Z(_N)-symmetric field theories and the thermodynamic Bethe ansatz

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$\mathbb{Z}_N$-SYMMETRIC FIELD THEORIES

AND THE

THERMODYNAMIC BETHE ANSATZ

by

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Abstract

This thesis is concerned with perturbed conformal field theory, the thermodynamic Bethe ansatz technique and applications to statistical mechanics. In particular, the phase space of two dimensional $\mathbb{Z}_N$-symmetric statistical models is examined using these techniques.

The aim of the first two chapters is to review some general material concerning statistical mechanics, perturbed conformal field theory, integrable two-dimensional quantum field theory and the thermodynamic Bethe ansatz (TBA) technique. In the third chapter $\mathbb{Z}_N$-symmetric statistical theories are discussed and the known features of the phase space of such models are surveyed. The field content of the conformal models in this space (called parafermionic models) is investigated in order to determine which perturbations can be used to investigate the phase space.

In the fourth and fifth chapters TBA equations are proposed to describe massless and massive renormalisation flows from the $\mathbb{Z}_N$-symmetric conformal theories under self-dual $\mathbb{Z}_N$-symmetric perturbations. According to the sign of the perturbation parameter the infrared limits are shown to be either conformal $c = 1$ or massive theories. The ground state energies of these models can be discovered in all perturbative regimes via the TBA method and the results agree with perturbation theory in ultraviolet and infrared limits. Results from detailed studies of the $N = 5, 6..10$ models are presented throughout. It is also deduced that the parafermionic models lie exactly at the bifurcation point of the first-order transition region into the Kosterlitz-Thouless region of the $\mathbb{Z}_N$-symmetric phase space.

The sixth and seventh chapters deal solely with massive perturbations. In chapter six, results from the TBA equations are used to deduce the mass spectrum and the vacuum structure of the underlying scattering theory. In chapter seven, proposals for the massive S-matrices are made. For N odd the mass spectra proposed by the TBA method and that predicted by the S-matrix approach (using the minimality principle) differ. It is suggested therefore, that the N odd S-matrices contain zeroes in the physical strip, violating the minimality principle.

Declaration

This thesis is the result of research carried out between April 1994 and October 1996. The special study described has been carried out and composed by me and this work has not been accepted in fulfilment of any other degree or professional qualification.

The material of the first three chapters is a review of material from several authors and, my interpretation apart, no claim is made of originality. Chapters four, five and six arose from a collaboration between myself, P. Dorey and R. Tateo, resulting in the publication *Massive and massless phases in self-dual $\mathbb{Z}_N$ spin models: Some exact results from the thermodynamic Bethe ansatz*: Nuclear Physics B470 (1996) 317-368. These chapters also contain several calculations and explanations not presented in this paper. Chapter seven contains independent work concerning the S-matrix for the massive $\mathbb{Z}_N$-symmetric perturbed theory. The appendices also contain original material. In particular, Appendix A contains a more detailed justification for the $\mathbb{Z}_N$-symmetric TBA systems than has appeared thus far.

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For my parents

Contents

Chapter 1

Statistical Mechanics and Perturbed Conformal Field Theory

The theory of statistical mechanics is based upon the hypothesis that a model describing the interactions between a few individual components on a microscopic level can be used to infer the macroscopic behaviour of a real system. Through prudent choice of parameters in the model one hopes to describe all the observable phases of the macroscopic system and the nature of transitions between those phases. We should be able to predict how thermodynamic quantities like the spontaneous magnetisation or susceptibility behave as the experimenter varies laboratory parameters such as temperature, pressure, or the external magnetic field. Furthermore, we should be able to predict the set of numbers called critical exponents which characterise these quantities near a phase transition. To do this, we must first evaluate the partition function: the sum of all the Boltzmann weights associated with each system configuration. In practice, this is a sum which must be approximated. The renormalisation group method provides a means of doing so by relating the parameters of the model on two scales via a renormalisation group map. The fixed points of this map represent critical theories which are not only scale invariant but conformally invariant.

In two dimensions, using a quantum field theoretical approach, the conformal symmetry at a fixed point can often be exploited to deduce the field content and critical exponents completely. A quantity called the central charge provides a partial classification of such conformal field theories (the classification is only partial because several

such theories can possess the same central charge). Perturbations of conformal models are important because much of the symmetry which exists at a fixed point persists away from that point, particularly for integrable theories where several conserved charges can be found. As a consequence many models which lie between phases (of order and disorder say) can be described precisely in terms of perturbed conformal models. It is therefore very useful to be able to construct a central charge function, a simple function of the ground state energy, so that small scale (ultraviolet) and large scale (infrared) limits of a perturbed theory may be investigated. Additionally, since perturbations into massive and massless phases are simply characterised by either a zero or nonzero central charge, a central charge function is useful when trying to distinguish which phase the perturbation describes.

For integrable theories the thermodynamic Bethe ansatz (TBA) method lets us construct a central charge function which gives exact details about the theory in all regimes including those inaccessible to perturbation theory. The renormalisation group method, perturbed conformal field theory and integrability are reviewed in this chapter. Scattering theory and the basic TBA idea are presented in the next. In the remaining chapters the TBA method is developed and applied to construct central charge functions for perturbations of $\mathbb{Z}_N$-symmetric conformal models to investigate the phases of $\mathbb{Z}_N$-symmetric statistical models.

1.1 Statistical mechanics and the renormalisation group

Consider N spins positioned over a square lattice, the spin s_i at site i taking values in a discrete range. The Hamiltonian $H_N(K, \{s_i\})$ describes the interaction between these spins with a set of couplings K. In an idealised state of statistical equilibrium all macroscopic behaviour may be determined once the classical partition function

$$Z(H_N(K)) = Tr_{\{s_i\}} e^{-\frac{1}{k_\beta T} H_N(K,\{s_i\})} \tag{1.1}$$

is evaluated. This is the sum of Boltzmann weights taken over all spin configurations. k_β is the Boltzmann constant and T the heatbath temperature. The trace consists of N sums, one for each of the lattice sites. If, for example, spins can only point up or down, the trace would be $Tr_{\{s_i\}} = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1}$.

The probability that the system will be found in a configuration $\{t_i\}$ is the Boltzmann weight for that configuration divided by the partition function. If $F(s_i)$ is some functional of the spins, then its averaged thermal expectation value is

$$\langle F(s_i) \rangle = \frac{1}{Z(H_N(K))} Tr_{\{s_i\}} F(s_i) e^{-\frac{1}{k_\beta T} H_N(K,\{s_i\})} . \tag{1.2}$$

Of central interest is the way thermodynamic quantities behave near criticality where our system undergoes some phase transition. Almost all such quantities can be expressed as derivatives of the Helmholtz free energy

$$f(H_N) = -\frac{1}{N} \log Z(H_N) \tag{1.3}$$

with respect to the various coupling parameters. Evaluation of the partition function is then our main objective if we want to predict the form of these functions. We are particularly interested in the thermodynamic limit where N becomes very large. The singular behaviour that an experimenter observes in thermodynamic quantities can only be described by letting $N \to \infty$ in the model, since neither the free energy nor its derivatives can exhibit singularities or discontinuities for finite N. Given this, it is still possible to make predictions about the nature of any singularities using finite-size scaling methods where one attempts to split the free energy into two parts, one which becomes singular in the large N limit and a part which remains non-singular. However, for a finite lattice with N sites there are 2^N possible configurations and therefore 2^N Boltzmann weights to be summed. In a realistic model one might wish to discuss correlations over a physical distance of at least 10^3 lattice spacings, that is $N = 10^6$ sites in two spatial

dimensions. Therefore, even for the simplest models summing over all configurations is practically impossible. Renormalisation group theory may be viewed as arising from an attempt to evaluate the partition function through an iterative method where at each stage of the iteration part of the sum is evaluated approximately. A discussion of the physical basis for the renormalisation group method follows (see [1] for more details).

1.1.1 Order parameters, phase transitions and correlation length

The correlation between spins at sites i and j on the lattice is given by

$$G_{ij} = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \,. \tag{1.4}$$

G_{ij} measures the variation of the spin field between points i and j a distance x apart. Consider, for example, a ferromagnet which above some critical temperature has a disordered phase of unbroken symmetry with average magnetisation zero: $|\langle s_i \rangle| = 0$. Below this temperature $|\langle s_i \rangle|$ is non-zero. Such a quantity, whose thermal average vanishes on one side of a phase transition but not on the other, is called an order parameter. For large distances and away from criticality the correlation will behave like a decreasing power of x multiplied by an exponential

$$G_{ij} \sim x^{-\eta} e^{-|x|/\xi} \,. \tag{1.5}$$

The constant η is a critical exponent and ξ is the correlation length. This is the width of the largest area where a group of spins as a whole all disagree with the mean. As the temperature increases from zero so does the correlation length until at some critical temperature it becomes either infinite or takes its maximum if the system is finite.

A crucial point to note is that fluctuations occur at all wavelengths between the atomic (lattice) spacing and the correlation length. This means that in the statistical limit one must account for all fluctuations of size less than ξ and cannot ignore the smaller oscillations. Furthermore, length scales are locally coupled, which means that if the lattice spacing is a units, then fluctuations of size $1000a - 2000a$ are primarily influenced by fluctuations in the ranges $500a - 1000a$ and $2000a - 4000a$. The result is a cascade where fluctuations over $1a - 2a$ influence those over $2a - 4a$, which in turn effect fluctuations on $4a - 8a$ and so on. The implications are:

1. Scaling: Fluctuations at scales between the lattice spacing a and the correlation length ξ behave identically, that is, in terms of fluctuations the system is scale invariant between these barriers.

2. Amplification and de-amplification: A small change in temperature, which has little effect on the atomic scale, can create huge differences in the macroscopic system (very large wavelengths) due to cascading. This is amplification. The opposite can occur where two dissimilar materials with quite different atomic structures can have the same macroscopic behaviour near criticality. This is called de-amplification or universality.

The renormalisation group method exploits the physical properties of the system to replace the summing of the partition function with the summing over all fluctuations in the cascade. The idea is to evaluate the contribution from the smallest fluctuations in a way which leaves a distribution of unsummed spins looking like a scaled version of the original. The smallest fluctuation interactions are described by a coupling vector K for the Hamiltonian $H(K) = k_1 \sum + k_2 \sum + ...$ so that each coupling uniquely labels a Hamiltonian. If we assume only local neighbour interactions we need only consider a finite dimensional K. After performing the first sum the next smallest fluctuations will interact locally with coupling K'. The renormalisation group method gives a map for K' in terms of K and the scaling property can be used to apply this map iteratively as an alternative to performing more and more sums. Specifically, the method comes in three steps:

1. Divide the lattice into neighbourhoods of side b lattice units and pick out one spin s_i' in each neighbourhood so that the set $\{s_i'\}$ forms a new lattice which looks exactly like a scaled up version of the original, the new spin variables being a distance b apart.

2. Approximate the contribution to the partition function from the spins s_i in the neighbourhood of s_i'. Do this for each neighbourhood so that the contribution from the smallest wavelengths over the lattice is estimated as

$$e^{-N'g(K)-H_{N'}(K',s_i')}, \tag{1.6}$$

where K' is the new coupling between spins s_i' and $g(K)$ is some function of the coupling K appearing $N' = b^{-d}N$ times (once for each neighbourhood). The trace is a sum over all configurations or, equivalently, all fluctuation scales. Therefore, write the trace as separate sums: over smallest fluctuations and a sum over all other fluctuations

$$Tr_{\{s_i\}} e^{-H_N(K,\{s_i\})} = \sum_{l \geq ba} \sum_{a \leq l \leq ba} e^{-H_N(K,\{s_i\})}, \tag{1.7}$$

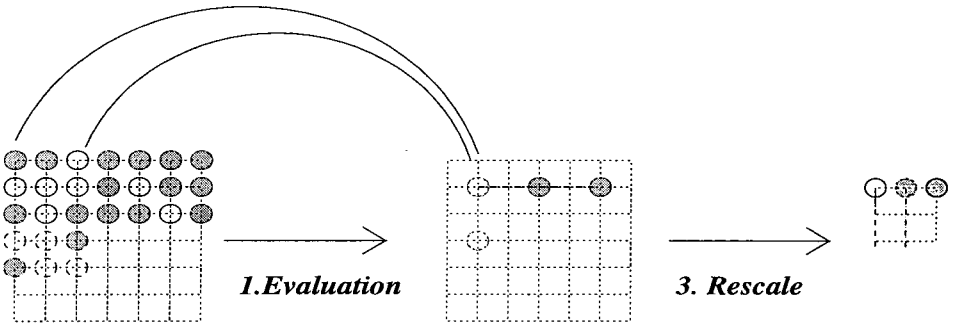


Figure 1.1: The evaluation and rescaling steps of renormalisation

where the sums are intended to be over fluctuations of size, l, less than and greater than b lattice units. If new couplings K' can be chosen so that the sum over smallest fluctuations can be written

$$\sum_{a\leq l\leq ba} e^{-H_N(K,\{s_i\})} = e^{-N'g(K)} e^{-H_{N'}(K',\{s'_i\})}\,, \tag{1.8}$$

then the partition function can be written covariantly

$$\begin{aligned} Z(H_N(K)) &= \sum_{l\geq ba} e^{-N'g(K)} e^{-H_{N'}(K',\{s'_i\})} \\ &= e^{-N'g(K)} Tr_{\{s'_i\}} e^{-H_{N'}(K',\{s'_i\})} \\ &= e^{-N'g(K)} Z(H_{N'}(K'))\,. \end{aligned} \tag{1.9}$$

The couplings K' are nonlinear functions of the couplings K, giving the discrete renormalisation group map

$$K' = R(K)\,. \tag{1.10}$$

The set of possible transformations $\{R\}$ actually forms monoid, rather than a group. This is because it is impossible to define an inverse of a given map since the renormalisation idea implicitly involves a loss of a certain amount of information at each step which cannot be regained by applying another renormalisation.

3. Re-scale the whole lattice by a scale factor b^{-1}, the aim being to set the system up for a reapplication of the first two steps (which is allowed by the scaling property of the physical system). Consequently, all physical lengths will be scaled by this factor. In particular, the correlation length on the new lattice ξ' is

$$\xi' = b^{-1}\xi\,. \tag{1.11}$$

Equation (1.9) implies the following important relationship satisfied by the free energy

$$f(K) = b^{-d}g(K) + b^{-d}f(K')\,, \tag{1.12}$$

which can be used to describe the behaviour of thermodynamical quantities near criticality. Barber [2] shows how iterations of this relation can be used to derive the critical exponent set for a given free energy by splitting iterations of (1.12) into analytic and singular parts and then deriving the singular part with respect to the relevant scaling parameters. The renormalisation method was used in this way to solve many long-standing problems in statistical mechanics which stemmed from an inability to describe these quantities when the correlation length becomes infinite.

Due to the scaling principle, (1)-(3) can be repeated to obtain

$$Z(H_N(K)) = e^{-N'g(K)-N''g(R(K))}Z(H_{N''}(R^2(K))\,, \tag{1.13}$$

where $N'' = b^{-2d}N$. Then, because we carry out the steps in exactly the same way, we begin to see the benefits of partly evaluating our partition function in this manner: *we can replace the evaluation of all the sum in the original partition function by studying the iterations of the map R instead.* When the lattice is infinite a partition function with microscopic coupling K is described by the asymptotics of the sequence $R^m(K)$ for large m.

There are several methods of approximating the first sum in step (2). The most basic are those of decimation and blocking. Decimation has been used above. In the blocking method we let each s_i' be the mode or mean spin of the neighbourhood.

1.1.2 The geometry of the renormalisation group flow

The first step toward understanding the flow (1.10) is to find the fixed points $K^\star$ satisfying $R(K^\star) = K^\star$. The flow in neighbourhoods of these points is described by the linearised version of the map. For K near $K^\star$ (so that $|dK|^2$ is small where $dK = K - K^\star$), the flow is well approximated by the linearised equation

$$K' = R(K^\star + dK) = K^\star + DR_{K^\star}dK\,. \tag{1.14}$$

Eigenvalues b^{y_i} of the Jacobi matrix $DR_{K^\star} = \left(\frac{\partial R_\alpha}{\partial K_\beta}\right)_{K^\star}$ determine the local nature of the flow. For each non-zero scaling dimension y_i there must exist a curve which arrives at or leaves the fixed point tangentially to the direction determined by the eigenvector $\hat{\varepsilon}_i$ of $DR_{K^\star}$. These eigenvectors define linear combinations of the interactions called scaling

fields whose couplings ζ_i in the Hamiltonian will have a simple power law behaviour $\zeta_i \longrightarrow b^{y_i}\zeta_i$ under the renormalisation transformation. If $y_i > 0$ ($y_i < 0$) the flow in the direction $\hat{\varepsilon}_i$ is said to be relevant (irrelevant), whilst in the marginal case $y_i = 0$ the linear approximation is insufficient to determine the local behaviour and one must resort to a nonlinear examination. A surface on which all flows are away from a fixed point may be called a relevant manifold of that point. Similar definitions hold for the irrelevant and marginal cases.

One can use the idea of the relevant, irrelevant and marginal manifolds in renormalisation theory to visualise phase transitions and critical surfaces and their relationship with the correlation length. The phase space of couplings is typically dissected by critical and non-critical manifolds flowing toward or away from fixed points of the renormalisation group map. As an illustration, consider a system dependent on two scaled couplings t and h. Suppose $h = h'$ is fixed and t is continuously varied so that we move along curve (a) in Figure (1.2) until, at $t = t_c$, we hit a critical point where the correlation length becomes infinite and where thermodynamic quantities look singular. Let us suppose the flow is restricted to the surface (A), then according to (1.11) the coupling (t_c, h') will map to another critical point under renormalisation. Repeated iterations take us toward or away from a fixed point according to the signs of the scaling parameters y_i. In the example shown in the figure, (t_c, h') flows on an irrelevant manifold of (t_c, h_c) to that fixed point.

In general, the Hamiltonian must contain sufficient terms to be consistent with the symmetries of the problem so that the renormalisation group flow remains in the space of chosen couplings. If an additional term is taken into consideration giving a three dimensional flow, (t_c, h_c) is no longer necessarily a fixed point. Instead, we might flow to the fixed point P^*.

A relevantly perturbed fixed point can either wander endlessly or will asymptotically arrive at another fixed point of the map under successive iterations (as the relevant perturbation of Q^* in the direction of P^* does). Both theories (t_c, h_c) and K_3 (just off the manifold (B)) flow to the same fixed point P^*. Renormalisation 'washes out' the smallest range fluctuations of each model, so the fact that different systems can flow to the same point means they share common large scale characteristics. This explains how systems consisting of very different molecular compositions (microscopically described by different Hamiltonians) can share exactly the same critical exponents when they undergo phase transitions (a macroscopic feature). This universality is due to the fact that the

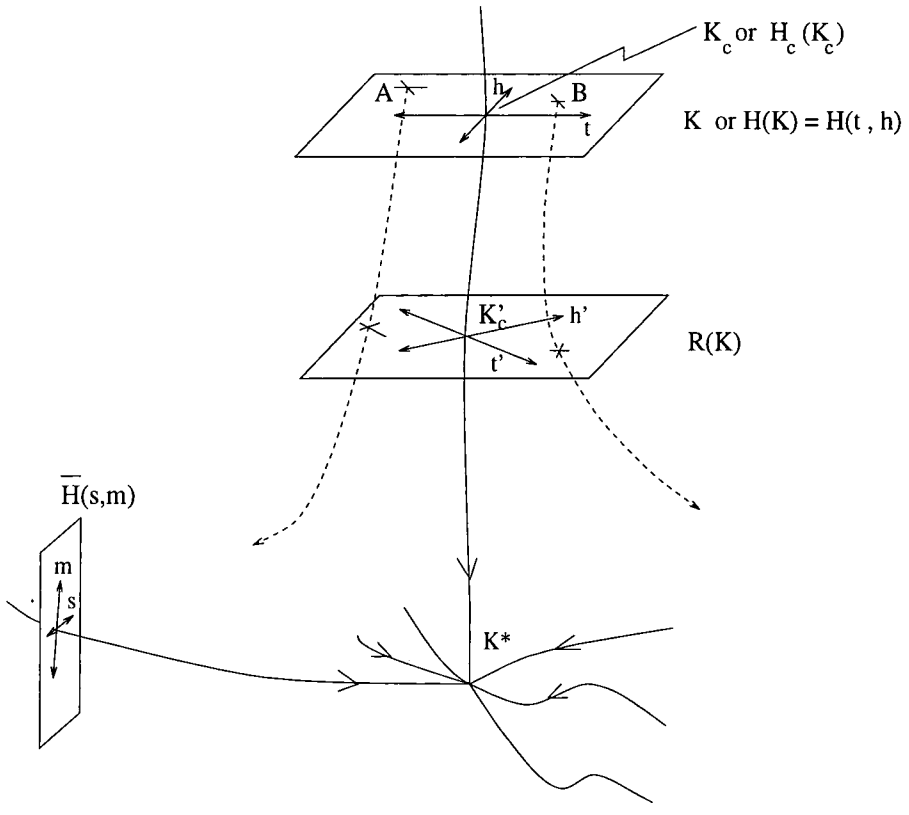


Figure 1.2: An example of a typical renormalisation group flow. Theories K_1 and K_2 flow apart under renormalisation demonstrating amplification, while (t_c, h_c) and K_3 flow to the same fixed point P^* demonstrating universality.

critical behaviour of two different Hamiltonians is described by the same fixed point of the renormalisation group map. The diagram also explains the contrasting amplification phenomenon: points K_1 and K_2 represent models originally close together in phase space but with very different large scale behaviours.

1.1.3 The renormalisation group with continuous fields

Replacing the set of spins over a lattice with a field ϕ over the continuum, the Hamiltonian becomes a sum of fields $\psi_i(r)$ (each of which may be a function of ϕ and its derivatives) integrated over all space with corresponding couplings g_i. Assuming the couplings have no spatial dependence, the partition function for a continuous field $\phi(r)$ is given by the reduced functional integral

$$Z[\phi] = \int [d\phi] e^{-\int d^d r \sum_i g_i \psi_i(r)} . \tag{1.15}$$

If it is possible to preserve the form of the partition function by some transformation of coupling parameters when we integrate over shortest wavelengths between a and $(1+\delta l)a$

and rescale the system by $r \to (1+\delta l)^{-1} r$, then in the limit $\delta l \to 0$ this will generate a system of nonlinear differential equations for the couplings given by

$$\frac{dg_i}{dl} = R_i(\{g_j\}), \tag{1.16}$$

where the scaling parameter l is an autonomous variable. Consider the behaviour of the renormalisation group flow in the neighbourhood of one of the fixed points of this system. Writing couplings g_i in terms of scaling couplings $\{\zeta_i\}$, each ζ_i will appear as a factor in front of a combination of fields which we label ϕ_i (ϕ_i may be any function of the original field ϕ). The Hamiltonian becomes

$$H(\phi) = H_{FP} + \sum_i \int d^d r \zeta_i \phi_i(r) \tag{1.17}$$

with $\zeta_i = 0$ representing the fixed point theory. If we assume that under the rescaling $r \longrightarrow r' = b^{-1} r$ the field ϕ_i transforms as

$$\phi_i(r) \longrightarrow \phi_i'(r') = \phi_i(br') = b^{-x_i} \phi_i(r'), \tag{1.18}$$

where x_i is called the scaling dimension of ϕ_i, the action for the perturbed theory will scale as

$$H \longrightarrow H' = H_{FP} + \sum_i b^{d-x_i} \int d^d r' \zeta_i \phi_i(r'). \tag{1.19}$$

Since specifying a coupling is equivalent to specifying a Hamiltonian, this transformation can equally be described in terms of the couplings as $\zeta_i \longrightarrow \zeta_i' = b^{y_i} \zeta_i$ so that $x_i + y_i = d$. Again, the invariant flow on relevant and irrelevant manifolds is determined by the scaling dimensions y_i.

Correlation functions of the fields $\phi_1(r_1), ..\phi_n(r_n)$ defined by

$$\langle \phi_1(r_1), ..\phi_n(r_n) \rangle = \frac{1}{Z} \int [d\phi] \phi_1(r_1)..\phi_n(r_n) e^{-H_{FP} - \sum_i \int d^d r \zeta_i \phi_i(r)}, \tag{1.20}$$

scale as $\langle \phi_1(r_1)...\phi_n(r_n) \rangle_H \longrightarrow \langle \phi_1'(r_1')...\phi_n'(r_n') \rangle_{H'}$. At criticality, the theory is scale invariant i.e. $H' = H = H_{FP}$ and the correlation functions satisfy

$$\langle \phi_1(r_1)...\phi_n(r_n) \rangle = b^{-x_1} b^{-x_2} \ldots b^{-x_n} \langle \phi_1(r_1')...\phi_n(r_n') \rangle, \tag{1.21}$$

where both sides are evaluated at the fixed point Hamiltonian.

1.1.4 Conformal invariance at fixed points

The physical properties of a system at fixed point criticality lead to the scaling hypothesis: the physical system is invariant under a scaling and renormalisation leaves the

Hamiltonian invariant. Polyakov [3] suggested that such fixed point critical theories can be considered to be invariant under local rescalings, not just global ones. The idea of scale invariance, together with assumed rotational and translational invariance, implies the fixed point theory is then invariant under the wider set of conformal transformations. We denote a conformally invariant Hamiltonian by H_{CFT}, so that $H_{FP} \equiv H_{CFT}$.

A general coordinate transformation $x^\mu \longrightarrow x^{\mu'}$ is said to be conformal if it leaves the arc length $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ invariant up to the scale change $ds^2 \longrightarrow ds'^2 = \Omega(x)ds^2$. In terms of the metric $g_{\mu\nu}$, a theory is conformally invariant if

$$g_{\mu\nu}(x)dx^\mu dx^\nu \longrightarrow g_{\alpha'\beta'}dx^{\alpha'}dx^{\beta'} = (g_{\alpha'\beta'}\partial^{\alpha'}_\mu \partial^{\beta'}_\nu)dx^\mu dx^\nu = \Omega(x)g_{\mu\nu}(x)dx^\mu dx^\nu \,. \quad (1.22)$$

The infinitesimal version of a general coordinate change is written $x^\mu \to x^{\mu'} = x^\mu + \varepsilon^\mu$, so that with a flat Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu}$ (with $\mu, \nu = 1, ..d$) the resulting change in the arc length is

$$\begin{aligned} ds^2 \longrightarrow ds'^2 = \eta_{\mu\nu}dx^{\mu'}dx^{\nu'} &= \eta_{\mu\nu}(\delta^\mu_\alpha + \varepsilon^\mu_{,\alpha})(\delta^\nu_\beta + \varepsilon^\nu_{,\beta})dx^\alpha dx^\beta \\ &= ds^2 + \eta_{\mu\nu}(\varepsilon^\nu_{,\beta}dx^\mu dx^\beta + \varepsilon^\mu_{,\alpha}dx^\alpha dx^\nu) \\ &= ds^2 + (\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu)dx^\mu dx^\nu \,. \end{aligned}$$

The infinitesimal transformation $x^\mu \to x^\mu + \varepsilon^\mu$ is therefore conformal if ε^μ satisfies the conformal Killing equation

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = \frac{2}{d}(\partial \cdot \varepsilon)\eta_{\mu\nu} \,. \quad (1.23)$$

When $d > 2$, the general solution, given by

$$\varepsilon^\mu = a^\mu + \omega^\mu_\nu x^\nu + b^{-1}x^\mu + (\lambda^\mu x^2 - 2x^\mu \lambda \cdot x) \,, \quad (1.24)$$

consists of parts corresponding to translations, rotations, scalings and what are called the special conformal transformations (the finite versions of which are combinations of a translation and inversion: $x'^\mu/x'^2 = x^\mu/x^2 + \lambda^\mu$). This solution has $d + d(d-1)/2 + 1 + d = (d+1)(d+2)/2$ free parameters in total.

The situation is significantly different in two dimensions. Here the Killing equation for infinitesimal conformal transformations is equivalent to the Cauchy-Riemann equations

$$\begin{aligned} \partial_1\varepsilon_1 &= \partial_2\varepsilon_2 \\ \partial_2\varepsilon_1 &= -\partial_1\varepsilon_2 \end{aligned} \quad (1.25)$$

which, defining $z = x^1 + ix^2$ and $\overline{z}$ its complex conjugate, imply that the functions $\varepsilon(z) = \varepsilon_1 + i\varepsilon_2$ and $\overline{\varepsilon}(\overline{z}) = \varepsilon_1 - i\varepsilon_2$ are analytic in z and $\overline{z}$ respectively. Considering

$\varepsilon(z) = -z^{n+1}$ and $\bar{\varepsilon}(\bar{z}) = -\bar{z}^{n+1}$ we observe that the conformal algebra is infinite. The corresponding generators of infinitesimal conformal transformations

$$l_n = -z^{n+1}\partial_z\,, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,, \tag{1.26}$$

satisfy the following algebra

$$\begin{aligned} \left[l_m, l_n\right] &= (m-n)l_{m+n} \\ \left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m+n} \\ \left[\bar{l}_m, l_n\right] &= 0. \end{aligned} \tag{1.27}$$

called the *local* conformal algebra; local because the generators involved yield transformations which are not necessarily analytic everywhere (not at the origin or infinity). If an infinitesimal conformal transformation is given by $f(z) = (-\sum_n a_n l_n)z$, then this is analytic everywhere only if $a_n = 0$ for all $n \neq 0, \pm 1$. The global generators $l_{+1,-1,0}$ (and $\bar{l}_{+1,-1,0}$) form a subalgebra. l_{-1} and $\bar{l}_{-1}$ are generators of translations, while the pair l_1 and $\bar{l}_1$ give the special conformal transformations. The dilatation and rotation generators are $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$.

The finite form of the infinitesimal global transformations is $f(z) = (az+b)/(cz+d)$, where $ad - bc \neq 0$. These are simply the invertible analytic maps on the Riemann sphere. The finite form of the local transformations are meromorphic functions which may have poles at zero or infinity. Together the global and local transformations

$$z \longrightarrow f(z)\,, \qquad\qquad \bar{z} \longrightarrow \bar{f}(\bar{z}) \tag{1.28}$$

form the direct product of holomorphic and anti-holomorphic transformations called the conformal group.

A field transforming like $\phi_i(r_i) \longrightarrow b^{-x_i}e^{is_i\theta}\phi_i(r_i')$ under a rotation through an angle θ and scaling by a factor b^{-1} is said to have scaling dimension x_i and spin s_i which are eigenvalues of $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ respectively. Furthermore, if the scaling transformation $r \to r' = b^{-1}r$ is replaced by a general conformal transformation written $r \to r' = b^{-1}(r)r$, local scaling couplings and fields ϕ_i renormalise as

$$\zeta_i'(r_i') = b(r_i)^{y_i}\zeta_i(r_i)\,, \qquad \phi_i'(r_i') = b(r_i)^{-x_i}\phi_i(r_i')\,, \tag{1.29}$$

and in contrast to (1.21) correlation functions satisfy the more general relation

$$\langle\phi_1(r_1), ..\phi_n(r_n)\rangle = b(r_1)^{-x_1}...b(r_n)^{-x_n}\langle\phi_1(r_1')...\phi_n(r_n')\rangle \tag{1.30}$$

at a fixed point.

1.2 Conformal field theory

A thorough understanding of a two dimensional conformally invariant statistical model can be achieved if we view the system in the continuum limit as a two dimensional quantum field theory with spacetime coordinates (x,t) replacing the spatial coordinates of statistical mechanics. Being critical, with infinite correlation lengths, these are field theories of massless particles, and correlations decay as power laws. Belavin, Polyakov and Zamolodchikov [4] demonstrated that the existence of the infinite dimensional conformal symmetry can heavily restrict the field content and the correlation functions of the model.

The energy-momentum tensor and generators of conformal transformations on fields

If we make an infinitesimal general coordinate transformation $x^\nu \longrightarrow x^\nu + \epsilon^\nu$ the response in the Hamiltonian action of the two dimensional theory may be written

$$\delta H = -\int \frac{d^2x}{2\pi} T_{\mu\nu}\partial^\mu \epsilon^\nu . \tag{1.31}$$

We may use this form to define the energy-momentum or stress-energy tensor $T^{\mu\nu}$. A theory invariant under infinitesimal rotations ($\epsilon^\nu = \omega^\nu_\kappa x^\kappa$) and translations ($\epsilon^\nu = a^\nu$) will have $T^{\mu\nu}$ symmetric, satisfying $\partial^\mu T_{\mu\nu} = 0$. Additionally, the theory is invariant under scaling ($\epsilon^\nu = \lambda x^\nu$) if the stress-energy tensor is traceless. In terms of correlation functions, equation (1.31) is equivalent to

$$\sum_{k=1}^{n} \langle \phi_1(x_1)..\delta_\varepsilon \phi_k(x_k)..\phi_n(x_n)\rangle = -\int \frac{d^2x}{2\pi}\partial^\mu \epsilon^\nu \langle T_{\mu\nu}\phi_1(x_1)...\phi_n(x_n)\rangle, \tag{1.32}$$

where $\delta_\varepsilon \phi(x)$ is the variation of the field under the general infinitesimal coordinate transformation and both correlation functions are evaluated at the conformal point.

Suppose the variation ε^ν is conformal in a domain R (containing the points x_i) and non-conformal in the complement R', the two regions being separated by contour C. Ensuring ε^ν vanishes sufficiently fast at infinity, integration by parts is permitted and the generator of the variations in (1.32) becomes

$$\begin{aligned} Q &= -\frac{1}{2\pi}\int_{R'} d^2x\ \partial^\mu \varepsilon^\nu(x)\, T_{\mu\nu} \\ &= -\frac{1}{2\pi}\left(\int_{R'} d^2x\ \varepsilon^\nu(x)\,\partial^\mu T_{\mu\nu} - \int_C ds\ n^\mu \varepsilon^\nu(x)\, T_{\mu\nu}\right) \\ &= \frac{1}{2\pi}\int_C ds\ n^\mu \varepsilon^\nu(x)\, T_{\mu\nu}\,, \end{aligned} \tag{1.33}$$

where n^μ is normal to C and we have used $\partial^\mu T_{\mu\nu} = 0$. For conformal transformations this conservation rule, together with the traceless and symmetric conditions, means only two independent combinations of components of $T_{\mu\nu}$ exist:

$$T = T_{11} - iT_{12}\,, \qquad \overline{T} = T_{11} + iT_{12}\,. \tag{1.34}$$

In complex coordinates $z = t + ix$ and $\overline{z} = t - ix$, $\partial^\mu T_{\mu\nu} = 0$ becomes $\partial_z \overline{T} = 0$ and $\partial_{\overline{z}} T = 0$, which means T and $\overline{T}$ have pure holomorphic and anti-holomorphic dependence

$$T = T(z)\,, \qquad \overline{T} = \overline{T}(\overline{z})\,. \tag{1.35}$$

The infinitesimal variations become $\varepsilon^z = \varepsilon^t + i\varepsilon^x$ and $\varepsilon^{\overline{z}} = \varepsilon^t - i\varepsilon^x$, therefore the generator of two dimensional conformal transformations (1.33) may be written

$$Q = \frac{1}{2\pi i}\oint_C dz\; \varepsilon(z)T(z) + \frac{1}{2\pi i}\oint_C d\overline{z}\; \overline{\varepsilon}(\overline{z})\overline{T}(\overline{z})\,, \tag{1.36}$$

where the contour integration around C is taken in an anti-clockwise direction enclosing the arguments z_i or $\overline{z}_i$ of the fields ϕ_i on which Q acts.

Operators in correlation functions have to be time ordered. Under the transformation $z' = e^z$ ($\overline{z}' = e^{\overline{z}}$) this becomes radial ordering in the complex plane. The origin represents the theory at $t = -\infty$ and the spatial direction becomes 2π periodic. In the operator formalism of quantum field theory the variation of field $\phi(w, \overline{w})$ under conformal transformation $(\varepsilon(z), \overline{\varepsilon}(\overline{z}))$ is expressed in terms of the equal time commutator

$$\delta_{(\varepsilon,\overline{\varepsilon})}\phi(w,\overline{w}) = [Q, \phi] \tag{1.37}$$

$$= \frac{1}{2\pi i}\oint_C [dz T(z)\epsilon(z), \phi(w,\overline{w})], \tag{1.38}$$

where we have omitted the barred contribution. The operator product $A(z)B(w)$ is only convergent for $|z| > |w|$, therefore the equal time commutator is better expressed using the radial ordering

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & if \quad |z| > |w| \\ B(w)A(z) & if \quad |z| < |w| \end{cases} \tag{1.39}$$

Fermionic operators pick up a minus sign when interchanged, whilst parafermionic operators pick up a nontrivial phase factor. With cautious contour manipulation the integral (1.38) is equal to

$$\delta_{(\epsilon,\overline{\epsilon})}\phi(w,\overline{w}) = \frac{1}{2\pi i}\oint_{\Gamma_w} dz\epsilon(z)R(T(z)\phi(w,\overline{w})) \tag{1.40}$$

where Γ_w denotes the contour of a small circle enclosing only the point w.

Primary fields

A field Φ is said to be primary if under all conformal transformations $\omega \longrightarrow f(\omega), \overline{\omega} \longrightarrow \overline{f}(\overline{\omega})$ it transforms as

$$\Phi(\omega,\overline{\omega}) \longrightarrow \left(\frac{\partial f}{\partial \omega}\right)^{h} \left(\frac{\partial \overline{f}}{\partial \overline{\omega}}\right)^{\overline{h}} \Phi(f,\overline{f}). \tag{1.41}$$

The pair $(h,\overline{h})$ are said to be the conformal weights of Φ. The change in Φ due to the infinitesimal conformal transformation $\omega \longrightarrow \omega + \epsilon(\omega), \overline{\omega} \longrightarrow \overline{\omega} + \overline{\epsilon}(\overline{\omega})$ is

$$\delta_{\epsilon(\omega),\overline{\epsilon}(\overline{\omega})}\Phi(\omega,\overline{\omega}) = \left(h\partial\epsilon + \overline{h}\,\overline{\partial}\overline{\epsilon} + \epsilon\partial + \overline{\epsilon}\overline{\partial}\right)\Phi(w,\overline{w}), \tag{1.42}$$

where ∂ ($\overline{\partial}$) is the derivative with respect to w ($\overline{w}$) only. A field is called quasi-primary if it transforms in this way under global conformal transformations and can transform in some other way under the local conformal transformations.

Primary fields are so fundamental because they scale in exactly the same way as the fields conjugate to the coupling parameters mentioned earlier in the discussion of statistical mechanics theory. Conformally invariant theories perturbed by primaries are equivalent to the statistical Hamiltonians in the neighbourhood of a fixed point, both sharing the scaling (1.29) under conformal transformations. (The scaling properties determine the behaviour of the theories near criticality.) Consequently, if we could determine the set of weights $(h,\overline{h})$ from the conformal symmetry these would give precisely the critical exponents of some fixed point theory (and of all theories in its universality class).

Making a rotation through angle θ and scaling by a factor b so that $\frac{df}{dw} = b^{-1}e^{i\theta}$ and $\frac{d\overline{f}}{d\overline{w}} = b^{-1}e^{-i\theta}$ we find that the conformal weights are related to the scaling and spin dimensions via

$$h = \frac{1}{2}(x+s), \qquad \overline{h} = \frac{1}{2}(x-s). \tag{1.43}$$

According to our previous definition of relevant fields, a primary Φ is therefore relevant if its scaling dimension is less than 2. In terms of conformal weights this means

$$h + \overline{h} < 2 \quad \Rightarrow \quad \Phi \text{ is relevant}. \tag{1.44}$$

In particular, a spinless primary Φ is relevant if $h < 1$ and irrelevant if $h > 1$. $h = 1$ is the marginal case.

Via Cauchy's Integral Theorem the variation in primary Φ with weight $(h,\overline{h})$ under the variation $\epsilon(w)$ is

$$\delta_{(\epsilon,\overline{\epsilon})}\Phi(w,\overline{w}) = \frac{1}{2\pi i}\oint_{\Gamma_w}\left[dz\epsilon(z)\left(\frac{h}{(z-w)^2} + \frac{\partial}{(z-w)}\right)\right]\Phi(w,\overline{w}),$$

where we continue to omit the contribution from the barred coordinates for simplicity. Comparing this equation with (1.40) we find an alternative way of expressing that $\Phi(z,\overline{z})$ is conformal

$$R(T(z)\Phi(\omega,\overline{\omega})) = \left(\frac{h}{(z-w)^2} + \frac{\partial}{(z-w)}\right)\Phi(w,\overline{w}) + \Lambda(z,\overline{\omega})\,, \tag{1.45}$$

where Λ is an analytic function. From now on we shall drop the symbol for radial ordering and assume all operator products are radially ordered.

Note that under the conformal transformations $z \to w = f(z)$ and $\overline{z} \to \overline{w} = \overline{f}(\overline{z})$ the correlation function of n primary fields $\phi_1(z_1,\overline{z}_1),..,\phi_n(z_n,\overline{z}_n)$ are covariant. This means

$$\langle\phi_1(z_1,\overline{z}_1)..\phi_n(z_n,\overline{z}_n)\rangle = \prod_j \left(\frac{\partial w}{\partial z_j}\right)^{h_j}\left(\frac{\partial\overline{w}}{\partial\overline{z_j}}\right)^{\overline{h}_j}\langle\phi_1(w_1,\overline{w}_1)..\phi_n(w_n,\overline{w}_n)\rangle\,, \tag{1.46}$$

where each primary field $\phi_j(z_j,\overline{z}_j)$ has weights $(h_j,\overline{h_j})$. This is simply (1.30) encountered earlier, which is to be expected if the above claim is true that primary fields are nothing more than the fields conjugate to the coupling parameters of statistical theories near a fixed point.

The Virasoro algebra

The algebra of the generators of infinitesimal conformal transformations may be found by considering the form of Q_{ϵ_3} in

$$Q_{\epsilon_3} = [Q_{\epsilon_2}, Q_{\epsilon_1}]\,. \tag{1.47}$$

Since $Q_{\epsilon_i} = \frac{1}{2\pi i}\oint\limits_{C_i} dz_i\epsilon_i(z_i)T(z_i)$, Q_{ϵ_3} may be written

$$Q_{\epsilon_3} = \frac{1}{(2\pi i)^2}[\oint\limits_{C_2} dz_2\epsilon_2(z_2)T(z_2)\oint\limits_{C_1} dz_1\epsilon_1(z_1)T(z_1) - \oint\limits_{C_1} dz_1\epsilon_1(z_1)T(z_1)\oint\limits_{C_2} dz_2\epsilon_2(z_2)T(z_2)].$$

In the first half of this expression C_2 encloses z_1, in the second C_1 must enclose z_2. Fixing z_1 and deforming the inner and outer contours as in Figure (1.3) Q_{ϵ_3} becomes

$$Q_{\epsilon_3} = \frac{1}{(2\pi i)^2}\oint\limits_{C_2'} dz_2\epsilon_2(z_2)T(z_2)\oint\limits_{C_1} dz_1\epsilon_1(z_1)T(z_1)\,. \tag{1.48}$$

where C_2' encloses z_1. Via (1.37) and (1.42), $\delta_{\epsilon_3} = h\partial\epsilon_3 + \epsilon_3\partial = \delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}$, so that

$$\epsilon_3 = \epsilon_1\partial\epsilon_2 - \epsilon_2\partial\epsilon_1\,. \tag{1.49}$$

Therefore

$$Q_{\epsilon_3} = \frac{1}{2\pi i}\oint\limits_{C_3} dz_1\epsilon_3(z_1)T(z_1) = \frac{1}{2\pi i}\oint\limits_{C_3} dz_1\left[\epsilon_1(z_1)\partial\epsilon_2(z_1) - \epsilon_2(z_1)\partial\epsilon_1(z_1)\right]T(z_1)\,. \tag{1.50}$$

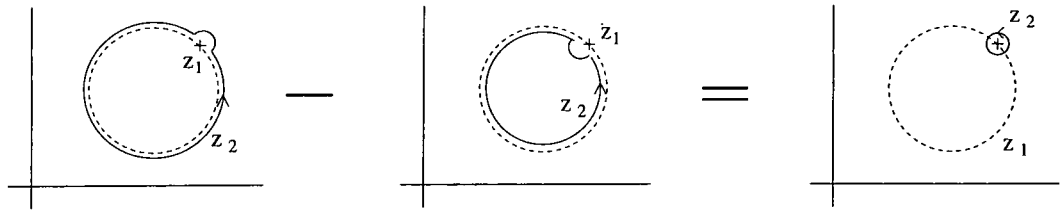


Figure 1.3: Contour choice for evaluation of the generator algebra

Integration by parts and an application of Cauchy's Integral Theorem gives

$$Q_{\epsilon_3} = \frac{1}{(2\pi i)^2} \oint_{C_2'} dz_2 \epsilon_2(z_2) \oint_{C_3} dz_1 \epsilon_1(z_1) \left[\frac{2T(z_1)}{(z_2 - z_1)^2} + \frac{\partial T(z_1)}{(z_2 - z_1)} \right] . \tag{1.51}$$

This operator is equal to $[Q_{\epsilon_2}, Q_{\epsilon_1}]$. Choosing $C_3 = C_1$, the product $T(z_2)T(z_1)$ can be determined since (1.48) and (1.51) are equal. By dimensional analysis, the most general form this product could have is

$$T(z_2)T(z_1) = \frac{A(z_1)}{(z_2 - z_1)^4} + \frac{B(z_1)}{(z_2 - z_1)^3} + \frac{C(z_1)}{(z_2 - z_1)^2} + \frac{D(z_1)}{(z_2 - z_1)} + \ldots \tag{1.52}$$

$A(z_1)$ must be constant because, according to (1.45), T has dimension 2 (if length has dimension -1). Define this constant to be $c/2$. For $B(z_1)$ to appear it must have dimension 1 e.g. $B(z_1) = (z_1 + k)^{-1}$ (B is not a spin 1 field, if it were this would be apparent in expression (1.51)), however this can be absorbed into the leading term, therefore $A(z_1) = c/2, B(z_1) = 0, C(z_1) = 2T(z_1)$ and $D(z_1) = \partial_{z_1} T(z_1)$. The same is true for the $\overline{T}(z_2)\,\overline{T}(z_1)$ expansion. So the generator algebra is

$$T(z_2)T(z_1) = \frac{c/2\ I}{(z_2 - z_1)^4} + \frac{2T(z_1)}{(z_2 - z_1)^2} + \frac{\partial T(z_1)}{(z_2 - z_1)} + \Lambda(z_1, z_2) \tag{1.53}$$

$$\overline{T}(\overline{z}_2)\overline{T}(\overline{z}_1) = \frac{\overline{c}/2\ I}{(\overline{z}_2 - \overline{z}_1)^4} + \frac{2\overline{T}(\overline{z}_1)}{(\overline{z}_2 - \overline{z}_1)^2} + \frac{\partial \overline{T}(\overline{z}_1)}{(\overline{z}_2 - \overline{z}_1)} + \Lambda(\overline{z}_1, \overline{z}_2) \tag{1.54}$$

$$\overline{T}(\overline{z}_2)T(z_1) = 0, \tag{1.55}$$

where I is the identity operator. The central charge c describes a particular set of realisations of conformal symmetry. Knowledge of the central charge is not enough to identify a specific model since several different stress-energy tensors can have the same central charge, but it does allow us to restrict our attention to a subset of theories.

Expanding $T(z)$ in a Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \tag{1.56}$$

the modes L_n satisfy

$$L_n = \frac{1}{2\pi i} \oint_C dz\, z^{n+1} T(z), \tag{1.57}$$

where the contour C encloses the argument of the field on which the modes act. Right modes $\overline{L}_n$ are defined similarly through $\overline{T}(\overline{z}) = \sum_{-\infty}^{\infty} \overline{L}_n \overline{z}^{-n-2}$. These components then form the direct product of a pair of Virasoro algebras

$$\begin{aligned} \left[L_m, L_n\right] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} && (1.58) \\ \left[\overline{L}_m, \overline{L}_n\right] &= (m-n)\overline{L}_{m+n} + \frac{\overline{c}}{12}(m^3 - m)\delta_{m,-n} && (1.59) \\ \left[L_m, \overline{L}_n\right] &= 0. && (1.60) \end{aligned}$$

Representations of the Virasoro algebra

In order to develop the quantum field theory we need to define what is meant by a state and correlation function. Correlation functions are radially ordered products of operators between vacuum states

$$\langle \phi_1(z_1, \overline{z}_1) ... \phi_n(z_n, \overline{z}_n) \rangle = \langle 0 | \mathcal{R} \hat{\phi}_1(z_1, \overline{z}_1) ... \hat{\phi}_n(z_n, \overline{z}_n) | 0 \rangle \,. \tag{1.61}$$

where $|0\rangle$ is the vacuum state. A primary state $|h, \overline{h}\rangle$ is defined by the action of the operator corresponding to the primary field $\phi_{h,\overline{h}}$ on the vacuum:

$$|h, \overline{h}\rangle = \hat{\phi}_{h,\overline{h}}(0)|0\rangle = |\phi_{in}\rangle = \lim_{z,\overline{z} \to 0} \hat{\phi}_{h,\overline{h}}(z, \overline{z})|0\rangle \,. \tag{1.62}$$

Through this definition the notions of a representation, degeneracy or unitarity can be formulated in terms of the primary fields or states. From here on let us drop the hat notation, which should strictly be used over all operators (including the stress energy-tensor and its modes).

Regularity of $T(z)|0\rangle$ at $z = 0$ and $z = \infty$ leads to the annihilation formulae

$$\begin{aligned} L_m|0\rangle &= 0 \qquad m \geq -1 \,, \\ \langle 0|L_m &= 0 \qquad m \leq 1 \,. \end{aligned} \tag{1.63}$$

Now return to equation (1.42) for the variation of a primary field under an infinitesimal conformal transformation. Choosing $\varepsilon = z^{n+1}$, (1.38) gives a more useful expression for the variation formula when dealing with primary fields

$$[L_n, \phi_{h,\overline{h}}(z, \overline{z})] = (h(n+1)z^n + z^{n+1}\partial_z)\phi_{h,\overline{h}}(z, \overline{z}) \,. \tag{1.64}$$

Then by definition (1.62) these results imply the primary state $|h, \overline{h}\rangle$ satisfies the highest weight conditions

$$\begin{aligned} L_0|h, \overline{h}\rangle &= h|h, \overline{h}\rangle \,, & \overline{L}_0|h, \overline{h}\rangle &= \overline{h}|h, \overline{h}\rangle \,, \\ L_n|h, \overline{h}\rangle &= 0 \,, & \overline{L}_n|h, \overline{h}\rangle &= 0 \,, \qquad \forall n > 0 \,. \end{aligned} \tag{1.65}$$

The central extension term proportional to c in (1.58) vanishes for the commutators of $L_{0,\pm 1}$ and $\overline{L}_{0,\pm 1}$. Therefore these generators, which form a subalgebra, have the same interpretation as the global conformal transformation generators $l_{0,\pm 1}$ and $\bar{l}_{0,\pm 1}$. The dilatation generator $L_0 + \overline{L}_0$ has eigenvalue $h + \overline{h} = x$ and the rotation generator $L_0 - \overline{L}_0$ has $h - \overline{h} = s$. If $|0\rangle$ is annihilated by $L_{1,0,-1}$ and $\overline{L}_{1,0,-1}$ as above, we conclude the vacuum is conformally invariant.

Each primary field $\phi_{h,\overline{h}}(z,\overline{z})$ and its descendants

$$\phi_{h,\overline{h}}^{\{-k_1,..,-k_m;-\overline{k}_1,..,-\overline{k}_n\}}(z,\overline{z}) \equiv L_{-k_1}..L_{-k_m}\overline{L}_{-\overline{k}_1}..\overline{L}_{-\overline{k}_n}\phi_{h,\overline{h}}(z,\overline{z}), \tag{1.66}$$

with $k_i, \overline{k}_i \geq 0$, form a conformal family $[\phi_{h,\overline{h}}]$. If the theory in question allows only primaries with highest weights $(h_j, \overline{h}_j)$ then all fields lie among the set $\oplus_j[\phi_{h_j,\overline{h}_j}]$. In the state picture a general descendant is formed by successive applications of operators L_{-m} and $\overline{L}_{-n}(m,n>0)$ on the primary states

$$|\Psi\rangle = \phi_{h,\overline{h}}^{\{-k_1,-k_2,..,-k_m;-\overline{k}_1,-\overline{k}_2,..,-\overline{k}_n\}}(0)|0\rangle = L_{-k_1}..L_{-k_m}\overline{L}_{-\overline{k}_1}..\overline{L}_{-\overline{k}_n}|h,\overline{h}\rangle\,. \tag{1.67}$$

The weight of this descendant is given by

$$L_0|\Psi\rangle = (\sum_{i=1}^{m} k_i + h)|\Psi\rangle\,, \qquad \overline{L}_0|\Psi\rangle = (\sum_{j=1}^{n} \overline{k}_j + \overline{h})|\Psi\rangle\,, \tag{1.68}$$

where the sums $M = \sum_{i=1}^{m} k_i$ and $N = \sum_{j=1}^{n} \overline{k}_j$ are called the levels of $|\Psi\rangle$ with respect to the left and right algebras. The set of $|h,\overline{h}\rangle$ and its left descendants is called a Verma module V_h (for right modes the module is $V_{\overline{h}}$). We say V_h forms a representation of the left Virasoro algebra. The isomorphism due to the operator-state correspondence (1.62) is then $[\phi_{h,\overline{h}}] \cong V_h \otimes V_{\overline{h}}$. Since left and right modes L_m and $\overline{L}_n$ commute, it is usually necessary to consider only the left modes.

Degeneracy, the Kac determinant and minimal models

A representation is degenerate or reducible if there exists a non-highest weight state $|\chi\rangle \neq |h,\overline{h}\rangle$ which is a linear combination of the descendants in V_h such that

$$\begin{aligned} L_0|\chi\rangle &= (h+k)|\chi\rangle\,, \\ L_n|\chi\rangle &= 0 \qquad\qquad \forall n>0\,. \end{aligned} \tag{1.69}$$

Then $|\chi\rangle$, which is orthogonal to all states, is said to be a null state of $|h,\overline{h}\rangle$ with respect to the left algebra. One can obtain an irreducible representation if we formally put the null state to zero: $|\chi\rangle = 0$.

A degeneracy of the Verma module V_h is predicted by the Kac matrix $M_l(c,h)$. This matrix has entries which are the inner products of descendants of $|h,\overline{h}\rangle$ at level l. For instance, states at level 3 can be ordered in $\mathcal{S} = (L_{-3}|h,\overline{h}\rangle \quad L_{-1}L_{-2}|h,\overline{h}\rangle \quad L_{-1}^3|h,\overline{h}\rangle)^T$ so that $M_3(c,h) \equiv \mathcal{S}^\dagger \otimes \mathcal{S}$, where $\mathcal{S}^\dagger$ is the transposed adjoint of S. The implication is that h or $\overline{h}$ is a root of $\det(M_l(c,h)) = 0$ if, and only if, there exists a linear combination of states $|\chi\rangle$ with zero norm and $V_{h,\overline{h}}$ is degenerate at level l. All h which are not roots give irreducible representations. Kac [5] proposed the following formula for $\det(M_L(c,h))$:

$$\det M_L(c,h) = \alpha \prod_{rs \leq L} (h - h_{r,s}(c))^{P(L-rs)}, \tag{1.70}$$

where α is some constant, $P(X)$ is the number of distinct partitions of X boxes and the central charge dependent weights are given by

$$h = h_{r,s}(c) = \frac{[r(\widetilde{m}+1) - s\widetilde{m}]^2 - 1}{4\widetilde{m}(\widetilde{m}+1)} \quad , \tag{1.71}$$

where

$$c(\widetilde{m}) = 1 - \frac{6}{\widetilde{m}(\widetilde{m}+1)}, \tag{1.72}$$

with $\widetilde{m}$ any real or even complex valued number. Note that the representation with highest weight $h_{r,s}$ is degenerate at level rs where the first null vector has dimension $h_{r,s} + rs$. The determinant has $P(L-rs)$ zeros for $L > rs$, because this is the number of states at level L descendant from a null state $|\chi\rangle$ at level rs.

If $\widetilde{m}$ is rational with $\widetilde{m} = \frac{m}{m-\overline{m}}$ for coprime integers m and $\overline{m}$, then the following theories are degenerate

$$c_{m,\overline{m}} = 1 - \frac{6(m-\overline{m})^2}{\overline{m}m} \tag{1.73}$$

and weights (of which there can be $\frac{1}{2}(m-1)(\overline{m}-1)$)

$$h_{r,s} = \frac{(mr - \overline{m}s)^2 - (m-\overline{m})^2}{4\overline{m}m}. \tag{1.74}$$

These theories are called rational models. Once a central charge is specified, the above set of dimensions determine which primary fields constitute the degenerate conformal representations. Belavin *et al.* [4] demonstrated that the algebra of the operators in these theories is closed, i.e. the operator product expansions of any two operators in these models is expressible in terms of the sum of a finite number of primaries and their descendants.

Unitarity

Hermiticity of T and $\overline{T}$ implies the modes satisfy $L_m^\dagger = L_{-m}$, $\overline{L}_m^\dagger = \overline{L}_{-m}$. Consequently, a typical inner product is written

$$\begin{aligned} || L_{-n}|h\rangle||^2 \equiv \langle h|L_{-n}^\dagger L_{-n}|h\rangle &= \langle h|L_n L_{-n}|h\rangle \\ &= \langle h|[L_n, L_{-n}]|h\rangle \\ &= 2nh + \frac{c}{12}(n^3 - n)\,, \end{aligned}$$

where (1.58) and (1.65) are used to obtain the last two lines. Unitarity is the requirement that the inner product of any state is non-negative for all combinations of c and $(h, \overline{h})$. Choosing $n = 1$ we find a necessary requirement for unitarity is

$$h \geq 0. \tag{1.75}$$

For such a theory c cannot be negative either, because regardless of the size of h one could increase n until the inner product becomes negative.

Kac's determinant is a crucial tool to establish which representation (c, h) is unitary. Should the determinant be negative at some level it implies the matrix has a negative eigenvalue which means there exist negative normed states denying unitarity. The Kac determinant can therefore be used to discover which models are unitary. Friedan, Qiu and Shenker ([6]) proved

- For $c > 1$ and $h > 0$ the determinant has no zeros and the theory can be unitary.
- For $c < 1$ and $h > 0$ a necessary requirement for unitarity is given by (1.71) and (1.72) where $\widetilde{m}$ takes integer values greater than or equal to 3.

The unitary cases $\widetilde{m} = 3, 4, 5, 6$, often denoted $\mathcal{M}_{\widetilde{m}}$, were identified immediately with various statistical models such as the critical Ising (c=1/2), tri-critical Ising (c=7/10), 3-state Potts (c=4/5) and the tri-critical 3-state Potts (c=6/7) statistical models by comparing the critical exponents of operators in these models with $h + \overline{h}$ and $h - \overline{h}$ from list (1.71,1.72). Friedan et al. did not prove that all the theories (1.71,1.72) with integer $\widetilde{m}$, are unitary. This was done by Goddard, Kent and Olive [7] through their explicit coset construction. This method allows us to build unitary conformal models which necessarily have rational central charges.

Briefly, the idea is to build a stress energy tensor $T^g(z)$ from the modes $T_m^a \equiv T^a z^m$ appearing in the level k_g Kac-Moody algebra

$$[T_m^a, T_n^b] = i f_c^{ab} T_{m+n}^c + \frac{k_g}{2} m \delta^{ab} \delta_{m,-n} \tag{1.76}$$

of the Lie algebra g with structure constants f_c^{ab} for the compact connected Lie group G of dimension $dim(g)$. It can be shown that the stress-energy tensor gives rise to a Virasoro algebra with central charge

$$c_G^{(k_g)} = \frac{k_g dim(g)}{k_g + \tilde{h}_g}, \tag{1.77}$$

where $\tilde{h}_g$ is the dual Coxeter number of g. The dual Coxeter number is conveniently given by the formula $\tilde{h}_g = \frac{1}{rank(g)}[n_L + (\frac{S}{L})^2 n_S]$. $rank(g)$ is the rank of the algebra (the dimension of the Cartan subalgebra) and there are $dim(g) - rank(g)$ roots of which n_L are long and n_S are short, with lengths L and S respectively (determined by the Dynkin diagram of g). Any Virasoro algebra found this way necessarily has $c \geq 1$.

If h is a subalgebra of g, one can similarly construct a corresponding stress-energy tensor $T^h(z)$, now with central charge

$$c_H^{(k_h)} = \frac{k_h dim(h)}{k_h + \tilde{h}_h}. \tag{1.78}$$

where $\tilde{h}_h$ is the dual Coxeter of h and k_h is the level of the Kac-Moody algebra. The modes $K_m \equiv L_m^g - L_m^h$ also form a Virasoro algebra with central charge given by the difference

$$c_{G/H} = c_G^{(k_g)} - c_H^{(k_h)}. \tag{1.79}$$

K_m is a mode of $T^g - T^h$ which is the stress energy tensor of corresponding to the coset G/H. Theories constructed this way are unitary because unitarity is built in from the onset. The minimal conformal series with $\tilde{m} = 3, 4, 5, ...$ is constructed by considering the coset $G/H = SU(2)^{(k)} \otimes SU(2)^{(1)}/SU(2)^{(k+1)}$. $SU(2)$ has rank 1, dimension 3 and dual Coxeter number 2, therefore the associated central charge is $c_{G/H} = 3k/(k+2) + 1 + 3(k+1)/(k+1+2) = 1 - 6(k+2)/(k+3)$, giving the $c < 1$ series of conformal theories identifying $\tilde{m} = k + 1$. A more relevant example here is the quotient group

$$\frac{SO(p)^{(2)} \otimes SO(p)^{(1)}}{SO(p)^{(3)}}. \tag{1.80}$$

For $p = 2l + 1$ odd, $SO(p)$ has dimension $l(2l+1)$ and dual Coxeter number $2l - 1$. If $p = 2l$ the dimension is $l(2l-1)$ and the dual Coxeter number is $2l - 2$. Therefore, the level k corresponding Virasoro algebra has central charge

$$c_{SO(p)}^{(k)} = \begin{cases} kl(2l+1)/(k+2l-1) & \text{for } p = 2l+1, \\ kl(2l-1)/(k+2l-2) & \text{for } p = 2l. \end{cases} \tag{1.81}$$

In both cases the coset central charge for the coset (1.80) is the same

$$c = c_{SO(p)}^{(2)} + c_{SO(p)}^{(1)} - c_{SO(p)}^{(3)} = \frac{2p-1}{p+1}. \tag{1.82}$$

1.3 Perturbed conformal field theories and integrability

The conformally invariant field theory with Hamiltonian H_{CFT} corresponds to a fixed point of a renormalisation group flow. Suppose we consider relevant perturbations of such a theory

$$H_{PCFT} = H_{CFT} + \sum_{i=1}^{F} \lambda_i \int d^2x\, \Phi_i(x)\,, \tag{1.83}$$

where $\Phi_i(x)$ is a spinless relevant primary, keeping rotational invariance, with dimensions (h_i, h_i) less than unity. The dimensionful coupling λ_i transforms as $\lambda_i \to b^{2-2h_i}\lambda_i$ under the scaling $x \longrightarrow b^{-1}x$ and therefore has dimensions $(1-h_i, 1-h_i)$. At least in a local CFT, where the generators like T and their modes have integer spin, all fields are either primary or have dimension at least one greater than that of a primary. Therefore, in a unitary local theory, where $h_i > 0$, the only relevant fields are primaries. So (1.83) is actually the most general relevant perturbation we can consider. When $c < 1$ there are a finite number of primaries and F is finite. There are an infinite number of possible irrelevant perturbations.

Relevant perturbations break conformal symmetry and the theory assumes either a finite or infinite correlation length, according to which the perturbation is said to be massive or massless (because the inverse correlation lengths ξ_i^{-1} are naturally interpreted as the masses m_i of a relativistic quantum field theory). The two-dimensional perturbed theory (1.83) is super-renormalisable which means correlation functions in a perturbative expansion can be rendered finite by introducing a finite number of fields to kill off all ultraviolet divergences. It is assumed that the field content of the perturbed conformal field theory is the same as the unperturbed theory so that, in particular, the fields in the perturbed theory have the same dimensions as those in the conformal.

The existence of integrals of motion or conserved charges is particularly important for two dimensional quantum field theories because such quantities not only deny particle production or creation for a specific massive theory, but they also force the factorisation of the scattering matrix for n-particles into a product of two particle amplitudes which may be determined. A review is left until the next section, meanwhile here we discuss exactly what is meant by integrability, beginning with a review of how to find integrals of motion by trial and error and the more systematic counting argument of Zamolodchikov.

Consider a perturbation of a conformal field theory H_{CFT} by a single spin-zero relevant

primary field Φ of weight $h < 1$

$$H = H_{CFT} + \lambda \int d^2x\, \Phi(x)\,. \tag{1.84}$$

We begin with an initial comparison of the field content of the conformal families of the identity $\Lambda = [I]$ and of the perturbing primary $[\Phi]$. Both families have a natural decomposition according to spin: $\Lambda = \oplus_{s=0}^{\infty}\Lambda_s$ and $[\Phi] = \oplus_{s=0}^{\infty}\Phi_s$ which persists when $\lambda \neq 0$.

Λ_s consists of all level s descendants T_s^{α} and Φ_s of all level s descendants of weight $(s+h, h)$. These fields are not necessarily algebraically independent because all fields are expressible in terms of Virasoro modes, nor are they linearly independent, as some fields in Λ_s are total ∂_z derivatives. Linear independence can be imposed by considering the factor spaces

$$\widehat{\Lambda}_s = \Lambda_s / L_{-1}\Lambda_{s-1} \qquad\qquad \widehat{\Phi}_s = \Phi_s / L_{-1}\Phi_{s-1}. \tag{1.85}$$

For $\lambda = 0$ all fields $T_s^{(\kappa)} \in \widehat{\Lambda}_s$ then satisfy

$$\partial_{\bar{z}} T_s^{(\kappa)} = 0. \tag{1.86}$$

If $\lambda \neq 0$, $\partial_{\bar{z}} T_s^{(\kappa)}$ has a $\bar{z}$ dependence

$$\partial_{\bar{z}} T_s^{(\kappa)} = \lambda R_{s-1}^{(\kappa)1} + \lambda^2 R_{s-1}^{(\kappa)2} + ... + \lambda^n R_{s-1}^{(\kappa)n} + ... \tag{1.87}$$

where $R_{s-1}^{(\kappa)i}$ are assumed to be fields existing in the CFT, the dimensions of which can be thus deduced:

$T_s^{(\kappa)}$	$(s, 0)$
$\partial_{\bar{z}} T_s^{(\kappa)}$	$(s, 1)$
λ^n	$(n(1-h),\, n(1-h))$
$R_{s-1}^{(\kappa)n}$	$(s - n(1-h),\, 1 - n(1-h))$
$R_{s-1}^{(\kappa)1} \equiv R_{s-1}^{(\kappa)}$	$(s - 1 + h,\, h)$

For large n the dimension of $R_{s-1}^{(\kappa)n}$ becomes negative, but $\oplus_h[\Phi_h]$ contains only fields with zero or positive dimension, therefore the sum (1.87) must terminate. Furthermore, the right dimension of $R_{s-1}^{(\kappa)n}$ is less than 1 and the only fields with this dimension in a local unitary theory are the primaries, therefore all fields other than the first vanish unless

$$1 - n(1-h) = h_r\,, \tag{1.88}$$

where h_r is a highest weight of another primary Φ_r. If this condition holds a more general action (1.83), which includes a term proportional to Φ_r, should be considered instead of (1.84) which is no longer sufficient to describe this case. If $h_r = 0$, (1.88) tells us a second field $\lambda^n R_{s-1}^{(\kappa)n}$ appears when h satisfies the resonance condition $1 - h = 1/n$. When this resonance condition is not satisfied (1.87) becomes

$$\partial_{\overline{z}} T_s^{(\kappa)} = \lambda R_{s-1}^{(\kappa)} . \tag{1.89}$$

Only if $R_{s-1}^{(\kappa)} \in \Phi_s$ is a total z derivative

$$\partial_{\overline{z}} T_{s+1}^{\kappa} = \partial_z \Theta_{s-1}(\overline{z}) , \tag{1.90}$$

does there exist a local integral of motion

$$P_s = \oint_C T_{s+1}^{\kappa} dz + \Theta_{s-1} d\overline{z} . \tag{1.91}$$

This integral is said to be local because the fields T_{s+1}^{κ} and Θ_{s-1} are local with integer spin. Also strictly we should have included an additional index on this conserved charge since there could exist several conserved charges with the same spin.

The derivative $\partial_{\overline{z}} T_s^{(\kappa)}$ can be determined once the modes $D_n : \Lambda \longrightarrow \Phi$ ($n = 0, \pm 1, \pm 2, ..$) are defined:

$$D_n \Lambda(z, \overline{z}) = \frac{1}{2\pi i} \oint_z d\xi \, (\xi - z)^n \Phi(\xi, \overline{\xi}) \Lambda(z, \overline{z}) . \tag{1.92}$$

The relation (1.64) can be used to relate these and the Virasoro modes

$$[L_n, D_m] = -\left((1 - h)(n + 1) + m\right) D_{n+m} . \tag{1.93}$$

Another important relation, obtained from (1.92), is

$$D_{-n-1} I = \frac{1}{n!} \partial_z^n \Phi(z, \overline{z}) . \tag{1.94}$$

Via (1.64) we can equally replace the derivative ∂_z by L_{-1} here (though we can not equate ∂_z and L_{-1} in general). Under the perturbation (1.84) the $\overline{z}$ derivative of T_s^{κ} satisfies

$$\partial_{\overline{z}} T_s^{(\kappa)}(z, \overline{z}) = \lambda \oint_z \frac{d\xi}{2\pi i} T_s^{(\kappa)}(z) \Phi(\xi, \overline{z}) . \tag{1.95}$$

This relationship is derived by expanding the expectation value of the stress-energy tensor and an arbitrary set of fields in a perturbative series involving correlation functions evaluated at the conformal point and powers of the coupling. Applying the $\partial_{\overline{z}}$ derivative to the result and incorporating the identity $\partial_z \partial_{\overline{z}} \log((z - z')(\overline{z} - \overline{z}')) = -2\pi i \delta^{(2)}(z - z')$

one obtains the above result. This gives the relation $\partial_{\overline{z}} = \lambda D_0$. Then writing $T_s^{(\kappa)}$ in terms of $\{L_{-n_i}\}$ acting on the identity operator I, we can use (1.93) and (1.94) to reduce $\partial_{\overline{z}} T_s^{(\kappa)}$ to a linear combination of L_{-p}'s acting on Φ. The result of this process may or may not be a total ∂_z derivative. The simplest integral of motion is constructed from $\partial_{\overline{z}} T$:

$$\partial_{\overline{z}} T(z,\overline{z}) = \lambda D_0(L_{-2}I) = \lambda(h-1)D_{-2}I = \lambda(h-1)L_{-1}\Phi(z,\overline{z}) \tag{1.96}$$

which gives an integral of motion $P_1 = \int dz T + d\overline{z}\Theta$ with $\Theta = \lambda(h-1)\Phi$ for all field theories. In general $R_{s-1}^{(\kappa)}$ is not a total derivative, for instance $T_4 = L_{-2}^2 I$ satisfies

$$\begin{aligned} \partial_{\overline{z}} T_4 = \lambda D_0 (L_{-2})^2 I &= \lambda(h-1)(D_{-2}L_{-2} + L_{-2}D_{-2})I \\ &= \lambda(h-1)(2L_{-2}L_{-1} + \frac{h-3}{6}L_{-1}^3)\Phi \end{aligned} \tag{1.97}$$

which is not, for general Φ, a total derivative. However, if Φ should be a degenerate field then the right hand side of (1.97) is a total derivative.

Rather than testing every $T_{s+1}^{(\kappa)}$ by trial and error, a dimensional argument, proposed by Zamolodchikov [8], is used to establish a necessary condition for the existence of integrals of motion. Stated most simply, a spin s integral of motion exists if the dimension of the space $\widehat{\Lambda}_{s+1}$ is greater than the dimension of $\widehat{\Phi}_s$ [1]. The dimensions of the subspaces for a given representation are contained in the character of the representation in question. For instance, the dimensions $d_h(s) = dim(\Phi_s)$ may be assembled into a character formula:

$$\chi_h(q) = q^{-c/24} Tr_h q^{L_0} \tag{1.99}$$

where the prefactor is conventional. The trace Tr_h is $\sum_{s=0}^{s=\infty} Tr_{h,s}$ with $Tr_{h,s}$ representing the trace of matrix $\langle i|q^{L_0}|j\rangle$, where $|i\rangle$ and $|j\rangle$ are normalized states of level s which have L_0 eigenvalue $s+h$, so that

$$\chi_h(q) = q^{-c/24} \sum_s q^{h+s} \sum_i \langle i|i\rangle = q^{h-c/24} \sum_s d_h(s) q^s. \tag{1.100}$$

[1] Given that $\widehat{\Phi}_s$ and $\widehat{\Lambda}_s$ are level s spaces with total derivatives factored out, let us define the projection operator $\Pi_s : \Phi_s \longrightarrow \widehat{\Phi}_s$ and define $D_{0,s} : \widehat{\Lambda}_s \longrightarrow \Phi_{s-1}$ to be the restriction of D_0 on subspace $\widehat{\Lambda}_s$. The composite of both these operators $B_s = \Pi_{s-1} D_{0,s} : \widehat{\Lambda}_s \longrightarrow \widehat{\Phi}_{s-1}$ gives the following basic result:

$$dim(\widehat{\Lambda}_{s+1}) > dim(\widehat{\Phi}_s) \Rightarrow \text{there exists an integral of motion with spin } s\,. \tag{1.98}$$

In order to see this, suppose there exists $T_{s+1} \in \widehat{\Lambda}_{s+1}$ such that $B_{s+1}T_{s+1} = 0$ i.e. $dim(ker(B_{s+1})) > 0$. Then $\Pi_s(D_{0,s+1}T_{s+1}) = 0$ and so $D_{0,s+1}T_{s+1}$ is in the kernel of Π_s i.e. is a total L_{-1} derivative of a field in Φ_{s-1} and we can write $D_{0,s+1}T_{s+1} = \partial_z \Theta_{s-1}$ with $\Theta_{s-1} \in \Phi_{s-1}$. Finally multiply by λ to obtain $\partial_{\overline{z}} T_{s+1} = \partial_z(\lambda\Theta_{s-1})$.

$d_h(s)$ here simply counts states which are not linearly independent. The dimensions of $\widehat{\Phi}_s$ are easily seen to be $\widehat{d}_h(s) = d_h(s) - d_h(s-1)$ from (1.85). So the character for the dimensions of linearly independent states is

$$\widehat{\chi}_h(q) = (1-q)\chi_h(q) \tag{1.101}$$

(for the Λ conformal family $\widehat{\chi}_0(q) = (1-q)\chi_0(q) + q$ since $L_{-1}I = 0$). When the representation is degenerate one must factor out the null states. Demonstrating integrability thereby reduces to the construction of the appropriate character formulae.

The character formulae for several classes of conformal theories have been established, but the formulae are generally very complicated when $c > 1$. In particular, the dimensions of the level subspaces necessary to prove integrability for the perturbations of the $\mathbb{Z}_N$-symmetric conformal field theories to be studied here have not yet been found (see [9] for details concerning character formulae of parafermionic models).

The generalisation of the above trial and error method, to the case where additional currents exist (which generate an extended conformal algebra), can be stated as follows. Whether the additional currents $\{G\}$, satisfying $\partial_{\bar{z}}G = 0$, have integer or fractional spin the principle is the same. Suppose we consider an extended symmetry conformal theory, perturbed by the spinless field $\epsilon(x)$ of dimension (h,h):

$$H_{PCFT} = H_{CFT} + \lambda \int d^2x\, \epsilon(x)\,. \tag{1.102}$$

If G_{s+1} (s not necessarily integer) is a descendant in the same conformal family as the generator G, then a spin s integral of motion will exist if *any* combination of fields R_s can be found to satisfy

$$\partial_{\bar{z}}G_{s+1} = \lambda R_s \tag{1.103}$$

such that R_s is a total ∂_z derivative. This may involve demonstrating R_s can be written as L_{-1} acting on some combination of fields, but more generally any combination of modes corresponding to ∂_z must factor out of R_s so that $R_s = \partial_z(R'_{s-1})$, implying

$$Q_s = \int G_{s+1}dz + \lambda R'_{s-1}d\bar{z}\,, \tag{1.104}$$

is the spin s integral of motion. We point out that candidates for R_s do not just include descendants of ϵ, any combination of fields with the correct spin value will do. The perturbing field does not necessarily appear explicitly in the integral of motion, but the dimensionful coupling λ does and this is how the integral of motion is 'attached' to the specific perturbation in question.

Chapter 2

Scattering Theory and the Thermodynamic Bethe Ansatz

2.1 Conserved charges and exact S-matrices

Each conserved charge restricts the particle dynamics of the perturbed theory. In particular, the form of the S-matrix describing the scattering of asymptotically well-separated particles is heavily restrained by each conservation law. However, in $d > 2$ spacetime dimensions integrability places too heavy a restriction on the possible scattering. Coleman and Mandula [10] argued that when there exist many symmetries of the scattering it is so restrictive that only collisions at an angle of 0 or π degrees can occur. Thus integrability is not compatible with real scattering in $d > 2$ dimensions. In $1 + 1$ dimensions these are the only possible collisions anyway, so integrability does not completely exclude some interesting scattering. In this section we review how the existence of conserved charges allows the determination of the massive S-matrix up to the Castillejo-Dalitz-Dyson or CDD ambiguity [11].

Integrals of motion and massive scattering theory

Consider a relativistic scattering theory consisting of N particles A_a of mass m_a and relativistic momenta $\vec{p}_a = (p_a^0, p_a^1)$ in Minkowski co-ordinates (where $a = 1, ..N$). In the infrared regime there are asymptotic regions where particles are well separated and on mass-shell so that $(p_a^0)^2 - (p_a^1)^2 = m_a^2$. This allows a parametrisation in terms of the rapidity coordinate θ_a

$$(p_a^0, p_a^1) = (E_a, p_a) = (m_a cosh\theta_a, m_a sinh\theta_a) \qquad a = 1, ..N \qquad (2.1)$$

so that E_a and p_a are the energy and momentum of particle a. Our aim is to determine the scattering amplitude which relates well-separated incoming particle states to outgoing ones; asymptotic multiparticle states being

$$|A_{a_1}(\theta_1), A_{a_2}(\theta_2), .., A_{a_N}(\theta_N)\rangle_{in(out)} \,. \tag{2.2}$$

In-states have ordered rapidities $\theta_1 > \theta_2 > .. > \theta_N$ at $t = -\infty$. The leftmost particle has the largest momenta, the particle to its right the next largest and so on. As time progresses interactions occur as the particles successively collide. Between collisions we assume the motion is well approximated by some wave function for free particles. Each a_i is intended to label all quantum numbers including mass number and internal quantum numbers, so that every type of particle is represented by its own label.

Suppose the theory has conserved charges $P_{\overline{s}}$ which exist in all perturbation ranges, in particular in the infrared [1]. The rotation generator $L_0 - \overline{L}_0$ is the Lorentz boost generator M in Minkowski co-ordinates such that $[M, P_{\overline{s}}] = \overline{s} P_{\overline{s}}$. Under Lorentz transformation $L_\alpha : \theta \to \theta + \alpha$, $P_{\overline{s}} \to P'_{\overline{s}} = e^{\overline{s}\alpha} P_{\overline{s}}$. If a charge $P_{\overline{s}}$ acts on a one particle state $|A_a(\theta)\rangle$ as

$$P_{\overline{s}}|A_a(\theta_a)\rangle = \omega_{\overline{s}}(\theta_a)|A_a(\theta_a)\rangle, \tag{2.3}$$

then the behaviour of the charge under Lorentz transformation implies $\omega_{\overline{s}}(\theta_a) = \gamma_{\overline{s}}^a e^{\overline{s}\theta_a}$, where $\gamma_{\overline{s}}^a$ are constants. Alternatively, we could write $\omega_{\overline{s}}(\theta_a) = \kappa_{\overline{s}}^a m_a e^{\overline{s}\theta_a}$, with $\kappa_{\overline{s}}^a$ constant (no summation is intended) so that in light cone co-ordinates the momenta take the form $(p, \overline{p}) = (me^{\theta}, me^{-\theta})$ i.e. $P_1 = p = me^{\theta}$, so that $\kappa_1^a = 1$ for all particles.

If they are local, the operators $P_{\overline{s}}$ are diagonalised by the asymptotic states and their action on well-separated particles with localized wave packets is

$$P_{\overline{s}}|A_{a_1}(\theta_1), .., A_{a_N}(\theta_N)\rangle = \left(\sum_{i=1}^{N} \gamma_{\overline{s}}^{a_i} e^{\overline{s}\theta_i}\right) |A_{a_1}(\theta_1), .., A_{a_N}(\theta_N)\rangle \,. \tag{2.4}$$

Non-local integrals of motion $Q_{\overline{s}}$, formed from fractional spin fields, have a different, non-distributive, action on asymptotic states (to be discussed later).

For the time being consider the case where the integrals of motion are local. The conservation of $P_{\overline{s}}$ implies the following set of selection rules for an N particle in-state scattering to an M particle out-state

$$\sum_{i=1}^{N} \gamma_{\overline{s}}^{a_i} e^{\overline{s}\theta_i} = \sum_{j=1}^{M} \gamma_{\overline{s}}^{b_j} e^{\overline{s}\theta'_j} \,. \tag{2.5}$$

[1] To avoid confusion between the spin index and Mandelstam variable to be introduced shortly we label the spin on a given conserved charge by $\overline{s}$. As pointed out earlier several distinct conserved charges can have the same spin

θ_i and θ'_j are initial and final rapidities while $\{\gamma_{\overline{s}}^{a_i}\}$ and $\{\gamma_{\overline{s}}^{b_j}\}$ are the sets of quantum numbers associated to $P_{\overline{s}}$ for in and out states respectively. The explicit mass version of this equation with $\overline{s} = 1$ implies $\sum_{i=1}^{N} m_{a_i} e^{\theta_i} = \sum_{j=1}^{M} m_{b_j} e^{\theta'_j}$ which is the statement of conservation of momentum that we expect from the existence of P_1 in all theories.

All scattering processes in an integrable theory are necessarily elastic. The existence of just a couple of non-trivial conserved charges is enough to forbid particle production or annihilation and give $N = M$. However to demonstrate that the initial and final sets of asymptotic rapidities are identical it is necessary to have an infinite number of conservation laws. In practice, this property is assumed once several integrals of motion have been identified. After a suitable re-ordering of momenta, integrability implies the equality of the selection coefficients

$$\gamma_{\overline{s}}^{a_i} = \gamma_{\overline{s}}^{b_i} \qquad \forall \overline{s}. \tag{2.6}$$

These numbers give us the action of $P_{\overline{s}}$ on the asymptotic states, but tell us nothing about the exchange of internal quantum numbers. The only other freedom allowed in an integrable theory are time delays. Consequently, the asymptotic *in* state is a superposition of *out* states as

$$|A_{a_1}(\theta_1), .., A_{a_N}(\theta_N)\rangle_{in} = S_{a_1 a_2 .. a_N}^{b_1 b_2 .. b_N}(\theta_1, .., \theta_N)|A_{b_1}(\theta_1), .., A_{b_N}(\theta_N\rangle_{out} \tag{2.7}$$

where summation is implied.

The determination of the S-matrix amplitudes is still highly non-trivial but is made possible because integrability allows the factorisation of the N particle S-matrix in terms of $N(N-1)/2$ two particle S-matrices. As an example, consider a three particle collision. The three ways this scattering can occur are shown in Figure (2.1). By applying a power of a higher order charge one can translate all world lines by an amount proportional to

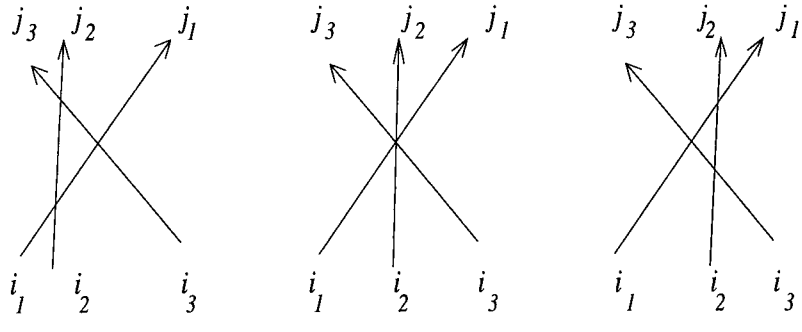


Figure 2.1: The possible three particle collisions

a power of each particle's momenta. Since each particle has a different momenta, the trajectories are shifted by different amounts and because such a charge is conserved, each of the processes related by translation have the same amplitude [12]. This fact is expressed via the Yang-Baxter factorisation equations

$$\sum_{k_1,k_2,k_3} S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 i_3}^{j_1 k_3}(\theta_{13}) S_{k_2 k_3}^{j_2 j_3}(\theta_{23}) = \sum_{k_1,k_2,k_3} S_{i_2 i_3}^{k_2 k_3}(\theta_{23}) S_{i_1 k_3}^{k_1 j_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) \qquad (2.8)$$

where the sums are over all internal colours consistent with (2.6). This system of consistency equations implies whatever factorisation we choose in order to evaluate $S_{i_1 i_2 i_3}^{j_1 j_2 j_3}$, the resulting amplitude must be the same. If there are no internal quantum numbers (no multiplet states) this equation is automatically satisfied. A similar argument is used to demonstrate the factorisation of the N particle S-matrix.

General restrictions on the scattering amplitudes

We first discuss analyticity, crossing and unitarity properties of the two particle amplitude $S_{ab}^{cd}(\theta)$ of the forward channel scattering process $A_a A_b \to A_d A_c$ in terms of the invariant energy squared, that is the Mandalstam variable s:

$$s = (\vec{p}_a + \vec{p}_b)^2 = m_a^2 + m_b^2 + 2 m_a m_b \cosh(\theta)\,, \qquad (2.9)$$

where $\theta = \theta_{ab} = \theta_a - \theta_b$ is the rapidity difference of the incoming particles. For physical processes θ is real and $s \geq (m_a + m_b)^2$. The crossed version of this collision $A_{\bar{d}} A_a \to A_c A_{\bar{b}}$ has associated the Mandelstam variable t:

$$\begin{aligned} t = (\vec{p}_a - \vec{p}_c)^2 &= 2(m_a^2 + m_b^2) - s \\ &= m_a^2 + m_b^2 + 2 m_a m_b \cosh(i\pi - \theta) \end{aligned}$$

The analyticity postulate states that the domain of s can be enlarged by continuation up from this part of the real axis so that S(s) becomes a complex valued matrix on the complex plane and takes complex conjugate values at conjugate values of s. This implies S is real when s is. In order to keep this function single-valued, two cuts are necessary. The first is along the real axis above threshold, the second is also on the real axis, and corresponds to the crossed dynamics being above threshold i.e. where t becomes physical: $t \geq (m_a + m_b)^2$, and is equivalent to continuing s to the region $s \leq (m_a - m_b)^2$. The resulting Riemann surface is called the physical sheet.

We take the convention that the forward channel above threshold physical process occurs when s is on the upper edge of $s \geq (m_a + m_b)^2$ and the crossed channel above

threshold process occurs when on the lower edge of $s \leq (m_a - m_b)^2$. Crossing symmetry and unitarity read:

$$S_{ab}^{cd}(s) = S_{da}^{\bar{b}c}(2m_a^2 + 2m_b^2 - s)$$
$$S_{ab}^{cd}(s^+)S_{cd}^{ef}(s^-) = \delta_a^e \delta_b^f$$

where $S(s^+)$ and $S(s^-)$ are the values the S-matrix takes on the upper and lower edges of the $s > (m_a + m_b)^2$ cut. $S(s)$ is a meromorphic function of s on the physical sheet. We assume bound state poles occur on the real axis between thresholds $(m_a - m_b)^2$ and $(m_a + m_b)^2$ with no other singularities in the physical strip.

Through (2.9) we can map the first sheet of the cut s plane to the strip $0 \leq Im\theta \leq \pi$. The other covers map to one of the strips $n\pi \leq \theta \leq (n+1)\pi$. The region of the real s-axis below threshold maps to the pure imaginary θ axis. Therefore, $S(\theta)$ is real on $Re(\theta) = 0$ and the only singularities occur on this axis. These are interpreted as bound states in either the forward or crossed channels. Crossing and unitarity become

$$S_{ab}^{cd}(\theta) = S_{da}^{\bar{b}c}(i\pi - \theta)\,, \qquad S_{ab}^{cd}(\theta)S_{cd}^{ef}(-\theta) = \delta_a^e \delta_b^f\,. \tag{2.10}$$

Physical scattering amplitudes in the forward channel are given by the values of $S_{ab}^{cd}(\theta)$ at $Im(\theta) = 0, Re(\theta) > 0$, while $Im(\theta) = \pi, Re(\theta) < 0$ describes the crossed channel physical scattering.

For purely elastic scattering processes, where there are no particle multiplets, things are further simplified. A simple pole $\theta = iu_{ab}^c$ in the amplitude for A_aA_b scattering corresponds to bound state particle A_c in the forward or crossed channels. A positive (negative) residue implies the bound state occurs in the forward (crossed) channel, and the mass of the bound state must be given by

$$m_c^2 = m_a^2 + m_b^2 + 2m_am_b\cos(u_{ab}^c)\,. \tag{2.11}$$

Crossing symmetry means there is a pole at $\theta = i\overline{u}_{ab}^c$ ($\overline{u}_{ab}^c = \pi - u_{ab}^c$) corresponding to the same particle in the crossed channel. The u_{ab}^c are called fusing angles and trivially satisfy $u_{ab}^c + u_{bc}^a + u_{ca}^b = 2\pi$. It is usually assumed that the bound states exist as stable asymptotic states and that their scattering is also described by the S-matrix.

The principle behind the Yang-Baxter equation was that the amplitudes for two processes equivalent but for the translation of one world line, should be equal. The *bootstrap principle* supposes this can be done when bound states are involved. For a diagonal scattering, the Yang-Baxter factorisation equations are trivially satisfied and as

an alternative the bootstrap equations are used to determine the form of the S-matrix. A bound state residue $\theta_{ab} = iu^c_{ab}$ occurring in the three particle scattering $A_dA_aA_b$: $S_{dab}(\theta_a, \theta_b, \theta_d) = S_{da}(\theta_{da})S_{db}(\theta_{db})S_{ab}(\theta_{ab})$, where particles A_aA_b form a bound state A_c, must give the same amplitude as the two particle scattering $S_{cd}(\theta)$. This condition gives the bootstrap equation

$$S_{cd}(\theta) = S_{ad}(\theta + i\overline{u}^b_{ca})S_{bd}(\theta - i\overline{u}^a_{bc})\,. \tag{2.12}$$

When combined with (2.4) the bootstrap equation gives the following condition on the coefficients $\gamma^a_{\overline{s}}$:

$$\gamma^c_{\overline{s}} = e^{-i\overline{s}\,\overline{u}^b_{ca}}\gamma^a_{\overline{s}} + e^{i\overline{s}\,\overline{u}^a_{bc}}\gamma^b_{\overline{s}}\,. \tag{2.13}$$

One begins with the assumption that the theory contains at least two particles, and that fusing angles $\overline{u}^1_{22}$ and $\overline{u}^2_{11}$ exist for the formation of bound states. Using the system (2.13) those values of spin for which such fusions exist are determined and (2.11) gives mass m_2 in terms of m_1. Once two angles are known, it is easy to use the bootstrap equation (2.12) to show there exists another angle giving a pole in the physical strip corresponding to another particle with mass m_3 given by (2.11). One continues cranking the bootstrap handle until no new masses are produced. The set of fusion angles $\overline{u}^c_{ab}$ when used in (2.13) restrict the values of $\overline{s}$ for which we should look for integrals of motion. It is possible to write down the non-diagonal version of this equation, though it is not usually used as a practical alternative to the Yang-Baxter equation.

In summary, for non-diagonal scattering, analyticity, crossing, unitarity, the factorisation equations, and the commutations $[P_{\overline{s}}, S] = 0$ pin down the S-matrix $S^{cd}_{ab}(\theta)$ up to the CDD ambiguity

$$S^{cd}_{ab} \to \Psi(\theta)S^{cd}_{ab}\,. \tag{2.14}$$

The CDD factor itself satisfies crossing and unitarity requirements:

$$\Psi(\theta) = \Psi(i\pi - \theta)\,, \qquad \Psi(\theta)\Psi(-\theta) = 1\,. \tag{2.15}$$

The importance of this ambiguity will be discussed in Chapter (7).

2.2 The sine-Gordon S-matrix

The massive two-dimensional sine-Gordon model is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m_0^2}{\beta^2}\cos\beta\varphi\,. \tag{2.16}$$

m_0 is dimensionful constant which essentially defines the scale of the theory, whereas fixing β dictates the physical particle content. (2.16) therefore describes a family of relativistic scattering theories. An alternative parametrisation of this model is through h or ξ, where

$$\frac{h}{2} = \frac{8\pi}{\beta^2} - 1\,, \tag{2.17}$$

$$\xi = \frac{\beta^2/8\pi}{1-\beta^2/8\pi}\,. \tag{2.18}$$

Pre-empting the discussion of $\mathbb{Z}_N$ models we are particularly interested in the theories with $\beta^2 = 32\pi/N$, where $N \geq 5$ is an integer, in which case

$$h = \frac{N-4}{2}\,, \qquad \xi = \frac{4}{N-4}\,. \tag{2.19}$$

The sine-Gordon S-matrix is the minimal $O(2)$-symmetric solution of the unitary, crossing and factorization equations (see [13] for more details). As a quantum field theory this model contains two types of particle: the soliton (A) and antisoliton ($\overline{A}$). Using the compact notation of [13] the two-particle scatterings are given by

$$\begin{aligned} A(\theta_1)A(\theta_2) &= S\,(\theta_1-\theta_2,\xi)\,A(\theta_2)A(\theta_1) \\ \overline{A}(\theta_1)\overline{A}(\theta_2) &= S\,(\theta_1-\theta_2,\xi)\,\overline{A}(\theta_2)\overline{A}(\theta_1) \\ A(\theta_1)\overline{A}(\theta_2) &= S_T(\theta_1-\theta_2,\xi)\,\overline{A}(\theta_2)A(\theta_1) + S_R(\theta_1-\theta_2,\xi)\,A(\theta_2)\overline{A}(\theta_1)\,, \end{aligned} \tag{2.20}$$

the amplitudes for which are encapsulated in the non-diagonal scattering matrix $\hat{S}$

$$\hat{S}\,(\theta,\xi) = \begin{pmatrix} S(\theta,\xi) & & & \\ & S_R(\theta,\xi) & S_T(\theta,\xi) & \\ & S_T(\theta,\xi) & S_R(\theta,\xi) & \\ & & & S(\theta,\xi) \end{pmatrix} = S_0\,(\theta,\xi)\,\hat{R}\,(\theta,\xi)$$

where all entries not shown are zero, S_0 is a scalar:

$$S_0(\theta,\xi) = \frac{1}{\sinh\left(\frac{1}{\xi}(\theta-\imath\pi)\right)}\exp\left[-i\int_0^\infty \frac{dk}{k}\frac{\sin(k\theta)\sinh\left(\frac{\pi-\xi\pi}{2}k\right)}{\cosh\left(\frac{\pi k}{2}\right)\sinh\left(\frac{\xi\pi k}{2}\right)}\right] \tag{2.21}$$

and $\hat{R}$ is a simple matrix:

$$\hat{R}(\theta,\xi) = \begin{array}{c} \\ AA \\ A\overline{A} \\ \overline{A}A \\ \overline{AA} \end{array} \begin{array}{c} \begin{array}{cccc} AA & A\overline{A} & \overline{A}A & \overline{AA} \end{array} \\ \begin{pmatrix} \sinh\left(\frac{1}{\xi}(\theta - \imath\pi)\right) & & & \\ & -\sinh\left(\frac{\imath\pi}{\xi}\right) & -\sinh\left(\frac{\theta}{\xi}\right) & \\ & -\sinh\left(\frac{\theta}{\xi}\right) & -\sinh\left(\frac{\imath\pi}{\xi}\right) & \\ & & & \sinh\left(\frac{1}{\xi}(\theta - \imath\pi)\right) \end{pmatrix} \end{array}$$

For convenience later we introduce the shorthand notation

$$a(\theta) = \sinh(\frac{1}{\xi}(\theta - i\pi))\,, \qquad b(\theta) = -\sinh(\frac{\theta}{\xi})\,, \qquad c = -\sinh(\frac{i\pi}{\xi})\,. \tag{2.22}$$

The poles in S_0 at $\theta = \imath\pi - \imath k\xi\pi$ for $k = 1, 2, \ldots < [1/\xi]$ are interpreted as $A\overline{A}$ bound states or breathers. According to (2.11) the breather masses are

$$M_k = 2M \sin\left(\frac{k\pi}{2}\xi\right) \qquad k = 1, 2, \ldots \text{n-1}\,, \tag{2.23}$$

where A and $\overline{A}$ have renormalised mass M, and n is the integer part of $(N-1)/4$ (when N is an integer). The soliton labels A and $\overline{A}$ will often be replaced by n and $n{+}1$, so that

$$M_n = M_{n+1} = M. \tag{2.24}$$

Although the scattering among solitons and antisolitons is non-diagonal, once a bound state is involved the scattering becomes diagonal. Introducing the block notation

$$(x)\,(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{\imath\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{\imath\pi x}{2h})} \qquad , \qquad \{x\} = (x-1)\,(x+1)\,,$$

the diagonal amplitudes are

$$S_{jk} = \prod_{\substack{|j-k|+1 \\ \text{step } 2}}^{j+k-1} \{l\}\{h-l\} \qquad (j,k = 1 \ldots n{-}1) \tag{2.25}$$

and

$$S_{kn} = S_{k,n+1} = (-1)^k \prod_{\substack{h/2-k+1 \\ \text{step } 2}}^{h/2+k-1} \{l\} \qquad (k = 1 \ldots n{-}1)\,. \tag{2.26}$$

When $h = 2n$ $(N = 4n+4)$, the entire S-matrix is diagonal and the soliton amplitudes are

$$\begin{aligned} \text{n even} \;:\; & S_{nn} = S_{n+1,n+1} = \prod_{\substack{l=1 \\ \text{step } 4}}^{2n-3} \{l\} \;\; ; \;\; S_{n,n+1} = \prod_{\substack{l=3 \\ \text{step } 4}}^{2n-1} \{l\}\,; \\ \text{n odd} \;:\; & S_{nn} = S_{n+1,n+1} = \prod_{\substack{l=1 \\ \text{step } 4}}^{2n-1} \{l\} \;\; ; \;\; S_{n,n+1} = -\prod_{\substack{l=3 \\ \text{step } 4}}^{2n-3} \{l\}\,. \end{aligned} \tag{2.27}$$

2.3 The TBA technique

There are two distinct ways to develop a Hamiltonian approach to a relativistic field theory on a toroidal geometry given by two circles of circumference R and L. Periodicity of one axis, in the L-direction leads to the quantisation of states with momenta taking values $\frac{2\pi m}{L}$ (for integer m). These states evolve in the time-like R-direction under the influence of Hamiltonian H_L. With this description, the partition function is $Z(R,L) = Tr(e^{-RH_L})$ so that $1/R$ is proportional to the temperature. In the limit $L \to \infty$ the free energy per unit length is $Rf(R) = \lim_{L\to\infty} -\frac{1}{L}\log Z(R,L)$. In contrast, quantisation in the R-direction means states representing particles with momenta $\frac{2\pi n}{R}$ (with integer n) evolve in the L-direction under Hamiltonian H_R. In this picture the partition function is dominated by $E_0(R)$ (the ground state energy of H_R) as L becomes large, so that $Z(R,L) \sim e^{-LE_0(R)}$. Consequently, the free energy per unit length and ground state energy are equal in the $L \to \infty$ limit:

$$E_0(R) = Rf(R). \tag{2.28}$$

In a unitary conformal theory $E_0(R)$ is related to the central charge c by $E_0(R) = -\pi c/6R$ [14, 15]. This result is obtained by considering the variation in the free energy per unit length in response to an incremental increase in the radius R. For the non-conformal theory we may take the following relationship to define the effective central charge $\widetilde{c}(r)$

$$E_0(R) = f_0 R - \frac{\pi\widetilde{c}(r)}{6R}, \tag{2.29}$$

where f_0 is the bulk term. $\widetilde{c}(r)$ depends on the finite-size scaling parameter $r = MR$, where M is the lightest mass or inverse correlation length and has the property that in the small r limit $\widetilde{c}(r) \longrightarrow c$: the central charge of the limiting theory (c is replaced by $c - 12\Delta_0$ for a non-unitary theory with lowest conformal weight Δ_0).

The thermodynamic Bethe ansatz method allows the determination of the function $E_0(R)$ and therefore $\widetilde{c}(r)$. To begin with, consider a theory consisting of N identical neutral particles with two-particle scattering amplitude $S(\theta)$. In the space of all configurations there are regions where the relativistic particles are spatially well separated with $|x_i - x_j| > M^{-1}$ (provided L is chosen to be much greater than M^{-1}) and have on mass-shell energies and momenta $E_i = M\cosh\theta_i$ and $p_i = M\sinh\theta_i$ so that $E_i^2 - p_i^2 = M^2$. In these regions all off shell contributions may be ignored and the N-particle state can be described by the Bethe wave function $\Psi(x_1, x_2, .., x_N) = A(Q)e^{i\sum_j p_j x_j}$, where $A(Q)$ is some a configuration dependent function.

If a pair of particles, which are originally well separated at x_i and x_j, interact and pass to another free configuration where the Bethe wave function description can also be used, the S-matrix can be used to match the wave functions of the two configurations. That is, the encounter is described by the multiplication of the wave function by $S(\theta_i - \theta_j)$. If the ith particle, originally at x_i, is passed in the spacelike direction through distance L until we regain the same free configuration that we started with, the resulting wave function is simply the original multiplied by a phase $(-1)^F$ which accounts for the fermionic (F odd) or bosonic (F even) nature of the particles involved:

$$\Psi(x_1, .., x_i, x_{i+1}, .., x_N) = (-1)^F \Psi(x_1, .., x_i - L, x_{i+1}, .., x_N) \,. \tag{2.30}$$

The original wave function picks up an S-matrix amplitude for every interaction. Repeating this process for every particle results in N quantisation equations for the momenta $p_i : e^{ip_i L} \prod_{j \neq i} S(\theta_i - \theta_j) = 1$. Taking logarithms the following set of admissibility conditions for the rapidities $\{\theta_i\}$ in the free regions is discovered

$$ML \sinh\theta_i - i \sum_{j \neq i} \log S(\theta_i - \theta_j) = 2\pi n_i \,. \tag{2.31}$$

where n_i are integers, one for each of the N particles. (An additional amount π should appear in the right hand side of this equation when the particles are fermionic. In the thermodynamic limit to be considered shortly this contribution can be neglected.) This equation is valid provided the distance L is not comparable with the correlation length ($L \gg M^{-1}$), otherwise off mass-shell effects appear and the wave function description of the separated particles is not permitted. In terms of rapidities the state $(\theta_1, .., \theta_N)$ has energy $H = \sum_{i=1}^{N} M \cosh\theta_i$. These Bethe ansatz equations are extremely complicated but yield very useful results in the thermodynamic limit as $L \to \infty$ with the ratio L/N constant.

The statistics are a combination of wavefunction and generalised S-matrix statistics. A particle is said to be of fermionic type if $(-1)^F S(0) = -1$ and bosonic type if $(-1)^F S(0) = 1$. Now suppose we can re-arrange the above equations so that the sequence $\{n_i\}$ is strictly monotonic if the particles are of fermionic type (e.g. $(n_i)_F = (1, 3, 4, 7, 18, ...)$) and monotonic, but not strictly, if of bosonic type (e.g. $(n_i)_B = (2, 4, 5, 5, 7, ...)$). These give rise to a density of rapidities or *roots* denoted $\rho_r(\theta)$ so that $d = \rho_r(\theta) L \Delta\theta$ rapidities lie in the interval $(\theta, \theta + \Delta\theta)$. The above system becomes

$$ML \sinh\theta_i - iL \int d\theta' \, \log S(\theta_i - \theta') \rho_r(\theta') = 2\pi n_i \,. \tag{2.32}$$

Here, and in the following, the integration range is assumed from $-\infty$ to ∞ unless specified otherwise. With no integers missing from the sequence $\{n_i\}$, the rapidity density is $\rho(\theta)$ defined so that at most $D = \rho(\theta)L\Delta\theta$ rapidities can appear in the interval $(\theta, \theta + \Delta\theta)$. The number of different distributions with the same number of roots in this interval is $\frac{D!}{(D-d)!d!}$ for fermionic particles and $\frac{(D+d-1)!}{(D-1)!d!}$ for bosonic particles (when two or more particles may have the same rapidity value). As $L \to \infty$ the product of these expressions over all θ ranges gives the number of densities which we are unable to distinguish from $\rho_r(\theta)$. The logarithm of this quantity divided by L defines the entropy $\mathcal{S}$ per unit length. For fermionic particles this is

$$\mathcal{S}(\rho, \rho_r) = \int d\theta \, [\rho \log \rho - (\rho - \rho_r) \log(\rho - \rho_r) - \rho_r \log \rho_r] \,, \tag{2.33}$$

($\mathcal{S}(\rho + \rho_r, \rho)$ for bosonic particles). In this limit (2.32) gives the following important integral equation which relates the root density $\rho_r(\theta)$ and level density $\rho(\theta)$

$$M \cosh\theta + \int d\theta' \, \varphi(\theta - \theta')\rho_r(\theta') = 2\pi\rho(\theta) \,. \tag{2.34}$$

The level density at θ is thus dependent on all roots via the kernel $\varphi(\theta) = -i\frac{d}{d\theta}\log S(\theta)$. The total energy per unit length of the system is $H(\rho_r) = \int d\theta \, M\cosh\theta\rho_r(\theta)$.

The thermodynamic relation $F = E - T\mathcal{S}$ connects the free energy, energy, temperature ($T = 1/R$) and entropy. For rapidity densities this becomes the functional in ρ and ρ_r

$$Rf(\rho, \rho_r) = RH(\rho_r) - \mathcal{S}(\rho, \rho_r) \,. \tag{2.35}$$

This expression must be minimised to determine which root density ρ_r gives the equilibrium configuration. Through functional differentiation of the energy and entropy with respect to root density (and using the important dependency equation (2.34)) we can obtain a condition for the system to be in equilibrium. Introducing the pseudo-energy $\varepsilon(\theta)$ and L-function $L(\theta)$

$$\frac{\rho_r}{\rho} = \frac{e^{-\varepsilon(\theta)}}{1 \pm e^{-\varepsilon(\theta)}} \quad , \qquad L(\theta) = \pm \log(1 \pm e^{-\varepsilon(\theta)}) \,, \tag{2.36}$$

where the upper and lower signs correspond to fermionic and bosonic type particles respectively, it is easy to verify that the derivative of the free energy is zero provided

$$MR\cosh\theta = \varepsilon(\theta) + \frac{1}{2\pi}\int d\theta' \, \varphi(\theta - \theta')L(\theta') \,. \tag{2.37}$$

This thermodynamic Bethe ansatz equation must be solved for $\varepsilon(\theta)$ which is then substituted into the general free energy expression to determine the equilibrium free energy f.

The result is

$$Rf(R) = -\frac{1}{2\pi}\int d\theta\, M\cosh(\theta)\, L(\theta). \tag{2.38}$$

In the rest of this work we shall deal only with theories consisting of particles of fermionic type.

The generalisation to the case of P distinct types of particles of masses M_k which have reflectionless scattering (described by the diagonal $P^2 \times P^2$ matrix of amplitudes $S_{ij}^{ij}(\theta)$) is straightforward. We obtain a system of P integral equations relating the respective root densities $\rho_{r,k}(\theta)$ to level densities $\rho_k(\theta)$, for $k = 1, .., P$, and these will be required in order to write down a condition which gives the equilibrium free energy. The most general thermodynamic Bethe ansatz system for P types of particle with diagonal scattering is then

$$RM_i\cosh\theta = \varepsilon_i(\theta) + \sum_{j=1}^{P}(\varphi_{ij} * L_j)(\theta) \tag{2.39}$$

where $\varphi_{ij}(\theta) = -i\frac{d}{d\theta}\log S_{ij}(\theta)$. We consider the fermionic case $L_j(\theta) = \log(1 + e^{-\varepsilon_j(\theta)})$ and $*$ denotes the convolution

$$f*g(\theta) = \frac{1}{2\pi}\int f(\theta')g(\theta - \theta')d\theta'. \tag{2.40}$$

The corresponding generalisation of (2.38) is

$$Rf(R) = E_0(R) = -\sum_{i=1}^{P}\frac{1}{2\pi}\int d\theta\, L_i(\theta)M_i\cosh\theta\,. \tag{2.41}$$

The TBA method gives bulk free energy f_0 zero, so $E_0(R) = -\frac{\pi\widetilde{c}(r)}{6R}$ and the effective central charge or finite-size scaling function is

$$\widetilde{c}(r) = \frac{3}{\pi^2}\sum_{i=1}^{P}\int_{-\infty}^{\infty} d\theta\, L_i(\theta)m_i r\cosh\theta\,, \tag{2.42}$$

where $r = RM_1$ and $m_i = M_i/M_1$ are normalised masses with $m_1 = 1$, and $\varepsilon_i(\theta)$, the solutions to the system (2.39), give the required L-functions.

In all derivations of TBA systems the idea is basically the same. One must find equations like (2.34) which relate root and level densities. These will always come from the appropriate Bethe ansatz equations and are necessary because they provide the functional derivatives of level densities with respect to root densities which are required to extremize the free energy.

In the next chapter we define the class of two dimensional statistical models which are $\mathbb{Z}_N$-symmetric. Among this class are models which are conformally invariant. In order to

describe the neighbourhood of these points we need to study relevant perturbations which remain in the $\mathbb{Z}_N$-symmetric phase space. Non-diagonal generalisations of the system (2.39) are required to describe these perturbations. In such situations the derivation is far more complicated because it is necessary to diagonalise the Bethe equations. The full non-diagonal calculation is carried out in Appendix (A) for a system of solitons which interact via the sine-Gordon S-matrix written above. Because the derivation is very complicated it is often more practicable to make a conjecture based on knowledge of the ultraviolet limit of the theory in question. One can then use the TBA method to discover exact results which may be compared to perturbed theory.

Many of the TBA techniques to be used here are extensions or applications of results from the work of Zamolodchikov [16, 17, 18] and Klassen and Melzer [47, 53].

Chapter 3

$\mathbb{Z}_N$-symmetric Models and Parafermionic Field Theories

Consider a class of theories with spins $\sigma_j = e^{i\theta_j}, \theta_j \in [0, 2\pi)$, located at sites j on a two-dimensional square lattice interacting according to the reduced Hamiltonian

$$\mathcal{H}[\{\theta_i\}] = \sum_{\langle ij \rangle} V(\theta_i - \theta_j) - h_N \sum_i \cos N\theta_i \,. \tag{3.1}$$

The corresponding partition function is $Z(\{\sigma_i\}) = Tr_{\{\sigma_i\}} \exp(-\mathcal{H}[\{\theta_i\}])$, where the inverse temperature factor $1/k_\beta T$ is implicit in the Hamiltonian. $V(\theta)$ is an even, 2π-periodic potential describing the interaction of neighbouring spins. Summation is over nearest neighbours $\langle ij \rangle$ with the supposition that $V(0) < V(\theta \neq 0)$.

When $h_N = 0$ the model has a continuous $O(2)$-symmetry which, according to the Mermin-Wagner theorem [19], forbids the formation of an ordered phase no matter how low the temperature becomes provided the spins have a finite range of interaction. However, a phase transition can occur via the Kosterlitz-Thouless mechanism: above some temperature T_{KT} vortices are free, correlations are exponential and the system is disordered (the order parameter has zero expectation value). Below T_{KT} vortices become bound together and correlations are algebraic. The latter phase is called the Kosterlitz-Thouless phase, also known as the soft or massless phase.

For a slightly increased h_N and low enough temperature, below T_c say, we expect to observe spontaneous symmetry breaking to an ordered phase [20]. Interestingly, José *et al.* [21] have shown that in the subset of models of Villain form (defined by (3.7)), the massless phase persists for small positive values of h_N, as long as $N \geq 5$. The reason is that $T_c < T_{KT}$ for $N \geq 5$, therefore at temperatures T satisfying $T_c < T < T_{KT}$

the Kosterlitz-Thouless phase remains as a third phase between massive low and high temperature domains. For $N < 5$, $T_c \not< T_{KT}$ so an intermediate phase cannot exist.

3.1 $\mathbb{Z}_N$-symmetric statistical theories

The limit $h_N \to \infty$ pins the classical spin variable to N discrete angles $\theta_j = \frac{2\pi}{N} n(j)$, where $n(j) \in \{0, 1, 2, ..N-1\}$. The result is the pure $\mathbb{Z}_N$ model. If only nearest neighbours interact, the partition function may be written

$$Z(\{\sigma_i\}) = Tr_{\{\sigma_i\}} e^{-\sum_{\langle i,j \rangle} H(\sigma_i, \sigma_j)} = Tr_{\{\sigma_i\}} \prod_{\langle i,j \rangle} W(\sigma_i, \sigma_j)\,, \tag{3.2}$$

where

$$W(\sigma_i, \sigma_j) \equiv e^{-H(\sigma_i, \sigma_j)} = \sum_{k=0}^{N-1} w_k (\sigma_i^\dagger \sigma_j)^k = \sum_{k=0}^{N-1} w_k e^{\frac{2\pi i}{N}[n(j)-n(i)]k}\,. \tag{3.3}$$

The real positive coefficients w_k satisfy $w_k = w_{N-k}$ and $w_0 \equiv 1$ so that the Hamiltonian is real. Therefore, there are $[N/2]$ (the integer part of $N/2$) parameters required to define a $\mathbb{Z}_N$ model and this is the dimensionality of the phase space of $\mathbb{Z}_N$-symmetric theories. The point $w_1 = w_2 = ... = w_{[N/2]} = 0$ corresponds to zero temperature theory and $w_1 = w_2 = ... = w_{[N/2]} = 1$ to the high temperature limit.

The $\mathbb{Z}_N$-symmetric models have been the subject of a great deal of research for several reasons. This set of models generalises the well known Ising model ($N = 2$) which exhibits an order-disorder phase transition typifying that of the ferromagnetic to paramagnetic universality class. The $\mathbb{Z}_N$ generalisations could describe the phases of systems with $\mathbb{Z}_N$-symmetry with equal success. Phase transitions in such systems arise naturally in the condensation and melting of two-dimensional crystals in condensed matter theory. Transitions from solid to liquid have been observed for idealised symmetric molecules on a smooth substrate e.g. experiments with He^4 on exfoliated graphite have been carried out which reveal exact order-disorder phase transitions expected from a system with underlying $\mathbb{Z}_3$-symmetry (an obvious symmetry of the graphite substrate which could be described by the large h_3 limit of (3.1) above) [22]. Transitions from order to disorder via an intermediate liquid-crystal phase characterised by a power law decay of the 'orientation' order parameter have also been found (see [23] for a discussion of these features in real systems with $\mathbb{Z}_6$-symmetry). Such intermediate phases have correlations agreeing with those expected in the Kosterlitz-Thouless regions mentioned above.

When starting with a potential such as V, a convenient parametrisation of the phase

space is

$$x_r = \exp\{-V(2\pi r/N) + V(0)\} \qquad r = 0, 1, 2, ..[N/2]\,, \tag{3.4}$$

where $x_r = x_{N-r}$ because V is even and $x_0 = 1$. Equivalently, as mentioned above, we can use the numbers w_k. The two are related through

$$x_r = (\sum_{q=0}^{N-1} w_q e^{\frac{2\pi i}{N}qr})/(\sum_{q=0}^{N-1} w_q) \qquad r = 0, 1, 2, ..[N/2]\,. \tag{3.5}$$

It is now useful to introduce the idea of a duality transformation, which may be defined as follows. Using parametrisation (3.3), the dual transformation for this model is carried out by replacing the summation over spin variables σ_i by the same summation over dual spins $\mu_{\tilde{i}}$ positioned at sites $\tilde{i}$ on a dual lattice which interact via the same form of Hamiltonian but with the coupling w_k replaced by

$$\widetilde{w}_k = (1 + \sum_{q=1}^{N-1} w_q e^{\frac{2\pi i}{N}qk})/(1 + \sum_{q=1}^{N-1} w_q)\,. \tag{3.6}$$

The effect of this transformation is to exchange low and high temperature fixed points. The theories defined by (3.3) and (3.4) are dual i.e. $\widetilde{w}_k = x_k$. Either description is equally valid as a starting point for the description of the phase space. Under the dual transformation there is always an invariant $[N/4]$-dimensional hyperplane given by $w_k = \widetilde{w}_k$ for $k = 1, ..[N/2]$ on which systems are said to be self-dual. With $\sigma_i^k = (\sigma_i)^k$ and $\mu_{\tilde{i}}^l = (\mu_{\tilde{i}})^l$, order parameters $\langle\sigma^k\rangle$ and their dual disorder parameters $\langle\mu^l\rangle$ define the different phases of the theory:

- $\langle\sigma^k\rangle \neq 0$, $\langle\mu^l\rangle = 0$ for an ordered phase,
- $\langle\sigma^k\rangle = 0$, $\langle\mu^l\rangle \neq 0$ for a disordered phase,
- $\langle\sigma^k\rangle = \langle\mu^l\rangle = 0$ in a Kosterlitz-Thouless phase.

It has been conjectured that provided N is prime these are the only phases [24]. (This is based upon the assumption that the product of expectation values of powers of order and disorder fields might satisfy the relation $\langle\sigma^k\rangle\langle\mu^l\rangle = e^{\frac{2\pi i}{N}kl}\langle\mu^l\rangle\langle\sigma^k\rangle$, so that the expectation values cannot both be nonzero for N prime.) For non-prime N other 'partially ordered' phases exist where expectation values of certain powers $\langle\sigma^k\rangle$ or $\langle\mu^l\rangle$ can both be non-zero (provided $kl = N$). It is necessary to investigate these regions for a complete understanding of the $\mathbb{Z}_N$-symmetric phase space, but here we are interested in the three phases above. In particular, we ask: does the three-phase picture outlined above for h_N small

continue for the *pure* $\mathbb{Z}_N$ model, as universality might suggest? This has been shown to be so in [24] for the subset of theories which have a potential of Villain form [25], defined by

$$\mathrm{H}(\{\theta_i\}) = \sum_{\langle ij \rangle} H_\beta^V(\theta_i - \theta_j) \qquad e^{-H_\beta^V(\theta)} = \sum_{l=-\infty}^{\infty} e^{-\frac{\beta}{2}(\theta - 2\pi l)^2}, \tag{3.7}$$

where spin variables θ_i take the same discrete values as above and $\beta \in [0, \infty)$, which plays a role like temperature, specifies a particular model. In terms of parameters x_r, each Villain model corresponds to a point on the line given by the equations

$$x_r = \Big(\sum_{l=-\infty}^{\infty} e^{\frac{-\beta}{2}(\frac{2\pi r}{N} - 2\pi l)^2} \Big) \Big/ \Big(\sum_{l=-\infty}^{\infty} e^{\frac{-\beta}{2}(2\pi l)^2} \Big) \qquad r = 1, ..[N/2]\,. \tag{3.8}$$

Cardy [26] succeeded in showing that a wider class of theories also have this intermediate massless phase behaviour. Much earlier, Baxter [27] studied the N-state Potts model (defined by (3.2) with $H(\sigma_i, \sigma_j) = \epsilon \delta_{\sigma_i, \sigma_j}$, for constant ϵ, or as $w_1 = w_2 = ...w_{[N/2]}$ in terms of $\mathbb{Z}_N$ parameters) and showed that a first-order transition exists between ordered and disordered phases when $N \geq 5$. This means that for $N \geq 5$ the Kosterlitz-Thouless region does not keep order and disorder domains apart everywhere.

Most is known about the simplest case N=5. Using the w_k-parametrisation, the phase space is hypothesised to be that shown in Figure (3.1). Four phases are shown: (1) and (2) are both massless Kosterlitz-Thouless phases (the case N=5 has the symmetry $w_1 \longleftrightarrow w_2$), (3) is the ordered and (4) the disordered phase. The dashed line (a) represents the subset of 5-state Potts models given by $w_1 = w_2$ and (b) denotes the set of Villain models. Theories along AB are self-dual. The transition from order to disorder can occur in two ways: via a first-order transition, as happens along the Potts line as it intersects AB, or through an intermediate massless phase as is seen for the Villain line. All these features have been confirmed numerically. For instance, the susceptibility per unit spin diverges across the borders of (1) and (2) with domains (3) and (4) [28].

C and C' label those points on the self-dual line where the line of first-order transitions are *believed* to bifurcate into pairs of Kosterlitz-Thouless transitions. Fateev and Zamolodchikov [31] identified sets of self-dual Boltzmann weights w_k for the general $\mathbb{Z}_N$ model and conjectured there are critical points located at C and C' for N=5 and their generalisations to higher N. In 1985, they found parafermionic or $\mathbb{Z}_N$-symmetric conformal field theories which are natural candidates to describe these special points [32, 33]. The Fateev-Zamolodchikov Boltzmann weights and their critical exponents were matched with those of the parafermionic theory both analytically [29] and numerically [30]. In

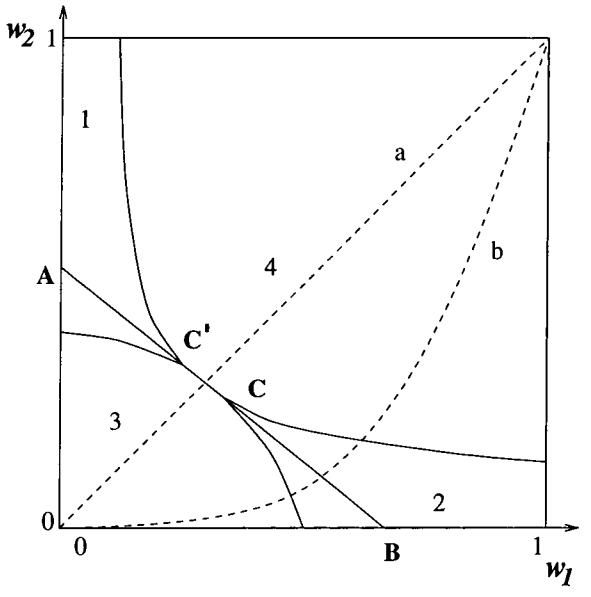


Figure 3.1: Phases for the general $\mathbb{Z}_5$ model. The Kosterlitz-Thouless regions 1 and 2 are massless, while regions 3 (ordered) and 4 (completely disordered) are massive. The line AB is self-dual; a labels the line of Potts models, and b is the Villain line.

other words the exact locations of the conformal points are known.

However, it is not proved that the conformal field theories are positioned at the opening of the first-order transition line into the massless Kosterlitz-Thouless phase. The conformal theories could lie within the massless regions, rather than at their opening. For the $\mathbb{Z}_5$ case, Alcaraz [30] used numerical evidence to suggest that the point at which the first-order transition line bifurcates into the Kosterlitz-Thouless region coincides with the parafermionic theory. Apart from this work, it remains unproven that the conformal theories are located at the opening of the Kosterlitz-Thouless regions. One of the main aims of this work is to demonstrate that this claim is true for all values of $N \geq 5$. The positions would be established if we could prove that perturbing the conformal theories in the self-dual directions gives a *massless* phase in one direction and a *massive* phase in the other. This would almost certainly pin down the conformal points because if they lay entirely within the massless region both perturbations would have to be massless.

Actually, another scenario is possible. The conformal points could lie on the self-dual line between the Kosterlitz-Thouless openings. We later show that perturbing the parafermionic models by the appropriate field, with a positive coupling, drives us along the self-dual line to another conformal theory which has to be at the opening of the massless phase because in this direction the infrared limiting theory has effective central

charge consistent only with the massless neighbourhood of the Kosterlitz-Thouless phase. This would mean there exists a segment on the self-dual line between the opening of massless phases which has infinite correlation length. Although not rigourously proved, there is numerical evidence to suggest, at least for $N = 5$, that the whole line between the openings of the Kosterlitz-Thouless regions is first order (see [28]) which would discount this possibility.

The space of fields in the parafermionic models provide a natural candidate for the perturbing field, namely $\epsilon^{(2)}$ of dimension $6/(N+2)$, which is both $\mathbb{Z}_N$-symmetric and self-dual. We therefore consider the $\epsilon^{(2)}$ perturbation of the parafermionic theory

$$\widehat{Z}_N^{(\pm)} = Z_N + \lambda \int \epsilon^{(2)}(x) d^2x \tag{3.9}$$

with positive and negative signs of the coupling. We shall show that for one sign of coupling ($\lambda < 0$) this perturbation forces us into a first-order surface, while for the other ($\lambda > 0$) the flow is from the conformal theory with central charge

$$c_N = \frac{2(N-1)}{(N+2)} \tag{3.10}$$

into the massless Kosterlitz-Thouless phase, to a conformal theory with $c = 1$.

The integrability of (3.9), proven in [34], suggests there could well exist a set of TBA equations which encode the exact flow of the theory from ultraviolet to infrared via the effective central charge. As mentioned earlier, the derivation of these equations is by no means simple when the theory is non-diagonal (as the massive $\mathbb{Z}_N$-symmetric theories are). This difficulty is compounded by the fact that the S-matrix for the massless flow is unknown. In such circumstances one can postulate a system of TBA equations and then make several checks on the validity of that conjecture. This problem is addressed for the flows (3.9) in the next chapter. In the next section we discuss the parafermionic algebra and show how the fields $\epsilon^{(2)}$ arise.

3.2 $\mathbb{Z}_N$-symmetric conformal field theories

The two-dimensional pure $\mathbb{Z}_N$-symmetric lattice model described in the last section has spins σ_j which can only take values

$$\sigma_j \in \{w^q : w = e^{2\pi i/N}, q = 0, 1, 2.., N-1\}. \tag{3.11}$$

Following [32, 33] we let $\sigma_{k,j} \equiv (\sigma_j)^k, k = 1, 2, .., N-1$ be the spin at site j raised to power k, so that $\sigma_{k,j}$ can only take values w^{qk}. The charge conjugate of $\sigma_{k,j}$ is written

$\sigma^{\dagger}_{k,j} = \sigma_{N-k,j}$. Let us suppose there are critical fixed-point models in the phase space of all $\mathbb{Z}_N$ models where we assume it is possible to replace the spins on the lattice by a conformal continuous spin field $\sigma_k(x)$ with scaling dimension $2d_k$. The critical theory is $\mathbb{Z}_N$ symmetric if, and only if, all correlation functions are invariant under

$$\sigma_k(x) \longrightarrow w^{mk}\sigma_k(x) \qquad \forall m\,. \tag{3.12}$$

This defines a $\mathbb{Z}_N$-charge k for the field $\sigma_k(x)$. The system is said to possess the Kramers-Wannier duality if the order fields $\sigma_k(x)$ are partnered by disorder fields $\mu_k(x)$ of the same dimensions such that all correlation functions are invariant under duality transformation $\sigma_k \longleftrightarrow \mu_k$. The self-dual theory also has the $\widetilde{\mathbb{Z}}_N$ symmetry $\mu_l(x) \longrightarrow w^{nl}\mu_l(x) \quad \forall n$, and the disorder fields are said to have a $\widetilde{\mathbb{Z}}_N$-charge l. σ_k has $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N$-charge $\{k,0\}$, while μ_k has $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N$-charge $\{0,k\}$. The $\mathbb{Z}_N$ statistical model is not self-dual in general, and self-duality is not restricted to fixed point or even critical theories.

The notion of mutual locality is very useful when describing the relative statistics of the fields in $\mathbb{Z}_N$-symmetric conformal theories. If $\langle\alpha_1(z_1,\overline{z}_1)\ldots\,\alpha_M(z_M,\overline{z}_M)\rangle$ is some M-point correlation function which picks up a phase $w^{-kl} = e^{2\pi i(\frac{-kl}{N})}$ as the coordinate z_k is analytically continued around z_l in the anti-clockwise direction, then the number $\gamma_{kl} = -kl/N$ is called the mutual locality exponent (MLE) of the two fields α_k and α_l. If γ_{kl} is an integer, α_k and α_l are said to be mutually local, otherwise mutually semilocal. A non-integer valued MLE simply means the correlation function is not single-valued. The field α_i is local (semilocal) if it is local (semilocal) with respect to itself. In particular fields σ_k and $\sigma_{k'}$ are mutually local.

For $N > 2$ the operator product expansions of $\sigma_k(z,\overline{z})\mu_k(0,0)$ and $\sigma_k(z,\overline{z})\mu^{\dagger}_k(0,0)$ begin

$$\sigma_k(z,\overline{z})\mu_k(0,0) = \frac{\psi_k(0,0)}{z^{2d_k-\Delta_k}\;\;\overline{z}^{2d_k-\overline{\Delta}_k}} + \ldots\,, \tag{3.13}$$

$$\sigma_k(z,\overline{z})\mu^{\dagger}_k(0,0) = \frac{\overline{\psi}_k(0,0)}{z^{2d_k-\overline{\Delta}_k}\;\;\overline{z}^{2d_k-\Delta_k}} + \ldots\,, \tag{3.14}$$

where the fields ψ_k and $\overline{\psi}_k$ are conformal with non-negative dimensions $(\Delta_k,\overline{\Delta}_k)$ and $(\overline{\Delta}_k,\Delta_k)$.

At this point we know neither d_k nor Δ_k. It is possible to fix the latter immediately since the MLE of the fields σ_k and μ_k is known. The MLE is $-k^2/N$ so analytic continuation $z \to e^{2\pi i}z$ ($\overline{z} \to e^{-2\pi i}\overline{z}$) gives a phase $e^{2\pi i(\Delta_k-\overline{\Delta}_k)}$ inside some correlation function due to (3.13). Hence

$$\Delta_k - \overline{\Delta}_k = -\frac{k^2}{N} \quad mod(\mathbb{Z})\,, \qquad k = 1,..N-1\,. \tag{3.15}$$

By the same argument, the MLE of σ_k and $\mu_k^\dagger$ is $\overline{\Delta_k} - \Delta_k$. We *assume* that there exist self-dual critical theories such that the pair $\psi_k, \overline{\psi}_k$ satisfy:

1. $\overline{\Delta}_k = 0$
2. $\partial_{\overline{z}}\psi_k = 0$ and $\partial_z\overline{\psi}_k = 0$, implying these fields are purely holomorphic and anti-holomorphic: $\psi_k = \psi_k(z)$ and $\overline{\psi}_k = \overline{\psi}_k(\overline{z})$.

Unlike the local fields encountered earlier, this pair in general have *fractional* spins. ψ_k and $\overline{\psi}_k$ are parafermionic currents and their modes will generate some extension of the Virasoro algebra. With this pair of assumptions the spins of the fields ψ_k are simply the left dimensions $\Delta_k = m_k - k^2/N$ for integer m_k. Suppose we *select* this spin so that $\Delta_{N-k} = \Delta_k$, then the parafermionic currents ψ_k have spin

$$\Delta_k = \frac{k(N-k)}{N}. \tag{3.16}$$

The fields $\psi_k, \overline{\psi}_k$ are themselves semilocal with $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N$-charges $\{k, k\}$ and $\{k, -k\}$.

The MLE for fields ψ_k and $\psi_{k'}$ is $-2kk'$, implying the identity $\Delta_{k+k'} - \Delta_k - \Delta_{k'} = -2kk'$ via (3.16). Omitting $\overline{z}$ dependence, the operator product expansion therefore looks like

$$\psi_k(z)\psi_{k'}(z') = c_{k,k'}\ (z-z')^{\Delta_{k+k'}-\Delta_k-\Delta_{k'}} \sum_{n=0}^{\infty}(z-z')^n\ \Psi^{(n)}_{k+k',k}(z') \tag{3.17}$$

where $k+k'$ is modulo N. $c_{k,k'}$ are parafermionic structure constants which are found by imposing associativity on the algebra once the normalisation $\langle\psi_k(z)\psi_{k'}(0)\rangle = \delta_{kk'}z^{-2\Delta_k}$ is fixed. In particular, when $k+k'=0$, $\Psi^{(0)}_{0,k}$ is the identity operator, and $\Psi^{(2)}_{0,k}(z)$ which has spin 2, is identified $(2\Delta_k/c)T(z)$, where we scale by this constant to have conformal invariance. There is also $\Psi^{(1)}_{0,k}$, a spin 1 field, which is ignored because it generates a spin $U(1)$ symmetry, greater than the $\mathbb{Z}_N$ symmetry. The algebra of the parafermion currents looks like

$$\psi_k(z)\psi_k^\dagger(z') = (z-z')^{\frac{-2k(N-k)}{N}}\ [I + (2\Delta_k/c)T(z')(z-z')^2 + O((z-z')^3)]; \tag{3.18}$$

$$\psi_k(z)\psi_{k'}(z') = c_{k,k'}\ (z-z')^{\frac{-2kk'}{N}}\ [\psi_{k+k'}(z') + O(z-z')]; \qquad k+k' \neq N, \tag{3.19}$$

$$\psi_k(z)\psi_{k'}^\dagger(z') = c_{k,N-k'}\ (z-z')^{\frac{-2k(N-k')}{N}}\ [\psi_{k-k'}(z') + O(z-z')]; \qquad k' < k. \tag{3.20}$$

Additionally, the ψ_k satisfy

$$T(z)\psi_k(z') = \frac{\Delta_k\psi_k(z')}{(z-z')^2} + \frac{\partial_{z'}\psi_k(z')}{(z-z')} + O(1) \tag{3.21}$$

and are thus primary. Fateev and Zamolodchikov [32] have shown that the central charge for this theory must take values given by (3.10).

The space of all parafermionic mutually semilocal fields is denoted $\{F\}$. We observed earlier that the space of conformal fields can be split into conformal families, each of which is characterised by a single highest weight field and forms a representation of the Virasoro algebra. In order to do the same for the parafermionic algebra it is convenient to first decompose $\{F\}$ into subspaces $F_{[q,\bar{q}]}$ of different charges $[q,\bar{q}] = \{k+l, k-l\}$, so that $\psi_k \in F_{[2k,0]}, \overline{\psi}_k \in F_{[0,2k]}, \sigma_k \in F_{[k,k]}$ and $\mu_k \in F_{[k,-k]}$. With this reorganisation all fields of $F_{[k,k]}$ and $F_{[k,-k]}$ are local. Specifically, the fields $I, T(z), \overline{T}(\bar{z})$ are in $F_{[0,0]}$.

The modes $A, A^\dagger, \overline{A}$ and $\overline{A}^\dagger$ of parafermionic currents can be introduced, in a manner analogous to (1.57), through operator expansions with arbitrary semilocal fields of dimension $(\Delta, \overline{\Delta})$

$$\psi_1(z)\phi_{[k,\bar{k}]}(0,0) \equiv \sum_{m=-\infty}^{\infty} z^{-\frac{k}{N}+m-1} A_{(1+k)/N-m}\phi_{k,\bar{k}}(0,0)\,, \tag{3.22}$$

$$\psi_1^\dagger(z)\phi_{[k,\bar{k}]}(0,0) \equiv \sum_{m=-\infty}^{\infty} z^{\frac{k}{N}+m-1} A^\dagger_{(1-k)/N-m}\phi_{k,\bar{k}}(0,0)\,. \tag{3.23}$$

We have similar equations for the barred components which introduce modes $\overline{A}_{(1+\bar{k})/N-m}$ and $\overline{A}^\dagger_{(1-\bar{k})/N-m}$. Note that each mode is dependent on the charges of the fields on which they act. The reason for looking only at the currents $\psi_1, \psi_1^\dagger, \overline{\psi}_1, \overline{\psi}_1^\dagger$ is that according to the above operator products the full algebra of the parafermionic currents $\psi_k, \overline{\psi}_k$ is uniquely determined by the algebra generated by the operators $A, A^\dagger, \overline{A}, \overline{A}^\dagger$. The dimensions of the fields in the right hand side above are

$$A_{(1+k)/N-m}\phi_{k,\bar{k}} \in F_{[k+2,\bar{k}]}: \qquad (\Delta + m - \frac{1+k}{N}, \overline{\Delta})\,, \tag{3.24}$$

$$A^\dagger_{(1-k)/N-m}\phi_{k,\bar{k}} \in F_{[k-2,\bar{k}]}: \qquad (\Delta + m - \frac{1-k}{N}, \overline{\Delta})\,. \tag{3.25}$$

Expressions (3.22,3.23) can be inverted to express the modes in terms of integrals of the parafermionic currents. These can then be used to find the parafermionic algebra generated by these modes.

If $\nu = \frac{1+k}{N}$ (modulo $\mathbb{Z}$), the fields obtained by applying the parafermionic modes to the fields $\phi_{[k,\bar{k}]}$ have dimensions

$$A_\nu\phi_k : (\Delta - \nu, \overline{\Delta}) \qquad \overline{A}_\nu\phi_k : (\Delta, \overline{\Delta} + \nu) \tag{3.26}$$

and unlike the action of the Virasoro modes L_n and L_{-n}, these parafermionic modes decrease (or increase) the spin by a fractional amount. The action of the conjugate

modes $A^\dagger_\nu$ is exactly the same with $-k$ replacing k. If we assume the dimensions of these fields are bounded below, there must exist fields which are annihilated by all operators A_ν with $\nu > 0$, i.e.

$$A_{(1+k)/N+n}\sigma_k = 0\,, \qquad \overline{A}_{(1+k)/N+n}\sigma_k = 0 \qquad \forall n > 0 \tag{3.27}$$

(and $A^\dagger_{(1-k)/N+n+1}\sigma_k = 0, \overline{A}^\dagger_{(1-k)/N+n+1}\sigma_k = 0 \quad \forall n > 0$, for the conjugate modes). This highest weight condition for the parafermionic algebra defines N *scalar* fields (order parameters) $\sigma_k \in \{F_{[k,k]}\}$. The algebra of the modes can be used to show that these fields (and the μ_k) have dimension [1]

$$d_k = \frac{k(N-k)}{2N(N+2)}\,. \tag{3.28}$$

All the fields spanning $\{F\}$ can be obtained by applying the operators $A, A^\dagger, \overline{A}, \overline{A}^\dagger$ to the parafermionic highest weight fields σ_k defined by (3.27). Each order field and its descendants give an irreducible representation $[\sigma_k]_A$ of the parafermionic current algebra and $\{F\} = \oplus_{k=0}^{N-1}[\sigma_k]_A$.

One simple example is the conformal $\mathbb{Z}_3$ or 3-state Potts model which coincides with $\widetilde{m} = 5$ in the minimal series. This model has Virasoro primaries σ_s with L_0 weights $s = 0, \frac{1}{8}, \frac{2}{3}, \frac{13}{8}, 3, \frac{2}{5}, \frac{1}{40}, \frac{1}{15}, \frac{21}{40}$ and $\frac{7}{5}$. Each of these fit into representations of the parafermionic algebra, though not all of these fields are parafermionic primaries. $\sigma_{1/8}$ and $\sigma_{1/40}$ are parafermionic primaries and descendants $A_{-3/2}\sigma_{1/8}$ and $A_{-1/2}\sigma_{1/40}$ are the fields with dimension $\frac{13}{8}$ and $\frac{21}{40}$ respectively.

We now briefly describe a way of organising the descendants of the fields σ_k and their dimensions. A special set of descendants of σ_k are the *principal semilocal fields* $\phi^{(k)}_{[q,k]}$ (where the index q runs from $-k$ to $-k+2(N-1)$ in steps of 2):

$$\begin{aligned}
\phi^{(k)}_{[k+2l,k]} &= A_{(k-1+2l)/N-1}A_{(k-1+2l-2)/N-1}A_{(k-1+2l-4)/N-1}\cdots A_{(k-1+2)/N-1}\sigma_k\,, \\
&\quad (d_k + \frac{l(N-k-l)}{N}, d_k) \qquad\qquad l = 0, 1, ..N-k \qquad (3.29)\\
\phi^{(k)}_{[k-2l,k]} &= A^\dagger_{-(k+1-2l)/N}A^\dagger_{-(k+1+2l+2)/N}A^\dagger_{-(k+1+2l+4)/N}\cdots A^\dagger_{(k+1-2)/N}\sigma_k \\
&\quad (d_k + \frac{l(k-l)}{N}, d_k) \qquad\qquad l = 0, 1, ..k \qquad (3.30)
\end{aligned}$$

[1] The algebra of the parafermionic modes is stated in [32] to give

$$\begin{aligned}
&\sum_{l=0}^{\infty} C^l_{-(N+2)/N}[A_{-(1-k)/N+n-l-1}A^\dagger_{(1-k)/N+m+l+1} - A^\dagger_{-(k+1)/N+m-l}A_{(1+k)/N+n+l}] \\
&= \frac{N+2}{N}L_{m+n} + \frac{1}{2}(n+\frac{k}{N})(n-1+\frac{k}{N})\delta_{n+m,0}.
\end{aligned}$$

where $C^l_\lambda = \frac{\Gamma(1-\lambda)}{l!\Gamma(-\lambda)}$. Just set $m = n = 0$ to obtain $\frac{N+2}{N}L_0\sigma_k = (\frac{k}{2N})(\frac{N-k}{N})\sigma_k$, giving (3.28).

where $\phi^{(k)}_{[q+2N,k]} = \phi^{(k)}_{[q,k]}$. Note that $\phi^{(k)}_{[k,k]}$ are the spin fields σ_k, $\phi^{(k)}_{[-k,k]}$ are the duals μ_k and the parafermionic currents ψ_k are $\phi^{(0)}_{[2k,0]} = \phi^{(N)}_{[N+2k,N]}$. If we apply the barred modes $\overline{A}, \overline{A}^\dagger$ in a similar way we can construct the principal semilocal fields $\phi^{(k)}_{[q,\overline{q}]}$ which have dimensions $(d^{(q)}_k, d^{(\overline{q})}_k)$. These dimensions are

$$d^{(q)}_k = \begin{cases} d_k + \frac{(k-q)(k+q)}{4N}, & -k \leq q \leq k\,, \\ d_k + \frac{(q-k)(2N-k-q)}{4N} & k \leq q \leq 2N-k\,. \end{cases} \tag{3.31}$$

The dimensions of all other fields of $\{F\}$ differ from those of the principal semilocal fields only by integers.

For this work the most important principal semilocal fields are those which are charge neutral and self-dual. This is because we want to examine perturbations of the self-dual conformal $\mathbb{Z}_N$ theories which preserve that $\mathbb{Z}_N$ invariance and 'aim' the perturbation along the self-dual direction. If we were to perturb by a field with non-zero $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N$-charge we would leave the $\mathbb{Z}_N$-invariant subspace.

$\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N$ neutral fields are obtained from a highest weight field σ_k through application of $\overline{A}$ or $\overline{A}^\dagger$ as well as A or $A^\dagger$ through (3.29) and (3.30). We therefore require the principal semilocal descendant field $\phi^{(k)}_{[k-2l,k-2j]}$ to have $k = 2l = 2j$. The resulting fields $\epsilon^{(j)} = \phi^{(2j)}_{[0,0]}$ are called thermal operators and have dimension $(d^{(0)}_{2j}, d^{(0)}_{2j})$ where

$$d^{(0)}_{2j} = \frac{j(j+1)}{N+2} \qquad j = 1, .., [N/2]\,. \tag{3.32}$$

There are $[N/2]$ such operators because the spin field index k can only run up to $N-1$. This number coincides with the dimensionality of the $\mathbb{Z}_N$ phase space. Under the duality transformation $\mu_k \longleftrightarrow \sigma_k$ the thermal operators can change parity when j is odd: $\epsilon^{(j)} \longrightarrow (-1)^j \epsilon^{(j)}$. Consequently, there are $[N/4]$ self-dual spinless thermal operators coinciding with the dimensionality of the self-dual hypersurface in phase space.

We note that $\epsilon^{(j)}$ is a relevant field only when $N > (j-1)(j+2)$, so the thermal operator $\epsilon^{(j)}$ first appears as a relevant field in model $\mathbb{Z}_N$ where $N = (j-1)(j+2)+1$. Specifically, the first self-dual thermal operator $\epsilon^{(2)}$ becomes relevant in the $\mathbb{Z}_5$ model. For $N = 6$ and 7 the phase spaces are three-dimensional but the self-dual hypersurface remains one dimensional. In the case $N = 8$ the self-dual region becomes two dimensional and another self-dual field appears, namely $\epsilon^{(4)}$, which is irrelevant. This field remains irrelevant until $\mathbb{Z}_{19}$. As we increase N more self-dual fields appear, each being irrelevant until N becomes sufficiently large. $\epsilon^{(2)}$ has dimension

$$\Delta = \frac{6}{N+2}\,. \tag{3.33}$$

Since this is the only self-dual $\mathbb{Z}_N$-symmetric field by which we can make a self-dual perturbation for $N = 5$, it could be that we can drive the theory into the first order phase or into the massless phase according to the sign of the coupling of this field. It will be shown that the field $\epsilon^{(2)}$ has this property for all values of N.

Chapter 4

The $\mathbb{Z}_N$ TBA Systems

TBA equations for the flows from self-dual $\mathbb{Z}_N$-symmetric conformal models in the self-dual directions are presented in this chapter. We begin with massless flows which have infinite correlation length for all values of the perturbing parameter (see Section (1.1.2)).

4.1 Known massless equations

It has been conjectured by Fateev [34] that the massless flows $\widehat{Z}^{(+)}_{N=2m}$ ($m \geq 3$) coincide with the perturbed coset theories

$$\frac{SO(m)^{(2)} \times SO(m)^{(1)}}{SO(m)^{(3)}} + \lambda \phi_{1,1,\mathrm{Adj}} \quad \longrightarrow \quad \frac{SO(m)^{(1)} \times SO(m)^{(1)}}{SO(m)^{(2)}}, \tag{4.1}$$

as λ increases from zero to infinity, where the relevant spinless perturbing field $\phi_{1,1,\mathrm{Adj}}$ has dimension $\Delta = 6/(N+2)$. This flow is said to be b_n-related if $m = 2n+1$, or d_{n+1}-related if $m = 2n+2$. The basis for this observation is that the dimension of $\phi_{1,1,\mathrm{Adj}}$ coincides with that of the self-dual energy field $\epsilon^{(2)}$ and the central charge for the nonperturbed theory is, via (1.82), $c = (2m-1)/(m+1) = c_{N=2m}$. The infrared limiting central charge is $c = 1$ which characterises another massless theory which could be in the Kosterlitz-Thouless phase. This evidence alone is not enough to conclude that the flows (4.1) coincide with $\widehat{Z}^{(+)}_{N=2m}$ because the fact that two models have the same ground state energy does not mean they are the same theory. However, in this and the next section we show that the infrared limits of the TBA equations for the flows (4.1) agree perfectly with perturbation theory about a conformal theory which lies in the Kosterlitz-Thouless region of $\mathbb{Z}_N$ phase space.

TBA equations have been found for the flows (4.1) without explicit reference to $\mathbb{Z}_{2m}$-symmetric models. We state these and later give strong evidence that they give the

ground state energy for the $\widehat{Z}^{(+)}_{N=2m}$ models. The only other TBA equations stated for a $\widehat{Z}^{(+)}_N$ theory are for $N = 5$. This apart, no odd-N TBA systems have been proposed. We present these shortly, but first explain what is known about the even and $N = 5$ theories beginning with the d_n-related or $N = 4n + 4$ cases. In Section (4.5) we carry out the first check on the TBA systems by calculating the ultraviolet and infrared limits of the effective central charge.

4.1.1 $\mathbb{Z}_{4n+4}$ TBA

Equations have been hypothesised to explain the flows between a sequence of coset models, ending in the flow (4.1) with $m = 2n+2$ ($N = 4n+4$). The equations may be written as [36]

$$\varepsilon_i^{(\alpha)}(\theta) = \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n+1} \Big[\phi_{ij} * L_j^{(\alpha)}(\theta) - \psi_{ij} * L_j^{(\widetilde{\alpha})}(\theta) \Big] \qquad (\alpha = 1, 2; i = 1, ...n+1)\,, \quad (4.2)$$

where $\widetilde{\alpha} = 3-\alpha$, $*$ denotes the convolution already defined and L-functions have an additional upper index with $L_i^{(\alpha)}(\theta) = \log(1 + e^{-\varepsilon_i^{(\alpha)}(\theta)})$. In this massless theory each pseudoenergy $\varepsilon_i^{(\alpha)}(\theta)$ has an associated energy according to which the corresponding particle is called a left or right mover:

$$\begin{aligned} \nu_i^{(1)}(\theta) &= m_i r e^{\theta}/2 \quad \text{for right movers}\,, \\ \nu_i^{(\tilde{1})}(\theta) &= m_i r e^{-\theta}/2 \quad \text{for left movers}\,, \end{aligned} \qquad (4.3)$$

where i runs from 1 to $n+1$ for this d_{n+1} case and m_i is the mass ratio of the i-th particle M_i and the lightest mass of the theory. (Strictly we should refer to crossover scales rather than masses for these infinite correlation length theories, but we shall continue with this usage for convenience.) The right and left moving massless 'particles' have S-matrix elements

$$S_{ij}^{LL}(\theta) = S_{ij}^{RR}(\theta) = S_{ij}(\theta)\,, \qquad S_{ij}^{RL}(\theta) = S_{ij}^{LR}(\theta) = (T_{ij}(\theta))^{-1}\,,$$

where $S_{ij}(\theta)$ is the d_n sine-Gordon S-matrix for $h = 2n$ of section (2.2) and $T_{ij}(\theta)$ is obtained by replacing each block $\{x\}$ in (2.25–2.26) by (x). Both the kernels ϕ_{ij}, ψ_{ij} and the masses M_i come from the sine-Gordon data of Section (2.2). The kernels are $(-i)$ times the derivative of the logarithm of $S_{ij}(\theta)$ and $T_{ij}(\theta)$

$$\phi_{jk} = -i\frac{d}{d\theta}\log S_{jk}\,, \qquad \psi_{jk} = -i\frac{d}{d\theta}\log T_{jk}\,. \qquad (4.4)$$

One must interpret these amplitudes for massless 'scattering' with care. It is not clear that the massless S-matrix has a proper physical meaning in its own right: it is difficult

to produce a convincing argument that two massless particles travelling in the same direction can scatter. Also the assertion that integrability implies S-matrix factorisation is questionable. Nonetheless, quantities like central charge or ground state energy, which are derived using the massless S-matrices do make sense and can be verified in a variety of ways.

The solutions $\varepsilon_i^{(\alpha)}(\theta)$ to (4.2) allow the calculation of the effective central charge, which is naturally given by

$$\tilde{c}(r) = \frac{3}{\pi^2} \sum_{\substack{i=1 \\ \alpha=1,2}}^{n+1} \int_{-\infty}^{\infty} d\theta \, \nu_i^{(\alpha)}(\theta) L_i^{(\alpha)}(\theta) \,. \tag{4.5}$$

Note the massless TBA systems have the following $\mathbb{Z}_2$ symmetry under reversal of spatial momenta and simultaneous interchange of right and left movers

$$\varepsilon_i^{(\alpha)}(\theta) \longrightarrow \varepsilon_i^{(\tilde{\alpha})}(-\theta) \,, \tag{4.6}$$

which simplifies the calculation of the central charge function and reduces the amount of computation required in any numerical calculations.

In the large r limit $L_i^{(\tilde{1})}(\theta)$ becomes vanishingly small at any values of θ for which $L_i^{(1)}(\theta)$ is non-zero, and vice versa, so the system (4.2) decouples into

$$\varepsilon_i^{(\alpha)}(\theta) = \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n+1} \phi_{ij} * L_j^{(\alpha)}(\theta) \qquad (\alpha = 1,2; i = 1, \ldots n+1) \,. \tag{4.7}$$

These are exactly the TBA equations expected for the ultraviolet limit massive sine-Gordon model with $\beta^2 = \frac{8\pi}{n+1}$ or

$$\beta^2 = \frac{32\pi}{N} \qquad \text{(N=4n+4)} \tag{4.8}$$

which can be easily derived since the theory has diagonal scattering. Note that there is a complete absence of mass scale in the $r \longrightarrow \infty$ limit where we can make an arbitrary shift of the θ variable without deviating from the fixed point.

4.1.2 $\mathbb{Z}_{4n+2}$ TBA

For the b_n-related flows, with $m = 2n+1$ ($N = 4n+2$), the TBA equations have also already been discovered [37, 38]. The underlying massless scattering theory is not diagonal and consequently this means the introduction of additional pseudoenergies $(\varepsilon_n^{(0)}, \varepsilon_n^{(2)}, \varepsilon_n^{(4)})$ corresponding to pseudoparticles or *magnons* which have zero energy. These have a real

effect on the entropy and therefore on the final TBA equations. The TBA system can then be written as

$$\begin{aligned}
\varepsilon_i^{(\alpha)}(\theta) &= \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n} \Big[\phi_{ij} * L_j^{(\alpha)}(\theta) - \psi_{ij} * L_j^{(\widetilde{\alpha})}(\theta) \Big] \\
&\quad - \delta_{i,n} \sum_{\beta=0}^{4} l_{\alpha\beta}^{[a_5]} \phi_1 * L_n^{(\beta)}(\theta)\,, \qquad (\alpha = 1,3;\ i = 1,..n)\,, \\
\varepsilon_n^{(\alpha)}(\theta) &= -\sum_{\beta=0}^{4} l_{\alpha\beta}^{[a_5]} \phi_1 * L_n^{(\beta)}(\theta)\,, \qquad (\alpha = 0,2,4)\,.
\end{aligned} \tag{4.9}$$

Here $l_{\alpha\beta}^{[a_5]}$ represents the incidence matrix of an a_5 Dynkin diagram so that $l_{23}^{[a_5]} = 1$, $l_{24}^{[a_5]} = 0$ etc. This and the incidence matrices for other Dynkin diagrams ($l^{[a_n]}, l^{[d_n]}, l^{[e_6]}$) are a convenient way of encoding the TBA equations. Setting $\widetilde{\alpha} = 4-\alpha$, the energy terms are again given by equation (4.3), now for $i = 1,..n$. Parity symmetry is implemented as in (4.6). The masses M_i and the kernels ϕ_{ij}, ψ_{ij} can be found from the sine-Gordon data of Section (2.2) with $h = 2n-1 = N/2-2$ and

$$\phi_1 = \frac{h}{\cosh h\theta}\,. \tag{4.10}$$

For this and the other non-diagonal cases $N = 4n+1, 4n+3$ the soliton-soliton kernels ϕ_{nn} and ψ_{nn} are defined by

$$\phi_{nn}(\theta) = \chi_\rho * \phi_{nn-1}(\theta) \qquad , \qquad \psi_{nn}(\theta) = \chi_\rho * \psi_{nn-1}(\theta)\,, \tag{4.11}$$

where $\chi_\rho(\theta) = 2h/(\rho\cosh(2h\theta/\rho))$ for $N = 4n+\rho$, while ϕ_{nn-1} and ψ_{nn-1} are given in Section (2.2). For $N \leq 7$, $\phi_{n,n-1}$ and $\psi_{n,n-1}$ are not defined and we set $\phi_{nn} = \psi_{nn} = 0$. Again, the effective central charge is given by

$$\widetilde{c}(r) = \frac{3}{\pi^2} \sum_{\substack{i=1 \\ \alpha=1,3}}^{n} \int_{-\infty}^{\infty} d\theta\, \nu_i^{(\alpha)}(\theta) L_i^{(\alpha)}(\theta)\,. \tag{4.12}$$

In the infrared limit of this theory $r \to \infty$, the equations separate into two sets as before, one where the index α takes the values $0,1,2$ and a second in which it takes the values $2,3,4$. (Due to the symmetry (4.6) only one of these need appear in the calculation of the central charge.) The first set is

$$\begin{aligned}
\varepsilon_i^{(1)}(\theta) &= \nu_i^{(1)}(\theta) - \sum_{j=1}^{n} \phi_{ij} * L_j^{(1)}(\theta) - \delta_{i,n} \sum_{\beta=0,2} \phi_1 * L_n^{(\beta)}(\theta) \qquad (i = 1,..n)\,, \\
\varepsilon_n^{(\alpha)}(\theta) &= -\phi_1 * L_n^{(1)}(\theta) \qquad\qquad (\alpha = 0,2).
\end{aligned} \tag{4.13}$$

Again we observe that these equations coincide with the ultraviolet limit of the massive sine-Gordon TBA system (A.38) with coupling $\beta^2 = \frac{8\pi}{n+1/2}$ or

$$\beta^2 = \frac{32\pi}{N} \qquad \text{(N=4n+2)}\,. \tag{4.14}$$

which are derived in Section (A.1) of Appendix (A). In this appendix we explain how magnonic densities arise in theories with non-diagonal scattering.

4.1.3 $\mathbb{Z}_5$ TBA

The only previously known N-odd case was a conjecture for the $N = 5$ flow from c=8/7 to c=1 found by Ravanini *et al.* [39]. The TBA equations, with solutions $\varepsilon^{(1)}, .., \varepsilon^{(6)}$, are

$$\varepsilon^{(\alpha)}(\theta) = \nu^{(\alpha)}(\theta) - \sum_{\beta=1}^{6} l_{\alpha\beta}^{[e_6]} \phi * L^{(\beta)}(\theta) \qquad (\alpha = 1, ..6)\,, \tag{4.15}$$

where $\nu^{(\alpha)}(\theta) = \frac{1}{2}mre^{\theta}\delta_{1\alpha} + \frac{1}{2}mre^{-\theta}\delta_{6\alpha}$, $\phi(\theta) = 1/\cosh\theta$ and $l_{\alpha\beta}^{[e_6]}$ is the incidence matrix of the e_6 Dynkin diagram, labelled so that 1 and 6 are the nodes which are farthest apart. This system also has the parity symmetry $\varepsilon^{(1)}(\theta) = \varepsilon^{(6)}(-\theta)$, which again helps in the evaluation of the central charge function

$$\widetilde{c}(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \left[\nu^{(1)}(\theta)L^{(1)}(\theta) + \nu^{(6)}(\theta)L^{(6)}(\theta)\right]\,. \tag{4.16}$$

In this case the limit $r \to \infty$ removes all trace of $\varepsilon^{(6)}$ from the equations satisfied by $\varepsilon^{(1)}$, and vice versa. So

$$\varepsilon^{(\alpha)}(\theta) = \nu^{(\alpha)}(\theta) - \sum_{\beta=1}^{5} l_{\alpha\beta}^{[d_5]} \phi * L^{(\beta)}(\theta) \qquad (\alpha = 1, ..5)\,, \tag{4.17}$$

where $l_{\alpha\beta}^{[d_5]}$ is the incidence matrix of the d_5 Dynkin diagram. Analogous equations determine the infrared form of $\varepsilon^{(6)}(\theta)$. Again this is exactly the TBA for the ultraviolet limit of the

$$\beta^2 = \frac{32\pi}{5} \tag{4.18}$$

sine-Gordon model.

4.2 The $\mathbb{Z}_N$ Y-systems

Unfortunately, there is nothing at all known about the S-matrices of the massless scattering for the flow $\widehat{Z}_N^{(+)}$. A direct derivation of the $\mathbb{Z}_N$ TBA equations is therefore currently impossible. However, we know that the infrared limits of the $N = 4n + 2$, $N = 4n + 4$

and $N = 5$ TBA equations for $\widehat{Z}_N^{(+)}$ coincide with those of the ultraviolet limit of the massive sine-Gordon for β^2 given by (4.8),(4.14) and (4.18) respectively. This leads to the conjecture that (apart from mass terms) *the $\mathbb{Z}_2$-symmetric version of the $\beta^2 = 32\pi/N$ massive sine-Gordon TBA system is the TBA system for the flows $\widehat{Z}_N^{(\pm)}$ when $N \geq 5$.*

In Section (A.1) we derive a set of sine-Gordon TBA equations for $\beta^2 = 32\pi/(4n+2)$, that is we prove (4.13). Though complicated by the fact that it is necessary to diagonalise a very intricate set of Bethe equations, the derivation follows along similar lines to the diagonal case developed earlier.

The pseudoenergy solutions to each TBA system are also solutions to a set of functional equations obtained from the TBA system following the removal of any energy terms and the substitution

$$Y_i^{(\alpha)}(\theta) \equiv e^{\varepsilon_i^{(\alpha)}(\theta)}. \tag{4.19}$$

The resulting Y-system which actually represents the TBA equations of both massless and massive directions is useful for several reasons. First, the stationary (θ independent) version of the Y-system enables one to calculate the central charge of the ultraviolet and infrared limiting theories. Also, the Y-system allows us to determine the dimension of the perturbing field of the corresponding perturbative theory. Both provide a means of checking that the TBA system describes the perturbed conformal theory it is meant to. Also, the Y-systems are basically simpler to manipulate than the individual TBA systems.

For these reasons, and because it is known how to write down the Y-systems of sine-Gordon models at rational values of $\beta^2/32\pi$ [40], we carry out the above symmetrisation using the Y-systems rather than the TBA equations. Therefore, the next step in the $\mathbb{Z}_N$ TBA derivation is to find the Y-systems for $\beta^2 = 32\pi/(4n+2)$. This is explained in Section (A.1). Then in Section (A.2) we apply the rules laid out in [40] to obtain the sine-Gordon Y-system for $\beta^2 = 32\pi/N$ for all integers $N \geq 5$. These functional equations fall into four distinct classes, each represented by one of the diagrams in Figure (A.2). These can be made symmetric almost immediately, and the result is a set of Y-systems which are believed to represent the flows $\widehat{Z}_N^{(\pm)}$. A summary of the whole scheme is given in Figure (4.1).

The *symmetrised* version of the sine-Gordon Y-systems of Section (A.2) are the $\mathbb{Z}_N$ Y-systems represented by the graphs in Figure (4.2). Using the functions defined by

$$\begin{aligned} \overline{Y}_i^{(\alpha)}[\theta, r] &= Y_i^{(\alpha)}(\theta - \tfrac{\imath\pi\xi r}{2})\, Y_i^{(\alpha)}(\theta + \tfrac{\imath\pi\xi r}{2}) \\ \overline{Y}_i^{(\alpha)}\{\theta, r\} &= \left(1 + Y_i^{(\alpha)}(\theta - \tfrac{\imath\pi\xi r}{2})\right)\left(1 + Y_i^{(\alpha)}(\theta + \tfrac{\imath\pi\xi r}{2})\right) \end{aligned} \tag{4.20}$$

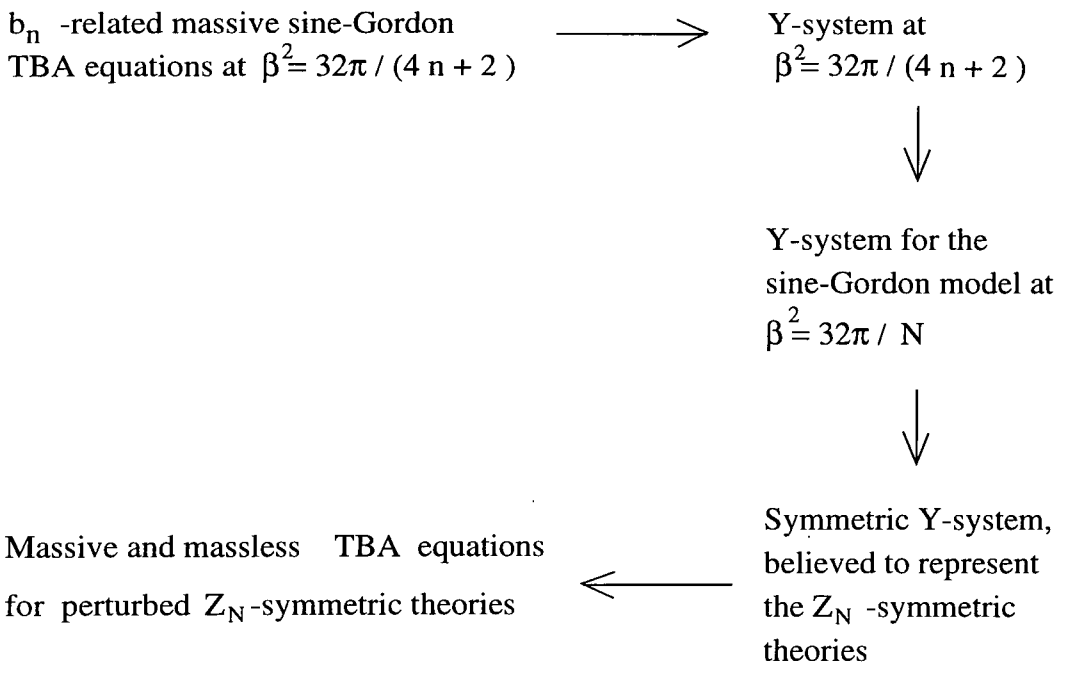


Figure 4.1: Summary of the strategy behind the derivation of the $\mathbb{Z}_N$ TBA equations.

$$V_i^{(\alpha)}\{\theta,r\} \;=\; \Big(1+Y_i^{(\alpha)}(\theta-\tfrac{\imath\pi\xi r}{2})^{-1}\Big)^{-1}\Big(1+Y_i^{(\alpha)}(\theta+\tfrac{\imath\pi\xi r}{2})^{-1}\Big)^{-1},$$

the four types of $\mathbb{Z}_N$ Y-systems may be written as follows.

1) $N = 4n+1$

Nodes ($i<n-1, \alpha = 1,6$), with $\widetilde{\alpha} = 7-\alpha$:

$$\overline{Y}_i^{(\alpha)}[\theta,1] \;=\; \Big(1+Y_i^{(\widetilde{\alpha})}(\theta)^{-1}\Big)^{-1}\prod_{j=1}^{n-1}\Big(1+Y_j^{(\alpha)}(\theta)\Big)^{I_{ij}^{[a_{n-1}]}} \tag{4.21}$$

Nodes ($n-1, \alpha=1,6$) :

$$\begin{aligned}\overline{Y}_{n-1}^{(1)}[\theta,1] \;=\; &\Big(1+Y_{n-1}^{(6)}(\theta)^{-1}\Big)^{-1}\Big(1+Y_{n-2}^{(1)}(\theta)\Big)\Big(1+Y_n^{(4)}(\theta)\Big)\Big(1+Y_n^{(5)}(\theta)\Big)\\ &\times \overline{Y}_n^{(1)}\{\theta,3/4\}\overline{Y}_n^{(2)}\{\theta,2/4\}\overline{Y}_n^{(3)}\{\theta,1/4\}\end{aligned}$$

$$\begin{aligned}\overline{Y}_{n-1}^{(6)}[\theta,1] \;=\; &\Big(1+Y_{n-1}^{(1)}(\theta)^{-1}\Big)^{-1}\Big(1+Y_{n-2}^{(6)}(\theta)\Big)\Big(1+Y_n^{(4)}(\theta)\Big)\Big(1+Y_n^{(2)}(\theta)\Big)\\ &\times \overline{Y}_n^{(6)}\{\theta,3/4\}\overline{Y}_n^{(5)}\{\theta,2/4\}\overline{Y}_n^{(3)}\{\theta,1/4\}\end{aligned}$$

Nodes ($n, \alpha=1\ldots6$) :

$$\overline{Y}_n^{(\alpha)}[\theta,1/4] \;=\; \Big(1+Y_{n-1}^{(\alpha)}(\theta)\Big)\prod_{\beta}\Big(1+Y_n^{(\beta)}(\theta)^{-1}\Big)^{-I_{\alpha\beta}^{[e_6]}}$$

with $Y_{n-1}^{(\alpha)}(\theta) \equiv 0$ when $\alpha \neq 1,6$.

2) $N = 4n{+}2$ (b_n)

Nodes $(i{<}n{-}1, \alpha{=}1,3)$: equation (4.21) holds with $\widetilde{\alpha} = 4{-}\alpha$.

Nodes $(n{-}1, \alpha{=}1,3)$:

$$\begin{aligned}\overline{Y}^{(\alpha)}_{n-1}[\theta,1] &= \left(1+Y^{(\widetilde{\alpha})}_{n-1}(\theta)^{-1}\right)^{-1}\left(1+Y^{(\alpha)}_{n-2}(\theta)\right)\left(1+Y^{(\alpha-1)}_{n}(\theta)\right)\\ &\quad\times\left(1+Y^{(\alpha+1)}_{n}(\theta)\right)\overline{Y}^{(\alpha)}_{n}\{\theta,1/2\}\end{aligned}$$

Nodes $(n, \alpha{=}0\ldots4)$ (with $Y^{(\alpha)}_{n-1}(\theta) \equiv 0$ when $\alpha \neq 1,3$) :

$$\overline{Y}^{(\alpha)}_{n}[\theta,1/2] = \left(1+Y^{(\alpha)}_{n-1}(\theta)\right)\prod_{\beta}\left(1+Y^{(\beta)}_{n}(\theta)^{-1}\right)^{-I^{[\alpha_5]}_{\alpha\beta}}$$

3) $N = 4n{+}3$

Nodes $(i{<}n{-}1, \alpha{=}1,3)$: equation (4.21) holds with $\widetilde{\alpha} = 7{-}\alpha$.

Nodes $(n{-}1, \alpha{=}1,6)$:

$$\begin{aligned}\overline{Y}^{(1)}_{n-1}[\theta,1] &= \left(1+Y^{(1)}_{n-2}(\theta)\right)\left(1+Y^{(2)}_{n}(\theta)\right)\left(1+Y^{(6)}_{n-1}(\theta)^{-1}\right)^{-1}\overline{Y}^{(1)}_{n}\{\theta,1/4\}\\ \overline{Y}^{(6)}_{n-1}[\theta,1] &= \left(1+Y^{(6)}_{n-2}(\theta)\right)\left(1+Y^{(5)}_{n}(\theta)\right)\left(1+Y^{(1)}_{n-1}(\theta)^{-1}\right)^{-1}\overline{Y}^{(6)}_{n}\{\theta,1/4\}\end{aligned}$$

Nodes $(n, \alpha = 1\ldots6)$:

$$\begin{aligned}\overline{Y}^{(1)}_{n}[\theta,3/4] &= \left(1+Y^{(1)}_{n-1}(\theta)\right)\left(1+Y^{(4)}_{n}(\theta)^{-1}\right)^{-1}\left(1+Y^{(5)}_{n}(\theta)^{-1}\right)^{-1}\\ &\quad\times V^{(2)}_{n}\{\theta,2/4\}\,V^{(3)}_{n}\{\theta,1/4\}\\ \overline{Y}^{(2)}_{n}[\theta,1/4] &= \left(1+Y^{(3)}_{n}(\theta)\right)\left(1+Y^{(1)}_{n}(\theta)^{-1}\right)^{-1}\\ \overline{Y}^{(3)}_{n}[\theta,1/4] &= \left(1+Y^{(4)}_{n}(\theta)\right)\left(1+Y^{(2)}_{n}(\theta)\right)\left(1+Y^{(5)}_{n}(\theta)\right)\\ \overline{Y}^{(4)}_{n}[\theta,1/4] &= \left(1+Y^{(3)}_{n}(\theta)\right)\\ \overline{Y}^{(5)}_{n}[\theta,1/4] &= \left(1+Y^{(3)}_{n}(\theta)\right)\left(1+Y^{(6)}_{n}(\theta)^{-1}\right)^{-1}\\ \overline{Y}^{(6)}_{n}[\theta,3/4] &= \left(1+Y^{(6)}_{n-1}(\theta)\right)\left(1+Y^{(4)}_{n}(\theta)^{-1}\right)^{-1}\left(1+Y^{(2)}_{n}(\theta)^{-1}\right)^{-1}\\ &\quad\times V^{(5)}_{n}\{\theta,2/4\}\,V^{(3)}_{n}\{\theta,1/4\}\end{aligned}$$

4) $N = 4n{+}4$ (d_{n+1})

In this case the scope of equation (4.21), with $\widetilde{\alpha} = 3{-}\alpha$, can be extended to cover all of the nodes $(i{=}1\ldots n{+}1, \alpha{=}1,2)$:

$$\overline{Y}^{(\alpha)}_{i}[\theta,1] = \left(1+Y^{(\widetilde{\alpha})}_{i}(\theta)^{-1}\right)^{-1}\prod_{j=1}^{n+1}\left(1+Y^{(\alpha)}_{j}(\theta)\right)^{I^{[d_{n+1}]}_{ij}}$$

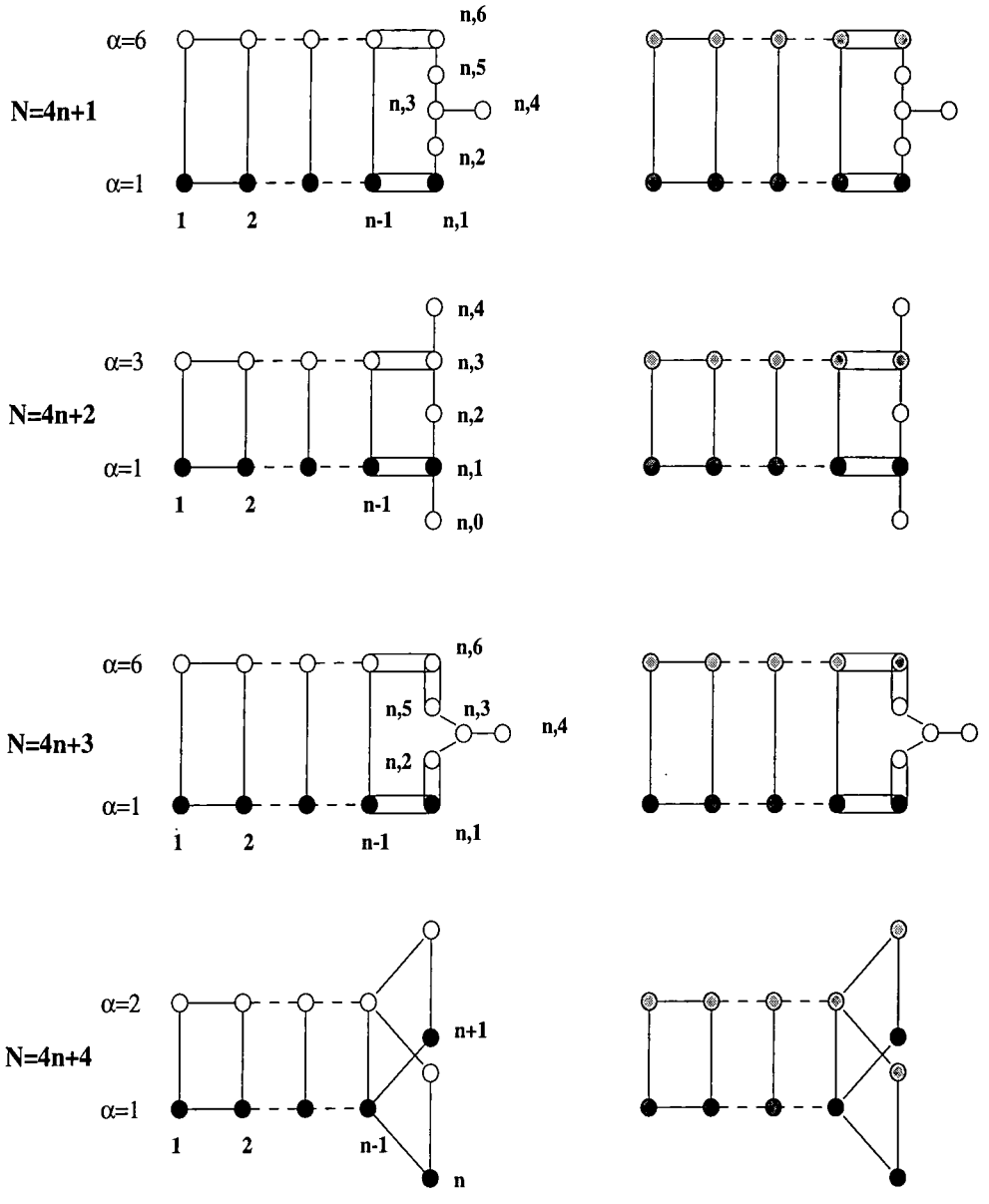


Figure 4.2: Diagrammatic representations of the $\mathbb{Z}_N$ Y-systems. The left (right) column represents the massive (massless) theories. The darkest nodes label energies $\nu_i^{(1)}(\theta) = m_i r \cosh\theta$ and $\nu_i^{(1)}(\theta) = \frac{1}{2} m_i r e^{\theta}$ respectively. The lighter nodes correspond to the energies $\nu_i^{(\bar{1})} = 0$ and $\nu_i^{(\bar{1})} = \frac{1}{2} m_i r e^{-\theta}$ respectively. Nodes without shading have zero associated energy.

To write down the massless TBA equations from the $\mathbb{Z}_N$ Y-systems one must carry out the following steps.

1. Take the logarithm of all equations and then Fourier transform with respect to θ. Every pair of shifts in the Y-system gives a hyperbolic cosine term.

2. Re-arrange the result to express each Fourier transformed pseudoenergy in terms of the transformed L-functions $L_i^{(\beta)}(k)$ and K-functions $K_i^{(\beta)}(k) = \log(1 + e^{\varepsilon_i^{\beta}(k)})$.

3. Fourier invert to obtain a set of integral equations for the pseudoenergies $\varepsilon_i^{(\alpha)}(\theta)$. This expression involves a convolution for every product in the above step.

4. Introduce the appropriate scale-dependent energy terms $\nu_i^{(1)}$ and $\nu_i^{(\tilde{1})}$ of (4.3) by hand.

We should not find the last step so surprising. The ultraviolet limit of the sine-Gordon model has lost all the scale dependence therefore the symmetric version of the corresponding system also has the same feature. All $\mathbb{Z}_N$ TBA equations can be derived from the Y-systems listed above via these steps. The $\mathbb{Z}_N$ Y-systems for $N = 4n+4$ and $N = 4n+2$ give exactly the equations (4.7) and (4.9). The results for odd-N are stated next.

The diagrams in Figure (4.2) help when constructing the TBA equations. The Y-system diagrams do not encode any explicit scattering information; they simply represent conditions on pseudoenergies which give an equilibrium state. If nodes of $\varepsilon_i^{(1)}$ and $\varepsilon_j^{(1)}$ are on the same horizontal level then a convolution $-\phi_{ij} * L_j^{(1)}$ appears in the TBA equation for $\varepsilon_i^{(1)}$. The same is true for the pair $\varepsilon_i^{(\tilde{1})}$ and $\varepsilon_j^{(\tilde{1})}$. On the other hand, a vertical link from the node of $\varepsilon_i^{(1)}$ simply means that $\varepsilon_j^{(\tilde{1})}$ appears via the convolution $-\psi_{ij} * L_j^{(\tilde{1})}$. As for the magnonic nodes, any direct horizontal connection between $\varepsilon_n^{(\alpha)}$ and $\varepsilon_n^{(\beta)}$ gives rise to the convolution $\phi_2 * K_n^{(\alpha)}$ in the TBA equation for $\varepsilon_n^{(\beta)}$. If the direct magnonic link is vertical, the convolution $-\phi_2 * L_n^{(\alpha)}$ appears instead. The only contribution which is difficult to discover directly from the Y-system diagrams is the additional contribution to $\varepsilon_n^{(1)}$ (and $\varepsilon_n^{(\tilde{1})}$) from the magnons. This is denoted by a double link which signifies the contribution from the negative convolution of an L-function and a kernel for each magnon to which $\varepsilon_n^{(1)}$ is connected without passing through $\varepsilon_n^{(\tilde{1})}$.

4.3 New massless systems

4.3.1 $\mathbb{Z}_7$ TBA

The first new massless system comes from applying the above steps to the $N = 7$ Y-system. The notation used here to simplify things a little is $\varepsilon^{(1)} \equiv \varepsilon_1^{(1)}, \varepsilon^{(2)} \equiv \varepsilon_1^{(2)}, \ldots \varepsilon^{(6)} \equiv \varepsilon_1^{(6)} = \varepsilon_1^{(\bar{1})}$.

$$\begin{aligned}
\varepsilon^{(1)}(\theta) &= \nu^{(1)}(\theta) - \phi_3 * (L^{(4)}(\theta) + L^{(5)}(\theta)) - \phi_4 * L^{(2)}(\theta) - \phi_5 * L^{(3)}(\theta) \\
\varepsilon^{(2)}(\theta) &= \phi_2 * (K^{(3)}(\theta) - L^{(1)}(\theta)) \\
\varepsilon^{(3)}(\theta) &= \phi_2 * (K^{(2)}(\theta) + K^{(4)}(\theta) + K^{(5)}(\theta)) \\
\varepsilon^{(4)}(\theta) &= \phi_2 * K^{(3)}(\theta) \\
\varepsilon^{(5)}(\theta) &= \phi_2 * (K^{(3)}(\theta) - L^{(6)}(\theta)) \\
\varepsilon^{(6)}(\theta) &= \nu^{(6)}(\theta) - \phi_3 * (L^{(4)}(\theta) + L^{(2)}(\theta)) - \phi_4 * L^{(5)}(\theta) - \phi_5 * L^{(3)}(\theta)
\end{aligned} \tag{4.22}$$

Here $n = 1$, so there exist no bound states and the lower index is not required. The L- and K-functions are $L^{(\alpha)}(\theta) = \log(1 + e^{-\varepsilon^{(\alpha)}(\theta)})$ and $K^{(\alpha)}(\theta) = \log(1 + e^{\varepsilon^{(\alpha)}(\theta)})$, while the energy terms are $\nu^{(1)}(\theta) = mre^{\theta}/2$ and $\nu^{(6)}(\theta) = mre^{-\theta}/2$. The kernels are obtained by substituting $h = 3/2$ into the following functions

$$\phi_2(\theta) = \frac{2h}{\cosh 2h\theta} \tag{4.23}$$

$$\phi_3(\theta) = \frac{2h}{3\cosh \frac{2h}{3}\theta} \tag{4.24}$$

$$\phi_4(\theta) = \frac{8h\cosh \frac{2h}{3}\theta}{3(4\cosh^2(\frac{2h}{3}\theta) - 3)} \tag{4.25}$$

$$\phi_5(\theta) = \frac{8h\cosh \frac{2h}{3}\theta}{\sqrt{3}(4\cosh^2(\frac{2h}{3}\theta) - 1)}, \tag{4.26}$$

where more generally h is equal to $N/2 - 2$.

In the $r \to \infty$ limit, the system (4.22) decouples and coincides with the ultraviolet limit of the $\beta^2 = 32\pi/7$ massive sine-Gordon theory

$$\begin{aligned}
\varepsilon^{(1)}(\theta) &= \nu^{(1)}(\theta) - \phi_3 * (L^{(4)}(\theta) + L^{(5)}(\theta)) - \phi_4 * L^{(2)}(\theta) - \phi_5 * L^{(3)}(\theta) \\
\varepsilon^{(2)}(\theta) &= \phi_2 * (K^{(3)}(\theta) - L^{(1)}(\theta)) \\
\varepsilon^{(3)}(\theta) &= \phi_2 * (K^{(2)}(\theta) + K^{(4)}(\theta) + K^{(5)}(\theta)) \\
\varepsilon^{(4)}(\theta) &= \phi_2 * K^{(3)}(\theta) \\
\varepsilon^{(5)}(\theta) &= \phi_2 * K^{(3)}(\theta)
\end{aligned} \tag{4.27}$$

which has energy term $\nu^{(1)}(\theta) = mr\cosh\theta$. This is seen most clearly by noting $L_1^{(6)}$ is vanishingly small when $L_1^{(1)}$ is not, and vice-versa, so that (4.22) decouples. Also (4.22)

clearly has the parity invariance mentioned above so that left and right moving particles are treated on an equal footing. The central charge for the $\mathbb{Z}_7$ TBA system is

$$\widetilde{c}(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \left[\nu^{(1)}(\theta)L^{(1)}(\theta) + \nu^{(6)}(\theta)L^{(6)}(\theta)\right] . \tag{4.28}$$

4.3.2 $\mathbb{Z}_{4n+1}$ TBA

Applying the above steps to the $N = 4n + 1$ Y-systems we obtain the following TBA equations.

$$\begin{aligned}
\varepsilon_i^{(\alpha)}(\theta) &= \nu_i^{(\alpha)}(\theta) - \sum_{j=1}^{n}\left[\phi_{ij} * L_j^{(\alpha)}(\theta) - \psi_{ij} * L_j^{(\widetilde{\alpha})}(\theta)\right] \\
&\quad - \delta_{i,n}\sum_{\beta=1}^{6} l_{\alpha\beta}^{[e_6]}\phi_2 * L_n^{(\beta)}(\theta)\,, \qquad (\alpha{=}1,6) \\
\varepsilon_n^{(\alpha)}(\theta) &= \sum_{\beta=1}^{6} l_{\alpha\beta}^{[e_6]}\phi_2 * L_n^{(\beta)}(\theta)\,, \qquad (\alpha = 2,..5)\,,
\end{aligned} \tag{4.29}$$

where $\widetilde{\alpha} = 7 - \alpha$. The energy terms are $\nu_i^{(1)}(\theta) = \nu_i^{(\bar{1})}(-\theta) = \frac{1}{2}m_i r e^{\theta}$, where m_i is the ith mass M_i of (2.23,2.24) divided by that of the lightest particle. For any N=$4n$+1 the magnonic structure is identical to that of N=5, as given in equation (4.17). The additional terms come from the $n-1$ breathers present in the spectrum in addition to the fundamental soliton-antisoliton doublet. Since each $\mathbb{Z}_N$ Y-system involves different shift factors, the kernels appearing in each system differ and are specified by (4.23) with $h = N/2 - 2$.

4.3.3 $\mathbb{Z}_{4n+3}$ TBA

Finally, for N=$4n$+3 the magnonic structure is identical to that of N=7. In general there are $n - 1$ breathers, and the TBA equations are

$$\begin{aligned}
\varepsilon_i^{(1)}(\theta) &= \nu_i^{(1)}(\theta) - \sum_{j=1}^{n}\left[\phi_{ij} * L_j^{(1)}(\theta) - \psi_{ij} * L_j^{(6)}(\theta)\right] \\
&\quad - \delta_{i,n}\left[\phi_3 * (L_n^{(4)}(\theta) + L_n^{(5)}(\theta)) + \phi_4 * L_n^{(2)}(\theta) + \phi_5 * L_n^{(3)}(\theta)\right] \\
\varepsilon_n^{(2)}(\theta) &= \phi_2 * (K_n^{(3)}(\theta) - L_n^{(1)}(\theta)) \\
\varepsilon_n^{(3)}(\theta) &= \phi_2 * (K_n^{(2)}(\theta) + K_n^{(4)}(\theta) + K_n^{(5)}(\theta)) \\
\varepsilon_n^{(4)}(\theta) &= \phi_2 * K_n^{(3)}(\theta) \\
\varepsilon_n^{(5)}(\theta) &= \phi_2 * (K_n^{(3)}(\theta) - L_n^{(6)}(\theta)) \\
\varepsilon_i^{(6)}(\theta) &= \nu_i^{(6)}(\theta) - \sum_{j=1}^{n}\left[\phi_{ij} * L_j^{(6)}(\theta) - \psi_{ij} * L_j^{(1)}(\theta)\right]
\end{aligned} \tag{4.30}$$

$$- \delta_{i,n} \left[\phi_3 * (L_n^{(4)}(\theta) + L_n^{(2)}(\theta)) + \phi_4 * L_n^{(5)}(\theta) + \phi_5 * L_n^{(3)}(\theta) \right] .$$

The kernels here are given by (4.23)-(4.26), again with $h = N/2 - 2$ and here we take $\widetilde{\alpha} = 7 - \alpha$.

The tilde notation introduced throughout allows the effective central charge to be written

$$\widetilde{c}(r) = \frac{3}{\pi^2} \sum_{i=1}^{n} \int_{-\infty}^{\infty} d\theta \; \nu_i^{(1)}(\theta) L_i^{(1)}(\theta) + \nu_i^{(\tilde{1})}(\theta) L_i^{(\tilde{1})}(\theta) . \tag{4.31}$$

As usual the sum runs from $i = 1, ..n + 1$ in the d_{n+1} case.

4.4 Adaptation for the massive perturbations

The TBA equations believed to represent the massive perturbation $\widehat{Z}_N^{(-)}$ are obtained from the $\mathbb{Z}_N$ Y-systems by taking the same steps as above, apart from the last where the energy terms

$$\nu_i^{(\alpha)} = \begin{cases} m_i r \cosh\theta & \text{if } \alpha = 1, \quad i = 1, ..n \qquad (i = 1, ..n + 1 \text{ in } d_{n+1} \text{ case}) \\ 0 & \text{else} \end{cases} \tag{4.32}$$

are inserted to break the scale invariance. It is a noteworthy feature of the TBA method that by changing only the energy terms (4.3) to those of (4.32) a system is obtained which describes the self-dual perturbation in the massive direction. In Section (5.1) we show there is precise agreement between these two types of TBA system and the perturbed theories (3.9) with positive and negative couplings in the ultraviolet regime. In the next section it is shown that the different energy terms give infrared limiting central charges consistent with massless and massive theories respectively.

4.5 Ultraviolet and infrared limits of $\widetilde{c}(r)$

The ultraviolet and infrared limits of $\widetilde{c}(r)$ may be found as follows, beginning with the massless direction. The effective central charge is (4.31) with energy terms given by (4.3) and pseudoenergies $\varepsilon_i^{(\alpha)}(\theta)$ appearing in the L-functions satisfy any of the massless TBA systems in Section (4.1) or (4.3). The parity symmetry $\varepsilon_i^{(\alpha)}(\theta) = \varepsilon_i^{(\tilde{\alpha})}(-\theta)$ means the integrand is even, which implies

$$\widetilde{c}(r) = \frac{6}{\pi^2} \sum_i \int_{-\infty}^{\infty} d\theta \; m_i r e^{\theta} / 2 \log(1 + e^{-\varepsilon_i^{(1)}(\theta)}) . \tag{4.33}$$

For small r, the pseudoenergies and L-functions become constant in the region given by $-\log(2/r) \ll \theta \ll \log(2/r)$. For $\theta \ll -\log(2/r)$ and $\theta \gg \log(2/r)$ the pseudoenergies are also constant with the exception of those whose energy terms grow exponentially (see Figure (4.4)). Strictly we should include masses in all such estimates, but they make no essential difference to any of the asymptotic calculations where contributions of the form $\log(m_i)$ are insignificant. With r small, $\nu_i^{(1)}(\theta) = \frac{1}{2} m_i r e^{\theta}$ is negligible for θ on $(-\infty, \theta_c)$ provided $\theta_c \ll \log(2/r)$, therefore we only need consider the integral (4.33) over (θ_c, ∞). $\nu_i^{(\tilde{1})}(\theta)$ can be neglected on this range and the pseudoenergies $\varepsilon_i^{(\tilde{1})}(\theta)$ may be treated as magnonic. This leads to a slightly modified TBA equation in which $\nu_i^{(\tilde{1})}(\theta)$ are assumed zero on (θ_c, ∞), say

$$\nu_i^{(\alpha)} = \varepsilon_i^{(\alpha)} + \sum_j \Psi_{ij} * L_j^{(\alpha)}, \tag{4.34}$$

where $\Psi_{ij}(\theta)$ represents all convolution terms.

For computational purposes one could easily discretise (4.33) and substitute the numerical solutions to the TBA equations to evaluate $\widetilde{c}(r)$ at any r, but we would like an exact expression for the ultraviolet limit. To achieve this write the sum in (4.33) over all nodes

$$\begin{aligned} \widetilde{c}(0) &= \lim_{r\to 0} \frac{6}{\pi^2} \sum_i \int_{\theta_c}^{\infty} d\theta \; m_i r e^{\theta}/2 \, \log(1 + e^{-\varepsilon_i^{(1)}(\theta)}) \\ &= \lim_{r\to 0} \frac{6}{\pi^2} \sum_{i,\alpha} \int_{\theta_c}^{\infty} d\theta \; \nu_i^{(\alpha)}(\theta) \log(1 + e^{-\varepsilon_i^{(\alpha)}(\theta)}). \end{aligned} \tag{4.35}$$

Several of the nodes have zero energy term, but this expression allows us to use the following trick. Differentiate the modified TBA equations with respect to θ to obtain a system of the general form

$$\frac{d}{d\theta}\nu_i^{(\alpha)} = \frac{d}{d\theta}\varepsilon_i^{(\alpha)} + \frac{d}{d\theta}\Big[\sum_j \Psi_{ij} * L_j^{(\alpha)}\Big]. \tag{4.36}$$

Since $\frac{d}{d\theta}\nu_i^{(\alpha)} = \nu_i^{(\alpha)}$, the right hand side may be substituted into (4.35) and provided Ψ_{ij} is even, the result can be written

$$\widetilde{c}(0) = \lim_{r\to 0} \frac{6}{\pi^2} \sum_{i,\alpha} \int_{\theta_c}^{+\infty} d\theta \left(\frac{d\varepsilon_i^{(\alpha)}}{d\theta} L_i^{(\alpha)}(\theta) + \frac{1}{2}\frac{d}{d\theta}\Big[L_i^{(\alpha)}(\theta) \sum_j \Psi_{ij} * L_j^{(\alpha)}(\theta)\Big] \right) \tag{4.37}$$

giving

$$\widetilde{c}(0) = \lim_{r\to 0} \frac{6}{\pi^2} \sum_{i,\alpha} \left(\int_{\varepsilon_i^{(\alpha)}(\theta_c)}^{\varepsilon_i^{(\alpha)}(\infty)} dy \log(1 + e^{-y}) + \frac{1}{2}\Big[L_i^{(\alpha)}(\theta)(\nu_i^{(\alpha)}(\theta) - \varepsilon_i^{(\alpha)}(\theta))\Big]_{\theta_c}^{\infty} \right). \tag{4.38}$$

The exponential nonzero energy terms $\nu_i^{(1)}$ are damped by the doubly exponentially decaying $L_i^{(1)}(\theta)$ and $\nu_i^{(\tilde{1})} = \frac{1}{2}m_i r e^{-\theta}$ are negligible as above. Therefore

$$\tilde{c}(0) = \frac{6}{\pi^2}\sum_{i,\alpha}\left(\int_{\varepsilon_i^{(\alpha)}(\theta_c)}^{\varepsilon_i^{(\alpha)}(\infty)} dy \log(1+e^{-y}) - \frac{1}{2}\left[L_i^{(\alpha)}(\infty)\varepsilon_i^{(\alpha)}(\infty) - L_i^{(\alpha)}(\theta_c)\varepsilon_i^{(\alpha)}(\theta_c)\right]\right). \tag{4.39}$$

At this point we introduce Rogers' dilogarithm function

$$\mathcal{L}(x) = -\frac{1}{2}\int_0^x dy\left[\frac{\log y}{1-y} + \frac{\log(1-y)}{y}\right], \qquad 0 < x < 1, \tag{4.40}$$

which satisfies the identity

$$\mathcal{L}\left(\frac{1}{1+e^{\varepsilon}}\right) = \int_{\varepsilon}^{\infty} dy \log(1+e^{-y}) + \frac{1}{2}\varepsilon\log(1+e^{-\varepsilon}). \tag{4.41}$$

This can be verified by differentiating with respect to ε. If we define the stationary Y-function values

$$\Upsilon_i^{(\alpha)} = \lim_{r\to 0} Y_i^{(\alpha)}(\theta) \qquad (\theta \text{ finite}), \tag{4.42}$$

$$\mathcal{X}_i^{(\alpha)} = \lim_{\theta\to\infty} Y_i^{(\alpha)}(\theta) \qquad (r \text{ finite}), \tag{4.43}$$

where θ takes any value in $-\log(2/r) \ll \theta \ll \log(2/r)$ because all pseudoenergies are constant in this region, the ultraviolet central charge can be expressed as

$$\tilde{c}(0) = \frac{6}{\pi^2}\sum_{i,\alpha}\mathcal{L}\left(\frac{1}{1+\Upsilon_i^{\alpha}}\right) - \mathcal{L}\left(\frac{1}{1+\mathcal{X}_i^{\alpha}}\right). \tag{4.44}$$

All pseudoenergies are finite for $\theta \gg \log(2/r)$ except for $\varepsilon_i^{(1)}(\theta)$ which has $\mathcal{X}_i^{(1)} = \infty$.

Turning to the large r limit, $\nu_i^{(1)}(\theta)$ is small on $(-\infty, -\theta_c)$ provided $-\theta_c \ll -\log(r/2)$. Thus as r increases, the range of integration should be over $(-\infty, \infty)$. When θ increases through 0 from below $\varepsilon_i^{(1)}(\theta)$ grows exponentially, which means $\nu_i^{(1)}(\theta)$ becomes doubly exponentially suppressed by $L_i^{(1)}(\theta)$ as θ approaches 0^+ near zero. Consequently we may write

$$\tilde{c}(\infty) = \lim_{r\to\infty}\frac{6}{\pi^2}\sum_i\int_{-\infty}^{0^+} d\theta\; m_i r e^{\theta}/2\,\log(1+e^{-\varepsilon_i^{(1)}(\theta)}) \tag{4.45}$$

In the large r limit the pseudoenergy solutions of the TBA equations are constant in the region $-\log(r/2) \ll \theta \ll \log(r/2)$ apart from $\varepsilon_i^{(1)}(\theta)$ and $\varepsilon_i^{(\tilde{1})}(\theta)$ as displayed in Figure (4.5). The form of the energy functions $\nu_i^{(\alpha)}(\theta)$ and pseudoenergies $\varepsilon_i^{(\alpha)}(\theta)$ are exactly those needed to make the product $\nu_i^{(\alpha)}(\theta)L_i^{(\alpha)}(\theta)$ negligible near $\theta = 0$ and $\theta = -\infty$. As a result, the calculation of $\lim_{r\to\infty}\tilde{c}(r)$ follows as above to give the infrared central charge

$$\tilde{c}(\infty) = \frac{6}{\pi^2}\sum_{i,\alpha}\mathcal{L}\left(\frac{1}{1+\mathcal{X}_i^{\alpha}}\right) - \mathcal{L}\left(\frac{1}{1+\mathcal{Z}_i^{\alpha}}\right), \tag{4.46}$$

where the exponentials of pseudoenergies near $\theta = 0$ as $r \longrightarrow \infty$ are introduced

$$\mathcal{Z}_i^{(\alpha)} = \lim_{r\to\infty} Y_i^{(\alpha)}(\theta) \qquad (\theta \,\text{finite})\,. \tag{4.47}$$

As for the massive direction, the ultraviolet and infrared limits work in much the same way. For the massive theories the pseudoenergy solutions are symmetric (see Figure (4.6)) so that the central charge becomes

$$\begin{aligned} \widetilde{c}(r) &= \frac{3}{\pi^2}\sum_i \int_{-\infty}^{\infty} d\theta\; m_i r \cosh\theta\, L_i^{(1)}(\theta) \\ &= \frac{6}{\pi^2}\sum_i \int_{0}^{\infty} d\theta\; m_i r \cosh\theta\, L_i^{(1)}(\theta)\,. \end{aligned} \tag{4.48}$$

Therefore $\lim_{r\to 0}\widetilde{c}(r)$ gives the same ultraviolet central charge obtained above. In the massive direction, when r is small, the energy $\nu_i^{(1)}(\theta)$ and corresponding pseudoenergy 'blows up' near $\theta = \pm\log(2/r)$. For θ outside these walls the L-functions kill off all contributions to the integral. Consequently, as r is increased, the walls of $\varepsilon_i^{(1)}(\theta)$ approach each other and meet. Therefore the contribution to the central charge decreases. We conclude $\lim_{r\to\infty}\widetilde{c}(r) = 0$.

The finite values of the stationary Y-functions $\mathcal{X}_i^{(\alpha)}$, $\Upsilon_i^{(\alpha)}$ and $\mathcal{Z}_i^{(\alpha)}$ were found numerically by iterating the Y-systems for fixed values of θ. The results suggest dilogarithm sum-rules similar to those stated in [41]. A few of the stationary values are given by

$$\Upsilon_i^{(\alpha)} = \frac{\sin((i+3)\eta)\sin(i\eta)}{\sin((2\eta))\sin(\eta)}\,,$$

$$\mathcal{X}_i^{(\tilde{1})} = (i+2)i\,,$$

for $i = 1\ldots n{-}1$ and $\eta = \frac{\pi}{N+2}$. All other stationary values can be found by substituting these (and some simple expressions for $\mathcal{Z}_i^{(\alpha)}$) into the relevant full Y-system (see Appendix (B)). The following sum rules were verified numerically, for all cases up to N=30:

$$c_\Upsilon = \frac{6}{\pi^2}\sum_{i,\alpha} L\left(\frac{1}{1+\Upsilon_i^{(\alpha)}}\right) = \frac{2(N-1)}{N+2} + \begin{cases} 4 & \text{for } N = 4n{+}1 \\ 5/2 & \text{for } N = 4n{+}2 \\ 2 & \text{for } N = 4n{+}3 \\ 1 & \text{for } N = 4n{+}4 \end{cases}$$

$$c_\mathcal{X} = \frac{6}{\pi^2}\sum_{i,\alpha} L\left(\frac{1}{1+\mathcal{X}_i^{(\alpha)}}\right) = \begin{cases} 4 & \text{for } N = 4n{+}1 \\ 5/2 & \text{for } N = 4n{+}2 \\ 2 & \text{for } N = 4n{+}3 \\ 1 & \text{for } N = 4n{+}4 \end{cases}$$

$$c_{\mathcal{Z}} = \frac{6}{\pi^2}\sum_{i,\alpha} L\left(\frac{1}{1+\mathcal{Z}_i^{(\alpha)}}\right) = \begin{cases} 3 & \text{for } N = 4n{+}1 \\ 3/2 & \text{for } N = 4n{+}2 \\ 1 & \text{for } N = 4n{+}3 \\ 0 & \text{for } N = 4n{+}4 \end{cases}$$

The ultraviolet and infrared central charges are therefore

$$c_{UV} = c_{\Upsilon} - c_{\mathcal{X}} = \frac{2(N-1)}{(N+2)}, \tag{4.49}$$

$$c_{IR} = c_{\mathcal{X}} - c_{\mathcal{Z}} = 1, \tag{4.50}$$

and for the massive TBA,

$$c_{IR} = c_{\mathcal{X}} - c_{\mathcal{Z}} = 0. \tag{4.51}$$

It is quite remarkable that the symmetric version of the sine-Gordon TBA equations lead to the ultraviolet charge c_N conjectured. These results imply the proposed TBA equations could certainly represent the perturbed theories $\widehat{Z}_N^{(\pm)}$. Furthermore, the infrared limit has the central charge of a $c = 1$ conformal theory if massless energy terms are used whereas massive energy terms give $c = 0$, which is also expected for a massive theory. It is therefore plausible that the $\mathbb{Z}_N$-symmetric conformal theories are positioned at the opening of the Kosterlitz-Thouless region on the self-dual line e.g. at the points C and C' in figure (3.1) for $N = 5$. Further evidence will be given in the next chapter.

4.6 Numerical solutions

Being such a complicated system of coupled nonlinear equations, with exact solution only possible at $r = 0$ and $r = \infty$, a numerical method proves extremely useful in the study of the $\mathbb{Z}_N$ TBA systems. Our aim is to solve each system to discover the form of the effective central charge $\tilde{c}(r)$ for all values of r. Among other things this would allow us to compare the TBA results with the predictions from perturbation theory. Here we discuss the particular numerical method used to solve the equations and leave the main results until the next chapter.

Ignoring upper indices, the generic TBA system may be written

$$\varepsilon_a(\theta) = \nu_a(\theta) - \sum_b \int d\theta' \psi_{ab}(\theta - \theta') L_b(\theta'). \tag{4.52}$$

For each value r, discretise the θ axis in steps of $\Delta\theta$ so that at $\theta_i \equiv i\Delta\theta$ the values of the pseudoenergies are given by

$$\varepsilon_a(\theta_i) = \nu_a(\theta_i) - \sum_b \Big(\sum_{j=-imax}^{imax} \Delta\theta\psi_{ab}((i-j)\Delta\theta) L_b(\theta_j) \Big) \equiv f_a(\{\varepsilon_b(\theta_i)\}). \tag{4.53}$$

The integer $imax$ is taken sufficiently large and $\Delta\theta > 0$ sufficiently small so that the sum is a very good approximation to the integral (4.52). Note that the value of ε_a at θ_i is dependent on all other pseudoenergies at all other points θ_j.

If $\varepsilon_a(i\Delta\theta) = \varepsilon_a^{[0]}(i\Delta\theta)$ is an initial guess, we find the iteration scheme

$$\varepsilon_a^{[n+1]}(i\Delta\theta) = f_a(\varepsilon_b^{[n]}(i\Delta\theta)) \tag{4.54}$$

does not in general converge. This problem is easily rectified if we instead consider

$$\varepsilon_a^{[n+1]}(i\Delta\theta) = \frac{1}{2} f_a(\varepsilon_b^{[n]}(i\Delta\theta)) + \frac{1}{2}\varepsilon_a^{[n]}(i\Delta\theta)\,, \tag{4.55}$$

(or some similar modification) which shares the same solution as (4.54) in the limit $n \to \infty$ but is now convergent. Although there are other methods which give a faster rate of convergence, the simple scheme (4.55) proves quite sufficient to solve all the cases of interest in this work (more generally, whether or not a method is convergent depends on the eigenvalues of the matrix $\partial_{\varepsilon_b} f_a$: the iteration will diverge if any have magnitude greater than unity and the fastest convergence occurs when eigenvalues are as close to zero as possible). Appendix (D) contains programs solving the $\mathbb{Z}_5$ and $\mathbb{Z}_7$ TBA systems. The numerical method was used to solve the $\mathbb{Z}_5, \mathbb{Z}_6, ...\mathbb{Z}_{10}$ TBA systems in both massive and massless directions. Figure (4.3) displays the effective central charges for the massless models over the range $0.0001 < r < 70$. To obtain these graphs between two and four hundred iterations of the above scheme were required at every value of r. The program is terminated when the central charge value produced is the same (to within 10^{-14}: the computer numerical accuracy) on two consecutive iterations. The first result from this numerical data (for the theories $N = 5, 6, .., 10$) is that ultraviolet and infrared limits of the effective central charge are $c_N = 2(N-1)/(N+2)$ and $c = 1$ to within ten significant figures. The $\mathbb{Z}_5$ central charge function has a slower convergence to $c = 1$ than all others. This is explained in the next chapter. Another initial observation from the graphs is that all the $\tilde{c}(r)$ satisfy Zamolodchikov's c-theorem: that the effective central charge $\tilde{c}(r)$ is a monotonic decreasing function of r.

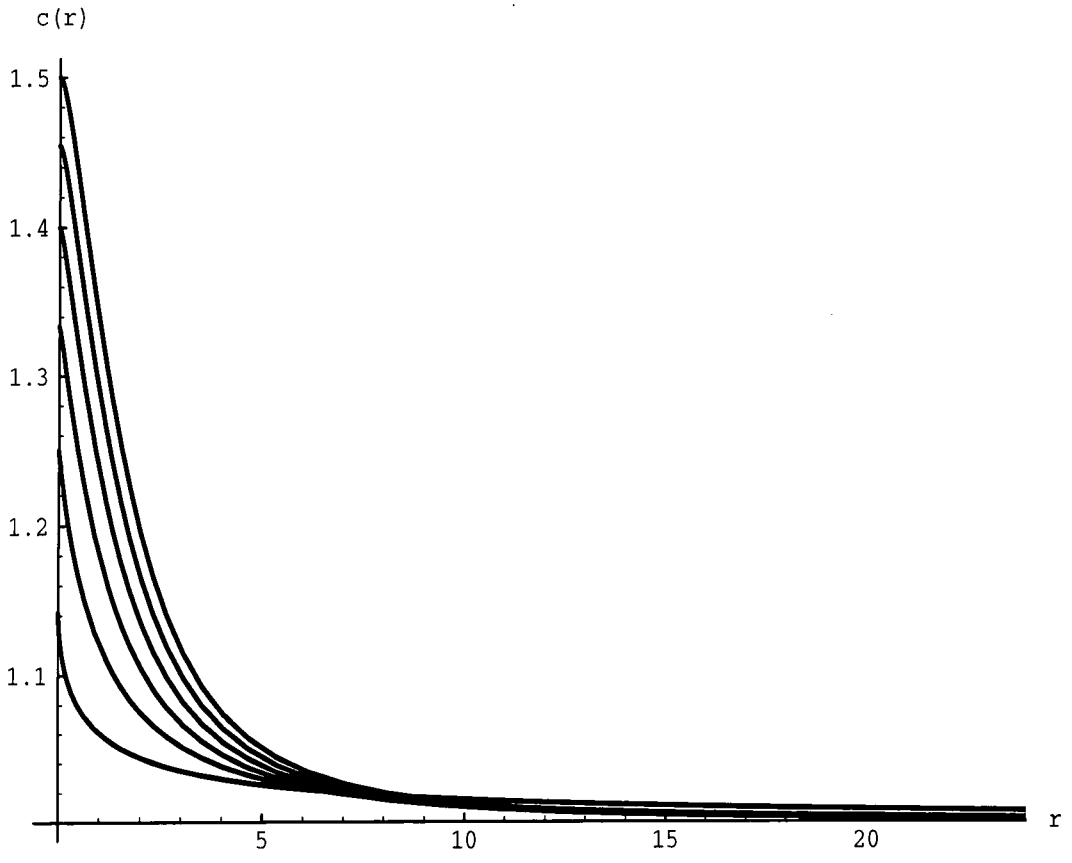


Figure 4.3: The effective central charge $\widetilde{c}(r)$ for the perturbations $\widehat{Z}_N^{(+)}$ described by the $\mathbb{Z}_N$ TBA systems for $N = 5, 6, ...10$. The ultraviolet and infrared limits are $c = c_N$ and $c = 1$ respectively.

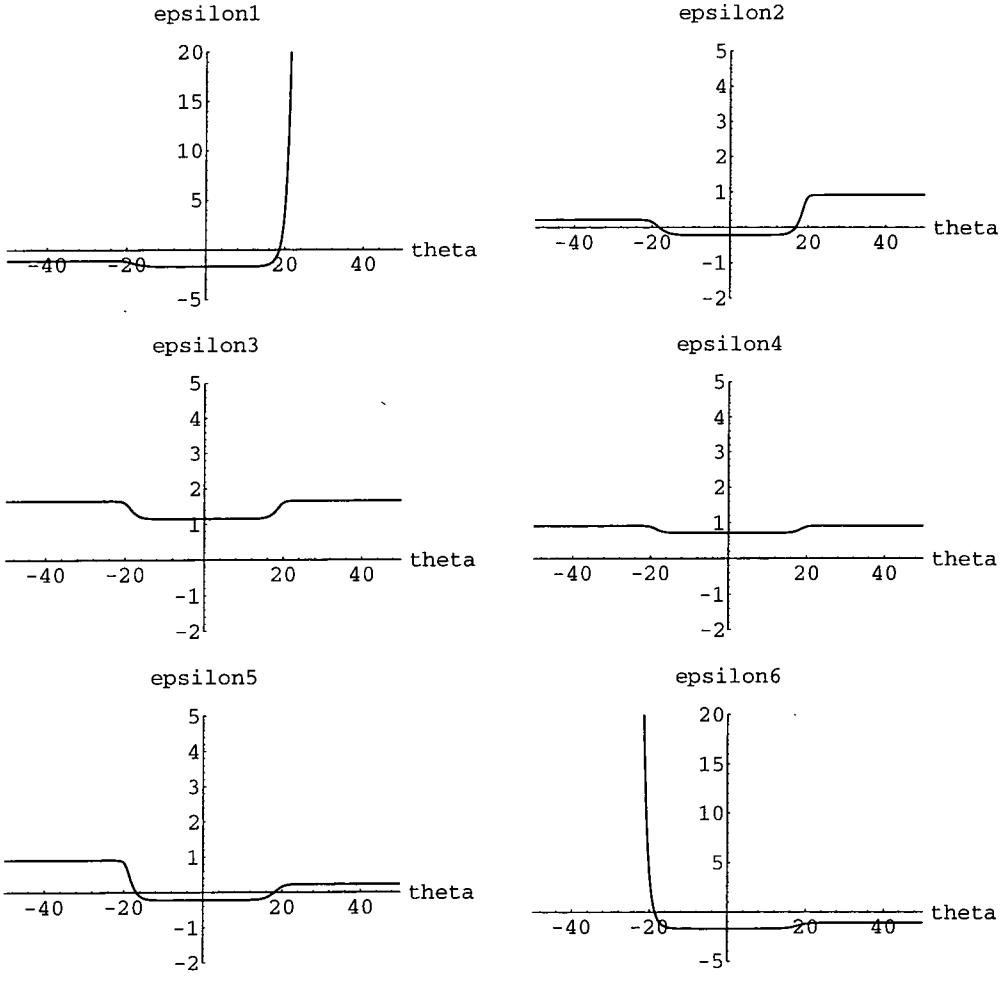


Figure 4.4: The six pseudoenergy solutions for the *massless* $\mathbb{Z}_7$ TBA system in the *ultraviolet* regime at $r = 10^{-7}$. The pseudoenergy labels used here are $epsilon1 \equiv \varepsilon_1^{(1)}, epsilon2 \equiv \varepsilon_1^{(2)}, ...epsilon6 \equiv \varepsilon_1^{(6)}$.

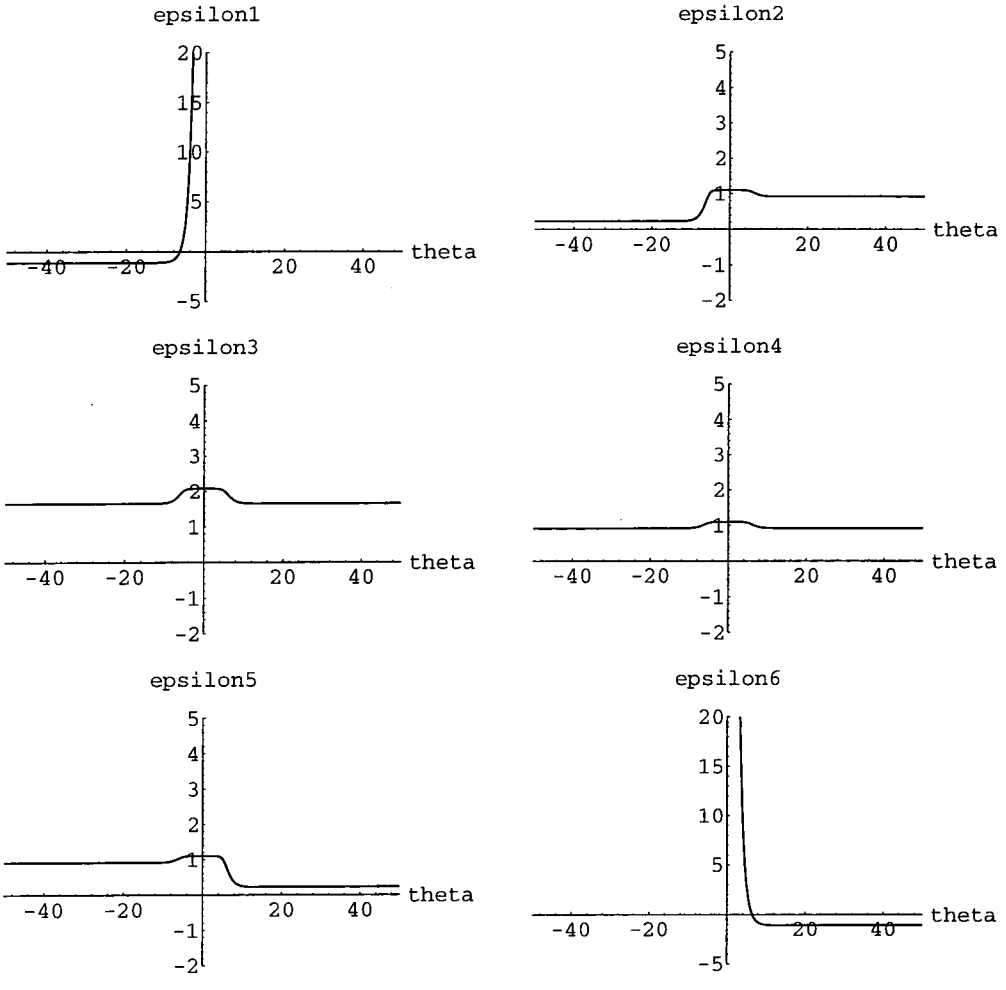


Figure 4.5: The six pseudoenergy solutions for the *massless* $\mathbb{Z}_7$ TBA system in the *infrared* regime at $r = 10^4$.

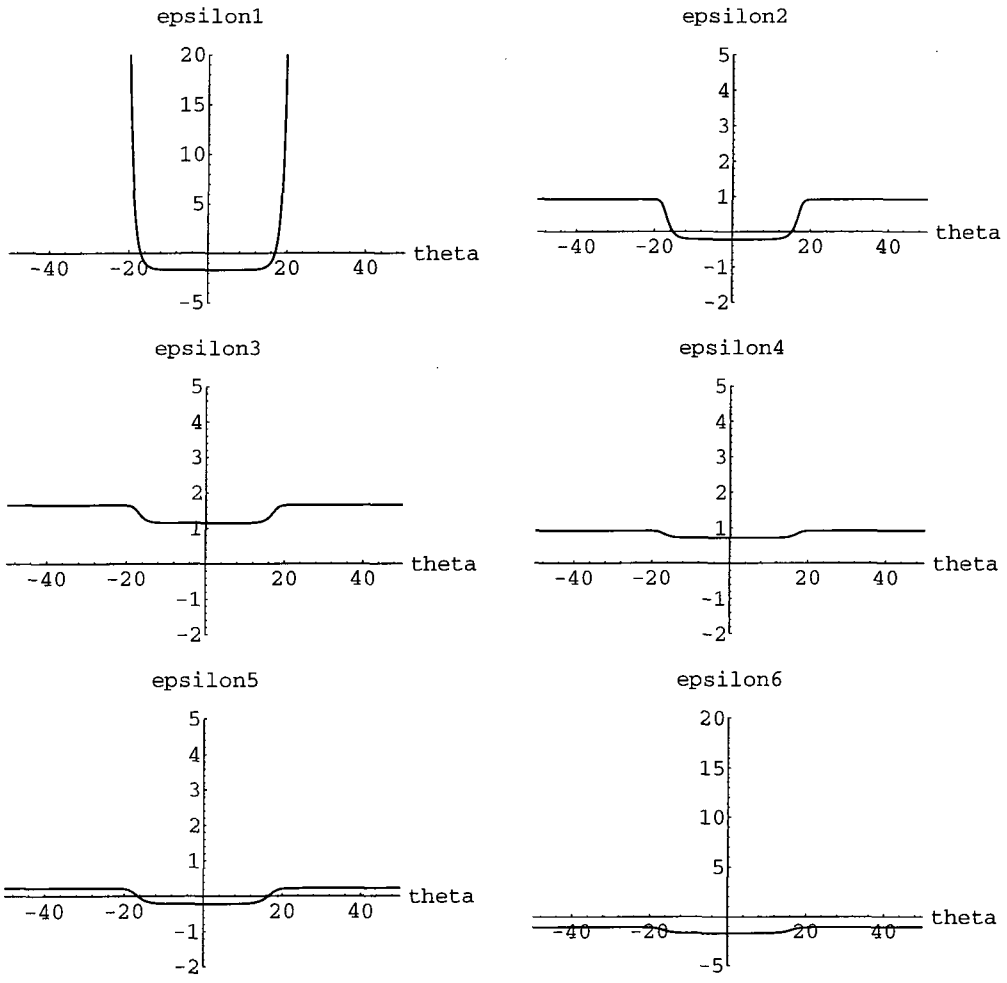


Figure 4.6: The six pseudoenergy solutions for the *massive* $\mathbb{Z}_7$ TBA system in the *ultra-violet* regime at $r = 10^{-7}$.

Chapter 5

TBA Equations and Perturbation Theory

The ultraviolet limit of massive and massless $\mathbb{Z}_N$ TBA equations gives effective central charge c_N. This is certainly a good indication that the proposed TBA equations could represent the flows $\widehat{Z}_N^{(\pm)}$. If the TBA equations stated really do describe these perturbed conformal theories then the ground state energy predicted by the TBA method must agree with conformal perturbation theory about the ultraviolet limit described by small λ in (3.9). The same must be true in the infrared regime. We begin this chapter with an analysis of the ground state energy in ultraviolet regime.

5.1 The ultraviolet region

Mapping from the ξ-parametrised cylinder to the infinite z-plane via the map $z = e^{-2\pi\xi/R}$, the small λ expansion of the ground state energy for a unitary conformal theory perturbed by a spinless primary field ϵ, of dimension (Δ, Δ), may be found. The integral measure on the cylinder $d^2\xi_i$ is replaced by $(R^2 dz_i d\overline{z}_i)/((2\pi)^2 z_i \overline{z}_i)$, while via (1.41) the integrand receives a factor $(R^2/((2\pi)^2 z_i \overline{z}_i))^{-\Delta}$ for each primary field in the correlation function. Temporarily dropping the suffix from the ground state energy, the result is

$$E^{(\mathrm{pert})}(\lambda, R) = -\frac{\pi c}{6R} + \frac{2\pi}{R} \sum_{m=2}^{\infty} B_m t^m \ , \tag{5.1}$$

where coupling λ appears in the dimensionless parameter $t = -2\pi\lambda\,(R/2\pi)^{2-2\Delta}$ and the normalised coefficients B_m are

$$B_m = \frac{-1}{(2\pi)^{m-1} m!} \int \left(\prod_{k=2}^{m} \frac{d^2 z_k}{(z_k \overline{z}_k)^{1-\Delta}} \right) \langle \epsilon(1,1)\epsilon(z_2, \overline{z}_2) \ldots \epsilon(z_m, \overline{z}_m) \rangle_{conn} \, . \tag{5.2}$$

The connected correlation functions of the perturbing field ϵ are evaluated at the conformal point. Conformal invariance means $B_1 = 0$ and translational invariance allows us to fix one of the field arguments to unity. The two-dimensional theory is super-renormalisable, therefore we expect any ultraviolet expansion to have only a finite number of divergent terms which can be renormalised. The series (5.1) is assumed to converge in some finite region near $t = 0$.

For any conformal theory the two and three point correlation functions are fixed up to a normalisation constant by conformal invariance to be

$$\langle \epsilon(1,1)\epsilon(z_1,\overline{z}_1)\rangle = \frac{1}{(1-z_1)^{2\Delta}(1-\overline{z}_1)^{2\Delta}},$$

and

$$\langle \epsilon(1,1)\epsilon(z_1,\overline{z}_1)\epsilon(z_2,\overline{z}_2)\rangle = \frac{C_{\epsilon\epsilon\epsilon}}{[(1-z_1)(1-\overline{z}_1)(1-z_2)(1-\overline{z}_2)(z_1-z_2)(\overline{z}_1-\overline{z}_2)]^{\Delta}},$$

where $C_{\epsilon\epsilon\epsilon}$ is the three point correlation coefficient. The identities [35]

$$\int \frac{d^2z_1}{(z_1\overline{z}_1)^{1-\Delta}\left[(1-z_1)(1-\overline{z}_1)\right]^{2\Delta}} = \frac{\pi\gamma^2(\Delta)}{\gamma(2\Delta)},$$

$$\int \frac{d^2z_1 d^2z_2}{(z_1\overline{z}_1z_2\overline{z}_2)^{1-\Delta}\left[(1-z_1)(1-\overline{z}_1)(1-z_2)(1-\overline{z}_2)(z_1-z_2)(\overline{z}_1-\overline{z}_2)\right]^{\Delta}} = \frac{\pi^2\gamma^3(\Delta/2)}{2\gamma(3\Delta/2)}$$

(the integration ranges here are over the entire plane) allow B_2 and B_3 to be calculated explicitly as

$$B_2 = -\frac{1}{4}\frac{\gamma(\Delta)^2}{\gamma(2\Delta)} \quad , \quad B_3 = -\frac{C_{\epsilon\epsilon\epsilon}}{48}\frac{\gamma(\Delta/2)^3}{\gamma(3\Delta/2)} \tag{5.3}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. These are coefficients which depend only on Δ and $C_{\epsilon\epsilon\epsilon}$.

For the self-dual $\mathbb{Z}_N$ models under discussion the perturbing dimension of $\epsilon^{(2)}$ is $\Delta = 6/(N+2)$ and the values of $C_{\epsilon\epsilon\epsilon}$ for the unperturbed conformal theory were found by Zamolodchikov and Fateev [32] to be

$$C^{(N)}_{\epsilon\epsilon\epsilon} = \frac{2}{35}\frac{\Pi^{(N)}(7)\Pi^{(N)}(2)^3}{\Pi^{(N)}(4)^3}\sqrt{5^3\frac{\Gamma((N+3)/(N+2))\Gamma(1-5/(N+2))^3}{\Gamma((N+1)/(N+2))\Gamma(1+5/(N+2))^3}} \tag{5.4}$$

where $\Pi^{(N)}(j) = \prod_{k=1}^{j}\left(\Gamma(1+\frac{k}{N+2})/\Gamma(1-\frac{k}{N+2})\right)$.

An obvious difference between the TBA and perturbation theory approaches is that in the TBA picture $E(R)$ depends on R and mass scale M, while the perturbation theory depends on R and the coupling constant λ. The relationship between λ and M can be determined by dimensional analysis, via (1.19), to be of the form

$$\lambda = \kappa M^y \ , \tag{5.5}$$

where $y = 2-2\Delta$ is the scaling dimension of λ and κ is a dimensionless constant to be determined later. The relationship (5.5) can be used to check that the TBA equations proposed in the last chapter do actually correspond to the $\epsilon^{(2)}$ perturbation of the $c = 2(N-1)/(N+2)$ theory, rather than some other, as follows. If each of the $\mathbb{Z}_N$ Y-systems has the periodicity P given by

$$Y_i^{(\alpha)}(\theta + 2\pi i P) = Y_i^{(\alpha)}(\theta) \tag{5.6}$$

then we should be able to expand any solution of this system in positive and negative powers of $u(\theta) = e^{\theta/P}$. When MR is small, but not zero, the pseudoenergy solutions may be thought of as a sum of left and right kink-like solutions (the kinks become singular at either $\theta = -\infty$ or $\theta = \infty$ for the non-magnonic pseudoenergies), so that we may view the ultraviolet corrections as coming from the presence of the other kink a distance $2\log(2/MR)$ away. These appear in powers of $u(2\log(2/MR))$ so that the first corrections to $RE(R)$ form a regular series for small MR

$$RE(R) = -\frac{\pi c}{6} + 2\pi \sum_{m=1}^{\infty} F_m (MR)^{\frac{2}{P}m} + \text{further terms}. \tag{5.7}$$

The further terms are related to the bulk contribution to be discussed shortly. Comparing coefficients of $R^{2-2\Delta}$ in this expansion with those in the perturbative expansion for $\Delta = 6/(N+2)$ (the dimension of the perturbing field $\epsilon^{(2)}$) we find that the periodicity of the Y-system must satisfy

$$\frac{2}{P} = y = \frac{2(N-4)}{N+2}, \tag{5.8}$$

if it genuinely represents the perturbed theory $\widehat{Z}_N^{(\pm)}$. The periodicity of the $\mathbb{Z}_N$ Y-systems listed in Section (4.2) was verified numerically for all values of N up to N=30 and found to be given by

$$Y_i^{(\alpha)}\left(\theta + 2\pi i \frac{N+2}{N-4}\right) = Y_i^{(\alpha)}(\theta), \tag{5.9}$$

which implies $P = (N+2)/(N-4)$, in full agreement with (5.8).

The quantity $RE^{(pert)}(\lambda, R)$ is dominated by the bulk free energy for large λ, so that asymptotically $RE^{(\text{pert})}(\lambda, R) \sim \mathcal{E}(\lambda)R^2$. The ground state energy obtained from the TBA method does not have any infrared bulk term (at large MR, $RE(R)$ becomes constant), therefore to compare the perturbative and TBA expansions it is necessary to subtract the bulk term from the perturbative series. One should compare the small M TBA expansion with the small λ expansion of

$$RE(\lambda, R) = RE^{(\text{pert})}(\lambda, R) - \mathcal{E}(\lambda)R^2, \tag{5.10}$$

that is,

$$RE(\lambda, R) = -\frac{\pi c}{6} - \mathcal{E}(\lambda)R^2 + 2\pi \sum_{m=2}^{\infty} B_m t^m \,. \tag{5.11}$$

Additional care must be exercised when a power t^m is proportional to the bulk related term $\mathcal{E}(\lambda)R^2$, in which case logarithmic terms appear.

In the non-resonant case the bulk term (or more correctly the anti-bulk term) proportional to R^2 may be extracted exactly from the TBA equations. Fateev [43] has given expressions for the functions $\mathcal{E}$ in terms of the mass scale for $N = 4n+4$. In massive and massless directions these are

$$\mathcal{E}_{\text{massive}} = -\frac{M^2}{2}\frac{\sin\theta_N \sin 2\theta_N}{\sin 3\theta_N}\,, \qquad \mathcal{E}_{\text{massless}} = -\frac{M^2}{2}\frac{\sin^2\theta_N}{\sin 3\theta_N}\,, \tag{5.12}$$

where $\theta_N = 2\pi/(N-4)$. It is believed these hold for all values of $N \geq 5$, which is backed up by numerical evidence in several cases.

When $N = 5, 6, 7$ and 10 the terms t^7, t^2, t^3 and t^2 are respectively proportional to R^2. These are the only cases where we might expect an extra contribution to the ultraviolet expansion given above. In actual fact, there is no resonance for $N = 5$ and $N = 6$ because by l'Hopital's rule the bulk related terms (5.12) vanish. The $N = 7$ and $N = 10$ theories are therefore the only resonant cases. The ultraviolet expansion corrections are given, in the massive cases, by considering the difference

$$\lim_{N \to 2\frac{2m+1}{m-1}} \left[-\frac{\sin\theta_N \sin 2\theta_N}{2\sin 3\theta_N} \left((MR)^2 - (MR)^{2\frac{N-4}{N+2}m} \right) \right] . \tag{5.13}$$

This expression may be evaluated by considering $m\left(\frac{N-4}{N+2}\right) = 1 + \varepsilon$ where ε is small as $N \to 2\left(\frac{2m+1}{m-1}\right)$. The term in round brackets becomes

$$\begin{aligned} (MR)^2 - (MR)^{2(1+\varepsilon)} &= (MR)^2(1 - e^{2\varepsilon \log(MR)}) \\ &= -2\varepsilon(MR)^2 \log(MR) + O(\varepsilon^2). \end{aligned}$$

and $-2\sin(3\theta_N) = -2\sin[6\pi/(N-4)]$ is $2\pi m\varepsilon\cos[6\pi/(N-4)]$ to $O(\varepsilon)$. Noting that $\sin(\theta_N)\sin(2\theta_N)$ needs no expansion, the final result is

$$\mathcal{E}_{\text{massive}} = -M^2 \frac{\sin\theta_N \sin 2\theta_N}{\pi\cos 3\theta_N} \frac{N-4}{N+2} \log MR \,. \tag{5.14}$$

In the massless case $\sin\theta_N$ appears instead of $\sin 2\theta_N$ and the results are

$$\begin{aligned} \mathcal{E}^{(N=7)}_{\text{massive}} &= \tfrac{1}{4\pi} M^2 \log MR\,, & \mathcal{E}^{(N=10)}_{\text{massive}} &= \tfrac{3}{8\pi} M^2 \log MR \\ \mathcal{E}^{(N=7)}_{\text{massless}} &= -\tfrac{1}{4\pi} M^2 \log MR\,, & \mathcal{E}^{(N=10)}_{\text{massless}} &= \tfrac{3}{8\pi} M^2 \log MR \end{aligned} \tag{5.15}$$

It is also possible to find the logarithmic terms directly from perturbation theory. These appear when one considers the divergences in the coefficients B_3 at order λ^3 for $N = 7$ and in B_2 at λ^2 for $N = 10$. After introducing a cutoff scale a to cope with short distance divergences the logarithmic terms can be calculated as

$$B_3(7) = \frac{-1}{12} C_{\epsilon\epsilon\epsilon}^{(N=7)} \gamma(\tfrac{1}{3})^3 \log \frac{R}{2\pi a} + \text{non-singular terms} \tag{5.16}$$

$$B_2(10) = \frac{-1}{2} \log \frac{R}{2\pi a} + \text{non-singular terms} \tag{5.17}$$

By non-singular terms we mean the coefficients of the pure R^2 part in the analytic expansion. Substituting (5.16) and (5.17) into the general series (5.1) we rediscover the full logarithmic bulk terms, but now in terms of λ rather than in terms of the mass scale. Equating perturbative and TBA expressions allows κ to be found exactly. The results in both massless and massive directions are

$$\begin{aligned} N{=}7: \quad \kappa^3 &= 3/(4\pi^3 \gamma\,(\tfrac{1}{3})^3\, C_{\epsilon\epsilon\epsilon}^{(N=7)})\,, \\ N{=}10: \quad \kappa^2 &= 3/(8\pi^2)\,. \end{aligned} \tag{5.18}$$

The constant of proportionality κ in (5.5) has also been determined exactly for the massive flows $N = 4n + 4$ by Fateev [43]. We conjecture that the formula therein also holds for all N (for massless as well as massive perturbations), i.e. κ^2 is given by

$$\kappa^2 = \frac{4}{9\pi^2} \frac{\gamma(\frac{4}{N+2})^2 \gamma(\frac{5}{N+2})}{\gamma(\frac{1}{N+2})} \left[\frac{\pi \Gamma\left(\frac{N+2}{N-4}\right)}{\Gamma\left(\frac{2}{N-4}\right) \Gamma\left(\frac{N}{N-4}\right)} \right]^{4\frac{(N-4)}{(N+2)}} . \tag{5.19}$$

This formula can be checked against values obtained from the numerical solution to the TBA equations as described below.

So far we have asserted that the massive and massless TBA equations do describe the perturbed conformal theories differing only in the sign of a coupling constant λ. If F_m denotes the TBA coefficients for the massless flow, and $\widetilde{F}_m$ those for the massive flow, then assuming the mass scales are equal in both directions, we would expect

$$F_m = (-1)^m \widetilde{F}_m \,. \tag{5.20}$$

Using the iterative method described earlier, the TBA equations in massless and massive directions were solved to find $RE(R)$ for $N = 5, 6, 7, 8, 9$ and 10 concentrating on $0.003 < (MR)^y < 0.5$ in the ultraviolet region. For convenience we set $M = 1$. The θ-axis was descretised in step sizes between 0.08 and 0.125, depending on the value of N. The coefficients were determined by fitting the numerical data to the expansion (5.11). To

ensure the solution was really of the form above we included various test powers of (MR) in the fit and found their coefficients were negligible. Using less than three hundred iterations in each case we obtained highly accurate results: the constant term in the fit agreeing with $-c\pi/6$, where c is the ultraviolet central charge (3.10), with an error of the order 10^{-13}. The exact constant piece, bulk terms from equations (5.12) and (5.15) and the non-perturbative contributions to bulk terms for N=7 and 10 given in (5.18) were then subtracted. After performing this subtraction for massless and massive cases we should have a regular expansion in $(MR)^y$, with coefficients that satisfy the general rule (5.20).

Values of massless and massive coefficients F_m and $\widetilde{F}_m$ obtained from these fits are listed in Appendix (C) for $m = 1, 2, ..8$. The m=0 and m=1 coefficients were left free; the difference between their measured values and exact predictions gives a further check on the numerical accuracy. The agreement with (5.20) is surprisingly good. Coefficients with odd m clearly have opposite sign, whilst coefficients with even m match up to small numerical error. Note also that the $\mathbb{Z}_5$ TBA has $\widetilde{F}_3 = F_3 = 0$ up to numerical error. This agrees with the perturbative expansion where the coefficient $C^{(5)}_{\epsilon\epsilon\epsilon}$ vanishes, via (5.4), since $\Pi^{(5)}(7) = 0$.

Equating coefficients of $(MR)^y$ in expansions (5.1) and (5.7), then substituting (5.5) we find

$$\kappa^n = \left(\frac{-(2\pi)^{2-2\Delta}}{2\pi}\right)^n \frac{F_n}{B_n} \tag{5.21}$$

Equating $(\kappa^2)^3$ and $(\kappa^3)^2$ provides a further test that the TBA equations are consistent with conformal perturbation theory. We should have the following coefficient ratio equal-

N	B_2^3/B_3^2	$\widetilde{F}_2^3/\widetilde{F}_3^2$	F_2^3/F_3^2
6	0.046260423	0.046260427	0.046260424
8	0.379827746	0.379827746	0.379827746
9	14.448969	14.448967	14.448991

Table 5.1: Comparison of perturbative and TBA coefficient ratios for the $\mathbb{Z}_6$, $\mathbb{Z}_8$ and $\mathbb{Z}_9$ systems.

ities: $F_2^3/F_3^2 = \widetilde{F}_2^3/\widetilde{F}_3^2 = B_2^3/B_3^2$ (provided B_2 and B_3 are finite). This can be verified numerically for N=6, 8 and 9 (not for $N = 5, 7, 10$ because $F_3 = \widetilde{F}_3 = 0, B_3 = \infty$ and $B_2 = \infty$ respectively.) Using equations (5.3) and (5.4) to evaluate B_2^3/B_3^2, the results are shown in Table (5.1). The N=7 and 10 results enforce formulae (5.15) and (5.18) because any errors in the subtracted bulk terms would destroy the agreement that the fits show with the prediction (5.20).

Numerical values of κ^2 using TBA data, are given by

$$\kappa^2_{\text{num}} = (2\pi)^{2y-2}\frac{F_2}{B_2} = -\frac{(2\pi)^{2y}}{\pi^2}\frac{\gamma\left(\frac{12}{N+2}\right)}{\gamma\left(\frac{6}{N+2}\right)}F_2 \tag{5.22}$$

and similarly for $\widetilde{\kappa}^2_{\text{num}}$ obtained from $\widetilde{F}_2$. The results are stated in Table (5.2), with excellent agreement. Note that for N=10 the second-order (m=2) term is obscured by the logarithm, but a prediction can still be obtained in an analogous way from κ^3_{num}. These results justify the exact TBA conjecture (5.19) for all $N \geq 5$.

N	κ^2_{exact}	$\widetilde{\kappa}^2_{\text{num}}$	κ^2_{num}
5	0.0235204664	0.0235204667	0.0235204663
6	0.0371477546	0.0371477547	0.0371477547
7	0.0434800500	0.0434800501	0.0434800501
8	0.0442207130	0.0442207131	0.0442207131
9	0.0418094000	0.0418094001	0.0418094003
10	0.0379954439	0.0379954451	0.0379954487

Table 5.2: Comparison of exact and numerical values for κ^2. κ relates the perturbation theory coupling λ and TBA mass parameter M via $\lambda = \kappa M^y$. κ^2_{num} denotes the numerical prediction for the massless direction and $\widetilde{\kappa}^2_{num}$ for the massive direction.

5.2 The massless infrared region

As far as the ultraviolet asymptotics are concerned the predictions from the TBA equations, proposed at the start of Chapter (4), agree very well with perturbation theory for $\widehat{Z}_N^{(\pm)}$. The infrared behaviours are very different according to the sign of the coupling. Here we discuss the massless direction, where the TBA method predicts the infrared limiting central charge value $\tilde{c}(\infty) = 1$.

To compare the predictions of TBA equations with perturbation theory in the massless infrared regime we must first identify which (irrelevantly) perturbed conformal theory describes the infrared limit of $\widehat{Z}_N^{(+)}$. There are several conformal theories with unit central charge, and we should consider only those which contain self-dual, $\mathbb{Z}_N$-symmetric fields if they are to lie in the Kosterlitz-Thouless phase of the $\mathbb{Z}_N$-symmetric phase space discussed earlier. A class of $c = 1$ theories are described by the action of a free boson compactified on a circle of radius $\tilde{r}$ ($\Phi \equiv \Phi + 2\pi\tilde{r}$)

$$S_{IR}[\Phi_{\tilde{r}}] = \frac{1}{2\pi}\int d^2x \sum_{\mu=1,2} (\partial_\mu \Phi)^2 . \tag{5.23}$$

In terms of complex coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, the equations of motion in complex coordinates imply $\Phi(z,\overline{z})$ may be split into holomorphic and anti-holomorphic parts $\Phi(z,\overline{z}) = \frac{1}{2}(\phi(z) + \overline{\phi}(\overline{z}))$. The fields $\phi(z)$ and $\overline{\phi}(\overline{z})$ are not primary, however the vertex operators

$$V^+_{nm}(z,\overline{z}) = \sqrt{2}\cos(p\phi(z) + \overline{p}\overline{\phi}(\overline{z})) \quad , \quad V^-_{nm}(z,\overline{z}) = \sqrt{2}\sin(p\phi(z) + \overline{p}\overline{\phi}(\overline{z})) \ ,$$

are, with $(p,\overline{p}) = (\frac{n}{2\tilde{r}} + m\tilde{r}, \frac{n}{2\tilde{r}} - m\tilde{r})$ and $n, m \in \mathbb{Z}$. These have conformal weights [44]

$$(\Delta_{nm}, \overline{\Delta}_{nm}) = (\tfrac{1}{2}p^2, \tfrac{1}{2}\overline{p}^2) = (\tfrac{1}{2}(\frac{n}{2\tilde{r}} + m\tilde{r})^2, \tfrac{1}{2}(\frac{n}{2\tilde{r}} - m\tilde{r})^2) \ .$$

Two types of field are of particular interest to us: the spin zero N-fold symmetry breaking fields $V^{\pm}_{N0}$ and the spin zero vortex fields $V^{\pm}_{0M}$ with conformal weights

$$\Delta_{N0}(\tilde{r}) = \overline{\Delta}_{N0}(\tilde{r}) = \frac{N^2}{8\tilde{r}^2} \quad , \quad \Delta_{0M}(\tilde{r}) = \overline{\Delta}_{0M}(\tilde{r}) = \frac{M^2\tilde{r}^2}{2} \ .$$

If a prefactor of $(1/\tilde{r})^2$ is inserted before the integral (5.23) one can see that $V^{\pm}_{N0}$ are the N-fold symmetry breaking fields in (3.1). These are therefore associated with the order fields σ. In contrast the $V^{\pm}_{0M}$ fields are interpreted as vortex-like objects (see that $p \neq \overline{p}$ for these fields so they must represent some sort of topological distortion). In particular we suppose fields $V^{\pm}_{01}$ represent single vortices. As pointed out in Chapter

(3) vortices are present in the disorder phase where μ took nonzero expectation values, so duality between order and disorder fields earlier may be extended to the Kosterlitz-Thouless region (even though that region is expected to have zero values of expectation value of order and disorder parameter). In particular, these facts allow us to determine a value of the compactified radius $\tilde{r}$ which describes a conformal model in the self-dual surface (in the Kosterlitz-Thouless region). In the $\mathbb{Z}_N$ model the duality transformation which exchanges the order and disorder fields exchanges the N-fold symmetry breaking field with the unit vortex field. We see from the dimensions of the fields involved that the duality $V_{N0} \longleftrightarrow V_{01}$ can be implemented by the 'radius duality' $\tilde{r} \longleftrightarrow \frac{N}{2\tilde{r}}$. Therefore, the system is self-dual only at $\tilde{r} = \tilde{r}_{sd} = \sqrt{N/2}$, giving

$$\Delta_{N0}(\tilde{r}_{sd}) = \Delta_{01}(\tilde{r}_{sd}) = N/4. \tag{5.24}$$

Since $\tilde{r}_{min} < \tilde{r}_{sd} < \tilde{r}_{max}$ for $N \geq 5$, we see that the self-dual unit central charge theory compactified at radius $\tilde{r} = \tilde{r}_{sd}$ consists of irrelevant primaries and is therefore the candidate for the infrared limit of the self-dual perturbation into the Kosterlitz-Thouless region of the $\mathbb{Z}_N$-symmetric phase space.

Pure $c = 1$ behaviour can not be attained for finite MR; there must be corrections from irrelevant fields consistent with the symmetry of the $\mathbb{Z}_N$ theory. In this infrared regime the theory is non-renormalisable which simply means an infinite number of corrections are necessary to describe the 'arrival' of the perturbed theory $\widehat{Z}_N^{(+)}$ at the $c = 1$ infrared limit. Any divergent terms in ultraviolet perturbation expansions may be viewed as arising because of quantum effects which become huge as we approach smaller and smaller length scales. In contrast, the infrared divergences appear for the non-physical reason that we have not included sufficient terms in the action to make the correlations finite. However, it is still possible to approximate an asymptotic expansion of $RE(R)$. The irrelevant fields with the dominant contributions in the infrared regime are the primary fields of dimension $N/4$ mentioned above and the first spinless descendant in the conformal family $[I]$, namely $T\overline{T}$. An approximate asymptotic description is therefore given by the action

$$S = S_{IR}[\Phi_{\tilde{r}_{sd}}] + \mu_1 \int \psi_{N/4} d^2x + \mu_2 \int T\overline{T} d^2x + \text{further corrections}, \tag{5.25}$$

where the self-dual $\psi_{N/4}$ consists of a combination of the unit vortex and N-fold symmetry breaking fields. $T\overline{T}$ is a descendant with non-vanishing one-point correlation function. Even though the renormalisation group eigenvalue of T and $\overline{T}$ is $y = 2(1 - \Delta) = -2$, $T\overline{T}$ has a one point function proportional to $\mu_2 R^{-2}$ on the cylinder. The two point function

will be proportional to $\mu_2^2 R^{-4}$ and so on. In contrast, $\psi_{N/4}$ is primary and so has vanishing one-point correlation function. $\psi_{N/4}$ has $y = 2(1 - N/4)$ and so the two-point correlation functions are proportional to $\mu_1^2 R^{-1}, \mu_1^2 R^{-2}, \mu_1^2 R^{-3}, ...$ for $N = 5, 6, 7..$ respectively. In general, corrections to $RE(R)$ come from $\psi_{N/4}$ in powers of $\mu_1^2 R^{4-N}$.

The corrections to pure $c = 1$ behaviour from these fields can be extracted from the TBA equations. In contrast to the ultraviolet limit, when MR is large, the non-magnonic pseudoenergies are dominated by their energy terms in the central region between kinks at $\theta = -\log(MR/2)$ and $\theta = \log(MR/2)$ (see Figure (4.5) for an example). The corresponding L-functions have double exponential decay in the mid region and so the non-magnonic pseudoenergies play no direct role in the expressions for the corrections to the infrared central charge. By completely neglecting the influence of the two equations with energy terms on the magnonic equations in the central region $-\log(MR/2) \ll \theta \ll \log(MR/2)$, we are left with magnonic diagrams only. From the Y-systems listed in Section (4.2) we find one node remains for $N = 4n + 2$

$$Y_n^{(2)}(\theta - \tfrac{\pi i}{2h}) Y_n^{(2)}(\theta + \tfrac{\pi i}{2h}) = 1\,, \tag{5.26}$$

which implies the periodicity $Y_n^{(2)}(\theta + \frac{2\pi i}{h}) = Y_n^{(2)}(\theta)$. For the odd cases, $N = 4n+1$ and $N = 4n+3$, the magnonic pseudoenergies form d_4-type Y-systems. These are

$$Y_n^{(\alpha)}(\theta - \tfrac{\pi i}{4h})\, Y_n^{(\alpha)}(\theta + \tfrac{\pi i}{4h}) = \prod_{\beta=2}^{5} \left(1 + Y_n^{(\beta)}(\theta)^{-1}\right)^{-I_{\alpha\beta}^{[d_4]}}\,, \tag{5.27}$$

for $N = 4n + 1$, and

$$Y_n^{(\alpha)}(\theta - \tfrac{\pi i}{4h})\, Y_n^{(\alpha)}(\theta + \tfrac{\pi i}{4h}) = \prod_{\beta=2}^{5} \left(1 + Y_n^{(\beta)}(\theta)\right)^{I_{\alpha\beta}^{[d_4]}} \tag{5.28}$$

for $N = 4n+3$. α runs from 2 to 5 in both cases. These Y-systems also have the periodicity [42] $Y_n^{(\alpha)}(\theta + \frac{2\pi i}{h}) = Y_n^{(\alpha)}(\theta)$ (where $h = N/2 - 2$). This result implies corrections to $RE(R)$ are in powers of $(MR)^{-2h} = (MR)^{4-N}$. The exponent is exactly that expected from the irrelevant $\psi_{N/4}$ perturbation of the Kosterlitz-Thouless $c = 1$ point. The TBA method thus suggests the asymptotic infrared expansion

$$RE(R) = \frac{-\pi}{6} + \sum_{k=1}^{\infty} D_k (MR)^{(4-N)k}\,. \tag{5.29}$$

where D_k are constant. However, since the infrared limit is not renormalisable, we expect additional counterterms to appear (with dimensions different to $N/4$) because (5.29) cannot generally provide a good description of the infrared asymptotics. For N=$4n$+4, where

there are no magnonic pseudoenergies in the TBA system at all, the leading corrections are identified as corresponding to the $T\overline{T}$ field (as discussed below). This suggests that the direct interactions between the non-magnonic pseudoenergies are responsible for the $T\overline{T}$ corrections to scaling in all theories.

Resonances between powers of $\mu_1^2 R^{4-N}$ from $\psi_{N/4}$ and $\mu_2 R^{-2}$ from $T\overline{T}$ are expected to appear as logarithmic corrections. For $N = 5$ the dominant correction comes from the $\psi_{N/4}$ field. In contrast when $N = 6$ the first correction is expected to be logarithmic and for all other values of N the first correction is from $T\overline{T}$. In all $N = 4n + 4$ theories, any sign of corrections to $\psi_{N/4}$ is delayed until a term proportional to R^{-8n} (There is no term proportional to R^{-4n} because the coefficient B_2 vanishes for $\Delta = n$.)

We now concentrate on the $T\overline{T}$ corrections, beginning with the $N = 4n + 4$ case first. The kinks near $\theta = -\log(MR/2)$ and $\theta = \log(MR/2)$ only interact via the tails of the kernel functions $\psi_{ij}(\theta)$. The corrections from $T\overline{T}$ in $\frac{1}{2\pi}RE(R)$ appear as

$$\tfrac{1}{2\pi}RE(R) = -\tfrac{1}{12}\tilde{c}(\infty) + \frac{C_1}{(MR)^2} + \frac{C_2}{(MR)^4} + O\Big((MR)^{-6}\Big)\,. \tag{5.30}$$

(for large MR). Coefficients C_1 and C_2 are given by

$$C_1 = -\frac{\tilde{c}(\infty)^2}{12}\,\frac{\pi}{3}\frac{M^2}{M_1^2}\psi_{11}^{(1)} \quad , \quad C_2 = -\frac{\tilde{c}(\infty)^3}{6}\left(\frac{\pi}{3}\frac{M^2}{M_1^2}\psi_{11}^{(1)}\right)^2 \tag{5.31}$$

(this is found via an iterative procedure similar to that explained on p.535 of [47]). Here M_1 is the mass of the lightest particle in the theory, and $\psi_{11}^{(1)}$ is the first coefficient of an expansion of the kernel $\psi_{11}(\theta)$ where

$$\psi_{ij}(\theta) = -\sum_{k=1}^{\infty} \psi_{ij}^{(s)} e^{-k|\theta|}\,. \tag{5.32}$$

The coefficients in this expression are nonzero only when s, taken modulo h, is an exponent of the relevant non-affine algebra $g = d_{n+1}$. (Generalisations exist for coset flows $g^{(2)} \times g^{(1)}/g^{(3)} \longrightarrow g^{(1)} \times g^{(1)}/g^{(2)}$ where g is an a, d or e -related algebra.) The coefficients are given by

$$\psi_{ij}^{(s)} = \frac{h}{\sin\frac{\pi s}{h}}\, q_i^{(s)} q_j^{(s)}\,, \tag{5.33}$$

where $q_i^{(s)}$ and $q_j^{(s)}$ are components of a unit-normalised eigenvector of the Cartan matrix of g with eigenvalue $2 - 2\cos(\frac{\pi}{h}s)$ (see [45, 46]).

The TBA expansion (5.30) can be compared with the perturbative expansion of $\frac{1}{2\pi}RE(R)$ for action $S_{IR}[\Phi_{\tilde{r}_{sd}}] + \mu_2 \int T\overline{T} d^2x$ [18]:

$$\tfrac{1}{2\pi}RE(R) = -\tfrac{1}{12}\tilde{c}(\infty) + \frac{(2\pi)^3\mu_2}{R^2}\left(\frac{\tilde{c}(\infty)}{24}\right)^2 - \frac{(2\pi)^6\mu_2^2}{R^4}\left(\frac{\tilde{c}(\infty)}{24}\right)^3 + O\left(\frac{\mu_2^3}{R^6}\right), \tag{5.34}$$

because this much of the expansion is uncontaminated by counter terms. If the TBA results above really are compatible with perturbation theory the coefficients (5.31) should satisfy $C_1^2/C_2 = -\tilde{c}(\infty)/24$, which can be easily verified. A comparison of the two series also allows the coupling μ_2 to be expressed in terms of the crossover scale M:

$$\mu_2 = -\frac{2M^2}{\pi^2 M_1^2}\psi_{11}^{(1)}\, M^{-2}\,. \tag{5.35}$$

Returning to the TBA result (5.31), it was found that $q_1^{(1)} = (2/\sqrt{h})\sin(\pi/h)$ for $N = 4n+4$ where $h = (N-4)/2$. Therefore $\psi_{11}^{(1)} = 4\sin\frac{2\pi}{N-4}$. Supposing this formula to hold for all $N \geq 5$ and substituting $\tilde{c}(\infty) = 1$ and $M_1 = 2M\sin\frac{2\pi}{N-4}$, we find

$$C_1 = -\frac{1}{12}\left(\frac{\pi}{3\sin\frac{2\pi}{N-4}}\right), \tag{5.36}$$

$$C_2 = -\frac{1}{6}\left(\frac{\pi}{3\sin\frac{2\pi}{N-4}}\right)^2. \tag{5.37}$$

Furthermore the coupling relation becomes

$$\mu_2 = -\frac{2}{\pi^2\sin\frac{2\pi}{N-4}}\, M^{-2}\,. \tag{5.38}$$

This derivation fails for $N \leq 7$, since the direct interaction kernels $\psi_{ij}(\theta)$ are not present. However, numerical results show R^{-2} corrections do appear in these cases suggesting these formulae, valid for $N > 7$, can also be used for $N = 5, 6$ and 7. When N=7 the predicted coefficients agree with the numerical data below. For N=5 and N=6 the situation is very similar to the ultraviolet case where bulk terms are resonant with terms in an analytic expansion. The $T\overline{T}$-related divergences of (5.36) match divergences in other parts of the expansion, giving a finite result. There is an additional contribution proportional to R^{-2} which comes from the term of order $\mu_1^{2/(N-4)}$ which can cancel the pole in C_1 (furthermore, the regularised B_2 itself diverges when N=6). The difference is found on setting $m = 2/(N-4)$ then evaluating

$$\lim_{N\to 4+\frac{2}{m}}\left[\frac{-1}{12}\,\frac{\pi}{3\sin\frac{2\pi}{N-4}}\left(\frac{1}{(MR)^2} - \frac{1}{(MR)^{(N-4)m}}\right)\right] = \frac{(N-4)}{36\cos\frac{2\pi}{N-4}}\,\frac{\log MR}{(MR)^2}\,. \tag{5.39}$$

in a manner similar to (5.13) by considering $N = 4+\frac{2}{m}+\varepsilon$ here. This gives the coefficients

$$C_1^{(N=5)} = \frac{1}{36}\log MR + \mathrm{A} \quad , \quad C_1^{(N=6)} = -\frac{1}{18}\log MR + \mathrm{B} \tag{5.40}$$

with constant A and B. As for C_2, two additional terms must be considered at orders μ_1^{2m} and $\mu_1^m\mu_2$ since both become proportional to R^{-4} when $N = 5$ or 6. The result is that

$$C_2^{(N=5,6)} = -\frac{(N-4)^2}{54}\log^2 MR + A'\log MR + B'. \tag{5.41}$$

where A' and B' constant. Similar infrared logarithms appearing to a single power have previously been observed numerically, in the interpolating flows between the minimal models $\mathcal{M}_p \to \mathcal{M}_{p-1}$ [47] mentioned in the first chapter.

The $\mathbb{Z}_N$ TBA equations were solved numerically for large MR to investigate the infrared region and it was found that the results did not coincide with the coefficients expected from an analytic expansion alone. When logarithmic corrections were introduced the fit proved exceedingly good. Through perturbation theory one can justify the logarithmic terms as resonances and then use the TBA equations to determine the above coefficients exactly. The numerical solution also gives several coefficients which are as yet unaccessible by the methods above.

For R ranging from approximately 50 to 11000, $RE(R)/2\pi$ was found for $N = 5$, 6 and 7 using the same normalisation M=1 adopted earlier. To begin with, the exponents of the leading correction terms were estimated by finding the limiting slopes of plots of $\log(RE(R)/2\pi + 1/12)$ against $\log R$, with the results -0.9993, -1.9967 and -2.0010 respectively (to be compared with the predictions of -1, -2 and -2). Thus reassured that at least the leading order behaviour was as expected, the data was fit to expansions in powers of R and $\log R$, leaving all the coefficients unconstrained. The following results were obtained for $\frac{1}{2\pi}RE(R)$:

$$N{=}5 \quad : \quad \frac{RE(R)}{2\pi} = \frac{-(1+\delta_5)}{12} - \frac{0.0177380}{R} + \frac{0.027733\log R}{R^2} - \frac{0.01951}{R^2} + \frac{0.097}{R^3} + \ldots$$

$$N{=}6 \quad : \quad \frac{RE(R)}{2\pi} = \frac{-(1+\delta_6)}{12} - \frac{0.05555546\log R}{R^2} + \frac{0.033585}{R^2} - \frac{0.0715\log^2 R}{R^4} + \ldots$$

$$N{=}7 \quad : \quad \frac{RE(R)}{2\pi} = \frac{-(1+\delta_7)}{12} - \frac{0.1007662}{R^2} + \frac{0.153}{R^3} + \ldots$$

As in the ultraviolet case, the constant terms δ_5, δ_6 and δ_7, were used to measure the difference between numerical and exact values for $\tilde{c}(\infty)/12$. These had modulus less that 10^{-11}. Note the first correction for the $N = 5$ central charge function explains the observation of slower convergence to $c = 1$ in the infrared region for the $\mathbb{Z}_5$ system observed earlier (see Figure (4.3)).

For N=5 and 6, the coefficients of $R^{-2}\log R$ match well with the predicted values of 0.02777... and $-0.05555...$ respectively. The N=7 the formula (5.37) predicts the coefficient of R^{-2} to be $-0.1007663...$, again in good agreement with the numerical results. The other predictions are more difficult to verify as one runs up against the limitations in numerical accuracy. For N=7, a coefficient of 0.24369... is predicted for the R^{-4} term, while for N=5 and N=6 the coefficients of $R^{-4}\log^2 R$ should be $-0.0185185...$

and $-0.074074...$ respectively. The agreement for N=6 is reasonable, and improved when a restricted fit (using the exact values for $\tilde{c}(\infty)$ and C_1) was performed. The estimate for this coefficient then becomes -0.0743 to three significant figures. For the other cases, no such convergence was observed for any coefficients beyond those reported above. This is not particularly surprising, since for N=5 there are more unknowns to be fixed before the term of interest can be determined. Note particularly that these solutions provide strong evidence for the hypothesis that formulae (5.36)-(5.38) be used for all N.

In conclusion, the perfect agreement of TBA and perturbative results in both the ultraviolet and infrared regimes suggests that the proposed TBA systems provide an exact description of the ground state energies for the perturbed theories $\hat{Z}_N^{(\pm)}$.

Chapter 6

The Massive Direction

An argument is presented here to find the energy levels of a finite volume quantum field theory possessing a vacuum which is degenerate in infinite volume. The argument is quantum mechanical but may be applied to a finite volume quantum field theory once transverse degrees of freedom have been summed [49]. The idea can be applied to the TBA theory which has finite circumference R.

6.1 Degenerate vacua and energy spectra

The quantum mechanical problem of identifying the ground and first excited states of a symmetric quartic potential theory with doubly degenerate classical ground state, in a way that can be applied to quantum field theory, was addressed by Coleman [48].

In terms of the action S_0 for an instanton travelling once between two neighbouring vacua, the energy levels $E_\pm$ were found to be

$$E_\pm = \hbar\omega/2 \pm A(\hbar S_0/\pi)^{\frac{1}{2}} e^{-S_0/\hbar}\,, \tag{6.1}$$

where $\hbar$ is Planck's constant, A is a constant determined by the asymptotic solution of the saddle-point equation of motion and ω^2 is the second derivative of the quartic potential V at either minima. This result is obtained by considering the Euclidean path integral of an instanton which moves from position x_i to x_j in a time T

$$\langle x_j|e^{-HT/h}|x_i\rangle = N\int [dx]e^{-S/h}\,, \tag{6.2}$$

where N is a normalisation factor, H is the Hamiltonian and S the action for the instanton which can make several journeys between the vacua in the time T. This integral can be evaluated by summing over all journeys between vacua i and j located at x_i and x_j. For

the double well problem $x_i = -a$ and $x_j = a$ are the locations of the minima of quartic potential V. (6.2) is then evaluated by summing over all ways that the particle can pass from $-a$ to a in time T.

If H has energy eigenstates $|E_n\rangle$ such that $H|E_n\rangle = E_n|E_n\rangle$ then inserting the identity $\sum_n |E_n\rangle\langle E_n| = 1$ into the left hand side of (6.2) gives

$$\langle x_j|e^{-HT/\hbar}|x_i\rangle = \sum_n e^{-E_nT/\hbar}\langle x_j|E_n\rangle\langle E_n|x_i\rangle\,. \tag{6.3}$$

Coleman obtained the result

$$\langle a|e^{-HT/h}|-a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{odd\, n} \frac{(Ke^{-S_0/\hbar}T)^n}{n!}[1+O(\hbar)] \tag{6.4}$$

where $(\frac{\omega}{\pi\hbar})^{1/2}e^{-\omega T/2}(K)^n$, with constant K, comes from the evaluation of a determinant which arises in a semiclassical approximation. $\frac{T^n}{n!}$ comes from integrating over all jump times. S_0 is the action for the instanton making a single journey from $-a$ to a (proportional to the mass of the instanton). If we carry out the same procedure to evaluate $\langle -a|e^{-HT/h}|-a\rangle$, a similar expression is found, but now with summation over an even number of steps. Comparing the results

$$\langle \pm a|e^{-HT/h}|-a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\frac{1}{2}[exp(Ke^{-S_0/\hbar}) \mp exp(-Ke^{-S_0/\hbar})] \tag{6.5}$$

with (6.3), the result (6.1) is obtained since it can be shown that K is a multiple of $(S_0/2\pi\hbar)^{1/2}$.

The generalisation of (6.1) is achieved by representing an m-fold degenerate vacuum structure by the $m \times m$ incidence matrix I_{ab}, so that entry (a, b) is non-zero only when an instanton may tunnel from the minima a to b. This allows us to write down a matrix of amplitudes for an instanton travelling between two vacua in n steps via the matrix I^n. The appropriate generalisation is therefore

$$\begin{aligned}\langle j|e^{-HT/\hbar}|i\rangle &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\sum_n \left(\frac{(KTe^{-S_0/\hbar})^n}{n!}(I^n)_{ij}\right)[1+O(\hbar)] \\ &= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\left(exp(KTe^{-S_0/\hbar}I)\right)_{ij}[1+O(\hbar)].\end{aligned} \tag{6.6}$$

Diagonalisation of the matrix (6.6) gives the dominant contributions to the allowed energies in terms of eigenvalues of I:

$$E_i = \hbar\omega/2 - \lambda_i A(\hbar S_0/\pi)^{\frac{1}{2}}e^{-S_0/\hbar}\,. \tag{6.7}$$

This allows us to make predictions about the energy differences $E_i - E_0$. λ_0, the largest eigenvalue, is unique by the Perron-Frobenius theorem (an incidence matrix representing

a connected graph has a unique largest eigenvalue corresponding to an eigenvector with all entries positive) and λ_{m-1} is the least, so that $-\lambda_0 < -\lambda_1 \leq -\lambda_2 \leq ... \leq -\lambda_{m-1}$. Each of the eigenvalues controls those energies which become degenerate with the same ground state in the infinite volume limit. In particular, the ground state energy is controlled by -1 times the largest eigenvalue of the incidence matrix.

An infinite volume field theory with a quartic potential has fields with two ground states $|0_+\rangle$ and $|0_-\rangle$ each with nonzero expectation value. For a finite volume system, barrier penetration is possible and the resulting symmetric ground state $|0_s\rangle = \frac{1}{\sqrt{2}}(|0_+\rangle + |0_-\rangle)$ is unique. The antisymmetric state $|0_a\rangle = \frac{1}{\sqrt{2}}(|0_+\rangle - |0_-\rangle)$ is separated from $|0_s\rangle$ with an energy E_{0a} proportional to $R^{d/2-1}e^{-\tilde{\sigma}R^d}$ in d+1 dimensions, where $\tilde{\sigma}$ is called the surface tension [49]. This energy difference includes a factor $R^{d/2}$ from zero modes, a contribution R^{-1} from 1-loop fluctuations and an exponential which is the Boltzmann factor of an instanton travelling through domain walls. When $d = 1$ the splitting looks like $R^{-1/2}e^{-\tilde{\sigma}R}$ and $\tilde{\sigma}$ can be interpreted as the action of an instanton which is in perfect agreement with the semiclassical prediction above (with $\hbar$ replaced by R^{-1}), provided R is large. Consequently, the above general result applies to the finite volume quantum field theory.

In $1+1$ dimensional field theory this instanton of mass m is called a kink. Precisely, a kink is an instanton solution with energy, momenta, mass and quantum numbers which interpolates between two (not necessarily distinct) vacua. Kinks form nontrivial multiparticle states. Unlike most multiparticle soliton states we are not free to order kinks in an arbitrary configuration; a two-kink state may only be formed if the kinks share a common vacuum. Kink scattering theories with this curious feature will be discussed more fully in the next chapter. Here we concentrate on the energy levels of their excited states.

A simple theory with degenerate vacuum is the tricritical Ising model perturbed by the subleading energy field [52]. This model possesses a symmetric vacuum with three minima. If the above claim is true, the ratio of energies $(E_2 - E_0)/(E_1 - E_0)$ must equal $\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1}$. For this theory, the vacuum is represented by incidence matrix A_3 which has eigenvalues $\lambda_0 = \sqrt{2}, \lambda_1 = 0$ and $\lambda_2 = -\sqrt{2}$. Consequently, we expect the ratio to be 2, which was verified numerically.

6.2 Phase coexistence in the massive perturbation

For the negative sign of coupling, the self-dual perturbing operator $\epsilon^{(2)}$ drives the theory onto a surface of first-order transitions where ordered and disordered phases coexist. There the vacuum structure consists of N ordered ground states with $\langle\sigma\rangle = e^{\frac{2\pi i m}{N}}$ $(m = 0, 1, ..N-1)$ and a ground state with $\langle\sigma\rangle = 0$.

For the instanton argument above it was assumed that all kinks interpolating between vacua were of the same mass. Just as in the case of non-kink particles, 'fundamental' kinks give rise to bound states so that we generally have a tower of particles all of which interpolate between the vacua in question with their own characteristic pattern. Each bound state has its own tower of excitations, so that an excited state of the full system in equilibrium comes from the combinations of fundamental and bound state kink excitations.

Depending on the value of N in question, the masses may be ordered $M_{\pi(1)} \leq M_{\pi(2)} \leq ... \leq M_{\pi(n)}$ where π gives a permutation of mass indices $1, .., n$. In the $\mathbb{Z}_N$ theories the mass spectra is given by (2.23), where M_n is the soliton mass. For $N < 16$ the soliton is the lightest mass, whereas it is the first bound state M_1 which is lightest when $N > 16$. The inequalities are strict, apart from the special cases $N = 16$ where $M_1 = M_n < M_2$ and for those higher N where one of the bound states has a mass value which coincides with the soliton mass. For each mass M_i let the incidence matrix $I^{(i)}_{ab}$ encode, whether or not kinks of that type join vacua a and b. Provided these matrices commute they can be simultaneously diagonalised, and we can write

$$(\lambda^{(1)}_0, \lambda^{(2)}_0, \ldots \lambda^{(n)}_0) > (\lambda^{(1)}_1, \lambda^{(2)}_1, \ldots \lambda^{(n)}_1) \geq \ldots \geq (\lambda^{(1)}_{m-1}, \lambda^{(2)}_{m-1}, \ldots \lambda^{(n)}_{m-1}) , \qquad (6.8)$$

where $\lambda^{(1)}_0, \lambda^{(1)}_1, ..$ is the decreasing sequence of eigenvalues of the incidence matrix for the least mass M_1 etc. Specifically, $\lambda^{(j)}_i$ is the ith eigenvalue in descending order of the incidence matrix of mass M_j. The asymptotic behaviour of the k^{th} energy level will therefore be

$$E_k(R) \sim -\lambda^{(1)}_k \Lambda_1(R) - \lambda^{(2)}_k \Lambda_2(R) \ldots - \lambda^{(n)}_k \Lambda_n(R) \ , \quad k = 0, 1, \ldots m-1 , \qquad (6.9)$$

where $\Lambda_i(R)$, of order $R^{-1/2} \exp(-M_i R)$, gives the leading energy splitting for two vacua and kink mass M_i. The lowest energy state comes from a sum of terms proportional to the largest eigenvalues of each incidence matrix i.e. from the largest eigenvalue of $I^{(1)}$ (for the lightest mass), the largest of $I^{(2)}$ and so on. However, if a heavier kink has more than

twice the mass of the lightest, its leading contributions above will be less important than sub-leading contributions from the lightest kink, which have been omitted. The ground state energy from the massive TBA equations can be used to find both the maximum eigenvalues of the as yet undetermined incidence matrices $I^{(j)}$ and the mass spectrum of the kinks in the massive phase.

The massive pseudoenergies are given by energy terms $\varepsilon_i^{(1)}(\theta) \sim \nu_i^{(1)}(\theta) = M_i R \cosh\theta$ to order $O(e^{-MR})$, while the magnonic pseudoenergies are approximated by the stationary values $\log \mathcal{X}_i^{(\alpha)}$ given in Section (4.5). The general TBA system then becomes

$$\varepsilon_i^{(1)}(\theta) = M_i R \cosh\theta - \sum_{j,\beta} N_{ij}^{1\beta} \log\left(1 + \mathcal{X}_j^{(\beta)^{-1}}\right) , \tag{6.10}$$

where $N_{ij}^{\alpha\beta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_{ij}^{\alpha\beta}(\theta)\, d\theta$, with $\psi_{ij}^{\alpha\beta}$ denoting some generic kernel linking $\varepsilon_i^{(\alpha)}$ to $\varepsilon_j^{(\beta)}$. Substituting this expression into the ground state energy

$$E(R) = -\frac{1}{2\pi}\sum_i \int_{-\infty}^{\infty} d\theta\, M_i \cosh\theta \log\left(1 + e^{-\varepsilon_i^{(1)}(\theta)}\right) \tag{6.11}$$

gives the asymptotic approximation to order $O(e^{-2MR})$ of

$$E(R) \sim -\frac{1}{2\pi}\sum_i \int_{-\infty}^{\infty} d\theta\, M_i \cosh\theta\, e^{-M_i R\cosh\theta} \prod_{j,\beta}\left(1 + \mathcal{X}_j^{(\beta)^{-1}}\right)^{N_{ij}^{1\beta}} . \tag{6.12}$$

Identifying $\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta M_i \cosh\theta e^{-M_i R\cosh\theta}$ with $\Lambda_i(R)$ allows us to read off the eigenvalues $\lambda_0^{(i)}$:

$$\lambda_0^{(i)} = \prod_{j,\beta}\left(1 + \mathcal{X}_j^{(\beta)^{-1}}\right)^{N_{ij}^{1\beta}} . \tag{6.13}$$

To calculate these values it is easier to obtain the correction term in (6.10) by substitution into the relevant Y-system. Thus, there are n kinks of masses M_i, $i = 1 \ldots n$, equal to the corresponding sine-Gordon masses (2.23,2.24) i.e. the soliton mass $M = M_n$ and

$$M_l = 2M \sin\left(\frac{2\pi l}{N-4}\right); \qquad l = 1, ..n-1 , \tag{6.14}$$

where $n = [\frac{N-1}{4}]$ and we find

$$\lambda_0^{(i)} = i{+}1 \qquad (i = 1 \ldots n{-}1) , \tag{6.15}$$

$$\lambda_0^{(n)} = \sqrt{N} . \tag{6.16}$$

The fact that the kinks have the masses given by (2.23) and (2.24) will turn out to be very important when we address the problem of the massive $\mathbb{Z}_N$ S-matrix in the next chapter.

It is generally true that integrability implies these incidence matrices must commute. Consider states which interpolate between vacuum a at $x = -\infty$ and c at $x = \infty$. Conservation of topological charge means this sector is invariant under time evolution. The fact that the theory is integrable means we can concentrate on the subspace of two-particle states to get our result. Suppose at time $t = -\infty$ the two particles of masses M_i and M_j with $i \neq j$ are positioned with M_i to the left of M_j and $\theta_i > \theta_j$. The dimension of this two particle subspace at $t = -\infty$ is $D_{in} = \sum_b I^{(i)}_{ab} I^{(j)}_{bc}$. For $M_i \neq M_j$ there is no reflection in the integrable theory and so the space of out states must be spanned by states in which the kink of mass M_i is to the right of kink of mass M_j when $t = \infty$. The dimension of the final state space is $D_{out} = \sum_b I^{(j)}_{ab} I^{(i)}_{bc}$, but the unitarity of the S-matrix means $D_{in} = D_{out}$. This means $[I^i, I^j] = 0$ and the incidence matrices commute.

This fact helps prove that the eigenvalue $\lambda^{(j)}_0$ is the largest of matrix $I^{(j)}$ for each $j = 1, ..n$ as follows. If the so far undetermined incidence matrices are combined to form a connected R-dependent matrix $I_{ab}(R) = \sum_i \Lambda_i(R) I^{(i)}_{ab}$ we find that for large R there is a unique largest eigenvalue of $I_{ab}(R)$ corresponding to an eigenvector with only positive entries. Furthermore, the commutativity derived above implies this is a simultaneous eigenvector of the individual $I^{(j)}$ and so corresponds to the maximum eigenvalue of each of the $I^{(j)}$, even though they are not individually connected.

For the time being we still have to identify the set of commuting, symmetric and $\mathbb{Z}_N$-symmetric $(N+1)\times(N+1)$ matrices $I^{(1)}, .., I^{(n)}$ which give the largest set of simultaneous eigenvalues (6.8). We propose two of the matrices immediately

$$I^{(1)} = \left(\begin{array}{c|c} [\widehat{A}_{N-1}]_{ab} & \begin{array}{c}0\\ \cdot\\ \cdot\end{array} \\ \hline 0 \;\; \cdot \;\; \cdot & 2 \end{array}\right) \quad , \quad I^{(n)} = \left(\begin{array}{c|c} 0 & \begin{array}{c}1\\ 1\\ \cdot\end{array} \\ \hline 1 \;\; 1 \;\; \cdot & 0 \end{array}\right) \tag{6.17}$$

where $\widehat{A}_{N-1}$ is the $N\times N$ incidence matrix of the affine $a^{(1)}_{N-1}$ Dynkin diagram and entry $(N+1, N+1)$ of $I^{(1)}$ is fixed by the commutation $[I^{(1)}, I^{(n)}] = 0$. Provided N is not a perfect square, $I^{(n)}$ is fixed uniquely (N square would mean $\lambda^{(n-1)}_0$ and $\lambda^{(n)}_0$ could be equal). The forms of $I^{(1)}$ and $I^{(n)}$ are consistent with the idea that the kinks of mass M_1 are the (lowest mass) bound states of a set of fundamental kinks with mass M_n=M. The idea parallels the related sine-Gordon model, where M_n is the mass of the soliton and M_1 the mass of its first bound state or breather. For the remaining kinks we make the conjecture that those of mass M_j in the $\mathbb{Z}_N$ model should be bound states of j kinks of

mass M_1. Accordingly, the allowed topological charges should be found among the paths with j steps on the graph of $I^{(1)}$. This fact suggests

$$I^{(j)} = \left(\begin{array}{c|c} \min\left([\widehat{A}^{\,j}_{N-1}]_{ab}\,,\,1\right) & \begin{array}{c}0\\ .\\ .\end{array} \\ \hline 0\ .\ .\ . & \text{j+1} \end{array}\right) \qquad (j = 1 \ldots n{-}1)\,. \tag{6.18}$$

Here the commutation of $I^{(j)}$ with $I^{(n)}$ is used to pin down the entry $(N+1, N+1)$. Alternatively, we could have conjectured that the breather-related incidence matrices also satisfy the fusion rule

$$I^{(j)} I^{(1)} = I^{(j-1)} + I^{(j+1)} \tag{6.19}$$

where $I^{(0)} = \mathbb{1}$, and $I^{(1)}$ is as in equation (6.17). Bound state incidence matrices are found iteratively. $I^{(2)}$ is found putting $j = 1$, $I^{(3)}$ from $j = 2$ and so on. For N even this prescription overcounts the number of links between vacua not connected to the central disordered vacuum. With an additional assumption that multiplicities of links are forbidden we do regain the expression (6.18). This indeed gives the same incidence matrices listed above.

Figures (6.1) and (6.2) illustrate the kink structures for N=9 and N=15. Tunnelling to and from the disordered phase only occurs via the fundamental-kink instantons, while instantons associated with bound states connect the ordered phases. Note also the presence of tadpoles on the disordered vacuum in all but the fundamental set of kinks. Their appearance might be understood as being due to virtual particle-like excitations above the various vacua.

Interestingly, for N>16 some kinks appear in more than one incidence matrix from the set $I^{(1)} \ldots I^{(n-1)}$ i.e some pairs of vacua are joined not only by a simple kink of minimal mass, but also by excitations of this kink with higher masses (breathers with nonzero

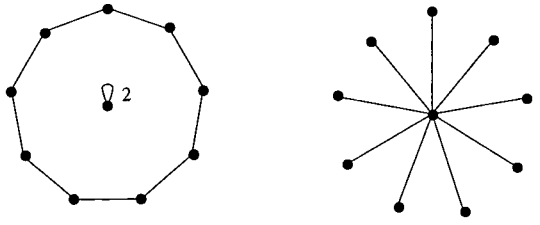


Figure 6.1: The vacua represented by the incidence matrices $I^{(1)}$ and $I^{(2)}$ for the massive $\mathbb{Z}_9$ theory

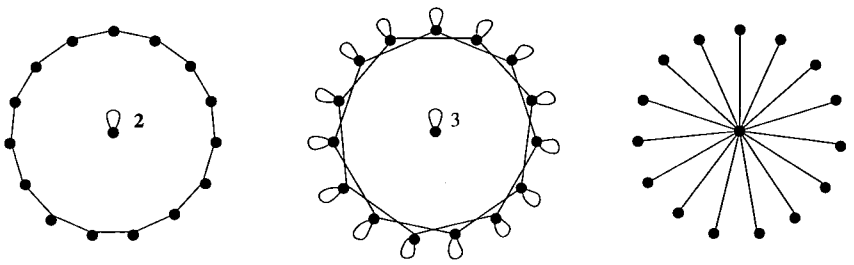


Figure 6.2: The vacua represented by the incidence matrices $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$ for the massive $\mathbb{Z}_{15}$ theory

topological charge). From the topological charge point of view, the information about the asymptotic one-particle states encoded in the incidence matrices may be summarised as follows. Let $a, a+i$ $(a, i = 1, ..N-1)$ always label ordered vacua $1 \ldots N$, with $N+1$ the disordered vacuum. The massive flow has the following features.

1. A kink of mass M_i joins each ordered vacuum a to $a+i$ for $i = 1, ..n-1$. The same vacua are joined by kinks of mass M_j where $j = i+2, i+4, .. \leq n-1$. The excitations of both would be expected.

2. Kinks of mass M_n join $N+1$ to each ordered vacua a.

3. Excitations of each ordered vacuum with masses $M_j, j = 2, 4, \ldots \leq n-1$ appear (indicated by tadpoles on outer part of vacuum diagrams).

4. Excitations of the disordered vacuum appear with masses $M_1, M_2, \ldots M_{n-1}$ and multiplicities $2, 3, ..n$ respectively.

To complete this section we address the case where N factorises. If $N=PQ$, then it is possible to use an orbifold construction [55] to produce alternative patterns of vacua, which might also be described by the self-dual $\mathbb{Z}_N$ TBA systems. In this case, the incidence matrices split into blocks of sizes P and Q rather than the previous N and 1:

$$I^{(1)} = \left(\begin{array}{c|c} [\widehat{A}_{P-1}]_{ab} & 0 \\ \hline 0 & [\widehat{A}_{Q-1}]_{cd} \end{array}\right), \quad I^{(n)} = \left(\begin{array}{c|c} 0 & 1_{ad} \\ \hline 1^{\mathrm{t}}_{cb} & 0 \end{array}\right). \tag{6.20}$$

$\widehat{A}_{P-1}$ and $\widehat{A}_{Q-1}$ are incidence matrices of the affine $a^{(1)}_{P-1}$ and $a^{(1)}_{Q-1}$ Dynkin diagrams, and $\mathbf{1}_{ad}$ is a $P \times Q$ matrix with all entries equal to 1. The remaining incidence matrices for

the bound state kink vacua, $I^{(2)} \dots I^{(n-1)}$, follow from $I^{(1)}$ via the fusion relation (6.19), with $I^{(0)} = \mathbb{1}$. As before it is assumed that after using this fusion rule we let no entry be larger than unity (an assumption which is actually only necessary for even values of N). An example of factorised vacua for $N = 28$ are shown in Figure (6.3). For this theory we expect $[\frac{N-1}{4}] - 1 = 5$ bound states which seems to be consistent with these figures. (The $\widehat{A}_3$ part of the $I^{(3)}$ incidence matrix is omitted, since it is simply the same as the four-node part of $I^{(1)}$.) We return to such systems of vacua in the next chapter.

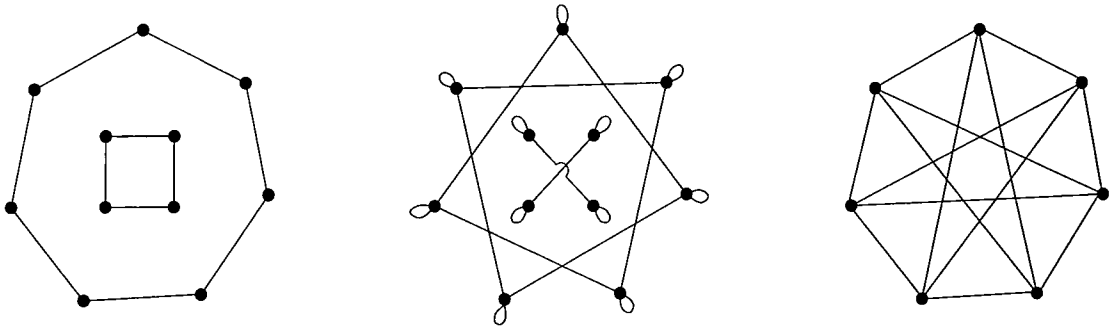


Figure 6.3: The vacua represented by the incidence matrices $I^{(1)}, I^{(2)}$ and $I^{(3)}$ for the massive $\mathbb{Z}_{28}$ theory

Chapter 7

The Massive $\mathbb{Z}_N$ S-matrix

In this chapter, we turn to the S-matrix of the theory perturbed in the massive direction. This was considered by Fateev [34], who suggested S-matrices for three distinct cases, namely N odd, $N = 4n + 2$ and $N = 4n + 4$, the latter two being identified as b_n and d_n related theories via (4.1). We claim that for the b_n and d_n related theories the spectrum deduced in [34] is correct, but cannot be inferred from the S-matrices stated therein. Finding the correct S-matrix is important in this work because it would verify both the mass spectrum and vacuum picture proposed in the last section.

The problem is as follows. The commutation of non-local integrals of motion with the S-matrix, together with analyticity, crossing and unitarity symmetries, allows the determination of the scattering amplitudes as a product of sine or cosine functions which have zeroes in the physical strip multiplied by some prefactor. It is then standard to assume the minimality principle: that the prefactor is chosen to cancel all zeroes. The outcome of this cancellation is the introduction of poles in the physical strip which are interpreted as bound state particles in forward or cross channels. For the b_n (d_n) related theories the $\mathbb{Z}_N$ symmetry of the S-matrix factorises into two copies of $\mathbb{Z}_{N/2}$-symmetric S-matrices (or four copies of $\mathbb{Z}_{N/4}$-symmetric matrices in the case of the d_n-related models) which may be thought of as factorisations distinguished by different quantum numbers. The S-matrices stated in [34] imply the number of particles implied is too large because factors with too many poles were introduced to cancel the zeroes.

Here the corrected S-matrices are presented which (up to an orbifolding) will be seen to agree perfectly with the vacuum structure and the mass spectrum properties derived using the TBA approach for the b_n and d_n related cases. For odd values of N the standard S-matrix calculation predicts twice as many particles predicted by the TBA method *if* the

minimality principle is assumed and the vacuum structure is completely irregular. What is intriguing is that the vacuum structure predicted in the previous chapter is embedded within that suggested by the N odd S-matrix. Furthermore, if one considers only the subset of poles giving this structure the corresponding particle spectrum is exactly that expected from the TBA method. The implication is that the minimality principle is violated in these cases.

This chapter is thus divided into four parts. The first is a review of the N odd S-matrix proposed by Fateev. The second and third concern modifications to Fateev's b_n and d_n related S-matrices and the resulting spectra and vacuum structures. In the final part we return to the N odd case and discuss the problem of extracting the correct vacuum structure and mass spectrum for that case. We adhere to the notation of [34] wherever possible.

7.1 The N odd S-matrix

The existence of a set of non-local integrals of motion $\{Q, \overline{Q}, Q^\dagger, \overline{Q}^\dagger\}$, each formed from fractional spin fields, is used to restrict the particle scattering in the massive $\hat{Z}_N^{(-)}$ theory for $N = 2n + 1$. These integrals satisfy the commutation rules

$$Q\overline{Q} - \overline{\omega}^4 \overline{Q}Q = t\,, \qquad Q\overline{Q}^\dagger - \omega^4 \overline{Q}^\dagger Q = \tilde{t}\,, \tag{7.1}$$

where $\omega = e^{2\pi i/N}$. The spinless topological charges t and $\tilde{t}$ are composed of fields with $\mathbb{Z}_N$ or $\widetilde{\mathbb{Z}}_N$ charges 4. Similar conjugate equations hold with ω replaced by $\overline{\omega}$ and all fields replaced by their conjugates e.g. $\sigma_4 \to \sigma_4^\dagger = \sigma_{N-4}$. It is reasonable to suppose that the vacua may be characterised by expectation values σ of the order parameter σ_4 and by 0 which represents the disordered phase

$$\sigma \in \{0, s : s = \omega^{4q}, q = 0, 1, 2, ..N-1\}. \tag{7.2}$$

s labels the broken symmetry phase and N, being odd, means s runs over the group $\mathbb{Z}_N$.

In this degenerate vacuum theory the particle content consists of N kinds of fundamental kink (K_{0s}) and anti-kink (K_{s0}). Asymptotic states of L kinks with individual rapidities θ_i may be written

$$|K_{\sigma_0\sigma_1}(\theta_1)...K_{\sigma_{i-1}\sigma_i}(\theta_i)...K_{\sigma_{L-1}\sigma_L}(\theta_L)\rangle_{in(out)}\,, \tag{7.3}$$

but the labels on each kink cannot be arbitrary: one of the labels must be 0 and the other takes a value s, so that $|\sigma_{i-1} - \sigma_i| = 1$. These asymptotic states are eigenstates of local

integrals of motion P_s ($s = 1, 3, 5, ..$) with eigenvalues $\sum_{i=1}^{L} M^s \gamma_s^i e^{s\theta_i}$, where γ_s^i are some constants. The existence of these integrals means the scattering theory can be factorised into sequences of two-particle interactions. In contrast, the integrals $Q, \overline{Q}$ ($Q^\dagger, \overline{Q}^\dagger$) with spin $\delta = (N-4)/N$ ($-\delta$) are not diagonalised by the asymptotic kink states. Rather, their action is given by

$$Q|K_{\sigma_0,\sigma_1}(\theta_1)...K_{\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L)\rangle_{in(out)} = \sum_{i=1}^{L}(Me^{\theta_i})^\delta \beta(\sigma_{i-1},\sigma_i)|K_{\omega^4\sigma_0,\omega^4\sigma_1}(\theta_1)...K_{\omega^4\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L)\rangle_{in(out)} \tag{7.4}$$

$$Q^\dagger|K_{\sigma_0,\sigma_1}(\theta_1)...K_{\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L)\rangle_{in(out)} = \sum_{i=1}^{L}(Me^{\theta_i})^\delta \beta^\dagger(\sigma_{i-1},\sigma_i)|K_{\overline{\omega}^4\sigma_0,\overline{\omega}^4\sigma_1}(\theta_1)...K_{\overline{\omega}^4\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L)\rangle_{in(out)} \tag{7.5}$$

where the ω^4 ($\overline{\omega}^4$) factors are determined by the commutations (7.1). The power of the mass term is fixed by the identity $Q^N = P_{N-4}$ and $\beta, \beta^\dagger$ are functions of neighbouring vacua which, by equating the actions of SQ and QS on a two kink state, must be of the form

$$\beta(\sigma,\sigma') = (\sigma + \overline{\omega}^2\sigma')\,, \qquad \beta^\dagger(\sigma,\sigma') = (\sigma^\dagger + \overline{\omega}^2\sigma'^\dagger)\,. \tag{7.6}$$

The ω^4 factors in (7.4) are explained as follows. The charge Q is non-local, this means the fields from which it is constructed are semilocal with respect to the fields on which they act. Another way of stating this non-locality is via the equal-time commutation relation which, for a pair of semilocal fields, states that $\Psi(x,t)\sigma(y,t) = R\,\sigma(y,t)\Psi(x,t)$ if $x > y$, where R is some phase which depends on the algebra of the fields in question ($R = 1$ if $x < y$). For the present case Ψ is the charge and the phase picked up is ω^4. Since a typical kink $K_{\sigma_{i-1}\sigma_i}(\theta_i)$ travels between two neighbouring vacua this definition of the action of a non-local charge explains why only the leftmost vacuum label is multiplied by the phase factor (it appears when a charge 'acts' to the right of the kink with rapidity θ_i, but to the left of the vacuum σ_i.) Considering this action on all kinks in a given L-kink state leads to (7.4). The barred integrals $\overline{Q}$ and $\overline{Q}^\dagger$ have a similar action on the asymptotic states, the results in this case are given by the same formulae but with the substitutions $\delta \to -\delta$, $\omega \to \overline{\omega}$ and $\overline{\beta}(\sigma,\sigma') = (\sigma + \omega^2\sigma'), \overline{\beta}^\dagger(\sigma,\sigma') = (\sigma^\dagger + \omega^2\sigma'^\dagger)$.

The theory is elastic with scattering matrix completely described by the two-particle

amplitudes $A_{s,s'}$ and $B_{s,s'}$ given by

$$|K_{0,s}(\theta_1)K_{s,0}(\theta_2)\rangle_{in} = \sum_{s'} A_{s,s'}(\theta_1/\pi i)|K_{0,s'}(\theta_2)K_{s',0}(\theta_1)\rangle_{out}\,, \tag{7.7}$$

$$|K_{s,0}(\theta_1)K_{0,s'}(\theta_2)\rangle_{in} = B_{s,s'}(\theta_1/\pi i)|K_{s,0}(\theta_2)K_{0,s'}(\theta_1)\rangle_{out}\,, \tag{7.8}$$

where $\theta_{12} = \theta_2 - \theta_1$. The functions A and B depend only on the ratio s/s' and, introducing $u = \frac{\theta_{12}}{\pi i}$ for convenience, we may write

$$\begin{aligned} A_{s,s'}(u) &= A_{s/s'}(u) \equiv A_k(u) = A_{N-k}(u) \\ B_{s,s'}(u) &= B_{s/s'}(u) \equiv B_k(u) = B_{N-k}(u) \end{aligned} \qquad \text{with } s/s' = \omega^{4k}\,. \tag{7.9}$$

Crossing and unitarity symmetries are expressed as

$$A_k(u) = B_k(1-u)\,, \tag{7.10}$$

$$\sum_{k=0}^{N-1} A_k(u)A_k(-u) = B_k(u)B_k(-u) = 1\,. \tag{7.11}$$

The S-matrix and integrals of motion Q, $Q^\dagger$ ($\overline{Q}, \overline{Q}^\dagger$) must commute. In other words, the action of SQ and QS on any asymptotic kink states must give the same result. This implies the following conditions on the amplitudes $A_k(u)$ and $B_k(u)$

$$\begin{aligned} A_{k+1}(u)\sin((2k+2-u)\tfrac{\pi\delta}{2}) &= A_k(u)\sin((2k+u)\tfrac{\pi\delta}{2})\,, \\ B_{k+1}(u)\sin((2k+1+u)\tfrac{\pi\delta}{2}) &= B_k(u)\sin((2k+1-u)\tfrac{\pi\delta}{2})\,. \end{aligned} \tag{7.12}$$

Introducing the functions $S(x) = \sin(\frac{\pi\delta}{2}x)$ and $C(x) = \cos(\frac{\pi\delta}{2}x)$, the solutions for the $N = 2n+1$ theory may be written

$$A_k(u) = a(u)\prod_{i=0}^{k-1} S(2i+u)\prod_{i=k+1}^{n} S(2i-u)\,, \tag{7.13}$$

$$B_k(u) = b(u)\prod_{i=0}^{k-1} S(2i+1-u)\prod_{i=k+1}^{n} S(2i-1+u)\,. \tag{7.14}$$

Crossing and unitarity symmetries force the functions $a(u)$ and $b(u)$ to satisfy

$$a(u) = b(1-u)\,, \qquad Na(u)a(-u) = b(u)b(-u) = 2^{2n}\frac{C(u)}{C(Nu)}\,. \tag{7.15}$$

The products in $A_k(u)$ and $B_k(u)$ contain zeroes. In the left hand diagram of Figure (7.1) the distribution of zeroes in the amplitudes B_k are shown for $N = 17$.

The solution has the form

$$a(u) = N^{(u-1)/2}R(u)\,, \qquad b(u) = N^{-u/2}R(1-u)\,, \tag{7.16}$$

where

$$R(u) = 2^n b_{n-1}(u) \frac{F\left((2n+1)/(2n-3), -2/(2n-3), u\right)}{F\left(1/(2n-3), (n-2)/(2n-3), u\right)} \tag{7.17}$$

and

$$F(a,b,u) = \prod_{l=1}^{\infty} \frac{\Gamma(al+b+u/2)\Gamma(al+b-u/2+1/2)\Gamma(al+b-1/2)}{\Gamma(al+b-u/2)\Gamma(al+b+u/2-1/2)\Gamma(al+b+1/2)}. \tag{7.18}$$

Such solutions are built, step by step, beginning with a ratio of sine functions by repeatedly using the requirements of crossing and unitarity to form an infinite product which can then be written as a product of gamma functions (see [12] for example). The resulting amplitudes satisfy the Yang-Baxter equations provided the $b_n(u)$ are crossing symmetric and satisfy unitarity.

Apart from having to satisfy crossing and unitarity, the amplitudes $b_n(u)$ cannot be determined. This is the CDD ambiguity. At this point it is conventional to assume that the solutions (7.13,7.14) have no zeroes in the physical strip $u \in (0,1)$. There seems to be no physical reason why this minimality assumption should hold, even though over a period of many years this principle has led to successful predictions of bound state spectra for several theories. If this principle is assumed one must introduce poles via the $b_n(u)$ factor to cancel zeroes, at u_z say. This gives rise to poles in some of the amplitudes, which did not have zeroes at u_z, and these are interpreted as bound states of the theory. The resulting bound state spectrum might then be verified by some other means.

For the theory in question, the minimality assumption implies $b_n(u)$ must have the form

$$b_{n-1}(u) = \prod_{l=1}^{n-2} \frac{\sin[\pi(u + \frac{2l}{2n-3})]}{\sin[\pi(u - \frac{2l}{2n-3})]} \tag{7.19}$$

Note that this is the only part of $R(u)$ which contains poles (at $u = \frac{2l}{2n-3}$ for $l = 1,..n-2$) since the F functions in (7.17) contain no zeroes or poles. The outcome is that *some* of the amplitudes B_k have poles at $u = 1 - \frac{2l}{2n-3}$ for $l = 1,..n-2$. For $N = 17$ the resulting distribution is shown in the right hand diagram of Figure (7.1). These poles would usually be interpreted as forward channel bound states which, via (2.11), have masses $M_l = 2M\sin(\frac{\pi l}{2n-3})$, for $l = 1,..n-2$, where M is the soliton kink mass. In terms of N, this is $M_l = 2M\sin(\frac{\pi l}{N-4})$ for $l = 1,..[\frac{N-1}{2}] - 2$. However, this spectrum directly contradicts (6.14) predicted in the last chapter. In particular, the minimality assumption implies approximately twice as many particles appear as predicted using the TBA approach. This indicates that we should use caution when attempting to apply the

minimality principle here. We return to this problem in Section (7.4) after we have built up more evidence based on the vacuum structure of the theory.

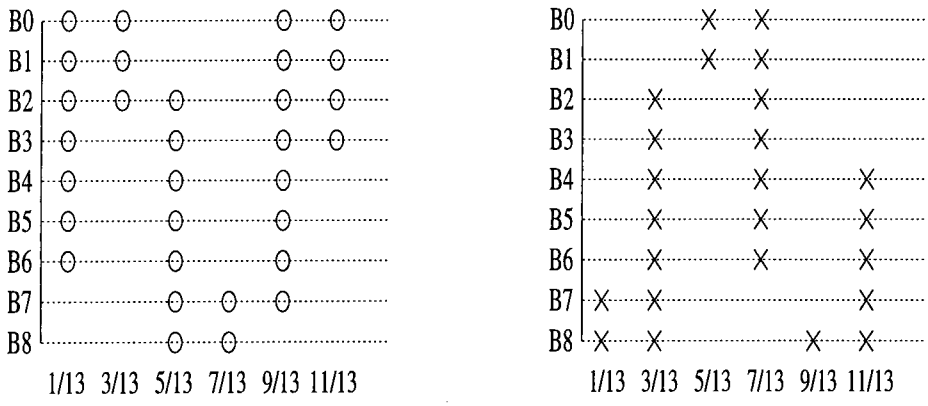


Figure 7.1: The left hand diagram shows the distribution of zeroes in the amplitudes $B_k(u)$ for $N = 17$. The *minimal* CDD function $b_{n-1}(u)$ cancels all these zeroes, leaving a distribution of poles displayed in the right hand diagram.

7.2 b_n-related S-matrices

For the $N = 2(2n+1)$ theories, the algebra has further non-local integrals of motion B and $\overline{B}$ with spins $n - 1/2$ and $1/2 - n$ respectively. B commutes with Q and $Q^\dagger$, while $\overline{B}$ commutes with $\overline{Q}$ and $\overline{Q}^\dagger$. The commutation rule for these additional integrals looks like $B\overline{B} + \overline{B}B = \tau$, where τ is another topological charge. The algebra of these integrals has the symmetry $Z_{N/2} \times Z_2$ and the vacua are expected to have the same symmetry. Therefore, the vacua corresponding to the ordered phase are assumed to be characterised by two variables $\varepsilon \in Z_2$ and $s = \omega^{4q}$, for $q = 0, 1, .., 2n$. So the vacua are identified by

$$\sigma \in \{0, \varepsilon s : s = \omega^{4q} q = 0, 1, .., 2n; \varepsilon = \pm 1\}\,, \tag{7.20}$$

where 0 labels the unbroken symmetric state. The fundamental particles in this theory are kinks $K_{0,\varepsilon s}$ and $K_{\varepsilon s,0}$, which separate the vacua. The actions of $Q, Q^\dagger$ (and $\overline{Q}, \overline{Q}^\dagger$) on asymptotic states is similar to (7.4) and (7.5), whereas the action of B is

$$\begin{aligned} &B|K_{\sigma_0,\sigma_1}(\theta_1)...K_{\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L))_{in(out)} \\ &\quad = \sum_{i=1}^{L} (Me^{\theta_i})^{N\delta/4} \gamma(\sigma_{i-1}, \sigma_i)|K_{-\sigma_0,-\sigma_1}(\theta_1)...K_{-\sigma_{i-1},\sigma_i}(\theta_i)...K_{\sigma_{L-1},\sigma_L}(\theta_L))_{in(out)} \end{aligned}$$

where $\gamma(\sigma, \sigma') = (\sigma^{N/2} - i\sigma'^{N/2})$ with a similar relation for $\overline{B}$.

The scattering amplitudes $A_{\varepsilon s,\varepsilon' s'}$ and $B_{\varepsilon s,\varepsilon' s'}$ are naturally defined by

$$|K_{0,\varepsilon s}(\theta_1)K_{\varepsilon s,0}(\theta_2)\rangle_{in} = \sum_{\varepsilon',s'} A_{\varepsilon s,\varepsilon' s'}(\theta_1/\pi i)|K_{0,\varepsilon' s'}(\theta_2)K_{\varepsilon' s',0}(\theta_1)\rangle_{out}\,, \quad (7.21)$$

$$|K_{\varepsilon s,0}(\theta_1)K_{0,\varepsilon' s'}(\theta_2)\rangle_{in} = B_{\varepsilon s,\varepsilon' s'}(\theta_1/\pi i)|K_{\varepsilon s,0}(\theta_2)K_{0,\varepsilon' s'}(\theta_1)\rangle_{out}\,. \quad (7.22)$$

The amplitudes split naturally into two sectors distinguished by values of the quantum number ε

$$\begin{aligned} A_{\pm s/s'}(u) &\equiv A_k^{\pm}(u) = A_{N/2-k}^{\pm}(u) \\ B_{\pm s/s'}(u) &\equiv B_k^{\pm}(u) = B_{N/2-k}^{\pm}(u) \end{aligned} \qquad \text{with } s/s' = \omega^{4k}\,. \quad (7.23)$$

Again, the S-matrix must commute with the non-local integrals of motion which gives a system similar to (7.12) with solutions

$$A_k^+(u) = a^+(u)\,S(\frac{N}{2}(2-u))\prod_{i=0}^{k-1} S(2i+u)\prod_{i=k+1}^{n} S(2i-u)\,, \quad (7.24)$$

$$A_k^-(u) = a^-(u)\,S(\frac{N}{2}u)\prod_{i=0}^{k-1} S(2i+u)\prod_{i=k+1}^{n} S(2i-u)\,, \quad (7.25)$$

$$B_k^+(u) = b^+(u)\,S(\frac{N}{2}(1+u))\prod_{i=0}^{k-1} S(2i+1-u)\prod_{i=k+1}^{n} S(2i-1+u)\,, \quad (7.26)$$

$$B_k^-(u) = b^-(u)\,S(\frac{N}{2}(1-u))\prod_{i=0}^{k-1} S(2i+1-u)\prod_{i=k+1}^{n} S(2i-1+u)\,. \quad (7.27)$$

The functions $a^{\pm}(u), b^{\pm}(u)$ are chosen to cancel the zeroes in these products. Crossing and unitarity requirements impose conditions similar to (7.15):

$$a^{\pm}(u) = b^{\pm}(1-u)\,, \qquad Na^{\pm}(u)a^{\pm}(-u) = b^{\pm}(u)b^{\pm}(-u) = 2^{(2n+1)}\frac{C(u)}{C^2(\frac{N}{2}u)}\,. \quad (7.28)$$

In [34] the prefactors $a^-(u)$ and $a^+(u)$ were assumed to be the same function. However, this assumption is unnecessarily restrictive. If the functions $a^-(u)$ and $a^+(u)$ are chosen to individually cancel the zeroes in the amplitudes A_k^- and A_k^+ above, the minimal solution is given by

$$a^+(u) = N^{(u-1)/2}R_1^+(u)\,, \qquad b^+(u) = N^{-u/2}R_1^+(1-u)\,, \quad (7.29)$$

$$a^-(u) = N^{(u-1)/2}R_1^-(u)\,, \qquad b^-(u) = N^{-u/2}R_1^-(1-u)\,, \quad (7.30)$$

where

$$R_1^{\pm}(u) = b_n^{\pm}(u)2^{(n+1/2)}\frac{F((2n+1)/(2n-1),-2/(2n-1),u)}{F^2(1/(2n-1),(n-1)/(2n-1),u)}\,. \quad (7.31)$$

The ratio of F factors contains neither zeroes nor poles. The individual meromorphic functions $b_n^-(u)$ and $b_n^+(u)$ are chosen to cancel the zeroes of (7.24-7.27):

$$b_n^-(u) = \frac{1}{S(\frac{N}{4}(u))}\prod_{l=1}^{n-1}\frac{\sin[\pi(u+\frac{2l}{2n-1})]}{\sin[\pi(u-\frac{2l}{2n-1})]}\,, \quad (7.32)$$

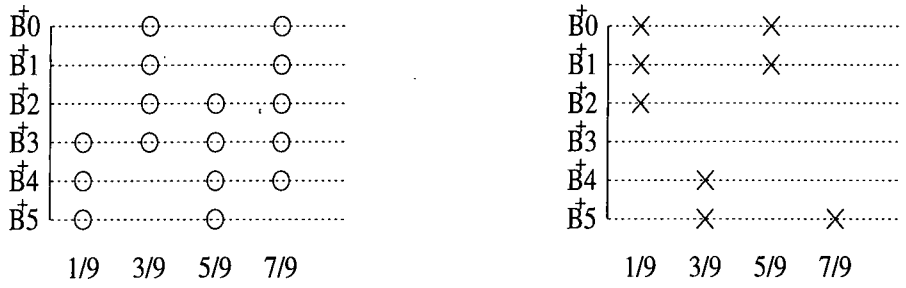


Figure 7.2: The left hand diagram shows the distribution of zeroes in amplitude $B_k^+(u)$ for the $N = 22$ theory. The zeroes from $S(\frac{N}{2}(1+u))$ are not shown. All zeroes are cancelled by the minimal CDD functions $b_{n-1}^+(u)$ which leave the distribution of poles shown in the right hand diagram. The same pole distributions appear in the amplitudes of $B_k^-(u)$.

$$b_n^+(u) = \frac{1}{S(\frac{N}{4}(2-u))} \prod_{l=1}^{n-1} \frac{\sin[\pi(u+\frac{2l}{2n-1})]}{\sin[\pi(u-\frac{2l}{2n-1})]}. \tag{7.33}$$

Wherever a pole occurs we suppose this represents a bound state in a forward or cross channel. The poles in the CDD functions $b^+(u)$ and $b^-(u)$ are at $u = 1 - \frac{2l}{2n-1}$ for $l = 1, ..n-1$ and correspond to bound state particles with masses

$$M_l = 2M \sin\left(\frac{2\pi l}{2(2n-1)}\right); \qquad l = 1, ..n-1. \tag{7.34}$$

This is in perfect agreement with (6.14) for $N = 2(2n+1)$. Let us consider the $N = 22$ scattering theory, choosing N sufficiently large to see the structure of the zeroes, poles and resulting vacuum picture clearly (see Figure (7.2)). We can reconstruct the underlying vacuum structure from the position of the poles in the scattering amplitudes as follows. If one of the poles is to correspond to a bound state described by the incidence matrix $I^{(1)}$, it is $u = 7/9$ in the amplitude B_5^+ since this is the only pole associated with a single topological value k. All higher incidence matrices $(I^{(2)}, I^{(3)}, ...)$ have more than one link from each vacuum and therefore must be associated to more than one amplitude. It is natural to identify each column of poles in Figure (7.2) with one of the graphs in Figure (7.3).

In order to interpret the graphs as part of the incidence diagrams for the $N = 22$ vacuum structure we must show that the higher graphs may be derived from the first. Note also that we cannot yet determine how many points are contained in each of the graphs.

The amplitude B_5^+ which represents a link between two adjacent points in the first figure was defined to be the amplitude relating vacuum s and vacuum $\omega^{20}s$ via (7.23).

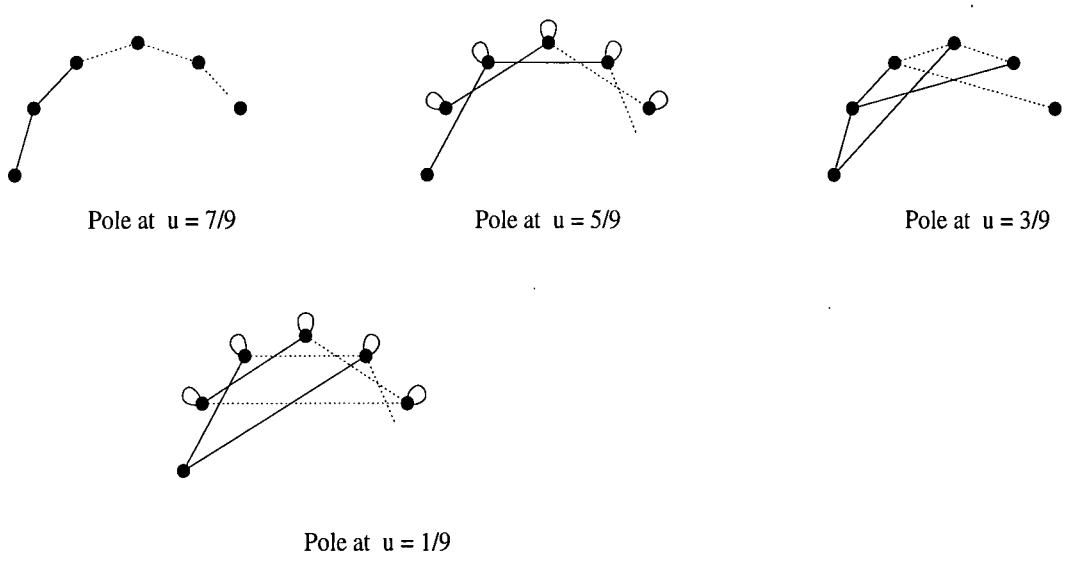


Figure 7.3: Reconstruction of the bound state vacuum diagrams for $N = 22$. Each of the diagrams closes as in Figures (6.2,6.3). The dots simply indicate that *a priori* we do not know the number of nodes in each picture (see text for full discussion).

Let the map $(B_5^+) : s \to \omega^{20}s$ indicate that B_5^+ links vacua separated by the factor ω^{20} (note $\omega^{-20}s$ is also connected to s by this map). If the graphs of Figure (7.3) genuinely represent vacuum incidence matrices then we expect (B_1^+) (which links next to nearest nodes) to be equivalent to applying (B_5^+) twice. Since $N = 22$, we find $(B_5^+)^2 : s \to \omega^{40}s = \omega^{18}s = \omega^{-4}s$. Therefore $(B_5^+)^2$ does connect the same vacua as (B_1^+). We then write

$$(B_5^+)^2 \sim (B_1^+)\,. \tag{7.35}$$

Similarly $(B_5^+)^3 \sim (B_4^+)$ and $(B_5^+)^4 \sim (B_2^+)$. Furthermore, the periodicity

$$(B_5^+)^{12} = (B_5^+)\,, \tag{7.36}$$

implies each of the graphs has exactly 11 nodes. For a general $N = 2(2n+1)$ there are $n-1$ poles distributed in a pattern like that of Figure (7.2); the resulting $n-1$ graphs have $N/2$ nodes and look like the sequence begun in Figure (7.3). The figures obtained coincide with those predicted in the last chapter for a $\mathbb{Z}_N$ symmetric theory where N factorises into the product $N = N/2 \times 2$.

7.3 d_n-related S-matrices

For the $N = 4(2n+1)$ theories, as in those considered already, states are formed by kink pairs which are restricted to travel from one of N vacua to the disordered vacua. If we

let $\varepsilon = \pm 1$ and $s = \omega^{4q} q = 0, 1, .., 2n$, the particle states may be written

$$|K^{\varepsilon}_{0,s}(\theta_1)K^{\varepsilon}_{s,0}(\theta_2)\rangle\,, \qquad |K^{\varepsilon}_{0,s}(\theta_1)K^{-\varepsilon}_{s,0}(\theta_2)\rangle \tag{7.37}$$

and

$$|K^{\varepsilon}_{s,0}(\theta_1)K^{\varepsilon}_{0,s'}(\theta_2)\rangle\,, \qquad |K^{\varepsilon}_{s,0}(\theta_1)K^{-\varepsilon}_{0,s'}(\theta_2)\rangle\,. \tag{7.38}$$

As expected integrals $\{Q, \overline{Q}, Q^\dagger, \overline{Q}^\dagger\}$ appear, but there is also another local integral of motion with even spin denoted D_{2n}. This acts on asymptotic states as

$$\begin{aligned} &D_{2n}|K^{\varepsilon_1}_{\sigma_0,\sigma_1}(\theta_1)..K^{\varepsilon_i}_{\sigma_{i-1},\sigma_i}(\theta_i)..K^{\varepsilon_L}_{\sigma_{L-1},\sigma_L}(\theta_L)_{in(out)} \\ &= \sum_{i=1}^{L} \varepsilon_i (Me^{\theta_i})^{2n} |K^{\varepsilon_1}_{\sigma_0,\sigma_1}(\theta_1)..K^{\varepsilon_i}_{\sigma_{i-1},\sigma_i}(\theta_i)..K^{\varepsilon_L}_{\sigma_{L-1},\sigma_L}(\theta_L)_{in(out)} \end{aligned} \tag{7.39}$$

and since this integral commutes with Q and $Q^\dagger$ it gives a restriction on the S-matrix amplitudes

$$|K^{\varepsilon}_{0,s}(\theta_1)K^{\varepsilon}_{s,0}(\theta_2)\rangle_{in} = \sum_{s'} A^{\varepsilon}_{s/s'}(u)|K^{\varepsilon}_{0,s'}(\theta_2)K^{\varepsilon}_{s',0}(\theta_1)\rangle_{out} \tag{7.40}$$

$$|K^{\varepsilon}_{0,s}(\theta_1)K^{-\varepsilon}_{s,0}(\theta_2)\rangle_{in} = \sum_{s'} \overline{A}^{\varepsilon}_{s/s'}(u)|K^{-\varepsilon}_{0,s'}(\theta_2)K^{\varepsilon}_{s',0}(\theta_1)\rangle_{out} \tag{7.41}$$

$$|K^{\varepsilon}_{s,0}(\theta_1)K^{\varepsilon}_{0,s'}(\theta_2)\rangle_{in} = B^{\varepsilon}_{s/s'}(u)|K^{\varepsilon}_{s,0}(\theta_2)K^{\varepsilon}_{0,s'}(\theta_1)\rangle_{out} \tag{7.42}$$

$$|K^{\varepsilon}_{s,0}(\theta_1)K^{-\varepsilon}_{0,s'}(\theta_2)\rangle_{in} = \overline{B}^{\varepsilon}_{s/s'}(u)|K^{-\varepsilon}_{s,0}(\theta_2)K^{\varepsilon}_{0,s'}(\theta_1)\rangle_{out} \tag{7.43}$$

Fateev has claimed that the system also has the $\mathbb{Z}_2$ charge symmetry $\varepsilon \leftrightarrow -\varepsilon$ described by the operator C : $CD_{2n} = -D_{2n}C$. If this is so, we may drop the upper index on $A, \overline{A}, B$ and $\overline{B}$. Now proceeding as before, with

$$A_k(u) = A_{N/4-k} = A_{s/s'}(u)\,, \qquad s/s' = \omega^{4k} \tag{7.44}$$

and similarly for $\overline{A}, B$ and $\overline{B}$. These amplitudes satisfy

$$\begin{aligned} A_{k+1}(u)S(2k+2-u) &= -A_k(u)S(2k+u) \\ \overline{A}_{k+1}(u)C(2k+2-u) &= \overline{A}_k(u)C(2k+u) \\ B_{k+1}(u)C(2k+1+u) &= B_k(u)C(2k+1-u) \\ \overline{B}_{k+1}(u)S(2k+1+u) &= -\overline{B}_k(u)S(2k+1-u) \end{aligned}$$

with $S(x)$ and $C(x)$ are defined earlier. The solutions in this case are

$$A_k(u) = a(u)(-1)^k \prod_{i=0}^{i=k-1} S(2i+u) \prod_{i=k+1}^{n} S(2i-u)\,, \tag{7.45}$$

$$\overline{A}_k(u) = \overline{a}(u) \prod_{i=0}^{i=k-1} C(2i+u) \prod_{i=k+1}^{n} C(2i-u)\,, \tag{7.46}$$

$$B_k(u) = b(u) \prod_{i=0}^{k-1} C(2i+1-u) \prod_{k+1}^{n} C(2i-1+u)\,, \tag{7.47}$$

$$\overline{B}_k(u) = \overline{b}(u)(-1)^k \prod_{i=0}^{k-1} S(2i+1-u) \prod_{k+1}^{n} S(2i-1+u)\,. \tag{7.48}$$

Crossing and unitarity conditions read

$$A_k(u) = \overline{B}_k(1-u), \qquad \overline{A}_k(u) = B_k(1-u)\,,$$
$$\sum_{k=0}^{2n} A_k(u)A_k(-u) = \sum_{k=0}^{2n} \overline{A}_k(u)\overline{A}_k(-u) = B_k(u)B_k(-u) = \overline{B}_k(u)\overline{B}_k(-u) = 1\,,$$

which impose the following conditions on $a(u), \overline{a}(u), b(u)$ and $\overline{b}(u)$

$$a(u) = \overline{b}(1-u)\,, \qquad \overline{a}(u) = b(1-u)\,, \tag{7.49}$$

$$\frac{N}{4}a(u)a(-u) = b(u)b(-u) = 2^{2n}\frac{C(u)}{C(\frac{N}{4}u)}\,, \tag{7.50}$$

$$\frac{N}{4}\overline{a}(u)\overline{a}(-u) = \overline{b}(u)\overline{b}(-u) = 2^{2n}\frac{S(u)}{S(\frac{N}{4}u)}\,. \tag{7.51}$$

The minimal solution (different to that stated in [34]) is

$$\overline{a}(u) = \left(\frac{N}{4}\right)^{\frac{(u-1)}{2}} \overline{R}_2(u)\,, \tag{7.52}$$

$$a(u) = \left(\frac{N}{4}\right)^{\frac{(u-1)}{2}} R_2(1-u)\,, \tag{7.53}$$

so that

$$b(u) = \left(\frac{N}{4}\right)^{-\frac{u}{2}} \overline{R}_2(1-u)\,, \tag{7.54}$$

$$\overline{b}(u) = \left(\frac{N}{4}\right)^{-\frac{u}{2}} R_2(u)\,, \tag{7.55}$$

where

$$R_2(u) = \left(\frac{2}{\pi}\right)^n d_n(u)P(u)\,, \qquad \overline{R}_2(u) = \left(\frac{2}{\pi}\right)^n \overline{d}_n(u)P(u)\,, \tag{7.56}$$

with

$$P(u) = \prod_{m=1}^{\infty} \frac{\Gamma(\frac{m+nu}{2n+1})\Gamma(\frac{1}{2}+\frac{m-nu}{2n+1})\Gamma(\frac{1}{2}+\frac{n(2m-u)}{2n+1})\Gamma(\frac{1}{2}+\frac{n(2m-1+u)}{2n+1})}{\Gamma(\frac{m+2n-nu}{2n+1})\Gamma(\frac{1}{2}+\frac{m+nu}{2n+1})\Gamma(\frac{1}{2}+\frac{n(2m+u)}{2n+1})\Gamma(\frac{1}{2}+\frac{n(2m-1-u)}{2n+1})}\,. \tag{7.57}$$

Again, the factors $d_n(u)$ and $\overline{d}_n(u)$ must be chosen to cancel any zeroes which would otherwise be present in the solutions. The minimal choice is

$$d_n = \prod_{l=1}^{n-1} \frac{\sin[\pi(u+\frac{2l}{2n})]}{\sin[\pi(u-\frac{2l}{2n})]}\,, \tag{7.58}$$

$$\overline{d}_n = \prod_{l=1}^{n} \frac{\sin[\pi(u+\frac{2l-1}{2n})]}{\sin[\pi(u-\frac{2l-1}{2n})]}\,. \tag{7.59}$$

These factors give rise to poles in some of the amplitudes B_k and $\overline{B}_k$ at $u = 1 - \frac{l}{2n}$ for $l = 1, ..2n-1$ and are expected to correspond to forward channel bound states with masses

$$M_l = 2M \sin\left(\frac{\pi l}{4n}\right); \qquad l = 1, ..2n-1 \tag{7.60}$$

which is again in perfect agreement with the mass formula (6.14) derived in the last section. As a specific example, consider the $N = 28$ scattering theory. Figure (7.4) shows the distribution of poles in the amplitudes B_k and $\overline{B}_k$: the result when we have multiplied through by the meromorphic functions $b(u)$ and $\bar{b}(u)$ respectively.

The pole at $u = 5/6$ in amplitude B_3 is supposed to connect adjacent nodes in the graph corresponding to part of the incidence matrix $I^{(1)}$ because all other poles correspond to more than one value of the label k. Together with the distribution of the other poles among the amplitudes B_k and $\overline{B}_k$ this allows the construction of the graphs in Figure (7.5). In the notation of the last section we write $(B_3) : s \to \omega^{12}s$. As in the b_n-related case, if the graphs shown do represent incidence matrices we expect $(B_3)^2 \sim (\overline{B}_1)$ because $\overline{B}_1$ clearly relates next to nearest nodes. For N=28, $(B_3)^2 : s \to \omega^{24}s = \omega^{-4}s$, so $(B_3)^2$ relates nodes separated by a factor ω^4 as expected. Similarly, one finds $(B_3)^3 \sim (B_2)$, $(B_3)^4 \sim (\overline{B}_2)$ and $(B_3)^5 \sim (B_1)$. Finally, we observe the periodicity

$$(B_3)^8 = (B_3)\,, \tag{7.61}$$

which tells us that seven nodes appear in the graphs of Figure (7.5). This is exactly one of the possibilities predicted in the last chapter for a theory where N factorises as $N = N/4 \times 4$. The graphs obtained therefore describe the $\mathbb{Z}_{N/4}$ part of the incidence matrices for the $\mathbb{Z}_N$ symmetric theory.

Thus if we exploit additional conserved charges, S-matrices which are $\mathbb{Z}_{N/2}$ or $\mathbb{Z}_{N/4}$ symmetric are obtained and from these we can only reconstruct vacuum diagrams which

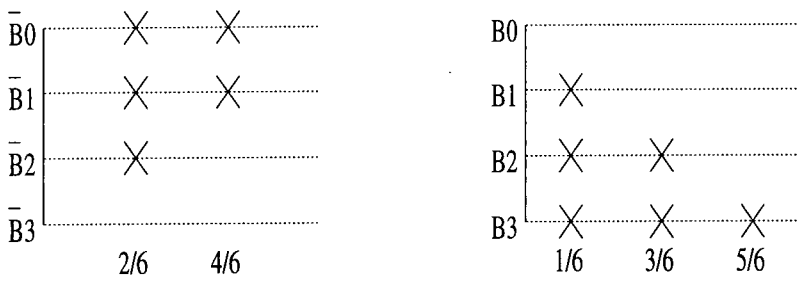


Figure 7.4: The distribution of poles in the minimal solutions $B_k^+(u)$ and $B_k^-(u)$ when $N = 28$.

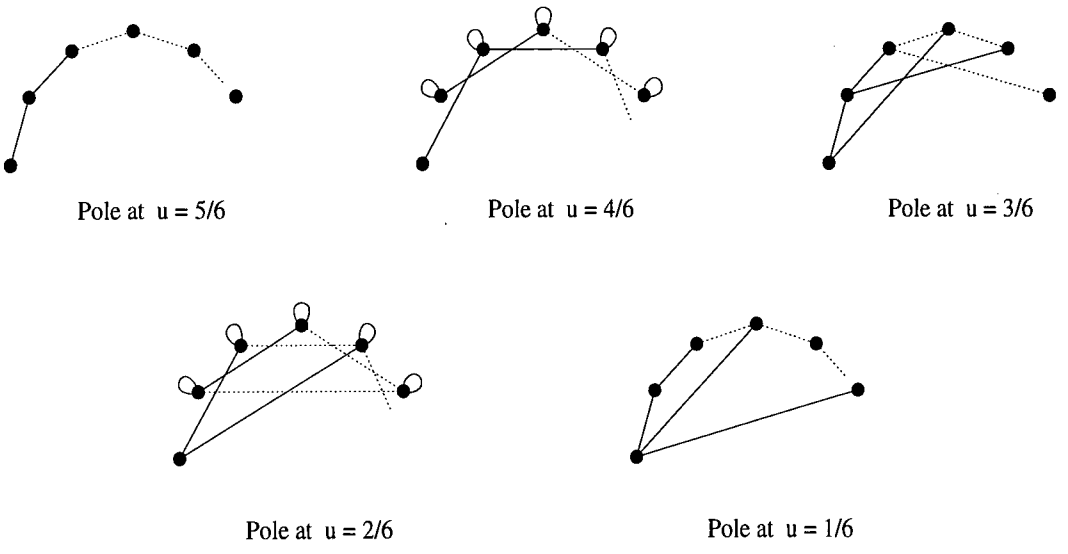


Figure 7.5: Reconstruction of the vacuum diagrams for $N = 28$.

appear to be related to orbifolds of the full $\mathbb{Z}_N$-symmetric vacuum. For even N it is likely that an orbifold the full $\mathbb{Z}_N$-symmetric theory would give S-matrices which would also predict bound states which give rise to the subdiagrams $\widehat{A}_{2-1}$ and $\widehat{A}_{4-1}$ for the b_n and d_n related theories respectively as well as the vacuum diagrams found above.

7.4 Hidden vacuum structure in the N odd S-matrix

Returning to Figure (7.1) we see that the distribution of the poles is very much more complicated in the N odd cases compared to the N even. However, considering only the poles at

$$u = 1 - \frac{4l}{N-4}, \qquad l = 1,.. \left[\frac{N-1}{4}\right] - 1 \tag{7.62}$$

gives the mass spectrum $M_l = 2M \sin(\frac{2l}{N-4})$ which agrees with the prediction (6.14). The poles $u = 1/13, 5/13$ and $9/13$ in the right hand graph of Figure (7.1) give exactly this spectrum when $N = 17$.

Another interesting fact is that if one considers only these poles and their corresponding topological values, we can reconstruct the vacuum picture derived in the last section. The argument is once again based on finding the single pole-charge pair which is likely to correspond to a single link on the $I^{(1)}$ incidence matrix. The pole in amplitude B_8 at $u = 9/13$, which links adjacent nodes via $(B_8) : s \to \omega^{32}s$, is the obvious candidate. To accept the graphs of Figure (7.6) as being parts of the incidence matrices $I^{(1)}, I^{(2)}$ and $I^{(3)}$ we expect $(B_8)^2 \sim (B_1)$ and $(B_8)^3 \sim (B_7)$. Indeed with $N = 17$, $(B_8)^2 : s \to \omega^{64}s = \omega^{-4}s$

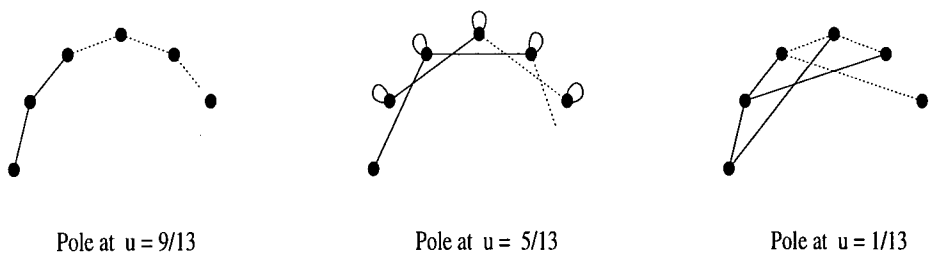


Figure 7.6: Suggestion for the reconstruction of the vacuum diagram for $N = 17$.

relates vacua s and $\omega^{\pm 4}s$, just as (B_1) does. Also, $(B_8)^3 : s \to \omega^{96}s = \omega^{28}s$ relates exactly those nodes connected by (B_7). Additionally, we find that the least value of m for which $(B_8)^m = (B_8)$ is $m = 18$ implying there are 17 nodes in the Figure (7.6) which is expected for the $\mathbb{Z}_{17}$ symmetric part of the incidence matrices because there is no factorisation here. The same pattern is true for all N odd, provided we exclude the spurious poles which were introduced because the S-matrix amplitude has zeroes at $u = 1 - \frac{4k-2}{N-4}$ $(k = 1, ..[\frac{N-1}{4}] - 1)$ for theories $N = 4p+3$, and at $u = 1 - \frac{4k}{N-4}$ for those with $N = 4p+1$ (for integer $p \geq 1$).

In summary, the above evidence suggests that if we leave these zeroes in the physical strip and only introduce poles at (7.62), then we would obtain a spectrum and vacuum structure which agrees with the TBA prediction. The problem is that if we allow these zeroes we effectively lose the power of the minimality assumption.

Chapter 8

Conclusions

Armed with the Y-systems summarized by the diagrams of Section (4.2), we can write down TBA systems (4.2), (4.9), (4.29) and (4.30) which describe a set of flows into both massless and massive directions from conformal theories with $\mathbb{Z}_N$ symmetry under the self-dual $\mathbb{Z}_N$-symmetric perturbation operator $\epsilon^{(2)}$. The ultraviolet and infrared limiting behaviour of the ground state energy and several leading corrections from perturbed conformal theory can easily be obtained through analysis of the TBA equations and corresponding Y-systems. The TBA method is constrained only by the numerical accuracy of the computer routine which solves the TBA system in question.

The phase space of all two-dimensional $\mathbb{Z}_N$-symmetric statistical models has at least three phases when $N \geq 5$. Low and high temperature limits are identified with ordered and disordered phases, while the existence of a massless perturbed theory with infrared limit identified as being conformal, self-dual and $\mathbb{Z}_N$-symmetric, demonstrates the existence of the third Kosterlitz-Thouless phase. It has also been established that the parafermionic theories are most likely located at the opening of the Kosterlitz-Thouless region from the first order transition region.

New dilogarithm identities appear naturally when we attempt to evaluate ultraviolet and infrared limits of the effective central charge. Such identities add to a huge class of new identities already found using the TBA approach which suggests a deep relationship between integrable models, Y-systems and sum rules of the transcendental dilogarithm functions [57]. That the Y-systems contain the information to calculate the ultraviolet features like the perturbing dimension and central charge of a perturbed conformal theory, as well as all the infrared behaviour in both massive and massless directions, seems to suggests that they embody many of the features expected from an integrable theory. One

of the most ambitious projects would be to develop a scheme for the classification of all integrable systems via the Y-system approach. This program has recently begun in the work of Bazhanov, Lukyanov and Al.B. Zamolodchikov [58] who showed that it is possible to pass directly from the integrable system to the Y-system, thus bypassing any need for an S-matrix derivation.

In Chapter (6) the energy level problem for a degenerate vacuum quantum theory is discussed. Results are obtained which can be applied to quantum field theory in finite volume i.e. to the TBA systems with a finite circumference R. The relationship (6.9) was found, which expresses excited state energies in terms of eigenvalues of incidence matrices which describe the paths of kinks of mass M_i travelling between degenerate vacua. A general expression has also been found for the incidence matrices which describe vacua between which bound state kinks may travel. The mass spectrum can be found in this case because the ground state energy is accessible from the TBA method and the degenerate vacuum structure can be deduced. Both facts are useful when checking a hypothesis for the scattering matrix of the massive theory.

In Chapter (7) the S-matrices are proposed for the $\mathbb{Z}_N$-symmetric massive scattering. When additional conserved charges are involved, as in the cases $N = 4n + 2$ and $N = 4n + 4$, the S-matrices have explicit $\mathbb{Z}_{N/2}$ and $\mathbb{Z}_{N/4}$ symmetry. For N odd the vacuum diagrams suggested by these matrices agree exactly with the predictions of Chapter (6). In the even cases, where N factorises, using the additional conserved charges means we only find $N/2$ and $N/4$ node vacuum pictures. It is believed that the full incidence structure (6.20) could be reconstructed from an S-matrix with the full $\mathbb{Z}_N$ symmetry by a suitable orbifold process. One of the main results of Chapter (7) is that the minimality principle must be violated if the S-matrix obtained is to agree with expected mass spectra and vacuum structures.

In this study we have concentrated on perturbations of $\mathbb{Z}_N$-symmetric conformal field theories by a single self-dual field $\epsilon^{(2)}$. However in any $\mathbb{Z}_N$ phase space there exists a self-dual subspace of dimension $[N/4]$. It would be interesting to investigate this subspace and all other perturbations of parafermionic models which remain in the $\mathbb{Z}_N$ phase space. More desirable still would be a non-integrable method which could be used to examine those regions of phase space off the relevant and irrelevant manifolds. This is currently a distant prospect, in the meantime it would be interesting to investigate just how far the TBA program can be developed.

Certainly, the quantum field theory and TBA techniques used to investigate the phases of the of the $\mathbb{Z}_N$-symmetric systems models could be applied to many other statistical mechanics systems. Though derivations of the TBA equations are generally difficult, we have shown here that it is possible to bypass this problem in some cases when the infrared limit of one perturbed theory coincides with the ultraviolet limit of another, effectively getting two sets of Y-systems for the price of one.

Appendix A

sine-Gordon TBA and Y-systems

In this appendix we derive the TBA equations for the b_n-related massive sine-Gordon theory with $\beta^2 = 32\pi/N$ (N=$4n+2$) using the algebraic Bethe ansatz. Then the corresponding Y-systems are stated along with the generalisations for all $N \geq 5$. The symmetric versions of these are believed to be the Y-systems for the perturbed parafermionic field theories $\widehat{Z}_N^{(\pm)}$.

A.1 Massive b_n-related sine-Gordon TBA

Suppose that the b_n-related sine-Gordon theory consists of a collection of N_s solitons or antisolitons, labelled by A and $\overline{A}$ respectively, each of mass M_n and N_b breathers of mass M_i, $i = 1, \ldots, n-1$. Positioning these on a circle of circumference L with periodic boundary conditions we intend taking the thermodynamic limit $N_s, N_b, L \to \infty$ as before. The wavefunction Ψ should remain invariant if a particle of rapidity θ is passed through all the others on the ring before returning to its original position. This condition may be stated in the form of three sets of equations, the first when we consider invariance when we move a breather around the circumference and the other two when we move a soliton or antisoliton. Introducing a breather or soliton label on the rapidities these systems become

$$e^{iLM_1 \sinh\theta_i^1} \prod_{k=1}^{N_b} S_{1,k}(\theta_i^1 - \theta_k) \prod_{k=1}^{N_s} S_{1,n}(\theta_i^1 - \theta_k^n)\Psi = \Psi \qquad \text{(A.1)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$e^{iLM_{n-1} \sinh\theta_i^{n-1}} \prod_{k=1}^{N_b} S_{n-1,k}(\theta_i^{n-1} - \theta_k) \prod_{k=1}^{N_s} S_{n-1,n}(\theta_i^{n-1} - \theta_k^n)\Psi = \Psi \qquad \text{(A.2)}$$

$$e^{\imath LM_n \sinh\theta_i^n} \prod_{k=1}^{N_b} S_{A,k}(\theta_i^n - \theta_k) \prod_{k=1,k\neq i}^{N_s} S_0(\theta_i^n - \theta_k^n) T_{AA}(\theta_i^n)\Psi = \Psi \qquad \text{(A.3)}$$

$$e^{\imath LM_n \sinh\theta_i^n} \prod_{k=1}^{N_b} S_{\overline{A},k}(\theta_i^n - \theta_k) \prod_{k=1,k\neq i}^{N_s} S_0(\theta_i^n - \theta_k^n) T_{\overline{AA}}(\theta_i^n)\Psi = \Psi\,, \qquad \text{(A.4)}$$

where $\theta_i^1, \theta_i^2, ..\theta_i^{n-1}$ are rapidities of breathers and θ_i^n the rapidities of solitons or antisolitons. In particular, the notation is such that a rapidity with lower index k is intended to be the rapidity of a particle in a sequence 1 to N_b or 1 to N_s. The upper index always labels the particle type. The significant difference between this and the diagonal case discussed earlier is that the wavefunctions Ψ of non-diagonal scattering are more complicated objects, being labelled by a set of N_s colour indices $\{\alpha_1, .., \alpha_{N_s}\}$ where each α_i is a soliton or antisoliton label A or $\overline{A}$ (in contrast to the scalar wavefunctions of the diagonal scattering). The number of configurations is therefore 2^{N_s}. T_{AA} and $T_{\overline{AA}}$ are elements of the transfer matrix

$$T(\theta) = \begin{bmatrix} T_{AA}(\theta) & T_{A\overline{A}}(\theta) \\ T_{\overline{A}A}(\theta) & T_{\overline{AA}}(\theta) \end{bmatrix}. \qquad \text{(A.5)}$$

Each element $T_{ab}(\theta)$ is itself a matrix of amplitudes for the process where particle a of rapidity θ passes N_s other solitonic particles of rapidity $\theta_1, .., \theta_{N_s}$ and returns as particle b of the same rapidity. This does not leave the other particles on the ring unaffected. If the initial configuration consists of the particle a and a combination of N_s solitons or antisolitons labelled $\{c_1, c_2, .., c_{N_s}\}$, then the final configuration is particle b together with another combination of N_s solitons or antisolitons labelled $\{d_1, d_2, .., d_{N_s}\}$ where c_i and d_j can only take two values A or $\overline{A}$. Therefore, the entries of $T_{ab}(\theta)$ are

$$T_{ab}(\theta|\theta_1^n, \ldots, \theta_{N_s}^n)_{\{c_j\}}^{\{d_j\}} = \sum_{k_1,..k_{N_s}} \hat{R}_{a,c_1}^{d_1,k_1}(\theta - \theta_1^n)\hat{R}_{k_1,c_2}^{d_2,k_2}(\theta - \theta_2^n)\ldots\hat{R}_{k_{N_s-1},c_{N_s}}^{d_{N_s},b}(\theta - \theta_{N_s}^n) \quad \text{(A.6)}$$

where each matrix $\hat{R}_{ij}^{kl}(\theta)$ is the 4×4 non-diagonal part of the soliton scattering matrix (2.22). (Note that the scalar part of the solitonic S-matrix for these interactions has already been taken care of above through the products of S_0 factors.) Since the labels a, b and all the k_i are restricted in this sum we note that each $T_{ab}(\theta)$ is a $2^{N_s} \times 2^{N_s}$ matrix. The action of $T_{ab}(\theta)$ on a wavefunction Ψ is

$$(T_{ab}(\theta)\Psi)^{\alpha_1,..\alpha_{N_s}} = \sum_{\alpha_1',..\alpha_{N_s}'} T_{ab}(\theta|\theta_1^n, \ldots, \theta_{N_s}^n)_{\alpha_1',..\alpha_{N_s}'}^{\alpha_1,..\alpha_{N_s}} \Psi^{\alpha_1',..\alpha_{N_s}'}. \qquad \text{(A.7)}$$

Note that $S_{A,k}(\theta) = S_{\overline{A},k}(\theta) \equiv S_{n,k}(\theta)$ so (A.3) and (A.4) can be combined. Our aim is to diagonalise the resulting equation. The Yang-Baxter factorisation equations ensure that

$(T_{AA}+T_{\overline{AA}})(\theta)$ commute with different values of θ. The set of these matrices can therefore be diagonalised for all θ by a set of θ independent eigenvectors Ψ_l ($l = 1, 2, ..2^{N_s}$), with θ dependent eigenvalues.

$$Tr\,T(\theta|\{\theta_k^n\})\Psi_l = \lambda_l(\theta|\{\theta_k^n\})\Psi_l \tag{A.8}$$

where $Tr\,T = T_{AA} + T_{\overline{AA}}$ is the trace of $T(\theta)$.

Now consider $\Omega = |\overline{A}(\theta_1^n), ..., \overline{A}(\theta_{N_s}^n)\rangle$. This is an eigenstate of both $T_{AA}(\theta)$ and $T_{\overline{AA}}(\theta)$, because by (2.22) and (2.22) we have

$$T_{AA}(\theta)\Omega = \prod_{i=1}^{N_s} b(\theta - \theta_i^n)\Omega\,, \tag{A.9}$$

$$T_{\overline{AA}}(\theta)\Omega = \prod_{i=1}^{N_s} a(\theta - \theta_i^n)\Omega\,. \tag{A.10}$$

The first equation comes about as follows. As we pass soliton A through the first anti-soliton we pick up a phase $b(\theta - \theta_1^n)$ for the process $A(\theta)\overline{A}(\theta_1^n) \longrightarrow \overline{A}(\theta_1^n)A(\theta)$. The next encounter is $A(\theta)\overline{A}(\theta_2^n) \longrightarrow \overline{A}(\theta_2^n)A(\theta)$, which picks up another factor, now of $b(\theta - \theta_2^n)$. We continue until we end up with the original rapidity configuration. The second equation arises in the same way.

To proceed we make the following *algebraic Bethe ansatz*: that all eigenstates of the trace above can be written as

$$\Psi = \prod_{j=1}^{r} T_{\overline{A}A}(y_j)\Omega \tag{A.11}$$

for some y_j and r. Since $T_{\overline{A}A}$ is the matrix for the process where antisoliton $\overline{A}$ passes through all other particles and ends up as soliton A, we see Ψ_l has r solitons and $N_s - r$ antisolitons. The Yang-Baxter equations yield [56]

$$R_{ac}^{a'c'}(\theta - y_j)T_{a'b}(\theta)T_{c'd}(y_j) = T_{cd'}(y_j)T_{ab'}(\theta)R_{b'd'}^{db}(\theta - y_j) \tag{A.12}$$

and these give the commutation relations for the transfer matrices

$$T_{AA}(\theta)T_{\overline{A}A}(y_j) = \frac{a(\theta - y_j)}{b(\theta - y_j)}T_{\overline{A}A}(y_j)T_{AA}(\theta) - \frac{c(\theta - y_j)}{b(\theta - y_j)}T_{\overline{A}A}(\theta)T_{AA}(y_j) \tag{A.13}$$

$$T_{\overline{AA}}(\theta)T_{\overline{A}A}(y_j) = -\frac{a(\theta - y_j)}{b(\theta - y_j)}T_{\overline{A}A}(y_j)T_{\overline{AA}}(\theta) + \frac{c(\theta - y_j)}{b(\theta - y_j)}T_{\overline{A}A}(\theta)T_{\overline{AA}}(y_j) \tag{A.14}$$

provided $b(\theta - y_j) \neq 0$. The result is that the action of the trace of the transfer matrix on the state Ψ is

$$\begin{aligned}[T_{AA}(\theta) + T_{\overline{AA}}(\theta)]\,\Psi &= \prod_{j=1}^{r}\frac{a(\theta - y_j)}{b(\theta - y_j)}\left(\prod_{k=1}^{N_s} b(\theta - \theta_k^n) + (-1)^r\prod_{k=1}^{N_s} a(\theta - \theta_k^n)\right)\Psi \\ &+ \quad \text{terms proportional to } [T_{AA}(y_j) + (-1)^r T_{\overline{AA}}(y_j)]\,.\end{aligned} \tag{A.15}$$

We find that the trace is diagonalised by Ψ only if

$$\prod_{k=1}^{N_s} \frac{b(y_j - \theta_k^n)}{a(y_j - \theta_k^n)} = (-1)^{r+1}. \tag{A.16}$$

Obtaining $a(\theta)$ and $b(\theta)$ from equation (2.22) and using the identity $\frac{1}{\xi} = \frac{h}{2}$, this condition becomes

$$\prod_{k=1}^{N_s} \frac{\sinh\left(\frac{h}{2}(\theta_k^n - y_j)\right)}{\sinh\left(\frac{h}{2}(y_j - \theta_k^n) - \frac{\imath\pi}{2}\right)} = (-1)^{r+1}. \tag{A.17}$$

When satisfied for some y_j and r, the corresponding eigenvalue from (A.15) looks like

$$\lambda(\theta|\{\theta_k^n\},\{y_j\}) = \Lambda \prod_{j=1}^{r} \sinh\left(\frac{h}{2}(\theta - y_j - i\pi)\right) \prod_{r+1}^{N_s} \sinh\left(\frac{h}{2}(y_j - \theta)\right), \tag{A.18}$$

where

$$\Lambda = \frac{\left[\prod_{k=1}^{N_s} \sinh\left(\frac{h}{2}(\theta_k^n - \theta)\right) + (-1)^r \prod_{k=1}^{N_s} \sinh\left(\frac{h}{2}(\theta - \theta_k^n - \imath\pi)\right)\right]}{\prod_{j=1}^{N_s} \sinh\left(\frac{h}{2}(y_j - \theta)\right)}. \tag{A.19}$$

After diagonalisation of the transfer matrix equations, (A.1)-(A.4) become

$$e^{\imath L M_1 \sinh\theta_i^1} \prod_{k=1}^{N_b} S_{1,k}(\theta_i^1 - \theta_k) \prod_{k=1}^{N_s} S_{1,n}(\theta_i^1 - \theta_k^n) = 1 \tag{A.20}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$e^{\imath L M_{n-1} \sinh\theta_i^{n-1}} \prod_{k=1}^{N_b} S_{n-1,k}(\theta_i^{n-1} - \theta_k) \prod_{k=1}^{N_s} S_{n-1,n}(\theta_i^{n-1} - \theta_k^n) = 1 \tag{A.21}$$

$$e^{\imath L M_n \sinh\theta_i^n} \prod_{k=1}^{N_b} S_{n,k}(\theta_i^n - \theta_k^n) \prod_{k=1}^{N_s} S_0(\theta_i^n - \theta_k^n)\ \lambda(\theta_i^n|\{\theta_k^n\},\{y_j\}) = 1. \tag{A.22}$$

We can now let the number of particles N_b and N_s increase with the circumference L and so examine the thermodynamics of this theory.

Diagonalisation is only possible when (A.17) is satisfied. The solutions to this equation are of the form

$$y_j = y_j^{(0)} + i\pi/2h, \qquad y_j = y_j^{(2)} - i\pi/2h \tag{A.23}$$

where $y_j^{(0)}, y_j^{(2)}$ are real, with the upper index chosen for convenience later. The real parts of these solutions may themselves be viewed as rapidities, because although they do not correspond directly to real particles (they have no associated energy) we may still talk about their statistics and densities exactly as we do for real rapidities. The pseudoparticles to which they would correspond are called magnons. In the limit where the number of particles becomes large, equation (A.17) relates the level density of the magnons ($\rho_n^{(0)}(\theta)$ and $\rho_n^{(2)}(\theta)$) to the soliton root density $\rho_{r,n}(\theta)$:

$$\int_{-\infty}^{\infty} \frac{h\rho_{r,n}(\theta')}{\cosh(h(\theta - \theta'))} dz = (\phi_1 * \rho_{r,n})(\theta) = 2\pi\rho_n^{(\alpha)}(\theta) \qquad \alpha = 0, 2. \tag{A.24}$$

where $\phi_1(\theta)$ is the kernel

$$\phi_1(\theta) = \frac{h}{\cosh(h\theta)}\,. \tag{A.25}$$

Since the magnonic particles are only related to the solitons, we attach the soliton index to their rapidities. This equation is derived in the same manner as before. We begin by taking the logarithm of the system (A.17) and equating the imaginary parts. In the large N_s limit we can first replace the sum over soliton rapidities θ_k^n by an integral which introduces the root density for solitons, and then, by considering two consecutive equations in the large particle limit, we obtain the level density of the magnons.

As for the relationships between breather and soliton densities, these are found directly from (A.20-A.22) in much the same way. Again we look at the imaginary part of the equations after taking logarithms. Then each sum over index k can be replaced by an integral over either breather or soliton rapidities. In the first $n-1$ equations, as mentioned above, k runs over all varieties of breather types, so in these equations we have a sum of root densities (one for each breather type). A soliton root density appears from the other sum in these equations. In the large N_b, N_s, L limit these are related to the level density of the breather in question. The soliton system (A.22) gives a similar integral equation in this limit, only now the $\lambda(\theta_i^n|\{\theta_k^n\},\{y_j\})$ gives rise to the root densities of the magnonic particles $\rho_{r,n}^{(0)}(\theta)$ and $\rho_{r,n}^{(2)}(\theta)$. There is also a contribution from the scalar piece of the soliton-antisoliton S-matrix S_0. Hence the integral equations relating magnon, soliton and breather root and level densities, together with (A.24), are

$$\rho_j(\theta) = \frac{M_j}{2\pi}\cosh\theta + \sum_{m=1}^{n-1}(\phi_{jm} * \rho_{r,m})(\theta) + (\phi_{nj} * \rho_{r,n})(\theta) \qquad j = 1,..n-1 \tag{A.26}$$

$$\begin{aligned}\rho_n(\theta) &= \frac{M_n}{2\pi}\cosh\theta + \sum_{k=1}^{n-1}(\phi_{nk} * \rho_{r,k})(\theta) + (\phi_0 * \rho_{r,n})(\theta) \\ &\quad + \frac{1}{2}\sum_{\alpha=0,2}\left(\phi_1 * (\rho_{r,n}^{(\alpha)} - \rho_{h,n}^{(\alpha)})\right)(\theta)\,.\end{aligned} \tag{A.27}$$

The kernels

$$\phi_{ij}(\theta) = \frac{d}{d\theta} Im \log S_{ij}(\theta) \qquad i,j = 1,..n-1 \tag{A.28}$$

$$\phi_j(\theta) = \frac{d}{d\theta} Im \log S_{nj}(\theta) \qquad j = 1,..n-1 \tag{A.29}$$

$$\phi_0(\theta) = \frac{d}{d\theta} Im \log S_0(\theta) \tag{A.30}$$

can be found explicitly since $S_{ij}(\theta), S_{nj}(\theta), S_0(\theta)$ are those stated in Section (2.2). The hole and root densities for magnonic particles which appear in the final term in (A.27)

originate in the limit of equation (A.18). When we take the logarithm of the equations we would have $2\pi n_i$ on the right hand side not n_j, so *level* densities $\rho_n^{(0)}$ and $\rho_n^{(2)}$ do not arise naturally in the usual way. Therefore, we can simply define the densities $\rho_{r,n}^{(0)}$ and $\rho_{h,n}^{(0)}$ of y_j from the two terms in (A.18) to be the root and hole densities respectively. Similarly, for $\rho_{r,n}^{(2)}$ and $\rho_{h,n}^{(2)}$. We define $\rho_n^{(0)} = \rho_{r,n}^{(0)} + \rho_{h,n}^{(0)}$ and $\rho_n^{(2)} = \rho_{r,n}^{(2)} + \rho_{h,n}^{(2)}$.

With these relations in place the TBA equations can be determined. The free energy for this system may be written

$$Rf(\rho, \rho_r, \rho_h) = R\sum_{i=1}^{n} H(\rho_{r,i}) - \left(\sum_{i=1}^{n} \mathcal{S}(\rho_i, \rho_{r,i}) + \mathcal{S}(\rho_n^{(0)}, \rho_{r,n}^{(0)}) + \mathcal{S}(\rho_n^{(2)}, \rho_{r,n}^{(2)})\right), \quad \text{(A.31)}$$

where energy and entropy terms are defined

$$H(\rho_{r,i}) = \int d\theta \rho_{r,i} M_i \cosh\theta\,, \quad \text{(A.32)}$$

$$\mathcal{S}(\rho, \rho_r) = \int d\theta \left[\rho \log\rho - (\rho - \rho_r)\log(\rho - \rho_r) - \rho_r \log\rho_r\right]. \quad \text{(A.33)}$$

Here we have omitted the θ dependence of the rapidities in order to simplify things a little. Note that there is no energy term for the magnonic particles. The system is said to be in equilibrium if all derivatives of the free energy with respect to root densities (which represent actual configurations) vanish:

$$\frac{\delta f}{\delta \rho_{r,j}} = 0 \qquad j = 1, ..n-1 \quad \text{(A.34)}$$

$$\frac{\delta f}{\delta \rho_{r,n}} = 0 \quad \text{(A.35)}$$

$$\frac{\delta f}{\delta \rho_{r,n}^{(0)}} = \frac{\delta f}{\delta \rho_{r,n}^{(2)}} = 0\,. \quad \text{(A.36)}$$

Introducing the pseudoenergies

$$\frac{\rho_{r,j}(\theta)}{\rho_j(\theta)} = \frac{1}{1 + e^{\varepsilon_j(\theta)}}, \qquad \frac{\rho_{r,n}^{(i)}(\theta)}{\rho_n^{(i)}(\theta)} = \frac{1}{1 + e^{\varepsilon_n^{(i)}(\theta)}}, \quad \text{(A.37)}$$

conditions (A.34-A.36) become

$$\begin{aligned}
RM_j \cosh\theta &= \varepsilon_j(\theta) + \sum_{k=1}^{n-1} (\phi_{jk} * \log(1 + e^{-\varepsilon_k}))(\theta) + (\phi_{nj} * \log(1 + e^{-\varepsilon_n}))(\theta) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n-1 \\
RM_n \cosh\theta &= \varepsilon_n(\theta) + \sum_{k=1}^{n-1} (\phi_{nk} * \log(1 + e^{-\varepsilon_k}))(\theta) + (\tilde{\phi}_0 * \log(1 + e^{-\varepsilon_n}))(\theta) \\
&\quad + \sum_{\alpha=0,2} (\phi_1 * \log(1 + e^{-\varepsilon_n^{(\alpha)}}))(\theta) \qquad \text{(A.38)} \\
0 &= \varepsilon_n^{(\alpha)}(\theta) + (\phi_1 * \log(1 + e^{-\varepsilon_n}))(\theta) \qquad\qquad \alpha = 0, 2
\end{aligned}$$

which are the TBA equations for the sine-Gordon b_n-related theories. The kernel $\tilde{\phi}_0$ is given by $\tilde{\phi}_0 = \phi_0 - \phi_1 * \phi_1$ and is labelled ϕ_{nn} in equation (4.13).

The Y-system corresponding to this set of TBA equations is obtained through the following steps, generalising the derivations of [39]. First write down the TBA equations for particles $j = 1, ..n-1$ evaluated with arguments $\theta + i\pi/h$ and $\theta - i\pi/h$. The same should be done for the mass M_n but with the arguments $\theta + i\pi/2h$ and $\theta - i\pi/2h$. Any magnonic equations are also shifted by $\pm i\pi/2h$. Each shifted pair of pseudoenergies is then summed so that the combination of energy terms becomes $2\cos(\pi/h)\nu_j(\theta)$ or $2\cos(\pi/2h)\nu_j(\theta)$. h appears in the mass spectrum identities

$$\begin{aligned} M_2 &= 2\cos(\pi/h)M_1 \\ M_{i-1} + M_{i+1} &= 2\cos(\pi/h)M_i \qquad i = 2, ..n-2 \\ M_{n-2} + 2\cos(\pi/2h)M_n &= 2\cos(\pi/h)M_{n-1} \\ M_{n-1} &= 2\cos(\pi/2h)M_n\,. \end{aligned} \tag{A.39}$$

The next step is to write down the following set of n equations. The first is simply the TBA equation involving mass M_1. The next $n-3$ are sums of TBA equations associated with masses M_{i-1} and M_{i+1} for $i = 2, ..n-2$. The next is the sum of the TBA equations for M_{n-2} and $2\cos(\pi/2h)$ times that for M_n. The final equation is the TBA equation for mass M_{n-1}. By subtracting this system from that obtained via the above shifts one should find that all energy terms when collected together vanish, leaving only pseudoenergies and L-functions. After using the appropriate kernel identities to simplify the result, taking exponentials of all equations gives a set of Y-systems on the substitution $Y_k^{(\alpha)}(\theta) = e^{\varepsilon_k^{(\alpha)}}$. The energy independent system becomes

$$\begin{aligned} Y_i^{(1)}(\theta + \imath\frac{\pi}{h})Y_i^{(1)}(\theta - \imath\frac{\pi}{h}) &= (1 + Y_{i-1}^{(1)}(\theta))(1 + Y_{i+1}^{(1)}(\theta)) \qquad i = 1, \ldots n-2 \\ Y_{n-1}^{(1)}(\theta + \imath\frac{\pi}{h})Y_{n-1}^{(1)}(\theta - \imath\frac{\pi}{h}) &= (1 + Y_{n-2}^{(1)}(\theta))(1 + Y_n^{(0)}(\theta))(1 + Y_n^{(2)}(\theta)) \\ &\quad \times\Big(1 + Y_n^{(1)}(\theta + \imath\frac{\pi}{2h})\Big)\Big(1 + Y_n^{(1)}(\theta - \imath\frac{\pi}{2h})\Big) \\ Y_n^{(1)}(\theta + \imath\frac{\pi}{2h})Y_n^{(1)}(\theta - \imath\frac{\pi}{2h}) &= (1 + Y_{n-1}^{(1)}(\theta))(1 + Y_n^{(0)}(\theta)^{-1})^{-1}(1 + Y_n^{(2)}(\theta)^{-1})^{-1} \\ Y_n^{(\alpha)}(\theta + \imath\frac{\pi}{2h})Y_n^{(\alpha)}(\theta - \imath\frac{\pi}{2h}) &= (1 + Y_n^{(1)}(\theta)^{-1})^{-1} \qquad \alpha = 0, 2\,. \end{aligned} \tag{A.40}$$

At this point it is convenient to introduce new functions

$$\begin{aligned} \overline{Y}_i^{(\alpha)}[\theta, r] &= Y_i^{(\alpha)}(\theta - \tfrac{\imath\pi\xi r}{2})\, Y_i^{(\alpha)}(\theta + \tfrac{\imath\pi\xi r}{2}) \\ \overline{Y}_i^{(\alpha)}\{\theta, r\} &= \Big(1 + Y_i^{(\alpha)}(\theta - \tfrac{\imath\pi\xi r}{2})\Big)\Big(1 + Y_i^{(\alpha)}(\theta + \tfrac{\imath\pi\xi r}{2})\Big) \\ V_i^{(\alpha)}\{\theta, r\} &= \Big(1 + Y_i^{(\alpha)}(\theta - \tfrac{\imath\pi\xi r}{2})^{-1}\Big)^{-1}\Big(1 + Y_i^{(\alpha)}(\theta + \tfrac{\imath\pi\xi r}{2})^{-1}\Big)^{-1} \end{aligned}$$

to express the Y-systems in a more compact form. With this notation the b_n-related Y-system can be expressed

$$\begin{aligned}
\overline{Y}_i^{(1)}[\theta,1] &= \prod_{j=1}^{n-1}\left(1+Y_j^{(1)}(\theta)\right)^{l_{ij}^{a_{n-1}}} \qquad i=1...n-2 \\
\overline{Y}_{n-1}^{(1)}[\theta,1] &= \left(1+Y_{n-2}^{(1)}(\theta)\right)\overline{Y}_n^{(1)}\{\theta,1/2\}\left(1+Y_n^{(0)}(\theta)\right)\left(1+Y_n^{(2)}(\theta)\right) \\
\overline{Y}_n^{(1)}[\theta,1/2] &= \left(1+Y_{n-1}^{(1)}(\theta)\right)\left(1+Y_n^{(0)}(\theta)^{-1}\right)^{-1}\left(1+Y_n^{(2)}(\theta)^{-1}\right)^{-1} \\
\overline{Y}_n^{(\alpha)}[\theta,1/2] &= \left(1+Y_n^{(1)}(\theta)^{-1}\right)^{-1} \qquad \alpha=0,2
\end{aligned} \tag{A.41}$$

where $l_{ij}^{a_{n-1}}$ is entry (i,j) of the incidence matrix for the a_{n-1} Dynkin diagram.

A.2 sine-Gordon Y-systems

Tateo [40] has given a relatively simple set of rules which enable us to write down the Y-systems for a general sine-Gordon model.

1. Define the parameter

$$\xi = (\beta^2/8\pi)/(1-\beta^2/8\pi), \tag{A.42}$$

where β^2 specifies any sine-Gordon model so that ξ is rational and may be written as the continued fraction

$$\xi = \hat{\xi}(n_1,n_2,..n_F) \equiv \cfrac{1}{n_1+\cfrac{1}{n_2...+\cfrac{1}{\cdots\cfrac{1}{n_F}}}} \tag{A.43}$$

For this theory there is one soliton, n_1 bound states and $\sum_2^F n_i$ magnonic pseudoparticles. An exceptional case is the pure d_{n+1} theory where there are two solitons. Each theory corresponds to a diagram like that shown in Figure (A.1).

2. Define the shifts

$$s_1 = i\pi\frac{\xi_1}{2}, \quad s_2=\xi_2 s_1, \quad s_3=\xi_3\xi_2 s_1,.. \qquad s_F=\xi_F\xi_{F-1}\ldots\xi_2 s_1 \tag{A.44}$$

where $\xi_i = \hat{\xi}(n_i,n_{i+1},...,n_F)$. Also introduce the shift label a where $a=1$ for nodes $\{1,2,..,n_1\}$, $a=2$ for nodes $\{n_1+1,n_1+2,..,n_1+n_2\}$, and so on, so that the shift associated with node i is simply

$$\mathcal{S}_i = s_a. \tag{A.45}$$

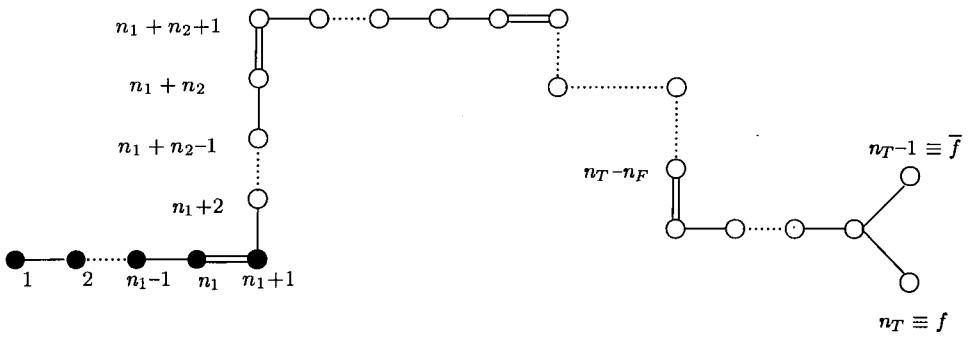


Figure A.1: The Y-system graph associated to the sine-Gordon model for rational ξ. The first n_1 black nodes represent breathers , node n_1+1 corresponds to the soliton and all remaining nodes are magnonic.

3. Define the link exponent

$$c_{i,j} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are neighbours in the same horizontal chain} \\ -1 & \text{if } i \text{ and } j \text{ are neighbours in the same vertical chain} \\ 0 & \text{for non-adjacent nodes} \end{cases} \tag{A.46}$$

4. With this notation in place the Y-system equation for any of the nodes $k \in \{n_1, n_1+n_2, .., n_T-n_F-n_{F-1}\}$ is

$$\begin{aligned} Y_k(\theta+\mathcal{S}_k)\ Y_k(\theta-\mathcal{S}_k) \ &= \ (1+Y_{k-1}(\theta)^{c_{k,k-1}})^{c_{k,k-1}}\ (1+Y_{k+n_{a+1}+1}(\theta)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\prod_{j=k+1}^{k+n_{a+1}} (1+Y_j(\theta+(k+n_{a+1}-j)\mathcal{S}_j+\mathcal{S}_{k+n_{a+1}+1})^{\tilde{c}_k})^{\tilde{c}_k} \\ &\prod_{j=k+1}^{k+n_{a+1}} (1+Y_j(\theta-(k+n_{a+1}-j)\mathcal{S}_j-\mathcal{S}_{k+n_{a+1}+1})^{\tilde{c}_k})^{\tilde{c}_k} \ , \end{aligned} \tag{A.47}$$

with $\tilde{c}_j = c_{j,j+1}$. Node (n_T-n_F) has equation

$$\begin{aligned} Y_k(\theta+\mathcal{S}_k)\ Y_k(\theta-\mathcal{S}_k) \ &= \ (1+Y_{k-1}(\theta)^{c_{k,k-1}})^{c_{k,k-1}}(1+Y_f(\theta)^{\tilde{c}_k})^{\tilde{c}_k}\ (1+Y_{\overline{f}}(\theta)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\prod_{j=k+1}^{n_T-2} (1+Y_j(\theta+(n_T-1-j)\mathcal{S}_j)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\prod_{j=k+1}^{n_T-2} (1+Y_j(\theta-(n_T-1-j)\mathcal{S}_j)^{\tilde{c}_k})^{\tilde{c}_k} \ , \end{aligned} \tag{A.48}$$

where f and $\overline{f}$ label the final magnonic nodes. All other nodes have associated equations

$$Y_i(\theta+\mathcal{S}_i)\ Y_i(\theta-\mathcal{S}_i) \ = \ \prod_{j\in adj(i)} (1+Y_j^{c_{j,i}}(\theta))^{c_{j,i}}\,, \tag{A.49}$$

where the product in (A.49) is intended over adjacent nodes i.

We can now list the Y-systems for the TBA of the sine-Gordon models with $\beta^2 = 32\pi/N$ ($N \geq 5$). Each of the systems corresponds to one of the diagrams in Figure (A.2).

1. N=4n+1

$$\xi = \frac{1}{(n-1)+\frac{1}{4}}, \qquad s_1 = \frac{i\pi\xi}{2}, \quad s_2 = (\frac{1}{4})s_1 \tag{A.50}$$

tells us there are $(n-1)$ breathers, four magnons and a soliton (see Figure (A.2)). With a more convenient labelling for the work of the main text, the Y-systems are

$$Y_i^{(1)}(\theta+s_1)Y_i^{(1)}(\theta-s_1) \ = \ \prod_j (1+Y_j^{(1)}(\theta))^{I_{ij}^{[a_{n-1}]}} \qquad i=1\ldots n-2$$

$$\begin{aligned} Y_{n-1}^{(1)}(\theta+s_1)Y_{n-1}^{(1)}(\theta-s_1) \ = \ & (1+Y_{n-2}^{(1)}(\theta)) \\ & \times(1+Y_n^{(1)}(\theta+3s_2))(1+Y_n^{(1)}(\theta-3s_2)) \\ & \times(1+Y_n^{(2)}(\theta+2s_2))(1+Y_n^{(2)}(\theta-2s_2)) \\ & \times(1+Y_n^{(3)}(\theta+s_2)(1+Y_n^{(3)}(\theta-s_2) \\ & \times(1+Y_n^{(4)}(\theta))(1+Y_n^{(5)}(\theta)) \end{aligned}$$

$$Y_n^{(1)}(\theta+s_2)Y_n^{(1)}(\theta-s_2) \ = \ (1+Y_{n-1}^{(1)}(\theta))(1+Y_n^{(2)}(\theta)^{-1})^{-1}$$

$$Y_n^{(\alpha)}(\theta+s_1)Y_n^{(\alpha)}(\theta-s_1) \ = \ \prod_\beta (1+Y_n^{(\alpha)}(\theta)^{-1})^{-I_{\alpha\beta}^{[e_6]}} \qquad (\alpha=2,..5)\,.$$

These can be simplified further through the definitions (4.20) introduced earlier to obtain the more compact form

$$\overline{Y}_i^{(1)}[\theta,1] \ = \ \prod_{j=1}^{n-1} \left(1+Y_j^{(1)}(\theta)\right)^{I_{ij}^{[a_{n-1}]}} \qquad i=1,\ldots n-2 \tag{A.51}$$

$$\begin{aligned} \overline{Y}_{n-1}^{(1)}[\theta,1] \ = \ & \left(1+Y_{n-2}^{(1)}(\theta)\right)\left(1+Y_n^{(4)}(\theta)\right)\left(1+Y_n^{(5)}(\theta)\right) \\ & \times \overline{Y}_n^{(1)}\{\theta,3/4\}\overline{Y}_n^{(2)}\{\theta,2/4\}\overline{Y}_n^{(3)}\{\theta,1/4\} \end{aligned}$$

$$\overline{Y}_n^{(1)}[\theta,1/4] = \Big(1+Y_{n-1}^{(1)}(\theta)\Big)\Big(1+Y_n^{(2)}(\theta)^{-1}\Big)^{-1}$$

$$\overline{Y}_n^{(\alpha)}[\theta,1/4] = \prod_\beta \Big(1+Y_n^{(\beta)}(\theta)^{-1}\Big)^{-I_{\alpha\beta}^{[d_5]}} \qquad \alpha=2,\ldots 5$$

2. N=4n+2

$$\xi = \frac{1}{(n-1)+\frac{1}{2}}, \qquad s_1 = \frac{i\pi\xi}{2}, \quad s_2 = (\frac{1}{2})s_1 \tag{A.52}$$

tells us there are $(n-1)$ breathers and two magnons, as well as a soliton. The compact form of the Y-system consists of equations (A.51) for $Y_i^{(1)}(\theta)$ $(i=1,..n-2)$, and

$$\begin{aligned}\overline{Y}_{n-1}^{(1)}[\theta,1] &= \Big(1+Y_{n-2}^{(1)}(\theta)\Big)\Big(1+Y_n^{(0)}(\theta)\Big)\Big(1+Y_n^{(2)}(\theta)\Big)\\ &\quad\times\overline{Y}_n^{(1)}\{\theta,1/2\}\end{aligned}$$

$$\begin{aligned}\overline{Y}_n^{(1)}[\theta,1/2] &= \Big(1+Y_{n-1}^{(1)}(\theta)\Big)\Big(1+Y_n^{(0)}(\theta)^{-1}\Big)^{-1}\Big(1+Y_n^{(2)}(\theta)^{-1}\Big)^{-1}\\ \overline{Y}_n^{(\alpha)}[\theta,1/2] &= \prod_\beta \Big(1+Y_n^{(\beta)}(\theta)^{-1}\Big)^{-I_{\alpha\beta}^{[a_3]}} \qquad \alpha=0,2.\end{aligned}$$

This is exactly the Y-system (A.41) we derived earlier, which suggests the above prescription is correct.

3. N=4n+3

$$\xi = \frac{1}{(n-1)+\frac{1}{1+\frac{1}{3}}}, \qquad s_1 = \frac{i\pi\xi}{2}, \quad s_2 = (\frac{3}{4})s_1, \quad s_3 = (\frac{1}{4})s_1 \tag{A.53}$$

tells us there should be $(n-1)$ breathers and four magnons, as well as a soliton. Now the compact form of the Y-system consists of equations (A.51) for $Y_i^{(1)}(\theta)$ $(i=1,..n-2)$, and

$$\begin{aligned}\overline{Y}_{n-1}^{(1)}[\theta,1] &= \Big(1+Y_{n-2}^{(1)}(\theta)\Big)\Big(1+Y_n^{(2)}(\theta)\Big)\,\overline{Y}_n^{(1)}\{\theta,1/4\}\\ \overline{Y}_n^{(1)}[\theta,3/4] &= \Big(1+Y_{n-1}^{(1)}(\theta)\Big)\Big(1+Y_n^{(4)}(\theta)^{-1}\Big)^{-1}\Big(1+Y_n^{(5)}(\theta)^{-1}\Big)^{-1}\\ &\quad\times\, V_n^{(2)}\{\theta,2/4\}\,V_n^{(3)}\{\theta,1/4\}\\ \overline{Y}_n^{(2)}[\theta,1/4] &= \Big(1+Y_n^{(3)}(\theta)\Big)\Big(1+Y_n^{(1)}(\theta)^{-1}\Big)^{-1}\\ \overline{Y}_n^{(3)}[\theta,1/4] &= \Big(1+Y_n^{(4)}(\theta)\Big)\Big(1+Y_n^{(2)}(\theta)\Big)\Big(1+Y_n^{(5)}(\theta)\Big)\\ \overline{Y}_n^{(4)}[\theta,1/4] &= \Big(1+Y_n^{(3)}(\theta)\Big)\\ \overline{Y}_n^{(5)}[\theta,1/4] &= \Big(1+Y_n^{(3)}(\theta)\Big),\end{aligned}$$

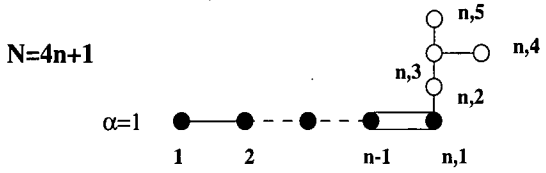
4. N=4n+4

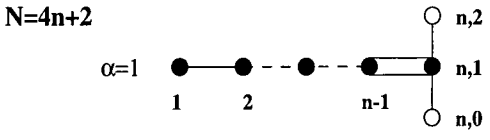
$$\xi = \frac{1}{(n-1)+1}, \qquad s_1 = \frac{i\pi\xi}{2}. \tag{A.54}$$

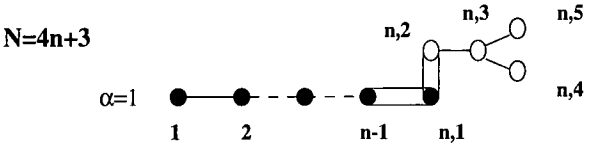
Here there are no magnonic particles, only $(n-1)$ breathers and two solitons. The Y-system is simply

$$\overline{Y}_i^{(1)}[\theta, 1] = \prod_{j=1}^{n+1} \left(1 + Y_j^{(1)}(\theta)\right)^{l_{ij}^{[d_{n+1}]}} \qquad i = 1, ..n+1 \tag{A.55}$$

The symmetric version of these equations, which represent the perturbed theories $Z_N^{(\pm)}$, are stated in Section (4.2).







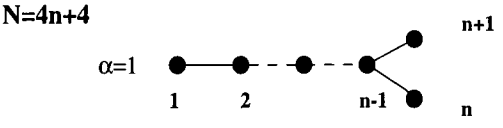


Figure A.2: The sine-Gordon Y-systems for $\beta^2 = 32\pi/N$ with $N = 4n+1, 4n+2, 4n+3, 4n+4$ respectively. Dark nodes denote energy terms $\nu_i^{(1)} = m_i r \cosh\theta$. Those with no shading are magnonic with zero associated energy.

Appendix B

Stationary Y-values for Dilogarithm Sum Rules

The stationary values of the functions at the remaining nodes for the sum rules stated in the Section (4.5) are given as follows, where $\eta = \pi/(N+2)$ and

$$\Upsilon_i^{(\alpha)} = \frac{\sin((i+3)\eta)\sin(i\eta)}{\sin((2\eta))\sin(\eta)} ,$$

$$\mathcal{X}_i^{(\tilde{1})} = (i+2)i ,$$

for $i = 1 \ldots n-1$.

1) $N = 4n+1$

$$\Upsilon_n^{(1)} = \Upsilon_n^{(6)} = \frac{\sin^2(n\eta)}{\sin((2n+2)\eta)\sin(2\eta)}$$

$$\Upsilon_n^{(2)} = \Upsilon_n^{(5)} = \frac{(\Upsilon_n^{(1)})^2}{1+\Upsilon_{n-1}^{(1)} - (\Upsilon_n^{(1)})^2}$$

$$\Upsilon_n^{(4)} = \frac{(\Upsilon_n^{(1)})^{3/2}(1+\Upsilon_n^{(1)})^{1/2}}{(1+\Upsilon_{n-1}^{(1)} - (\Upsilon_n^{(1)})^2)}$$

$$\Upsilon_n^{(3)} = \frac{(\Upsilon_n^{(1)})^3(1+\Upsilon_n^{(1)})}{1+2\Upsilon_{n-1}^{(1)} + (\Upsilon_{n-1}^{(1)})^2 - 2(\Upsilon_n^{(1)})^2 - 2\Upsilon_{n-1}^{(1)}(\Upsilon_n^{(1)})^2 - (\Upsilon_n^{(1)})^3}$$

$$\mathcal{X}_n^{(6)} = \frac{n^2}{2n+1}$$

$$\mathcal{X}_n^{(5)} = \frac{n^2}{(3n+1)(n+1)}$$

$$\mathcal{X}_n^{(4)} = \mathcal{X}_n^{(2)} = \frac{n}{3n+1}$$

$$\mathcal{X}_n^{(3)} = \frac{n^2}{8n^2+1+6n}$$

$$\mathcal{Z}_n^{(4)} = \mathcal{Z}_n^{(2)} = \mathcal{Z}_n^{(5)} = \frac{1}{3}, \quad \mathcal{Z}_n^{(3)} = \frac{1}{8}.$$

2) $N = 4n+2$

$$\Upsilon_n^{(1)} = \Upsilon_n^{(3)} = \frac{\sin^2(n\eta)}{\sin((2n+2)\eta)\sin(2\eta)}$$
$$\Upsilon_n^{(0)} = \Upsilon_n^{(4)} = \frac{\sin(n\eta)}{\sin((n+2)\eta)}$$
$$\Upsilon_n^{(2)} = \frac{\sin^2(n\eta)}{\sin^2((n+2)\eta)}$$

$$\mathcal{X}_n^{(3)} = \frac{n^2}{2n+1}, \qquad \mathcal{X}_n^{(2)} = \mathcal{X}_n^{(4)} = \frac{n}{n+1}, \qquad \mathcal{X}_n^{(0)} = 1$$
$$\mathcal{Z}_n^{(0)} = \mathcal{Z}_n^{(2)} = \mathcal{Z}_n^{(4)} = 1.$$

3) $N = 4n+3$

$$\Upsilon_n^{(1)} = \Upsilon_n^{(6)} = \frac{\sin^2(n\eta)}{\sin((2n+2)\eta)\sin(2\eta)}$$
$$\Upsilon_n^{(4)} = \frac{\sin((n+2)\eta)\sin((n+1)\eta) + \sin^2((2n+2)\eta)}{\sin^2((n+2)\eta)}$$
$$\Upsilon_n^{(3)} = (\Upsilon_n^{(4)})^2 - 1$$
$$\Upsilon_n^{(2)} = \Upsilon_n^{(5)} = \frac{(\Upsilon_n^{(4)})^2 - 1}{(1+\Upsilon_n^{(4)})^{1/2}} - 1$$

$$\mathcal{X}_n^{(6)} = \frac{n^2}{2n+1}$$
$$\mathcal{X}_n^{(5)} = \frac{n(3n+2)}{(n+1)^2}$$
$$\mathcal{X}_n^{(3)} = \frac{(4n+3)(2n+1)}{(n+1)^2}$$
$$\mathcal{X}_n^{(4)} = \mathcal{X}_n^{(2)} = \frac{3n+2}{n+1}$$

$$\mathcal{Z}_n^{(4)} = \mathcal{Z}_n^{(2)} = \mathcal{Z}_n^{(5)} = 3, \qquad \mathcal{Z}_n^{(3)} = 8.$$

4) $N = 4n+4$

$$\Upsilon_s^{(\alpha)} = \Upsilon_{\bar{s}}^{(\alpha)} = \frac{\sin(n\eta)}{2\sin(\eta)\cos((n+1)\eta)} \qquad (\alpha = 1, 2)\ ,$$

$$\mathcal{X}_s^{(2)} = \mathcal{X}_{\bar{s}}^{(2)} = n.$$

Appendix C

Regular Ultraviolet Expansion Coefficients

The following tables list the coefficients $\widetilde{F}_m$ and F_m obtained from fits of numerical solutions for the regular part of the small R expansion of $RE(R)$ for massive and massless directions respectively. The results are very important because they confirm that the massive and massless TBA equations give expansion coefficients which match those of perturbation theory with two different signs of the coupling.

$N=5$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
0	1.98×10^{-13}	-3.86×10^{-14}
1	-7.78×10^{-12}	3.23×10^{-12}
2	0.0112031704	0.0112031702
3	-3.37×10^{-09}	1.35×10^{-09}
4	0.00038484	0.00038477
5	0.019215	-0.019217
6	0.02056	0.02055
7	0.0004	-0.0006

$N=6$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
0	2.15×10^{-18}	4.10×10^{-17}
1	1.01×10^{-13}	1.22×10^{-13}
2	0.026663544239	0.026663544230
3	0.0202428666	− 0.0202428658
4	0.00088616	0.00088614
5	− 0.0061087	0.0061090
6	− 0.002800	− 0.002803
7	0.00033	− 0.00031

$N=7$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
0	3.98×10^{-15}	8.02×10^{-16}
1	2.42×10^{-14}	$- 5.84 \times 10^{-15}$
2	0.04302032556	0.04302032557
3	0.0128965993	− 0.0128965995
4	0.000972018	0.000972029
5	0.00120282	− 0.00120286
6	0.0000682	0.0000690
7	− 0.0000085	0.0000077

$N=8$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
2	0.06589308648	0.06589308648
3	− 0.02744520208	− 0.02744520208
4	0.00073623262	0.00073623264
5	0.00102222	− 0.00102216
6	− 0.00016842	− 0.00016851
7	− 0.00004770	0.00004710
8	− 0.0000268	− 0.0000226

$N=9$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
0	7.27×10^{-14}	2.89×10^{-13}
1	-4.21×10^{-12}	-1.97×10^{-11}
2	0.124757776	0.124757776
3	− 0.01159266	0.01159265
4	0.0004886	0.0004887
5	0.000574	− 0.000575
6	0.000088	0.000089
7	− 0.000028	0.000025

$N=10$:

m	$\widetilde{F}_m$ (massive)	F_m (massless)
0	1.17×10^{-14}	-2.94×10^{-14}
1	-8.58×10^{-13}	2.40×10^{-12}
2	0.0247185731	0.0247185730
3	−0.006211643	0.006211644
4	0.00031116	0.00031115
5	0.00030406	−0.00030402
6	0.0000402	0.0000401
7	−0.0000137	0.0000138

Appendix D

Numerical Routines

This section contains listings of the two Fortran routines used to obtain numerical solutions of the thermodynamic Bethe ansatz equations for the $\mathbb{Z}_5$ and $\mathbb{Z}_7$ flows developed in Section (4.1). All $\mathbb{Z}_N$ TBA equations were solved using similar programs.

```
C-------------------------------------------------------------------------C
C                      PROGRAM : Z5area.for                               C
C               THE FLOW FROM c=2(N-1)/(N+2) = 8/7   TO  c=1              C
C-------------------------------------------------------------------------C
      implicit real*8 (a-h,o-z)
      real*8 phi(-4500:4500),mass
      real*8 eps(0:10,-2020:2020),L(0:10,-2020:2020)
      integer A(0:10,0:10), bound
      real*8  s1(0:10),rcoexp(-2020:2020),coexp(-2020:2020)
C     real*8 co(-2020:2020),rco(-2020:2020)
      common r,f2,f3,f4,f5,f6
      data A /121*0.d00/
      external f1
      open(128,file="cr.m",status="unknown")
      write(128,*) '{'
10    Format('{',F15.9, ',' F120.14, '}')
20    Format('{',F15.9, ',' F50.14, '}')
      pi=dacos(-1.d00)
      dth=.1d00
      iter1=250
      iter2=200
      rmin=0.0001d00
      istep=40
```

```
      con = 0
C  istep is then the number of radial points
C  con is a count, after con = istep we'll put a } at end of file
      mass=1.d00
      imax=1000
      nnn=6
      bound=700
      do 1 i=-imax,imax
         phi(i)=(dth/(2.d00*pi))/dcosh(i*dth)
    1  continue
          r=rmin
       do 9 i=-imax-bound,imax+bound
       do 22 X=1,nnn
          L(X,i)=dlog(2.d00)
 22   continue
C           co(i)=dcosh(dth*i)     (for massive direction)
            coexp(i)=dexp(dth*i)/2.d00
    9   continue
            ijk=0
C-------------------------------------------------------------------------
       iter=iter1
       do 3 rrr=rmin,rmax,rstep
       do 3 cnt=1,istep
       con = con + 1
       if( con .ne. 1) then
       iter=iter2
       endif
       r=mass*rmin*1.4**con
C-------------------------------------------------------------------------
       do 7 i=-imax-bound,imax+bound
C        rco(i)=r*dcosh(dth*i)
         rcoexp(i)=r*dexp(dth*i)/2.d00
    7    continue
         do 5 n=1,iter
            do 6 i=-imax,imax
               s1(1)=0.d00
               s1(2)=0.d00
               s1(3)=0.d00
```

```
                s1(4)=0.d00
                s1(5)=0.d00
                s1(6)=0.d00
                do 14 j=i-bound,i+bound
C-------------------------------------------------------------------------
                 s1(1)=s1(1)+phi(i-j)*L(2,j)
                 s1(2)=s1(2)+phi(i-j)*(L(1,j)+L(3,j))
                 s1(3)=s1(3)+phi(i-j)*(L(2,j)+L(4,j)+L(5,j))
                 s1(4)=s1(4)+phi(i-j)*L(3,j)
                 s1(5)=s1(5)+phi(i-j)*(L(3,j)+L(6,j))
                 s1(6)=s1(6)+phi(i-j)*L(5,j)
C-------------------------------------------------------------------------
  14             continue
                   eps(1,i)=(rcoexp(i)-s1(1)+eps(1,i))/2.d00
                   eps(2,i)=(-s1(2)+eps(2,i))/2.d00
                   eps(3,i)=(-s1(3)+eps(3,i))/2.d00
                   eps(4,i)=(-s1(4)+eps(4,i))/2.d00
                   eps(5,i)=(-s1(5)+eps(5,i))/2.d00
                   eps(6,i)=(rcoexp(-i)-s1(6)+eps(6,i))/2.d00
                   L(1,i)=dlog(1.d00+dexp(-eps(1,i)))
                   L(2,i)=dlog(1.d00+dexp(-eps(2,i)))
                   L(3,i)=dlog(1.d00+dexp(-eps(3,i)))
                   L(4,i)=dlog(1.d00+dexp(-eps(4,i)))
                   L(5,i)=dlog(1.d00+dexp(-eps(5,i)))
                   L(6,i)=dlog(1.d00+dexp(-eps(6,i)))
    6         continue

    5         continue
              c1=0.d00
              do 8 i=-imax,imax
                 c1=c1+rcoexp(i)*L(1,i)+rcoexp(-i)*L(6,i)
  8         continue
              c = (3.d00/(pi**2))*c1*dth
              ener=-c*pi/6.d00/rrr
              print*,r,c
              write(128,20) r, c
             if( con .ne. istep ) then
             write(128,*) ','
```

```
            endif
  3         continue
            write(128,*) '}'

          astore=0.d00
          do 60 i=-imax,imax
          astore=astore+phi(i)
  60      continue
          area=astore
          print*,'area = ',area
print*,'Z5area.for finished'
            end
```

```
C-----------------------------------------------------------------------C
C                                                                       C
C                          PROGRAM : Z7area.for                         C
C              THE FLOW FROM c=2(N-1)/(N+2) = 4/3   TO  c=1             C
C-----------------------------------------------------------------------C
      implicit real*8 (a-h,o-z)
      real*8 mass
      real*8 phi1(-2020:2020),phi2(-2020:2020),stoa1(-2020:2020)
      real*8 phi3(-2020:2020),phi4(-2020:2020),stoa2(-2020:2020)
      real*8 eps(0:10,-2020:2020),L(0:10,-2020:2020),K(0:10,-2020:2020)
      integer A(0:10,0:10), bound
      real*8  s1(0:10),rcoexp(-2020:2020),coexp(-2020:2020)
C     real*8 co(-2020:2020),rco(-2020:2020)
      common r,f2,f3,f4,f5,f6
      data A /121*0.d00/
c      external f1
       open(128,file="cr.m",status="unknown")
       open(141,file="ireps1.m",status="unknown")
       open(142,file="ireps2.m",status="unknown")
       open(143,file="ireps3.m",status="unknown")
       open(144,file="ireps4.m",status="unknown")
       open(145,file="ireps5.m",status="unknown")
       open(146,file="ireps6.m",status="unknown")
c      write(550,*) '{'
       write(128,*) '{'
       write(141,*) '{'
       write(142,*) '{'
       write(143,*) '{'
       write(144,*) '{'
       write(145,*) '{'
       write(146,*) '{'
      pi=dacos(-1.d00)
10    Format('{',F15.9, ',' F120.14, '}')
20    Format('{',F15.9, ',' F50.14, '}')
C      dth=0.026068d00  (0.125 gives accuracy of ten figures in all areas)
       dth=0.125d00
      iter1=20
      iter2=180
```

```
C     for log data
      rmin =1.d00
      rmax =1.5d00
      rstep=0.5d00
      istep= (rmax-rmin)/rstep + 1
      con = 0
      mass=1.d00
      imax=1000
      nnn=6
      bound=600

C bound = K/dth where e^-K desired accuracy

C----------------------------  KERNEL DEFINITIONS ----------------------C
CCC                                                                    CCC
       do 1 i=-imax,imax
         phi2(i)=dth*(1.5d00/pi)/dcosh(i*dth*3.d00)
         phi1(i)=(dth/(2.d00*pi))/dcosh(i*dth)
c
         phi4(i)=dth*dcos(pi/6.d00)*dcosh(i*dth)
         stoa1(i)=(dcos(pi/6.d00)*dcosh(i*dth))**2
         stoa2(i)=(dsin(pi/6.d00)*dsinh(i*dth))**2
         phi4(i)=phi4(i)/(pi*(stoa1(i)+stoa2(i)))
c
         phi3(i)=dth*dcos(pi/3.d00)*dcosh(i*dth)
         stoa1(i)=(dcos(pi/3.d00)*dcosh(i*dth))**2
         stoa2(i)=(dsin(pi/3.d00)*dsinh(i*dth))**2
         phi3(i)=phi3(i)/(pi*(stoa1(i)+stoa2(i)))
    1  continue
          r=rmin
       do 9 i=-imax-bound,imax+bound
       do 22 X=1,nnn
          L(X,i)=dlog(3.d00)
          K(X,i)=dlog(3.d00)
 22   continue
c          co(i)=dcosh(dth*i)
           coexp(i)=dexp(dth*i)/2.d00
    9   continue
```

```
        iter=iter1
        do 3 rrr=rmin,rmax,rstep
        con = con + 1
        if( con .ne. 1) then
        iter=iter2
        endif
C-------------------------------------------------------------------------
          r=rrr*mass
        do 7 i=-imax-bound,imax+bound
c         rco(i)=r*dcosh(dth*i)
          rcoexp(i)=r*dexp(dth*i)/2.d00
    7     continue
          do 5 n=1,iter
C print*,'iteration ',n
             do 6 i=-imax,imax
                s1(1)=0.d00
                s1(2)=0.d00
                s1(3)=0.d00
                s1(4)=0.d00
                s1(5)=0.d00
                s1(6)=0.d00
C                s1(7)=0.d00
C                s1(8)=0.d00
                do 14 j=i-bound,i+bound
C-------------------------------------------------------------------------
                s1(1)=s1(1)+phi1(i-j)*(L(4,j)+L(5,j))
     &                      +phi3(i-j)*L(2,j)+phi4(i-j)*L( 3,j)
                s1(2)=s1(2)+phi2(i-j)*(L(1,j)-K(3,j))
                s1(3)=s1(3)+phi2(i-j)*(-K(2,j)-K(4,j)-K(5,j))
                s1(4)=s1(4)+phi2(i-j)*(-K(3,j))
                s1(5)=s1(5)+phi2(i-j)*(-K(3,j)+L(6,j))
                s1(6)=s1(6)+phi1(i-j)*(L(2,j)+
     &           L(4,j))+phi3(i-j)*L(5,j)+phi4(i-j)*L(3,j)
C-------------------------------------------------------------------------
  14            continue
                  eps(1,i)=(rcoexp(i)-s1(1)+eps(1,i))/2.d00
                  eps(2,i)=(-s1(2)+eps(2,i))/2.d00
                  eps(3,i)=(-s1(3)+eps(3,i))/2.d00
```

```
                eps(4,i)=(-s1(4)+eps(4,i))/2.d00
                eps(5,i)=(-s1(5)+eps(5,i))/2.d00
                eps(6,i)=(rcoexp(-i)-s1(6)+eps(6,i))/2.d00
C   rco or rcoexp for a mass like node
                L(1,i)=dlog(1.d00+dexp(-eps(1,i)))
                L(2,i)=dlog(1.d00+dexp(-eps(2,i)))
                L(3,i)=dlog(1.d00+dexp(-eps(3,i)))
                L(4,i)=dlog(1.d00+dexp(-eps(4,i)))
                L(5,i)=dlog(1.d00+dexp(-eps(5,i)))
                L(6,i)=dlog(1.d00+dexp(-eps(6,i)))
                K(2,i)=dlog(1.d00+dexp(+eps(2,i)))
                K(3,i)=dlog(1.d00+dexp(+eps(3,i)))
                K(4,i)=dlog(1.d00+dexp(+eps(4,i)))
                K(5,i)=dlog(1.d00+dexp(+eps(5,i)))
    6         continue
    5         continue
              c1=0.d00
              do 8 i=-imax,imax
                 c1=c1+rcoexp(i)*L(1,i)+rcoexp(-i)*L(6,i)
  8         continue
                 c = (3.d00/(pi**2))*c1*dth
                 ener=-c*pi/6.d00/rrr
                 print*,r,c
                 write(128,20) r ,c
             if( con .ne. istep ) then
             write(128,*) ','
             endif
  3         continue
              do 97 i=-imax,imax
             write(141,10) i*dth,eps(1,i)
             write(142,10) i*dth,eps(2,i)
             write(143,10) i*dth,eps(3,i)
             write(144,10) i*dth,eps(4,i)
             write(145,10) i*dth,eps(5,i)
             write(146,10) i*dth,eps(6,i)
             write(141,*) ','
             write(142,*) ','
             write(143,*) ','
```

```
            write(144,*) ','
            write(145,*) ','
            write(146,*) ','
             endif
 97         continue
            write(128,*) '}'
            write(141,*) '}'
            write(142,*) '}'
            write(143,*) '}'
            write(144,*) '}'
            write(145,*) '}'
            write(146,*) '}'
            astore1=0.d00
            astore2=0.d00
            astore3=0.d00
            astore4=0.d00
            do 60 i=-bound,bound
            astore1=astore1+phi1(i)
          astore2=astore2+phi2(i)
          astore3=astore3+phi3(i)
          astore4=astore4+phi4(i)
  60      continue
          print*,'area1 = ',astore1
          print*,'area2 = ',astore2
          print*,'area3 = ',astore3
          print*,'area4 = ',astore4
          close(128)
          close(141)
          close(142)
          close(143)
          close(144)
          close(145)
          close(146)
print*,'Z7area finished'
            end
```

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