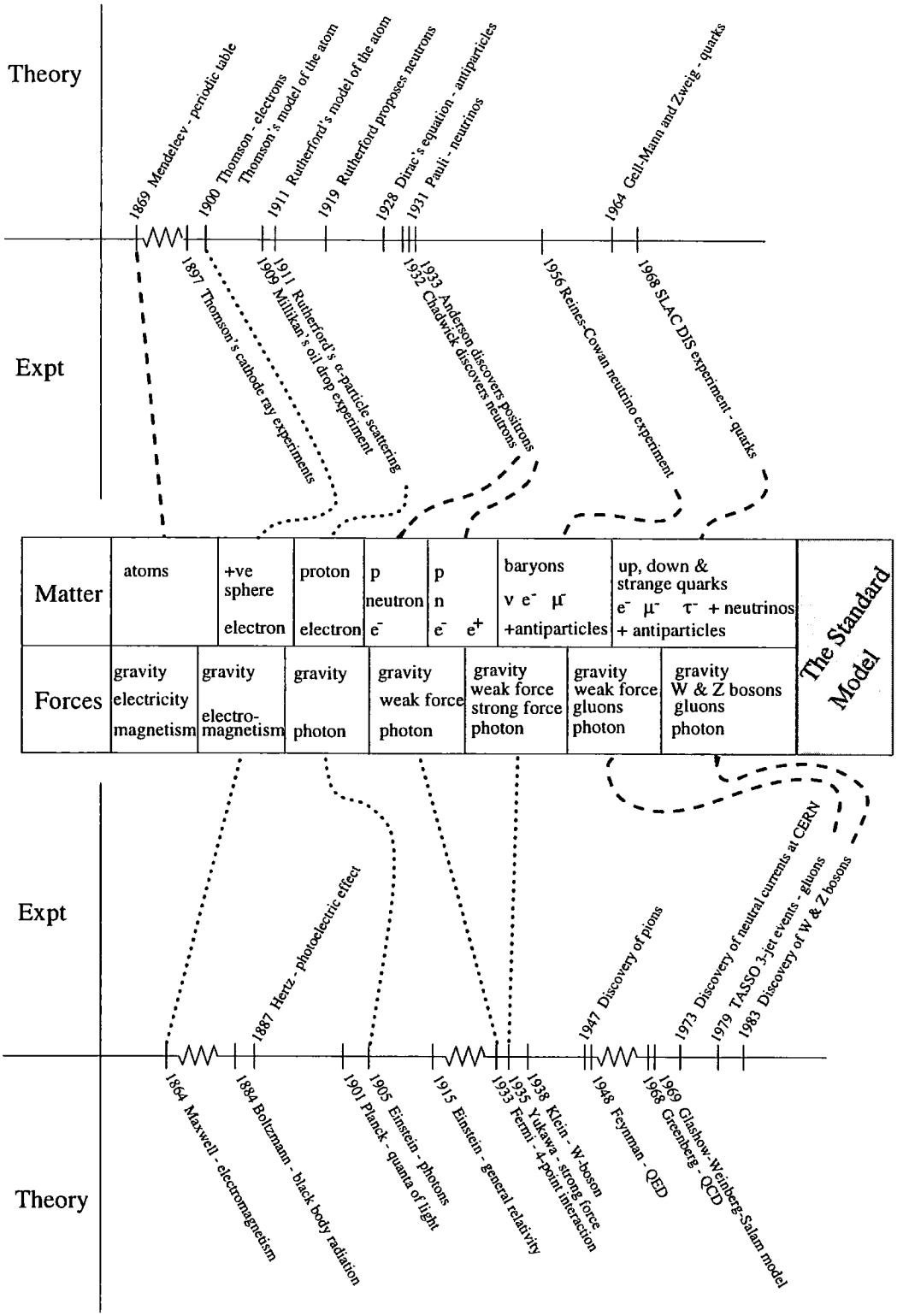


Figure 1.1: Modern history of particle physics.

final point to be made concerning Figure 1.1 is that, for sake of simplicity, only the most salient discoveries and ideas have been listed. For a fuller description and for a history of the discovery of the second and third generations of quarks and leptons see references [2].

In Figure 1.2 the fundamental matter and force carrying particles, together with some of their properties, are shown. This list of particles plus gravity, which at present cannot be explained satisfactorily in terms of a quantum field theory, represents succinctly the current state of particle physics. Its simplicity does, of course, conceal both the complexity of the dynamics of particle interactions and our lack of understanding of a great number of the properties exhibited by the particles listed.

This thesis focuses on Quantum Chromodynamics (QCD), the fundamental field theory of quarks and gluons. In order to find the origin of the quest for a fundamental quantum field theory to describe the strong interaction, one has to look back to the late 1940's and early 1950's. After the Second World War much attention was focused on the study of cosmic rays which at the time, in the absence of high energy particle accelerators, were seen as a rich source of new information about fundamental particles and their properties.

With the improving detector technology, new particles, both mesons and baryons, were discovered with increasing frequency and particle physicists began to draw up tables of particles and their properties in much the same way as had been done nearly a century earlier with the elements. As with the periodic table of elements, it was generally mooted that the proliferation of particles and the patterns which emerged belied a more fundamental structure.

In the early 1960's Gell-Mann and Ne'eman achieved several remarkable predictions by grouping mesons and baryons into octets and nonets according to their observed quantum numbers, a system which became known by Gell-Mann's phrase "The Eightfold Way" [3]. Finally in 1964, it was proposed separately by Gell-Mann and Zweig that the structure of the Eightfold Way could be explained if baryons and mesons were composed of fractionally charged spin-half objects which Gell-Mann named "quarks" [4].

At first quarks were treated as convenient calculational tools rather than

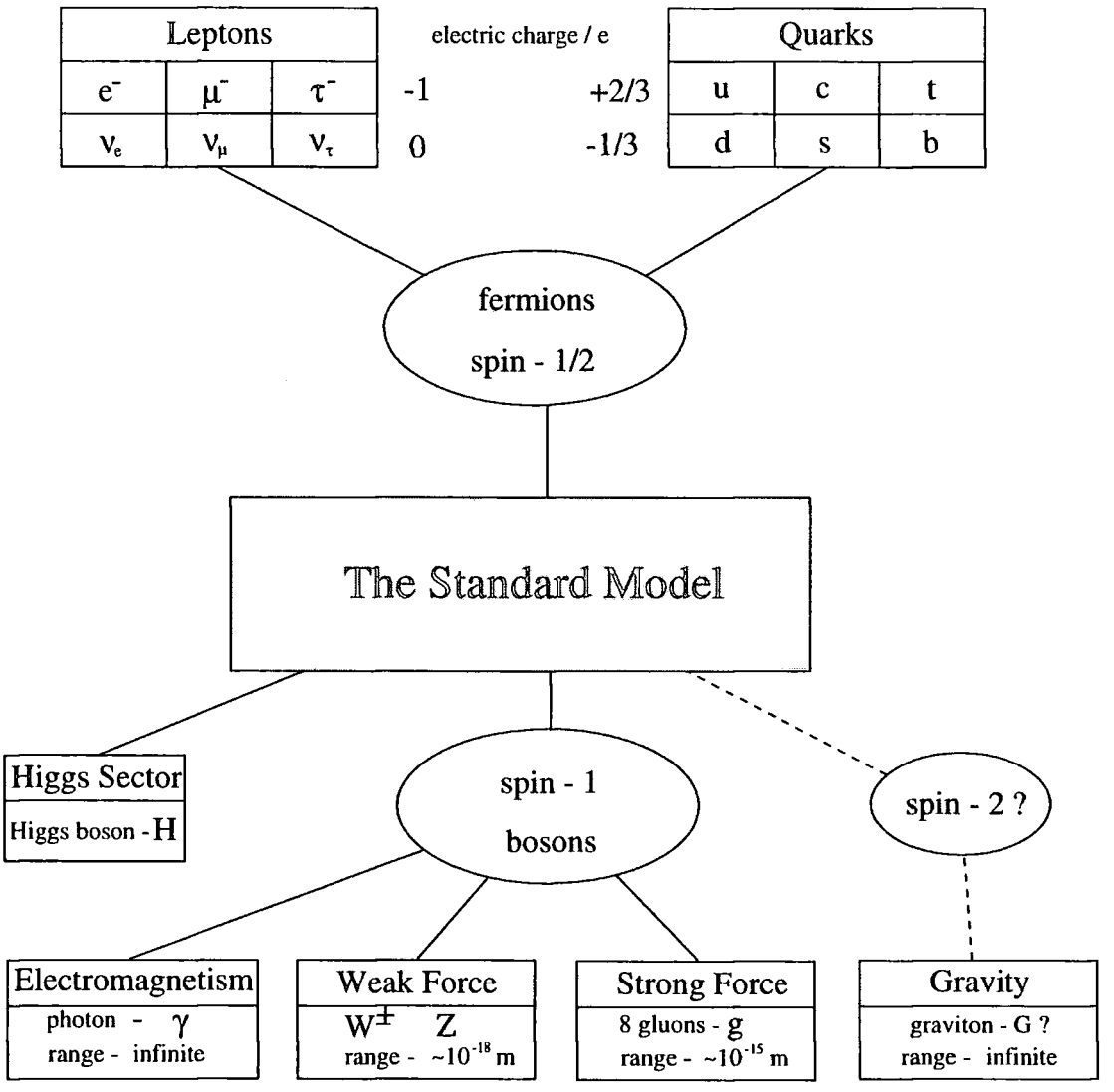


Figure 1.2: The Standard Model of particle physics.

real physical objects; but deep inelastic scattering (DIS) experiments at SLAC [5], which effectively revealed point-like scattering centres within the proton put quarks on a much firmer phenomenological footing. In fact, the SLAC DIS experiments laid the foundation stone for the development of QCD as the fundamental field theory of the strong interaction. A detailed history of QCD would be extraneous here and the reader is referred to the plentiful literature on the subject [2, 6].

We shall in subsequent chapters be addressing some of the problems faced when using perturbation theory to turn the concepts of QCD into practical predictions of physical phenomena.

1.2 Thesis Outline

As stated above, the aim of this thesis is to provide some insight into the workings of QCD as a quantum field theory and in doing this we shall concentrate primarily on the use of perturbation theory at high orders to make contact between the perturbative and non-perturbative aspects of the theory. Ultimately, we would like to find a way to express QCD in such a way as to unite these two regimes; but we are currently hindered by a great lack of understanding of long-distance QCD effects.

First, in Chapter 2, by demonstrating some of the properties of gauge theories, we will highlight the parts of the Standard Model relevant to subsequent discussion. Then, by considering the path integral formalism, we will motivate the use of perturbation theory as a calculational framework in which to study QCD observables. There follows a discussion of renormalization, in the context first of a simple model, and then with respect to Quantum Electrodynamics (QED) and QCD.

Having motivated the usefulness of perturbative expansions, we shall proceed in Chapter 3 to exemplify how the techniques introduced can be applied, by calculating explicitly the 3-jet rate in e^+e^- annihilation, resummed to all orders in the leading and next-to-leading large logarithm expansion.

In Chapter 4 we return to the mechanics of perturbation theory itself. We start with a brief resumé of the study of the behaviour of perturbation theory at

large orders and present an argument for why we expect large order perturbation theory to be divergent, not just for QCD but also for other more simple field theories. In this chapter the Borel transform is introduced as a means of encoding the divergent behaviour of perturbative expansions. The chapter concludes by relating this large order behaviour to the non-perturbative regime of QCD and the operator product expansion (OPE).

Chapter 5 contains a discussion of some of the large order results already available in QED through approximations such as expansions in large numbers of quark flavours (N_f). These expansions relate to a very specific class of Feynman diagrams and we shall show how they contain the origins of divergent behaviour. Having discussed the large-N_f expansion in QED, we shall see how such principles can be extended to QCD (specifically for the case of the Adler D-function, which is closely related to the QCD vacuum polarization). We will motivate the use of a new type of expansion (in $b=\beta_0/2$, the first coefficient of the QCD beta-function), which, despite not having a ready interpretation in terms of a class of Feynman diagrams, has a number of attractive qualities which make it a natural choice for studying large order behaviour in QCD.

With the framework in place for obtaining large order results in QCD, Chapter 6 will show how, by using Borel transform techniques, one can attempt to resum, to all orders, parts of the perturbative series for a number of QCD observables. In this chapter the dependence of these resummed results on the renormalization scheme is discussed and comparisons are made with the results from fixed order perturbation theory.

Finally, Chapter 7 ties up some loose ends and offers some concluding remarks.

Chapter 2

A Calculational Framework

2.1 QCD as a Gauge Theory

It has long been recognised that a deep insight into the nature of particle interactions can be gained by studying the symmetry principles that underlie the field theories governing the interactions. A convenient method for exposing the symmetry principles of a field theory is to extend classical ideas of mechanics to continuous systems and to examine the properties of the Lagrangian density under various transformations. That is, we make the transition from a discrete system, with coordinates $q_i(x)$, to one with continuously varying coordinates $\phi(x_\mu)$. The Lagrangian

$$\begin{aligned} L &\equiv \text{K.E.} - \text{P.E.} \\ &= L(q_i, \dot{q}_i, t) \\ &\rightarrow \mathcal{L}(\phi, \partial_\mu \phi, x_\mu) \end{aligned} \tag{2.1}$$

and the equations of motion become

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \,. \tag{2.2}$$

L and $\mathcal{L}$ are related by

$$L = \int \mathcal{L} d^3x \tag{2.3}$$

but from now on we shall refer to $\mathcal{L}$ as the Lagrangian, as is standard practice.

The concept that electromagnetism should be invariant under so-called "gauge transformations" is not a new one. It was recognised by Maxwell in the late nineteenth century and stems from the fact that making an overall change in the

phase of an electromagnetic wave should have no effect on the way the theory is formulated. In modern parlance we say that, if a Dirac spinor field ψ transforms by

$$\psi \rightarrow \psi' = \mathrm{e}^{i\alpha}\psi \ , \tag{2.4}$$

where α is constant for all spacetime points, then the Lagrangian for a free fermion, for example, an electron,

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi \ , \tag{2.5}$$

where the γ^{μ}'s are Dirac matrices, should be invariant under such a transformation. This is easy to verify. However, we should be able to restrict the Lagrangian further. That is, instead of considering α identical at all spacetime points, we should consider a phase transformation which is a function of position, $\alpha \rightarrow \alpha(x)$. The requirement that the Lagrangian be invariant under transformations of this type corresponds to the transition from a global to a local gauge symmetry. When we apply the transformation

$$\psi \rightarrow \psi' = \mathrm{e}^{i\alpha(x)}\psi \tag{2.6}$$

to the Lagrangian in equation (2.5) we find that the derivative of ψ transforms as

$$\partial_{\mu}\psi \rightarrow \mathrm{e}^{i\alpha(x)}\partial_{\mu}\psi + i\mathrm{e}^{i\alpha(x)}\psi\partial_{\mu}\alpha \ , \tag{2.7}$$

thus breaking the invariance of the Lagrangian. To remedy this, we are forced to modify the derivative such that

$$D_{\mu}\psi \rightarrow \mathrm{e}^{i\alpha(x)}D_{\mu}\psi \ , \tag{2.8}$$

where D_{μ} is known as the "covariant derivative". In order to find D_{μ} we are obliged to introduce a vector field, A_{μ}, such that

$$D_{\mu} \equiv \partial_{\mu} - ieA_{\mu} \ , \tag{2.9}$$

with A_{μ} transforming as

$$A_{\mu} \rightarrow A_{\mu} + \frac{1}{e}\partial_{\mu}\alpha \ . \tag{2.10}$$

It is now easy to check that the new Lagrangian, with ∂_{μ} replaced by D_{μ},

$$\begin{aligned} \mathcal{L} &= i\overline{\psi}\gamma^{\mu}D_{\mu}\psi - m\overline{\psi}\psi \\ &= \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \ , \end{aligned} \tag{2.11}$$

is invariant under the local gauge transformation of equation (2.6). Thus, by demanding local gauge invariance, we are obliged to introduce a vector "gauge field". Noting that the second term in equation (2.11) corresponds to a coupling between this gauge field and a Dirac fermion with strength e, it is proposed that this new field is in fact the photon field. If such a proposal is justified then we must also add a kinetic energy term to the Lagrangian. Since the kinetic term must also be invariant under local gauge transformations, the only possibility is that it involves the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{2.12}$$

The resulting Lagrangian,

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi + e\overline{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{2.13}$$

describes a field theory in which mass m, charge e fermions interact with massless photons (a mass term involving the gauge fields is not gauge invariant). In other words this is the Lagrangian of QED.

The success of this approach as applied to QED leads us to attempt to formulate QCD in a similar manner. First, let us consider the nature of the transformation in equation (2.6). The phase transformations $U(\alpha) \equiv \mathrm{e}^{i\alpha}$, where α runs continuously over the real numbers, form a unitary Abelian group known as the $U(1)$ group. The single parameter of the transformation is related to the existence of only one type of electric charge described by QED. When QCD was first formulated it was designed to explain the apparent violation of the Pauli exclusion principle by the Δ^{++} particle, in which the three quarks appeared to have exactly the same quantum numbers. The idea was that, by introducing a colour charge which came in three types, 'red-ness', 'green-ness' and 'blue-ness', the required antisymmetric form of the total wavefunction of the Δ^{++} was regained.

The implication of this proposal is that we should, when formulating a Lagrangian for QCD, consider phase transformations in three-dimensional colour space. The group naturally associated with such a transformation is SU(3). Thus we require that the free Lagrangian,

$$\mathcal{L}_0 = \overline{q}_j(i\gamma^\mu \partial_\mu - m)q_j \tag{2.14}$$

(shown with one quark flavour for simplicity and with j a colour index), be invariant under an SU(3) transformation,

$$q \to Uq \equiv \mathrm{e}^{i\alpha_a(x)T_a} q \; . \tag{2.15}$$

Here U is an arbitrary 3×3 unitary matrix, the T_a, with $a = 1, \ldots, 8$, are a set of linearly independent traceless 3×3 matrices and α_a are the group parameters. The group is non-Abelian since the generators, T_a, obey the commutation relation

$$[T_a, T_b] = i f_{abc} T_c \; , \tag{2.16}$$

with f_{abc} the (real) structure constants, some non-zero, of the group.

We now proceed in exactly the same way as for QED, defining a covariant derivative,

$$D_\mu = \partial_\mu + i g T_a G^a_\mu \; , \tag{2.17}$$

where the G^a_μ are eight gauge fields which transform as

$$G^a_\mu \to G^a_\mu - \frac{1}{g} \partial_\mu \alpha_a \; . \tag{2.18}$$

Substituting D_μ for ∂_μ in $\mathcal{L}_0$ we obtain

$$\mathcal{L} = \overline{q}(i\gamma^\mu \partial_\mu - m)q - g(\overline{q}\gamma^\mu T_a q) G^a_\mu \; . \tag{2.19}$$

However, closer inspection of the last term reveals that, under an SU(3) transformation, due to the non-Abelian nature of the group,

$$\begin{aligned} (\overline{q}\gamma^\mu T_a q) \;\; &\to \;\; (\overline{q}\gamma^\mu T_a q) + i\alpha_b \overline{q}\gamma^\mu (T_a T_b - T_b T_a) q \\ &= \;\; (\overline{q}\gamma^\mu T_a q) - f_{abc} \alpha_b \overline{q}\gamma^\mu T_c q \; ; \end{aligned} \tag{2.20}$$

and we are obliged to add an extra term to equation (2.18) to restore gauge invariance, giving

$$G^a_\mu \to G^a_\mu - \frac{1}{g} \partial_\mu \alpha_a - f_{abc} \alpha_b G^c_\mu \; . \tag{2.21}$$

Finally, for consistency, we should add a kinetic term for each of the gauge fields. Implementing this we arrive at the QCD Lagrangian,

$$\mathcal{L}_{QCD} = \overline{q}(i\gamma^\mu \partial_\mu - m)q - g(\overline{q}\gamma^\mu T_a q) G^a_\mu - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \; . \tag{2.22}$$

Now, to be invariant under SU(3) transformations, the field strength tensor, $G^a_{\mu\nu}$, also requires an extra term to become

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu \ . \tag{2.23}$$

This extra term imparts a crucial new property to QCD. That is, not only does the QCD Lagrangian describe propagation of quarks, propagation of the eight gauge fields, identified with gluons, and the interaction between quarks and gluons; but it also contains terms such as

$$g f_{abc} (\partial_\mu G^a_\nu) G^b_\mu G^c_\nu$$

$$\text{and} \qquad \frac{g^2}{4} f_{abc} f_{aef} G^b_\mu G^c_\nu G^\mu_e G^\nu_f \ .$$

These represent three- and four-point gluon interactions respectively. This reflects the fact that the gauge fields of QCD carry colour charge whereas, in QED, the photon has no electric charge.

As in QED, it is impossible to construct a gauge invariant mass term for the gauge fields in QCD, implying that the gluons are massless. This seems to be in contradiction of the observation, noted in the last chapter, that the strong force seems to have a range of $\sim 10^{-15}$m. The explanation of this can be thought of in terms of van der Waals forces in molecular physics. As we shall see later, the strength of the colour force increases with increasing separation between coloured objects and so, despite the theoretically infinite range of gluons, the force between the quarks and gluons in a proton, for example, restricts them to within the proton itself. The force between the neutrons and protons in a nucleus is due to the exchange of Yukawa's pions, which are fields produced by the non-uniform distribution of colour charge within the colour-neutral protons and neutrons.

2.2 Path Integrals and Perturbation Theory

The Lagrangians which we have proposed for QED and QCD in equations (2.13) and (2.22) were derived by extending classical ideas. They therefore describe classical field theories. To describe the behaviour of particles and fields in the real world we must quantize the theory. To do this we use the path integral formalism, derived originally by Dirac [7] and Feynman [8] from quantum mechanics. The

derivation and procedures involved are both lengthy and complicated; hence we shall give a concise review of the essential ideas, using the simplest case of a scalar field theory, and conclude the section with a few words about how these concepts can be applied to more realistic field theories. For greater detail the reader is referred to any one of a number of books dedicated to the subject, some of which can be found in references [9, 10, 11].

Let us first consider our objectives; we are aiming to describe the nature of the interactions between particles. The experimentally measured quantity that yields the most information about the dynamics of particle interactions is the cross-section for a process in which two particles scatter off each other. From quantum mechanics, this is related to a probability amplitude, the scattering amplitude, which in turn is closely related to the Green's functions of the theory. We shall see that, using the path integral formalism, we can arrive at a method for relating the Green's functions, albeit indirectly, to the Lagrangian of the theory. The intractable nature of this latter relation will ultimately motivate the use of perturbation theory as a calculational tool.

Let us begin by considering a Lagrangian which is no more than quadratic in the time derivatives of the field, $\mathcal{L}=\mathcal{L}(\phi, \partial_\mu \phi)$. Using the path integral formalism we can write down a vacuum to vacuum transition amplitude in the presence of a source $J(x)$:

$$Z[J] = N \int D\phi \exp\left[i \left(S(\phi) + \int d^4x J(x)\phi(x) \right) \right] , \tag{2.24}$$

where the normalization N is chosen such that $Z[0]=1$. That is, $Z[J]$ is the probability amplitude for the transition from the vacuum at $t = -\infty$ to the vacuum at $t = +\infty$. Here $D\phi$ denotes a path integral over all possible functions $\phi(x)$. From the definition of the classical action,

$$S(\phi) = \int d^4x \, \mathcal{L} , \tag{2.25}$$

equation (2.24) can be rewritten as

$$Z[J] = N \int D\phi \exp\left[i \int d^4x (\mathcal{L}(\phi, \partial_\mu \phi) + J\phi) \right] . \tag{2.26}$$

The most immediate problem faced when attempting to evaluate this integral is that the argument of the exponential is imaginary. Thus the integral is oscillatory

and convergence is not guaranteed. To overcome this problem we use the technique of Wick rotation, which corresponds to transforming from Minkowski spacetime to Euclidean spacetime, $t \rightarrow -i\tau$, performing the integral (which we now hope is convergent) and then transforming back to Minkowski spacetime. This process is justified providing that there are no poles above the positive real axis or below the negative real axis. As we shall see below, the use of a Feynman prescription for the treatment of poles helps to ensure that this is indeed the case. We shall subsequently assume that a Wick rotation has been implemented without stating so explicitly.

Let us consider the simple case of a free massive scalar field, defined by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) + \frac{1}{2} m \phi_0^2 \,, \tag{2.27}$$

the subscript '0' denoting a free theory. Substituting this Lagrangian into the classical Euler-Lagrange equations, (2.2), gives the classical equation of motion for a free neutral massive scalar field:

$$(\partial_\mu \partial^\mu + m)\phi_0 = 0 \,, \tag{2.28}$$

which is known as the Klein-Gordon equation. Using this Lagrangian, we can solve for $Z[J]$ in equation (2.26) exactly:

$$Z_0[J] = \exp\left[-\frac{i}{2}\int d^4x' \int d^4x \, J(x')\Delta_F(x'-x)J(x)\right] \,, \tag{2.29}$$

where $\Delta_F(x'-x)$ is the Feynman propagator and is defined by

$$\Delta_F(x'-x) = \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{-ip.(x'-x)} \frac{1}{p^2 - m^2 + i\epsilon} \,. \tag{2.30}$$

The prescription $\epsilon \rightarrow 0^+$ displaces the poles at $p = \pm m$ away from the real axis. This exactly calculable result will be of use below.

How can we, in general, relate $Z[J]$ to the Green's functions of the theory? What follows is a non-rigorous outline of how this connection is made. Let us start by taking the functional derivative of $Z[J]$ with respect to the source, $J(x)$:

$$\frac{\delta Z[J]}{\delta J(x)} = iN \int D\phi \, \phi(x) \exp\left[i \int d^4x (\mathcal{L} + J\phi)\right] \,, \tag{2.31}$$

where N is a normalization factor. Performing this step n times gives

$$\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)} = i^n N \int D\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \exp\left[i\int d^4x(\mathcal{L}+J\phi)\right] . \tag{2.32}$$

Now, the form of $Z[J]$ was obtained originally by taking the continuum limit of a set of paths between some initial spacetime point, x_i, and a final one, x_f, with the interval divided into m intermediate points. That is,

$$\begin{aligned}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)} \propto \lim_{m\to\infty}\lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} i^n \int d\phi_1 d\phi_2\cdots d\phi_m\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \\ \langle\phi_f x_f|\phi_m x_m\rangle\cdots\langle\phi_2 x_2|\phi_1 x_1\rangle\langle\phi_1 x_1|\phi_i x_i\rangle .\end{aligned} \tag{2.33}$$

We now interpret $\phi(x_1), \phi(x_2), \ldots$ as eigenvalues of some quantum mechanical operators, $\hat{\phi}(x_1), \hat{\phi}(x_2), \ldots$, and write

$$\begin{aligned}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)} \propto \lim_{m\to\infty}\lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} i^n \int d\phi_1 d\phi_2\cdots d\phi_m\, \langle\phi_f x_f|\phi_m x_m\rangle\cdots \\ \langle\phi_n x_n|\hat{\phi}(x_n)|\phi_{n-1}x_{n-1}\rangle\cdots\langle\phi_2 x_2|\hat{\phi}(x_2)|\phi_1 x_1\rangle \\ \langle\phi_1 x_1|\hat{\phi}(x_1)|\phi_i x_i\rangle .\end{aligned} \tag{2.34}$$

The right hand side of equation (2.34) is now just the path integral interpretation of the expectation value of the product of the operators $\hat{\phi}(x_1), \hat{\phi}(x_2), \ldots$ between the initial and final configurations of the system. That is,

$$\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)} \propto \lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} i^n \langle\phi_f x_f|\hat{\phi}(x_n)\cdots\hat{\phi}(x_2)\hat{\phi}(x_1)|\phi_i x_i\rangle , \tag{2.35}$$

where we require for the purposes of causality that $t_f > t_n > \cdots > t_2 > t_1 > t_i$. In other words, applying the limits gives

$$\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)} \propto i^n \langle 0, t=+\infty|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)]|0, t=-\infty\rangle , \tag{2.36}$$

where T is the time ordering operator, which has the effect of placing earlier times to the right. These vacuum expectations of the time ordered products precisely define the n-particle Green's functions for which we were searching:

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)]|0\rangle . \tag{2.37}$$

Thus $Z[J]$ generates all the n-particle Green's functions of the theory via the expression

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \, G^{(n)}(x_1, \ldots, x_n) \, J(x_1) \cdots J(x_n) \, . \tag{2.38}$$

Let us now turn to the (more interesting) case where the Lagrangian contains an interaction term. That is,

$$\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_{\text{int}}(\phi) \, . \tag{2.39}$$

We now write

$$\begin{aligned} Z[J] &= \int D\phi \, \exp\left[i \int d^4x (\mathcal{L}_0 + g\mathcal{L}_{\text{int}}(\phi) + J\phi) \right] \\ &= \int D\phi \, \exp\left[ig \int d^4x \mathcal{L}_{\text{int}}(\phi) \right] \exp\left[i \int d^4y (\mathcal{L}_0 + J\phi) \right] \, . \end{aligned} \tag{2.40}$$

Using the result of equation (2.31), this can be rewritten as

$$Z[J] = \int D\phi \, \exp\left[ig \int d^4x \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(x)} \right) \right] \exp\left[i \int d^4y (\mathcal{L}_0 + J\phi) \right] \, . \tag{2.41}$$

Now, since $\mathcal{L}_{\text{int}}$ is no longer a function of ϕ, the interaction term may be taken outside the functional integral, leaving us with

$$Z[J] = \exp\left[ig \int d^4x \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] \, . \tag{2.42}$$

This is still a rigorous expression and, since $Z_0[J]$ is known exactly, we have eliminated all functional integrals. However, this expression is uncalculable and it is due to this that we motivate a perturbative expansion in g. So, making the step from a non-perturbative representation of the theory to a perturbative one, we write

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n g^n}{n!} \int d^4x_1 \cdots d^4x_n \, \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(x_1)} \right) \cdots \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(x_n)} \right) Z_0[J] \, . \tag{2.43}$$

From this expression it is possible, using equation (2.36), to make calculations of the Green's functions at any order in the perturbative expansion. For example, we could do this in the case of ϕ^4-theory where $\mathcal{L}_{\text{int}}=\phi^4$ and $\mathcal{L}_0$ and $Z_0[J]$ are given by equations (2.27) and (2.29) respectively. However, this is an extremely

laborious process on which we shall not elaborate beyond quoting as an example the two-point Green's function:

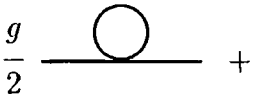
$$G^{(2)}(x_1,x_2) = i\Delta_F(x_1-x_2)\left(1-\frac{ig}{8}\int d^4x(i\Delta_F(x-x))^2\right) - \frac{ig}{2}\int d^4x\Delta_F(x_1-x)\Delta_F(x-x)\Delta_F(x-x)\Delta_F(x-x_2)+O(g^2)\,. \tag{2.44}$$

In the free theory g=0 and the expression above reduces to its first term, the free-field Green's function, $G_0^{(2)}(x_1,x_2)$, which describes the propagation of a scalar particle from x_1 to x_2. The terms generated by the interaction when $g \neq 0$ contain propagators with zero argument, $\Delta_F(x-x)$, and, since x is an arbitrary point of which the Green's functions are independent, we must integrate over it. From the definition of $\Delta_F(x-y)$ in equation (2.30) we can see that

$$\Delta_F(0) \sim \int \frac{d^4p}{p^2} \tag{2.45}$$

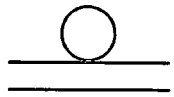
and so these integrals diverge at large values of momentum. It is the removal of such divergences that is known as renormalization and is discussed in the next section.

In a step which revolutionised calculations in scattering theory, Feynman [12] devised a means of representing the Green's functions of the theory diagrammatically and associating with each topologically different part of the diagram an algebraic expression. Thus a simple diagram can be used to encode a large amount of algebraic information. In our example of the two-point function in ϕ^4-theory the propagation of a particle from x_1 to x_2 is represented by a line and the propagator $\Delta_F(x-x)$ is represented by a loop at x. Thus we can rewrite equation (2.44) as

$G^{(2)}(x_1,x_2) =$ i  $-\frac{g}{2}$  $+O(g^2)\,.$ (2.46)

The contributions from the different elements of these Feynman diagrams are known as Feynman rules.

Now consider the four-point function; one contribution to $G^{(4)}(x_1,\ldots,x_4)$ at order g is the following:



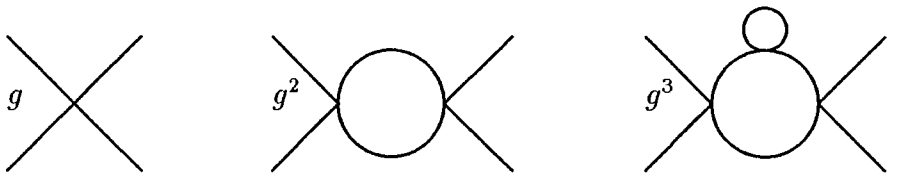
In this diagram the two particles do not interact with each other and the diagram is therefore of little interest when studying two particle scattering. This concept extends to higher order Green's functions where there will be analogous diagrams in which a subset of the initial state particles scatter into a subset of the final state particles, wholly independently of the rest of the diagram. In order to eliminate such diagrams we define a connected Green's function in which all external lines are connected to all other external lines. These connected functions can be generated via a new functional $X[J]$, where

$$X[J] = \frac{1}{i} \ln Z[J] \,. \tag{2.47}$$

The connected Green's functions, $G_c^{(n)}(x_1, \ldots, x_n)$, are then generated by an expression analogous to equation (2.38),

$$X[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \, G_c^{(n)}(x_1, \ldots, x_n) \, J(x_1) \cdots J(x_n) \,. \tag{2.48}$$

By restricting the Green's functions further we obtain the "one particle irreducible" (OPI) Green's functions of the theory, which describe diagrams that cannot be split into two by the cutting of any one particle line. For example, the OPI four-point function has up to $O(g^3)$ the following contributions:



A further property of the OPI Green's functions is that the external legs carry no propagator factors. The OPI Green's functions are generated by what is known as the classical field, ϕ_c, where

$$\phi_c = \frac{\delta X[J]}{\delta J(x)} \,. \tag{2.49}$$

We now define an effective action in terms of the classical field:

$$\Gamma[\phi_c] = Z[J] - \int d^4x\, J(x)\phi_c\,. \tag{2.50}$$

Again this is, in general, uncalculable for an interacting theory and so a perturbative expansion is made. That is, we write

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, \Gamma^{(n)}(x_1, \ldots, x_n)\phi_c(x_1)\cdots\phi_c(x_n)\,, \tag{2.51}$$

where the $\Gamma^{(n)}(x_1, \ldots, x_n)$'s are now the n-point OPI Green's functions. It is then these quantities that are most closely related to the scattering amplitudes that we wish to calculate. We will subsequently consider the OPI Green's functions converted to momentum space by taking their Fourier transform and drop the "OPI" prefix.

We have so far only been considering the simple case of ϕ^4 theory. Ideally we would like to quantize the Lagrangian of the Standard Model,

$$\begin{aligned}
\mathcal{L}_{\text{SM}} \;=\; & \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R - g(\overline{q}\gamma^\mu T_a q)G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \\
& -\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu}_a - \frac{1}{4}B^a_{\mu\nu}B^{\mu\nu}_a \\
& -\overline{\psi}_L\gamma^\mu\left(g'\frac{1}{2}\tau_a W^a_\mu + g''\frac{Y}{2}B_\mu\right)\psi_L - \overline{\psi}_R\gamma^\mu g''\frac{Y}{2}B_\mu\psi_R \\
& +\left|\left(i\partial_\mu - g'\frac{1}{2}\tau_a W^a_\mu - g''\frac{Y}{2}B_\mu\right)\phi\right|^2 - V(\phi) \\
& -(G_1\overline{\psi}_L\phi\psi_R + G_2\overline{\psi}_L\phi_c\psi_R + \text{hermitian conjugate})\,,
\end{aligned} \tag{2.52}$$

where

$$W^a_{\mu\nu} \;=\; \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g' f'_{abc} W^b_\mu W^c_\nu\,, \tag{2.53}$$

$$B_{\mu\nu} \;=\; \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.54}$$

and $G^a_{\mu\nu}$ is given by equation (2.23). This Lagrangian contains all the particles in Figure 1.2 except the graviton. As we have stated above, we are primarily interested in QCD and, since only the quarks and gluons have colour quantum numbers, we can safely ignore most of equation (2.52) and concentrate solely on $\mathcal{L}_{\text{QCD}}$, given in equation (2.22).

The quantization of theories containing fermions and gauge bosons is inevitably more complex and we shall say only that, once the somewhat counter-intuitive

laws of Grassman variables have been grasped, the difficulties are mainly technical and certainly do not warrant further detail here. Again the reader is referred to books on the subject [9, 10, 11]. The main physical principles of the path integral formalism have been laid out and we now progress by discussing the problem, mentioned earlier, of divergences.

2.3 Divergences and Renormalization

Let us return to our example of ϕ^4 theory. We saw above that, as soon as we attempt to calculate any term in the perturbation series for the interacting theory, we encounter divergences, exemplified by the so-called "tadpole diagram" contribution to the two-point Green's function. We must look for some way to treat these divergences in order to arrive at a finite result and hence for the theory to have any predictive power.

The method for tackling this problem comes in three stages. First we regularize the theory; that is, we isolate the divergent loop integrals and impose on them some prescription (for example, a high-momentum cutoff or an arbitrary spacetime dimension) such that they are finite. The idea is that at the end of the calculation the divergences will have been absorbed and we will be able to take the physical limit (cutoff $\rightarrow \infty$ or $d \rightarrow 4$ in our examples). There should ultimately be no dependence on the method of regularization.

We will then be in a position to renormalize the theory. There are several equivalent ways of doing this, two of which are multiplicative renormalization and the method of counterterms. The first method involves summing the infinite series of loop diagrams for some fixed number of external lines. This divergent sum is then absorbed into a redefinition of the coupling constant and the mass in the "bare" Lagrangian, under the assumption that the bare coupling and mass are unmeasurable quantities. In the second method, counterterms are added directly to the Lagrangian in such a way as to knock out precisely the divergent diagrams. A renormalizable theory will require only a finite number of counterterms to render it finite to any order. Moreover, the counterterms are proportional to terms in the original Lagrangian, so adding the two just gives the multiplicative redefinitions of the coupling and the mass as in the first method.

There are two simple criteria that these methods imply are necessary for renormalizability:

- The degree of divergence, $\mathcal{D}$, of any diagram must be a function only of the number of external lines. That is, $\mathcal{D}$ should not be increased by the addition of more loops. We may then collect all n-point loop diagrams into one term.

- The number of classes of divergent n-point diagrams must be finite. These divergences must cancel the divergences in the bare parameters of the original Lagrangian.

The final part of the renormalization process is to use an inductive argument to extend the renormalizability at nth order to all orders in perturbation theory. If one can prove renormalizability at $(n+1)$th order too then recurrence relations such as the Schwinger-Dyson equations imply that the whole theory is finite order by order. All such proofs hinge ultimately on Weinberg's theorem [13]: "A Feynman diagram is convergent if the degree of divergence of it and all its subdiagrams is negative."

By simple power counting arguments, it turns out that in ϕ^4 theory the degree of divergence of a diagram is given by

$$\mathcal{D} = 4 - \mathcal{E} \,, \tag{2.55}$$

with $\mathcal{E}$ the number of external lines. Hence only the two- and four-point diagrams are divergent. We shall start by considering the two-point diagrams derived from the bare Lagrangian

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu \phi_B)(\partial^\mu \phi_B) + \frac{1}{2} m_B \phi_B^2 - g_B \phi_B^4 \,. \tag{2.56}$$

We now define $\Sigma(p^2)$ to be the sum of all OPI two-point diagrams. For example, the one-loop contribution is the tadpole diagram,

$$-i\Sigma(p^2) = -i\frac{g_B}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m_B^2 + i\epsilon} + \cdots \,. \tag{2.57}$$

Now let us consider the full propagator, $\Delta'(p)$. That is, $\Delta'(p)$ is the sum over all two-point diagrams (in the free theory $\Delta'(p) \equiv \Delta_F(p) = (p^2 - m_B^2 + i\epsilon)^{-1}$). $\Delta'(p)$

can be represented as a sum over an infinite chain of OPI diagrams:

$$\begin{aligned} i\Delta'(p) &= i\Delta_F(p) + i\Delta_F(p)[-i\Sigma(p^2)]i\Delta_F(p) + \cdots \\ &= i\Delta_F(p)\left(\frac{1}{1-\Sigma(p^2)\Delta_F(p)}\right) \\ &= \frac{i}{p^2 - m_B^2 - \Sigma(p^2) + i\epsilon} \,. \end{aligned} \tag{2.58}$$

Expanding $\Sigma(p^2)$ around $p^2{=}m^2$, where m is finite but arbitrary, gives

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \tilde{\Sigma}(p^2) \,, \tag{2.59}$$

where $\tilde{\Sigma}(p^2) \sim O((p^2 - m^2)^2)$ and $\Sigma(m^2)$ and $\Sigma'(m^2)$ are divergent. So now

$$i\Delta'(p) = \frac{i}{(1-\Sigma'(m^2))(p^2 - m^2) - \tilde{\Sigma}(p^2) + i\epsilon} \,. \tag{2.60}$$

Recall that m_B is infinite but arbitrary. Since $\Sigma(m^2)$ is also divergent, we define m_B and m such that m_B cancels against the divergent part of $\Sigma(m^2)$, giving a finite m:

$$m_B^2 + \Sigma(m^2) = m^2 \,. \tag{2.61}$$

Now our expression for the full propagator becomes

$$i\Delta'(p) = \frac{iZ_\phi}{p^2 - m^2 - \tilde{\Sigma}'(p^2) + i\epsilon} \,, \tag{2.62}$$

with

$$\begin{aligned} m^2 &= m_B^2 + \Sigma(m^2) = m_B^2 + \delta m^2 \,, \\ Z_\phi &= \frac{1}{1-\Sigma'(m^2)} \,, \\ \tilde{\Sigma}'(p^2) &= \frac{\tilde{\Sigma}(p^2)}{1-\Sigma'(m^2)} = Z_\phi \tilde{\Sigma}(p^2) \,. \end{aligned} \tag{2.63}$$

We see then that a new renormalized propagator, $\Delta'_R(p)$, has been defined:

$$\Delta'(p) = Z_\phi \Delta'_R(p) \,. \tag{2.64}$$

The effect of introducing interactions has been to "dress" the infinite bare mass such that it is shifted to a new finite mass, m.

This renormalization procedure could equally well have been expressed in terms of the Green's functions, $\Gamma^{(n)}$. In the free theory

$$i\Gamma_B^{(2)}(p) = p^2 - m_B^2 \,. \tag{2.65}$$

That is, the bare two-point function is just the inverse of the bare propagator. Introducing interactions and summing to all orders in perturbation theory, the two-point function becomes infinite. Dividing out the infinite factor Z_ϕ and choosing p=0 gives the renormalized two-point function:

$$i\Gamma^{(2)}(0) = -m^2 \; . \tag{2.66}$$

What is the effect of renormalization on the bare coupling? To understand this we consider the four-point function summed over all possible diagrams:

$$i\Gamma_B^{(4)}(p_i) = g_B + O(g_B^2) \;\; , \;\; i = 1, \ldots, 4 \; . \tag{2.67}$$

We are not able to evaluate this to all orders in perturbation theory but we do know that it must be Lorentz invariant. It can therefore be written in terms of the Mandelstam variables,

$$s = (p_1 + p_2)^2 \; ; \; t = (p_1 - p_3)^2 \; ; \; u = (p_1 - p_4)^2 \; . \tag{2.68}$$

So

$$i\Gamma_B^{(4)}(p_i) = g_B + f(s) + f(t) + f(u) \tag{2.69}$$

with f some divergent function. For p_i=0 let Z_1^{-1} be the overall infinite factor contained in f. Dividing this factor out gives

$$i\Gamma^{(4)}(0) = g \; , \tag{2.70}$$

where g is the physical renormalized coupling constant. In fact we take equations (2.66) and (2.70) to be the *definitions* of the mass and the coupling constant, measured at p=0. Now, the choice of p=0 is arbitrary: we could have taken the definition to be at p=μ. So, in general, the mass and coupling constant are functions of μ, which is known as the renormalization point or subtraction point. Then

$$\begin{aligned} i\Gamma^{(2)}(\mu) &= p^2 - m^2(\mu) \; , \\ i\Gamma^{(4)}(\mu) &= g(\mu) \; . \end{aligned} \tag{2.71}$$

This dependence on the renormalization point will be discussed further below.

Let us summarise the results of our renormalization procedure:

$$\begin{aligned} \phi &= Z_\phi^{-\frac{1}{2}} \phi_B \ , \\ g &= Z_1^{-1} Z_\phi^2 g_B \ , \\ m^2 &= m_0^2 + \delta m^2 \ . \end{aligned} \tag{2.72}$$

To reach this set of redefinitions the renormalization constants have been reshuffled so as to absorb any extra divergent factors into a renormalization of the (unmeasurable) wavefunction.

As was stated above, this multiplicative method is equivalent to the method of counterterms in which the Lagrangian (for the massless theory) is written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\phi^4 + \frac{1}{2}(Z_\phi - 1)(\partial_\mu \phi)(\partial^\mu \phi) - (1 - Z_1) g\phi^4 \ , \tag{2.73}$$

where the third and fourth terms are counterterms introduced to kill the divergent diagrams. This method is widely used and very powerful as it removes divergences order by order in perturbation theory. The μ-dependence of the renormalized quantities found above corresponds to the fact that the nature of the counterterms is unchanged under the addition to them of arbitrary finite quantities.

2.4 The Renormalization Group

Let us now consider the implications of the μ-dependence of our renormalized Green's functions. The rearrangement of renormalization constants and the fact that the bare OPI Green's functions have no propagators on the external legs means that there is a "deficit" of $Z_\phi^{\frac{1}{2}}$ for each external leg. So, for an n-point Green's function,

$$\Gamma^{(n)}(p_i, g, \mu) = Z_\phi^{\frac{n}{2}} \Gamma_B^{(n)}(p_i, g_B) \ . \tag{2.74}$$

Note that we are considering, and shall from now on consider, the massless theory. (We shall eventually study QCD with massless quarks.) The implication of equation (2.74) is that Z_ϕ is a function of g and μ. The bare Green's function is independent of the renormalization point and so we can write

$$\mu \frac{d}{d\mu} \Gamma_B^{(n)} = 0 \tag{2.75}$$

and therefore

$$\mu\frac{d}{d\mu}\left(Z_\phi^{-\frac{n}{2}}\Gamma^{(n)}(p_i,g,\mu)\right)=0\ . \tag{2.76}$$

Using the chain rule on the left hand side gives us

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)\right]\Gamma^{(n)}(p_i,g,\mu)=0\ , \tag{2.77}$$

where

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}\ , \tag{2.78}$$

$$\gamma(g) = \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi}\ . \tag{2.79}$$

Equation (2.77) is known as the renormalization group equation (RGE).

We can derive a further constraint on the renormalized Green's functions. Let us scale the momenta such that

$$p_i\rightarrow p_i\,\mathrm{e}^t\ . \tag{2.80}$$

Then, using dimensional arguments and the fact that $\Gamma^{(n)}$ is Lorentz invariant, we have

$$\Gamma^{(n)}(p_i\,\mathrm{e}^t,g,\mu)=\mu^D f\left(\frac{p_i\cdot p_j}{\mu^2}\mathrm{e}^{2t},g\right)\ , \tag{2.81}$$

where D is the mass dimension of $\Gamma^{(n)}$. This, in turn, implies that $\Gamma^{(n)}$ satisfies the differential equation

$$\left[\mu\frac{\partial}{\partial\mu}+\frac{\partial}{\partial t}-D\right]\Gamma^{(n)}(p_i\,\mathrm{e}^t,g,\mu)=0 \tag{2.82}$$

which, using the RGE (equation (2.77)), gives

$$\left[-\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}+D-n\gamma(g)\right]\Gamma^{(n)}(p_i\,\mathrm{e}^t,g,\mu)=0\ , \tag{2.83}$$

the inhomogeneous Callan-Symanzik equation. This has the solution

$$\Gamma^{(n)}(p_i\,\mathrm{e}^t,g,\mu)=\Gamma^{(n)}(p_i,\overline{g}(t),\mu)\exp\left[tD-n\int_0^t dt'\gamma(\overline{g}(t'))\right]\ , \tag{2.84}$$

where $\exp[tD]$ is known as the canonical, or engineering, dimension. The extra term in the exponent is the anomalous dimension.

Note that we have introduced a running coupling constant,

$$\frac{d\overline{g}(t)}{dt} = \beta(\overline{g}) \,, \tag{2.85}$$

which enables the rewriting of the anomalous dimension as

$$\exp\left[-n\int_g^{\overline{g}} dg' \frac{\gamma(g')}{\beta(g')}\right] \,.$$

Knowing β and γ for the theory enables the evaluation of the momentum dependence of all the Green's functions. These functions are, at least theoretically, calculable in perturbation theory. Equation (2.85) shows that the way in which $\overline{g}(t)$ runs with momentum is entirely governed by the beta-function of the theory; and so by studying the beta-function we can discover in which momentum region(s) the coupling becomes large. In these regions, since we are making a series expansion in the coupling constant, perturbation theory breaks down.

In ϕ^4-theory and in QED the beta-function is positive and therefore the coupling increases with increasing energy. The energy scale at which perturbation theory breaks down in QED is $\sim 10^{277}$GeV, explaining perhaps why perturbation theory has been so spectacularly successful at describing QED effects at current 'low' energies. In QCD, however, the beta function is given by

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(11 - \frac{2N_f}{3}\right) + O(g^5) \,, \tag{2.86}$$

where N_f is the number of fermion flavours. So, for $N_f < 17$, the beta-function is negative and the coupling decreases with increasing energy. This feature of QCD, and non-Abelian gauge theories in general, is known as asymptotic freedom. Conversely, at low energies ($\lesssim$100–200 MeV), the coupling becomes large and perturbation theory breaks down.

2.5 Labelling Renormalization Schemes

We have seen the importance of the beta-function in determining the behaviour of the coupling constant. Through its dependence on the renormalization point it also has a role to play in the labelling of renormalization schemes. Let us start by defining a "renormalization group improved coupling" a:

$$a(\mu) = \frac{\alpha_s(\mu)}{\pi} = \frac{\overline{g}^2(\mu)}{4\pi^2} \tag{2.87}$$

and redefining our beta-function accordingly,

$$\mu \frac{da(\mu)}{d\mu} \equiv \frac{da(\mu)}{d\ln\mu} = \beta(a(\mu)) \,. \tag{2.88}$$

As we shall see later, a is a convenient parameter with which to perform perturbative expansions. Then, if we integrate up equation (2.88), we get a transcendental equation for $a(\mu)$. We now turn the μ-dependence of the beta-function to our advantage and find that choosing a form for the beta-function and selecting a renormalization point enables us to specify a unique renormalization scheme (RS).

For many perturbative applications the beta-function is truncated at a fixed order. For example, if we take the beta-function truncated to its one-loop form,

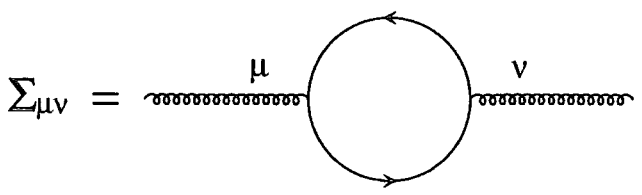
$$\beta(a) = -ba^2 \,, \tag{2.89}$$

then integrating equation (2.88) gives

$$a(\mu) = \frac{1}{b\ln(\mu/\Lambda)} \,, \tag{2.90}$$

where the integration constant Λ is a fundamental constant of the theory to be determined from experiment. It is the scale which characterizes the limit of validity of perturbation theory.

We shall discuss the choice of RS in more detail in later chapters. First though, let us introduce some of the more commonly used RS's. The first of these is the minimal subtraction (MS) scheme. It is based on the renormalization of the gluon self-energy diagram:



Applying the Feynman rules to this diagram and regularising the loop integrals using dimensional regularisation, where the number of spacetime dimensions is given by $d = 4 - 2\epsilon$, we obtain an expression which has a $1/\epsilon$ pole. In the MS scheme we simply remove this pole. Note that this does not define a unique scheme; there is still a choice of renormalization point to be made.

More widely used is the modified minimal subtraction scheme ($\overline{\text{MS}}$). In MS the $1/\epsilon$ pole always appears in conjunction with a group of constants, $\ln 4\pi - \gamma_E$, where γ_E is the Euler-Mascheroni constant. In $\overline{\text{MS}}$ these factors are removed along with the pole. The conciseness of this scheme has led to its almost universal popularity amongst phenomenologists but it should be stressed that there is no compelling physical argument to suggest that it should be preferred over any other scheme.

We shall in later chapters refer to the momentum space subtraction (MOM) scheme. This is based on the renormalization of a vertex (for example, the triple-gluon or quark-quark-gluon vertex) rather than of a self-energy. Its relation to the $\overline{\text{MS}}$ scheme will be discussed later.

The final point to be made in this section is that the schemes above do not exploit the ability to specify a scheme by choosing a beta-function. Rather they are defined purely in terms of a calculational procedure and, for example, the coefficients of the $\overline{\text{MS}}$ scheme beta-function are only known to a few orders in perturbation theory. (The first two coefficients are RS-invariant, as can be easily shown by comparing the beta-functions of two different arbitrary schemes.)

2.6 Summary

The aim of this chapter has been to outline some of the fundamental principles which underlie the application of field theory to the understanding of particle interactions. We saw that to describe physical processes requires that the Lagrangian of the field theory be invariant under local phase transformations; and how this leads to the introduction of the covariant derivative and gauge fields and, in the case of QCD, to self-interactions of the gauge fields.

The theory was then quantized using the path integral formalism and it was shown how this technique enables the calculation of the Green's functions, which are closely related to the scattering amplitudes that can be investigated experimentally. The fact that, in the case of a field theory which involved interactions, it was impossible to evaluate the Green's functions exactly finally motivated the use of a perturbative series in the coupling constant.

It became apparent, through studying the loop diagrams which enter the per-

turbative expansions for the Green's functions, that there was a problem with divergences at large momenta. This demanded that we should renormalize the theory in order for it to make any sense. The renormalization procedure was sketched for the simple case of ϕ^4-theory and it was shown that this involved the introduction of a dependence of the renormalized Green's functions on the choice of renormalization point.

Studying the nature of this dependence resulted in the RGE. Examining the scale dependence of the renormalized Green's functions produced the inhomogeneous Callan-Symanzik equation and led to the introduction of a running coupling constant. The behaviour of this running coupling was defined by the beta-function. Finally we saw the importance of the beta-function in providing information about the range of validity of perturbation theory and in choosing a renormalization scheme.

Chapter 3

Jet Rates in e^+e^- Annihilation

3.1 Introduction

In this chapter we study in some detail the application of perturbation theory to e^+e^- annihilation into hadronic jets, one of the most informative processes we have for studying QCD and in fact the process which provided the first conclusive evidence for QCD.

In this process an electron and a positron are collided at high energies. They annihilate to produce an energetic photon which itself decays into a back-to-back quark and antiquark, one or other of which may emit a hard gluon. Figures 3.1(a) and (b) show two tree-level Feynman diagrams for this process. The quark, antiquark and gluon (the "parent partons") are subject to the colour force of QCD which, as we have seen, increases with increasing separation; so, via a fragmen-

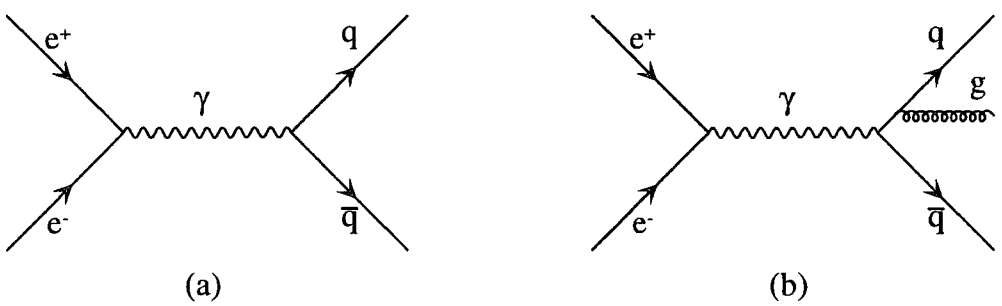


Figure 3.1: Feynman diagrams for (a) a two-jet event and (b) a three-jet event.

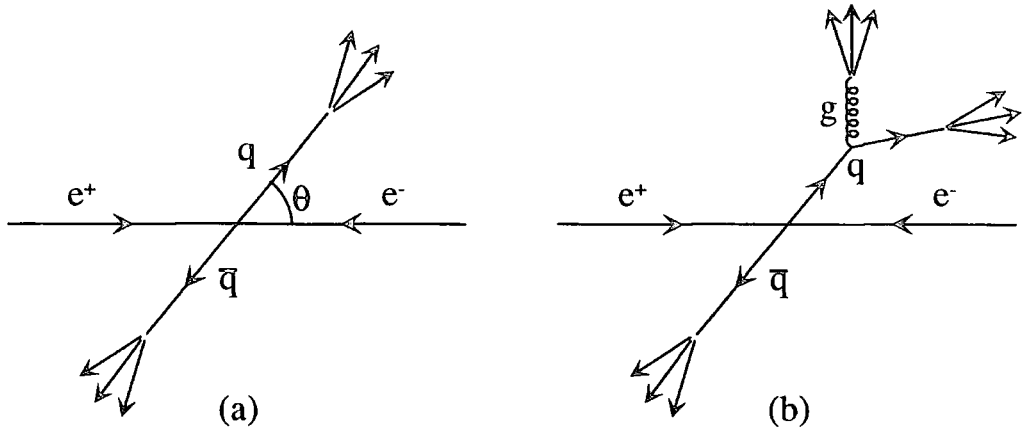


Figure 3.2: Centre of mass picture for (a) a two-jet event and (b) a three-jet event.

tation process, the parent partons produce jets of colourless hadrons. It is these hadrons which are observed in the final state. Theoretically any number of jets is possible but in practice higher jet multiplicities are suppressed as powers of the strong coupling constant; and we shall be interested primarily in two- and three-jet events. These are shown schematically in the e^+e^- centre of mass frame in Figures 3.2(a) and (b).

The process $e^+e^- \rightarrow$ hadrons is of vital importance in the study and testing of QCD. The angular distribution of two-jet events (differential cross-section as a function of θ in Figure 3.2(a)), first observed by the SPEAR collaboration at SLAC (at $\sqrt{s} = 7.4\text{GeV}$) in 1975 [14], conformed to the predictions made assuming spin-$\frac{1}{2}$ quarks (that is, $d\sigma/d\Omega \sim (1 + \cos^2\theta)$). In the three-jet case either the quark or the antiquark emits a hard gluon. At that vertex there enters a factor of the strong coupling constant, so we expect that

$$\frac{\sigma_{3-\text{jet}}}{\sigma_{2-\text{jet}}} \sim \alpha_s \, . \tag{3.1}$$

Measuring the two- and three-jet rates thus gives us a precise means by which to evaluate α_s (and hence Λ_{QCD}). The purpose of this chapter is to show how various perturbative techniques are applied to this problem.

3.2 Defining Jets

As mentioned in the last section, the final state particles in e^+e^- annihilation are hadrons, since the parent partons hadronize with probability one. There is an immediate problem: the parent partons will emit large numbers of soft gluons and, due to the non-Abelian nature of QCD, we may expect many $g \to gg$ interactions. These lead to the hadronic jets being roughly conical and this creates a serious problem in the tracing of final state hadrons back to original partons. (It is unusual to see such a "clean" event as the one portrayed in Figure 3.2(b).)

The best solution to this problem available so far is the introduction of a dimensionless jet resolution parameter, y, which is based on some measurable quantity: invariant mass, for example. We then use a jet algorithm (see later for details) with some recombination scheme to group together final state hadrons into jets. Such an algorithm should be finite in the limit of infrared (IR) or collinear gluon emission off the parent partons.

In calculating the total cross-section for $e^+e^- \to$ hadrons, divergences from real and virtual gluons cancel each other leaving a finite result [15]. Since the divergences for real gluons occur when the emitted gluons are either soft or collinear, this is precisely the region where three-jet and two-jet events become indistinguishable. By introducing the jet resolution parameter and choosing a minimum value, y_{cut}, for jets to be considered resolved, we overcome this problem; essentially we sidestep the issue by introducing an infrared cutoff. Hence, jet algorithms are by definition IR and collinear safe.

The other requirement of a jet algorithm is that it should be subject to small hadronization corrections. There are some successful models of the process by which the parent partons fragment into hadrons, such as preconfinement [16] and local parton-hadron duality (LPHD) [17]. These models are successful phenomenologically but we have no deep theoretical understanding of the fragmentation process. Preconfinement and LPHD both suggest that hadron jets maintain fairly closely the kinematic features of the underlying partons. One encouraging feature is that, provided the hadronization process is local in phase space (that is, it involves limited momentum transfers), then hadronization corrections should decrease like inverse powers of the hard scattering scale, Q, which in e^+e^-

annihilation is just the centre of mass energy.

The first jet algorithm to be applied with phenomenological success was the JADE algorithm [18]. Let us consider an n-hadron final state:

$$e^+(p) + e^-(\overline{p}) \rightarrow h_1(p_1) + \cdots + h_n(p_n) \ , \ p + \overline{p} = Q \ . \tag{3.2}$$

The JADE algorithm is then defined according to the following iterative procedure:

1. Define a resolution parameter, $y_{\rm cut}$.

2. For every pair of hadrons h_k, h_l evaluate the corresponding invariant mass, M_{kl}^2, and define
$$y_{kl} = \frac{M_{kl}^2}{Q^2} \ . \tag{3.3}$$

3. If y_{ij} is the smallest value of y_{kl} computed in step 2 and $y_{ij} < y_{\rm cut}$ then combine the momenta p_i and p_j into a single "pseudoparticle", p_{ij}, according to some recombination prescription.

4. Repeat this procedure from step 2 until all pairs of objects (particles and/or pseudoparticles) have $y_{kl} > y_{\rm cut}$. Whatever objects now remain are called jets.

There are various choices for the recombination prescription, some of which are listed in Table 3.1 along with the definition of M_{kl}^2. The E scheme would appear the most natural but it does in fact lead to massive pseudoparticles. Since most theoretical work is concerned with massless partons, it is generally accepted that one should work with massless particles throughout [19]. The various ways of achieving this are reflected in the diversity of recombination schemes in Table 3.1.

The JADE algorithm satisfies the conditions that it should be IR and collinear safe and that it should be subject to small hadronization corrections. However, as we shall discuss in the next section, it can lead to "unnatural" recombination of particles into jets and this is sufficient to destroy the phase space factorization property which is essential for the treatment of the small $y_{\rm cut}$ region. This will motivate the introduction of a new "QCD-inspired" algorithm based on the transverse momenta of the hadrons rather than on their invariant mass.

Scheme	M_{ij}^2/Q^2	Recombination
E	$(p_i+p_j)^2/Q^2$	$p_{ij}=p_i+p_j$
E_0	$2E_iE_j(1-\cos\theta_{ij})/Q^2$	$E_{ij}=E_i+E_j$ $p_{ij}=\frac{E_{ij}}{\lvert p_i+p_j\rvert}(p_i+p_j)$
P		$p_{ij}=p_i+p_j$ $E_{ij}=\lvert p_{ij}\rvert$
JADE		$p_{ij}=p_i+p_j$

Table 3.1: Recombination procedures for the JADE algorithm.

3.3 Large Logarithms and Exponentiation

Studying the perturbative expansion for the n-jet rate, we find that, in the region of small $y_{\rm cut}$, the couplant, a, is enhanced by a factor of $L^2=\ln^2(1/y_{\rm cut})$. The perturbative series now effectively becomes an expansion in aL^2; obviously, at $y_{\rm cut}\ll 1$ these large logarithms destroy the applicability of a truncated series and we are obliged to resum them to all orders in the couplant in order to obtain any reliable predictions.

We can see how large logarithms appear even at one-loop level by considering the brehmsstrahlung spectrum for emission of a soft (with energy fraction $\epsilon\ll 1$) and collinear (with angle $\theta\ll 1$) gluon from the $q\overline{q}$ pair. In the soft and collinear limit the single gluon emission probability is

$$dw(1)=C_F a\frac{d\epsilon}{\epsilon}\frac{d\theta^2}{\theta^2}\,, \tag{3.4}$$

where $C_F=(N^2-1)/2N$, the fundamental Casimir of SU(N). Integrating this over the phase space for the JADE algorithm ($y_{qg}\simeq\epsilon\theta^2$) gives

$$\int dw(1)\Theta(\epsilon\theta^2-y_{\rm cut}) \;=\; C_F a\int_0^1\frac{d\epsilon}{\epsilon}\int_0^1\frac{d\theta^2}{\theta^2}\Theta(\epsilon\theta^2-y_{\rm cut})$$

$$= \frac{C_F a}{2} L^2 \,. \tag{3.5}$$

The appearance of these large logarithmic terms is a common feature of any hard scattering process in the semi-inclusive or Sudakov region [20, 21]. In this region emission of radiation is constrained by the kinematics and for jet cross-sections at $y_{\rm cut} \ll 1$ the jet invariant mass is so restricted that only gluons which are soft and collinear with respect to the parent parton of the jet may be emitted.

The procedure by which the large logarithms are resummed hinges on the property of exponentiation, which is displayed by many quantities (for example, thrust and energy-energy correlation) in e^+e^- annihilation. We shall be interested in the exponentiation of jet fractions. Let us say what we mean by exponentiation. Recall that the emission of a single gluon introduced a factor of aL^2 into the three-jet rate at one-loop level. In general the n-jet rate will have the form

$$R_n(y_{\rm cut}) = \delta_{2n} + a[A_2^{(n)} L^2 + A_1^{(n)} L + O(1)] + a^2[B_4^{(n)} L^4 + B_3^{(n)} L^3 + O(L^2)] + \cdots \tag{3.6}$$

It should be noted that the n-jet rate first contributes at $O(a^n)$ so some of the perturbative coefficients in equation (3.6) will be zero. Now $R_2(y_{\rm cut})$, for example, is expected to exponentiate. It can then be written as

$$R_2(y_{\rm cut}) = \exp[L g_1(aL) + g_2(aL) + a g_3(aL) + \cdots] \,. \tag{3.7}$$

In other words all terms $a^n L^m$ with $m > n+1$ vanish from $\ln R_2(y_{\rm cut})$ whereas they do appear in $R_2(y_{\rm cut})$ itself. The function Lg_1 resums all the leading logarithmic (LL) contributions, $a^n L^{n+1}$, while g_2 contains the next-to-leading logarithmic (NLL) terms, $a^n L^n$, and the g_i, $i > 2$, represent subdominant logarithmic corrections. Calculating the functions g_1 and g_2 in the exponent is then a far more straightforward procedure than calculating the full $R_2(y_{\rm cut})$.

In order to understand how exponentiation arises let us consider the simpler case of multiple soft photon emission in QED. We have already seen that photons are not charged and have no self interactions. This means that the probability $dw(1,\ldots,n)$ for the emission of n soft photons factorizes into a product of single photon emission probabilities, $dw(i)$, analogous to equation (3.4), giving

$$dw(1,\ldots,n) \;=\; \frac{1}{n!} \prod_{i=1}^{n} dw(i)$$

$$= \frac{1}{n!}\prod_{i=1}^{n}\frac{\alpha_{\rm em}}{\pi}\frac{d\epsilon_i}{\epsilon_i}\frac{d\theta_i^2}{\theta_i^2}\,. \tag{3.8}$$

Using this result we can obtain the corresponding contribution to the cross-section by integrating over the relevant phase space, $\Theta(1,\ldots,n;y_{\rm cut})$. Exponentiation occurs if and only if the phase space also factorizes in the soft limit, that is,

$$\Theta(1,\ldots,n;y_{\rm cut}) \simeq \prod_{i=1}^{n}\Theta(i;y_{\rm cut})\,. \tag{3.9}$$

(For the JADE algorithm $\Theta(i;y_{\rm cut}) = \Theta(\epsilon_i\theta_i^2 - y_{\rm cut})$.) This enables us to express the cross section as

$$\begin{aligned} 1+\sum_{n=1}^{\infty}\int dw(1,\ldots,n)\Theta(1,\ldots,n;y_{\rm cut}) &\simeq 1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int dw(i)\Theta(i;y_{\rm cut}) \\ &= \exp\left[\int dw(i)\Theta(i;y_{\rm cut})\right]\,. \end{aligned} \tag{3.10}$$

It is important to note that the factorization of emission probabilities is a consequence of QED dynamics whereas the factorization of the phase space depends on the kinematic definition of the cross section. The cross section exponentiates in the semi-inclusive region only if its definition does not induce kinematic correlations between soft photons.

In QCD we need to investigate multiple gluon emission. Of course, gluons have colour charge and they can therefore radiate in cascade. Strong correlations between gluons are enforced by QCD dynamics and as a result multiple soft gluon emission probabilities will not factorize into a product of single gluon emission probabilities. The consequence of this is that simple exponentiation is not guaranteed. However, in the case of highly inclusive cross sections, like many two-jet dominated quantities, a simple exponentiation structure is still valid.

It is at this point that we encounter a disadvantage of the JADE algorithm. Calculating the n-jet rates to $O(a^2)$ using the JADE algorithm with the E recombination scheme gives an expression of the form of equation (3.6). Evaluating the LL coefficients $A_2^{(n)}$ and $B_4^{(n)}$, Brown and Stirling [22, 23] found that the LL contributions do not exponentiate. This is true even in the Abelian limit ($C_A \to 0$) where one should have no correlations between gluons. This absence of exponentiation is due to the unnatural way in which the JADE algorithm treats a situation such as that in Figure 3.3. Here the two gluons are soft yet their combined invariant

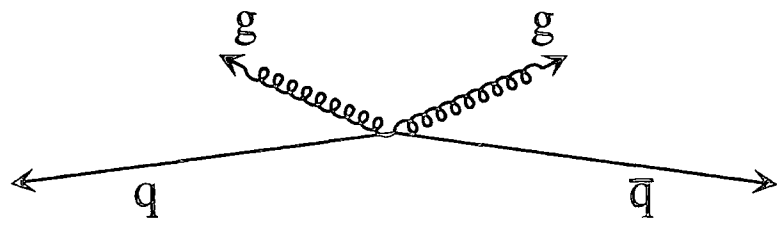


Figure 3.3: Example of a configuration which the JADE algorithm may assign to a three-jet final state.

mass is greater than that of each gluon combined with its respective (anti)quark and also greater than $y_{\rm cut}$. Thus the JADE algorithm clusters the gluons together and describes the process as a three-jet event rather than the two-jet event that it ought to be. This evidently introduces unnatural correlations between gluons and destroys the factorization property which leads to the exponentiation of the jet rates.

In order to ensure that this problem was not just a pathology of the E recombination scheme, which we mentioned above was not necessarily the most appropriate scheme, Catani et al. calculated the jet rates using other schemes (JADE, E0, P, P0 – see Table 3.1) and found that exponentiation did not occur for any of these schemes either.

It appears then that we must modify the JADE algorithm in order to remove this inherent flaw which destroys the exponentiation of jet fractions in the small $y_{\rm cut}$ region. It was suggested by Dokshitzer [24, 25] that a more natural way of clustering pairs of partons would be by their relative transverse momentum. This would overcome the problem of unnatural clustering, as can be seen intuitively from Figure 3.3. Thus the test variable for the new algorithm would be defined by

$$y_{ij} = \frac{(k_\perp^2)_{ij}}{Q^2} = \frac{\min(E_i^2, E_j^2)}{Q^2} \sin^2\theta_{ij} \; . \tag{3.11}$$

Unfortunately this definition causes a slight technical problem in that y_{ij} has a zero when the two partons are back-to-back ($\theta_{ij} = \pi$). This can be remedied by redefining y_{ij} as

$$y_{ij} = \frac{2\min(E_i^2, E_j^2)}{Q^2}(1 - cos\theta_{ij}) \; . \tag{3.12}$$

For $\theta_{ij} \ll 1$ this is equivalent to equation (3.11).

This new algorithm is known as the $k_\perp$ algorithm[1]. Its most significant advantage over the JADE algorithm is that it avoids introducing correlations between soft gluons and hence the jet rates may be expected to demonstrate a generalised exponentiation structure which will allow the resummation of leading and next-to-leading logarithms. As with the JADE algorithm there still remains a choice of recombination scheme, although in the region of small $y_{\rm cut}$ all the common schemes give the same LL and NLL contributions to the jet rates. Thus what follows in the next section is essentially recombination scheme independent.

3.4 LL and NLL Resummations

It has been found [26] that the $k_\perp$ algorithm fulfils the conditions that the algorithm should be infrared and collinear safe and that it should be subject to small hadronization corrections. Work by Catani, Webber et al. [25] has given implicit integral expressions for the jet rates $R_n(y_{\rm cut})$ which will exactly sum leading and next-to-leading powers of $\ln(1/y_{\rm cut})$ to all orders of perturbative QCD. They obtained the n-jet rates by evaluating

$$R_n(y_{\rm cut}) = \sum_{m \geq n} \frac{1}{\sigma_{\rm TOT}} \int d\sigma_m^{\rm excl.} \Theta_{n-{\rm jet}}(y_{\rm cut}) \,, \tag{3.13}$$

where $d\sigma_m^{\rm excl.}$ is the m-parton exclusive cross section and $\Theta_{n-{\rm jet}}(y_{\rm cut})$, the n-jet phase space, is given by the clustering procedure (equation 3.12 for the $k_\perp$ algorithm). This result was achieved by using the coherent branching formalism [21, 27, 28, 29] (which stems from the application of the Altarelli-Parisi splitting functions to the parent partons) to evaluate the exclusive cross sections and calculating the n-jet phase space to next-to-leading order.

The aim of the work presented in this section is to obtain a set of easily applicable explicit expressions which will be of use in jet phenomenology. Numerical evaluation of the integral equations mentioned above provides some useful information and also acts as a valuable check for our results. The n-jet rates $R_n(y_{\rm cut})$

[1]This algorithm was discussed at the Durham Workshop on Jet Studies at LEP and HERA in December 1990 and is sometimes referred to as the Durham algorithm.

are most readily expressed in terms of the generating function $\phi(Q, Q_0; u)$, where

$$Q_0^2 = y_{\rm cut} Q^2 \tag{3.14}$$

and u is a jet label. The n-jet rate is then given by

$$R_n(y_{\rm cut}) = \frac{1}{n!}\left(\frac{\partial}{\partial u}\right)^n \phi(Q, Q_0; u)\Big|_{u=0} . \tag{3.15}$$

The generating function obeys the following implicit coupled equations [25]:

$$\phi(Q, Q_0; u) = u^2 \exp\left(2\int_{Q_0}^{Q} dq \Gamma_q(Q, q)[\phi_g(q, Q_0; u) - 1]\right) \tag{3.16}$$

and

$$\begin{aligned}\phi_g(Q, Q_0; u) \;=\; & u \exp\Big(\int_{Q_0}^{Q} dq\{\Gamma_g(Q, q)[\phi_g(q, Q_0; u) - 1] - \Gamma_f(q)\}\Big) \\ & \Big(1 + u\int_{Q_0}^{Q} dq\, \Gamma_f(q) \exp\Big(\int_{Q_0}^{q} dq'\{[2\Gamma_q(q, q') - \Gamma_g(q, q')] \\ & [\phi_g(q', Q_0; u) - 1] + \Gamma_f(q')\}\Big)\Big) ,\end{aligned} \tag{3.17}$$

where the emission probabilities are defined as

$$\Gamma_q(Q, q) = \frac{2C_F}{\pi}\frac{\alpha_s(q)}{q}\left(\ln\frac{Q}{q} - \frac{3}{4}\right) , \tag{3.18}$$

$$\Gamma_g(Q, q) = \frac{2C_A}{\pi}\frac{\alpha_s(q)}{q}\left(\ln\frac{Q}{q} - \frac{11}{12}\right) , \tag{3.19}$$

$$\Gamma_f(q) = \frac{N_f}{3\pi}\frac{\alpha_s(q)}{q} . \tag{3.20}$$

In these expressions C_A and C_F are the fundamental Casimirs of SU(N) QCD, $C_A = N$ and $C_F = (N^2 - 1)/2N$; and N_f is the number of quark flavours. For $R_2(y_{\rm cut})$ one has a simple exponentiation structure and an explicit expression for the resummed leading and next-to-leading logarithms [23, 25] is straightforward to obtain:

$$R_2 = \exp\left[\frac{C_F a L}{2}(3 - L) - b\frac{C_F a^2 L^3}{6}\right] , \tag{3.21}$$

where $L = \ln(1/y_{\rm cut})$, $a = \alpha_s(Q)/\pi$ and $b = \beta_0/2$, the first coefficient of the QCD beta-function (see equation (2.89)). Applying the formalism to the case of $R_3(y_{\rm cut})$ one finds that

$$R_3 = 2R_2 \int_{Q_0}^{Q} dq \Gamma_q(Q, q) \exp\left[-\int_{Q_0}^{q} dq'(\Gamma_g(q, q') + \Gamma_f(q'))\right] . \tag{3.22}$$

Substituting the emission probabilities into equation (3.22) and bearing in mind the running of α_s gives a virtually intractable nested integral. There are also no obvious means by which to isolate before integrating terms which will contribute to LL or NLL parts of R_3. Our approach centres on performing the calculation in two easily tractable stages.

First we evaluate the integral equation with b set to zero to give the terms in R_3 which are independent of b and obtain

$$R_3^0 = 4C_F a R_2 \int_{Q_0}^{Q} \frac{dq}{q}\left(\ln\frac{Q}{q} - \frac{3}{4}\right)\exp\left[-2C_A a\int_{Q_0}^{q} dq'\frac{1}{q'}\ln\frac{q}{q'}\right] \tag{3.23}$$

$$= -\frac{C_F aL}{2}\exp\left[-\frac{C_F aL^2}{2}\right]\left\{\frac{1}{2}\sqrt{\frac{\pi}{A}}\mathrm{erf}(\sqrt{A})(3(1-C_F aL^2)-2L)\right.$$
$$\left.+\frac{2}{C_A}(1-\exp[-A])\left(\frac{2}{aL}+3C_F\right)\right\}, \tag{3.24}$$

where $A = C_A aL^2/4$ and $\mathrm{erf}(x)$ is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\,. \tag{3.25}$$

Note that we have used $b = (11C_A - 2N_f)/6$ to write Γ_f in terms of b.

Secondly we evaluate

$$R_3' = \left.\frac{\partial R_3}{\partial b}\right|_{b=0}$$
$$= \left.\frac{\partial}{\partial b}\left(2R_2\int_{Q_0}^{Q} dq\frac{2C_F\alpha_s(q)}{\pi q}\left(\ln\frac{Q}{q}-\frac{3}{4}\right)\right.\right.$$
$$\left.\left.\exp\left[\int_{Q_0}^{q} dq'\frac{2C_A\alpha_s(q')}{\pi q'}\ln\frac{q}{q'} - \frac{b\alpha_s(q')}{\pi q'}\right]\right)\right|_{b=0} \tag{3.26}$$

with

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1+\frac{b\alpha_s(\mu)}{2\pi}\ln\frac{Q}{\mu}}\,, \tag{3.27}$$

thus giving the coefficients of the terms in R_3 which are proportional to b. This approach relies on the assumption that R_3 at NLL level is linear in b. In order to justify this we calculate

$$R_3'' = \left.\frac{\partial^2 R_3}{\partial b^2}\right|_{b=0} \tag{3.28}$$

and find that R_3'' contains only subleading terms (that is, terms $\sim a^n L^{2n-p}$ with $p > 1$), hence validating our method.

Any subleading terms which appear as a result of this procedure are discarded since they will necessarily be incomplete. Combining the results of the two stages according to

$$R_3 = R_3^0 + bR_3' \tag{3.29}$$

gives

$$\begin{aligned} R_3(y_{\rm cut}) \;=\; & -\frac{C_F aL}{2}\exp\left[-\frac{C_F aL^2}{2}\right] \\ & \left\{\frac{1}{2}\sqrt{\frac{\pi}{A}}\,\mathrm{erf}(\sqrt{A})\left[3-2L+baL^2\left(\frac{C_F aL^2}{3}-\frac{1}{2}\right)-3C_F aL^2+\frac{b}{C_A}\right]\right. \\ & \left.+\frac{1}{C_A}\Big(\exp[-A]-1\Big)\left[\frac{2C_F baL^2}{3}-\frac{b}{3}-\frac{4}{aL}-6C_F\right]-\frac{b}{C_A}\right\}. \end{aligned} \tag{3.30}$$

This has been expressed in terms of error functions and exponentials for purposes of familiarity; but it should be noted that an equally succinct representation in terms of degenerate hypergeometric functions is possible and work on LL expressions for higher jet rates has shown that these functions appear to occur naturally.

We might speculate on the possibility of making a change of renormalization scale such that we could absorb any terms proportional to b. This can be done for R_2 where the b term in equation (3.21) can be removed by replacing $\alpha_s(Q)$ by $\alpha_s(y_{\rm cut}^{1/3}Q)$. A closer look at equation (3.22), however, reveals that b enters R_3 through a combination of the emission probabilities as well as through the renormalization scale; so it seems likely that no scale can be chosen so as to remove all b dependence.

3.5 Summary

In this chapter we have studied one example of perturbation theory in action. We highlighted the important features of the process $e^+e^- \rightarrow$ hadrons and stressed its usefulness as a test of QCD. We discussed the mechanics of jet algorithms and how they are applied to this particular process.

We saw how a traditional fixed order perturbative expansion in the coupling constant breaks down in the region where two- and three-jet events become indistinguishable. We investigated the cause of this and found that the emission of soft gluons from the parent partons introduced large logarithms of the inverse of the jet resolution parameter into the perturbative expansion.

In attempting to overcome this problem we discussed the property of exponentiation of jet observables and motivated the use of a new jet algorithm based on the relative transverse momenta of the final state hadrons. We then used this algorithm to resum leading and next-to-leading large logarithms for the two- and three-jet rates.

Not discussed was how this result can be utilised to achieve better quantitative predictions. The general aim would be to combine the resummed result, which is applicable at small $y_{\rm cut}$, with fixed order results, which are valid in the higher $y_{\rm cut}$ region where there is no problem with large logarithms. For more details on this and on the application of leading and next-to-leading large logarithm resummations to other jet observables we refer the reader to references [25, 30].

Chapter 4

Perturbation Theory at Large Orders

4.1 Introduction

In the previous chapter we presented an example of how perturbation theory works in a practical situation. Specifically, we concentrated on the experimental region of the process $e^+e^- \rightarrow$ hadrons in which the traditional perturbative expansion in the strong coupling constant broke down due to the presence of large logarithms. We overcame this problem by resumming the series in the effective expansion parameter, aL^2, to all orders.

This resummation only worked because the series in which we were interested was convergent. Equally, we assumed that, in the region where large logarithms were not a problem, truncating the perturbation series at a fixed order was legitimate.

In this chapter we investigate perturbation theory in more detail and show that in fact, for most practical field theories, the perturbative expansions, which are our best tools for making phenomenological predictions, demonstrate divergent behaviour at large orders. This property must cause concern since perturbation theory is at present one of our most developed methods of extracting "useful" information from a field theory.

Within our discussion we must clarify how it is that the application of fixed order perturbation theory to QED and QCD has had such spectacular success when this technique appears to ignore an infinite number of unquantifiable and divergent higher order terms. We will attempt also to turn a knowledge of the

large order behaviour to our advantage and to use it to make contact with non-perturbative effects. A final purpose of this chapter will be to introduce some of the techniques and notation that will be used in the following chapters.

4.2 Vacuum Instability of QED

To illustrate the mechanism through which divergences may occur, we start by considering QED. This argument was first presented by Dyson in 1951 [31]. A perturbative expansion for some generic QED observable may be written as

$$R(e^2) = \sum_{k=0}^{\infty} r_k (e^2)^k \ , \tag{4.1}$$

where e is the electron charge. This series may be calculated from the original field theory along the lines discussed in Chapter 2. Let us suppose, for the moment, that the series in equation (4.1) converges for some positive value of e^2. This implies that $R(e^2)$ is an analytic function of e^2 at e=0. Given this assumption, then, for sufficiently small values of e, $R(-|e^2|)$ will also be a well-behaved analytic function with a convergent power series expansion.

Let us now make a physical interpretation of $R(-|e^2|)$. When e^2 is positive $R(e^2)$ describes the real world in which like charges repel; but when e^2 is negative the interaction between like charges is attractive. This has a serious implication for the vacuum state; that is, in relativistic quantum mechanics, virtual electron-positron pairs may be created from the vacuum due to the uncertainty principle. Let us imagine creating N electron-positron pairs. Now let us ignore the positrons for the time being and consider the total energy of all the N electrons: in the real world this would be

$$E \sim NT + \frac{1}{2} e^2 V N^2 \ , \tag{4.2}$$

where T is the mean kinetic energy and V characterizes the mean Coulomb potential. The factor of $N^2/2$ counts the number of interacting pairs (assuming N is large). This system has a perfectly stable vacuum; as one increases N so the total energy of the system increases and the ground state of the system is achieved for N=0.

Now let us consider the fictitious case of $e^2 < 0$. Then equation (4.2) becomes

$$E \sim NT - \frac{1}{2} |e^2| V N^2 \ . \tag{4.3}$$

As N increases E starts to increase until N reaches some critical value, N_{crit};

$$N_{\text{crit}} = \frac{T}{V|e^2|} \,. \tag{4.4}$$

Then for $N > N_{\text{crit}}$ the total energy starts to decrease and for large N it exhibits the behaviour

$$E \sim -N^2 \,. \tag{4.5}$$

The same is true for the positrons created from the vacuum and so if we were to postulate the creation of N_{crit} electron-positron pairs then the vacuum would become catastrophically unstable with the energy of the system decreasing with the creation of each new pair.

Because of quantum barrier penetration effects there is a finite probability for the creation of N_{crit} pairs and, once this state is realised, an irreversible process of pair creation will set in until an infinite number of pairs has been created. Since we consider the field theory of QED as acting over all time, this explosive disintegration of the vacuum is inevitable for any finite N_{crit}. Hence our observable $R(e^2)$ cannot be analytic for any value of $e^2 < 0$ and its series in equation (4.1) cannot be convergent.

Despite the fact that this analysis is not entirely rigorous, the general conclusion that QED perturbative series have zero radius of convergence raises an important question: how do truncated perturbative series give such spectacular agreement with experiment when the series themselves formally diverge for any physical value of the expansion parameter?

To answer this question we must assume that QED perturbative series are in general asymptotic. In our example this means that the sum on the right hand side of equation (4.1) does not necessarily reproduce $R(e^2)$ uniquely but rather it converges asymptotically towards $R(e^2)$ for a finite number of terms, after which it begins to diverge.

To express this more formally let us consider a general function $f(z)$ which is analytic in the domain $\mathcal{D}$ (see Figure 4.1);

$$\mathcal{D}: \quad |\arg z| \leq \theta/2, \;\; |z| \leq \rho \;\; . \tag{4.6}$$

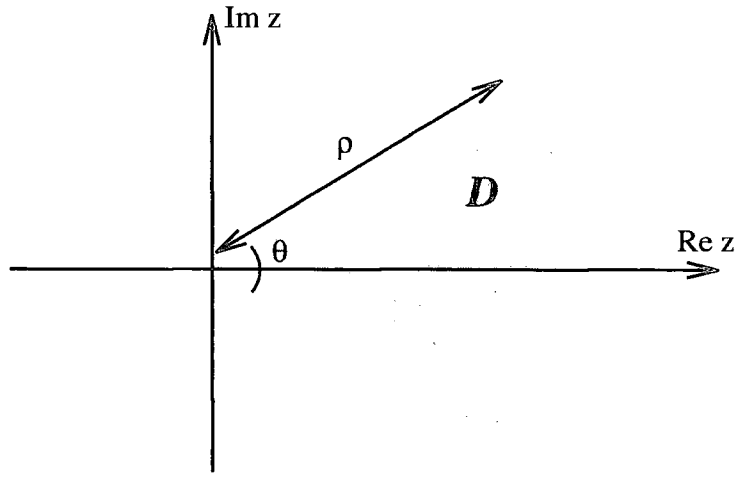


Figure 4.1: The domain of analyticity of $f(z)$.

Let us assume that $f(z)$ can be expanded as a power series at the origin:

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \; ; \tag{4.7}$$

and that this expansion is asymptotic to $f(z)$ in $\mathcal{D}$. That is, the right hand side of equation (4.7) diverges for all $z \neq 0$ and in $\mathcal{D}$ it satisfies the bound

$$\left| f(z) - \sum_{k=0}^{K} f_k z^k \right| \leq f_{K+1} |z|^{K+1} \tag{4.8}$$

for all K. So, despite the fact that the infinite series is divergent for all non-zero values of z, the partial sum of K terms can be used to estimate $f(z)$ providing $|z|$ is small. For a particular fixed value of $|z|$ we can minimize the bound on the error in this estimate by minimizing the right hand side of equation (4.8) with respect to K. Hence we can find the optimum number of terms, K_{opt}, to take in the partial sum. Up to K_{opt} terms the partial sum will converge; beyond it the series will begin to diverge.

Dyson's argument leads to the coefficients for QED perturbative series exhibiting a behaviour characterized by

$$f_k \sim k! \; . \tag{4.9}$$

In QED the expansion parameter is the fine structure constant, $\alpha \simeq 1/137$ and minimizing $f_k \alpha^k$ gives $K_{\text{opt}} \sim 1/\alpha \sim 137$. Thus QED starts to diverge only at

$\sim$137th order in the perturbative expansion, way beyond any order one could presently dream of calculating by summing Feynman diagrams.

Let us consider a slightly more general choice of form for the coefficients, f_k, which will lead to similar behaviour (that is, convergence up to $K_{\rm opt}$ terms and divergence beyond $K_{\rm opt}$ terms). So, for example,

$$f_k = AC^{-k}k! \,. \tag{4.10}$$

We can then write down a function of z which will characterize the maximum accuracy of the partial sum to K terms:

$$\begin{aligned} \epsilon(z) &= \min_{\{K\}} \left(f_K |z|^K \right) \\ &= f_{K_{\rm opt}} |z|^{K_{\rm opt}} \,. \end{aligned} \tag{4.11}$$

For f_k of the form given in equation (4.10), $K_{\rm opt} \sim C/|z|$. Using the Stirling formula for $k!$ at large k,

$$k! \sim \sqrt{2\pi k}\, \mathrm{e}^{k \ln k - k} \,, \tag{4.12}$$

we find that

$$\epsilon(z) \sim \exp[-C/|z|] \,. \tag{4.13}$$

The fact that the asymptotic series can approximate $f(z)$ only to finite accuracy means that it is in fact asymptotic to an infinite number of functions. For example, if we have

$$\sum_{k=0}^{\infty} f_k z^k \approx f(z) \,, \tag{4.14}$$

where '$\approx$' means '*is asymptotic to*', then it is also true that

$$\sum_{k=0}^{\infty} f_k z^k \approx f(z) + P \exp[-Q/z] \,, \tag{4.15}$$

provided that

$$Q \cos(\theta/2) > C \tag{4.16}$$

and that $|P|$ is sufficiently small. This new function is also analytic in $\mathcal{D}$ and satisfies the bound given in equation (4.8).

However, there is one situation in which the asymptotic series defines a unique function. If $\theta \geq \pi$ then, for some z such that $|\arg\, z| \geq \pi/2$, $Q \cos(\arg\, z) \leq 0$ and the only way in which the right hand side of equation (4.15) can be bounded by

$\epsilon(z)$ is if $P \equiv 0$. Then the right hand side of equation (4.15) reduces to $f(z)$ over the whole domain and the asymptotic series defines $f(z)$ uniquely.

The lesson we should learn from this is that not only must we investigate whether or not a perturbative series diverges but also, when it does diverge, if it still determines the relevant physical quantity uniquely. If we can find divergent series which do define physical quantities uniquely then we would like to find some way of reconstructing our generic $f(z)$ from its divergent expansion.

4.3 Exploiting Divergent Series

As an initial example of a divergent series with some physical interpretation, let us investigate the large order behaviour of the following partition function [10, 11, 32, 33]:

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, \mathrm{e}^{-x^2/2 - gx^4} \, . \tag{4.17}$$

This function is of interest because its perturbative coefficients in an expansion in g count the number of vacuum diagrams at each order in ϕ^4 field theory. The integral in equation (4.17) cannot be performed exactly so we follow the usual procedure of making a perturbative expansion. This gives

$$Z(g) = \sum_{k=0}^{\infty} Z_k g^k \tag{4.18}$$

and, by inspection, the coefficients are given by

$$\begin{aligned} Z_k &= \frac{(-1)^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{x^{4k}}{k!} \mathrm{e}^{-\frac{1}{2}x^2} \\ &= \frac{(-1)^k 4^k}{\sqrt{\pi}} \frac{\Gamma(2k + \frac{1}{2})}{k!} \, . \end{aligned} \tag{4.19}$$

To see how the Z_k behave at large orders, we again use the Stirling approximation formula and find

$$\begin{aligned} Z_k &\sim \frac{(-16)^k}{\sqrt{\pi k}} \mathrm{e}^{k \ln k - k} \\ &\sim \frac{1}{\pi\sqrt{2}} (-16)^k \, k^{-1} \, k! \, . \end{aligned} \tag{4.20}$$

We see then that the proliferation of Feynman diagrams at large orders leads to $k!$ growth of the coefficients. As we shall see later, in the case of more realistic

field theories, this is not the only way in which factorial growth can enter the perturbative coefficients.

Before we discuss this result further, let us concentrate on how we might go about extracting useful information from divergent series. To do this we use the technique of the Borel transform.

Consider a function $F(g)$ which has a divergent series expansion,

$$F(g) = \sum_{k=0}^{\infty} f_k g^k = \infty \, . \tag{4.21}$$

Although the original power expansion of $F(g)$ makes no sense, it is possible to define a new series that has much improved convergence properties [32]. To see this let us divide each coefficient by $k!$ in order to obtain a more convergent series:

$$B[F](z) = \sum_{k=0}^{\infty} \frac{f_k}{k!} z^k \, . \tag{4.22}$$

We can then reconstruct $F(g)$ via

$$F(g) = \int_0^{\infty} dz \, \mathrm{e}^{-z/g} B[F](z) \, . \tag{4.23}$$

Inserting the sum in equation (4.22) and performing the integral over z returns the original series of equation (4.21). (There is a caveat regarding this procedure which we will discuss later.)

The function $B[F](z)$ is known as the Borel transform of $F(g)$ and, unlike $F(g)$ itself, it may have a non-zero radius of convergence (in the z-plane). If $B[F](z)$ exists and $F(g)$ can be reconstructed through equation (4.23) then we say that the series expansion of $F(g)$ is Borel summable. In general, for any physically interesting quantities, there will be singularities in $B[F](z)$ and it will be our aim to isolate these and to find some method of regulating them such that we might be able to use equation (4.23) to extract some useful information about the function $F(g)$.

Let us investigate the case of an extremely simplistic field theory [33, 34]. Essentially we look at a field theory with one field and acting at one spacetime point. In such a theory a generic Green's function, which we shall call $G(g)$ and which has the divergent expansion

$$G(g) = \sum_{k=0}^{\infty} c_k g^k \, , \tag{4.24}$$

is defined by a path integral,

$$G(g) = \int_{-\infty}^{\infty} d\phi \, \mathrm{e}^{-S(\phi)/g} \, . \tag{4.25}$$

In our simple example the functional integral has reduced to a normal integral. Here $S(\phi)$ is the scaled action of the theory, which we will consider in Euclidean spacetime.

Now, $G(g)$ can also be written in the form of equation (4.23), giving

$$G(g) = \int_{0}^{\infty} dz \, \mathrm{e}^{-z/g} B[G](z) \, . \tag{4.26}$$

Comparing this with equation (4.25), we can see immediately that the Borel transform of $G(g)$ is

$$\begin{aligned} B[G](z) &= \int_{-\infty}^{\infty} d\phi \, \delta(z - S(\phi)) \\ &= \sum_i \left[\frac{dS(\phi)}{d\phi} \right]^{-1} \Bigg|_{\phi=\phi_i} \, . \end{aligned} \tag{4.27}$$

The sum is over all solutions of $z = S(\phi)$, labelled ϕ_i. It is evident then that the Borel transform will have singularities when

$$\frac{dS(\phi)}{d\phi} = 0 \, . \tag{4.28}$$

This equation is just the classical Euclidean equation of motion for the field theory. Finite action solutions of this equation exist and are called instantons. Singularities appear in the Borel transform for any field theory in which instantons are present; moreover, their location in the Borel plane is universal within each theory, since the interconnectedness, through Schwinger-Dyson equations, of all Green's functions ensures that an instanton singularity in the Borel transform of one Green's function propagates through to the Borel transform of all others [34].

In subsequent discussion we shall not elaborate much further on the role of instanton singularities for three reasons. First, instantons are a fairly well understood effect (even in more realistic field theories like QED and QCD). This means that one can calculate their effects separately without worrying about how to define the integral of equation (4.23) over the singular points. Secondly, instantons do not contribute the leading singularity in the Borel transform anyway; as we

shall discuss in much greater detail in subsequent chapters, there are other singularities in the Borel transform which dominate its structure to the exclusion of instantons and of which we do not currently have a full understanding. Finally, we shall primarily be interested in large-N_f and large-b expansions of perturbative coefficients and, due to the universality of the location of the instanton singularities, they become invisible in these limits.

Before we move on to discuss other properties of the Borel transform, let us look at an interesting feature of the instanton singularities. Let us denote the finite action solutions of equation (4.28) by z_i. So we have

$$\left.\begin{aligned} S(\phi) &= z\ , \\ S(\phi_i) &= z_i\ , \\ S'(\phi_i) &= 0\ . \end{aligned}\right\} \tag{4.29}$$

Expanding in a Taylor series around one solution of $z = S(\phi)$, ϕ_i, gives

$$S(\phi) \simeq z_i + \frac{1}{2}S''(\phi_i)(\phi - \phi_i)^2\ . \tag{4.30}$$

Rearranging this and differentiating with respect to $S(\phi)$ gives

$$\left[\frac{dS(\phi)}{d\phi}\right]^{-1} = \frac{d\phi}{dS(\phi)} = \frac{1}{\sqrt{2S''(\phi)}}\frac{1}{\sqrt{z - z_i}}\ . \tag{4.31}$$

So the singularity in the Borel transform is of a square root branch point type. We would like to know how this arises from the perturbative coefficients of $G(g)$. To do this let us expand the branch point as a series:

$$\begin{aligned} B[G](z) &\sim \frac{1}{\sqrt{1 - z/z_i}} \\ &= \sum_{k=0}^{\infty}\left(-\frac{1}{z_i}\right)^k \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - k)}\frac{z^k}{k!}\ . \end{aligned} \tag{4.32}$$

Re-expressing the gamma functions using $\Gamma(p)\Gamma(1-p) = \pi/\sin\pi p$, we find that

$$B[G](z) \sim \sum_{k=0}^{\infty}\left(\frac{1}{z_i}\right)^k \Gamma(k + \tfrac{1}{2})\frac{z^k}{k!}\ . \tag{4.33}$$

Comparing this expression with equation (4.22) we see that

$$c_k \sim \left(\frac{1}{z_i}\right)^k \Gamma(k + \tfrac{1}{2}) \tag{4.34}$$

and at large orders

$$c_k \sim \left(\frac{1}{z_i}\right)^k k^{-1/2} k! \,. \tag{4.35}$$

Comparing this with the large order behaviour, given in equation (4.20), of the coefficients Z_k of the partition function $Z(g)$ we see that both results have the large order behaviour $\sim A^k k^{-\gamma} k!$, indicating that the instanton singularities are intimately related to the proliferation of Feynman diagrams at large orders, at least for simple theories [35]. The reason that this relation must hold is that, for finite, renormalized diagrams, the diagrams themselves are bounded by a pure power law behaviour. Hence the only way in which factorial growth can manifest itself in the perturbative coefficients is through a proliferation in the number of diagrams.

4.4 Renormalon Singularities

The singularities in the Borel transform that we have encountered so far were shown to be due to a combinatorical increase in the number of Feynman diagrams at large orders. We stated that for simple theories this must be the origin of the factorial growth of the coefficients, since the diagrams themselves exhibit a power law behaviour. It turns out that this is only the case for quantum mechanics and super-renormalizable field theories (that is, theories in which there are only a finite number of divergent diagrams in the entire perturbation theory, an example being ϕ^4 in three dimensions).

In renormalizable theories, like ϕ^4 in four dimensions, QED and QCD, there may be additional large order divergences associated with single diagrams. These will contribute extra singularities to the Borel transform in addition to the usual instanton singularities. This feature was first identified in QED by Lautrup [36] and 't Hooft [34] and accordingly we will begin our discussion of these singularities by looking at a particular class of contributions to the anomalous magnetic moment of the electron.

The anomalous magnetic moment of the electron arises from corrections to the electron-photon vertex, the lowest order of which is represented by the Feynman diagram in Figure 4.2. For a full calculation we must take into account vacuum polarization insertions in the photon propagator, which take the form of a vacuum

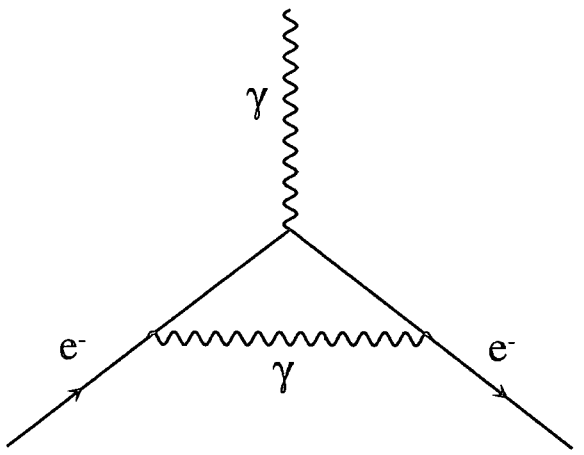


Figure 4.2: The lowest order contribution to the electron anomalous magnetic moment.

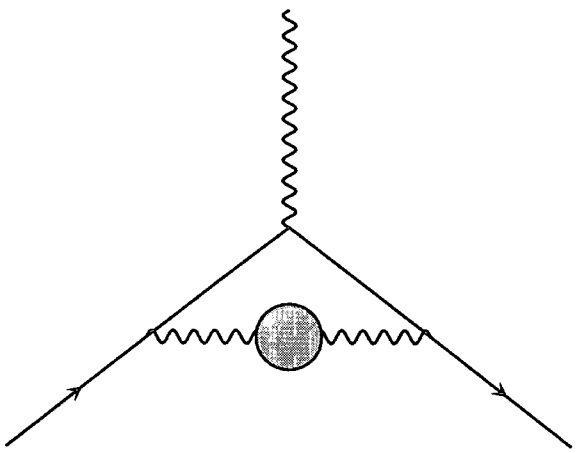


Figure 4.3: A general vacuum polarization blob insertion in the photon propagator.

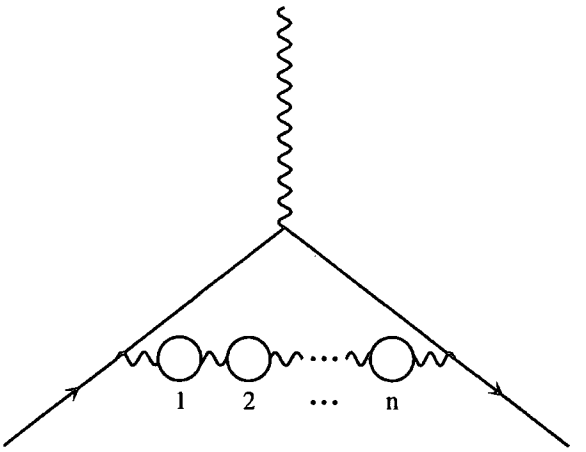


Figure 4.4: The n-bubble diagram which contributes to the electron anomalous magnetic moment at nth order in perturbation theory.

polarization blob, shown in Figure 4.3. For such a vacuum polarization insertion the anomalous magnetic moment can be written as

$$A(a) = \int_0^\infty \frac{dt}{t}\frac{1}{\pi}\,\mathrm{Im}\,\Pi(t)K(t)\;. \tag{4.36}$$

Here $\mathrm{Im}\,\Pi(t)/\pi$ is the spectral function of the blob and $K(t)$ is the anomalous magnetic moment due to the diagram shown in Figure 4.2,

$$K(t) = a\int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)t/m_e^2}\;, \tag{4.37}$$

where $a = \alpha/\pi$. By using the dispersion relation satisfied by the vacuum polarization amplitude $\Pi(k^2)$ and inserting $K(t)$, we can write

$$A(a) = a\int_0^1 dx(1-x)\left[-\Pi\left(\frac{-x^2}{1-x}m_e^2\right)\right]\;. \tag{4.38}$$

The class of diagrams in which we shall be interested is the contribution at nth order, shown in Figure 4.4, of an n-bubble diagram. For the n-bubble insertion we have

$$-\Pi(k^2) = (-a\Pi_2(k^2))^n\;, \tag{4.39}$$

where $\Pi_2(k^2)$ is the second order vacuum polarization. So the sum of the n-bubble contributions gives

$$A(a) = a\sum_{n=0}^{\infty} A_n a^n\;, \tag{4.40}$$

where

$$A_n = \int_0^1 dx(1-x)\left[-\Pi_2\left(\frac{-x^2}{1-x}m_e^2\right)\right]^n . \tag{4.41}$$

Now, for $k^2 < 0$, $\Pi_2(k^2)$ has a known functional form and, moreover, is negative definite; so we can write

$$A_n = \int_0^1 dx(1-x)[f(x)]^n , \tag{4.42}$$

where $f(x)$ is positive definite over the range of integration. Lautrup used a saddle point approximation to evaluate the integral and found that, for large n,

$$A_n \sim 6^{-n}n! . \tag{4.43}$$

So each n-bubble diagram on its own exhibits factorial growth. We can sum the series in equation (4.40) to obtain

$$A(a) = a\int_0^1 dx\frac{(1-x)}{1-af(x)} . \tag{4.44}$$

We know that $f(x)$ is positive definite, so in the physical region where $a > 0$ there will be a pole in the integrand and $A(a)$ will have a cut on the positive real axis.

This pole is directly related to the Landau singularity of QED, where the running coupling diverges. This occurs at very high momenta and is due to the same bubble insertions in the photon propagator. In the context of the Borel transform the non-Borel-summability of the perturbative series signals an ambiguity in $A(a)$ at very high momenta. It is possible to deform the contour in the Borel reconstruction integral of equation (4.23) around the pole by introducing a term $\sim e^{-1/a}$ which vanishes in a perturbative expansion. It is the choice of the exact method of avoiding the pole that leads to the ambiguity in $A(a)$. Terms $\sim e^{-1/a}$ correspond to non-perturbative effects and so discovering the correct method for negotiating the pole is necessarily equivalent to finding a non-perturbative solution of the theory.

There is no doubt about the asymptotic behaviour of these n-bubble diagrams. However, they represent only a subset of all the diagrams which contribute to $A(a)$ and we must ask whether or not their contributions may be cancelled in the full calculation. This question was tackled by Parisi in a series of papers [37, 38]. He

found that the divergences of the n-bubble diagrams had a much deeper origin in the renormalization process.

Parisi's argument was centred on the renormalization group equation (see equation (2.77)) satisfied by some generic n-point Green's function $\Gamma^{(n)}(p;a)$. He took a simple form for the beta-function, $\beta(a) = -\beta_0 a^2$.[1] Borel transforming the RGE gives

$$\left(-p\frac{\partial}{\partial p} - \beta_0 z + n\right) B[\Gamma^{(n)}](p,z) = 0 \,, \tag{4.45}$$

which has the solution

$$B[\Gamma^{(n)}](p,z) = r(z)p^n \left(\frac{p}{\mu}\right)^{-\beta_0 z} \,. \tag{4.46}$$

As we have already taken care to point out, the Green's functions of a theory are interrelated by Schwinger-Dyson equations. As an illustration let us consider ϕ^4 theory and write a schematic form for the Schwinger-Dyson equation connecting $\Gamma^{(6)}$ and $\Gamma^{(4)}$:

$$\Gamma^{(6)}(0,a) = \int \frac{d^4p}{(p^2+m^2)^3} \left|\Gamma^{(4)}(p,-p,0,0;a)\right|^3 \,. \tag{4.47}$$

We then write the Borel transform of this equation:

$$B[\Gamma^{(6)}](0,z) \sim \int \frac{d^4p}{(p^2+m^2)^3} \left(\frac{p}{\mu}\right)^{-\beta_0 z} \,, \tag{4.48}$$

where the factor of $(p/\mu)^{-\beta_0 z}$ comes from the expression in equation (4.46). The convergence of the integral in equation (4.48) depends crucially on the number of powers of p in the integrand. For $B[\Gamma^{(6)}](0,z)$ this p-dependence will be $p^{-2-\beta_0 z}$, so for $z = -2/\beta_0$ the Borel transform will have a pole. The interdependence of Green's functions through Schwinger-Dyson equations such as equation (4.47) means that poles in the Borel transform of any one Green's function will propagate to all others [34], giving the Borel transform poles at

$$z = z_\ell = -\frac{2\ell}{\beta_0} \,, \tag{4.49}$$

where ℓ is a positive integer. These poles, which come directly from renormalization, are known for short as *renormalons*. Divergences of this kind reflect

[1] We use β_0 rather than b to represent the first coefficient of the beta-function in order to avoid confusion with the b defined explicitly in previous chapters.

the ultraviolet behaviour of the theory and are therefore called ultraviolet renormalons.

In QED $\beta_0 < 0$ and so the UV renormalons lie evenly spaced along the positive real z-axis. They represent a genuine ambiguity in the reconstruction of a function from its divergent perturbative expansion. In his series of papers [37] Parisi showed in addition that the location of the UV renormalons is independent of the choice of beta-function, although their strengths have a weak dependence on the second beta-function coefficient. Most importantly, Parisi also derived a result which connected each pole with a local operator in the theory. This was based on the BPH theorem which states that all UV divergences can be removed by the introduction into the Lagrangian of counterterms of local operators.

4.5 Infrared Renormalons

In the previous section the presence of UV renormalon poles in the Borel transform was established for a non-asymptotically free theory. The same results carry over to an asymptotically free theory but with one crucial difference: in SU(3) QCD with $N_f \leq 16$, for example, the first coefficient of the beta-function is positive ($b = (11C_A - 2N_f)/6$) and the UV renormalon poles now lie on the negative real z-axis. Thus these poles do not present any problems in the reconstruction of QCD quantities from their divergent perturbative expansions.

However, one finds that a new set of poles appear on the positive real axis. These are the infrared renormalons and they arise essentially from infrared divergences of bubble diagrams similar to that in Figure 4.4. In a procedure analogous to the BPH treatment of UV divergences in QED, Parisi [38] attempted to relate the IR renormalons to counterterms of non-local operators, a process about whose validity he himself expressed doubts.

Nonetheless, significant progress has been made, notably by Grunberg [39] and Mueller [40], in relating IR renormalons to non-perturbative effects at low momenta through the operator product expansion (OPE). The question of how we deal with poles on the positive real axis is very important. At the very least we would like to know what degree of ambiguity these poles introduce into perturbative calculations, since large ambiguities would call into question the value

of continuing the programme of extending fixed order perturbative calculations of QCD observables. Less pessimistically, we might hope to obtain further insight into the non-perturbative regime of QCD.

As we shall see, there is indeed much to be learnt from studying the consequences of using a short-distance OPE to quantify some of the non-perturbative effects which we believe are intimately related to the presence of IR renormalons. Let us begin by introducing a generic correlator of two currents [39]:

$$\Pi(Q^2) = \int d^4x \, \mathrm{e}^{iqx} \langle 0|T[j(x)j(0)]|0\rangle \, , \tag{4.50}$$

where $Q^2 = -q^2$, the spacelike Euclidean momentum transfer; and Lorentz indices have been dropped. The associated RG invariant quantity is

$$D(Q^2) = Q^2 \frac{d\Pi}{dQ^2} \, . \tag{4.51}$$

We shall, in later chapters, give an explicit form to Π and D but for the time being these two functions should be regarded as generic QCD quantities.

The OPE, as proposed by Wilson [41] and implemented by Shifman, Vainshtein and Zakharov [42], enables one to attach some meaning to the limit in which the separation of two fields in coordinate space goes to zero. Formally, we write

$$T[\phi(x)\phi(0)] \sim \sum_i C_i(x)\mathcal{O}(0) \, . \tag{4.52}$$

Here the sum is over a set of local renormalized composite fields, $\mathcal{O}$, and the C_i's, known as Wilson coefficients, are complex coefficient functions.

Applying a short distance OPE to $D(Q^2)$ yields the representation

$$D(Q^2) = D_{PT}(a) + G_0(a) + \text{higher dimensional condensates} \, . \tag{4.53}$$

In this representation we have

$$D_{PT}(a) = a \sum_{n=0}^{\infty} d_n a^n \, , \tag{4.54}$$

the usual perturbative contribution. $G_0(a)$ is the leading (lowest dimensional) condensate contribution. For example, in QCD this will be the gluon condensate,

$$G_0(a) = \frac{1}{Q^4} \langle 0|GG|0\rangle(\mu) C_{GG}(Q/\mu, a) \, , \tag{4.55}$$

where $C_{GG}(Q/\mu, a)$ is the Wilson coefficient for the gluon condensate term in the OPE. In anticipation of later calculations we shall assume massless quarks.

Now, we expect $D_{PT}(a)$ and $G_0(a)$ to be separately RG invariant. If we consider a condensate with dimension d (that is, with scaling behaviour $\sim Q^{-d}$) we can determine the a-dependence of $G_0(a)$ by requiring that $G_0(a)$ satisfies the renormalization group equation (see equation (2.77)). Let us take the following form for the QCD beta-function:

$$\beta(a) = \frac{da}{d\ln\mu} = -ba^2(1+ca)\,. \tag{4.56}$$

Then we find that $G_0(a)$ must take the form

$$G_0(a) = C\left(\frac{\mu^2}{Q^2}\right)^{d/2} \exp[-d/ba]\, a^\delta\,(1+O(a))\,, \tag{4.57}$$

where

$$\delta = \frac{2\gamma_0}{b} - \frac{dc}{b} \tag{4.58}$$

with γ_0 the one-loop anomalous dimension of the corresponding operator. C is a scale-independent constant which contains the truly non-perturbative information. This form for $G_0(a)$ has an essential singularity at $a = 0$, so that the OPE motivated expression in equation (4.53) is only meaningful when we have a resummation prescription for $D_{PT}(a)$. This will be provided by the Borel transform, in which the IR renormalon poles will be negotiated by performing the reconstruction integral along a contour displaced above or below the real z-axis.

Let us write $D_{PT}(a)$ in its Borel representation:

$$D_{PT}(a) = \int_0^\infty dz\, \mathrm{e}^{-z/a} B[D_{PT}](z)\,, \tag{4.59}$$

where the Borel transform is defined, as before, as

$$B[D_{PT}](z) = \sum_{n=0}^{\infty} \frac{d_n}{n!} z^n\,. \tag{4.60}$$

Now let us assume that the first IR renormalon, arising from a bubble insertion as discussed above, occurs at $z = z_0$. By analogy with the case of UV renormalons, we can write down a form for $B[D_{PT}](z)$ in the region of this first renormalon pole:

$$B[D_{PT}](z) \underset{z=z_0}{\sim} B[D_{PT,0}](z) \equiv K\left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \frac{1}{(1-z/z_0)^\gamma}\left[1+O\left(1-\frac{z}{z_0}\right)\right]\,, \tag{4.61}$$

where K is a scale independent factor. This yields (disregarding any UV renormalons on the negative real z-axis) the large order behaviour

$$d_n \sim K \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \left(\frac{1}{z_0}\right)^n \frac{\Gamma(n+\gamma)}{\Gamma(\gamma)} \left[1 + O\left(1+\frac{1}{n}\right)\right] . \tag{4.62}$$

As we did for $D_{PT}(a)$ in equation (4.59), we can write the "renormalon contribution" to $D_{PT}(a)$ in terms of its Borel representation:

$$D_{PT,0}(a) = \int_0^\infty dz\, \mathrm{e}^{-z/a} B[D_{PT,0}](z) . \tag{4.63}$$

For $z > z_0$ the $(1 - z/z_0)^{-\gamma}$ factor in equation (4.61) implies that this contribution has an imaginary part (for $a > 0$). So

$$\mathrm{Im}\, D_{PT,0}(a) = \int_{z_0}^\infty dz\, \mathrm{e}^{-z/a} B[\mathrm{Im}\, D_{PT,0}](z) , \tag{4.64}$$

where

$$B[\mathrm{Im}\, D_{PT,0}](z) = K \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \frac{1}{|1 - z/z_0|^\gamma} \sin\pi(1-\gamma) \left[1 + O(1 - \frac{z}{z_0})\right] . \tag{4.65}$$

To evaluate the integral in equation (4.64) we assume temporarily that $\gamma < 1$, though the result will have general validity. We find that

$$\begin{aligned} \mathrm{Im}\, D_{PT,0}(a) &= \pm K\, z_0^\gamma\, \Gamma(1-\gamma) \sin\pi(1-\gamma) \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \mathrm{e}^{-z_0/a}\, a^{1-\gamma} \left[1 + O(a)\right] \\ &= \pm K \frac{\pi z_0^\gamma}{\Gamma(\gamma)} \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \mathrm{e}^{-z_0/a}\, a^{1-\gamma} \left[1 + O(a)\right] . \end{aligned} \tag{4.66}$$

The sign of this result depends on whether the contour is taken above or below the real z-axis.

If this ambiguity is to be avoided we must obtain some compensating factor from the non-perturbative part of the theory. One must assume that the constant C in equation (4.57), which we stated contained the non-perturbative information, is in fact complex. So we will have $C = C_R \pm iC_I$, with the ambiguous imaginary part reflecting the renormalon induced ambiguity in $D_{PT}(a)$. We can motivate the presence of this ambiguous imaginary part in the condensate contribution by noting that there exists an additional solution to the RGE if we replace a^δ in equation (4.57) by $(-a)^\delta$. Then, for $a > 0$ and $\delta \neq$ integer, the condensate contribution has a double-valued complex coefficient.

The critical step is now to require cancellation of these ambiguities between the perturbative (equation (4.66)) and non-perturbative (equation (4.57)) parts of the theory. This leads to the following relations:

$$z_0 = \frac{d}{b} \,, \tag{4.67}$$

$$1 - \gamma = \delta \,, \tag{4.68}$$

$$K = -\frac{C_I \Gamma(1-\delta)}{\pi z_0^{1-\delta}} \,. \tag{4.69}$$

There are a number of important points to raise about these relations. First, the renormalon residue is related to the condensate parameter, which leads us to expect that it is a process independent quantity. Moreover, this implies that one can obtain (by making an all-orders perturbative calculation) a "perturbation theory determined" part of the condensate contribution! However, this is slightly misleading in that exponentially small terms may be shifted between $D_{PT}(a)$ and $G_0(a)$ [39].

Secondly, equation (4.67) tells us the location of the IR renormalon poles on the positive real z-axis. The condensates obtained from performing a short distance OPE have dimensions $d = 4, 6, 8, \ldots$, implying that the Borel transform will have IR renormalon singularities at

$$z_\ell = \frac{2\ell}{b} \quad , \quad \ell = 2, 3, 4, \ldots \,. \tag{4.70}$$

Notice that the lowest dimension condensate in the OPE is the gluon condensate with $d = 4$. Hence we expect that there will be no IR renormalon at $z = 2/b$ since there is no dimension two condensate to compensate for it. We shall discuss this issue in more detail in later chapters.

The final point to be made is that equation (4.68) gives the structure of the renormalon singularities. In general they will be branch point singularities; but in the case of a large-b expansion (which we shall motivate in the next chapter) they will reduce (mainly) to simple or double poles.

4.6 Summary

Perhaps the most efficient way to summarise the results of this chapter is to refer the reader to the diagram in Figure 4.5. This shows the singularity structure

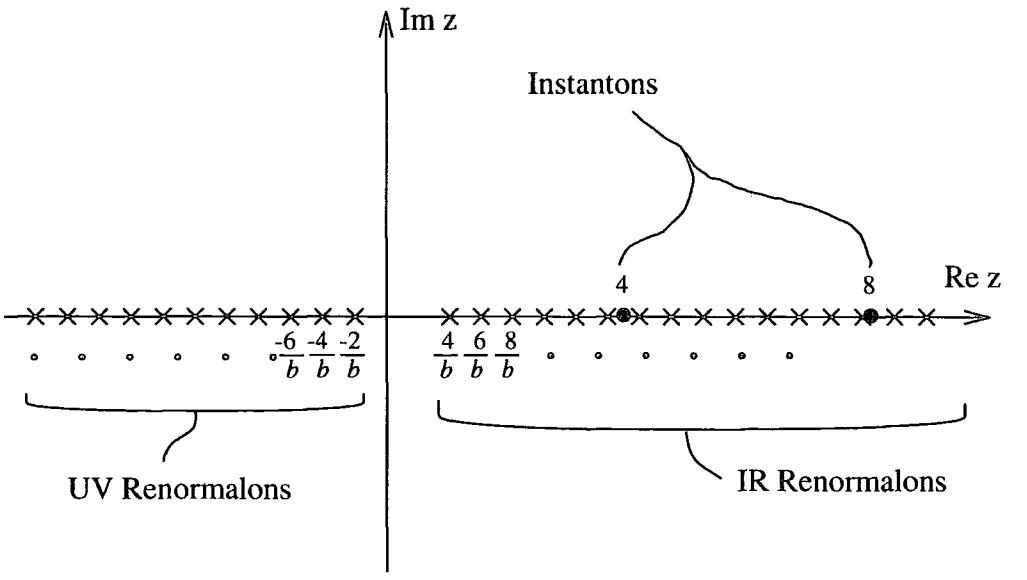


Figure 4.5: The singularity structure of the Borel transform of the QCD quantity $D(a)$, as discussed in the text. The relative positions of the singularities are based on $b = 23/6$, corresponding to SU(3) QCD with $N_f = 5$.

in the Borel plane for the Borel transform of the QCD quantity $D(a)$ which we discussed in the last section.

We began this chapter by looking at how perturbation theory behaves at large orders and found that in most cases the normal perturbative expansion was at best an asymptotic series. Furthermore, we found that the $n!$ growth of the perturbative coefficients due to the proliferation of Feynman diagrams was related to instanton singularities in the Borel transform.

We went on to discover that, in QED, not only are there $n!$ contributions from the growth in the number of Feynman diagrams but there also exist $n!$ contributions from single diagrams, notably from the UV divergences of n-bubble insertions into the photon propagator. These enter the Borel transform as UV renormalon singularities, whose existence and location are a product of the renormalization procedure.

In QCD the UV renormalons are confined to the negative real z-axis and do not pose problems for the reconstruction of QCD observables from their divergent perturbative expansions. However, a new set of singularities appears on the pos-

itive real z-axis. These are the IR renormalons, which reflect the IR divergences of similar bubble diagrams. Following a procedure similar to that for the UV renormalons in QED, which may be compensated by introducing counterterms of local operators into the Lagrangian, we attempted to relate the IR renormalons to non-perturbative effects. The mechanism for achieving this was the application of the OPE and, using this technique, we saw how the structure shown in Figure 4.5 arises.

Having set up some of the formalism, in the next chapter we will move on to examining some of the practical applications of these techniques.

Chapter 5

Singularity Structure of the QCD Vacuum Polarization Function

5.1 Introduction

In the last chapter we introduced some of the formalism and ideas associated with investigating large order behaviour in perturbation theory. In this chapter we go on to utilise recent progress [43, 44] in exact all-orders QED calculations to leading order in the N_f expansion, with N_f the number of fermions, to determine explicitly the singularity structure of the QCD vacuum polarization function in the Borel plane.

On very general grounds one anticipates branch point singularities evenly spaced along the positive and negative real axis in the Borel variable [45]. As we have already seen, those on the positive axis, the infrared (IR) renormalons, are supposedly correlated with the absence from the formal perturbation series of infrared non-perturbative effects, vacuum condensates, present in the operator product expansion. They are responsible for fixed-sign factorial growth of the series coefficients and represent a genuine ambiguity in reconstructing the physical observable from the formal perturbation series. Those on the negative axis, the ultraviolet (UV) renormalons, correspond to alternating-sign factorial growth of the series coefficients and do not prevent the reconstruction of the observable by Borel summation.

Whilst the above singularity structure is well-motivated theoretically, there

have been various problematic issues. In particular the connection with the OPE suggests that the leading IR renormalon should be absent for the case of the QCD vacuum polarization function since there is no relevant operator of dimension two; the first contribution being the gluon condensate of dimension four. This conclusion has been questioned on various grounds by several authors [46, 47].

The leading asymptotic growth of the perturbative coefficients will be determined by the Borel plane singularity nearest the origin; for the case of the QCD vacuum polarization function this is the first UV renormalon. We point out that the conventionally expected structure of this singularity, with a single branch point exponent, would enable one to obtain the asymptotic growth of the coefficients to all orders in the N_f expansion given an exact large-N_f result. We suggest that this is unlikely and indicate a more complicated structure for the first UV renormalon in accordance with recent results of Vainshtein and Zakharov obtained using their "UV renormalon calculus" [48].

We shall show that the actual singularity structure of the QCD vacuum polarization function is precisely as expected in the large-N_f limit; in particular the leading IR renormalon singularity is indeed absent. This has also been noted for the singularities in the QED vacuum polarization function in reference [43]. We further demonstrate that there is an unexpected symmetry between the third and higher UV and IR singularities. We are able to sum up the UV and IR contributions separately to obtain a closed form result involving ζ-functions.

We finally show that in SU(3) QCD, with N_f=15 or 16, the IR renormalon singularities are possibly absent [49, 50]; and that they first vanish when the instanton/anti-instanton singularity becomes leading. The requirement that this happens for an SU(N) theory uniquely selects $N = 3$.

5.2 The Adler D-function and the Expansion in N_f

We shall be interested in the SU(N) QCD vacuum polarization function with N_f flavours of massless quarks,

$$\Pi(-q^2)(q_\mu q_\nu - g_{\mu\nu}q^2) = 16\pi^2 i \int d^4x\, \mathrm{e}^{iq.x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle \ . \qquad (5.1)$$

In order to avoid an unspecified constant, we shall actually focus on the related Adler D-function,

$$D(Q^2) = -\frac{3}{4}Q^2\frac{d}{dQ^2}\Pi(Q^2)\,, \tag{5.2}$$

where $Q^2 = -q^2$ is the spacelike Euclidean squared momentum transfer. This quantity is related to the experimentally-relevant R-ratio in e^+e^- annihilation,

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,. \tag{5.3}$$

Taking s to be the physical timelike Minkowski squared momentum transfer, R and Π may be related by using the optical theorem, which implies that

$$R(s) = \frac{3}{4\pi}\text{Im}\,\Pi(-s)\,. \tag{5.4}$$

In QCD perturbation theory we have

$$D(Q^2) = d(R)\sum_f Q_f^2\left(1 + \frac{3}{4}C_F\tilde{D}\right) + \left(\sum_f Q_f\right)^2\tilde{\tilde{D}}\,, \tag{5.5}$$

where Q_f denotes the electric charge of the quarks and the summation is over the flavours accessible at a given energy. $d(R)$ is the dimension of the quark representation of the colour group (here $d(R) = N$). As in Chapter 3 the SU(N) Casimirs are defined as $C_A = N$, $C_F = (N^2-1)/2N$.

The correction to the parton model result has the perturbative expansion

$$\tilde{D} = a + d_1a^2 + d_2a^3 + \cdots + d_ka^{k+1} + \cdots\,, \tag{5.6}$$

with $a = \alpha_s(\mu^2)/\pi$ the renormalization group (RG) improved coupling. The $\tilde{\tilde{D}}$ contribution first enters at $O(a^3)$ due to the existence of diagrams of the "light-by-light" type (see Figure 5.1). Our interest here is in the asymptotic growth of the d_k coefficients in large orders.

The RG-improved coupling $a(\mu^2)$ will evolve with renormalization scale μ^2 according to the beta-function equation

$$\frac{da}{d\ln\mu} = -ba^2(1 + ca + c_2a^2 + \cdots)\,, \tag{5.7}$$

where b and c are universal with [51]

$$\begin{aligned} b &= \frac{(11C_A - 2N_f)}{6}\,, \\ c &= \left[-\frac{7}{8}\frac{C_A^2}{b} - \frac{11}{8}\frac{C_AC_F}{b} + \frac{5}{4}C_A + \frac{3}{4}C_F\right] \end{aligned} \tag{5.8}$$

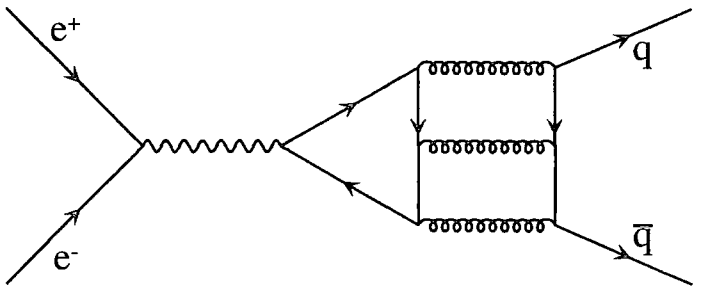


Figure 5.1: "Light-by-light" type contribution to $D(Q^2)$.

and where c_2 and higher coefficients are renormalization scheme (RS) dependent. We shall usually consider the $\overline{\rm MS}$ scheme with $\mu^2 = Q^2$. For the R-ratio there is an analogous expansion for the quantity $\tilde{R}$ with perturbative coefficients r_k, defined as in equations (5.5) and (5.6). The dispersion relation in equation (5.4) means that the r_k are directly related to the d_k. For instance $r_1 = d_1$ and $r_2 = d_2 - \pi^2 b^2/12$. The π^2 terms arise due to analytic continuation. Given knowledge of the asymptotic growth of the d_k one can obtain that of the r_k using equation (5.4). We shall continue to focus on the d_k for the moment.

The coefficients d_k can themselves be expanded as polynomials of degree k in N_f:

$$d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \cdots + d_k^{[0]} \,, \tag{5.9}$$

where each term is a sum of multinomials in C_A, C_F and N_f of degree k so that $d_k^{[r]}$ has the structure $C_A^{k-r-s} C_F^s$ (note the prefactor of C_F in equation (5.5)). The first two coefficients d_1 and d_2 have been computed [52] and the result using the $\overline{\rm MS}$ scheme with $\mu^2 = Q^2$, expanded in N_f as in equation (5.9), is

$$\begin{aligned}
d_1 &= \left(-\frac{11}{12} + \frac{2}{3}\zeta_3\right) N_f + C_A \left(\frac{41}{8} - \frac{11}{3}\zeta_3\right) - \frac{1}{8} C_F \\
d_2 &= \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 + C_A \left(-\frac{970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5\right) N_f \\
&\quad + C_F \left(-\frac{29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5\right) N_f + C_A^2 \left(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5\right) \\
&\quad + C_A C_F \left(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5\right) + C_F^2 \left(-\frac{23}{32}\right) \,.
\end{aligned} \tag{5.10}$$

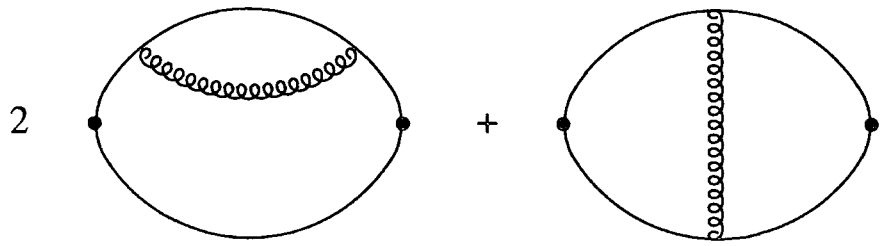


Figure 5.2: Zeroth order contribution to $\tilde{D}$.

Here ζ_p denotes the Riemann zeta function,

$$\zeta_p \equiv \sum_{m=1}^{\infty} m^{-p} . \tag{5.11}$$

Large-N_f expansions as in equation (5.9) have been widely used in the past in the investigation of large-order behaviour and renormalons [43]. Let us sketch how these calculations work. First, let us examine the diagrams which contribute to the large-N_f expansion of $\tilde{D}$. The zeroth coefficient in the expansion of $\tilde{D}$ (that is, the coefficient of a in equation (5.6), which is normalized to 1) is given by the diagrams in Figure 5.2, where a blob stands for a photon line.

The diagrams which contribute to d_1 are shown in Figure 5.3 [53]. In the large-N_f expansion we have that

$$d_1 = d_1^{[1]} N_f + d_1^{[0]} . \tag{5.12}$$

So diagrams A–J in Figure 5.3 contribute to $d_1^{[0]}$ and diagrams K and L, which depend on the number of quark flavours, contribute at leading order in the large-N_f expansion, that is, to $d_1^{[1]}$. In general, when the nth coefficient is considered, the leading contribution from the large-N_f expansion will come from the diagrams in Figure 5.4, since these diagrams contribute $\sim g^{2n+2} N_f^n \sim a^{n+1} N_f^n$.

We should note that these single bubble chains alone contribute the leading large-N_f behaviour. For example, if we consider the behaviour of a two chain diagram (see Figure 5.5) then this will contribute $\sim g^{2k+2} g^{2\ell+2} N_f^{k+\ell} \sim a^{k+\ell+2} N_f^{k+\ell}$, which is $O(1/N_f)$ in the large-N_f expansion and hence subleading. Some doubts have been raised as to the validity of the assumption that the subleading terms make a negligible contribution to the asymptotics [48]. This point will be discussed in more detail below.

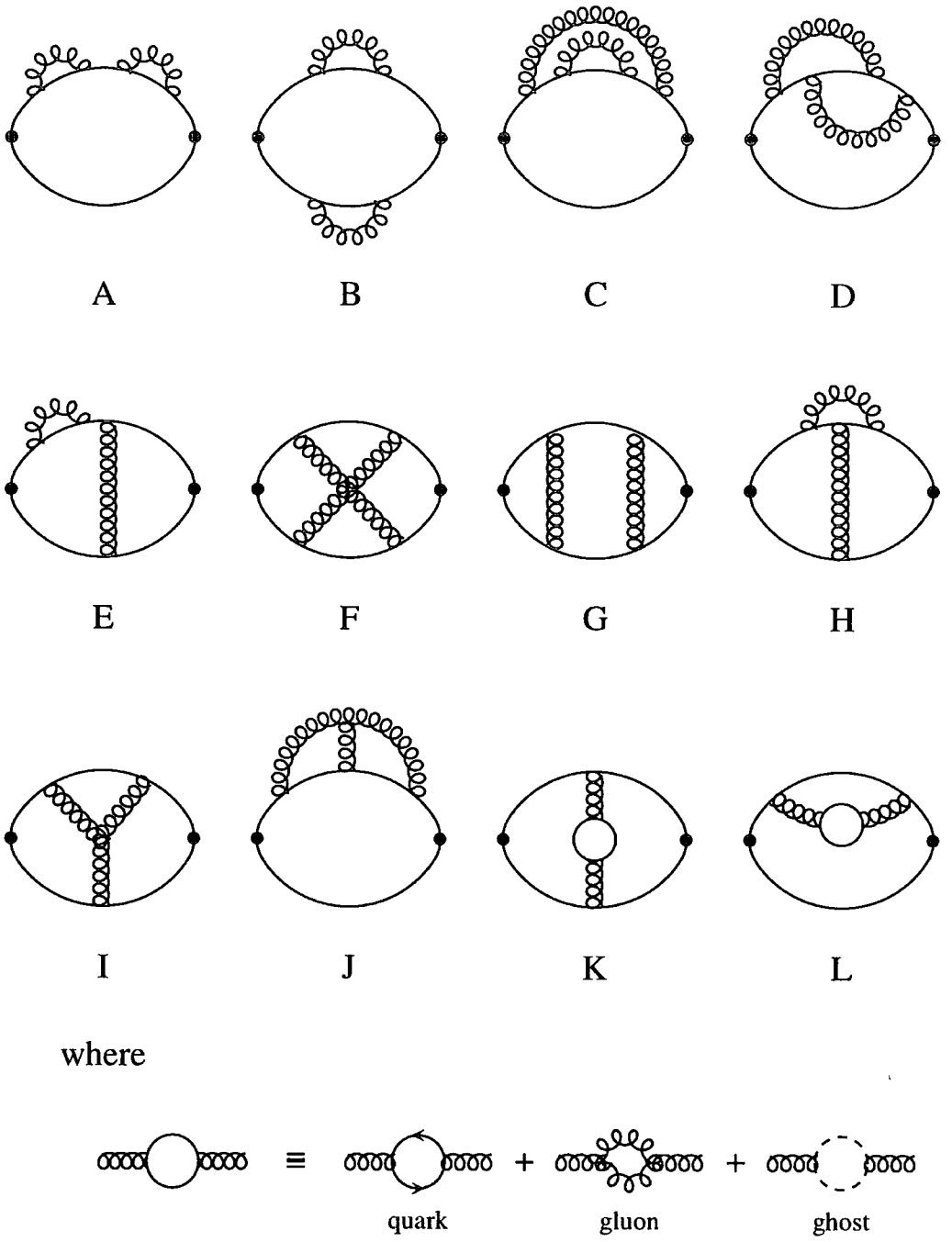


Figure 5.3: The topologically inequivalent diagrams contributing to d_1 (combinatoric weights not shown).

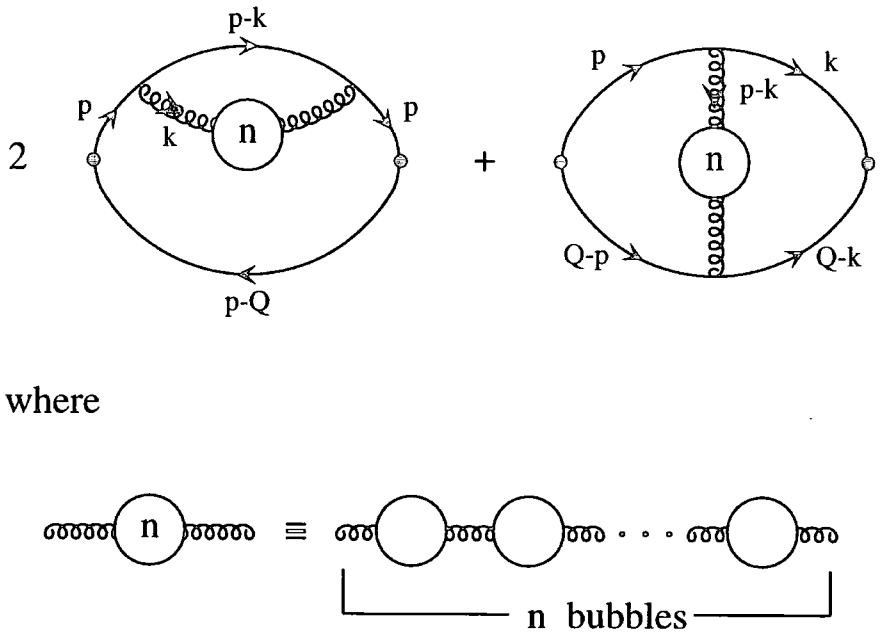


Figure 5.4: Leading large-N_f contribution at nth order in perturbation theory.

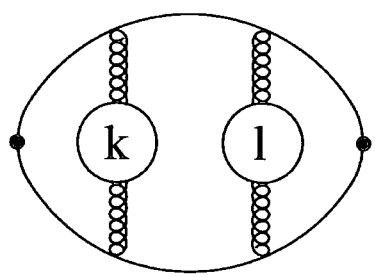


Figure 5.5: A generic two chain diagram.

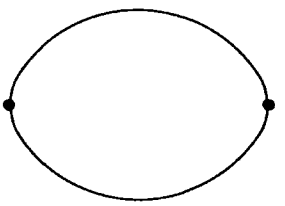
Having identified the diagrams which will be of interest, we now proceed to sketch the calculation of these diagrams, in particular demonstrating how factorial growth occurs. Let us turn our attention again to the diagrams in Figure 5.4. Applying the Feynman rules to these diagrams, one finds [43] that the leading coefficient in the large-N_f expansion, $d_n^{[n]}$, is of the form

$$\begin{aligned} d_n^{[n]} \sim & \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \Bigg\{ B_{\rho\sigma}^{(n)}(p-k) \mathrm{Tr} \left[\gamma_\mu \frac{1}{\not k} \gamma_\rho \frac{1}{\not p} \gamma_\mu \frac{1}{\not p - \not Q} \gamma_\sigma \frac{1}{\not k - \not Q} \right] \\ & + 2 B_{\rho\sigma}^{(n)}(k) \mathrm{Tr} \left[\gamma_\mu \frac{1}{\not p} \gamma_\rho \frac{1}{\not p - \not k} \gamma_\sigma \frac{1}{\not p} \gamma_\mu \frac{1}{\not p - \not Q} \right] \Bigg\} . \end{aligned} \tag{5.13}$$

Here $B_{\rho\sigma}^{(n)}$ is the renormalized sum of n bubbles:

$$B_{\rho\sigma}^{(n)}(k) = \frac{(k_\rho k_\sigma - k^2 g_{\rho\sigma})}{(k^2)^2} \left[-\Pi_0(k^2) \right]^n , \tag{5.14}$$

where $\Pi_0(k^2)$ is the contribution of one bubble,



$$= \left[\ln \frac{k^2}{\mu^2} + C \right] ; \tag{5.15}$$

and C depends on the renormalization scheme. So, for $\overline{\mathrm{MS}}$ $C = -5/3$; and for the V scheme ($\overline{\mathrm{MS}}$ with $\mu = \exp[-5/6]Q$ - see later) $C = 0$.

We can then investigate the behaviour of $d_n^{[n]}$ by counting powers of momenta in the loop integrals. This approach (bearing in mind the problems associated with overlapping divergences) gives

$$\begin{aligned} d_n^{[n]} & \sim \int d^4k (\ln k)^n \, k^{-6} \\ & \sim \int dk (\ln k)^n \, k^{-3} . \end{aligned} \tag{5.16}$$

Then, making the substitution $k = \mathrm{e}^t$, we see that

$$d_n^{[n]} \sim \int dt \, t^n \mathrm{e}^{-2t} \tag{5.17}$$

and, recalling that $\int_0^\infty dx \, x^p \mathrm{e}^{-x} = p!$, we find that

$$d_n^{[n]} \sim n! . \tag{5.18}$$

In practice the loop integrals are calculated using the Gegenbauer polynomial x-space technique [43].

As we shall emphasise, it is actually more useful for our purposes to consider an expansion in powers of b.[1] We can write

$$d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} + \cdots + d_k^{(0)} \ . \tag{5.19}$$

The leading coefficient in the N_f expansion is exactly related to that in the "b expansion" with

$$d_k^{[k]} = (-1/3)^k d_k^{(k)} \ . \tag{5.20}$$

As before, we can write out the known d_1 and d_2 coefficients now expanded according to equation (5.19):

$$\begin{aligned} d_1 &= \left(\frac{11}{4} - 2\zeta_3\right) b + \frac{C_A}{12} - \frac{C_F}{8} \\ d_2 &= \left(\frac{151}{18} - \frac{19}{3}\zeta_3\right) b^2 + C_A \left(\frac{31}{6} - \frac{5}{3}\zeta_3 - \frac{5}{3}\zeta_5\right) b \\ &\quad + C_F \left(\frac{29}{32} - \frac{19}{2}\zeta_3 + 10\zeta_5\right) b + C_A^2 \left(-\frac{799}{288} - \zeta_3\right) \\ &\quad + C_A C_F \left(-\frac{827}{192} + \frac{11}{2}\zeta_3\right) + C_F^2 \left(-\frac{23}{32}\right) \ . \end{aligned} \tag{5.21}$$

We note in passing that the "b expansion" has a somewhat more compact structure in relation to ζ-functions than that in N_f. In particular the ζ_3, present in all orders of the N_f-expansion for d_1, is present only in the leading term in the b-expansion. For d_2 the ζ_5, present in all but the leading term in the N_f-expansion, is now present only in $d_2^{(1)}$.

In both cases the highest ζ-function present cancels and is absent in the 'conformal' $b \to 0$ limit [54]; so $d_1^{(0)}$ does not involve ζ_3 and $d_2^{(0)}$ does not involve ζ_5. Since the $\overline{\rm MS}$ beta-function coefficients do not involve ζ-functions, this is presumably not an artefact of the particular RS chosen but may well be of more fundamental significance. It may ultimately be connected with the fact that in the $b \to 0$ limit the renormalon singularities in the Borel plane move off to infinity, leaving only instanton singularities. As we shall discuss below, the ζ-functions are intimately linked with the presence of renormalon singularities. It is amusing

[1] Although this expansion in b is convenient, we should emphasise that it does not have such a ready interpretation in terms of a class of Feynman diagrams as does the large-N_f expansion.

to notice that in the original NNLO result for d_2 [55], which was subsequently found to be in error [52], $d_2^{(0)}$ does contain a non-vanishing $-\frac{20}{9}\zeta_5$ term for SU(3) QCD. If a fundamental result about the absence of ζ-functions in the conformal limit could be established it would have enabled the incorrect result to have been dismissed at once.

Continuing progress in applying the N_f expansion in QED [56] has led Broadhurst to an elegant generating function for the leading order (large-N_f) coefficients of the QED Gell-Mann–Low function (MOM scheme beta-function) [44]:

$$\Psi_n^{[n]} = \frac{3^{2-n}}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-2} P(x)\Big|_{x=1}, \tag{5.22}$$

where

$$P(x) = \frac{32}{3(1+x)}\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k^2-x^2)^2}. \tag{5.23}$$

$\Psi_n^{[n]}$ can be explicitly evaluated in closed form [44]:

$$\begin{aligned}\frac{\Psi_n^{[n]}}{(n-2)!} &= \frac{(n-1)}{(-3)^{n-1}}\Big[-2n+4-\frac{n+4}{2^n} \\ &\quad +\frac{16}{n-1}\sum_{\frac{n}{2}>s>0} s(1-2^{-2s})(1-2^{2s-n})\zeta_{2s+1}\Big].\end{aligned} \tag{5.24}$$

Using this result one can then obtain the leading-order large-N_f result for the QCD Adler D-function. In the $\overline{\mathrm{MS}}$ scheme with $\mu^2 = Q^2$ one has [57]

$$d_k^{[k]} = 2T_f^k k!\sum_{m=0}^{k}\frac{(-\frac{5}{9})^m}{m!}\frac{\Psi_{k+2-m}^{[k+2-m]}}{(k-m)!}, \tag{5.25}$$

where T_f is a group theory factor; $T_f = 1/2$ for the standard fermion representation. The $(-5/9)^m$ factors enter since one is converting from the MOM scheme Adler function to that in the $\overline{\mathrm{MS}}$ scheme. Again we shall defer to the next chapter a fuller discussion of the RS dependence. The results of equations (5.25) and (5.24) are in agreement with the exactly known coefficients $d_1^{[1]}$ and $d_2^{[2]}$ in equation (5.10).

Our aim is to make use of the exact large-N_f result of equation (5.25) to obtain as much information as possible about the singularity structure of the QCD D-function in the Borel plane; and hence about the large-order behaviour of its perturbative coefficients. To this end we shall begin by reviewing what can be inferred on very general grounds about this structure; and then we shall compare the exact result with these expectations.

5.3 The Structure of the Borel Transform of the Adler D-function

We discussed in some detail in the previous chapter how factorial growth of perturbative coefficients leads to singularities in the Borel transform. Let us now review very briefly some of what we found.

Consider a general quantity F, calculated in perturbative field theory with coupling a,

$$F = (a + f_1 a^2 + f_2 a^3 + \cdots + f_k a^{k+1} + \cdots) . \tag{5.26}$$

Suppose, for example, that $f_k = (-1)^k k!$; then $B[F](z) = 1 - z + z^2 - z^3 + \cdots = 1/(1+z)$, where we have assumed analytic continuation along the whole real line. More generally, if $f_k = (1/z_i)^k k^\gamma k!$ ($\gamma > 0$), then $B[F](z)$ has a singularity proportional to $(z - z_i)^{-\gamma-1}$; so if γ is a positive integer we have a pole; and for non-integer γ a branch point in the z-plane at $z = z_i$. If all the singularities are located off the positive z-axis it may be possible, if certain analyticity properties of F are satisfied, to reconstruct $F(a)$ from its formal divergent series using the Borel sum which we defined in equation (4.23). This, however, is not our interest here. We want to use the z-plane singularities to encode the large-order behaviour of the perturbative coefficients.

In the specific case of the Adler D-function in QCD, let us now ask on rather general grounds where the singularities z_i could be located. In the large-N_f limit we know that, if the d_k have factorial growth, asymptotically we must have $d_k \sim N_f^k k!$. Similarly we can consider a large-N limit where necessarily $d_k \sim N^k k!$. This follows since the k^{th} order Feynman diagrams for d_k necessarily have factors which are multinomials in N_f, C_A and C_F of degree k. We therefore have singularities at positions $z_i \sim 1/N_f$ in the large-N_f limit and $z_i \sim 1/N$ in the large-N limit. If singularities are present and visible in both limits then the simplest possibility is $z_i \sim 1/(AN + BN_f)$ involving some unspecified linear combination of N and N_f. These are the renormalon singularities and in fact they lie at $z_\ell = 2\ell/b$ where $\ell = \pm 1, \pm 2, \pm 3, \ldots$. There are also singularities due to instanton/anti-instanton solutions of the classical equations of motion [34, 38, 45]. These lie at $z_\ell = 4\ell$, $\ell = 1, 2, 3, \ldots$. Since their positions are independent of N and N_f, the above

arguments suggest that they are invisible in the large-N and large-N_f limits. In fact they are invisible at all orders of the N and N_f expansions, so we shall not learn about them from the exact large-N_f result.

In QED, one-loop vacuum polarization diagrams with a chain of vacuum polarization bubbles inserted lead to fixed-sign factorial growth and UV renormalon singularities [34]. These are associated with the Landau pole in QED. For the QCD vacuum polarization function one can also consider one-loop diagrams with a single gluon line inserted. Applying a cut-off on the momentum of this line, inserting the running QCD coupling and integrating over the high-momentum region, Mueller [40] has shown that one can explicitly derive the form of the leading UV renormalon at $z = -2/b$; and, by integrating over momenta less than the cut-off, of the leading IR renormalon at $z = 4/b$.

The final conclusion is that one expects UV renormalon singularities, UV_ℓ, at $z = z_\ell = -2\ell/b$, $\ell = 1, 2, 3, \ldots$; and IR renormalon singularities, IR_ℓ, at $z = z_\ell = 2\ell/b$, $\ell = 1, 2, 3, \ldots$. For the specific case of QCD vacuum polarization one expects IR_1 to be absent since, as discussed in the last chapter, there is no dimension two condensate in the corresponding OPE.

The singularities at $z = z_\ell$ in the Borel plane should be branch points of the form [40]

$$B[\tilde{D}](z) = \frac{A_0 + A_1(1 - z/z_\ell) + O((1 - z/z_\ell)^2)}{(1 - z/z_\ell)^{\gamma + cz_\ell}}\,, \tag{5.27}$$

where, as we have discussed in section 4.5, for IR renormalons,

$$\gamma = p + \frac{\gamma^{(1)}}{b} + \frac{\gamma^{(2)}}{b^2} + \cdots \tag{5.28}$$

is related to the anomalous dimension of the relevant OPE condensate operator associated with the renormalon singularity [39]. For the UV renormalons, as we discussed in section 4.4, the exponent γ is connected with the anomalous dimension of the local operators, which, as Parisi showed [37], are associated with each singularity via the introduction of counterterms which are needed to remove UV divergences. The exponent cz_ℓ is fixed by the renormalization group in order to give the required Q^2 scaling, taking into account a realistic two-term beta-function with $c \neq 0$. Here p, the large-N_f limit of γ, is a pure number; and the $\gamma^{(i)}$'s are multinomials in C_A and C_F of degree i. UV_1 should be the singularity closest to

the origin (assuming that IR_1 is absent) and so will give the overall asymptotic large-order behaviour of the d_k coefficients. Corresponding to a branch point in the Borel plane of the form of equation (5.27), one has the large-order behaviour

$$d_k = \frac{A_0}{\Gamma(\gamma + cz_\ell)} \left(\frac{1}{z_\ell}\right)^k \Gamma(k + \gamma + cz_\ell) \,. \tag{5.29}$$

So for UV_1 we have the leading behaviour

$$d_k \approx \frac{A_0}{\Gamma(\gamma - 2c/b)} \left(-\frac{b}{2}\right)^k k^{\gamma-1} k^{-2c/b} k! (1 + O(1/k)) \,, \tag{5.30}$$

where we have expanded the second Γ-function of equation (5.29) for large k. The A_1 and higher terms in the numerator Taylor series about $z = z_\ell$ in equation (5.27) are $\mathrm{O}(1/k)$ sub-asymptotic effects.

To make contact with the N_f and b expansions of equations (5.9) and (5.19) we note that from equation (5.8)

$$\frac{c}{b} = \left[-\frac{7}{8}\frac{C_A^2}{b^2} + \frac{5}{4}\frac{C_A}{b} - \frac{11}{8}\frac{C_A C_F}{b^2} + \frac{3}{4}\frac{C_F}{b}\right] \,. \tag{5.31}$$

We shall define

$$\frac{A_0}{\Gamma(\gamma - 2c/b)} = A_{01} + \frac{A_{02}}{b} + \frac{A_{03}}{b^2} + \cdots \,, \tag{5.32}$$

where A_{0r} is a sum of multinomials in C_A and C_F of degree $r-1$ and A_{01} is a pure number. Then writing $k^{-2c/b}$ as $e^{-2c \ln k/b}$ and expanding we have

$$d_k \approx \left(A_{01} + \frac{A_{02}}{b} + \cdots\right) \left(-\frac{b}{2}\right)^k k^{p-1} k! \left[1 - \left(\frac{5}{2}C_A + \frac{3}{2}C_F - \gamma^{(1)}\right)\frac{\ln k}{b} + \cdots\right] \,. \tag{5.33}$$

One can then obtain the asymptotic behaviour of the coefficients to all orders in the N_f and b expansions. The b expansion corresponds to the successive terms in equation (5.33). So we have (up to $O(1/k)$ corrections)

$$\begin{aligned} d_k^{(k)} &\approx A_{01} \left(-\frac{1}{2}\right)^k k^{p-1} k! \\ d_k^{(k-1)} &\approx \left(-\frac{1}{2}\right)^k k^{p-1} k! \left[A_{01}\left(-\frac{5}{2}C_A - \frac{3}{2}C_F + \gamma^{(1)}\right) \ln k + A_{02}\right] \\ &\vdots \end{aligned} \tag{5.34}$$

For the N_f expansion asymptotics, the leading terms will come from the binomial expansion of $b^k = (\frac{11C_A - 2N_f}{6})^k$,

$$d_k^{[k-r]} \approx A_{01} \frac{6^{-k}}{r!} \left(-\frac{11}{2}C_A\right)^r k^{p+r-1} k! \tag{5.35}$$

where $r = 0, 1, 2, \ldots$. Note that an additional factor of k is gained for each additional order in the N_f expansion. The leading term in the large-N expansion, $d_k^{[0]}$, is given by

$$d_k^{[0]} \approx \tilde{A}_{01} 12^{-k} (-11 C_A)^k k^{p-1} k^{-102/121} k^{\tilde{\gamma}} k! \,, \tag{5.36}$$

where $\tilde{A}_{01}$ is the large-N limit of equation (5.32) and $\tilde{\gamma}$ that of the anomalous dimension of equation (5.28). Of course $\tilde{A}_{01}$ cannot be determined from the large-N_f result. The $\gamma^{(i)}$ coefficients for UV_1 of the D-function are unknown but for IR_2 it is known that the $\gamma^{(i)} = 0$ (a vanishing anomalous dimension with our conventions) [40]. A_{02} is also unknown; so the $d_k^{(k-r)}$ asymptotics are not obtainable for the D-function.

However, assuming a UV_1 singularity as in equation (5.27), we apparently have the remarkable conclusion that, given an exact large-N_f result for $d_k^{[k]}$, we can obtain A_{01} and p; and hence the asymptotics (up to an $\mathrm{O}(1/k)$ correction) of the coefficients to *all orders* in the N_f expansion are determined by equation (5.35). We shall check in due course that the exact large-N_f result of equation (5.25) gives $A_{01} = 4/9$ and $p = 2$ (V scheme) in agreement with other evaluations [40, 43, 48].

The conclusion that the large-N_f result can determine the full asymptotics beyond leading order in N_f seems too good to be true; and indeed recent work by Vainshtein and Zakharov [48] casts doubt on it. These authors have systematically developed a "UV renormalon calculus" in which the leading ultra-violet behaviour of loop diagrams with different numbers of chains of vacuum polarization graphs inserted is extracted using an operator product expansion. Including one chain they reproduce a UV_1 result of the form of equation (5.27) in agreement with the result quoted above. Explicitly evaluating the two-chain (three-loop) result in a simplified U(1) model they find a contribution to the UV renormalon asymptotics which is $1/N_f$ down on the one-chain result; but has additional powers of k and so dominates the one-chain result. Conventional wisdom would have expected additional chains to have a suppression by powers of k and hence not to contribute to the A_0 coefficient in equation (5.27). Combinatoric factors conspire to modify this, however.

The UV renormalon calculus results [48] imply that the Borel plane singularity

at $z = -2/b$ in equation (5.27) should be modified to

$$B[\tilde{D}](z) = \sum_{m=1}^{\infty} \frac{A_0^{(m)} + A_1^{(m)}(1 + bz/2) + O((1 + bz/2)^2)}{(1 + bz/2)^{\gamma_m - 2c/b}} \tag{5.37}$$

where m labels the number of vacuum polarization chains, $m = 1$ corresponding to the previous one-chain results. Counting powers of N_f, one expects each additional chain to give a $1/N_f$ suppression and hence equation (5.32) should generalise to

$$\frac{A_0^{(m)}}{\Gamma(\gamma_m - 2c/b)} = \frac{A_{01}^{(m)}}{b^{m-1}} + \frac{A_{02}^{(m)}}{b^m} + \cdots \tag{5.38}$$

The exponent γ_m will be related to the anomalous dimensions of the operators appearing in the operator product expansion of reference [48]. There will be a number of operators and hence a number of contributions to equation (5.37) for any m. Each such γ_m will have an N_f, b expansion and we can define the leading term p_m. We select for each m the γ_m contribution with largest p_m; and this is the single term displayed in equation (5.37). Providing that $p_{m+1} > p_m + 1$ for any m, then equations (5.37) and (5.38) lead to the N_f, b expansion asymptotics

$$\begin{aligned} d_k^{[k-r]} &\approx A_{01}^{(r+1)} \frac{(-3)^r}{6^k} k^{p_{r+1}-1} k! \\ d_k^{(k-r)} &\approx A_{01}^{(r+1)} \left(-\frac{1}{2}\right)^k k^{p_{r+1}-1} k! \end{aligned} \tag{5.39}$$

with $r = 0, 1, 2, \ldots$. The $A_{01}^{(r+1)}$ will consist of multinomials in C_A, C_F of degree r. The inequalities on p_m are required since, from equation (5.35), we see that, at fixed m, one gains an extra factor of k for each additional $1/N_f$ order. The results of equation (5.39) imply that one does not get something for nothing after all; but requires $O((1/N_f)^r)$ results (diagrams with $r + 1$ chains) to obtain the leading asymptotics to this order in the N_f expansion.

Having discussed the general expectations for the Borel plane singularity structure we return to the exact large-N_f result of equation (5.25) and exhibit what its singularity structure actually is.

5.4 Singularity Structure of the Exact Large-N_f Result

Returning to equation (5.22) we can use Taylor's theorem to write

$$\Psi_n^{[n]} = \frac{3^{2-n}}{2}(n-2)!\,\mathbb{C}_{n-2}\tilde{P}(u)\,, \tag{5.40}$$

where $\mathbb{C}_n F(x)$ denotes the coefficient of x^n in the expansion of $F(x)$ as a power series. Here

$$\tilde{P}(u) = \frac{32}{3(2+u)}\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k^2-(1+u)^2)^2}\,; \tag{5.41}$$

this is obtained from equation (5.23) by writing $u = x-1$. Using partial fractions on equation (5.41) and isolating the required coefficient one finds the rather simple result

$$\begin{aligned}\frac{\Psi_n^{[n]}}{(n-2)!} &= 12\sum_{\ell=1}^{\infty}(-1)^{\ell+1}\Big\{n\Big(\frac{1}{\ell+1}-\frac{1}{\ell+2}\Big)+\Big(-\frac{1}{(\ell+1)^2}+\frac{2}{(\ell+2)^2}\Big)\Big\}\Big(\frac{1}{3\ell}\Big)^n \\ &\quad +12\sum_{\ell=3}^{\infty}(-1)^{\ell+1}\Big\{n\Big(\frac{1}{\ell-1}-\frac{1}{\ell-2}\Big)+\Big(\frac{1}{(\ell-1)^2}-\frac{2}{(\ell-2)^2}\Big)\Big\}\Big(\frac{-1}{3\ell}\Big)^n \\ &\quad -\frac{3}{(-6)^{n-1}}\,.\end{aligned} \tag{5.42}$$

Putting in factors of $T_f^n N_f^n = (1/2)^n N_f^n$ and replacing N_f by $(-3b)$, we see that successive terms in equation (5.42) are proportional to $(-b/2\ell)^n$, $(b/2\ell)^n$ and $(b/4)^n$ corresponding exactly to the poles UV_ℓ, IR_ℓ and IR_2 respectively in the Borel plane. There is no term involving $(1/(-3))^n$ and so IR_1 is indeed absent as anticipated. The linear factor proportional to n in all but IR_2 means that all the poles are double poles ($p=2$) except for IR_2 which is a simple pole ($p=1$). Notice that the branch point exponent cz_ℓ is sub-leading in the N_f, b expansion and so one sees poles and not branch points in the large-N_f limit.

We can then obtain the leading-b result for d_k. Rather than using the $\overline{\mathrm{MS}}$ scheme with $\mu = Q$ we find that considerable simplification in the form of the asymptotic behaviour results from choosing the so-called V scheme [58], $\overline{\mathrm{MS}}$ with $\mu = \mathrm{e}^{-5/6}Q$, which is directly related to the QED MOM scheme. With this choice of scheme equation (5.25) becomes

$$d_k^{[k]} = 2T_f^k k!\Psi_{k+2}^{[k+2]} \tag{5.43}$$

and hence

$$d_k^{(k)} = 2\left(-\frac{3}{2}\right)^k k!\Psi_{k+2}^{[k+2]} . \tag{5.44}$$

Then, from equation (5.42), one obtains directly

$$\begin{aligned} d_k^{(k)} b^k &= k!\Big\{ \sum_{\ell=1}^{\infty}[A_0(\ell)(k+1) + A_1(\ell)]\Big(-\frac{b}{2\ell}\Big)^k \\ &\quad + \sum_{\ell=1}^{\infty}[B_0(\ell)(k+1) + B_1(\ell)]\Big(\frac{b}{2\ell}\Big)^k\Big\} , \end{aligned} \tag{5.45}$$

where

$$\begin{aligned} A_0(\ell)(\ell \geq 1) &= \frac{8}{3}\frac{1}{\ell^2}\left(\frac{1}{\ell+1} - \frac{1}{\ell+2}\right)(-1)^{\ell+1} ; \\ A_1(\ell)(\ell \geq 1) &= \frac{8}{3}\frac{1}{\ell^2}\Big[\left(\frac{1}{\ell+1} - \frac{1}{\ell+2}\right) + \left(-\frac{1}{(\ell+1)^2} + \frac{2}{(\ell+2)^2}\right)\Big](-1)^{\ell+1} ; \\ B_0(1) &= B_1(1) = 0; \\ B_0(2) &= 0;\ B_1(2) = 1 ; \\ B_0(\ell) &= -A_0(-\ell);\ B_1(\ell) = -A_1(-\ell)\ (\ell \geq 3) . \end{aligned} \tag{5.46}$$

The first sum in equation (5.45) generates UV_ℓ singularities of the form of equation (5.27) and the second sum the IR_ℓ singularities. IR_1 is absent ($B_0(1) = B_1(1) = 0$) as required from the OPE; and all poles are double ($p = 2$) except IR_2 for which $B_0(2) = 0$ giving a simple pole ($p = 1$). For $\ell \geq 3$ there is a curious and unexplained symmetry between the residues of UV_ℓ and IR_ℓ with $A(\ell) = -B(-\ell)$. As promised the coefficient $A_0(1) = \frac{4}{9}$ is in agreement with other calculations of UV_1 [43, 48]. The coefficient $B_1(2) = 1$ is consistent with the result of [40] for the first IR renormalon, IR_2.

Since equation (5.42) is split into contributions from UV and IR renormalons we can sum up these contributions separately. Performing the first (UV) sum in equation (5.42) one obtains

$$\begin{aligned} \frac{\Psi_n^{[n]}}{(n-2)!}(\mathrm{UV}) &= -\frac{2(\zeta_2 - 2)}{(-3)^{n-1}} - \frac{2\zeta_2 - 3}{(-6)^{n-1}} \\ &\quad + \frac{4}{(-3)^{n-1}}\sum_{m=1}^{n-1}(-1)^m m(1-2^{-m})(1-2^{m-n})\zeta_{m+1}; \end{aligned} \tag{5.47}$$

and summing the final two IR terms gives

$$\frac{\Psi_n^{[n]}}{(n-2)!}(\mathrm{IR}) = -\frac{2(n^2 - 3n + 4 - \zeta_2)}{(-3)^{n-1}} - \frac{\frac{1}{2}(n^2 + 3n + 2 - 4\zeta_2)}{(-6)^{n-1}}$$

$$+\frac{4}{(-3)^{n-1}}\sum_{m=1}^{n-1} m(1-2^{-m})(1-2^{m-n})\zeta_{m+1}. \qquad (5.48)$$

The UV and IR pieces separately contain even and odd ζ-functions but the even ζ-functions cancel in the sum of equations (5.47) and (5.48) to reproduce equation (5.24) which contains only odd ζ-functions.

We finally discuss the connection between the Borel plane singularity structure of $\tilde{D}$, which we have discussed extensively, and that of the more experimentally-relevant $\tilde{R}$, related to it by the equation (5.4). Brown and Yaffe have shown that in the large-b limit ($c = 0$) [47]

$$B[\tilde{R}](z) = \frac{\sin(\pi bz/2)}{\pi bz/2} B[\tilde{D}](z)\,. \qquad (5.49)$$

For details of this derivation see Appendix A. Since $\sin(\pi bz/2)$ has single zeros $\sim (z - z_\ell)$ at the same positions as the renormalon singularities one finds that renormalon poles of order p in $B[\tilde{D}]$ are converted to poles of order $p-1$ in $B[\tilde{R}]$. This implies that the poles in $B[\tilde{R}]$ are simple poles except for IR_2 which was a simple pole in $B[\tilde{D}]$ and hence apparently vanishes [43]. The absence of the IR_1 singularity at $z = 2/b$ in $B[\tilde{D}](z)$ implies from equation (5.49) that $B[\tilde{R}](z)$ must have a compensating zero at this position. Brown and Yaffe [47] considered this unlikely and hence cast doubt on the absence of IR_1. The exact large-N_f result shows that there is indeed no IR_1 singularity in $B[\tilde{D}]$ and hence such a zero is present in $B[\tilde{R}]$.

The leading asymptotics of the coefficients r_k will be given by UV_1 for $B[\tilde{R}](z)$. Expanding around $z = -2/b$ we have

$$\frac{\sin(\pi bz/2)}{\pi bz/2} = \left(1+\frac{bz}{2}\right) + O\left(\left(1+\frac{bz}{2}\right)^2\right) \qquad (5.50)$$

and hence from equation (5.49) we find that the asymptotic behaviour of r_k is given by changing $p \to p-1$ in the results for d_k (equations (5.34) and (5.35)). Even assuming the more complicated UV_1 structure of equation (5.37) one simply changes $p_m \to p_m - 1$. This implies that on very general grounds one expects

$$\frac{r_k}{d_k} \approx \frac{1}{k}(1 + O(1/k)) \qquad (5.51)$$

so that the r_k coefficients grow more slowly asymptotically.

We conclude this section by noting that exact large-N_f results also exist for other QCD observables. In particular, reference [57] contains a leading-N_f result for the perturbative coefficients of the radiative corrections to the Gross–Llewellyn-Smith (GLS) sum rule. This reveals that in the Borel plane the GLS corrections have simple poles at $z = \pm 2/b, \pm 4/b$. So the first two UV and IR renormalons are present but the remaining renormalons are absent to leading order in N_f. We will discuss the GLS sum rule (and also the tau-decay ratio) in much greater detail in the next chapter where we explore the possibilities of constructing all-orders resummations from our knowledge of the Borel transforms of these quantities.

5.5 Vanishing IR Renormalons

In this section, as a footnote to our previous, phenomenologically-orientated discussion, we investigate some of the theoretically-interesting features of the Borel plane singularity structure of the Adler D-function.

The singularity structure in the Borel plane for SU(N) QCD will change as N and N_f are varied. In particular, if N_f approaches $11N/2$ from below then $b = (11N - 2N_f)/6$ will approach zero from above; and, as more flavours of quark are added, the IR_ℓ and UV_ℓ singularities, which are spaced at intervals of $2/b$, will move outwards away from the origin in the z-plane. The instanton/anti-instanton ($\mathrm{I\bar{I}}$) singularities remain fixed at $z = 4, 8, 12, \ldots$ independent of N and N_f. When $b = 1/2$ ($N_f = 15$ for SU(3)), IR_1 will be at the same position as the leading $\mathrm{I\bar{I}}$ singularity at $z = 4$; and for flavour saturation (maximum N_f for which $b > 0$, $N_f = 16$ for SU(3)) $b = 1/6$ and the leading singularity on the positive axis will be the $\mathrm{I\bar{I}}$ at $z = 4$.

For SU(3) QCD a remarkable phenomenon first occurs at $b = 1/2$ ($N_f = 15$). One finds that the RG-predictable part of the branch point exponent of IR_ℓ, $2c\ell/b$ in equation (5.27), becomes a negative integer and the structure of IR_ℓ in the Borel plane is then (neglecting the anomalous dimension part)

$$B[\tilde{D}](z) = A_0\Big(1 - \frac{z}{4\ell}\Big)^{88\ell - p}. \tag{5.52}$$

The IR_ℓ singularity disappears provided that $p < 88\ell$. For the particular case of

the D-function we have apparently $p = 1$ or 2 and so all of the IR renormalons disappear at $b = 1/2$ and the only singularities on the positive z-axis are those due to instantons. Notice that the UV renormalons are still present but become poles.

As first noted by White [49, 50] the IR renormalon singularities also disappear for $N_f = 16$ flavour-saturated SU(3). The exponent $2c\ell/b$ again becomes a negative integer and the IR_ℓ structure in the Borel plane is

$$B[\tilde{D}](z) = A_0\Big(1 - \frac{z}{12\ell}\Big)^{906\ell - p} . \tag{5.53}$$

So for $p < 906\ell$ the IR_ℓ singularities disappear.

Of course, the disappearance of the IR renormalons depends on suitable behaviour of the anomalous dimension piece of the branch point exponent as well. For the D-function, as mentioned earlier, we know that there is a vanishing anomalous dimension for IR_2 and hence we can conclude that, for the D-function, this singularity does disappear for $N_f = 15$ and $N_f = 16$ SU(3). The status of the other IR_ℓ singularities is, however, an open question as is the situation for other physical quantities.

The implication is that $N_f = 15$ and 16 SU(3) QCD may be very special instanton-dominated theories. At precisely the point when instantons become the leading singularities all the other IR renormalon singularities on the positive z-axis vanish. It is interesting to ask if this scenario is unique to SU(3) or can be realised for other values of N. $b = 1/2$ will occur for integer N_f only for N odd. For N even the $\mathrm{I}\bar{\mathrm{I}}$ singularity becoming leading and flavour saturation are telescoped into the single value $b = 1/3$.

Considering N odd first, we require that c/b is integer for $b = 1/2$;

$$\Big(\frac{c}{b}\Big)\Big|_{b=\frac{1}{2}} = \frac{(-25N^3 + 13N^2 + 11N - 3)}{4N} . \tag{5.54}$$

A necessary condition for this to be an integer is that $3 \equiv 0 \pmod{N}$, which uniquely fixes $N = 3$. Then $(c/b)|_{b=\frac{1}{2}} = -44$.

For N odd and flavour saturation $b = 1/6$;

$$\Big(\frac{c}{b}\Big)\Big|_{b=\frac{1}{6}} = \frac{(-225N^3 + 39N^2 + 99N - 9)}{4N} . \tag{5.55}$$

A necessary condition for this to be an integer is that $9 \equiv 0 \pmod{N}$ so $N = 3$ or 9. The $N = 9$ case gives $(c/b)|_{b=\frac{1}{6}} = -4444$; and, for $N = 3$, $(c/b)|_{b=\frac{1}{6}} = -453$.

For N even and $b = 1/3$

$$\left(\frac{c}{b}\right)\Big|_{b=\frac{1}{3}} = \frac{(-225N^3 + 78N^2 + 99N - 18)}{4N} . \tag{5.56}$$

A necessary condition for this to be an integer is that $18 \equiv 0 \pmod{N}$ and so $N = 2, 6$ or 18. $N = 6$ is the only case for which c/b is an integer and then $(c/b)|_{b=\frac{1}{3}} = -1884$.

So for flavour-saturated SU(N) IR renormalons are (possibly) absent only for $N = 3, 6$ or 9. For SU(9) the $b = 1/2$ case where the $I\bar{I}$ singularity becomes leading still has IR renormalons.

We conclude that SU(3) QCD is a very special theory from yet another point of view.

5.6 Summary

In this chapter we have discussed and sought to extend our knowledge of the Borel plane singularity structure of the Adler D-function (QCD vacuum polarization). This singularity structure succinctly encodes the large-order asymptotic behaviour of the perturbation theory coefficients.

We pointed out that an expansion of the perturbative coefficients in powers of b, the first QCD beta-function coefficient, rather than in N_f, was natural when comparing with QCD renormalon expectations. We further noted that, if the leading UV renormalon indeed has the expected structure of a simple branch point, then knowledge of the perturbative coefficients to leading order in N_f allows the large-order behaviour to all-orders in N_f to be inferred, equation (5.35). This seems unlikely and a more complicated structure was proposed, equation (5.37), which is consistent with the UV renormalon calculus results of Vainshtein and Zakharov [48].

Using the exact large-N_f result for the D-function of reference [57] we exhibited the explicit singularity structure in the z-plane and found the expected UV and IR renormalon singularities. They appear as poles in the large-N_f limit. In particular, the first IR renormalon, which would correspond to Q^{-2} behaviour not

present in the OPE, is absent. We gave explicit expressions for the residues at all of these poles (equations (5.46)) and unexpectedly found those for the third and higher UV and IR renormalons to be symmetrically related. We were also able to sum up separately the UV and IR renormalon contributions in closed form (equations (5.47) and (5.48)) and obtained expressions containing even and odd ζ-functions. When UV and IR contributions are combined the even ζ-functions cancel.

We finally noted that in flavour-saturated SU(N) QCD, considering only the RG-predictable part of the branch point exponents, the IR renormalons are absent for $N = 3, 6$ and 9. These theories then have ambiguities dominated by instantons. For SU(3) the IR renormalons first disappear when $N_f = 15$ ($b = 1/2$), at which point the $I\bar{I}$ singularity is in the same position as IR_1 and becomes leading.

Chapter 6

All-Orders Resummations for some QCD Observables

6.1 Introduction

In the last chapter we discussed at length the singularity structure in the Borel plane of the Adler D-function, which is closely related to the QCD vacuum polarization. In this chapter we explore the possibility of resumming to all orders the part of perturbative corrections contributed by QCD renormalons, a topic explored in several recent papers [59, 60, 61]. Let us first recap quickly on some of the points raised in the last chapter.

As we saw in the previous chapter, the QCD perturbative corrections to some generic QCD Green's function or current correlator (to the Adler D-function of QCD vacuum polarization for instance) can be written:

$$D = a + d_1 a^2 + d_2 a^3 + \cdots + d_k a^{k+1} + \cdots \; ; \tag{6.1}$$

and the perturbative coefficients d_k can themselves be written as polynomials of degree k in the number of quark flavours, N_f; again we shall assume massless quarks:

$$d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \cdots + d_k^{[0]} \; . \tag{6.2}$$

The N_f-expansion coefficients, $d_k^{[k-r]}$, will consist of sums of multinomials in the adjoint and fundamental Casimirs, C_A=N, C_F=$(N^2$–$1)/2N$, of SU(N) QCD; and will have the structure $C_A^{k-r-s} C_F^s$. The terms in this N_f-expansion will correspond to Feynman diagrams with differing numbers of vacuum polarization loops. By explicit evaluation of diagrams with chains of such loops inserted, it has been

possible to obtain the leading $d_k^{[k]}$ coefficient exactly to all orders for the Adler D-function [43, 44, 57] (and hence its Minkowski continuations, the e^+e^- QCD R-ratio and the τ-decay ratio, R_τ); the Gross Llewellyn-Smith (GLS) sum rule corrections [57]; and heavy quark decay widths and pole masses [62]. A general procedure enabling $d_k^{[k]}$ to be obtained from knowledge of the one-loop correction with a fictitious gluon mass has been developed [59, 60].

Furthermore, we pointed out that the large order behaviour of perturbative coefficients is most transparently discussed in terms of an expansion of the perturbative coefficients in powers of $b=(11C_A-2N_f)/6$, the first QCD beta-function coefficient:

$$d_k = d_k^{(k)}b^k + d_k^{(k-1)}b^{k-1} + \cdots + d_k^{(0)} \; . \tag{6.3}$$

This b-expansion is uniquely obtained by substituting $N_f=(\frac{11}{2}C_A - 3b)$ in equation (6.2). $d_k^{[k]}=(-1/3)^k d_k^{(k)}$ and so exact knowledge of the leading-N_f $d_k^{[k]}$ to all orders implies exact knowledge of the $d_k^{(k)}$.

In QCD one expects the large-order growth of perturbative coefficients to be driven by Borel plane singularities at $z=z_\ell=2\ell/b$, with $\ell=\pm1,\pm2,\pm3,\ldots$. The singularities on the negative real axis are the ultraviolet renormalons, UV_ℓ, and those on the positive real axis are the infrared renormalons, IR_ℓ. These singularities result in the large-order behaviour of the coefficients $d_k \sim b^k k!$. Indeed we showed that, given a set of renormalon singularities at the expected positions in the Borel plane, the leading terms in the b-expansion, $d_k^{(k)}b^k$, should, if expanded in powers of N_f, asymptotically reproduce the $d_k^{[k-r]}$ coefficients of equation (6.2) up to $O(1/k)$ accuracy. We conversely checked that the exact $d_k^{(k)}b^k$ results corresponded to a set of renormalon singularities at the expected positions. We shall demonstrate in section 6.2 that, for the Adler D-function and the GLS sum rule, the N_f-expansion coefficients obtained by expanding $d_1^{(1)}b$ and $d_2^{(2)}b^2$ are in good (10–20% level) agreement with those of the exact $O(a^3)$ next-to-next-to-leading order (NNLO) perturbative calculations for these quantities, so that the anticipated asymptotic dominance of the leading-b term is already apparent in low orders.

Given the dominance of the leading-b terms, an obvious proposal is to sum

them to all orders. That is to split D into two components:

$$D = D^{(L)} + D^{(NL)} , \tag{6.4}$$

where 'L' and 'NL' superscripts refer to leading and non-leading terms in the b-expansion.

$$D^{(L)} \equiv \sum_{k=0}^{\infty} d_k^{(k)} b^k a^{k+1} \tag{6.5}$$

and

$$D^{(NL)} \equiv \sum_{k=1}^{\infty} a^{k+1} \sum_{\ell=0}^{k-1} d_k^{(\ell)} b^\ell . \tag{6.6}$$

The summation of terms can be achieved by using the Borel sum. The Borel integral can itself be split into two components and is well defined for the UV_ℓ singularities on the negative axis, which contribute poles to the Borel transform of $D^{(L)}$. The integral can be performed explicitly in terms of exponential integral functions and other elementary functions. The piece of the Borel integral for $D^{(L)}$ involving the IR_ℓ singularities on the positive real axis is formally divergent; but a principal value or other prescription can be used to go around the poles. The specification of this prescription is intimately linked to the procedure needed to combine the non-perturbative vacuum condensates in the operator product expansion (OPE) with the perturbation theory in order to arrive at a well-defined result for D [39].

In recent papers by Neubert [61] and by Ball, Beneke and Braun [59, 60] a summation of the leading-b terms has also been considered. In these papers it has been motivated as a generalisation of the BLM scale fixing prescription [58] and termed "naïve non-abelianization" [63]. The Neubert procedure uses weighted integrals over a running coupling. For the Euclidean Adler D-function this representation is equivalent to splitting the Borel integral into ultraviolet renormalon and infrared renormalon singularities and principal value regulating the latter. When one continues to Minkowski space to obtain the e^+e^- R-ratio and the τ-decay ratio, R_τ, there are several inequivalent ways to perform the continuation of the running coupling representation; and hence apparent additional non-perturbative ambiguities are claimed. In reference [60] the resummation is defined by using the principal value regulated Borel integral, as we shall do. They concentrate on

R_τ and heavy quark pole masses. We agree with reference [60] that only a consideration of the singularities in the Borel integral provides a satisfactory way of combining perturbative effects with non-perturbative condensates, along the lines discussed in reference [39]; and that the extra uncertainties claimed in reference [61] are spurious.

Our intention in this chapter is to focus on the Adler D-function, the e^+e^- R-ratio, R_τ and the GLS sum rule (the latter was not considered in references [59, 60, 61]). For all of these quantities there exist exact NNLO fixed order perturbative calculations and our interest is in comparing the leading-b resummation with these exact fixed order results. The large-b results provide partial information about the Borel transform and the question is how this can best be utilised. We discuss the renormalization scheme (RS) dependence of the split between $D^{(L)}$ and $D^{(NL)}$ in equation (6.4), the relative contribution of $D^{(L)}$ being RS-dependent. This RS uncertainty needs to be kept in mind and is carefully discussed.

6.2 The b-expansions for $\tilde{D}$ and $\tilde{K}$

We begin by defining some Deep Inelastic Scattering sum rules. First let us consider the polarized Bjorken sum rule (PBjSR):

$$\begin{aligned} K_{PBj} &\equiv \int_0^1 g_1^{\text{ep-en}}(x,Q^2)\mathrm{d}x \\ &= \frac{1}{3}\left|\frac{g_A}{g_V}\right|\left(1-\frac{3}{4}C_F\tilde{K}\right) . \end{aligned} \tag{6.7}$$

Here $\tilde{K}$ denotes the perturbative corrections to the zeroth order parton model sum rule,

$$\tilde{K} = a + K_1 a^2 + K_2 a^3 + \cdots + K_k a^{k+1} + \cdots . \tag{6.8}$$

We can also consider the GLS sum rule,

$$\begin{aligned} K_{GLS} &\equiv \frac{1}{6}\int_0^1 F_3^{\bar{\nu}\mathrm{p}+\nu\mathrm{p}}(x,Q^2)\mathrm{d}x \\ &= \left(1-\frac{3}{4}C_F\tilde{K}+\tilde{\tilde{K}}\right) . \end{aligned} \tag{6.9}$$

The perturbative corrections, $\tilde{K}$, are the same as for the PBjSR; but there are additional corrections of "light-by-light" type, $\tilde{\tilde{K}}$, analogous to $\tilde{\tilde{D}}$ of equation (5.5).

These will similarly enter at $O(a^3)$ and be subleading in N_f and we shall assume once again that they are small.

For both $\tilde{D}$ and $\tilde{K}$ the first two perturbative coefficients, d_1, d_2 and K_1, K_2, are known from the exact perturbative calculations [52, 53, 63, 64, 65]. We shall assume $\overline{\rm MS}$ renormalization with renormalization scale μ=Q for the present; but will later discuss RS dependence more generally.

We gave the exact large-N_f results for d_1 and d_2 in the last chapter (equations (5.10)). For later comparisons it will be useful to write these results numerically for SU(N) QCD:

$$d_1 = -.115N_f + \left(.655N + \frac{.063}{N}\right) , \tag{6.10}$$

$$d_2 = .086N_f^2 + N_f\left(-1.40N - \frac{.024}{N}\right) + \left(2.10N^2 - .661 - \frac{.180}{N^2}\right) . \tag{6.11}$$

The corresponding results for the N_f-expansion for the deep inelastic scattering sum rules are:

$$K_1 = -\frac{1}{3}N_f + \left(\frac{23}{12}C_A - \frac{7}{8}C_F\right) , \tag{6.12}$$

$$\begin{aligned} K_2 = {} & N_f^2\left(\frac{115}{648}\right) + N_f\left(-\frac{3535}{1296} - \frac{\zeta_3}{2} + \frac{5}{9}\zeta_5\right)C_A + N_f\left(\frac{133}{864} + \frac{5}{18}\zeta_3\right)C_F \\ & + C_A^2\left(\frac{5437}{648} - \frac{55}{18}\zeta_5\right) + C_AC_F\left(-\frac{1241}{432} + \frac{11}{9}\zeta_3\right) + C_F^2\left(\frac{1}{32}\right) . \end{aligned} \tag{6.13}$$

In numerical form, for SU(N) QCD,

$$K_1 = -.333N_f + \left(1.48N + \frac{.438}{N}\right) , \tag{6.14}$$

$$K_2 = .177N_f^2 + N_f\left(-2.51N - \frac{.244}{N}\right) + \left(4.53N^2 + .686 + \frac{.008}{N^2}\right) . \tag{6.15}$$

Expanding in powers of b as in equation (6.3),

$$K_1 = b + \left(\frac{C_A}{12} - \frac{7}{8}C_F\right) , \tag{6.16}$$

$$\begin{aligned} K_2 = {} & b^2\left(\frac{115}{72}\right) + b\left(\frac{335}{144} + \frac{3}{2}\zeta_3 - \frac{15}{9}\zeta_5\right)C_A + b\left(-\frac{133}{288} - \frac{5}{6}\zeta_3\right)C_F \\ & + C_A^2\left(-\frac{179}{144} - \frac{11}{4}\zeta_3\right) + C_AC_F\left(-\frac{389}{192} + \frac{11}{4}\zeta_3\right) + C_F^2\left(\frac{1}{32}\right) . \end{aligned} \tag{6.17}$$

Again we see that $K_2^{(0)}$ does not contain ζ_5 as also noted for $d_2^{(0)}$.

We now wish to demonstrate that the leading term in the b-expansion, when expanded in N_f, approximates the N_f-expansion coefficients well, even in rather low orders.

For d_1 and d_2 we have

$$d_1^{(1)}b = .345b = -.115N_f + .634N \ , \tag{6.18}$$

$$d_2^{(2)}b^2 = .776b^2 = .086N_f^2 - .948N_fN + 2.61N^2 \ . \tag{6.19}$$

The subleading, N, N_fN and N^2, coefficients approximate well in sign and magnitude those in the exact expressions in equations (6.10) and (6.11). The leading, N_f and N_f^2, coefficients of course agree exactly.

For K_1 and K_2 we have

$$K_1^{(1)}b = b = -.333N_f + 1.83N \ , \tag{6.20}$$

$$K_2^{(2)}b^2 = 1.59b^2 = .177N_f^2 - 1.95N_fN + 5.37N^2 \ . \tag{6.21}$$

The agreement with the exact N, N_fN and N^2 coefficients in equations (6.14) and (6.15) is again rather good.

In fairness it should be noted that, whilst the leading-b term reproduces the sub-leading coefficients in the N_f-expansion at the $\sim 20\%$ level, there are significant cancellations between large terms and as a result the overall NNLO perturbative coefficients for $\tilde{D}$ and $\tilde{K}$ are significantly overestimated by the leading-b term. For N_f=3 and SU(3) QCD one has the exact ($\overline{\rm MS}$, μ=Q) coefficient d_2=6.37, to be compared with the leading-b term $d_2^{(2)}b^2$=$0.776b^2$=15.7; and an exact coefficient K_2=20.2, to be compared with $K_2^{(2)}b^2$=$1.60b^2$=32.4. In each case the leading-b piece is a factor ~ 2 larger than the exact coefficient.

Notice that the level of accuracy with which the sub-leading coefficients are reproduced is far in excess of that to be anticipated from the asymptotic expectation of equation (5.35). This is a rather weak statement which implies only that $d_k^{[k-r]}$ should be reproduced to $O(1/k)$ accuracy for *fixed* r and *large* k on expanding $d_k^{(k)}b^k$; whereas the $d_k^{[0]}$ ($r = k$) leading-N term is reproduced remarkably accurately already for $k = 1$ and $k = 2$.

This clearly suggests that there is some far more powerful effect at work which guarantees that the leading-b term reproduces large-N_f *and* large-N coefficients simultaneously. A clue as to how this might operate comes from noting that we could formulate a second version of the b-expansion based on the large-N expansion of the coefficients by making the replacement $N \to \frac{2}{11}(3b + N_f)$. For

simplicity assume that we replace C_F by $N/2$, hence throwing away some $1/N$ terms, this yields an expansion

$$d_k = d_k^{<k>} b^k + d_k^{<k-1>} b^{k-1} + \cdots + d_k^{<0>} \ , \tag{6.22}$$

where $d_k^{<k>}$ is a pure number and $d_k^{<k-r>} \sim N_f^r$. We would then hope to be able to explain why $d_k^{(k)} \simeq d_k^{<k>}$. This "large-$N_f$–large-$N_C$ duality", which is found empirically from comparison with exact calculations is clearly an intriguing feature of QCD; and it provides a motivation for resumming the leading-b terms to all orders. A clearer understanding of its origins will of course be crucial in assessing the value of such resummations.

Notice that the more complicated UV renormalon structure suggested by Vainshtein and Zakharov, discussed in Chapter 5, would apparently wreck even the weak asymptotic estimate of equation (5.35) and would make it hard to explain the above observations. Evidently much remains to be clarified. Nonetheless we shall go ahead and resum the leading-b terms for some phenomenologically important QCD observables and derive simple numerically tractable formulae for the resummations which we believe will in any case be of use in further studies.

We now turn to a consideration of the RS dependence of the N_f and b expansions. In variants of minimal subtraction, where the $1/\epsilon$ pole in dimensional regularization is subtracted along with an N_f-independent finite part, K, the QCD perturbative coefficients will have the form of polynomials in N_f as in equation (6.2). Modified minimal subtraction ($\overline{\rm MS}$), corresponding to $K{=}(\ln 4\pi{-}\gamma_E)$, with $\gamma_E{=}0.5722\ldots$, Euler's constant, is most commonly employed. We can consider $\overline{\rm MS}$ with renormalization scale $\mu{=}{\rm e}^u Q$, where u is an N_f-independent number. The most general subtraction procedure which will result in perturbative coefficients polynomial in N_f, however, can be regarded as $\overline{\rm MS}$ with scale $\mu{=}{\rm e}^{u+v/b} Q$, where v is again N_f-independent. We shall refer to such renormalization schemes as 'regular' schemes. Of course the renormalization scheme is not specified by the scale and subtraction procedure alone but by higher order beta-function coefficients as well. Any variant of minimal subtraction with an N_f-independent renormalization scale will have $v{=}0$. Momentum space subtraction (MOM) based on the *ggg* vertex at a symmetric subtraction point $\mu^2{=}Q^2$ [66] corresponds to $u{=}2.56$ and $v{=}C_A f(\xi)$, where f is a cubic polynomial in the gauge parameter ξ.

For the Landau gauge, ξ=0, v=$-2.49C_A$. For other versions of MOM based on the qqg or ghost vertices, v will involve C_A and C_F.

Let us denote the perturbative coefficients in the $\overline{\rm MS}$ scheme with $\mu = Q$ (u=v=0) by d_k; and those with general u and v by d'_k. Then

$$\begin{aligned} d'_1 &= (d_1^{(1)} + u)b + (d_1^{(0)} + v) \\ &= d_1 + bu + v \ . \end{aligned} \tag{6.23}$$

Changing v, one can make the $d_1^{(0)}$ coefficient as large as one pleases and hence destroy the dominance of the leading-b term noted above for the D-function and the sum rules in low orders; although the leading-b term should still reproduce asymptotically the $d_k^{[k-r]}$ coefficients to $O(1/k)$ accuracy.

For d_2 and higher coefficients the specification of the RS will involve higher beta-function coefficients as well as the scale and subtraction procedure. The RG-improved coupling, $a(\mu^2)$, will evolve with renormalization scale according to the beta-function equation of equation (5.7). Integrating equation (5.7) with a suitable choice of boundary condition [67], one obtains a transcendental equation for a:

$$b \ln \frac{\mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{1+ca} + \int_0^a {\rm d}x \left[-\frac{1}{x^2 B(x)} + \frac{1}{x^2(1+cx)} \right] \ , \tag{6.24}$$

where $B(x)$=$(1+cx+c_2x^2+c_3x^3+\cdots+c_kx^k+\cdots)$. The beta-function coefficients, c_2, c_3, ... together with $b \ln \frac{\mu}{\Lambda}$ label the RS. In a fixed order perturbative calculation one would truncate the beta-function. For the all-orders resummations of the next section, however, one requires an all-orders definition of the coupling. In the $\overline{\rm MS}$ scheme the higher beta-function coefficients, $c_2^{\overline{\rm MS}}$, $c_3^{\overline{\rm MS}}$, ..., presumably exhibit factorial growth, $c_k^{\overline{\rm MS}} \sim k!$; and the '$a$' coupling in the Borel integral would not be defined, since $B(x)$ would itself need to be defined by a Borel integral or other summation. One therefore needs to use a finite scheme [34] where $B(x)$ has a finite radius of convergence and can be summed. An extreme example is the so-called 't Hooft scheme [34] where c_2=c_3=$\cdots$=c_k=$\cdots$=0, $B(x)$=$1 + cx$. This results in the all-orders definition of the coupling,

$$b \ln \frac{\mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{1+ca} \ . \tag{6.25}$$

In such a finite scheme, where c_2, c_3, ... are N_f-independent, the b-expansion of equation (6.3) will contain additional inverse powers of b and the d_k will no longer be polynomials in N_f. The leading-b coefficients, $d_k^{(k)}$, in regular schemes are independent of c_2, c_3, ..., since these beta-function coefficients, c_k, are $O(1/N_f)$ relative to d_k.

In order to avoid complications with RS-dependence, one can formulate perturbation theory in terms of RS invariants, using the effective charge formalism [68, 69]. For a generic dimensionless observable D dependent on a single energy scale Q and having a perturbative series as in equation (6.1), we have the evolution equation

$$\frac{dD}{d\ln Q} = -b\rho(D) = -bD^2(1 + cD + \rho_2 D^3 + \cdots + \rho_k D^{k+1} + \cdots) , \qquad (6.26)$$

which is just the beta-function equation (equation (5.7)) in the scheme in which $D = a$. The coefficients ρ_k are Q independent RS invariant combinations of the perturbative coefficients d_k in any arbitrary RS with beta-function coefficients c_k. For instance,

$$\rho_2 = d_2 + c_2 - d_1 c_1 - d_1^2 . \qquad (6.27)$$

Defining $\hat{\rho}_k = b\rho_k$, one can show that these invariants have a polynomial b-expansion analogous to equation (6.3),

$$\hat{\rho}_k = \hat{\rho}_k^{(k+1)} b^{k+1} + \hat{\rho}_k^{(k)} b^k + \cdots + \hat{\rho}_k^{(0)} . \qquad (6.28)$$

The leading-b coefficient in this expansion is an RS invariant combination of the $d_k^{(k)}$ leading-b coefficients. For instance,

$$b^3 \hat{\rho}_2^{(3)} = b^3 (d_2^{(2)} - (d_1^{(1)})^2) \qquad (6.29)$$

and so, given exact large-N_f results, one could resum to all orders the leading-b pieces, defining

$$\rho^{(L)}(D) = D^2 \left(1 + cD + \sum_{k=2}^{\infty} \hat{\rho}_k^{(k+1)} b^k\right) . \qquad (6.30)$$

The leading-b piece of these invariants, as for the d_k coefficients, reproduces remarkably well the subleading coefficients in the b-expansion of $\hat{\rho}_k$. For the vacuum polarization $\tilde{D}$ we have an exact result for ρ_2 based on the exact coefficients

for $d_1^{\overline{\rm MS}}$, $d_2^{\overline{\rm MS}}$ and $c_2^{\overline{\rm MS}}$ [52],

$$\begin{aligned}\hat{\rho}_2\,(\text{exact}) &= -0.0243N_f^3 + \left(0.553N - 0.00151\frac{1}{N}\right)N_f^2 \\ &+ \left(-3.32N^2 + 0.344 + 0.0612\frac{1}{N^2}\right)N_f \\ &+ \left(3.79N^3 - 1.45N - 0.337\frac{1}{N}\right) . \end{aligned} \tag{6.31}$$

This is to be compared with the leading-b piece

$$\begin{aligned} b^3\hat{\rho}_2^{(3)} &= b^3(d_2^{(2)} - (d_1^{(1)})^2) = 0.656b^3 \\ &= -0.0243N_f^3 + 0.401NN_f^2 - 2.21N^2N_f + 4.04N^3 . \end{aligned} \tag{6.32}$$

Notice the good agreement of the subleading NN_f^2, N^2N_f and in particular the leading-N N^3 coefficients of the RS invariant. This gives us some confidence that the remarkable "large-N_f–large-N_C duality" discussed above for d_1 and d_2 is not just an artefact of the particular RS choice of $\overline{\rm MS}$ with $\mu = Q$. In order to further strengthen this conviction, we can perform a similar exercise for the GLS sum rule, $\tilde{K}$, where we have that

$$\begin{aligned}\hat{\rho}_2(\tilde{K})\,(\text{exact}) &= -0.0221N_f^3 + \left(0.513N + 0.00665\frac{1}{N}\right)N_f^2 \\ &+ \left(-3.29N^2 + 0.505 + 0.0143\frac{1}{N^2}\right)N_f \\ &+ \left(3.85N^3 - 1.73N - 0.337\frac{1}{N}\right) \end{aligned} \tag{6.33}$$

to be compared with

$$\begin{aligned} b^3\hat{\rho}_2^{(3)}(\tilde{K}) &= b^3(K_2^{(2)} - (K_1^{(1)})^2) = 0.597b^3 \\ &= -0.0221N_f^3 + 0.365NN_f^2 - 2.01N^2N_f + 3.68N^3 . \end{aligned} \tag{6.34}$$

Again the agreement is seen to be fairly good.

6.3 Leading-b Resummations

For the Euclidean quantities, $\tilde{D}$ and $\tilde{K}$, defined earlier we shall deduce from the exact large-N_f results that, in the $\overline{\rm MS}$ scheme with μ=e$^{-5/6}Q$, the Borel transform,

$B[D](z)$, is of the form

$$B[D](z) = \sum_{\ell=1}^{\infty} \frac{A_0(\ell) + A_1(\ell)z + \overline{A}_1(\ell)z + \overline{A}_2(\ell)z^2 + \cdots}{\left(1+\frac{z}{z_\ell}\right)^{\alpha_\ell+\overline{\alpha}_\ell}} + \sum_{\ell=1}^{\infty} \frac{B_0(\ell) + B_1(\ell)z + \overline{B}_1(\ell)z + \overline{B}_2(\ell)z^2 + \cdots}{\left(1-\frac{z}{z_\ell}\right)^{\beta_\ell+\overline{\beta}_\ell}} + \cdots , \quad (6.35)$$

where $z_\ell=2\ell/b$. The two terms correspond to a summation over the ultraviolet renormalons, UV_ℓ, and infrared renormalons, IR_ℓ, respectively. $A_0(\ell)$, $A_1(\ell)$, α_ℓ and $B_0(\ell)$, $B_1(\ell)$, β_ℓ will be obtained from the large-N_f results. The barred terms are sub-leading in N_f and remain unknown. The use of the so-called 'V-scheme' [58], $\overline{\mathrm{MS}}$ with $\mu=\mathrm{e}^{-5/6}Q$, means that only the constant and $O(z)$ terms in the numerator polynomials are leading in N_f. For a general $\overline{\mathrm{MS}}$ scale, $\mu=\mathrm{e}^{u}Q$, an overall factor $\mathrm{e}^{bz(u+5/6)}$ should multiply the unbarred leading-N_f terms in the numerator. The presence of this exponential factor, when it is expanded in powers of z, can mask the presence of the UV and IR renormalons in low orders of perturbation theory.

It is hoped that no confusion will arise from the different definitions of the $A_{0,1}$ and $B_{0,1}$ coefficients of equations (5.44) and (5.45) and those of equation (6.35). In Chapter 5, when considering the asymptotic behaviour of the perturbative coefficients, it was useful to consider a numerator expansion around $z = z_\ell$, whereas in the present context we have chosen to expand around $z = 0$. For the remainder of this thesis we shall adopt the definition of the coefficients given in equation (6.35).

Notice that, whilst the residue at each renormalon singularity (the A_0, B_0 coefficients of Chapter 5) is only known to leading order in N_f, the $A_0(\ell)$ and $B_0(\ell)$ constant terms in the numerator polynomials in equation (6.35) are known exactly; indeed $\sum_{\ell=1}^{\infty}(A_0(\ell) + B_0(\ell))=1$, as is required to reproduce the unit coefficient of the $O(a)$ term in D in equation (6.1). We now turn to the explicit determination of the coefficients and exponents for $\tilde{D}$ and $\tilde{K}$.

Using the expressions of equations (5.44) and (5.45) obtained in Chapter 5 it is straightforward to deduce that the coefficients and exponents in equation (6.35) for $B[\tilde{D}](z)$ are

$$A_0(\ell) = \frac{8}{3}\frac{(-1)^{\ell+1}(3\ell^2+6\ell+2)}{\ell^2(\ell+1)^2(\ell+2)^2}, \quad A_1(\ell) = \frac{8}{3}\frac{b(-1)^{\ell+1}(\ell+\frac{3}{2})}{\ell^2(\ell+1)^2(\ell+2)^2}$$
$$\ell = 1,2,3,\ldots$$

$$
\begin{aligned}
&B_0(1)=0, \quad B_0(2)=1, \quad B_0(\ell)=-A_0(-\ell) \quad \ell \geq 3 \\
&B_1(1)=0, \quad B_1(2)=0, \quad B_1(\ell)=-A_1(-\ell) \quad \ell \geq 3 \\
&\alpha_\ell = 2 \;\; \ell = 1,2,3,\ldots, \qquad \beta_2 = 1, \; \beta_\ell = 2 \quad \ell \geq 3 \, .
\end{aligned} \tag{6.36}
$$

We point out again that IR_1 is absent, as required from the absence of a dimension two condensate in the OPE (see Chapter 5 and reference [43]). IR_2 is a single pole. All the other singularities are double poles. Not only are the coefficients for the UV_ℓ and IR_ℓ singularities related by the curious symmetry $B_{0,1}(\ell)=-A_{0,1}(-\ell)$, noted in Chapter 5; but the form of $A_0(\ell)$ means that there is an additional relation, $A_0(\ell)=-B_0(\ell+2)$, so that the constant term in the numerator polynomial for UV_ℓ exactly cancels that for $\mathrm{IR}_{\ell+2}$. This ensures that

$$
\sum_{\ell=1}^{\infty}(A_0(\ell)+B_0(\ell)) = B_0(2) = 1 \, ,
$$

which, as noted earlier, is required to reproduce the unit coefficient of the $O(a)$ term in the perturbative expansion. The precise origin of these relations between UV and IR renormalons remains unclear and deserves further study. They have also been noted and discussed in reference [70].

For the D-function the singularity nearest the origin is UV_1 and from the $A_0(1)$, $A_1(1)$ in equation (6.36) this should correspond to

$$
d_n^{(n)}|_{UV_1} = \frac{12n+22}{27} n! \left(-\frac{1}{2}\right)^n \, . \tag{6.37}
$$

In Table 6.1 we compare the exact leading-b, $d_n^{(n)}$, coefficients with the contribution from UV_1 of equation (6.37). UV and IR denote the separate sums over the UV_ℓ and IR_ℓ singularities. The $\overline{\mathrm{MS}}$ scheme with $\mu=\mathrm{e}^{-5/6}Q$ (V-scheme) is assumed. With this choice of scheme UV_1 dominates even in low orders and the alternating factorial behaviour is apparent.

We now consider the coefficients and exponents in equation (6.35) for $B[\tilde{K}](z)$. The generating function for the leading-b coefficient in the V-scheme is [57]

$$
K_n^{(n)} = \frac{1}{3}\left(\frac{1}{2}\right)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{(3+x)}{(1-x^2)(1-\frac{x^2}{4})}\bigg|_{x=0} \, . \tag{6.38}
$$

This results in

$$
B[\tilde{K}](z) = \frac{\frac{4}{9}}{(1+\frac{bz}{2})} - \frac{\frac{1}{18}}{(1+\frac{bz}{4})} + \frac{\frac{8}{9}}{(1-\frac{bz}{2})} - \frac{\frac{5}{18}}{(1-\frac{bz}{4})} \, . \tag{6.39}
$$

n	$d_n^{(n)}$	UV_1	UV	IR
0	1	.8148148	.7198242	.28018
1	-.4874471	-.6296296	-.5921448	.10470
2	.8938293	.8518519	.8258924	.06794
3	-1.525257	-1.611111	-1.586113	.06086
4	3.927235	3.888889	3.858335	.06890
5	-11.24973	-11.38889	-11.34378	.09405
6	39.23893	39.16667	39.08871	.15022
7	-154.1541	-154.5833	-154.4291	.27499
8	688.5574	688.3333	687.9894	.56801
9	-3410.339	-3412.500	-3411.647	1.3079
10	18638.50	18637.50	18635.17	3.3243

Table 6.1: Leading-b coefficients, $d_n^{(n)}$, for the Adler D-function, $\tilde{D}$, compared with the contribution of the first UV renormalon, 'UV_1', (equation (6.37)). The V-scheme, $\overline{\mathrm{MS}}$ with μ=$\mathrm{e}^{-5/6}Q$, is assumed. 'UV' and 'IR' denote the separate sums over the UV_ℓ and IR_ℓ singularities.

The terms correspond to UV_1, UV_2, IR_1, IR_2 respectively. Each numerator and exponent will contain in addition $O(1/N_f)$ corrections corresponding to the barred terms in equation (6.35). The constant terms in the numerators sum to 1, again ensuring a unit $O(a)$ coefficient in $\tilde{K}$. It would be interesting to try to understand the fact that only the first two UV and IR renormalons are leading in N_f in the context of the OPE for the deep inelastic sum rules, a topic discussed in reference [71].

$K_n^{(n)}$ is then given by (in the V-scheme)

$$K_n^{(n)} = \frac{8}{9}n!\left(\frac{1}{2}\right)^n + \frac{4}{9}n!\left(-\frac{1}{2}\right)^n - \frac{5}{18}n!\left(\frac{1}{4}\right)^n - \frac{1}{18}n!\left(-\frac{1}{4}\right)^n . \qquad (6.40)$$

Table 6.2 shows that $K_n^{(n)}$ is dominated, even in low orders, by the combined UV_1+IR_1 contributions of the two singularities nearest the origin, the first two terms of equation (6.40).

We now wish to use the Borel integrals to perform the leading-b resummation defined in equations (6.5, 6.6). For the Adler D-function we have

$$\begin{aligned}\tilde{D}^{(L)}(a) &= \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z/a}\mathrm{e}^{bz(u-5/6)} \sum_{\ell=1}^{\infty} \frac{A_0(\ell)+A_1(\ell)z}{(1+\frac{z}{z_\ell})^2} \qquad (6.41)\\ &+ \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z/a}\mathrm{e}^{bz(u-5/6)} \left(\frac{B_0(2)}{(1-\frac{z}{z_2})} + \sum_{\ell=3}^{\infty} \frac{B_0(\ell)+B_1(\ell)z}{(1-\frac{z}{z_\ell})^2} \right) ,\end{aligned}$$

n	$K_n^{(n)}$	$\mathrm{UV}_1 + IR_1$	UV	IR
0	1	1.333333	.3888889	.6111111
1	.1666667	.2222222	-.2083333	.3750000
2	.6250000	.6666667	.2152778	.4097222
3	.3125000	.3333333	-.3281250	.6406250
4	1.968750	2.000000	.6614583	1.307292
5	1.640625	1.666667	-1.660156	3.300781
6	14.94141	15.00000	4.990234	9.951172
7	17.43164	17.50000	-17.48291	34.91455
8	209.7949	210.0000	69.96582	139.8291
9	314.6924	315.0000	-314.9231	629.6155
10	4723.846	4725.000	1574.808	3149.039

Table 6.2: As for Table 6.1 but for the Deep Inelastic Scattering sum rules, $\tilde{K}$, $K_n^{(n)}$. 'UV_1+IR_1' denotes the contribution of the singularities nearest the origin, the first two terms of equation (6.40).

where the coefficients A_0, A_1, B_0, B_1 are summarised in equation (6.36) and we have assumed $\overline{\mathrm{MS}}$ subtraction with $\mu = \mathrm{e}^{u+v/b}Q$. Notice that v does not appear in the expression for $\tilde{D}^{(L)}(a)$ since the leading-b terms are v-independent. Assuming that a finite RS with beta-function $B(x)$ has been used, then the coupling, a, is defined by the integrated beta-function equation (equation (6.24)); thus,

$$\frac{1}{a} = b\ln\left(\frac{\mathrm{e}^{u+v/b}Q}{\Lambda_{\overline{\mathrm{MS}}}}\right) - c\ln\frac{ca}{1+ca} - \int_0^a \mathrm{d}x\left[-\frac{1}{x^2B(x)} + \frac{1}{x^2(1+cx)}\right] . \quad (6.42)$$

Substituting from equation (6.42) for $1/a$ in equation (6.41) one then obtains

$$\begin{aligned}\tilde{D}^{(L)}(a) &= \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-F(a)z}\sum_{\ell=1}^{\infty}\frac{A_0(\ell)+A_1(\ell)z}{(1+\frac{z}{z_\ell})^2} \\ &\quad + \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-F(a)z}\left(\frac{B_0(2)}{(1-\frac{z}{z_2})} + \sum_{\ell=3}^{\infty}\frac{B_0(\ell)+B_1(\ell)z}{(1-\frac{z}{z_\ell})^2}\right) , \quad (6.43)\end{aligned}$$

where

$$F(a) = b\ln\frac{Q}{\Lambda_{\overline{\mathrm{MS}}}} - \frac{5}{6}b - c\ln\frac{ca}{1+ca} + v - \int_0^a \mathrm{d}x\left[-\frac{1}{x^2B(x)} + \frac{1}{x^2(1+cx)}\right] . \quad (6.44)$$

Notice that the u-dependence in equation (6.41) has cancelled exactly, since it is compensated by the leading-b scale dependence of a. The subleading in b scale dependence is of course not compensated and hence the partial resummation of

the perturbation series is RS dependent. We shall return to this RS dependence in a moment.

The first, UV renormalon, term in equation (6.43) is a completely well-defined integral. It may be performed in terms of the exponential integral function (with negative argument),

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \mathrm{d}t \frac{\mathrm{e}^{-t}}{t} \,. \tag{6.45}$$

The first term yields

$$\begin{aligned} \tilde{D}^{(L)}(a)|_{UV} &= \sum_{\ell=1}^{\infty} z_\ell \{ \mathrm{e}^{F(a)z_\ell} \mathrm{Ei}(-F(a)z_\ell) \left[F(a)z_\ell (A_0(\ell) - z_\ell A_1(\ell)) - z_\ell A_1(\ell) \right] \\ &\quad + (A_0(\ell) - z_\ell A_1(\ell)) \} \,. \end{aligned} \tag{6.46}$$

To evaluate the second, IR renormalon, term we shall use a principal value prescription; correspondingly we need to define $\mathrm{Ei}(x)$ with a positive argument as a principal value. We find

$$\begin{aligned} \tilde{D}^{(L)}(a)|_{IR} &= \mathrm{e}^{-F(a)z_2} z_2 B_0(2) \mathrm{Ei}(F(a)z_2) \\ &\quad + \sum_{\ell=3}^{\infty} z_\ell \{ \mathrm{e}^{-F(a)z_\ell} \mathrm{Ei}(F(a)z_\ell) \left[F(a)z_\ell (B_0(\ell) + z_\ell B_1(\ell)) - z_\ell B_1(\ell) \right] \\ &\quad - (B_0(\ell) + z_\ell B_1(\ell)) \} \,. \end{aligned} \tag{6.47}$$

Finally

$$\tilde{D}^{(L)}(a) = \tilde{D}^{(L)}(a)|_{UV} + \tilde{D}^{(L)}(a)|_{IR} \,. \tag{6.48}$$

For the Deep Inelastic Scattering sum rules we will have, analogously, using equation (6.39),

$$\begin{aligned} \tilde{K}^{(L)}(a) &= \int_0^{\infty} \mathrm{d}z \, \mathrm{e}^{-F(a)z} \left(\frac{\frac{4}{9}}{(1 + \frac{z}{z_1})} - \frac{\frac{1}{18}}{(1 + \frac{z}{z_2})} \right) \\ &\quad + \int_0^{\infty} \mathrm{d}z \, \mathrm{e}^{-F(a)z} \left(\frac{\frac{8}{9}}{(1 - \frac{z}{z_1})} - \frac{\frac{5}{18}}{(1 - \frac{z}{z_2})} \right) \,. \end{aligned} \tag{6.49}$$

These integrals may be expressed once again in terms of $\mathrm{Ei}(x)$:

$$\tilde{K}^{(L)}(a)|_{UV} = \left[-\frac{4}{9} \mathrm{e}^{F(a)z_1} z_1 \mathrm{Ei}(-F(a)z_1) + \frac{1}{18} \mathrm{e}^{F(a)z_2} z_2 \mathrm{Ei}(-F(a)z_2) \right] \tag{6.50}$$

and

$$\tilde{K}^{(L)}(a)|_{IR} = \left[\frac{8}{9} \mathrm{e}^{-F(a)z_1} z_1 \mathrm{Ei}(F(a)z_1) - \frac{5}{18} \mathrm{e}^{-F(a)z_2} z_2 \mathrm{Ei}(F(a)z_2) \right] \,. \tag{6.51}$$

Similarly

$$\tilde{K}^{(L)}(a) = \tilde{K}^{(L)}(a)|_{UV} + \tilde{K}^{(L)}(a)|_{IR} \,. \tag{6.52}$$

Before we numerically evaluate these results and comment further on RS dependence, we shall derive the analogous resummations for the Minkowski continuations of the D-function, the e^+e^- annihilation R-ratio and the analogous quantity in τ-decay, R_τ. As we discussed in the last chapter, the R-ratio is related to D by the optical theorem as in equation (5.4). Using equations (5.2) and (5.4) we can obtain the succinct relation

$$R(s) = \frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} \mathrm{d}Q^2 \frac{D(Q^2)}{Q^2} \,. \tag{6.53}$$

Here s is the physical timelike Minkowski squared momentum transfer. A perturbative result for R of the form of equation (5.5) can be written down involving a quantity $\tilde{R}$ with perturbative coefficients r_k. The r_k may be directly related to the d_k using the relation (6.53): $r_1{=}d_1$, $r_2{=}d_2{-}\pi^2b^2/12$. The π^2 term arises due to analytical continuation. Inserting the Borel representation of D into equation (6.53) one can directly obtain in the large-N_f limit

$$B[\tilde{R}](z) = \frac{\sin(\pi bz/2)}{\pi bz/2} B[\tilde{D}](z) \,, \tag{6.54}$$

as in equation (5.49) (see also Appendix A). The leading-b resummation is then obtained from equation (6.43) simply by adding an extra $\frac{\sin(\pi bz/2)}{\pi bz/2}$ factor in the integrand.

$$\begin{aligned}\tilde{R}^{(L)}(a) \;=\; & \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-F(a)z} \frac{\sin(\pi bz/2)}{\pi bz/2} \sum_{\ell=1}^{\infty} \frac{A_0(\ell)+A_1(\ell)z}{(1+\frac{z}{z_\ell})^2} \\ & + \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-F(a)z} \frac{\sin(\pi bz/2)}{\pi bz/2} \left(\frac{B_0(2)}{(1-\frac{z}{z_2})} + \sum_{\ell=3}^{\infty} \frac{B_0(\ell)+B_1(\ell)z}{(1-\frac{z}{z_\ell})^2} \right) \,.\end{aligned} \tag{6.55}$$

Writing the 'sin' as a sum of complex exponentials and using partial fractions, the UV integrals can be explicitly performed in terms of the generalised exponential integral functions $\mathrm{Ei}(n,w)$, with complex argument w, defined for $\mathrm{Re}\, w > 0$ by

$$\mathrm{Ei}(n,w) = \int_1^\infty \mathrm{d}t \frac{\mathrm{e}^{-wt}}{t^n} \,. \tag{6.56}$$

One also needs

$$\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-F(a)z}\, \frac{\sin(\pi b z/2)}{z} = \arctan\left(\frac{\pi b}{2F(a)}\right) . \tag{6.57}$$

One finds

$$\begin{aligned} \tilde{R}^{(L)}(a)|_{UV} &= \frac{2}{\pi b}\left(\frac{8\zeta_2}{3} - \frac{11}{3}\right)\arctan\left(\frac{\pi b}{2F(a)}\right) \\ &\quad + \frac{2}{\pi b}\sum_{\ell=1}^{\infty}\Big\{ A_0(\ell)\phi_+(1,\ell) + (A_0(\ell) - A_1(\ell)z_\ell)\phi_+(2,\ell)\Big\} , \end{aligned} \tag{6.58}$$

where

$$\phi_+(p,q) = \mathrm{e}^{F(a)z_q}(-1)^q \mathrm{Im}[\mathrm{Ei}(p, F_+ z_q)] \tag{6.59}$$

with $F_\pm = F(a) \pm \frac{i\pi b}{2}$.

To evaluate the principal value of the IR contribution in equation (6.55) one needs to continue $\mathrm{Ei}(n,w)$, defined by equation (6.56) for $\mathrm{Re}\, w > 0$, to $\mathrm{Re}\, w < 0$. With the standard continuation one then arrives at a function analytic everywhere in the cut complex w-plane, except at w=0; and with a branch cut running along the negative real axis. Explicitly [72]

$$\mathrm{Ei}(n,w) = \frac{(-w)^{n-1}}{(n-1)!}\left[-\ln w - \gamma_E + \sum_{m=1}^{n-1}\frac{1}{m}\right] - \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty} \frac{(-w)^m}{(m-n+1)m!} , \tag{6.60}$$

with γ_E=0.5722..., Euler's constant. The $\ln w$ term in equation (6.60) means that $\mathrm{Ei}(n,w)$ is not a real function. For instance, for negative real w one has $\mathrm{Ei}(1, -x \pm i\epsilon) = \mathrm{Ei}(x) \mp i\pi$, where $\mathrm{Ei}(x)$ is the principal value of equation (6.45) used to define the IR renormalon contribution for the Euclidean quantities.

In order to evaluate the IR renormalon contribution correctly, one in fact needs to continue $\mathrm{Ei}(n,w)$ as a real function, so that for $\mathrm{Re}\, w < 0$ one makes the replacement $\ln w \to \ln w + i\pi\, \mathrm{sign}(\mathrm{Im}\, w)$. Correspondingly, one should define the IR analogue of equation (6.59),

$$\phi_-(p,q) = \mathrm{e}^{-F(a)z_q}(-1)^q \mathrm{Im}[\mathrm{Ei}(p, -F_+ z_q)] - \frac{\mathrm{e}^{-F(a)z_q}(-1)^q z_q^{p-1}}{(p-1)!}\pi \mathrm{Re}[(F_+)^{p-1}] , \tag{6.61}$$

where $\mathrm{Ei}(p, -F_+ z_q)$ is defined by equation (6.60). The principal value of the IR renormalon contribution is then given by

$$\tilde{R}^{(L)}(a)|_{IR} = \frac{2}{\pi b}\left(\frac{14}{3} - \frac{8\zeta_2}{3}\right)\arctan\left(\frac{\pi b}{2F(a)}\right) + \frac{2B_0(2)}{\pi b}\phi_-(1,2)$$

$$+\frac{2}{\pi b}\sum_{\ell=3}^{\infty}\Big\{B_0(\ell)\phi_-(1,\ell)+(B_0(\ell)+B_1(\ell)z_\ell)\phi_-(2,\ell)\Big\}. \quad (6.62)$$

Then

$$\tilde{R}^{(L)}(a)=\tilde{R}^{(L)}(a)|_{UV}+\tilde{R}^{(L)}(a)|_{IR}\,. \quad (6.63)$$

The τ-decay analogue of the R-ratio, R_τ, can be defined in terms of the R-ratio by the integral representation [73]

$$\begin{aligned} R_\tau &= 2\int_0^{M_\tau^2}\frac{\mathrm{d}s}{M_\tau^2}(1-s/M_\tau^2)^2(1+2s/M_\tau^2)\hat{R}(s) \\ &= d(R)(|V_{ud}|^2+|V_{us}|^2)\left[1+\frac{3}{4}C_F\tilde{R}_\tau\right]\,. \end{aligned} \quad (6.64)$$

Here $\hat{R}(s)$ denotes $R(s)$ with the $\sum_f Q_f^2$ replaced by $|V_{ud}|^2+|V_{us}|^2\approx 1$, where the V's are KM mixing matrix elements. $\tilde{R}_\tau$ has the form

$$\tilde{R}_\tau = a+r_1^\tau a^2+r_2^\tau a^3+\cdots+r_k^\tau a^{k+1}+\cdots\,. \quad (6.65)$$

It is then straightforward to show that

$$B[\tilde{R}_\tau](z)=\frac{\sin(\pi bz/2)}{\pi bz/2}\left[\frac{2}{(1-\frac{bz}{2})}-\frac{2}{(1-\frac{bz}{6})}+\frac{1}{(1-\frac{bz}{8})}\right]B[\tilde{D}](z)\,. \quad (6.66)$$

Proceeding in a manner analogous to that for the R-ratio, we find

$$\begin{aligned} \tilde{R}_\tau^{(L)}(a)|_{UV} &= \frac{2}{\pi b}\left(\frac{8\zeta_2}{3}-\frac{11}{3}\right)\arctan\left(\frac{\pi b}{2F(a)}\right) \\ &\quad +\frac{4}{\pi b}\sum_{\ell=1}^{\infty}[(A_0(\ell)(G(\ell)+H(\ell))-z_\ell A_1(\ell)G(\ell))\phi_+(1,\ell) \\ &\quad +H(\ell)(A_0(\ell)-z_\ell A_1(\ell))\phi_+(2,\ell)] \end{aligned} \quad (6.67)$$

and

$$\begin{aligned} \tilde{R}_\tau^{(L)}(a)|_{IR} &= \frac{2}{\pi b}\left(\frac{14}{3}-\frac{8\zeta_2}{3}\right)\arctan\left(\frac{\pi b}{2F(a)}\right) \\ &\quad +\frac{4}{\pi b}\left(-\frac{14}{3}+\frac{64}{3}\ln 2-8\zeta_3+bz_1\left(\frac{23}{3}-\frac{32}{3}\ln 2\right)\right)\phi_-(1,1) \\ &\quad -\frac{12}{\pi b}\phi_-(1,2)+\frac{4}{\pi b}\left(-\frac{703}{18}+64\ln 2+bz_3\left(\frac{245}{36}-\frac{32}{3}\ln 2\right)\right)\phi_-(1,3) \\ &\quad +\frac{4}{\pi b}\left(-\frac{1627}{972}-\frac{128}{81}\ln 2+bz_4\left(\frac{2035}{7776}+\frac{16}{81}\ln 2\right)\right)\phi_-(1,4) \\ &\quad +\frac{4}{\pi b}\left(-\frac{11}{27}+\frac{bz_3}{6}\right)\phi_-(2,3)+\frac{4}{\pi b}\left(\frac{247}{648}-\frac{5bz_4}{162}\right)\phi_-(2,4) \end{aligned}$$

$$-\frac{8}{\pi b}\left(-\frac{11}{27}+\frac{bz_3}{18}\right)\phi_-(3,3)+\frac{8}{\pi b}\left(\frac{13}{432}-\frac{5bz_4}{1728}\right)\phi_-(3,4)$$
$$+\frac{4}{\pi b}\sum_{\ell=5}^{\infty}[(B_0(\ell)(G(-\ell)+H(-\ell))+z_\ell B_1(\ell)G(-\ell))\phi_-(1,\ell)$$
$$+H(-\ell)(B_0(\ell)+z_\ell B_1(\ell))\phi_-(2,\ell)]\,, \qquad (6.68)$$

where

$$G(\ell)=\frac{6\ell(3\ell^2+16\ell+19)}{(\ell+1)^2(\ell+3)^2(\ell+4)^2},\quad H(\ell)=\frac{6}{(\ell+1)(\ell+3)(\ell+4)}\,.$$

Then, as before,

$$\tilde{R}_\tau^{(L)}(a)=\tilde{R}_\tau^{(L)}(a)|_{UV}+\tilde{R}_\tau^{(L)}(a)|_{IR}\,. \qquad (6.69)$$

Before we proceed to discuss RS dependence further and to evaluate numerically these resummed expressions, we would like to make some remarks. The first concerns the ease of evaluation of both the Euclidean resummed expressions, equations (6.46), (6.47) and (6.50), (6.51), and those for the Minkowski quantities, equations (6.58), (6.62) and (6.67), (6.68). Even though these expressions contain infinite summations over the contributions for the UV_ℓ and IR_ℓ singularities, successive terms in the sums are strongly damped, with the result that, in order to obtain the three significant figure accuracy of the resummed results to be tabulated in the next section, it is only necessary to retain terms up to and including ℓ=7 in each sum. The resummations can then be straightforwardly and rapidly evaluated.

The second remark concerns the connection between these explicit expressions for the principal value of the Borel sum and the inequivalent continuations of the running coupling representation to the Minkowski region in reference [61]. It is straightforward to show that procedure '1' of reference [61] for $\tilde{R}$ and $\tilde{R}_\tau$ corresponds exactly to evaluating equations (6.58), (6.62) and (6.67), (6.68) using $\phi_-(p,q)$ defined by equation (6.61) with the second term omitted, i.e. using the standard continuation of $\mathrm{Ei}(n,w)$ defined in equation (6.60). This does not produce the principal value of the Borel sum. Worse still, the Borel sum contains pieces involving single IR renormalon poles together with a $\sin\frac{\pi bz}{2}$ factor, which are well-defined and finite due to the compensating zero contained in the 'sin'. These contributions, which do not require regulation, are evaluated incorrectly

with the standard continuation. Procedure '2' of reference [61] corresponds to evaluating the Borel sum, incorrectly omitting the second term in equation (6.61) for some of the IR singularities and, correctly, retaining it for others. In our view the inequivalent continuations of the running coupling representation of reference [61] to the Minkowski region correspond to various ways of wrongly evaluating the Borel sum. We see no reason to believe that these discrepancies have a physical relevance, or that they reflect inadequacies in the definition of the OPE in the Minkowski region, as suggested in reference [61]. We agree with reference [60] that, with our present state of knowledge, the regulated Borel sum provides a satisfactory framework for combining IR renormalon ambiguities with the vacuum condensate ambiguities in the OPE.

6.4 RS-Dependence of the Resummed Results

Armed with these resummed expressions, we now return to the question of the RS dependence of $D^{(L)}(a)$ before presenting the numerical results.

For a generic quantity we can write

$$D = D^{(L)}(a) + D^{(NL)}(a) \,. \tag{6.70}$$

'D' on the left of the equation denotes the full Borel sum, i.e. equation (6.35) with barred and unbarred terms included and the IR singularities principal value regulated; and we assume that this exists. Crucially, the full Borel sum is RS independent and so will not depend on 'a'. The L and NL components do depend on 'a', however. We can consider 'a' varying between a=0 and a=+∞, labelling possible RS's. From equation (6.44), as $a \to 0$ so $F(a) \to +\infty$, resulting from the $-c \ln \frac{ca}{1+ca}$ term, and we have assumed $c > 0$, which is true for $N_f \leq 8$ in SU(3) QCD. One then has $D^{(L)}(0)$=0. Correspondingly, from equation (6.70), $D^{(NL)}(0)$=D and so, as $a \to 0$, the NL component contributes the whole resummed D. As 'a' increases the $-c \ln \frac{ca}{1+ca}$ term in equation (6.44) decreases and as $a \to \infty$ it vanishes, resulting in a finite limit $F(\infty)$:

$$F(\infty) = b \ln \frac{Q}{\Lambda_{\overline{\rm MS}}} - \frac{5}{6} b + v - \int_0^\infty \mathrm{d}x \left[-\frac{1}{x^2 B(x)} + \frac{1}{x^2(1+cx)} \right] \,. \tag{6.71}$$

We assume that $B(x)$ is such that the integral exists. Thus $D^{(L)}(a)$ increases from $D^{(L)}(0)=0$ to a finite maximum value, $D^{(L)}(\infty)$, as 'a' increases; correspondingly, $D^{(NL)}(a)$, which provides the whole resummed D at $a=0$, decreases as 'a' increases.

This RS dependence of $D^{(L)}(a)$ is clearly problematic. It is monotonic and hence there is no basis for choosing a particular scheme. There is also a dependence on the particular finite scheme, characterised by the choice of $B(x)$, and on the parameter v. The maximum value, $D^{(L)}(\infty)$, does perhaps minimize the relative contribution of the unknown $D^{(NL)}$ component but there is no guarantee that $D^{(NL)}(\infty)$ is positive; and it is entirely possible that $D^{(L)}(\infty)$ overestimates D.

We shall choose $v=0$, corresponding to a variant of minimal subtraction, a choice which one can motivate by the observed dominance of the leading-b term in $\overline{\rm MS}$ noted in section 6.2. For reasons of simplicity we shall choose the 't Hooft scheme, corresponding to $B(x)=1+cx$. With these choices one has

$$F(\infty) = b\ln\frac{Q}{\Lambda_{\overline{\rm MS}}} - \frac{5}{6}b\,; \tag{6.72}$$

and we shall use this in the resummations. It corresponds simply to taking 'a' as the one-loop coupling in the $\overline{\rm MS}$ scheme with $\mu=e^{-5/6}Q$ (the V-scheme):

$$a_{\text{1-loop}} = \frac{1}{b\ln\frac{Q}{\Lambda_V}}\,, \tag{6.73}$$

where $\Lambda_V=e^{5/6}\Lambda_{\overline{\rm MS}}$. This is in fact the same choice for a as in references [59, 60, 61], where it is motivated by noting that using the one-loop form for $a(\mu^2)$ makes the leading-b summation μ-independent. We stress once again that in our view its significance is that it maximizes $D^{(L)}(a)$ for a given choice of finite scheme, $B(x)$, and parameter v. In Figure 6.1 we show $\tilde{D}^{(L)}(a)$ plotted versus 'a' ('t Hooft scheme with $v=0$). In the figures we have plotted versus $\tilde{a}=1-e^{-a}$, so that the full RS variation can be fitted in a unit interval in $\tilde{a}$. We have taken $Q=M_Z=91$GeV and $\Lambda_{\overline{\rm MS}}(N_f=5)=111$MeV. The solid curve gives the overall $\tilde{D}^{(L)}(a)$, split into $\tilde{D}^{(L)}(a)|_{UV}$ (dashed) and $\tilde{D}^{(L)}(a)|_{IR}$ (dashed-dot) contributions. Similar curves for $\tilde{K}^{(L)}(a)$ with $Q^2=2.5\text{GeV}^2$ and $\Lambda_{\overline{\rm MS}}(N_f=3)=201$MeV are given in Figure 6.2; and for $\tilde{R}^{(L)}(a)$ with $Q=91$GeV in Figure 6.3. The corresponding curve for $\tilde{R}_\tau^{(L)}(a)$ is given in Figure 6.4, $Q=M_\tau=1.78$GeV and $\Lambda_{\overline{\rm MS}}(N_f=3)=201$MeV. The qualitative behaviour is as we described earlier. The relative sizes of the UV and IR contri-

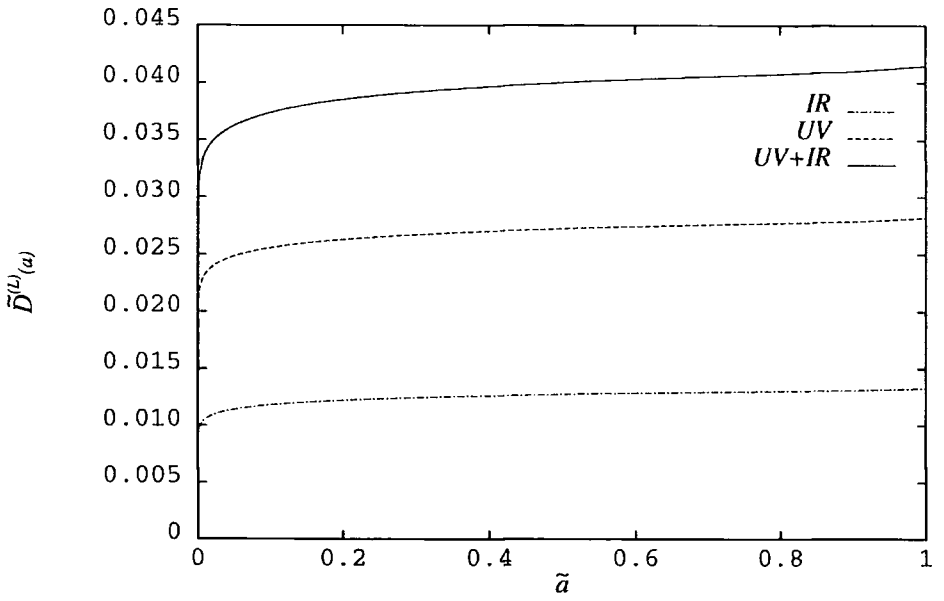


Figure 6.1: The leading-b resummation, $\tilde{D}^{(L)}(a)$, plotted versus $\tilde{a}=1-e^{-a}$ ('t Hooft scheme $v=0$), $Q=M_Z=91\text{GeV}$ and $\Lambda_{\overline{\text{MS}}}(N_f=5)$ is as in the text. The solid curve is the overall result split into $\tilde{D}^{(L)}(a)|_{UV}$ (dashed) and $\tilde{D}^{(L)}(a)|_{IR}$ (dashed-dot) contributions.

butions reflect the disposition of the UV and IR singularities described above for the different quantities.

The $a \to \infty$ limits obtained using $F(\infty)$ as in equation (6.72) are tabulated in Table 6.3. Specifically, the 'Resummed' column contains $1+\tilde{D}^{(L)}(\infty)$, $1+\tilde{R}^{(L)}(\infty)$, $1+\tilde{R}_\tau^{(L)}(\infty)$ and $1-\tilde{K}^{(L)}(\infty)$ for each observable and we have added some extra energies, $Q_0=20\text{GeV}$ and $Q_1^2=2.5\text{GeV}^2$. The values of $\Lambda_{\overline{\text{MS}}}$ for $N_f=5,3$ are as noted above. In each case we have taken care to include sufficient terms in the summation over UV_ℓ and IR_ℓ singularities to guarantee accuracy to the quoted number of significant figures.

As noted, using $F(\infty)$ is equivalent to the one-loop definition of the coupling used in references [59, 60, 61]; and the same value of $\Lambda_{\overline{\text{MS}}}(N_f=3)$ has been used. Our resummed results for $D(m_\tau^2)$, $R(m_\tau^2)$ agree with the principal value of the Borel sum for these quantities quoted in reference [61]; and the resummed result for R_τ agrees with the results quoted in references [60, 61]. As discussed at the

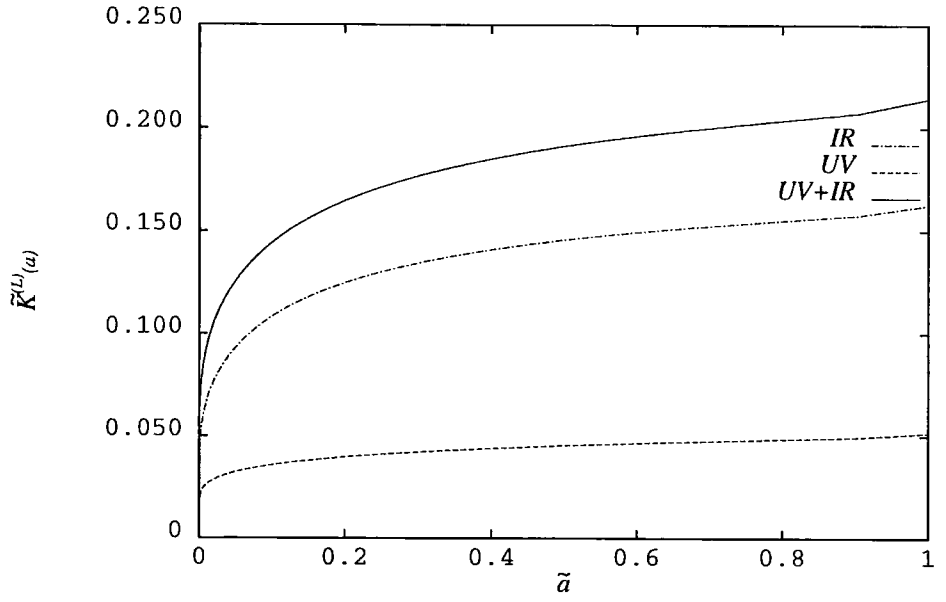


Figure 6.2: As for Figure 6.1 but for $\tilde{K}^{(L)}(a)$ with Q^2=2.5GeV^2.

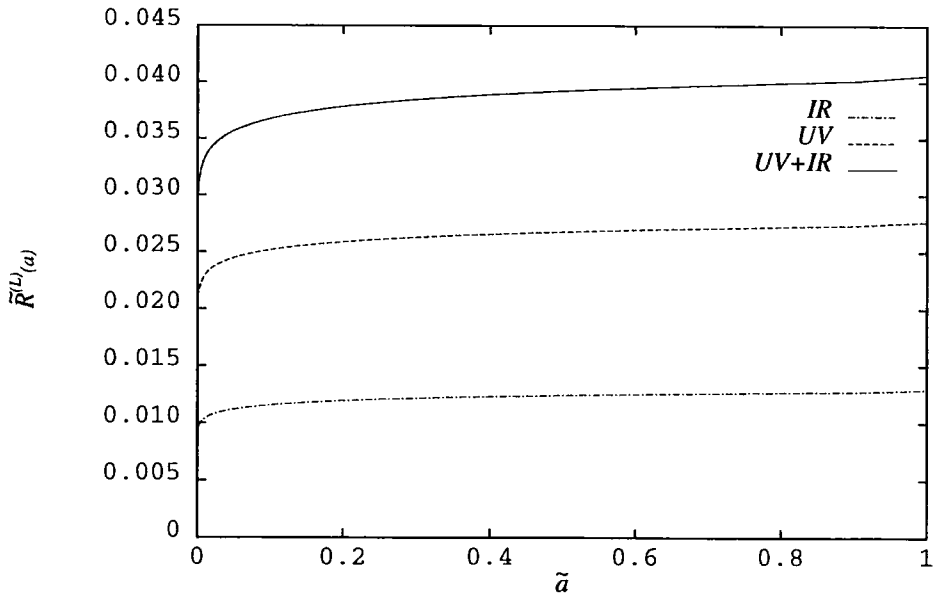


Figure 6.3: As for Figure 6.1 but for $\tilde{R}^{(L)}(a)$ with Q=M_Z=91GeV.

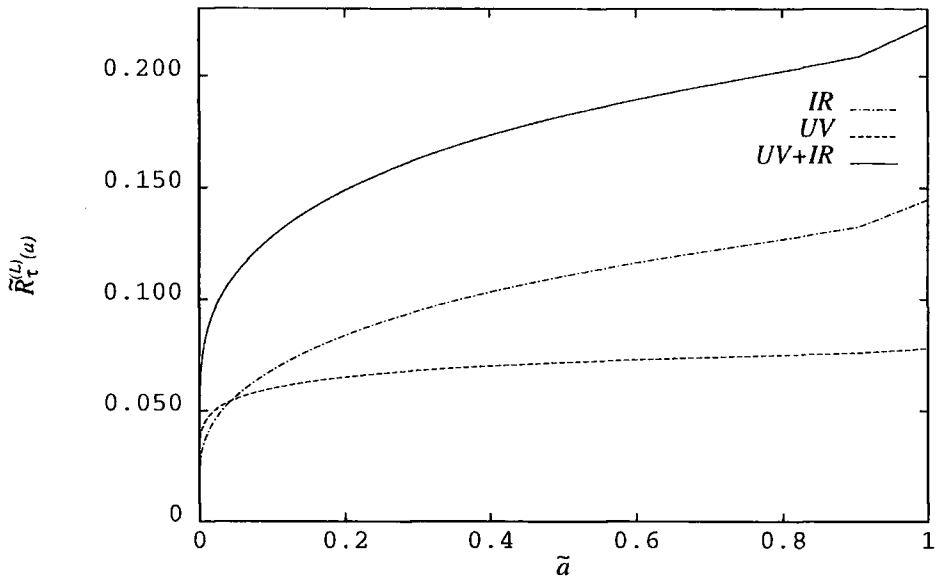


Figure 6.4: As for Figure 6.1 but for $\tilde{R}_\tau^{(L)}(a)$ with $Q{=}M_\tau{=}1.78$GeV.

Observable	Energy	Resummed	FOPT	Expt
D	m_τ	1.151	1.087	—
	Q_0	1.055	1.045	—
	m_Z	1.042	1.035	—
R	m_τ	1.105	1.080	—
	Q_0	1.053	1.044	—
	m_Z	1.041	1.035	1.040 ± 0.004
R_τ	m_τ	1.228	1.115	1.183 ± 0.010
K	Q_1	.784	.889	$.768 \pm 0.09$

Table 6.3: Comparison of resummed results ('Resummed') of section 6.3 using $F(\infty)$, equation (6.72) (see text). $Q_0^2{=}(20\text{GeV})^2$, $Q_1^2{=}2.5\text{GeV}^2$. 'FOPT' gives the exact NNLO perturbative results with $\mu{=}Q$ $\overline{\text{MS}}$ scheme and the $\Lambda_{\overline{\text{MS}}}$ values noted in the text. 'Expt' gives the experimentally deduced values for some of these quantities [74, 75, 76].

end of section 6.3 we can reproduce the results of procedures '1' and '2' for R and R_τ in reference [61] by incorrectly omitting the second term in equation (6.61) for some of the IR renormalon singularities.

There is clearly an ambiguity associated with the IR renormalons or, correspondingly, with vacuum condensates in the OPE. For a leading IR_ℓ singularity in which there is a single pole one would expect an ambiguity in the principal value $\sim \tilde{B}\,\mathrm{e}^{-z_\ell/a}$, where $\tilde{B}$ is the residue of the renormalon. For $\tilde{D}$ the leading IR singularity is IR_2. For the Minkowski quantities, $\tilde{R}$ and $\tilde{R}_\tau$, the $\frac{\sin(\pi bz/2)}{\pi bz/2}$ factor apparently removes the single pole at $z{=}z_2$ but there is presumably still a branch point singularity at $z{=}z_2$ beyond the leading-N_f approximation [43]. The determination of the residue $\tilde{B}$ would require a resummation of the numerator polynomial and so it is only known to leading-N_f (for $\tilde{D}$ $B_0(2){=}1$). Taking $\tilde{B}{=}1$ and putting $N_f{=}3,5$ values for b and 'a' values corresponding to the energies and $\Lambda_{\overline{\mathrm{MS}}}$ considered in Table 6.3 yields an ambiguity $\lesssim 10^{-4}$ for $\tilde{D}$, $\tilde{R}$, $\tilde{R}_\tau$, so the significant figures quoted in the resummed result do not change. For $\tilde{K}$, however, the leading singularity is IR_1 and one would estimate the IR ambiguity $\sim 10^{-2}$ for $N_f{=}3$ and $Q^2{=}2.5\mathrm{GeV}^2$, which is clearly significant.

The column labelled 'FOPT' gives the fixed order perturbation theory results obtained at NNLO (up to $O(a^3)$) using the exact perturbative coefficients in the $\overline{\mathrm{MS}}$ scheme with $\mu{=}Q$ [52, 53, 63, 64, 65]. The coupling 'a' is defined using the NNLO truncated beta-function, $B(x){=}1 + cx + c_2^{\overline{\mathrm{MS}}}x^2$, in equation (6.24), with the values of $\Lambda_{\overline{\mathrm{MS}}}$ as above.

The column labelled 'Expt' gives the values determined from experimental data for $(1{+}\tilde{R})$ [74], $(1{+}\tilde{R}_\tau)$ [75] and $(1{-}\tilde{K})$ [76]. The results of adjusting $\Lambda_{\overline{\mathrm{MS}}}$ to fit the FOPT and resummed predictions to these experimental values are summarised in Table 6.4. Both fixed order perturbation theory and the leading-b resummed results exhibit RS-dependence and it is not at all obvious which procedure gives the closest approximation to the all-orders sum. Work is currently in progress [77] to resum the leading-b part of the effective charge RS invariants discussed briefly at the end of section 6.2. Achieving this would solve the problems of dependence of the results upon $B(x)$ and v.

To conclude this section let us consider how the leading-b resummation might

Observable	N_f	$\Lambda_{\overline{\rm MS}}$/MeV fitted to experimental data	
		Resummed	NNLO FOPT
R	5	98^{+93}_{-55}	287^{+226}_{-145}
R_τ	3	159^{+9}_{-10}	386^{+21}_{-26}
K	3	375^{+20}_{-33}	495^{+15}_{-16}

Table 6.4: Values of $\Lambda_{\overline{\rm MS}}$ adjusted to fit the predictions of NNLO FOPT and the resummed results to the experimental data for $(1+\tilde{R})$ [74], $(1+\tilde{R}_\tau)$ [75] and $(1-\tilde{K})$ [76].

be improved. An obvious improvement would be to include the full branch point structure of the renormalon singularities by incorporating the subleading in N_f, $\overline{\alpha}_\ell$ and $\overline{\beta}_\ell$, pieces of the exponents in equation (6.35) into the resummations. One expects

$$\begin{aligned} \overline{\alpha}_\ell &= -cz_\ell + \gamma_\ell \,, \\ \overline{\beta}_\ell &= cz_\ell + \gamma'_\ell \,. \end{aligned} \tag{6.74}$$

As we have already noted in Chapter 5, the first term can be deduced from RG considerations but the γ_ℓ and γ'_ℓ are the one-loop anomalous dimensions of the relevant operators [39, 48]. For $\tilde{D}$ it is known that IR_2 has a corresponding OPE operator with vanishing one-loop anomalous dimension, $\gamma'_2{=}0$ [40], so $\overline{\beta}_2{=}cz_2$ is known. To the best of our knowledge the remaining γ_ℓ and γ'_ℓ's are not known. One could nonetheless include the first RG-predictable terms in equation (6.74) in the resummations and see by how much the results change. The problematic RS-dependence of $D^{(L)}(a)$ would be qualitatively unchanged, however.

Further improvement could be achieved by including some of the subleading in N_f, barred, coefficients in the numerator polynomials in equation (6.35). A complete fixed order perturbative calculation for D up to $O(a^n)$ would enable the

series coefficients of $B[D](z)$ up to $O(z^n)$ to be determined exactly; but to obtain the coefficients of the numerator polynomials for each singularity up to $O(z^n)$, even given knowledge of the full branch point exponents discussed above, would still be very difficult. The improvement of perturbation series by developing a representation of the form of equation (6.35) with truncated numerator polynomials was suggested in reference [40].

6.5 Summary

In this chapter we have investigated the possibility of resummation to all orders of the leading term in the 'b-expansion' of QCD perturbative coefficients in equation (6.3). This expansion was introduced and motivated in Chapter 5 by a consideration of renormalon singularities in the Borel plane; and, if such singularities are present, then the $d_k^{(k)}b^k$ term should, when expanded in N_f, reproduce the N_f-expansion coefficients of equation (6.2) to all orders in N_f with asymptotic accuracy $O(1/k)$. We checked explicitly in section 6.2 that for the QCD Adler D-function ($\tilde{D}$) and Deep Inelastic Scattering sum rules ($\tilde{K}$) asymptotic dominance of the leading-b terms was already evident in comparisons with the exact NLO and NNLO perturbative coefficients for those quantities. This dominance was beyond that to be anticipated from equation (5.35) and we speculated as to why such a striking result might hold. The RS dependence of the leading-b coefficient and the need to give an all-orders definition of the coupling in order to perform resummations was discussed. In this light we also discussed the possibility of resumming to all orders the leading-b parts of RS invariant quantities, calculated in the effective charge formalism. This procedure would ultimately overcome some of the difficulties associated with trying to make comparisons between all-orders resummations and fixed order perturbation theory.

In section 6.3 we used exact large-N_f results [43, 44, 57] to obtain partial information about the Borel transforms $B[\tilde{D}](z)$ and $B[\tilde{K}](z)$. Ultra-violet and infrared renormalon singularities are present and we obtained the constant coefficients in the numerator polynomials exactly; and the exponents, single and double poles, to leading-N_f (equations (6.36), (6.39)). We showed that in the V-scheme ($\overline{\rm MS}$ with μ=e$^{-5/6}Q$) the leading-b coefficients, $d_n^{(n)}$ and $K_n^{(n)}$, are dominated, even

in low orders, by the renormalon singularities nearest the origin, respectively UV_1 and UV_1+IR_1 combined (see Tables 6.1 & 6.2).

For each quantity we split the leading-b Borel sum into UV and IR poles. The first contribution could be evaluated exactly in terms of the exponential integral function $\mathrm{Ei}(x)$ (equation (6.45)) and elementary functions; and the second, IR, contribution could be obtained as a principal value, in terms of a principal value of $\mathrm{Ei}(x)$. We showed how to modify the Borel transform for the Minkowski continuations of $\tilde{D}$, the e^+e^- R-ratio ($\tilde{R}$) and the R-ratio for τ-decay ($\tilde{R}_\tau$), and a similar resummation was performed for these Minkowski quantities in terms of a generalised $\mathrm{Ei}(n,w)$ function (equations (6.56) and (6.60)).

In this way we obtained the $D^{(L)}$ component of the split defined in equation (6.4) for the above quantities. Unfortunately the result obtained by summing the leading-b terms is RS-dependent, $D^{(L)}(a)$. In section 6.4 we showed that one can maximize $D^{(L)}(a)$ and hence perhaps minimize $D^{(NL)}(a)$ for any given choice of finite scheme and subtraction procedure. Maximizing $D^{(L)}(a)$ whilst choosing the 't Hooft scheme and a variant of minimal subtraction (v=0) was equivalent to using the one-loop coupling in the V-scheme, the choice also made in references [59, 60, 61].

We compared our resummed results with the principal values of the Borel sum for $D(m_\tau^2)$ and $R(m_\tau^2)$ quoted in reference [61] and with that for R_τ in references [60, 61] and found agreement. They are tabulated in Table 6.3. The procedures '1' and '2' for continuing the running coupling representation of reference [61], for R and R_τ, to the Minkowski region were shown to correspond to using different continuations of the $\mathrm{Ei}(n,w)$ function from $\mathrm{Re}\,w > 0$ to $\mathrm{Re}\,w < 0$. Only a continuation of $\mathrm{Ei}(n,w)$ as a real function enables one to evaluate correctly the well-defined pieces of the Borel sum involving single IR renormalon poles with a $\sin\frac{\pi bz}{2}$ factor. The inequivalent procedures of reference [61] are seen to correspond to various ways of wrongly evaluating the Borel sum and are therefore spurious. We stress that the Borel sum with IR singularities identified and principal value regulated provides a unique result and, in our view, a firm foundation for combining vacuum condensates in the OPE with IR renormalons to achieve a well-defined overall result (see reference [39]). There is, of course, ambiguity due

to the IR poles and we estimated this to be $\lesssim 10^{-4}$ for the $\tilde{D}$, $\tilde{R}$, $\tilde{R}_\tau$ resummed results in Table 6.3, so the displayed significant figures should be valid; but much larger, $\sim 10^{-2}$, for $\tilde{K}$ due to the presence of an IR_1 singularity. However, let us reiterate that, in a full calculation, ambiguities associated with IR renormalons would cancel exactly against non-perturbative ambiguities from the condensate operators in the OPE to give a well defined result for the quantities discussed.

We also compared with fixed order perturbation theory up to NNLO for these quantities. Since both $D^{(L)}(a)$ and the NNLO fixed order result suffer from RS dependence it is unclear which is the more reliable and further investigation of this question is required.

We finally considered how the leading-b resummation might be improved by including more exact information about the Borel transform.

Chapter 7

Conclusion

The broad aim of this thesis is to look at ways in which we can improve our understanding of QCD. On the one hand we need to increase the accuracy and reliability of perturbative predictions, since only by doing this can we strengthen our confidence in QCD as the fundamental quantum field theory of strong interactions. Moreover, by refining the predictive capabilities of QCD, we enhance our ability to uncover new phenomena. On the other hand we would like to clarify the links between the perturbative and non-perturbative aspects of the theory.

These two goals are of course related. We saw in Chapter 3 how resumming QCD perturbation series can help us achieve greater predictive power; but this is tempered by the realisation that, for perturbative series in the strong coupling constant, we find there is necessarily zero radius of convergence.

Thus, in order to use the technique of resummation to all orders for most QCD quantities, we require a knowledge of the large order behaviour of perturbation theory. This ultimately leads us to the connection with the non-perturbative regime.

We find that the problem is somewhat more pressing in QCD than in QED as a result of the relative sizes of the coupling constants. That is, divergence of the perturbative series (which are assumed to be asymptotic) appears to set in at an order $\sim$(coupling constant)$^{-1}$, which for QED means at approximately 130th order but for QCD around 8th order (or lower for energies $< M_Z$).

It is obvious that we must find ways to cope with such problematic behaviour and it is in this context that we introduce the Borel transform as a means of ameliorating the situation. By investigating the structure of the Borel transform

for various QCD quantities, we hope at least to encode the large order behaviour and at best actually to reconstruct physical quantities from their divergent perturbative expansions.

Using Borel methods to analyse the behaviour of perturbative series at large orders, we find that, although instantons, related to the proliferation of Feynman diagrams and first identified as the culprits in causing divergences, are of interest, the overwhelming contribution to the divergent nature of QCD perturbative expansions comes from 'renormalon' singularities in the Borel transform. They correspond to the factorial divergences induced by a specific class of Feynman diagrams, namely those with a number of fermion bubble insertions.

The renormalon singularities are divided into UV and IR singularities. In QCD, due to asymptotic freedom and the resulting sign of the first coefficient of the beta-function, the UV renormalons lie on the negative real axis of the Borel variable and pose no problems in the reconstruction of a physical quantity from its divergent perturbative expansion. The IR renormalons, however, are of more interest since they introduce a genuine ambiguity into the reconstruction process.

Despite the hindrance to the resummation programme caused by IR renormalons, we are obliged to accept that they are necessary for QCD to be consistent. That is, we believe that IR renormalons are directly connected to non-perturbative effects characterised by the operator product expansion and that the only way in which one can unite perturbation theory with the non-perturbative regime is by systematic cancellation between IR renormalon induced ambiguities and ambiguities originating in the OPE.

It was in this light that we focused on the Adler D-function in Chapter 5, where we demonstrated, by using exact large-N_f results from QED, that the singularity structure in the Borel plane was precisely in line with expectations from the OPE. Most notably this involved the absence of the first IR renormalon at $z = 2/b$, corresponding to the absence from the OPE of an operator with dimension two.

In Chapter 6 we implemented the resummation programme in earnest for a number of QCD quantities. We asserted that, with our current state of knowledge, the principal-value regulated Borel sum provides the most satisfactory framework

for combining IR renormalon ambiguities with the vacuum condensate ambiguities in the OPE. This is not to say that this method will not be superceded, since there is little compelling physical motivation for any one particular method. The other problem associated with the 'leading-b resummation' approach is that, by neglecting pieces of the full resummed result which are subleading in b, we introduce a renormalization scheme dependence into our result. We explored some ways of attempting to minimize the contribution of subleading terms but, without a full calculation, RS-dependence will always be a problem. Of course, this RS-dependence is qualitatively no worse than that inherent in any fixed order perturbative calculation; but it renders comparison with any other results difficult, especially since we find no basis for choosing a particular scheme.

There are approaches to the problem of RS-dependence such as the effective charge formalism and optimized perturbation theory which may bear fruit when applied in this particular context. There is also scope for improving the resummed results by considering a more detailed branch point structure for the singularities (which is already known to some extent from renormalization group considerations) and by incorporating next-to-leading order results into the numerator polynomials in equation 6.35.

In conclusion, we have provided a framework for obtaining resummed results for a range of QCD observables from their Borel transforms, which should provide a convenient starting point for the tackling of the various problems highlighted above. With the recent resurgence of interest in the subject, we expect the near future to hold some exciting developments.

Appendix A

Borel Transforms for Minkowski Quantities

The following argument was used by Brown and Yaffe [47] to relate the Borel transforms of a function in the physical and deep Euclidean regions.

Let us start with the RG-improved coupling $a(Q^2)$ defined via a simple beta-function, $\beta(a) = -ba^2$. Now consider a scalar function $f(-t)$ which is analytic across most of the $t = -q^2$ plane, such that in the deep Euclidean region, where $t \to \infty$, it has a perturbative expansion in $a(-t)$ of the form

$$f(-t) = \sum_{n=0}^{\infty} \tilde{f}_n a(-t)^{n+1} , \tag{A.1}$$

where the coefficients $\tilde{f}_n$ are real.

Now let us analytically continue this expansion from the region where t is real and negative to the region where $t = s + i0^+$, with s real and negative. The expansion then has the form:

$$f(s) = \sum_{n=0}^{\infty} f_n a(s)^{n+1} . \tag{A.2}$$

The running coupling is given, with our simple beta-function, by

$$a(-t) = \frac{a(\mu^2)}{1 + \frac{1}{2}ba(\mu^2)\ln(-t/\mu^2)} . \tag{A.3}$$

This gives us the relation between the couplants before and after continuation:

$$\frac{1}{a(-t)} = \frac{1}{a(s)} + \frac{b}{2}\ln\left(\frac{-t}{s}\right) , \tag{A.4}$$

$$-t = se^{-i\pi} . \tag{A.5}$$

We could then use this to rewrite equation (A.1) as an expansion in $a(s)$ and, by comparing this with equation (A.2) we could derive an expression for the f_n coefficients in terms of the $\tilde{f}_n$'s.

This, however, involves some messy and complicated algebra. Instead we note that Brown and Yaffe expressed the relation between the coefficients in a much more succinct way. They introduced a dummy variable, z, and wrote

$$\sum_{m=0}^{\infty} \frac{f_m z^m}{m!} = \exp[i\pi bz/2] \sum_{m=0}^{\infty} \frac{\tilde{f}_m z^m}{m!} \,. \tag{A.6}$$

This equation simply relates the Borel transforms for the expansion of f in the physical and deep Euclidean regions. Extracting the imaginary part (remembering that the $\tilde{f}_n$ are real), we find that

$$B[f(-t)](z) = \frac{B[\mathrm{Im}f(s)](z)}{\sin \pi bz/2} \,. \tag{A.7}$$

This general result gives a set of poles at the usual positions in the z-plane. It can also be used to relate the Borel transform of physically relevant quantities, such as the R-ratio for e^+e^- annihilation into hadrons, to the Borel transform of the Adler D-function.

To do this we consider f to be Π, the scalar part of the photon polarization tensor, as defined in Chapter 5. Then, from the optical theorem we know that

$$R = 12\pi \mathrm{Im}\Pi \,. \tag{A.8}$$

So, using Brown and Yaffe's result, we have

$$\begin{aligned} B[\Pi](z) &= \frac{B[\mathrm{Im}\Pi](z)}{\sin \pi bz/2} \\ \Rightarrow B[\Pi](z) &\sim \frac{B[R](z)}{\sin \pi bz/2} \,. \end{aligned} \tag{A.9}$$

Then

$$\begin{aligned} B[D](z) &\sim B\left[-Q^2 \frac{d\Pi}{dQ^2}\right](z) \\ &\sim \frac{\pi bz}{2} B[\Pi](z) \end{aligned} \tag{A.10}$$

and finally

$$B[R](z) = \frac{\sin(\pi bz/2)\, B[D](z)}{\pi bz/2} \,, \tag{A.11}$$

the result which we quote in equation (5.49).

Bibliography

[1] H.Yukawa, Proc. Phys. Math. Soc. Japan **17** (1935) 48.

[2] F.Halzen and A.D.Martin, *Quarks and Leptons*, John Wiley & Sons, New York, 1984;
I.J.R.Aitchison and A.J.Hey, *Gauge Theories in Particle Physics*, Adam Hilger, Bristol, 1989;
D.H.Perkins, *Introduction to High Energy Physics*, Addison-Wesley, Reading, *Mass.*, 1987;

[3] M.Gell-Mann and Y.Ne'eman, *The Eightfold Way*, Benjamin, New York, 1964.

[4] M.Gell-Mann, Phys. Lett. **8** (1964) 214;
G.Zweig, CERN report 8182/TH401.

[5] For a review of the early experiments see J.Friedman and H.Kendall, Ann. Rev. Nucl. Sci. **22** (1972) 203.

[6] T.Muta, *Foundations of Quantum Chromodynamics*, World Scientific Publishing, Singapore, 1987.

[7] P.A.M.Dirac, Phys. Zeit. der Sowjet Union **3** (1933) 64.

[8] R.P.Feynman, Reviews of Mod. Phys. **20** (1948) 367.

[9] L.H.Ryder, *Quantum Field Theory*, CUP, Cambridge, 1985.

[10] C.Itzykson and J.-B.Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980.

[11] M.Kaku, *Quantum Field Theory*, OUP, New York, 1993.

[12] R.P.Feynman, *The Theory of Fundamental Processes*, Benjamin, New York, 1962.

[13] S.Weinberg, Phys. Rev. **118** (1960) 838.

[14] G.Hanson et al., Phys. Rev. Lett. **35** (1975) 1609.

[15] T.Kinoshita, J. Math. Phys. **3** (1962) 650;
T.D.Lee and M.Nauenberg, Phys. Rev. **B133** (1964) 1549;
M.R.Pennington, Rep. Prog. Phys. **46** (1983) 393.

[16] D.Amati and G.Veneziano, Phys. Lett. **B83** (1979) 87;
A.Bassetto, M.Ciafaloni and G.Marchesini, Phys. Lett. **B83** (1979) 207;
G.Marchesini, L.Trentadue and G.Veneziano, Nucl. Phys. **B181** (1980) 335.

[17] Ya.I.Azimov, Yu.L.Dokshitzer, V.A.Khoze and S.I.Troyan, Phys. Lett. **B165** (1985) 147; Zeit. Phys. **C27** (1985) 65.

[18] JADE Collaboration, W.Bartel et al., Zeit. Phys. **C33** (1986) 23;
S.Bethke et al., Phys. Lett. **B213** (1988) 235.

[19] S.Bethke, J. Phys. **G17** (1991) 1455.

[20] Yu.L.Dokshitzer, D.I.Dyakonov and S.I.Troyan, Phys. Rep. **58** (1980) 270.

[21] A.Bassetto, M.Ciafaloni and G.Marchesini, Phys. Rep. **100** (1983) 202.

[22] N.Brown and W.J.Stirling, Phys. Lett. **B252** (1990) 657.

[23] N.Brown and W.J.Stirling, Zeit. Phys. **C53** (1992) 629.

[24] Yu.L.Dokshitzer: Contribution to the Workshop on Jets at LEP and HERA, Durham, December 1990.

[25] S.Catani, Yu.L.Dokshitzer, M.Olsson, G.Turnock and B.R.Webber, Phys. Lett. **B269** (1991) 432.

[26] S.Bethke, Z.Kunszt, D.E.Soper and W.J.Stirling, Nucl. Phys. **B370** (1992) 310.

[27] A.H.Mueller, Phys. Lett. **B104** (1981) 161;
B.I.Ermolaev and V.S.Fadin, JETP Lett. **33** (1981) 285.

[28] Yu.L.Dokshitzer, V.S.Fadin and V.A.Khoze, Zeit. Phys. **C15** (1982) 325; **C18** (1983) 37;
Yu.L.Dokshitzer and S.Troyan, Leningrad Report LNPI-922 (1984).

[29] Yu.L.Dokshitzer, V.A.Khoze and S.Troyan, in "Perturbative QCD", ed. A.H.Mueller (World Scientific, Singapore, 1989).

[30] G.Turnock, Cavendish Laboratory preprint CAVENDISH-HEP-92-3 (1992).

[31] F.Dyson, Phys. Rev. **85** (1952) 631.

[32] For a review see: Current Physics – Sources and Comments, vol.7, "Large-Order Behaviour of Perturbation Theory", eds. J.C.Gillou and J.Zinn-Justin, (1990).

[33] A.Duncan and S.Pernice, Phys. Rev. **D51** (1995) 1956.

[34] G.'t Hooft, in "The Whys of Subnuclear Physics", Erice 1977, ed. A.Zichichi (Plenum, New York 1977).

[35] C.M.Bender and T.T.Wu, Phys. Rev. Lett. **37** (1976) 117.

[36] B.Lautrup, Phys. Lett. **B69** (1977) 109.

[37] G.Parisi, in "Hadron Structure and Lepton-Hadron Interactions", Proceedings, Cargèse, 1977, ed. M.Lévy, Plenum, 1979; Phys. Lett. **B76** (1978) 65; Phys. Rep. **49** (1979) 215.

[38] G.Parisi, Nucl. Phys. **B150** (1979) 163.

[39] G.Grunberg, Phys. Lett. **B325** (1994) 441.

[40] A.H.Mueller, Nucl. Phys. **B250** (1985) 327.

[41] K.G.Wilson, Phys. Rev. **179** (1969) 1499.

[42] M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Nucl. Phys. **B147** (1979) 385, 448, 519.

[43] M.Beneke, Nucl. Phys. **B405** (1993) 424.

[44] D.J.Broadhurst, Zeit. Phys. **C58** (1993) 339.

[45] A.H.Mueller, in "QCD - Twenty Years Later", Aachen, vol.1, pp.162-171, edited by P.M.Zerwas and H.A.Kastrup. (World Scientific 1992).

[46] G.P.Lepage and P.Mackenzie, Nucl. Phys. (Proc. suppl.) **20** (1991) 173.

[47] L.S.Brown and L.G.Yaffe, Phys. Rev. **D45** (1992) R398;
L.S.Brown, L.G.Yaffe and C.Zhai, Phys. Rev. **D46** (1992) 4712.

[48] A.I.Vainshtein and V.I.Zakharov, Phys. Rev. Lett. **73** (1994) 1207.

[49] A.R.White, in "Physics Simulations at High Energy", Madison, pp.77-88, edited by V.Barger, T.Gottschalk and F.Halzen, (World Scientific 1986).

[50] A.R.White, Int. Journal of Mod. Phys. **A11** (1991) 1859.

[51] D.R.T.Jones, Nucl. Phys. **B75** (1974) 537;
W.E.Caswell, Phys. Lett. **33** (1974) 244.

[52] S.G.Gorishny, A.L.Kataev, S.A.Larin, Phys. Lett. **B259** (1991) 144;
L.R.Surguladze and M.A.Samuel, Phys. Rev. Lett. **66** (1991) 560; **66** (1991) 2416 (Erratum).

[53] M.Dine and J.Sapirstein, Phys. Rev. Lett. **43** (1979) 668.

[54] S.J.Brodsky and H.J.Lu, Phys. Rev. **D51** (1995) 3652.

[55] S.G.Gorishny, A.L.Kataev and S.A.Larin, Phys. Lett. **B212** (1988) 238.

[56] H.Kawai, T.Kinoshita and Y. Okamoto, Phys. Lett. **B260** (1991) 193.

[57] D.J.Broadhurst and A.L.Kataev, Phys. Lett. **B315** (1993) 179.

[58] S.J.Brodsky, G.P.Lepage and P.B.Mackenzie, Phys. Rev. **D28** (1983) 228.

[59] M.Beneke and V.M.Braun, Phys. Lett. **B348** (1995) 513.

[60] P.Ball, M.Beneke and V.M.Braun, CERN–TH/95–26, [hep–ph/9502300].

[61] M.Neubert, CERN–TH7524/94, [hep–ph/9502264];
M.Neubert, Phys Rev. **D51** (1995) 5924.

[62] M.Beneke and V.M.Braun, Nucl. Phys. **B426** (1994) 301.

[63] D.J.Broadhurst and A.G.Grozin, OUT–4102–52, [hep–ph/9410240].

[64] K.G.Chetyrkin, A.L.Kataev and F.V.Tkachov, Phys. Lett. **B85** (1979) 277;
W.Celmaster and R.J.Gonsalves, Phys. Rev. Lett. **44** (1980) 560.

[65] S.G.Gorishny and S.A.Larin, Phys. Lett. **B172** (1986) 109;
E.B.Zijlstra and W. van Neerven, Phys. Lett. **B297** (1992) 377.

[66] W.Celmaster and R.J.Gonsalves, Phys. Rev. **D20** (1979) 1420.

[67] P.M.Stevenson, Phys. Rev. **D23** (1981) 2916.

[68] G.Grunberg, Phys. Lett. **B95** (1980) 70;
G.Grunberg, Phys. Rev. **D29** (1984) 2315.

[69] D.T.Barclay, C.J.Maxwell and M.T.Reader, Phys. Rev. **D49** (1994) 3480.

[70] M.Beneke, Ph.D. Thesis "Die Struktur der Störungsreihe in hohen Ordnungen", München, 1993.

[71] X.Ji, MIT–CTP–2381 (1994), [hep–ph/9411312].

[72] Handbook of Mathematical Functions, ed. Milton Abramowitz and Irene A.Stegan (Dover), Section 5.1, p228.

[73] E.Braaten, S.Narison and A.Pich, Nucl. Phys. **B373** (1992) 581.

[74] LEP Collaborations Joint Report No. CERN-PPE/93-157 (1993) .

[75] This value is quoted in the review by A.Pich, preprint FTUV–94–71 (1994), [hep–ph/9412273].

[76] J.Ellis and M.Karliner, Phys. Lett. **B341** (1995) 397.

[77] C.J.Maxwell and D.G.Tonge, work in progress.