Classical sigma models in 2+1 dimensions

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CLASSICAL SIGMA MODELS IN 2+1 DIMENSIONS

by

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To the Memory of

Ιωαννίδης Αλέξιος

Teacher and Father

Preface

This thesis is based on work done by the author between May 1994 and August 1996 at the University of Durham. No part of it has been previously submitted for any degree, either in this or any other university.

With the exception of the reviews in chapter 1 and 2, and Appendix A, it is believed that the material in this thesis is original work. Chapter 3 is based on a paper [1], written jointly with R. S. Ward, published in *Physics Letters A*. Chapter 4 is based on a paper [2] published in *Journal of Mathematical Physics*. Finally, chapter 5 is based on a paper [3] submitted for publication in *Nonlinearity*.

I would like to warmly thank my supervisor Prof. R. S. Ward for his guidance and encouragement. I would also like to thank the late E. J. Squires for his constant support, Prof. W. J. Zakrzewski and Dr. P. E. Dorey for many helpful conversations and, finally, Sharry Borgan for reading this thesis and making valuable comments. Financial support was provided by the *"Pontium Female Care"* (i.e., *Μέριμνα Ποντίων Κυριών*) for the first year of my studies; *"Federation of West Germany Pontium"* (i.e., *Ομοσπονδία Ποντίων Δυτ. Γερμανίας*) for the second year of my studies; *"Dept. of Math. Science, Durham University"* and *"European Commission"* for my final year; whom I acknowledge with thanks. Last but not least, I gratefully acknowledge, the late *John Papadopoulos*, whose valuable support and assistance made this study possible.

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Abstract

The work in this thesis is concerned with the study of dynamics, scattering and stability of solitons in planar models, i.e. where spacetime is (2+1)-dimensional. We consider both integrable models, where exact solutions can be written in closed form, and nonintegrable models where approximations and numerical methods must be employed. For theories that possess a topological lower bound on the energy, there is a useful approximation in which the kinetic energy is assumed to remain small. All these approaches are used at various stages of the thesis.

Chapters 1 and 2 review the planar models which are the subjects ot this thesis. Chapters 3 and 4 are concerned with integrable chiral equations. First we exhibit an infinite sequence of well-defined conserved quantities and then we construct exact soliton and soliton-antisoliton solutions using analytical methods. We find that there exist solitons that scatter in a different way to those previously found in integrable models. Furthermore, this soliton scattering resembles very closely that found in nonintegrable models, thereby providing a link between the two classes. Chapter 5 develops a numerical simulation based on topological arguments, which is used in a study of soliton stability in the (unmodified) O(3) model. This confirms that the solitons are unstable, in the sense that their size is subject to large changes. The same results are obtained by using the slow-motion approximation.

Contents

Chapter 1

Introduction

An exciting and extremely active area of research during the past thirty years has been the study of a certain class of nonlinear partial differential equations known as *soliton equations.* The key of these equations is that they possess special types of elementary solutions (taking the form of localized disturbances, or pulses), which retain their shape even after interactions among themselves (at least in the integrable systems), and thus they act like particles. These localized disturbances have come to be known as *solitons.*

Before any detailed discussion, and to avoid confusion later on, it is worthwhile clearing up a point of terminology: the word *soliton* was introduced by mathematicians to describe lumps of energy which were stable to perturbations and did not change either velocity or shape when colliding with each other. However, in recent literature all sorts of localized energy configurations have been called solitons. We shall go with this looser definition. By a soliton we shall mean a lump of energy that moves but we shall not imply stability of the shape or the velocity or a simple behaviour in collision.

The theory of solitons is attractive, since not only are they interesting mathematical structures but their applications in the natural sciences are immense. Solitons occur in many areas of physics including nonlinear optics, hydrodynamics, superconductivity, cosmology, plasma and particle physics, and even in biophysics. The major breakthrough in these theories is the discovery of the so-called inverse scattering method [4] which provides a recipe for writing down soliton solutions to a large number of equations.

The majority of equations possessing soliton solutions occur in one space dimension, so that soliton dynamics is confined to motion along a line. In fact in the integrable systems, they occur when dispersion effects are exactly balanced by nonlinearities, for it is only then that a lump moves without changing shape or velocity. It should be stressed that, for an equation picked at random, this is very much the exception rather than the rule. The simplest equation with the above property is the wave equation, which is both linear and dispersionless, but in general the balance is much more delicate. In any case, the wave equation has only wave-like solitons (as its name implies) and not lump-like ones.

In more than one dimension the situation is far less understood. There are examples of equations possessing soliton solutions but most are simple extensions of familiar examples from one space dimension. Some higher dimensional systems which possess soliton solutions have originated in the area of elementary particle physics. The localized structure of solitons together with their collision properties make them ideally suited to describe elementary particles.

The aim of this thesis is to discuss the important properties of sigma models in (2+1) dimensions, in which an important features is the existence of solitons. In particular, we will investigate the dynamics of localized solitons in the plane, i.e. in three-dimensional spacetime. Systems admitting such solitons may be grouped into two distinct classes:

- Systems admitting topological solitons, the stability of which depend on nontrivial topology; these includes vortices in the Abelian Higgs model [5], and lump solutions of sigma models (with various possible modifications) [6]. These solutions can be studied using topological methods. (In topology, two geometric surfaces are considered equivalent if they can be smoothly deformed into each other without cutting). Moreover, they may be assigned an integer valued topological charge, which is conserved as the solitons evolve in time and also in a collision of two or more solitons.

  The topological charge has a natural physical interpretation. One can think of the solitons as subatomic particles, and of the topological charge as one of the conserved quantities of particle physics. For example, in (3+1) dimensions the soliton solutions of the Skyrme model [7, 8] are thought of as baryons, in particular protons and neutrons; and the topological charge is taken to be the conserved baryon number.

  These are not solitons in the strictest sense; for example, the collision of two solitons is not elastic (some radiation is emitted). The feature of topological systems which is relevant here, is that a head-on collision results in 90^0 scattering. In other words, if two solitons approach each other along the x-axis and collide at the origin, then two solitons emerge, traveling in opposite directions along the y-axis (with slightly less speed, because of inelasticity). When they overlap at the origin, they form a ring rather than a single lump.

- The integrable systems which admit localized solitons. The earliest examples of integrable systems admitting soliton solutions are considered in (1+1) [9, 10] and (2+0) dimensions. Solutions in (2+0) dimensions are the static configurations for the same theory in (2+1) dimensions. In (1+1) dimensions some of the known systems, such as the sine-Gordon equation and the principal chiral field equation [9] are Lorentz invariant, which means that they are invariant under the action of the group SO(1,1).

  Up to now, there has been much investigation of finding analogous systems, that are both integrable and Lorentz invariant, in higher dimensions. In (2+1) dimensions, there are examples of integrable systems such as the Kadomtsev-Petviashvili [11], the Konopelchenko-Rogers [12] and the Davey-Stewartson [13, 14] equations, but all these are long away from being Lorentz invariant.

  There are some examples of systems with both properties; they arise from the self-dual Yang-Mills (sdYM) equations in four dimensions, which are Lorentz invariant (relativistic) and integrable [15]. As a result on the one hand, it is a beautiful example of extending totally integrable systems into four dimensions and on the other hand, many integrable systems in lower dimensions; specially in (1+1) dimensions, may be obtained as reductions (dimensional or algebraic) of these equations by using a unifying framework [16].

  In integrable systems the scattering of solitons is usually trivial, with a phase shift being the only (if any) effect upon solitons which collide. Such a trivial elastic collision behaviour is one of the properties of solitons in integrable systems that allows the analytic construction of exact multisoliton solutions. In integrable planar systems the possibilities for soliton dynamics are much greater than in (1+1) dimensions, where solitons are confined to motion in a line. The inelastic scattering of solitons in nonintegrable systems, is far from simple, and although it usually involves radiation components this can be extremely small. Whether this type of nontrivial soliton scattering can occur in integrable models is an interesting question, which lies at the heart of connecting solitons of integrable and nonintegrable systems.

This thesis deals exclusively with classical theory of sigma models. Since most physical applications employ quantum theories, the most direct relevance will not be in these areas, but in more mathematical areas, especially the theory of solitons. On the other hand, classical theories may be thought of as just the first approximation to the corresponding quantum theory; so it is possible that the results presented here might well turn out to have consequences for physics.

The main body of the thesis is laid as follows. In chapter 2 we shall review particular examples of two modified sigma models; which are dimensional reductions of the sdYM equations, and therefore are integrable; in particular that procedures exist for writing down explicit multisoliton solutions. These are the integrable chiral models in (2+1) dimensions. Although these models do not have any obvious application in physics they are very interesting from the soliton theoretic point of view. To begin with, the equation of motions are presented along with known procedures for constructing multisoliton solutions. Roughly speaking, these configurations represent *rational* solitons which look like lumps, and *exponential* solitons which look like waves. In fact, each lump-like soliton moves with an independent speed, undergoes multiple collisions and emerges intact with an unchanged speed; while extended wave-like solutions suffer a phase shift upon scattering although again there is no change in velocity. To close the chapter we shall discuss the stability of the one-soliton solution and also the soliton head-on collision. By connecting the aforementioned model with the O(3) sigma model and using numerical simulations, it was observed that no unstable modes can be found with many varied perturbations. This is compelling evidence for the stability of these solitons under radially symmetric perturbations. Also, nontrivial scattering may occur between two solitons and between a soliton an antisoliton.

In chapter 3 we will exhibit infinite sequences of well-defined conserved quantities that exist for the planar integrable chiral models, and have a simple explicit form. Infinite sequences of conservation laws involving nonlocal densities have been known for some time; but do not necessarily yield conserved quantities, since the relevant spatial integrals may diverge. We also discuss some local conserved quantities, including Noether charges, arising from symmetries of the Lagrangian.

In chapter 4 it is described how exact soliton and soliton-antisoliton solutions may be obtained for one of the aforementioned planar models, by using analytic methods. The behaviour of solitons in integrable theories is strongly constrained by the integrability of the theory; i.e. by the existence of an infinite number of conserved quantities which these theories are known to possess. One usually expects the scattering of solitons in such theories to be rather simple, i.e. trivial. By contrast, in this chapter we generate new soliton solutions for the planar integrable chiral model whose scattering properties are highly nontrivial; more precisely, in head-on collisions of N indistinguishable solitons the scattering angle (of the emerging structures relative to the incoming ones) is π/N. The indication seems to be that the internal degrees of freedom in the chiral field allow the behaviour of its solitons to be rather rich. We also generate soliton-antisoliton solutions with elastic scattering; in particular, a head-on collision of a soliton and an antisoliton resulting in 90^0 scattering.

Chapter 5 concentrates on lump-like solitons in the (unmodified) O(3) sigma model. A central question is that of soliton stability. Since there is no natural scale in the model, the solitons can take any size (although they have the same general shape). There is the possibility that as a result of small perturbations they could either expand indefinitely, eventually covering the whole plane, or else shrink, so becoming infinitely tall spikes. It seems that to answer these questions, one must evolve the soliton configuration numerically on a lattice, since the O(3) model is not integrable. An important consideration in any numerical evolution is the choice of initial data. Specifically, to look at small perturbations, one wants to begin with a *discrete* (lattice) static solution. This chapter shows that by taking proper account of the topological aspects in the theory (i.e., Bogomol'nyi bound on the energy of the configuration), one is led to a natural evolution scheme containing explicit static lattice solitons. Using them as the basis for a study of soliton stability, we find that the O(3) lumps are unstable. In fact, numerical simulations show that these lattice solitons tend to change (shrink or expand) linearly, with time.

Finally, chapter 6 outlines work currently in progress, and also suggest some possible avenues for future research.

Chapter 2

Planar Integrable Chiral Models

2.1 The Self-Duality Equations and their Reductions

Considerable progress has recently been made towards understanding various nonlinear systems which are integrable. They are very special; roughly speaking, almost every equation is not integrable. They turn up in the study of nonlinear phenomena, and are associated with beautiful mathematics.

The word *integrability* refers to a special property which certain equations have. In the case of classical mechanics, for example, it implies that one can transform to action-angle variables. In addition to classical mechanics (ordinary differential equations), partial differential equations will also be considered; examples of these are the integrable field-theory equations such as the self-dual Yang-Mills and the two-dimensional sigma models.

Let us first recall the situation with regard to ordinary differential equations. For Hamiltonian systems with n degrees of freedom the classical Liouville definition of integrability is in terms of the existence of sufficiently many constants of motion. Namely, *there should exist on phase space* $(n-1)$ *independent functions which Poisson-commute with the Hamiltonian and with each other.* Consequently, a continuum system must have an infinite number of conserved quantities in order to be integrable. Hamiltonian systems which satisfy the above hypotheses are called *completely integrable.*

Let us now study the meaning of integrability for a system of partial differential equations (cf. Ward [17]).

- By analogy with the Liouville definition, one may specify that the system admits an infinite number of conserved currents.

- One may require that the system has the Painlevé property; which, roughly speaking, would say that its solutions should be meromorphic functions of the complexified independent variables.

- Many integrable systems are closely associated with Lie algebras, and in particular with infinite-dimensional (Kac-Moody) Lie algebras. Algebraic structure is also important in integrable quantum field theories and statistical-mechanics systems. But it is not clear whether this algebraic background applies to a specific class of integrable systems with specific boundary conditions.

- Finally, one could require that the system of equations be the consistency condition for an overdetermined system of linear equations. However, this linear system has to have some special property.

Hence, when one attempts to formulate a precise definition for integrability, many possibilities appear, each with a certain theoretic interest. For our purposes we shall consider a system to be integrable if it can be written as the compatibility condition for an overdetermined linear system of a suitable type. Such systems of nonlinear differential equations can be solved exactly.

A basic example of an integrable system is that of the self-dual Yang-Mills equations in four dimensions. Its solutions can be described in terms of holomorphic vector bundles, or, equivalently, in terms of a *Riemann-Hilbert problem.*

It is standard to define the Yang-Mills field as

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n], \tag{2.1}$$

where x_m, $m = 1, 2, 3, 4$, are the coordinates in Euclidean space $\mathbf{R}^4$, $\partial_m = \partial/\partial x_m$; A_m are the Yang-Mills potentials and take values in some Lie algebra g (associated with a Lie group $\mathcal{G}$). From now on, take $\mathcal{G} = \mathrm{SU}(2)$ for simplicity.

Consider a flat space in which one defines new spacetime coordinates, i.e. σ, $\bar{\sigma}$, τ, $\bar{\tau}$, in such a way that the spacetime metric is

$$ds^2 = d\sigma\, d\bar{\sigma} + d\tau\, d\bar{\tau}. \tag{2.2}$$

The overbar may denote complex conjugate, but equally may not, depending on the spacetime signature.

Therefore, if the signature is $++++$ (Euclidean space $\mathbf{R^4}$) then

$$\begin{aligned} \tau &= x_1 + i x_2, & \sigma &= x_3 + i x_4, \\ \bar{\tau} &= x_1 - i x_2, & \bar{\sigma} &= x_3 - i x_4, \end{aligned} \tag{2.3}$$

which can well be considered in $++--$ signature by substituting $(\partial_\tau, \partial_{\bar{\tau}}) \to (i\partial_\tau, i\partial_{\bar{\tau}})$. On the other hand, if the signature is $++--$ ($\mathbf{R^{2+2}}$) then

$$\begin{aligned} \tau &= x_1 + x_2, & \sigma &= x_3 + x_4, \\ \bar{\tau} &= x_1 - x_2, & \bar{\sigma} &= x_3 - x_4. \end{aligned} \tag{2.4}$$

The corresponding gauge potentials are

$$\begin{aligned} A_\tau &= A_1 + i A_2, & A_\sigma &= A_3 + i A_4, \\ A_{\bar{\tau}} &= A_1 - i A_2, & A_{\bar{\sigma}} &= A_3 - i A_4, \end{aligned} \tag{2.5}$$

for the Euclidean space, and

$$\begin{aligned} A_\tau &= A_1 + A_2, & A_\sigma &= A_3 + A_4, \\ A_{\bar{\tau}} &= A_1 - A_2, & A_{\bar{\sigma}} &= A_3 - A_4, \end{aligned} \tag{2.6}$$

for the $\mathbf{R^{2+2}}$ space. For the coordinates (2.3), the self-duality equations (sdYM) are given by

$$\begin{aligned} &F_{\tau\sigma} = 0, \\ &F_{\bar{\tau}\bar{\sigma}} = 0, \\ &F_{\tau\bar{\tau}} + F_{\sigma\bar{\sigma}} = 0. \end{aligned} \tag{2.7}$$

The above equations are a set of three independent Lie-algebra-valued equations and the inverse scattering transformation [18] can be applied to them in order to evaluate their local solutions. In fact, they are invariant under the gauge transformation

$$A_m \to \Lambda\, A_m\, \Lambda^{-1} - (\partial_m \Lambda)\, \Lambda^{-1}, \qquad F_{mn} \to \Lambda^{-1} F_{mn} \Lambda, \tag{2.8}$$

for any $\Lambda(x^m) \in \mathrm{SU}(2)$.

The inverse scattering transform is a nonlinear analogue of the Fourier transform, and relies upon the fact that the sdYM equations can be written as the compatibility condition of an overdetermined linear system. Consider the following system

$$\begin{aligned}(\partial_\tau + \lambda\,\partial_{\bar\sigma})\,\Psi &= (A_\tau + \lambda\,A_{\bar\sigma})\,\Psi,\\ (\partial_\sigma - \lambda\,\partial_{\bar\tau})\,\Psi &= (A_\sigma - \lambda\,A_{\bar\tau})\,\Psi,\end{aligned} \tag{2.9}$$

where λ is a complex constant known as *spectral parameter.* The compatibility condition is expressed as a polynomial in λ, i.e.

$$(\partial_\tau + \lambda\,\partial_{\bar\sigma})\,(A_\sigma - \lambda\,A_{\bar\tau}) - (\partial_\sigma - \lambda\,\partial_{\bar\tau})\,(A_\tau + \lambda\,A_{\bar\sigma}) = [A_\tau + \lambda\,A_{\bar\sigma}\,,\,A_\sigma - \lambda\,A_{\bar\tau}]. \tag{2.10}$$

Equating coefficients of powers of λ yields (2.7). Equations (2.9) are known as the *Lax pair* [19].

It has been conjectured by Ward (cf. [16]) that every completely integrable equation arises as a reduction of some main equations, i.e.

"*many of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.*"

Recently it has been shown that the sdYM equations admit reductions to well known soliton equations in (1+1)-dimensions, e.g., the Sine-Gordon and the Toda Lattice equations [16, 20]; additionally it has also been shown that several classical systems of ordinary differential equations including the Euler-Arnold equations for free motion of an n-dimensional rigid body about a fixed point, and a generalization of the Nahm equation which is related to a classical third order differential equation possessing a movable natural boundary in the complex plane, arise as one-dimensional reductions of the sdYM equations [21]. Moreover, Mason *et al* [22] have shown that the Korteweg de Vries and the nonlinear Schrödinger equations arise as reductions of the sdYM equations; increasing the validity of the conjecture.

There are two types of reduction that may be performed on the sdYM equations (cf. [17] and [23]):

(i) First, one can reduce the number of independent variables by factoring out by a subgroup of the Poincare group, i.e. *dimensional reduction.*

(ii) Secondly, one can reduce the number of dependent variables by imposing algebraic conditions on the gauge potential A_μ in a consistent way, i.e. *algebraic reduction.*

In general, in order to obtain a particular integrable system requires a combination of both of these reductions. A complete analysis of all reductions of the system (sdYM) looks like a large problem, because there are so many possibilities for both (i) and (ii).

The sdYM equations are often written in the following form [24, 25]

$$\partial_{\bar{\tau}}\left(J^{-1}\partial_\tau J\right)+\partial_{\bar{\sigma}}\left(J^{-1}\partial_\sigma J\right)=0, \tag{2.11}$$

which is called the *left sdYM-J equations*; or equivalently can be written as

$$\partial_\tau\left(\partial_{\bar{\tau}}J\,J^{-1}\right)+\partial_\sigma\left(\partial_{\bar{\sigma}}J\,J^{-1}\right)=0, \tag{2.12}$$

which is called the *right sdYM-J equations*, where $J \in g$. To show this, suppose that

$$\begin{array}{ll} A_\tau = D^{-1}\partial_\tau D, & A_{\bar{\tau}} = \bar{D}^{-1}\partial_{\bar{\tau}}\bar{D}, \\ A_\sigma = D^{-1}\partial_\sigma D, & A_{\bar{\sigma}} = \bar{D}^{-1}\partial_{\bar{\sigma}}\bar{D}, \end{array} \tag{2.13}$$

with D and $\bar{D}$ in the complexified gauge group SL$(2,C)$. It is readily seen that the first two equations of (2.7) are identically satisfied (i.e., integrability condition of (2.13)). Substituting (2.13) in the last equation of (2.7) and setting $J \equiv D\bar{D}^{-1}$ yields (2.11).

Although, the J-formulation is a neat form of the equations, it is not as general as the original form (2.7) since the SO(4)-invariance, which was present in (2.7), and the geometrical interpretation (in terms of connections and curvatures) have been lost [SO(4) is the Lie group of matrices A such that $\det A = 1$ and $AA^T = 1$, where A^T is the transpose of A].

Before procceding any further, let us mention a subtle difference between the signature $++++$ and $++--$ which stems from the fact that the gauge potentials A_m may be thought of as 2×2 anti-hermitian matrices. In $++++$, this means that $A_\tau^\dagger = -A_{\bar{\tau}}$ (where † denotes the complex conjugate transpose matrix) and, therefore, $\bar{D} = (D^\dagger)^{-1}$. This has the result that J is hermitian, in addition to having a unit determinant. In contrast, for $++--$, A_τ, $A_{\bar{\tau}}$, A_σ and $A_{\bar{\sigma}}$ are themselves anti-hermitian, so D and $\bar{D}$, and hence also J, may be taken to lie in SU(2).

In order to obtain 3-dimensional static systems (such as the monopole equations), the metric is taken to have signature $++++$. For example, the solutions of the simplest reduction from $\mathbb{R}^4$ to $\mathbb{R}^{3+0}$ are the Bogonomolny-Prasad-Sommerfield (BPS) monopoles. The remaining possibility is therefore to consider a metric with signature $+++-$ ($\mathbb{R}^{3+1}$). With this choice the self-duality equations only allow complex gauge groups, and not real forms such as SU(N). One consequence of this is that they do not appear to admit a positive-definite, conserved energy functional, and this makes the situation rather hard to interpret. All the equations obtained in this way are integrable, by virtue of the fact that the self-duality equations from which they are derived are themselves integrable.

2.2 Chiral Models with Torsion Term

Start with (2.11) in $++--$ and assume that the field is invariant under a non-null translation in $\mathbb{R}^{2+2}$: this then yields an *integrable chiral equation in (2+1) dimensions.* But there is more than one way of doing this, since the original equations in (2+2) dimensions are not SO(2,2)-invariant. The reduced equation involves a choice of unit vector V_α, and has the form

$$\partial^\mu(J^{-1}J_\mu) - \frac{1}{2}\varepsilon^{\alpha\mu\nu}V_\alpha\,[J^{-1}J_\mu, J^{-1}J_\nu] = 0. \tag{2.14}$$

Here Greek indices range over the values 0, 1, 2, $x^\mu = (t, x, y)$ are the space coordinates, J is a $2{\times}2$ matrix function of the coordinates x^μ with $\det J = 1$, $J_\mu \equiv \partial_\mu J$ denotes partial derivatives, $\varepsilon^{\mu\nu\alpha}$ is the totally skew tensor with $\varepsilon^{012} = 1$, and V_α is a constant unit vector in spacetime, i.e. $V^\alpha V_\alpha = 1$. Indices are raised and lowered using the (inverse) Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1)$. If there are no further conditions on J, then solutions of (2.14) would correspond to sdYM fields with gauge group SL(2,C). To reduce to the gauge group SU(2), we need to impose a reality condition on J, the precise nature of which depends on the choice of V_α.

Before any further discussion notice that, when $V_\alpha = (0, 0, 0)$ (2.14) corresponds to the *(unmodified) chiral model in (2+1) dimensions*, i.e.

$$\eta^{\mu\nu}\partial_\mu(J^{-1}\partial_\nu J) = 0, \tag{2.15}$$

which is Lorentz invariant but not integrable. In addition, if J is restricted to be a diagonal matrix $\mathrm{diag}[e^{(i\phi)}, e^{(-i\phi)}]$, (2.14) reduces to the planar wave equation. A more important consequence is that this equation may be thought of as a nonlinear generalization of the wave equation in (2+1) dimensions.

The existence of the non-zero vector V_α explicitly breaks the Lorentz invariance of the chiral model by picking out a particular direction in spacetime. Due to the fact that the vector V_α is unit, two cases of particular interest occur when either V_α is a real spacelike unit vector, or when it is an imaginary unit vector. The two cases we shall deal with in the next two chapters, are as follows:

- Take V_α to be spacelike, specifically $V_\alpha = (0,1,0)$; and require J to be unitary, i.e. $J \in \mathrm{SU}(2)$. With $u = (t+y)/2$ and $v = (t-y)/2$, then (2.14) can be rewritten as

$$S\ :\ \ \partial_v(J^{-1}J_u) - \partial_x(J^{-1}J_x) = 0. \tag{2.16}$$

  This follows from (2.11) by setting $\partial_4 = 0$ and relabeling the other three coordinates (2.4) so that $x^1 \to y$, $x^2 \to t$ and $x^3 \to x$. It is the *(2+1)-dimensional SU(2) modified chiral model*, formulated by Ward [26].

  Taking V_α to be spacelike means that the symmetry which remains is an SO(1,1) symmetry. Equation (2.16) has many of the properties of an integrable system. For example, it arises as the consistency condition for a pair of linear equations, and this description can be used to generate multi-soliton solutions [26, 27]; an inverse scattering transform [28] can be set up; it satisfies the Painlevé property for integrability [29]. A stricter characterization of integrability involves the existence of sufficiently many conserved quantities in involution, and hence a description of action-angle variables. But such an infinite set of conserved quantities is not known.

- Take V_α to be i times a timelike unit vector, specifically $V_\alpha = (-i,0,0)$; and require J to be hermitian (with positive eigenvalues). Putting $z = (x+iy)/2$, we can write the resulting equation as

$$T\ :\ \ \partial_t(J^{-1}J_t) - \partial_{\bar z}(J^{-1}J_z) = 0, \tag{2.17}$$

  where bar denotes complex conjugate. This equation is a reduction of (2.11) by setting $\partial_3 = 0$ and relabeling (2.3) (after the translation $\mathbf{R}^4 \to \mathbf{R}^{2+2}$) so that $x^1 \to x$, $x^2 \to y$ and $x^4 \to t$.

  Due to the fact that the vector V_α is now a timelike vector the residual group is SO(2). This model has been proposed by Manakov and Zakharov [30]. These authors have found localized soliton solutions (see below) which do not scatter. Moreover, they have tackled the initial value problem by employing the well known inverse scattering method.

Note that each equation is equivalent to its hermitian conjugate; in other words, the reality condition on J is consistent with the equation. We impose the boundary condition,

$$J = J_0 + J_1(\theta)\, r^{-1} + O(r^{-2}), \qquad \text{as} \qquad r \to \infty, \tag{2.18}$$

where $x + iy = r\exp(i\theta)$. Here J_0 denotes a constant matrix and J_1 depends, only on θ (no time dependence). This implies finite energy.

The energy-momentum tensor of the unmodified chiral model (2.15), is

$$T^{\mu\nu} = (-\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta})\, \mathrm{tr}(J^{-1}J_\alpha J^{-1}J_\beta), \tag{2.19}$$

and its divergence, for the modified equation (2.14) is

$$\partial_\mu T^{\mu\nu} = -\frac{1}{3}\, V^\nu\, \varepsilon^{\alpha\beta\gamma}\, \mathrm{tr}(J^{-1}J_\alpha J^{-1}J_\beta J^{-1}J_\gamma), \tag{2.20}$$

where tr denotes the trace. So $T^{\mu\nu}$ is not conserved and neither, in general, is the energy-momentum vector $P^\mu = T^{\mu 0}$. Clearly, the divergence of this energy-momentum vanishes if and only if $V_0 = 0$. For (2.17) one needs something else (see section 3.3). By contrast, the aforementioned energy is conserved for (2.16): it is the integral over x and y of the energy density

$$P^0 = -\frac{1}{2}\mathrm{tr}\left[(J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2\right]. \tag{2.21}$$

Note that, the energy for the modified chiral model (2.16) is the same as the one for the unmodified chiral model, hence the additional term in (2.14) proportional to V_α (so-called *torsion term*), does not affect the energy. In fact, it is analogous to a background magnetic field in classical mechanics [31].

The differences in the inverse scattering transform for equations (2.16) and (2.17) are:

- for equation (2.16) it is formulated as a Riemmann-Hilbert problem on the real line [28], whereas for equation (2.17) it is formulated as a Riemmann-Hilbert problem on the unit circle [32].

- for equation (2.16) the evolution of the scattering data is pure imaginary, whereas for equation (2.17) the evolution is real exponential decay (or growth).

2.3 Multisoliton Solutions

The method for generating soliton solutions of (2.16) is that of the *Riemann problem with zeros* [26]. Let λ be a complex parameter, and (u, v, x) real variables; i.e. coordinates on $\mathbf{R}^{2+1}$. Let A and B be 2×2 anti-hermitian trace-free matrices, depending on u, v, x but not on λ. Consider the set of linear equations

$$\begin{aligned} L\psi &\equiv (\lambda\partial_x - \partial_u)\psi = A\psi, \\ M\psi &\equiv (\lambda\partial_v - \partial_x)\psi = B\psi, \end{aligned} \tag{2.22}$$

where $\psi(\lambda, u, v, x)$ is an unimodular 2×2 matrix function satisfying $\det \psi = 1$, and the reality condition

$$\psi(\lambda, u, v, x)\, \psi(\bar{\lambda}, u, v, x)^{\dagger} = I, \tag{2.23}$$

where I is the 2×2 identity matrix. It is easy checked that (2.23) is consistent with (2.22). However, the system (2.22) is overdetermined, and in order for a solution ψ to exist, A and B have to satisfy the integrability conditions, which are

$$B_x = A_v, \qquad A_x - B_u - [A, B] = 0. \tag{2.24}$$

If we put $J(u, v, x) = \psi(\lambda = 0, u, v, x)^{-1}$ where ψ is a solution of the system (2.22), we get by comparing (2.22) and (2.24) that

$$A = J^{-1}J_u, \qquad B = J^{-1}J_x. \tag{2.25}$$

Therefore, the integrability condition for (2.22) implies that there exists a field J which satisfies the equation of motion (2.16); and moreover, the reality condition on ψ ensures that J is unitary.

This means that solutions of (2.16) can be generated with the aid of the overdetermined linear system (2.22). To obtain the solution that we want, one may assume that $\psi(\lambda)$ has the form

$$\psi(\lambda) = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k}, \tag{2.26}$$

where M_k are 2×2 matrices independent of the complex parameter λ, n is the number of solitons, and the complex parameter μ_k determines the velocity of the k-th soliton. The components of the matrix M_k are given in terms of a rational function f_k of the complex variable

$$\omega_k = x + \mu_k u + \mu_k^{-1} v. \tag{2.27}$$

Roughly speaking, $f_k(\omega_k)$ describes the shape of the k-th soliton. In fact, the matrix M_k has the form

$$M_k = -\sum_{l=1}^{n} (\Gamma^{-1})^{kl} \bar{m}_a^l m_b^k, \tag{2.28}$$

with Γ^{-1} the inverse of

$$\Gamma^{kl} = \sum_{a=1}^{2} (\bar{\mu}_k - \mu_l)^{-1} \, \bar{m}_a^k m_a^l. \tag{2.29}$$

Here m_a^k are holomorphic functions of ω_k, given by $m_a^k = (m_1^k, m_2^k) = (1, f_k)$.

The solution ψ therefore depends on n constants μ_k (which must all be different and nonreal) and n holomorphic functions $f_k = f_k(\omega_k)$. The above is not quite in its final form, since, as it stands,

$$\delta(\lambda) \equiv \det \psi = \prod_{k=1}^{n} \frac{(\lambda - \bar{\mu}_k)}{(\lambda - \mu_k)}. \tag{2.30}$$

Therefore, by dividing by the square root of this function $\delta(\lambda)$ achieves $\det \psi = 1$.

So, finally, the expression for the inverse of J may be obtained by evaluating $\psi(\lambda)$ at $\lambda = 0$, i.e.

$$J^{-1} = \delta(0)^{-1/2} \left(I - \sum_{k=1}^{n} \frac{M_k}{\mu_k} \right). \tag{2.31}$$

Clearly, the expression for (2.31) becomes very complicated very quickly as n is increased. Such a solution corresponds to n solitons, each moving at constant velocity and experiencing no scattering (not even a phase shift) when it interacts. For these solutions the field J is a rational function of t, x, y. For example, one may take f_k to be

$$f_k(\omega_k) = \alpha_k \, \omega_k + c_k. \tag{2.32}$$

Here $\alpha_k \in \mathbf{R}$ and $c_k \in \mathbf{C}$. Another way is to choose a solution so that α_k will take values also in $\mathbf{C}$, but this is a little trickier to analyze it as introduces many complications. The parameters α_k, μ_k, c_k have simple physical interpretations: $\mu_k = m_k \, e^{i\theta_k}$ specifies the soliton velocity via the formula

$$(v_x, v_y) = \left(-\frac{2m_k \cos \theta_k}{1 + m_k^2}, \frac{1 - m_k^2}{1 + m_k^2}\right), \tag{2.33}$$

c_k determines the position of the peak at $t = 0$ and, finally, α_k fixes the ratio of the height of the lump to its width. Evidently, the soliton speed is

$$v^2 = 1 - \frac{4m_k^2 \sin^2 \theta_k}{(1+m_k^2)^2}. \tag{2.34}$$

The simplest family of lump solutions may be obtained by considering $n = 1$, in which case the solutions are specified only by a complex number μ and a meromorphic function $f(\omega)$. Hence, the inverse of (2.31) simplifies to give

$$J = \frac{1}{|\mu|(1+|f|^2)} \begin{pmatrix} \mu + \bar{\mu}|f|^2 & (\mu - \bar{\mu})f \\ (\mu - \bar{\mu})\bar{f} & \bar{\mu} + \mu|f|^2 \end{pmatrix}, \tag{2.35}$$

which represent the one-soliton solution. Writing $\mu = m\, e^{i\theta}$, the energy density becomes

$$P^0 = \frac{2(1+m^2)^2 \sin^2\theta}{m^2} \frac{|f'|^2}{(1+|f|^2)^2}, \tag{2.36}$$

where f' denotes the derivative of f as a function of ω. Keeping things simple, let us choose $f(\omega) = \alpha\omega + c$, thus the factor $|f'|^2$ in the numerator becomes just α^2. So it is seen that the solution looks like a single lump at the point where $f(\omega) = 0$. Note that in the static case ($\mu = i$) one may easily integrate P^0 over x and y to obtain the total energy. The result is

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^0\, dx\, dy = 8\pi, \tag{2.37}$$

independent of α.

A family of extended wave solitons have been constructed for (2.16), by Leese [33]. Ward showed that taking f to be any rational function of degree N leads to a configuration with N peaks, which in the static case has energy $8N\pi$. An extended wave must have infinite energy and so no function of finite degree will do. Thus, by taking f to be an exponential function of ω, i.e.

$$f(\omega) = \exp(b\omega + c), \tag{2.38}$$

with $c \in \mathbf{R}$ and $b \in \mathbf{C}$, leads to an energy density

$$\begin{aligned} P^0 &= \frac{2(1+m^2)^2 \sin^2\theta}{m^2} \frac{|b|^2|f|^2}{(1+|f|^2)^2} \\ &= \frac{(1+m^2)^2 \sin^2\theta}{2m^2} |b|^2 \operatorname{sech}^2(\Re(b\omega) + c). \end{aligned} \tag{2.39}$$

The energy density P^0 is constant along each of the lines $\Re(b\omega) + c = \text{const}$. In fact, the wavefront lies along $\Re(b\omega) + c = 0$; which is the equation of a straight line in the xy-

plane, for any t. As t varies, the wave maintains its shape and simply moves at constant velocity.

It has, also, been found that as two waves interact, they do not change shape or velocity, but each has a phase shift across the region of intersection, which may be dependent upon internal parameters. In addition, a wave and a soliton interact in a non-trivial way, but after the interaction both the wave and the soliton recover their initial shapes with no radiation emission.

We conclude this section by mentioning the corresponding results for (2.17). Equation (2.17) arises as the consistency conditions for a pair of linear equations, which are

$$\begin{aligned} \widehat{L}\widehat{\psi} &\equiv (\widehat{\lambda}\partial_{\bar{z}} + \partial_t)\widehat{\psi} = -\widehat{A}\widehat{\psi}, \\ \widehat{M}\widehat{\psi} &\equiv (\widehat{\lambda}\partial_t + \partial_z)\widehat{\psi} = -\widehat{B}\widehat{\psi}, \end{aligned} \tag{2.40}$$

with $\widehat{A} = J^{-1}J_t$ and $\widehat{B} = J^{-1}J_z$. Therefore, one can apply the inverse scattering transform to it.

This system possesses a great variety of soliton solutions having quite unusual properties. These solitons can be constructed in the same way as above (cf. [34, 35]). A soliton is described by a single-pole function $\widehat{\psi}(\widehat{\lambda})$, i.e.

$$\widehat{\psi}(\widehat{\lambda}) = I - \frac{\bar{\widehat{\lambda}}^{-1} - \widehat{\mu}}{\widehat{\lambda} - \widehat{\mu}} R(t, z, \bar{z}), \tag{2.41}$$

with $R^2 = R$. For such a function to be a solution of (2.40) it is necessary and sufficient that the vector π_i which define a one-dimensional hermitian projector R,

$$R_{ik} = \bar{\pi}_i \pi_k (\sum |\pi_l|^2)^{-1}, \tag{2.42}$$

satisfy the system of linear equations

$$(\partial_z + \widehat{\mu}\partial_t)\pi_i = 0, \qquad (\partial_t + \widehat{\mu}^{-1}\partial_{\bar{z}})\pi_i = 0. \tag{2.43}$$

In fact, π_i are entire functions of the complex variable $\xi = \widehat{\mu} z + \widehat{\mu}^{-1}\bar{z} - t$, i.e. $\pi_i = \pi_i(\xi)$.

A solution J of (2.17) is given by the expression

$$J = |\widehat{\mu}|^{-1}\left(I + (|\widehat{\mu}|^2 - 1)R\right). \tag{2.44}$$

Here $|\widehat{\mu}| > 1$ is the complex parameter which determines the soliton velocity via the relation

$$v = \frac{2|\widehat{\mu}|}{1 + |\widehat{\mu}|^2}, \tag{2.45}$$

whereas $-\arg \widehat{\mu}$ is an angle between v and x-axis.

Without loss of generality, for the one-soliton solution one can put $\pi_1 = 1$ and $\pi_2 = \pi(\xi)$. The requirement that J should be regular on the whole plane implies that π is a rational function of the complex variable ξ, that is

$$\pi(\xi) = c' \frac{(\xi - a_1')...(\xi - a_n')}{(\xi - b_1')...(\xi - b_m')}. \tag{2.46}$$

The parameters c', a_1',..., a_n', b_1', ..., b_m' are arbitrary, and correspond to certain intrinsic degrees of freedom. It is easy to produce other interesting solutions by choosing proper expressions for the function $\pi(\xi)$. For example, it is easy to produce the extended wave solitons by choosing

$$\pi(\xi) = c \exp(a\xi), \tag{2.47}$$

where again c and a are arbitrary complex parameters.

Explicit multisoliton solutions of (2.17), can also be constructed by a standard procedure. They are described by a factorized function $\psi(\lambda)$ of the form $\psi(\lambda) = \prod \psi_i(\lambda)$, where $\psi_i(\lambda)$ are functions like (2.41). Note that in this model, two-soliton solutions interact trivially. That means, on scattering the solitons suffer no change in velocity and no phase shift. Unfortunately, as we are going to see in the next chapter, there are no static solutions of (2.17).

2.4 Soliton Stability and Nontrivial Scattering

One of the most important questions about solitons is whether or not they are stable. The question is: if one starts with a soliton at a fixed point and perturbs its shape, does the solution stay close to the initial configuration for all t? In this section the stability of the one-soliton solution (2.35) under radially symmetric perturbations, is discussed.

The static soliton solutions of (2.16) are simply the embedding of the static lump solutions of the $\mathbf{CP^1}$ sigma model, and so it is worth comparing the stability of this model with that of lumps in the $\mathbf{CP^1}$ model. Before going any further, let us briefly describe the $\mathbf{CP^1}$ model which is an alternative description of the O(3) sigma model.

The field of the O(3) model is a real three vector which is constrainted to have unit length, i.e. $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ with the constraint $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = 1$. The target manifold is therefore a two sphere, i.e. $M = \mathbf{S}^2$. The name O(3) refers to the symmetry of the field $\boldsymbol{\phi}$ under rotation by a constant O(3) matrix. The equation of motion is

$$\partial_\mu \partial^\mu \boldsymbol{\phi} + (\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi})\, \boldsymbol{\phi} = 0, \tag{2.48}$$

which is derived from the free field Lagrangian

$$\mathcal{L} = \frac{1}{4} \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi}, \tag{2.49}$$

and the nonlinearities are due to the constraint $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$. This model will be the subject of chapter 5.

The static lump solutions are most easily written in terms of a complex field W, which is the stereographic projection of $\boldsymbol{\phi}$ from the point $\phi_3 = 1$ onto the complex plane, i.e.

$$W = \frac{\phi_1 + i\phi_2}{1 - \phi_3}, \tag{2.50}$$

which is an element of the coset space $\mathbf{CP^1}$; where

$$\mathbf{CP^n} = \frac{\mathrm{SU}(n+1)}{\mathrm{SU}(n) \times \mathrm{U}(1)}. \tag{2.51}$$

This alternative $\mathbf{CP^1}$ description of the O(3) model is possible because $\mathbf{CP^1}$ is isomorphic to $\mathbf{S^2}$. The Lagrangian in the $\mathbf{CP^1}$ formulation is

$$\mathcal{L} = \frac{\partial_\mu W \partial^\mu \bar{W}}{(1 + |W|^2)^2}. \tag{2.52}$$

The static solitons are the lumps (anti-lumps) of the O(3) model and are given by W a holomorphic (anti-holomorphic) function of $z = x + iy$ (cf. [36, 37]).

The $\mathbf{CP^1}$ model is conformally invariant in (2+0) dimensions, which is reflected in the fact that the static one-lump can have an arbitrary size. The total energy is independent of this size. Lumps of the $\mathbf{CP^1}$ model in (2+1) dimensions, possess a topological stability, due to the topological nature of the target manifold. Only field configurations with finite energy are considered, which requires that the field must take the same value at all points of spatial infinity. The upshot of this is that the space may be compactified from $\mathbf{R^2}$ to $\mathbf{S^2}$, so at any fixed time the field configuration may be considered as a map from $\mathbf{S^2}$ into $\mathbf{CP^1}$. The homotopy group relation

$$\pi_2(\mathbf{S^2}) = \mathbf{Z}, \tag{2.53}$$

then implies that to each field configuration there may be associated an integer, known as the topological charge, which is conserved and represents the winding number of the field as a map from space to the target manifold. An n-lump configuration is defined to be a field configuration with topological charge n. This means that the one-lump solution cannot decay to vacuum, since the vacuum has zero topological charge. This topological stability implies that the lumps of the $\mathbf{CP^1}$ model by themselves have no negative modes. However, it was found [38] that the lumps do possess zero modes (due to the conformal invariance of the model), which are modes of instability in which the width of the lumps become either infinite or zero. In other words, under small perturbations the size of the soliton tends to expand or shrink, depending on the exact form of the initial disturbance.

Now for the (2+1)-dimensional modified chiral model (2.16) there is no topological stability. This is due to the fact that the field J takes values in the gauge group SU(2), which has group manifold $\mathbf{S^3}$. The corresponding homotopy relation in this case is that

$$\pi_2(\mathbf{S^3}) = 0. \tag{2.54}$$

There is no winding number for such a map, and hence no topological charge. This means that the solitons of this model may possibly possess both negative modes (i.e. they may decay to the vacuum), and zero modes in a way similar to those found for the $\mathbf{CP^1}$ lumps.

By using the method of discretization (i.e. replacing derivatives by symmetric finite difference), Sutcliffe [39] found that there are no negative modes present for the one-soliton solution (2.35) for (2.16). Under small perturbations, the width of the soliton oscillates around its initial value with the amplitude of the oscillation decaying exponentially. This oscillation is accompanied by a ring of radiation (i.e. moving with the speed of light) that spreads from the centre of the soliton. Not only are negative modes not excited by these perturbations but there are also no zero modes excited, so it appears that the soliton is stable.

As shown in the previous section multisoliton solutions of (2.16) have been found which correspond to solitons that interact in a trivial way. On scattering the solitons suffer no change in velocity and no phase shift. The static solitons of this model (2.16) are the embeddings of the $\mathbf{CP^1}$ static lumps, so it would be interesting to see the relationship between the trivial scattering of multisoliton solutions and the lumps of the $\mathbf{CP^1}$ model in (2+1) dimensions. The lumps of the latter model have nontrivial scattering [40, 41] in

Let us consider soliton collisions governed by the modified chiral model (2.16). The known soliton solutions (inverse of) (2.31) pass each other without any change of velocity or shape. However, the chiral field has internal degrees of freedom, and these can affect the scattering of solitons: one can get nontrivial scattering, despite the fact that the system is integrable. This was discovered in numerical experiments [42].

The numerical procedure was tested on the two-soliton exact solution, and this confirmed its reliability (and the absence of scattering). Then different initial data (still representing two solitons fired at each other) was used: this time, the two solitons collided to form a ring, and separated at 90^0. In other words, one gets highly nontrivial scattering, similar in nature to that occurring in nonintegrable systems. In addition, numerical simulations of soliton-antisoliton collisions in the integrable system also reveal 90^0 scattering. In this case, the integrability preserves the solitons (unlike in nonintegrable models, where they annihilate, i.e. after head-on collision, only radiation remains) but permits nontrivial scattering.

In principle, one should be able to use the techniques of inverse scattering to analyze the problem, but this does not appear to be straightforward. In contrast, Ward [44] recently, using an analytic method, has generated an explicit solution (since the model is integrable) representing nontrivial soliton interaction; in particular, in a head-on collision two solitons undergo 90^0 scattering. In chapter 4 we will further investigate this scheme, by studying the nontrivial interaction of multisoliton solution; and modify it. In fact, we will see how to deduce families of soliton solutions with nontrivial scattering as well as soliton-antisoliton solutions with elastic scattering (no radiation is emitted).

Chapter 3

Conserved Quantities

3.1 Introduction

The four dimensional self-dual Yang-Mills equations (2.7) have many of the properties of an integrable system: Bäcklund tranformations [25]; nonlocal conservation laws [45]; and the corresponding linear system [24]. Infinite sequences of nonlocal conservation laws (continuity equations) have been known for many years [24, 25], but these do not necessarily yield conserved quantities, since the relevant integrals may diverge.

This chapter deals with integrable chiral equations in (2+1) dimensions. Recall that these are equivalent to the self-dual Yang-Mills equation, reduced from (2+2) to (2+1) dimensions. This two dimensional version is known to possess an infinite number of conservation laws since it admits a Lax pair formulation as well as an auto- and non-auto-Bäcklund transformation (see later). Up to now, most of the known conserved densities do not yield finite conserved charges: the spatial integrals either diverge, or when they do converge they are equal to zero. In fact, for these equations the solitons are localized in space, but the localization is only polynomial, and the conserved densities referred to above do not fall off fast enough at spatial infinity, and so cannot be integrated to give finite conserved quantities. Here we exhibit many conserved quantities that do exist, and have a simple explicit form.

In this chapter, we demonstrate the existence of infinite sequences of well-defined conserved quantities for the integrable chiral equations. These may help to throw further light on the dynamics of solitons in this system, such as their stability [39] and the fact that they can scatter either trivially [26] or nontrivially [42, 44]. We also discuss local conserved quantities, including Noether charges, arising from symmetries of the Lagrangian. For simplicity, we take the gauge group to be SU(2); but the infinite sequences of conserved quantities extend automatically to arbitrary gauge group.

3.2 Integrable Chiral Equations

Recall the two versions of the integrable chiral equation (2.14) of section 2.2:

$$S: \qquad \partial_v(J^{-1}J_u) - \partial_x(J^{-1}J_x) = 0, \tag{3.1}$$

with $u = (t+y)/2$ and $v = (t-y)/2$.

$$T: \qquad \partial_t(J^{-1}J_t) - \partial_{\bar{z}}(J^{-1}J_z) = 0, \tag{3.2}$$

with $z = (x+iy)/2$.

The issue of boundary conditions is crucial, and we shall be using boundary conditions which are natural for the chiral equation form, rather than the gauge-theory form. We impose the boundary condition given by (2.18), i.e.

$$J = J_0 + J_1(\theta)\, r^{-1} + O(r^{-2}), \tag{3.3}$$

as $r \to \infty$, where $x + iy = r\exp(i\theta)$. This condition allows the existence of finite-energy soliton solutions.

Equation (2.14) has a global symmetry, in that J can be multiplied on both sides by constant matrices. If we require the reality conditions to be preserved, then this global symmetry is SO(4) in the case of the S-equation (3.1), and SO(1,3) in the case of the T-equation (3.2).

To proceed further, let us remark on the static solutions of (3.1). Static solutions of (3.1) are harmonic maps from $\mathbf{R}^2$ into SU(2)$\cong$ $\mathbf{S}^3$ (due to boundary condition) and they look like n localized solitons (lumps of energy) in the xy-plane. Given the boundary condition (3.3), these are all known [46]: up to the global symmetry mentioned above, they are

$$J = \frac{i}{(1+|f|^2)}\begin{pmatrix} 1-|f|^2 & 2f \\ 2\bar{f} & |f|^2-1 \end{pmatrix}, \tag{3.4}$$

where J is smooth everywhere and satisfies the boundary condition at infinity, provided f is a rational function of either z or $\bar{z}$. [This follows from (2.35) with $\mu = i$.]

The corresponding energy-momentum tensor $T^{\mu\nu}$ (2.19) relevant to (3.1), as already mentioned, is not conserved, and neither in general is the energy-momentum vector $P^{\mu} = T^{\mu 0}$. More precisely, from (2.20) it is obvious that the energy and the y-momentum are conserved. Although, in general, the x-momentum is not conserved. For the one-soliton solution (2.35), however, the x-momentum happens to be conserved as will now be shown. The x-momentum can be computed directly by the formula

$$\mathcal{X} = \iint T^{10}\, dx\, dy, \tag{3.5}$$

and we find that $d\mathcal{X}/dt = 0$. Furthermore, we can compute it explicitly, i.e. its final form turns out to be

$$\mathcal{X} = -\frac{8\alpha|\beta|\pi}{\alpha^2+\beta^2}, \qquad \beta \neq 0, \tag{3.6}$$

where $\mu = a + i\beta$ is the complex parameter which determines the soliton velocity. Note that the x-momentum is independent of t, as expected and depends, only, on the soliton velocity.

Let us present an analysis of the two-soliton interaction, pointing out the main features. It seems like that the general case will be very similar. The two-soliton solution given by (2.31) for $n = 2$, depends on the two complex constants μ_1 and μ_2, and on the two holomorphic functions f_1 and f_2. Take these functions to be of the form (2.32). Then, the corresponding solution represent two lumps L_1 and L_2, where L_k travels at a velocity determined by μ_k. Specifically, L_k travels along a straight line determined by μ_k and c_k: but the two lumps do not scatter off each other. There is no change of direction or phase shift when they pass each other. One can see this as follows.

Let us consider the case of a soliton L_2 incident on a stationary lump L_1. In terms of the input to the two-soliton solution, let us take $\alpha_1 = \alpha_2 = 1$; $c_1 = c_2 = 0$; $\mu_1 = e^{(i\pi/2)}$ and $\mu_2 = e^{(i\pi/4)}$. Therefore, by taking $f_k(\omega_k) = \omega_k$ then the unitary matrix J is given in terms of the two holomorphic functions ω_1 and ω_2 as

$$J_{11} = (\sqrt{i}\Delta)^{-1}[\sqrt{2}\,(1-|\omega_1|^2)(1-i|\omega_2|^2)-(1+|\omega_1|^2)(1+i|\omega_2|^2)+\sqrt{2}\,(1+i)(\bar{\omega}_2\omega_1+\bar{\omega}_1\omega_2)],$$
$$J_{12} = (i-1)(\sqrt{i}\Delta)^{-1}[\omega_2(1+|\omega_1|^2)-\sqrt{2}\,\omega_1(1+|\omega_2|^2)],$$
$$J_{22} = \bar{J}_{11}, \qquad J_{21} = -\bar{J}_{12}, \tag{3.7}$$

where

$$\Delta = (\sqrt{2}-1)(1+|\omega_1|^2)(1+|\omega_2|^2)+2|\omega_1-\omega_2|^2. \tag{3.8}$$

Since Δ is nowhere zero, J is smooth everywhere; and it satisfies the required boundary condition (3.3) at spatial infinity, as can be easily checked .

Figure 3.1 shows a series of snapshots of the two-soliton interaction. The physical picture is this: the shape and velocity of the solitons are the same long before and long after the collision, and they suffer no phase shift. In fact, the first lump L_1 (taller one) remains stationary at the origin, while L_2 moves along the x-axis with speed $1/2$. At $t=0$ the two lumps coincide, and form a single sharp peak with height (maximum of the energy density) approximately eight times the original height of the stationary lump. Although, the total energy remains unchanged. However, this is the only effect of the interaction (no phase shift; no radiation). In spite of the fact that the model is not rotationally symmetric, the same features occur when the second lump L_2 moves along the y-axis while the first lump L_1 remains stationary at the origin (cf. [26]).

3.3 Lagrangian and Local Conserved Quantities

The classical conserved quantities arise, via Noether's theorem, from symmetries of a Lagrangian. As we are going to see, a Lagrangian can be obtained for the chiral form of the sdYM equations, but then the global symmetry [24] of the model is broken. Let us concentrate on the T-equation (3.2) for the time being, to see the explicit expressions.

Following a well-established technique, we parametrize J with the help of Poincaré coordinates

$$J = \phi^{-1}\begin{pmatrix} 1 & \bar{\rho} \\ \rho & \phi^2+|\rho|^2 \end{pmatrix}. \tag{3.9}$$

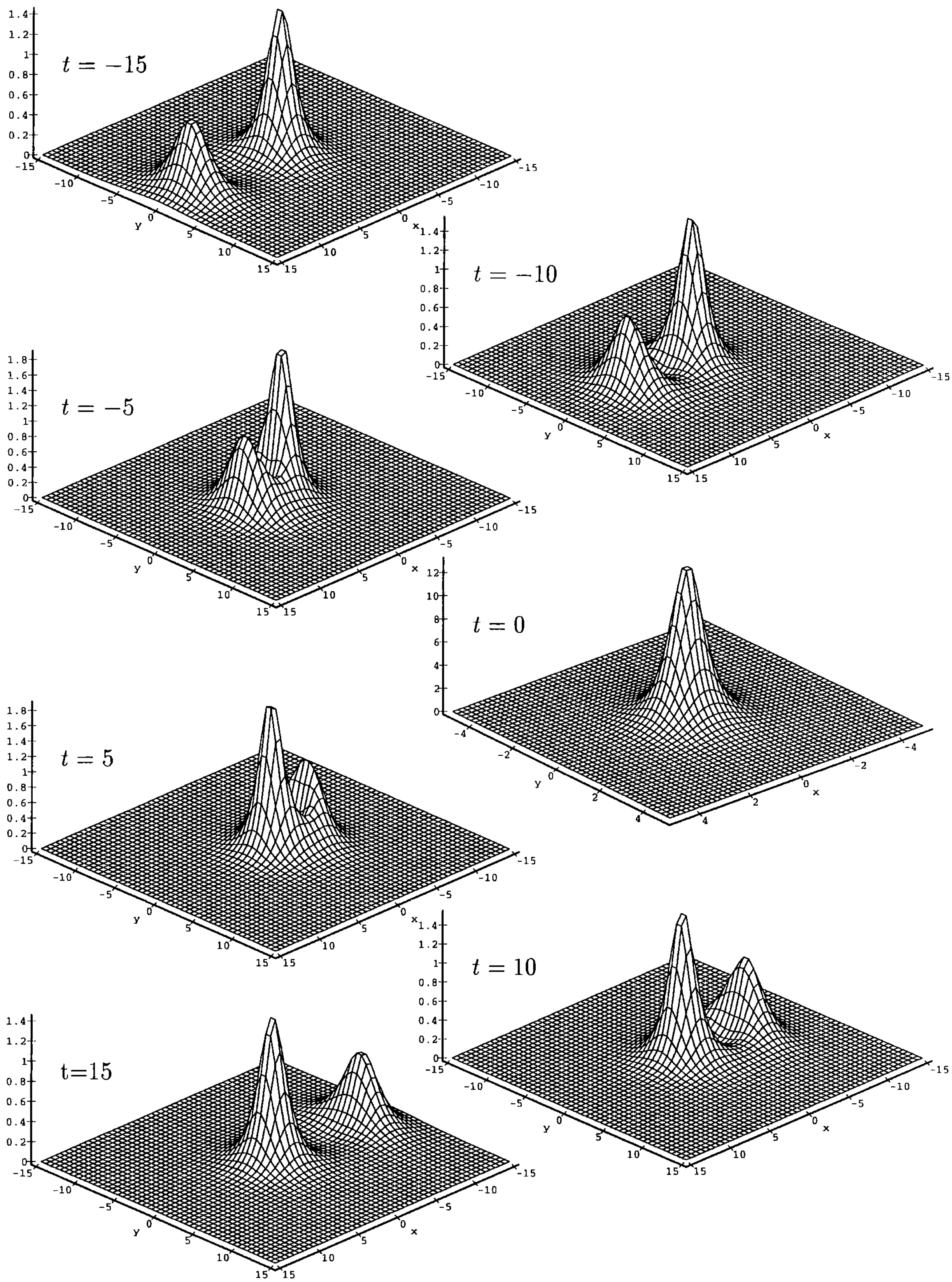


Figure 3.1: A series of snapshots of the energy density for the two-soliton interaction. The one lump is stationary in the origin (in the middle of the square) and the other one is moving towards it along x-axis.

Note that ϕ is well-defined and real, owing to the positivity of J, while ρ is a complex function of the coordinates $(z, \bar{z}, t)$. Substituting this parametrized formula of J to the T-equation (3.2), the latter transforms to the following set

$$\begin{aligned} &\phi(\phi_{tt} - \phi_{\bar{z}z}) - \phi_t^2 + |\phi_z|^2 + |\rho_t|^2 - |\rho_z|^2 = 0, \\ &\phi(\bar{\rho}_{tt} - \bar{\rho}_{\bar{z}z}) - 2\phi_t\bar{\rho}_t + 2\phi_z\bar{\rho}_{\bar{z}} = 0, \\ &\phi(\rho_{tt} - \rho_{\bar{z}z}) - 2\phi_t\rho_t + 2\phi_{\bar{z}}\rho_z = 0, \end{aligned} \tag{3.10}$$

where the subscripts denote differentiation and $|\rho_t|^2 = \rho_t\bar{\rho}_t$ denotes the norm-squared of ρ_t. Equations (3.10) are the Euler-Lagrange equations for the following Lagrangian

$$\mathcal{L} = \phi^{-2}(\phi_t^2 - |\phi_z|^2 + |\rho_t|^2 - |\rho_z|^2). \tag{3.11}$$

An obvious symmetry is the four-parameter family

$$\phi \to |b|\phi, \qquad \rho \to b\rho + c, \tag{3.12}$$

where b and c are complex constants. This is part of the global symmetry noted previously, after the dimensional reduction. It follows from the fact that although in the (unmodified) chiral model we have an SO(1,2) spacetime symmetry, for the T-equation (3.2) this symmetry no longer exists due to the timelike vector V_α; instead, there is a residual symmetry group, which is SO(2).

The corresponding conserved Noether densities are components of $J^{-1}J_t$. In fact, it is already clear that $J^{-1}J_t$ is a matrix of conserved densities, since (3.2) has the form of a conservation law. However these densities go like $O(r^{-2})$ as $r \to \infty$, and so the corresponding charges are not, in general, well-defined.

The next obvious symmetries of (3.11) are the spacetime translations, and these lead to a conserved energy-momentum tensor, which is

$$T^{\mu\nu} = g^{\mu\alpha}\phi^{-2}[\delta_0^\nu(2\phi_\alpha\phi_t + \rho_\alpha\bar{\rho}_t + \rho_t\bar{\rho}_\alpha) - \delta_1^\nu(\phi_{\bar{z}}\phi_\alpha + \bar{\rho}_{\bar{z}}\rho_\alpha] - \delta_2^\nu(\phi_z\phi_\alpha + \rho_z\bar{\rho}_\alpha)) - g^{\mu\nu}L, \tag{3.13}$$

where $x'^\mu = (x'^0, x'^1, x'^2) = (t, z, \bar{z})$ and the components of the metric tensor are $g^{00} = 1$, $g^{12} = g^{21} = -(1/2)$ and all the rest are equal to zero. Clearly, we find an energy-momentum tensor which is conserved; and also is the corresponding energy-momentum vector, i.e. $\partial_\mu P^\mu = \partial_\mu T^{\mu 0} = 0$. Consequently, the energy density is

$$P^0 = \phi^{-2}(\phi_t^2 + |\rho_t|^2 + |\phi_z|^2 + |\rho_z|^2), \tag{3.14}$$

which is $O(r^{-4})$ as $r \to \infty$. So the energy (the spatial integral of P^0) is a well-defined positive-definite functional of the field. The momentum is also well-defined; the x-momentum and y-momentum densities are

$$\begin{aligned} P^1 &= -\mathrm{tr}(J^{-1}J_xJ^{-1}J_t) \\ &= -\phi^{-2}(2\phi_t\phi_x + \bar{\rho}_t\rho_x + \rho_t\bar{\rho}_x). \\ P^2 &= -\mathrm{tr}(J^{-1}J_yJ^{-1}J_t) \\ &= -\phi^{-2}(2\phi_t\phi_y + \bar{\rho}_t\rho_y + \rho_t\bar{\rho}_y). \end{aligned} \tag{3.15}$$

Note that the momentum densities are invariant under the full SO(1,3) global symmetry, whereas the energy density is not; although it is invariant under the reduced symmetry (3.12).

By way of example, let us examine the one-soliton solution. The field J (2.44) takes the simple form

$$J = \frac{1}{|\widehat{\mu}|(1+|\pi|^2)} \begin{pmatrix} |\widehat{\mu}|^2 + |\pi|^2 & (|\widehat{\mu}|^2 - 1)\pi \\ (|\widehat{\mu}|^2 - 1)\bar{\pi} & 1 + |\widehat{\mu}|^2|\pi|^2 \end{pmatrix}, \tag{3.16}$$

where π is a rational meromorphic function of

$$\xi = \widehat{\mu}\, z + \widehat{\mu}^{-1}\, \bar{z} - t, \tag{3.17}$$

and $\widehat{\mu}$ is a complex constant with $|\widehat{\mu}| > 1$. This solution represents a single lump located at $\xi = 0$. This locus is a point in space which moves in a straight line with constant speed $v = 2|\widehat{\mu}|/(1 + |\widehat{\mu}|^2)$. The direction of the motion is determined by the phase of $\widehat{\mu}$.

Note that there are no static solitons for the T-equation (3.2), contrary to the S-equation (3.1). The velocity v is equal to zero (static solitons), when the complex parameter $\widehat{\mu}$ is either infinite or zero; but then, in both cases the complex variable ξ and accordingly the field J are ill-defined. Additionally, note that (3.2) with $J_t = 0$ implies that $J^{-1}J_z = 0$, since $J^{-1}J_z$ tending to zero at infinity; and this is true only when $J = J_0$ constant; by Liouville's theorem.

By equating (3.9) and (3.16) the values of the fields ϕ, ρ are

$$\begin{aligned} \phi &= \frac{|\widehat{\mu}|(1+|\pi|^2)}{(|\widehat{\mu}|^2 + |\pi|^2)}, \\ \rho &= \frac{\bar{\pi}(|\widehat{\mu}|^2 - 1)}{(|\widehat{\mu}|^2 + |\pi|^2)}. \end{aligned} \tag{3.18}$$

For the sake of simplicity, take $\pi(\xi) = \xi$. Then

$$P^0 = \frac{(|\hat{\mu}|^2 - 1)^2(|\hat{\mu}|^2 + 1)}{|\hat{\mu}|^2(|\hat{\mu}|^2 + |\xi|^2)(1 + |\xi|^2)}. \tag{3.19}$$

The energy of the soliton, expressed as a function of v, is $E = 8\pi\,\mathrm{sech}^{-1}v$. This has the *anti-relativistic* feature of decreasing as v increases: $E \to 0$ as $v \to 1$, and $E \to \infty$ as $v \to 0$, which is consistent with the absence of static solitons. [Note that, we have set the speed of light, c, equal to unity, in order to use dimensionless quantities in all our calculations].

There is also a conserved angular momentum, corresponding to the rotation symmetry $z \to z\exp(i\chi)$. This, together with the energy and momentum, are the only local conserved quantities of which we are aware; and their existence is not really connected with the integrability of the equation.

In [47] a process is described for deriving infinite sets of local conserved densities for the sdYM fields with arbitrary gauge group. These are constructed from differential operators acting on $J^{-1}J_\mu$. The analogous process utilizes a set of constructed Bäcklund transformations for the S-equation (3.1). [Similar process exists for the T-equation (3.2)]. This particular construction of auto-Bäcklund transformations is possible due to the fact that the chiral equations can be placed in the form of conservation laws.

The set of eight parametric Bäcklund transformations, which upon integration, produce *new* solutions J' of (3.1) from *old* ones J, is

$$\begin{aligned}
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda\partial_\mu(J^{-1}J_u), \\
J^{-1\prime}J'_x &= J^{-1}J_x + \lambda\partial_\mu(J^{-1}J_x), \qquad \mu = u, v, x;
\end{aligned}$$

$$\begin{aligned}
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda(1 + u\partial_u + x\partial_x)(J^{-1}J_u), \\
J^{-1\prime}J'_x &= J^{-1}J_x + \lambda(u\partial_u + x\partial_x)(J^{-1}J_x),
\end{aligned}$$

$$\begin{aligned}
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda(v\partial_v + x\partial_x)(J^{-1}J_u), \\
J^{-1\prime}J'_x &= J^{-1}J_x + \lambda(v\partial_v + x\partial_x)(J^{-1}J_x),
\end{aligned}$$

$$\begin{aligned}
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda(1 + u\partial_v + x\partial_x)(J^{-1}J_u), \\
J^{-1\prime}J'_x &= J^{-1}J_x + \lambda(u\partial_v + x\partial_x)(J^{-1}J_x),
\end{aligned}$$

$$\begin{aligned}
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda(x\partial_v + u\partial_x)(J^{-1}J_u) + \lambda(J^{-1}J_x),\\
J^{-1\prime}J'_x &= J^{-1}J_x + \lambda(x\partial_v + u\partial_x)(J^{-1}J_x),\\
J^{-1\prime}J'_u &= J^{-1}J_u + \lambda(x\partial_v + v\partial_x)(J^{-1}J_u) + \lambda(J^{-1}J_x),\\
J^{-1\prime}J'_x &= J^{-1}J_u + \lambda(J^{-1}J_u) + \lambda(x\partial_v + v\partial_x)(J^{-1}J_x).
\end{aligned} \tag{3.20}$$

Evidently, all eight Bäcklund transformations are of the general form

$$\begin{aligned}
J^{-1}J'_u &= J^{-1}J_u + \lambda\,[L_1(J^{-1}J_u) + L_2(J^{-1}J_x)],\\
J^{-1}J'_x &= J^{-1}J_x + \lambda\,[L_3(J^{-1}J_u) + L_4(J^{-1}J_x)],
\end{aligned} \tag{3.21}$$

where $L_1, ..., L_4$ are local linear operators. Thus we have eight families of equations of the above form, the members of each family having the same set of operators $L_1, ..., L_4$, but different values of λ. The above equation can be written in matrix form of the Bäcklund transformations $\mathbf{B}^\lambda$:

$$\begin{bmatrix} J'^{-1}J'_u \\ J'^{-1}J'_x \end{bmatrix} = \mathbf{M}^\lambda \begin{bmatrix} J^{-1}J_u \\ J^{-1}J_x \end{bmatrix}, \tag{3.22}$$

where $\mathbf{M}^\lambda$ is the operator-valued matrix

$$\begin{bmatrix} 1+\lambda L_1 & \lambda L_2 \\ \lambda L_3 & 1+\lambda L_4 \end{bmatrix}. \tag{3.23}$$

Let us denote with $\Psi(J)$ the two-dimensional column vector with components

$$B_u \equiv J^{-1}J_u, \qquad B_x \equiv J^{-1}J_x. \tag{3.24}$$

Multiple application of the Bäcklund transformations $\mathbf{B}^\lambda$ on J, will produce a new function $\Psi(\mathcal{K})$ such that

$$\Psi(\mathcal{K}) = (\mathbf{M}^\lambda)^n\,\Psi(J). \tag{3.25}$$

It follows from this that the quantities $\mathcal{K}^{-1}\mathcal{K}_u$ and $\mathcal{K}^{-1}\mathcal{K}_x$ are expansions in power of λ, i.e.

$$\mathcal{K}^{-1}\mathcal{K}_u = \sum_{r=0}^{n} \lambda^r P_r, \qquad \mathcal{K}^{-1}\mathcal{K}_x = \sum_{r=0}^{n} \lambda^r R_r. \tag{3.26}$$

Now, since $\mathcal{K}$ is a chiral field, it satisfies the continuity equation (3.1), i.e. $\partial_v(\mathcal{K}^{-1}\mathcal{K}_u) - \partial_x(\mathcal{K}^{-1}\mathcal{K}_x) = 0$. Substituting from (3.26) and equating coefficients of powers of λ to zero, we obtain the $(n+1)$ local continuity equation,

$$\partial_v(P_r) - \partial_x(R_r) = 0, \qquad r = 0, 1, ..., n. \tag{3.27}$$

By letting $n \to \infty$ we obtain an infinite set of local conservation laws from each family of Bäcklund transformations, $\mathbf{B}^{\lambda}$.

The corresponding number of conserved charges are

$$Q_r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_r \, dx \, dy, \qquad r = 0, 1, ..., n. \tag{3.28}$$

The local property of these conservation laws follows from the fact that the densities P_r and R_r are obtained directly from (3.26) for all values of r; and thus their derivation does not require knowledge of lower-order charges.

By virtue of (3.25) we find that $P_0 = B_u$, for each family of Bäcklund transformations. In addition, for $r = 1, 2, ..., n$ we obtain

$$\begin{aligned}
P_r &= \begin{pmatrix} n \\ r \end{pmatrix} L_1^{(r)} B_u, \\
P_r &= \begin{pmatrix} n \\ r \end{pmatrix} L_1^{(r-1)}(L_1 B_u + r B_x), \\
P_r &= \tfrac{1}{2}\begin{pmatrix} n \\ r \end{pmatrix} [(1 + L_1)^r (B_u + B_x) + (L_1 - 1)^r (B_u - B_x)],
\end{aligned} \tag{3.29}$$

where the binomial is given by the expression

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n\,(n-1)....(n-r-1)}{r\,!}. \tag{3.30}$$

The first equation of the set (3.29) corresponds to the first six families of Bäcklund transformations, while the other two correspond to the seventh and eight family, respectively.

Consequently, the first conserved charge for all Bäcklund transformations is

$$Q_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_u \, dx \, dy. \tag{3.31}$$

By using the boundary condition (3.3) and integrating by parts, we deduce that the remaining conserved charges $\{Q_r\}$ (for $r = 1, ..., n$) are equal to zero for each family of transformations, except: the seventh and eight. For these cases, the conservation laws are given by the inductive relations

$$\begin{aligned}
Q_r &= -\frac{2(n-r+1)}{r} Q_{r-1}, \\
Q_r &= \frac{(n-r+1)}{r} Q_{r-1}, \qquad r \geq 1,
\end{aligned} \tag{3.32}$$

respectively. Unfortunately, if we attempt to verify the above set for the soliton solutions, they diverge in the same way as the integral of $J^{-1}J_u$. Even if a boundary condition were chosen which ensured convergence, integration by parts gives relations between these local conserved charges which indicate that very few, if any, of them are independent and new. Although, one still has the obvious local conservation laws, irrespective of convergence.

We conclude this section by mentioning the corresponding results for the S-equation (3.1). If we parametrize the chiral field J, as

$$J = \phi^{-1}\begin{pmatrix} 1 & \tilde{\rho} \\ \rho & \phi^2 + \rho\tilde{\rho} \end{pmatrix}, \tag{3.33}$$

by regarding ϕ, ρ, $\tilde{\rho}$ as arbitrary functions of the variables (t,x,y), (3.1) reads

$$\begin{aligned} &G^{\kappa\lambda}(\phi\phi_{\kappa\lambda} + \tilde{\rho}_\kappa\rho_\lambda - \phi_\kappa\phi_\lambda) = 0, \\ &G^{\kappa\lambda}(\phi\rho_{\kappa\lambda} - 2\phi_\kappa\rho_{\lambda)} = 0, \\ &G^{\kappa\lambda}(\phi\tilde{\rho}_{\kappa\lambda} - 2\phi_\lambda\tilde{\rho}_{\kappa)} = 0. \end{aligned} \tag{3.34}$$

Here $G^{\kappa\lambda} = \eta^{\kappa\lambda} + \varepsilon^{\kappa\lambda}$ is the metric with $\eta^{\kappa\lambda} = \mathrm{diag}(-1,1,1)$ and the constant tensor $\varepsilon^{\kappa\lambda}$ is the dual vector, i.e. $\varepsilon^{\kappa\lambda} = \varepsilon^{\alpha\kappa\lambda}V_\alpha$. The above equations are the Euler-Lagrange equations for a variational problem with Lagrangian

$$\mathcal{L} = \phi^{-2}G^{\kappa\lambda}(\phi_\kappa\phi_\lambda + \tilde{\rho}_\kappa\rho_\lambda). \tag{3.35}$$

Accordingly, the energy-momentum tensor relevant to the original system (3.34) is

$$T^{\mu\nu} = \eta^{\mu\nu}L - \phi^{-2}(2\eta^{\kappa\nu}\eta^{\mu\alpha}\phi_\kappa\phi_\alpha + G^{\kappa\nu}\eta^{\mu\alpha}\tilde{\rho}_\kappa\rho_\alpha + G^{\nu\kappa}\eta^{\mu\alpha}\rho_\kappa\tilde{\rho}_\alpha). \tag{3.36}$$

Obviously, this tensor is non-symmetric in μ and ν but it is conserved.

It is evident from the S-equation that $J^{-1}J_u$ is a conserved density; but as before, the corresponding charges diverge. The energy and y-momentum are well-defined, their densities are

$$\begin{aligned} P^0 &= -\frac{1}{2}\mathrm{tr}\left[(J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2\right] \\ &= -\phi^{-2}(\phi_x^2 + \phi_y^2 + \phi_t^2 + \rho_x\tilde{\rho}_x + \rho_y\tilde{\rho}_y + \rho_t\tilde{\rho}_t), \\[2ex] P^2 &= \mathrm{tr}\,(J^{-1}J_yJ^{-1}J_t) \\ &= \phi^{-2}(2\phi_t\phi_y + \tilde{\rho}_y\rho_t + \tilde{\rho}_t\rho_y). \end{aligned} \tag{3.37}$$

A conserved x-momentum density can also be obtained from the Lagrangian (3.35), i.e.

$$P^1 = \phi^{-2}(2\phi_x\phi_t + \tilde{\rho}_t\rho_x + \tilde{\rho}_x\rho_t - \tilde{\rho}_y\rho_x + \tilde{\rho}_x\rho_y). \tag{3.38}$$

But the functions appearing in it have singularities in general (in particular, this is the case for the soliton solution); and as a consequence, the x-momentum is divergent. The problem occurs because of singularities in a parametrization such as (3.33) in this case. It is not obvious whether one can find a parametrization which avoids these singularities. For the one-soliton solution (2.35), however, it happens to be conserved as will now be shown.

Compared to the densities which correspond to the energy-momentum tensor (2.19) of the (unmodified) chiral model; notice that, although the energy and the y-momentum density are the same, the x-momentum density $P^1 = \phi^{-2}(2\phi_x\phi_t + \tilde{\rho}_t\rho_x + \tilde{\rho}_x\rho_t)$ is different from the one given by (3.38) by the function $X(x, y, t) = \phi^{-2}(\tilde{\rho}_x\rho_y - \tilde{\rho}_y\rho_x)$.

It is easy to verify that the fields ϕ, ρ and $\tilde{\rho}$, for the one-soliton solution (2.35), take the values

$$\begin{aligned} \phi &= \frac{|\mu|(1+|f|^2)}{\mu+\bar{\mu}|f|^2}, \\ \rho &= \frac{(\mu-\bar{\mu})\bar{f}}{\mu+\bar{\mu}|f|^2}, \\ \tilde{\rho} &= \frac{(\mu-\bar{\mu})f}{\mu+\bar{\mu}|f|^2}, \end{aligned} \tag{3.39}$$

where f is a rational meromorphic function of $\omega = x + \mu u + \mu^{-1}v$. In the remainder of this section and for the sake of simplicity, let us take $f(\omega) = \omega$ and substitute the complex parameter μ by the analytic form $\mu = \alpha + i\beta$.

In this case, the x-momentum which is the integral of the density P^1 given by (3.38) over the spacelike plane $x^0 = \text{const}$, becomes

$$\mathcal{X} = -\frac{8\alpha|\beta|\pi}{\alpha^2+\beta^2} + \iint X(x, y, t)\, dx\, dy, \tag{3.40}$$

with

$$X(x, y, t) = \frac{(\bar{\mu}-\mu)^3(|\mu|^2+1)(\mu-\bar{\mu}|f|^2)}{2|\mu|^4(1+|f|^2)^2(\mu+\bar{\mu}|f|^2)}. \tag{3.41}$$

In order to simplify things, let us introduce the new variables (w, v) so that the function f becomes: $f = w + iv$, with

$$w = x + \alpha\frac{(\alpha^2+\beta^2-1)}{2(\alpha^2+\beta^2)}y + \alpha\frac{(\alpha^2+\beta^2+1)}{2(\alpha^2+\beta^2)}t,$$

$$v = \frac{\beta}{2}\left(\frac{(\alpha^2+\beta^2+1)}{\alpha^2+\beta^2}y + \frac{(\alpha^2+\beta^2-1)}{\alpha^2+\beta^2}t\right). \tag{3.42}$$

Hence, (3.40) yields

$$\mathcal{X} = -\frac{8\alpha|\beta|\pi}{(\alpha^2+\beta^2)} + \frac{8\beta\,|\beta|\,i}{(\alpha^2+\beta^2)}\iint \frac{(\alpha+i\beta-(\alpha-i\beta)(w^2+v^2))\;dw\,dv}{(1+w^2+v^2)^2\,(\alpha+i\beta+(\alpha-i\beta)(w^2+v^2))}. \tag{3.43}$$

Eventually, to make the situation even more transparent, one can replace (w, v) with polar coordinates.

- By way of illustration, if $\alpha = 0$ then (3.40) transforms to

  $$\mathcal{X} = 16\pi i\frac{|\beta|}{\beta}\int_{-\infty}^{\infty}\frac{r\,dr}{(1-r^4)}, \tag{3.44}$$

  and therefore the x-momentum diverges. Hence the x-momentum density becomes discontinuous when μ is a pure imaginary number.

- On the other hand, if $\alpha \neq 0$ the x-momentum may be readily calculated explicitly, i.e.

  $$\mathcal{X} = -8\pi\frac{|\beta|}{\beta}\arctan\left(\frac{\beta}{\alpha}\right), \qquad \beta \neq 0. \tag{3.45}$$

  Thus the momentum is well-defined, real and independent of t, as was expected.

The upshot is that, in the one-soliton sector, one has soliton solution with definite and finite x-momentum when the complex parameter μ which determines the velocity has a real non-zero part and arbitrary imaginary part. In general though, the x-momentum diverges.

3.4 Nonlocal Conserved Quantities

An infinite sequence of nonlocal conserved currents for the sdYM equations was first exhibited by Prasad *et al* [25, 45], and independently by Pohlmeyer [24]. This was motivated by an analogous sequence for the two-dimensional chiral model. A different sequence of nonlocal currents was later given by Leznov (cf. [48]), and independently by Papachristou [49]. A third sequence was mentioned by Sutcliffe [39]. For the time being, let us concentrate on the S-equation (3.1) to see the explicit expressions.

∘ Prasad *et al* exhibited a set of nonlocal conservation laws for the sdYM equations, by using an inductive procedure. Let us describe the one-dimensional reduction of this procedure.

Consider the B_u and B_x of (3.24) to be the first conserved currents

$$\mathcal{V}_u^{(1)} \equiv B_u = \partial_x \Psi^{(1)}, \qquad \mathcal{V}_x^{(1)} \equiv B_x = \partial_v \Psi^{(1)}. \tag{3.46}$$

Here $\Psi^{(1)}$ exists because of (3.1). Suppose that $\mathcal{V}_u^{(n)}$ and $\mathcal{V}_x^{(n)}$ are chiral fields, therefore they satisfy the S-equation (3.1), for arbitrary n. That means that the n-th current exist, i.e.

$$\partial_v(\mathcal{V}_u^{(n)}) - \partial_x(\mathcal{V}_x^{(n)}) = 0, \qquad n = 1, 2, ... \tag{3.47}$$

So this implies that there exist a 2×2 complex matrix function $\Psi^{(n)}(u, v, x)$ such that

$$\mathcal{V}_u^{(n)} = \partial_x \Psi^{(n)}, \qquad \mathcal{V}_x^{(n)} = \partial_v \Psi^{(n)}, \qquad n = 1, 2, ... \tag{3.48}$$

Then the $(n+1)$-th currents, which imposed as an induction hypothesis, is defined as

$$\mathcal{V}_u^{(n+1)} = D_u \Psi^{(n)}, \qquad \mathcal{V}_x^{(n+1)} = D_x \Psi^{(n)}, \qquad n = 0, 1, ..., \tag{3.49}$$

where D_k is the covariant derivative, defined as $D_k = \partial_k + B_k$, $k = u, x$. Observe that, the induction starts with $\Psi^{(0)} = I$, and thus $\mathcal{V}_u^{(1)} = J^{-1} J_u$ and so forth.

It is a matter of algebra to prove that $\mathcal{V}^{(n+1)}$ is conserved, and satisfies the S-equation. This is true since,

$$\begin{aligned} \partial_v(\mathcal{V}_u^{(n+1)}) - \partial_x(\mathcal{V}_x^{(n+1)}) &= (\partial_v D_u - \partial_x D_x)\Psi^{(n)} \quad \text{from (3.49)} \\ &= (D_u \partial_v - D_x \partial_x)\Psi^{(n)} \quad \text{due to (3.1)} \\ &= D_u \mathcal{V}_x^{(n)} - D_x \mathcal{V}_u^{(n)} \quad \text{using (3.48)} \\ &= [D_u, D_x]\Psi^{(n-1)} \quad \text{from (3.49)} \\ &= 0 \quad \text{due to (3.46).} \end{aligned} \tag{3.50}$$

Therefore, we can linerize the S-equation (3.1), using the nonlocal currents (3.48) and (3.49) in the following way,

$$\begin{aligned} \partial_x \Psi^{(n)} &= D_u \Psi^{(n-1)}, \\ \partial_v \Psi^{(n)} &= D_x \Psi^{(n-1)}, \qquad n = 1, 2, ..., \end{aligned} \tag{3.51}$$

with consistency equation

$$\partial_v(D_u \Psi^{(n)}) - \partial_x(D_x \Psi^{(n)}) = 0, \qquad n = 1, 2, ... \tag{3.52}$$

Multiplying both equations of the set (3.51) by λ^{-n}, summing over n, and defining

$$\Psi \equiv \sum_{n=0}^{\infty} \lambda^{-n} \Psi^{(n)}, \tag{3.53}$$

we obtain the linear set

$$\begin{aligned} (\lambda\partial_x - D_u)\Psi &= 0, \\ (\lambda\partial_v - D_x)\Psi &= 0, \end{aligned} \tag{3.54}$$

which is the *Lax pair* formulation given by (2.22).

- Pohlmeyer's procedure based on the fact that the field J is invariant under rotation around an angle $2 \cdot \theta$ in the (1,2)-plane, i.e.

$$J(u,v,x) \to J^{(\theta)}(u',v',x') = \Psi^{(\theta)}\, J(u,v,x)\Psi^{(\theta)^\dagger}, \tag{3.55}$$

where $\Psi^{(\theta)}$ is an SU(2)-matrix-valued function of u, v, x and of λ (here $\lambda \equiv \tan(\theta)^{-1}$). The hermitian adjoint of this matrix satisfies the following system of linear differential equations

$$\begin{aligned} (\lambda\partial_v - D_x)\Psi^{(\theta)^\dagger} &= 0, \\ (\lambda\partial_x - D_u)\Psi^{(\theta)^\dagger} &= 0. \end{aligned} \tag{3.56}$$

The approach employed is to expand the function $\Psi^{(\theta)^\dagger}$ in powers of the spectral parameter λ, i.e.

$$\Psi^{(\theta)^\dagger} = \sum_{n=0}^{\infty} \lambda^{n} \Psi^{(n)}, \tag{3.57}$$

and then, insert this expansion into the left hand sides of the system (3.56), collect all terms of the same order in λ and set the resulting coefficient separately equal to zero. As a consequence, the following set of equations have been obtained

$$\begin{aligned} D_x\Psi^{(0)} &= 0, \\ D_u\Psi^{(0)} &= 0, \\ \partial_x\Psi^{(n-1)} &= D_u\Psi^{(n)}, \\ \partial_v\Psi^{(n-1)} &= D_x\Psi^{(n)}, \qquad n = 1,2,\ldots \end{aligned} \tag{3.58}$$

with consistency equation identically equal to (3.52).

The conserved currents arising from (3.52) may then be used to construct the following infinite number of conserved charges

$$a_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(D_u\Psi^{(n)}\right)\, dx\, dy, \qquad n = 1,2,\ldots \tag{3.59}$$

Clearly, these currents are nonlocal as in order to determine $\Psi^{(n)}$ from $\Psi^{(n-1)}$ requires the integration of either (3.51) or (3.58); and are divergent in general.

As already mentioned, using the method of *Riemann problem with zeros*, $\Psi(\lambda)$ is known explicitly (see Ward [26]). In fact, for the one-soliton solution is given by (2.26) for $n = 1$ and possess the simple form

$$\Psi(\lambda) = \left(\frac{\lambda-\mu}{\lambda-\bar{\mu}}\right)^{1/2}\left(I + \frac{1}{(\lambda-\mu)}\frac{(\mu-\bar{\mu})}{(1+|f|^2)}\begin{pmatrix} 1 & f \\ \bar{f} & |f|^2 \end{pmatrix}\right). \tag{3.60}$$

Note that, $\Psi(\lambda)$ is not determined uniquely (cf. [45]) due to the fact that it can be multipled by an arbitrary function $\Lambda(\lambda, x + \lambda u + \lambda^{-1}v)$ on the right, so that $\Psi(\lambda) \to I$ at spatial infinity. By expanding $\Psi(\lambda)$ in terms of λ (e.g., Taylor's expansion), one may evaluate the components $\Psi^{(n)}$, and thus the conserved charges $\{a_n\}$. Unfortunately, $\{a_n\}$ are zero for (3.60).

- Papachristou's process yields an infinite number of nonlocal currents, the densities of which depend on an increasing number of nonlocal charges. These currents are obtained by an inductive process which involves various integrability conditions and successive introductions of nonlocal charges. To be more specific, one starts with the S-equation (3.1) and finds a simple non-auto-Bäcklund transformation that relates the equation of motion with a nonlocal conservation law depending on a nonlocal charge. Then another Bäcklund transformation is introduced which relates the aforementioned conservation law with a new one depending on an additional charge, and so forth. The above described progress can be continued infinitely; but no recursion relation seems to exist that allows expressions of currents densities in term of lower order charges.

  Let us start with the observation that, a new conservation law can be found for (3.1) by employing the simple Bäcklund transformation

$$J^{-1}J_u = \Phi^{(1)}_x, \qquad J^{-1}J_x = \Phi^{(1)}_v. \tag{3.61}$$

  Note that, $\Phi^{(1)}_x$ and $\Phi^{(1)}_v$ are equal to $\mathcal{V}^{(1)}_u$ and $\mathcal{V}^{(1)}_x$ of (3.67), respectively. The integrability condition $(\Phi^{(1)}_x)_v = (\Phi^{(1)}_v)_x$ of the system (3.61) yields the S-equation. On the other hand, the integrability condition $(J_u)_x = (J_x)_u$ or equivalent

$$\partial_u(J^{-1}J_x) - \partial_x(J^{-1}J_u) + [J^{-1}J_u, J^{-1}J_x] = 0, \tag{3.62}$$

  yields a nonlinear equation for $\Phi^{(1)}$

$$\Phi^{(1)}_{uv} - \Phi^{(1)}_{xx} + [\Phi^{(1)}_x, \Phi^{(1)}_v] = 0. \tag{3.63}$$

Equation (3.63) also arises from a Lagrangian (cf. [48]), but the corresponding energy functional is not positive-definite. Papachristou [50] called (3.63) the *potential sdYM equations.* (The reason is that, according to (3.61), $\Phi^{(1)}$ is a potential to the corresponding law, which is the sdYM equations). He pointed out that new conservation laws could be derived from symmetries of (3.63), and that such symmetries are in effect solutions of a novel linear system for the sdYM equations [51].

With the observation that

$$[\Phi^{(1)}_v, \Phi^{(1)}_x] = \frac{1}{2}\left(\partial_v[\Phi^{(1)}, \Phi^{(1)}_x] - \partial_x[\Phi^{(1)}, \Phi^{(1)}_v]\right), \tag{3.64}$$

equation (3.63) is written in the form of the continuity equation

$$\partial_v\left(\Phi^{(1)}_u - \frac{1}{2}[\Phi^{(1)}, \Phi^{(1)}_x]\right) - \partial_x\left(\Phi^{(1)}_x - \frac{1}{2}[\Phi^{(1)}, \Phi^{(1)}_v]\right) = 0. \tag{3.65}$$

Substituting for $\Phi^{(1)}_v$ and $\Phi^{(1)}_x$ from (3.61), the above equation becomes

$$\partial_v\left(\Phi^{(1)}_u + \frac{1}{2}[J^{-1}J_u, \Phi^{(1)}]\right) - \partial_x\left(\Phi^{(1)}_x + \frac{1}{2}[J^{-1}J_x, \Phi^{(1)}]\right) = 0. \tag{3.66}$$

Therefore, (3.66) is a nontrivial, nonlocal (due to $\Phi^{(1)}$) conservation law which is satisfied on all chiral solutions J. In fact, the first conserved quantity is

$$b_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\Phi^{(1)}_u + \frac{1}{2}[J^{-1}J_u, \Phi^{(1)}]\right) dx\, dy. \tag{3.67}$$

Note that, the densities of the conserved current depend explicitly on the nonlocal charge $\Phi^{(1)}$.

The next Bäcklund transformation should be such that, one of the integrability conditions of which yields (3.66) while another yields a higher order continuity equation, i.e.

$$\begin{aligned} \Phi^{(1)}_u - \tfrac{1}{2}[\Phi^{(1)}, \Phi^{(1)}_x] &= \Phi^{(2)}_x, \\ \Phi^{(1)}_x - \tfrac{1}{2}[\Phi^{(1)}, \Phi^{(1)}_v] &= \Phi^{(2)}_v. \end{aligned} \tag{3.68}$$

The consistency condition $(\Phi^{(1)}_x)_u = (\Phi^{(1)}_u)_x$, yields, after some calculation

$$\Phi^{(2)}_{uv} - \Phi^{(2)}_{xx} + \frac{1}{2}([\Phi^{(2)}_x, \Phi^{(1)}_v] - [\Phi^{(2)}_v, \Phi^{(1)}_x]) + \frac{1}{4}[\Phi^{(1)}, [\Phi^{(1)}_v, \Phi^{(1)}_x]] = 0. \tag{3.69}$$

Therefore, with the observation that

$$\begin{aligned} &[\Phi^{(2)}_x, \Phi^{(1)}_v] - [\Phi^{(2)}_v, \Phi^{(1)}_x] = \partial_x[\Phi^{(2)}, \Phi^{(1)}_v] - \partial_v[\Phi^{(2)}, \Phi^{(1)}_x], \\ &[\Phi^{(1)}, [\Phi^{(1)}_v, \Phi^{(1)}_x]] = \tfrac{1}{3}\left(\partial_v[\Phi^{(1)}, [\Phi^{(1)}, \Phi^{(1)}_x]] - \partial_x[\Phi^{(1)}, [\Phi^{(1)}, \Phi^{(1)}_v]]\right), \end{aligned} \tag{3.70}$$

(3.69) takes the form of a continuity equation, i.e.

$$\begin{aligned}&\partial_v\left(\Phi_u^{(2)}+\tfrac{1}{2}[J^{-1}J_u,\Phi^{(2)}]-\tfrac{1}{12}[\Phi^{(1)},[J^{-1}J_u,\Phi^{(1)}]]\right)-\\&\partial_x\left(\Phi_x^{(2)}+\tfrac{1}{2}[J^{-1}J_x,\Phi^{(2)}]-\tfrac{1}{12}[\Phi^{(1)},[J^{-1}J_x,\Phi^{(1)}]]\right)=0,\end{aligned}\tag{3.71}$$

where we use (3.61) to eliminate $\Phi_x^{(1)}$ and $\Phi_u^{(1)}$. Hence, the second nonlocal conservation for the S-equation (3.1) is,

$$b_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\Phi_u^{(2)}+\frac{1}{2}[J^{-1}J_u,\Phi^{(2)}]-\frac{1}{12}[\Phi^{(1)},[J^{-1}J_u,\Phi^{(1)}]]\right)\,dx\,dy. \tag{3.72}$$

To find the next conservation law, return to the continuity equation (3.71) and notice that this expression is a consistency condition for the Bäcklund transformation

$$\begin{aligned}&\Phi_u^{(2)}-\tfrac{1}{2}[\Phi^{(2)},\Phi_x^{(1)}]+\tfrac{1}{12}[\Phi^{(1)},[\Phi^{(1)},\Phi_x^{(1)}]]=\Phi_x^{(3)},\\&\Phi_x^{(2)}-\tfrac{1}{2}[\Phi^{(2)},\Phi_v^{(1)}]+\tfrac{1}{12}[\Phi^{(1)},[\Phi^{(1)},\Phi_v^{(1)}]]=\Phi_v^{(3)}.\end{aligned}\tag{3.73}$$

Then, apply the other integrability condition $(\Phi_x^{(2)})_u=(\Phi_u^{(2)})_x$ and use (3.61), (3.68) and (3.73). After a very lengthy calculation, the result is rewritten in the form of a continuity equation, from which may be verified the third nonlocal conservation law for the chiral equation.

Thus, this process generates nonlocal conservation laws for the S-equation. An infinite number of currents can be obtained in this fashion, although a rigorous proof of this statement requires further investigation. The constructing of higher order conservation laws is an increasingly hard task since it becomes excessively difficult to express the ensuing relations in the form of continuity equations.

Intuitively, it is of interest to compare Papachristou's nonlocal conservation laws with those of Prasad *et al.* By expressing $\mathcal{V}_u^{(n)}$ and $\mathcal{V}_x^{(n)}$ in terms of $\Psi^{(n-1)}$ by virtue of (3.49), then (3.46) becomes a system of equations which play a role analogous to that of the aforementioned Bäcklund transformations. The only difference with Papachristou's laws is that these transformations are essentially the same for all steps of recursive process (that is, for all values of the index n). Thus, Prasad's *et al* conservation laws can be evaluated via a recursion relation for all values of n, which is not the case with Papachristou's currents. Moreover, the latter are much more complicated than the former since their densities depend on an increasing number of charges rather than on one charge at a time.

The nonlocal conserved densities of the charges a_1, a_2, ... of (3.59) and those of b_1, b_2, ... of (3.67) and (3.72) fall off as $O(r^{-2})$ for soliton solution, and so these charges diverge; or when they do converge they are equal to zero (e.g., for the one-soliton solution (2.35)). But it turns out that the differences $a_1 - b_1$, $a_2 - b_2$, ..., yield nontrivial conserved quantities, i.e. Q_1, Q_2, ...

Let us derive the first nonlocal conservation law Q_1. In order to find the correspond equation of motion subtract (3.66) from (3.52) with $n = 1$ and observe that the first nonlocal charges coincide, i.e. $\Psi^{(1)} = \Phi^{(1)}$. Thus, we obtain

$$\partial_v\{J^{-1}J_u, \Psi^{(1)}\} - \partial_x\{J^{-1}J_x, \Psi^{(1)}\} = 0, \tag{3.74}$$

where the curly brackets denote anticommutators. Clearly, the conservation law (3.74) is not trivially related (i.e. equivalent) to the familiar conservation laws of Pohlmeyer and Prasad *et al.* Moreover, it corresponds to a conserved charge of the form

$$Q_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{J^{-1}J_u, \Psi^{(1)}\}\,dx\,dy. \tag{3.75}$$

But the *old* nonlocal conserved density $\Psi^{(1)}$ involves the integral operator ∂_x^{-1} which we take to be

$$\partial_x^{-1}F(x) = \int_{-\infty}^{x} F(x')\,dx'. \tag{3.76}$$

In fact, $\Psi^{(1)} = \partial_x^{-1}B_u$ (recall, $B_u = J^{-1}J_u$); and therefore, (3.75) simplifies to

$$Q_1 = \int_{-\infty}^{\infty} dy\,\left[\int_{-\infty}^{\infty} J^{-1}J_u\,dx\right]^2. \tag{3.77}$$

Note that, we used the relation

$$\int_{-\infty}^{\infty} B_u\,dx\left(\partial_x^{-1}B_u\right)^{n-1} = \frac{1}{n!}\left[\int_{-\infty}^{\infty} J^{-1}J_u\,dx\right]^n. \tag{3.78}$$

Obviously, the first conserved charge derivation does not require knowledge of lower-order charge. In fact, all the conserved quantities Q_n are been characterized by the above property.

The second conserved law of our sequence is obtained, by subtracting (3.71) from (3.52) with $n = 2$. After a lengthy calculation, the following continuity equation has been obtained

$$\begin{aligned}&\partial_v\left(\{\Psi^{(1)}, \Psi_x^{(1)}\Psi^{(1)}\} - \tfrac{1}{2}([\Psi^{(1)}, \Psi_x^{(1)}] - [\Psi^{(1)}, \{\Psi_x^{(1)}, \Psi^{(1)}\}]) + \tfrac{1}{6}[\Psi^{(1)}, [\Psi^{(1)}, \Psi_x^{(1)}]]\right) = \\ &\partial_x\left(\{\Psi^{(1)}, \Psi_v^{(1)}\Psi^{(1)}\} - \tfrac{1}{2}([\Psi^{(1)}, \Psi_v^{(1)}] - [\Psi^{(1)}, \{\Psi_v^{(1)}, \Psi^{(1)}\}]) + \tfrac{1}{6}[\Psi^{(1)}, [\Psi^{(1)}, \Psi_v^{(1)}]]\right),\end{aligned} \tag{3.79}$$

and so the corresponding conserved quantity is,

$$Q_2 = \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{\infty} J^{-1} J_u \, dx\right]^3 . \tag{3.80}$$

It is easy to see how to generalize Q_1 and Q_2 to obtain a sequence $\{Q_n\}$. Therefore, we conjecture that the Q_n are essentially obtained from $a_n - b_n$, but the calculations involved in this seem rather complicated. The expression for Q_n is, as we can deduce, very simple; namely,

$$Q_n = \textstyle\int_{-\infty}^{\infty} M^{n+1} \, dy, \qquad n = 1, 2, ..., \tag{3.81}$$

where

$$M(t, y) = \int_{-\infty}^{\infty} J^{-1} J_u \, dx. \tag{3.82}$$

So the $\{Q_n\}$ form an infinite sequence of well-defined quantities, for any fixed value of t. This follows from the boundary condition which implies that there exists a positive constant K such that each component of the matrix B_u, satisfies $|(B_u)_{\alpha\beta}| \leq K/(r^2 + 1)$ and so $|M_{\alpha\beta}| \leq \pi K/\sqrt{y^2+1}$. As a result $M^{n+1} = O(y^{-n-1})$ as $|y| \to \infty$, and so the integral (3.81) converges. The case $n = 0$ corresponds to the local Noether density $J^{-1}J_u$ but diverges (as remarked in the previous section).

In addition, the quantities $\{Q_n\}$ are conserved. Note that

$$\partial_v M = \int_{-\infty}^{\infty} \partial_v (J^{-1} J_u) \, dx = \int_{-\infty}^{\infty} \partial_x (J^{-1} J_x) \, dx = 0. \tag{3.83}$$

Hence $\partial_v M^{n+1} = 0$, and so

$$\frac{dQ_n}{dt} = \int_{-\infty}^{\infty} \partial_t M^{n+1} \, dy = \int_{-\infty}^{\infty} \partial_y M^{n+1} \, dy = 0, \tag{3.84}$$

as claimed. These two remarks, apply to any gauge group.

Finally, let us investigate how many independent conserved quantities there are among the $\{Q_n\}$. Since M takes values in the Lie-algebra su(2), we have

$$\begin{aligned} M^{2p} &= (-1)^p ||M||^{2p} I, \\ M^{2p+1} &= (-1)^p ||M||^{2p} M, \end{aligned} \tag{3.85}$$

where $||M||^2 = -\mathrm{tr}(M^2)/2$. So for n odd, there is one real conserved quantity, since Q_n equals, the number $\int ||M||^{n+1} \, dy$ times I; and for n even, three conserved quantities exist, the components of $\int ||M||^n\, M\, dy$. All these conserved quantities are independent; since M is essentially arbitrary su(2)-valued function of y.

By way of example, let us find the corresponding expressions of M and $\{Q_n\}$ for the perturbed static one-soliton solution. To do so, let us take J to be of the form (3.4) with $f = z + v\,t$, evaluate the matrix $B_u = J^{-1}J_u$ and finally, set $t = 0$. (Roughly speaking, the above configuration represents a moving soliton which emits radiation in order to be stable). Thus, the matrix $M(t, y)$ is time-independent, and equal to

$$M(y) = \frac{2v\pi}{(y^2) + 1)^{3/2}} \begin{pmatrix} iy & 1 \\ -1 & -iy \end{pmatrix}, \tag{3.86}$$

while the first four conserved quantities $\{Q_n\}$ are

$$\begin{aligned} Q_1 &= -2v^2\pi^3 I, & Q_2 &= -128v^3\pi^3\mathcal{I}/15, \\ Q_3 &= 5v^4\pi^5 I, & Q_4 &= 8192v^5\pi^5\mathcal{I}/315, \\ Q_5 &= -63v^6\pi^7 I/4, & &\text{etc}, \end{aligned} \tag{3.87}$$

where $\mathcal{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Although, all these conserved quantities are related to the nonlocal currents of [24, 25], they do not involve repeated integrations. They are nontrivial and independent, but by no means complete. For example, the matrix M vanishes for the one-soliton solution (3.4) and its moving version, and therefore so does $\{Q_n\}$.

There is another set of well-defined conserved charges which are complementary to the $\{Q_n\}$. They are related to the (nonintegrable) conserved densities given in [39]. The key point is that (3.1) which is the dimensional reduction of the *left sdYM-J equations* is equivalent to

$$\partial_u(J_vJ^{-1}) - \partial_x(J_xJ^{-1}) = 0, \tag{3.88}$$

which is the dimensional reduction of the *right sdYM-J equations.* From this, it is easy to see that the following quantities are conserved

$$\widehat{Q}_n = \int_{-\infty}^{\infty} \widehat{M}^{n+1}\, dy, \qquad n = 1, 2, ..., \tag{3.89}$$

where

$$\widehat{M}(t, y) = \int_{-\infty}^{\infty} J_vJ^{-1}\, dx. \tag{3.90}$$

Unfortunately, for the T-equation (3.2), the situation is rather different. This is not unexpected since, for example, the inverse scattering transform for the T-equation differs significantly from that of the S-equation [28, 30, 32]. Once more, the conserved densities of [24, 25] turn out to be of order r^{-2} as $r \to \infty$; in particular, this is the case for the soliton solution described in the previous section. So the corresponding conserved charges do not exist. However, one can define charges analogous to the Q_n above. These are

$$R_n = \int_{\mathbf{R}^2} \partial_{\bar{z}}(\Psi^{n+1})\, dz \wedge d\bar{z}, \qquad n = 1, 2, ..., \tag{3.91}$$

where Ψ is the solution of the $\bar{\partial}$-problem

$$\partial_{\bar{z}}\Psi = J^{-1}J_t, \qquad \Psi \to 0 \qquad \text{as} \quad |z| \to \infty. \tag{3.92}$$

The conservation of the R_n follows from $\partial_t\Psi = J^{-1}J_z$. In fact,

$$\partial_t\Psi = \int_{\partial R} \partial_t(J^{-1}J_t)\, d\bar{z} = \int_{\partial R} \partial_{\bar{z}}(J^{-1}J_z)\, d\bar{z} = 0. \tag{3.93}$$

Hence $\partial_t\Psi^{n+1} = 0$, and so $dR_n/dt = 0$.

3.5 Concluding Remarks

We have exhibited infinitely many well-defined conserved charges for the (2+1)-dimensional reduction of the self-dual Yang-Mills equations. We are not aware of any conserved quantities that are local (in the usual sense), other than the energy-momentum vector; a systematic search for generalized symmetries of the Lagrangian might well uncover many more.

Our (slightly) nonlocal conserved quantities, have a particularly simple form, not involving repeated integration. They are related to nonlocal conservation laws known previously; however the latter do not yield well-defined conserved quantities, and so cannot contribute to *a complete set of action variables.* The present sequences certainly do not make up a complete set: for example, the matrix M vanishes for the one-soliton solution (2.35) and its moving version, and therefore so does $\{Q_n\}$.

It seems likely that the stability of the solitons in the chiral model (or sigma model) is due to the existence of the infinite number of conserved quantities (as it happen in lower dimensions). There is at present no proof of this conjecture, although the reasoning is: if the initial configuration consists of n solitons then, for large t, the infinite number of conserved charges impose such constraints on the system so that the configuration is again n solitons.

Chapter 4

Nontrivial Scattering

4.1 Introduction

This chapter studies certain exact soliton solutions of an integrable system. An interesting problem is to look at the scattering properties of two or more solitons colliding. In some known systems with nontrivial topology, the collision of two solitons is inelastic (some radiation is emitted) and nontrivial (a head-on collision results in 90^0 scattering); all this has been observed analytically [41], [52]-[56] and numerically [57]-[61]. One can construct explicit time-dependent solutions only in very special, so-called integrable models. Usually in these models extended objects interact trivially, in the sense that they pass through each other with no lasting change in velocity or shape (i.e., they behave as genuine solitons). Some examples in (2+1) dimensions are the Kadomtsev-Petviashvili equation [62] and the modified chiral model [26]. The last system is the subject of this chapter and will be described below.

Until now, nontrivial scattering of solitons occurs mostly in nonintegrable systems which is far from simple. The question that arises is whether this type of scattering can occur in integrable models too. There are some limited examples of integrable systems where soliton dynamics can be nontrivial. In (1+1) dimensions there many models which possess nontrivial soliton-like solutions (cf. [63]); like the boomeron solutions [64], which are solitons with time dependent velocities. In (2+1) dimensions there are the dromion solutions [65] of the Davey-Stewartson equation, which decay exponentially in both spatial coordinates and interact in a nontrivial manner [66, 67]; and the soliton solutions [68] of the Kadomtsev-Petviashvili equations, whose scattering properties are highly nontrivial.

In the present work we are going to construct families of soliton solutions for the (2+1)-dimensional modified chiral model and observe the occurrence of different types of behaviour. This happens since the solitons in this system have internal degrees of freedom which determine their orientation in space; do not affect the initial energy density; and are important in understanding the evolution as a whole. Therefore, they can interact either trivially or nontrivially, depending on the orientation of these internal parameters and on the values of the impact parameter defined as the distance of closest of approach between their centres in the absence of interaction. Namely, if two initial soliton-like structures are sent towards each other at zero impact parameter, then, as most numerical simulations have shown, the outgoing structures emerge at 90^0.

To proceed further let us recall the system. The modified SU(2) chiral model studied by Ward is given by the field equation

$$(\eta^{\mu\nu} + \varepsilon^{\alpha\mu\nu} V_\alpha)\partial_\mu(J^{-1}\partial_\mu J) = 0, \tag{4.1}$$

where $V_\alpha = (0, 1, 0)$. This is an alternative expression of (2.16). Recall that, this is the chiral equation with torsion term and has the same conserved energy-momentum vector as the chiral field equation. In fact, the corresponding energy density is

$$\mathcal{E} = -\frac{1}{2}\mathrm{tr}\left[(J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2\right], \tag{4.2}$$

which is identical to (2.21). It should be emphasized that $\mathcal{E}$ is a positive-defined functional of J, and hence a conserved energy exists which is the integral of the energy density over the spacelike plane x^0 = const. The boundary conditions are chosen so that the field configuration has finite energy. Hence, we require that J be everywhere smooth and of the form (2.18), i.e.

$$J = J_0 + J_1(\theta)r^{-1} + O(r^{-2}), \tag{4.3}$$

at spatial infinity, with $x + iy = r\,e^{i\theta}$.

As we have already shown in chapter 1, equation (4.1) admits solitons, localized in two dimensions, with trivial scattering, i.e. each soliton suffers no change in velocity and no phase shift upon scattering. It is the purpose of this chapter to construct new soliton solutions for (4.1), and investigate their scattering behaviour. Such solutions are localized along the direction of motion; they are not however, of constant size: their height, which corresponds to the maximum of the energy density $\mathcal{E}$, is time dependent.

The rest of the chapter is arranged as follows. In the next section we shall briefly discuss the integrability properties of (4.1), and write down a family of multisoliton solutions as configurations that are the limiting cases of the ones already obtained using the standard method of *Riemann problem with zeros* [26]. In section 4.3 we construct two families of multisoliton solutions with nontrivial scattering; in particular, for the first one we prove that in all head-on collisions the N moving structures undergo π/N scattering. In section 4.4 we construct a mixture of soliton-antisoliton solutions, and in section 4.5 we discuss their dynamics and scattering properties. We finish the chapter with a short section containing our conclusions.

4.2 Construction of Soliton Solutions

The integrable nature of equation (4.1) means that there is a variety of methods for constructing exact solutions. Together with *Riemann problem with zeros* [26], both *twistor techniques* [27] and a full *inverse scattering formalism* [28] have been applied to the model. This section indicates a general method for constructing soliton solutions of the modified chiral model (4.1). The technique is a variation of that in [26, 44], where Ward (following a pioneering idea of Zakharov and his collaborators [34, 35]), has generated an explicit solution representing a head-on collision of two solitons which undergo 90^0 scattering.

We have seen that the nonlinear equation (4.1) is integrable in a sense that it may be written as the compatibility condition for the following linear system

$$\begin{aligned} L\psi &\equiv (\lambda\partial_x - \partial_u)\psi = A\psi, \\ M\psi &\equiv (\lambda\partial_v - \partial_x)\psi = B\psi, \end{aligned} \tag{4.4}$$

which is identical to (2.22). Recall that, $\psi(\lambda, u, v, x)$ is an unimodular 2×2 matrix function satisfying the reality condition given by (2.23), i.e.

$$\psi(\lambda, u, v, x)\,\psi(\bar{\lambda}, u, v, x)^\dagger = I, \tag{4.5}$$

and A and B are 2×2 anti-hermitian trace-free matrices depending on (u, v, x). The integrability conditions for (4.4) implies that there exists a J such that

$$A = J^{-1}J_u, \qquad B = J^{-1}J_x, \tag{4.6}$$

and that this J satisfies the equation of motion (4.1). Comparing (4.4) and (4.6), we see that J can be identified with $\psi(0)^{-1}$. The reality condition on J follows from an analogous conditions on $\psi(\lambda)$, namely (4.5). So the idea is that if we can find a $\psi(\lambda)$ such that the reality condition (4.5) holds, and such that $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ be independent of λ, then $J = \psi(0)^{-1}$ is a unitary solution of (4.1).

In order to construct multi-soliton solution one may assume that the function ψ has simple poles in λ, or in other words must possess the form (2.26), i.e.

$$\psi(\lambda) = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k}, \tag{4.7}$$

where M_k are 2×2 matrices independent of λ. This leads to an n-soliton solution, in which the velocity of the k-th soliton is determined by the complex constant μ_k; one consequence is that there is no scattering [26]. The components of the matrix M_k are given in terms of a rational function f_k of the complex variable $\omega_k = x + \mu_k u + \mu_k^{-1} v$. Roughly speaking, $f_k(\omega_k)$ describes the shape of the k-th soliton. (For more details, see section 2.3.)

All this assumes that the parameters μ_k are distinct, and also $\bar{\mu}_k \neq \mu_l$ for all k, l. In this chapter examples are given of two generalizations of these constructions: one involving higher-order poles in μ_k, and the other where $\bar{\mu}_k \neq \mu_l$.

Let us look at an example in which the function ψ has a double pole in λ, and no other poles. So we take ψ to have the form

$$\psi = I + \sum_{k=1}^{2} \frac{R_k}{(\lambda - \mu)^k}, \tag{4.8}$$

where R_k are 2×2 matrices independent of λ. [This hypothesis can be generalized by taking the function ψ to have a pole of order n in λ.]

It has been proved [44] that ψ given by (4.8) satisfies the reality condition (4.5) if and only if it factorizes as

$$\psi(\lambda) = \left(I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} \frac{q_1^\dagger \otimes q_1}{\|q_1\|^2}\right) \left(I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} \frac{q_2^\dagger \otimes q_2}{\|q_2\|^2}\right), \tag{4.9}$$

where q_k are two-dimensional row vectors and $\|q_k\|^2 = q_k \cdot q_k^\dagger$. The same is true if $\psi(\lambda)$ has a pole of order n: the reality condition is satisfied if and only if ψ factorizes into n simple factors of the type appearing in (4.9). This is an example of a two-uniton.

The idea of n-uniton was introduced in connection with finding SU(N) chiral fields on $\mathbf{R}^2$ [46], and it extends naturally to the corresponding system on $\mathbf{R}^{2+1}$ [27]. For SU(2) chiral fields on $\mathbf{R}^2$, i.e. the static version of this system, one-uniton is enough (the static soliton is a one-uniton). The original reason for introducing n-unitons was that they are needed for static SU(N+1) chiral fields [46]. Higher unitons are also needed for the time-dependent SU(2) case.

The q_k have to satisfy a condition, which amounts to saying the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ are independent of λ. One way of obtaining q_k with this property is as a limit of the simple-pole case (4.7) with $n = 2$. The idea is to take a limit $\mu_k \to \mu$. In order to end up with a smooth solution ψ for all (u, v, x), it is necessary that $f_2(\omega_2) - f_1(\omega_1) \to 0$ in this limit. If we arrange things carefully the limit gives a solution of the double-pole type (4.9).

In our case, with $n = 2$, we put $\mu_1 = \mu + \varepsilon$, $\mu_2 = \mu - \varepsilon$ and write $f_1(\omega_1) = f(\omega_1)$, $f_2(\omega_2) = f(\omega_2)$, with f being a rational function of one variable. In the limit $\varepsilon \to 0$, ψ has the form (4.9), with

$$\begin{aligned} q_1 &= (1 + |f|^2)(1, f) + \varphi\,(\bar{\mu} - \mu)(\bar{f}, -1), \\ q_2 &= (1, f), \end{aligned} \tag{4.10}$$

where

$$\varphi = (u - \mu^{-2} v)\, f'(\omega). \tag{4.11}$$

Here f is a rational function of $\omega = x + \mu u + \mu^{-1} v$, while $f'(\omega)$ denotes the derivative of $f(\omega)$ with respect to its argument. As a result, we have a solution $J = \psi(\lambda = 0)^{-1}$ depending on the complex parameter μ and on the arbitrary function f. In fact, it has the form of the following product

$$J = \left(I + \frac{(\bar{\mu} - \mu)}{\mu} \frac{q_2^\dagger \otimes q_2}{\|q_2\|^2} \right) \left(I + \frac{(\bar{\mu} - \mu)}{\mu} \frac{q_1^\dagger \otimes q_1}{\|q_1\|^2} \right), \tag{4.12}$$

with q_k given by (4.10). Notice that J takes values in SU(2); is smooth everywhere on $\mathbf{R}^{2+1}$ (mainly because, the two vectors q_1 and q_2 are nowhere zero since they are orthogonal); it satisfies the boundary condition (4.3); and the equation of motion (4.1).

To start with, and in order to illustrate the above family of soliton solutions, let us examine two simple cases, by giving specific values to the parameters μ and $f(\omega)$. [The complex parameter μ determines the velocity of the "centre-of-mass" of the system.]

- Let us take $\mu = i$ (which corresponds to the "centre-of-mass" of the system being stationary) and $f(\omega) = \omega$, thus $\omega = z$ and $\varphi = t$, where $z = x + iy$; $r^2 = z\bar{z}$. Therefore the row vectors (4.10), become

$$\begin{aligned} q_1 &= (1 + r^2)(1, z) - 2it(\bar{z}, -1), \\ q_2 &= (1, z). \end{aligned} \tag{4.13}$$

In this time-dependent solution, for t negative, a ring structure with reducing radius is obtained, which deforms to a single peak at $t = 0$ and thereafter expands again to a ring. Figure 4.1 presents few pictures of the corresponding energy density at some representative values of time. Ring structures occur in the soliton scattering of many nonintegrable planar systems [57, 59] and are an approximation of two solitons.

This picture can be confirmed by looking at the energy density of the solution, which is

$$\mathcal{E} = 16\frac{r^4 + 2r^2 + 4t^2(2r^2 + 1) + 1}{[r^4 + 2r^2 + 4t^2 + 1]^2}. \tag{4.14}$$

Notice that the energy density is time-reversible and rotationally symmetric (see below); and also that, it goes like $\mathcal{E} = O(r^{-4})$ as $r \to \infty$, which is the case for all the solutions described in this chapter.

For large (positive) t, the height of the ring (maximum of $\mathcal{E}$) is proportional to $1/t$, while its radius is proportional to $\sqrt{t}$. This is obvious since, for $r^2 << t^2$, the energy density becomes

$$\mathcal{E} \sim 128\frac{t^2 r^2}{(r^4 + 4t^2)^2}, \tag{4.15}$$

and its local maximum, i.e. $d\mathcal{E}/dr = 0$ gives that $r_{max} \propto \sqrt{t}$. Substituting, then, r_{max} in (4.14) leads to $\mathcal{E}_{max} \propto 1/t$ (which corresponds to the height of the soliton); while by solving $\mathcal{E} = \mathcal{E}_{max}/2$ one may find that the ring radius is proportional to $\sqrt{t}$.

- Accordingly, let us take $\mu = i$ and $f(\omega) = \omega^2$. Thus, the row vectors (4.10) are

$$\begin{aligned} q_1 &= (1 + r^4)(1, z^2) - 4itz(\bar{z}^2, -1), \\ q_2 &= (1, z^2). \end{aligned} \tag{4.16}$$

Here, for negative t, a single peak occurs with an additional ring, which changes to a ring structure at $t = 0$ and reverts back to the original form, for positive t

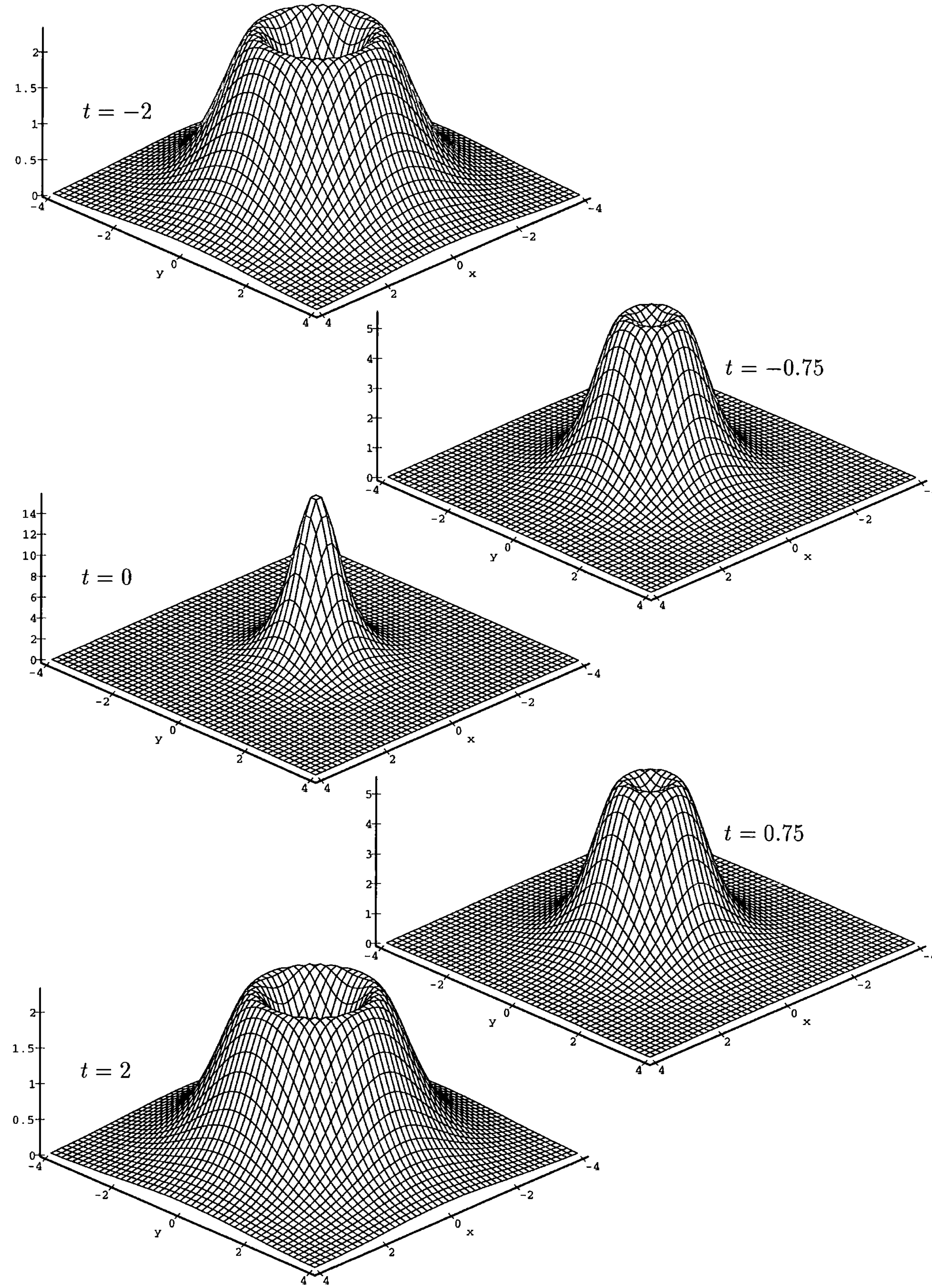


Figure 4.1: The energy density $\mathcal{E}$ (4.14) at increasing time.

(see Figure 4.2). However, these rings are not radiation since they travel with speed less than that of light. In fact, for large (positive) t, their velocity is approximately proportional to $t^{-2/3}$. [Note that we have set the velocity of the light, c, equal to the unity, so that in all our calculations we can use dimensionless quantities.]

This leads to an energy density, which is

$$\mathcal{E} = 64\frac{r^{10} + 18t^2r^8 + 2r^6 + 4t^2r^4 + r^2 + 2t^2}{\left[r^8 + 2r^4 + 16t^2r^2 + 1\right]^2}. \tag{4.17}$$

Again, $\mathcal{E}$ has the same symmetries as in (4.14). For large (positive) t, the height of the soliton peak is proportional to t^2 and its radius is proportional to $1/t$; while the soliton ring spread out, becoming broader and broader, with height proportional to $t^{-2/3}$ and radius proportional to $t^{1/3}$.

Finally, a general concluding remark should be made. Although (4.1) is not rotationally symmetric in the xy-plane; when $f(z) = z^p$ the field J (4.10,4.12) is invariant under the transformation $z \to e^{i\phi}z$, since

$$J \to J' = \begin{pmatrix} e^{i\phi p/2} & 0 \\ 0 & e^{-i\phi p/2} \end{pmatrix} J \begin{pmatrix} e^{-i\phi p/2} & 0 \\ 0 & e^{i\phi p/2} \end{pmatrix}. \tag{4.18}$$

This transformation does not affect the equation of motion (4.1) due to the chiral symmetry $J \to \kappa J \tau$ where κ and τ are constant SU(2) matrices. The main features of this time-dependent solution may be inferred as follow. If r is large, the field J is close to its asymptotic value J_0, as long as $2tf'/|f|^2 \to 0$. But as $2t|f'|/|f|^2 \approx 1$, J departs from its asymptotic value J_0 and a ring structure emerge with radius proportional to $(2tp)^{1/(p+1)}$.

4.3 Soliton-Soliton Scattering

We now move on to the more interesting question of scattering processes. In fact, we will use the method of section 2 to construct solutions of (4.1) representing scattering solitons. We will see that, in all head-on collisions of N moving solitons the scattering angle is π/N. Moreover, when the N solitons are very close together, and in particular, when they are on top of each other the N lumps which represent them merge together to

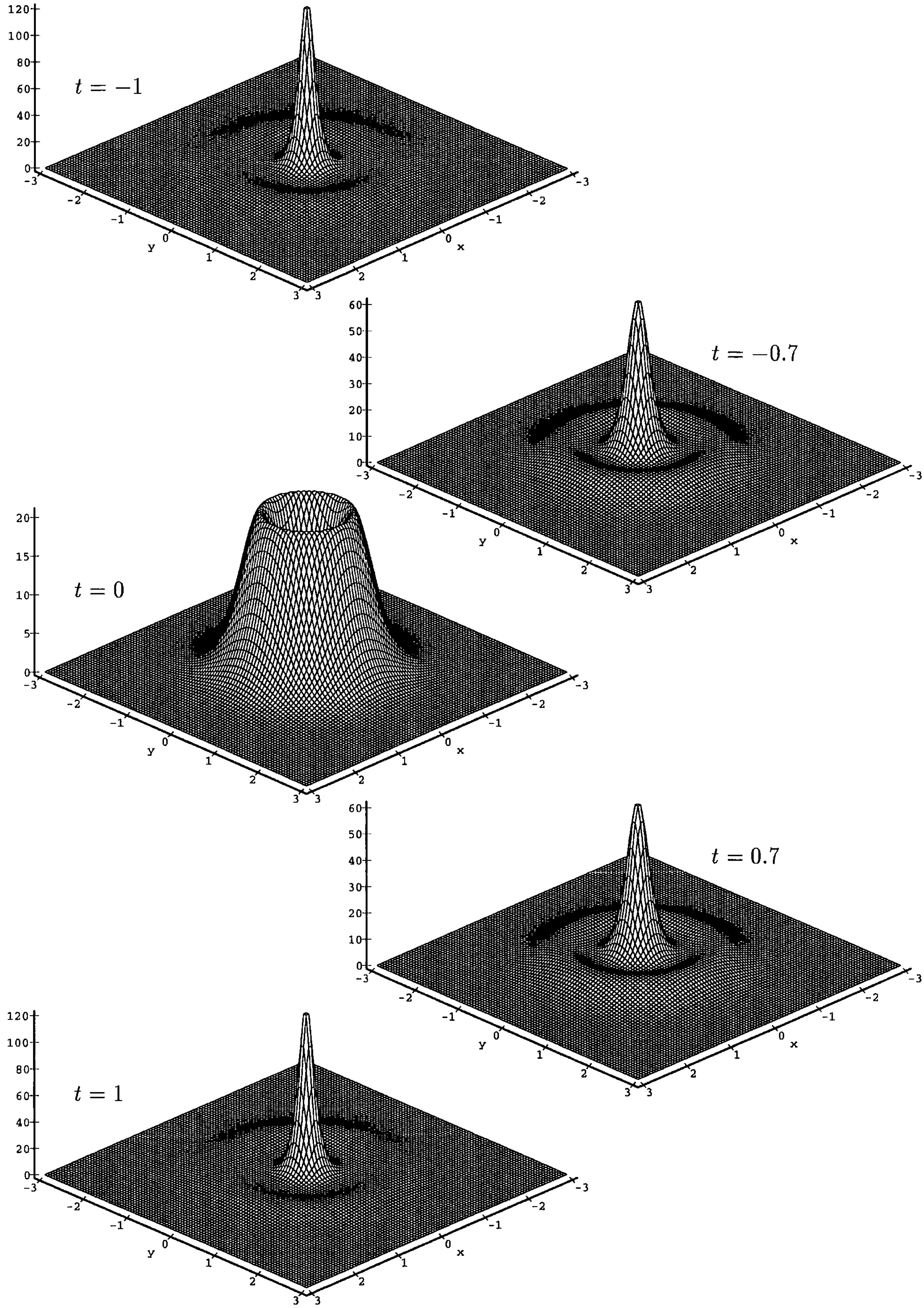


Figure 4.2: The density $\mathcal{E}$ (4.17) at various times.

form a ring-like structure. Then, instead of moving towards the centre, they emerge from the ring in a direction that bisects the angle formed by the incoming ones. As we have already mentioned this nontrivial scattering is not usual in an integrable theory, but is exceptional.

The scattering solutions arise if we take a solution of the simple-pole case (4.7) with $n = 2$, put $\mu_1 = \mu+\varepsilon$, $\mu_2 = \mu-\varepsilon$ and take the limit $\varepsilon \to 0$. The constraint $f_2(\omega_2) - f_1(\omega_1) \to 0$ as $\varepsilon \to 0$ has to be imposed, in order for the resulting solution ψ to be smooth for all (u, v, x). So let us write $f_1(\omega_1) = f(\omega_1) + \varepsilon h(\omega_1)$, $f_2(\omega_2) = f(\omega_2) - \varepsilon h(\omega_2)$, where f and h are both rational functions of one variable (the examples of the previous section had $h = 0$). Once again J is given by (4.12), with the 2-vectors q_k given by

$$\begin{aligned} q_1 &= (1+|f|^2)(1,f) + \vartheta\,(\bar{\mu}-\mu)(\bar{f},-1), \\ q_2 &= (1,f), \end{aligned} \tag{4.19}$$

where $\vartheta = \varphi + h(\omega)$ with φ given by (4.11). So this solution belongs to a large family, since one may take f and h to be any rational meromorphic functions of ω. Note that J is smooth on $\mathbf{R}^{2+1}$ and satisfies its boundary condition, irrespective of the choice of f and h.

It may seem strange that one can take the limit of a family of soliton solutions with trivial scattering, and obtain a new one with nontrivial scattering. Thus, it is interesting to study how the solitons are affected by varying ε. To do so, let us take a solution of the simple-pole case (4.7) with $n = 2$, put $\mu_1 = i+\varepsilon$, $\mu_2 = i-\varepsilon$, while taking $f_k = \omega_k$; and study how the configuration of the two initial well separated solitons changes as $\varepsilon \to 0$ at a fixed time ($t = -15$). Figure 4.3 shows that as $\varepsilon \to 0$ the solitons disperse, shift and interact with each other. In other words, their internal degrees of freedom as well as the impact parameter change in this limit, making the process highly nontrivial.

As an example, let us present two typical cases.

- Let us take $\mu = i$, $f(\omega) = \omega$ and $h(\omega) = \omega^3$; thus $\vartheta = t + z^3$. For r large, J is equal to its asymptotic value J_0, as long as $\vartheta/z^3 = 1 + t/z^3 \approx 1$, but as z approaches any of the three cube roots of $-t$ then $\vartheta \to 0$, while J departs from its asymptotic value J_0, and three localized solitons emerge. For t negative, the three solitons are approximately at the points: $\left((-t)^{1/3}, 0\right)$, $\left(-(-t)^{1/3}, \pm\sqrt{3}\,(-t)^{1/3}\right)$; while for t positive, the solitons are at $\left(-t^{1/3}, 0\right)$, $\left(t^{1/3}, \pm\sqrt{3}\,t^{1/3}\right)$.

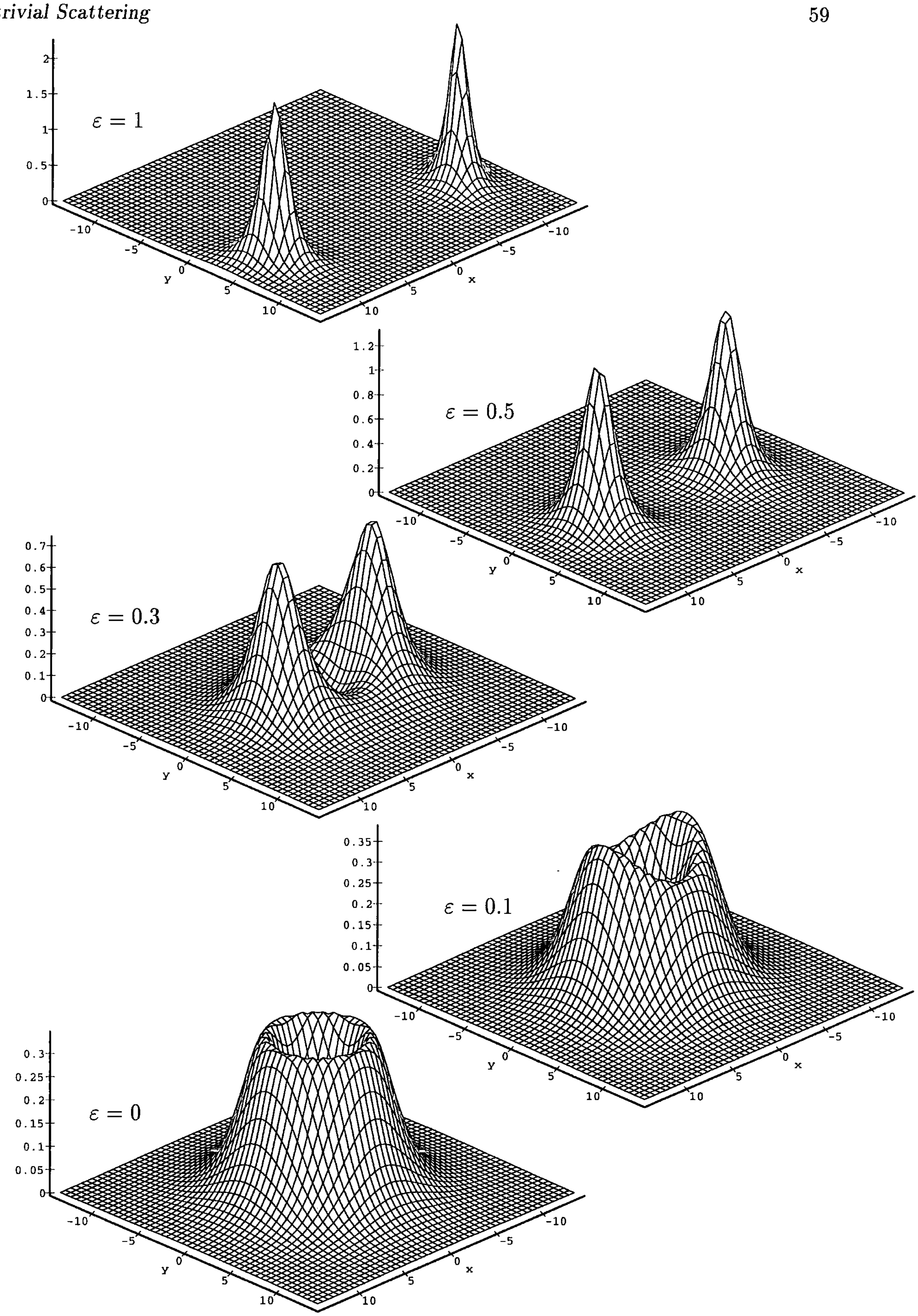


Figure 4.3: Energy density $\mathcal{E}$ at $t = -15$, for soliton-soliton interaction by varying ε.

More information can be deduced from the energy density, which is

$$\begin{aligned}\mathcal{E} = \ & 16\,[2r^8 + 16r^6 + 19r^4 + 2r^2(1 + 8xy^2t) + 4t^2(1 + 2r^2) + 1 + 8xy^4t \\ & -8x^5t - 16tx(x^2 - y^2)]/[4r^6 + r^4 + 2r^2 + 4t^2 + 1 + 8tx(x^2 - 3y^2)]^2.\end{aligned} \tag{4.20}$$

The density $\mathcal{E}$ is symmetric under the interchange $t \mapsto -t$, $x \mapsto -x$ and $y \mapsto -y$. For small (negative) t, the solitons form an intermediate state having the shape of a ring with three maxima on the direction of the incoming solitons which deforms to a circularly-symmetric ring at $t = 0$ and then energy seems to flow around, until three other maxima are formed in the transverse direction, for small (positive) t. Figure 4.4 shows clearly the intermediate states with three maxima. The three new maxima then give rise to three new solitons emerging at 60^0 to the original direction of motion. During the intermediate phase solitons lose their identity.

Finally something has to be said about their size. For large (positive) t, their height is proportional to $t^{-4/3}$, their radius is proportional to $t^{1/3}$, while their speed is proportional to $t^{-2/3}$: therefore, they spread out and slow down.

- Accordingly, let us take $\mu = i$ while choose $f(\omega) = \omega^2$ and $h(\omega) = \omega^3$. Here J departs from its asymptotic value J_0, when z approaches the values $\pm\sqrt{-2t}$ or zero (since $\vartheta = z(2t + z^2) \to 0$); and (again) three localized solitons emerge. In this case though, if t is negative, all three of them are on the x-axis at $x \approx \pm\sqrt{-2t}$ and at the origin; while if t is positive, they are on the y-axis at $y \approx \pm\sqrt{2t}$ and at the origin. So the picture consists of three solitons: a static one at the origin, with the other two accelerating towards the origin, scattering at right angles and then decelerating as they separate.

This can be observed from the energy density, which is

$$\begin{aligned}\mathcal{E} = \ & 32[r^{12} + 2r^2(r^8 + r^6 + 1) + 36t^2r^8 + 4r^6 + 9r^4 + 8t^2r^4 + 4t^2 + 12t(x^{10} - y^{10}) \\ & +4t(x^2 - y^2)(3 + 2x^2y^2 + 6x^4y^4) + 4t(x^6 - y^6)(9x^2y^2 - 2) - y^{10})]/[r^8 + 4r^6 \\ & +2r^4 + 16tr^2(t + x^2 - y^2) + 1]^2.\end{aligned} \tag{4.21}$$

Here $\mathcal{E}$ is symmetric under the interchange $t \mapsto -t$, $x \rightleftharpoons y$; therefore the collision is time symmetric, with the only effect the 90^0 scattering (no phase shift; no radiation). For large (positive) t, the height of the static soliton is proportional to t^2 and its radius is proportional to $1/t$; while the moving solitons expand with height proportional to $t^{-2/3}$ and radius proportional to $t^{1/3}$.

In Figure 4.5 we present some pictures of the total energy densities of three solitons during a typical nontrivial evolution.

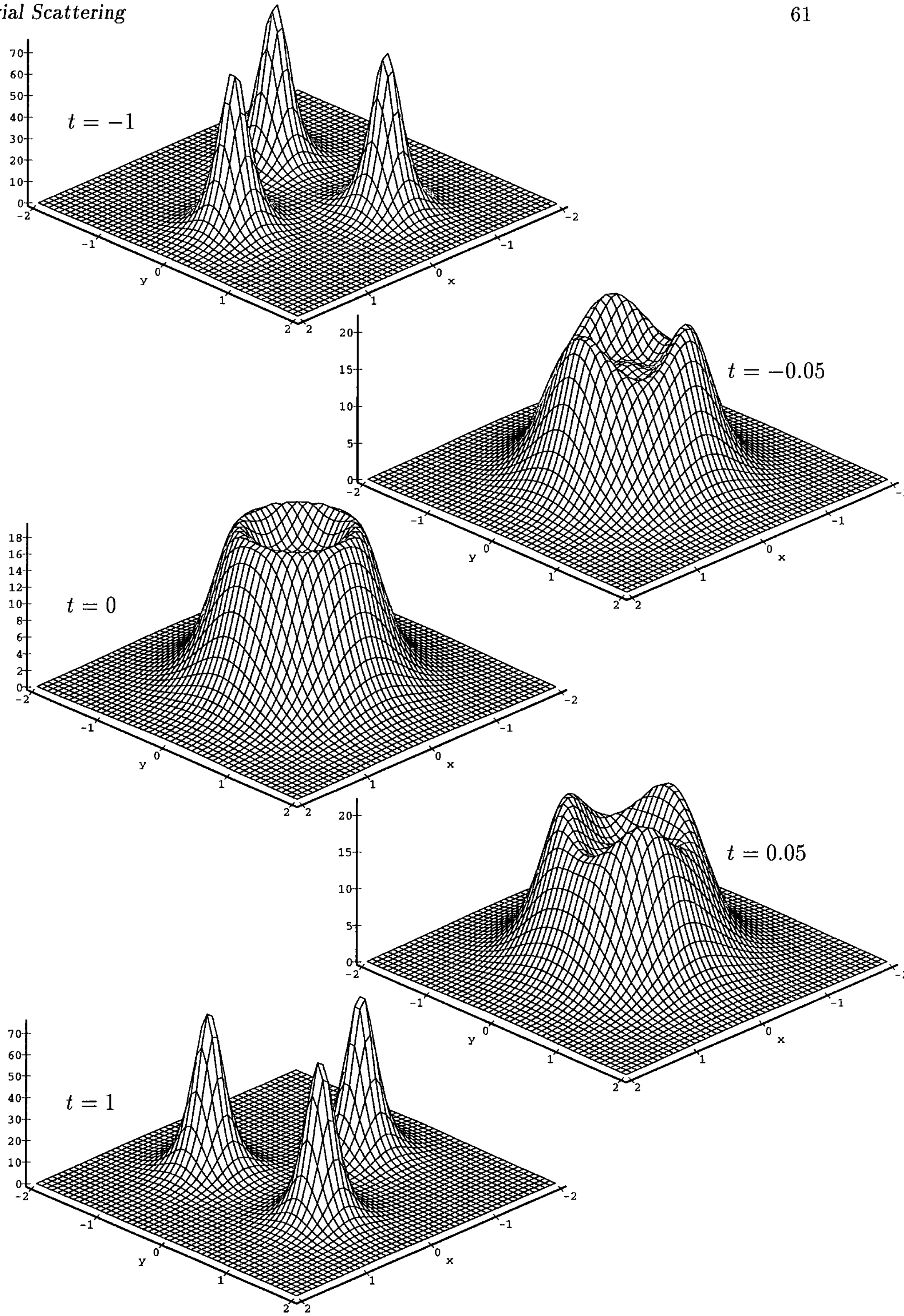


Figure 4.4: Energy density at increasing time of the three-soliton system with 60^0 angle scattering.

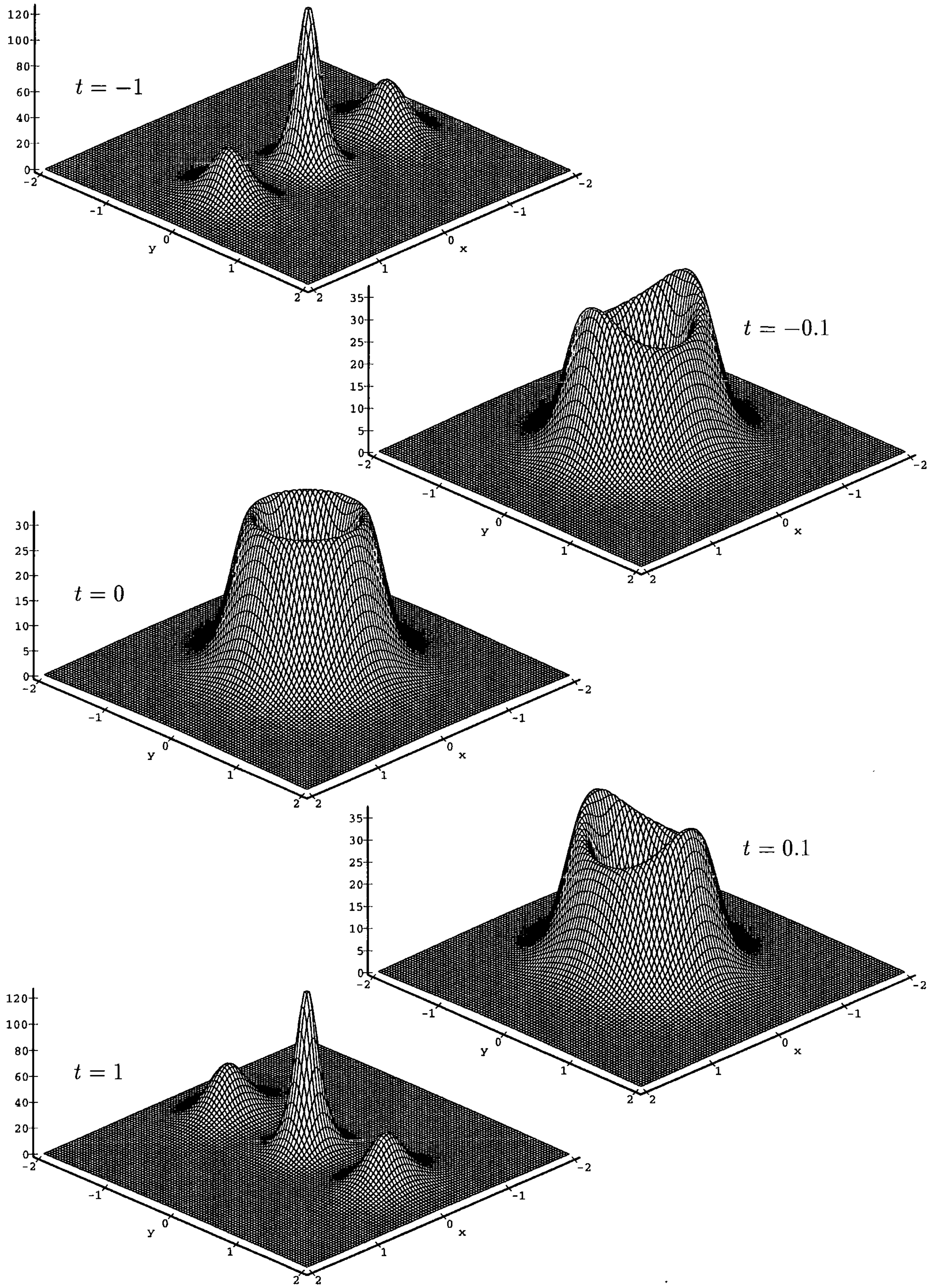


Figure 4.5: Energy density at various times of three-soliton system, with one being static at the origin.

In principle one should be able to visualize the emerging soliton structures when $f(\omega) = \omega^p$ and $h(\omega) = \omega^q$, i.e. are rational of degree $p, q \in \mathbb{N}$, respectively. In fact, for $q > p$ the configuration consists of $(p-1)$ static solitons at the "centre-of-mass" of the system (if more than one, a ring structure is formed) accompanied by $N = q - p + 1$ solitons accelerating towards the ones in the middle, scattering at an angle of π/N, and then decelerating as they separate. This follows from the fact that the field J departs from its asymptotic value J_0 when $\vartheta = \omega^{(p-1)}(p\,(u - \mu^{-2}v) + \omega^N) \to 0$, which is true when either $\omega^{(p-1)} = 0$ or $\omega^N + p\,(u - \mu^{-2}v) = 0$; and this is approximately where the solitons are located.

We conclude this section by investigating the corresponding case where $\psi(\lambda)$ has a triple pole (and no others). Therefore, it is taken to have the form

$$\psi(\lambda) = I + \sum_{k=1}^{3} \frac{\mathcal{R}_2}{(\lambda - \mu)^k}. \tag{4.22}$$

As we have already mentioned, the reality condition (4.5) is satisfied if and only if ψ factorizes into three simple factors of the following type

$$\psi(\lambda) = i \left(I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} \frac{q_1^\dagger \otimes q_1}{\|q_1\|^2} \right) \left(I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} \frac{q_2^\dagger \otimes q_2}{\|q_2\|^2} \right) \left(I - \frac{(\bar{\mu} - \mu)}{(\lambda - \mu)} \frac{q_3^\dagger \otimes q_3}{\|q_3\|^2} \right), \tag{4.23}$$

for some 2-vectors q_k. The requirement that the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ should be independent of λ imposes differential equations on q_k; which are three nonlinear equations, and it seems difficult to find their general solution.

One way of proceeding is to take a solution for the simple-pole case (4.7) with $n = 3$, put $\mu_1 = i + \varepsilon$, $\mu_2 = i$, $\mu_3 = i - \varepsilon$ and take the limit $\varepsilon \to 0$. In order to obtain a smooth solution ψ for all (u, v, x), it is necessary that $f_1(\omega_1) - f_2(\omega_2) \to 0$, $f_1(\omega_1) - f_3(\omega_3) \to 0$, $f_2(\omega_2) - f_3(\omega_3) \to 0$ as $\varepsilon \to 0$. So let us write $f_1(\omega_1) = f(\omega_1) + \varepsilon\, h(\omega_1) + \varepsilon^2\, g(\omega_1)$, $f_2(\omega_2) = f(\omega_2)$, $f_3(\omega_3) = f(\omega_3) - \varepsilon\, h(\omega_3) + \varepsilon^2\, g(\omega_3)$, where f, h and g are rational functions of one variable. On taking the limit, we obtain a ψ of the form (4.23), smooth on $\mathbf{R}^{2+1}$ and such that the matrices A and B be independent of λ.

Consequently, $J = \psi(0)^{-1}$ is a smooth solution of (4.1) of the form

$$J = i \left(I - \frac{2q_3^\dagger \otimes q_3}{\|q_3\|^2} \right) \left(I - \frac{2q_2^\dagger \otimes q_2}{\|q_2\|^2} \right) \left(I - \frac{2q_1^\dagger \otimes q_1}{\|q_1\|^2} \right), \tag{4.24}$$

with q_k being in terms of $f(z)$, $h(z)$ and $g(z)$ by

$$\begin{aligned} q_1 &= (1+|f|^2)^2(1,f) - 4i(b+id)(1+|f|^2)(\bar{f},-1) - 4b^2(\bar{f}^2,-\bar{f}-2i\bar{b}) - 8id\bar{b}(1,f), \\ q_2 &= (1+|f|^2)(1,f) - 2ib(\bar{f},-1), \\ q_3 &= (1,f), \end{aligned} \tag{4.25}$$

where $b = tf'(z) + h(z)$ and $d = t^2 f''(z)/2 + i(t-y)f'(z)/2 + th'(z) + g(z)$. Note that the 2-vectors q_2, q_3 here correspond to the ones given by (4.19) for $\mu = i$, respectively.

Let us examine a sample example of this solution, since we may take f, h and g to be any rational meromorphic function of z.

- Let us take $f(z) = 0$, $h(z) = z$ and $g(z) = z^2$; thus $b = z$ and $d = t + z^2$. This solution consists of two solitons coming in along the y-axis merging to form a peak at the origin and then two new solitons emerging along the x-axis. Figure 4.6 illustrates what happens near $t = 0$.

  The energy density of the system is,

$$\mathcal{E} = 32\frac{80r^4 + 32(r^2+t^2) + 256t^2r^2 - 64t(x^2-y^2) + 128tyr^2 - 8y + 3}{[32r^4 + 12r^2 - 16yr^2 + 16t^2 + 16ty + 32t(x^2-y^2) + 1]^2}, \tag{4.26}$$

  which has a reflection symmetry around the x-axis. For large (positive) t, $\mathcal{E}$ is peaked at two points on the y-axis, namely $y \approx \pm\sqrt{t}$. Moreover, the height of the corresponding solitons is proportional to $1/t$, and their radius is proportional to $\sqrt{t}$; which means that the y-axis asymmetry vanishes at $t \to \infty$.

4.4 Construction of Soliton-Antisoliton Solutions

In this section we construct a large family of solutions which as we will argue later, can be thought of as representing soliton-antisoliton filed configurations. Roughly speaking, solitons correspond to f being a function of the variable z, and antisolitons correspond to a function of $\bar{z}$.

One way to generate a soliton-antisoliton solution of (4.1), is to assume that $\psi(\lambda)$ has the form

$$\psi(\lambda) = I + \frac{n^1 \otimes m^1}{(\lambda - i)} + \frac{n^2 \otimes m^2}{(\lambda + i)}. \tag{4.27}$$

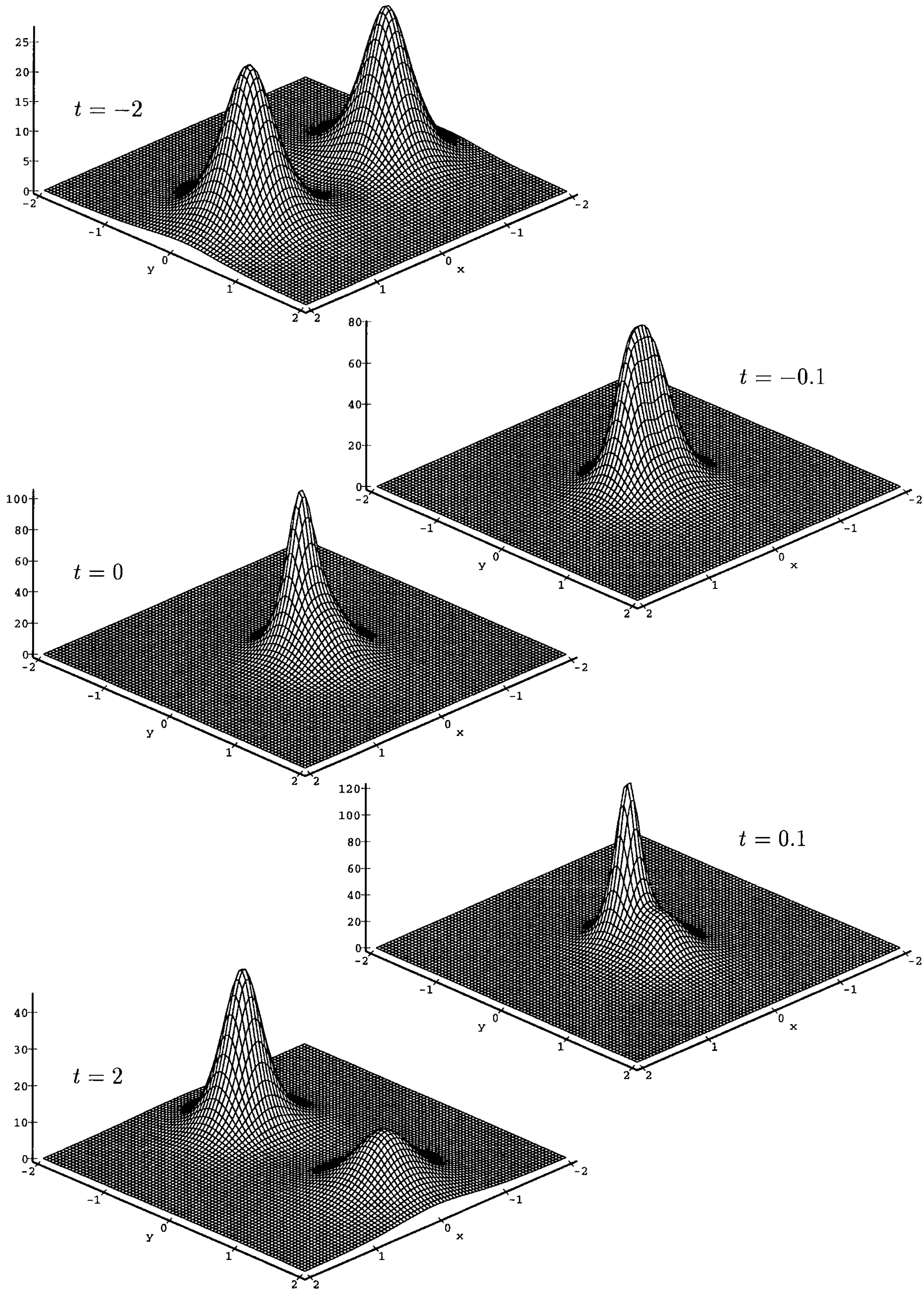


Figure 4.6: Energy density at increasing time when $\psi(\lambda)$ has a triple pole (and no others).

Here n^k, m^k for $k = 1, 2$ are complex-valued 2-vector functions of $(t, z, \bar{z})$ (not depending on λ).

The idea is to find the $n_1^1, ..., m_1^1, ...$ such that the reality condition (4.5) holds, and such that the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ are independent of λ. One way of proceeding is to take the solution (4.7) with $n = 2$, put $\mu_1 = i + \varepsilon$, $\mu_2 = -i - \varepsilon$ and take the limit $\varepsilon \to 0$. In order for the resulting ψ to be smooth on $\mathbf{R}^{2+1}$ it is necessary to take $f_1 = f(\omega_1)$, $f_2 = -1/\bar{f}(\omega_2) - \varepsilon\, h(\omega_2)$, where f and h are rational functions of one variable. On taking the limit $\varepsilon \to 0$, we then obtain a ψ as in (4.27) with $m^k = (m_1^k, m_2^k)$ being holomorphic functions of z (or $\bar{z}$), through the relations $m^1 = (1, f)$, $m^2 = (-\bar{f}, 1)$, while

$$
\begin{aligned}
n^1 &= \frac{2i(1+|f|^2)}{(1+|f|^2)^2+|w|^2}\bar{m}^1 + \frac{2\bar{w}}{(1+|f|^2)^2+|w|^2}\bar{m}^2, \\
n^2 &= -\frac{2w}{(1+|f|^2)^2+|w|^2}\bar{m}^1 - \frac{2i(1+|f|^2)}{(1+|f|^2)^2+|w|^2}\bar{m}^2,
\end{aligned}
\tag{4.28}
$$

with

$$
w \equiv \bar{h}f^2 + 2tf'. \tag{4.29}
$$

So we generate a solution $J = \psi(\lambda = 0)^{-1}$, which depends on the two arbitrary rational functions $f = f(z)$ and $h = h(\bar{z})$. This solution has the form

$$
J = \frac{1}{(1+|f|^2)^2+|w|^2}\begin{bmatrix} |w|^2 + 2i(f\bar{w}+\bar{f}w) - (1+|f|^2)^2 & -2i\,(w - f^2\bar{w}) \\ -2i\left(\bar{w} - \bar{f}^2 w\right) & |w|^2 - 2i(f\bar{w}+\bar{f}w) - (1+|f|^2)^2 \end{bmatrix}, \tag{4.30}
$$

with w given by (4.29). In general, by taking $f(z) = z^p$ and $h(\bar{z}) = \bar{z}^q$ where p is a positive integer and q is a non-negative integer; the energy, obtained by integrating (4.2), is $E = (2p+q)8\pi$. Roughly speaking, the solution looks like $(2p+q)$ lumps at arbitrary positions in the xy-plane; which as we are going to see are a combination of solitons and antisolitons.

A topological charge may be defined for the field J (4.30) by exploiting the connection of it with the O(3) sigma model. Recall that, the (unmodified) chiral model (2.15) is equivalent to the O(4) sigma model through the relation

$$
J = I\,\phi_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{\phi}, \tag{4.31}
$$

as we have already mentioned in (2.55). The only static finite energy solutions of the O(4) sigma model correspond to the embedding of the O(3) sigma model [70]. Therefore the only static solutions of (4.1) are the O(3) embeddings that we shall describe. This is because for the one-soliton solution (static or Lorentz boosted in the y-axis) the term in (4.1) proportional to V^α is zero, so the system behaves like the O(4) model, for which the O(3) embedding is totally geodesic. [However, for time-dependent configurations, the term proportional to V^α is non-zero and will affect the evolution of the field, which will in general not lie in an O(3) subspace of O(4).]

To proceed further, let us mention the topological aspects of the O(3) and O(4) sigma models. In studying soliton-like solutions, we require that the field configuration has finite energy. If we compactify the fixed-time surfaces to a sphere by requiring the fields to tend to a constant value at spatial infinity, the classical field configurations at fixed time, are maps from $\mathbf{S}^2$ into the target space and thus fall into disconnected homotopy classes. Now for the O(3) model, the field is a map $\boldsymbol{\phi} : \mathbf{S}^2 \to \mathbf{S}^2$, and due to the homotopy relation

$$\pi_2(\mathbf{S}^2) = \mathbf{Z}, \tag{4.32}$$

fields in different homotopy classes cannot be deformed continuously into each other. Though, they are classified by an integer winding number $\mathcal{N}$ which is a conserved topological charge and counts how many times ϕ spans $\mathbf{S}^2$ as x runs all space. An expression for this charge is given by

$$\mathcal{N} = (8\pi)^{-1} \int \epsilon_{ij}\, \boldsymbol{\phi} \cdot (\partial_i \boldsymbol{\phi} \wedge \partial_j \boldsymbol{\phi})\, d^2x, \tag{4.33}$$

where $i = 1, 2$ with $x^i = (x, y)$.

Although, for the O(4) model [the same argument is valid for (4.1) due to the topological aspects of the theory] the field at fixed time is a map $(\phi_0, \boldsymbol{\phi}) : \mathbf{S}^2 \to \mathbf{S}^3$ and the corresponding homotopy relation is

$$\pi_2(\mathbf{S}^3) = 0, \tag{4.34}$$

so there is no winding number. However, for soliton solutions that correspond to some initial embedding of O(3) space into O(4), there is a useful topological quantity, as we are going to see.

Consider the O(4) configuration which at some time corresponds to an O(3) embedding, which we choose to be $\phi_0 = 0$ for definiteness. At this time the field is restricted to an $\mathbf{S}^2$ equator of the possible $\mathbf{S}^3$ target space. Suppose that the field never maps to the anti-podal points $\{\mathcal{A}_1, \mathcal{A}_2\} = \{\phi_0 = 1, \phi_0 = -1\}$ at any time, so the target space is $\mathbf{S}^3_0 = \mathbf{S}^3 - \{\mathcal{A}_1, \mathcal{A}_2\}$. Now $\mathbf{S}^3_0 \approx \mathbf{S}^2 \times \mathbf{R}$, and thus we have the homotopy relation

$$\pi_2(\mathbf{S}^3_0) = \pi_2(\mathbf{S}^2 \times \mathbf{R}) = \pi_2(\mathbf{S}^2) \oplus \pi_2(\mathbf{R}) = \mathbf{Z}, \tag{4.35}$$

and therefore a topological winding number exists. An expression for this winding number is easy to give, since it is the winding number of the map after projection onto the chosen $\mathbf{S}^2$ equator, i.e.

$$\mathcal{N}' = (8\pi)^{-1} \int \epsilon_{ij}\, \boldsymbol{\phi}' \cdot (\partial_i \boldsymbol{\phi}' \wedge \partial_j \boldsymbol{\phi}')\, d^2x, \tag{4.36}$$

where $\boldsymbol{\phi}' = \boldsymbol{\phi}/|\boldsymbol{\phi}|$. If the field does map to the anti-podal points $\{\mathcal{A}_1, \mathcal{A}_2\}$ at some time the winding number is ill-defined at this time and if considered as a function of time $\mathcal{N}'$ will be integer valued but may suffer discontinuous jumps as the field moves through the anti-podal points. In the following examples, before comparing the solution J given by (4.30) with the O(3) embedding it is convenient to perform the transformation $J \to M\,J$ with $M = (\sqrt{2})^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ so that the evolution of the field remains close to the O(3) embedding.

4.5 Soliton-Antisoliton Scattering

Usually in the nonintegrable models, there is an attractive force between solitons of opposite topological charge. In fact, if the solitons and antisolitons are well separated, then they attract each other and eventually annihilate into a wave of pure radiation which spreads with the velocity of light [57, 58]. However, the interaction forces between solitons and antisolitons do depend on their configuration; in particular, they depend on the relative orientation between them in the internal space. Therefore, the cross section for the soliton-antisoliton elastic scattering is non-zero. [In the real world, the proton-antiproton elastic scattering is seen in a reasonable fraction of the cases.] This is the first example for which there has been constructed an explicit (since the system is integrable) solution of elastic soliton-antisoliton scattering in either integrable or nonintegrable model. As a result, it provides a major link between soliton dynamics in integrable and nonintegrable systems.

The picture was, however, undetermined by some numerical solutions obtained through the connection of the modified chiral model (4.1) with the O(3) sigma model [42]. This reveal that soliton-antisoliton solution can also scatter at right angles. Since the system is integrable, one might expect there to be explicit solutions which exhibit nontrivial scattering. This section provided examples of such solutions. In particular, we will see that when a soliton and an antisoliton are moving along the x-axis towards each other at an accelerating rate, they merge at the origin and form a peak. Note that a peak is formed rather than a ring since the energy is mainly kinetic when a soliton and an antisoliton merge. However, rather than the peak dissipating in a wave of radiation it now reforms into two new structures which undergo 90^0 scattering. In general, in all head-on collisions of N moving soliton and antisoliton objects, the scattering angle is π/N degrees relative to the initial direction of motion.

Next we looked at two cases corresponding to the mixtures of solitons and antisolitons. [The configurations given by (4.30) when $h(\bar{z}) = 0$ are equivalent to the ones obtained from (4.10,4.12) when $f(z) = z^p$.]

- First, let us take $f(z) = z$ and $h(\bar{z}) = 1$. Roughly speaking, if r is large, J is close to its asymptotic value J_0, as long as $w/z^2 = 1 + 2t/z^2 \approx 1$; but as z approaches $\pm\sqrt{-2t}$ then $w \to 0$, and J departs from its asymptotic value: this is where the two structures are located. More precisely, for negative t, the two objects are on the x-axis, approximately at $x \approx \pm\sqrt{-2t}$; while for positive t, they are on the y-axis, approximately at $y \approx \pm\sqrt{2t}$. Figure 4.7 illustrates what happens near $t = 0$.

  The picture is consistent with the properties of the energy density of the solution, which is
$$\mathcal{E} = 16\frac{2r^4 + 4r^2 + 4t^2(1 + 2r^2) - 4t(x^2 - y^2) + 1}{[2r^4 + 2r^2 + 4t(x^2 - y^2) + 4t^2 + 1]^2}. \tag{4.37}$$
  Note the symmetry of $\mathcal{E}$ under the interchange $t \mapsto -t$, $x \rightleftharpoons y$; the time symmetry of the density confirms the lack of radiation. The corresponding localized structures are not however of constant size: for large (positive) t, their height is proportional to $1/t$, while their radius is proportional to $\sqrt{t}$.

  The projected topological charge $\mathcal{N}'$ is zero throughout the scattering process and so at first sight it appears that the peak has reformed into a soliton-antisoliton pair. Indeed, this is what happens. If we examine the projected topological charge density q', i.e.
$$\mathcal{N}' = \int q'\, dx\, dy, \tag{4.38}$$

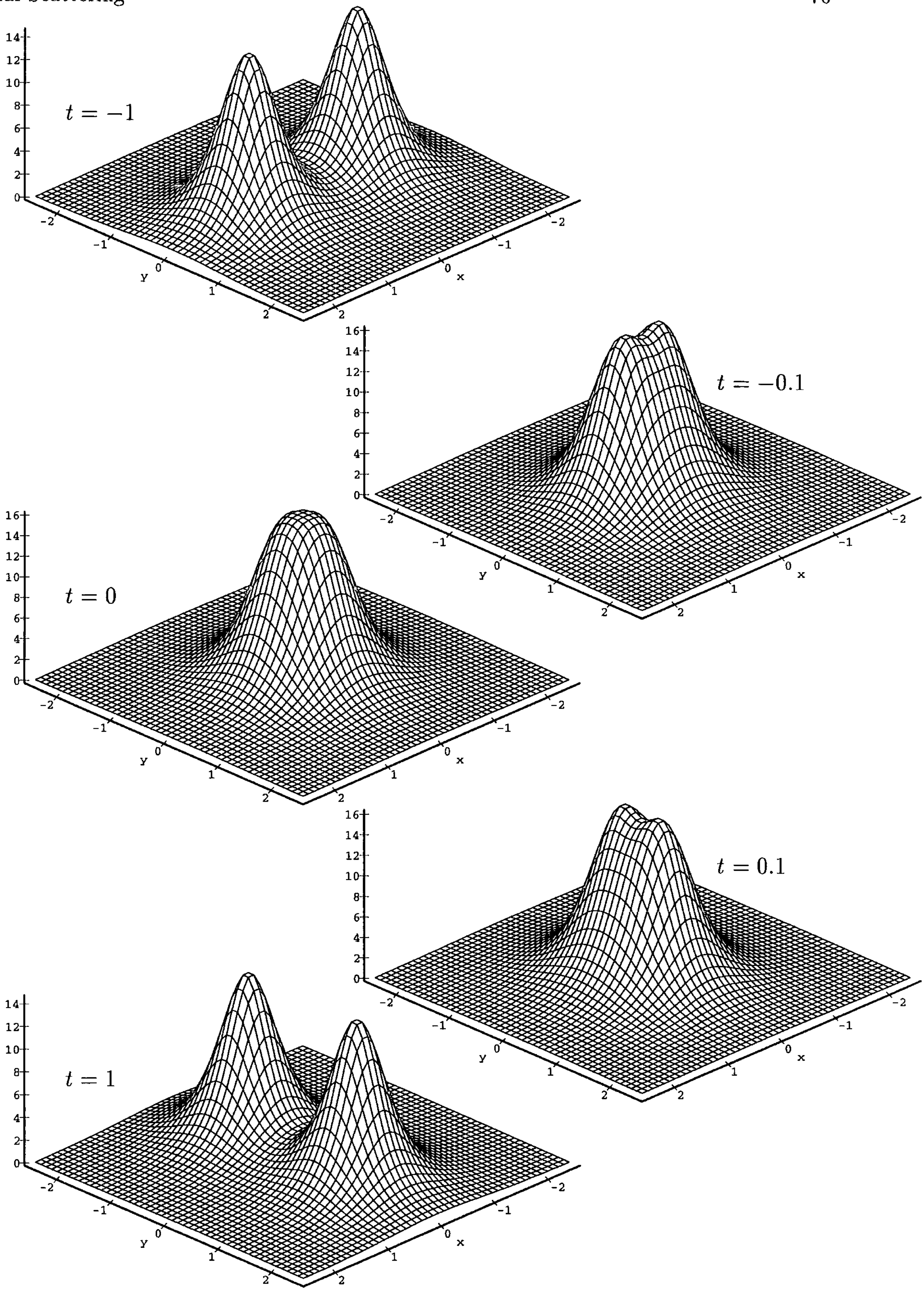


Figure 4.7: Energy density at increasing time showing a 90^0 scattering between a soliton and an antisoliton.

we find that it has an almost identical distribution (up to a scale) to that of the energy density (see Figure 4.8(a)). Therefore, the configuration represents a soliton and an antisoliton which are clearly visible as distinct structures having respectively +1 and -1 units of topological charge concentrated in a singe lump.

One reason that the incoming and outcoming structures of the topological charge density appears quite different may perhaps be related to the asymmetry between x and y coordinates in the equation of motion (4.1).

Equation (4.1) is not Lorentz invariant and indeed is not even radially symmetric due to the presence of the vector V_a which picks out a particular direction in space, and therefore one may expect to find different scattering behaviour for more general solutions; e.g., when the soliton and the antisoliton are moving along the x-axis rather that the y-axis. However, this is not true since (4.1) is a reduction of the self-dual Yang-Mills equation in $\mathbf{R}^{2+2}$ which does have an SO(1,2) symmetry. Therefore, the SO(2) symmetry of the Yang-Mills system means that any given solution J, can in principle be converted to gauge fields by performing a coordinate rotation (together with a gauge transformation) and then recover the corresponding J' which will describe the same solution as J but with a rotated coordinate system. Indeed, this is what happens by taking

$$f(z) = e^{(2i\phi)}z, \qquad h(\bar{z}) = 1, \tag{4.39}$$

where ϕ is an angle in the xy-plane. This picture presents a rotated version through any angle ϕ in the xy-plane of the original one (i.e., Figure 4.7).

- Finally, let us take $f(z) = z$ and $h(\bar{z}) = \bar{z}$. The corresponding configuration consists of two antisolitons and one soliton (see Figure 8(b)).

  It is interesting to look at the time dependence of various energies in each process. The total energy, of course, is constant and it is the spatial integral of the following energy density

$$\begin{aligned}\mathcal{E} \;=\;& 8[r^8 + 8r^6 + 11r^4 + 4r^2 - 8x^5t + 16ty^2(x^3 + t) + 8t^2 + 48xy^2t + 2\\ &-16x^2t(x - t) + 24xty^4]/[r^6 + r^4 + 2r^2 + 4t^2 + 4tx^3 - 12xy^2t + 1]^2.\end{aligned} \tag{4.40}$$

  Obviously, the energy density $\mathcal{E}$ is symmetric under the interchange $t \mapsto -t$, $x \mapsto -x$ and $y \mapsto -y$, only. Again all three structures come together forming a bell-like

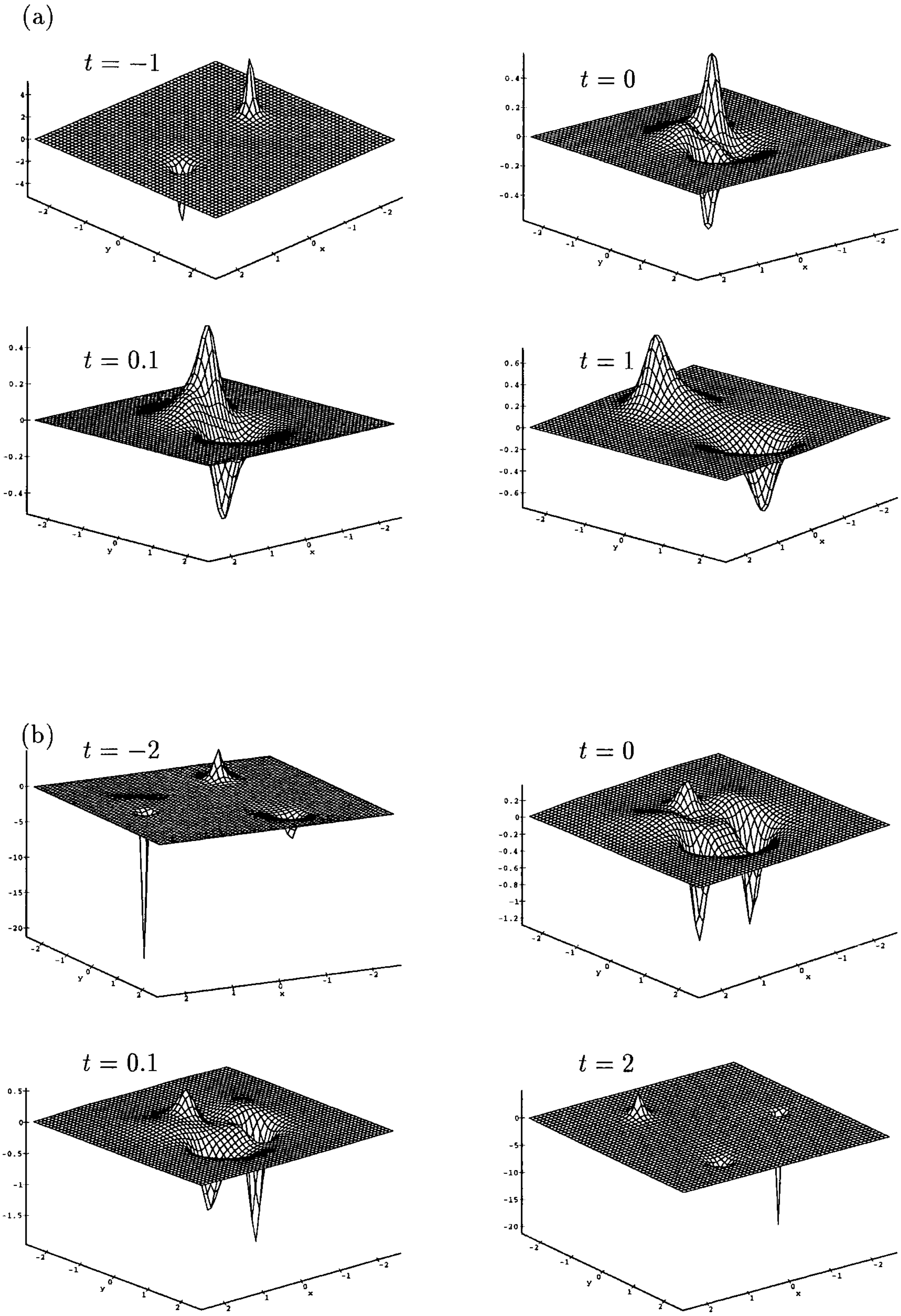


Figure 4.8: Topological charge density at increasing times for (a) soliton-antisoliton scattering, (b) two-antisoliton one-soliton scattering.

structure and then emerge at an angle of 60^0 with respect to the original direction. However, by looking at the maximum of $\mathcal{E}$ we observe that, for large (positive) t, the height of the localized structures is proportional to $t^{-4/3}$, while their radius is proportional to $t^{1/3}$; thus they spread out as they move apart.

Figure 4.9 shows the results of a head-on collision of one-soliton two-antisoliton system.

Let us conclude with the observation that, by taking $f(z) = z^p$ and $h(\bar{z}) = \bar{z}^q$, J departs from its asymptotic value J_0 when $w = z^{p-1}(2tp + z^N) \to 0$ with $N = p + q + 1$, which is true when either $z^{(p-1)} = 0$ or $2tp + z^N = 0$: this is approximately where the lumps are located. Therefore, J represents a family of soliton-antisoliton solution which consists of $(p - 1)$ static soliton-like objects at the origin, with N others accelerating towards them, scattering at an angle of π/N, and then decelerating as they separate.

4.6 Conclusion

The infinite number of conservation laws associated with a given integrable system place severe constraints upon possible soliton dynamics. The construction of exact analytic multisoliton solutions with trivial scattering properties is a result of such integrability properties. In this chapter new soliton and soliton-antisoliton solutions have been obtained for the planar modified chiral model (4.1). These structures travel with non-constant velocity; their size is non-constant; and they interact non-trivially. Such results might be useful for connecting integrable and nonintegrable systems which possess soliton solutions. In addition, they indicate the likely occurrence of new phenomena in higher dimensional soliton theory that are not present in (1+1) dimensions.

It seems likely that there are many more interesting solutions still to be found; an open question being what is the general form of the function ψ when it has a higher-order pole in λ. One could, for example, investigate the case $n = 3$ for $\psi(\lambda)$ with a single and a double pole; and determine the scattering properties of the emerging structures, in terms of their initial velocity and of the values of the impact parameter. Finally, it would be of great interest to deduce the general form of the function $\psi(\lambda)$ for the soliton-

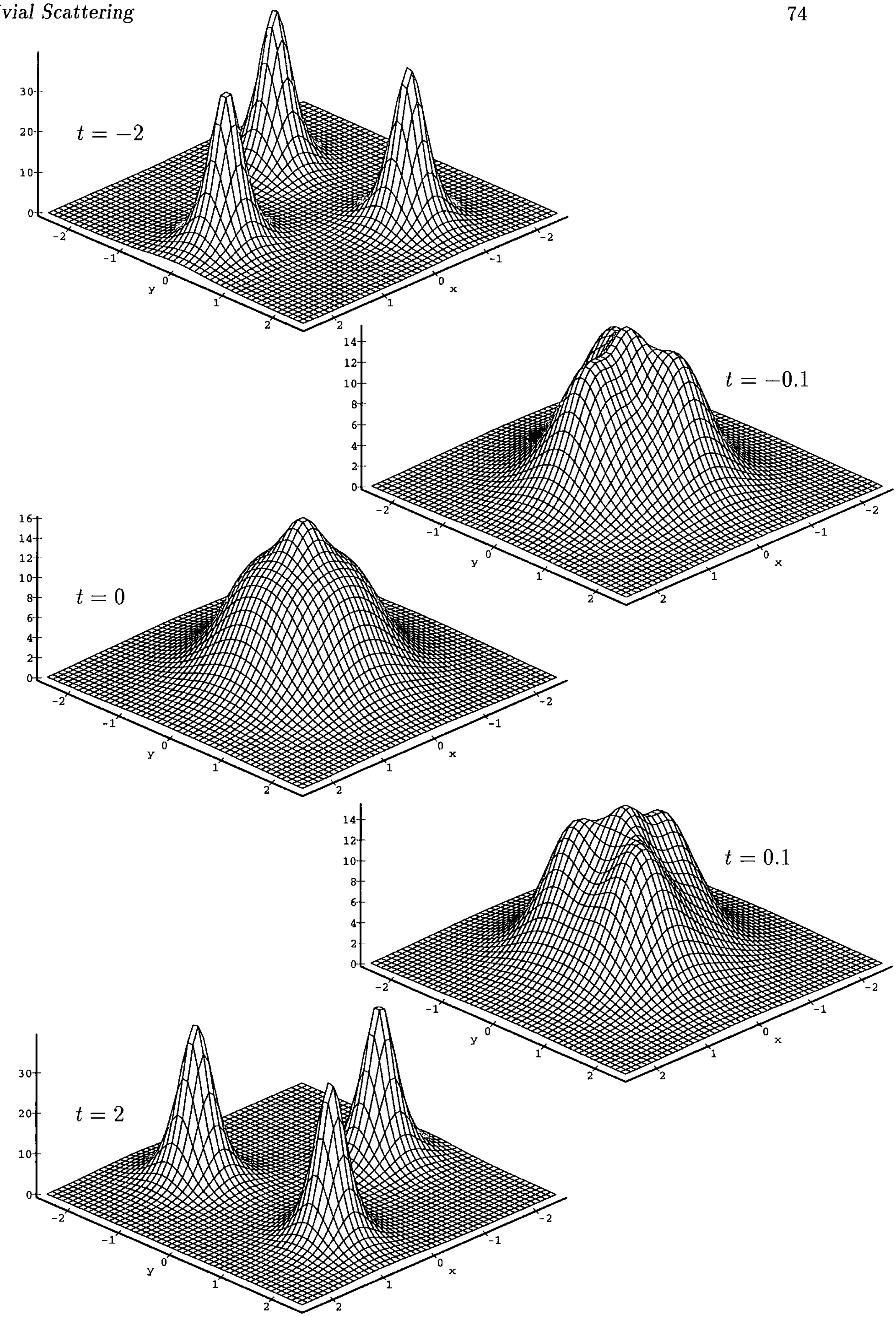


Figure 4.9: Energy density of two-antisoliton one-soliton system at various times.

antisoliton case (4.27) with the only constraint to satisfy the reality condition (4.5) and the requirement that the matrices $A = (L\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ be independent of λ.

Chapter 5

A Novel Discrete O(3) Sigma Model

5.1 Introduction

The nonlinear O(3) sigma model in (2+1) dimensions is a popular model in theoretical physics; the static system is integrable and of Bogomol'nyi type (all minimal energy solutions can be obtained by solving the Bogomol'nyi equations). As a result, one can explicitly write down soliton solutions of arbitrary degree in term of rational functions [36]; but the model is scale invariant and therefore, its solitons have no fixed size and so their stability is a central question. There is a possibility that under small perturbations they could shrink towards infinitely tall spikes of zero width or may spread out, with this expansion continuing indefinitely. That this indeed happens is confirmed by numerical experements [38, 71]. General time-dependent solutions cannot be constructed explicitly, and so it is natural to investigate numerical evolution techniques which discretize the partial differential equations.

Given a continuum field theory, there are many different lattice systems which reduce to it in the continuum limit. In systems where there are topological configurations (instantons, monopoles, etc) one often has a Bogomol'nyi bound which is related to the stability of the topological objects in question. If the bound is maintained on the lattice, the topological objects will be well-behaved even when their size is comparable to the lattice spacing. Lattice versions of these systems are important for purposes of numerical computations but they have, generally, not preserved the Bogomol'nyi bound. The object of this chapter is to present a lattice version of the O(3) sigma model in two space dimensions, in which the Bogomol'nyi bound is maintained. The primary aim is not to simulate the continuum system, but rather to define an alternative, *genuine* lattice system with similar properties but more convenient to study numerically.

Few years ago, Leese [72] discretized the (unmodified) O(3) sigma model in $(2+1)$ dimensions. He imposed radial symmetry, made the radial coordinate r discrete and found a *reduced* lattice system with Bogomol'nyi bound. But, although the topological lower bound can be attained, the minimum-energy configurations are not explicit. On the other hand, Ward [73, 74] described a lattice version of this model with Bogomol'nyi bound, without any symmetry constraint. In this general case, however, the lower bound cannot be attained.

In this chapter, we describe an alternative discrete O(3) sigma model in $(2+1)$ dimensions which maintains an important feature of the continuum model, i.e. the Bogomol'nyi bound, and admits explicit minimum-energy configurations. So this lattice O(3) sigma model is quite different from the ones described above. Following Leese, only field configurations for which the energy density (and not necessarily the fields) is radially symmetric will be considered here, so that in effect one obtains a one-dimensional system and therefore, the construction of the discrete Bogomol'nyi equations is less complicated. Solutions of these equations (which were obtained analytically) are then used as the basis for a numerical study of soliton stability.

The rest of this chapter is arranged as follows. In the next section we describe initially, the familiar continuum O(3) sigma model in $(2+1)$ dimensions then reparametrize the fields in order to impose radial symmetry and, finally, discretize the model. In section 5.3 we study the dynamics of the 2-soliton configuration at low shrinking velocities using the slow-motion (or geodesic) approximation, make approximate analytic predictions of its behaviour, and compare these with numerical results. In section 5.4 we investigate the properties of the lattice O(3) solitons numerically, i.e. using numerical procedures for solving the evolution scheme. Unfortunately, due to the imposed radial symmetry in the xy-plane, we could not study scattering processes; but the scheme is still profitable on studying the soliton stability. We finish the chapter with a short section containing our conclusions.

5.2 The Lattice O(3) Sigma Model

Let us begin with a brief review of the continuum O(3) sigma model in two space dimensions. The field ϕ is a unit 3-vector field on $\mathbb{R}^2$ (i.e. a smooth function from $\mathbb{R}^2$ to the target space $\mathbb{S}^2$), with the boundary condition $\phi \to \phi_o$ as $r \to \infty$ in $\mathbb{R}^2$ (sufficiently fast for the energy to converge). Here ϕ_o is some fixed point on the image sphere $\mathbb{S}^2$. Hence there are distinct topological sectors classified by an integer k (topological charge), which represents the number of times $\mathbb{R}^2$ is wrapped around $\mathbb{S}^2$. Roughly speaking, k is the number of solitons. The potential energy of the field is

$$E_p = (8\pi)^{-1} \int [(\partial_x \phi)^2 + (\partial_y \phi)^2]\, dx\, dy, \tag{5.1}$$

and the appropriate Bogomol'nyi argument gives the bound $E_p \geq |k|$. There are fields which attain this lower bound (such minimum-energy fields will be called solitons in what follows). Since E_p is invariant under the scaling transformation $\phi(x^j) \mapsto \phi(\lambda x^j)$ these configurations are metastable rather than stable (their size is not fixed).

From now on we will restrict attention to fields which are invariant under simultaneous rotations and reflections in space and target space. Thus we assume that $\phi = (\phi^\alpha, \phi^3)$ with $\alpha = 1, 2$ is of the so-called hedgehog form

$$\phi^\alpha = \sin g(r,t)\, k^\alpha, \qquad \phi^3 = \cos g(r,t), \tag{5.2}$$

characterized by its topological charge k, defining the unit vector $k^\alpha = (\cos k\theta, \sin k\theta)$ in (5.2) in terms of the azimuthal angle θ; and by the real function g of the polar coordinates and t (so-called profile function) which satisfies certain boundary conditions. The corresponding potential energy of the field (5.2) is

$$E_p = \frac{1}{4} \int_0^\infty (r\, g'^2 + \frac{k^2}{r} \sin^2 g)\, dr, \tag{5.3}$$

where $g' = dg/dr$. (This is normalized so that a static configuration has k energy). The boundary conditions are $g(0,t) = \pi$, in order to ensure a unique definition of ϕ at the origin and $g(r,t) \to 0$ as $r \to \infty$, so that E_p converges.

The standard Bogomol'nyi argument [75] is

$$0 \;\leq\; \frac{1}{4} \int_0^\infty (\sqrt{r} g' + \frac{k}{\sqrt{r}} \sin g)^2\, dr$$

$$= E_p - \frac{k}{2}\int_0^\infty \partial_r(\cos g)\,dr$$
$$= E_p - k. \tag{5.4}$$

So the energy E_p is bounded below by k; and E_p equals k if and only if $g' = -k\sin g/r$, the solution of which is the static k-soliton configuration

$$g(r) = 2\arctan\left(\frac{a}{r^k}\right), \tag{5.5}$$

located at the origin, with a being a positive real constant which determines the soliton size. If a is large, the soliton configuration is flat and broad; while if a is small, it is tall but narrow. In fact, the height of the configuration (maximum of the energy density) is proportional to $a^{-2/k}$; while its radius (width) is proportional to $a^{1/k}$. Notice that, for $k = 0$ the field is constant and the energy density is zero everywhere; while for $k = 1$ the configuration looks like a lump peaked at the origin; and for $k > 1$ it is a ring centered at the origin. In what follows, we will assume that in all cases $k > 0$, since taking $k = 0$ does not test the ability of the model to handle nontrivial topologies.

So far all we have done is to re-express the k-soliton solution in terms of a real field g, which is a function of the polar radius r. It will now been seen how this description is useful in constructing discrete analogues of the Bogomol'nyi equations. From now on, r becomes a discrete variable, with lattice spacing h. More precisely, $r = nh$ with $n \geq 0$; h may be regarded as a dimensionless parameter in the model; while the real field g is defined at each lattice site. The subscript $+$ denotes forward shift, i.e. $g_+(r,t) = g(r+h,t) = g((n+1)h,t)$; and so the forward difference is given by $\Delta g = (g_+ - g)/h$. To obtain a lattice version of the Bogomol'nyi bound, we may begin with the same function $\cos g$ as appears in (5.4), and reconstruct the inequality. One way, (motivated by [76]) is to choose a factorization

$$k(\Delta\cos g) = -D_n F_n, \tag{5.6}$$

where $D_n \to \sqrt{r}g'$ and $F_n \to k\sin g/\sqrt{r}$ in the continuum limit $h \to 0$. Then define the potential energy of the lattice O(3) sigma model field to be

$$E_p = \frac{h}{4}\sum_{n=0}^{\infty}\left(D_n^2 + F_n^2\right). \tag{5.7}$$

As in the continuum case, it follows that E_p is bounded below by k (with soliton boundary conditions); and the minimum is attained if and only if $D_n + F_n = 0$.

Due to the fact that $k(\Delta \cos g) = -2k/h \sin(\frac{g_+ - g}{2}) \sin(\frac{g_+ + g}{2})$, the most natural choice seems to be

$$\begin{aligned} D_n &= \frac{2f(h)\sqrt{n}}{\sqrt{h}} \sin\left(\frac{g_+ - g}{2}\right), \\ F_n &= \frac{k}{f(h)\sqrt{hn}} \sin\left(\frac{g_+ + g}{2}\right), \qquad n > 0, \end{aligned} \tag{5.8}$$

where $f(h)$ is an arbitrary function of the lattice spacing with constraints $f(h) \to 1$ as $h \to 0$ and $f(h) > \sqrt{k/2}$ (see below). Remark: this implies that $k \leq 2$. The origin must be treated in a special way since (5.8) are undefined when $n = 0$. One possibility is to arrange that $D_0 + F_0 = 0$ identically. So choose

$$D_0 \equiv -F_0 = \sqrt{\frac{2k}{h}} \cos\left(\frac{g(h,t)}{2}\right). \tag{5.9}$$

Substituting these into (5.7) gives

$$E_p = k \cos^2\left(\frac{g(h,t)}{2}\right) + \sum_{n=1}^{\infty} [\, f^2\, n \sin^2\left(\frac{g_+ - g}{2}\right) + \frac{k^2}{4f^2\, n} \sin^2\left(\frac{g_+ + g}{2}\right)], \tag{5.10}$$

which reduces to (5.3) in the continuum limit. [Notice that, for models with different lattice spacing the effect of $f(h)$ is to decrease the importance of the $\sin^2 g$ term in the energy density, although the total energy is still the same as in the continuum limit, i.e. k in our units.]

So the real-valued field $g(r,t)$ depends on the continuous variable t, and the discrete variable r. The kinetic energy can be defined by the simple choice

$$E_k = \frac{h^2}{4} \sum_{n=1}^{\infty} n\, \dot{g}^2, \tag{5.11}$$

where $\dot{g} = dg/dt$. The boundary condition on g is that it should tend to zero at spatial infinity; this guarantees finite energy. For such fields, the total energy $E_t = E_p + E_k$ is bounded below by k ; and this lower bound is attained if and only if $\dot{g} = 0$, and $D_n + F_n = 0$ for $n > 0$. [Recall that $D_0 + F_0 = 0$ identically.]

This latter condition, i.e. $D_n + F_n = 0$, is called the Bogomol'nyi equation. It is a first-order difference equation, whose solutions (for the aforementioned boundary conditions) minimize the potential energy, and therefore, are also solutions (static ones) of the Euler-Lagrange equations

$$\sum_{n=1}^{\infty} n\, \ddot{g}(nh,t) = -\frac{2}{h^2} \frac{\partial E_p}{\partial g}, \tag{5.12}$$

since $\partial E_p/\partial g = 0$ at a minimum. So using the discrete Bogomol'nyi equations one gets first-order equations whose solutions are also static solutions of the second-order equations of motion. Moreover, these solutions have energy which is at its topological minimum value.

The Bogomol'nyi equation $D_n + F_n = 0$, may also be written as

$$\tan\frac{g_+}{2} = \frac{2f^2\,n - k}{2f^2\,n + k}\tan\frac{g}{2}, \qquad n > 0, \tag{5.13}$$

from which one sees that the function $f(h)$ should be greater than $\sqrt{k/2}$ if one is to obtain a well-behaved solution. For $k > 2$, one may choose $f(h)$ to have an appropriate form such that a well-behaved solution to exist; however, the corresponding lattice system does not have the correct $h \to 0$ limit (i.e. does not reduce to the continuum system). An alternative way is to take, again, $f(h) = 1+h$ with the additional constraint $h > \sqrt{k/2}-1$.

The solution of (5.13) can be written down explicitly, (by setting $g(nh) = 2\arctan z_n$ and solving the reduced linear difference equations with variable coefficients) one gets that

$$g(nh) = \begin{cases} \pi, & n = 0, \\ 2\arctan(z_1\mathcal{Z}_n), & n > 0, \end{cases} \tag{5.14}$$

where

$$\mathcal{Z}_n = \frac{\Gamma(n - k/2f^2)\,\Gamma(1 + k/2f^2)}{\Gamma(n + k/2f^2)\,\Gamma(1 - k/2f^2)}, \tag{5.15}$$

and z_1 is an arbitrary positive constant which specifies, as in the continuum case, the soliton size. This is a static lattice k-soliton solution located at the origin; which corresponds to a minimum of the energy in the k sector and thus, it is stable under perturbations which remain in that sector. So there are a large family of models having static solutions of the form (5.14) since one may take f to be any function of h. Although, the corresponding lattice profile function tends to $g(nh) \to 2\arctan(z_1/n^{k/f^2})$ as $n \to \infty$ which reduces to (5.5) in the continuum limit if and only if $f(h)$ is close to unity; i.e. $f(h) = 1 + O(h)$ for small h. Hence, we will take in the $k = 1$ sector the function $f(h)$ to be constant and equal to unity, for any h; while in the $k = 2$ sector to be $f(h) = 1 + h$, for small h.

It would be nice to have a lattice analogue of the configuration width $a^{1/k}$, which appeared in (5.5). One possibility is to set

$$a_n = (nh)^k \tan\frac{g(nh)}{2}, \tag{5.16}$$

and then to define $a = \lim_{n\to\infty} a_n$, provided this limit exists. Indeed, a is proportional to $z_1\, h^k$. If $E_p = \sum_{n=0}^{\infty} E_{p_n}$, the energy density at the origin is $E_{p_0} = k(1+z_1^2)^{-1}$; therefore, E_{p_0} is close to the Bogomol'nyi bound as $z_1 \to 0$ (i.e. $E_{p_0} \mapsto k$) while at all other sites $E_{p_n} = 0$ (highly localized soliton). In addition, $E_{p_n} \mapsto 0$ as $z_1 \to \infty$. A diagram illustrating the profiles of the function $g(nh)$ and the energy densities profiles are represented in Figure 5.1 for $k=1$ and $k=2$ with $z_1 = 15$ and $h = 0.19$.

The situation we wish to study is that of an isolated perturbed static k-soliton configuration and investigate the effects of the perturbation. As it costs them no energy to shrink or expand they can shrink to almost a zero width configuration in the energy density plot. As the soliton configuration is described by a few points on a lattice it is difficult to decide what is meant by its width and how to calculate it. In the continuum systems, is defined as the radius of the soliton. In this case, the field configuration can shrink to almost a delta function in the energy density plot. The lattice analogue will be a field configuration with $g(0,t) = \pi$ (due to the boundary conditions) and $g(nh,t) = 0$, for $n > 0$. In fact, this corresponds to a spike soliton (of almost zero width) in the continuum.

Since, there is no explicit solution in this case, one has to resort to approximation, or to numerical solutions of the equations of motion (5.12), namely

$$\begin{aligned} \ddot{g} &= \frac{1}{h^2}[\, k\sin g(h,t) + f^2 \sin(g(2h,t) - g(h,t))] - \frac{k^2}{4f^2\, h^2}\sin(g(2h,t) + g(h,t)), \qquad n = 1, \\ n\ddot{g} &= \frac{f^2}{h^2}[n\sin(g_+ - g) - (n-1)\sin(g - g_-)] - \frac{k^2}{4f^2\, h^2}[\frac{\sin(g_+ + g)}{n} + \frac{\sin(g + g_-)}{n-1}], \qquad n > 1. \end{aligned} \tag{5.17}$$

5.3 The Slow-Motion Approximation

The most important question about a k-soliton configuration is whether or not it is stable. Since the energy in the k topological sector is bounded below by k, and the static configuration energy satisfies this bound, the only way that g can change (roughly speaking) corresponds to a (or z_1) changing for the continuum (or lattice) model. With this observation in mind, one may write down a more general family of radially symmetric configurations, by allowing a (or z_1) to be a function of t. So the question is if one starts

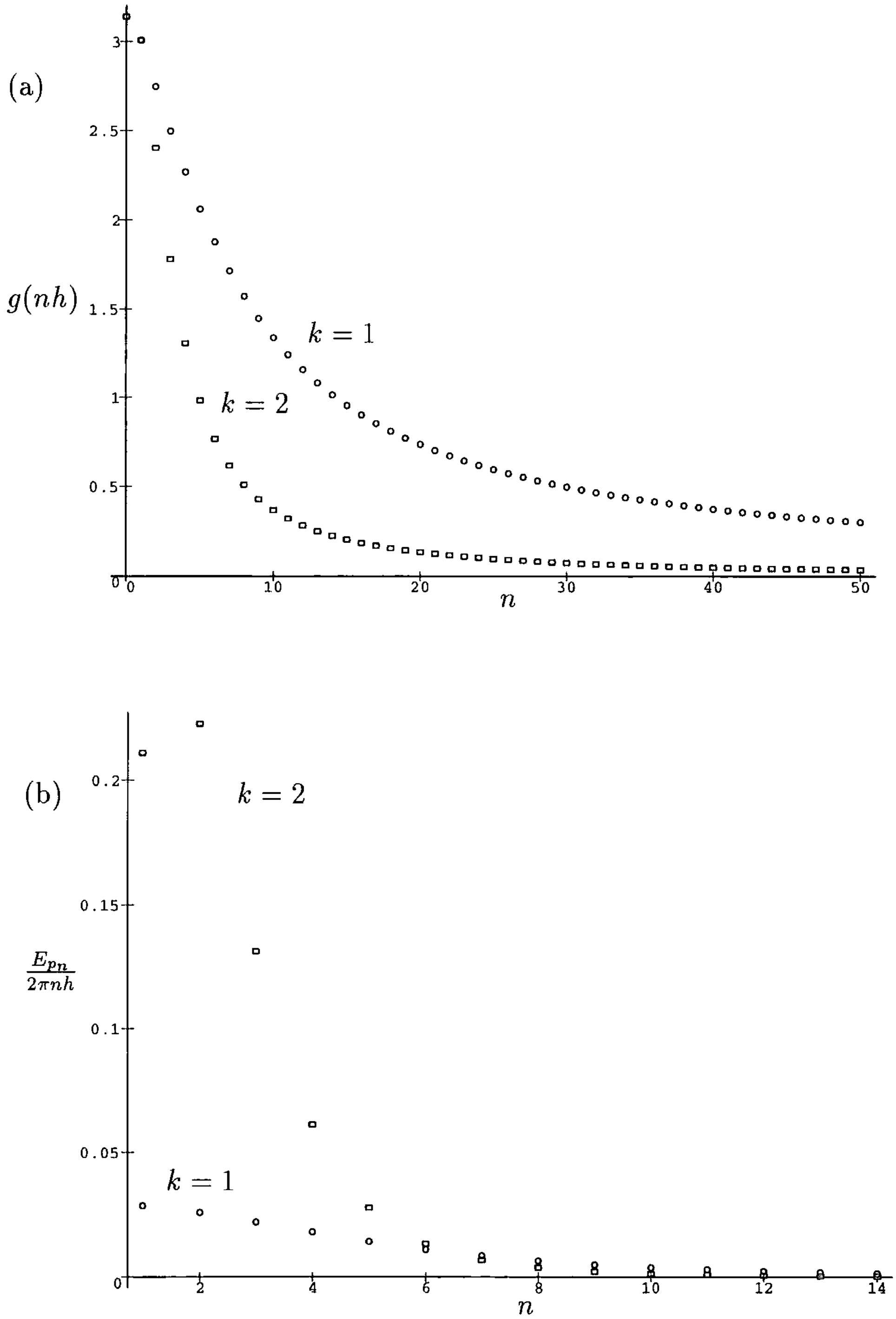


Figure 5.1: (a) Profiles of g for topological charges $k = 1, 2$. (b) Profiles of the energy densities $E_{p_n}/(2\pi nh)$ for the charges in (a). The $k = 2$ energy density is ring shaped.

with a configuration located at the origin and perturbs its shape, does the soliton stay close to the initial configuration for all t? And if it does not, what is the rate of changing?

There is a fundamental difference between the cases $k = 1$ and $k > 1$, which becomes apparent when one considers the so-called slow-motion (or geodesic) approximation; originally proposed in connection with monopole scattering [77, 78]. In this scheme one assumes that the field g is a static solution like (5.14), but slightly perturbed. More precisely, since the energy is conserved, and due to the existence of the Bogomol'nyi bound, we may assume that a k-soliton dynamics is obtained by restricting g to have the form of (5.14), with z_1 now becoming a dynamical variable $z_1(t)$. [So the number of degrees of freedom is reduced to one.] These static solutions form a manifold, which is equipped with a natural metric coming from the kinetic energy, and the evolution is given by the resulting geodesics. Since every configuration of the form (5.14) has the same potential energy, the kinetic energy may be taken as the Lagrangian; thus, the corresponding Euler-Lagrange equations are precisely the geodesic equations associated with the aforementioned metric. This approximation is a good one if the speeds are small (i.e. if E_k is small compared to $E_p = k$).

For $k = 1$, the requirement of finite kinetic energy means that z_1 should be independent of t at spatial infinity, so ruling out the slow-motion approximation. In other words, taking z_1 to be a function only of t leads to a divergent kinetic energy. But when $k > 1$ there are sufficient powers of n in the denominator of (5.14) to keep the energy finite. For this case the slow-motion approximation has been considered in [40, 41] in order to study the dynamics of $\mathbf{CP}^1$ (or equivalently O(3)) lumps in (2+1) dimensions. Let us concentrate on the $k = 2$ topological sector, where two solitons are sitting on top of each other at the origin, forming a ring structure. The Lagrangian is

$$\begin{aligned} L &= E_k - E_p \\ &= l(z_1)\, \dot{z}_1^2 - 2, \end{aligned} \tag{5.18}$$

where

$$l(z_1) = h^2 \sum_{n=1}^{\infty} \frac{n\mathcal{Z}_n^2}{(1 + z_1^2 \mathcal{Z}_n^2)^2}. \tag{5.19}$$

In order for (5.19) to converge, one needs $f(h) \approx 1$ (since $\mathcal{Z}_n \mapsto n^{-2/f^2}$ at spatial infinity). So let us write $f(h) = 1 + h$ with a relative small lattice step, i.e. $h \in (0, 0.2)$.

[In fact, when the lattice spacing is small compare to the size of the topological solitons, then one is close to the continuum limit.] Graphs of $l(z_1)$ for various values of h are given in Figure 5.2.

The Euler-Lagrange equation of the system is

$$2\,l(z_1)\,\ddot{z}_1 + l'(z_1)\,\dot{z}_1^2 = 0 \qquad \Rightarrow \qquad l(z_1)\,\dot{z}_1^2 = \text{const.} \tag{5.20}$$

This may be reduced to quadratures:

$$\begin{aligned} vt &= \int_c^{z_1(t)} \sqrt{\frac{l(\tilde{z}_1)}{l(c)}}\, d\tilde{z}_1 \\ &\equiv \Lambda_h(z_1), \end{aligned} \tag{5.21}$$

where $z_1(0) = c$, $\dot{z}_1(0) = v$. Recall that, z_1 determines the configuration size which evolves with t; more precisely, $\sqrt{c}h$ is the initial width of the configuration and, v is the initial rate of change of the configuration width in each lattice site per unit time. In fact, $v < 0$ corresponds to an initial contraction and $v > 0$ to an initial expansion.

The function $\Lambda_h(z_1)$ decreasing or increasing depending on the value of $z_1(t)$; which corresponds to contraction or expansion of the configuration. It is easily inverted to give the time variation of the configuration size (Figure 5.3), i.e.

$$z_1(t) = \Lambda_h^{-1}(vt). \tag{5.22}$$

Recall that, the lattice analogue of the configuration width is proportional to $\sqrt{z_1}h$. In fact, the time taken for the configuration to shrink from the initial width to zero, i.e. to become a spike, is

$$t_c = \frac{\Lambda_h(0)}{v}. \tag{5.23}$$

Notice that, t_c depends on the initial conditions, i.e. on the values of c and v. We believe that this approximation of k-soliton dynamics is accurate for small $|v|$.

The accuracy of the approximation has been tested numerically using a fully-explicit fourth-order Runge-Kutta algorithm with fixed time step 0.0053. The initial condition was a static 2-soliton profile whose width we perturbed to shrink with initial velocity 0.1 lattice site per unit time ($v = -0.1h$). Simulations of duration 2985 time units were performed for $h = 0.01$. Inspection of the rate of change of $z_1(t) = \tan(g(h,t)/2)$ reveals close agreement with $\dot{z}_1(t)$ calculated from (5.22) (see Figure 5.4).

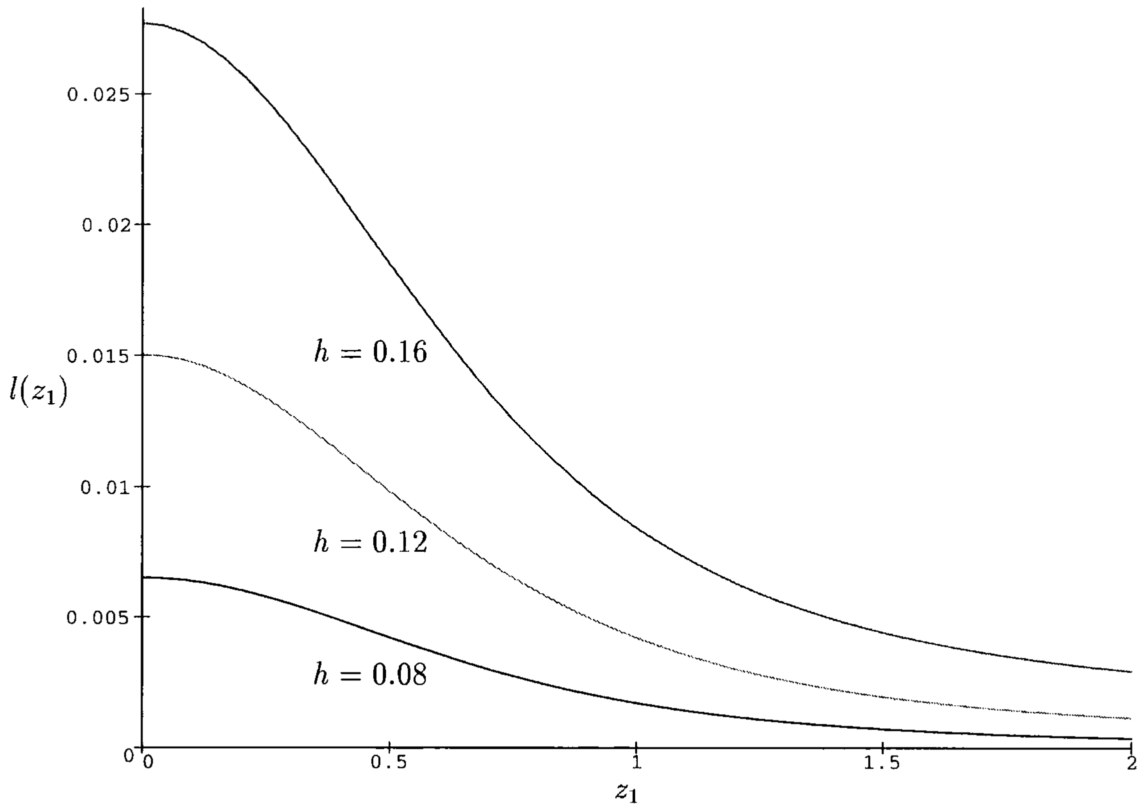


Figure 5.2: The function $l(z_1)$ of equation (19).

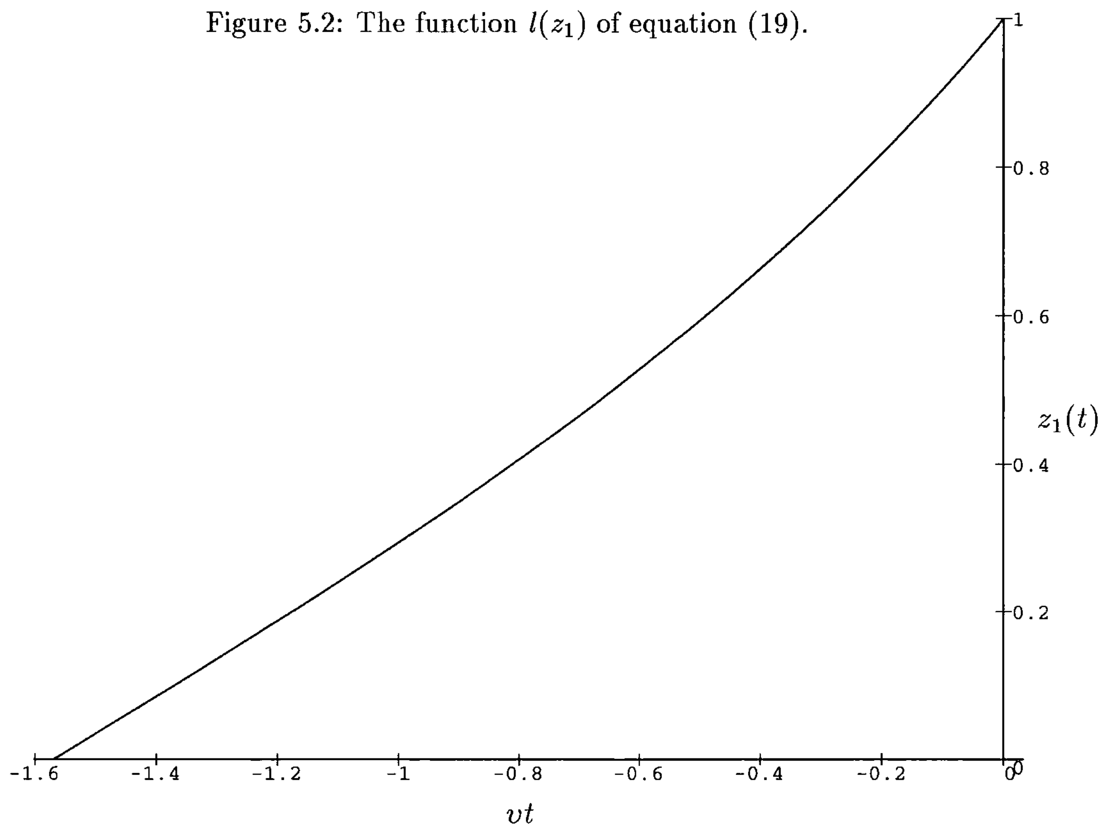


Figure 5.3: The function $z_1(t)$ showing the shrinking of the configuration for $c = 1$ and $h \in (0, 0.2)$.

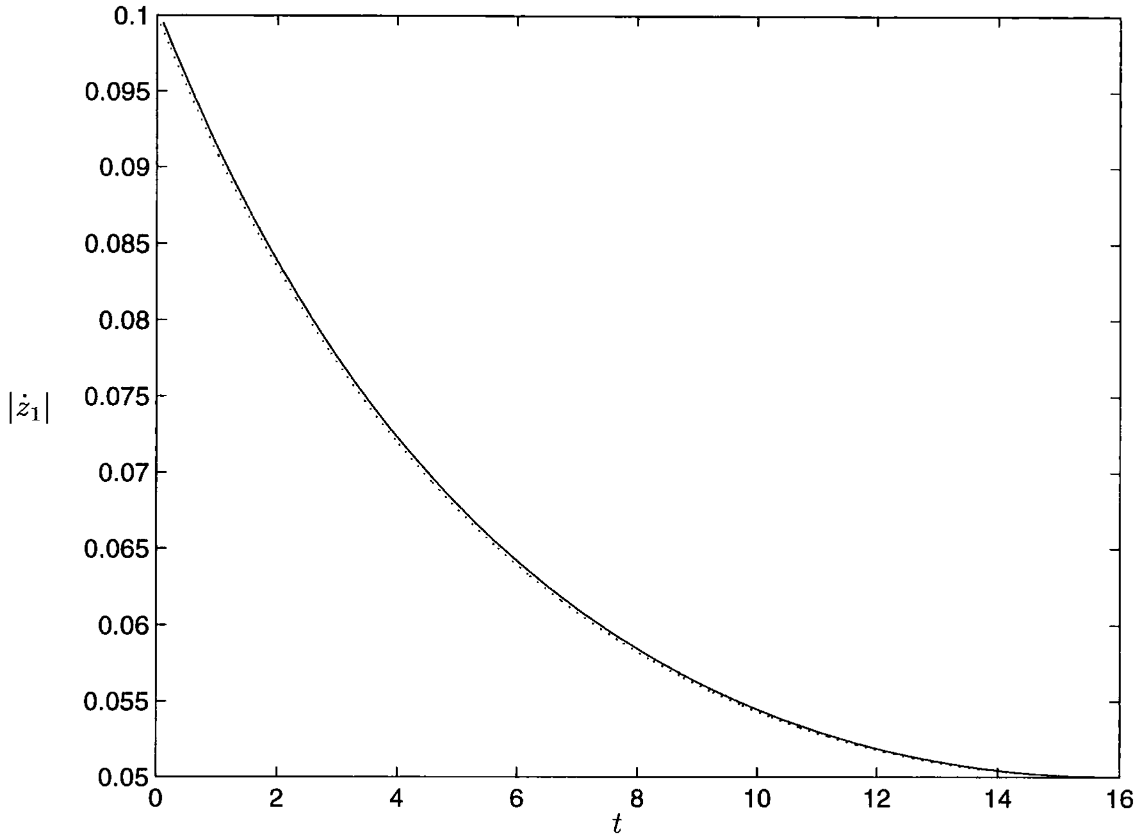


Figure 5.4: The time variation of $|\dot{z}_1|$ in the analytic slow-motion approximation (solid line), and also for the numerical evolution (dotted line).

5.4 Dynamics of the Lattice O(3) Solitons

The slow-motion approximation is expected to fail at high velocities (except for small h). Therefore, we incorporate the notion of lattice soliton in a full numerical evolution scheme and compare the results with the continuum behaviour as well as with the ones obtained in the slow-motion approximation, for a single soliton and a 2-soliton ring. Throughout the simulations, the extensive use of the difference equations (5.17) have not revealed any instabilities (i.e. the total energy is conserved).

The lattice formulation necessarily has a spatial boundary at $n = n_{max}$, say. That means that, all the quantities we are going to use in order to study the soliton dynamics will be calculated within some radius (n_{max}). Moreover, the infinite sums on the energies will be truncated. In fact, for $k = 1$ the finiteness of the grid imposes an artificial cutoff and, therefore, provides a finiteness in the energy.

On the boundary though, the fields are taken to be fixed in time, i.e.

$$g((n_{max}+1)\,h,t) = g((n_{max}+1)\,h,0). \tag{5.24}$$

These boundary conditions (so-called fixed boundary conditions) may appear severe, but they seem to be a sensible choice, especially, in the $k=1$ sector. Recall that, for $k=1$ the expressions for the kinetic energy are divergent and so the slow-motion approximation cannot be used. This feature also occurs in numerical evolutions in the sense that, if one attempts to apply boundary conditions which allow the field to change with time at arbitrary distances, then the total energy of the system for $k=1$, grows rapidly and without bound. Although, there are other options. For example, one may choose absorbing boundary conditions or may place the boundary far enough from the configuration; and, therefore, they will be no radiation effects. But, in this scheme, the choice of the boundary conditions do not affect the numerical results.

Moving on to the question of initial data, there are clearly many different types of perturbation which we could apply to the configuration, the only restriction being that we do not perturb the field close to the boundary. Since the evolution equations (5.17) are second order the initial data must specify the field values $g(nh,t)$ and its time derivatives $\dot{g}(nh,t)$ at $t=0$. So the field configuration at $t=0$ is taken to be the static lattice configuration (5.14) but slightly perturbed, that means $z_1 \mapsto z_1 + vt$ in the $k=1,2$ sector. In fact, the imposed perturbation is

$$\dot{g}(nh,t)\,|_{t=0} = \frac{2v\mathcal{Z}_n}{1+z_1^2\mathcal{Z}_n^2}. \tag{5.25}$$

Physically the picture is this: there is a continuous interpolation between the inner region where v is the amplitude of the perturbation (same as in the slow-motion approximation) and the outer one where there is no perturbation at all. This class of perturbation reveals all the qualitative types of behaviour that can occur.

So we have a k-soliton configuration whose centre remains fixed, but whose radius decreases to a minimum (close to zero) and then increases again. More precisely, the initial perturbation (for $v<0$) tends to shrink the configuration, while large burst of radiation travel outwards at the speed of light (see Figure 5.5), together with a residual motion in the central region occupied by the soliton. When the radiation reaches the boundary, is reflected back (due to the fixed boundary conditions); reabsorbed by the

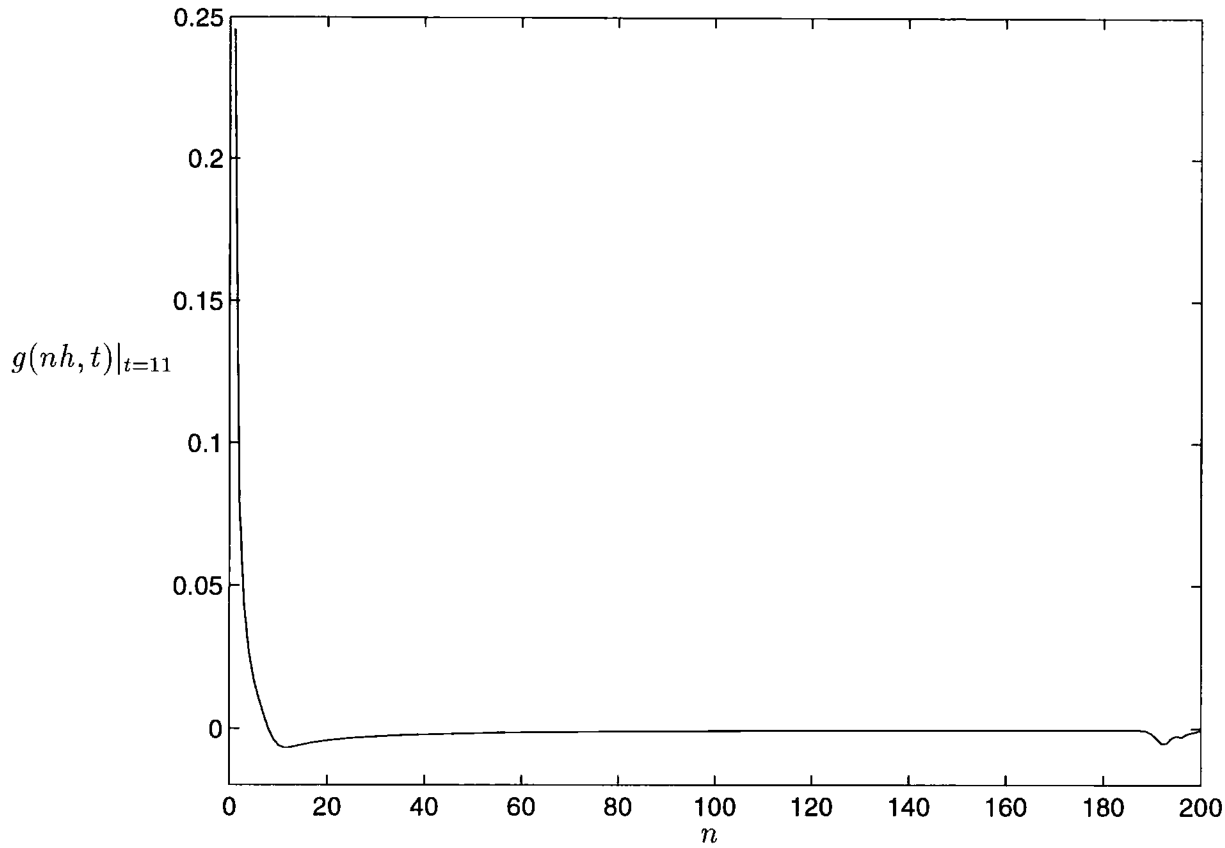


Figure 5.5: Radiation emitted by the 1-soliton solution.

configuration (which expands) and then another pulse is emitted a short time later; and so the process repeats. The data were produced by the aforementioned Runge-Kutta algorithm for $v = -0.1$ and $z_1 = 1$, on a lattice of unit spacing ($h = 1$) in the $k = 1$ sector. In this case, the lattice spacing can be relatively large, without compromising the behaviour of the solitons.

We are interested in the speed of the shrinking in detail. This has been partially done for the continuum model (cf. [71]); therefore, it would be interesting to compare our results with these ones. In order to analyze the results of the numerical simulations, we use the dynamical quantity, which corresponds to the value of the field at the first lattice point, i.e. $g(h,t)$. Since we are on a lattice, the soliton will be highly localized (i.e. a spike) when the profile function at the first site ($n = 1$) will become zero. Then, the soliton occupies essentially only one lattice site.

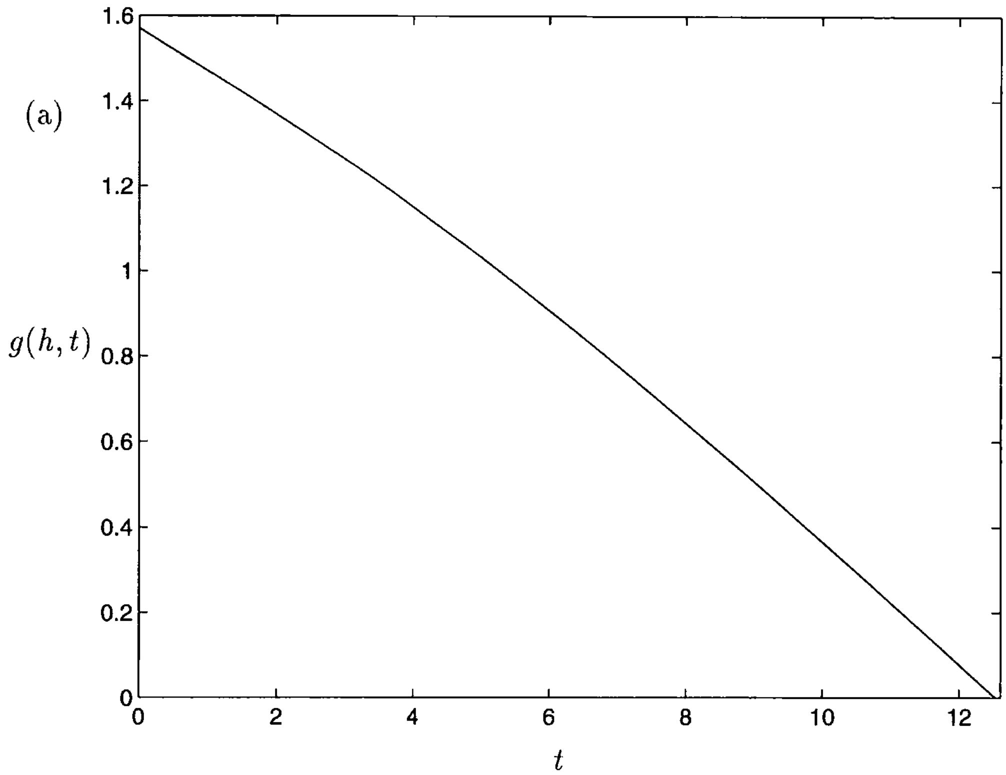
In Figure 5.6 we present the time dependence of the field $g(h,t)$, for a single soliton and a 2-soliton configuration. The results are derived from a relatively small mesh $n_{max} = 200$ (in fact, they do not change for larger mesh sizes), for $v = -0.1$ and $z_1 = 1$. In the single soliton ($k = 1$) case, we make the simple choice $f(h) = 1$ whereas $h = 1$; while in the 2-soliton case, we take $f(h) = 1 + h$ with $h = 0.01$. [Note that, Figure 6(b) corresponds to Figure 3 and Figure 4.] The field configuration saturates the Bogomol'nyi bound throughout the numerical evolution. Looking at the graphs we note that the two cases give very similar results, i.e. the curves are nearly straight, confirming the power law for the rate of shrinking; which are consistent with the ones obtained from the slow-motion approximation, and from the continuum model. Note that, since the lattice spacing h in Figure 6(b) is small compare to the size of the topological soliton (in contrast, with Figure 6(a) where the 1-soliton size is comparable to the spacing) the corresponding curve looks straighter. This follows from the fact that the model is closer to the continuum one.

5.5 Conclusions

This study has revealed that, in the context of classical soliton dynamics the lattice O(3) solitons behave very much like the ones of the continuum model; which means that they are unstable under small perturbations due to the absence of a natural scale. However, the lattice model is a much better approximation of the continuum one; since the size of the lattice does not affect the time dependence of the dynamical quantities $g(h,t)$. Moreover, the results show that the slow-motion approximation works very well for small velocities and for small lattice sizes (i.e. for small h).

In the continuum, a more *physical* model can be obtained by adding a (2+1)-dimensional version of the Skyrme term to stabilize against configuration collapse, and a potential to stabilize against spread. Then the Bogomol'nyi bound remains valid but unsaturable. The lattice analogue model was discussed in [79]. This involves adding next-to-nearest-neighbour couplings between lattice sites; but a fairly small perturbation can induce the corresponding configuration to decay. Clearly, the technique of discretizing the Bogomol'nyi bound, in order to obtain static solutions on the lattice, may be applied to other models that have nontrivial topologies; like the Maxwell-Higgs model in (2+1) dimensions and the Skyrme model in (3+1) dimensions. Although, in higher dimensions it seems to be much more difficult to find a lattice version of (5.4).

(a)



(b)

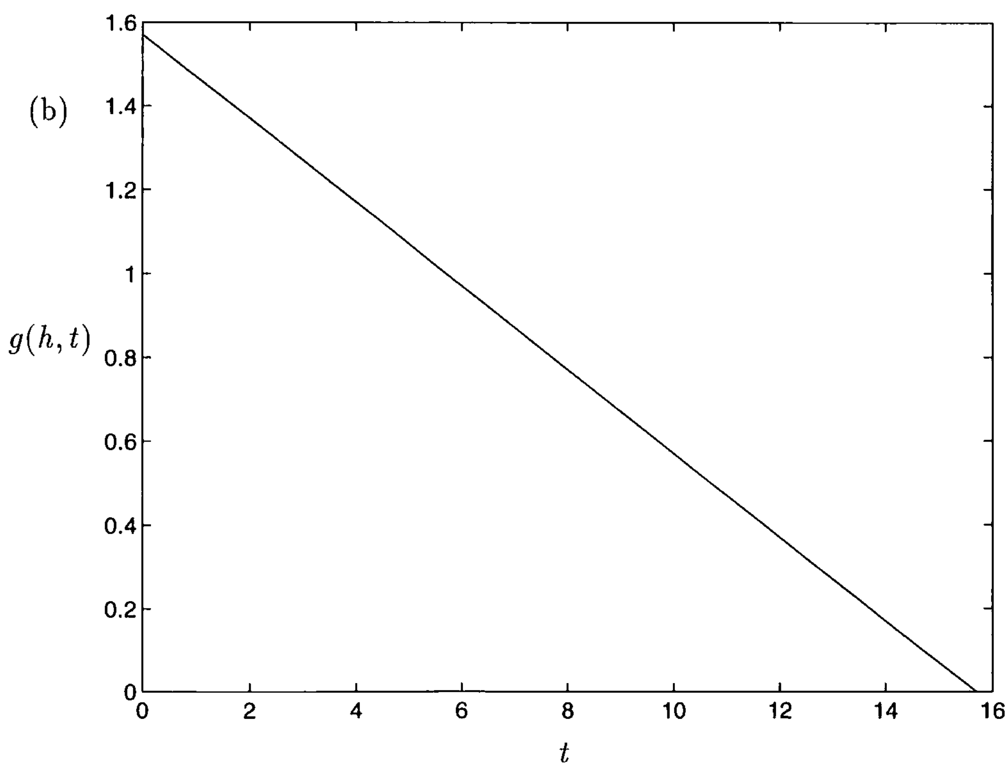


Figure 5.6: The variation of $g(h,t)$ over the range (a) $0 \leq t \leq 12.6$ for a slowly shrinking 1-soliton lump and (b) $0 \leq t \leq 16$ for a slowly shrinking 2-soliton ring, for the numerical evolution.

Chapter 6

Outlook

The possibility of the modified chiral model to be *completely integrable*, in the sense of there being a sufficient number of conserved quantities in involution; is discussed here. The modified chiral model (2.16) is Hamiltonian, as may be seen by the identification

$$
\begin{aligned}
&\text{coordinates } (J): && J(x,y,t),\\
&\text{momenta } (P): && P(x,y,t) \equiv J^{-1}J_tJ^{-1},\\
&\text{Hamiltonian } (H): && \tfrac{1}{2}\textstyle\iint[(PJ)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2]\,dx\,dy,
\end{aligned}
$$

where the Hamiltonian is the integral of the energy density (2.21) expressed in the phase-space coordinates.

Recall that, the torsion term in this model is analogous to a background magnetic field in classical mechanics. Hence, we define a Poisson bracket on the space $\mathcal{F}(\mathcal{M})$ of matrix functions on the xy-plane $\mathcal{M}$, as

$$\{A,B\} = \frac{\delta A}{\delta P} \bullet \frac{\delta B}{\delta J} - \frac{\delta A}{\delta J} \bullet \frac{\delta B}{\delta P} - (J^{-1}J_y) \bullet [J^{-1}\frac{\delta A}{\delta P}, J^{-1}\frac{\delta B}{\delta P}], \tag{6.1}$$

which corresponds to the Poisson bracket for a charged particle in an *external magnetic field* (cf. [80]) The scalar product $\bullet$ on $\mathcal{F}$, is defined as

$$K \bullet N = \iint \mathrm{tr}(KN)\,dx\,dy. \tag{6.2}$$

The Hamiltonian equations then assume the following form

$$\frac{dJ}{dt} \equiv \{H,J\} = JPJ,$$
$$\frac{dP}{dt} \equiv \{H,P\} = (J^{-1}J_t)^2J^{-1} - (J^{-1}J_x)_xJ^{-1} + (J^{-1}J_y)_yJ^{-1} + [J^{-1}J_t, J^{-1}J_y]J^{-1}, \tag{6.3}$$

which describe the equation of motion of the modified chiral model (2.16).

In chapter 3, we deduced an infinite number of conserved quantities $\{Q_n\}$ for (2.16). This discussion applies specifically in the SU(2) case; where these quantities $\{Q_n\}$ (3.81) (due to the fact that the matrix $M \in \mathrm{su}(2)$ given by (3.85)) take the form

$$\tilde{Q}_p \equiv Q_n = \int \Gamma^p \, dy, \qquad \text{for } n \text{ odd},$$
$$\tilde{Q}_{p\alpha} \equiv Q_n = \int \Gamma^p \, \mathrm{tr}(M\sigma_\alpha) \, dy, \qquad \text{for } n \text{ even}, \tag{6.4}$$

where

$$\Gamma = -2||M||^2 \equiv \mathrm{tr}(M^2),$$
$$M(t,y) = \int (J^{-1}J_u) \, dx \equiv \int (PJ + J^{-1}J_y) \, dx.$$

Then, the functional variations of the conserved quantities $\{Q_n\}$, for n odd, are

$$\frac{\delta \tilde{Q}_p}{\delta P} = 2p\Gamma^{p-1}JM,$$
$$\frac{\delta \tilde{Q}_p}{\delta J} = 2p\Gamma^{p-1}\left(MP + [M, J^{-1}J_y]J^{-1}\right) - 2p\,(\Gamma^{p-1}M)_yJ^{-1}, \tag{6.5}$$

while, for n even, are

$$\frac{\delta \tilde{Q}_{p\alpha}}{\delta P} = 2p\Gamma^{p-1}JM\,\mathrm{tr}(M\sigma_\alpha) + \Gamma^p J\sigma_\alpha,$$
$$\frac{\delta \tilde{Q}_{p\alpha}}{\delta J} = 2p\,\mathrm{tr}(M\sigma_\alpha)\left(MP + [M, J^{-1}J_y]J^{-1}\right) - 2p\left(\Gamma^{p-1}M\,\mathrm{tr}(M\sigma_\alpha)\right)_y J^{-1} + \Gamma^p\left(\sigma_\alpha P + [\sigma_\alpha, J^{-1}J_y]J^{-1}\right) - (\Gamma^p\sigma_\alpha)_y J^{-1}. \tag{6.6}$$

So the question arises: are these quantities in involution? That is, do the Poisson brackets of these quantities with one another vanish? Note that, the Hamiltonian and the quantities $\widetilde{Q}_p$ and $\widetilde{Q}_{p\alpha}$ commute. In fact, after lengthy calculations we found that their Poisson brackets (for soliton boundary conditions (2.18)) are zero, i.e.

$$\begin{aligned} \{H, \widetilde{Q}_p\} &= -\int \partial_y \Gamma^p \, dy = 0, \\ \{H, \widetilde{Q}_{p\alpha}\} &= \int \partial_y \left(\Gamma^p \operatorname{tr}(M\sigma_\alpha)\right) dy = 0. \end{aligned} \tag{6.7}$$

The system (6.7) is simply a statement of the fact that the $\{Q_n\}$'s are conserved, i.e. $dQ_n/dt = 0$. By contrast, although a simple calculation of the Poisson bracket between the two components of $\widetilde{Q}_p$ yields a vanishing result, i.e. $\{\widetilde{Q}_p, \widetilde{Q}_q\} = 0$, the Poisson brackets of $\widetilde{Q}_{p\alpha}$ and $\widetilde{Q}_p$, i.e. $\{\widetilde{Q}_{p\alpha}, \widetilde{Q}_{q\beta}\}$ and $\{\widetilde{Q}_p, \widetilde{Q}_{q\alpha}\}$ yield nonvanishing results. More precisely, they are divergent integrals; that is, they are undefined. This follows from the fact that the $\{Q_n\}$ are nonlocal and, therefore, their functional derivatives are not functionals on the phase-space $\mathcal{F}$.

Therefore, the conserved quantities $\{Q_n\}$ are not in involution and so the complete integrability of the model (2.16) remains an open question. One needs to search for more conserved quantities in order to fully investigate this problem.

The modified chiral model (2.16) of the first four chapters seems to be very different from the (unmodified) O(3) model (2.48) considered in the last chapter; although in fact they are very closely related. Recall that, they possess essentially the same static solitons. In the formalism of the integrable model the static solutions correspond to a matrix J of the form (3.4). It is easy to check that any such matrix satisfies $J^2 = -I$, i.e. they all lie on the equator of SU(2), which is precisely the condition for reduction to the O(3) model. The similarity becomes even more explicit when one considers the expressions for the potential energy in each case. In the modified model, (2.36) shows that the static configuration (3.4) has potential energy

$$E_f = \int \frac{8|f'|^2}{(1+|f|^2)^2} \, d^2x, \tag{6.8}$$

while for the O(3) model the corresponding expression (parametrizing the two-sphere by using the complex field W) is

$$E_W = \int \frac{|\partial_x W|^2 + |\partial_y W|^2}{(1+|W|^2)^2} \, d^2x. \tag{6.9}$$

So for static solutions, f in the modified chiral model plays the same role as W in the O(3) model. Therefore it is natural to ask the question: what are the stability properties of lumps in the modified chiral model?

In the modified chiral model, a static lump will have the same zero modes as in the O(3) model; i.e. it may shrink or expand under small perturbations. But, the absence of topological stability suggests the existence of negative modes (i.e. the solitons may decay to the vacuum), which corresponds to the field J moving off the equator of $SU(2)$. Because the model is integrable, one might expect that these modes could be constructed. Some work has been done investigating the stability of the one-soliton solution under radially symmetric perturbations. No negative modes are excited by such perturbations, which suggest that there may be no negative modes present for the one-soliton solution. Although, numerical simulation can not rule out this possibility. At the moment this remains an open question.

It would, also, be of great interest to investigate the stability of the soliton and the soliton-antisoliton solutions given in chapter 4. Bearing in mind that the scattering process do not appear to excite any negative mode, it is conceivable that none exist. By contrast, Figure 4.2 shows that the static soliton located at the origin change its shape as time passes. This implies that the static soliton shrinks towards tall spikes, probably, of zero width. Therefore, it would be of great interest to study the potential energy density of this configuration and investigate its time dependence. This will through further light in undertsanding the dynamics of the O(3) solitons (since they are connected to these solitons).

The work described in chapter 4 is, to my knowledge, the first example in which explicit elastic soliton-antisoliton scattering solutions have been constructed, at least in (2+1)-dimensions. Although, the present solutions do not make a complete set. It is of great interest to find the function $\psi(\lambda)$ which corresponds to families of soliton-antisoliton solutions using only the properties that $\psi(\lambda, u, v, x)\psi(\bar{\lambda}, u, v, x)^{\dagger} = I$ and that the anti-hermitian matrices $A = (\Lambda\psi)\psi^{-1}$ and $B = (M\psi)\psi^{-1}$ are independent of λ. This remains an open question.

It seems likely that there are many more interesting solutions still to be revealed. One could, for example, ask whether a parameter exists which determines the solitons velocity. Then, one can study the behaviour of the corresponding static solitons, which in this case they are not known. In addition, it would be interesting to look for breather soliton-antisoliton solutions. One way to do so, is by taking the function $f(\omega) = e^{\omega}$ and study the dynamics of the corresponding configurations. Finally, one may ask whether the sizes of the interacting solitons must necessarily be nonconstant.

Appendix A

The $\bar{\partial}$ Problem

In this section we give an elementary introduction to the $\bar{\partial}$-problem [81]. We consider the general case of complex functions, whose domain of non-analyticity may be two-dimensional, and, therefore, they may be even nowhere analytic in the plane. In fact, the main mathematical tool will be the usual differential and integral calculus on the xy-plane. It is convenient to replace the cartesian coordinates x and y with the complex variables z and $\bar{z}$. Though, z and $\bar{z}$ are complex numbers, they should be regarded merely as a new coordinate system in the plane, the coordinate transformation being

$$\begin{array}{ll} z = x + iy, & \bar{z} = x - iy, \\ x = (z + \bar{z})/2, & y = -i(z - \bar{z})/2. \end{array} \tag{A.1}$$

By means of this transformation, all formulas can be rewritten in the $(z, \bar{z})$ coordinate system. Thus a complex function $g(x, y)$, defined in a domain $\mathcal{A}$ of the plane, can also expressed in terms of the two complex variables $(z, \bar{z})$ as

$$g(x, y) = f(z, \bar{z}), \quad z \in \mathcal{A} \subset \mathbf{C}. \tag{A.2}$$

The corresponding partial differential operators are

$$\begin{array}{ll} \partial \equiv \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right), & \bar{\partial} \equiv \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right), \\ \partial_x \equiv \dfrac{\partial}{\partial x} = \partial + \bar{\partial}, & \partial_y \equiv \dfrac{\partial}{\partial y} = i(\partial - \bar{\partial}). \end{array}$$

The differential operator $\bar{\partial}$ is the so-called *DBAR* operator; and its name originated from the fact that $\bar{\partial}$ is the complex conjugate of ∂, with the notation convention that the bar indicates the complex conjugate. This operator plays a special role in connection with the theory of analytic functions. In fact, if $f(z, \bar{z})$ is analytic in a domain $\mathcal{A}$, then

$$\bar{\partial} f(z, \bar{z}) = 0, \qquad z \in \mathcal{A}, \tag{A.3}$$

which means that f does not depend on the coordinates $\bar{z}$. Indeed, if $f(z, \bar{z}) = u(x, y) + iv(x, y)$, the above equation coincides with the Cauchy-Riemann condition of analyticity expressed by the elliptic system of two partial differential equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{A.4}$$

In general though, the function $f(z, \bar{z})$ is not analytic, i.e $\bar{\partial} f(z, \bar{z}) \neq 0$.

The Gauss-Green integral formulae

$$\iint_{\mathcal{A}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \oint_{\partial \mathcal{A}} P\, dy - Q\, dx, \tag{A.5}$$

or in special cases

$$\oint_{\partial \mathcal{A}} g(x, y)\, dx = - \iint_{\mathcal{A}} \partial_y g(x, y)\, dx \wedge dy,$$

$$\oint_{\partial \mathcal{A}} g(x, y)\, dy = \iint_{\mathcal{A}} \partial_x g(x, y)\, dx \wedge dy,$$

are of basic importance. Here, $\mathcal{A}$ is a simply domain of the plane, its boundary $\partial \mathcal{A}$ being a differentiable and clockwise oriented curve, while $dx \wedge dy$ is the Lebesgue measure on the plane. In the complex coordinates z and $\bar{z}$, the Gauss-Green formulae take the form

$$\oint_{\partial \mathcal{A}} f(z, \bar{z})\, dz = - \iint_{\mathcal{A}} \bar{\partial} f(z)\, dz \wedge d\bar{z},$$

$$\oint_{\partial \mathcal{A}} f(z, \bar{z})\, d\bar{z} = \iint_{\mathcal{A}} \partial f(z)\, dz \wedge d\bar{z}, \tag{A.6}$$

where we have set $f(z, \bar{z}) = g(x, y)$. Note that the first equation of (A.6) generalizes the Cauchy theorem to the class of non-analytic functions

$$\oint_{\partial \mathcal{A}} f(z, \bar{z})\, dz = 0, \tag{A.7}$$

for analytic functions, to which it reduces if $f(z)$ is analytic in $\mathcal{A}$. Indeed, these formulas (A.6) shows that the contribution to the closed curved integral in the left hand side originates the *departure of analyticity* measured by the $\bar{\partial}$ derivative in the right hand side. The familiar residue theorem, which applies when $f(z)$ has simple pole singularities in $\mathcal{A}$, can be recover this way by computing the $\bar{\partial}$ derivative of such singular functions.

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