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**Aspects of the  
Gauged, Twisted,  $SL(2|1)/SL(2|1)$   
Wess-Zumino-Novikov-Witten Model**

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Gauged, Twisted,  $SL(2|1)/SL(2|1)$   
Wess-Zumino-Novikov-Witten Model**

Submitted for the degree of  
Doctor of Philosophy  
at  
The Department of Mathematical Sciences  
The University of Durham

by

Rachel-Louise Kvertus Kuktava

September 1996

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- 3 JUL 1997

I dedicate this work to my grandfather

Josef Kokta (1920-1992)

*When your spirt left this world*

*The light went out*

*Of many lives.*

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A great vote of thanks goes to those who were instrumental in the development of this work, and my education. Of these people Dr. Anne Taormina is prominent. Thanks are owed to her for the willingness to take on a student and together begin the study of an unfamiliar and difficult field. For her patience and guidance, and willingness to help, I say thank you.

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## Preface

The following work is presented by myself for the degree of Doctor of Philosophy at the University of Durham. The content of this thesis is a blend of known work, research undertaken with my supervisor, and research of my own. Only chapter one contains no original work. Chapter two is an even split between work of my own and work jointly undertaken with Dr. Anne Taormina and Dr. Peter Bowcock, except where references indicate otherwise.

Chapter three is composed of work undertaken jointly with Dr. Taormina and Dr. Bowcock.

Chapter four is largely my own work, with some assistance in the calculations from Dr. Taormina.

The work in this dissertation does not follow the full rigour that the reader may have anticipated. I make no apologies for this. As someone whose intuition and background lies in the realm of physics, this thesis has been a great edification in study. In attempting to understand the work of mathematicians I have repeatedly been struck by the difficulty I, as a physicist, have encountered in trying to comprehend what should, and could, be simple. I have therefore specifically set out to keep the nature of this work readily accessible to physicists, postgraduate students, and the knowledgeable alike. I assume that if the reader is qualified enough to see the failings in the lack of full rigour, then the reader is qualified enough to supply those deficits for themselves.

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## Abstract

In this thesis we examine some of the interesting aspects of the Wess-Zumino-Novikov-Witten model when this model has been gauged and its energy tensor twisted by the addition of the derivative of one of its Cartan subalgebra valued currents. Specifically we consider the group valued model with the group taken as  $SL(2|1)$  which is the Lie super group used to describe  $N = 2$  supersymmetry.

This model is advocated as being a good and natural description of the  $N = 2$  superstring (also known as the charged spinning string, or  $N = 2$  fermionic string) when it tensors an additional topological system of ghosts. The evidence for this assertion is presented by gauging and twisting the model and then extracting the  $N = 2$  super Liouville action by the method of Hamiltonian reduction.

The connection between the  $SL(2|1)/SL(2|1)$  Wess-Zumino-Novikov-Witten model and field theory is made through its current algebra. As is true of many super groups there exists more than one interpretation of the Dynkin diagram for the algebra of  $SL(2|1)$  and this results in more than one set of currents for this model. The classical and quantum currents in free field form are found in both cases, as is the highly non-linear transformation by which the two sets of currents are related.

An analysis of a section of the cohomology of physical states of the model is undertaken. It is shown that the additional topological ghost system that tensors the gauged, twisted  $SL(2|1)$  model when it describes the  $N = 2$  string only contributes a vacuum state to the overall cohomology, so reducing the analysis. As the  $SL(2|1)/SL(2|1)$  Wess-Zumino-Novikov-Witten model is a topological field theory its spectrum of physical states lie in the cohomology class defined with respect to the BRST charge. The spectrum formed from the free field currents composes the so called Wakimoto module and this is calculated via the BRST formalism.

# Introduction

*Mathematics is where the answer is right and everything is nice and you can look out of the window and see the blue sky — or the answer is wrong and you have to start all over and try again and see how it comes out this time.*

From *Complete Poems*

by Carl Sandburg

It is one of those curious patterns of history which tells us to expect a revolution in physics, on average, every quarter of a century. It is perhaps this expectation that caused many individuals to see the coming of string theory as the overdue fulfilment of this cycle.

Whether the theory of strings is the long awaited golden goose of modern theoretical physics — or the golden duck — is as unknown to us as the future. Yet one

thing is clear. During its development it has shown itself to be vast undiscovered country for both the mathematician and physicist alike as they have struggled to uncover the rich mathematical structures, and have been forced to apply themselves to the solutions of new and stimulating problems.

In recent years the flow of ideas has been stemmed by the sheer complexity of the analysis needed. As a consequence the remaining problems in string theory are likely to remain problems for some time to come. This work is a peep at one such problem — the solving, off the critical dimension, of the Liouville sector for the model of the fermionic string, otherwise known as the  $N = 2$  superstring or the charged spinning string.

It is inevitable that the traditional province of physics should be invaded by mathematicians in such a difficult subject, and as such much of the material in the following pages is perhaps more to the taste of the mathematician. In particular this work concentrates on the gauged, twisted,  $SL(2|1)/SL(2|1)$  Wess-Zumino-Novikov-Witten model which, it is suggested, is the most likely candidate to offer a successful path to the quantization of the supersymmetric Liouville model.

In chapter one some of the ancestry of the work to be presented is covered. This is done to provide a degree of context for an otherwise abstract and seemingly unphysical theory. As string theory was the spark of originality that led to this field of study it is only natural that this chapter should examine string theory and explain how this leads to the gauged, twisted, Wess-Zumino-Novikov-Witten model.

Chapter two takes a look at the  $SL(2|1)$  Wess-Zumino-Novikov-Witten model and in particular this chapter calculates aspects of the theory which will be needed in later pages. For Lie super algebras there are, usually, more than one interpretation of the Dynkin diagram and this leads in turn to more than one interpretation of the currents. Both sets of currents that are present for the Lie super group  $SL(2|1)$  are calculated — both in the classical case and in the quantum case, and the transformation that relates the two is also presented in classical and quan-

tum form. The method of extracting this transformation is quite general and can be used as a prescription for finding similar transformations in other Lie super algebras.

The third chapter begins to analyze the  $SL(2|1)/SL(2|1)$  Wess-Zumino-Novikov-Witten model when it has been gauged and twisted by adding a term,  $-\partial J^3$ . Specifically it is demonstrated that this model is a valid description of  $N = 2$  non-critical string theory when it tensors a system of topological ghosts. These ghosts are shown to contribute only a vacuum state to the overall space of physical states, which means that the non-trivial cohomology of physical states can be found by analyzing the gauged, twisted, Wess-Zumino-Novikov-Witten model.

The final chapter of this thesis begins an analysis of the cohomology of physical states. From the perspective of string theory the full cohomology of physical states is composed of smaller cohomologies. One of these is the so called Wakimoto module and this is the cohomology of states described by the gauged, twisted, Wess-Zumino-Novikov-Witten model. This module is examined, providing a first step toward a full understanding of the total space of states.

# Chapter One

## Ancestry of the G/G Models

*True, I talk of dreams,  
Which are the children of an idle brain.  
Begot of nothing but vain fantasy;  
Which is as thin of substance as the air,  
And more inconstant than the wind,*

*From Romeo and Juliet  
by William Shakespeare*

## 1.1 Introduction

This chapter serves to lay the foundations on which this thesis is set. The well-spring of ideas of modern times, of which the gauge, twisted, Wess-Zumino-Novikov-Witten model is but one, began with the advent of string theory. It is therefore hardly surprising that the best backdrop to this thesis is the very same

theory of strings. The following twelve pages are a brief review of the development of string theory from its simple birth through to current thinking. The details are not expanded upon since the main focus is toward the gauged, twisted, Wess-Zumino-Novikov-Witten model, but it is hoped that the reader should gain some insight into the results gathered by knowing their origins.

Section two of this chapter introduces the ancestry of string theory starting with the dual models of the strong interaction. It charts the rise of this concept to prominence as a possible grand unified field theory and highlights the potential that string theory has to fulfil this role.

Section three contains a little more mathematical detail upon the methodology of Polyakov's path integral formalism for random surfaces. Apart from the power and beauty of this route, it results in the conclusion that to solve string theory when the number of spacetime dimensions is other than the critical value, we must quantize the Liouville action.

Section four looks at this task in brief for the case of the bosonic string. This thesis is an extension of these ideas to the  $N = 2$  supersymmetric string with this section providing the reader with a degree of anticipation of the work to come.

## 1.2 The Beginnings of String Theory

The birth of what is now called string theory — or in more modern times, superstring theory — can trace its origins as far back as 1968. Around this time the world of theoretical and experimental physics was going through a period of great excitement and even greater turmoil. One of the many conundrums of the time (and today still a major area of research) was the nature and origin of the strong nuclear interaction. This was confused by the sudden discovery of a whole host of hadronic particles (hadronic particles being those interacting primarily through the nuclear force), or resonances, as they have become known. One of the peculiarities of many of these particles was the very large values of spin that they could take,

with the value of the spin linearly related to the mass squared. High values of spin were seen as a problem because such theories had proven unrenormalizable.

The experimental evidence of the time also suggested an approximate equivalence in the scattering amplitudes for the  $s$  and  $t$  channels of cross sections [1], where  $s$  and  $t$  are Mandelstam variables defined through the energy momentum of particles, used in calculating cross sections for reasons of convenience. This form of duality was suggested as being fact for small values of  $s$  and  $t$  in 1968. Following this suggestion Veneziano proposed a model that might explain some of these features. The original proposition was barely more than an *ad hoc* idea to attempt to shed some light on the problem, since it was explicitly designed to exhibit a duality in the  $s$  and  $t$  channels while predicting the relationship between the mass of hadrons and the spin. This initiative led to the birth of the so called dual models as an attempt to quantify the strong interaction.

The ideas of Veneziano stood in the main lime light for only a short while since other areas of physics started to present new ideas and experimental confirmations. Yet during its brief appearance on stage the dual models had evoked much interest and provided an area for intense research. As a result various features had been uncovered, the first of which was the fact that the Veneziano amplitude belonged to a theory of relativistic extended quantum mechanical objects. The string was born.

The problems encountered in studying a theory of strings are many, the first and most obvious being a consequence of the special theory of relativity. Different points on the string are following trajectories in different reference frames, rather than being stationary in the same frame, which added complications to the analysis of the problem. The initial approach to this was one of covariant canonical quantization, which previously had been so successful in developing field theories. This approach led to an understanding of many of the features of string theory, and for a while the subject flourished. One exciting result was that it appeared that a wide variety of particles can be considered no more than an excitation of a

ground state string, the properties of which can be calculated.

The early quantization also revealed other facts. One of most dominant of these was that a string description of bosonic particles could only be easily solved when the number of spacetime dimensions was twenty six, the so called critical dimension. The need to include fermions lead to the introduction of supersymmetry, with the  $N = 1$  supersymmetric string only easily solvable in the canonical quantization of the day in ten spacetime dimensions. These spurious dimensions, seen at the time as a problem, can be reasoned away by 'compactification'; a process originally proposed by Kaluza in 1921 as a way of unifying classical electromagnetism with general relativity in a five dimensional model. Today the extra dimensions are considered as one of the positive aspects of string theory as they can be interpreted as extra degrees of freedom.

The extra dimensions of string theory are thought of, with the hindsight of modern understanding, to be a big hint at the role that string theory was to play. Today's opinion tends to believe that these extra dimensions are a must if a unification of the fundamental interactions is to be attained — a goal which has long been the wish of physicists. The pursuit in Grand Unified field theories was nothing more than a belief in natural beauty until the 1960's when the electro-weak theory was proposed, and then experimentally verified nearly two decades later. The electro-weak theory is a gauge theory that unifies the electromagnetic force with the weak nuclear force. They are, so to speak, two sides of the same coin. Hopes that the strong force could soon have been unified to the electro-weak force in the same manner have so far lacked the authority of the electro-weak model and the necessary experimental confirmation. It seems that the inclusion of gravity is as far away from the unification with the other three forces as it could be possible to get.

The reasons for the problems in quantizing gravity by the same procedure as the other three fundamental forces is basically due to the spin of the proposed propagator — the graviton. The graviton is a massless spin two particle, and as



was pointed out earlier no one had yet found a way of renormalizing a spin two field theory. The dual models predicted a large number of massless particles one of which had spin two. As the strong interaction did not contain a spin two particle, this was originally seen as a failing for the dual model. Yet it was a triumph for the search for a quantum theory of gravity. The theory of strings was resurrected as a unifying theory with the massless spin two particle given the role of the graviton. Initially the idea was given a wide berth until Green and Schwarz showed that this theory was renormalizable. String theory, in its modern guise, had arrived at last.

We shall not bother to augment the history of the following few years. Needless to say the physics community became very excited by the prospects of attaining its holy grail. Large amounts of effort went into the study of string theory and each time a problem came along that threatened to destroy the ambitions of string theorists, a solution would be found and the string model would emerge the stronger for it. However, it did face one very big hindrance. Very quickly the easier aspects of the analysis of the model was undertaken leaving only the less tractable mathematics of the theory to be confronted. The canonical quantization became a messy tool for such complexity and for a while interest waned. An easier method was needed.

The fresh approach was provided by Polyakov in 1981 [2] who succeeded in finding a path integral approach to quantization which hinted at the possibility of producing solutions off the critical dimension, and it is his methods that have been the cornerstone of much of the analysis of recent times. His work is also the point at which this thesis starts, for the work contained in the following pages can be seen as the natural progression in a chain of steps which began with Polyakov's summation over random surfaces.

### 1.3 Polyakov's Path Integral Formalism

Polyakov brought to the field his special blend of clarity of thought and simplicity of argument. We shall follow the chain of reasoning that he initiated as it concludes with the Liouville equation, the study of which is our concern. As Polyakov pointed out, there are many reasons for urgency in addressing the problem of a sum over random surfaces. It has long been known that the amplitude for a free particle, in the calculation of the partition function, is described by an action that appears in an exponential. Feynman determined the form of this to be :

$$K(x_1, x_2) = \lim_{i \rightarrow \infty} \sum_{l_i} \delta l_i \exp(-mL(l_i)) \quad (1.3.1)$$

where  $l_i$  is the path connecting two points,  $x_1$  and  $x_2$ , while  $m$  is a scaling parameter connected to the mass and  $L(l_i)$  is the classical action for the motion of the particle along the path  $l_i$ .

The arguments for the generalization of (1.3.1) to the amplitude for a world sheet rather than a point particle are many [3]. Yet it is not too unreasonable to assume that if the action for the trajectory of a particle is that which minimizes the path taken, then the action for the trajectory of a string, when that trajectory carves out an area in spacetime, is given by an action that minimizes that area. In this manner we are led to view with prospects an amplitude of the form :

$$K(C_1, C_2) = \lim_{i \rightarrow \infty} \sum_{s_i} \delta s_i \exp(-m^2 L(s_i)) \quad (1.3.2)$$

and now we have  $C_1$  and  $C_2$  as loops that form the boundaries of a surface  $s_i$ , and  $L(s_i)$  is the action formed from the area of the surface  $s_i$ .

Before we proceed to the mathematical analysis of equation (1.3.2) let us examine some of the interesting symmetries that are present in this action, and which will be of use to us later in the evaluation of (1.3.2).

Perhaps the most important aspect of the actions (1.3.1) and (1.3.2) is that both are reparameterization invariant. Reparameterization invariance is the mathematical expression of gauge invariance and signifies the independence in how we

can formulate a description of a geometric object [4]. Naturally it is a very powerful symmetry. A more formal expression of reparameterization invariance is the following. Let us parameterize the general action  $L$  by a parameter  $x_\alpha(t)$ , then the action is invariant under

$$x_\alpha(t) \mapsto x_\alpha(f(t)) \quad (1.3.3)$$

for a general function  $f(t)$ .

The second major symmetry of the actions (1.3.1) and (1.3.2) is not *a priori* present, although, in two dimensions, it is often there. Weyl invariance is an indication of an underlying conformal symmetry that is of great use in solving the model exactly. So important has the role of conformal invariance become that many physicists and mathematicians look to its existence as being a key ingredient in all the fundamental formulations of physics, particularly in two dimensions.

The Weyl transformation is described by the mappings :

$$g_{ab}(z) \mapsto \exp(\phi(z))g_{ab}(z) \quad X_\alpha(z) \mapsto X_\alpha(z) \quad (1.3.4)$$

which manifestly demonstrates that Weyl transformations are a rescaling of the metric. Actions that are invariant under these transformations are conformal invariant, conformal invariance being a combination of scale, rotational and translational invariance; three invariances which together preserve angles but not lengths.

The original formulation of (1.3.2) for random surfaces was done for the case of the bosonic string [2]. It was suggested by Nambu in 1969 that the action  $L(s_i)$  could take the form

$$L(s_i) = - \int d^2x \sqrt{\det\{\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu\}} \quad (1.3.5)$$

with  $\eta_{\mu\nu}$  being the metric of spacetime and  $\mu, \nu$  taking values  $1 \dots d$  where  $d$  is the number of dimensions of spacetime. This action was suggested as it describes the area of a world sheet which is mapped out by a string moving through a space parameterized by coordinates  $(x, t)$ . Unfortunately the appearance of the square

root makes the evaluation of the partition function, in which (1.3.5) describes the amplitude, difficult. It is therefore more useful to use the alternative action

$$L(s_i) = \frac{1}{2} \int d^2x \sqrt{g} g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (1.3.6)$$

which has the same classical equation of motion for  $X^\mu$  when this action is varied with respect to  $g^{ab}$  — an additional metric for the worldsheet — and  $X^\mu$ , but without the problems caused by the extreme nonlinearity of equation (1.3.5). Notice that we have also introduced a new metric  $g_{\mu\nu}$  which is the metric of the world sheet — the world sheet being the two dimensional surface produced by the motion of the string.

In truth the action (1.3.6) is for a bosonic string. Our thesis is concerned with the  $N = 2$  superstring which describes a string theory with a fermionic character. Despite this difference we shall proceed with the analysis of (1.3.6) for the following reasons. Firstly, in a superfield formalism the action and method of solving the partition function mirrors the procedure for the bosonic case with only minor differences. Secondly, in the work to follow the results of this section can be considered as a reduction of a Wess-Zumino-Novikov-Witten model valued for some group  $G$ . In our case we choose this group to be the Lie supergroup  $SL(2|1)$  and this choice will naturally impart the fermionic nature into the results.

To return to the matter at hand, we need to find a method to deal with the following partition function [2]:

$$Z = \int \frac{\mathcal{D}g \mathcal{D}X}{V_{diff}} \exp\left(-L(s_i) - \frac{\mu_0}{2} \int d^2x \sqrt{g}\right). \quad (1.3.7)$$

The additional term appearing on the right in equation (1.3.7) is a cosmological constant which has its value set by a choice of  $\mu_0$ . Although such a term is not required for the classical study of the theory, it is important for the renormalizability of the quantum version. We have also normalized the integral by the factor  $\frac{1}{V_{diff}}$  which is necessary since the action and the measures are reparameterization invariant which contributes an infinite factor from the volume of the group of these diffeomorphisms under the integral.

The action  $L(s_i)$  of equation (1.3.6) is conformally invariant as can clearly be seen since we have, under Weyl rescalings

$$g_{ab} \mapsto g_{ab} \exp\{\phi\} \quad , \quad \sqrt{g} \mapsto \exp\{\phi\} \sqrt{g}, \quad (1.3.8)$$

but the measures in equation (1.3.7) are not invariant under this scaling. As was shown in [2,5] the measure for  $\mathcal{D}X$  maps in the following way under Weyl rescalings,  $g_{ab} \mapsto g_{ab} \exp \phi$  :

$$\mathcal{D}X \mapsto \mathcal{D}X \exp\left(\frac{D}{48\pi} S_L(\phi)\right). \quad (1.3.9)$$

The number of spacetime dimensions  $D$  appears in this expression as does the Liouville action  $S_L(\phi)$  given by :

$$S_L(\phi) = \int d^2x \sqrt{g} \left( \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + R\phi + \mu \exp \phi \right) \quad (1.3.10)$$

where  $R$  is the scalar curvature in two dimensions [2,6], and  $\mu$  is the quantum version of the classical cosmological constant  $\mu_0$  that was present in equation (1.3.7) and its value is determined by the choice of  $\mu_0$  and the additional shift introduced through renormalization of the quantum theory that is described by equation (1.3.10).

The handling of the measure  $\mathcal{D}g$  is a little more involved. The full details can be found elsewhere [3,4,6,7] — only a short summary is given now. It is mathematically difficult to deal with a measure for a tensor quantity. The resolution of this difficulty uses the fundamental theorem of a 2 dimensional geometry which states that, given any metric  $g_{ab}(x)$  there is always a reparameterization  $x \mapsto y(x)$  which, at least locally, makes the metric conformally Euclidean. I.e.,

$$g_{ab}(x) dx^a dx^b = \sigma(y) dy^a dy^b \delta_{ab} \quad (1.3.11)$$

although global topology however usually prevents the existence of an everywhere Euclidean metric.

The fundamental integral (1.3.7), as mentioned above, is invariant under the local gauge group of reparameterization. We can fix the gauge freedom by restricting the integral to a gauge slice, i.e., to a subspace of metrics which meets each orbit of the local gauge group exactly once.

A good global gauge slice must involve a family of inequivalent non-Euclidean conformal classes parameterized by a finite number of complex variables called the moduli  $\tau_i$  with  $i = 1, \dots, k$ . If  $\hat{g}_{ab}$  is the reference metric, we can write

$$[\hat{g}(\tau_1, \dots, \tau_k)] = \{g_{ab}(x) = \exp \sigma(x) \hat{g}_{ab}(x)\} \quad (1.3.12)$$

for the conformal class parameterized by the moduli.

The treatment of the integration over all metrics now breaks down into an integration over the moduli and over some conformal class :

$$\int_{\text{all metrics}} \mathcal{D}g = \int_{\hat{g}} [\mathcal{D}\tau] \mathcal{D}\sigma J(g) \quad (1.3.13)$$

where  $J(g)$  is a Faddeev-Popov determinant which takes into account the variable volume of the orbits of the reparameterization group. It is represented as a Grassmannian integral over anti-commuting ghost fields. Although the ghost action is Weyl invariant, the measure for the ghost fields, like the measure  $\mathcal{D}X$ , is not Weyl invariant and it will pick up a contribution of [3] :

$$\mathcal{D}b\mathcal{D}c \mapsto \mathcal{D}b\mathcal{D}c \exp\left(\frac{-26}{48\pi} S_L(\phi)\right); \quad (1.3.14)$$

again we have the appearance of the Liouville action of equation (1.3.9). If we collect the various results together we conclude that the original partition function of equation (1.3.7) reduces to

$$Z = \int [\mathcal{D}\tau] \mathcal{D}\sigma \exp\left\{-\left(\frac{D-26}{48\pi}\right) S_L(\sigma)\right\} \quad (1.3.15)$$

which clearly shows that in the critical dimension of  $D = 26$  the contribution to the partition function of the Liouville action decouples. Here  $S_L(\sigma)$  has the

same form as in equation (1.3.10), but the metric is the reference metric  $\hat{g}$ , and the scalar curvature  $\hat{R}$  is calculated with  $\hat{g}$ .

Thus we have recovered the same results that were known from the old covariant quantization approach. Yet, in addition, we see that if string theory is to be solved off the critical dimension it is vital to find a method for quantizing the Liouville action. It is the tackling of this task that is the focus of the next section.

## 1.4 Development of the G/G Models

In the last section Polyakov's path integral method was examined and its results presented in a concise form. This condensed format will continue in this section. The development of the study of the Liouville section off the critical dimension has a long and thorough history. Even a simplified explanation could involve many pages of detail which would largely be immaterial to the results of this thesis. Due to this a brief examination of the progress in this line of research will be presented for the bosonic string in this section. As the main aim of this dissertation is to extend the results of the bosonic string to the  $N = 2$  superstring it is hoped that any necessary points will be dealt with at the time they arise. This sections primary objective is to supply an overview of the work to come, from the perspective of the past results of other authors.

The point at which the last section ended was with the partition function for a free string at tree level (1.3.15) with

$$S_L(\sigma) = \int d^2x \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{ab} \partial_a \sigma \partial_b \sigma + \hat{R} \sigma + \mu \exp \sigma \right) \quad (1.4.1)$$

If the reference metric is flat, i.e.,  $\hat{g}_{ab} = \delta_{ab}$ , the dilaton term  $\hat{R} \sigma$  vanishes.

The action of (1.3.10) without the dilaton and exponential terms has the form of a class of integrable models known as sigma models. The general form for such a sigma model being :

$$S_\sigma \sim \int d^2x \sqrt{g} g^{\mu\nu} \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \quad (1.4.2)$$

which is mainly characterized by the familiar kinetic term, but where additional constraints can be introduced, or alternatively we can have a non-flat metric on our  $d$  dimensional space for the coordinates  $\phi$  on the manifold. Clearly the string action of equation (1.3.6) is a free sigma model with no constraints and a flat metric  $\eta_{\mu\nu}$ .

When we consider equation (1.3.10) with just the dilaton term removed then we are examining a special case of a Toda action, which is usually composed of several scalar fields with an interaction between them of exponential form i.e.,

$$S_{\text{toda}} \sim \int d^2z \sum_i \partial_\nu \phi^i \partial^\nu \phi^i + \sum_i \mu^i \exp \phi^i. \quad (1.4.3)$$

These two exactly solvable models suggest that if we wish to generalize the Liouville model, and thus find a route to quantizing it, then perhaps we might consider an action of the following form :

$$S = \frac{1}{2} \int_M d^2x \text{Tr} \{g^{-1} \partial_\mu g g^{-1} \partial^\mu g\} \quad (1.4.4)$$

which again has a simple kinetic term. As in equation (1.4.2) we have a matrix field  $g(x)$  that is defined on some manifold  $M$ . The fundamental problem which plagues equation (1.4.4) is that it does not extend to the case of non-abelian groups in a way that is appealing, in the sense that it preserves many features that are desirable in a field theory such as symmetries and the dual of the currents. In fact the extension to (1.4.4) so as to make it applicable to non abelian groups was found by Witten who added the so called Wess-Zumino-Novikov-Witten term (hereafter referred to as WZNW) [8]. The details about the resulting action can be found in section 2.2. For the time being it is only necessary to observe that it results in the integrability of the Toda field theories, and so provides the first realistic approach to the quantization of a whole host of integrable models, including the Liouville action, each of which can be gained from the WZNW model through the process of Hamiltonian reduction, and this is examined for the case of the  $N = 2$  super Liouville action in section 3.5.



The next step was to turn these results into a field theory — and all modern field theories are gauge invariant — and this was immediately addressed by a number of authors [9,10,11,12,13] with the conclusion that a subgroup  $H$  of  $G$  could be gauged in the manner indicated in section (3.3), although it was the authors of [13] who first showed that this gauging was through the usual method of minimal coupling, even if this was not readily apparent due to the nature of the WZNW term.

It was rapidly appreciated that the study of the gauging of the WZNW model had many useful and subtle consequences. Perhaps the most important of these is the fact that the gauged  $G/H$  model is a natural realization of the Goddard-Kent-Olive coset construction. As the Goddard-Kent-Olive (hereafter referred to as GKO) coset construction will be required later we shall now make a small digression to explain it.

The WZNW model possesses non-abelian currents which form a current algebra, specifically, a Kac Moody algebra, and it is the exploitation of this Kac-Moody algebra which allows us to solve the model exactly. This Kac-Moody algebra is a consequence of the inclusion of a group representation into the model, and is accompanied by a Virasoro algebra that is a consequence of the reparameterization invariance mentioned earlier.

The theory of Kac-Moody and Virasoro algebras contains a useful result. For any Kac-Moody algebra there is a Virasoro algebra associated with it in such a way that a semi-direct product is formed, a result that we shall now reiterate.

Kac-Moody algebras are vital to the framework of current algebras as we shall see in section 2.2. The theory of current algebras contains a definition of an energy tensor expressed in bilinears of the generators, and we shall denote these generators by  $J$  because in the theory of current algebras the generators of the model are the currents. The form of this tensor was first proposed by Sugawara and Sommerfield in 1968, and the generalizations and completion of its consequences and nature was given later by Goddard, Kent, and Olive [14,15,16]. The energy

tensor takes the form :

$$L^G(z) = \sum_n z^{-n-2} L_n^G = \frac{1}{k + c_G} \sum_{a,b=1}^{\dim G} : h_{ab} J^a J^b : (z) \quad (1.4.5)$$

where normal ordering moves positive modes of  $J_n$  to the right of negative modes, and  $n$  and  $m$  takes integer values. We also have  $h_{ab}$  which is the inverse metric of the group and the dual Coxeter number  $c_G$ , both of which are defined in Appendix A. This expression also introduces the level  $k$  which specifies the Kac-Moody algebra by being the central extension for the algebra in question (see section 2.2). When the Sugawara form is defined in this manner it is possible to show that the following result is true :

$$[L_m, J_n^a] = -m J_{m+n}^a \quad [L_m, k] = 0 \quad (1.4.6)$$

with a Virasoro central charge of :

$$c = \frac{k \dim G}{k + c_G} . \quad (1.4.7)$$

The important result of [14,15,16] is that if a subgroup  $H \subseteq G$  is chosen, with a common basis between the first  $\dim H$  generators of  $H$  and the first  $\dim H$  generators of  $G$ , then an energy tensor for the subgroup can be defined in a similar way:

$$L^H(z) = \sum_n z^{-n-2} L_n^H = \frac{1}{k + c_H} \sum_{a,b=1}^{\dim H} : h_{ab} \tilde{J}^a \tilde{J}^b : (z) \quad (1.4.8)$$

Each energy tensor forms a Virasoro algebra, and more importantly, so does the difference  $\mathcal{K}_n = L_n^G - L_n^H$ . The central charge of the Virasoro algebra composed of generators  $\mathcal{K}_n$  is

$$c = \frac{k \dim G}{k + c_G} - \frac{k \dim H}{k + c_H} . \quad (1.4.9)$$

It was discovered that if the  $G$  WZNW model was gauged in a subgroup  $H \subseteq G$  then the action splits into WZNW models. The first of these was a level  $k$ ,

$G$  valued WZNW model and the second was a  $H$  valued WZNW model at level  $-(k + 2c_H)$ , with the central charge for the full gauged model given by equation (1.4.9) and each sector having Virasoro algebras generated by energy tensor defined through equations (1.4.5) and (1.4.8).

The second major realization was that the coseted  $G/G$  models, which are obtained by taking  $H = G$ , are topological conformal field theories. The nature of such a topological conformal field theory is discussed in section 3.7 where the results of this thesis will be used to demonstrate the details. In short a topological field theory has attributes that are independent of the metric and depend only on global quantities.

The breakthrough that followed was the discovery that if the full group  $G$  is gauged and that if this group  $G = SL(2, \mathfrak{R})$  then a twisted version of the model is equivalent to  $c \leq 1$  bosonic string theory. The twisting procedure is outlined in section 3.3 but in short it involves the altering of the energy tensor by the addition of the derivative of the  $J^3$  current. This changes some properties of the model such as the central charge and conformal weights of some of the fields, so making it equivalent to the non-critical string — subject to various conditions.

This then is the history of the study of the non-critical bosonic string. The next logical step to take is to extend this analysis to the case of the non-critical fermionic string. The study of the bosonic string had the advantage of a two pronged attack ; initially it was approached from the perspective of matrix model calculations which were able to indicate various aspects of the solution by discretizing two dimensional random surfaces. This information was then acted upon, and proven correct through the continuum method as detailed above, with the agreement between results from the matrix and continuum methods seen as a triumph for the string theorists.

Unfortunately the matrix model approach could not be used in the development of the fermionic string. The reasons for this is, in essence, quite simple. There are understandable difficulties encountered while trying to account for the spin

statistics of fermionic particles on a lattice. Thus in the study of supersymmetric strings there is no guidance available from matrix models at this time.

None the less, the continuum approach has recently been started by [17,18] for the  $N = 1$  superstring. This supplies us with a template to generalize the results to  $N = 2$  superstrings, and confidently speculate about the extensions to further degrees of supersymmetry. Some work on the  $SL(2|1)$  WZNW has been done [19] but new features, and the relation to non-critical fermionic strings, are presented in this thesis for the first time.

# Chapter Two

## The $SL(2|1)$ WZNW Model

*An ill favoured thing, sir, but mine own.*

From *As You Like It*  
by *William Shakespeare*

### 2.1 Introduction

Details of the WZNW model are not in short supply in the literature, which is fortunate as the WZNW model is of such leading importance to this thesis. Although some authors have presented specific details of the  $SL(2|1)$  WZNW model in the past [53,54], the picture is far from complete. The following chapter is a blend of information available in the literature and of original work on the classical and quantum currents.

Section two is a brief description of the WZNW model and its role in the study of Kac-Moody algebras. Section three will present the reader with details

about the Lie super algebra  $sl(2|1)$  and some of its interesting features such as the existence of more than one interpretation of the Dynkin diagram — a fact that will be clearly analyzed later on.

Section four will examine the Wakimoto construction, which is a method often used in applications of the WZNW model to extract a classical bosonization of the currents. This bosonization can be used to calculate a Fock space cohomology which is a large component of the complete space of physical states. There are in fact two possible bosonizations — one for each interpretation of the algebra, and both are presented for the first time.

Section five will extend on the work in section four. A new, highly non-linear field transformation is given which maps the free field representation of one set of currents onto the other set of currents. The method and theory behind this transformation is neat ; providing a general method that could be applied for all Lie super algebras which have more than one interpretation of their Dynkin diagram.

Section six extends the work of section four to the quantum case, and the quantum version of the free field currents. These results are of vital importance later on in this thesis. Section seven continues this theme by presenting the quantum version of the newly discovered field transformation. This is important for the following reason. For the first time we have established solidly that different interpretations of Lie super algebras do indeed describe the same model.

## 2.2 Introducing the Wess-Zumino-Novikov-Witten Model

The natural group invariant actions to consider in the study of string theory are the sigma models. Unfortunately they seem to be incomplete as a full and useful description on which to build solutions. For example, in their naive — but understandable — extension to non-abelian groups they do not readily conjure up the Kac-Moody symmetry that is needed to render the model integrable [8,14,20].

It was Witten who was the first to show how non-abelian groups could be included in the modelling of the symmetries of sigma models [8]. Witten added the so called Wess-Zumino term to the basic sigma model, which resulted in the following action :

$$\begin{aligned}
 S(g)_k = & -\frac{k}{8\pi} \int_M d^2x \operatorname{Tr} \left\{ (g^{-1} \partial g)(g^{-1} \bar{\partial} g) \right\} \\
 & + \frac{k}{8\pi} \int_B d^3x \operatorname{Tr} \left\{ g^{-1} \partial_3 g [g^{-1} \partial g, g^{-1} \bar{\partial} g] \right\}
 \end{aligned}
 \tag{2.2.1}$$

This action is worthy of a detailed explanation. First note that it is a classical action and that it will be necessary to quantize its field equations if it is to be used in any quantum field theory. The following material is first presented in classical form, and after in its quantum version.

The matrix field  $g$  in (2.2.1) is an element of group  $G$  with a maximally non-compact real Lie algebra  $\mathcal{G}$ . The trace is defined to be the non-degenerate, bi-linear form on the algebra of the group, normalized so that it cancels the square of the length of the highest root of  $\mathcal{G}$ , making it independent of the basis chosen. It will be the convention in this thesis to take an unspecified group to be symbolized by  $G$  and its algebra by  $\mathcal{G}$ .

In (2.2.1) we have made use of light cone coordinates, and this convention will be rigidly adhered to throughout the following pages. Let  $x_1$  and  $x_2$  be coordinates on our two dimensional target space (the manifold  $M$  in the above action). The light cone coordinates are defined by  $x_{\pm} = x_1 \pm x_2$ . Due to the large number of subscripts in the following work the notation  $\partial$ ,  $\bar{\partial}$  for  $\partial_+$ ,  $\partial_-$  respectively, will be commonly used, but where necessary the other notations listed may be employed. The WZNW model is a conformally invariant theory and the use of complex coordinates is a cornerstone of its integrability.

The Wess-Zumino term is locally a total derivative and so Gauss's theorem may be applied to write this term as an ordinary two dimensional action [8]. The 2d manifold  $M$  may be taken as space-time and is interpreted as the boundary of a 3d manifold  $B$ . Coordinates on  $B$  extend in a smooth manner through the

interior volume of  $B$ . This is not an unique definition and the Wess-Zumino term is well defined for integer multiples of  $2\pi$  i.e.,  $2n\pi$ . We may fix at a convention that  $n = 1$  [8,14].

The WZNW action can be varied to yield the following currents and equations of motion

$$\mathcal{J} = J^b \tau^b = \kappa g^{-1} \partial g \quad \text{with} \quad \bar{\partial} \mathcal{J} = 0, \quad (2.2.2a)$$

$$\bar{\mathcal{J}} = \bar{J}^a \tau^a = \kappa \bar{\partial} g g^{-1} \quad \text{with} \quad \partial \bar{\mathcal{J}} = 0 \quad (2.2.2b)$$

where we have put  $\kappa = \frac{-k}{4\pi}$ . The generators of the group are given by  $\tau^c$  with a commutator of  $[\tau^a, \tau^b] = f_c^{ab} \tau^c$  and a normalization of  $\text{Tr}(\tau^a \tau^b) = \frac{1}{2} h^{ab}$ , where  $h^{ab}$  is the metric of the group (see Appendix A). For the group  $SL(2|1)$  under consideration there are eight generators with  $c = 1, \dots, 8$ . When  $c = 1, \dots, 4$  the generators are members of an ordinary Lie algebra, and are representative of bosonic fields, and when  $c = 5, \dots, 8$  the generators are members of a Grassmann algebra, representative of fermionic fields.

An alternative expression of equation (2.2.2) is :

$$J^a = -\text{Tr}(\tau^a \mathcal{J}) \quad , \quad \bar{J}^b = \text{Tr}(\tau^b \bar{\mathcal{J}}) . \quad (2.2.3)$$

These currents are representative of the invariance of the action (2.2.1) under the symmetry of the form :

$$g(x_+, x_-) \rightarrow \Omega^{-1}(x_+) g(x_+, x_-) \bar{\Omega}(x_-), \quad g, \Omega, \bar{\Omega}^{-1} \in G. \quad (2.2.4)$$

It is this symmetry that makes it possible to solve exactly the WZNW model [6,20]. The currents (2.2.3) satisfy a classical version (Poisson brackets) of two commuting Kac-Moody algebras :

$$\{J^a(x), J^b(y)\} = f_c^{ab} J^c(y) \delta(x-y) + \frac{1}{2} \kappa h^{ab} \delta'(x-y) \quad (2.2.5)$$

$$\{J^a(x), \bar{J}^b(y)\} = 0$$



with a similar expression for the anti-holomorphic currents which generate a second Kac-Moody symmetry.

At the quantum level — using the variables  $z = \exp ix_+$  and  $\bar{z} = \exp ix_-$  — one defines the Laurent modes of the operators  $J(z)$  and  $\bar{J}(\bar{z})$  as follows :

$$\mathcal{J}(z) = \sum_{n=-\infty}^{\infty} \mathcal{J}_n z^{-n-1} \quad , \quad \bar{\mathcal{J}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\mathcal{J}}_n \bar{z}^{-n-1} \quad (2.2.6)$$

which finally gives us the quantum Kac-Moody algebra

$$[J_n^a, J_m^b] = f_c^{ab} J_{m+n}^c + \frac{1}{2} knh^{ab} \delta_{m+n,0} . \quad (2.2.7)$$

The manner in which the Kac-Moody algebra is used to solve the theory exactly is not important for this thesis, and the interested reader is referred to the original reference [20] .

### 2.3 The Lie Super Algebra $sl(2|1)$

The Lie super group  $SL(2|1)$  ( $A(1|0)$  in the notation of Kac) is a rank 2,  $N = 2$  supersymmetric extension of the group  $SL(2, \mathfrak{R})$ , and is isomorphic to the group  $OSP(2|2)$  ( $C(2)$  in the notation of Kac). The algebra  $sl(2|1)$  of this group is a direct sum of two familiar simple Lie algebras,  $sl(2|\mathfrak{R})$  and  $sp(1)$  — which are associated to the groups  $SL(2, \mathfrak{R})$  and  $SP(1)$  — and in addition there is the  $\mathcal{Z}_2^{\otimes 2}$  graded extension that forms the fermionic sector.

It has long been established that all the information about a Lie group can be summarized in a Dynkin diagram unique to the algebra of the group. The structure of the algebra is encoded in the Cartan subalgebra, step operators, and a set of non-zero roots associated to the step operators, and these contain all the details that are needed to construct the algebra.

The roots can all be obtained by forming linear combinations of  $r$  simple roots (where  $r$  is the rank of the group), with either all positive or all negative

coefficients. For each simple root  $\alpha_i$  ( $i = 1 \dots r$ ), it is possible to form a Weyl reflection  $\sigma$  acting on a weight vector  $\lambda$  by using the transformation

$$\sigma_{\alpha_i}(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \tag{2.3.1}$$

for the scalar product  $(a, b)$ , induced by the Cartan Killing metric  $g_{ij} = f_{ib}^c f_{jc}^b$ . The weight vector  $\lambda$  is an eigenvector of Cartan subalgebra generators  $\{H_1, \dots, H_r\}$  taken in a basis where they are diagonal. The group of transformations generated by these reflections about the simple roots is called the Weyl group, and an application of the Weyl group to a set of simple roots generates another set of simple roots. For ordinary Lie algebras all the possible sets of simple roots are related in this fashion — they are said to be Weyl equivalent.

Many Lie super algebras differ from ordinary Lie algebras since they may have more than one interpretation of their Dynkin diagram. As a consequence two sets of simple roots are said to be Weyl inequivalent. In short this is because Lie super algebras also have fermionic roots in addition to their bosonic ones. Specifically they may have a purely fermionic interpretation of their Dynkin diagram, and as can be clearly seen from equation (2.3.1) a purely fermionic root system cannot have a Weyl group according to the standard definition of the Weyl reflections when the norm square of such roots is  $(\alpha_i, \alpha_i) = 0$ . Each Weyl inequivalent root system corresponds to a different interpretation of the Dynkin diagram.

In the case under consideration — the algebra  $sl(2|1)$  — there are two such interpretations. Following [21,22,23] we shall call them Type A and Type B. Type A has one bosonic simple root (symbolized by  $\odot$ ) and one fermionic simple root (symbolized by  $\otimes$ ), and we can use the useful basis provided by [21] to detail the algebra.

We introduce an orthonormal basis  $e_i, i = 1, 2$  with a positive metric,  $\delta_j$  with  $j = 1$  and negative metric. This basis obeys :

$$e_i \cdot e_j = \delta_{ij} \quad \delta_i \cdot \delta_j = -\delta_{ij} \quad e_i \cdot \delta_j = 0 \tag{2.3.2}$$

The Type A algebra has simple roots that are expressed as :

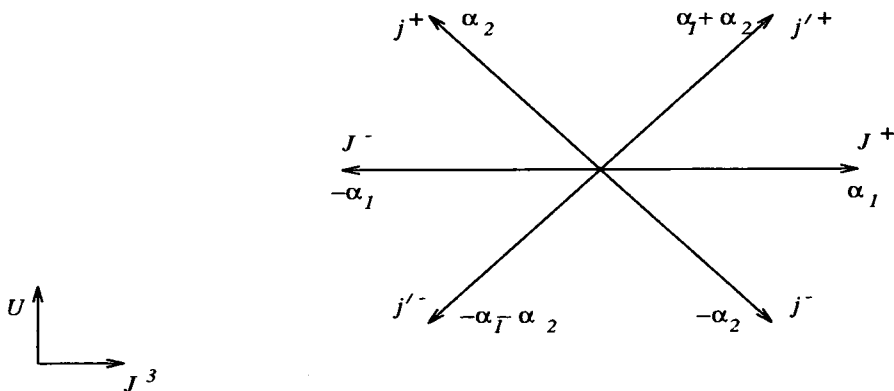
$$\alpha_1 = e_1 - e_2 \quad , \quad \alpha_2 = e_2 - \delta_1 \tag{2.3.3}$$

with a corresponding Dynkin diagram of  $\ominus \text{---} \otimes$ . For Type A the simple roots satisfy  $(\alpha_1)^2 = 2, (\alpha_1 + \alpha_2)^2 = 0$  and  $\alpha_1 \cdot \alpha_2 = -1$ . It is possible to find a convenient choice of components for  $\alpha_1$  and  $\alpha_2$  if a Minkowski 2-metric is used — and these components, which reproduce the positive roots — take the form :

$$\alpha_1 = (\sqrt{2}, 0) \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad \alpha_1 + \alpha_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \tag{2.3.4}$$

with a root diagram of :

Figure 2a



The Type B interpretation composes the purely fermionic root system and has the Dynkin diagram  $\otimes \text{---} \otimes$ , with simple roots (expressed in the basis of equation (2.3.2) ) :

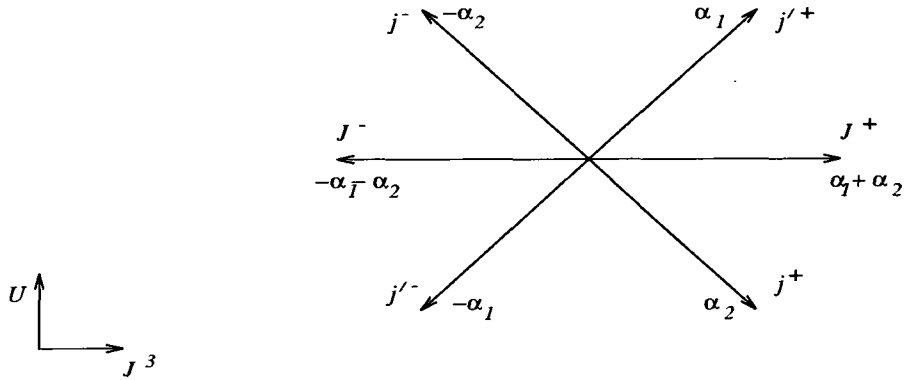
$$\tilde{\alpha}_1 = e_1 - \delta_1 \quad , \quad \tilde{\alpha}_2 = \delta_1 - e_2 \tag{2.3.5}$$

with  $(\tilde{\alpha}_1)^2 = 0, (\tilde{\alpha}_2)^2 = 0$  and  $\tilde{\alpha}_1 \cdot \tilde{\alpha}_2 = 1$ . In component form, with a Minkowski 2 metric, the positive roots are :

$$\tilde{\alpha}_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad \tilde{\alpha}_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \quad \tilde{\alpha}_1 + \tilde{\alpha}_2 = (\sqrt{2}, 0) \tag{2.3.6}$$

with a root diagram of :

Figure 2b



The two root systems are clearly related by  $\alpha_1 = \tilde{\alpha}_1 + \tilde{\alpha}_2, \alpha_2 = -\tilde{\alpha}_2$ .

The algebra  $sl(2|1)$  has a matrix representation in terms of eight  $(2 + 1) \times (2 + 1)$  generators, each of which obeys<sup>†</sup>  $\text{STr}(M) = M_{11} + M_{22} + (-1)^{d(M)} M_{33} = 0$ . The algebra is composed of the  $sl(2, \mathbb{R})$  generators  $J^3, J^+, J^-$ , four fermionic generators that form  $so(2)$  doublets,  $(j^+, j^-), (j'^+, j'^-)$ , and a  $u(1)$  generator  $U$ . The explicit form of these matrices is presented here for completeness and future reference. Type A and Type B share the same Cartan subalgebra and bosonic generators, which are :

$$\begin{aligned}
 J^3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & J^+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 U &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} & J^- &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned} \tag{2.3.7a}$$

<sup>†</sup> We have here introduced the definition of the degree of a supermatrix, which is defined in Appendix A, and is taken to mean that  $d(M)=0$  if the element  $M_{33}$  is bosonic in nature, with  $d(M)=1$  if the element  $M_{33}$  is fermionic in nature.

In addition to these the fermionic generators for Type A are :

Type A

$$\begin{aligned}
 j^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & j'^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & (2.3.7b) \\
 j^- &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & j'^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

and for Type B the fermionic generators are :

Type B

$$\begin{aligned}
 j^+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & j'^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & (2.3.7c) \\
 j^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & j'^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Note that the Type B generators can be gained from the Type A generators by interchanging  $j^+$  and  $j^-$ .

Naturally these generators satisfy sets of commutator/anti-commutator relations that are different for the two types of algebra and these too are also presented here for future reference :

Type A

$$\begin{aligned}
 [J^3, J^\pm] &= \pm J^\pm & [J^+, J^-] &= 2J^3 & (2.3.8a) \\
 [U, j^\pm] &= \pm \frac{1}{2}j^\pm & [U, j'^\pm] &= \pm \frac{1}{2}j'^\pm \\
 [J^\pm, j^\pm] &= \mp j'^\pm & [J^\pm, j'^\mp] &= \pm j^\mp \\
 [J^3, j^\pm] &= \mp \frac{1}{2}j^\pm & [J^3, j'^\pm] &= \pm \frac{1}{2}j'^\pm \\
 \{j^+, j^-\} &= J^3 + U & \{j^\pm, j'^\mp\} &= +J^\mp \\
 \{j'^+, j'^-\} &= -J^3 + U
 \end{aligned}$$

### Type B

$$\begin{aligned}
 [J^3, J^\pm] &= \pm J^\pm & [J^+, J^-] &= 2J^3 & (2.3.8b) \\
 [U, j^\pm] &= \mp \frac{1}{2} j^\pm & [U, j'^\pm] &= \pm \frac{1}{2} j'^\pm \\
 [J^\pm, j^\mp] &= \mp j'^\pm & [J^\pm, j'^\mp] &= \pm j^\pm \\
 [J^3, j^\pm] &= \pm \frac{1}{2} j^\pm & [J^3, j'^\pm] &= \pm \frac{1}{2} j'^\pm \\
 \{j^+, j^-\} &= J^3 + U & \{j^\pm, j'^\pm\} &= +J^\pm \\
 \{j'^+, j'^-\} &= -J^3 + U
 \end{aligned}$$

Again we note that Type B can be gained from Type A by interchanging  $j^+$  and  $j^-$ . This is a common feature throughout the following work.

## 2.4 The Wakimoto Construction

It was Wakimoto [24] who first developed a method to calculate the currents of a WZNW model in terms of free fields, and used his method to find the Fock space representation for the currents. It is this procedure for bosonization that will be outlined now.

It is generally true that any group element  $g \in G$  considered in the neighbourhood of the identity can be decomposed as a Gauss decomposition. For  $SL(2|1)$  the form of this decomposition is  $g = g_< g_0 g_>$ , where

$$\begin{aligned}
 g_< &= \exp\{\lambda E^- + \psi_i F^i\} &= \exp\{\xi J^- + \psi_- j^- + \psi'_- j'^-\} & (2.4.1) \\
 g_0 &= \exp\{\phi_j H^j\} &= \exp\{\phi U + \theta J^3\} \\
 g_> &= \exp\{\tilde{\lambda} \tilde{E}^+ + \tilde{\psi}_m \tilde{F}^m\} &= \exp\{\gamma J^+ + \psi_+ j^+ + \psi'_+ j'^+\}
 \end{aligned}$$

with  $E$  and  $F$  ( $\tilde{E}$ ,  $\tilde{F}$ ) being step operators associated to negative (positive) bosonic and fermionic roots respectively. This decomposition is in terms of the Borel subalgebras  $g_<$  and  $g_>$  [25,26], and a Cartan subalgebra  $g_0$  with generators  $H^1, H^2$ .

The decomposition introduces bosonic fields denoted by  $\gamma, \xi, \phi, \theta$ , and fermionic fields denoted by  $\psi_-, \psi'_-, \psi_+, \psi'_+$ . If we use the representation given in equation (2.3.7) then we can calculate the group elements as given by (2.4.1) to be :

Type A

$$\begin{aligned}
 g_{<} &= \begin{pmatrix} 1 & 0 & \psi_- \\ \xi & 1 & \psi'_- + \frac{1}{2}\xi\psi_- \\ 0 & 0 & 1 \end{pmatrix} \quad g_{>} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ \psi_+ & \psi'_+ + \frac{1}{2}\gamma\psi_+ & 1 \end{pmatrix} \\
 g_0 &= \begin{pmatrix} \exp \frac{1}{2}\{\phi + \theta\} & 0 & 0 \\ 0 & \exp \frac{1}{2}\{\phi - \theta\} & 0 \\ 0 & 0 & \exp \phi \end{pmatrix} \tag{2.4.2a}
 \end{aligned}$$

Type B

$$\begin{aligned}
 g_{<} &= \begin{pmatrix} 1 & 0 & 0 \\ \xi - \frac{1}{2}\psi_-\psi'_- & 1 & \psi'_- \\ \psi_- & 0 & 1 \end{pmatrix} \quad g_{>} = \begin{pmatrix} 1 & \gamma + \frac{1}{2}\psi_+\psi'_+ & \psi_+ \\ 0 & 1 & 0 \\ 0 & \psi'_+ & 1 \end{pmatrix} \\
 & \tag{2.4.2b}
 \end{aligned}$$

with the Cartan subalgebra  $g_0$  remaining the same for both the Type A and Type B decompositions.

Once we have found an expression for the element  $g$  we are at liberty to calculate the currents as defined earlier (2.2.2) and (2.2.3), namely  $J^a = -\text{STr}(\tau^a \mathcal{J})$  with  $\mathcal{J} = \kappa g^{-1} \partial g$ , along with many other useful results. Note that the extension to Lie super groups is easy to follow, and obvious, with the trace being replaced by the super trace.

Throughout this work only the holomorphic currents will be presented, but there exists the set of anti-holomorphic currents (also given by calculation of the above) which satisfy a similar theory. As the holomorphic and anti-holomorphic currents commute they may be considered separately. Unless it is important to the mathematics the anti-holomorphic sector will not be presented.

An alternative method for finding the currents is to explicitly calculate the action (2.2.1) and to find the currents by varying this action. Although this route

is not as straight forward as the approach mentioned above, this method does have some advantages that will become apparent.

We can calculate the action in terms of the fields  $\gamma, \xi, \phi, \theta, \psi_-, \psi'_-, \psi_+, \psi'_+$ , which will hereafter be referred to as the decomposition fields. This can be done by using the Wakimoto construction of introducing a Gauss decomposition of the group element as mentioned above, and following the steps laid down in [26,27]. This makes use of the remarkable Polyakov-Wiegmann identity [28] for the expression of the WZNW action in terms of a product of three matrices :

$$\begin{aligned}
 S(ABC) &= S(A) + S(B) + S(C) & (2.4.3a) \\
 &- \frac{k}{4\pi} \int d^2z \text{STr} [(A^{-1}\partial_+A)(\partial_-B)B^{-1} + (B^{-1}\partial_+B)(\partial_-C)C^{-1} \\
 &\quad + (A^{-1}\partial_+A)B(\partial_-C)C^{-1}B^{-1}]
 \end{aligned}$$

where  $S(A), S(B), S(C)$  are WZNW action of  $A, B, C$  at the same level, modulo local terms.

Substituting the representation (2.4.2), as given by the Gauss decomposition, into (2.2.1) we gain the action in terms of the decomposition fields. In calculating the action we exploit the useful fact that  $S(g_<)$  and  $S(g_>)$  are actions of elements taking values in the nilpotent subalgebras only, and as such the contributions from these actions vanish.

Locally the actions for both Type A and Type B are :

#### Type A

$$\begin{aligned}
 S(g) &= \frac{-k}{4\pi} \int d^2z \left( \left( \frac{1}{4}\psi_-\psi_+\partial_+\xi\partial_-\gamma + \frac{1}{4}\partial_-\psi_+\psi_-\gamma\partial_+\xi - \frac{1}{2}\partial_-\psi'_+\psi_-\partial_+\xi \right. \right. \\
 &\quad - \frac{1}{4}\partial_+\psi_-\psi_+\xi\partial_-\gamma + \frac{1}{4}\partial_+\psi_-\partial_-\psi_+\gamma\xi + \partial_+\psi_-\partial_-\psi_+\exp\{-\theta\} \\
 &\quad - \frac{1}{2}\xi\partial_+\psi_-\partial_-\psi'_+ + \frac{1}{2}\partial_+\psi'_-\psi_+\partial_-\gamma - \frac{1}{2}\gamma\partial_+\psi'_-\partial_-\psi_+ + \partial_+\psi'_-\partial_-\psi'_+ \\
 &\quad + \partial_+\xi\partial_-\gamma\exp\frac{1}{2}\{\theta - \phi\} + \frac{1}{4}\partial_+\theta\partial_-\theta\exp\left\{-\frac{1}{2}\{\theta + \phi\}\right\} \\
 &\quad \left. \left. - \frac{1}{4}\partial_+\phi\partial_-\phi\exp\left\{-\frac{1}{2}\{\theta + \phi\}\right\}\right) \exp\frac{1}{2}\{\theta + \phi\} \right) & (2.4.3b)
 \end{aligned}$$



Type B

$$\begin{aligned}
 S(g) = \frac{-k}{4\pi} \int d^2z & \left( \left( -\frac{1}{2} \partial_- \psi_+ \psi'_+ \partial_+ \xi - \frac{1}{2} \partial_- \psi'_+ \psi_+ \partial_+ \xi + \frac{1}{2} \partial_+ \psi_- \psi'_- \partial_- \gamma \right. \right. \\
 & + \frac{1}{4} \partial_+ \psi_- \partial_- \psi_+ \psi'_- \psi'_+ - \partial_+ \psi_- \partial_- \psi_+ \exp\left\{-\frac{1}{2}\{\theta + \phi\}\right\} \\
 & + \frac{1}{4} \partial_+ \psi_- \partial_- \psi'_+ \psi'_- \psi_+ + \frac{1}{2} \partial_+ \psi'_- \psi_- \partial_- \gamma + \frac{1}{4} \partial_+ \psi'_- \partial_- \psi_+ \psi_- \psi'_+ \\
 & + \frac{1}{4} \partial_+ \psi'_- \partial_- \psi'_+ \psi_- \psi_+ + \partial_+ \psi'_- \partial_- \psi'_+ \exp\frac{1}{2}\{\phi - \theta\} \\
 & \left. \left. + \partial_+ \xi \partial_- \gamma + \frac{1}{4} \partial_+ \theta \partial_- \theta \exp\{-\theta\} - \frac{1}{4} \partial_+ \phi \partial_- \phi \exp\{-\theta\} \right) \exp \theta \right)
 \end{aligned} \tag{2.4.3c}$$

Varying these actions with respect to the individual fields will give the currents required. Unfortunately these currents are not best expressed in terms of the decomposition fields. Instead the currents take the most appealing form in a so called free field representation which is a parameterization of the decomposition fields. The free field representation naturally divides into pairs, of fields and their canonically conjugate momenta, which in turn form canonical Poisson Brackets.

Not surprisingly the actions (2.4.3) can be used to find the precise parameterization by using the variation principle. It is well known that for a general Lagrangian  $\mathcal{L}(\phi_i)$  composed of fields  $\phi_i$ , then the canonical momenta  $\Pi_i$  conjugate to each  $\phi_i$  is given by:

$$\Pi_i = \frac{\partial \mathcal{L}(\phi_i)}{\partial \dot{\phi}_i} \tag{2.4.4}$$

It is important to note that the partial differentiation is with respect to  $\dot{\phi}_i$ , the time derivative of  $\phi_i$ , as there is a subtlety involved if the actions (2.4.3) are to be used in their presented form. We have made use of light cone coordinates  $x_{\pm} = x_1 \pm x_2$  which are linear combinations of space and time coordinates. In such coordinates we should be careful about whether it is  $x_+$  or  $x_-$  which gives a valid correspondence to time in our equations. Calculations in Hamiltonian mechanics

are performed on a surface (usually of constant time). When using light cone coordinates, as in our case, our constant surface is taken to be a choice of constant  $x_-$ . Then  $x_-$  plays a role equivalent to time in our light cone coordinates, and can be used to give a valid definition of the canonical momenta.

For example, in the actions of equations (2.4.3b) and (2.4.3c), the field canonically conjugate to  $\psi_+$  is found by differentiating with respect to  $\partial_- \psi_+$ , and so on. In this manner we can determine the canonical momenta to the fields  $\gamma$ ,  $\psi_+$ ,  $\psi'_+$  for Type A to be :

$$\begin{aligned} \frac{\partial S}{\partial(\partial_- \gamma)} &= \beta = -\kappa \partial \xi \exp \theta + \frac{1}{2} [\psi'_+]^\dagger \psi_+ & (2.4.5) \\ \frac{\partial S}{\partial(\partial_- \psi_+)} &= [\psi_+]^\dagger = -\kappa \partial \psi_- \exp \frac{1}{2} \{\phi - \theta\} - \frac{1}{2} \gamma [\psi'_+]^\dagger \\ \frac{\partial S}{\partial(\partial_- \psi'_+)} &= [\psi'_+]^\dagger = -\kappa (\partial \psi'_- - \frac{1}{2} \partial \psi_- \xi + \frac{1}{2} \psi_- \partial \xi) \exp \frac{1}{2} \{\theta + \phi\} \end{aligned}$$

respectively.

The Noether currents associated with the Kac-Moody symmetries of the action (2.4.3) are constructed as

$$J(\lambda) = -\kappa S \text{Tr} (\lambda g^{-1} \partial g) \quad (2.4.6)$$

where the matrices  $\lambda$  are given in (2.3.6). For type A, they take the form :

Type A

$$\begin{aligned} J^+ &= -\beta - \frac{1}{2} [\psi'_+]^\dagger \psi_+ & (2.4.7) \\ J^3 &= \beta \gamma - \frac{1}{2} [\psi_+]^\dagger \psi_+ + \frac{1}{2} [\psi'_+]^\dagger \psi'_+ + \frac{1}{2} \kappa \partial \theta \\ U &= \frac{1}{2} [\psi_+]^\dagger \psi_+ + \frac{1}{2} [\psi'_+]^\dagger \psi'_+ - \frac{1}{2} \kappa \partial \phi \\ J^- &= \beta \gamma^2 + \kappa \gamma \partial \theta + \kappa \partial \gamma - \frac{1}{4} \gamma^2 [\psi'_+]^\dagger \psi_+ + \frac{1}{2} \gamma [\psi'_+]^\dagger \psi'_+ \\ &\quad - \frac{1}{2} \gamma [\psi_+]^\dagger \psi_+ - [\psi_+]^\dagger \psi'_+ \\ j^- &= \beta \psi'_+ + \frac{1}{2} [\psi'_+]^\dagger \psi_+ \psi'_+ - \frac{1}{2} \beta \gamma \psi_+ - \frac{1}{2} \kappa \psi_+ (\partial \theta - \partial \phi) + \kappa \partial \psi_+ \\ j'^- &= -\frac{1}{2} \gamma^2 \beta \psi_+ + \beta \gamma \psi'_+ + [\psi_+]^\dagger \psi'_+ \psi_+ + \kappa \partial \psi'_+ + \frac{1}{2} \kappa \psi'_+ (\partial \theta + \partial \phi) + \frac{1}{2} \kappa \gamma \partial \psi_+ \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}\kappa\psi_+\partial\gamma - \frac{3}{4}\kappa\gamma\psi_+\partial\theta + \frac{1}{4}\kappa\gamma\psi_+\partial\phi \\
 j^+ &= -[\psi_+]^\dagger + \frac{1}{2}\gamma[\psi'_+]^\dagger \\
 j'^+ &= -[\psi'_+]^\dagger
 \end{aligned}$$

These currents will satisfy Poisson Brackets, which correspond in the quantum theory to operator product expansions. These Poisson Brackets take the form :

$$\begin{aligned}
 \text{Type A} \quad \{J^+(x), J^-(y)\} &= 2J^3(y)\delta(x-y) + \kappa\delta'(x-y) & (2.4.8) \\
 \{J^3(x), J^\pm(y)\} &= \pm J^\pm(y)\delta(x-y) \\
 \{J^3(x), J^3(y)\} &= -\{U(x), U(y)\} = \frac{1}{2}\kappa\delta'(x-y) \\
 \{U(x), j^\pm(y)\} &= \pm\frac{1}{2}j^\pm(y)\delta(x-y) \\
 \{U(x), j'^\pm(y)\} &= \pm\frac{1}{2}j'^\pm(y)\delta(x-y) \\
 \{J^\pm(x), j^\pm(y)\} &= \mp j'^\pm(y)\delta(x-y) \\
 \{J^\pm(x), j'^\mp(y)\} &= \pm j^\mp(y)\delta(x-y) \\
 \{J^3(x), j^\pm(y)\} &= \mp\frac{1}{2}j^\pm(y)\delta(x-y) \\
 \{J^3(x), j'^\pm(y)\} &= \pm\frac{1}{2}j'^\pm(y)\delta(x-y) \\
 \{j^+(x), j^-(y)\} &= (J^3(y) + U(y))\delta(x-y) - \kappa\delta'(x-y) \\
 \{j^\pm(x), j'^\mp(y)\} &= +J^\mp(y)\delta(x-y) \\
 \{j'^+(x), j'^-(y)\} &= (-J^3(y) + U(y))\delta(x-y) - \kappa\delta'(x-y)
 \end{aligned}$$

where  $x$  and  $y$  are coordinates on our surface of  $x_- = \text{constant}$ . In a similar fashion the Type B canonical fields can be found to be :

$$\begin{aligned}
 \beta &= -\kappa\left\{\partial\xi + \frac{1}{2}\partial\psi_-\psi'_- + \frac{1}{2}\partial\psi'_-\psi_-\right\} \exp\theta & (2.4.9) \\
 [\psi_+]^\dagger &= \kappa\partial\psi_-\exp\frac{1}{2}\{\theta - \phi\} + \frac{1}{2}\beta\psi'_+ \\
 [\psi'_+]^\dagger &= -\kappa\partial\psi'_-\exp\frac{1}{2}\{\phi + \theta\} + \frac{1}{2}\beta\psi_+
 \end{aligned}$$

with classical currents in free field form :

Type B

$$J^+ = -\beta \quad (2.4.10)$$

$$J^3 = \gamma\beta + \frac{1}{2}\kappa\partial\theta + \frac{1}{2}[\psi_+]^\dagger\psi_+ + \frac{1}{2}[\psi'_+]^\dagger\psi'_+$$

$$U = -\frac{1}{2}\kappa\partial\phi - \frac{1}{2}[\psi_+]^\dagger\psi_+ + \frac{1}{2}[\psi'_+]^\dagger\psi'_+$$

$$J^- = \gamma^2\beta + \frac{1}{2}\kappa\psi'_+\psi_+\partial\phi + \frac{1}{2}\kappa\partial\psi_+\psi'_+ + \frac{1}{2}\kappa\partial\psi'_+\psi_+ + \gamma[\psi_+]^\dagger\psi_+ \\ + \gamma[\psi'_+]^\dagger\psi'_+ + \kappa\gamma\partial\theta + \kappa\partial\gamma$$

$$j^- = \frac{1}{2}\gamma\beta\psi_+ + \frac{1}{2}\kappa\psi_+(\partial\theta - \partial\phi) + \kappa\partial\psi_+ + \frac{1}{2}[\psi'_+]^\dagger\psi'_+\psi_+ + \gamma[\psi'_+]^\dagger$$

$$j'^- = \frac{1}{2}\gamma\beta\psi'_+ + \frac{1}{2}\kappa\psi'_+(\partial\theta + \partial\phi) + \kappa\partial\psi'_+ - \frac{1}{2}[\psi_+]^\dagger\psi'_+\psi_+ + \gamma[\psi_+]^\dagger$$

$$j^+ = \frac{1}{2}\beta\psi'_+ + [\psi_+]^\dagger$$

$$j'^+ = -\frac{1}{2}\beta\psi_+ - [\psi'_+]^\dagger$$

The Type B currents also satisfy a set of Poisson Brackets, which can be gained from (2.4.8) by interchanging  $j^+$  and  $j^-$ .

The verification that the currents do indeed satisfy (2.4.7) requires the following free field Poisson Brackets

$$\{\gamma(x), \beta(y)\} = \delta(x - y) \quad (2.4.11)$$

$$\{\psi_+(x), [\psi_+(y)]^\dagger\} = \{\psi'_+(x), [\psi'_+(y)]^\dagger\} = \delta(x - y)$$

$$\{\partial\theta(x), \partial\theta(y)\} = -\{\partial\phi(x), \partial\phi(y)\} = \frac{2}{\kappa}\delta'(x - y)$$

which are satisfied by both the Type A and Type B free fields.

## 2.5 The Non-linear Transformations Between Currents

Having presented the two currents that come from the two interpretations of the algebra, we are in a position to ask how, if in any way, they are related. It has been suggested in [19] that there exists a non-linear Bogolubov transformation of the fields relating the two sets of currents, but that the nature of this transformation was such that it had proved difficult to find.

Contrary to this there will now be presented a simple argument and proof that the fundamental transformation is not at the level of the currents, but at the level of the group elements. In this way the field transformations that map the two types of currents onto each other will be found [29,30].

As the currents are described by  $\mathcal{J} = \kappa g^{-1} \partial g$ , their simplest component is the group element  $g$ . If we were to consider a transformation at the level of  $g$  then we might guess at a transformation of the form  $g^A = \gamma g^B \gamma^{-1}$ , which is familiar from throughout physics. In the context of the WZNW model such a transformation corresponds to a gauge transformation.

Yet the WZNW model can also be described in a chirally gauge invariant manner. That is to say that it can have gauge transformations of the form  $g \mapsto \alpha g \beta$  provided  $\alpha, \beta \in H^-, H^+ \subseteq G$  where  $H^-, H^+$  are nilpotent Borel subalgebras generated by negative and positive step operators, respectively (see section 3.2). Thus we may be in a position to consider a wider class of transformations of the form

$$g^A = \mathcal{A} g^B \mathcal{B}. \tag{2.5.1}$$

This is clearly not a transformation of the fields as suggested in [19]. If we draw upon an analogy with many areas of physics then the transformation suggested by equation (2.5.1) could be compared with a familiar passive transformation, which is a transformation of the coordinates in a system; in actuality equation (2.5.1) changes one matrix representation into another.

The forms of  $\mathcal{A}$  and  $\mathcal{B}$  are easy to determine. The first step is to express  $g^A$  and  $g^B$  in the form of a Gauss decomposition as presented in Section 2.4. Since the Type A generators can be gained from the Type B generators by interchanging  $j^-$  and  $j^+$ , we are at liberty to express our calculations entirely in terms of the generators of Type B. In this way  $g^A$  and  $g^B$ , when written as expansions in the step operators of Type B, take the form :

$$g^A = \exp(\psi_- j^+ + \psi'_- j'^- + \xi J^-) \exp(\phi U + \theta J^3) \exp(\psi_+ j^- + \psi'_+ j'^+ + \gamma J^+) \quad (2.5.2a)$$

$$g^B = \exp(\psi_- j^- + \psi'_- j'^- + \xi J^-) \exp(\phi U + \theta J^3) \exp(\psi_+ j^+ + \psi'_+ j'^+ + \gamma J^+) \quad (2.5.2b)$$

Note that we are considering this in analogy to a passive transformation and as such the fields do not change, so the same fields appear in both expressions. In other words, equations (2.5.2a) and (2.5.2b) offer two differing interpretations of the element  $g$ . Both provide a distinct and complete description of the group  $SL(2|1)$  in terms of the same fields. Usually such a situation would be the consequence of a symmetry. As the transformation has a form very similar to a gauge transformation it was initially thought that the two interpretations of the algebra were manifestations of a gauge symmetry and could be related by a gauge choice. However, at the time of writing this remains a speculation.

Upon examination we can note a useful fact, that the Cartan subalgebra for both decompositions is the same. In such a case we can map (2.5.2b) to (2.5.2a) if we take our transformation matrices in (2.5.1) as being of the form

$$\mathcal{A} = \exp(\psi_- j^+ + \psi'_- j'^- + \xi J^-) \exp -(\psi_- j^- + \psi'_- j'^- + \xi J^-) \quad (2.5.3a)$$

$$\mathcal{B} = \exp -(\psi_+ j^+ + \psi'_+ j'^+ + \gamma J^+) \exp(\psi_+ j^- + \psi'_+ j'^+ + \gamma J^+) \quad (2.5.3b)$$

Since  $\mathcal{A}$  and  $\mathcal{B}$  are composed of elements of the group  $G$  then  $\mathcal{A}$  and  $\mathcal{B}$  will correspondingly be elements of  $G$ , although they may not be members of the Borel subalgebras.

We may make use of the identity :

$$\exp A \exp -B = \exp(A - B - \frac{1}{2}[A, B]), \quad (2.5.4)$$

which holds provided  $[A, [A, B]] = [B, [A, B]] = 0$  and if we take  $[A, B]$  to be an anticommutator when both  $A$  and  $B$  are fermionic, to calculate the explicit matrix representation of  $\mathcal{A}$  and  $\mathcal{B}$ . In this way equations (2.5.3a) and (2.5.3b) may be written as

$$\begin{aligned} \mathcal{A} &= \exp(\psi_- j^+ - \psi_- j^- + \frac{1}{2}\xi\psi_- [J^-, j^+] - \frac{1}{2}\psi'_- \psi_- \{j'^-, j^-\}) \\ &= \exp(\psi_- j^+ - \psi_- j^- + \frac{1}{2}\xi\psi_- j'^- - \frac{1}{2}\psi'_- \psi_- J^-) \end{aligned} \quad (2.5.5a)$$

$$\begin{aligned} \mathcal{B} &= \exp(-\psi_+ j^+ + \psi_+ j^- - \frac{1}{2}\psi_+ \psi'_+ \{j^+, j'^+\} - \frac{1}{2}\gamma\psi_+ [J^+, j^-]) \\ &= \exp(-\psi_+ j^+ + \psi_+ j^- - \frac{1}{2}\psi_+ \psi'_+ J^+ + \frac{1}{2}\gamma\psi_+ j'^+) \end{aligned} \quad (2.5.5b)$$

and now the exponentials may be expanded to give :

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & \psi_- \\ \frac{1}{2}\psi_- \psi'_- & 1 & \frac{1}{2}\xi\psi_- \\ -\psi_- & 0 & 1 \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 1 & -\frac{1}{2}\psi_+ \psi'_+ & -\psi_+ \\ 0 & 1 & 0 \\ \psi_+ & \frac{1}{2}\gamma\psi_+ & 1 \end{pmatrix} \quad (2.5.6)$$

This transformation can be verified as correct by applying them to the explicit matrix representations of equations (2.4.2). The relationship between the currents for the Type A and Type B interpretations of the algebra can also be found by substituting (2.5.1) into the definition of the currents (2.2.2) to give

$$\mathcal{J}^A = \mathcal{B}^{-1}(g^B)^{-1} \mathcal{A}^{-1} \partial \mathcal{A} g^B \mathcal{B} + \mathcal{B}^{-1} \mathcal{J}^B \mathcal{B} + \mathcal{B}^{-1} \partial \mathcal{B} \quad (2.5.7)$$

However, the above transformation is not as useful as a mapping of the fields themselves. Such a field transformation should exist, as suggested in [19]. In this respect equations (2.5.1) and (2.5.7) have a greater value. In our analogy equation (2.5.1) represents a passive transformation. Similarly the field transformation would correspond to an active transformation, and (2.5.1) suggests that this would

take place as a mapping of the decomposition fields at the level of  $g$ , and not primarily a mapping between the currents. As such it is enough to compare the Gauss decompositions of the Type A and Type B group elements to deduce eight simultaneous equations in terms of the fields, which can then be solved.

The extraction of this transformation is not as difficult as one might expect since the decomposition in terms of the Borel subalgebras drastically simplifies the situation as does the Grassmann nature of four of the fields. In this way the field transformations that map the group element of Type B onto that of Type A have been found to be :

$$\begin{aligned}
 \exp \frac{1}{2} \theta &\mapsto \exp \frac{1}{2} \theta + \frac{1}{2} \psi_- \psi_+ \exp \frac{1}{2} \phi & (2.5.8) \\
 \exp \frac{1}{2} \phi &\mapsto \exp \frac{1}{2} \phi + \frac{1}{2} \psi_- \psi_+ \exp \{ \phi - \frac{1}{2} \theta \} \\
 \psi_- &\mapsto \psi_+ \exp \frac{1}{2} \{ \phi - \theta \} \\
 \psi_+ &\mapsto \psi_- \exp \frac{1}{2} \{ \phi - \theta \} \\
 \psi'_- &\mapsto \psi'_- - \frac{1}{2} \xi \psi_- \\
 \psi'_+ &\mapsto \psi'_+ - \frac{1}{2} \gamma \psi_+ \\
 \xi &\mapsto \xi + \frac{1}{2} \psi'_- \psi_+ \exp \frac{1}{2} \{ \phi - \theta \} - \frac{1}{4} \xi \psi_- \psi_+ \exp \frac{1}{2} \{ \phi - \theta \} \\
 \gamma &\mapsto \gamma + \frac{1}{2} \psi_- \psi'_+ \exp \frac{1}{2} \{ \phi - \theta \} - \frac{1}{4} \gamma \psi_- \psi_+ \exp \frac{1}{2} \{ \phi - \theta \}
 \end{aligned}$$

To complete the transformations above the remaining free field mappings, (the free fields being given in (2.4.5) and (2.4.9) ), can be calculated as :

$$\begin{aligned}
 [\psi_+]^\dagger &\mapsto k \partial \psi_+ - \frac{1}{2} k \psi_+ (\partial \phi - \partial \theta) + \frac{1}{2} \beta \psi'_+ - \frac{1}{4} \beta \gamma \psi_+ + \frac{1}{4} [\psi'_+]^\dagger \psi_+ \psi'_+ \\
 [\psi'_+]^\dagger &\mapsto [\psi'_+]^\dagger - \frac{1}{2} \beta \psi_- \exp \frac{1}{2} \{ \phi - \theta \} - \frac{1}{4} [\psi'_+]^\dagger \psi_+ \psi_- \exp \frac{1}{2} \{ \phi - \theta \} \\
 \beta &\mapsto \beta + \frac{1}{2} [\psi'_+]^\dagger \psi_+ & (2.5.9)
 \end{aligned}$$

Clearly other Lie Super-groups for which there is more than one interpretation of the Dynkin diagram — and hence more than one set of currents — would have correspondingly similar field mappings that may be determined in the same way.



Following the initiative of equation (2.5.1), a more elegant proof of the above field transformation was later found by Peter Bowcock and Anne Taormina. This is presented in Appendix B for the more mathematically inclined reader. To conclude, the inverse field transformations, mapping from Type A to Type B are :

$$\begin{aligned}
 \exp \frac{1}{2}\theta &\mapsto \exp \frac{1}{2}\theta + \frac{1}{2}\psi_-\psi_+ \exp\{\theta - \frac{1}{2}\phi\} & (2.5.10) \\
 \exp \frac{1}{2}\phi &\mapsto \exp \frac{1}{2}\phi + \frac{1}{2}\psi_-\psi_+ \exp \frac{1}{2}\theta \\
 \psi_- &\mapsto \psi_+ \exp \frac{1}{2}\{\theta - \phi\} \\
 \psi_+ &\mapsto \psi_- \exp \frac{1}{2}\{\theta - \phi\} \\
 \psi'_- &\mapsto \psi'_- + \frac{1}{2}\xi\psi_+ \exp \frac{1}{2}\{\theta - \phi\} - \frac{1}{4}\psi'_-\psi_-\psi_+ \exp \frac{1}{2}\{\theta - \phi\} \\
 \psi'_+ &\mapsto \psi'_+ + \frac{1}{2}\gamma\psi_- \exp \frac{1}{2}\{\theta - \phi\} - \frac{1}{4}\psi'_+\psi_-\psi_+ \exp \frac{1}{2}\{\theta - \phi\} \\
 \xi &\mapsto \xi + \frac{1}{2}\psi_-\psi'_- \\
 \gamma &\mapsto \gamma - \frac{1}{2}\psi_+\psi'_+ \\
 [\psi'_+]^\dagger &\mapsto [\psi'_+]^\dagger + \frac{1}{2}\beta\psi_+ \\
 [\psi_+]^\dagger &\mapsto -k\partial\psi_+ - \frac{1}{2}k\psi_+(\partial\theta - \partial\phi) - \frac{1}{2}\gamma[\psi'_+]^\dagger - \frac{1}{4}\beta\gamma\psi_+ - \frac{1}{4}[\psi'_+]^\dagger\psi'_+\psi_+ \\
 \beta &\mapsto \beta - \frac{1}{2}[\psi'_+]^\dagger\psi_- \exp \frac{1}{2}\{\theta - \phi\} - \frac{1}{4}\beta\psi_+\psi_- \exp \frac{1}{2}\{\theta - \phi\}
 \end{aligned}$$

## 2.6 The Quantum Currents

If our study of the Liouville sector is to be directed towards the quantum theory then the classical currents found so far are not sufficient. What we require is their quantum versions. The first step towards the quantum model is to perform a Wick rotation from Minkowski space to Euclidean space. In doing so our Poisson brackets become operator product expansions of complex coordinates [21], which are summarized here :

Type A

$$\begin{aligned}
 J^+(z)J^-(w) &\sim 2\frac{J^3(w)}{z-w} + \frac{k}{(z-w)^2} & (2.6.1) \\
 J^3(z)J^\pm(w) &\sim \pm\frac{J^\pm(w)}{z-w} \\
 J^3(z)J^3(w) &\sim -U(z)U(w) \sim \frac{1}{2}\frac{k}{(z-w)^2} \\
 U(z)j^\pm(w) &\sim \pm\frac{1}{2}\frac{j^\pm(w)}{z-w} \\
 U(z)j'^\pm(w) &\sim \pm\frac{1}{2}\frac{j'^\pm(w)}{z-w} \\
 J^3(z)j^\pm(w) &\sim \mp\frac{1}{2}\frac{j^\pm(w)}{z-w} \\
 J^3(z)j'^\pm(w) &\sim \pm\frac{1}{2}\frac{j'^\pm(w)}{z-w} \\
 J^\pm(z)j^\pm(w) &\sim \mp\frac{j'^\pm(w)}{z-w} \\
 J^\pm(z)j'^\mp(w) &\sim \pm\frac{j'^\mp(w)}{z-w} \\
 j^+(z)j^-(w) &\sim \frac{U(w)}{z-w} + \frac{J^3(w)}{z-w} - \frac{k}{(z-w)^2} \\
 j'^+(z)j'^-(w) &\sim \frac{U(w)}{z-w} - \frac{J^3(w)}{z-w} - \frac{k}{(z-w)^2} \\
 j^\pm(z)j'^\mp(w) &\sim +\frac{J'^\mp(w)}{z-w}.
 \end{aligned}$$

As before the Type B operator product expansions can be obtained by interchanging the  $j^+$  and  $j^-$  currents, and bearing in mind that the free field Poisson Brackets (2.4.11) now become anti/commutator relations with small distance operator product expansions of :

$$\begin{aligned}
 \partial\theta(z)\partial\theta(w) &= \frac{1}{(z-w)^2} \quad ; \quad \partial\phi(z)\partial\phi(w) = -\frac{1}{(z-w)^2} & (2.6.2) \\
 \psi_+(z)[\psi_+(w)]^\dagger &= \psi'_+(z)[\psi'_+(w)]^\dagger = \frac{1}{z-w} \\
 \gamma(z)\beta(w) &= \frac{1}{z-w}
 \end{aligned}$$

The quantum currents can now be found by adjusting the coefficients in the classical currents (2.4.7) and (2.4.9) until they satisfy the above operator product expansions. If we define  $\alpha_+ = \sqrt{2k+2}$  then it is not difficult to discover the quantum currents.

### Type A

$$J^+ = -\beta - \frac{1}{2}[\psi'_+]^\dagger \psi_+ \quad (2.6.3a)$$

$$J^3 = \beta\gamma - \frac{1}{2}[\psi_+]^\dagger \psi_+ + \frac{1}{2}[\psi'_+]^\dagger \psi'_+ + \frac{\alpha_+}{2}\partial\theta$$

$$U = \frac{1}{2}[\psi_+]^\dagger \psi_+ + \frac{1}{2}[\psi'_+]^\dagger \psi'_+ - \frac{\alpha_+}{2}\partial\phi$$

$$J^- = \beta\gamma^2 + \alpha_+\gamma\partial\theta + \left(k - \frac{1}{2}\right)\partial\gamma - \frac{1}{4}\gamma^2[\psi'_+]^\dagger \psi_+ + \frac{1}{2}\gamma[\psi'_+]^\dagger \psi'_+ \\ - \frac{1}{2}\gamma[\psi_+]^\dagger \psi_+ - [\psi_+]^\dagger \psi'_+$$

$$j^- = \beta\psi'_+ + \frac{1}{2}[\psi'_+]^\dagger \psi_+\psi'_+ - \frac{1}{2}\beta\gamma\psi_+ - \frac{\alpha_+}{2}\psi_+(\partial\theta - \partial\phi) + \left(k + \frac{1}{2}\right)\partial\psi_+$$

$$j'^- = -\frac{1}{2}\gamma^2\beta\psi_+ + \beta\gamma\psi'_+ + [\psi_+]^\dagger \psi'_+\psi_+ + k\partial\psi'_+ + \frac{\alpha_+}{2}\psi'_+(\partial\theta + \partial\phi) \\ + \left(\frac{k+1}{2}\right)\gamma\partial\psi_+ - \left(\frac{k-1}{2}\right)\psi_+\partial\gamma - \frac{3\alpha_+}{4}\gamma\psi_+\partial\theta + \frac{\alpha_+}{4}\gamma\psi_+\partial\phi$$

$$j^+ = -[\psi_+]^\dagger + \frac{1}{2}\gamma[\psi'_+]^\dagger$$

$$j'^+ = -[\psi'_+]^\dagger$$

### Type B

$$J^+ = -\beta \quad (2.6.3b)$$

$$J^3 = \gamma\beta + \frac{\alpha_+}{2}\partial\theta + \frac{1}{2}[\psi_+]^\dagger \psi_+ + \frac{1}{2}[\psi'_+]^\dagger \psi'_+$$

$$U = -\frac{\alpha_+}{2}\partial\phi - \frac{1}{2}[\psi_+]^\dagger \psi_+ + \frac{1}{2}[\psi'_+]^\dagger \psi'_+$$

$$J^- = \gamma^2\beta + \frac{\alpha_+}{2}\psi'_+\psi_+\partial\phi + \left(\frac{k+1}{2}\right)\partial\psi_+\psi'_+ + \left(\frac{k+1}{2}\right)\partial\psi'_+\psi_+ + \gamma[\psi_+]^\dagger \psi_+ \\ + \gamma[\psi'_+]^\dagger \psi'_+ + \alpha_+\gamma\partial\theta + k\partial\gamma$$

$$j^- = \frac{1}{2}\gamma\beta\psi_+ + \frac{\alpha_+}{2}\psi_+(\partial\theta - \partial\phi) + \left(k + \frac{1}{2}\right)\partial\psi_+ + \frac{1}{2}[\psi'_+]^\dagger \psi'_+\psi_+ + \gamma[\psi'_+]^\dagger$$

$$\begin{aligned}
 j'^- &= \frac{1}{2}\gamma\beta\psi'_+ + \frac{\alpha_+}{2}\psi'_+(\partial\theta + \partial\phi) + \left(k + \frac{1}{2}\right)\partial\psi'_+ - \frac{1}{2}[\psi_+]^\dagger\psi'_+\psi_+ + \gamma[\psi_+]^\dagger \\
 j^+ &= \frac{1}{2}\beta\psi'_+ + [\psi_+]^\dagger \\
 j'^+ &= -\frac{1}{2}\beta\psi_+ - [\psi'_+]^\dagger
 \end{aligned}$$

The extraction of the quantum field transformations is a little more tricky.

## 2.7 The Quantum Field Transformation

Having succeeded in finding the non-linear transformation between the classical currents it is desirable to achieve the same for their quantum versions. In the classical theory of Lie super algebras there is a trivial equivalence between the WZNW model using the Type A interpretation, and the WZNW model using the Type B interpretation. This is due to the action being defined for a general element of the group  $G$  (see equation (2.2.1)). For the quantum case this equivalence is not so assured. As the physical space of the gauged, twisted,  $SL(2|1)/SL(2|1)$  WZNW model has not been calculated for either the Type A or Type B case, then it is unknown whether the quantization procedure will result in differences between the two alternatives.

Peter Bowcock and Anne Taormina pointed out that the following provided a mapping between the Type B and Type A quantum currents of the  $SL(2|1)$  WZNW model provided that the normal ordering of non-abelian groups [31] is carefully considered.

$$\begin{aligned}
 \theta &\mapsto \theta + \frac{1}{2\alpha_-}\psi_-\psi_+ \exp(\alpha_-\{\phi - \theta\}) & (2.7.1) \\
 \phi &\mapsto \phi + \frac{1}{2\alpha_-}\psi_-\psi_+ \exp(\alpha_-\{\phi - \theta\}) \\
 \psi_- &\mapsto \psi_+ \exp(\alpha_-\{\phi - \theta\}) \\
 \psi_+ &\mapsto \psi_- \exp(\alpha_-\{\phi - \theta\})
 \end{aligned}$$

$$\begin{aligned}
 \psi'_- &\mapsto \psi'_- - \frac{1}{2}\xi\psi_- \\
 \psi'_+ &\mapsto \psi'_+ - \frac{1}{2}\gamma\psi_+ \\
 \xi &\mapsto \xi + \frac{1}{2}\psi'_-\psi_+ \exp(\alpha_-\{\phi - \theta\}) - \frac{1}{4}\xi\psi_-\psi_+ \exp(\alpha_-\{\phi - \theta\}) \\
 \gamma &\mapsto \gamma + \frac{1}{2}\psi_-\psi'_+ \exp(\alpha_-\{\phi - \theta\}) - \frac{1}{4}\gamma\psi_-\psi_+ \exp(\alpha_-\{\phi - \theta\}) \\
 [\psi_+]^\dagger &\mapsto (k + \frac{1}{2})\partial\psi_+ - \frac{\alpha_+}{2}(\partial\phi - \partial\theta)\psi_+ + \frac{1}{2}\beta\psi'_+ - \frac{1}{4}\beta\gamma\psi_+ + \frac{1}{4}[\psi'_+]^\dagger\psi_+\psi'_+ \\
 [\psi'_+]^\dagger &\mapsto [\psi'_+]^\dagger - \frac{1}{2}\beta\psi_-\exp(\alpha_-\{\phi - \theta\}) - \frac{1}{4}[\psi'_+]^\dagger\psi_+\psi_-\exp(\alpha_-\{\phi - \theta\}) \\
 \beta &\mapsto \beta + \frac{1}{2}[\psi'_+]^\dagger\psi_+
 \end{aligned}$$

where we have defined

$$\alpha_- = \frac{\sqrt{k+1}}{k + \frac{1}{2}}, \quad \alpha_+ = \sqrt{2k+2} \quad (2.7.2)$$

where normal ordering is defined as moving positive nodes to the right of negative nodes (e.g. :  $A_n B_{-n} := B_{-n} A_n \forall n \geq 0$ ), with careful accounting for the non-abelian nature of the group [31], and where products  $ABC$  and  $ABCD$  are taken as :

$$ABC \equiv \circ A : BC : \circ, \quad ABCD \equiv \times A \circ B : CD : \circ \times \quad (2.7.3)$$

with colons, circles, and crosses simply indicating different levels of nesting of normal ordered pairs.

For example, consider the term  $k\partial\psi'_+$  in the Type A,  $j'^-$  quantum current. This is mapped to from the terms  $(k + \frac{1}{2})\partial\psi'_+$  and  $\frac{1}{2}\gamma\beta\psi'_+$  that appear in the  $j'^-$  current of the Type B algebra :

$$\begin{aligned}
 &(k + \frac{1}{2})\partial\psi'_+ + \frac{1}{2}\gamma\beta\psi'_+ \\
 &= -n(k + \frac{1}{2})\psi'_{+,n} + \frac{1}{2}\circ\gamma_m : \beta_p \psi'_{+,n-m-p} : \circ \\
 &= -nk\psi'_{+,n} \\
 &= k\partial\psi'_+
 \end{aligned} \quad (2.7.4)$$

where at each stage we have dropped terms that don't contribute to the final result, and used the modal decomposition of the currents listed in Appendix C, with results from equations (4.2.4), (4.2.5), and (4.3.9).

We also need to redefine the canonically conjugate momenta to the fields to be :

Type A

$$\begin{aligned}
 [\psi_+]^\dagger &= -\left(k + \frac{1}{2}\right)\partial\psi_- \exp(\alpha_-\{\phi - \theta\}) - \frac{1}{2}\gamma[\psi'_+]^\dagger & (2.7.5a) \\
 [\psi'_+]^\dagger &= -\left(k + \frac{1}{2}\right)\left\{\partial\psi'_- - \frac{1}{2}\partial\psi_- \xi + \frac{1}{2}\psi_- \partial\xi\right\} \exp(\alpha_-\{\theta + \phi\}) \\
 \beta &= -\left(k + \frac{1}{2}\right)\partial\xi \exp\{2\alpha_-\theta\} + \frac{1}{2}[\psi'_+]^\dagger \psi_+
 \end{aligned}$$

Type B

$$\begin{aligned}
 [\psi_+]^\dagger &= \left(k + \frac{1}{2}\right)\partial\psi_- \exp(\alpha_-\{\theta - \phi\}) + \frac{1}{2}\beta\psi'_+ & (2.7.5) \\
 [\psi'_+]^\dagger &= -\left(k + \frac{1}{2}\right)\partial\psi'_- \exp(\alpha_-\{\theta + \phi\}) + \frac{1}{2}\beta\psi_+ \\
 \beta &= -\left(k + \frac{1}{2}\right)\left\{\partial\xi + \frac{1}{2}\partial\psi_- \psi'_- + \frac{1}{2}\partial\psi'_- \psi_-\right\} \exp\{2\alpha_-\theta\} .
 \end{aligned}$$

The sections 2.5, 2.6 and 2.7 together imply a powerful, although seemingly obvious, conclusion. Since the action of equation (2.2.1) is defined for a general group element this would clearly mean that the same model can be described classically by either the Type A or Type B algebras.

Now the quantum field transformation of this section establishes for the first time that the different interpretations of the algebra are connected in the quantum model, even though the precise details of the Fock spaces might differ.

# Chapter Three

The Gauged, Twisted

$SL(2|1)/SL(2|1)$  WZNW Model

*What is now proved was once only imagined.*

From *The Marriage of Heaven and Hell*

by *William Blake*

## 3.1 Introduction

Having now explored the WZNW model for the group  $SL(2|1)$  we can survey the gauged, twisted, WZNW model. A brief history and explanation of this theory has been presented in section 1.4 but these points shall now be expanded upon due to their association to the particular case of the  $SL(2|1)$  WZNW model. In the following pages it will be revealed that the  $N = 2$  superstring can be described by the gauged, twisted, WZNW model when it tensors an additional ghost system. In establishing these features we recover constraints on the specific characteristics of the matter sector for an  $N = 2$  fermionic string theory.

Section two of this chapter gauges the WZNW model of  $SL(2|1)$  and shows how the gauged model realizes the GKO coset construction, and predicts a zero central charge, which is the first indication that the model is a topological field theory. In addition to this, the gauging is demonstrated to introduce its own set of ghosts, the significance of which is illustrated later in the chapter.

Section three twists the energy tensors of the gauged WZNW model and in doing so reveals that the ghost sector that was found in section two must have its conformal dimensions altered if the conformal invariance of the theory is to be maintained.

Section four starts an analysis of the  $N = 2$  fermionic string by the path integral method and introduces another set of ghost fields, as well as the  $N = 2$  super Liouville action, while in the following section it is shown how it is possible to recover the same  $N = 2$  super Liouville action from the gauged WZNW model by constraining the currents in a Hamiltonian reduction. The Hamiltonian reduction is best performed by enlarging the space on which the theory is defined. In comparing the results of the Hamiltonian reduction and the  $N = 2$  string we conclude that the equivalency of these two models can only take place if the gauged, twisted WZNW model tensors a system of four fermions.

Section seven shows that the theory is topological in nature and that it contains an inherent  $N = 2$  superconformal algebra. The presence of an  $N = 2$  superconformal algebra has been a characteristic demonstrated by all topological conformal field theories found to date.

Finally the chapter concludes with a preliminary examination of the BRST cohomology of the gauged, twisted, WZNW model. This section shows that the tensoring system of fermions only makes minimal contributions in the form of vacuum states to the model, so reducing the examination of the cohomology of the physical states to a study of just the cohomology of the gauged, twisted, WZNW model. This serves as a precursor to the following chapter where this cohomology is calculated.



## 3.2 The Gauged WZNW Model

The standard gauging of the WZNW model is the natural progressive step to take if the WZNW model is to be advanced as a modern quantum field theory and this issue has been addressed by a number of authors in the past from many differing perspectives [9,10,11,12,13].

The symmetry of the WZNW model (equation (2.2.4)) is representative of a much larger  $G_L \times G_R$  Kac-Moody symmetry which results in the Kac-Moody algebra of equation (2.2.7). Kac-Moody algebras are usually the consequence of gauge symmetry (just as Virasoro algebras are the consequence of conformal symmetry) and as such we should look to gauge a subgroup of the  $G \times G$  symmetry of (2.2.4) — let us call this subgroup  $H$ .

Consider the functional

$$I(g, h, \tilde{h}) = S(hg\tilde{h}) - S(h\tilde{h}) \quad (3.2.1)$$

where  $S$  are WZNW actions defined in equation (2.2.1), and where  $h, \tilde{h}$  are group elements of the subgroup  $H$  to be gauged. If we have a gauge transformation of the form  $g \rightarrow \gamma g \gamma^{-1}$  for  $\gamma(x_+, x_-) \in H$ , then we can clearly see from equation (3.2.1) that we have a gauge invariance of  $I(g, h, \tilde{h})$  if we take

$$h \rightarrow h\gamma^{-1} \quad \tilde{h} \rightarrow \gamma\tilde{h} \quad \text{when } g \rightarrow \gamma g \gamma^{-1}. \quad (3.2.2)$$

An explicit form for the WZNW action given by equation (2.2.1) may be found by making use of the famous Polyakov-Wiegmann identity [28] for a product of group elements defined for a WZNW action :

$$S(AB) = S(A) + S(B) - \frac{k}{4\pi} \int d^2z \operatorname{Tr} \left\{ A^{-1} \partial A \bar{\partial} B B^{-1} \right\} \quad (3.2.3)$$

to gain the gauge invariant action :

$$I(g, A, \bar{A}) = S(g) - \frac{k}{4\pi} \int d^2z \operatorname{Tr} \left\{ A \bar{\partial} g g^{-1} + g^{-1} \partial g \bar{A} + A g \bar{A} g^{-1} - A \bar{A} \right\} \quad (3.2.4)$$

where we have introduced gauge fields  $\bar{A}$ ,  $A$  parameterized as :

$$\bar{A} = \bar{\partial}\tilde{h}\tilde{h}^{-1} \quad , \quad A = h^{-1}\partial h. \quad (3.2.5)$$

The gauge fields  $\bar{A}$ ,  $A$  take their values in the adjoint representation of  $H$ , and from equation (3.2.2) we can see that these fields will have gauge transformations of the form

$$\bar{A} \rightarrow \gamma\bar{A}\gamma^{-1} + \partial\gamma\gamma^{-1} \quad , \quad A \rightarrow \gamma A\gamma^{-1} - \bar{\partial}\gamma\gamma^{-1}. \quad (3.2.6)$$

The gauging of the action (3.2.1) in this manner, for a Kac-Moody symmetry, is only permissible if the subgroup being gauged is an anomaly free diagonal vector subgroup [13]. However, if the  $G$  valued model under consideration does not contain such a subgroup, the action (3.2.4) and the partition function (3.2.11) (below) can always be taken as the definition of the gauged WZNW model [32].

To proceed further let us consider the functional (3.2.1) under integration :

$$Z = \int \mathcal{D}g \mathcal{D}h \mathcal{D}\tilde{h} \det\{\partial - [A, \bullet]\} \det\{\bar{\partial} - [\bar{A}, \bullet]\} \exp(-kS(hg\tilde{h}) + kS(h\tilde{h})). \quad (3.2.7)$$

The determinants have been introduced because of the parameterization of the measures  $\mathcal{D}A, \mathcal{D}\bar{A}$  in terms of the elements  $h, \tilde{h}$ . The handling of functional integrals will not be dealt with rigourously in this thesis as such a treatment is irrelevant to the work presented, but the interested reader is referred to [3] for a thorough treatment.

It has been shown in [28,33] that the contribution from the determinants can be written as an action and two chiral determinants :

$$\det\{\partial - [A, \bullet]\} \det\{\bar{\partial} - [\bar{A}, \bullet]\} = \exp(2c_H S(h\tilde{h})) \det \partial \det \bar{\partial} \quad (3.2.8)$$

where  $c_H$  is the dual Coxeter number of the subgroup  $H$  in the adjoint representation. The action in (3.2.8) can be summed with the second action on the left hand side of equation (3.2.7) to produce a WZNW action  $S(h\tilde{h})$  at level  $-(k+2c_H)$ , and

we then need to fix the gauge. This may be done by making the convenient choice  $\tilde{h} = 1$  ( $\bar{A} = 0$ ). This gauge fixing, and a change of variables from  $hg\tilde{h} \rightarrow g$ , has two effects. The first is that the level  $-(k + 2c_H)S(h\tilde{h})$  action may be interpreted as a WZNW action for the  $H$  valued element  $h$  at level  $-(k + 2c_H)$  i.e.,

$$\exp(2c_H S(h\tilde{h})) \exp(kS(h\tilde{h})) \Big|_{\tilde{h}=1} \equiv \exp((k + 2c_H)S(h)). \quad (3.2.9)$$

The second effect is that the gauge fixing, along with the contribution from the chiral determinants in equation (3.2.8), may be written as an action of ghost fields  $(b^\alpha(z), c_\alpha(z))$  which take their values in the adjoint representation of  $H$  [12]; the action of the ghost fields being

$$\exp(S_{ghost}(b, c)) = \det \partial \det \bar{\partial} = \exp\left(-\int d^2z \text{STr}(b^\alpha \bar{\partial} c_\alpha + \bar{b}^\alpha \partial c_\alpha)\right), \quad (3.2.10)$$

with the currents from this action forming a Kac-Moody algebra of level  $2c_H$ .

So our original partition function of equation (3.2.7) now takes the following form :

$$Z = \int \mathcal{D}g \mathcal{D}h \mathcal{D}b \mathcal{D}c \exp(-kS(g)) \exp((k + 2c_H)S(h)) \exp(S_{gh}(b, c)). \quad (3.2.11)$$

Equation (3.2.11) is an important result. It has been shown [12] that the  $G/H$  gauged WZNW model as given by equation (3.2.11) is a natural realization of the GKO construction of section 1.4 in the following sense.

We are looking to gauge the full diagonal group  $G$  resulting in a  $G/G$  model. The results from [12] state that the three actions of equation (3.2.11) have currents  $J^\alpha(z)$  at level  $k$ ,  $\tilde{J}^\alpha(z)$  at level  $-(k + 2c_G)$ , and  $J_{gh}^\alpha(z)$ , ghost currents at level  $2c_G$ , from which we can form a total current

$$J_{tot}^\alpha(z) = J^\alpha(z) + \tilde{J}^\alpha(z) + J_{gh}^\alpha(z) \quad (3.2.12)$$

with

$$J_{gh}^\alpha(z) = (-1)^{d(\gamma)} f_\gamma^{\alpha\beta} : c^\gamma(z) b_\beta(z) :, \quad (3.2.13)$$

and Kac-Moody algebras for individual sectors being :

$$J^\alpha(z)J^\beta(w) = f_\gamma^{\alpha\beta} \frac{J^\gamma(w)}{z-w} + \frac{1}{2} \frac{kh^{\alpha\beta}}{(z-w)^2} \quad (3.2.14a)$$

$$\tilde{J}^\alpha(z)\tilde{J}^\beta(w) = f_\gamma^{\alpha\beta} \frac{\tilde{J}^\gamma(w)}{z-w} - \frac{1}{2} \frac{(k+2c_G)h^{\alpha\beta}}{(z-w)^2} \quad (3.2.14b)$$

$$J_{gh}^\alpha(z)J_{gh}^\beta(w) = f_\gamma^{\alpha\beta} \frac{J_{gh}^\gamma(w)}{z-w} + \frac{1}{2} \frac{2c_G h^{\alpha\beta}}{(z-w)^2}. \quad (3.2.14c)$$

From these last two equations we see that  $J_{tot}^\alpha(z)$  forms a Kac-Moody algebra at level  $k_{tot} = k - (k+2c_G) + 2c_G = 0$ . The energy tensor  $T(z)$  for the  $G/G$  model is a sum of Sugawara forms for  $J^\alpha(z)$  and  $\tilde{J}^\alpha(z)$ , and the usual energy tensor for the ghost sector

$$T(z) = T_k(z) + T_{-(k+2)}(z) + T_{gh}(z) \quad (3.2.15a)$$

$$= \frac{h_{\alpha\beta}}{k+c_G} : J^\alpha J^\beta : - \frac{h_{\alpha\beta}}{k+c_G} : \tilde{J}^\alpha \tilde{J}^\beta : + : \partial c^\alpha b_\alpha \quad (3.2.15b)$$

which is to be expected from equation (3.2.12) and equations (1.4.5) and (1.4.8). The results of Appendix A allow us to calculate  $T_k(z)$  explicitly in terms of the currents :

$$T_k(z) = \frac{1}{k+1} : (J^3 J^3 - UU + \frac{1}{2} J^+ J^- + \frac{1}{2} J^- J^+ - \frac{1}{2} j^+ j^- + \frac{1}{2} j^- j^+ + \frac{1}{2} j'^+ j'^- - \frac{1}{2} j'^- j'^+) : \quad (3.2.16a)$$

which is the energy tensor for Type B — the energy tensor for Type A is gained by interchanging  $j^+$  and  $j^-$ . In terms of the free fields this tensor is :

$$T_k(z) = -\beta\partial\gamma - [\psi_+]^\dagger \partial\psi_+ - [\psi'_+]^\dagger \partial\psi'_+ - \frac{1}{2} (\partial\phi)^2 - \frac{i}{\alpha_+} \partial^2\phi + \frac{1}{2} (\partial\theta)^2 + \frac{1}{\alpha_+} \partial^2\theta \quad (3.2.16b)$$

and there are similar forms for  $T_{-(k+2)}(z)$  which are found through the involution of equation (C.7) in Appendix C.

As in the manner of equation (1.4.6) there is a Virasoro algebra associated to the Kac-Moody algebra produced by  $J_{tot}^\alpha(z)$ . The central charge of this Virasoro algebra is given by the GKO construction (equation (1.4.9) ):

$$c_{tot} = \frac{k \dim G}{k + c_G} + \frac{(k + 2c_G) \dim G}{k + c_G} - 2 \dim G. \quad (3.2.17)$$

This is a first indication that the  $G/G$  WZNW model is a topological field theory. Note that when  $G$  is a Lie supergroup, the Virasoro central charge appearing in the operator product expansion of the Sugawara energy-momentum tensor with itself is :

$$c = \frac{k \text{sdim } G}{k + c_G} \quad (3.2.18)$$

where  $\text{sdim}$  is the difference between the number of the bosonic and fermionic generators of  $G$ . In particular it is important to note that the super dimension of  $SL(2|1)$  is zero.

For each field associated to a generator we introduce a ghost, which arises through the gauge fixing of that field under the functional integral, see equation (3.2.10). For a super group rather than an ordinary Lie group the adaptations are obvious and easy. The trace is replaced by the super trace which is defined in Appendix A. The gauge fields  $\bar{A}, A$  and ghost fields  $(b, c)$  take spin  $(1, 0)$ , with these ghosts have fermionic statistics when the corresponding gauge field is bosonic in nature, and bosonic statistics if the corresponding gauge field is fermionic in nature. Following the conventions presented in Chapter 2 and Appendix A the ghosts will be fermionic for  $\alpha = 1 \dots 4$  and bosonic for  $\alpha = 5, \dots, 8$ , with a contraction

$$\langle b^\alpha(z) c^\beta(w) \rangle = \frac{h^{\alpha\beta}}{z - w} \quad \text{where } c^\beta = h^{\alpha\beta} c_\alpha, b^\beta = h^{\alpha\beta} b_\alpha \quad (3.2.19)$$

and since the ghost fields take values in the adjoint of  $G$  it is possible to write them in terms of generators with  $b = \tau^\alpha b_\alpha$  and  $c = \tau^\alpha c_\alpha$ ; the metric is defined by

$$h^{\alpha\beta} = (-1)^{d(\alpha)} f_\lambda^{\alpha\gamma} f_\gamma^{\beta\lambda}. \quad (3.2.20)$$

Table 3a summarizes the ghost fields introduced in gauging the  $G/G$  model.

Table 3a

ghosts	spin	ghosts	spin
$(b_+, c^+)$	$(1, 0)$	$(\beta_{\frac{1}{2}}, \gamma^{\frac{1}{2}})$	$(1, 0)$
$(b_-, c^-)$	$(1, 0)$	$(\beta_{-\frac{1}{2}}, \gamma^{-\frac{1}{2}})$	$(1, 0)$
$(b_3, c^3)$	$(1, 0)$	$(\beta'_{\frac{1}{2}}, \gamma'^{\frac{1}{2}})$	$(1, 0)$
$(b_0, c^0)$	$(1, 0)$	$(\beta'_{-\frac{1}{2}}, \gamma'^{-\frac{1}{2}})$	$(1, 0)$

The above table shows that all the ghosts are spin  $(1, 0)$  although we have used Greek symbols to denote ghosts corresponding to gauge fixed fermionic fields.

The gauging of the diagonal vector subgroup  $H$  of the Kac-Moody group  $G_L \times G_R$  described above can be generalized. As first described in [34] the more general transformation is

$$g \mapsto \alpha(x_+, x_-)g\beta^{-1}(x_+, x_-) \quad (3.2.21)$$

where  $\alpha$  and  $\beta$  are elements of the subgroups  $H^+$  and  $H^-$  of  $G$  generated by the raising and lowering operators respectively. This transformation leaves the action

$$I(g, A, \bar{A}) = S(g) - \frac{k}{4\pi} \int d^2x \text{STr} \left\{ A\bar{\partial}g g^{-1} + g^{-1}\partial g\bar{A} + Ag\bar{A}g^{-1} - \bar{A}\mu - A\nu \right\}. \quad (3.2.22)$$

invariant when the gauge fields  $A_{\pm}$  transform as

$$A \mapsto \alpha A \alpha^{-1} - \bar{\partial} \alpha \alpha^{-1}, \quad \bar{A} \mapsto \beta \bar{A} \beta^{-1} + \partial \beta \beta^{-1}. \quad (3.2.23)$$

The constant matrices  $\mu$  and  $\nu$  are elements of  $H^-$  and  $H^+$  respectively, while  $A$  and  $\bar{A}$  take values in the adjoint representation of  $H^{\pm}$ . The gauge fields  $A$  and  $\bar{A}$

play the role of Lagrange multipliers which can be used to incorporate constraints on Kac-Moody currents through the constant matrices  $\mu$  and  $\nu$ . Although the  $\mu$  and  $\nu$  terms may look as if they are not gauge invariant terms, they are only defined in nilpotent subalgebras and as such they change by a total derivative under gauge transformations. The use of such matrices will become apparent in section 3.5 when we discuss Hamiltonian reduction.

Let us end this section by mentioning that there exists a hidden twisted  $N = 2$  superconformal algebra in this model, as predicted in [19]. The identification of this  $N = 2$  algebra is very important and its recovery and significance is presented in section 3.7.

### 3.3 The Twisted $G/G$ Model

Having established the details of the gauged  $SL(2|1)/SL(2|1)$  WZNW model we can follow references [35,36,37,38] and twist the energy momentum tensor of this theory. The twisting process is accomplished by adding a term  $-\partial J^3$  to the stress energy tensor (3.2.15). The  $-\partial J^3$  term is BRST exact and so does not alter the space of physical states that lie in the cohomology of  $Q$ . As we shall see in sections 3.4, 3.5, and 3.6, the twisting allows us to equate the  $SL(2|1)/SL(2|1)$  model with non-critical string theory in a natural fashion.

The twisting of the energy-momentum tensor (3.2.15) is explicitly given by

$$T_{HWS}^{twisted}(z) = T(z) - \partial J_{tot}^3(z) \quad (3.3.1)$$

where  $J_{tot}^3(z)$  is given in (3.2.12).

The twisting of  $T(z)$  given in equation (3.3.1) is that for the Highest Weight States (HWS) of the model being studied. When the Lowest Weight States (LWS) are under consideration the twist has the opposite sign i.e.,

$$T_{LWS}^{twisted} = T(z) + \partial J_{tot}^3(z) \quad (3.3.2)$$

The currents of the three sectors after the twist will have modified dimensions and thus contribute differently to the respective central charges. However, the total central charge after the twist remains zero. Indeed, because  $T(z)$  and  $J_{tot}^3(z)$  are BRST exact the twisted energy-momentum tensor  $T^{twisted}(z)$  is also BRST exact, and as we shall see in section 3.8 the twisting does not significantly effect the space of physical states.

Table 3b

ghost	spin	$j^3$ isospin	new spin
$(b_+, c^+)$	(1, 0)	(-1, 1)	(0, 1)
$(b_-, c^-)$	(1, 0)	(1, -1)	(2, -1)
$(b_3, c^3)$	(1, 0)	(0, 0)	(1, 0)
$(b_0, c^0)$	(1, 0)	(0, 0)	(1, 0)
$(\beta_{\frac{1}{2}}, \gamma^{\frac{1}{2}})$	(1, 0)	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$(\beta_{-\frac{1}{2}}, \gamma^{-\frac{1}{2}})$	(1, 0)	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2})$
$(\beta'_{\frac{1}{2}}, \gamma'^{\frac{1}{2}})$	(1, 0)	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$(\beta'_{-\frac{1}{2}}, \gamma'^{-\frac{1}{2}})$	(1, 0)	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2})$

The comparison with the  $N = 2$  non-critical string which we shall discuss in the following sections is based on the ghost content of the two theories. In the twisted  $SL(2|1)/SL(2|1)$  theory, the spins of the ghosts obtained in Table 3a are altered. If  $j^3$  is the eigenvalue of the  $J^3$  component of a ghost of spin  $\Delta$  then the twisting modifies the spin thus [43] :

$$\Delta \mapsto \Delta + j^3. \tag{3.3.3}$$

If we examine the ghosts of the  $G/G$  model as given in Table 3a, we can easily establish the effect of the twist and the spins after the twist are shown in table 3b. This table is the basis of our comparison in section 3.6, although we see that



we have a residue set of ghosts  $(\beta_{\frac{1}{2}}, \gamma^{\frac{1}{2}}), (\beta'_{\frac{1}{2}}, \gamma'^{\frac{1}{2}})$  which need to be accounted for. This suggests that the gauged, twisted WZNW model may be equal to the  $N = 2$  string when the former also tensors an additional ghost sector. This statement will now be made more exact.

### 3.4 The Non-critical $N=2$ Superstring

The initial interest in the study of strings with  $N \geq 2$  supersymmetry was motivated by the discovery that the supersymmetric extension of the bosonic string — the  $N = 1$  superstring — had a critical dimension of  $D = 10$ . Naturally, the fact that a degree of supersymmetry could lower the critical dimension from  $D = 25$  to  $D = 10$  introduced the prospect that the critical dimension could be lowered yet further by adding more degrees of supersymmetry.

Although  $N = 2$  strings naturally live in four dimensions (two complex dimensions), with the signatures of spacetime either  $(2, 2)$  or  $(4, 0)$  [39,40], it seemed that critical  $N = 2$  strings — although mathematically interesting — were of little relevance to nature. Yet they have provided a greater insight into string theory because of their relative simplicity.

Strings, it turns out, can be consistently defined in any dimension, as long as the conformal factor of the worldsheet metric is treated appropriately. As explained in section 1.3 the introduction of this factor does not decouple when the theory is defined away from the criticality, and it becomes a new dynamic degree of freedom. The spacetime dimension is thus described by the central charge of the matter system, which is coupled to 2d gravity or in the case of superstrings, 2d supergravity.

The non-critical  $N = 2$  strings were the first non-trivial example of theories with extended worldsheet symmetry, and as a consequence they have been the subject of some investigation [41,42,43]. Above their critical dimension of  $D = 2$  (where  $D$  is complex), they could potentially lead to new string theories which have

yet to be discovered. Also, as was first pointed out in [40], there is no intermediate regime where the critical exponent becomes complex for the  $N = 2$  string. This is in contrast to regimes of  $1 < D < 25$  and  $1 < D < 9$  for the bosonic and  $N = 1$  string respectively, which otherwise collapses at  $D = 1$ . It is therefore believable to continue the theory smoothly from the  $D < 1$  region, where it is exactly solvable, to the physically interesting  $D > 1$  region.

In this section we examine the  $N = 2$  superstring using  $N = 2$  chiral superfields in an  $N = 2$  superspace formalism [42], and via this approach derive the  $N = 2$  super Liouville equations which we shall require later on in this chapter. We then explore how the  $N = 2$  super Liouville action emerges as a by-product of the path integration over a matter action for the  $N = 2$  string.

To build up an  $N = 2$  string theory, one couples  $N = 2$  supergravity in two dimensions to some  $N = 2$  matter, as was first considered by Brink and Schwarz [44]. The  $N = 2$  supergravity multiplet consists of a zweibein  $e_\alpha^a$ , a complex gravitino  $\chi_\alpha$  and an  $SO(2)$  gauge field  $A_\alpha$ . The action of the superconformally invariant fermionic string (also referred to as the charged spinning string) is

$$S_{matter} = \frac{1}{2} \int d^2 z \sqrt{g} \left\{ g^{\mu\nu} \partial_\mu X \partial_\nu X^* - \frac{1}{2} i \bar{\lambda} \gamma^\alpha \partial_\alpha \lambda + A_\alpha \bar{\lambda} \gamma^\alpha \lambda \right. \\ \left. + (\partial_\alpha X^* + \lambda \chi_\alpha) \bar{\chi}_\beta \gamma^\alpha \gamma^\beta \lambda + (\partial_\alpha X + \bar{\chi}_\alpha \lambda) \bar{\lambda} \gamma^\beta \gamma^\alpha \chi_\beta \right\} \quad (3.4.1)$$

In the above,  $X$  and  $\lambda$  are complex combinations of two bosonic fields and two fermionic fields respectively, and  $X$  carries a suppressed superscript for the number of dimensions of the space-time — which is two (two complex dimensions) —  $X^a = X_1^a + i X_2^a$ , and  $\lambda^j = \lambda_1^j + i \lambda_2^j$ . All other Greek superscripts and subscripts take values 1, 2, while arabic indices indicate the dimensions of the target space, which is this model also run over 1, 2. An  $SO(2)$  symmetry exists that rotates  $\lambda^1$  and  $\lambda^2$  into one-another, and the gauge field  $A_\alpha$  is present to ensure that this symmetry is local, and separate from the symmetries of the other fields.

In addition to these there are two more gauge fields. A graviton  $e_\mu$  ( $e_\mu^a e_\mu^b = g^{ab}$  and with  $a = 1, 2$ ) with conformal weight 2 which serves as a zweibein for the

metric, and two gravitinos  $\chi_\mu$  of conformal weight  $\frac{3}{2}$ , both of which are required if local supersymmetry is to be maintained. As usual we have defined the two dimensional gamma matrices by :

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (3.4.2)$$

which admits the identity  $\gamma_\beta \gamma_\alpha \gamma^\beta = 0$ , and as expected  $\bar{\lambda} = \lambda^\dagger \gamma^0$ .

All supergravity fields can be locally gauged away via the gauge symmetries of the theory : the zweibein  $e_\alpha^a$  is removed by general coordinate invariance, local Lorentz invariance and local Weyl transformations, the gravitinos  $\chi_\alpha$  by the  $N = 2$  supersymmetry and super Weyl transformations, and the  $SO(2)$  gauge field  $A_\alpha$  by vector and chiral  $U(1)$  gauge symmetries on the worldsheet [45].

A common choice of gauge fixing is the superconformal gauge [41,46,47],

$$g_{\alpha\beta} = \delta_{\alpha\beta} \exp \phi_1 = e_\alpha^a e_\beta^a \quad (3.4.3)$$

$$\chi^\alpha = \frac{1}{2} \gamma^\alpha \psi$$

$$A_\mu = \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu \phi_2$$

$$\text{with } \epsilon^{10} = 1 \quad \text{and } \psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix},$$

where the conformal field  $\phi_1$ , the Dirac field  $\psi$ , and  $\phi_2$  the axial component of the gauge field have been introduced.

To proceed let us start with a manifestly  $N = 2$  superspace formalism. We introduce on this  $N = 2$  superspace the coordinates  $(z, \bar{z}; \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$  with  $\theta, \bar{\theta}$  denoting holomorphic and anti-holomorphic Grassmann odd quantities respectively [48,49,42]. In this space there are four super-derivatives defined through :

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\mp \partial \quad \{D_+, D_-\} = 2\partial \quad (3.4.4a)$$

$$\bar{D}_\pm = \frac{\partial}{\partial \bar{\theta}^\pm} + \bar{\theta}^\mp \bar{\partial} \quad \{\bar{D}_+, \bar{D}_-\} = 2\bar{\partial} \quad (3.4.4b)$$

and chiral coordinates of the form

$$z^\pm = z \pm \theta^+ \theta^- \quad \bar{z}^\pm = \bar{z} \pm \bar{\theta}^+ \bar{\theta}^-. \quad (3.4.5)$$

On this space we further introduce  $D$  superfields that are defined as a function of these chiral coordinates, and as such the calculus of many variables is used in deriving the action and equations of motion for any  $N = 2$  supersymmetric theory. Explicitly these  $\mu = 1 \dots D$  superfields are :

$$\begin{aligned} \Phi^{\mu+}(z^+, \bar{z}^+) &= \frac{1}{2}\phi^{\mu+}(z^+, \bar{z}^+) + \theta^+\psi^{\mu+}(z^+, \bar{z}^+) \\ &\quad + \bar{\theta}^+\bar{\psi}^{\mu+}(z^+, \bar{z}^+) + \theta^+\bar{\theta}^+F^{\mu+}(z^+, \bar{z}^+) \end{aligned} \quad (3.4.6a)$$

$$\begin{aligned} \Phi^{\mu-}(z^-, \bar{z}^-) &= \frac{1}{2}\phi^{\mu-}(z^-, \bar{z}^-) + \theta^-\psi^{\mu-}(z^-, \bar{z}^-) \\ &\quad + \bar{\theta}^-\bar{\psi}^{\mu-}(z^-, \bar{z}^-) + \theta^-\bar{\theta}^-F^{\mu-}(z^-, \bar{z}^-) \end{aligned} \quad (3.4.6b)$$

where we have complex scalar fields  $\phi^\pm$ , complex spinor fields  $\psi^\pm, \bar{\psi}^\pm$ , and auxiliary fields  $F^\pm$ . These superfields are chiral in the sense that :

$$D_-\Phi^{\mu+} = \bar{D}_-\Phi^{\mu+} = 0 \quad D_+\Phi^{\mu-} = \bar{D}_+\Phi^{\mu-} = 0. \quad (3.4.7)$$

As discussed by Antoniadis et al [42] the gauge renormalizable  $\sigma$  model coupled to  $N = 2$  supergravity can be described by a Kähler potential  $K(\Phi^{\mu+}, \Phi^{\mu-})$  which is a real function of the coordinate superfields, and by a superpotential  $W(\Phi^{\mu+})$  and dilaton field  $\Phi(\Phi^{\mu+})$  both of which are analytic functions.

The action for the  $\sigma$  model can be obtained from the  $N = 1$  supergravity action by dimensional reduction from four dimensions down to two, and this action takes the form :

$$S = \frac{1}{4\pi} \int d^2\xi \left\{ d^2\theta d^2\bar{\theta} K + \int d^2\theta W E + \int d^2\theta \Phi R + h.c. \right\} \quad (3.4.8)$$

with  $E$  being the chiral super-determinant and  $R$  the chiral supersymmetric generalization of  $\sqrt{g}R^{(2)}$  — the scalar curvature in two dimensions. Note that the only degree of freedom is the Kähler potential  $K$  (the super-potential  $W$  and the dilaton  $\Phi$  are analytical functions, and in contrast with the bosonic and  $N = 1$  cases, where they correspond to tachyon and dilaton field backgrounds, they do not correspond in  $N = 2$  to physical degrees of freedom).

We can chose the superconformal gauge, which in this prescription is

$$\begin{aligned} E &= \exp \Sigma \quad , \quad R = \bar{D}^2 \log \Sigma \quad , \quad \bar{D}R = 0 \\ \Sigma &= \sigma + i\theta\eta + \frac{1}{2}i\theta\bar{\theta}H \quad , \quad \bar{D}\Sigma = 0 \end{aligned} \quad (3.4.9)$$

i.e.,  $\Sigma$  and  $R$  are chiral superfields. The conformal factor  $\sigma$  is complex ; its real part is the Liouville model with its imaginary part related to the axial  $SO(2)$  gauge field component. In addition,  $\eta$  is the gravitino trace and  $H$  the auxiliary field which can be set to zero when the theory is super Weyl invariant.

In this gauge the equations of motion for the auxiliary fields  $H$  and  $F^\mu$  give the constraints [42]

$$W^* = \Phi^{\mu+} K_{\mu\bar{\nu}}^{-1} W^{\bar{\nu}*} \quad (3.4.10)$$

where the indices  $\mu, \bar{\nu}$  on  $K^{-1}$  and  $W^*$  denote derivation with respect to the superfields  $\Phi^{\mu+}$  and  $\Phi^{\mu-}$ .

The  $N = 2$  super Liouville action is classically determined by taking a single superfield  $\Phi^+$  which is called the Liouville mode with the dilaton being able to be taken as linear in the Liouville mode thanks to analytic field redefinition

$$\Phi = Q\Phi^+ \quad , \quad Q = \text{background charge.} \quad (3.4.11)$$

The constraint (3.4.10) can then be solved by  $K = \frac{1}{2}\Phi^-\Phi^+$  and  $W = \mu \exp\{\frac{1}{2Q}\Phi^+\}$  so that the  $N = 2$  supersymmetric generalization of the Liouville action takes the form

$$\begin{aligned} S_{super \ Liouville} &= \frac{1}{4\pi} \int d^2\xi \left\{ \frac{\epsilon}{4} \int d^2\theta d^2\bar{\theta} \Phi^-\Phi^+ \right. \\ &\quad \left. + \int d^2\theta Q\Phi^+ R + \mu \int d^2\theta \exp\left\{\frac{\epsilon}{2Q}\Phi^+\right\} E + h.c. \right\} \end{aligned} \quad (3.4.12)$$

where the last term before the hermitian conjugate is the cosmological constant term.

The Virasoro central charge of the above theory is given by  $C_{super \ Liouville} = 3(1 + 2Q^2)$ . For future reference (in section 3.5) we will now derive the super-Liouville equations of motion starting with a slightly less general form of the

action (3.4.12), namely,

$$S_{super\ Liouville} = \int d^2z \left\{ d^2\theta^+ d^2\theta^- \frac{1}{4} \Phi^+ \Phi^- - d^2\theta^+ \exp \frac{1}{2} \Phi^+ - d^2\theta^- \exp \frac{1}{2} \Phi^- \right\}. \quad (3.4.13)$$

Varying this action and carefully using the correct calculus allows us to calculate the superfield equations of motion :

$$D_+ \bar{D}_+ \Phi^+ + \exp \frac{1}{2} \Phi^- = 0 \quad (3.4.14a)$$

$$D_- \bar{D}_- \Phi^- + \exp \frac{1}{2} \Phi^+ = 0. \quad (3.4.14b)$$

By integrating out the fermionic coordinates in equation (3.4.13) and substituting for the auxiliary fields  $F^+, F^-$  from their equations of motion, we are able to deduce the super Liouville action in component form :

$$S_{superLiouville} = \int d^2z \left\{ \frac{1}{4} (\partial\phi^+ \bar{\partial}\phi^- + \partial\phi^- \bar{\partial}\phi^+) - \psi^+ \bar{\partial}\psi^- - \bar{\psi}^+ \partial\bar{\psi}^- \right. \quad (3.4.15) \\ \left. + \psi^+ \bar{\psi}^+ \exp\left(\frac{1}{2}\phi^+\right) + \psi^- \bar{\psi}^- \exp\left(\frac{1}{2}\phi^-\right) + 2 \exp\left(\frac{1}{2}(\phi^+ + \phi^-)\right) \right\}.$$

it is now easy to derive the equations of motion which we shall reference later on :

$$\partial\bar{\partial}\phi^- = \frac{1}{2} \psi^+ \bar{\psi}^+ \exp\left(\frac{1}{2}\phi^+\right) + \exp\left(\frac{1}{2}(\phi^+ + \phi^-)\right) \quad (3.4.16a)$$

$$\partial\bar{\partial}\phi^+ = \frac{1}{2} \psi^- \bar{\psi}^- \exp\left(\frac{1}{2}\phi^-\right) + \exp\left(\frac{1}{2}(\phi^+ + \phi^-)\right) \quad (3.4.16b)$$

$$\bar{\partial}\psi^- = \bar{\psi}^+ \exp\left(\frac{1}{2}\phi^+\right) \quad (3.4.16c)$$

$$\partial\bar{\psi}^- = -\psi^+ \exp\left(\frac{1}{2}\phi^+\right) \quad (3.4.16d)$$

$$\bar{\partial}\psi^+ = \bar{\psi}^- \exp\left(\frac{1}{2}\phi^-\right) \quad (3.4.16e)$$

$$\partial\bar{\psi}^+ = -\psi^- \exp\left(\frac{1}{2}\phi^-\right). \quad (3.4.16f)$$

The quantization of the  $N = 1$  superstring by the path integral method was originally tackled by Polyakov [50] with the quantization of  $N = 2$  superstring by the path integral approach undertaken by Distler, Hlousek and Kawai,

and separately by Antoniadis, Bachas and Kounnas [42,41]. In the path integral formulation of the  $2d$  quantum supergravity the partition function is given by

$$Z = \int \frac{\mathcal{D}g\mathcal{D}\chi\mathcal{D}A\mathcal{D}X\mathcal{D}\lambda}{\text{Vol}[\text{Diff}]\text{Vol}[N=2\text{ SUSY}]\text{Vol}[SO(2)]} \exp\{-S_{\text{matter}}\}. \quad (3.4.17)$$

We adopt here the standard definitions for the measures appearing in (3.4.17) (see for instance [51] and references there in ), and where the action  $S_{\text{matter}}$  (equation (3.4.1) ) is invariant under reparameterizations of the world sheet, there also exists an  $N = 2$  supersymmetry in the action in addition to a  $SO(2)$  gauge symmetry between the fermion doublets.

Unfortunately the measures  $\mathcal{D}X\mathcal{D}\lambda\mathcal{D}g$  are not invariant under Weyl rescalings, superconformal symmetry and chiral  $SO(2)$  gauge transformations and so these have to be treated in turn. Let us first discuss the domain of integration for the space of metrics  $g_{\alpha\beta}$ . In evaluating the path integral over the metrics in (3.4.17) one writes the integral as an integral along a gauge orbit (parameterized by diffeomorphisms  $g, \bar{g}$ ) multiplying an integral over a gauge slice (parameterized by degrees of freedom left after using reparameterization invariance to fix the gauge). A convenient choice is the superconformal gauge given earlier in equation (3.4.3).

This gauge choice however cannot be made globally on surfaces with genus  $h > 0$ . On such surfaces it is possible to build infinitesimally about the identity a set of  $3h - 3$  complex parameters (the moduli  $\tau_i$ ) which are global obstructions to the choice of coordinates  $g_{\alpha\beta} = \delta_{\alpha\beta} \exp \phi_1$ . For Each 2d surface with  $h > 1$  there exists reparameterizations which would change the value of the moduli without changing the shape of the surface.

The change of path integration variables that implements the splitting between the gauge orbits and the gauge slice is given by

$$\int \mathcal{D}g \mapsto \int \mathcal{D}g\mathcal{D}\bar{g}\mathcal{D}\phi_1\mathcal{D}\phi_2\mathcal{D}\psi \left\{ \prod_{i=1}^{3h-3} d^2\tau_i \right\}, \quad (3.4.18)$$

and the associated Jacobian can be represented in terms of Fadeev-Popov ghosts

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}\xi \mathcal{D}\bar{\xi} \exp\{-S_{ghost}\} \equiv \int \mathcal{D}[ghosts] \exp\{-S_{ghosts}\} \quad (3.4.19)$$

where

$$S_{ghosts} = \int d^z \{ \eta^i \bar{\partial} \xi_i + \bar{\eta}^i \partial \bar{\xi}_i \} \quad (3.4.20)$$

with  $i = 1 \dots 4$ . The ghost fields that appear in (3.4.20) come in pairs, of spin  $(j, 1 - j)$ , each pair corresponding to one of the gauged fixed fields of equation (3.4.1) and carrying conformal weight  $j$ . The ghosts are fermionic when representing the graviton and  $SO(2)$  fields, and bosonic when representing the gravitinos. A summary of the fields and their ghosts appears in Table 3c.

Gauged Field	Ghost	Spin	Statistics	Origin
$e_\alpha^a$	$(\eta^1, \xi_1)$	$(2, -1)$	fermionic	reparam.
$\chi^1$	$(\eta^2, \xi_2)$	$(\frac{3}{2}, -\frac{1}{2})$	bosonic	$N = 2$
$\chi^2$	$(\eta^3, \xi_3)$	$(\frac{3}{2}, -\frac{1}{2})$	bosonic	$N = 2$
$A_\alpha$	$(\eta^4, \xi_4)$	$(1, 0)$	fermionic	$SO(2)$

The partition function is therefore of the form

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}\psi \left\{ \prod_{i=1}^{3h-3} d^2 \tau_i \right\} \mathcal{D}[ghosts] \mathcal{D}X \mathcal{D}\lambda \exp\{-S_{matter} - S_{ghosts}\} \quad (3.4.21)$$

Choosing the gauge slice of equation (3.4.3) one gets (with  $\hat{g}_{\alpha\beta} = \delta_{\alpha\beta}$ )

$$\begin{aligned} & \mathcal{D}_{e^{\phi_1 \hat{g}}} \phi_1 \mathcal{D}_{e^{\phi_1 \hat{g}}} \phi_2 \mathcal{D}_{e^{\phi_1 \hat{g}}} \psi \mathcal{D}_{e^{\phi_1 \hat{g}}} X \mathcal{D}_{e^{\phi_1 \hat{g}}} \lambda \mathcal{D}_{e^{\phi_1 \hat{g}}} [ghosts] \\ & = J(\phi_1, \phi_2, \psi, \hat{g}) \mathcal{D}_{\hat{g}} \phi_1 \mathcal{D}_{\hat{g}} \phi_2 \mathcal{D}_{\hat{g}} \psi \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} \lambda \mathcal{D}_{\hat{g}} [ghosts] \end{aligned} \quad (3.4.22)$$

where the Jacobian is assumed by analogy to the bosonic and  $N = 1$  cases to take the form of the  $N = 2$  super Liouville action  $S_{super Liouville}$  given earlier in equation (3.4.15).



Thus we have explored the connection between the  $N = 2$  supersymmetric string and the Liouville model, and have established that in quantizing the former, we not only extract the appropriate super Liouville action, but in addition we gain the ghost fields as given in Table 3c.

The next step is to show how the  $N = 2$  super Liouville theory can be obtained by gauge fixing the gauge invariant WZNW model based on  $SL(2|1)$  in order to compare the  $N = 2$  non-critical string theory with the gauged twisted  $SL(2|1)/SL(2|1)$  WZNW theory.

### 3.5 Hamiltonian Reduction

The WZNW model is a general template for a range of renormalizable, integrable, non-trivial, conformally invariant field theories. Let us first describe how the  $N = 2$  super Liouville equations can be derived by Hamiltonian reduction of the  $SL(2|1)$  WZNW model.

The Hamiltonian reduction [19,52,53,54] in essence is quite simple. Some of the currents in the WZNW model are constrained, and the rest are gauged away to zero. The model so constrained reduces to a Toda field theory, or in our case super Liouville field theory. The process of Hamiltonian reduction can be viewed from two perspectives. The direct recovery of the correct model by constraining the currents, as mentioned, or the effect on the space of physical states of the theory when the reduction is employed.

The imposing of constraints in order to recover the Liouville equations has been covered by many authors. The following work is heavily influenced by the paper by Zhang [55]. Let us recall that the currents of the WZNW model are defined by :

$$\mathcal{J} = g^{-1}\partial g \quad \bar{\mathcal{J}} = \bar{\partial}g g^{-1} \quad (3.5.1a)$$

$$\bar{\partial}\mathcal{J} = 0 \quad \partial\bar{\mathcal{J}} = 0 \quad (3.5.1b)$$

If we make a Gauss decomposition of the group element as before,  $g = g_< g_0 g_>$  then we may write the currents in the form :

$$\mathcal{J} = g_>^{-1} g_0^{-1} (g_<^{-1} \partial g_<) g_0 g_> + g_>^{-1} (g_0^{-1} \partial g_0) g_> + g_>^{-1} \partial g_> \quad (3.5.2a)$$

$$\mathcal{J} = \mathcal{J}_< + \mathcal{J}_0 + \mathcal{J}_>$$

$$\bar{\mathcal{J}} = \bar{\partial} g_< g_<^{-1} + g_< (\bar{\partial} g_0) g_0^{-1} g_<^{-1} + g_< g_0 (\bar{\partial} g_> g_>^{-1}) g_0^{-1} g_<^{-1} \quad (3.5.2b)$$

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}_< + \bar{\mathcal{J}}_0 + \bar{\mathcal{J}}_>$$

The method of Hamiltonian reduction is to constrain currents taking values in a nilpotent subalgebra. An inspection of equation (3.5.2a) indicates that we are at liberty to constrain the current  $\mathcal{J}_<$  by constraining  $J_< = g_<^{-1} \partial g_<$ , since the element  $g_<$  does not appear in the currents  $\mathcal{J}_0$  or  $\mathcal{J}_>$ . Similarly we can constrain  $\bar{\mathcal{J}}_>$  by constraining  $\bar{J}_> = \bar{\partial} g_> g_>^{-1}$ .

(As a point of interest  $J_<$  is all that is needed to completely calculate the currents  $J^-, j^-, j'^-$ . Thus we see that constraining  $J_< = g_<^{-1} \partial g_<$  is equivalent to placing constraints on the currents  $J^-, j^-, j'^- \in \mathcal{J}_<$  as described in [53,54]).

As our main aim is to reduce the WZNW model to the Liouville model, it is most useful to substitute the Gauss decomposition into equation (3.5.1a), and recover the following form for the field equations of the WZNW model :

$$\partial \bar{J}_> + \bar{\partial} (g_0^{-1} J_< g_0 + J_0) + [g_0^{-1} J_< g_0 + J_0, \bar{J}_>] = 0 \quad (3.5.3a)$$

$$\bar{\partial} J_< + \partial (g_0 \bar{J}_> g_0^{-1} + \bar{J}_0) - [g_0 \bar{J}_> g_0^{-1} + \bar{J}_0, J_<] = 0 \quad (3.5.3b)$$

where on this occasion we have defined  $J_0 = g_0^{-1} \partial g_0$  and  $\bar{J}_0 = \bar{\partial} g_0 g_0^{-1}$ .

It is at this point that we depart from the stages laid down in [55]. In [55] the Hamiltonian reduction is performed by making the following choice of constraints

$$J_< = g_0 \mu g_0^{-1} \quad \bar{J}_> = g_0^{-1} \nu g_0 \quad (3.5.4)$$

where  $\mu$  and  $\nu$  are elements taking values in nilpotent subalgebras of negative and positive roots respectively. However, the diligent reader who takes the trouble to

follow the stages in [55] will notice many points that detract from the sincerity of the argument. Instead it is proposed that constraints should be applied of the form :

$$\mu = J_0 + g_0^{-1} J_{<} g_0 \quad \nu = \bar{J}_0 + g_0 \bar{J}_{>} g_0^{-1} \quad (3.5.5)$$

which does not affect the process of Hamiltonian reduction since the fields in the Cartan sub-algebra remain separate from the nilpotent subalgebras in the following calculations. With such a choice of constraints equation (3.5.3) becomes :

$$\partial \bar{J}_{>} + \bar{\partial} \mu = [\bar{J}_{>}, \mu] \quad (3.5.6a)$$

$$-\bar{\partial} J_{<} - \partial \nu = [J_{<}, \nu] \quad (3.5.6b)$$

We can deduce the exact form that  $\mu$  and  $\nu$  should take to recover the super Liouville field equations by examining the Hamiltonian reduction of the  $SL(2, \mathfrak{R})$  WZNW model.

In the traditional approach to Hamiltonian reduction, as applied to the gauged, twisted,  $sl(2, \mathfrak{R})/sl(2, \mathfrak{R})$  WZNW model, constraints are imposed on the currents in the Borel subalgebras. The twisting procedure of section 3.3 alters the conformal dimensions of the currents relative to those of the energy tensor, so that it is now possible to constrain the  $J^-$  and  $\bar{J}^+$  currents to scalars, and we choose the normalization  $J^- = -\bar{J}^+ = 1$ . These form a set of first class constraints which are desirable for a BRST formalism.

In expanding the reduction to the gauged, twisted,  $SL(2|1)/SL(2|1)$  WZNW model it is possible to expand the Hamiltonian reduction to account for the fermionic currents which are also present in the Borel subalgebras. Due to the twist the fermionic currents carry conformal dimension  $\frac{1}{2}$ .

In the standard approach to Hamiltonian reduction a choice of  $j^- = j'^- = \bar{j}^+ = \bar{j}'^+ = 0$  would be made. These are a set of second class constraints, and in analyzing the model with a mixture of first class (the bosonic sector) and second class (the fermionic sector) constraints we need to be careful. It is possible to proceed by decomposing the fermionic currents into positive and negative frequency

parts, and impose the constraints  $j^- = j'^- = \bar{j}^+ = \bar{j}'^+ = 0$  on the positive frequency components. In doing so due consideration has to be given to the normal ordering of the zero frequency components.

Instead of pursuing this standard approach we follow [21,52,53,54,55,56] and expand our model to allow us to impose a set of maximal first class constraints on all the Borel valued currents. There are a number of appealing features to this approach :

- 1) It allows the full use and convenience of the BRST formulation.
- 1) First class constraints are manifestations of a gauge symmetry in a gauge theory.
- 3) Because of item 2, the resulting model is amenable to the path integral approach.
- 4) Because of items 2 and 3 the model admits a GKO construction.
- 5) Because of all the above, it is easy to recover the buried  $N = 2$  supersymmetry.

By expanding our model so as to make all constraints first class, we have present a gauge symmetry which we can use to fix gauge degrees of freedom to regain our original set of second class constraints on our fermionic currents [56], so illustrating that the traditional approach is buried in our alternative approach, while admitting the elegance and convenience of items 1 to 5 above.

The only remaining question is whether this expanded model contributes any undesirable physical features to our gauge, twisted, WZNW model. This is answered later in the section when we present the Kugo-Ojima quartets, and in section 3.8 where we show that only a vacuum state is added to the space of physical states. It has also been proven explicitly in [21] that the two separate approaches to Hamiltonian reduction are equivalent. Thus the traditional approach and the alternative method presented here and in [21,52,53,54,55] give identical results.

To understand the following points let us confine ourselves, for the time being, to the Type B interpretation of the algebra. The operator product expansions are

altered when we constrain  $J^- = 1$  and  $\bar{J}^+ = -1$ . In particular we have the following :

$$j^-(z)j'^-(w) \sim \frac{1}{z-w} \quad \bar{j}^+(z)\bar{j}'^+(w) \sim \frac{-1}{z-w} \quad (3.5.7)$$

We can form a maximal set of first class constraints by putting :

$$j^-(z) = \chi(z) \quad j'^-(z) = \chi'(z) \quad (3.5.8a)$$

$$\bar{j}^+(z) = \bar{\chi}(z) \quad \bar{j}'^+(z) = \bar{\chi}'(z) \quad (3.5.8b)$$

where we have introduced auxiliary fields  $\chi, \chi', \bar{\chi}, \bar{\chi}'$  which obey :

$$\partial\chi = \partial\chi' = 0 \quad ; \quad \chi(z), \chi'(w) \sim \frac{1}{z-w} \quad (3.5.9)$$

$$\bar{\partial}\bar{\chi} = \bar{\partial}\bar{\chi}' = 0 \quad ; \quad \bar{\chi}(z), \bar{\chi}'(w) \sim \frac{-1}{z-w} .$$

The introduction of the auxiliary fields  $\chi, \chi', \bar{\chi}, \bar{\chi}'$  makes our set of constraints first class. We are thus constraining the space

$$\mathcal{H}_{sl(2|1)} \otimes \mathcal{H}_{(\chi, \chi')} \otimes \mathcal{H}_{(\bar{\chi}, \bar{\chi}')} \quad (3.5.10)$$

with  $\mathcal{H}_{(x,y)}$  being the irreducible Fock space of fields  $(x, y)$ .

To continue with the matter at hand, we wish to implement the constraints of equation (3.5.8). If we constrain  $J^-, j^-, j'^-$  and  $\bar{J}^+, \bar{j}^+, \bar{j}'^+$  as specified in equations (3.5.7) and (3.5.8) then we need to set :

$$J_{<} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \chi' \\ \chi & 0 & 0 \end{pmatrix} \quad \bar{J}_{>} = \begin{pmatrix} 0 & -1 & \bar{\chi} \\ 0 & 0 & 0 \\ 0 & \bar{\chi}' & 0 \end{pmatrix} \quad (3.5.11)$$

It is now a simple calculation to find  $\mu$  and  $\nu$  :

$$\mu = \begin{pmatrix} -\frac{1}{2}(\partial\phi + \partial\theta) & 0 & 0 \\ \exp \theta & -\frac{1}{2}(\partial\phi - \partial\theta) & \chi' \exp \frac{1}{2}\{\theta + \phi\} \\ \chi \exp \frac{1}{2}\{\theta - \phi\} & 0 & -\partial\phi \end{pmatrix} \quad (3.5.12a)$$

$$\nu = \begin{pmatrix} \frac{1}{2}(\bar{\partial}\phi + \bar{\partial}\theta) & -\exp \theta & \bar{\chi} \exp \frac{1}{2}\{\theta - \phi\} \\ 0 & \frac{1}{2}(\bar{\partial}\phi - \bar{\partial}\theta) & 0 \\ 0 & \bar{\chi}' \exp \frac{1}{2}\{\theta + \phi\} & \bar{\partial}\phi \end{pmatrix} \quad (3.5.12b)$$

We can now make the necessary substitutions into equation (3.5.6) in order to recover the field equations of the  $N = 2$  super-Liouville model. These field follow from an action

$$S_{super-Liouville} = \int d^2z \left\{ \frac{1}{4} \partial\theta\bar{\partial}\theta - \frac{1}{4} \partial\phi\bar{\partial}\phi + \exp\theta + \chi\bar{\chi} \exp\frac{1}{2}\{\theta - \phi\} - \chi'\bar{\chi}' \exp\frac{1}{2}\{\theta + \phi\} + \chi'\bar{\partial}\chi - \bar{\chi}'\partial\bar{\chi} \right\}. \quad (3.5.13)$$

If the following associations are used

$$\begin{aligned} \theta &= \frac{1}{2}(\phi^+ + \phi^-) & \phi &= \frac{1}{2}(\phi^+ - \phi^-) \\ \psi^+ &= -\sqrt{2}\chi' & \bar{\psi}^+ &= \sqrt{2}\bar{\chi}' \\ \psi^- &= \sqrt{2}\chi & \bar{\psi}^- &= \sqrt{2}\bar{\chi} \end{aligned} \quad (3.5.14)$$

then we recover the super Liouville action of equation (3.4.14), whose stress-energy tensor is *not* traceless.

For the Type A reduction reference [19] is followed where a single spin  $\frac{3}{2}$  auxiliary field  $\lambda$  is introduced and the constraints take the form :

$$J_{<} = \begin{pmatrix} 0 & 1 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{J}_{>} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \bar{\lambda} & 0 & 0 \end{pmatrix} \quad (3.5.15)$$

with the same reduction technique of this section applied.

The imposing of the constraints mentioned above can be best encapsulated in the BRST formalism. A nilpotent BRST charge  $Q_{BRST}$  is introduced which is defined through the currents of the model. The constraining of these currents can be easily included in this BRST charge in the following manner :

$$Q_{BRST} = \frac{1}{2\pi i} \oint d^2z \left\{ (J^- - 1)\rho + (j^- - \chi)\varrho + (j'^- - \chi')\varrho' \right\} \quad (3.5.16)$$

where yet another set of ghosts have been introduced to account for the constraints. These ghosts  $(\sigma, \rho), (\varsigma, \varrho), (\varsigma', \varrho')$  having conformal dimensions  $(1, 0)$  and  $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$  respectively.

The amazing result of Hamiltonian reduction is that the introduction of these ghosts does not contribute to the cohomology of physical states for the WZNW model provided that the currents are expressed in a free field parameterization, explicitly we have the result

$$\mathbf{H}(\mathcal{H}_{sl(2|1)}^k \otimes \mathcal{H}_{(\rho^i, \sigma_i)}, Q_{BRST}) \simeq \mathcal{H}_{super\ Virasoro}^k \quad (3.5.17)$$

where  $\mathbf{H}(\bullet, Q_{BRST})$  is the cohomology class defined with respect to a BRST charge  $Q_{BRST}$ .

Such a free field parameterization was introduced in section 2.4. This parameterization was in terms of fields  $(\beta, \gamma), (\psi_+, [\psi_+]^\dagger), (\psi'_+, [\psi'_+]^\dagger)$  which now pair with the ghosts  $(\sigma, \rho), (\varsigma, \varrho), (\varsigma', \varrho')$  into so called Kugo-Ojima quartets. The contributions from the fields forming each quartet then cancel each other out. In short, provided a free field realization of the currents can be found, then the constraining of currents via the Hamiltonian reduction does not introduce any additional ghost contributions into the model. For the full proof of this the reader is referred to the original work [52].

In section 3.8 it will be shown that this remarkable property means that the enlarging of our space does not affect the space of physical states.

The constraints (3.5.11) can be incorporated in the gauge invariant action

$$\begin{aligned} I(g, A, \bar{A}, \chi, \chi') &= I^{twist}(g, A, \bar{A}) + S(\chi, \chi') \\ &= S_{WZNW}^{twist} - \frac{k}{4\pi} \int d^2z \left\{ \chi \bar{\partial} \chi + \bar{\chi} \partial \bar{\chi} + \chi' \bar{\partial} \chi' + \bar{\chi}' \partial \chi' \right\} \\ &\quad - \frac{k}{4\pi} \int_{twist} d^2z \text{STr} \left\{ A \bar{\partial} g g^{-1} + g^{-1} \partial g \bar{A} + A g \bar{A} g^{-1} - \bar{A} \mu - A \nu \right\} \end{aligned} \quad (3.5.18)$$

where the term  $S(\chi, \chi')$ , which is added to the action (3.2.22), encodes the enlarging of the space (3.5.10) to include the auxiliary fields  $\chi$  and  $\chi'$ . In the above action,  $S_{WZNW}^{twist}$  refers to the WZNW action based on  $SL(2|1)$  with a twisting of the energy-momentum tensor, required to keep the conformal invariance of the constrained WZNW action, and recover the  $N = 2$  superLiouville theory as it appears in the quantization of the  $N = 2$  string.

The partition function of equation (3.5.18) has the following gauge symmetries under which it is invariant :

$$\begin{aligned}
 g &\mapsto \alpha g \beta^{-1} \\
 A &\mapsto \alpha A \alpha^{-1} - \bar{\partial} \alpha \alpha^{-1} \\
 \bar{A} &\mapsto \beta \bar{A} \beta^{-1} + \partial \beta \beta^{-1} \\
 \chi &\mapsto \chi + \psi_- \\
 \chi' &\mapsto \chi' + \psi'_- \\
 \bar{\chi} &\mapsto \bar{\chi} + \psi_+ \\
 \bar{\chi}' &\mapsto \bar{\chi}' + \psi'_+ \\
 \alpha &= \exp(\xi \tau^- + \psi_- \tau^{-\frac{1}{2}} + \psi'_- \tau^{-\frac{1}{2}'}) \\
 \beta &= \exp(\gamma \tau^+ + \psi_+ \tau^{\frac{1}{2}} + \psi'_+ \tau^{\frac{1}{2}'})
 \end{aligned} \tag{3.5.19}$$

and we must fix the gauge to avoid an infinite contribution to the volumes in the path integral formed from equation (3.5.18). A convenient choice is the gauge  $A = \bar{A} = 0$ .

This gauge fixing introduces the set of ghosts  $(\sigma, \rho), (\varsigma, \varrho), (\varsigma', \varrho')$  with spin  $(1, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$  respectively. The gauge fixed path integral representation of the  $N = 2$  super Liouville theory is :

$$Z_{Liouville} = \int \mathcal{D}g \mathcal{D}\sigma \mathcal{D}\rho \dots Z_k \tag{3.5.20}$$

where

$$Z_k = \exp\left\{-k S_{WZNW}^{twisted} - S_{gh}(\sigma, \rho, \varsigma, \varrho, \varsigma', \varrho') - S(\chi, \chi')\right\} \tag{3.5.21}$$

and  $S_{WZNW}^{twist}$  is the twisted  $SL(2|1)$  WZNW model at level  $k$ .

Non-critical strings describe the coupling of 2d gravity and supergravity to minimal matter. In this thesis, we take the matter in a  $N = 2$  super Coulomb gas formulation, i.e., we represent the  $N = 2$  supermatter by the action

$$S_{matter} = \frac{1}{4\pi} \int d^2 \xi d^4 \theta \left\{ X \bar{X} + 2i\alpha_0 R(X + \bar{X}) \right\} \tag{3.5.22}$$



where  $X$  and  $\bar{X}$  are chiral and anti-chiral superfields respectively. Or equivalently we could constrain the WZNW theory based on  $SL(2|1)$  at level  $\tilde{k}$ . The contribution to the partition function is therefore similar to (3.5.20).

$$Z_{matter} = \int \mathcal{D}\tilde{g}\mathcal{D}\sigma\mathcal{D}\rho \dots \tilde{Z}_{\tilde{k}} \quad (3.5.23)$$

where

$$Z_{\tilde{k}} = \exp\left\{-kS_{WZNW}^{twisted} - S_{gh}(\tilde{\sigma}, \tilde{\rho}, \tilde{\zeta}, \tilde{\varrho}, \tilde{\zeta}', \tilde{\varrho}') - S(\tilde{\chi}, \tilde{\chi}')\right\} \quad (3.5.24)$$

The partition function for the  $N = 2$  superstring is then taken to be

$$Z = Z_{Liouville} \times Z_{matter} \times Z_{ghost} \quad (3.5.25)$$

with

$$Z_{ghost} = \exp\{-S_{gh}(\eta^i, \xi_i)\}. \quad (3.5.26)$$

$Z_{ghost}$  is the partition function for the gauge fixing ghosts  $(\eta^i, \xi_i)$  with  $i = 1, 2, 3, 4$  which takes spins  $(2, 1), (\frac{3}{2}, -\frac{1}{2}), (\frac{3}{2}, -\frac{1}{2})$  and  $(1, 0)$  respectively (see Table 3c). Their contributions to the central charge is  $C_{gh} = -6$ . The other two sets of ghosts are  $(\sigma, \rho), (\varsigma, \varrho), (\varsigma', \varrho')$  which are then introduced in constraining the currents in the Hamiltonian reduction, with  $(\tilde{\sigma}, \tilde{\rho}), (\tilde{\zeta}, \tilde{\varrho}), (\tilde{\zeta}', \tilde{\varrho}')$  being their counterparts for the reduction in the matter sector.

Table 3d

Role	ghost	spin
gauge fixing	$(\sigma, \rho)$	$(1, 0)$
ghosts in N=2	$(\varsigma, \varrho)$	$(\frac{1}{2}, \frac{1}{2})$
super Liouville	$(\varsigma', \varrho')$	$(\frac{1}{2}, \frac{1}{2})$
gauge fixing	$(\tilde{\sigma}, \tilde{\rho})$	$(1, 0)$
ghosts in N=2	$(\tilde{\zeta}, \tilde{\varrho})$	$(\frac{1}{2}, \frac{1}{2})$
super Liouville	$(\tilde{\zeta}', \tilde{\varrho}')$	$(\frac{1}{2}, \frac{1}{2})$

Let us end this section by giving the relations between the coupling  $\alpha_0$  and the level  $\tilde{k}$  in the matter sector, which is

$$\tilde{k} + 1 = 4\alpha_0^2 \quad (3.5.27)$$

as can be seen as follows. The central charge of the supermatter sector is given by

$$C_M = 3(1 - 8\alpha_0^2) \quad (3.5.28)$$

with  $Q = 2|\alpha_0|$  since  $C_{SL} + C_{matter} + C_{gh} = 0$ , (recall from section 3.4 that  $C_{SL} = 3(1 + 2Q^2)$ ).

On the other hand, the total energy-momentum tensor of the reduced  $N = 2$  theory is

$$\begin{aligned} T^{total} = & T_{Sugawara}^{SL(2|1)} + \partial J^3 - \frac{1}{2}\chi^1\partial\chi^1 - \frac{1}{2}\chi^2\partial\chi^2 \\ & - \rho\partial\sigma + \frac{1}{2}(\varsigma\partial\varrho - \partial\varsigma\varrho) + \frac{1}{2}(\varsigma'\partial\varrho' - \partial\varsigma'\varrho') \end{aligned} \quad (3.5.29)$$

with central charge

$$C^{total} = 0 - 6\tilde{k} + \frac{1}{2} + \frac{1}{2} - 2 - 1 - 1 = -6\tilde{k} - 3 \quad (3.5.30)$$

Since  $C^{total} = C_M$  one obtains equation (3.5.27) (where we have used the well know contribution to the central charge from a pair of ghosts of spin  $(j, 1-j)$

$$C^j = 2(-1)^{2j+1}(6j^2 - 6j + 1) \quad (3.5.31)$$

and the fact that the super-dimension  $\text{sdim } G$  for  $G = SL(2|1)$  is zero).

There also exists a relation between the levels  $k$  and  $\tilde{k}$  of the fundamental  $SL(2|1)$  algebras, stemming from the condition that the total conformal anomaly of the  $N = 2$  non-critical string vanishes :

$$\tilde{k} = -(k + 2). \quad (3.5.32)$$

This establishes yet another link between the gauged, twisted WZNW model and non-critical  $N = 2$  strings. Let us now make this equivalence more concrete.

### 3.6 The Equivalence between the Gauged, Twisted WZNW Model the Fermionic String

We are now in a position to tidy the results of the preceding two sections. In section 3.2 we showed that the gauge fixing of the gauged WZNW model introduces a set of spin  $(1, 0)$  ghosts which are tabulated in Table 3a. Section 3.3 showed that the twisting of the energy-momentum tensor in the model adjusts the spins of the  $(1, 0)$  ghosts to new values given in Table 3b.

On the other hand, in section 3.4 it was described how the quantization of the  $N = 2$  superstring had led us to introduce various ghosts in order to make the path integral finite when fixing the superconformal gauge. These ghosts are collected in Table 3c. Then we showed in section 3.5 how to recover the  $N = 2$  super-Liouville action and the  $N = 2$  minimal matter from constrained gauge invariant WZNW  $SL(2|1)$ ; an action of the form (3.5.143). The gauge fixing  $A = \bar{A} = 0$  introduces three pairs of ghosts in the Liouville and matter sections, listed in Table 3d.

Our comparisons between the gauged  $SL(2|1)/SL(2|1)$  WZNW model when the energy tensor is twisted, and the  $N = 2$  non-critical string theory, consists of matching the ghost content of the former theory (Table 3b) and the ghost content of the latter (Tables 3c and 3d).

The correspondence between the various ghosts leads us to conjecture that the space of states of the non-critical  $N = 2$  string can be considered as the space of states generated by four fermions and four ghosts, tensoring the space of states

of the gauged, twisted  $SL(2|1)/SL(2|1)$  WZNW model. I.e.,

$$\langle \text{non-critical } N=2 \text{ string} \rangle \simeq \left\langle \left\langle \begin{matrix} (\Delta^+, \Delta^-), (\varsigma, \varrho) \\ (\Delta'^+, \Delta'^-), (\varsigma', \varrho') \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} \text{twisted} \\ SL(2|1)/SL(2|1) \end{matrix} \right\rangle \right\rangle \quad (3.6.1)$$

where we have parameterized the auxiliary fields of equation (3.5.8) in the following way :

$$\begin{aligned} \Delta^+ &= \frac{1}{\sqrt{2}}(\chi + i\tilde{\chi}) & \Delta'^- &= \frac{1}{\sqrt{2}}(\chi' - i\tilde{\chi}') \\ \Delta^- &= \frac{1}{\sqrt{2}}(\chi - i\tilde{\chi}) & \Delta'^+ &= \frac{1}{\sqrt{2}}(\chi' + i\tilde{\chi}') \end{aligned} \quad (3.6.2)$$

which have the contractions

$$\Delta^-(z)\Delta^+(w) = \Delta'^-(z)\Delta'^+(w) = \frac{1}{z-w} \quad (3.6.3)$$

and of course the contractions for the ghosts  $(\sigma, \rho), (\varsigma, \varrho), (\varsigma', \varrho')$  are the same as those for  $(b, c), (\beta, \gamma), (\beta', \gamma')$ . However, the matter sectors for the two models are not *a priori* identical. One of the remarkable and useful aspects of the quantization of string theory is clearly illustrated in equation (3.4.21). Namely the fact that the matter, super Liouville and ghost sectors are independent of one another — for the non-critical string. This allows us freedom of calculation, since we may equate the super Liouville and ghost sectors of equation (3.4.21) to the Liouville and ghost sectors of the twisted, gauged WZNW model respectively, equation (3.2.11). Thus the only remaining detail is to equate the matter sectors.

We have shown at the end of the previous section that, in the matter sector of the  $N = 2$  non-critical string

$$C^{total} = -6\tilde{k} - 3 = 3(1 - 8\alpha_0^2) = C_{matter} . \quad (3.6.4)$$

The conformal anomaly for unitary minimal  $N = 2$  superstring theory has been shown to be [54]

$$C_{matter} = 3\left(1 - \frac{2}{M}\right) \quad M = 3, 4, 5, \dots \quad (3.6.5)$$

so if the matter sectors for the two models are to concur we must have :

$$\tilde{k} + 1 = \frac{1}{M} \quad M = 3, 4, \dots \quad (3.6.6a)$$

$$\text{or} \quad k + 1 = \frac{-1}{M} . \quad (3.6.6b)$$

We can clearly see that the level  $\tilde{k}$  of the Kac-Moody algebra  $SL(2|1)$  for the matter sector is going to take fractional values.

Therefore, the corresponding representations are non-unitary and general non-integrable. However, for carefully chosen values of the highest weight state quantum numbers, the representations still have an interesting structure. For instance their corresponding characters transform in an appealing fashion under modular transformations. Such representations are called admissible and are discussed in [25]. These representations are believed to play a crucial role in the calculation of the physical states of the  $N = 2$  non-critical string theory.

### 3.7 Topological Conformal Field Theories

This section is a short aside that is intended to point out the topological nature of the  $G/G$  WZNW model. One of the most important aspect of this is shown in equation (3.2.17) where one can clearly see that the total central charge is zero. It is this feature that hallmarks the  $G/G$  WZNW model as a topological conformal field theory.

The definition of a topological field theory will be taken as that presented in [58,59]. A topological field theory has the following :

- i) A nilpotent operator  $Q$  which is odd with respect to a Grassmann grading.
- ii) A set of fields  $\Phi$  taking values on a Riemannian manifold. These fields are Grassmann graded.
- iii) An energy tensor that is  $Q$ -exact so that if there is a functional of fields  $V$

then

$$T(z) = \{Q, V(\Phi)\} \quad (3.7.1)$$

iv) The physical states of the model lie in the cohomology of  $Q$ .

It so happens that in all the known topological field theories the operator  $Q$  has corresponded with the BRST charge defined by :

$$Q_{BRST} = \sum_{n,\alpha} c_{\alpha,n} \left( J_{-n}^{\alpha} + \tilde{J}_{-n}^{\alpha} + \frac{1}{2} (J_{gh}^{\alpha})_{-n} \right) \quad (3.7.2)$$

It has been established [58] that topological conformal field theories contain a twisted  $N = 2$  superconformal symmetry. Following [17] we can check that the form of this symmetry includes the energy tensor of equation (3.2.15) with additional terms that originate from the tensoring ghost sector

$$\begin{aligned} T(z) = & \frac{h_{\alpha\beta}}{k+c_G} : J^{\alpha} J^{\beta} : - \frac{h_{\alpha\beta}}{k+c_G} : \tilde{J}^{\alpha} \tilde{J}^{\beta} : + \partial c^{\alpha} b_{\alpha} \\ & + \frac{1}{2} (\partial \Delta^{+} \Delta^{-} - \Delta^{+} \partial \Delta^{-} + \partial \Delta'^{+} \Delta'^{-} - \Delta'^{+} \partial \Delta'^{-}) \\ & + \frac{1}{2} (\varsigma \partial \varrho - \partial \varsigma \varrho + \varsigma' \partial \varrho' - \partial \varsigma' \varrho') . \end{aligned} \quad (3.7.3)$$

In addition we can define its super-symmetric partners to be

$$G(z) = (J^{\alpha}(z) + \tilde{J}^{\alpha}(z) + \frac{1}{2} J_{gh}^{\alpha}) c^{\beta}(z) h_{\alpha\beta} + \Delta^{-} \varsigma + \Delta'^{-} \varsigma' \quad (3.7.4a)$$

$$\begin{aligned} \bar{G}(z) = & \frac{(-1)^{d(\alpha)}}{k+c_G} (J^{\alpha}(z) - \tilde{J}^{\alpha}(z)) b_{\alpha}(z) \\ & + \frac{1}{2} (\Delta^{+} \partial \varrho - \partial \Delta^{+} \varrho + \Delta'^{+} \partial \varrho' - \partial \Delta'^{+} \varrho') \end{aligned} \quad (3.7.4b)$$

and include a  $U(1)$  current defined as

$$J^{U(1)} = c^{\alpha}(z) b_{\alpha}(z) + \frac{1}{2} (\Delta^{-} \Delta^{+} - \varsigma \varrho + \Delta'^{-} \Delta'^{+} - \varsigma' \varrho') \quad (3.7.5)$$

With these definitions it is easy to check that the operator product expansions between the supersymmetry generators satisfy :

$$G(z) \bar{G}(w) = \frac{\text{sdim } G}{(z-w)^3} + \frac{J^{U(1)}(w)}{(z-w)^2} + \frac{T(w)}{z-w} \quad (3.7.6)$$

$$G(z)G(w) = \bar{G}(z)\bar{G}(w) = 0 \quad (3.7.7)$$

Yet another important result is that we may use the above formula to show that  $Q_{BRST}^2 = 0$  with

$$Q_{BRST} = \oint G(z)dz . \quad (3.7.8)$$

So defined the four requirements of a topological field theory can be completed, since it is a simple matter to show that

$$T(z) = \{Q_{BRST}, \bar{G}(z)\} \quad (3.7.9a)$$

$$J_{tot}^\alpha = \{Q_{BRST}, b^\alpha(z)\} \quad (3.7.9b)$$

which satisfy items (iii) and (iv) on the list of properties for a topological field theory, because the currents generate the physical states of the model.

### 3.8 BRST Cohomology of the Non-Critical String

This chapter ends with a small discussion concerning the space of physical states. We have shown the suggestive ways in which a gauged, twisted WZNW model for the group  $SL(2|1)$  could be proposed as a candidate for modelling a  $N = 2$  superstring. Ultimately the exact equivalence is established by showing that the space of physical states and the correlation functions for the two theories are identical (or in the least give identical results) and the final chapter of this thesis will calculate the space of physical states which lie in the cohomology class of  $Q_{BRST}$  — the BRST charge. However we shall make some simple introductory calculations in this section because they reduce the work needed subsequently, and they relate to the Hamiltonian reduction of section 3.5 .

The total BRST charge for our proposed model of the  $N = 2$  superstring can be written :

$$Q_{BRST}^{tot} = Q_{BRST}^{G/G, twisted} + \hat{Q}_{BRST} . \quad (3.8.1)$$

However, because  $J_{tot}^3$  is BRST exact the Q cohomology representatives of the twisted  $G/G$  theory are the same as in the untwisted ones. We shall therefore study the cohomology of

$$Q_{BRST}^{tot} = Q_{BRST}^{G/G} + \widehat{Q}_{BRST}. \quad (3.8.2)$$

which is a sum of two BRST charges

$$Q_{BRST}^{G/G} = \oint d^2z (J^\alpha + \tilde{J}^\alpha + \frac{1}{2} J_{gh}^\alpha) c^\beta h_{\alpha\beta} \quad (3.8.3a)$$

$$\widehat{Q}_{BRST} = \oint d^2z (\Delta^- \varsigma + \Delta'^- \varsigma'). \quad (3.8.3b)$$

and where  $Q_{BRST}^{G/G}$  is the BRST charge for the untwisted model.

A comparison with equations (3.7.4a) and (3.7.8) shows that this definition is just a simple breakdown of the BRST charge, and it is easy to show by direct calculations that both BRST charges commute with each other and hence the states in each cohomology class for  $Q_{BRST}^{G/G}$  and  $\widehat{Q}_{BRST}$  are unrelated. As a consequence it is possible to deduce that the stress energy tensor that accompanies  $\widehat{Q}_{BRST}$  can be defined by :

$$\begin{aligned} T^{(\Delta^+, \Delta^-, \varsigma, \varrho, \Delta'^+, \Delta'^-, \varsigma', \varrho')} &= \frac{1}{2} \{ \widehat{Q}_{BRST}, (\Delta^+ \partial \varrho + \Delta'^+ \partial \varrho' - \partial \Delta^+ \varrho - \partial \Delta'^+ \varrho') \} \\ \text{or } T^\otimes &= \{ \widehat{Q}_{BRST}, \Delta^\otimes \} \\ &= + \frac{1}{2} (\partial \Delta^+ \Delta^- - \Delta^+ \partial \Delta^- + \partial \Delta'^+ \Delta'^- - \Delta'^+ \partial \Delta'^-) \\ &\quad + \frac{1}{2} (\varsigma \partial \varrho - \partial \varsigma \varrho + \varsigma' \partial \varrho' - \partial \varsigma' \varrho') \end{aligned} \quad (3.8.4)$$

where we have written  $T^\otimes = T^{(\Delta^+, \Delta^-, \varsigma, \varrho, \Delta'^+, \Delta'^-, \varsigma', \varrho')}$  and  $\Delta^\otimes = (\Delta^+ \partial \varrho + \Delta'^+ \partial \varrho' - \partial \Delta^+ \varrho - \partial \Delta'^+ \varrho')$  as abbreviations.

It is possible to use this definition to show that only the vacuum state exists in the cohomology class of  $\widehat{Q}_{BRST}$  [17]. To do this we need to recall some simple facts about BRST charges and states. A general BRST charge is defined so that when acting on any physical state it vanishes i.e.,

$$Q |phys\rangle = 0 \quad \text{with } Q^2 = 0 \quad (3.8.5)$$



with the nilpotency of  $Q$  demanded by conformal invariance. A physical operator in a general formulation is defined so that it produces another physical state when applied to a physical state, and we may choose a basis of eigenvectors to describe this space i.e.,

$$W |phys\rangle = \omega |phys\rangle \quad (3.8.6)$$

which is true when

$$\{Q, W\} = 0 \quad (3.8.7)$$

subject to the condition that the same operator cannot be a BRST projection of another operator,  $W \neq \{Q, W'\}$ , i.e.,  $W$  is a nontrivial physical operator.

In our case let us take  $W \equiv T^\otimes$ . From the above we have

$$\begin{aligned} T^\otimes |phys\rangle &= \{\widehat{Q}_{BRST}, \Delta^\otimes\} |phys\rangle & (3.8.8) \\ &= \widehat{Q}_{BRST} \Delta^\otimes |phys\rangle + \Delta^\otimes \widehat{Q}_{BRST} |phys\rangle \\ &= \widehat{Q}_{BRST} \Delta^\otimes |phys\rangle \\ \text{or } |phys\rangle &= \frac{\widehat{Q}_{BRST} \Delta^\otimes}{\omega} |phys\rangle, \quad \omega \neq 0 \end{aligned}$$

but then  $|phys\rangle$  would be in the image of  $\widehat{Q}$  which contradicts the hypothesis that  $|phys\rangle$  is a nontrivial state. Thus  $\omega$  must equal zero and the only possible solution to equation (3.8.8) occurs if the cohomology class of  $\widehat{Q}_{BRST}$  contains only zero mode excitations of the fields  $\Delta^+, \Delta^-, \varsigma, \varrho, \Delta'^+, \Delta'^-, \varsigma', \varrho'$ .

Zero mode excitations occur when the spin  $\frac{1}{2}$  fields above are antiperiodic around equal time circles on the complex plane. This corresponds to the Ramond sector of the theory (note that section 3.5 on Hamiltonian reduction implies that  $\Delta^+, \Delta^-, \varsigma, \varrho, \Delta'^+, \Delta'^-, \varsigma', \varrho'$  have the same boundary conditions as  $j^\pm, j'^\pm$ ).

As was pointed out in [60] the choice of vacuum states is arbitrary. One must specify a Fermi and a Bose sea level, i.e., an energy level below which all levels are filled. Since  $\Delta^\pm, \Delta'^\pm$  are fermions of spin  $\frac{1}{2}$  and  $(\varsigma, \varrho), (\varsigma', \varrho')$  are bosons of spin

$(\frac{1}{2}, \frac{1}{2})$ , their prescription for vacua in the Ramond sector gives :

$$\varsigma_n |q\rangle_R = 0 \quad \forall n \geq q + \frac{1}{2} \quad (3.8.9a)$$

$$\varrho_n |q\rangle_R = 0 \quad \forall n \geq -q - \frac{1}{2} \quad (3.8.9b)$$

$$\Delta_n^+ |q\rangle_R = 0 \quad \forall n \geq q - \frac{1}{2} \quad (3.8.9c)$$

$$\Delta_n^- |q\rangle_R = 0 \quad \forall n \geq -q + \frac{1}{2} \quad (3.8.9d)$$

where  $q \in \mathcal{Z} + \frac{1}{2}$  is the vacuum charge and

$$\Delta^\pm(z) = \sum_n \Delta_n^\pm z^{-n-\frac{1}{2}} \quad , \quad \{\Delta_n^+, \Delta_n^-\} = \delta_{n+m,0} \quad (3.8.10a)$$

$$\varsigma(z) = \sum_n \varsigma_n z^{-n-\frac{1}{2}} \quad (3.8.10b)$$

$$\varrho(z) = \sum_n \varrho_n z^{-n-\frac{1}{2}} \quad , \quad [\varsigma_n, \varrho_m] = \delta_{n+m,0} \quad (3.8.10c)$$

and similarly for the primed fields. When choosing the vacuum with charge  $q = -\frac{1}{2}$ , we have

$$n_{\Delta_0^-} + n_{\varrho_0} = \Delta_0^- \Delta_0^+ + \varrho_0 \varsigma_0 = \{\widehat{Q}_{BRST}, \Delta_0^+ \varrho_0\} \quad (3.8.11a)$$

$$n_{\Delta_0'^-} + n_{\varrho_0'} = \Delta_0'^- \Delta_0'^+ + \varrho_0' \varsigma_0' = \{\widehat{Q}_{BRST}, \Delta_0'^+ \varrho_0'\} \quad (3.8.11b)$$

and we can apply the same argument that was used for the deduction in equation (3.8.8) to conclude that

$$n_{\Delta_0^-} + n_{\varrho_0} = 0 \quad (3.8.12a)$$

$$n_{\Delta_0'^-} + n_{\varrho_0'} = 0 \quad (3.8.12b)$$

from which we deduce that  $n_{\Delta_0^-} = n_{\varrho_0} = n_{\Delta_0'^-} = n_{\varrho_0'}$ , and the only nontrivial BRST state is the vacuum state. Thus we are led to conclude that

$$H\left(\mathcal{H}^{SL(2,1)} \otimes \left(\frac{\Delta'^+, \Delta'^-, \varsigma', \varrho'}{\Delta^+, \Delta^-, \varsigma, \varrho}\right), Q_{BRST}^{tot}\right) \simeq H\left(\mathcal{H}^{SL(2,1)}, Q_{BRST}^{G/G}\right) \otimes |vacuum\rangle \quad (3.8.13)$$

where  $H$  denotes the cohomology class of  $Q_{BRST}$  for a Fock space  $\mathcal{H}$  where in this case  $\mathcal{H}^{SL(2,1)}$  is the Fock space for the full gauged, twisted WZNW model which tensors the additional ghosts, and  $|vacuum\rangle$  denotes the vacuum state of this ghost system.

Having determined this result we are left only with the need to solve the BRST cohomology of  $Q_{BRST}^{G/G}$ . This is attempted in the next chapter.

# Chapter Four

## The Space of Physical States

*“it is not really difficult to construct a series of inferences,  
each dependent upon its predecessor and each simple in itself”*

From *The Adventure of the Dancing Men*  
by Sir Arthur Conan Doyle

### 4.1 Introduction

This thesis concludes in this chapter with a study of the remaining cohomology of the BRST charge which was defined in section 3.8. The procedure for attempting the analysis of these states is now well established [61,17,18,32,35,36,37,38] and for greater details the reader can refer to these.

The analysis of the cohomology takes place on the whole Fock space  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$  which is generated by the negative modes of the currents at levels  $k$  and  $-(k+2)$  and parameterized by free fields. All discussions in section 4.3 refer to the Ramond sector of the theory. However, section 4.2 establishes the important fact that

the Ramond and Neveu-Schwarz sectors are isomorphic — and this fact can be exploited to reduce the number of calculations that we are required to perform.

To extract the physical states one must study the cohomology of a tensor product of two conjugate Wakimoto modules, i.e., one uses a free field parameterization for the currents at level  $k$  and its conjugate (in the sense described in Appendix C) is formed from the currents at level  $-(k+2)$ . This is attempted in the final section of this chapter.

It should be noted that the spectrum of the  $SL(2|1)/SL(2|1)$  theory is also included in the irreducible highest weight modules, and one must project the BRST cohomology of a tensor product of two conjugate Wakimoto modules onto a tensor product of an irreducible (admissible) module and a Wakimoto module. This is not attempted in this thesis as such admissible representations are still under investigation [25].

## 4.2 The Isomorphism Between Ramond and Neveu-Schwarz Sectors

As was described in section 3.7 the physical states lie in the cohomology defined with respect to the BRST charge of equation (3.7.2) which obeys the properties detailed in section 3.8. Although the space of physical states is quite intricate it can be viewed as being constructed from three modules. These three modules are referred to as the Verma, Wakimoto, and irreducible modules, which we shall denote by  $V(j, u, k)$ ,  $W(j, u, k)$  and  $I(j, u, k)$  respectively, where  $j$  denotes  $J_0^3$  — the eigenvalue of the highest weight state (HWS) in each module — and  $u$  is the eigenvalue of the bosonic generator  $U_0$  on the HWS, with  $k$  the level algebra.

The  $SL(2|1)$  WZNW model has a contribution to the space of physical states from the matter, Liouville and ghost sectors - which correspond to the Verma, Wakimoto, and irreducible modules respectively. The algebra of the currents which generate these modules divides into two sectors referred to as the Ramond and

Neveu-Schwarz sectors. The two sectors arise because of the possible choices of boundary conditions that a string may have. Periodic boundary conditions result in the Ramond sector and anti-periodic boundary conditions result in the Neveu-Schwarz sector. Each type of algebra produces a different representation of the Fock space, and the main result of the following pages is that these two representations are isomorphic.

In Section 2.6 the operator product expansions for the currents were presented (equation (2.6.1)). To extract the affine Kac-Moody algebra we can make use of equation (2.2.6) to decompose the currents into modes by making appropriate Laurent expansions of the currents. If we do this we gain the following anti/commutators :

Type A (4.2.1a)

$$\begin{aligned}
 [J_m^+, J_n^-] &= 2J_{m+n}^3 + km\delta_{m+n,0} & [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm \\
 [U_m, U_n] &= -\frac{1}{2}nk\delta_{m+n,0} & [J_m^3, J_n^3] &= \frac{1}{2}nk\delta_{m+n,0} \\
 [U_m, j_r^\pm] &= \pm \frac{1}{2}j_{m+r}^\pm & [U_m, j_r^{\prime\pm}] &= \pm \frac{1}{2}j_{m+r}^{\prime\pm} \\
 [J_m^\pm, j_r^\pm] &= \mp j_{m+r}^{\prime\pm} & [J_m^\pm, j_r^{\prime\mp}] &= \pm j_{m+r}^{\mp} \\
 [J_m^3, j_r^\pm] &= \mp \frac{1}{2}j_{m+r}^\pm & [J_m^3, j_r^{\prime\pm}] &= \pm \frac{1}{2}j_{m+r}^{\prime\pm} \\
 \{j_r^+, j_s^-\} &= J_{r+s}^3 + U_{r+s} - kr\delta_{r+s,0} & \{j_r^\pm, j_s^{\prime\mp}\} &= +J_{r+s}^{\mp} \\
 \{j_r^{\prime+}, j_s^{\prime-}\} &= -J_{r+s}^3 + U_{r+s} - kr\delta_{r+s,0} & &
 \end{aligned}$$

Type B (4.2.1b)

$$\begin{aligned}
 [J_m^+, J_n^-] &= 2J_{m+n}^3 + km\delta_{m+n,0} & [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm \\
 [U_m, U_n] &= -\frac{1}{2}nk\delta_{m+n,0} & [J_m^3, J_n^3] &= \frac{1}{2}nk\delta_{m+n,0} \\
 [U_m, j_r^\pm] &= \mp \frac{1}{2}j_{m+r}^\pm & [U_m, j_r'^\pm] &= \pm \frac{1}{2}j_{m+r}'^\pm \\
 [J_m^\pm, j_r^\mp] &= \mp j_{m+r}'^\pm & [J_m^\pm, j_r'^\mp] &= \pm j_{m+r}^\pm \\
 [J_m^3, j_r^\pm] &= \pm \frac{1}{2}j_{m+r}^\pm & [J_m^3, j_r'^\pm] &= \pm \frac{1}{2}j_{m+r}'^\pm \\
 \{j_r^+, j_s^-\} &= J_{r+s}^3 + U_{r+s} + kr\delta_{r+s,0} & \{j_r^\pm, j_s'^\pm\} &= +J_{r+s}^\pm \\
 \{j_r'^+, j_s'^-\} &= -J_{r+s}^3 + U_{r+s} - kr\delta_{r+s,0} .
 \end{aligned}$$

Although both the Type A and Type B set are written above for completeness, this chapter will only concern itself, for the larger part, with the Type B algebra.

The anti/commutators (4.2.1) have modings denoted by  $m$  and  $n$  for bosonic currents with  $m$  and  $n$  valued over integers, while  $r$  and  $s$  are the modings of fermionic currents with the Ramond sector having  $r$  and  $s$  integer valued, and the Neveu-Schwarz sector having  $r$  and  $s$  taking half integer values. Following [16,17] we can confirm that the Type B isomorphism takes the form :

$$\begin{aligned}
 J_m^\pm &\mapsto -J_{m\pm 1}^\mp & J_m^3 &\mapsto -J_m^3 + \frac{k}{2}\delta_{m,0} & k &\mapsto k & (4.2.2) \\
 U_m &\mapsto -U_m & j_r^\pm &\mapsto \pm j_{r\pm \frac{1}{2}}^\mp & j_r'^\pm &\mapsto \mp j_{r\pm \frac{1}{2}}'^\mp
 \end{aligned}$$

with the isomorphism for the Type A algebra found by interchanging  $j^+$  and  $j^-$ , so that (4.2.2) remains the same except for the amendment  $j_r^\pm \mapsto \mp j_{r\mp \frac{1}{2}}^\mp$ .

(It is not entirely impossible to find a representation which does possess a single isomorphism between Ramond and Neveu-Schwarz sectors, valid for both Type A and Type B interpretations simultaneously. To do so we need to redefine fermionic generators thus :

$$\begin{aligned}
 \hat{j}^+ &= j^+ + ij'^+ & \hat{j}'^+ &= j^+ - ij'^+ & (4.2.3) \\
 \hat{j}^- &= j^- + ij'^- & \hat{j}'^- &= j^- - ij'^-
 \end{aligned}$$

from which the corresponding commutators and isomorphism may be calculated. We mention this point since the assignment of (4.2.3) is used in reference [19], and this point will allow the reader to understand the connection between this work and that of [19].

This isomorphism can also be represented as a map between free fields. To do this let us introduce the following Laurent expansions :

$$\begin{aligned}
 \beta(z) &= \sum_n \frac{\beta_n}{z^{n+1}} & \gamma(z) &= \sum_n \frac{\gamma_n}{z^n} & i\partial\phi &= \sum_n \frac{\phi_n}{z^{n+1}} & (4.2.4) \\
 [\psi_+]^\dagger &= \sum_n \frac{[\psi_+]_n^\dagger}{z^{n+1}} & \psi_+ &= \sum_n \frac{\psi_{+,n}}{z^n} & \partial\theta &= \sum_n \frac{\theta_n}{z^{n+1}} \\
 [\psi'_+]^\dagger &= \sum_n \frac{[\psi'_+]_n^\dagger}{z^{n+1}} & \psi'_+ &= \sum_n \frac{\psi'_{+,n}}{z^n}
 \end{aligned}$$

which allow us to rewrite the short order free field contractions as :

$$\begin{aligned}
 [\gamma_n, \beta_m] &= \delta_{n+m,0} & \{\psi_{+,n}, [\psi_+]_m^\dagger\} &= \delta_{n+m,0} & \{\psi'_{+,n}, [\psi'_+]_m^\dagger\} &= \delta_{n+m,0} \\
 [\theta_n, \theta_m] &= n\delta_{n+m,0} & [\phi_n, \phi_m] &= n\delta_{n+m,0} . & & (4.2.5)
 \end{aligned}$$

With these Laurent expansions we can decompose the free field representations into modal form. The modal form for the Type B currents is shown in Appendix C with the following abbreviation in notation :

$$\begin{aligned}
 \psi_+ &\leftrightarrow \psi & [\psi_+]^\dagger &\leftrightarrow \psi^\dagger & (4.2.6) \\
 \psi'_+ &\leftrightarrow \psi' & [\psi'_+]^\dagger &\leftrightarrow \psi'^\dagger \\
 \partial\theta &\leftrightarrow \theta & i\partial\phi &\leftrightarrow \phi .
 \end{aligned}$$

This should cause no confusion. In chapter two the introduction of the subscripts  $+$ ,  $-$  was employed so that the reader could clearly understand the origin of each field, but this is no longer necessary. The terms in the currents are normal ordered in the fashion indicated by equation (2.7.3), and Appendix C.



From these currents and equation (4.2.2) it is now possible to deduce the following mapping of the fields which preserves the isomorphism between the Wakimoto module  $W^R(j_R, u_R, k)$  and its conjugate under (4.2.2)  $W^{*NS}(j_{NS}, u_{NS}, k)$

$$\begin{aligned}
 \beta_n^R &\mapsto \beta_{n+1}^{NS} & [\psi_+^R]_n^\dagger &\mapsto [\psi_+^{NS}]_{n+\frac{1}{2}}^\dagger \\
 \psi_{+,n}^{\prime R} &\mapsto \psi_{+,n-\frac{1}{2}}^{\prime NS} & [\psi_{+,n}^{\prime R}]_n^\dagger &\mapsto [\psi_{+,n-\frac{1}{2}}^{\prime NS}]_{n+\frac{1}{2}}^\dagger \\
 \psi_{+,n}^R &\mapsto \psi_{+,n-\frac{1}{2}}^{NS} & \gamma_n^R &\mapsto \gamma_{n-1}^{NS} \\
 \phi_n^R &\mapsto \phi_n^{NS} & \theta_n^R &\mapsto \theta_n^{NS} + \frac{1}{2}\alpha_+\delta_{n,0} + \frac{1}{2\alpha}\delta_{n,0} .
 \end{aligned} \tag{4.2.7}$$

From the isomorphism (4.2.2) one sees that the isospin of the HWS of a Ramond module,  $j_R$ , is related to the isospin  $j_{NS}$  of the HWS of a Neveu-Schwarz module by the formula

$$j_R = \frac{k}{2} - j_{NS} . \tag{4.2.8a}$$

Similarly,

$$u_R = -u_{NS} \tag{4.2.8b}$$

Now consider the ghost algebra, whose currents were defined in (3.2.13) as

$$J_{gh}^\alpha = (-1)^{d(\gamma)} f_\gamma^{\alpha\beta} : c^\gamma b_\beta : . \tag{4.2.9}$$

The ghosts,  $c^\pm, c^3, c^0$  and  $b_\pm, b_3, b_0$  are fermionic but with integer modes while  $c^{\pm\frac{1}{2}}, c^{\pm\frac{1}{2}'}$  and  $b_{\pm\frac{1}{2}}, b_{\pm\frac{1}{2}'}$  are bosonic ghosts, but can be Ramond (integer modes) or Neveu-Schwarz (half integer mode). Normal ordering in the Ramond sector is with respect to the vacuum  $|0\rangle_R$  satisfying :

$$\begin{aligned}
 b_{\pm\frac{1}{2},n}|0\rangle_R &= b_{\pm\frac{1}{2}',n}|0\rangle_R = 0 & \forall n \geq 0 & \tag{4.2.10} \\
 b_{\pm,n}|0\rangle_R &= b_{3,n}|0\rangle_R = b_{0,n}|0\rangle_R = 0 & \forall n \geq 0 & \\
 c_n^{\pm\frac{1}{2}}|0\rangle_R &= c_n^{\pm\frac{1}{2}'}|0\rangle_R = 0 & \forall n > 0 & \\
 c_n^\pm|0\rangle_R &= c_n^3|0\rangle_R = c_n^0|0\rangle_R = 0 & \forall n > 0 &
 \end{aligned}$$

while in the Neveu-Schwarz sector normal ordering is with respect to the vacuum  $|\frac{1}{2}\rangle_{NS}$  satisfying :

$$\begin{aligned}
 b_{\pm, n}|\frac{1}{2}\rangle_{NS} &= b_{3, n}|\frac{1}{2}\rangle_{NS} = b_{0, n}|\frac{1}{2}\rangle_{NS} = 0 & \forall n \geq 0 & \quad (4.2.11) \\
 b_{\pm\frac{1}{2}, n+\frac{1}{2}}|\frac{1}{2}\rangle_{NS} &= b_{\pm\frac{1}{2}', n+\frac{1}{2}}|\frac{1}{2}\rangle_{NS} = 0 & \forall n \geq 0 & \\
 c_n^{\pm}|\frac{1}{2}\rangle_{NS} &= c_n^3|\frac{1}{2}\rangle_{NS} = c_n^0|\frac{1}{2}\rangle_{NS} = 0 & \forall n \geq 0 & \\
 c_{n+\frac{1}{2}}^{\pm\frac{1}{2}}|\frac{1}{2}\rangle_{NS} &= c_{n+\frac{1}{2}}^{\pm\frac{1}{2}'}|\frac{1}{2}\rangle_{NS} = 0 & \forall n \geq 0 & .
 \end{aligned}$$

By using the isomorphism of the currents as given in equation (4.2.2) for level  $k = 2$ , it is possible to deduce that the ghost fields have the following mapping between Ramond and Neveu-Schwarz sectors :

$$(4.2.12a)$$

$$\begin{array}{cccc}
 c_+ \mapsto z^{-1}c_- & c_- \mapsto zc_+ & c_3 \mapsto c_3 & c_0 \mapsto c_0 \\
 c_{\frac{1}{2}} \mapsto -z^{-\frac{1}{2}}c_{-\frac{1}{2}} & c_{-\frac{1}{2}} \mapsto z^{\frac{1}{2}}c_{\frac{1}{2}} & c_{\frac{1}{2}'} \mapsto z^{-\frac{1}{2}}c_{-\frac{1}{2}'} & c_{-\frac{1}{2}'} \mapsto -z^{\frac{1}{2}}c_{\frac{1}{2}'}
 \end{array}$$

$$(4.2.12b)$$

$$\begin{array}{cccc}
 b^+ \mapsto zb^- & b^- \mapsto z^{-1}b^+ & b^3 \mapsto b^3 & b^0 \mapsto b^0 \\
 b^{\frac{1}{2}} \mapsto -z^{\frac{1}{2}}b^{-\frac{1}{2}} & b^{-\frac{1}{2}} \mapsto z^{-\frac{1}{2}}b^{\frac{1}{2}} & b^{\frac{1}{2}'} \mapsto z^{\frac{1}{2}}b^{-\frac{1}{2}'} & b^{-\frac{1}{2}'} \mapsto -z^{-\frac{1}{2}}b^{\frac{1}{2}'}
 \end{array}$$

which we define with respect to a ghost vacuum  $|0\rangle_R$ .

The effect of these free field mappings is to map the BRST charge of equation (3.7.2)  $Q^R \mapsto -Q^{NS}$  which means that the resulting cohomology of  $Q$  is, as expected, unchanged regardless of whether the Ramond or Neveu-Schwarz sector is used. Thus we have the following isomorphism of modules :

$$V^R(j^R, u^R, k) \simeq V^{NS}(j^{NS}, u^{NS}, k) \quad (4.2.13)$$

$$W^R(j^R, u^R, k) \simeq W^{*NS}(j^{NS}, u^{NS}, k)$$

$$I^R(j^R, u^R, k) \simeq I^{NS}(j^{NS}, u^{NS}, k)$$

where the identity of superscripts is self evident.

A final feature to note is that the twisting procedure needed in the Hamiltonian reduction not only alters the conformal dimensions of the fields but also changes the modings. Thus a reduced Ramond sector algebra results in a theory in the Neveu-Schwarz sector and *vice versa*.

### 4.3 The BRST Cohomology

We now wish to turn the information so far gathered to good use by using it to calculate the physical states of our model. The work of sections 3.8 and 4.2 has allowed us to develop a firm foundation on which to progress to another calculation, namely the analysis of the cohomology of the BRST charge of the gauged, twisted, WZNW model when the group is  $SL(2|1)$ . Since the Ramond and Neveu-Schwarz sectors are isomorphic we are at liberty to concentrate our efforts on the Ramond sector, and mention relevant points about the Neveu-Schwarz sector as appropriate.

The analysis of the space of physical states has been undertaken by many authors [24,32,35,36,37,38,61] for other groups, in the context of 2d gravity as well as the  $G/G$  models. A modern method of analysis is via the BRST quantization, and as such the physical states lie in the cohomology of  $Q_{BRST}$ .

As we saw in section 3.8 the cohomology of physical states of the non-critical  $N = 2$  superstring in effect reduces to that of the  $G/G$  model due to the fact that the ghost system that tensors the  $G/G$  model only contributes a vacuum state, equation (3.8.13). Our total BRST charge is

$$Q_{BRST}^{tot} = Q_{BRST}^{G/G, twisted} + \hat{Q} \quad (4.3.1)$$

where  $\hat{Q}$  is the BRST charge topological ghost sector  $(\Delta^+, \Delta^-), (\Delta'^+, \Delta'^-)$  which commutes with  $Q_{BRST}^{G/G, twisted}$ . As was emphasized in section 3.8 it is sufficient to

study the cohomology of the untwisted BRST charge  $Q_{BRST}^{G/G}$  defined through equation (3.7.2)

$$Q_{BRST}^{G/G} = \sum_{n=-\infty}^{\infty} \left( c_{\alpha,n} (J_{-n}^{\alpha} + \tilde{J}_{-n}^{\alpha}) - \frac{1}{2} (-1)^{d(\alpha)} f_{\lambda}^{\alpha\beta} \sum_{m=-\infty}^{\infty} : c_{\alpha,n} c_{\beta,m} b_{-n-m}^{\lambda} : \right) \quad (4.3.2)$$

where we have normal ordering of ghost fields in such a manner that negative subscripts are placed to the left of positive ones, and  $b_0^{\alpha}$  is to the right of  $c_0^{\alpha}$ . Hereafter, unless otherwise stated, we will denote  $Q_{BRST}^{G/G}$  by  $Q$ , and the cohomology class of  $Q$  by

$$\mathbf{H} \equiv H\left(\mathcal{H}^{SL(2|1)}, Q\right). \quad (4.3.3)$$

From equation(3.7.9b) we see that  $J_0^{tot,3}$  and  $J_0^{tot,0}$ , where  $J^{tot}$  is the total current of equation (3.2.12), are BRST exact. I.e., we have

$$\{Q, b_0^0\} = J_0^{tot,0} \quad , \quad \{Q, b_0^3\} = J_0^{tot,3} \quad , \quad \{Q, \bar{G}_o\} = L_0. \quad (4.3.4)$$

As  $J_0^{tot,0}$  and  $J_0^{tot,3}$  and the energy tensor  $T(z)$  are BRST exact it follows that on the space of physical states defined with respect to a vacuum  $|\text{phys}\rangle$  we have

$$L_0 |\text{phys}\rangle = 0 \quad , \quad J_0^{tot,3} |\text{phys}\rangle = 0 \quad , \quad J_0^{tot,0} |\text{phys}\rangle = 0. \quad (4.3.5)$$

To assist the analysis of  $Q$  further we will first perform the BRST calculations on the subspace of  $\mathbf{H}$  where  $b_0^3 = 0$ , in addition to the requirements of equation (4.3.6). Explicitly we may write  $Q$  as :

$$Q = c_{3,0} J_0^{tot,3} + c_{0,0} J_0^{tot,0} + M b_0^3 + M' b_0^0 + \hat{Q} \quad (4.3.6)$$

where

$$M = -\frac{1}{2} (-1)^{d(\alpha)} f_3^{\alpha\beta} \sum_m : c_{\alpha,m} c_{\beta,-m} : \quad (4.3.7a)$$

$$M' = -\frac{1}{2} (-1)^{d(\alpha)} f_0^{\alpha\beta} \sum_m : c_{\alpha,m} c_{\beta,-m} : \quad (4.3.7a)$$

and we shall refer to the cohomology of  $\widehat{Q}$  as the relative cohomology and denote it by  $\mathbf{H}_{rel}$ .

We shall now proceed and identify the relative cohomology of  $\widehat{Q}$  in the Ramond sector. The corresponding states are built on a vacuum  $|j, u, \tilde{j}, \tilde{u}\rangle$  satisfying

$$J_{n>0}^a |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad j_{n>0}^a |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad (4.3.8)$$

$$j'_{n>0} |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad J_0^+ |j, u, \tilde{j}, \tilde{u}\rangle = 0$$

$$j_0^+ |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad j_0'^+ |j, u, \tilde{j}, \tilde{u}\rangle = 0$$

$$c_{n>0}^a |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad b_{n\geq 0}^a |j, u, \tilde{j}, \tilde{u}\rangle = 0$$

with  $J_0^3 |j, u, \tilde{j}, \tilde{u}\rangle = j |j, u, \tilde{j}, \tilde{u}\rangle$  and  $U_0 |j, u, \tilde{j}, \tilde{u}\rangle = u |j, u, \tilde{j}, \tilde{u}\rangle$ , with similar relations existing for the  $\tilde{J}^\alpha$  currents (the Liouville sector).

Using the free field representation (2.6.3b) and Appendix C equation (C.6) for the matter sector (algebra with currents  $J^\alpha$  and at level  $k$ ) and the conjugate free field representation (C.8) for the Liouville sector (algebra with currents  $\tilde{J}^\alpha$  and level  $\tilde{k} = -k - 2$ , required for nilpotency of  $Q$ ), one sees that the conditions (4.3.8) impose :

$$\beta_n |j, u, \tilde{j}, \tilde{u}\rangle = \psi_n^\dagger |j, u, \tilde{j}, \tilde{u}\rangle = \psi_n'^\dagger |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad \forall n \geq 0 \quad (4.3.9a)$$

$$\gamma_n |j, u, \tilde{j}, \tilde{u}\rangle = \psi_n |j, u, \tilde{j}, \tilde{u}\rangle = \psi_n' |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad \forall n > 0 \quad (4.3.9b)$$

$$\tilde{\beta}_n |j, u, \tilde{j}, \tilde{u}\rangle = \tilde{\psi}_n^\dagger |j, u, \tilde{j}, \tilde{u}\rangle = \tilde{\psi}_n'^\dagger |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad \forall n \geq 0 \quad (4.3.9c)$$

$$\tilde{\gamma}_n |j, u, \tilde{j}, \tilde{u}\rangle = \tilde{\psi}_n |j, u, \tilde{j}, \tilde{u}\rangle = \tilde{\psi}_n' |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad \forall n > 0 \quad (4.3.9d)$$

with

$$\phi_0 |j, u, \tilde{j}, \tilde{u}\rangle = \frac{2}{i\alpha_+} u |j, u, \tilde{j}, \tilde{u}\rangle \quad (4.3.10a)$$

$$\theta_0 |j, u, \tilde{j}, \tilde{u}\rangle = \frac{2}{\alpha_+} j |j, u, \tilde{j}, \tilde{u}\rangle \quad (4.3.10b)$$

$$\tilde{\phi}_0 |j, u, \tilde{j}, \tilde{u}\rangle = \frac{2}{\alpha_+} \tilde{u} |j, u, \tilde{j}, \tilde{u}\rangle \quad (4.3.10c)$$

$$\tilde{\theta}_0 |j, u, \tilde{j}, \tilde{u}\rangle = -\frac{2}{i\alpha_+} \tilde{j} |j, u, \tilde{j}, \tilde{u}\rangle \quad (4.3.10d)$$

The Wakimoto module  $W^R(j, u, k)$  is the Fock space built on the vacuum  $|j, u\rangle$  with the negative modes of the free fields  $\beta, \gamma, \psi, \psi'$  above as well as  $\gamma_0, \psi_0$  and  $\psi'_0$ . Similarly, the Wakimoto module  $\tilde{W}^R(\tilde{j}, \tilde{u}, \tilde{k})$  is the Fock space built on the vacuum  $|\tilde{j}, \tilde{u}\rangle$  with the negative modes of the free fields  $\tilde{\beta}, \tilde{\gamma}, \tilde{\psi}, \tilde{\psi}', \tilde{\psi}^\dagger, \tilde{\psi}'^\dagger$  as well as  $\tilde{\beta}_0, \tilde{\psi}_0^\dagger$  and  $\tilde{\psi}'_0^\dagger$ .

In order to follow the prescription of [61] for the calculation of the relative cohomology  $\mathbf{H}_{rel}$  we assign to the vacuum state a quantity we shall refer to as the degree, and define the vacuum state to have degree zero. Each field can then be assigned a degree of  $-1$  or  $+1$  depending upon whether it will or will not annihilate the vacuum, respectively.

With this definition the BRST charge  $\hat{Q}$  can be decomposed as a finite sum of terms  $\hat{Q}^{(i)}, i = 1, 2, 3, 4$  of degree  $i$ , and such that  $\hat{Q}^{(0)}$  is nilpotent. The assignment of degree's to fields is as follows :

Table 4a

Degree 1	Degree -1	Degree 1	Degree -1
$c$	$b$	$\tilde{c}$	$\tilde{b}$
$\gamma$	$\beta$	$\tilde{\gamma}$	$\tilde{\beta}$
$\theta^+$	$\theta^-$	$\phi^+$	$\phi^-$
$\psi$	$\psi^\dagger$	$\tilde{\psi}$	$\tilde{\psi}^\dagger$
$\psi'$	$\psi'^\dagger$	$\tilde{\psi}'$	$\tilde{\psi}'^\dagger$

In addition to the abbreviations of equation (4.2.6), we have defined

$$\phi^\pm = \frac{1}{\sqrt{2}}(\tilde{\phi} \mp i\phi) \quad \theta^\pm = \frac{1}{\sqrt{2}}(\theta \pm i\tilde{\theta}) \tag{4.3.11}$$

the zero modes of which are assigned a degree of zero, and non-zero modes having the assignments given in Table 4a. The vacuum has degree zero.

We are now positioned to commence the analysis. We may start by using the

modal decomposition of the currents as given in Appendix C to write  $\hat{Q}$  as a sum of BRST operators of differing degrees :

$$\begin{aligned}
 (\hat{Q}^{(0)})^2 &= (\hat{Q}^{(4)})^2 = 0 & (4.3.12) \\
 \{\hat{Q}^{(0)}, \hat{Q}^{(1)}\} &= \{\hat{Q}^{(3)}, \hat{Q}^{(4)}\} = 0 \\
 \{\hat{Q}^{(0)}, \hat{Q}^{(3)}\} + \{\hat{Q}^{(1)}, \hat{Q}^{(2)}\} &= 0 \\
 \{\hat{Q}^{(1)}, \hat{Q}^{(4)}\} + \{\hat{Q}^{(2)}, \hat{Q}^{(3)}\} &= 0 \\
 \{\hat{Q}^{(0)}, \hat{Q}^{(2)}\} + (\hat{Q}^{(1)})^2 &= 0 \\
 \{\hat{Q}^{(2)}, \hat{Q}^{(4)}\} + (\hat{Q}^{(3)})^2 &= 0 \\
 \{\hat{Q}^{(0)}, \hat{Q}^{(4)}\} + \{\hat{Q}^{(1)}, \hat{Q}^{(3)}\} + (\hat{Q}^{(2)})^2 &= 0
 \end{aligned}$$

The explicit form for  $\hat{Q}^{(0)}$  is :

$$\begin{aligned}
 \hat{Q}^{(0)} &= \sum_n \left( c_{-,n} \tilde{\beta}_{-n} - c_{+,n} \beta_{-n} + c_{\frac{1}{2},n} \psi_{-n}^\dagger + c_{-\frac{1}{2},n} \tilde{\psi}_{-n}^\dagger \right. & (4.3.13) \\
 &\quad \left. - c_{\frac{1}{2}',n} \psi_{-n}'^\dagger + c_{-\frac{1}{2}',n} \tilde{\psi}_{-n}'^\dagger \right) + \sum_{n \neq 0} \left( \frac{\alpha_+}{\sqrt{2}} c_{0,n} \phi_{-n}^- + \frac{\alpha_+}{\sqrt{2}} c_{3,n} \theta_{-n}^- \right)
 \end{aligned}$$

The method described in [61] is only valid if there is a finite number of degrees for each ghost number, which is the case here. Indeed, the states in the Fock space built on the vacuum  $|j, u, \tilde{j}, \tilde{u}\rangle$  called  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$ , are annihilated by  $L_0$ ,  $J_0^{tot,3}$  and  $J_0^{tot,0}$  (see equation 4.3.6). The zero mode  $L_0$  of the energy-momentum tensor for the  $SL(2|1)/SL(2|1)$  theory, given in equation (3.2.16) can be rewritten as

$$L_0 = \hat{L}_0 + \frac{1}{k+1} \left\{ (j-u)(j+u) - (\tilde{j}-\tilde{u})(\tilde{j}+\tilde{u}) \right\} \quad (4.3.14)$$

where

$$\begin{aligned}
 \hat{L}_0 &= \sum_n : n(\beta_{-n} \gamma_n + \tilde{\beta}_{-n} \tilde{\gamma}_n + \psi_{-n}^\dagger \psi_n + \psi_{-n}'^\dagger \psi_n' & (4.3.15) \\
 &\quad + \tilde{\psi}_{-n}^\dagger \tilde{\psi}_n + \tilde{\psi}_{-n}'^\dagger \tilde{\psi}_n' + c_{-n}^\alpha b_{\alpha,n}) : + \sum_{n \neq 0} : (\phi_{-n}^+ \phi_n^- + \theta_{-n}^+ \theta_n^-) :
 \end{aligned}$$

is the contribution to  $L_0$  of excitations and is a degree zero quadratic combination of fields. We also have

$$J_0^{tot,3} = j + \tilde{j} + \sum_n \left( : \gamma_n \beta_{-n} : + \frac{1}{2} : \psi_n^\dagger \psi_{-n} : + \frac{1}{2} : \psi_n^{\prime\dagger} \psi'_{-n} : - : \tilde{\gamma}_n \tilde{\beta}_{-n} : \right. \\ \left. - \frac{1}{2} : \tilde{\psi}_n^\dagger \tilde{\psi}_{-n} : - \frac{1}{2} : \tilde{\psi}_n^{\prime\dagger} \tilde{\psi}'_{-n} : - f_\gamma^{3\beta} : c_{\beta,n} b_{-n}^\gamma : \right) \quad (4.3.16a)$$

and

$$J_0^{tot,0} = u + \tilde{u} - \sum_n \left( \frac{1}{2} : \psi_n^\dagger \psi_{-n} : - \frac{1}{2} : \psi_n^{\prime\dagger} \psi'_{-n} : - \frac{1}{2} : \tilde{\psi}_n^\dagger \tilde{\psi}_{-n} : \right. \\ \left. + \frac{1}{2} : \tilde{\psi}_n^{\prime\dagger} \tilde{\psi}'_{-n} : + f_\gamma^{0\beta} : c_{\beta,n} b_{-n}^\gamma : \right). \quad (4.3.16b)$$

For a given HWS  $|j, u, \tilde{j}, \tilde{u}\rangle$  the expression for  $L_0$  restricts the amount of excitations and therefore the degree they carry is finite. Furthermore, the conditions derived in (4.3.5)

$$J_0^{tot,3} |j, u, \tilde{j}, \tilde{u}\rangle = J_0^{tot,0} |j, u, \tilde{j}, \tilde{u}\rangle = 0 \quad (4.3.17)$$

also restrict the contributions of the zero modes  $\gamma_0, \psi_0, \psi'_0$  and  $\tilde{\beta}_0, \tilde{\psi}_0^\dagger, \tilde{\psi}'_0$ . So, on  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$ , the degree carried by any state is bounded, and the method of [61] may apply.

To find the cohomology of  $\hat{Q}^{(0)}$  on  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$  note that

$$\hat{L}_0 = \{ \hat{Q}^{(0)}, \bar{G}_0^{(0)} \} \quad (4.3.18)$$

where we have the zero degree operator

$$\bar{G}_0^{(0)} = \sum_n \left( : nb_{+,n} \tilde{\gamma}_{-n} - nb_{-,n} \gamma_{-n} + nb_{-\frac{1}{2},n} \psi_{-n} - nb_{\frac{1}{2},n} \tilde{\psi}_{-n} + nb_{-\frac{1}{2},n} \psi'_{-n} + \right. \\ \left. nb_{\frac{1}{2},n} \tilde{\psi}'_{-n} : \right) + \frac{\sqrt{2}}{\alpha_+} \sum_n \left( : b_{3,n} \theta_{-n}^+ - b_{0,n} \phi_{-n}^+ : \right) \quad (4.3.19)$$

So, by a previous argument (section 3.8), the non-trivial  $\hat{Q}^{(0)}$  states must be annihilated by  $\hat{L}_0$  i.e., they only contain zero mode excitations, which we now identify.



The most general physical state annihilated by  $\hat{L}_0$  is

$$(\gamma_0)^{n_\gamma} (\psi_0)^{n_\psi} (\psi'_0)^{n_{\psi'}} (\tilde{\beta}_0)^{n_{\tilde{\beta}}} (\tilde{\psi}_0^\dagger)^{n_{\tilde{\psi}^\dagger}} (\tilde{\psi}'_0^\dagger)^{n_{\tilde{\psi}'^\dagger}} (c_\alpha)^{n_{c_\alpha}} | j, u, \tilde{j}, \tilde{u} \rangle \quad (4.3.20)$$

where  $n_\psi, n_{\psi'}, n_{\tilde{\psi}^\dagger}, n_{\tilde{\psi}'^\dagger}, n_{c_+}, n_{c_-}, n_{c_3}$  and  $n_{c_0}$  are restricted to the values 0 and 1 (the zero modes being fermionic), while  $n_\gamma, n_{\tilde{\beta}}, n_{c_{\frac{1}{2}}}, n_{c_{\frac{1}{2}'}} , n_{c_{-\frac{1}{2}}}, n_{c_{-\frac{1}{2}'}}$  are positive integers. They are the eigenvalues of the following number operators

$$n_\gamma = -\gamma_0 \beta_0 \quad , \quad n_{\tilde{\beta}} = \tilde{\beta}_0 \tilde{\gamma}_0 \quad (4.3.21)$$

$$n_\psi = \psi_0 \psi_0^\dagger \quad , \quad n_{\tilde{\psi}^\dagger} = \tilde{\psi}_0^\dagger \tilde{\psi}_0$$

$$n_{\psi'} = \psi'_0 \psi_0'^\dagger \quad , \quad n_{\tilde{\psi}'^\dagger} = \tilde{\psi}'_0^\dagger \tilde{\psi}'_0$$

$$n_{c_\alpha} = c_{\alpha,0} b_0^\alpha \quad , \quad \alpha = \pm, 3, 0, \pm \frac{1}{2}, \pm \frac{1}{2}'$$

where we use the commutation relations

$$[\gamma_n, \beta_m] = [\tilde{\gamma}_n, \tilde{\beta}_m] = \delta_{n+m,0} \quad (4.3.22)$$

$$\{\psi_n^\dagger, \psi_m\} = \{\psi_n'^\dagger, \psi_m'\} = \{\tilde{\psi}_n^\dagger, \tilde{\psi}_m\} = \{\tilde{\psi}_n'^\dagger, \tilde{\psi}_m'\} = \delta_{n+m,0}$$

$$\{b_n^\beta, c_{\alpha,m}\} = \delta_\alpha^\beta \delta_{n+m,0} \quad \alpha, \beta = \pm, 3, 0$$

$$\{b_n^\beta, c_{\alpha,m}\} = \delta_\alpha^\beta \delta_{n+m,0} \quad \alpha, \beta = \pm \frac{1}{2}, \pm \frac{1}{2}'$$

Drastic conditions are imposed on the general state (4.3.20) if it has to be  $\hat{Q}^{(0)}$  non-trivial. Indeed, from the following relations,

$$\{\hat{Q}^{(0)}, b_0^- \tilde{\gamma}_0\} = 1 - n_{c_-} + n_{\tilde{\beta}} \quad (4.3.23a)$$

$$\{\hat{Q}^{(0)}, b_0^+ \gamma_0\} = n_{c_+} + n_\gamma \quad (4.3.23b)$$

$$\{\hat{Q}^{(0)}, b_0^{\frac{1}{2}} \psi_0\} = n_{c_{\frac{1}{2}}} + n_\psi \quad (4.3.23c)$$

$$\{\hat{Q}^{(0)}, b_0^{-\frac{1}{2}} \tilde{\psi}_0\} = 1 + n_{c_{-\frac{1}{2}}} - n_{\tilde{\psi}^\dagger} \quad (4.3.23d)$$

$$\{\hat{Q}^{(0)}, b_0^{\frac{1}{2}'} \tilde{\psi}'_0\} = -n_{c_{\frac{1}{2}'}} - n_{\tilde{\psi}'^\dagger} \quad (4.3.23e)$$

$$\{\hat{Q}^{(0)}, b_0^{-\frac{1}{2}'} \tilde{\psi}'_0\} = 1 + n_{c_{-\frac{1}{2}'}} - n_{\tilde{\psi}'^\dagger} \quad (4.3.23f)$$



one must have  $n_{c_-} = n_{\tilde{\psi}^\dagger} = n_{\tilde{\psi}'^\dagger} = 1$  and all the others zero, so that (4.3.20) becomes

$$c_{-,0} \tilde{\psi}_0^\dagger \tilde{\psi}'_0^\dagger | j, u, \tilde{j}, \tilde{u} \rangle . \tag{4.3.24}$$

Moreover, since  $L_0 = 0$ , states in the cohomology of  $\hat{Q}^{(0)}$  are such that

$$(j - u)(j + u) = (\tilde{j} - \tilde{u})(\tilde{j} + \tilde{u}) \tag{4.3.25}$$

and we also have the conditions  $J_0^{tot,3} = 0$  and  $J_0^{tot,0} = 0$ , then we must impose

$$j + \tilde{j} = 0 \quad , \quad u + \tilde{u} = 0 . \tag{4.3.26}$$

We can therefore conclude that the relative cohomology of  $\hat{Q}^{(0)}$  on  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$  is given by

$$\mathbf{H}_{rel}(\hat{Q}^{(0)}) = \{ c_{-,0} \tilde{\psi}_0^\dagger \tilde{\psi}'_0^\dagger | j, u, -j, -u \} \tag{4.3.27}$$

and one obtains the absolute cohomology [61] by taking

$$\mathbf{H}_{abs}(\hat{Q}^{(0)}) = \mathbf{H}_{rel}(\hat{Q}^{(0)}) \oplus c_{3,0} \mathbf{H}_{rel}(\hat{Q}^{(0)}) \oplus c_{0,0} \mathbf{H}_{rel}(\hat{Q}^{(0)}) . \tag{4.3.28}$$

The cohomology of (4.3.27) is non-trivial for only one ghost number, and only one degree of  $\hat{Q}$  (degree zero), and so, according to the general results of [61], we conclude that there is a one to one correspondence between the cohomology states of  $\hat{Q}^{(0)}$  and those of  $\hat{Q}$ :

$$\mathbf{H}_{rel}(\hat{Q}^{(0)}) = \mathbf{H}_{rel}(\hat{Q}) . \tag{4.3.29}$$

This completes the analysis of the cohomology of  $Q$  on the whole Fock space  $\mathcal{J}(j, u, \tilde{j}, \tilde{u})$ . The cohomology of  $Q$  on the space of irreducible representations requires, for its study, detailed information on the singular vectors appearing in the Verma modules corresponding to  $SL(2|1)$  at fractional level, and is beyond the scope of this work. This will be touched upon in the conclusion.

# Conclusions and Speculations

*And what you thought you came for,  
Is only a shell, a husk of meaning  
From which the purpose breaks only when it is fulfilled,  
If at all. Either you had no purpose  
Or the purpose is beyond the end you figured  
And is altered in fulfilment.*

From *Little Gidding*  
by T. S. Eliot

Modern theoretical physics has branched into various lines of enquiry each of which is pursued by many talented individuals. Some of the major areas of work include solid state physics, particle physics, theoretical astronomy, Q.C.D., and the

quest for a grand unified field theory. Each of these subjects divides further, and each overlaps — to some degree — with the others.

String theory is a fine example of this behaviour. It was a theory originally conceived to explain the strong nuclear interaction, although this role was later taken on by Q.C.D., so it then found itself put forward as a grand unified field theory. Its usefulness is not just confined to abstract physics but is also readily demonstrated in solid state physics where string theory has been found to provide exact solutions to one and two dimensional problems — in addition to providing new pathways of discovery for the pure and applied mathematician alike.

Yet in its modern guise as a grand unified field theory string theory has reached a level of sophistication that makes the extraction of exact mathematical results difficult. To progress further many turned to matrix modelling which was successful in first providing results, and then guiding the analytical approach.

However, the extension of the matrix models to supersymmetry encountered problems which can be traced to the spin statistics of fermionic particles. As a result the study of the models of fermionic strings can only proceed via the continuum approach — at the time of writing.

The challenge of tackling the continuum method was taken up by Fan and Yu [17,18] for the  $N = 1$  supersymmetric case, to be followed — at least initially — by Bershadsky and Ooguri [53,54] for the  $N = 2$  model. Building on from this work the gauged, twisted  $SL(2|1)/SL(2|1)$  Wess-Zumino-Novikov-Witten model has been advocated — in the preceding pages — as a possible description of the  $N = 2$  non-critical supersymmetric string.

The results of this dissertation can be broken into three, corresponding to the last three chapters. In chapter two the details of the  $SL(2|1)$  WZNW model are calculated. Although this is a vital first step toward determining such features as the currents, which are required later on in chapter four, another result has been established for the first time that potentially could be of great use. This result is the free field transformation which connects the two types of currents that

accompany the two interpretations of the algebra. As the method of extraction of these transformations is simple — despite the highly non-linear nature of the currents — it is hard to believe that the same approach is not also valid for any Lie super group. The importance of the field transformation is not clear. It at least establishes for the first time that the quantum WZNW model is independent of the choice of interpretation of the algebra used, yet the transformation also allows us to easily calculate all free field representations, and so allow us to choose a free field parameterization that takes as simple a form as possible. At the present time it is thus a computational aid. However, we may speculate that it could be more than this. Since both types of currents describe the same model and presumably generate the same physical space then it is not inconceivable to assume that the field transformation is representative of an invariance of the physical space. A feature, which if quantified, could reduce the computational work required in the analysis of the cohomology of the physical states of the WZNW model.

The second major result of this thesis is that which is presented in chapter three — the assertion that the gauged, twisted  $SL(2|1)/SL(2|1)$  WZNW model can be presented as a description of the  $N = 2$  supersymmetric string. The procedure of gauging and twisting the WZNW model is well known and in sections 3.2, 3.3 and 3.4 it is shown that the gauging and twisting results in a match between the ghosts of the gauged, twisted WZNW model and the ghosts of the  $N = 2$  supersymmetric string. Some additional ghosts are also present which tensor the model. The Hamiltonian reduction then shows that if some additional fermions are introduced, also tensoring the model, then constraints are present on the currents which reduce the gauged, twisted WZNW model to a description of the  $N = 2$  superstring. This reduction naturally uncovers the  $N = 2$  super Liouville sector and so gives us a direction to pursue toward the quantization of the  $N = 2$  superstring off the critical dimension.

The process of Hamiltonian reduction also uncovers an  $N = 2$  superconformal algebra which is one of the defining features of a topological conformal field theory,

along with the vanishing of the central charge. This  $N = 2$  superconformal algebra is formed from the fermions and ghosts that are introduced as tensoring the gauged, twisted, WZNW model and the final section of chapter three shows that these fields do not make substantial contributions to the cohomology of physical states of the theory.

As in the case of the field transformation we are in a position to speculate about further generalizations to greater degrees of supersymmetry. This is possible because this work, along with the work of Fan and Yu [17,18] for the  $N = 1$  case, suggests patterns in the results of the gauged, twisted WZNW model, which are strengthened by the logic and consistency in the generalization. We may speculate that the gauging, twisting, and Hamiltonian reduction introduces tensoring system of fermions and ghosts  $(\Delta, \bar{\Delta}), (\beta, \gamma)$  for each additional degree of supersymmetry. In addition we can assert that each of these sets of fields do not affect the cohomology of physical states. Except for the existence of a vacuum state the cohomology of each set of additional fields is empty.

Chapter four takes a closer look at the space of physical states, and starts an analysis which, along with the results of [25], should complete the understanding of the cohomology of the  $N = 2$  string. This chapter extends a result from the  $N = 1$  case — namely that there exists an isomorphism between the Ramond and Neveu-Schwarz sectors, and then progresses in turn to a calculation of the BRST charge. This calculation is found to be relatively simple and concludes with the important results of equations (4.3.27) and (4.3.28).

This by itself is not enough for the full cohomology as we require information about the irreducible representations of  $Sl(2|1)_k$  which relies on some detailed knowledge of the admissible representations, and this is beyond the scope of this thesis.

In short, admissible modules for  $SL(2|1)$  are characterized by the fact that they contain an infinite number of singular vectors (which requires  $k + 1 = \frac{p}{q}$  for  $p$  and  $q$  relatively prime) and that the corresponding irreducible representa-

tions (usually non-integrable) have characters which transform as finite representations of the modular group. When the full spectrum of the irreducible module is known [25] then the complete cohomology of the gauged, twisted,  $SL(2|1)/SL(2|1)$  WZNW model can be determined by removing the spurious states.

When this is done we can claim to possess a firm understanding of the non-critical  $N = 2$  string and can perhaps then turn our attention to other topics such as the  $N = 4$  superstring which is represented by the  $D(2|1; \alpha)/D(2|1; \alpha)$  gauged, twisted WZNW model.

This draws to a close this investigation into the relationship between the gauged, twisted Wess-Zumino-Novikov-Witten model for the group  $SL(2|1)$  and the non-critical  $N = 2$  supersymmetric string. Some new results and some interesting avenues of exploration have been presented, and the task now is to pursue these to their end. One thing seems to be sure : the knowledge gained from much of the work that has in the past been undertaken on the topic of strings has found usefulness in other — often seemingly unrelated — fields of physics and mathematics.

It will undoubtedly be interesting to watch how the subject develops.

## Appendix A

### Details of the Algebra $sl(2|1)$

The following material presents specific details on the group  $SL(2|1)$  and its algebra  $sl(2|1)$ . The group can be formed out of eight generators  $\tau^\alpha$  with  $G = G_0 + G_1$  where  $G_0$  denotes the bosonic subgroup and  $G_1$  the fermionic sector. There are four bosonic generators  $\tau^\alpha \in G_0$  with  $\alpha = 1, 2, 3, 4$  and four fermionic generators  $\tau^\alpha \in G_1$  with  $\alpha = 5, 6, 7, 8$ . The generators satisfy the Lie bracket :

$$[\tau^\alpha, \tau^\beta] = f_\gamma^{\alpha\beta} \tau^\gamma \quad (A.1)$$

To define the Lie bracket we need to introduce the concept of a degree of a generator  $d(\tau^\alpha)$ , where :

$$d(\tau^\alpha) = 0 \quad \text{if } \tau \text{ is bosonic} \quad (A.2)$$

$$d(\tau^\alpha) = 1 \quad \text{if } \tau \text{ is fermionic}$$

which allows us to define the Lie bracket :

$$[A, B] = AB - (-1)^{d(A)d(B)} BA. \quad (A.3)$$

The generators also satisfy the super Jacobi identity

$$(-1)^{d(A)d(C)} [A, [B, C]] + (-1)^{d(B)d(A)} [B, [C, A]] + (-1)^{d(C)d(B)} [C, [A, B]] = 0. \quad (A.4)$$

There are other features that should be mentioned. The dimension of any group is defined as the number of generators, which is eight in the case of  $SL(2|1)$ , but the super dimension is defined as the difference between the dimension of the bosonic subalgebra and that of the fermionic sector :

$$\begin{aligned} \text{sdim } G &= \dim G_0 - \dim G_1 \\ &= 0 \quad \text{for } SL(2|1). \end{aligned} \quad (A.5)$$



In addition we can define the supertrace of a matrix  $M$  partitioned as  $(n + m) \times (n + m)$  to be :

$$\text{STr } M = \text{Tr } M_{(n \times n)} - (-1)^{d(M)} \text{Tr } M_{(m \times m)} \quad (\text{A.6})$$

where  $M_{(n \times n)}$  is the submatrix of the top left hand corner and  $M_{(m \times m)}$  the submatrix of the bottom right hand corner, with  $d(M)$  taking value 0 if both  $M_{(n \times n)}$  and  $M_{(m \times m)}$  have elements bosonic in character, and  $d(M)$  taking value 1 if both  $M_{(n \times n)}$  and  $M_{(m \times m)}$  have elements fermionic in character.

Having defined the structure constants in equation (A.1) we are now also free to define a metric for the group through the expression :

$$h^{ab} = (-1)^{d(e)} f_e^{ac} f_c^{be}. \quad (\text{A.7})$$

It is easy to read off the structure constants from the commutators of equation (2.3.8), and determine the elements of the metric for Type B as having zero components except for the following :

$$\begin{aligned} h^{33} &= 1 & h^{00} &= -1 & h^{+-} &= 2 & h^{-+} &= 2 & (\text{A.8}) \\ h^{\frac{1}{2}, -\frac{1}{2}} &= 2 & h^{-\frac{1}{2}, \frac{1}{2}} &= -2 & h^{\frac{1}{2}', -\frac{1}{2}'} &= -2 & h^{-\frac{1}{2}', \frac{1}{2}'} &= 2 \end{aligned}$$

The inverse metric for Type B is defined by  $h^{ab} h_{bc} = \delta_c^a$  and has non zero elements

$$\begin{aligned} h_{33} &= 1 & h_{00} &= -1 & h_{+-} &= \frac{1}{2} & h_{-+} &= \frac{1}{2} & (\text{A.9}) \\ h_{\frac{1}{2}, -\frac{1}{2}} &= -\frac{1}{2} & h_{-\frac{1}{2}, \frac{1}{2}} &= \frac{1}{2} & h_{\frac{1}{2}', -\frac{1}{2}'} &= \frac{1}{2} & h_{-\frac{1}{2}', \frac{1}{2}'} &= -\frac{1}{2}. \end{aligned}$$

As expected the components for the metric for the Type A algebra are obtained by interchanging  $j^+$  and  $j^-$  i.e.,

$$h^{33} = 1 \quad h^{00} = -1 \quad h^{+-} = 2 \quad h^{-+} = 2 \quad (\text{A.10})$$

$$h^{\frac{1}{2}, -\frac{1}{2}} = -2 \quad h^{-\frac{1}{2}, \frac{1}{2}} = 2 \quad h^{\frac{1}{2}', -\frac{1}{2}'} = -2 \quad h^{-\frac{1}{2}', \frac{1}{2}'} = 2.$$

With the information above it is possible to construct the adjoint representation that has generators define in the following way

$$(F^\alpha)_\gamma^\beta = f_\gamma^{\alpha\beta} \quad (\text{A.11})$$

with additional definitions for the adjoint being :

$$h^{\alpha\beta} = \text{STr}(F^\alpha F^\beta) \quad \text{STr} F = \sum_\alpha (-1)^{d(\alpha)} F_\alpha^\alpha \quad (\text{A.12})$$

which can be used to calculate the dual Coxeter number in the adjoint representation of  $SL(2|1)$  :

$$c_G \text{ for } SL(2|1) = 1 \quad (\text{A.13})$$

## Appendix B

### Proof of the Field Transformation

Presented in the following few pages is an alternative analytical method of deriving the field transformation, using a proof discovered by Peter Bowcock and Anne Taormina, who based it on the work presented in Section 2.5 .

Throughout this Appendix we will make use of a corollary to the Baker-Campbell-Hausdorff formula, equation (B.1), and its various related identities :

$$\exp(A + B) = \exp A \exp B \exp\left(-\frac{1}{2}[A, B]\right) \quad (B.1.a)$$

$$\exp A \exp B = \exp\left(A + B + \frac{1}{2}[A, B]\right) \quad (B.1.b)$$

which holds provided that  $[A, [A, B]] = [A, [A, B]] = 0$ , and where the bracket  $[A, B]$  is an anticommutator if both  $A$  and  $B$  are fermionic.

We will also require the following lemmas.

#### First Lemma

$$\exp(\tilde{\psi}_{-j^+}) \exp(\tilde{\phi}U + \tilde{\theta}J^3) = \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\alpha\tilde{\psi}_{-j^+})$$

where we have defined  $\alpha = \exp \frac{1}{2}\{\tilde{\phi} - \tilde{\theta}\}$ .<sup>†</sup>

#### Proof

Consider :

$$T = \exp(-\tilde{\phi}U - \tilde{\theta}J^3)\tilde{\psi}_{-j^+} \exp(\tilde{\phi}U + \tilde{\theta}J^3)$$

then :

$$\frac{\partial T}{\partial \tilde{\phi}} = \frac{1}{2}T \quad ; \quad \frac{\partial T}{\partial \tilde{\theta}} = -\frac{1}{2}T$$

---

<sup>†</sup> The notations used in this Appendix are the same as those used in chapter two of this thesis, with a tilde used to distinguish fields belong to the Type A interpretation from the fields of the Type B interpretation.

which implies :

$$T = \tilde{\psi}_- \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} j^+ = \alpha \tilde{\psi}_- j^+ .$$

Next consider :

$$\begin{aligned} & \exp(-\tilde{\phi}U - \tilde{\theta}J^3) \exp(\tilde{\psi}_- j^+) \exp(\tilde{\phi}U + \tilde{\theta}J^3) \\ &= \exp(-\tilde{\phi}U - \tilde{\theta}J^3) (1 + \tilde{\psi}_- j^+) \exp(\tilde{\phi}U + \tilde{\theta}J^3) \\ &= 1 + \alpha \tilde{\psi}_- j^+ = \exp(\alpha \tilde{\psi}_- j^+) \end{aligned}$$

and therefore :

$$\exp(\tilde{\psi}_- j^+) \exp(\tilde{\phi}U + \tilde{\theta}J^3) = \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\alpha \tilde{\psi}_- j^+) .$$

*Q.E.D.*

## Second Lemma

$$\exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^-) = \exp(\alpha \tilde{\psi}_+ j^-) \exp(\tilde{\phi}U + \tilde{\theta}J^3)$$

### Proof

Consider :

$$T = \exp(\tilde{\phi}U + \tilde{\theta}J^3) \tilde{\psi}_+ j^- \exp(-\tilde{\phi}U - \tilde{\theta}J^3)$$

then :

$$\frac{\partial T}{\partial \tilde{\phi}} = \frac{1}{2} T \quad ; \quad \frac{\partial T}{\partial \tilde{\theta}} = -\frac{1}{2} T$$

which implies :

$$T = \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} j^- = \alpha \tilde{\psi}_+ j^- .$$

Next consider :

$$\begin{aligned} & \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^-) \exp(-\tilde{\phi}U - \tilde{\theta}J^3) \\ &= 1 + \alpha \tilde{\psi}_+ j^- = \exp(\alpha \tilde{\psi}_+ j^-) \end{aligned}$$

and therefore :

$$\exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^-) = \exp(\alpha \tilde{\psi}_+ j^-) \exp(\tilde{\phi}U + \tilde{\theta}J^3) .$$

*Q.E.D.*

For convenience it is simplest to express the mathematics purely in terms of one set of generators. We are free to do this since the Type A interpretation, and commutators, are gained from Type B by interchanging the  $j^+$  and  $j^-$  generators. In the following the Type B set has been selected, and its commutator relations used (equation (2.3.8b)).

In this basis of generators the Gauss decomposition for the Type A group element has the form :

$$g^A = g_< g_0 g_>$$

$$g^A = \exp(\tilde{\psi}_- j^+ + \tilde{\psi}'_- j'^- + \tilde{\gamma} J^-) \exp(\tilde{\phi} U + \tilde{\theta} J^3) \exp(\tilde{\psi}_+ j^- + \tilde{\psi}'_+ j'^+ + \tilde{\xi} J^+).$$

We may now make use of equation (B.1.b) by defining  $A = \tilde{\psi}'_- j'^- + \tilde{\gamma} J^-$  and  $B = \tilde{\psi}_- j^+$  and considering the following :

$$\begin{aligned} \exp(\tilde{\gamma} J^- + \tilde{\psi}'_- j'^-) \exp(\tilde{\psi}_- j^+) &= \\ \exp(\tilde{\gamma} J^- + \tilde{\psi}'_- j'^- + \tilde{\psi}_- j^+ + \frac{1}{2}[\tilde{\gamma} J^- + \tilde{\psi}'_- j'^-, \tilde{\psi}_- j^+]) & \end{aligned}$$

which we can rewrite using the commutator relations (2.3.8b) as :

$$\exp(\tilde{\gamma} J^- + \tilde{\psi}'_- j'^-) \exp(\tilde{\psi}_- j^+) = \exp(\tilde{\gamma} J^- + \tilde{\psi}'_- j'^- + \tilde{\psi}_- j^+ + \frac{1}{2} \tilde{\gamma} \tilde{\psi}_- j'^-)$$

or, if we rewrite  $\tilde{\psi}'_- = \tilde{\psi}'_- - \frac{1}{2} \tilde{\gamma} \tilde{\psi}_-$  :

$$\exp(\tilde{\gamma} J^- + \tilde{\psi}'_- j'^- + \tilde{\psi}_- j^+) = \exp(\tilde{\gamma} J^- + (\tilde{\psi}'_- - \frac{1}{2} \tilde{\gamma} \tilde{\psi}_-) j'^-) \exp(\tilde{\psi}_- j^+)$$

which re-expresses the  $g_<$  sector.

Via a similar argument it is possible to rewrite the  $g_>$  sector as :

$$\exp(\tilde{\psi}_+ j^- + \tilde{\psi}'_+ j'^+ + \tilde{\xi} J^+) = \exp(\tilde{\psi}_+ j^-) \exp(\tilde{\xi} J^+ + (\tilde{\psi}'_+ - \frac{1}{2} \tilde{\xi} \tilde{\psi}_+) j'^+).$$

Thus the Gauss decomposition of  $g^A$  may be written as a product of five exponentials :

$$\begin{aligned} g^A &= \exp(\tilde{\gamma} J^- + (\tilde{\psi}'_- - \frac{1}{2} \tilde{\gamma} \tilde{\psi}_-) j'^-) \exp(\tilde{\psi}_- j^+) \exp(\tilde{\phi} U + \tilde{\theta} J^3) \times \\ &\exp(\tilde{\psi}_+ j^-) \exp(\tilde{\xi} J^+ + (\tilde{\psi}'_+ - \frac{1}{2} \tilde{\xi} \tilde{\psi}_+) j'^+). \end{aligned}$$

Consider the product of the central three exponentials, namely :

$$\exp(\tilde{\psi}_- j^+) \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^-).$$

Using the first lemma we can rewrite this as :

$$\exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\alpha\tilde{\psi}_- j^+) \exp(\tilde{\psi}_+ j^-).$$

and now make repeated use of equations (B.1.a) and (B.1.b) :

$$\begin{aligned} &= \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^- + \alpha\tilde{\psi}_- j^+ + \frac{1}{2}\alpha\tilde{\psi}_- \tilde{\psi}_+ \{j^+, j^-\}) \\ &= \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\tilde{\psi}_+ j^-) \exp(\alpha\tilde{\psi}_- j^+ + \alpha\tilde{\psi}_- \tilde{\psi}_+ \{j^+, j^-\}) \end{aligned}$$

and then use the second lemma, and the commutator relations to express this as :

$$\begin{aligned} &= \exp(\alpha\tilde{\psi}_+ j^-) \exp(\tilde{\phi}U + \tilde{\theta}J^3) \exp(\alpha\tilde{\psi}_- j^+ + \alpha\tilde{\psi}_- \tilde{\psi}_+ (J^3 + U)) \\ &= \exp(\alpha\tilde{\psi}_+ j^-) \exp((\tilde{\phi} + \alpha\tilde{\psi}_- \tilde{\psi}_+)U + (\tilde{\theta} + \alpha\tilde{\psi}_- \tilde{\psi}_+)J^3) \exp(\alpha\tilde{\psi}_- j^+) \end{aligned}$$

If we now gather these results, we have succeeded in writing  $g^A$  in the following form :

$$\begin{aligned} g^A &= \exp((\tilde{\psi}'_- - \frac{1}{2}\tilde{\gamma}\tilde{\psi}_-)j'^- + \tilde{\gamma}J^-) \times \\ &\quad \exp(\alpha\tilde{\psi}_+ j^-) \exp((\tilde{\phi} + \alpha\tilde{\psi}_- \tilde{\psi}_+)U + (\tilde{\theta} + \alpha\tilde{\psi}_- \tilde{\psi}_+)J^3) \exp(\alpha\tilde{\psi}_- j^+) \times \\ &\quad \exp((\tilde{\psi}'_+ - \frac{1}{2}\tilde{\xi}\tilde{\psi}_+)j'^+ + \tilde{\xi}J^+) \end{aligned}$$

and again we can use equation (B.1) to express this as :

$$\begin{aligned} g^A &= \\ &\exp((\tilde{\psi}'_- - \frac{1}{2}\tilde{\gamma}\tilde{\psi}_-)j'^- + \tilde{\gamma}J^- + \alpha\tilde{\psi}_+ j^- + \frac{1}{2}[(\tilde{\psi}'_- - \frac{1}{2}\tilde{\gamma}\tilde{\psi}_-)j'^- + \tilde{\gamma}J^-, \alpha\tilde{\psi}_+ j^-]) \times \\ &\exp((\tilde{\phi} + \alpha\tilde{\psi}_- \tilde{\psi}_+)U + (\tilde{\theta} + \alpha\tilde{\psi}_- \tilde{\psi}_+)J^3) \times \\ &\exp((\tilde{\psi}'_+ - \frac{1}{2}\tilde{\xi}\tilde{\psi}_+)j'^+ + \tilde{\xi}J^+ + \alpha\tilde{\psi}_- j^+ + \frac{1}{2}[\alpha\tilde{\psi}_- j^+, (\tilde{\psi}'_+ - \frac{1}{2}\tilde{\xi}\tilde{\psi}_+)j'^+ + \tilde{\xi}J^+]) \end{aligned}$$

or

$$\begin{aligned} g^A &= \exp((\tilde{\psi}'_- - \frac{1}{2}\tilde{\gamma}\tilde{\psi}_-)j'^- + (\tilde{\gamma} + \frac{1}{2}\alpha\tilde{\psi}'_- \tilde{\psi}_+ - \frac{1}{4}\alpha\tilde{\gamma}\tilde{\psi}_- \tilde{\psi}_+)J^- + \alpha\tilde{\psi}_+ j^-) \times \\ &\quad \exp((\tilde{\phi} + \alpha\tilde{\psi}_- \tilde{\psi}_+)U + (\tilde{\theta} + \alpha\tilde{\psi}_- \tilde{\psi}_+)J^3) \times \\ &\quad \exp((\tilde{\psi}'_+ - \frac{1}{2}\tilde{\xi}\tilde{\psi}_+)j'^+ + \alpha\tilde{\psi}_- j^+ + (\tilde{\xi} + \frac{1}{2}\alpha\tilde{\psi}_- \tilde{\psi}'_+ - \frac{1}{4}\alpha\tilde{\xi}\tilde{\psi}_- \tilde{\psi}_+)J^+) \end{aligned}$$

The Gauss decomposition for Type B has the form :

$$g^B = \exp(\psi_- j^- + \psi'_- j'^- + \gamma J^-) \exp(\phi U + \theta J^3) \exp(\psi_+ j^+ + \psi'_+ j'^+ + \xi J^+)$$

and by direct comparison it is easy to deduce the field transformations as those given in (2.5.8), namely :

$$\begin{aligned} \theta &\mapsto \tilde{\theta} + \tilde{\psi}_- \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \\ \phi &\mapsto \tilde{\phi} + \tilde{\psi}_- \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \\ \psi_- &\mapsto \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \\ \psi_+ &\mapsto \tilde{\psi}_- \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \\ \psi'_- &\mapsto \tilde{\psi}'_- - \frac{1}{2} \tilde{\gamma} \tilde{\psi}_- \\ \psi'_+ &\mapsto \tilde{\psi}'_+ - \frac{1}{2} \tilde{\xi} \tilde{\psi}_+ \\ \gamma &\mapsto \tilde{\gamma} + \frac{1}{2} \tilde{\psi}'_- \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} - \frac{1}{4} \tilde{\gamma} \tilde{\psi}_- \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \\ \xi &\mapsto \tilde{\xi} + \frac{1}{2} \tilde{\psi}_- \tilde{\psi}'_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} - \frac{1}{4} \tilde{\xi} \tilde{\psi}_- \tilde{\psi}_+ \exp \frac{1}{2} \{ \tilde{\phi} - \tilde{\theta} \} \end{aligned}$$

*Q.E.D.*

## Appendix C

### Modal Decomposition of Currents

This appendix lists the decomposition of the currents into modal form. Both the  $J^\alpha$  and  $\tilde{J}^\alpha$  currents are listed for the Type B interpretation. As before we have put  $\alpha_+ = \sqrt{2k+2}$ , while for compactness we have abbreviated the following symbols for the free fields :

$$\begin{aligned}
 \psi_+ &\leftrightarrow \psi & [\psi_+]^\dagger &\leftrightarrow \psi^\dagger \\
 \psi'_+ &\leftrightarrow \psi' & [\psi'_+]^\dagger &\leftrightarrow \psi'^\dagger \\
 \partial\theta &\leftrightarrow \theta & i\partial\phi &\leftrightarrow \phi
 \end{aligned}
 \tag{C.1}$$

This should cause no confusion. In chapter two the introduction of the subscripts  $+, -$  was employed so that the reader could clearly understand the origin of each field, but this is no longer necessary. The terms in the currents are normal ordered in the fashion indicated in equation (2.7.3), and with respect to the  $SL_2$  invariant vacuum i.e.,

$$\begin{aligned}
 \beta_n |j, u, \tilde{j}, \tilde{u}\rangle_R &= \psi_n^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_R = \psi_n'^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_R = \tilde{\beta}_n |j, u, \tilde{j}, \tilde{u}\rangle_R = \\
 \tilde{\psi}_n^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_R &= \tilde{\psi}_n'^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_R = 0 \quad \forall n \geq 0
 \end{aligned}
 \tag{C.2}$$

and

$$\begin{aligned}
 \gamma_n |j, u, \tilde{j}, \tilde{u}\rangle_R &= \psi_n |j, u, \tilde{j}, \tilde{u}\rangle_R = \psi_n' |j, u, \tilde{j}, \tilde{u}\rangle_R = \tilde{\gamma}_n |j, u, \tilde{j}, \tilde{u}\rangle_R = \\
 \tilde{\psi}_n |j, u, \tilde{j}, \tilde{u}\rangle_R &= \tilde{\psi}_n' |j, u, \tilde{j}, \tilde{u}\rangle_R = 0 \quad \forall n > 0
 \end{aligned}
 \tag{C.3}$$

in the Ramond sector.

In the Neveu-Schwarz sector, the normal ordering is with respect to the vacuum  $|j, u, \tilde{j}, \tilde{u}\rangle_{NS}$  which satisfies :

$$\beta_n |j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \psi_{n+\frac{1}{2}}^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \psi_{n+\frac{1}{2}}'^\dagger |j, u, \tilde{j}, \tilde{u}\rangle_{NS} =$$



$$\begin{aligned}
 \psi_{n+\frac{1}{2}}|j, u, \tilde{j}, \tilde{u}\rangle_{NS} &= \psi'_{n+\frac{1}{2}}|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \tilde{\beta}_n|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \\
 \tilde{\psi}_{n+\frac{1}{2}}^\dagger|j, u, \tilde{j}, \tilde{u}\rangle_{NS} &= \tilde{\psi}'_{n+\frac{1}{2}}^\dagger|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \tilde{\psi}_{n+\frac{1}{2}}|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \\
 \tilde{\psi}'_{n+\frac{1}{2}}|j, u, \tilde{j}, \tilde{u}\rangle_{NS} &= 0 \quad \forall n \geq 0
 \end{aligned} \tag{C.4}$$

and

$$\gamma_n|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = \tilde{\gamma}_n|j, u, \tilde{j}, \tilde{u}\rangle_{NS} = 0 \quad \forall n > 0 \tag{C.5}$$

We use the expansions for the free fields given in (4.2.4) and the  $sl(2|1)$  currents in (2.6.3) to find the currents in modal form :

### Currents $J^\alpha$ at level $k$

$$\begin{aligned}
 J_n^+ &= -\beta_n \tag{C.6} \\
 J_n^3 &= : \gamma_m \beta_{n-m} : + \frac{\alpha_+}{2} \theta_n + \frac{1}{2} : \psi_m^\dagger \psi_{n-m} : + \frac{1}{2} : \psi'_m \psi'_{n-m} : - \frac{\epsilon}{4} \delta_{n,0} \\
 U_n &= \frac{i\alpha_+}{2} \phi_n - \frac{1}{2} : \psi_m^\dagger \psi_{n-m} : + \frac{1}{2} : \psi'_m \psi'_{n-m} : \\
 J_n^- &= \circ \gamma_m : \gamma_p \beta_{n-m-p} : \circ - \frac{i\alpha_+}{2} \circ \phi_m : \psi'_p \psi_{n-m-p} : \circ + \alpha_+ : \theta_m \gamma_{n-m} : \\
 &\quad - \left( \frac{k+1}{2} \right) m : \psi_m \psi'_{n-m} : - \left( \frac{k+1}{2} \right) m : \psi'_m \psi_{n-m} : \\
 &\quad - \circ \gamma_m : \psi_p \psi_{n-m-p}^\dagger : \circ - \circ \gamma_m : \psi'_p \psi'_{n-m-p}^\dagger : \circ \\
 &\quad - kn \gamma_n - \frac{\epsilon}{2} \gamma_n \delta_{n,0} \\
 j_n^- &= \frac{1}{2} \circ \gamma_m : \beta_p \psi_{n-m-p} : \circ + \frac{\alpha_+}{2} : (\theta_m + i\phi_m) \psi_{n-m} : - \left( k + \frac{1}{2} \right) n \psi_n \\
 &\quad + \frac{1}{2} \circ \psi'_m \psi'_p : \psi'_p \psi_{n-m-p} : \circ + : \gamma_m \psi'_{n-m}^\dagger : \\
 j_n'^- &= \frac{1}{2} \circ \gamma_m : \beta_p \psi'_{n-m-p} : \circ + \frac{\alpha_+}{2} : (\theta_m - i\phi_m) \psi'_{n-m} : - \left( k + \frac{1}{2} \right) n \psi'_n \\
 &\quad - \frac{1}{2} \circ \psi_m^\dagger \psi'_p : \psi'_p \psi_{n-m-p} : \circ + : \gamma_m \psi_{n-m}^\dagger : \\
 j_n^+ &= \frac{1}{2} : \beta_m \psi'_{n-m} : + \psi_n^\dagger
 \end{aligned}$$

$$j_n'^+ = -\frac{1}{2} : \beta_m \psi_{n-m} : - \psi_n'^\dagger$$

where  $\epsilon = 0$  for the Ramond sector (i.e., when the modes of the fermionic currents are integer), and  $\epsilon = 1$  for the Neveu-Schwarz sector (modes of the fermionic currents half integer).

The following set of currents can be obtained from the previous set by using an automorphism of order 4 on the Kac-Moody subalgebra :

$$\begin{aligned} \sigma(J^\pm) &= -J^\mp ; \quad \sigma(J^3) = -J^3 ; \quad \sigma(U) = -U \\ \sigma(j^\pm) &= \pm j^\mp ; \quad \sigma(j'^\pm) = \mp j'^\mp ; \quad \sigma(k) = k \end{aligned} \quad (C.7)$$

and by subsequently replacing  $k$  by  $-(k+2)$  or, equivalently,  $\alpha_+$  by  $i\alpha_+$ .

Currents  $\tilde{J}^\alpha$  at level  $-(k+2)$

$$\begin{aligned} \tilde{J}_n^+ &= -\circ\tilde{\gamma}_m : \tilde{\gamma}_p \tilde{\beta}_{n-m-p} : \circ - \frac{\alpha_+}{2} \circ\tilde{\phi}_m : \tilde{\psi}'_p \tilde{\psi}_{n-m-p} : \circ \\ &\quad - i\alpha_+ : \tilde{\theta}_m \tilde{\gamma}_{n-m} : - \left(\frac{k+1}{2}\right) m : \tilde{\psi}_m \tilde{\psi}'_{n-m} \\ &\quad - \left(\frac{k+1}{2}\right) m : \tilde{\psi}'_m \tilde{\psi}_{n-m} : + \circ\tilde{\gamma}_m : \tilde{\psi}_p \tilde{\psi}_{n-m-p}^\dagger : \circ + \\ &\quad \circ\tilde{\gamma}_m : \tilde{\psi}'_p \tilde{\psi}_{n-m-p}^\dagger : \circ - (k+2)n\tilde{\gamma}_n + \frac{\epsilon}{2}\tilde{\gamma}_n\delta_{n,0} \\ \tilde{J}_n^3 &= - : \tilde{\gamma}_m \tilde{\beta}_{n-m} : - \frac{i\alpha_+}{2} \tilde{\theta}_n - \frac{1}{2} : \tilde{\psi}_m^\dagger \tilde{\psi}_{n-m} : - \frac{1}{2} : \tilde{\psi}'_m^\dagger \tilde{\psi}'_{n-m} : + \frac{\epsilon}{4}\delta_{n,0} \\ \tilde{U}_n &= \frac{\alpha_+}{2} \tilde{\phi}_n + \frac{1}{2} : \tilde{\psi}_m^\dagger \tilde{\psi}_{n-m} : - \frac{1}{2} : \tilde{\psi}'_m^\dagger \tilde{\psi}'_{n-m} : \\ \tilde{J}_n^- &= \tilde{\beta}_n \\ \tilde{j}_n^- &= \frac{1}{2} : \tilde{\beta}_m \tilde{\psi}'_{n-m} : + \tilde{\psi}_n^\dagger \\ \tilde{j}'_n^- &= \frac{1}{2} : \tilde{\beta}_m \tilde{\psi}_{n-m} : + \tilde{\psi}'_n^\dagger \\ \tilde{j}_n^+ &= -\frac{1}{2} \circ\tilde{\gamma}_m : \tilde{\beta}_p \tilde{\psi}_{n-m-p} : \circ - \frac{i\alpha_+}{2} : (\tilde{\theta}_m + i\tilde{\phi}_m) \tilde{\psi}_{n-m} : - \left(k + \frac{3}{2}\right) n \tilde{\psi}_n \end{aligned} \quad (C.8)$$

$$\begin{aligned}
& -\frac{1}{2} \circ \tilde{\psi}_m^\dagger : \tilde{\psi}'_p \tilde{\psi}_{n-m-p} : \circ - : \tilde{\gamma}_m \tilde{\psi}_{n-m}^\dagger : \\
\tilde{j}_n'^+ & = \frac{1}{2} \circ \gamma_m : \tilde{\beta}_p \tilde{\psi}'_{n-m-p} : \circ + \frac{i\alpha_+}{2} : (\tilde{\theta}_m - i\tilde{\phi}_m) \tilde{\psi}_{n-m} : + \left(k + \frac{3}{2}\right) n \tilde{\psi}'_n \\
& - \frac{1}{2} \circ \tilde{\psi}_m^\dagger : \tilde{\psi}'_p \tilde{\psi}_{n-m-p} : \circ + : \tilde{\gamma}_m \tilde{\psi}_{n-m}^\dagger :
\end{aligned}$$

## Appendix D

### REDUCE Program used for Calculations

There are various mathematical packages on the market which may be useful in calculating the currents and field transformations of Lie super algebras. However, it has been discovered that none of these could handle Grassmannian variables as part of their intrinsic programming, although many could be programmed — with some difficulty — to do so.

By far the easiest package to use and to customize to our requirements was the package REDUCE. The following pages list the REDUCE commands necessary to calculate the currents of equations (2.4.6) and (2.4.9), and with a little further effort, these same commands could be supplemented with others to perform any calculation required in chapter 2.

This information is provided to assist others endeavouring to produce similar work. Unfortunately REDUCE does not use Greek symbols. In the following program the fields valued in the negative Borel subalgebra,  $\xi, \psi_-, \psi'_-$  are represented by operators  $f, m(), n()$  respectively. Fields in the positive Borel subalgebra  $\gamma, \psi_+, \psi'_+$  are represented by  $h, a(), b()$  respectively, and  $\theta$  and  $\phi$  — fields in the Cartan subalgebra — are denoted by  $t$  and  $p$  respectively.

A derivative of a field has a  $d$  in front of its operator, i.e.,  $dm()$  is the derivative of the field  $m()$ . A  $\%$  symbol denotes a comment line.

### The REDUCE Program

```
% linelength changes the size of the display
```

```
linelength 130 ;
```

```
% define the following to be operators
```

```
operator a,b,h,p,t,m,n,f,da,db,dh,dm,dn;
```

```
% define the operators to be noncommuting, this is needed to prevent a program-
```

```

ming loop while trying to define the operators to be anticommuting noncom a,b ;
noncom a,da;
noncom a,db;
noncom a,m ;
noncom a,dm;
noncom a,n ;
noncom a,dn;
noncom b,da;
noncom b,db;
noncom b,m ;
noncom b,dm;
noncom b,n ;
noncom b,dn;
noncom n,m ;
noncom n,dn;
noncom n,dm;
noncom n,da;
noncom n,db;
noncom m,da;
noncom m,dm;
noncom m,db;
noncom m,dn;

```

```

% define the fermionic fields to have zero norm

```

```

let a()2 => 0;
let a()*a()=> 0;
let b()2 => 0;
let b()*b()=> 0;
let m()2 => 0;

```

```
let m()*m()=> 0;
```

```
let n()^2 => 0;
```

```
let n()*n()=> 0;
```

```
% define anticommutation ( in conjunction with the noncom commands) listed
```

```
% above
```

```
let a()*b() => - b()*a() ;
```

```
let a()*da() => -da()*a();
```

```
let a()*db() => -db()*a() ;
```

```
let a()*m() => - m()*a() ;
```

```
let a()*dm() => -dm()*a() ;
```

```
let a()*n() => - n()*a() ;
```

```
let a()*dn() => -dn()*a() ;
```

```
let b()*da() => -da()*b();
```

```
let b()*db() => -db()*b() ;
```

```
let b()*m() => -m()*b() ;
```

```
let b()*dm() => -dm()*b() ;
```

```
let b()*n() => -n()*b() ;
```

```
let b()*dn() => -dn()*b();
```

```
let m()*da() => -da()*m() ;
```

```
let m()*dm() => -dm()*m();
```

```
let m()*db() => -db()*m() ;
```

```
let m()*n() => -n()*m() ;
```

```
let m()*dn() => -dn()*m() ;
```

```
let n()*da() => -da()*n();
```

```
let n()*dn() => -dn()*n() ;
```

```
let n()*db() => -db()*n() ;
```

```
let n()*dm() => -dm()*n() ;
```

% define the matrices for Type B. nrb  $\equiv$  negative roots of Type B, cab  $\equiv$  cartan  
% subalgebra of Type B and so on.

```
nrb:=mat((1,0,0),((f-(1/2)*m()*n()),1,n()),(m(),0,1));
cab:=mat((exp((1/2)*(p+t)),0,0),(0,exp(-(1/2)*(t-p)),0),(0,0,exp(p)));
prb:=mat((1,(h+(1/2)*a()*b()),a()),(0,1,0),(0,b(),1));
```

% calculate  $g^B = g_{<}^B g_o^B g_{>}^B$

```
gb:=nrb*cab*prb;
```

% define matrices for Type A

```
nra:=mat((1,0,m()),(f,1,(1/2)*f*m()+n()),(0,0,1));
caa:=mat((exp((1/2)*(p+t)),0,0),(0,exp(-(1/2)*(t-p)),0),(0,0,exp(p)));
pra:=mat((1,h,0),(0,1,0),(a(),(1/2)*a()*h+b(),1));
```

% calculate  $g^A$

```
ga:=nra*caa*pra;
```

% calculate  $\partial g_{<}, \partial g_o, \partial g_{>}$  for Types A and B

```
dnrb:=mat((0,0,0),((k*df-k*dm()*n())/2-k*m()* dn()/2),0,k*dn()),
(k*dm(),0,0));
dcab:=mat((k*(dt+dp)*exp((1/2)*(t+p))/2,0,0),
(0,(-k*(dt-dp)*exp(-(1/2)*(t-p))/2),0),
(0,0,k*dp*exp(p)));
dprb:=mat((0,(k*dh+k*da()*b())/2+k*a()*db()/2), k*da()),(0,0,0),
(0,k*db(),0));
```

```
dnra:=mat((0,0,k*dm()),(k*df,0,(k*df*m()/2+k*f*dm()/2+k*dn())),
(0,0,0));
```

```
dcaa:=mat((k*(dt+dp)*exp((1/2)*(t+p))/2,0,0),
```

```

(0,(-k*(dt-dp)*exp(-(1/2)*(t-p))/2),0), (0,0,k*dp*exp(p)));
dpra:=mat((0,k*dh,0),(0,0,0),(k*da(), k*da()*h/2+k*a()*dh/2+k*db(),0));

% calculate  $g_{<}^{-1}, g_o^{-1}, g_{>}^{-1}$  for Types A and B
iprb:= 1/prb ;
icab:= 1/cab;
inrb:= 1/nrb;

ipra:= 1/prb ;
icaa:= 1/caa;
inra:= 1/nra;

% calculate  $(g^B)^{-1}, (g^A)^{-1}$ 
igb:=iprb*icab*inrb;
iga:=ipra*icaa*inra;

% calculate  $\partial g^B, \partial g^A$ 
dgb:= dnrb*cab*prb+nrb*dcab*prb+nrb*cab *dprb;
dga:= dnra*caa*pra+nra*dcaa*pra+nra *caa*dpra;

% the desired currents can now be calculated in matrix form
jnew:= igb*dgb;
jnew:= iga*dga;

% To extract the currents as defined by  $\mathcal{J} = J^a \tau^a = g^{-1} \partial g$  we need the
% generators. They are, in the order  $J^3, U, J^+, J^-, j^+, j^-, j'^+, j'^-$  :
% generators for Type B
one:=mat(((1/2),0,0),(0,(1/2),0),(0,0,1));
two:=mat(((1/2),0,0),(0,(-1/2),0),(0,0,0));

```



```

three:=mat((0,1,0),(0,0,0),(0,0,0));
four:=mat((0,0,0),(1,0,0),(0,0,0));
five:=mat((0,0,1),(0,0,0),(0,0,0));
six:=mat((0,0,0),(0,0,0),(1,0,0));
seven:=mat((0,0,0),(0,0,0),(0,1,0));
eight:=mat((0,0,0),(0,0,1),(0,0,0));

```

% generators for Type A

```

one:=mat(((1/2),0,0),(0,(1/2),0),(0,0,1));
two:=mat(((1/2),0,0),(0,(-1/2),0),(0,0,0));
three:=mat((0,1,0),(0,0,0),(0,0,0));
four:=mat((0,0,0),(1,0,0),(0,0,0));
five:=mat((0,0,0),(0,0,0),(1,0,0));
six:=mat((0,0,1),(0,0,0),(0,0,0));
seven:=mat((0,0,0),(0,0,0),(0,1,0));
eight:=mat((0,0,0),(0,0,1),(0,0,0));

```

% multiply the currents in matrix form by the generators

```

xone:=-one*jnew;
xtwo:=-two*jnew;
xthree:=-three*jnew;
xfour:=-four*jnew;
xfive:=-five*jnew;
xsix:=-six*jnew;
xseven:=-seven*jnew;
xeight:=-eight*jnew;

```

% take the trace, defined by  $\text{STr}(M) = M_{11} + M_{22} - M_{33}$  for bosonic currents, and

%  $\text{STr}(M) = M_{11} + M_{22} + M_{33}$  for fermionic currents

```
% the currents are
Unew:=- (xone(1,1)+xone(2,2)-xone(3,3));
jthreenew:=- (xtwo(1,1)+xtwo(2,2)-xtwo(3,3));
JPnew:=- (xthree(1,1)+xthree(2,2)-xthree(3,3));
JMnew:=- (xfour(1,1)+xfour(2,2)-xfour(3,3));
jplusnew:=- (xfive(1,1)+xfive(2,2)+xfive(3,3));
jminusnew:=- (xsix(1,1)+xsix(2,2)+xsix(3,3));
jpluspnew:=- (xseven(1,1)+xseven(2,2)+xseven(3,3));
jminuspnew:=- (xeight(1,1)+xeight(2,2)+xeight(3,3));
```

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