Geometry of arithmetic surfaces

Aghasi, Mansour

How to cite:
Aghasi, Mansour (1996) Geometry of arithmetic surfaces, Durham theses, Durham University. Available at Durham E-Theses Online: http://etheses.dur.ac.uk/5270/

Use policy
The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a link is made to the metadata record in Durham E-Theses
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.
Please consult the full Durham E-Theses policy for further details.
Geometry of Arithmetic

Surfaces

Mansour Aghasi

A thesis submitted in partial
fulfilment of the requirement for
the degree of Doctor of Philosophy

July 1996

Department of Mathematical Sciences
University of Durham
South Road, Durham DH1 3LE
ENGLAND

The copyright of this thesis rests with the author.
No quotation from it should be published without
his prior written consent and information derived
from it should be acknowledged.
Abstract

In this thesis my emphasis is on the resolution of the singularities of fibre products of Arithmetic Surfaces.

In chapter one as an introduction to my thesis some elementary concepts related to regular and singular points are reviewed and the concept of tangent cone is defined for schemes over a discrete valuation ring. The concept of arithmetic surfaces is introduced briefly in the end of this chapter.

In chapter 2 my new procedures namely the procedure of Mojgan\(^1\) and the procedure of Mahtab\(^2\) and a new operator called Moje are introduced. Also the concept of tangent space is defined for schemes over a discrete valuation ring.

In chapter 3 the singularities of schemes which are the fibre products of some surfaces with ordinary double points are resolved. It is done in two different methods. The results from both methods are consistent.

In chapter 4, I have tried to resolve the singularities of a special class of arithmetic three-folds, namely those which are the fibre product of two arithmetic surfaces, which were very helpful to achieve my final results about the resolution of singularities of fibre products of the minimal regular models of Tate.

Chapter 5 includes my final results which are about the resolution of singularities of the fibre product of two minimal regular models of Tate.

\(^1\)In the Persian Language Moje means eyelash and Mojgan is its plural.

\(^2\)In the Persian Language Mahtab means moonlight.
Preface

This thesis is the result of my research work in the Department of Mathematical Sciences at the University of Durham, between October 1993 and July 1996 under the supervision of Professor A.J. Scholl. No part of it has been previously submitted for any degree and no part is published anywhere. It is believed that the material presented in this thesis is entirely original.

I thank Prof. Scholl for his careful supervision and his encouragement. In addition I would like to thank all of the lecturers whose courses I attended. In particular I should thank Prof. Scholl and Dr Oxbury for their courses in my subject under the titles of schemes and algebraic geometry respectively, which were excellent.

I respect the previous heads of our Department Dr Armstrong, Prof. Squires (whose recent death is regretted by us all), Prof. Scholl and the present head of the Department Prof. E.F. Corrigan and wish continuing success for the Department in future.

I would deeply like to thank the Iranian Ministry of Culture and Higher Education and also the Isfahan University of Technology for their financial support during this period of my studies in England.

My special thanks go to my children for being so patient, tolerant and understanding. This thesis is dedicated to them.

I will never forget the memory of this period of my studies, especially for having very kind fellow-students.

The copyright of this thesis rests with me. No part of it should be published without my prior written consent and in the case of using any information of my thesis it should be acknowledged. My permanent address is as follows:

Department of Mathematics
Isfahan University of Technology
Isfahan, Iran
Contents

1 Introduction ............................................. 4
   1.1 Definitions + Conventions ........................... 6

2 Procedures .............................................. 9
   2.1 Introduction ......................................... 9
   2.2 The Procedure of Mojgan ............................ 9
   2.3 Generalization ........................................ 12
   2.4 The Procedure of Mahtab ............................ 18

3 Product of ordinary double points ...................... 30
   3.1 Introduction .......................................... 30
   3.2 Proposed schemes ..................................... 30
   3.3 First Method .......................................... 31
   3.4 Second Method ........................................ 38
   3.5 Over a dvr ............................................. 42

4 Desingularisation of a special class of arithmetic three-folds .... 50
   4.1 Introduction .......................................... 50
   4.2 One component is $I_1$ ................................ 52
   4.3 One component is $I_2$ ................................ 61
   4.4 One component is $I_3$ ................................ 92
   4.5 One component is $I_4$ ................................ 99
   4.6 One component is $I_5$ ................................ 103
5 Desingularisation of the fibre product of minimal regular models of Tate

5.1 Introduction ........................................ 110
5.2 Neron model and Tate's algorithm .................... 111

Bibliography ........................................... 120
Chapter 1

Introduction

The basis of this thesis is the resolution of singularities of the fibre product of Arithmetic Surfaces. The blowing-up of a point of a variety or a subvariety of a variety is discussed before, see for example [11], [4], [9], [10] and [20]. We can find about the blowing-up of a noetherian scheme with respect to a coherent sheaf of ideals in [11]. Some discussions about the blowing-up are purely algebraic, see for example [14], [28] and [11]. My research work started with proposition 2.0.2 and remark 2.1.2 in Scholl’s paper (see [21]), and lemma 5.5 in Deligne’s paper (see [7]) and I tried to find a resolution for the singularities of a fibre product of the arithmetic surfaces with ordinary double points and could find the answer when our surfaces are over Spec $k[t]$, where $k$ is an algebraically closed field. Later on, I realized that the results carry through without change if $k$ is replaced by an arbitrary regular ring $R$. Then there was an attempt for the substitution of $R[t]$ by a discrete valuation ring $R$ with an algebraically closed residue field $k = R/(\pi)$ and it was possible. These results are collected in chapter 3. The rest of my thesis is mainly about the resolution of singularities of the fibre product of arithmetic surfaces of genus one.

Before introducing more details about the research work, let’s have a quick review on the history of the resolution of singularities, the question which has always been interesting to mathematicians.
The resolution of curves was known earlier. Walker proved it for surfaces over $\mathbb{C}$ in 1935. The first algebraic proof for resolution of surfaces and then the embedded resolution for surfaces and resolution for three-folds over a field $k$ (Char. $k=0$) was given by Zariski, 1939 and 1944 respectively, see [28] for the first one.

Hironaka proved the resolution and embedded resolution in all dimensions in characteristic 0, see [12].

In 1966 Abhyankar proved resolution of three-folds in characteristic $p \geq 7$.

The resolution of singularities of arithmetic schemes (eg of schemes of finite type over $\mathbb{Z}$ or a dvr) is in general unknown. But it is known for arithmetic surfaces (see [23]). For arithmetic surfaces of genus one there is a nice algorithm to obtain the minimal regular model (due to Tate), see [25] for details.

The aim of this thesis is to investigate the case of those Arithmetic 3-folds which are fibre products of two arithmetic surfaces. The simplest case is when the singularities in the fibres of the surfaces are ordinary double points, so locally for the etale topology isomorphic to $\text{Spec } R[x,y]/(xy - \pi)$. In this case we can find an explicit desingularisation for the fibre products of an arbitrary number of surfaces. The existence of such a desingularisation (at least in the geometric case) has been known for a long time. In 1969 Deligne gave a sheaf of ideals which under blowing-up, would resolve such a singularity. In chapter 3, Deligne's method is reproduced with details. We also give another method to desingularise a product of double points, following Scholl's remark (see [21]) and prove that these two methods give the same answer.

In the last section of chapter 3 we imitate the methods of Deligne and Scholl in the arithmetic case, and prove that they give a (common) desingularisation over a dvr.

The final results in this thesis are in chapters 4 and 5, where we show that (under certain hypothesis) the fibre product of two arithmetic surfaces of genus one has a regular model, by taking the product of two Tate models and applying birational transformations.
1.1 Definitions + Conventions

Recall that a noetherian local ring $A$ of dimension $d$ and with maximal ideal $m$ and residue field $k = A/m$ is regular if $\dim_k m/m^2 = d$ (see [3]), or equivalently the maximal ideal of $A$ is generated by $d$ elements, which also means that the associated graded ring $G_m = \bigoplus_{n \geq 0} m^n/m^{n+1}$ is a polynomial ring in $d$ variables over the field $A/m$, see [27].

Also recall that for a commutative ring $A$ with identity and a prime ideal $P$ of $A$, the localization of $A$ at $P$ is called the local ring of $\text{Spec } A$ at $P$ and the field $A_P/PA_P$ is said to be the residue class field of $\text{Spec } A$ at $P$ and is denoted by $k(P)$ which is actually the field of fractions of $A/P$, see [13] and [11].

When $A$ is a commutative ring with identity and has the spectrum $(\text{Spec } A, \mathcal{O})$, the stalk $\mathcal{O}_P$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_P$, see [11]. A scheme $X$ is regular if all of its local rings are regular local rings. If for $P \in X$, $\mathcal{O}_P$ is not regular we say that $P$ is a singular point of $X$.

Recall that for a scheme $X$ the Zariski tangent space to $X$ at $P \in X$ is the dual of the $k(P)$-vector space $m_P/m_P^2$, i.e, $\text{Hom}_{k(P)}(m_P/m_P^2, k(P))$ which is described completely in [17]. If $k$ is an algebraically closed field and $X$ is a scheme of finite type over $k$ and $P$ is a closed point of $X$ and $U \subset X$ is an affine open neighbourhood of $P$, a closed immersion $i:U \rightarrow A^n_k$ makes $U$ isomorphic with a subscheme $\text{Spec } (k[x_1, \ldots, x_n]/A)$ of $A^n_k$. Then by using suitable translation we can assume that $x_1, \ldots, x_n \in I(P)$ or equivalently $i(P) = P_0$ the origin in $A^n_k$. For each $f \in k[x_1, \ldots, x_n]$ we use $f^*$ to denote the leading form of $f$, i.e, if $f = \sum_{i=0} f_i$, where $f_i$ is homogeneous of degree $i$ (and $f_r \neq 0$), then $f^* = f_r$ and we use $A^*$ for the ideal of $k[x_1, \ldots, x_n]$ generated by all polynomials $f^*$ (for $f \in A$) and call $\text{Spec } (k[x_1, \ldots, x_n]/A^*)$ the tangent cone of $X$ at $P$. It is the same as $\text{Spec } (\text{Gr } (\mathcal{O}_P))$, see [17], page 216. Recall that a point $P \in X$ is a regular point if the tangent cone and the Zariski tangent space at $P$ are the same.

For example if $k$ is an algebraically closed field of Char. 0 and $X = \text{Spec } k[x, y]/(y^2 - x^2(x + 1))$, then at $P_0=(0,0)$ the tangent cone consists of the line pair $\text{Spec } k[x, y]/(y^2 - x^2)$ and the tangent space is $A^2_k$. So $P_0$ is a singular point.
CHAPTER 1. Introduction

Remark 1.1 In the above discussion we cannot substitute \( k \) by an arbitrary ring, but if \( R \) is a discrete valuation ring with algebraically closed residue field \( k = R/\langle \pi \rangle \) we can define the tangent cone at \( P \in X \) in a similar way.

Here a closed point on the special fibre \( A^n_k \) of \( A^n_R \) say \( P = (a_1, \ldots, a_n) \) corresponds to the maximal ideal \( M_P = (x_1 - a_1, \ldots, x_n - a_n, \pi) \) of \( R[x_1, x_2, \ldots, x_n] \) where \( \alpha_i \in R \) and \( \bar{\alpha}_i = a_i, \text{ ie }, a_i \) is the reduction of \( \alpha_i \) \( \text{ (mod } \pi \). 

Now let \( X \) be a scheme of finite type over \( R \) and \( P \) be a closed point of \( X \) and \( U \subseteq X \) be an affine open neighbourhood of \( P \) and \( i : U \rightarrow A^n_R \) be a closed immersion making \( U \) isomorphic with the subscheme \( V = \text{Spec}(R[x_1, \ldots, x_n]/A) \) of \( A^n_R \) and use suitable translation such that \( i(P) = P_0 \), where \( P_0 \) is the origin of the special fibre of \( A^n_R \), ie, \( A^n_k \).

Let \( f \) be a non-zero polynomial in \( A \). Since \( i(P) = P_0 \in V \), we have \( f \equiv 0 \) (mod \( M \)) (ie, \( P_0 \in A^n_k \) lies on \( f = 0 \)), where \( M = (\pi, x_1, \ldots, x_n) \). So there exists \( r \geq 0 \) such that \( f \in M^r \) and \( f \notin M^{r+1} \). Suppose that \( f = (\Sigma_{i=1}^r g_i) + h \) where \( g_i \) is a monomial in \( R[x_1, \ldots, x_n] \) and \( g_i \in M^r \setminus M^{r+1} \) and \( h \in M^{r+1} \).

If \( g_i = u_i \pi^{a_i} x_1^{a_{i1}} \ldots x_n^{a_{in}} \) (where \( u_i \in R^* \)), then \( \alpha_{i_0} + \alpha_{i_1} + \ldots + \alpha_{i_n} = r \) and \( u_i \) is well-defined \( \text{ (mod } \pi \) as \( g_i \) is well-defined \( \text{ (mod } M^{r+1} \). Now replace \( \pi \) by an indeterminate say \( x_0 \). Let \( g_i^* = u_i x_0^{a_{i0}} x_1^{a_{i1}} \ldots x_n^{a_{in}} \in k[x_0, \ldots, x_n] \) and \( f^* = \Sigma_{i=1}^r g_i^* \). Then \( f^* \) is a homogeneous polynomial of degree \( r \) in \( k[x_0, x_1, \ldots, x_n] \) which is determined uniquely by \( f \). Let \( A^* \) be the ideal of \( k[x_0, x_1, \ldots, x_n] \) generated by all \( f^* \)'s (for \( f \in A \)). Now we define the tangent cone at \( P \in X \) as follows:

Definition 1.2 With the above notation

\[ \text{Tangent cone of } X \text{ at } P = \text{Spec}(k[x_0, x_1, \ldots, x_n]/A^*) \]

Remark 1.3 In definition (2.5) of chapter 2 we introduce the concept of tangent space and then we give an example about tangent cone and tangent space.

Convention 1.4 In this thesis valuation ring means a discrete valuation ring with uniformisor \( \pi \) and algebraically closed residue field \( k = R/\langle \pi \rangle \).

Remark 1.5 In [23] the concept of arithmetic surfaces is defined over a Dedekind domain, but since singularity is a local property and the localization of a Dedekind ring
domain at any of its non-zero primes is a discrete valuation ring, we study our arithmetic surfaces over a discrete valuation ring.

Convention 1.6 Let $R$ be a discrete valuation ring. By an arithmetic surface we mean a regular scheme $V$ purely of dimension 2, which is flat over $R$.

Remark 1.7 Let $X \to \text{Spec } R$ be a separated integral scheme which is flat and of finite type. By a desingularisation of $X$ we mean a birational, proper morphism $X' \to X$ such that $X'$ is flat over $\text{Spec } R$ and is a regular scheme.

Remark 1.8 For more details and several examples of arithmetic surfaces see pp 311-318 in [23]. The blowing-up of an arithmetic surface over a discrete valuation ring is discussed in remark 7.7, pp 345-347 in [23].

The theorem related to the resolution of singularities of arithmetic surfaces and also existence of minimal proper regular models for arithmetic surfaces is discussed in theorem 4.5, page 317 in [23]. For nice examples over a discrete valuation ring see Tate’s algorithm in [25].

All schemes which I have used in this thesis as components in the fibre products, are examples for arithmetic surfaces.
Chapter 2

Procedures

2.1 Introduction

To determine the singular points of a geometric scheme, one uses the Jacobian criterion. In the arithmetic case there is no Jacobian criterion, but we find an analogous procedure to determine the singular points, which we call the procedure of Mojgan (§2.2 and §2.3). For resolution of singularities we mainly use a sequence of blowings-up. To do these with a tidy method, we employ a procedure which we call Mahtab (§2.4).

2.2 The Procedure of Mojgan

As it was mentioned in chapter one, the tangent cones and tangent spaces are helpful for the determination of singular points. Here we try for another method. We start with another interpretation of Jacobian which would be generalized later on.

Let $k$ be an algebraically closed field and $X = \text{Spec } k[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ and $P = (a_1, \ldots, a_n) \in X$. Then define Moje$(P, f_i)$ as follows:
\( \text{Moje}(P, f_i) = \text{Linear form of } f_i(T_1 + a_1, ..., T_n + a_n), \)
say
\( \text{Moje}(P, f_i) = C_1 T_1 + C_2 T_2 + ... + C_n T_n \) and let \( C = (C_{ij}) \) where
\( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), and \( P \) corresponds to the maximal ideal
\[
\mathcal{M}_P = (x_1 - a_1, x_2 - a_2, ..., x_n - a_n)/(f_1, ..., f_m)
\] (2.1)
of \( \Gamma(X, \mathcal{O}_X) \) and
\[
\mathcal{M}_P/\mathcal{M}_P^2 = (T_1, ..., T_n)/((T_1, ..., T_n)^2 + \text{Span}(\text{Moje}(P, f_i))_{1 \leq i \leq m}).
\] (2.2)
Let \( r = \text{rank}(C_{ij}) \). Then \( \dim \mathcal{M}_P/\mathcal{M}_P^2 = n - r \). If \( n - r = \dim X \), then by Jacobian criterion \( P \) is a regular point, otherwise it is singular, see [11].

**Remark 2.1** For the calculation of singular points of schemes over a field of characteristic zero, or determination of singular points of the special fibre (when our schemes are over a dvr) and also for solving the system of equations, I have used WMAPLE53.

**Example 2.2** Let \( X = \text{Spec } k[x_1, y_1, x_2, y_2, x_3, y_3]/(x_1 y_1 - x_2 y_2, x_1 y_1 - x_3 y_3) \). Then
\[
f_1(x_1, y_1, x_2, y_2, x_3, y_3) = x_1 y_1 - x_2 y_2
\]
and
\[
f_2(x_1, y_1, x_2, y_2, x_3, y_3) = x_1 y_1 - x_3 y_3.
\]
If \( P = (a_1, b_1, a_2, b_2, a_3, b_3) \in X \), then it corresponds to
\[
\mathcal{M}_P = (x_1 - a_1, y_1 - b_1, ..., x_3 - a_3, y_3 - b_3)/(f_1, f_2)
\]
\[
= (T_1, S_1, ..., T_3, S_3)/(f_1(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3),
\]
\[
f_2(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3))
\] (2.3)
but
\[
f_1(T_1 + a_1, S_1 + b_1, ..., T_3 + a_3, S_3 + b_3) =
\]
\[
T_1 S_1 - T_2 S_2 + (a_1 S_1 + b_1 T_1 - a_2 S_2 - b_2 T_2),
\] (2.4)
CHAPTER 2. Procedures

\[ \text{Moje}(P, f_1) = b_1 T_1 + a_1 S_1 - b_2 T_2 - a_2 S_2. \]  
\[ \text{Moje}(P, f_2) = b_1 T_1 + a_1 S_1 - b_3 T_3 - a_3 S_3, \]

Similarly we can check that

\[ \text{Moje}(P, f_3) = b_1 T_1 + a_1 S_1 - b_4 T_4 - a_4 S_4, \]

so

\[ C(P) = \begin{pmatrix} b_1 & a_1 & -b_2 & -a_2 & 0 & 0 \\ b_1 & a_1 & 0 & 0 & -b_3 & -a_3 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ b_1 & a_1 & 0 & 0 & -b_n & -a_n \end{pmatrix}. \]

If we equate the minor determinants to zero and use them as the equations of a system, we see that \( P \) is singular if at least two pairs \((a_i, b_i)\) are zero \((1 \leq i \leq 3)\).

Example 2.3 Let

\[ X = \text{Spec } k[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 - x_i y_i)_{2 \leq i \leq n}, \]

and \( P = (a_1, b_1, \ldots, a_n, b_n) \in X \). Consider \( f_i = x_1 y_1 - x_i y_i \) where \( 2 \leq i \leq n \). Then

\[ \text{Moje}(P, f_i) = b_1 T_1 + a_1 S_1 - b_i T_i - a_i S_i, \]

\[ \text{i.e.,} \]

\[ C(P) = \begin{pmatrix} b_1 & a_1 & -b_2 & -a_2 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ b_1 & a_1 & 0 & 0 & -b_3 & -a_3 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_1 & a_1 & 0 & 0 & \ldots & \ldots & 0 & 0 & -b_n & -a_n \end{pmatrix}. \]

This matrix shows us that if at least two pairs \((a_i, b_i)\) are zero where \( 2 \leq i \leq n \), then \( P \) is a singular point.
2.3 Generalization

Let $R$ be a discrete valuation ring with algebraically closed residue field $k = \frac{R}{(\pi)}$ and

$$X = \text{Spec } R[x_1, ..., x_n]/(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)).$$  (2.11)

Then the closed points of $X$ are on the special fibre of $X$, i.e., on the scheme

$$X_\pi = \text{Spec } k[x_1, ..., x_n]/(\bar{f}_1(x_1, ..., x_n), ..., \bar{f}_m(x_1, ..., x_n)),$$  (2.12)

where $\bar{f}_i$ denotes the reduction of $f_i (\mod \pi)$. Let $P = (a_1, ..., a_n) \in X_\pi$ be a closed point of $X$ and choose $\alpha_1, ..., \alpha_n \in R$ such that $\bar{\alpha}_i \equiv a_i (\mod \pi)$. Then as a point of $X$, $P$ corresponds to the maximal ideal of $\Gamma(X, \mathcal{O}_X)$ generated by $x_1 - \alpha_1, ..., x_n - \alpha_n$ and $\pi$, i.e.,

$$\mathcal{M}_P = (x_1 - \alpha_1, x_2 - \alpha_2, ..., x_n - \alpha_n, \pi) / (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)).$$  (2.13)

By using the translation $T_i = x_i - \alpha_i$, we get

$$\mathcal{M}_P = (T_1, T_2, ..., T_n, \pi) / (f_1(T_1 + \alpha_1, ..., T_n + \alpha_n), ..., f_m(T_1 + \alpha_1, ..., T_n + \alpha_n)).$$  (2.14)

So

$$\mathcal{M}_P/\mathcal{M}_P^2 = (T_1, T_2, ..., T_n, \pi) / ((T_1, T_2, ..., \pi)^2 + (T_1, T_2, ..., T_n, \pi)\cap
(f_1(T_1 + \alpha_1, ..., T_n + \alpha_n), ..., f_m(T_1 + \alpha_1, ..., T_n + \alpha_n))),$$  (2.15)

ie,

$$\mathcal{M}_P/\mathcal{M}_P^2 = (T_1, T_2, ..., T_n, \pi) / ((T_1, T_2, ..., T_n, \pi)^2 + \{(\text{Moje}(P, f_i))_{1 \leq i \leq m})\),$$  (2.16)

where $\text{Moje}(P, f_i)$ is the linear part in $T_1, ..., T_n$ and $\pi$ of $f_i(T_1 + \alpha_1, ..., T_n + \alpha_n)$, say

$$\text{Moje}(P, f_i) = d_0 \pi + d_{i_1} T_1 + d_{i_2} T_2 + ... + d_{i_m} T_n,$$  (2.17)
where \( 1 \leq i \leq m \) and \( d_{ij} \in R \), i.e., \( \text{Moje}(P, f_i) = f_i(T_1 + \alpha_1, \ldots, T_n + \alpha_n) \mod (T_1, \ldots, T_n, \pi)^2 \).

Now consider the matrix

\[
D(P) = \begin{pmatrix}
\tilde{d}_{10} & \tilde{d}_{11} & \tilde{d}_{12} & \ldots & \ldots & \tilde{d}_{1n} \\
\tilde{d}_{20} & \tilde{d}_{21} & \tilde{d}_{22} & \ldots & \ldots & \tilde{d}_{2n} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\tilde{d}_{m0} & \tilde{d}_{m1} & \tilde{d}_{m2} & \ldots & \ldots & \tilde{d}_{mn}
\end{pmatrix}, \tag{2.18}
\]

where \( \tilde{d}_{ij} \) is the reduction of \( d_{ij} \) (mod \( \pi \)). If \( r = \text{rank} \ D(P) \), then

\[
\dim \mathcal{M}_P / \mathcal{M}_P^2 = n + 1 - r. \tag{2.19}
\]

Thus we obtain:

**Proposition 2.4 (Procedure of Mojgan)**

\( P \) is a regular point of \( X \) if \( n + 1 - r = \dim X \), otherwise it is singular.

**Definition 2.5** By using the same notation as above we consider

\[
\text{moje}(P, f_i) = \tilde{d}_{i0}T_0 + \tilde{d}_{i1}T_1 + \ldots + \tilde{d}_{in}T_n,
\]

corresponding to \( \text{Moje}(P, f_i) \) (\( T_0 \) corresponds to \( \pi \)), and define

Tangent space of \( X \) at \( P = \text{Spec} k[T_0, T_1, T_2, \ldots, T_n] / (\{\text{moje}(P, f_i)\}_{1 \leq i \leq m}) \)

**Example 2.6** Let \( X = \text{Spec} \, k[x_1, y_1, x_2, y_2] / (x_1y_1 - y_2^2 + x_2^3, x_1y_1 - \pi) \). Consider \( f_1 = x_1y_1 - y_2^2 + x_2^3 \), \( f_2 = x_1y_1 - \pi \). Then \( f_1^* = x_1y_1 - y_2^2 \) and \( f_2^* = -\pi = -x_0 \).

Hence

Tangent cone of \( X \) at \( P_0 = \text{Spec} \, k[x_0, x_1, y_1, x_2, y_2] / (x_1y_1 - y_2^2, x_0) \) =

\[ \text{Spec} \, k[x_1, y_1, x_2, y_2] / (x_1y_1 - y_2^2) \].

But \( \text{moje}(P_0, f_1) = 0 \) and \( \text{moje}(P_0, f_2) = -T_0 \), i.e,

Tangent space of \( X \) at \( P_0 = \text{Spec} \, k[T_0, T_1, S_1, T_2, S_2] / (0, T_0) \) =

\[ \text{Spec} \, k[T_1, S_1, T_2, S_2] = A_4^1 \], so \( P_0 \) is a singular point.
Lemma 2.7 Let $Y$ be a regular scheme. Then $\mathbb{A}_Y^n = \mathbb{A}^n \times_{\text{Spec} Z} Y$ is regular.

Proof: Let $Y = \text{Spec } R$, where $R$ is a regular ring. Then $\mathbb{A}_Y^n = \mathbb{A}^n \times_{\text{Spec} Z} \text{Spec } R = \text{Spec } (\mathbb{Z}[x_1, ..., x_n] \otimes \mathbb{Z} R) = \text{Spec } R[x_1, ..., x_n]$ which is regular (see theorem 40, page 126 in [15]).

If $Y$ is non-affine, then we consider $Y$ as a union of affine schemes and use the above result for the product of $\mathbb{A}^n$ with the affine pieces and then glue them together. □

Theorem 2.8 Let $f : X \to Y$ be etale. If $Y$ is regular, then $X$ is regular.

Proof: See page 27, prop. 3.17-(c) in [16]. □

Theorem 2.9 Let $Y$ be a regular scheme and $X$ be smooth over $Y$. Then $X$ is a regular scheme.

Proof: Let $f : X \to Y$ be the smooth morphism. Then by definition (see definition (1.1), page 128 in [1]), each $x \in X$ has an open neighbourhood $U$ such that $U \to A^2_Y$ is etale and $A^2_Y \to Y$ is the projection on the second factor and the following diagram is commutative:

$$
\begin{array}{ccc}
U & \to & A^2_Y = \mathbb{A}^n \times_{\text{Spec} Z} Y \\
\downarrow_{f|U} & & \downarrow \\
Y & & 
\end{array}
$$

Since $Y$ is regular by lemma (2.7), $A^2_Y$ is regular. By using theorem (2.8) we conclude that $U$ is regular. □

Remark 2.10 Note that the other definition of smooth morphism as it is stated in page 304 of [17] or page 305 in [23] is equivalent with definition (1.1), page 128 in [1] (for the proof see prop. 3.24, page 31 in [16]).

Corollary 2.11 Let $X$ and $Y$ be schemes over $S$. If $Y$ is regular and $X$ is smooth over $S$, then $X \times_SY$ is a regular scheme.
Proof: \( X \to S \) is smooth. After base extension we get \( X \times_S Y \to Y \). Now \( Y \) is regular and \( X \times_S Y \to Y \) is smooth (see [1], page 129, prop. (1.7)-(iii)). By using theorem (2.9) we conclude that \( X \times_S Y \) is regular. □

Here we will have an application of the procedure of Mojgan which is very useful in coming chapters.

**Lemma 2.12** Let \( R \) be a discrete valuation ring and \( f(x_1, x_2, \ldots, x_n) \) be a non-constant monomial in \( R[x_1, x_2, \ldots, x_n] \) such that \( f \) is not divisible by \( \pi \). Then \( X = \text{Spec } R[x_1, x_2, \ldots, x_n]/(f(x_1, x_2, \ldots, x_n) - \pi) \) is a regular scheme.

**Proof:** If \( f \) does not contain \( x_1, x_2, \ldots, x_l \) (for \( l < n \)), then we get

\[
X = A^1_R \times_{\text{Spec } R} \text{Spec } R[x_{l+1}, x_{l+2}, \ldots, x_n]/((f(x_{l+1}, x_{l+2}, \ldots, x_n) - \pi)).
\] (2.20)

So without loss of generality we can assume that \( f \) contains all \( x_i \)'s, say \( f(x_1, x_2, \ldots, x_n) = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \). Now let \( g(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) - \pi \), ie, \( g(x_1, x_2, \ldots, x_n) = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} - \pi \). Then the special fibre of \( X \) is

\[
X_\pi = \text{Spec } k[x_1, x_2, \ldots, x_n]/(g(x_1, x_2, \ldots, x_n)) = \text{Spec } k[x_1, x_2, \ldots, x_n]/(x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}).
\] (2.21)

We can find the singular points of \( X_\pi \) by using the Jacobian criterion. Let \( Q = (a_1, a_2, \ldots, a_n) \in X_\pi \). Then as a point of \( X \), \( Q \) corresponds to the ideal of \( \Gamma(X, \mathcal{O}_X) \) generated by \( x_1 - \alpha_1, x_2 - \alpha_2, \ldots, x_n - \alpha_n \) and \( \pi \), ie, \( Q \) corresponds to

\[
\mathcal{M}_Q = (x_1 - \alpha_1, x_2 - \alpha_2, \ldots, x_n - \alpha_n, \pi)/(g(x_1, x_2, \ldots, x_n)),
\] (2.22)

where \( \alpha_i \in R \) and \( \hat{\alpha}_i \equiv a_i \pmod{\pi} \). Since \( Q \in X_\pi \), we have \( \prod_{i=1}^n a_i = 0 \), which means that at least one \( a_i \) is zero. Without loss of generality assume \( a_1 = 0 \), ie, \( \pi | \alpha_1 \), say \( \alpha_1 = u_1 \pi \). According to the values of \( m_1 \) we consider two different cases as follows:

Case one \( m_1 = 1 \)
CHAPTER 2. Procedures

We get \( g(x_1, x_2, \ldots, x_n) = x_1 x_2^{m_2} \ldots x_n^{m_n} - \pi \). Hence

\[
g(T_1 + \alpha_1, T_2 + \alpha_2, \ldots, T_n + \alpha_n) =
\]

\[
(T_1 + u_1 \pi)(T_2 + \alpha_2)^{m_2} \ldots (T_n + \alpha_n)^{m_n} - \pi =
\]

\[
(u_1 \pi \alpha_2^{m_2} \ldots \alpha_n^{m_n} - \pi) + \alpha_2^{m_2} \ldots \alpha_n^{m_n} T_1 +
\]

\[
m_2(u_1 \pi)\alpha_2^{m_2-1} \alpha_3^{m_3} \ldots \alpha_n^{m_n} T_2 + m_3(u_1 \pi)\alpha_2^{m_2} \alpha_3^{m_3-1} \alpha_4^{m_4} \ldots \alpha_n^{m_n} T_3 +
\]

\[
\ldots + m_n(u_1 \pi)\alpha_2^{m_2} \ldots \alpha_{(n-1)}^{m_{(n-1)}} \alpha_n^{m_n-1} T_n + \ldots
\]

\( (2.23) \)

ie,

\[
\text{Moje}(Q, g) = (u_1 \alpha_2^{m_2} \ldots \alpha_n^{m_n} - 1)\pi + \alpha_2^{m_2} \ldots \alpha_n^{m_n} T_1 +
\]

\[
m_2(u_1 \pi)\alpha_2^{m_2-1} \alpha_3^{m_3} \ldots \alpha_n^{m_n} T_2 + m_3(u_1 \pi)\alpha_2^{m_2} \alpha_3^{m_3-1} \alpha_4^{m_4} \ldots \alpha_n^{m_n} T_3 +
\]

\[
\ldots + m_n(u_1 \pi)\alpha_2^{m_2} \ldots \alpha_{(n-1)}^{m_{(n-1)}} \alpha_n^{m_n-1} T_n (\text{mod} (T_1, \ldots, T_n, \pi)^2).
\]

\( (2.24) \)

So

\[
D(Q) = (\bar{u}_1 \alpha_2^{m_2} \ldots \alpha_n^{m_n} - 1 \quad \alpha_2^{m_2} \ldots \alpha_n^{m_n} 0 0 \ldots 0 0 )
\]

\( (2.25) \)

This means that \( Q \) is singular if \( \bar{u}_1 \alpha_2^{m_2} \ldots \alpha_n^{m_n} - 1 = \alpha_2^{m_2} \ldots \alpha_n^{m_n} = 0 \) which is impossible, so \( Q \) is a regular point.

Case two \( m_1 > 1 \)

In this case we get

\[
g(T_1 + \alpha_1, T_2 + \alpha_2, \ldots, T_n + \alpha_n) =
\]

\[
(T_1 + u_1 \pi)^{m_1}(T_2 + \alpha_2)^{m_2} \ldots (T_n + \alpha_n)^{m_n} - \pi =
\]

\[
(u_1^{m_1} \pi^{m_1} \alpha_2^{m_2} \ldots \alpha_n^{m_n} - \pi) + m_1(u_1 \pi)^{m_1-1} \alpha_2^{m_2} \ldots \alpha_n^{m_n} T_1 +
\]

\[
m_2(u_1 \pi)^{m_1} \alpha_2^{m_2-1} \alpha_3^{m_3} \ldots \alpha_n^{m_n} T_2 + m_3(u_1 \pi)^{m_1} \alpha_2^{m_2} \alpha_3^{m_3-1} \alpha_4^{m_4} \ldots \alpha_n^{m_n} T_3 +
\]

\[
\ldots + m_n(u_1 \pi)^{m_1} \alpha_2^{m_2} \ldots \alpha_{(n-1)}^{m_{(n-1)}} \alpha_n^{m_n-1} T_n + \ldots,
\]

\( (2.26) \)

ie,

\[
\text{Moje}(Q, g) = (\pi^{m_1-1}u_1^{m_1} \alpha_2^{m_2} \ldots \alpha_n^{m_n} - 1)\pi + m_1(u_1 \pi)^{m_1-1} \alpha_2^{m_2} \ldots \alpha_n^{m_n} T_1 +
\]

\[
m_2(u_1 \pi)^{m_1} \alpha_2^{m_2-1} \alpha_3^{m_3} \ldots \alpha_n^{m_n} T_2 + m_3(u_1 \pi)^{m_1} \alpha_2^{m_2} \alpha_3^{m_3-1} \alpha_4^{m_4} \ldots \alpha_n^{m_n} T_3 +
\]

\[
\ldots + m_n(u_1 \pi)^{m_1} \alpha_2^{m_2} \ldots \alpha_{(n-1)}^{m_{(n-1)}} \alpha_n^{m_n-1} T_n (\text{mod} (T_1, \ldots, T_n, \pi)^2).
\]

\( (2.27) \)
Hence

\[ D(Q) = (-1 \ 0 \ 0 \ \ldots \ \ldots \ 0 \ 0), \]

which has rank 1. So \( \dim \mathcal{M}_Q/\mathcal{M}_Q^2 = (n + 1) - 1 = n = \dim X \), i.e., \( Q \) is a regular point. □

**Corollary 2.13** The scheme \( X = \text{Spec } R[x_1, x_2, \ldots, x_{n-1}, x_n, y_n]/((x_1 x_2 \ldots x_{n-1})^2 x_n y_n - \pi) \) is a regular scheme.

Now we generalise lemma (2.12) as follows:

**Theorem 2.14** Let \( g_1(x_1, x_2, \ldots, x_n), \ldots, g_m(x_1, x_2, \ldots, x_n) \in R[x_1, x_2, \ldots, x_n] \) such that \( f(x_1, \ldots, x_2) = g_1(x_1, x_2) \ldots g_m(x_1, x_2) \) is not divisible by \( \pi \) and \( h(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \pi \). If for each \( i \), the polynomial \( g_i \) is a linear form or the \( k \)-variety \( W_i = \text{Spec } k[x_1, x_2]/(g_i(x_1, x_2)) \) is non-singular, then

\[ Y = \text{Spec } R[x_1, \ldots, x_n]/((g_1(x_1, \ldots, x_n) \ldots g_m(x_1, \ldots, x_n) - \pi) \]

is a regular scheme.

**Proof**: Let \( P = (a_1, \ldots, a_n) \in Y_\pi \) and \( \alpha_1, \ldots, \alpha_n \in R \) such that \( \alpha_i \equiv a_i \pmod{\pi} \), (for \( i = 1, \ldots, n \)). Surely \( f(\alpha_1, \ldots, \alpha_n) \equiv 0 \pmod{\pi} \) and \( \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) \equiv 0 \pmod{\pi} \) if \( P \in Y_\pi^{\text{Sing}} \).

Now let \( P \in Y_\pi^{\text{Sing}} \) and \( T_i = x_i - \alpha_i \). Then \( \pi \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) \) and so

\[ \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) T_i \equiv 0 \pmod{(T_1, \ldots, T_n, \pi)^2} \]

But

\[ f(T_1 + \alpha_1, \ldots, T_n + \alpha_n) = f(\alpha_1, \ldots, \alpha_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) T_i + \ldots \]

Hence

\[ \text{Moje}(P, h) = \{ f(\alpha_1, \ldots, \alpha_n) - \pi \} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) T_i \pmod{(T_1, \ldots, T_n, \pi)^2} \]

\( \equiv \{ f(\alpha_1, \ldots, \alpha_n) - \pi \} \pmod{\pi^2}. \]
So \( P \in Y^{\text{Sing}} \) if \( f(\alpha_1, \ldots, \alpha_n) - \pi \equiv 0 \pmod{\pi^2} \) which means that \( \pi | f(\alpha_1, \ldots, \alpha_n) \) and \( \pi^2 | f(\alpha_1, \ldots, \alpha_n) \). Without loss of generality assume that \( \pi | g_1(\alpha_1, \ldots, \alpha_n) \) and \( \pi | g_2(\alpha_1, \ldots, \alpha_n) \ldots g_m(\alpha_1, \ldots, \alpha_n) \). Note that

\[
\frac{\partial f}{\partial x_i} = \frac{\partial g_1}{\partial x_i} g_2 g_3 \ldots g_m + g_1 \left\{ (\frac{\partial g_2}{\partial x_i}) g_3 \ldots g_m + \ldots + g_2 \ldots g_{m-1} (\frac{\partial g_m}{\partial x_i}) \right\}.
\] (2.33)

So

\[
\frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) = \frac{\partial g_1}{\partial x_i}(\alpha_1, \ldots, \alpha_n) g_2(\alpha_1, \ldots, \alpha_n) \ldots g_m(\alpha_1, \ldots, \alpha_n) \pmod{\pi}. \tag{2.34}
\]

If \( P \in Y^{\text{Sing}} \), we get \( \frac{\partial f}{\partial x_i}(\alpha_1, \ldots, \alpha_n) \equiv 0 \pmod{\pi} \) which implies that \( \frac{\partial g_1}{\partial x_i}(\alpha_1, \ldots, \alpha_n) \equiv 0 \pmod{\pi} \) (recall that \( \pi | g_2(\alpha_1, \ldots, \alpha_n) \ldots g_m(\alpha_1, \ldots, \alpha_n) \)) and \( g_1(\alpha_1, \ldots, \alpha_n) \equiv 0 \pmod{\pi} \), hence \( P \in W_1^{\text{Sing}} \). \( \square \)

**Example 2.15** Let

\[
Y = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1^2 + x_2^3 + 1)(x_1^2 + y_2) - \pi). \tag{2.35}
\]

By using \( g_1(x_1, y_1, x_2, y_2) = g_2(x_1, y_1, x_2, y_2) = x_1 \) and \( g_3(x_1, y_1, x_2, y_2) = y_2^2 + x_2^3 + 1 \) and \( g_4(x_1, y_1, x_2, y_2) = x_1^2 + y_2 \) and applying theorem (2.14), we conclude that \( Y \) is a regular scheme.

### 2.4 The Procedure of Mahtab

Let \( X \) be a noetherian scheme and \( \mathcal{J} \) be a coherent sheaf of ideals on \( X \) and \( S = \bigoplus_{d \geq 0} \mathcal{J}^d \) (where \( \mathcal{J}^d \) is the \( d \)-th power of the ideal \( \mathcal{J} \) with \( \mathcal{J}^0 = \mathcal{O}_X \)). Then \( S \) is a quasi-coherent sheaf of \( \mathcal{O}_X \)-modules which has a structure of a sheaf of \( \mathcal{O}_X \)-algebras. The scheme \( \widetilde{X} = \text{Proj} S \) is called the blowing-up of \( X \) with respect to \( \mathcal{J} \). If \( Y \) is a closed subscheme of \( X \) corresponding to \( \mathcal{J} \), then \( \widetilde{X} \) is called the blowing-up of \( X \) along \( Y \) or with centre \( Y \). For more details see [11].

As a trivial example we can consider the case \( X = \mathbb{A}_k^n \) and blow-up \( P_0 = (0,0,\ldots,0,0) \in X \). Let \( \Lambda = \Gamma(X, \mathcal{O}_X) = k[x_1, x_2, \ldots, x_n] \). Then \( P_0 \) corresponds to the ideal \( I = (x_1, x_2, \ldots, x_n) \) of \( A \) and \( \widetilde{X} \) is isomorphic with a closed
subscheme of \(\text{Proj} A[\mu_1, \mu_2, \ldots, \mu_n] = \mathbb{P}^{n-1}_A = \text{Spec} A \times_{\text{Spec} k} \mathbb{P}^{n-1}_k\) defined by \(\text{Ker} \, \alpha = \langle \{x_i \mu_j - x_j \mu_i\} \rangle_{1 \leq i, j \leq n}\), where \(\alpha\) is the following epimorphism:

\[\alpha : A[\mu_1, \mu_2, \ldots, \mu_n] \rightarrow S = \bigoplus_{d \geq 0} I^d\]

\[\mu_i \mapsto x_i.\]

When we are involving with a scheme \(X\) over a field of characteristic zero and blow-up a suitable subscheme \(Y\) of \(X\), by Hironaka's proof there exists a regular scheme \(\widetilde{X}\) birational to \(X\) such that \(\widetilde{X} - E\) is isomorphic to \(X - Y\), where \(E\) is the exceptional divisor.

**Remark 2.16** After blowing-up a scheme \(X\) we get some open pieces for the covering of \(\widetilde{X}\) and then glue them together. For more details about the gluing of schemes see [11], [13], [20] vol II, and [8]. If we continue using the process of blowings-up, we use \(\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n\), for the result of gluing of the open pieces after first, second, ..., nth blowings-up.

**Remark 2.17** Let \(R\) be a ring and \(M\) be an \(R\)-module. A sequence \(x_1, x_2, \ldots, x_r\) of elements of \(R\) is called a regular sequence for \(M\) if \(x_1\) is not a zero divisor in \(M\), and for all \(i = 2, \ldots, r\), \(x_i\) is not a zero divisor in \(M/(x_1, \ldots, x_{i-1})M\). For more details see [11], page 184.

**Remark 2.18** Recall that a scheme \(X\) is normal if all of its local rings are integrally closed domains. Let \(X\) be an integral scheme and for each open affine subset \(U = \text{Spec} A\) of \(X\), let \(\tilde{A}\) be the integral closure of \(A\) in its quotient field and let \(\tilde{U} = \text{Spec} \tilde{A}\). By gluing the schemes \(\tilde{U}\) we obtain a normal integral scheme \(\overline{X}\) called the normalization of \(X\), see [11] and [13].

If a scheme is not normal after blowing-up the singular part would be worse than the singular part of the original scheme. So before each blowing-up, if the scheme is not normal we find its normalization and then blow-up this normalized scheme. In some cases just after normalization we get a regular scheme. For instance the normalization of the singular scheme \(X = \text{Spec} k[x, y]/(y^2 - x^3)\) is isomorphic with \(A^1_k\). Of course I don't use it in my examples except in example (2.21).
In coming chapters we will use several times the procedure of Mahtab. Here we introduce this procedure.

Let \( X = \text{Spec } R[x_1, x_2, \ldots, x_n]/(f_1, \ldots, f_m) \) such that \( X \) is flat over \( \text{Spec } R \), and \( Y \) be a subscheme of \( X \) and \( g_1, g_2, \ldots, g_r \) be a regular sequence generating the ideal \( I = I(Y) \) of \( Y \) in \( A = \Gamma(X, \mathcal{O}_X) \). Define

\[
\alpha : A[\mu_1, \ldots, \mu_r] \rightarrow S = \bigoplus_{d \geq 0} I^d \\
\mu_i \mapsto g_i.
\]

Then \( \alpha \) is an epimorphism and

\[
\ker \alpha = (\{\mu_i g_j - \mu_j g_i\})_{1 \leq i, j \leq r}
\]

and \( A[\mu_1, \mu_2, \ldots, \mu_r]/\ker \alpha \cong S \), which induces an isomorphism

\[
\phi : \bar{X} = \text{Proj } S \rightarrow \text{Proj } (A[\mu_1, \mu_2, \ldots, \mu_r]/\ker \alpha).
\]

The right hand side scheme is a closed subscheme of

\[
\text{Proj } A[\mu_1, \mu_2, \ldots, \mu_r] = \mathbb{P}^{r-1}_A = \text{Spec } A \times \text{Spec } R \mathbb{P}^{r-1}_R.
\] (2.36)

The standard open covering of \( \mathbb{P}^{r-1}_R \), i.e.,

\[
U_1^0 = \text{Spec } R[\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1}, \ldots, \frac{\mu_r}{\mu_1}], \\
= \ldots \\
= \ldots \\
= \ldots
\]

\[
U_r^0 = \text{Spec } R[\frac{\mu_1}{\mu_r}, \frac{\mu_2}{\mu_r}, \ldots, \frac{\mu_{r-1}}{\mu_r}],
\] (2.37)

induces an open covering for \( \bar{X} \), say \( \bar{X} = V_1^0 \cup \ldots \cup V_r^0 \) where

\[
V_1^0 = \text{Spec } (A \otimes_R \Gamma(U_1^0, \mathcal{O}) (\text{mod } \ker \alpha)) = \\
\text{Spec } R[x_1, \ldots, x_n, \frac{\mu_2}{\mu_1}, \ldots, \frac{\mu_r}{\mu_1}]/(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n), \\
\frac{\mu_2}{\mu_1} g_2 - \frac{\mu_3}{\mu_1} g_3 - \frac{\mu_r}{\mu_1} g_r, \ldots, \frac{\mu_1}{\mu_r} g_1 - \frac{\mu_1}{\mu_r} g_r) = \\
\text{Spec } R[x_1, x_2, \ldots, x_n, T_{12}, T_{13}, \ldots, T_{1r}]/(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n), \{T_{ij} g_i - g_j\}_{2 \leq j \leq r})).
\] (2.38)
Similarly for $2 \leq i \leq r$ we get

$$V_i^0 = \text{Spec} \left( A \otimes_R \Gamma(U_i^0, \mathcal{O}) \text{ (mod Ker)} \right) =$$

$$\text{Spec} R[x_1, ..., x_n, \frac{\mu_1}{\mu_i}, ..., \frac{\mu_r}{\mu_i}, \frac{f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n),}{(g_1, ..., g_n, ..., g_n)} =$$

$$\text{Spec} \frac{R[x_1, ..., x_n, T_{i1}, T_{i2}, ..., T_{ii-1}, T_{ii+1}, ..., T_{ir}]}{(f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n), \{T_{ij}g_i - g_j\})_{1 \leq j \leq r, \ j \neq i}}. \quad (2.39)$$

If all the pieces $V_i^0$ are regular, their gluing gives us the regular scheme $\overline{X}$. If any $V_i^0$ is singular after normalization we blow-up its singular part or a subscheme of its singular part and continue this process until getting all pieces regular and then glue them through their overlap.

We call the above procedure, the procedure of Mahtab.

**Example 2.19** Let $k$ be a field and $X = \text{Spec} k[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2)$. By using the Jacobian criterion we can check that $P_0 = (0, 0, 0, 0)$ is the only singular point of $X$ and as a point of $X$ it corresponds to the maximal ideal $I$ of $A = \Gamma(X, \mathcal{O}_X)$ generated by $x_1, y_1, x_2$ and $y_2$. Now consider the ring homomorphism

$$\alpha : A[\mu_1, \eta_1, \mu_2, \eta_2] \longrightarrow S = \bigoplus_{d \geq 0} I^d$$

$$\mu_i \longrightarrow x_i$$

$$\eta_i \longrightarrow y_i.$$ 

Then $\alpha$ is an epimorphism and Ker$\alpha$ is the ideal of $A[\mu_1, \eta_1, \mu_2, \eta_2]$ generated by

$$\{x_i\mu_j - x_j\mu_i, x_i\eta_j - y_j\mu_i, y_i\eta_j - y_j\eta_i\}_{1 \leq i, j \leq 2} \quad (2.41)$$

and we have

$$A[\mu_1, \eta_1, \mu_2, \eta_2] / \text{Ker}\alpha \cong S, \quad (2.42)$$

which induces an isomorphism

$$\overline{X} = \text{Proj} S \longrightarrow \text{Proj} A[\mu_1, \eta_1, \mu_2, \eta_2] / \text{Ker}\alpha. \quad (2.43)$$
The right hand side scheme is a closed subscheme of

\[ \text{Proj} A[\mu_1, \eta_1, \mu_2, \eta_2] \cong \text{Spec} A \times \text{Spec} k \text{Proj} k[\mu_1, \eta_1, \mu_2, \eta_2] = \text{Spec} A \times \text{Spec} k \mathbb{P}^3_k. \]  

(2.44)

So we have the following diagram for blowing-up:

\[ \begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \text{Spec} A \times \text{Spec} k \mathbb{P}^3_k \\
\downarrow & & \downarrow \\
X & = & \text{Spec} A 
\end{array} \]

The standard open covering of \( \mathbb{P}^3_k = \text{Proj} k[\mu_1, \eta_1, \mu_2, \eta_2] \), i.e.,

\[ U_1^0 = \text{Spec} k[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, \frac{\eta_2}{\mu_1}] \]  

(2.45)

\[ U_1^1 = \text{Spec} k[\frac{\mu_1}{\eta_1}, \frac{\mu_2}{\eta_1}, \frac{\eta_2}{\eta_1}] \]  

(2.46)

\[ U_2^0 = \text{Spec} k[\frac{\mu_1}{\mu_2}, \frac{\eta_1}{\mu_2}, \frac{\eta_2}{\mu_2}] \]  

(2.47)

\[ U_2^1 = \text{Spec} k[\frac{\mu_1}{\eta_2}, \frac{\eta_1}{\eta_2}, \frac{\mu_2}{\eta_2}] \]  

(2.48)

induces an open covering for \( \mathcal{X} \), say \( W_1^0, W_1^1, W_2^0 \) and \( W_2^1 \) where

\[ \mu_1 \neq 0, \quad W_1^0 = \text{Spec} k[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \]  

(2.49)

\[ \eta_1 \neq 0, \quad W_1^1 = \text{Spec} k[y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}] \]  

(2.50)

\[ \mu_2 \neq 0, \quad W_2^0 = \text{Spec} k[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2] \]  

(2.51)

\[ \eta_2 \neq 0, \quad W_2^1 = \text{Spec} k[\frac{x_1}{y_2}, \frac{y_1}{y_2}, y_2]. \]  

(2.52)

The schemes \( W_1^0, W_1^1, W_2^0 \) and \( W_2^1 \) are all regular schemes and are pieces for the covering of \( \mathcal{X} \). □
Example 2.20 Let $k$ be a field and

$$X = \text{Spec } k[x_1, y_1, x_2, y_2]/(y_1^2 - y_2^2 - y_1x_1^2 + y_2x_2^2).$$  \hfill (2.53)

By using the Jacobian criterion we can check that $P_0 = (0,0,0,0)$ is the only singular point of $X$ which corresponds to the ideal of $A = I(X, \mathcal{O}_X)$ generated by $x_1, y_1, x_2$ and $y_2$, ie,

$$I = \mathcal{M}_{P_0} = (x_1, y_1, x_2, y_2)/(y_1^2 - y_2^2 - y_1x_1^2 + y_2x_2^2).$$  \hfill (2.54)

Now we blow-up $P_0$. Consider the epimorphism

$$\alpha : A[\mu_1, \eta_1, \mu_2, \eta_2] \longrightarrow S = \bigoplus_{d \geq 0} I^d$$

$$\mu_i \mapsto x_i$$
$$\eta_i \mapsto y_i.$$ 

If we use the the procedure of Mahtab, we get four pieces for the covering of $\tilde{X}$, say $V_0^1, V_1^1, V_2^0$ and $V_2^1$ as follows:

Chart 1

$$V_1^0 = \text{Spec } k[x_1, y_1, x_2, y_2]/((\frac{y_1}{x_1})^2 - (\frac{y_2}{x_1})^2 - x_1(\frac{y_1}{x_1}) + x_1(\frac{y_2}{x_1})(\frac{x_2}{x_1})^2) = \text{Spec } k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2).$$  \hfill (2.55)

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - X_1Y_1 + X_1Y_2X_2^2$. If $Q = (a_1, b_1, a_2, b_2) \in V_1^0$, then

$$\text{Moje}(Q, f) = (a_2^2b_2 - b_1)T_1 + (2b_1 - a_1)S_1 + 2a_1a_2b_2T_2 + (a_1a_2^2 - 2b_2)S_2,$$  \hfill (2.56)

ie,

$$C(Q) = (a_2^2b_2 - b_1 - 2b_1 - a_1 - 2a_1a_2b_2 - a_1a_2^2 - 2b_2).$$  \hfill (2.57)

We can check that $C(Q)$ has rank zero if and only if $a_1 = b_1 = b_2 = 0$. Hence

$$\text{(2.58)}$$
Now we blow-up $S$. Let $A = \Gamma(V_1^0, \mathcal{O}_{X_1} | V_1^0)$ and $I$ be the ideal of $A$ generated by $x_1, y_1, y_2$ and define

$$\alpha : A[\mu_1, \eta_1, \eta_2] \to S = \bigoplus_{d \geq 0} I^d$$

$$\eta_i \mapsto y_i$$

$$\mu_1 \mapsto x_1.$$ 

If we use the rest of details about the procedure of Mahtab we get three pieces for the covering of $\widetilde{V}_1^0$, say $V_{11}^{00}, V_{11}^{01}$ and $V_{12}^{01}$ as follows:

**Chart 1.1**

$$V_{11}^{00} = \text{Spec } k[x_1, \frac{y_1}{x_1}, x_2, \frac{y_2}{x_1}] / \left( \left( \frac{y_1}{x_1} \right)^2 - \left( \frac{y_2}{x_1} \right)^2 - \left( \frac{y_1}{x_1} \right) + x_2 \left( \frac{y_2}{x_1} \right) \right) = \text{Spec } k[X_1, Y_1, X_2, Y_2] / (Y_1^2 - Y_2^2 - Y_1 - X_2^2Y_2).$$

(2.59)

Let $Q = (a_1, b_1, a_2, b_2) \in V_{11}^{00}$ and $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1 - X_2^2Y_2$. Then

$$\text{Moje}(Q, f) = (2b_1 - 1)S_1 - 2a_2b_2T_2 - (a_2^2 + 2b_2)S_2,$$

(2.60)

so

$$C(Q) = \begin{pmatrix} 0 & 2b_1 - 1 & -2a_2b_2 & -(a_2^2 + 2b_2) \end{pmatrix}.$$ 

(2.61)

Hence $C(Q)$ has rank zero if $b_1 = \frac{1}{2}, a_2 = b_2 = 0$ which is impossible. This means that $Q$ is a regular point and consequently $V_{11}^{00}$ is a regular scheme.

**Chart 1.2**

$$V_{11}^{01} = \text{Spec } k[X_1, Y_1, X_2, Y_2] / (1 - Y_2^2 - X_1 + X_1Y_1X_2^2),$$

(2.62)

which is a regular scheme (look at $C(Q)$).

**Chart 1.4**

$$V_{12}^{01} = \text{Spec } k[X_1, Y_1, X_2, Y_2] / (Y_1^2 - 1 - X_1Y_1 + X_1X_2^2)$$

(2.63)
which is a regular scheme (look at $C(Q)$). The gluing of $V_{11}^{00}$, $V_{11}^{01}$ and $V_{12}^{01}$ gives us the regular scheme $V_1^{00}$.

Chart 2

$$V_1^{1} = \text{Spec} \left\{ \frac{x_1}{y_1}, \frac{x_2}{y_2} \right\} / (1 - \left( \frac{y_2}{y_1} \right) - \frac{x_1}{y_1} + y_1(\frac{y_2}{y_1})(\frac{x_2}{y_1})^2 \right\} = \text{Spec} k[X_1, Y_1, X_2, Y_2] / (1 - Y_2^2 - Y_1 X_1^2 + Y_1 Y_2 X_2^2). \quad (2.64)$$

Let $f(X_1, Y_1, X_2, Y_2) = 1 - Y_2^2 - Y_1 X_1^2 + Y_1 Y_2 X_2^2$ and $Q = (a_1, b_1, a_2, b_2) \in V_1^{00}$. Then

$$\text{Moje}(Q, f) = -2a_1 b_1 T_1 + (b_2 a_2^2 - a_1^2) S_1 + 2b_1 a_2 b_2 T_2 + (b_1 a_2^2 - 2b_2) S_2, \quad (2.65)$$

so

$$C(Q) = (-2a_1 b_1 \quad b_2 a_2^2 - a_1^2 \quad 2b_1 a_2 b_2 \quad b_1 a_2^2 - 2b_2 ), \quad (2.66)$$

which has rank zero if $-2a_1 b_1 = b_2 a_2^2 - a_1^2 = 2b_1 a_2 b_2 = b_1 a_2^2 - 2b_2 = 0$. This system has the solution $a_1 = b_1 = b_2 = 0$ or $a_1 = a_2 = b_2 = 0$. Considering $f(a_1, b_1, a_2, b_2) = 0$, we find out that none of these solutions are acceptable. Hence $Q$ is a regular point and consequently $V_1^{1}$ is a regular scheme.

Chart 3

$$V_2^{0} = \text{Spec} \left\{ \frac{x_1}{x_2}, \frac{y_1}{x_2}, \frac{x_2}{x_2} \right\} / (\frac{y_1}{x_2})^2 - (\frac{y_2}{x_2})^2 - x_2(\frac{y_1}{x_2})(\frac{x_1}{x_2})^2 + x_2(\frac{y_2}{x_2}) = \text{Spec} k[X_1, Y_1, X_2, Y_2] / (Y_2^2 - Y_1^2 - X_2 Y_1 X_1^2 + X_2 Y_2). \quad (2.67)$$

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - X_2 Y_1 X_1^2 + X_2 Y_2$ and $Q = (a_1, b_1, a_2, b_2) \in V_2^{0}$. Then

$$\text{Moje}(Q, f) = (-2a_1 a_2 b_1) T_1 + (2b_1 - a_1^2 a_2) S_1 + (b_2 - a_1^2 b_1) T_2 + (a_2 - 2b_2) S_2, \quad (2.68)$$

ie,

$$C(Q) = (-2a_1 a_2 b_1 \quad 2b_1 - a_1^2 a_2 \quad b_2 - a_1^2 b_1 \quad a_2 - 2b_2 ). \quad (2.69)$$
So $Q$ is singular if $-2a_1a_2b_1 = 2b_1 - a_1^2a_2 = b_2 - a_1^2b_1 = a_2 - 2b_2 = 0$. This system has the solution $b_1 = a_2 = b_2 = 0$ (note that $a_1 = b_1 = a_2 = b_2 = 0$ is also a solution for the system, but actually it is a special case of $b_1 = a_2 = b_2 = 0$), i.e.,

$$(V^0_2)_{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in V^0_1 | b_1 = a_2 = b_2 = 0\}. \quad (2.70)$$

Now we blow-up $(V^0_2)_{\text{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $\tilde{V}^0_2$, say $V^{01}_2, V^{00}_2$ and $V^{01}_2$ as follows:

**Chart 3.2**

$$V^{01}_2 = \text{Spec } k[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - X_2X_1^2 + X_2Y_2), \quad (2.71)$$

which is a regular scheme (just check $C(Q)$).

**Chart 3.3**

$$V^{00}_2 = \text{Spec } k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1X_1^2 + Y_2). \quad (2.72)$$

Let $f(X_1, Y_1, X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1X_1^2 + Y_2$ and $Q = (a_1, b_1, a_2, b_2) \in V^{00}_2$. Then

$$\text{Moje}(Q, f) = -2a_1b_1T_1 + (2b_1 - a_1^2)S_1 + (1 - 2b_2)S_2, \quad (2.73)$$

so

$$C(Q) = (-2a_1b_1 \quad 2b_1 - a_1^2 \quad 0 \quad 1 - 2b_2). \quad (2.74)$$

So $Q$ is singular if $-2a_1b_1 = 2b_1 - a_1^2 = 1 - 2b_2 = 0$. But this system has the solution $a_1 = b_1 = b_2 = -\frac{1}{2} = 0$ which is not acceptable. Hence $Q$ is a regular point and as a result $V^{00}_2$ is a regular scheme.

**Chart 3.4**

$$V^{01}_2 = \text{Spec } k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - X_2Y_1X_1^2 + X_2), \quad (2.75)$$

which is a regular scheme (just check $C(Q)$). The gluing of $V^{01}_2, V^{00}_2$ and $V^{01}_2$ gives us the regular scheme $\tilde{V}_2$. 
CHAPTER 2. Procedures

Chart 4

\[ V_2^1 = \text{Spec} k[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - Y_2Y_1X_1^2 + Y_2X_2^2), \] (2.76)

which is a regular scheme (just check \( C(Q) \)). The gluing of \( \widetilde{V}_0^1, V_1^1, \widetilde{V}_2^0 \) and \( V_2^1 \) gives us the regular scheme \( \widetilde{X} \).

Example 2.21 Let \( R \) be a discrete valuation ring and \( U = \text{Spec} R[x, y]/(x^2 + \pi^2y^3 - \pi^4) \). Since \( (\frac{x}{\pi})^2 + y^3 - \pi^2 = 0 \), \( (\frac{x}{\pi}) \) is integral over \( \Gamma(U, \mathcal{O}_U) \). We get \( V = U^{\text{nor}} = \text{Spec} R[\frac{x}{\pi}, y]/((\frac{x}{\pi})^2 + y^3 - \pi^2) = \text{Spec} R[X, Y]/(X^2 + Y^3 - \pi^2) \).

By using the Jacobian criterion we can check that

\[ V_0^{\text{Sing}} = \{(a, b) \in V_0 | a = b = 0\}. \] (2.77)

Let \( f(X, Y) = X^2 + Y^3 - \pi^2 \) and \( P_0 = (0, 0) \in V_0 \). Then for \( u \) and \( v \) in \( R \) we get

\[ f(T + u\pi, S + v\pi) = (T + u\pi)^2 + (S + v\pi)^3 - \pi^2 \] (2.78)

So \( \text{Moje}(P_0, f) = 0 \). Hence

\[ D(P_0) = (0 \ 0 \ 0), \] (2.79)

which has rank zero, i.e., \( P_0 = (0, 0) \in V_0 \) is the only singular point of \( V \).

Let \( A = \Gamma(V, \mathcal{O}_V) \). Then as a point of \( V \), \( P_0 \) corresponds to the maximal ideal \( I = (x, y, \pi) \) of \( A \). Now we blow-up \( P_0 \). Consider the ring homomorphism

\[ \phi : A[\mu, \eta, \theta] \rightarrow S = \bigoplus_{d \geq 0} I^d \]

\[ \mu \mapsto x \]
\[ \eta \mapsto y \]
\[ \theta \mapsto \pi. \]

By using the rest details of the procedure of Mahtab, we get three pieces for the covering of \( \widetilde{V} \) as follows:
CHAPTER 2. Procedures

Chart 1

\[ V^0_1 = \text{Spec } R[x, y, \frac{\pi}{x}, \frac{\pi}{y}]/(1 + x(\frac{y}{x})^3 - (\frac{\pi}{x})^2) = \]
\[ \text{Spec } R[X, Y, Z]/(1 + XY^3 - Z^2, XZ - \pi). \] (2.80)

So

\[ (V^0_1)_\pi = \text{Spec } k[X, Y, Z]/(1 + XY^3 - Z^2, XZ). \] (2.81)

By using the Jacobian criterion, we can check that

\[ (V^0_1)^{\text{Sing}}_\pi = \phi. \] (2.82)

which means that \( (V^0_1)^{\text{Sing}} = \phi, \) ie, \( V^0_1 \) is a regular scheme.

Chart 2

\[ V^1_1 = \text{Spec } R[\frac{x}{\pi}, y, \frac{\pi}{y}]/((\frac{x}{\pi})^2 + y - (\frac{\pi}{y})^2) \]
\[ \text{Spec } R[X, Y, Z]/(X^2 + Y - Z^2, YZ - \pi) \]
\[ \text{Spec } R[X, Z]/(Z^3 - X^2Z - \pi), \] (2.83)

so

\[ (V^1_1)^{\text{Sing}}_\pi = \{ (a, c) \in (V^1_1)_\pi | x = z = 0 \}. \] (2.84)

Let \( g(X, Z) = Z^3 - X^2Z - \pi. \) Then

\[ g(T + u\pi, W + w\pi) = (W + w\pi)^3 - (T + u\pi)^2(W + w\pi) - \pi. \] (2.85)

So \( \text{Moje}(Q, g) = -\pi, \) ie,

\[ D(Q) = (\begin{array}{ccc} -1 & 0 & 0 \end{array}), \] (2.86)

which has rank one. Hence \( Q \) is a regular point of \( V^1_1 \) and consequently \( V^1_1 \) is a regular scheme.

Chart 3

\[ V^0_\theta = \text{Spec } R[\frac{x}{\pi}, y]/((\frac{x}{\pi})^2 + \pi(\frac{y}{\pi})^3 - 1) = \]
\[ \text{Spec } R[X, Y]/(X^2 + \pi Y^3 - 1), \] (2.87)
and we get
\[(V_{\theta}^0)_\pi = \text{Spec } k[X,Y]/(X^2 - 1).\] (2.88)

By using the Jacobian criterion we can check that \((V_{\theta}^0)^\text{Sing}_\pi = \phi\), so \((V_{\theta}^0)^\text{Sing} = \phi\), i.e., \(V_{\theta}^0\) is a regular scheme. The gluing of \(V^0_1, V^1_1\) and \(V^0_\theta\) gives us the regular scheme \(\tilde{V}\).

**Remark 2.22** In the rest of this thesis each time that we are involving with a scheme of the form \(X = \text{Spec } R[x_1,y_1,x_2,y_2]/(F(x_1,y_1,x_2,y_2), G(x_1,y_1,x_2,y_2))\), we use \(f_1\) and \(f_2\) rather than \(F\) and \(G\) respectively. If we have an additional indeterminate \(Z\) and
\[
X = \text{Spec } R[x_1,y_1,x_2,y_2,Z]/
\quad (F(x_1,y_1,x_2,y_2,Z), G(x_1,y_1,x_2,y_2,Z), H(x_1,y_1,x_2,y_2,Z)),
\] (2.89)
we use \(f_1, f_2\) and \(f_3\) instead of \(F, G\) and \(H\) respectively.

For the calculation of \(\text{Moje}\) at a point \(P = (a_1,b_1,a_2,b_2,c) \in X_\pi\), we use \(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma \in R\) such that \(\alpha_i \equiv a_i \pmod{\pi}\) and \(\beta_i \equiv b_i \pmod{\pi}\) and \(\gamma \equiv c \pmod{\pi}\). If \(a_i = 0\) or \(b_i = 0\) or \(c = 0\) we use \(\alpha_i = u_i\pi\) or \(\beta_i = v_i\pi\) or \(\gamma = w\pi\).

For new indeterminates after translation we use \(T, S, T\) and \(W\) corresponding to \(x_1, y_1, y_2\) respectively.

The lower and upper index show us the indeterminate involving with the blowing-up. To be more precise, we use \(X^0_1\) to show that the involving indeterminate in the blowing-up is \(x_1\), \(X^1_1\) when it is \(y_1\), \(X^0_2\) when it is \(x_2\), \(X^1_2\) when it is \(y_2\) and finally \(X^0_\theta\) when the involving indeterminate is \(Z\).
Chapter 3

Product of ordinary double points

3.1 Introduction

In this chapter $R$ always denotes an arbitrary regular ring (except in §3.5 where $R$ denotes a dvr). We introduce $Y^{12-n}$ in the second section and try to resolve its singularities later. We find this desingularisation in two different methods. The results from the both methods are the same. In the last section of this chapter, we introduce $X^{12-n}$ and consider our schemes over Spec$R$ where $R$ is a discrete valuation ring with algebraically closed residue field $k = \frac{R}{\pi}$.

3.2 Proposed schemes

Consider the surface

$$Y^i = \text{Spec } R[t][x_i, y_i]/(x_i y_i - t) \cong \mathbb{A}^2_k, \quad 1 \leq i \leq n,$$

over Spec $R[t]$. Then $Y^i$ over Spec $R[t]$ is smooth everywhere except at $x_1 = y_1 = 0$, where $Y^i$ has an ordinary double point on the fibre corresponding to $t = 0,$
but $Y^i$ is a regular scheme. The fibre product of

$$Y^1 = \text{Spec } R[t][x_1, y_1]/(x_1 y_1 - t)$$

(3.2)

and

$$Y^2 = \text{Spec } R[t][x_2, y_2]/(x_2 y_2 - t)$$

(3.3)

is

$$Y^{12} = \text{Spec } R[t][x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_1 y_1 - t)$$

(3.4)

$$= \text{Spec } R[x_1, y_1, x_2, y_2]/B_2$$

where $B_2$ is the ideal $(x_1 y_1 - x_2 y_2)$ of $A_2 = R[x_1, y_1, x_2, y_2]$. Inductively for each $n \in \mathbb{N}$ we get

$$Y^{12\ldots n} = \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 - x_2 y_2, x_1 y_1 - x_3 y_3, \ldots, x_1 y_1 - x_n y_n)$$

$$= \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 - x_i y_i)_{2 \leq i \leq n}$$

$$:= \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/B_n,$$

(3.5)

where

$$B_n = (x_1 y_1 - x_i y_i)_{2 \leq i \leq n}. \quad (3.6)$$

In fact in the affine space $\mathbb{A}_k^{2n}$, $Y^{12\ldots n}$ is determined by the equations

$$x_1 y_1 = x_2 y_2 = \ldots = x_n y_n.$$

(3.7)

**Remark 3.1** In the other sections of this chapter we will try to find singular points and also a desingularisation of $Y^{12\ldots n}$ in two different methods. These desingularisations would be the results of some successive blowings-up.

### 3.3 First Method

In this section we will find the singular points of the scheme $Y^{12\ldots n}$ and by successive blowings-up, we will try to resolve its singularities. This is the resolution described (without proof) in Scholl’s paper (see [21]).
**Convention 3.2** In the rest of this section we consider $Y^{1_{2...n}}$ as

$$Y^{1_{2...n}} = \text{Spec } R[x_1, y_1, ..., x_n, y_n]/B_n$$  \hspace{1cm} (3.8)

where

$$B_n = (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, ..., x_1y_1 - x_ny_n).$$  \hspace{1cm} (3.9)

We also use $A_n$ for the polynomial ring $k[x_1, y_1, ..., x_n, y_n]$ and $A_{1_{2...n}}$ for the quotient ring $A_n/B_n = \Gamma(Y^{1_{2...n}}, O_{Y^{1_{2...n}}})$.

**Lemma 3.3** Let $X$ and $Z$ be schemes over $S$ such that $Z$ is flat over $S$, and $Y$ is a closed subscheme of $X$. Then the blowing-up of $X \times_S Z$ at $Y \times_S Z$ is isomorphic to $\overline{X} \times_S Z$, where $\overline{X}$ is the blowing-up of $X$ at $Y$.

**Proof:** We prove it in the affine case. Let $X = \text{Spec } A$, $Y = \text{Spec } A/I$, $Z = \text{Spec } B$ and $S = \text{Spec } R_1$, ie,

\[
\begin{array}{c}
R_1 \\
\downarrow \downarrow \\
I \subset A \quad B
\end{array}
\]

which induces

\[
\begin{array}{c}
Y = \text{Spec } A/I \hookrightarrow X = \text{Spec } A \\
\downarrow \\
S = \text{Spec } R_1 \quad Z = \text{Spec } B
\end{array}
\]

The following diagram shows us the scheme of the proof.
Since $B$ is flat over $R_1$, $I \hookrightarrow A$ implies that $I \otimes_{R_1} B \hookrightarrow A \otimes_{R_1} B$ is an injection. Let $J$ be the image of $I \otimes_{R_1} B$ in $A \otimes_{R_1} B$. Then $J \cong I \otimes_{R_1} B$. Now notice that $\widetilde{X} \times_S Z = \left( \text{Proj}_{d \geq 0} I^d \right) \times_S \text{Spec } B$ and $X \times_S Z = \text{Proj}_{d \geq 0} J^d = \text{Proj}_{d \geq 0} (I \otimes_{R_1} B)^d \cong \text{Proj}_{d \geq 0} (I \otimes_{R_1} B) \cong (\text{Proj}_{d \geq 0} I^d) \otimes_{\text{Spec } R_1} \text{Spec } B = \widetilde{X} \times_{\text{Spec } R_1} Z$.

If $X$ and $Z$ are not affine, then we consider them as the union of affine schemes and use the above result for the affine pieces and then glue them.

**Example 3.4** Consider the scheme $Y^{123} = \text{Spec } R[x_1, y_1, \ldots, x_3, y_3]/(f_1, f_2)$ where

$$f_1(x_1, y_1, \ldots, x_3, y_3) = x_1 y_1 - x_2 y_2$$

and

$$f_2(x_1, y_1, \ldots, x_3, y_3) = x_1 y_1 - x_3 y_3.$$  

(3.10)  

(3.11)

By using the Jacobian criterion we can check that the singular points of $Y^{123}$ are points of the form $(a_1, b_1, a_2, b_2, a_3, b_3)$ such that at least two pairs $(a_i, b_i)$ are zero (see example(2.2)). In particular $P_0 = (0, 0, 0, 0, 0, 0)$ is a singular point. Now we blow-up $P_0$. The following diagram is related to this blowing-up:
By using the procedure of Mahtab we can check that the standard open covering for $\mathbb{P}^5_R = \text{Proj} R[\mu_1, \eta_1, ..., \mu_3, \eta_3]$, i.e.,

$$ U_1^0 = \text{Spec } R[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, ..., \frac{\mu_3}{\mu_1}, \eta_3], $$

induces an open covering for $\mathcal{Y}_{123}^3$, say $W_1^0, W_1^1, ..., W_3^0, W_3^1$, where

$$ U_3^1 = \text{Spec } R[\frac{\mu_1}{\eta_3}, \frac{\eta_1}{\eta_3}, ..., \frac{\eta_2}{\eta_3}, \frac{\mu_3}{\eta_3}], $$

(3.12)

and $Y_{123}^3$ has $P_0 = (0, 0, 0, 0)$ as its only singular point. Hence $(W_1^0)^{\text{Sing}} = A_R^1 \times \text{Spec } R \{ P_0 \}$ in $A_R^1 \times \text{Spec } R Y_{123}^3$. According to lemma (3.3), the blowing-up of $W_1^0$ along $(W_1^0)^{\text{Sing}}$ is isomorphic to

$$ A_R^1 \times \text{Spec } R Y_{23}^3 \cong A_R^1 \times \text{Spec } R (W_2^0 \cup W_2^1) \cup W_3^0 \cup W_3^1) \cong $$

$$ (A_R^1 \times \text{Spec } R W_2^0) \cup (A_R^1 \times \text{Spec } R W_2^1) \cup (A_R^1 \times \text{Spec } R W_3^0) \cup (A_R^1 \times \text{Spec } R W_3^1) \cong $$

$$ \text{Spec } R[x_1] \times \text{Spec } R \left[ \frac{x_3}{x_2}, \frac{y_3}{y_2} \right] \cup ... \cup \text{Spec } R[x_1] \times \text{Spec } R \left[ \frac{x_2}{y_3}, \frac{y_2}{y_3} \right] \cong $$

$$ W_{12}^{00} \cup W_{12}^{01} \cup W_{13}^{00} \cup W_{13}^{01}. $$

(3.14)
Using the same method for the resolution of singularities of $W_1^1, W_2^0, ..., W_3^1$ we get

$$Y_2^{123} = W_{12}^{00} \cup W_{12}^{01} \cup W_{13}^{00} \cup W_{13}^{01} \cup W_{31}^{10} \cup W_{31}^{11} \cup W_{32}^{10} \cup W_{32}^{11},$$

(3.15)

where $W_{ij}^{0\beta}$'s are 24 pieces for the covering of $Y_2^{123}$ and

$$W_{12}^{00} = \text{Spec } R[x_1, \frac{x_2}{x_1}, \frac{x_3}{x_2}, \frac{y_3}{x_2}],$$

$$W_{12}^{01} = \text{Spec } R[x_1, \frac{y_2}{x_1}, \frac{x_3}{y_2}, \frac{y_3}{y_2}],$$

$$\vdots = \ldots$$

$$\vdots = \ldots$$

$$\vdots = \ldots$$

$$W_{32}^{11} = \text{Spec } R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, \frac{y_2}{y_3}, y_3],$$

(3.16)

are all regular schemes. So after two successive blowings-up we get the regular scheme $Y_2^{123}$.

**Remark 3.5** By using the Jacobian criterion we can check that points of the form $P = (a_1, b_1, ..., a_n, b_n) \in Y_1^{12..n}$ such that at least two pairs $(a_i, b_i)$ are zero, are singular (see example (2.3)).

**Theorem 3.6** After $n - 1$ successive blowings-up of $Y_1^{12..n}$ we get a regular scheme $Y_n^{12..n}$ as the gluing of $2^{n-1}(n!)$ open pieces of its covering.

**Proof**: We use induction on $n$. For $n = 2$ and 3 we already have checked (examples 2.19 and 3.4). Suppose it is true for $n < m$ (with $m \geq 4$) and recall that $P_0 = (0,0, ..., 0) \in Y_1^{12..m}$ is a singular point of $Y_1^{12..m}$ which corresponds to the ideal $I_{12..m}$ of $\Gamma(Y_1^{12..m}, O_{Y_1^{12..m}})$ generated by $x_1, y_1, ..., x_m, y_m$, i.e, $m_{P_0}$ is the maximal ideal of $A_m$ generated by $x_1, y_1, ..., x_m, y_m$. We blow-up the origin in $Y_1^{12..m}$ to obtain a scheme $Y_1^{12..m}$. By the procedure of Mahtab, it has an open covering $W_1^0, W_1^1, ..., W_m^0, W_m^1$ where

$$\mu_1 \neq 0, \quad W_1^0 = \text{Spec } (R[\frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, ..., \frac{\mu_m}{\mu_1}, \frac{\eta_m}{\mu_1}] \otimes_R R[x_1, y_1, ..., x_m, y_m]/B_m) \text{ (mod Relations)}$$
\[ \begin{align*}
= \text{Spec } R\left[ \frac{\eta_1}{\mu_1}, \frac{\mu_2}{\mu_1}, \ldots, \frac{\mu_m}{\mu_1}, \frac{\eta_m}{\mu_1}, x_1, y_1, \ldots, x_m, y_m \right] &/ \\
& (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \ldots, x_1y_1 - x_my_m, \text{ Relations}) \\
= \text{Spec } R\left[ \frac{y_1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_m}{x_1}, \frac{y_m}{x_1}, x_1, y_1, \ldots, x_m, y_m \right] &/ \\
& (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \ldots, x_1y_1 - x_my_m) \\
= \text{Spec } R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}, \ldots, \frac{x_m}{x_1}, \frac{y_m}{x_1}] &/ \\
& \left( \frac{y_1}{x_1} - \left( \frac{x_2}{x_1} \right), \frac{y_2}{x_1} - \left( \frac{x_3}{x_1} \right), \ldots, \frac{y_1}{x_1} - \left( \frac{x_m}{x_1} \right) \right) \\
= \text{Spec } R[x_1, \frac{y_1}{x_1}, \frac{x_2}{x_1}, \frac{y_2}{x_1}, \ldots, \frac{x_m}{x_1}, \frac{y_m}{x_1}] &/ \\
& \left( \frac{y_2}{x_1} - \left( \frac{x_3}{x_1} \right), \frac{y_2}{x_1} - \left( \frac{x_4}{x_1} \right), \ldots, \frac{y_2}{x_1} - \left( \frac{x_m}{x_1} \right) \right)
\end{align*} \]

\( = \mathbb{A}_R \times \text{Spec } R \left[ X_1, Y_1, \ldots, X_m, Y_m \right]/(X_2Y_2 - X_3Y_3, \ldots, X_2Y_2 - X_mY_m) \)

\( = \mathbb{A}_R \times \text{Spec } R Y^{2 \ldots m} . \) (3.17)

Note that \((W^\circ_1)^{\text{Sing}} = \mathbb{A}_R \times \text{Spec } R (Y^{23 \ldots m})^{\text{Sing}} \) and \( P_0 = (0, 0, \ldots, 0, 0) \in Y^{23 \ldots m} \) is a singular point. Let \( S = \mathbb{A}_R \times \text{Spec } R \{ P_0 \} \subset (W^\circ_1)^{\text{Sing}} \). If we find the blow-up of \( W^\circ_1 \) at \( S \), by lemma (3.3), the blowing-up of \( W^\circ_1 \) is isomorphic to \( \mathbb{A}_R \times \text{Spec } R Y^{23 \ldots m} \).

By using the same calculations as above we can check that

\( W^\circ_1 = \mathbb{A}_R \times \text{Spec } R Y^{23 \ldots m} \)

\( W^\circ_1 = \mathbb{A}_R \times \text{Spec } R Y^{23 \ldots m} \)

\( \ldots = \ldots \)

\( \ldots = \ldots \)

\( W^\circ_0 = \mathbb{A}_R \times \text{Spec } R Y^{12 \ldots (m-1)} \)

\( W^\circ_1 = \mathbb{A}_R \times \text{Spec } R Y^{12 \ldots (m-1)} . \) (3.18)

If we use the assumption of the induction for \( Y^{23 \ldots m}, \ldots, Y^{12 \ldots (m-1)} \), after \( m - 2 \) successive blowings-up we can resolve the singularities of each piece (Note that all of these pieces are isomorphic) and for each of them we find \( 2^{m-2}(m-1)! \) open pieces for the covering, so totally we get \( 2m(2^{m-2}(m-1)!) = 2^{m-1}(m!) \) open pieces for the covering of \( Y^{12 \ldots m}_{m-1} \) and their gluing through their overlap gives us \( Y^{12 \ldots m}_{m-1} \). \( \square \)
CHAPTER 3. Products of double points

Remark 3.7 As we have seen in theorem (3.6), we can use successive blowings-up for desingularisation of $Y^{12...m}$ and get $2^{m-1}(m!)$ open pieces for the covering. To get an explicit formula for each piece we index the open pieces as $W^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}}_{n_1 n_2 \ldots n_{m-1}}$ where each $\alpha_i \in \{0,1\}$ and $n_i \in \{1,2,\ldots,m\}; = I_m$ and $n_i \neq n_j$. Let $\hat{n}$ be the remaining element of $I_m$ and $Z_i = x_i$ if $t = 0$, $y_i$ if $t = 1$. Then

$$W^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}}_{n_1 n_2 \ldots n_{m-1}} = \text{Spec } R[Z_{n_1}^{\alpha_1}, Z_{n_2}^{\alpha_2}, \ldots, Z_{n_{m-1}}^{\alpha_{m-1}}, Z_{n_{m-1}}^{\alpha_m}, x_\hat{n}, y_\hat{n}].$$

(3.19)

We find it convenient to express it by using the following matrix

$$W^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}}_{n_1 n_2 \ldots n_{m-1}} \leftrightarrow \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{m-1} & 0 & 1 \\ n_1 & n_2 & \ldots & n_{m-1} & \hat{n} & \hat{n} \end{pmatrix}$$

(3.20)

For instance if $\alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0$ and $n_t = t$ for $1 \leq t \leq m - 1$, then $\hat{n} = m$ and we get

$$W^{00...00}_{12...(m-2)(m-1)} \leftrightarrow \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 2 & \ldots & (m-1) & m & m \end{pmatrix}$$

(3.21)

and

$$W^{00...00}_{12...(m-2)(m-1)} = \text{Spec } R[x_1, x_2, x_3, \ldots, y_{m-1}, x_m, y_m].$$

(3.22)

Example 3.8 Let $n = 8$. Then

$$W^{0111001}_{4537218} \leftrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 4 & 5 & 3 & 7 & 2 & 1 & 8 & 6 & 6 \end{pmatrix},$$

(3.23)

hence

$$W^{0111001}_{4537218} = \text{Spec } R[x_4, y_5, y_3, y_7, x_2, x_1, x_8, y_8, x_5, y_7, x_2, x_1, y_8, y_8, x_3, y_3].$$

(3.24)

which is one of 5160960 (\(= 2^7(8!)\)) pieces which appears in the collection of the open pieces for the covering of 7th blowing-up of $Y^{12...8}$, *i.e.* $Y_7^{12...8}$. 
CHAPTER 3. Products of double points

3.4 Second Method

In this section we will try to blow-up a specific ideal and show that just one blowing-up of this ideal gives us the same result as we got in theorem (3.6). For case \( n = 2 \) this method gives us the same result as we had in example (2.19). This ideal was given by Deligne in [7].

Convention 3.9 Let \( J_n \) be the ideal of \( \Gamma(Y^{12...n}, O_{Y^{12...n}}) \) generated by the elements of the form \( Z^0_{\sigma(1)}Z^1_{\sigma(2)}...Z^{n-1}_{\sigma(n)} \) where \( Z^t_{\sigma(t)} \) means \( Z_{\sigma(t)} \) to power \( t - 1 \) and \( Z_{\sigma(t)} = x_{\sigma(t)} \) or \( y_{\sigma(t)} \), \( t = 1, 2, ..., n \) and \( \sigma \in S_n \). Since \( Z^0_{\sigma(1)} = 1 \) this term is not effective and so \( J_n \) is generated by \( 2^{n-1}(n!) \) elements of this form. There are a lot of relations between the generators of \( J_n \) which we will use later.

Theorem 3.10 Let \( Y^{12...n} = \text{Spec} R[x_1, y_1, ..., x_n, y_n]/B_n \). If we blow-up the ideal \( J_n \), then we get the regular scheme \( Y^{12...n} \).

Proof: Let \( N = 2^{n-1}(n!) \), \( \sigma \in S_n \) and \( i_1, i_2, ..., i_{n-1} \in \{0, 1\} \). We consider \( N \) variables \( \mu^i_{\sigma,1}, \mu^i_{\sigma,2}, ..., \mu^i_{\sigma,n-1} \) (corresponding to the generators of \( J_n \) say \( A^i_{\sigma,1}, A^i_{\sigma,2}, ..., A^i_{\sigma,n-1} \)'s) and define the ring homomorphism

\[
\alpha_n : A_{12...n}[\mu^{00...00}_{\sigma_1}, ..., \mu^{11...11}_{\sigma_n}] \rightarrow S^{12...n} = \bigoplus_{d \geq 0} J^d_n
\]

Then \( \alpha_n \) is an epimorphism and we have

\[
S^{12...n} = \bigoplus_{d \geq 0} J^d_n \cong A_{12...n}[\mu^{00...00}_{\sigma_1}, ..., \mu^{11...11}_{\sigma_n}]/\text{Ker} \alpha_n,
\]

where

\[
\text{Ker} \alpha_n = (\mu^{00...00}_{\sigma_1} A_{\sigma_2}^{00...00} - A_{\sigma_1}^{00...00} \mu^{00...00}_{\sigma_2}, ..., \mu^{11...11}_{\sigma_{n-1}} A_{\sigma_n}^{11...11} - A_{\sigma_{n-1}}^{11...11} \mu^{11...11}_{\sigma_n}),
\]

and \( \alpha_n \) induces an isomorphism

\[
Y^{12...n} = \text{Proj} S^{12...n} \cong \text{Proj}(A_{12...n}[\mu^{00...00}_{\sigma_1}, ..., \mu^{11...11}_{\sigma_n}]/\text{Ker} \alpha_n),
\]

which is a closed subscheme of

\[
P^{N-1}_{A_{12...n}} = \text{Proj} A_{12...n}[\mu^{00...00}_{\sigma_1}, ..., \mu^{11...11}_{\sigma_n}] = \text{Spec} A_{12...n} \times \text{Spec} R P^{N-1}_R.
\]
CHAPTER 3. Products of double points

The standard open covering of $\mathbb{P}^{N-1}_k$, i.e.,

$$\mu_{\sigma_1}^{00...00} \neq 0, \quad U_{\sigma_1}^{00...00} = \text{Spec } R[\frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_1}^{00...00}}, \ldots, \frac{\mu_{\sigma_{N-1}}^{11...11}}{\mu_{\sigma_1}^{00...00}}],$$

$$= \ldots,$$

$$= \ldots,$$

$$= \ldots,$$

$$\mu_{\sigma_{N-1}}^{11...11} \neq 0, \quad U_{\sigma_{N-1}}^{11...11} = \text{Spec } R[\frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_{N-1}}^{00...00}}, \ldots, \frac{\mu_{\sigma_{N-1}}^{11...11}}{\mu_{\sigma_{N-1}}^{11...11}}],$$

(3.29)

induces an open covering for $Y_{12...n}$, say

$$Y_{12...n} = V_{\sigma_1}^{00...00} \cup \ldots \cup V_{\sigma_{N-1}}^{11...11},$$

(3.30)

where

$$V_{\sigma_1}^{00...00} = \text{Spec } R[x_1, y_1, \ldots, x_n, y_n, \frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_1}^{00...00}}, \ldots, \frac{\mu_{\sigma_{N-1}}^{11...11}}{\mu_{\sigma_1}^{00...00}}]/(x_1y_1 - x_2y_2, \ldots, x_1y_1 - x_ny_n, \frac{1}{\mu_{\sigma_1}^{00...00}}(\text{Relations}))$$

$$= \ldots,$$

$$= \ldots,$$

$$= \ldots,$$

$$V_{\sigma_{N-1}}^{11...11} = \text{Spec } R[x_1, y_1, \ldots, x_n, y_n, \frac{\mu_{\sigma_2}^{00...00}}{\mu_{\sigma_{N-1}}^{11...11}}, \ldots, \frac{\mu_{\sigma_{N-1}}^{11...11}}{\mu_{\sigma_{N-1}}^{11...11}}]/(x_1y_1 - x_2y_2, \ldots, x_1y_1 - x_ny_n, \frac{1}{\mu_{\sigma_{N-1}}^{11...11}}(\text{Relations})),$$

(3.31)
By using the relations in our calculations as it is stated in Deligne’s paper (see lemma (5.5) in [7]), we get

\[ V_{\sigma_1}^{00} = \text{Spec } R[x_1, x_2, \ldots, x_n, y_1, y_2], \]  

(3.32)

The regularity of \( V_{\sigma_1}^{00} \) is also stated in lemma 5.5 in [7]. We can do the same calculations for the other pieces of the covering of \( Y_{12}^{12} \).

**Remark 3.11** To get an explicit formula for pieces of the covering of \( Y_{12}^{12} \) stated in theorem (3.10), we index the open pieces as \( V_{\sigma}^{\beta_1 \beta_2 \ldots \beta_{n-1}} \) where each \( \beta_i \in \{0, 1\} \) and \( \sigma \in S_n \). Then

\[ V_{\sigma}^{\beta_1 \beta_2 \ldots \beta_{n-1}} = \text{Spec } R[x^{\beta_1}_{\sigma(1)}, x^{\beta_2}_{\sigma(2)}, \ldots, x^{\beta_{n-1}}_{\sigma(n-1)}, x^{\beta_n}_{\sigma(n)}], \]  

(3.33)

where

\[ \beta_1^* = 1 - \beta_1, \]  

(3.34)

and

\[ T_{\sigma(j)}^{\beta_j} = \begin{cases} x_{\sigma(j)} & \text{if } \beta_j = 0 \\ y_{\sigma(j)} & \text{if } \beta_j = 1. \end{cases} \]  

(3.36)

We can use a diagram to get \( V_{\sigma}^{\beta_1 \beta_2 \ldots \beta_n} \) as follows:

Which means that we start from \( \beta_1 \) and consider \( T_{\sigma(1)}^{\beta_1} \) corresponding to \( \beta_1 \) and \( \sigma(1) \) and divide it by \( T_{\sigma(2)}^{\beta_1} \) corresponding to the arrow between \( \beta_1 \) and \( \sigma(2) \). Then we start with \( \beta_2 \) and divide \( T_{\sigma(2)}^{\beta_2} \) by \( T_{\sigma(3)}^{\beta_2} \) and continue this process up-to division of \( T_{\sigma(n-1)}^{\beta_{n-1}} \) by \( T_{\sigma(n)}^{\beta_{n-1}} \). Later on, we write the term \( T_{\sigma(n)}^{\beta_{n-1}} \) alone and finally we find the division of \( T_{\sigma(1)}^{\beta_1} \) by \( T_{\sigma(2)}^{\beta_1} \).
For instance if $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0$ and
\begin{equation}
\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\
1 & 2 & 3 & \ldots & n \end{pmatrix},
\end{equation}
then
\begin{equation}
V_{\sigma^{00\ldots00}} = \operatorname{Spec} R \left[ \frac{x_1}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_{n-1}}{x_n}, \frac{x_n}{x_1}, \frac{y_1}{x_2} \right] = \operatorname{Spec} R \left[ \frac{x_1}{x_n}, \frac{x_{n-1}}{x_n}, \frac{x_{n-2}}{x_{n-1}}, \ldots, \frac{x_2}{x_3}, \frac{x_1}{x_2}, \frac{y_1}{x_2} \right].
\end{equation}

Lemma 3.12 By using the same notation as it was used in theorem (3.6) and theorem (3.10), we get
\begin{equation}
W_{n_1 n_2 \ldots n_{m-1}}^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}} = V_{\sigma}^{\alpha_{m-1} \alpha_m - \alpha_1},
\end{equation}
where
\begin{equation}
\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\
\hat{n} & n_{m-1} & n_{m-2} & \ldots & n_1 \end{pmatrix}.
\end{equation}

Proof: Let $\sigma \in S_m$, say
\begin{equation}
\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\
\sigma(1) & n_{m-1} & n_{m-2} & \ldots & n_1 \end{pmatrix}
\end{equation}
and $\beta_i = \alpha_{m-i}$. Then
\begin{equation}
W_{n_1 n_2 \ldots n_{m-1}}^{\alpha_1 \alpha_2 \ldots \alpha_{m-1}} = V_{\sigma}^{\beta_1 \beta_2 \ldots \beta_{m-1}} = \operatorname{Spec} R \left[ \frac{T_{\sigma(1)}^{\beta_1}}{T_{\sigma(2)}^{\beta_1}}, \frac{T_{\sigma(2)}^{\beta_2}}{T_{\sigma(3)}^{\beta_2}}, \ldots, \frac{T_{\sigma(m-2)}^{\beta_{m-2}}}{T_{\sigma(m-1)}^{\beta_{m-2}}}, \frac{T_{\sigma(m-1)}^{\beta_{m-1}}}{T_{\sigma(m)}^{\beta_{m-1}}}, \frac{T_{\sigma(m)}^{\beta_{m}}}{T_{\sigma(1)}^{\beta_{m}}}, \ldots, \frac{T_{\sigma(2)}^{\beta_2}}{T_{\sigma(3)}^{\beta_2}}, \frac{T_{\sigma(3)}^{\beta_1}}{T_{\sigma(1)}^{\beta_1}}, \frac{T_{\sigma(1)}^{\beta_1}}{T_{\sigma(2)}^{\beta_1}} \right] = \ldots
\end{equation}

Example 3.13 Compare $W = W_{\{13\}2578\{11\}19\{6143\}}^{101001001011}$ and $V = V_{\sigma}^{111010001001}$, where
CHAPTER 3. Products of double points

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 9 & 12 & 3 & 4 & 1 & 6 & 10 & 11 & 8 & 7 & 5 & 2 & 13 \end{pmatrix}. \] (3.43)

Solution

We have

\[ W \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 13 & 2 & 5 & 7 & 8 & 11 & 10 & 6 & 1 & 4 & 3 & 12 & 9 & 9 \end{pmatrix}. \] (3.44)

Hence

\[ W = \text{Spec } R[y_{13}, \frac{x_2}{y_{13}}, \frac{x_5}{x_2}, \frac{y_7}{y_5}, \frac{x_8}{x_7}, \frac{x_{11}}{x_8}, \frac{x_{10}}{x_{11}}, \frac{y_6}{x_{10}}, \frac{x_1}{y_6}, \frac{y_4}{x_1}, \frac{y_3}{y_4}, \frac{y_{12}}{y_3}, \frac{x_9}{y_{12}}, \frac{y_9}{y_{12}}] \]

and

\[ \begin{array}{cccccccccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
9 & 12 & 3 & 4 & 1 & 6 & 10 & 11 & 8 & 7 & 5 & 2 & 13
\end{array} \]

so we get

\[ V = \text{Spec } R[y_9, y_{12}, y_3, y_4, x_1, y_6, x_{10}, x_{11}, x_8, y_7, x_5, x_2, y_{13}, y_{12}], \] (3.45)

ie,

\[ W = V. \]

3.5 Over a dvr

Convention 3.14 In this section we will introduce \( X^{12\ldots n} \) and try to resolve its singularities later. \( R \) always denotes a discrete valuation ring with field of fractions \( K \), prime ideal \( (\pi) \) and algebraically closed residue field \( k = \frac{R}{(\pi)} \).
Now consider the arithmetic surface $X^i = \text{Spec } R[x_i, y_i]/(x_i y_i - \pi)$, $1 \leq i \leq n$, over $\text{Spec } R$. Then $X^i$ over $\text{Spec } R$ is smooth everywhere except at $x_i = y_i = 0$ where the special fibre has an ordinary double point, but $X^i$ is a regular scheme. The fibre product of $X^1 = \text{Spec } R[x_1, y_1]/(x_1 y_1 - \pi)$ and $X^2 = \text{Spec } R[x_2, y_2]/(x_2 y_2 - \pi)$ is $X^{12} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi) := \text{Spec } R[x_1, y_1, x_2, y_2]/B'_2$ where $B'_2$ is the ideal $(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi)$ of $A'_2 = R[x_1, y_1, x_2, y_2]$.

Inductively for each $n \in \mathbb{N}$, $n > 2$ we get $X^{12 \ldots n} = \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 - x_2 y_2, \ldots, x_n y_n, x_1 y_1 - \pi) := \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/B'_n$ where $B'_n = (x_1 y_1 - x_2 y_2, \ldots, x_n y_n, x_1 y_1 - \pi)$. In fact in the affine space $A^n_R$, $X^{12 \ldots n}$ is determined by the equations

$$x_1 y_1 = x_2 y_2 = \ldots = x_n y_n = \pi. \quad (3.46)$$

**Remark 3.15** In the rest of this section we will try to find the singular points and also a desingularisation of $X^{12 \ldots n}$ in two different methods. These desingularisations are the results of some successive blowings-up, analogous to those performed in the geometric case.

**Lemma 3.16** The only singular point of the scheme $X^{12}$ is the point $P_0 \in (X^{12})_\pi$ (i.e., $x_1 = y_1 = x_2 = y_2 = 0$).

**Proof:** Recall that $X^{12} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_1 y_1 - \pi)$. Let $A = R[x_1, y_1]/(x_1 y_1 - \pi)$. By using theorem (2.14) (or just by using Moje) we can check that $A$ is a regular ring. Note that $X^{12} = \text{Spec } A[x_2, y_2]/(x_1 y_1 - x_2 y_2). \quad (3.47)$

Let $\phi(x_2, y_2) = x_1 y_1 - x_2 y_2$. Then $\frac{\partial \phi}{\partial x_2} = -y_2$ and $\frac{\partial \phi}{\partial y_2} = -x_2$. The system $\frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial y_2} = 0$ has the solution $x_2 = y_2 = 0$. So $X^{12}$ is smooth over $A$ everywhere except at those points where $x_2 = y_2 = 0$.

Now let $B = \text{Spec } R[x_2, y_2]/(x_2 y_2 - \pi)$. By using theorem (2.14) (or just by using Moje) we can check that $B$ is a regular ring. Note that $X^{12} = \text{Spec } B[x_1, y_1]/(x_1 y_1 - x_2 y_2)$. Let $\psi(x_1, y_1) = x_1 y_1 - x_2 y_2$. Then $\frac{\partial \psi}{\partial x_1} = y_1$ and $\frac{\partial \psi}{\partial y_1} = x_1$. The
system $\frac{\partial \psi}{\partial x_1} = \frac{\partial \psi}{\partial y_1} = 0$ has the solution $x_1 = y_1 = 0$. So $X^{12}$ is smooth over $B$ everywhere except at those points where $x_1 = y_1 = 0$. Considering the above discussion we can conclude that $X^{12}$ is regular everywhere except possibly at $x_1 = y_1 = x_2 = y_2 = 0$. But $\text{Moje}(P_0, f_1) = 0$, so $P_0$ is the only singular point of $X^{12}$, i.e.,

$$(X^{12})^{\text{Sing}} = \{P_0\}. \Box$$

**Lemma 3.17** After one blowing-up of $X^{12}$ at $P_0$ we get the regular scheme $\overline{X}^{12}$.

**Proof:** If we do exactly the same with whatever we did in example (2.19) for the scheme $X_2 = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2) = \text{Spec } A^2$ over $\text{Spec } R[t]$, with the structure morphism induced by the ring homomorphism

$$R[t] \longrightarrow R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2) = A^2 \quad (3.48)$$

$$t \longrightarrow x_1 y_1$$

we get a regular scheme $\overline{X}_2$ which is the gluing of the following open pieces of the covering:

$$\begin{align*}
\mu_1 \neq 0, & \quad W_1^0 = \text{Spec } R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \\
\eta_1 \neq 0, & \quad W_1^1 = \text{Spec } R[y_1, \frac{x_2}{y_1}, \frac{y_2}{y_1}] \\
\mu_2 \neq 0, & \quad W_2^0 = \text{Spec } R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, y_2] \\
\eta_2 \neq 0, & \quad W_2^1 = \text{Spec } R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, y_2]. \quad (3.49)
\end{align*}$$

Now consider the ring homomorphism

$$\phi: R[t] \longrightarrow R$$

$$t \longrightarrow \pi,$$

which induces the structure morphism

$$\phi^* : \text{Spec } R \longrightarrow \text{Spec } R[t].$$

So we have
which induces

$$X_2 = \text{Spec } A^2 \quad \text{Spec } R$$

and we get $X_{12} = X_2 \times_{\text{Spec } R[t], \phi} \text{Spec } R$.

By using the lemma (3.3), we have $\overline{X_{12}} = \overline{X_2} \times_{\text{Spec } R[t], \phi} \text{Spec } R$.

To find the affine open pieces of the covering of $\overline{X_{12}}$ we can do as follows:

$$V_1^0 = W_1^0 \times_{\text{Spec } R[t], \phi} \text{Spec } R = \text{Spec } R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}] \times_{\text{Spec } R[t], \phi} \text{Spec } R =$$

$$\text{Spec } R[x_1, \frac{x_2}{x_1}, \frac{y_2}{x_1}]/(x_2y_2 - \pi) = \text{Spec } R[X_1, X_2, Y_2]/(X_1^2X_2Y_2 - \pi),$$

which is a regular scheme (by corollary 2.13).

Using the same method, gives us

$$V_1^1 = \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^2X_2Y_2 - \pi)$$

$$V_2^0 = \text{Spec } R[X_1, Y_1, X_2]/(X_2^2X_1Y_1 - \pi)$$

$$V_2^1 = \text{Spec } R[X_1, Y_1, Y_2]/(Y_2^2X_1Y_1 - \pi). \quad (3.50)$$

which are all regular schemes and their gluing through their overlap, gives us the desired regular scheme $\overline{X_{12}}$. $\square$
Lemma 3.18 Let \( P = (a_1, b_1, ..., a_n, b_n) \in (X_{12^n})_p \) such that between \( a_1, b_1, ..., a_n, b_n \) at least two pairs \((a_i, b_i)\) are zero. Then \( P \) is a singular point of \( X_{12^n} \).

**Proof:** Without loss of generality let \( a_1 = b_1 = a_j = b_j = 0 \) (for a fixed \( j \), \( 2 \leq j \leq n \)).

Now consider

\[
X_{12^n} = \text{Spec } R[x_1, y_1, ..., x_n, y_n]/(f_1, f_2, ..., f_n),
\]

where

\[
f_i(x_1, y_1, ..., x_n, y_n) = x_iy_1 - x_{i+1}y_{i+1}, \text{ for } i = 1, 2, ..., n - 1
\]

and \( f_n(x_1, y_1, ..., x_n, y_n) = x_1y_1 - \pi \).

Note that for points of the form \( Q = (0, 0, a_2, b_2, ..., a_n, b_n) \in (X_{12^n})_{\text{Sing}} \), \( \text{Moje}(Q, f_i) = -\beta_{i+1}T_{i+1} - \alpha_{i+1}S_{i+1} \) (for \( 1 \leq i \leq n - 1 \)) and \( \text{Moje}(Q, f_n) = -\pi \), i.e.,

\[
D(Q) = \begin{pmatrix}
0 & 0 & 0 & -b_2 & -a_2 & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b_3 & -a_3 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & -b_n & -a_n \\
-1 & 0 & 0 & ... & ... & ... & ... & ... & 0 & 0
\end{pmatrix}
\]  

(3.53)

Now let \( Q = P \), i.e, \( a_j = b_j = 0 \). Then \( \text{Moje}(Q, f_{j-1}) = 0 \). So \( \text{rank } D(P) < n \). Hence \( P \) is a singular point. □

**Theorem 3.19** After \( n - 1 \) blowings-up of \( X_{12^n} \) starting with \( P_0 \) (i.e., \( x_1 = y_1 = ... = x_n = y_n = \pi = 0 \)) we get \( 2^{n-1}(n!) \) regular pieces such that their gluing through the overlap gives us the regular scheme \( \overline{X_{12^n}} \).
Proof: If we do exactly the same with whatever that we did in theorem (3.6) for the scheme

\[ X_n = \text{Spec } R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/ \]
\[ (x_1y_1 - x_2y_2, x_1y_1 - x_3y_3, \ldots, x_1y_1 - x_ny_n) := \text{Spec } A^n, \]  
\hspace{1cm} (3.54)

over \( \text{Spec } R[t] \), with the structure morphism induced by the ring homomorphism

\[ R[t] \longrightarrow A^n \]
\[ t \longmapsto x_1y_1, \]

after \( n - 1 \) successive blowings-up we get \( 2^{n-1}(n!) \) open pieces for the covering of the regular scheme \( \widetilde{X}_n \), say \( W_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}} \) where \( \alpha_i \in \{0,1\} \) and \( m_j \in \{1,2,\ldots,n\} \).

Consider \( \phi^\#: \text{Spec } R \longrightarrow \text{Spec } R[t] \) as it was defined in lemma (3.17). Then we have

\[ \begin{array}{ccc}
R[t] & \xrightarrow{\phi} & A^n \\
\downarrow & & \downarrow \phi^\#
\end{array} \]

\[ R \]

which induces

\[ \begin{array}{ccc}
X_n = \text{Spec } A^n & \xrightarrow{\phi^\#} & \text{Spec } R \\
\downarrow & & \downarrow \phi^\#
\end{array} \]

\[ \text{Spec } R[t] \]

By using lemma (3.3) we have

\[ X_{n-1}^{12,\ldots,n} = \widetilde{X}_n \times_{\text{Spec } R[t],\phi^\#} \text{Spec } R. \]  
\hspace{1cm} (3.55)
The affine pieces for the covering of $X_{n-1}^{12...n}$ are of the form

$$ V_{m_1 m_2 \ldots m_{n-1}}^{n_1 n_2 \ldots n_{n-1}} = W_{m_1 m_2 \ldots m_{n-1}}^{n_1 n_2 \ldots n_{n-1}} \times \text{Spec } R[t] \times \text{Spec } R $$

The gluing of these open pieces together through their overlaps, gives us the regular scheme $X_{n-1}^{12...n}$.

**Example 3.20** As it is shown in example (3.8) for the case $n = 8$ we get

$$ W_{4537218}^{011001} = \text{Spec } R[x_4, y_3, x_7, x_2, x_1, y_8, x_6, y_6]. $$

So

$$ V_{4537218}^{011001} = W_{4537218}^{011001} \times \text{Spec } R[t] \times \text{Spec } R = $$

$$ \text{Spec } R[x_4, y_3, x_7, x_2, x_1, y_8, x_6, y_6]/(x_6 y_6 - \pi) = $$

$$ \text{Spec } R[x_4, y_3, x_7, x_2, x_1, y_8, x_6, y_6]/(x_6 y_6 - \pi) = $$

$$ \text{Spec } R[x_4, y_3, x_7, x_2, x_1, y_8, x_6, y_6]/((x_4 y_5 y_7 y_8 x_2 x_1 x_6 y_8 - \pi), (3.58)$$

which is a regular scheme by corollary (2.13).

**Remark 3.21** Now we will try to blow-up a specific ideal and show that just one blowing-up of this ideal gives us the same result as we got in theorem (3.6). For the case $n = 2$ this method gives us the same result with what we had in example (2.19) (just by using $R$ rather than $k$).

**Remark 3.22** We use $J_n$ to show the ideal of $\Gamma(X_{12\ldots n}^{12\ldots n}, \mathcal{O}_{X_{12\ldots n}}^{12\ldots n})$ generated by the elements of the form $Z_{a(1)}^0, Z_{a(2)}^1, \ldots, Z_{a(n)}^{n-1}$ with the details stated in convention (3.9).

**Theorem 3.23** Let $I_n = (x_1 y_1 - x_2 y_2, x_1 y_1 - x_3 y_3, \ldots, x_1 y_1 - x_n y_n, x_1 y_1 - \pi)$ and $X_{12\ldots n}^{12\ldots n} = \text{Spec } R[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/I_n$. If we blow-up the ideal $J_n$, then we get $2^{n-1}(n!)$ open pieces of regular schemes and their gluing gives us the regular scheme $X_{12\ldots n}^{12\ldots n}$. 
Proof: Let

\[ X_n = \text{Spec } R[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 - x_2 y_2, x_1 y_1 - x_3 y_3, \ldots, x_1 y_1 - x_n y_n) := \text{Spec } A^n. \]  

(3.59)

If we do exactly the same with what we did in theorem (3.10) for the scheme \( X_n = \text{Spec } A^n \) over \( \text{Spec } R[t], \) with the structure morphism induced by the ring homomorphism

\[ R[t] \longrightarrow A^n \]

\[ t \mapsto x_1 y_1 \]

and blow-up the ideal \((J_n + B_n)/B_n\) of \( R[x_1, y_1, \ldots, x_n, y_n]/B_n, \) we get \( 2^{n-1}(n!) \) pieces for the covering of \( X_{12} \) which are all regular schemes, say \( V_{\sigma_1}^{\alpha_1 \alpha_2 \ldots \alpha_{n-1}} \) where \( \alpha_i \in \{0, 1\} \) and \( \sigma_i \in S_n, \) as they were calculated in theorem (3.10). Considering \( \phi \) and \( \phi^* \) as in lemma (3.17), we have \( X_{12} = X_n \times_{\text{Spec } R[t], \phi^*} \text{Spec } R. \) By using lemma (3.3) we have \( X_{12} = X_n \times_{\text{Spec } R[t], \phi^*} \text{Spec } R. \) Note that the open pieces for the covering of \( X_{12} \) (which are \( 2^{n-1}(n!) \) pieces) are of the form \( V_{\sigma_i}^{\alpha_1 \alpha_2 \ldots \alpha_{n-1}} \times_{\text{Spec } R[t], \phi^*} \text{Spec } R \) for \( i, j \in \{0, 1\} \) and \( l \in \{1, 2, \ldots, n!\}, \sigma_l \in S_n \) and each piece is isomorphic with one \( V_{m_1 m_2 \ldots m_{n-1}} \) calculated in theorem (3.19). Recall that \( V_{\sigma_i}^{\alpha_1 \alpha_2 \ldots \alpha_{n-1}} \) ’s are calculated in theorem (3.10). \( \Box \)

Example 3.24 As we have seen in example (3.13) for the given \( \sigma \) in that example we get

\[ V_{110110000100}^{110110000100} = \text{Spec } R[\frac{y_9}{y_{12}}, \frac{y_{12}}{y_3}, \frac{y_3}{y_4}, \frac{y_4}{y_{13}}, \frac{y_{13}}{y_9}, \frac{y_9}{y_{12}}] . \]  

(3.60)

So we have

\[ V' = \text{Spec } R[\frac{y_9}{y_{12}}, \frac{y_{12}}{y_3}, \frac{y_3}{y_4}, \frac{y_4}{y_{13}}, \frac{y_{13}}{y_9}, \frac{y_9}{y_{12}}]/(x_9 y_9 - \pi) = \]

\[ \text{Spec } R[y_{13}, \frac{x_2}{y_{13}}, \frac{x_5}{y_3}, \frac{y_{12}}{y_3}, \frac{x_9}{y_{12}}, \frac{y_9}{y_{12}}]/(y_3 (\frac{x_2}{y_{13}})^2 (\frac{x_5}{y_3})^2 (\frac{y_{12}}{y_3})^2 (\frac{x_9}{y_{12}})^2 (\frac{y_9}{y_{12}})) - \pi) = \]

\[ \text{Spec } R[Y_{13}, X_2, X_5, \ldots, Y_{12}, X_9, Y_9]/(Y_{13} X_2^2 X_5^2 \ldots Y_{12}^2 X_9 Y_9 - \pi), \]  

(3.61)

which is regular by corollary (2.13).
Chapter 4

Desingularisation of a special class of arithmetic three-folds

4.1 Introduction

As before $R$ denotes a discrete valuation ring with field of fractions $K$, maximal ideal $(\pi)$ and algebraically closed residue field $k = \frac{R}{(\pi)}$. By an arithmetic surface over $\text{Spec } R$ we mean a regular scheme $V$ purely of dimension 2, which is flat over $\text{Spec } R$. Our aim in this chapter will be to attempt to settle the resolution of singularities of a certain class of arithmetic three-folds, namely those which are the fibre product of two Arithmetic Surfaces. Let $V_1$ and $V_2$ be arithmetic surfaces over $\text{Spec } R$, where $R$ is a discrete valuation ring and consider the fibre product $X = V_1 \times_{\text{Spec } R} V_2$. This is an arithmetic three-fold. In general it will not be regular. It may have some singularities at points $(y_1, y_2)$ if $f_1(y_1) = f_2(y_2) = (\pi) := \xi$ (where $f_1 : V_1 \rightarrow \text{Spec } R$, $f_2 : V_2 \rightarrow \text{Spec } R$ are the structural morphisms and both $y_1$ and $y_2$ are singular points of the special fibres $(V_1)_y = f_1^{-1}(\xi)$ and $(V_2)_y = f_2^{-1}(\xi)$. In this Chapter we mainly try to resolve the singularities of some three-folds of this sort which are useful in chapter 5. The
arithmetic surfaces which we use in this chapter have one of the following forms:

\[
\begin{align*}
I_1 & \longleftrightarrow \text{Spec } R[x, y]/(xy - \pi) \\
I_2 & \longleftrightarrow \text{Spec } R[x, y]/(y^2 - x^3 - \pi) \\
I_3 & \longleftrightarrow \text{Spec } R[x, y]/(y(y - x^2) - \pi) \\
I_4 & \longleftrightarrow \text{Spec } R[x, y]/(xy(x - y) - \pi) \\
I_5 & \longleftrightarrow \text{Spec } R[x, y]/(x^n y^n - \pi)
\end{align*}
\]

We can summarize our results as follows:

**Theorem 4.1** Let \( R \) be a dvr and \( p = \text{char } k \neq 2, 3 \) and \( V_1 \) and \( V_2 \) be arithmetic surfaces of the forms \( I_i \) and \( I_j \) respectively, where either \( 1 \leq i < j \leq 4 \), \( (i,j) \neq (2,2), (3,4) \) or \( (i,j) = (1,5) \). Then there exists a desingularisation for \( X = V_1 \times_{\text{Spec} R} V_2 \).

**Lemma 4.2** Let \( \alpha, \beta \geq 0 \) and \( g(x_2, y_2) \in R[x_2, y_2] \) such that \( \pi | g \) and
\[
V = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - g(x_2, y_2), x_1 y_1 x_2^\alpha y_2^\beta - \pi).
\]
If \( A = R[x_2, y_2]/(x_2^\alpha y_2^\beta g(x_2, y_2) - \pi) \) is regular, then
\[
V^{\text{Sing}} = \{(0,0,a_2,b_2) \in V_\pi | \frac{\partial \bar{g}}{\partial x_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial y_2}(a_2, b_2) = 0\},
\]
where \( \bar{g} \) denotes the reduction of \( g \) (mod \( \pi \)).

**Proof**: We have
\[
V = \text{Spec } A[x_1, y_1]/(x_1 y_1 - g(x_2, y_2)). \tag{4.2}
\]
Let \( \phi(x_1, y_1) = x_1 y_1 - g(x_2, y_2) \). By using the Jacobian criterion we find out that \( V \) is smooth over \( A \) everywhere except at those points where \( \frac{\partial \phi}{\partial x_1} = \frac{\partial \phi}{\partial y_1} = 0 \), i.e., \( x_1 = y_1 = 0 \). Considering \( \phi(x_1, y_1) = 0 \), we get \( g(x_2, y_2) = 0 \). To have these points on the special fibre, we need \( \bar{g}(x_2, y_2) = 0 \). So
\[
V^{\text{Sing}} \subseteq \{ P = (a_1, b_1, a_2, b_2) \in V_\pi | a_1 = b_1 = \bar{g}(a_2, b_2) = 0 \} = \{ P = (0,0,a_2,b_2) \in V_\pi | \bar{g}(a_2, b_2) = 0 \} := S \subset V_\pi. \tag{4.3}
\]
Let $P = (0, 0, a_2, b_2) \in S$ and $\alpha_2, \beta_2 \in R$ such that $\alpha_2 \equiv a_2 (\text{mod } \pi)$ and $\beta_2 \equiv b_2 (\text{mod } \pi)$. Considering $f_1(x_1, y_1, x_2, y_2) = x_1y_1 - g(x_2, y_2)$ and $f_2(x_1, y_1, x_2, y_2) = x_1y_1x_2^2y_2^3 - \pi$ we get

$$f_1(T_1 + u_1 \pi, S_1 + v_1 \pi, T_2 + \alpha_2, S_2 + \beta_2) = f_1(u_1 \pi, v_1 \pi, \alpha_2, \beta_2) + \frac{\partial f_1}{\partial x_1}(u_1 \pi, v_1 \pi, \alpha_2, \beta_2)T_1 + \frac{\partial f_1}{\partial y_1}(u_1 \pi, v_1 \pi, \alpha_2, \beta_2)S_1 + \frac{\partial f_1}{\partial x_2}(u_1 \pi, v_1 \pi, \alpha_2, \beta_2)T_2$$
$$+ \frac{\partial f_1}{\partial y_2}(u_1 \pi, v_1 \pi, \alpha_2, \beta_2)S_2 + ... = (u_1 \pi)(v_1 \pi) - g(\alpha_2, \beta_2) + v_1 \pi T_1 + u_1 \pi S_1 - \frac{\partial g}{\partial x_2}(\alpha_2, \beta_2)T_2 - \frac{\partial g}{\partial y_2}(\alpha_2, \beta_2)S_2 + ...$$

So

$$\text{Moje}(P, f_1) = \{-g(\alpha_2, \beta_2) - \frac{\partial g}{\partial x_2}(\alpha_2, \beta_2)T_2 - \frac{\partial g}{\partial y_2}(\alpha_2, \beta_2)S_2\} \pmod{(T_2, S_2, \pi)^2}$$

and $\text{Moje}(P, f_2) = -\pi$, i.e.,

$$D(Q) = \begin{pmatrix} -\bar{g}(a_2, b_2) & 0 & 0 & -\frac{\partial \bar{g}}{\partial x_2}(a_2, b_2) & -\frac{\partial \bar{g}}{\partial y_2}(a_2, b_2) \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ (4.6)

So $Q$ is singular if $\frac{\partial \bar{g}}{\partial x_2}(a_2, b_2) = \frac{\partial \bar{g}}{\partial y_2}(a_2, b_2) = 0$ (since $P$ is on the special fibre, $\bar{g}(a_2, b_2) = 0$). $\square$

**Convention 4.3** For the most cases in this chapter the second row of $D(Q)$ corresponding to the point $Q$ is $(-1 \ 0 \ 0 \ 0 \ 0)$. For abbreviation we use $D(Q)_r_1$ for the first row of $D(Q)$. In the cases that the second row is different from $(-1 \ 0 \ 0 \ 0 \ 0)$ we write down $D(Q)$.

### 4.2 One component is $I_1$

In this section our arithmetic three-folds are those which are the fibre products of two arithmetic surfaces such that one of them is of the form $I_1$.

**Lemma 4.4** Let $V_1 = V_2 = \text{Spec } R[x, y]/(xy - \pi)$. Then $X = V_1 \times_{\text{Spec}\, R} V_2$ is singular and just after one blowing-up we can resolve its singularity.
Proof: See lemma (3.17). □

Lemma 4.5 Let $p = \text{char } k \neq 2$ and $V_1 = \text{Spec } R[x,y]/(xy - \pi)$ and $V_2 = \text{Spec } R[x,y]/(y^2 - x^3 - \pi)$. Then $X = V_1 \times_{\text{Spec } R} V_2$ is singular and after one blowing-up we can resolve its singularity.

Proof: In fact $X = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1y_1 - y_2 + x_2y_1 - \pi)$. By using corollary (2.11) we can show that $P_0$ is the only singular point of $X$. Now we blow-up $P_0$. Using the procedure of Mahtab gives us four open pieces for the covering of $X$ as follows:

Chart 1

$$V_1^0 = \text{Spec } R[X_1, X_2, Y_2]/(X_1^2Y_2^2 - X_1^3X_2^3 - \pi). \quad (4.7)$$

By using the Jacobian criterion we can check that

$$\left( V_1^0 \right)_\pi^{\text{Sing}} = \{(a_1, a_2, b_2) \in (V_1^0)_\pi | a_1 = 0 \text{ or } a_2 = b_2 = 0 \}. \quad (4.8)$$

Let $Q \in \left( V_1^0 \right)_\pi^{\text{Sing}}$. Then

$$D(Q) = (-1 \ 0 \ 0 \ 0 \ 0), \quad (4.9)$$

ie, $Q$ is a regular point. So $V_1^0$ is a regular scheme.

Chart 2

$$V_1^1 = \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^2Y_2^2 - Y_1^3X_2^3 - \pi) \cong V_1^0. \quad (4.10)$$

Chart 3

$$V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2^2 + X_2, X_2^2X_1Y_1 - \pi). \quad (4.11)$$

Let $A = R[X_2, Y_2]/(X_2^2(Y_2^2 - X_2) - \pi)$ and $g(X_2, Y_2) = Y_2^2 - X_2$. Then $A$ is a regular ring (by theorem (2.14) and $V_2^0 = \text{Spec } A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2))$. By
using lemma (4.2) we get
\[ (V^0_2)^{\text{Sing}} = \{ P = (0,0,a_2,b_2) \in (V^0_2)_* | \frac{\partial \bar{g}}{\partial X_2} (a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2} (a_2,b_2) = 0 \} = \]
\[ \{(0,0,a_2,b_2) \in (V^0_2)_* | 2b_2 = -1 \} = \{ P = (0,0,a_2,-\frac{1}{2}) | \bar{g}(0,-\frac{1}{2}) = 0 \} = \]
\[ \{ P = (0,0,a_2,-\frac{1}{2}) | (\frac{1}{2})^2 = 0 \} = \phi. \]  
(4.12)
So $V^0_2$ is a regular scheme.

**Chart 4**

\[ V^1_2 = \text{Spec } A[X_1,Y_1,X_2,Y_2]/(X_1Y_1 - 1 + Y_2X_2^3, Y_2^2X_1Y_1 - \pi). \]  
(4.13)

Let $A = R[X_2,Y_2]/(Y_2(1 - Y_2X_2^3) - \pi)$ and $g(X_2,Y_2) = 1 - Y_2X_2^3$. Then $A$ is regular (by theorem (2.14)) and

\[ V^1_2 = \text{Spec } A[X_1,Y_1]/(X_1Y_1 - g(X_2,Y_2)). \]

By using lemma (4.2) we get
\[ (V^1_2)^{\text{Sing}} = \{ P = (0,0,a_2,b_2) \in (V^1_2)_* | \frac{\partial \bar{g}}{\partial X_2} (a_2,b_2) = \frac{\partial \bar{g}}{\partial Y_2} (a_2,b_2) = 0 \} = \]
\[ \{ P = (0,0,a_2,b_2) \in (V^0_1)_* | a_2 = 0 \} = \]
\[ \{ P = (0,0,a_2,b_2) \in (V^0_1)_* | \bar{g}(0,b_2) = 0 \} = \phi. \]  
(4.14)
So $V^1_2$ is a regular scheme. The gluing of $V^0_1,V^1_1,V^0_2$ and $V^1_2$ gives us the regular scheme $\overline{X}$ which is the blowing-up of $X$ at $P_0$. □

**Lemma 4.6** Let $p = \text{char } k \neq 2$ and $V_1 = \text{Spec } R[x,y]/(xy - \pi)$ and $V_2 = \text{Spec } R[x,y]/(y(y - x^2) - \pi)$. Then $X = V_1 \times_{\text{Spec } R} V_2$ is singular and after two blowings-up we can resolve its singularity.

**Proof**: The fibre product of $V_1$ and $V_2$ is
\[ X = \text{Spec } R[x_1,y_1,x_2,y_2]/(x_1y_1 - y_2^2 + y_2x_2^2, x_1y_1 - \pi). \]  
(4.15)
By using corollary (2.11) we can check that $P_0$ is the only singular point of $X$. 
CHAPTER 4. Special class of Arithmetic three-folds

Now we blow-up $P_0$. By using the procedure of Mahtab we get four open pieces for the covering of $\tilde{X}$ as follows:

**Chart 1**

$$V_1^0 = \text{Spec } R[X_1, X_2, Y_2]/(X_1^2 Y_2^2 - X_1^3 X_2^2 Y_2 - \pi). \quad (4.16)$$

By using theorem (2.14) we can check that $V_1^0$ is a regular scheme.

**Chart 2**

$$V_1^1 = \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^2 Y_2^2 - Y_1^3 Y_2 X_2^2 - \pi) \cong V_1^0. \quad (4.17)$$

**Chart 3**

$$V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - Y_2^2 + X_2 Y_2, X_2^2 X_1 Y_1 - \pi). \quad (4.18)$$

Let $A = R[X_2, Y_2]/(X_2^2 Y_2(Y_2 - X_2) - \pi)$ and $g(X_2, Y_2) = Y_2(Y_2 - X_2)$. Then $A$ is regular (by using theorem (2.14)) and

$$V_2^0 = \text{Spec } A[X_1, Y_1]/(X_1 Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$(V_2^0)^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V_2^0), \partial g/\partial X_2 (a_2, b_2) = \partial g/\partial Y_2 (a_2, b_2) = 0 \} = \{ P = (0, 0, a_2, b_2) \in (V_2^0), b_2 = a_2 - 2b_2 = 0 \} = \{ P_0 \}. \quad (4.19)$$

Now we blow-up $P_0$. By using the procedure of Mahtab we get four open pieces for the covering of $\tilde{V}_2^0$ as follows:

**Chart 3.1**

$$V_{21}^{00} = \text{Spec } R[X_1, X_2, Y_2]/(X_1^4 X_2^2 Y_2^2 - X_1^3 X_2^3 Y_2 - \pi). \quad (4.20)$$
By using theorem (2.14) we can check that $V_{21}^{00}$ is a regular scheme.

Chart 3.2

$$V_{21}^{01} = \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^4X_2^4Y_2^2 - Y_1^4X_2^4Y_2 - \pi) \cong V_{21}^{00}. \quad (4.21)$$

Chart 3.3

$$V_{22}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2^2 + Y_2, X_2^4X_1Y_1 - \pi). \quad (4.22)$$

Let $A = R[X_2, Y_2]/(X_2^4Y_2(Y_2 - 1) - \pi)$ and $g(X_2, Y_2) = Y_2(Y_2 - 1)$. Then $A$ is regular (by theorem (2.14)) and

$$V_{22}^{00} = \text{Spec } A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)).$$

By using lemma (4.2) we get

$$(V_{22}^{00})^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{00})_{\pi} \mid \frac{\partial g}{\partial X_2}(a_2, b_2) = \frac{\partial g}{\partial Y_2}(a_2, b_2) = 0 \} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{00})_{\pi} \mid 2b_2 - 1 = 0 \} = \{ P = (0, 0, a_2, \frac{1}{2}) \mid g(a_2, \frac{1}{2}) = 0 \} = \phi. \quad (4.23)$$

So $V_{22}^{00}$ is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \text{Spec } R[X_1, Y_1, Y_2]/(Y_2^4X_1Y_1(1 + X_1^2Y_2^2 - 2X_1Y_1) - \pi). \quad (4.24)$$

By using theorem (2.14) we can check that $V_{22}^{00}$ is a regular scheme. The gluing of $V_{21}^{00}, V_{21}^{01}, V_{22}^{00}$ and $V_{22}^{01}$ gives us the regular scheme $V_2^0$.

Chart 4

$$V_2^1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - 1 + Y_2X_2^2, Y_2^2X_1Y_1 - \pi). \quad (4.25)$$
Let \( A = \mathbb{R}[X_2, Y_2]/(Y_2^2(1 - Y_2X^2_2) - \pi) \) and \( g(X_2, Y_2) = 1 - Y_2X^2_2 \). Then \( A \) is regular (by theorem (2.14)) and
\[
V^1_2 = \text{Spec } A[X_1, Y_1]/(X_1Y_1 - g(X_2Y_2)).
\]

By using lemma (4.2) we get
\[
(V^1_2)^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V^1_2)_x | \frac{\partial g}{\partial X_2}(a_2, b_2) = \frac{\partial g}{\partial Y_2}(a_2, b_2) = 0 \} = \\
\{ P = (0, 0, a_2, b_2) \in (V^1_2)_x | a_2 = 0 \} = \emptyset. \tag{4.26}
\]
So \( V^1_2 \) is a regular scheme. The gluing of \( V^0_1, V^1_1, V^0_2 \) and \( V^1_2 \) gives us the regular scheme \( \overline{X} \).

**Lemma 4.7** Let \( p = \text{char } k \neq 2 \) and \( V_1 = \text{Spec } R[x, y]/(xy - \pi) \) and \( V_2 = \text{Spec } R[x, y]/(xy(x - y) - \pi) \). Then \( X = V_1 \times_{\text{Spec } R} V_2 \) is singular and after three blowings-up we can resolve its singularity.

**Proof**: In fact
\[
X = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^2y_2 + x_2y_2^2, x_1y_1 - \pi). \tag{4.27}
\]

By using corollary (2.11) we can check that \( P_0 = (0, 0, 0, 0) \in X_s \) is the only singular point of \( X \). Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( \overline{X}_1 \) as follows:

**Chart 1**
\[
V^0_1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_1X^2_2Y_2 + X_1X_2Y^2_2, X_1^2Y_1 - \pi) = \\
\text{Spec } R[X_1, X_2, Y_2]/(X_1^3X^2_2Y_2 - X_1^3X_2Y^2_2 - \pi). \tag{4.28}
\]

By using corollary (2.14) we can check that \( V^0_1 \) is a regular scheme.

**Chart 2**
\[
V^1_1 = \text{Spec } R[Y_1, X_2, Y_2]/(Y^3_1X^2_2Y_2 - Y^3_1X_2Y^2_2 - \pi) \cong V^0_1. \tag{4.29}
\]
Chart 3

\[ V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - X_2 Y_2 + X_2 Y_2^2, X_2^2 X_1 Y_1 - \pi). \] (4.30)

Let \( A = R[X_2, Y_2]/(X_2^3 Y_2(1 - Y_2) - \pi) \) and \( g(X_2, Y_2) = X_2 Y_2(1 - Y_2). \) Then \( A \) is regular (by theorem (2.14)) and

\[ V_2^0 = \text{Spec } A[X_1, Y_1]/(X_1 Y_1 - g(X_2, Y_2)). \]

By using lemma (4.2) we get

\[ (V_2^0)^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V_2^0)_{\pi} | \frac{\partial g}{\partial X_2}(a_2, b_2) = \frac{\partial g}{\partial Y_2}(a_2, b_2) = 0 \} = \{ P = (0, 0, a_2, b_2) \in (V_2^0)_{\pi} | b_2(1 - b_2) = a_2(1 - 2b_2) = 0 \} = \{(0, 0, 0, 0), (0, 0, 0, 1)\}. \] (4.31)

We will try to resolve the singularities of \( V_2^0 \) later on.

Chart 4

\[ V_2^1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - Y_2 X_2^2 + X_2 Y_2, Y_2^2 X_1 Y_1 - \pi) \cong V_2^0. \] (4.32)

Recall that

\[ V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - X_2 Y_2 + X_2 Y_2^2, X_2^2 X_1 Y_1 - \pi) \] (4.33)

is singular and \( P_0 = (0, 0, 0, 0) \) is a singular point of \( V_2^0 \). Now we blow-up \( P_0 \).

By using the procedure of Mahtab we get four pieces for the covering of \( V_1^0 \) as follows:

Chart 3.1

\[ V_{21}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_2 Y_2 + X_1 X_2 Y_2^2, X_1^4 X_2^2 Y_1 - \pi) = \text{Spec } R[X_1, X_2, Y_2]/(X_1^4 X_2^3 Y_2 - X_1^5 X_2^3 Y_2^2 - \pi). \] (4.34)

By using theorem (2.14) we can check that \( V_{21}^{00} \) is a regular scheme.
Chart 3.2

\[ V_{21}^{01} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 - X_2Y_2 + Y_1X_2Y_2^2, Y_1^4X_1^2X_1 - \pi) = \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^4X_2^3Y_2 - Y_1^5X_3^4Y_2^2 - \pi) \cong V_{21}^{00}. \] (4.35)

Chart 3.3

\[ V_{22}^{00} = \text{Spec } [X_1, Y_1, X_2, Y_2]/(X_1Y_1 - Y_2 + X_2Y_2^2, X_2^4X_1Y_1 - \pi). \] (4.36)

Let \( A = R[X_2, Y_2]/(X_2^4Y_2(1 - X_2Y_2) - \pi) \) and \( g(X_2, Y_2) = Y_2(1 - X_2Y_2) \). Then \( A \) is regular (by theorem (2.14)) and

\[ V_{22}^{00} = \text{Spec } A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)). \]

By using lemma (4.2) we get

\[ (V_{22}^{00})^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{00}) \mid \frac{\partial g}{\partial X_2}(a_2, b_2) = \frac{\partial g}{\partial Y_2}(a_2, b_2) = 0 \} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{00}) \mid b_2 = 0 \text{ and } 2a_2b_2 = 1 \} = \phi. \] (4.37)

So \( V_{22}^{00} \) is a regular scheme.

Chart 3.4

\[ V_{22}^{01} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2 + X_2Y_2, Y_2^4X_2^3X_1Y_1 - \pi). \] (4.38)

Let \( A = R[X_2, Y_2]/(Y_2^4X_2^3(1 - Y_2) - \pi) \) and \( g(X_2, Y_2) = X_2(1 - Y_2) \). Then \( A \) is regular (by theorem (2.14)) and

\[ V_{22}^{01} = \text{Spec } A[X_1, Y_1]/(X_1Y_1 - g(X_2, Y_2)). \]

By using lemma (4.2) we get

\[ (V_{22}^{01})^{\text{Sing}} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{01}) \mid \frac{\partial g}{\partial X_2}(a_2, b_2) = \frac{\partial g}{\partial Y_2}(a_2, b_2) = 0 \} = \{ P = (0, 0, a_2, b_2) \in (V_{22}^{01}) \mid a_2 = 0 \text{ and } b_2 = 1 \} = \{(0, 0, 0, 1)\}. \] (4.39)
Hence the only singular point of $V^{01}_{22}$ is $Q = (0,0,0,1)$. Now we blow-up $Q \in \left(V^{01}_{22}\right)^{\text{Sing}}$. For convenience first we use the translation $Y_2 = y_2 + 1$ to get the scheme

\[ W^{01}_{22} = \]
\[ \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 + x_2 (y_2 + 1), (y_2 + 1)^4 x_2^2 x_1^2 y_1 - \pi) = \]
\[ \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 + x_2 y_2, (y_2 + 1)^4 x_2^2 x_1^2 y_1 - \pi). \] (4.40)

and surely

\[ (W^{01}_{22})^{\text{Sing}} \supset \{ P_0 \in (W^{01}_{22})_x \}. \] (4.41)

Now we blow-up $P_0$. By using the procedure of Mahtab we get four pieces for the covering of $\overline{W^{01}_{22}}$ as follows:

**Chart 3.4.1**

\[ W^{010}_{221} = \]
\[ \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 + (x_2)(y_2), x_1^4(x_1(y_2/x_1) + 1)^4 (y_1/x_1)(x_2^2 - \pi)) = \]
\[ \text{Spec } R[X_1, X_2, Y_2]/(-X_1^4(X_1 Y_2 + 1)^4 X_2^2 Y_2 - \pi) \] (4.42)

which is a regular scheme (use theorem (2.14)).

**Chart 3.4.2**

\[ W^{011}_{221} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 + (x_2)(y_2), y_1^4(y_1/y_1)(y_2/y_1) + 1)^4 (x_2^2 - \pi) = \]
\[ \text{Spec } R[Y_1, X_2, Y_2]/(-Y_1^4(Y_1 Y_2 + 1)^4 X_2^2 Y_2 - \pi) \cong W^{010}_{221}. \] (4.43)

**Chart 3.4.3**

\[ W^{010}_{222} = \]
\[ \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 + (x_2)(y_2), x_2^4(x_2(y_2/x_2) + 1)^4 (x_1 y_1/x_2 - \pi)) = \]
\[ \text{Spec } R[X_1, Y_1, X_2]/(-X_1 Y_1 X_2 + 1)^4 X_1 Y_1 - \pi) \] (4.44)
which is a regular scheme (use theorem (2.14)).

Chart 3.4.4

\[ W_{222}^{011} = \]  
\[ \text{Spec } R[\frac{x_1}{y_2}, y_1, \frac{x_2}{y_2}, y_2]/(\left(\frac{x_1}{y_2}\right) + \frac{x_2}{y_2}, y_2^4(y_2 + 1)^4\left(\frac{x_2}{y_2}\right)^2\left(\frac{x_1}{y_2}\right) - \pi) \]  
\[ = \text{Spec } R[X_1, Y_1, Y_2]/(Y_2^4(Y_2 + 1)^4X_1^3Y_1^3 - \pi) \]  
(4.45)

which is a regular scheme (use theorem (2.14)). The gluing of \( W_{221}^{010}, W_{221}^{011}, W_{222}^{010} \) and \( W_{222}^{011} \) gives us the the regular scheme \( \widetilde{W}_{22}^{01} \). If we glue all regular schemes which we have had so far, we get the desired regular scheme \( \tilde{X} \). □

4.3 One component is \( I_2 \)

In this section our arithmetic three-folds are the fibre products of two arithmetic surfaces such that one of them is of the form \( I_2 \) and apart from the cases discussed in section two. When \( V_1 \) and \( V_2 \) are both of the form \( I_2 \), the output of the MAPLE in calculations related to the singular points of the special fibre of \( X = V_1 \times_{\text{Spec } R} V_2 \) shows us that after two blowings-up the resolution of singularities of the charts is much more complicated than other cases. It is part (a) of open problem (4.21).

**Lemma 4.8** Let \( p = \text{char } k \neq 2,3 \) and \( V_1 = \text{Spec } R[x, y]/(y^2 - x^3 - \pi) \) and \( V_2 = \text{Spec } R[x, y]/(y^2 - xy^2 - \pi) \). Then \( X = V_1 \times_{\text{Spec } R} V_2 \) is singular and after some blowings-up we can resolve its singularity.

**Proof:** In fact

\[ X = \text{Spec } R[x_1, y_1, x_2, y_2]/(y_1^2 - x_1^3 - y_2^2 + y_2x_2^2, y_1^2 - x_1^3 - \pi). \]  
(4.46)

By using corollary (2.11) we can check that \( P_0 \) is the only singular point of \( X \). Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( \tilde{X} \) as follows:
CHAPTER 4. Special class of Arithmetic three-folds

Chart 1

\[ V_1^0 = \text{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 - Y_2^2 + X_1 Y_2 X_2^2, X_1^2 Y_1^2 - X_1^3 - \pi). \]  
(4.47)

Let \( A = R[X_1, Y_1]/(X_1^2(Y_1^2 - X_1) - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Note that

\[ V_1^0 = \text{Spec} A[X_2, Y_2]/(Y_1^2 - X_1 - Y_2^2 + X_1 Y_2 X_2^2). \]  
(4.48)

Let \( \phi(X_2, Y_2) = Y_1^2 - X_1 - Y_2^2 + X_1 Y_2 X_2^2 \). Then \( \frac{\partial \phi}{\partial X_2} = 2X_1 Y_2 X_2 \) and \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_1 X_2^2 \). Since \( p = \text{char } k \neq 2 \), the system \( \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( 2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0 \). So \( V_1^0 \) is smooth over \( A \) everywhere except at those points where \( 2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0 \). To have the points on the special fibre of \( V_1^0 \), we need \( X_1^2(Y_1^2 - X_1) = 0 \). Considering \( \phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^2(Y_1^2 - X_1) = 0 \) we get \( (V_1^0)^\text{Sing} \subset S \subset (V_1^0)_\pi \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_1^0)_\pi \) satisfying one of the following conditions:

1. \( a_1 = b_1 = b_2 = 0; \)
2. \( a_1 - b_1^2 = a_2 = b_2 = 0. \)

Let \( Q_1 = (0,0,a_2,0) \in S \). Then

\[ \text{Moje}(Q_1, f_1) = -u_1 \pi - T_1 \text{ and } \text{Moje}(Q_1, f_2) = -\pi, \]  
(4.49)

ie,

\[ D(Q_1)_{r_1} = \begin{pmatrix} -u_1 & -1 & 0 & 0 & 0 \end{pmatrix}. \]  
(4.50)

So \( \text{rank } D(Q_1) = 2 \). Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (a_1, b_1, 0, 0) \in S \) such that \( a_1 - b_1^2 = 0. \) Then \( \text{Moje}(Q_2, f_1) = -T_1 + 2\beta_1 S_1 \) and

\[ \text{Moje}(Q_2, f_2) = (2a_1 \beta_1^2 - 3a_1^2)T_1 + 2\beta_1 a_1^2 S_1 - \pi, \]  
(4.51)

ie,

\[ D(Q_2) = \begin{pmatrix} 0 & -1 & 2b_1 & 0 & 0 \\ -1 & 2a_1 b_1^2 - 3a_1^2 & 2b_1 a_1^2 & 0 & 0 \end{pmatrix}. \]  
(4.52)

So \( \text{rank } D(Q_2) = 2 \). Hence \( D(Q_2) \) is a regular point. This means that \( V_1^0 \) is a regular scheme.
CHAPTER 4. Special class of Arithmetic three-folds

Chart 2

\[ V_1^1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(1 - Y_1 X_1^3 - Y_2^2 + Y_1 Y_2 X_2^2, Y_1^2 - Y_1^3 X_1^3 - \pi). \]  
(4.53)

Now let \( A = R[X_1, Y_1]/(Y_1^2 - Y_1^3 X_1^3 - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Note that

\[ V_1^1 = \text{Spec } A[X_2, Y_2]/(1 - Y_1 X_1^3 - Y_2^2 + Y_1 Y_2 X_2^2). \]  
(4.54)

Let \( \phi(X_2, Y_2) = 1 - Y_1 X_1^3 - Y_2^2 + Y_1 Y_2 X_2^2 \). Then \( \frac{\partial \phi}{\partial X_2} = 2Y_1 Y_2 X_2 \) and \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + Y_1 X_2^2 \). Since \( p = \text{char } k \neq 2 \), the system \( \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( 2Y_1 Y_2 X_2 = -2Y_2 + Y_1 X_2^2 = 0 \). So \( V_1^1 \) is smooth over \( A \) everywhere except at those points where \( 2Y_1 Y_2 X_2 = -2Y_2 + Y_1 X_2^2 = 0 \). To have the points on the special fibre, we need \( Y_2^2(1 - Y_1 X_1^3) = 0 \). Considering \( \phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = Y_2^2(1 - Y_1 X_1^3) = 0 \) we get \( (V_1^1)^{\text{sing}} \subset S \subset (V_1^1)_\pi \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_1^1)_\pi \) satisfying the following condition:

\[ a_2 = b_2 = a_1^3 b_1 - 1 = 0. \]

Let \( Q = (a_1, b_1, 0, 0) \in S \) such that \( a_1^3 b_1 - 1 = 0 \). Then \( \text{Moje}(Q, f_1) = -3a_1^2 \beta_1 T_1 - \alpha_1^3 S_1 \) and

\[ \text{Moje}(Q, f_2) = -3a_1^2 \beta_1^2 T_1 + \beta_1(2 - 3a_1^2 \beta_1) S_1 - \pi, \]  
(4.55)

ie,

\[ D(Q) = \begin{pmatrix} 0 & -3a_1^2 b_1 & -a_1^3 & 0 & 0 \\ -1 & -3a_1^2 b_1^3 & b_1(2 - 3a_1^2 b_1) & 0 & 0 \end{pmatrix}. \]  
(4.56)

So \( Q \) is singular if \(-3a_1^2 b_1 = -a_1^3 = a_1^3 b_1 - 1 = 0 \) which is impossible. Hence \( Q \) is a regular point and as a result \( V_1^1 \) is a regular scheme.

Chart 3

\[ V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_2^2 - X_2 X_1^3 - Y_2^2 + X_2 Y_2, X_2^2 Y_2^2 - X_2^3 X_1^3 - \pi). \]  
(4.57)
CHAPTER 4. Special class of Arithmetic three-folds

From the first equation we get \( Y_1^2 - X_2X_1^2 = Y_2^2 - X_2Y_2 \). By using it in the other equation we can write \( V_2^0 \) as follows:

\[
V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2, X_2^2Y_2^2 - X_2Y_2 - \pi).
\]

(4.58)

Now suppose that \( A = R[X_2, Y_2]/(X_2^2Y_2(Y_2 - X_2) - \pi) \). We can check that \( A \) is a regular ring (by using theorem (2.14)). Note that

\[
V_2^0 = \text{Spec } A[X, Y]/(Y_2 - X_2X_1 - X_2Y_2 + X_2Y_2).
\]

(4.59)

Let \( \phi(X_1, Y_1) = Y_1^2 - X_2X_1^3 - Y_2^2 + X_2Y_2 \). Then \( \frac{\partial \phi}{\partial X_1} = -3X_2X_1^2 \) and \( \frac{\partial \phi}{\partial Y_1} = 2Y_1 \). Since \( p = \text{char } k \neq 2, 3 \), the system \( \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \(-3X_2X_1^2 = 2Y_1 = 0\). So \( V_2^0 \) is smooth over \( A \) everywhere except at those points where \(-3X_2X_1^2 = 2Y_1 = 0\). To have the points on the special fibre of \( V_2^0 \), we need \( X_2^3Y_1^2 - X_2^2X_1^3 = 0 \). Considering \( \phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = X_2^3Y_1^2 - X_2^2X_1^3 = 0 \) we get \( (V_2^0)_{\text{Sing}} \subset S \subset (V_2^0)_\pi \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_2^0)_\pi \) satisfying one of the following conditions:

1. \( a_1 = b_1 = b_2 = 0; \)
2. \( a_1 = b_1 = a_2 - b_2 = 0; \)
3. \( b_1 = a_2 = b_2 = 0. \)

Let \( Q_1 = (0, 0, a_2, 0) \in S \). Then \( \text{Moje}(Q_1, f_1) = \alpha_2v_2\pi + \alpha_2S_2 \) and \( \text{Moje}(Q_1, f_2) = -\pi, \) ie,

\[
D(Q_1)_{r_1} = (a_2v_2 \ 0 \ 0 \ 0 \ a_2).
\]

(4.60)

So \( Q_1 \) is singular if \( a_2 = 0, \) ie, \( Q_1 = P_0. \)

Let \( Q_2 = (0, 0, a_2, b_2) \in S \) such that \( a_2 - b_2 = 0. \) Then

\[\begin{align*}
\text{Moje}(Q_2, f_1) &= \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \\
\text{Moje}(Q_2, f_2) &= -\pi, \text{ ie,}
\end{align*}\]

\[
D(Q_2)_{r_1} = (0 \ 0 \ b_2 \ (a_2 - 2b_2)).
\]

(4.62)

So \( Q_2 \) is singular if \( a_2 = b_2 = 0 \) ie \( Q_2 = P_0. \)

Let \( Q_3 = (a_1, 0, 0, 0) \in S. \) Then

\[\begin{align*}
\text{Moje}(Q_3, f_1) &= -\alpha_1^3v_2\pi - \alpha_1^3T_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi,
\end{align*}\]

(4.63)
CHAPTER 4. Special class of Arithmetic three-folds

\[ D(Q_3)_{r_1} = (-a_1^2 \bar{u}_2 \ 0 \ 0 \ -a_1^2 \ 0) . \]  

(4.64)

So \( Q_3 \) is singular if \( a_1 = 0 \), \( i.e., Q_3 = P_0 \). Hence

\[ (V_2^0)^{\text{Sing}} = \{ P_0 \in (V_2^0)_\pi \} . \]

We will try to resolve this singular point later.

**Chart 4**

\[ V_2^1 = \text{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2 X_1^3 - 1 + Y_2 X_2^2, Y_2^2 Y_1^2 - Y_2^3 X_1^3 - \pi) . \]  

(4.65)

From the first equation we get \( Y_1^2 - Y_2 X_1^3 = 1 - Y_2 X_2^2 \). By using it in the other equation we get

\[ V_2^1 = \text{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2 X_1^3 - 1 + Y_2 X_2^2, Y_2^2(1 - Y_2 X_2^2) - \pi) . \]  

(4.66)

Now let \( A = R[X_2, Y_2]/(Y_2^2(1 - Y_2 X_2^2) - \pi) \). Considering theorem (2.14), we can check that \( A \) is a regular ring. Notice that

\[ V_2^1 = \text{Spec} A[X_1, Y_1]/(Y_1^2 - Y_2 X_1^3 - 1 + Y_2 X_2^2) . \]  

(4.67)

Let \( \phi(X_1, Y_1) = Y_1^2 - Y_2 X_1^3 - 1 + Y_2 X_2^2 \). Then \( \frac{\partial \phi}{\partial X_1} = -3 Y_2 X_1^2 \) and \( \frac{\partial \phi}{\partial Y_1} = 2 Y_1 \).

Since \( p = \text{char} k \neq 2, 3 \), the system \( \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \(-3 Y_2 X_1^2 = 2 Y_1 = 0 \). So \( V_2^1 \) is smooth over \( A \) everywhere except at those points where \(-3 Y_2 X_1^2 = 2 Y_1 = 0 \). To have the points on the special fibre of \( V_2^1 \), we need \( Y_2^2 Y_1^2 - Y_2^3 X_1^3 = 0 \). Considering \( \phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = Y_2^2 Y_1^2 - Y_2^3 X_1^3 = 0 \) we get \((V_2^1)^{\text{Sing}} \subset S \subset (V_2^1)_\pi \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_2^1)_\pi \) satisfying the following condition:

\[ a_1 = b_1 = a_2^2 b_2 - 1 = 0 . \]

Let \( Q = (0, 0, a_2, b_2) \in S \) such that \( a_2^2 b_2 - 1 = 0 \). Then Moje\((Q, f_1) = 2 \alpha_2 \beta_2 T_2 + \alpha_2^2 S_2 \) and Moje\((Q, f_2) = -\pi, i.e.,

\[ D(Q)_{r_1} = (0 \ 0 \ 0 \ 2 a_2 b_2 \ a_2^2) . \]  

(4.68)
CHAPTER 4. Special class of Arithmetic three-folds

So \( Q \) is singular if \( a_2 = 2a_2b_2 = a_2^2b_2 - 1 = 0 \) which is impossible. So \( Q \) is a regular point and as a result \( V_2^1 \) is a regular scheme.

Recall that

\[
V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1^2X_1^3 - Y_2^2 + X_2Y_2, X_2^2Y_1^2 - X_2^3X_1^3 - \pi)
\]

(4.69)

is singular and \( P_0 \) is its singular point. Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( V_2^0 \) as follows:

Chart 3.1

\[
V_{21}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1^2X_1^3 - Y_2^2 + X_2Y_2, X_1^4X_2^2Y_1^2 - X_1^6X_2^3 - \pi).
\]

(4.70)

From the first equation we get \( Y_1^2 - X_1^2X_1^3 = Y_2^2 - X_2Y_2 \). By using it in the other equation we can write \( V_{21}^{00} \) as follows:

\[
V_{21}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2, X_1^4X_2^2Y_1^2 - X_1^6X_2^3 - \pi).
\]

(4.71)

Suppose that \( A = R[X_1, X_2, Y_2]/(X_1^4X_2^3Y_2(Y_2 - X_2) - \pi) \). We can use theorem (2.14) to check that \( A \) is a regular ring. Note that

\[
V_{21}^{00} = \text{Spec } A[Y_1]/(Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2).
\]

(4.72)

Let \( \phi(Y_1) = Y_1^2 - X_1^2X_2 - Y_2^2 + X_2Y_2 \). Then \( \frac{\partial \phi}{\partial Y_1} = 2Y_1 \). Since \( p = \text{char } k \neq 2 \), the equation \( \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \( 2Y_1 = 0 \). So \( V_{21}^{00} \) is etale over \( A \) everywhere except at those points where \( Y_1 = 0 \). To have the points on the special fibre of \( V_{21}^{00} \), we need \( X_1^4X_2^3Y_1^2 - X_1^6X_2^3 = 0 \). Considering \( \phi(Y_1) = \frac{\partial \phi}{\partial Y_1} = X_1^4X_2^2Y_1^2 - X_1^6X_2^3 = 0 \) we get \( (V_{21}^{00})^{\text{Sing}} \subset S \subset (V_{21}^{00})_p \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_{21}^{00})_p \) satisfying one of the following conditions:

(1) \( a_1 = b_1 = b_2 = 0; \)

(2) \( a_1 = b_1 = a_2 - b_2 = 0; \)
(3) \( b_1 = a_2 = b_2 = 0 \).

Let \( Q_1 = (0, 0, a_2, 0) \in S \). Then

\[
\text{Moje}(Q_1, f_1) = \alpha_2 \nu_2 \pi + \alpha_2 S_2 \quad \text{and} \quad \text{Moje}(Q_1, f_2) = -\pi,
\]

ie,

\[
D(Q_1)_{r_1} = (a_2 \nu_2, 0, 0, 0, a_2).
\]

(4.73)

So \( Q_1 \) is singular if \( a_2 = 0 \), ie, \( Q_1 = P_0 \).

Let \( Q_2 = (0, 0, a_2, b_2) \in S \) such that \( a_2 - b_2 = 0 \). Then

\[
\text{Moje}(Q_2, f_1) = \beta_2 T_2 + (\alpha_2 - 2\beta_2) S_2 \quad \text{and} \quad \text{Moje}(Q_2, f_2) = -\pi,
\]

ie,

\[
D(Q_2)_{r_1} = (0, 0, 0, b_2, a_2 - 2b_2).
\]

(4.74)

(4.75)

So \( Q_2 \) is singular if \( a_2 = b_2 = 0 \), ie, \( Q_2 = P_0 \).

Let \( Q_3 = (a_1, 0, 0, 0) \in S \). Then

\[
\text{Moje}(Q_3, f_1) = -\alpha_1^2 u_2 \pi - \alpha_1^2 T_2 \quad \text{and} \quad \text{Moje}(Q_3, f_2) = -\pi,
\]

ie,

\[
D(Q_3)_{r_1} = (-a_1^2 \nu_2, 0, 0, -a_1^2, 0).
\]

(4.76)

(4.77)

(4.78)

So \( Q_3 \) is singular if \( a_1 = 0 \), ie, \( Q_3 = P_0 \). This means that

\[
(V_{21}^{00})^{\text{Sing}} = \{P_0\}.
\]

We will try to resolve this singular point later.

\[\text{Chart 3.2}\]

\[
V_{21}^{01} = \text{Spec } R[X_1, Y_1, X_2, Y_2] / (1 - Y_1^2 X_2 X_1^3 - Y_2^2 + X_2 Y_2, Y_1^4 X_2^2 - Y_1^6 X_2^3 X_1^3 - \pi).
\]

(4.79)
CHAPTER 4. Special class of Arithmetic three-folds

Let \( A = R[X, Y, Z]/(Y^4X_2^3(1 - Y_1^2X_2^2X_1^3) - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Note that

\[
V_{21}^{01} = \text{Spec } A[Y_2]/(1 - Y_1^2X_2^2X_1^3 - Y_2^2 + X_2Y_2).
\]

(4.80)

Let \( \phi(Y_2) = 1 - Y_1^2X_2^2X_1^3 - Y_2^2 + X_2Y_2 \). Then \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2 \). Since \( p = \text{char } k \neq 2 \), the equation \( \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \(-2Y_2 + X_2 = 0\). So \( V_{21}^{01} \) is etale over \( A \) everywhere except at those points where \( X_2 - 2Y_2 = 0 \). To have the points on the special fibre of \( V_{21}^{01} \), we need \( Y_1^4X_2^2(1 - Y_1^2X_2^2X_1^3) = 0 \). Considering \( \phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^4X_2^2(1 - Y_1^2X_2^2X_1^3) = 0 \), we get \( (V_{21}^{01})^{\text{Sing}} \subset S \subset (V_{21}^{01})_x \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_{21}^{01})_x \) satisfying the following condition:

\[
b_1 = a_2 - 2b_2 = 4 + a_2^2 = 0.
\]

Let \( Q = (a_1, 0, a_2, b_2) \in S \) such that \( a_2 - 2b_2 = 4 + a_2^2 = 0 \). Then \( \text{Moje}(Q, f_1) = \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \) and \( \text{Moje}(Q, f_2) = -\pi \), i.e.,

\[
D(Q)_{r_1} = (0 \ 0 \ 0 \ b_2 \ (a_2 - 2b_2)).
\]

(4.81)

So \( Q \) is singular if \( b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0 \) which is impossible. Hence \( Q \) is a regular point and as a result \( V_{21}^{01} \) is a regular scheme.

**Chart 3.3**

\[
V_{22}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2^2X_1^3 - Y_2^2 + Y_2, X_2^4Y_1^2 - X_2^6X_1^3 - \pi).
\]

(4.82)

From the first equation we get \( Y_1^2 - X_2^2X_1^3 = Y_2^2 - Y_2 \). By using it in the other equation we get

\[
V_{22}^{00} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2^2X_1^3 - Y_2^2 + Y_2, X_2^4(Y_2^2 - Y_2) - \pi).
\]

(4.83)

Now let \( A = R[X_2, Y_2]/(X_2^4(Y_2^2 - Y_2) - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Notice that

\[
V_{22}^{00} = \text{Spec } A[X_1, Y_1]/(Y_1^2 - X_2^2X_1^3 - Y_2^2 + Y_2).
\]

(4.84)
Let $\phi(X_1, Y_1) = Y_1^2 - X_2^2X_1^3 - Y_2^2 + Y_2$. Then $\frac{\partial \phi}{\partial X_1} = -3X_2^2X_1^2$ and $\frac{\partial \phi}{\partial Y_1} = 2Y_1$.

Since $p = \text{char } k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0$ can be written as $-3X_2^2X_1^2 = 2Y_1 = 0$. So $V^\infty_{22}$ is smooth over $A$ everywhere except at those points where $-3X_2^2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of $V^\infty_{22}$, we need $X_2^4Y_1^2 - X_2^6X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = X_2^4Y_1^2 - X_2^6X_1^3 = 0$, we get $(V^\infty_{22})^{\text{Sing}} \subset S \subset (V^\infty_{22})_\pi$ where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V^\infty_{22})_\pi$ satisfying one of the the following conditions:

1. $b_1 = a_2 = b_2 = 0$;
2. $b_1 = a_2 = b_2 - 1 = 0$;
3. $a_1 = b_1 = b_2 = 0$;
4. $a_1 = b_1 = b_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = S_2 + v_2 \pi \text{ and } \text{Moje}(Q_1, f_2) = -\pi,$$

ie,

$$D(Q_1)_r_1 = (v_2 \ 0 \ 0 \ 0 \ 1).$$

So $D(Q_1)$ has rank two. Hence $Q_1$ is a regular point.

Let $Q_2 = (a_1, 0, 0, 1) \in S$. Then $\text{Moje}(Q_2, f_1) = (1 - 2\beta_2)S_2$ where $\beta_2 \equiv 1(\text{mod } \pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, ie,

$$D(Q_2)_r_1 = (0 \ 0 \ 0 \ 0 \ -1).$$

So $D(Q_2)$ has rank two. Hence $Q_2$ is a regular point.

Let $Q_3 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = v_2 \pi + S_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi,$$

ie,

$$D(Q_3)_r_1 = (v_2 \ 0 \ 0 \ 0 \ 1).$$

So $D(Q_3)$ has rank two. Hence $Q_3$ is a regular point.

Let $Q_4 = (0, 0, a_2, 1) \in S$. Then $\text{Moje}(Q_4, f_1) = (1 - 2\beta_2)S_2$ where $\beta_2 \equiv 1(\text{mod } \pi)$ and $\text{Moje}(Q_4, f_2) = -\pi$, ie,

$$D(Q_4)_r_1 = (0 \ 0 \ 0 \ 0 \ -1).$$
So $D(Q_4)$ has rank two. Hence $Q_4$ is a regular point and as a result $V_{22}^{00}$ is a regular scheme.

Chart 3.4

$$V_{22}^{01} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 X_2 X_1^3 - 1 + X_2, Y_2^4 X_2^2 Y_1^2 - Y_2^6 X_2^3 X_1^3 - \pi).$$

(4.91)

From the first equation we get $Y_1^2 - Y_2^2 X_2 X_1^3 = 1 - X_2$. By using it in the other equation we get

$$V_{22}^{01} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - Y_2^2 X_2 X_1^3 - 1 + X_2, Y_2^4 X_2^2 (1 - X_2) - \pi).$$

(4.92)

Now suppose that $A = R[X_2, Y_2]/(Y_2^4 X_2^2 (1 - X_2) - \pi)$. We can check that $A$ is a regular ring (by using theorem (2.14)). Note that

$$V_{22}^{01} = \text{Spec } A[X_1, Y_1]/(Y_1^2 - Y_2^2 X_2 X_1^3 - 1 + X_2).$$

(4.93)

Let $\phi(X_1, Y_1) = Y_1^2 - Y_2^2 X_2 X_1^3 - 1 + X_2$. Then $\frac{\partial \phi}{\partial X_1} = -3Y_2^2 X_2 X_1^2$ and $\frac{\partial \phi}{\partial Y_1} = 2Y_1$. Since $p = char k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = 0$ can be written as $-3Y_2^2 X_2 X_1^2 = 2Y_1 = 0$. So $V_{22}^{01}$ is smooth over $A$ everywhere except at those points where $-3Y_2^2 X_2 X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of $V_{22}^{01}$, we need $Y_2^4 X_2^2 Y_2^2 - Y_2^6 X_2^3 X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = Y_2^4 X_2^2 Y_1^2 - Y_2^6 X_2^3 X_1^3 = 0$, we get $(V_{22}^{01})^{\text{sing}} \subset S \subset (V_{22}^{01})_p$ where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{22}^{01})_p$ satisfying one of the following conditions:

1. $b_1 = a_2 - 1 = b_2 = 0$;
2. $a_1 = b_1 = a_2 - 1 = 0$.

Let $Q_1 = (a_1, 0, 1, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = T_2 \text{ and } \text{Moje}(Q_1, f_2) = -\pi,$$

(4.94)

ie,

$$D(Q_1)_{r_1} = (0 \ 0 \ 0 \ 1 \ 0).$$

(4.95)
So $Q_1$ is a regular point.

Let $Q_2 = (0, 0, 1, b_2) \in S$. Then $\text{Moje}(Q_2, f_1) = T_2$ and $\text{Moje}(Q_2, f_2) = -\pi$ where $\alpha_2 \equiv 1 \pmod{\pi}$, i.e.,

$$D(Q_2)_{r_1} = (0 \ 0 \ 0 \ 1 \ 0). \quad (4.96)$$

So $\text{rank} D(Q_2) = 2$. Hence $Q_2$ is a regular point and as result $V_{22}^{01}$ is a regular scheme.

Recall that

$$V_{21}^{00} =$$

$$\text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - x_1^2 X_2 - Y_2^2 + X_2 Y_2, X_1^4 X_2 Y_1^2 - X_1^6 X_2^3 - \pi)$$

is singular and $P_0$ is its singular point. Now we blow-up $P_0$. By using the procedure of Mahtab we get four pieces for the covering of $V_{21}^{00}$ as follows:

**Chart 3.1.1**

$$V_{211}^{000} =$$

$$\text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2, X_1^4 X_2 Y_1^2 - X_1^6 X_2^3 - \pi).$$

(4.98)

From the first equation we get $Y_1^2 - X_1 X_2 = Y_2^2 - X_2 Y_2$. By using it in the equation we get

$$V_{211}^{000} =$$

$$\text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2, X_1^4 X_2 Y_2(Y_2 - X_2) - \pi).$$

(4.99)

Assume that $A = R[X_1, X_2, Y_2]/(X_1^8 X_2^2 Y_2(Y_2 - X_2) - \pi)$. We can use theorem (2.14) to check that $A$ is a regular ring. Note that

$$V_{211}^{000} = \text{Spec } A[Y_1]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2).$$

(4.100)
Let $\phi(Y_1) = Y_1^2 - X_1X_2 - Y_2^2 + X_2Y_2$. Then $\frac{\partial \phi}{\partial Y_1} = 2Y_1$. Since $p = \text{char } k \neq 2$, the equation $\frac{\partial \phi}{\partial Y_1} = 0$ can be written as $2Y_1 = 0$. So $V_{211}^{000}$ is etale over $A$ everywhere except at those points where $Y_1 = 0$. To have the points on the special fibre of $V_{211}^{000}$, we need $X_1^8X_2^2Y_1^2 - X_1^9X_2^3 = 0$. Considering $\phi(Y_1) = \frac{\partial \phi}{\partial Y_1} = X_1^8X_2^2Y_1 - X_1^9X_2^3 = 0$ we get $(V_{211}^{000})_{\text{Sing}} \subseteq S \subseteq (V_{211}^{000})_{\pi}$ where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{211}^{000})_{\pi}$ satisfying one of the following conditions:

1. $a_1 = b_1 = b_2 = 0$;
2. $a_1 = b_1 = a_2 - b_2 = 0$;
3. $b_1 = a_2 = b_2 = 0$.

Let $Q_1 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_1, f_1) = \alpha_2(v_2 - u_1)\pi - \alpha_2T_1 + \alpha_2S_2 \tag{4.101}$$

and $\text{Moje}(Q_1, f_2) = -\pi$, ie,

$$D(Q_1)_{r_1} = (a_2(\bar{v}_2 - \bar{u}_1) - a_2 \ 0 \ 0 \ a_2). \tag{4.102}$$

So $Q_1$ is singular if $a_2 = 0$, ie, $Q_1 = P_0$.

Let $Q_2 = (0, 0, a_2, b_2) \in S$ such that $a_2 - b_2 = 0$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_2u_1\pi - \alpha_2T_1 + \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \tag{4.103}$$

and $\text{Moje}(Q_2, f_2) = -\pi$, ie,

$$D(Q_2)_{r_1} = (-a_2\bar{u}_1 - a_2 \ 0 \ b_2 \ a_2 - 2b_2). \tag{4.104}$$

So $Q_2$ is singular if $a_2 - 2b_2 = b_2 = 0$, ie, $Q_2 = P_0$.

Finally let $Q_3 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_3, f_1) = -\alpha_1u_2\pi - \alpha_1T_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \tag{4.105}$$

ie,

$$D(Q_3)_{r_1} = (-a_1\bar{u}_2 \ 0 \ 0 \ -a_1 \ 0). \tag{4.106}$$

So $Q_3$ is singular if $a_1 = 0$, ie, $Q_3 = P_0$. Hence

$$(V_{211}^{000})_{\pi}^{\text{Sing}} = \{P_0\}.$$
CHAPTER 4. Special class of Arithmetic three-folds

We will try to resolve this singularity later.

Chart 3.1.2

\[ V_{211}^{001} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2, Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 - \pi). \] (4.107)

Now assume that \( A = R[X_1, Y_1, X_2]/(Y_1^8X_1^4X_2^2(1 - Y_1X_1^2X_2) - \pi). \) Considering theorem (2.14) we can check that \( A \) is a regular ring. Notice that

\[ V_{211}^{001} = \text{Spec } A[Y_2]/(1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2). \] (4.108)

Let \( \phi(Y_2) = 1 - Y_1X_1^2X_2 - Y_2^2 + X_2Y_2. \) Then \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2. \) Since \( p = \text{char } k \neq 2, \) the equation \( \frac{\partial \phi}{\partial Y_2} \) can be written as \( X_2 - 2Y_2 = 0. \) So \( V_{211}^{001} \) is etale over \( A \) everywhere except at those points where \( X_2 - 2Y_2 = 0. \) To have the points on the special fibre of \( V_{211}^{001}, \) we need \( Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 = 0. \) Considering the system \( \phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^8X_1^4X_2^2 - Y_1^9X_1^6X_2^3 = 0, \) we can check that \( (V_{211}^{001})^{\text{Sing}} \subset S \subset (V_{211}^{001})_\pi \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_{211}^{001})_\pi \) satisfying one of the following conditions:

1. \( a_1 = a_2 - 2b_2 = 4 + a_2^2 = 0; \)
2. \( b_1 = a_2 - 2b_2 = 4 + a_2^2 = 0. \)

Let \( Q_1 = (0, b_1, a_2, b_2) \in S \) such that \( a_2 = 2b_2 \) and \( 4 + a_2^2 = 0. \) Then

\[ \text{Moje}(Q_1, f_1) = \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \] (4.109)

and \( \text{Moje}(Q_1, f_2) = -\pi, \) ie,

\[ D(Q)_{r_1} = (0 \quad 0 \quad b_2 \quad a_2 - 2b_2). \] (4.110)

So \( Q_1 \) is singular if \( b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0 \) which is impossible. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (a_1, 0, a_2, b_2) \in S \) such that \( a_2 - 2b_2 = 4 + a_2^2 = 0. \) Then

\[ \text{Moje}(Q_2, f_1) = -\alpha_1^2\alpha_2v_1\pi - \alpha_1^2\alpha_2S_1 + \beta_2T_2 + (\alpha_2 - 2\beta_2)S_2 \] (4.111)
and \( \text{Moje}(Q_2, f_2) = -\pi \), i.e.,

\[
D(Q_2)_{\mathfrak{m}} = (-a_1^3 a_2 \alpha_2 0 - a_1^2 a_2 b_2 a_2 - 2b_2).
\] (4.112)

So \( Q_2 \) is singular if \( a_1^2 a_2 = b_2 = a_2 - 2b_2 = 4 + a_2^2 = 0 \) which is impossible. Hence \( Q_2 \) is a regular point and as a result \( V_{211}^{001} \) is a regular scheme.

**Chart 3.1.3**

\[
V_{212}^{000} = \text{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 X_1^2 - Y_2^2 + Y_2, X_2^8 X_1^4 Y_2^2 - X_2^9 X_1^8 - \pi).
\] (4.113)

From the first equation we get \( Y_1^2 - X_2 X_1^2 = Y_2^2 - Y_2 \). By using it in the other equation we can write

\[
V_{212}^{000} = \text{Spec} R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 X_1^2 - Y_2^2 + Y_2, X_2^8 X_1^4 (Y_2^2 - Y_2) - \pi).
\] (4.114)

Now suppose that \( A = R[X_1, X_2, Y_2]/(X_2^8 X_1^4 (Y_2^2 - Y_2) - \pi) \). We can check that \( A \) is a regular ring (by using theorem (2.14)). We can write

\[
V_{212}^{000} = \text{Spec} A[Y_1]/(Y_1^2 - X_2 X_1^2 - Y_2^2 + Y_2).
\] (4.115)

Let \( \phi(Y_1) = Y_1^2 - X_2 X_1^2 - Y_2^2 + Y_2 \). Then \( \frac{\partial \phi}{\partial Y_1} = 2Y_1 \). Since \( p = \text{char } k \neq 2 \), the equation \( \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \( 2Y_1 = 0 \). So \( V_{212}^{000} \) is etale over \( A \) everywhere except at those points where \( Y_1 = 0 \). To have the points on the special fibre of \( V_{212}^{000} \), we need \( X_2^8 X_1^4 Y_2^2 - X_2^9 X_1^8 = 0 \). Considering \( \phi(Y_1) = \frac{\partial \phi}{\partial Y_1} = X_2^8 X_1^4 Y_2^2 - X_2^9 X_1^8 = 0 \), we get \( (V_{212}^{000})^{\text{Sing}} \subset S \subset (V_{212}^{000})_{\pi} \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_{212}^{000})_{\pi} \) satisfying one of the following conditions:

1. \( b_1 = a_2 = b_2 = 0; \)
2. \( b_1 = a_2 = b_2 - 1 = 0; \)
3. \( b_1 = a_1 = b_2 = 0; \)
4. \( b_1 = a_1 = b_2 - 1 = 0. \)

Let \( Q_1 = (a_1, 0, 0, 0) \in S \). Then

\[
\text{Moje}(Q_1, f_1) = (v_2 - a_1^2 u_2) - a_1^2 T_2 + S_2 \text{ and Moje}(Q_1, f_2) = -\pi,
\] (4.116)
i.e.,

\[ D(Q_1)_{r_1} = (\bar{v}_2 - a_1^2 \bar{u}_2 \quad 0 \quad 0 \quad -a_1^2 \quad 1). \]  
(4.117)

So \( D(Q_1) \) has rank two. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (a_1, 0, 0, 1) \in S \). Then

\[ \text{Moje}(Q_2, f_1) = -\alpha_1^2 u_2 \pi - \alpha_1^2 T_2 + (1 - 2\beta_2) S_2 \]  
(4.118)

where \( \beta_2 \equiv 1 \mod \pi \) and \( \text{Moje}(Q_2, f_2) = -\pi \), i.e.,

\[ D(Q_2)_{r_1} = (-a_1^2 \bar{u}_2 \quad 0 \quad 0 \quad -a_1^2 \quad -1). \]  
(4.119)

So \( D(Q_2) \) has rank two. Hence \( Q_2 \) is a regular point.

Let \( Q_3 = (0, 0, a_2, 0) \in S \). Then

\[ \text{Moje}(Q_3, f_1) = v_2 \pi + S_2 \text{ and } \text{Moje}(Q_3, f_2) = -\pi, \]  
(4.120)

i.e.,

\[ D(Q_3)_{r_1} = (\bar{v}_2 \quad 0 \quad 0 \quad 0 \quad 1). \]  
(4.121)

So \( D(Q_3) \) has rank two. Hence \( Q_3 \) is a regular point.

Let \( Q_4 = (0, 0, a_2, 1) \in S \). Then

\[ \text{Moje}(Q_4, f_1) = (1 - 2\beta_2) S_2 \text{ and } \text{Moje}(Q_4, f_2) = -\pi, \]  
(4.122)

i.e.,

\[ D(Q_4)_{r_1} = (0 \quad 0 \quad 0 \quad 0 \quad -1). \]  
(4.123)

So \( D(Q_2) \) has rank two. Hence \( Q_2 \) is a regular point and as a result \( V_{212}^{000} \) is a regular scheme.

**Chart 3.1.4**

\[ V_{212}^{001} = \text{Spec } R[ X_1, Y_1, X_2, Y_2]/ \]

\[ (Y_1^2 - Y_2 X_1^2 X_2 - 1 + X_2, Y_2^8 X_1^4 X_2^2 Y_1^2 - Y_2^9 X_1^6 X_2^3 - \pi). \]  
(4.124)
From the first equation we get \( Y^2 - Y_2 X^2 X_2 = 1 - X_2 \). By using it in the other equation we get

\[
V_{212}^{001} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y^2 - Y_2 X^2 X_2 - 1 + X_2, Y^8 X^4 X_2^2 (1 - X_2) - \pi).
\]

(4.125)

Let \( A = R[X_1, X_2, Y_2]/(Y_2^8 X^4 X_2^2 (1 - X_2) - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Note that

\[
V_{212}^{001} = \text{Spec } A[Y_1]/(Y_1^2 - Y_2 X^2 X_2 - 1 + X_2).
\]

(4.126)

Let \( \phi(Y_1) = Y^2 - Y_2 X^2 X_2 - 1 + X_2 \). Then \( \frac{\partial \phi}{\partial Y_1} = 2Y_1 \). Since \( p = \text{char } k \neq 2 \), the system \( \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \( 2Y_1 = 0 \). So \( V_{212}^{001} \) is etale over \( A \) everywhere except at those points where \( Y_1 = 0 \). To have the points on the special fibre of \( V_{212}^{001} \), we need \( Y_2^8 X^4 X_2^2 Y_1^2 - Y_2^9 X^6 X_2^3 = 0 \). Considering \( \phi(Y_1) = \frac{\partial \phi}{\partial Y_1} = Y_2^8 X_1^4 X_2^2 Y_2^2 - Y_2^9 X_1^6 X_2^3 = 0 \), we can check that \( (V_{212}^{001})^{\text{Sing}} \subset S \subset (V_{212}^{001})_x \), where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_{212}^{001})_x \) satisfying one of the following conditions:

(1) \( b_1 = a_2 - 1 = b_2 = 0 \);
(2) \( a_1 = b_1 = a_2 - 1 = 0 \).

Let \( Q_1 = (a_1, 0, 1, 0) \in S \). Then

\[
\text{Moje}(Q_1, f_1) = -a_1^2 \alpha_2 v_2 \pi + T_2 - \alpha_1^2 \alpha_2 S_2
\]

(4.127)

where \( \alpha_2 \equiv 1(\text{mod } \pi) \) and \( \text{Moje}(Q_1, f_2) = -\pi \), \( i.e., \)

\[
D(Q_1)_{r_1} = (-a_1^2 v_2 \quad 0 \quad 0 \quad 1 \quad -a_1^2).
\]

(4.128)

SO \( D(Q_1) \) has rank two. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (0, 0, 1, b_2) \in S \). Then

\[
\text{Moje}(Q_2, f_1) = T_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi,
\]

(4.129)

\( i.e., \)

\[
D(Q_2)_{r_1} = (0 \quad 0 \quad 0 \quad 1 \quad 0).
\]

(4.130)

So \( D(Q_2) \) has rank two. Hence \( Q_2 \) is a regular point and as a result \( V_{212}^{001} \) is a regular scheme.
Recall that
\[ V_{2111}^{000} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - Y_2^2 + X_2 Y_2, X_1^8 X_2^2 Y_2^2 - X_1^9 X_2^3 - \pi) \] (4.131)
is singular and \( P_0 \) is its singular point. Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( V_{2111}^{000} \) as follows:

Chart 3.1.1.1

\[ V_{2111}^{000} = \text{Spec } R[Y_1, X_1, X_2, Y_2]/(Y_1^2 - X_2 - Y_2^2 + X_2 Y_2, X_1^{12} X_2^2 Y_1^2 - X_1^{12} X_2^3 - \pi). \] (4.132)

Let \( A = R[X_1, Y_1, X_2]/(X_1^{12} X_2^2 Y_1^2 - X_1^{12} X_2^3 - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Note that
\[ C_{\mathbb{R}} = \text{Spec } A[Y_2]/(Y_1^2 - X_2 - Y_2^2 + X_2 Y_2). \] (4.133)

Let \( \phi(Y_2) = Y_1^2 - X_2 - Y_2^2 + X_2 Y_2 \). Then \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2 \). Since \( p = \text{char } k \neq 2 \), the equation \( \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( X_2 - 2Y_2 = 0 \). So \( V_{2111}^{000} \) is etale over \( A \) everywhere except at those points where \( X_2 - 2Y_2 = 0 \). To have the points on the special fibre of \( V_{2111}^{000} \), we need \( X_1^{12} X_2^2 (Y_1^2 - X_2) = 0 \). Considering \( \phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = X_1^{12} X_2^2 (Y_1^2 - X_2) = 0 \), we can check that \((V_{2111}^{000})^{\text{Sing}} \subset S \subset (V_{2111}^{000})_\pi \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_{2111}^{000})_\pi \) satisfying one of the following conditions:

1. \( a_1 = a_2 - 2b_2 = 4b_1^2 - 4a_2 + a_2^2 = 0 \);
2. \( b_1 = a_2 = b_2 = 0 \).

Let \( Q_1 = (0, b_1, a_2, b_2) \in S \) such that \( a_2 - 2b_2 = 4b_1^2 - 4a_2 + a_2^2 = 0 \). Then
\[ \text{Moje}(Q_1, f_1) = 2\beta_1 S_1 + (\beta_2 - 1)T_2 + (\alpha_2 - 2\beta_2)S_2 \] (4.134)
and \( \text{Moje}(Q_1, f_2) = -\pi, \) ie,
\[ D(Q_1)_{\pi} = (0 \ 0 \ 2b_1 \ b_2 - 1 \ a_2 - 2b_2). \] (4.135)
So $Q_1$ is singular if $2b_1 = b_2 - 1 = 4b_1^2 - 4a_2 + a_2^2 = 0$ which is impossible. Hence $Q_1$ is a regular point.

Let $Q_2 = (a_1, 0, 0, 0) \in S$. Then

$$\text{Moje}(Q_2, f_1) = -u_2 \pi - T_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi,$$

(4.136)

ie,

$$D(Q_2)_{r_1} = \begin{pmatrix} -u_2 & 0 & 0 & -1 & 0 \end{pmatrix}.$$  

(4.137)

So $D(Q_2)$ has rank two. Hence $Q_2$ is a regular point and as a result $V_{2111}^{0000}$ is a regular scheme.

**Chart 3.1.1.2**

$$V_{2111}^{0001} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/$$

$$(1 - X_1 X_2 - Y_2^2 + X_2 Y_2, Y_1^{12} X_1^8 X_2^2 - Y_1^{12} X_1^8 X_2^3 - \pi).$$

(4.138)

Suppose that $A = R[X_1, Y_1, X_2]/(Y_1^{12} X_1^8 X_2^2(1 - X_1 X_2) - \pi)$. Considering theorem (2.14) we can check that $A$ is a regular ring. Notice that

$$V_{2111}^{0001} = \text{Spec } A[Y_2]/(1 - X_1 X_2 - Y_2^2 + X_2 Y_2).$$

(4.139)

Let $\phi(Y_2) = 1 - X_1 X_2 - Y_2^2 + X_2 Y_2$. Then $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2$. Since $p = \text{char } k \neq 2$, the equation $\frac{\partial \phi}{\partial Y_2} = 0$ can be written as $X_2 - 2Y_2 = 0$. So $V_{2111}^{0001}$ is etale over $A$ everywhere except at those points where $X_2 - 2Y_2 = 0$. To have the points on the special fibre of $V_{2111}^{0001}$, we need $Y_1^{12} X_1^8 X_2^2(1 - X_1 X_2) = 0$. Considering $\phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^{12} X_1^8 X_2^2(1 - X_1 X_2) = 0$, we can check that $(V_{2111}^{0001})_{\text{Sing}} \subset S \subset (V_{2111}^{0001})_{r}$

where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{2111}^{0001})_{r}$ satisfying one of the following conditions:

1. $b_1 = a_2 - 2b_2 - a_2^2 = 0$;
2. $a_1 = a_2 - 2b_2 = a_2^2 + 4 = 0$.

Let $Q_1 = (a_1, 0, a_2, b_2) \in S$ such that $a_2 - 2b_2 = 4a_1a_2 - 4 - a_2^2 = 0$. Then

$$\text{Moje}(Q_1, f_1) = -\alpha_2 T_1 + (\beta_2 - \alpha_1) T_2 + (\alpha_2 - 2\beta_2) S_2,$$

(4.140)
and \( \text{Moje}(Q_1, f_1) = -\pi, \) ie,

\[
D(Q_1)_{r_1} = (0 \quad -a_2 \quad 0 \quad b_2 - a_1 \quad 0).
\]

(4.141)

So \( Q_1 \) is singular if \( -a_2 = b_2 - a_1 = a_2 - 2b_2 = 4a_1a_2 - 4 - a_2^2 = 0 \) which is impossible. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (0, b_1, a_2, b_2) \in S \) such that \( a_2 - 2b_2 = a_2^2 + 4 = 0. \) Then

\[
\text{Moje}(Q_2, f_1) = -\alpha_2 u_1 \pi - \alpha_2 T_1 + \beta_2 T_2 + (\alpha_2 - 2\beta_2)S_2
\]

(4.142)

and \( \text{Moje}(Q_2, f_2) = -\pi, \) ie,

\[
D(Q_2)_{r_1} = (-a_2 \bar{u}_1 \quad -a_2 \quad 0 \quad b_2 \quad 0).
\]

(4.143)

So \( Q_2 \) is singular if \( -a_2 = b_2 = a_2 - 2b_2 = a_2^2 + 4 = 0 \) which is impossible. Hence \( Q_2 \) is a regular point and as a result \( V^{0000}_{2111} \) is a regular scheme.

\textbf{Chart 3.1.1.3}

\[
V^{0000}_{2112} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/\left(Y_1^2 - X_1 - Y_2^2 + Y_2, X_2^{12}X_1^8Y_1^2 - X_2^{12}X_1^9 - \pi \right).
\]

(4.144)

Suppose that \( A = R[X_1, Y_1, X_2]/(X_2^{12}X_1^8(Y_1^2 - X_1) - \pi) \). We can use theorem (2.14) to prove that \( A \) is a regular ring. Notice that

\[
V^{0000}_{2112} = \text{Spec } A[Y_2]/(Y_1^2 - X_1 - Y_2^2 + Y_2).
\]

(4.145)

Let \( \phi(Y_2) = Y_1^2 - X_1 - Y_2^2 + Y_2. \) Then \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + 1. \) Since \( p = \text{char } k \neq 2, \) the equation \( \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( 2Y_2 - 1 = 0. \) So \( V^{0000}_{2112} \) is etale over \( A \) everywhere except at those points where \( 2Y_2 - 1 = 0. \) To have the points on the special fibre of \( V^{0000}_{2112}, \) we need \( X_2^{12}X_1^8(Y_1^2 - X_1) = 0. \) Considering \( \phi(Y_2) = X_2^{12}X_1^8(Y_1^2 - X_1) = 0, \) we can check that \( (V^{0000}_{2112})_{\text{Sing}} \subset S \subset (V^{0000}_{2112})_\pi \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V^{0000}_{2112})_\pi \) satisfying one of the following conditions:

(1) \( a_2 = b_2 - \frac{1}{2} = a_1 - b_1^2 - \frac{1}{4} = 0; \)
CHAPTER 4. Special class of Arithmetic three-folds

(2) \[ a_1 = b_2 - \frac{1}{2} = 4b_1^2 + 1 = 0. \]

Let \( Q_1 = (a_1, b_1, 0, b_2) \in S \) such that \( b_2 - \frac{1}{2} = a_1 - b_1^2 - \frac{1}{4} = 0. \) Then

\[
\text{Moje}(Q_1, f_1) = -T_1 + 2\beta_1 S_1 + (1 - 2\beta_2) S_2
\]

(4.146)

and \( \text{Moje}(Q_1, f_2) = -\pi, \) ie,

\[
D(Q_1)_{r_1} = (0 -1 2b_1 0 0).
\]

(4.147)

So \( \text{rank } D(Q_1) = 2. \) Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (0, b_1, a_2, \frac{1}{2}) \in S \) such that \( b_2 - \frac{1}{2} = 4b_1^2 + 1 = 0. \) Then

\[
\text{Moje}(Q_2, f_1) = -u_1\pi - T_1 + 2\beta_1 S_1 + (1 - 2\beta_2) S_2
\]

(4.148)

where \( \beta_2 \equiv \frac{1}{2}(\text{mod } \pi) \) and \( \text{Moje}(Q_2, f_2) = -\pi, \) ie,

\[
D(Q_2)_{r_1} = ( -u_1 -1 2b_1 0 0).
\]

(4.149)

So \( D(Q_2) \) has rank two. Hence \( Q_2 \) is a regular point and as a result \( V_{2112}^{2000} \) is a regular scheme.

**Chart 3.1.1.4**

\[
V_{2112}^{2000} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - 1 + X_2, Y_2^{12} X_1^8 X_2^2 Y_1^2 - Y_2^{12} X_1^8 X_2^3 - \pi).
\]

(4.150)

From the first equation we get \( Y_1^2 - X_1 X_2 = 1 - X_2. \) By using it in the other equation we get

\[
V_{2112}^{2000} = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 X_2 - 1 + X_2, Y_2^{12} X_1^8 X_2^2 (1 - X_2) - \pi).
\]

(4.151)

Let \( A = R[X_1, X_2, Y_2]/(Y_2^{12} X_1^8 X_2^2 (1 - X_2) - \pi). \) By using theorem (2.14) we can check that \( A \) is a regular ring. Note that

\[
V_{2112}^{2000} = \text{Spec } A[Y_1]/(Y_1^2 - X_1 X_2 - 1 + X_2).
\]

(4.152)
CHAPTER 4. Special class of Arithmetic three-folds

Let \( \phi(Y_1) = Y_1^2 - X_1 X_2 - 1 + X_2 \). Then \( \frac{\partial \phi}{\partial Y_1} = 2Y_1 \). Since \( p = \text{char } k \neq 2 \), the equation \( \frac{\partial \phi}{\partial Y_1} = 0 \) can be written as \( 2Y_1 = 0 \). So \( V_{2112}^{0001} \) is etale over \( A \) everywhere except at those points where \( Y_1 = 0 \). To have the points on the special fibre of \( V_{2112}^{0001} \), we need \( Y_1^2 X_1^2 X_2 - Y_2^{12} X_1^3 X_2^3 = 0 \). Considering \( \phi(Y_1) = \frac{\partial \phi}{\partial Y_1} = Y_2^{12} X_1^3 X_2^2 Y_1 - Y_2^{12} X_1^3 X_2^3 = 0 \), we can check that \( (V_{2112}^{0001})_{\text{Sing}} \subseteq S \subseteq (V_{2112}^{0001})_v \), where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_{2112}^{0001})_v \) satisfying one of the following conditions:

1. \( b_1 = b_2 = a_1 a_2 + 1 - a_2 = 0 \);
2. \( a_1 = b_1 = a_2 - 1 = 0 \).

Let \( Q_1 = (a_1, 0, a_2, 0) \in S \) such that \( a_1 a_2 + 1 - a_2 = 0 \). Then \( \text{Moje}(Q_1, f_1) = -\alpha_2 T_1 + (1 - \alpha_1) T_2 \) and \( \text{Moje}(Q_1, f_2) = -\pi \), i.e.,

\[
D(Q_1)_r = (0 \quad -a_2 \quad 0 \quad 1 - a_1 \quad 0) \quad (4.153)
\]

So \( Q_1 \) is singular if \( -a_2 = 1 - a_1 = a_1 a_2 + 1 - a_2 = 0 \) which is impossible. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (0, 0, 1, b_2) \in S \). Then \( \text{Moje}(Q_2, f_1) = -\alpha_2 u_1 \pi - \alpha_2 T_1 + T_2 \) where \( \alpha_2 \equiv 1(\text{mod } \pi) \) and \( \text{Moje}(Q_2, f_2) = -\pi \), i.e.,

\[
D(Q_2)_r = (-a_2 u_1 \quad -a_2 \quad 0 \quad 1 \quad 0) \quad (4.154)
\]

So \( \text{rank } D(Q_2) = 2 \). Hence \( Q_2 \) is a regular point and as a result \( V_{2112}^{0001} \) is a regular scheme. The gluing of the regular schemes which we have had so far, gives us the regular scheme \( \overline{X} \).

**Lemma 4.9** Let \( p = \text{char } k \neq 2, 3 \) and \( V_1 = \text{Spec } R[x, y]/(y^2 - x^3 - \pi) \) and \( V_2 = \text{Spec } R[x, y]/(x^2 y - xy^2 - \pi) \). Then \( X = V_1 \times_{\text{Spec } R} V_2 \) is singular and after some blowings-up we get a regular scheme \( \overline{X} \).

**Proof:** We have

\[
X = \text{Spec } R[x_1, y_1, x_2, y_2]/(y_1^2 - x_1^3 - x_2^2 y_2 + x_2 y_2^2, y_1^2 - x_1^3 - \pi). \quad (4.155)
\]

By using corollary (2.11) we can check that \( P_0 \in X_\pi \) is the only singular point of \( X \). Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( \overline{X} \) as follows:
CHAPTER 4. Special class of Arithmetic three-folds

Chart 1

\[ V_1^0 = \text{Spec} \mathbb{R}[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2, \, X_1^2Y_2^2 - X_1^3 - \pi). \]  

(4.156)

Assume that \( \mathbb{A} = \mathbb{R}[X_1, Y_1]/(X_1^2(Y_1^2 - X_1) - \pi) \). We can check that \( \mathbb{A} \) is a regular ring (by using theorem (2.14)). Note that

\[ V_1^0 = \text{Spec} \mathbb{A}[X_2, Y_2]/((Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2)). \]  

(4.157)

Let \( \phi(X_2, Y_2) = Y_1^2 - X_1 - X_1X_2^2Y_2 + X_1X_2Y_2 \). Then \( \frac{\partial \phi}{\partial X_2} = -2X_1X_2Y_2 + X_1Y_2^2 \) and \( \frac{\partial \phi}{\partial Y_2} = -X_1X_2^2 + 2X_1X_2Y_2 \). Since \( p = \text{char} \, k \neq 2 \), the system \( \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( X_1Y_2(Y_2 - 2X_2) = X_1X_2(2Y_2 - X_2) = 0 \). So \( V_1^0 \) is smooth over \( \mathbb{A} \) everywhere except at the points where \( X_1Y_2(Y_2 - 2X_2) = X_1X_2(2Y_2 - X_2) = 0 \). To have the points on the special fibre of \( V_1^0 \), we need \( X_1^2(Y_1^2 - X_1) = 0 \). Considering \( \phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^2(Y_1^2 - X_1) = 0 \) we get \( (V_1^0)_{\text{Sing}} \subset S \subset (V_1^0)_x \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_1^0)_x \) satisfying one of the following conditions:

1. \( a_1 = b_1 = 0 \);
2. \( a_2 = b_2 = a_1 - b_1^2 = 0 \).

Let \( Q_1 = (0, 0, a_2, b_2) \in S \). Then

\[ \text{Moje}(Q_1, f_1) = (\alpha_2\beta_2^2 - \alpha_2^2\beta_2 - 1)u_1 \pi + (\alpha_2\beta_2^2 - \alpha_2^2\beta_2 - 1)T_1 \]  

(4.158)

and \( \text{Moje}(Q_1, f_2) = -\pi \), ie,

\[ D(Q_1)_r = (a_2b_2^2 - a_2^2b_2 - 1)\bar{a}_1 \quad (a_2b_2^2 - a_2^2b_2 - 1) \quad 0 \quad 0 \quad 0). \]  

(4.159)

So \( Q_1 \) is singular if \( a_2b_2^2 - a_2^2b_2 - 1 = 0 \).

Let \( Q_2 = (a_1, b_1, 0, 0) \in S \) such that \( a_1 - b_1^2 = 0 \). Then \( \text{Moje}(Q_2, f_1) = -T_1 + 2\beta_1S_1 \) and

\[ \text{Moje}(Q_2, f_2) = (2a_1\beta_1^2 - 3a_1^2)T_1 + 2a_1^2\beta_1S_1 - \pi, \]  

(4.160)

ie,

\[ D(Q_2) = \begin{pmatrix} 0 & -1 & 2b_1 & 0 & 0 \\ -1 & 2a_1\beta_1^2 - 3a_1^2 & 2a_1^2b_1 & 0 & 0 \end{pmatrix}. \]  

(4.161)
So $D(Q_2)$ has rank two. Hence $Q_2$ is a regular point. This means that

$$(V_1^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_1^0)_x \mid a_1 = b_1 = a_2b_2 - a_2b_2 - 1 = 0\}.$$  

Now we blow-up $(V_1^0)^{\text{Sing}}$. By using the procedure of Mahtab we get three pieces for the covering of $V_1^0$ as follows:

**Chart 1.1**

$$V_{110}^0 = \text{Spec } R[x_1, y_1, x_2, y_2, \frac{x_2y_2^2 - x_2^2y_2 - 1}{x_1}]/((\frac{y_1}{x_1})^2 + (\frac{x_2y_2^2 - x_2^2y_2 - 1}{x_1}),$$

$$x_1^3(\frac{y_1}{x_1})^2 - x_1^3 - \pi) = \text{Spec } R[x_1, X_1, X_2, Y_2, Z]/(Y_2^2 + Z, X_1^4Y_2 - X_1^3 - \pi, X_1Z - X_2Y_2^2 + X_2^2Y_2 + 1) = \text{Spec } R[x_1, X_1, X_2, Y_2]/$$

$$(-X_1Y_2^2 - X_2Y_2^2 + X_2^2Y_2 + 1, X_1^4Y_2^2 - X_1^3 - \pi). \quad (4.162)$$

Let $A = R[x_1, Y_1]/(X_1^3(X_1Y_1^2 - 1) - \pi)$. By using theorem (2.14) we can check that $A$ is a regular ring. Note that

$$V_{11}^0 = \text{Spec } A[X_2, Y_2]/(-X_1Y_2^2 - X_2Y_2^2 + X_2^2Y_2 + 1). \quad (4.163)$$

Let $\phi(X_2, Y_2) = -X_1Y_2^2 - X_2Y_2^2 + X_2^2Y_2 + 1$. Then $\frac{\partial \phi}{\partial X_2} = -Y_2^2 + 2X_2Y_2$ and $\frac{\partial \phi}{\partial Y_2} = -2X_2Y_2 + X_2^2$. Since $p = \text{char } k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $Y_2(2X_2 - Y_2) = X_2(X_2 - 2Y_2) = 0$. So $V_{11}^0$ is smooth over $A$ everywhere except at the points where $Y_2(2X_2 - Y_2) = X_2(X_2 - 2Y_2) = 0$. To have the points on the special fibre of $V_{11}^0$, we need $X_1^3(X_1Y_1^2 - 1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^3(X_1Y_1^2 - 1) = 0$, we can check that $(V_{11}^0)^{\text{Sing}} \subset S \subset (V_{11}^0)_x$ where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{11}^0)_x$ satisfying the following condition:

$$a_2 = b_2 = a_1b_1^2 - 1 = 0.$$  

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $a_1b_1^2 = 0$. Then $\text{Moje}(Q, f_1) = -\beta_1^2T_1 - 2a_1\beta_1S_1$ and

$$\text{Moje}(Q, f_2) = (4\alpha_1^3\beta_1^2 - 3\alpha_1^2)T_1 + 2a_1^4\beta_1S_1 - \pi. \quad (4.164)$$
ie,\[ D(Q) = \begin{pmatrix} 0 & -b_1^2 & -2a_1b_1 & 0 & 0 \\ -1 & (4a_1^2b_1^2 - 3a_1^2) & 2a_1^2b_1 & 0 & 0 \end{pmatrix}. \] (4.165)

So \( Q \) is singular if \(-b_1^2 = -2a_1b_1 = a_1b_1 - 1 = 0 \) which is impossible. Hence \( Q \) is a regular point and as a result \( V^{00}_{11} \) is a regular scheme.

**Chart 1.2**

\[
V^{01}_{11} = Spec R[x_1, y_1, x_2, y_2, x_2y_2^2 - x_2y_2 - 1]/(1 + x_1(x_2y_2^2 - x_2y_2 - 1)),
\]

\[
y_1^2(y_1^2 - x_1^3)^2 - y_1^3(y_1^2 - x_1^3)^3 - \pi) = Spec R[x_1, y_1, x_2, y_2, Z]/(1 + x_1Z),
\]

\[
Y_1^4x_2^2 - Y_1^3x_1^3 - \pi, Y_1Z - X_2y_1 + X_2y_2^2 + X_2^2y_2 + 1). \] (4.166)

Suppose that \( A = R[x_1, y_1]/(x_1^2y_1^3(y_1 - x_1) - \pi) \). By using theorem (2.14) we can check that \( A \) is a regular ring. Notice that

\[
V^{01}_{11} = Spec A[z, x_2, y_2]/(1 + x_1Z, y_1Z - X_2y_2^2 + X_2^2y_2 + 1) =
Spec A[x_1^{-1}][x_2, y_2]/(y_1 + x_1X_2y_2^2 - X_1X_2y_2 - x_1).
\] (4.167)

Note that \( B = A[x_1^{-1}] = A[Z]/(1 + X_1Z) \) and Spec \( B \) is etale over \( A \). So \( A[x_1^{-1}] \) is a regular ring.

Let \( \phi(x_2, y_2) = Y_1 + x_1x_2y_2^2 - x_1x_2^2y_2 - x_1 \). Then \( \frac{\partial \phi}{\partial x_2} = X_1y_2^2 - 2X_1x_2y_2 \) and \( \frac{\partial \phi}{\partial y_2} = 2X_1x_2y_2 - x_1x_2^2 \). Since \( p = char k \neq 2 \), and \( x_1 \neq 0 \), the system \( \frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial y_2} = 0 \) can be written as \( Y_2^2 - 2X_2y_2 = 2X_2y_2 - x_2^2 = 0 \). So \( V^{01}_{11} \) is smooth over \( A[x_1^{-1}] \) everywhere except at the points where \( Y_2^2 - 2X_2y_2 = 2X_2y_2 - x_2^2 = 0 \). To have the points on the special fibre of \( V^{01}_{11} \), we need \( X_1^3y_1^3(y_1 - x_1) = 0 \).

Considering \( \phi(x_2, y_2) = \frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial y_2} = X_1^3y_1^3(y_1 - x_1) = x_1Z + 1 = 0 \), we can check that \( (V^{01}_{11})_{\text{Sing}} \subset S \subset (V^{01}_{11})_\pi \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2, c) \in (V^{01}_{11})_\pi \) satisfying the following condition:

\[
a_1 - b_1 = a_2 = b_2 = a_1c + 1 = 0.
\]

Let \( Q = (a_1, b_1, 0, 0, c) \in S \). Then \( \text{Moje}(Q, f_1) = \alpha_1W + \gamma T_1 \) and

\[
\text{Moje}(Q, f_2) = (2\alpha_1\beta_1^4 - 3\alpha_1^2\beta_1^3)T_1 + (4\alpha_1^2\beta_1^3 - 3\alpha_1\beta_1^2)S_1 - \pi \] (4.168)
CHAPTER 4. Special class of Arithmetic three-folds

and \( \text{Moje}(Q, f_3) = \gamma S_1 + \beta_1 W \), i.e.,

\[
D(Q) = \begin{pmatrix}
0 & c & 0 & 0 & 0 & a_1 \\
-1 & -a_1^5 & a_5 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & b_1
\end{pmatrix}.
\]

(4.169)

The determinant consisting of the first three columns of \( D(Q) \) is equal to \( c^2 = a^{-2} \neq 0 \). So \( \text{rank} D(Q) = 2 \). Hence \( Q \) is a regular point and as a result \( V^1_{11} \) is a regular scheme.

Chart 1.5

Let \( Z = x_2 y_2^2 - x_2^2 y_2 - 1 \). Then

\[
V^0_{10} = \text{Spec} \mathbb{R}[x_1, y_1, x_2, y_2, Z]/((z_1)^2 + \frac{x_1}{Z}Z^4(\frac{x_1}{Z})^2 - Z^3(\frac{x_1}{Z})^3 - \pi),
\]

\[
Z - x_2 y_2^2 + x_2^2 y_2 + 1 = \text{Spec} \mathbb{R}[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 + X_1, Z^4 X_2^2 Y_1^2 - Z^3 X_1^3 - \pi, Z - X_2 Y_2^2 + X_2^2 Y_2 + 1 = \text{Spec} \mathbb{R}[Y_1, X_2, Y_2, Z]/(Z - X_2 Y_2^2 + X_2^2 Y_2 + 1, Z^4 Y_1^2 - Z^3 Y_1^3 - \pi) = \text{Spec} \mathbb{R}[Y_1, X_2, Y_2]/
\]

\[
((X_2 Y_2^2 - X_2^2 Y_2 - 1)^3 Y_1^2(X_2 Y_2^2 - X_2^2 Y_2 - 2) - \pi).
\]

(4.170)

Since \( X_2 Y_2^2 - X_2^2 Y_2 - 1 = 0 \) and \( X_2 Y_2^2 - X_2^2 Y_2 - 2 = 0 \) in \( A_k^3 \) define non-singular \( k \)-varieties (by Jacobian criterion), by using theorem (2.14) we conclude that \( V^0_{10} \) is a regular scheme.

Chart 2

\[
V^1_1 = \text{Spec} \mathbb{R}[(\frac{x_1}{y_1}, y_1, x_2, y_2)/(1 - y_1(\frac{x_1}{y_1})^3 - y_1(\frac{x_2}{y_1})^2 - y_1(\frac{x_2}{y_1})(\frac{y_2}{y_1})^2,
\]

\[
y_1^2 - y_1^3(\frac{x_1}{y_1})^3 - \pi) = \text{Spec} \mathbb{R}[X_1, Y_1, X_2, Y_2]/
\]

\[
(1 - Y_1 X_1^3 - Y_1 X_2^2 Y_2 + Y_1 X_2^2, Y_1^2 - Y_1^2 X_1^3 - \pi).
\]

(4.171)

Now let \( A = \mathbb{R}[X_1, Y_1]/(Y_1^2(1 - Y_3 X_3^3) - \pi) \). We can use theorem (2.14) to show that \( A \) is a regular ring. Notice that

\[
V^1_1 = \text{Spec} A[X_2, Y_2]/(1 - Y_1 X_3^3 - Y_1 X_2^2 Y_2 + Y_1 X_2 Y_2).
\]

(4.172)
Let \( \phi(X_2, Y_2) = 1 - Y_1 X_1^2 - Y_1 X_2^2 Y_2 + Y_1 X_2 Y_2^2 \). Then \( \frac{\partial \phi}{\partial X_2} = -2Y_1 X_2 Y_2 + Y_1 Y_2^2 \) and \( \frac{\partial \phi}{\partial Y_2} = -Y_1 X_2^2 + 2Y_1 X_2 Y_2 \). Since \( p = \text{char } k \not= 2 \), we can write the system \( \frac{\partial \phi}{\partial X_2} = 0 \) as \(-2Y_1 X_2 Y_2 + Y_1 Y_2^2 = -Y_1 X_2^2 + 2Y_1 X_2 Y_2 = 0\). So \( V_1^1 \) is smooth over \( A \) everywhere except at the points where \(-2Y_1 X_2 Y_2 + Y_1 Y_2^2 = -Y_1 X_2^2 + 2Y_1 X_2 Y_2 = 0\). To have the points on the special fibre of \( V_1^1 \), we need \( Y_1^2(1 - Y_1 X_1^2) = 0 \). Considering \( \phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = Y_1^2(1 - Y_1 X_1^2) = 0 \), we can check that \((V_1^1)^{\text{Sing}} \subset S \subset (V_1^1)_x\) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_1^1)_x\) satisfying the following condition:

\[
a_2 = b_2 = a_1^2 b_1 - 1 = 0.
\]

Let \( Q = (a_1, b_1, 0, 0) \in S \) such that \( a_1^2 b_1 - 1 = 0 \). Then \( \text{Moje}(Q, f_1) = -3a_1^2 \beta_1 T_1 - a_1^3 S_1 \) and

\[
\text{Moje}(Q, f_2) = -3a_1^2 \beta_1^3 T_1 + \beta_1(2 - 3a_1^2 \beta_1) S_1 - \pi,
\]

ie,

\[
D(Q) = \begin{pmatrix}
0 & -3a_1^2 b_1 & -a_1^3 & 0 & 0 \\
-1 & -3a_1^2 b_1 & b_1(2 - 3a_1^2 b_1) & 0 & 0
\end{pmatrix}.
\]

So \( Q \) is singular if \( 3a_1^2 b_1 = a_1^2 = 6a_1^2 b_1(a_1^2 b_1 - 1) = a_1^2 b_1 - 1 = 0 \) which is impossible. Hence \( Q \) is a regular point and as a result \( V_1^1 \) is a regular scheme.

**Chart 3**

\[
V_2^0 = \text{Spec } R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, \frac{y_2}{x_2}]/((\frac{y_1}{x_2})^2 - x_2(\frac{x_1}{x_2})^3 - x_2(\frac{y_2}{x_2})^2 + x_2(\frac{y_2}{x_2})^2),
\]

\[
x_2^2(\frac{y_1}{x_2})^2 - x_2^3(\frac{x_1}{x_2})^3 - \pi) = \text{Spec } R[X_1, Y_1, X_2, Y_2]/
\]

\[
(Y_1^2 - X_2 X_1^3 - X_2 Y_2 + X_2 Y_2^2, X_2^2 Y_1^2 - X_2^3 X_1^3 - \pi).
\]

From the first equation we get \( Y_1^2 - X_2 X_1^3 = X_2 Y_2(1 - Y_2) \). By using it in the other equation, we can write

\[
V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 X_1^3 - X_2 Y_2 + X_2 Y_2^2, X_2^3 Y_2(1 - Y_2) - \pi).
\]
CHAPTER 4. Special class of Arithmetic three-folds

Now assume that $A = R[X_2, Y_2]/(X^2Y_2(1 - Y_2) - \pi)$. Considering theorem (2.14) we can check that $A$ is a regular ring. Notice that

$$V^0_2 = \text{Spec } A[X_1, Y_1]/(Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2).$$

(4.177)

Let $\phi(X_1, Y_1) = Y_1^2 - X_2X_1^3 - X_2Y_2 + X_2Y_2^2$. Then $\frac{\partial \phi}{\partial X_1} = -3X_2X_1^2$ and $\frac{\partial \phi}{\partial Y_1} = 2Y_1$. Since $p = \text{char } k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0$ can be written as $-3X_2X_1^2 = 2Y_1 = 0$. So $V^0_2$ is smooth over $A$ everywhere except at the points where $-3X_2X_1^2 = 2Y_1 = 0$. To have the points on the special fibre of $V^0_2$, we need $X_2^2Y_1^2 - X_2X_1^3 = 0$. Considering $\phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = X_2^2Y_1^2 - X_2X_1^3 = 0$, we can check that $(V^0_2)^{\text{Sing}} \subset S \subset (V^0_2)$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V^0_2)$ satisfying one of the following conditions:

1. $b_1 = a_2 = 0$;
2. $a_1 = b_1 = b_2 = 0$;
3. $a_1 = b_1 = b_2 - 1 = 0$.

Let $Q = (a_1, 0, 0, b_2) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (\beta^2_2 - \beta_2 - \alpha^3_1)u_2\pi + (\beta^2_2 - \beta_2 - \alpha^3_1)T_2$$

(4.178)

and $\text{Moje}(Q_1, f_2) = -\pi$, i.e,

$$D(Q_1)_{r_1} = \begin{pmatrix} (b_2^2 - b_2 - a_1^3)u_2 & 0 & 0 & b_2^2 - b_2 - a_1^3 & 0 \end{pmatrix}.$$

(4.179)

So $Q_1$ is singular if $b_2^2 - b_2 - a_1^3 = 0$.

Let $Q_2 = (0, 0, a_2, 0) \in S$. Then

$$\text{Moje}(Q_2, f_1) = -\alpha_2v_2\pi - \alpha_2S_2$$

and $\text{Moje}(Q_2, f_2) = -\pi$, (4.180)

i.e,

$$D(Q_2)_{r_1} = \begin{pmatrix} -a_2u_2 & 0 & 0 & 0 & -a_2 \end{pmatrix}.$$

(4.181)

So $D(Q_2)$ is singular if $a_2 = 0$ i.e $Q_2 = P_0$. Note that $P_0$ satisfies the conditions $b_1 = a_2 = b_2^2 - b_2 - a_1^3 = 0$.

Finally let $Q_3 = (0, 0, a_2, 1) \in S$. Then

$$\text{Moje}(Q_3, f_1) = \beta_2(\beta_2 - 1)T_2 + \alpha_2(2\beta_2 - 1)S_2$$

(4.182)
and \( \text{Moje}(Q_3, f_2) = -\pi, \) \( \text{i.e.,} \)
\[
D(Q_3, \eta) = (0\ 0\ 0\ 0\ a_2).
\] (4.183)

So \( Q_3 \) is singular if \( a_2 = 0, \) \( \text{i.e.,} \) \( Q_3 = (0, 0, 0, 1). \) Note that \( Q_3 \) also satisfies the conditions \( b_1 = a_2 = b_2 - a_3^2 = 0. \) Hence
\[
(V_2^0)^{\text{Sing}} = \{(a_1, b_1, a_2, b_2) \in (V_2^0)_{\eta} \mid b_1 = a_2 = a_3^2 + b_2 - b_2^2 = 0\}.
\] (4.184)

Now we blow-up \((V_2^0)^{\text{Sing}}. \) By using the procedure of Mahtab we get three pieces for the covering of \( V_2^0 \) as follows:

**Chart 3.2**

\[
V_{21}^{01} = \text{Spec } R[x_1, y_1, x_2, y_2, x_1^3 + y_2 - y_2^2]/(1 - \frac{x_2}{y_1} (\frac{x_2}{y_1}^3 \frac{x_1^3}{y_1} - \pi),
\]
\[
y_1^4(x_2)^2 - y_1^3(\frac{x_2}{y_1}^3 \frac{x_1^3}{y_1} - \pi) = \text{Spec } R[X_1, Y_1, X_2, Y_2, Z]/(1 - X_2 Z, Y_1 X_2^3 - Y_1^3 X_2 - X_1^3 - Y_2 + Y_2^2).
\] (4.185)

Suppose that \( A = R[X_1, Y_1, X_2]/(Y_2^3 X_2^2 (Y_1 - X_2 X_1^3) - \pi). \) We can check that \( A \) is a regular ring (by using theorem (2.14)). We can write
\[
V_{21}^{01} = \text{Spec } A[Z, Y_2]/(1 - X_2 Z, Y_1 X_2 - X_1^3 - Y_2 + Y_2^2) = \text{Spec } A[X_2^{-1}][Y_2]/(Y_1 - X_1^3 X_2 - X_2 Y_2 + X_2 Y_2^2).
\] (4.186)

Note that \( B = A[X_2^{-1}] = A[Z]/(1 - X_2 Z) \) and \( \text{Spec } B \) is etale over \( A. \) So \( A[X_2^{-1}] \) is a regular ring.

Now let \( \phi(Y_2) = Y_1 - X_1^3 X_2 - X_2 Y_2 + X_2 Y_2^2. \) Then \( \frac{\partial \phi}{\partial Y_2} = -X_2 + 2X_2 Y_2. \) Since \( p = \text{char } k \neq 2 \) and \( X_2 \neq 0 \) we get \( 2Y_2 - 1 = 0. \) So \( V_{21}^{01} \) is etale over \( A[X_2^{-1}] \) everywhere except at the points where \( 2Y_2 - 1 = 0. \) To have the points on the special fibre of \( V_{21}^{01}, \) we need \( Y_1^3 X_2^2 (Y_1 - X_2 X_1^3) = 0. \) Considering \( \phi(Y_2) = \frac{\partial \phi}{\partial Y_2} = Y_1^3 X_2^2 (Y_1 - X_2 X_1^3) = 1 - X_2 Z = 0, \) we can check that \((V_{21}^{01})^{\text{Sing}} \subset S \subset (V_{21}^{01})_{\eta} \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2, c) \in (V_{21}^{01})_{\eta} \) satisfying the following condition:
\[
b_1 = b_2 - \frac{1}{2} = 1 + 4a_1^3 = ca_2 - 1 = 0.
\]
Let $Q = (a_1, 0, a_2, \frac{1}{2}, c) \in S$. Then $\text{Moje}(Q, f_1) = -\gamma T_2 - \alpha_2 W$ and $\text{Moje}(Q, f_2) = -\pi$ and

$$\text{Moje}(Q, f_3) = \gamma v_1 \pi - 3\alpha_1^2 T_1 + \gamma S_1 + (-1 + 2\beta_2)S_2$$

where $\beta_2 \equiv \frac{1}{2} (\mod \pi)$, i.e.,

$$D(Q) = \begin{pmatrix}
0 & 0 & 0 & -c & 0 & -a_2 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
c\bar{v}_1 & -3a_1^2 & c & 0 & 0 & 0
\end{pmatrix}.$$ (4.188)

The $3 \times 3$ matrix consisting of the first, second and fourth columns of $D(Q)$ has determinant $-3a_1^2 c \neq 0$ (note that $ca_2 - 1 = 0$ and $1 + 4a_1^3 = 0$). So $D(Q)$ has rank three. Hence $Q$ is a regular point and as a result $V_{21}^{01}$ is a regular scheme.

Chart 3.3

\[ V_{22}^{00} = \text{Spec} R[x_1, y_1, x_2, y_2, \frac{x_1^3 + y_2 - y_2^2}{x_2}] / ((y_1)^2 - (\frac{x_1^3 + y_2 - y_2^2}{x_2})) \]

\[ x_2^2(y_1^2 x_2 - x_2^3 x_1 - \pi) = \text{Spec} R[X_1, Y_1, X_2, Y_2, Z] / (Y_1^2 - Z, X_2^4 Y_1^2 - X_2^3 X_1^3 - \pi, X_2 Z - X_1^3 Y_2 - Y_2^2 = \text{Spec} R[X_1, Y_1, X_2, Y_2] / (X_2 Y_1^2 - X_1^3 Y_2 - Y_2^2, X_2^4 Y_1^2 - X_2^3 X_1^3 - \pi). \] (4.189)

From the first equation we get $X_2 Y_1^2 - X_1^3 = Y_2 - Y_2^2$. By using it in the other equation we get

$$V_{22}^{00} = \text{Spec} R[X_1, Y_1, X_2, Y_2] / (X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2, X_2^3 Y_2(1 - Y_2) - \pi).$$ (4.190)

Assume that $A = R[X_2, Y_2] / (X_2^2 Y_2(1 - Y_2) - \pi)$. By using theorem (2.14) we can check that $A$ is a regular ring. Note that

$$V_{22}^{00} = \text{Spec} A[X_1, Y_1] / (X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2).$$ (4.191)

Let $\phi(X_1, Y_1) = X_2 Y_1^2 - X_1^3 - Y_2 + Y_2^2$. Then $\frac{\partial \phi}{\partial X_1} = -3X_2^2$ and $\frac{\partial \phi}{\partial Y_1} = 2X_2 Y_1$. Since $p = \text{char} k \neq 2, 3$, the system $\frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = 0$ can be written as $-3X_2^2 = 2X_2 Y_1 = 0$. So $V_{22}^{00}$ is smooth over $A$ everywhere except at the points where
\[ -3X_1^2 = 2X_2Y_1 = 0. \] To have the points on the special fibre of \( V_{22}^{00} \), we need \( X_1^2Y_1^2 - X_2^2X_1^3 = 0 \). Considering \( \phi(X_1, Y_1) = \frac{\partial \phi}{\partial X_1} = \frac{\partial \phi}{\partial Y_1} = X_1^4Y_1^2 - X_2^3X_1^3 = 0 \), we can check that \( (V_{22}^{00})^\text{Sing} \subset S \subset (V_{22}^{00})_x \) where \( S \) contains points of the form \((a_1, b_1, a_2, b_2) \in (V_{22}^{00})_x \) satisfying one of the following conditions:

1. \( a_1 = a_2 = b_2 = 0 \);
2. \( a_1 = b_1 = b_2 = 0 \);
3. \( a_1 = b_1 = b_2 - 1 = 0 \);
4. \( a_1 = a_2 = b_2 - 1 = 0 \).

Let \( Q_1 = (0, b_1, 0, 0) \in S \). Then

\[
\text{Moje}(Q_1, f_1) = (\beta_1^2u_2 - v_2)\pi + \beta_1^2T_2 - S_2 \text{ and } \text{Moje}(Q_1, f_2) = -\pi, \tag{4.192}
\]

\[ \text{ie, } D(Q_1)_{r_1} = \begin{pmatrix} b_1^2u_2 - \bar{v}_2 & 0 & 0 & b_1^2 \end{pmatrix}. \tag{4.193} \]

So \( D(Q_1) \) has rank two. Hence \( Q_1 \) is a regular point.

Let \( Q_2 = (0, 0, a_2, 0) \in S \). Then

\[
\text{Moje}(Q_2, f_1) = -v_2\pi - S_2 \text{ and } \text{Moje}(Q_2, f_2) = -\pi, \tag{4.194}
\]

\[ \text{ie, } D(Q_2)_{r_1} = \begin{pmatrix} -\bar{v}_2 & 0 & 0 & 0 \end{pmatrix}. \tag{4.195} \]

So \( D(Q_2) \) has rank two. Hence \( Q_2 \) is a regular point.

Let \( Q_3 = (0, 0, a_2, 1) \in S \). Then \( \text{Moje}(Q_3, f_1) = (2\beta_2 - 1)S_2 \) where \( \beta_2 \equiv 1 \pmod{\pi} \) and \( \text{Moje}(Q_3, f_2) = -\pi, \text{ ie,} \)

\[
D(Q_3) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.196}
\]

which means that \( D(Q_3) \) has rank two. Hence \( Q_3 \) is a regular point.

Let \( Q_4 = (0, b_1, 0, 1) \in S \). Then \( \text{Moje}(Q_4, f_1) = \beta_1^2u_2\pi + \beta_1^2T_2 + (2\beta_2 - 1)S_2 \) where \( \beta_2 \equiv 1 \pmod{\pi} \) and \( \text{Moje}(Q_4, f_2) = -\pi, \text{ ie,} \)

\[
D(Q_4)_{r_1} = \begin{pmatrix} b_1^2\bar{u}_2 & 0 & 0 & b_1^2 \end{pmatrix}. \tag{4.197} \]

So \( D(Q_4) \) has rank two. Hence \( Q_4 \) is a regular point and as a result \( V_{22}^{00} \) is a regular scheme.
CHAPTER 4. Special class of Arithmetic three-folds

Chart 3.5

Let \( Z = x_1^3 + y_2 - y_2^2 \). Then

\[
V_{2i}^{06} = \text{Spec } R[x_1, \frac{y_1}{Z}, \frac{x_2}{Z}, y_2, Z]/(\left(\frac{y_1}{Z}\right)^2 - \frac{x_2}{Z}, Z^4 \left(\frac{y_1}{Z}\right)^2\left(\frac{x_2}{Z}\right)^2 - Z^3 (\frac{x_2}{Z})^3 x_1^3 - \pi,
\]

\( Z - x_1^3 - y_2 + y_2^2 \) = \text{Spec } R[X_1, Y_1, X_2, Y_2, Z]/(Y_1^2 - X_2, Z - X_1^3 - Y_2 + Y_2^2),

\( Z^4 Y_1^2 X_2^2 - Z^3 X_1^3 X_1^3 - \pi) = \text{Spec } R[X_1, Y_1, X_2, Z]/(Z - X_1^3 - Y_2 + Y_2^2, Y_1^6 Z^4 - X_1^3 Y_1^6 Z^3 - \pi)

(4.198)

Since \( Y_2 - Y_2^2 + X_1^3 = 0 \) and \( Y_2 - Y_2 = 0 \) in \( \mathbb{A}^2_k \) define non-singular \( k \)-varieties, by using the theorem (2.14) we conclude that \( V_{2i}^{06} \) is a regular scheme.

Chart 4

\( V_2^1 = \text{Spec } R[\frac{x_1}{y_2}, \frac{y_1}{y_2}, \frac{x_2}{y_2}, y_2]/(\left(\frac{y_1}{y_2}\right)^3 - y_2\left(\frac{x_1}{y_2}\right)^2 - y_2\left(\frac{x_2}{y_2}\right)^2 + y_2\left(\frac{x_2}{y_2}\right)^2,\)

\( y_2\left(\frac{y_1}{y_2}\right)^3 - y_2\left(\frac{x_1}{y_2}\right)^3 - \pi) = \text{Spec } R[X_1, Y_1, X_2, Y_2]/\)

\( (Y_2^2 - Y_2 X_1^3 - Y_2 X_2^2 + Y_2 X_2, Y_2^2 Y_1^2 - Y_2 X_1^3 - \pi). \) (4.199)

Recall that

\( V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1^2 - X_2 X_1^3 - X_2 Y_2 + X_2 Y_2^2, X_2^2 Y_1^2 - X_2 X_1^3 - \pi). \) (4.200)

The ring homomorphism

\[
\alpha : \Gamma(V_2^0, \mathcal{O}_{V_2^0}) \longrightarrow \Gamma(V_2^1, \mathcal{O}_{V_2^1})
\]

\[
X_1 \longmapsto -X_1 \\
Y_1 \longmapsto Y_1 \\
X_2 \longmapsto -Y_2 \\
Y_2 \longmapsto X_2
\]

is an isomorphism and induces \( V_2^1 \cong V_2^0 \). Hence \( \tilde{V}_2^1 \cong \tilde{V}_2^0 \). The gluing of the regular schemes which we have had so far, gives us the regular scheme \( \tilde{X} \). □
4.4 One component is $I_3$

In this section our arithmetic three-folds are the products of two arithmetic surfaces such that one of them is of the form $I_3$ and apart from the cases discussed before.

**Lemma 4.10** Let $p = \text{char } k \neq 2,3$ and $V_1 = V_2 = \text{Spec } R[x, y]/(y^2 - yx^2 - \pi)$. Then $X = V_1 \times_{\text{Spec } R} V_2$ is singular and after some blowings-up we get a regular scheme $\widetilde{X}$.

**Proof:** We have

$$X = \text{Spec } R[x, y, z]/(y^2 - yx^2 - y_1 x_1^2 + y_2 x_2^2, y_1^2 - y_1 x_1^2 - \pi). \quad (4.201)$$

By using corollary (2.11) we can check that $P_0 = (0, 0, 0, 0) \in X_1$ is the only singular point of $X$. We blow up $P_0$. By using the procedure of Mahtab we get four pieces for the covering of $\widetilde{X}_1$ as follows:

**Chart 1**

$$V_1^0 = \text{Spec } R[x, y, z]/(y_1^2 - y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2, X_1^2 Y_1^2 - X_1^2 Y_1 - \pi). \quad (4.202)$$

Suppose that $A = R[x, y]/(X_1^2 Y_1(Y_1 - X_1) - \pi)$. We can use theorem (2.14) to show that $A$ is a regular ring. Notice that

$$V_1^0 = \text{Spec } R[x, y, z]/(y_1^2 - y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2). \quad (4.203)$$

Let $\phi(X_2, Y_2) = Y_1^2 - Y_2^2 - X_1 Y_1 + X_1 Y_2 X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2X_1 Y_2 X_2$ and $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_1 X_2^2$. Since $p = \text{char } k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0$. So $V_1^0$ is smooth over $A$ everywhere except at the points where $2X_1 Y_2 X_2 = -2Y_2 + X_1 X_2^2 = 0$. To have the points on the special fibre of $V_1^0$, we need $X_1^2 Y_1(Y_1 - X_1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^2 Y_1(Y_1 - X_1) = 0$, we get $(V_1^0)^{\text{Sing}} \subset S \subset (V_1^0)^\pi$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)^\pi$ satisfying one of the following conditions:

1. $a_1 = b_1 = a_2 = b_2 = 0$;
CHAPTER 4. Special class of Arithmetic three-folds

(2) \( b_1 = a_2 = b_2 = 0; \)

(3) \( a_1 - b_1 = a_2 = b_2 = 0. \)

Let \( Q_1 = (0,0,a_2,0) \in S. \) Then \( \text{Moje}(Q_1,f_1) = 0 \) and \( \text{Moje}(Q_1,f_2) = -\pi. \) So \( D(Q_1) \) has rank one hence \( Q_1 \) is a singular point.

Let \( Q_2 = (a_1,0,0,0) \in S. \) Then \( \text{Moje}(Q_2,f_1) = -\alpha_1 v_1 \pi - \alpha_1 S_1 \)

\[
\text{Moje}(Q_2,f_2) = -\left(\alpha_1^3 v_1 + 1\right) \pi - \alpha_1^3 S_1, \tag{4.204}
\]

ie,

\[
D(Q_2) = \begin{pmatrix}
-a_1 \bar{v}_1 & 0 & -a_1 & 0 \\
-(a_1^3 \bar{v}_1 + 1) & 0 & -a_1^3 & 0
\end{pmatrix}. \tag{4.205}
\]

So \( Q_2 \) is singular if \( (a_1 \bar{v}_1)(a_1^3) - (a_1^3 \bar{v}_1 + 1)a_1 = 0 \) ie \( a_1 = 0 \) which means that \( Q_2 = P_0. \)

Let \( Q_3 = (a_1,b_1,0,0) \in S \) such that \( a_1 - b_1 = 0. \) Then \( \text{Moje}(Q_3,f_1) = -\beta_1 T_1 + (2\beta_1 - \alpha_1)S_1 \)

\[
\text{Moje}(Q_3,f_2) = -\pi + \alpha_1(2\beta_1^2 - 3\alpha_1 \beta_1)T_1 + \alpha_1^2(2\beta_1 - \alpha_1)S_1, \tag{4.206}
\]

ie,

\[
D(Q_3) = \begin{pmatrix}
0 & -b_1 & 2b_1 - a_1 & 0 \\
-1 & a_1(2b_1^2 - 3a_1 b_1) & a_1^2(2b_1 - a_1) & 0
\end{pmatrix}. \tag{4.207}
\]

Since \( a_1 = b_1 \) we get

\[
D(Q_3) = \begin{pmatrix}
0 & -a_1 & a_1 & 0 \\
-1 & -a_1^3 & a_1^3 & 0
\end{pmatrix}. \tag{4.208}
\]

So \( Q_3 \) is singular if \( a_1 = 0, \) ie, \( Q_3 = P_0. \) Hence

\[
(V_1^{0\text{Sing}}) = \{(a_1,b_1,a_2,b_2) \in (V_1^{0\text{Sing}}) \mid a_1 = b_1 = b_2 = 0 \}.
\]

Now we blow-up \( (V_1^{0\text{Sing}}). \) By using the procedure of Mahtab we get three pieces for the covering of \( V_1^{0\text{Sing}} \) as follows:

\section*{Chart 1.1}

\[
V_{11}^{00} = \text{Spec } R[X_1,Y_1,X_2,Y_2]/(Y_1^2 - Y_2^2 - Y_1 + Y_2 X_2^2, X_1^4 Y_1^2 - X_1^4 Y_1 - \pi). \tag{4.209}
\]
Now let $A = R[X_1, Y_1]/(X_1^4 Y_1 (Y_1 - 1) - \pi)$. By using theorem (2.14) we can check that $A$ is a regular ring. We can write

$$V_{11}^{00} = \text{Spec } A[X_2, Y_2]/(Y_1^2 - Y_2^2 - Y_1 + Y_2 X_2^2).$$

(4.210)

Let $\phi(X_2, Y_2) = Y_1^2 - Y_2^2 - Y_1 + Y_2 X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2X_2 Y_2$ and $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_2^2$. Since $p = \text{char } k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $2X_2 Y_2 = -2Y_2 + X_2^2 = 0$. So $V_{11}^{00}$ is smooth everywhere except at those points where $2X_2 Y_2 = -2Y_2 + X_2^2 = 0$. To have the points on the special fibre of $V_{11}^{00}$, we need $X_1^4 Y_1 (Y_1 - 1) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^4 Y_1 (Y_1 - 1) = 0$, we get $(V_{11}^{00})^{\text{Sing}} \subset S \subset (V_{11}^{00})_\pi$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{11}^{00})_\pi$ satisfying one of the following conditions:

1. $b_1 = a_2 = b_2 = 0$;
2. $b_1 - 1 = a_2 = b_2 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then $\text{Moje}(Q_1, f_1) = -v_1 \pi - S_1$ and

$$\text{Moje}(Q_1, f_2) = - (\alpha_1^4 v_1 + 1) \pi - \alpha_1^4 S_1,$$

(4.211)

ie,

$$D(Q_1) = \begin{pmatrix}
-a_1^4 v_1 & 0 & -1 & 0 & 0 \\
-(a_1^4 v_1 + 1) & 0 & -a_1^4 & 0 & 0
\end{pmatrix}.$$  

(4.212)

Since $a_1^4 v_1 - (a_1^4 v_1 + 1) = -1 \neq 0$, $D(Q_1)$ has rank two. Hence $Q_1$ is a regular point.

Let $Q_2 = (a_1, 1, 0, 0) \in S$. Then $\text{Moje}(Q_2, f_1) = (2b_1 - 1)S_1$ and

$$\text{Moje}(Q_2, f_2) = 4\alpha_1^3 \beta_1 (\beta_1 - 1) T_1 + \alpha_1^4 (2\beta_1 - 1) S_1 - \pi$$

(4.213)

where $\beta_1 \equiv 1(\text{mod } \pi)$, ie,

$$D(Q_2) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
-1 & 0 & a_1^4 & 0 & 0
\end{pmatrix}$$

(4.214)

which has rank two. Hence $Q_2$ is a regular point and as a result $V_{11}^{00}$ is a regular scheme.
CHAPTER 4. Special class of Arithmetic three-folds

Chart 1.2

\[ V_{11}^{01} = \text{Spec } \mathbb{R}[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - X_1 + X_1 Y_2 X_2^2, Y_1^4 X_1^2 - Y_1^4 X_1^3 - \pi). \]  \hspace{1cm} (4.215)

Now suppose that \( A = \mathbb{R}[X_1, Y_1]/(Y_1^4 X_1^2(1 - X_1) - \pi). \) By using theorem (2.14), we can check that \( A \) is a regular ring. Notice that

\[ V_{11}^{01} = \text{Spec } A[X_2, Y_2]/(1 - Y_2^2 - X_1 + X_1 Y_2 X_2^2). \]  \hspace{1cm} (4.216)

Let \( \phi(X_2, Y_2) = 1 - Y_2^2 - X_1 + X_1 Y_2 X_2^2. \) Then \( \frac{\partial \phi}{\partial X_2} = 2X_1 X_2 Y_2 \) and \( \frac{\partial \phi}{\partial Y_2} = -2Y_2 + X_1 X_2^2. \) Since \( p = \text{char } k \neq 2, \) the system \( \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0 \) can be written as \( 2X_1 X_2 Y_2 = -2Y_2 + X_1 X_2^2 = 0. \) So \( V_{11}^{01} \) is smooth over \( A \) everywhere except at the points where \( 2X_1 X_2 Y_2 = -2Y_2 + X_1 X_2^2 = 0. \) To have the points on the special fibre of \( V_{11}^{01}, \) we need \( Y_1^4 X_1^2(1 - X_1) = 0. \) Considering \( \phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = Y_1^4 X_1^2(1 - X_1) = 0, \) we get \( (V_{11}^{01})^\text{Sing} \subset S \subset (V_{11}^{01})^\pi, \) where \( S \) contains points of the form \( (a_1, b_1, a_2, b_2) \in (V_{11}^{01})^\pi \) satisfying the following condition:

\[ a_1 - 1 = a_2 = b_2 = 0 \]

Let \( Q = (1, b_1, 0, 0) \in S. \) Then \( \text{Moje}(Q, f_1) = -T_1 \) and

\[ \text{Moje}(Q, f_2) = -\pi + \alpha_1 \beta_1^4 (2 - 3\alpha_1) T_1 + 4\alpha_1^2 \beta_1^3 (1 - \alpha_1) S_1 \]  \hspace{1cm} (4.217)

where \( \alpha_1 \equiv 1 \pmod{\pi}, \) \( i.e., \)

\[ D(Q) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -\beta_1^4 & 0 & 0 \end{pmatrix}. \]  \hspace{1cm} (4.218)

So \( D(Q) \) has rank two. Hence \( Q \) is a regular point and as a result \( V_{11}^{01} \) is a regular scheme.

Chart 1.4

\[ V_{12}^{01} = \text{Spec } \mathbb{R}[X_1, Y_1, X_2, Y_2]/(Y_1^2 - 1 - X_1 Y_1 + X_1 X_2^2, Y_2^4 X_1^2 Y_1^2 - Y_2^4 X_1^3 Y_1 - \pi). \]  \hspace{1cm} (4.219)
CHAPTER 4. Special class of Arithmetic three-folds

Now suppose that $A = R[X_1,Y_1,Y_2]/(Y_2^4X_1^2Y_1(Y_1 - X_1) - \pi)$. Considering theorem (2.14), we can check that $A$ is a regular ring. Note that

$$V_{12}^{01} = \text{Spec } A[X_2]/(Y_1^2 - 1 - X_1Y_1 + X_1X_2^2).$$  \hspace{1cm} (4.220)

Let $\phi(X_2) = Y_1^2 - 1 - X_1Y_1 + X_1X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2X_1X_2$. Since $p = \text{char} k \neq 2$, the equation $\frac{\partial \phi}{\partial X_2} = 0$ can be written as $2X_1X_2 = 0$. So $V_{12}^{01}$ is etale over $A$ everywhere except at the points where $2X_1X_2 = 0$. To have the points on the special fibre of $V_{12}^{01}$, we need $Y_2^4X_1^2Y_1(Y_1 - X_1) = 0$. Considering $\phi(X_2) = \frac{\partial \phi}{\partial X_2} = Y_2^4X_1^2Y_1(Y_1 - X_1) = 0$, we get $(V_{12}^{01})^\text{Sing} \subset S \subset (V_{12}^{01})^\text{reg}$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_{12}^{01})^\text{reg}$ satisfying one of the following conditions:

1. $a_1 = b_1 - 1 = 0$;
2. $a_1 = b_1 + 1 = 0$;
3. $a_2 = b_2 = a_1b_1 - b_1^2 + 1 = 0$.

Let $Q_1 = (0, 1, a_2, b_2) \in S$. Then

$$\text{Moje}(Q_1, f_1) = (a_2^2 - \beta_1)u_1\pi + (a_2^2 - \beta_1)T_1 + 2\beta_1S_1$$  \hspace{1cm} (4.221)

where $\beta_1 \equiv 1(\text{mod } \pi)$ and $\text{Moje}(Q_1, f_2) = -\pi$, ie,

$$D(Q_1)_{r_1} = (a_2^2 - 1)\bar{u}_1 \hspace{0.5cm} a_2^2 - 1 \hspace{0.5cm} 2 \hspace{0.5cm} 0 \hspace{0.5cm} 0).$$  \hspace{1cm} (4.222)

So $D(Q_1)$ has rank two. Hence $Q_1$ is a regular point.

Let $Q_2 = (0, -1, a_2, b_2) \in S$. Then

$$\text{Moje}(Q_2, f_1) = (a_2^2 - \beta_1)u_1\pi + (a_2^2 - \beta_1)T_1 + 2\beta_1S_1$$  \hspace{1cm} (4.223)

where $\beta_1 \equiv -1(\text{mod } \pi)$ and $\text{Moje}(Q_2, f_2) = -\pi$, ie,

$$D(Q_2)_{r_1} = ((a_2^2 + 1)\bar{u}_1 \hspace{0.5cm} a^2 + 1 \hspace{0.5cm} -2 \hspace{0.5cm} 0 \hspace{0.5cm} 0).$$  \hspace{1cm} (4.224)

So $\text{rank } D(Q_2) = 2$. Hence $Q_2$ is a regular point.

Let $Q_3 = (a_1, b_1, 0, 0) \in S$ such that $a_1b_1 - b_1^2 + 1 = 0$. Then

$$\text{Moje}(Q_3, f_1) = -\beta_1T_1 + (2\beta_1 - a_1)S_1 \text{ and } \text{Moje}(Q_3, f_2) = -\pi,$$  \hspace{1cm} (4.225)

ie,

$$D(Q_3)_{r_1} = (0 \hspace{0.5cm} -b_1 \hspace{0.5cm} (2b_1 - a_1) \hspace{0.5cm} 0 \hspace{0.5cm} 0).$$  \hspace{1cm} (4.226)
So $Q_3$ is singular if $a_1 = b_1 = a_1 b_1 - b_1^2 + 1 = 0$, which is impossible. Hence $Q_3$ is a regular point and as a result $V_{12}^{01}$ is a regular scheme.

**Chart 2**

$$V_1^1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(1 - Y_2^2 - Y_1 X_1^2 + Y_1 Y_2 X_2^2, Y_1^2 - Y_1^3 X_1^2 - \pi).$$

(4.227)

Let $A = R[X_1, Y_1]/(Y_1^2(1 - Y_1 X_1^2) - \pi)$. We can check that $A$ is a regular ring (by using theorem (2.14)). Notice that

$$V_1^1 = \text{Spec } A[X_2, Y_2]/(1 - Y_2^2 - Y_1 X_1^2 + Y_1 Y_2 X_2^2).$$

(4.228)

Now suppose that $\phi(X_2, Y_2) = 1 - Y_2^2 - Y_1 X_1^2 + Y_1 Y_2 X_2^2$. Then $\frac{\partial \phi}{\partial X_2} = 2Y_1 Y_2 X_2$ and $\frac{\partial \phi}{\partial Y_2} = -2Y_2 + Y_1 X_2^2$. Since $p = \text{char } k \neq 2$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $2Y_1 Y_2 X_2 = -2Y_2 + Y_1 X_2^2 = 0$. So $V_1^1$ is smooth over $A$ everywhere except at the points where $2Y_1 Y_2 X_2 = -2Y_2 + Y_1 X_2^2 = 0$. To have the points on the special fibre of $V_1^1$, we need $V_1^2(1 - Y_1 X_1^2) = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = Y_1^2(1 - Y_1 X_1^2) = 0$, we get $(V_1^1)_{\text{Sing}} \subset S \subset (V_1^1)_\pi$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^1)_\pi$ satisfying the following condition:

$$a_2 = b_1 a_1^2 - 1 = b_2 = 0$$

Let $Q = (a_1, b_1, 0, 0) \in S$ such that $b_1 a_1^2 - 1 = 0$. Then $\text{Moje}(Q, f_1) = -2a_1 \beta_1 T_1 - a_1^2 S_1$ and

$$\text{Moje}(Q, f_2) = -2a_1 \beta_1^3 T_1 + \beta_1 (2 - 3a_1^2 \beta_1) S_1 - \pi,$$

(4.229)

ie,

$$D(Q) = \begin{pmatrix} 0 & -2a_1 b_1 & -a_1^2 & 0 & 0 \\ -1 & -2a_1 b_1^2 & b_1 (2 - 3a_1^2 b_1) & 0 & 0 \end{pmatrix}.$$  

(4.230)

So $Q$ is singular if $-2a_1 b_1 = -a_1^2 = b_1 a_1^2 - 1 = 0$ which is impossible. So $Q$ is a regular point and as a result $V_1^1$ is a regular scheme.
CHAPTER 4. Special class of Arithmetic three-folds

Chart 3

\[ V^0_2 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y^2_1 - Y^2_2 - X_2 Y_1 X^2_1 + X_2 Y_2, X^2_2 Y^2_1 - X^3_2 Y_1 X^2_1 - \pi). \]  
(4.231)

Recall that

\[ V^0_1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y^2_1 - Y^2_2 - X_1 Y_1 + X_1 Y_2 X^2_2, X^2_1 Y^2_1 - X^3_1 Y_1 - \pi) = \]
\[ \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y^2_1 - Y^2_2 - X_1 Y_1 + X_1 Y_2 X^2_2, X^2_1 Y^2_1 - X^3_1 Y_2 X^2_2 - \pi), \]  
(4.232)

(we used \( Y^2_1 - X_1 Y_1 = Y^2_2 - X_1 Y_2 X^2_2 \)). We can write

\[ V^0_2 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y^2_2 - Y^2_1 - X_2 Y_2 + X_2 Y_1 X^2_1, X^2_1 Y^2_1 - X^3_1 Y_1 X^2_1 - \pi). \]  
(4.233)

The ring homomorphism

\[ \alpha : \Gamma(V^0_1, \mathcal{O}_{V^0_1}) \longrightarrow \Gamma(V^0_2, \mathcal{O}_{V^0_2}) \]

\[ X_1 \mapsto X_2 \]
\[ Y_1 \mapsto Y_2 \]
\[ X_2 \mapsto X_1 \]
\[ Y_2 \mapsto Y_1 \]

is an isomorphism and induces \( V^0_2 \cong V^0_1 \).

Chart 4

\[ V^1_2 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/ \\
(Y^2_1 - 1 - Y_2 Y_1 X^2_1 + Y_2 X^2_2, Y^2_2 Y^2_1 - Y^3_2 Y_1 X^2_1 - \pi). \]  
(4.234)

Recall that

\[ V^1_1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(1 - Y^2_2 - Y_1 X^2_1 + Y_1 Y_2 X^2_2, Y^2_1 - Y^3_1 X^2_1 - \pi) = \]
\[ \text{Spec } R[X_1, Y_1, X_2, Y_2]/(1 - Y^2_2 - Y_1 X^2_1 + Y_1 Y_2 X^2_2, Y^2_2 Y^2_1 - Y^3_1 Y_2 X^2_2 - \pi), \]  
(4.235)
(we used $1 - Y_1 X_1^2 = Y_2^2 - Y_1 Y_2 X_2^2$). Note that $V_2^1$ can be written as

$$V_2^1 = \text{Spec} \mathcal{R}[X_1, Y_1, X_2, Y_2]/(1 - Y_1^2 - Y_2 X_2^2 + Y_2 Y_1 X_1^2, Y_2^2 Y_1^2 - Y_2^3 Y_1 X_1^2 - \pi).$$

(4.236)

The ring homomorphism

$$\alpha : \Gamma(V_1^1, \mathcal{O}_{V_1^1}) \longrightarrow \Gamma(V_2^1, \mathcal{O}_{V_2^1})$$

$$X_1 \mapsto X_2$$

$$Y_1 \mapsto Y_2$$

$$X_2 \mapsto X_1$$

$$Y_2 \mapsto Y_1$$

is an isomorphism and induces $V_2^1 \cong V_1^1$. The gluing of the regular schemes which we have had so far gives us the regular scheme $X$. □

### 4.5 One component is $I_4$

In this section our arithmetic three-folds are the products of two arithmetic surfaces such that one of them is of the form $I_4$ and apart from the cases discussed before.

**Lemma 4.11** Let $p = \text{char} k \neq 3$ and $V_1 = V_2 = \text{Spec} \mathcal{R}[x, y]/(x^2 y - x y^2 - \pi)$. Then $X = V_1 \times_{\text{Spec} \mathcal{R}} V_2$ is singular and after one blowing-up we get a regular scheme $\tilde{X}$.

**Proof:** By using corollary (2.11) we can check that $P_0$ is the only singular point of

$$X = \text{Spec} \mathcal{R}[x_1, y_1, x_2, y_2]/(x_1^2 y_1 - x_1 y_1^2 - x_2^2 y_2 + x_2 y_2^2, x_1^2 y_1 - x_1 y_1^2 - \pi).$$

(4.237)

Now we blow-up $P_0$. By using the procedure of Mahtab we get four pieces for the covering of $\tilde{X}$ as follows:
Chapter 1

$V \upsilon^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1 - Y_1^2 - X_2^2 Y_2 + X_2 Y_2^2, X_1^3 Y_1 - X_1^3 Y_1^2 - \pi).$ (4.238)

Now let $A = R[X_1, Y_1]/(X_1^3 Y_1 - X_1^3 Y_1^2 - \pi)$. Considering theorem (2.14), we can check that $A$ is a regular ring. We can write

$V \upsilon^0 = \text{Spec } A[X_2, Y_2]/(Y_1 - Y_1^2 - X_1^2 Y_2 + X_1 Y_2^2).$ (4.239)

Let $\phi(X_2, Y_2) = Y_1 - Y_1^2 - X_1^2 Y_2 + X_1 Y_2^2$. Then

$$\frac{\partial \phi}{\partial X_2} = -2X_2 Y_2 + Y_2^2 \quad \text{and} \quad \frac{\partial \phi}{\partial Y_2} = -X_2^2 + 2X_1 Y_2.$$ (4.240)

Since $p = \text{char } k \neq 3$, the system $\frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = 0$ can be written as $-2X_2 Y_2 + Y_2^2 = -X_2^2 + 2X_1 Y_2 = 0$. So $V_1^0$ is smooth over $A$ everywhere except at the points where $-2X_2 Y_2 + Y_2^2 = -X_2^2 + 2X_1 Y_2 = 0$. To have these points on the special fibre of $V_1^0$, we need $X_1^3(1 - Y_1)Y_1 = 0$. Considering $\phi(X_2, Y_2) = \frac{\partial \phi}{\partial X_2} = \frac{\partial \phi}{\partial Y_2} = X_1^3(1 - Y_1)Y_1 = 0$, we get $(V_1^0)_{\text{Sing}} \subset S \subset (V_1^0)^\pi$, where $S$ contains points of the form $(a_1, b_1, a_2, b_2) \in (V_1^0)_s$ satisfying one of the following conditions:

1. $b_1 = a_2 = b_2 = 0$;
2. $b_1 - 1 = a_2 = b_2 = 0$.

Let $Q_1 = (a_1, 0, 0, 0) \in S$. Then

$\text{Moje}(Q_1, f_1) = v_1 \pi + S_1$ and $\text{Moje}(Q_1, f_2) = \alpha_1^3 v_1 \pi + \alpha_1^3 S_1 - \pi,$ (4.241)

ie,

$D(Q_1) = \begin{pmatrix} \bar{v}_1 & 0 & 1 & 0 & 0 \\ (a_1^3\bar{v}_1 - 1) & 0 & a_1^3 & 0 & 0 \end{pmatrix},$ (4.242)

which has rank 2. So $Q_1$ is a regular point.

Let $Q_2 = (a_1, 1, 0, 0) \in S$. Then $\text{Moje}(Q_2, f_1) = (1 - 2\beta_1)S_1$ where $\beta_1 \equiv 1(\text{mod } \pi)$ and

$\text{Moje}(Q_2, f_2) = 3\alpha_1^4 \beta_1 (1 - \beta_1) T_1 + \alpha_1^3 (1 - 2\beta_1) S_1 - \pi,$ (4.243)
ie,
\[ D(Q_2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & -\alpha_1^3 & 0 & 0 \end{pmatrix}, \]
which has rank two. So \( Q_2 \) is a regular point. This means that \( V_1^0 \) is a regular scheme.

**Chart 2**

\[ V_1^1 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1^2 - X_1 - X_2^2 Y_2 + X_2 Y_2^2, Y_1^3 X_1^2 - Y_1^3 X_1 - \pi) \]
\[ = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 - X_1^2 - Y_2^2 X_2 + Y_2 X_2^2, -Y_1^3 X_1 + Y_1^3 X_1^2 - \pi). \]

The homomorphism
\[ \alpha : \Gamma(V_1^0, \mathcal{O}_{V_1^0}) \to \Gamma(V_1^1, \mathcal{O}_{V_1^1}) \]
\[ X_1 \longmapsto -Y_1 \]
\[ Y_1 \longmapsto X_1 \]
\[ X_2 \longmapsto Y_2 \]
\[ Y_2 \longmapsto X_2 \]

is an isomorphism of rings which induces

\[ V_1^1 \cong V_1^0 \]

**Chart 3**

\[ V_2^0 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/ \]
\[ (X_1^2 Y_1 - X_1 Y_1^2 - Y_2 + Y_2^2, X_2^3 X_1^2 Y_1 - X_2^3 X_1 Y_1^2 - \pi). \]

In \( \Gamma(V_2^0, \mathcal{O}_{V_2^0}) \) from the first equation we get \( Y_1 - Y_1^2 = X_2^3 Y_2 - X_2 Y_2^2 \). So in the other equation of \( V_2^0 \) we can use

\[ X_1^3(Y_1 - Y_1^2) = X_1^3(X_2^3 Y_2 - X_2 Y_2^2) = X_1^3 X_2^3 Y_2 - X_1^3 X_2 Y_2^2, \]
ie,

\[ V^0_i = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X^2Y_2 - X_2Y_2^2 - Y_1 + Y_1^2, X_1^3X^2_2Y_2 - X_1^3X_2Y_2^2 - \pi). \]  

(4.247)

The homomorphism

\[ \alpha : \Gamma(V^0_i, \mathcal{O}_{V^0_i}) \longrightarrow \Gamma(V^0_2, \mathcal{O}_{V^0_2}) \]

\[ X_1 \mapsto X_2 \]
\[ Y_1 \mapsto Y_2 \]
\[ X_2 \mapsto X_1 \]
\[ Y_2 \mapsto Y_1 \]

is an isomorphism of rings and induces

\[ V^0_2 \cong V^0_i. \]

Chart 4

\[ V^1_2 = \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X^2Y_1 - X_1Y_1^2 - X_2^2 + X_2, Y_2^3X^2_1Y_1 - Y_2^3X_1Y_1^2 - \pi). \]  

(4.248)

Considering \( V^0_i \) in the form which was used in chart 3, we can write

\[ V^0_i = \text{Spec } R[X_1, Y_1, X_2, Y_2]/\]
\[ (Y^2_2X_2 - Y_2X_2^2 - Y_1^2 + Y_1, -X_1^3X_2Y_2^2 + X_1^3X_2Y_2^2 - \pi). \]  

(4.249)

The homomorphism

\[ \alpha : \Gamma(V^0_i, \mathcal{O}_{V^0_i}) \longrightarrow \Gamma(V^1_2, \mathcal{O}_{V^1_2}) \]

\[ X_1 \mapsto -Y_2 \]
\[ Y_1 \mapsto X_2 \]
\[ X_2 \mapsto Y_1 \]
is an isomorphism and induces

\[ V^1_v \cong V^0_v. \]

The gluing of \( V^0_1, V^1_1, V^0_2 \) and \( V^1_2 \) gives us the regular scheme \( \tilde{X} \). \( \square \)

### 4.6 One component is \( I_5 \)

In this section we will discuss about some arithmetic three-folds and then about those arithmetic three-folds which are the products of two arithmetic surfaces such that one of them is of the form \( I_5 \).

**Convention 4.12** For non-negative integers \( a, b, m \) and \( n \), let

\[ V_{a,b,m,n} = \text{Spec} R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_2^m y_2^n - \pi). \]  

Then the fibre product of two arithmetic surfaces of the forms \( I_1 \) and \( I_5 \) is \( V_{m,n,m,n} \).

**Remark 4.13** We will try to find a desingularisation for \( V_{a,b,m,n} \) and finally we consider the case \( a = m \) and \( b = n \).

**Lemma 4.14** The three-fold \( V_{1,1,m,n} \) is singular and just after one blowing-up we can resolve its singularity.

**Proof:** Let \( Y = V_{1,1,m,n} = \text{Spec} R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2 y_2, x_2^m y_2^n - \pi) \). By using the procedure of Mojgan we can check that \( P_0 \in Y^{\text{Sing}} \). Now we blow-up \( P_0 \). By using the procedure of Mahtab we get four pieces for the covering of \( \tilde{Y} \) as follows:

**Chart 1**

\[ V^0_1 = \text{Spec} R[x_1, y_1, x_2, y_2]/(y_1 - (\frac{y_2}{x_1})^2, x_1 y_1^n - \pi) = \text{Spec} R[X_1, X_2, Y_2]/(X_1^{m+n} X_2^m Y_2^n - \pi) \]  

(4.251)
which is a regular scheme (by lemma (2.12)). By using similar calculation we can check that \( V'_1, V'_2 \) and \( V'_3 \) are regular schemes. The gluing of \( V'_0, V'_1, V'_2 \) and \( V'_3 \) gives us the regular scheme \( \tilde{Y} \). □

**Lemma 4.15** There exists a desingularisation for \( V_{a,0,m,n} \).

**Proof:** We discuss it in three different possible cases as follows:

**Case 1** If \( a = 0 \), then

\[
V_{0,0,m,n} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - 1, x_2^m y_2^n - \pi). \tag{4.252}
\]

Let \( A = R[x_2, y_2]/(x_2^m y_2^n - \pi) \). Then \( A \) is a regular ring. We can write

\[
V_{0,0,m,n} = \text{Spec } A[x_1, y_1]/(x_1 y_1 - 1). \tag{4.253}
\]

By using the Jacobian criterion we can check that \( V_{0,0,m,n} \) is smooth over \( A \) and so it is regular.

**Case 2** If \( a = 1 \), then

\[
V_{1,0,m,n} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2, x_2^m y_2^n - \pi) = \text{Spec } R[x_1, y_1, y_2]/(x_2^m y_2^n - \pi) \tag{4.254}
\]

which is a regular scheme (by lemma (2.12)).

**Case 3** If \( a > 1 \), then

\[
Y := V_{a,0,m,n} = \text{Spec } R[x_1, y_1, x_2, y_2]/(x_1 y_1 - x_2^a, x_2^m y_2^n - \pi). \tag{4.255}
\]

By using the procedure of Mojgan we can check that

\[
Y^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in Y_n \mid a_1 = b_1 = a_2 = 0 \} := S \tag{4.256}
\]

Now we find the blowing-up of \( Y \) with centre \( S \). By using the procedure of Mahtab we get three pieces for the covering of \( \tilde{Y} \) as follows:
CHAPTER 4. Special class of Arithmetic three-folds

Chart 1

\[ V^0_1 = \text{Spec } R[x_1, y_1, x_2, y_2]/(y_1 x_1 - x_1^a - 2(x_2/x_1)^a, x_1^m(x_2/y_1)^m y_1^n - \pi) = \]
\[ \text{Spec } R[X_1, Y_1, X_2, Y_2]/(Y_1 - X_1^a - 2X_2^a, X_1^m Y_2^m - \pi) = \]
\[ \text{Spec } R[X_1, X_2, Y_2]/(X_1^m X_2^m Y_2^n - \pi) \] (4.257)

which is a regular scheme.

Chart 2

\[ V^1_1 = \text{Spec } R[y_1/x_1, y_1, x_2, y_2]/(y_1 y_1 - y_1^a - 2(y_1/x_1), y_1^m(y_2/y_1)^m y_1^n - \pi) = \]
\[ \text{Spec } R[Y_1, X_2, Y_2]/(Y_1^m X_2^m Y_2^n - \pi) \cong V^0_1. \] (4.258)

Chart 3

\[ V^0_2 = \text{Spec } R[x_1/x_2, y_1, x_2, y_2]/(y_1/x_2)(x_1/x_2 - x_2^a - 2, x_2^m y_2^n - \pi) = \]
\[ \text{Spec } R[X_1, Y_1, X_2, Y_2]/(X_1 Y_1 - X_2^a - 2, X_2^m Y_2^n - \pi) = V_{a-2,0,m,n}. \] (4.259)

If \( a = 2 \), then

\[ V_{a-2,0,m,n} = V_{0,0,m,n} = \]
\[ \text{Spec } R[x_1, y_1, x_1, y_2]/(x_1 y_1 - 1, x_2^m y_2^n - \pi) \] (4.260)

which is regular (we discussed it in the case one).

If \( a = 3 \) then

\[ V_{a-2,0,m,n} = V_{1,0,m,n} = \text{Spec } R[x_1, y_1, y_2]/(x_1^m y_1^m y_2^n - \pi) \] (4.261)

which is a regular scheme.

If \( a > 3 \) then we continue the process of blowings-up of \( V_{a,0,m,n} \). Each blowings-up reduces \( a \) by two. So finally we get \( V_{0,0,m,n} \) or \( V_{1,0,m,n} \) which were discussed in cases 1 and 2. Hence there exists a desingularisation for \( V_{a,0,m,n} \).
**Remark 4.16** For the three-fold $V_{0,0,m,n}$ we can change the variables $x_2$ and $y_2$ with each other and use lemma (4.15) for $V_{0,0,m,n}$.

**Lemma 4.17** There exists a desingularisation for

$$V_{a,1,m,n} = \text{Spec } R[y_1, x_1, x_2, y_2]/(x_1y_1 - x_2y_2, x_2^a y_2^n - \pi).$$

**Proof:** We consider the following cases:

**Case 1** If $a = 0$, then

$$V_{0,1,m,n} = \text{Spec } R[y_1, x_1, x_2, y_2]/(x_1y_1 - y_2, x_2^m y_2^n) =$$

$$\text{Spec } R[x_1, y_1, x_2]/(x_2^m y_1^n - \pi)$$

which is regular.

**Case 2** If $a = 1$, then

$$V_{1,1,m,n} = \text{Spec } R[y_1, x_1, x_2, y_2]/(x_1y_1 - x_2y_2, x_2^m y_2^n - \pi)$$

which was discussed in lemma (4.14).

**Case 3** If $a > 1$, then we can check that

$$V_{a,1,m,n}^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in (V_{a,1,m,n})_x | a_1 = b_1 = a_2 = 0 \} := S.$$

Now we blow-up $S$. By using the procedure of Mahtab we get three pieces for the covering of $V_{a,1,m,n}$ as follows:

**Chart 1**

$$V_1^0 = \text{Spec } R[x_1, y_1, x_2]/(y_1 - x_1^{a_2} (x_2/y_1) y_2, x_1 (x_2/y_1)^m y_2^n - \pi) =$$

$$\text{Spec } R[X_1, X_2, Y_2]/(X_1^m X_2^m Y_2^n - \pi)$$

(4.265)
which is a regular scheme.

Chart 2

\[ V_1^1 \cong V_1^0. \]  \hspace{1cm} (4.266)

Chart 3

\[ V_2^0 = \text{Spec} R[\frac{x_1}{x_2}, \frac{y_1}{x_2}, x_2, y_2]/((\frac{x_1}{x_2})(\frac{y_1}{x_2}) - x_2^{a-2}y_2, x_2 y_2^n - \pi) = \]
\[ \text{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2^{a-2}Y_2, X_2^m Y_2^n - \pi) = V_{a-2,1,m,n}. \]  \hspace{1cm} (4.267)

For \( a = 2 \), \( V_{a-2,1,m,n} = V_{0,1,m,n} \) which is regular (see case one). For \( a = 3 \), \( V_{a-2,1,m,n} = V_{1,1,m,n} \) which was discussed in lemma (4.14). If \( a > 3 \) the process of blowings-up decreases \( a \) by two after each blowing-up. So after some blowings-up and gluing the regular pieces of the covering we get a regular scheme \( V_{a,1,m,n}. \)

Remark 4.18 For \( V_{1,b,m,n} \) we change \( x_1 \) and \( y_1 \) with each other and use lemma (4.17) for \( V_{b,1,m,n}. \)

Theorem 4.19 There exists a desingularisation for

\[ V_{a,b,m,n} = \text{Spec} R[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2^a y_2 x_2^m y_2^n - \pi). \]

Proof: Without loss of generality let \( b \geq a \). If \( a \geq 2 \) then by using the procedure of Mojgan we can check that

\[ V_{a,b,m,n}^{\text{Sing}} \supset \{(a_1, b_1, a_2, b_2) \in (V_{a,b,m,n})_\pi | a_1 = b_1 = a_2 b_2 = 0 \} := S. \]  \hspace{1cm} (4.268)

Now we find the blowing-up of \( V_{a,b,m,n} \) with centre \( S \). By using the procedure of Mahtab we get three pieces for the covering of \( V_{a,b,m,n} \) as follows:

Chart 1

\[ V_1^0 = \text{Spec} R[\frac{y_1}{x_1}, x_2, y_2, \frac{x_2 y_2}{x_1}]/((\frac{y_1}{x_1}) - x_2^{a-2} (\frac{x_2 y_2}{x_1})^{b-a}, x_2 y_2^n - \pi) = \]
\[ \text{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(Y_1 - X_2^{a-2} Z^a Y_2^{b-a}, X_1 Z - X_2 Y_2, X_2^m Y_2^n - \pi) = \]
\[ \text{Spec} R[X_1, X_2, Y_2, Z]/(X_1 Z - X_2 Y_2, X_2^m Y_2^n - \pi) \cong V_{1,1,m,n}. \]  \hspace{1cm} (4.269)
CHAPTER 4. Special class of Arithmetic three-folds

which was discussed in lemma (4.14).

Chart 2

\[ V_1^1 = \text{Spec} R[\frac{x_1}{y_1}, y_1, x_2, y_2, \frac{x_2y_2}{y_1}]/(\frac{x_1}{y_1} - y_1^{a-2}(\frac{x_2y_2}{y_1})^a y_2^{b-a}, x_2^my_2^n - \pi) = \]
\[ \text{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(X_1 - Y_1^{a-2}Z^aY_2^{b-a}, Y_1Z - X_2Y_2, X_2^mY_2^n - \pi) = \]
\[ \text{Spec} R[Y_1, X_2, Y_2, Z]/(ZY_1 - X_2Y_2, X_2^mY_2^n - \pi) \cong V_{1,m,n}. \quad (4.270) \]

Chart 5

Let \( Z = x_2y_2 \). Then

\[ V_2^0 = \text{Spec} R[\frac{x_1}{Z}, \frac{y_1}{Z}, x_2, y_2, Z]/ \]
\[ (\frac{x_1}{Z})(\frac{y_1}{Z}) - Z^{a-2}(\frac{x_2y_2}{Z})^a y_2^{b-a}, Z - x_2y_2, x_2^my_2^n - \pi) = \]
\[ \text{Spec} R[X_1, Y_1, X_2, Y_2, Z]/(X_1Y_1 - Z^{a-2}Y_2^{b-a}, Z - X_2Y_2, X_2^mY_2^n - \pi) = \]
\[ \text{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2^{a-2}Y_2^{b-2}, X_2^mY_2^n - \pi) \]
\[ \cong V_{a-2,b-2,m,n}. \quad (4.271) \]

By continuing the process of blowings-up with centre \( S \) after each blowing-up \( a \)
and \( b \) reduce by two. We continue the process of blowings-up until getting \( V_{a,b,m,n} \)
such that \( \beta = 0 \) or 1. So it is enough to discuss about \( V_{a,0,m,n} \) and \( V_{a,1,m,n} \). But
the first one is discussed in lemma (4.15) and the second one is discussed in lemma (4.17).

Corollary 4.20 Let \( V_1 = \text{Spec} R[x_1, y_1]/(x_1y_1 - \pi) \) and \( V_2 = \text{Spec} R[x_2, y_2]/(x_2^my_2^n - \pi) \). Then \( X = V_1 \times_{\text{Spec} R} V_2 \) is singular and after some blowings-up we
can resolve its singularities.

Proof: In theorem (4.19) put \( a = m \) and \( b = n \).

Open problem 4.21 There exists a desingularisation for \( X = V_1 \times_{\text{Spec} R} V_2 \) where
\( V_1 \) is of the form \( I_i \) and \( V_2 \) is of the form \( I_j \) in the following cases:
CHAPTER 4. Special class of Arithmetic three-folds

(a) \( i = j = 2; \)
(b) \( \begin{array}{c} i = 3 \text{ and } j = 4; \end{array} \)
(c) \( \begin{array}{c} i = 5 \text{ and } j = 2, 3, 4, 5. \end{array} \)
Chapter 5

Desingularisation of the fibre product of minimal regular models of Tate

5.1 Introduction

Any non-singular cubic with a $K$-rational point can be transformed into Weierstrass normal form. The affine equation of a Weierstrass normal form is

$$E = \text{Spec } K[x, y]/(y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6).$$  \hfill (5.1)

For more details about $b_2, b_4, b_8, c_4, c_6, \Delta$ and $j$ see page 36 in [25]. Recall that for a UFD, $R$ with $K = Q(R)$, $p \in R$ irreducible and $k(p) = Q(R/pR)$, the function

$$r_p : \mathbb{P}_n(R) \rightarrow \mathbb{P}_n(k(p))$$

$$[y_0, y_1, \ldots, y_n] \mapsto [\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_n]$$

(where $y_i \in R$ with no common factor), is called reduced map (mod$p$)-function. The representation $[\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_n]$ for $[y_0, y_1, \ldots, y_n]$ is called reduced. Note that the
reduction $\tilde{E}$ of $E$ is
\[
\tilde{E} = \text{Spec} k(p)[x,y]/(y^2 + \tilde{a}_1xy + \tilde{a}_3y - x^3 - \tilde{a}_2x^2 - \tilde{a}_4x - \tilde{a}_6). \quad (5.2)
\]
If $R$ is a discrete valuation ring with $Q(R) = K$, then $k(\xi) = Q(R/(\pi)) = k$, and we write
\[
\tilde{E} = \text{Spec} k[x,y]/(y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6). \quad (5.3)
\]
If for some choice of Weierstrass model (5.1) with $a_i \in R$, $\tilde{E}$ is non-singular, it is said that $E$ over $k$ has good reduction, otherwise $E$ has bad reduction.

### 5.2 Neron model and Tate's algorithm

Let $R$ be a Dedekind domain with $K = Q(R)$ and $E/K$ be an elliptic curve. A Neron model for $E/K$ is a smooth group scheme $E/R$ whose generic fibre is $E/K$ and satisfies the following universal property:

If $H/K$ is a smooth $R$-scheme (ie, $H$ is smooth over $R$) with generic fibre $X/K$ and $\phi_K : X/K \to E/K$ is a rational map defined over $K$, then there exists a unique $R$-morphism $\phi_R : H/R \to E/R$ extending $\phi_K$.

The algorithm of Tate computes the reduction type of an elliptic curve given by Weierstrass equation. He discusses about $\tilde{C}/k$ which is the special fibre of $\tilde{C}$ ( $\tilde{C}$ is a minimal proper regular model of $E$ over $R$), ie, $\tilde{C} = C \times_{\text{Spec } R} \text{Spec } k$ and would be one of the types $I_0, I_n, II, III, IV, I_0^*, I_n^*, IV^*, III^*$ and $II^*$, see page 46 in [25] or page 365 in [23].

In this algorithm Tate starts with the given Weierstrass equation and uses a sequence of blowings-up to produce a minimal regular model for $E$.

**Convention 5.1** We classify the minimal regular models of Tate as follows:

- $J_1 : \tilde{E}$ is of the form $I_n$ ($n > 0$);
- $J_2 : \tilde{E}$ is of the form $II$;
- $J_3 : \tilde{E}$ is of the form $III$;
- $J_4 : \tilde{E}$ is of the form $IV$;
- $J_5 : \tilde{E}$ has one of the forms $I_0^*, I_n^*$ for $n > 0$, $IV^*$, $III^*$ and $II^*$. 
CHAPTER 5. Products of minimal regular models of Tate

Remark 5.2 Let $X$ and $Y$ be noetherian schemes (over $R$) and $f : X \to Y$ be a morphism of finite type, $x \in X$, $y \in Y$ and $y = f(x)$, such that $k(x) \cong k(y)$. By theorem 3, page 249 in [17], $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$ if and only if $f$ is etale in a neighbourhood of $x$. If $X$ and $Y$ are regular schemes, to check that $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$, it is enough to show it for tangent spaces, i.e., to show that

$$m_x/m_x^2 \cong m_y/m_y^2.$$ 

For more details see pages 249-254 in [17] and page 116 in [1].

Definition 5.3 Let $X$ and $Y$ be schemes over $R$, $x \in X$ and $y \in Y$. We say $(X, x)$ is an etale neighbourhood of $(Y, y)$ if there exists a morphism $f : X \to Y$ such that $f(x) = y$ and $f$ is etale at $x$.

The smallest equivalence relation on pairs $(Y, y)$ such that if $(X, x)$ is an etale neighbourhood of $(Y, y)$ then $(X, x) \sim (Y, y)$, is called etale equivalence or sometimes is called local isomorphism for etale topology.

Remark 5.4 With the notation as in definition (5.3) let $(X, x) \sim (Y, y)$. Then $X$ is regular at $x$ if and only if $Y$ is regular at $y$.

In fact by using remark (5.2) we get $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$. But a noetherian local ring is regular if and only if its completion is regular (see [3], page 124, prop. 11.24).

Remark 5.5 In this section we want to show that

(a) Every minimal regular model of Tate at each singular point of its special fibre is etale equivalent to one $V_j$, $j = 1, 2, \ldots, 5$.

(b) The same chain of blowings-up used for resolution of singularities of the fibre products of $V_j$'s (used in chapter 4) does the same on the fibre products of the Tate's minimal regular models.

Theorem 5.6 Let $R$ be a dvr and $p=\text{char} k \neq 2, 3, d$ (where $d = (m, n)$ for the case $i=5$) and $W_i$ be a minimal regular model of Tate of the form $J_i$ for $(E/K)$ and $Q \in (W_i)_\pi$ be a singular point of $(W_i)_\pi$. Then $(W_i, Q)$ is etale equivalent to $(V_i, P)$, for a singular point $P \in (V_i)_\pi$ (where $V_i$ is an affine scheme of the form $I_i$).
Proof: Recall that \( k = \frac{R}{\pi} \) is algebraically closed. We prove the theorem in five possible cases as follows:

CASE ONE \((i = 1)\)

We can choose an affine neighbourhood \( Z_1 = \text{Spec} A_1 \) of the singular point \( Q \in (W_1)_\pi \) such that \( A_1 \) is a finitely generated \( R \)-algebra and \( Z_1 \cap (W_1)_\pi \) is the divisor generated by \( uv = 0 \), where \( u, v \in A_1 \). This means that \( \pi|uv, ie uv = \pi e \) for a unit element \( e \in A_1 \) and we get \( u(ve^{-1}) = \pi \).

Now let \( f: Z_1 \longrightarrow V_1 = \text{Spec} R[x, y]/(xy - \pi) \) be the morphism induced by the ring homomorphism

\[
f^\natural: R[x, y]/(xy - \pi) \longrightarrow A_1
\]

\[
x \mapsto u
\]

\[
y \mapsto \epsilon^{-1} v.
\]

Then \( f \) is of finite type and \( f(Q) = P_0 \). We can check that \( k(Q) = k(P_0) \).

Notice that \( Z_1 \) and \( V_1 \) are two-dimensional regular schemes. So \( T_{Z_1, Q} \cong T_{V_1, P_0} \) (we can also use definition (2.5) to check that tangent spaces are isomorphic), which means that \( m_{P_0}/m_{P_0}^2 \cong m_Q/m_Q^2 \). This implies that \( O_{V_1, P_0} \cong O_{Z_1, Q} \), which in turn implies that \( f \) is etale at \( Q \) (see corollary (4.5), page 116 in [1]). So it is etale in a neighbourhood of \( Q \) (see prop. (4.6), page 116 in [1]). Now we get the following diagram:

\[
\begin{array}{ccc}
Z_1 = \text{Spec} B_1 & \longrightarrow & V_1 \\
\downarrow & & \downarrow \\
& & W_1
\end{array}
\]

where \( Z_1 \longrightarrow W_1 \) is the inclusion. Note that if \( k \) is not algebraically closed, we cannot assume that the components of the special fibres and singular points are defined over \( k \). They are defined over a finite separable extension \( k'/k \) and there exists a discrete valuation ring \( R' \), etale over \( R \) with residue field \( k' \). But by making the base change \( R \longrightarrow R' \) we are reduced to the case \( k = k' \).
CHAPTER 5. Products of minimal regular models of Tate

CASE TWO \((i = 2)\)

In this case the special fibre has a cusp at \(Q \in (W_2)_x\). As it is shown in Tate’s algorithm (see [25] or [23]), the minimal regular model \(W_2\) has the following affine equation

\[
v^2 - u^3 - \pi(a_{2,1}u^2 - a_{1,1}uv + a_{4,1}u - a_{3,1}v + a_{6,1}) = 0. \tag{5.4}
\]

Let \(w = a_{2,1}u^2 - a_{1,1}uv + a_{4,1}u - a_{3,1}v + a_{6,1}\) and \(A = R[u, v]/(v^2 - u^3 - \pi w)\).

Since \(\pi^2|a_6\) we get \(\pi|a_{6,1}\), \(w\) is invertible in an open set \(U_2 = \text{Spec} A[w^{-1}]\) containing the singular point \(Q\). Let \(B = A[w^{-1}, \alpha]/(\alpha^6 - w)\) and \(Z_2 = \text{Spec} B\). Then \(Z_2 \rightarrow \text{Spec} A[w^{-1}] = U_2\) is etale (by Jacobian criterion). So locally for the etale topology \((U_2, Q) \cong (\text{Spec} B, Q')\) where \(Q'\) is the singular point of \((Z_2)_x\). Recall that \(B = R[u, v, w^{-1}, w^{\frac{1}{2}}]/(v^2 - u^3 - \pi w)\) and let \(C = R[x, y]/(y^2 - x^3 - \pi)\). Then the morphism \(f : Z_2 = \text{Spec} B \rightarrow \text{Spec} C = V_2\) induced by the ring homomorphism

\[
f^1 : R[x, y]/(y^2 - x^3 - \pi) \rightarrow R[u, v, w^{-1}, w^{\frac{1}{2}}]/(v^2 - u^3 - \pi)
\]

\[
x \mapsto w^{-\frac{1}{2}}u \\
y \mapsto w^{\frac{1}{2}}v
\]

is etale at \(Q'\) (just compare the tangent spaces), so singularities are locally isomorphic for the etale topology, \(ie\), \((\text{Spec} B, Q') \sim (\text{Spec} C, P_0)\). Now we have

\[
Z_2 = \text{Spec} B \\
V_2 = \text{Spec} C \\
U_2
\]

where both morphisms are etale. So we get \((V_2, P_0) \sim (Z_2, Q')\) and \((Z_2, Q') \sim (U_2, Q)\) which implies that \((V_2, P_0) \sim (U_2, Q)\), \(ie\), \((W_2, Q)\) is etale equivalent to \((V_2, P_0)\).

CASE THREE \((i = 3)\)

We can choose an affine neighborhood \(U_3 = \text{Spec} A_3\) (where \(A_3\) is a finitely generated \(R\)-algebra) of \(Q\) such that \(U_3 \cap (W_3)_x\) is the divisor generated by
$y'(y' - x'^2)$, where $x'$ and $y'$ are reductions (mod $\pi$) of some $u$ and $v$, so $v(v - u^2) \equiv 0$ (mod $\pi$), ie, $v(v - u^2) = \pi \epsilon$, $\epsilon \in A_3^*$ and we get

$$\pi = \frac{1}{\epsilon} v(v - u^2) = \frac{v}{\epsilon^2} \left( \frac{v}{\epsilon^2} - \left(\frac{u}{\epsilon^4}\right)^2 \right).$$

Now let $Z_3 = \text{Spec } A_3[\epsilon^4]$ and $f : Z_3 \longrightarrow V_3 = \text{Spec } R[x, y]/(y(y - x^2) - \pi)$ be the morphism induced by the ring homomorphism

$$f^4 : R[x, y]/(y(y - x^2) - \pi) \longrightarrow A_3[\epsilon^4]$$

(5.5)

Then $f$ is of finite type and $f(Q') = P_0$ where $Q'$ is the singular point of $(Z_3)_x$. Now we can check that $k(Q') = k(P_0)$ and by doing the same conclusion as it was done in case one, $T_{Z_3, Q'} \cong T_{V_3, P_0}$. Since $Z_3$ and $V_3$ are both regular, we get $\hat{O}_{V_3, P_0} \cong \hat{O}_{Z_3, Q'}$. This implies that $f$ is etale at $Q'$ and so it is etale in a neighbourhood of $Q'$. We can check that $Z_3 = \text{Spec } A_3[\epsilon^4] \longrightarrow \text{Spec } A_3 = U_3$ is etale (by Jacobian criterion). So we get

$$Z_3 \xrightarrow{f} V_3 \xleftarrow{g} U_3$$

where both morphisms are etale. Now we get $(V_3, P_0) \sim (Z_3, Q')$ and $(Z_3, Q') \sim (U_3, Q)$, which implies that $(V_3, P_0) \sim (U_3, Q)$, ie, $(W_3, Q)$ is etale equivalent to $(V_3, P_0)$.

**CASE FOUR ($i = 4$)**

Here the special fibre consists of three rational curves passing through a point $Q \in (W_4)_x$. We can choose an affine scheme $U_4 = \text{Spec } A_4$ (where $A_4$ is a finitely generated $R$-algebra) such that $(U_4) \cap (W_4)_x$ is the divisor generated by $x'y'(x' - y')$, where $x'$ and $y'$ are reductions (mod $\pi$) of some $u$ and $v$. So
$uv(u - v) \equiv 0 \pmod{\pi}$, i.e., $uv(u - v) = \pi \epsilon, \epsilon \in A_4^*$ and we get $\pi = \frac{1}{\epsilon}uv(u - v) = \left(\frac{u}{\epsilon^3}\right)\left(\frac{v}{\epsilon^3}\right)\left(\frac{u}{\epsilon^3}\right) - \left(\frac{v}{\epsilon^3}\right)$. Now let $A_4' = A_4[\epsilon^{\frac{1}{3}}], Z_4 = \text{Spec } A_4'$ and

$$f : Z_4 \rightarrow V_4 = \text{Spec } R[x, y]/(xy(x - y) - \pi)$$

be the morphism induced by the ring homomorphism

$$f^\#: R[x, y]/(xy(x - y) - \pi) \rightarrow A_4[\epsilon^{\frac{1}{3}}]$$

$$x \mapsto \frac{u}{\epsilon^{\frac{1}{3}}}$$

$$y \mapsto \frac{v}{\epsilon^{\frac{1}{3}}}.$$ 

Then $f$ is of finite type and $f(Q') = P_0$, where $Q'$ is the singular point of $(Z_4)_\pi$. As we showed in case 3, we can check that $f$ is etale in a neighbourhood of $Q'$. We can also check that $Z_4 = \text{Spec } A_4[\epsilon^{\frac{1}{3}}] \rightarrow \text{Spec } A_4 = U_4$ is etale (by Jacobian criterion). So we get

$$Z_4 \rightarrow V_4 \rightarrow U_4$$

where both morphisms are etale. By using the same discussion as we did in case 3, we can show that $(W_4, Q)$ is etale equivalent to $(V_4, P_0)$.

**CASE FIVE** ($i = 5$)

Here the special fibre is the union of some rational curves with multiplicities which is a normal crossing as a divisor. Let $Q \in (W_5)_\pi$ be the intersection of two of these rational curves. We can choose an affine scheme $U_5 = \text{Spec } A_5$ (where $A_5$ is a finitely generated $R$-algebra) of $Q$ such that $U_5 \cap (W_5)_\pi$ is the divisor generated by $x'^my'^n$, where $x'$ and $y'$ are reductions (mod $\pi$) of some $u$ and $v$. So $u'^m v'^n \equiv 0 \pmod{\pi}$, i.e., $u'^m v'^n = \pi \epsilon, \epsilon \in A_5^*$. Let $d=\gcd(m, n) = am + bn$ (which is a unit, by our assumption). Then we get $u'^m v'^n = (\epsilon^{\frac{1}{d}})^{\frac{d}{2}} \pi = \epsilon \pi$ which implies that

$$\left(\frac{u}{\epsilon^{\frac{1}{d}}} \right)^m \left(\frac{v}{\epsilon^{\frac{1}{d}}} \right)^n = \pi.$$
Let $A'_5 = A_5[\epsilon^\frac{1}{d}], Z_5 = \text{Spec } A'_5$ and $f : Z_5 \rightarrow V_5 = \text{Spec } R[x, y]/(x^m y^n - \pi)$ be the morphism induced by the ring homomorphism

$$f^\sharp : R[x, y]/(x^m y^n - \pi) \rightarrow A_5[\epsilon^\frac{1}{d}]$$

$$x \mapsto \frac{u}{\epsilon^\frac{1}{d}}$$
$$y \mapsto \frac{v}{\epsilon^\frac{1}{d}}.$$  

we can do the same conclusion as we did in cases 3 and 4, to show that $f$ is etale in a neighbourhood of $Q'$ ($Q'$ is the corresponding singular point of $(Z_5)_a$). We can also check that $Z_5 = \text{Spec } A_5[\epsilon^\frac{1}{d}] \rightarrow \text{Spec } A_5 = U_5$ is is etale (by Jacobian criterion). So we get

$$\begin{tikzcd}
Z_5 \ar[dr] \ar[dd] & \\
V_5 & U_5 \ar[dl]
\end{tikzcd}$$

where both morphisms are etale. As we discussed in cases 3 and 4, this diagram shows that $(W_5, Q)$ is etale equivalent to $(V_5, P_0)$. □

**Remark 5.7** The above discussion tells us that each minimal regular model of Tate at each singular point of its special fibre is etale equivalent to one $V_j$, $j = 1, 2, ..., 5$. In the following theorem $W_\alpha$ shows a minimal regular model of Tate of the form $J_\alpha$, $V_\alpha$ is an arithmetic surface of the form $I_\alpha$ and $Z_\alpha$ is as it was used in theorem (5.6). We also use $W = W_\alpha \times_{\text{Spec } R} W_\beta$, $V = V_\alpha \times_{\text{Spec } R} V_\beta$ and $Z = Z_\alpha \times_{\text{Spec } R} Z_\beta$, for $\alpha, \beta \in \{1, 2, ..., 5\}$.

Recall that so far $\overline{X}_n$ was used for the $n$-th blowing-up of $X$ with the centre $X^{\text{Sing}}$ or a subscheme of $X^{\text{Sing}}$. Here we use $X^{(n)} = \overline{X}_n, n \geq 0$ (where $X^{(0)} = X$) and $\phi_n : X^{(n)} \rightarrow X^{(n-1)}$ to show that $X_n$ is the blowing-up of $X^{(n-1)}$ with the centre $X^{\text{Sing}}_{(n-1)}$.

**Theorem 5.8** Let $p = \text{char } k \neq 2, 3, d$ (d the same as in theorem (5.6)). If we are not in the cases $W_2 \times_{\text{Spec } R} W_2, W_3 \times_{\text{Spec } R} W_4$ and $W_5 \times_{\text{Spec } R} W_j$ (for $j = 2, 3, 4, 5$), then there exists a desingularisation for $W = W_\alpha \times_{\text{Spec } R} W_\beta$, $(1 \leq \alpha, \beta \leq 5)$. 


Proof: Let $W = W(0) = W_\alpha \times \text{Spec}_R W_\beta$ and consider $\psi_n : W(n) \to W_{(n-1)}$ as the blowing-up of $W_{(n-1)}$ with centre $W^{\text{Sing}}_{(n-1)}$. Since the product of etale morphisms is etale, by using the results of the theorem (5.6), for each $Q \in W^{\text{Sing}}$ we get the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & V \\
\downarrow{h} & & \downarrow{g} \\
W & = & W
\end{array}
\]

where $f : Z \to V$ and $g : Z \to W$ are etale and $Q \in g(Z)$. We define $Z(n)$, $h_n$ and also $V(n)$, $\phi_n$ in the similar way that we defined $W(n)$ and $\psi_n$. Inductively we find $f(n)$ and $g(n)$ by base change and surely they are etale (note that the blow-ups commute with flat basechange, so the squares are cartesian and we can deduce that the top arrows are etale from the knowledge that the bottom ones are). For $n = 1$ we get the following diagram:

\[
\begin{array}{ccc}
Z_{(1)} & \xrightarrow{f_{(1)}} & V_{(1)} \\
\downarrow{h_1} & & \downarrow{g_{(1)}} \\
Z & \xrightarrow{f} & W \\
\downarrow{\psi_1} & & \downarrow{g} \\
W_{(1)} & = & W
\end{array}
\]

Since $f_{(1)}$ and $g_{(1)}$ are etale, we get $f_{(1)}^{-1}(V_{(1)}^{\text{Sing}}) = Z_{(1)}^{\text{Sing}} = g_{(1)}^{-1}(W_{(1)}^{\text{Sing}})$. If $V_{(1)}$ is regular we get $V_{(1)}^{\text{Sing}} = \phi$, so $g_{(1)}^{-1}(W_{(1)}^{\text{Sing}}) = \phi$, which means that $W_{(1)}^{\text{Sing}} = \phi$, i.e., $W_{(1)}$ is regular at points $P$ such that $\psi_1(P) \in g(Z)$.

Now we do the same for other $Z$'s so that $g(Z)$ covers $W$. If in all of such cases $W_{(1)}^{\text{Sing}} = \phi$, we are done and the gluing of these regular schemes is the answer. Otherwise we continue the process of blowings-up and inductively we get the following diagram:
Suppose that $V_{(n)}$ is not regular. Since products and base extensions preserve the etale property of morphisms, so $f_{(n)}$ and $g_{(n)}$ are etale and the above diagram is commutative. Hence

$$f_{(n)}^{-1}(V_{(n)}^{\text{Sing}}) = Z_{(n)}^{\text{Sing}} = g_{(n)}^{-1}(W_{(n)}^{\text{Sing}})$$

By doing similar discussion with what we did about $W_{(1)}$ we can conclude that if $V_{(n)}$ is regular, then $W_{(n)}$ is regular at points $P$ such that $\psi_1 \circ \psi_2 \circ \ldots \circ \psi_{n-1} \circ \psi_n(P) \in g(Z)$.

If $V_{(n)}$ is not regular we continue the process of blowings-up. Recall that in chapter 4 we have proved the existence of a desingularisation for $V$ (in the involving cases). So there exists $k_0 \in \mathbb{N}$ such that for $n \geq k_0$, $V_{(n)}^{\text{Sing}} = \phi$. By using the same discussion as above we conclude that for $n \geq k_0$, $W_{(n)}$ is regular at any point whose image in $W$ lies in $g(Z)$. Now we do it for other $Z$'s so that $g(Z)$ covers $W$. Hence for sufficiently large $n$, $W_{(n)}$ is regular. This means that there exists a desingularisation for $W$. $\square$
Bibliography


BIBLIOGRAPHY


BIBLIOGRAPHY

