The non-commutative standard model

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by

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A thesis presented for the degree
of Doctor of Philosophy at the
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September 1996

ABSTRACT

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In this work aspects of the classical Connes-Lott non-commutative standard model are examined. In particular the relationship between the chiral structure of the standard model and the condition of Poincaré Duality is investigated. Then the natural prediction of an additional force in the non-commutative standard model is explained and the consequences calculated. Finally the attempts at grand unification within the non-commutative framework are reviewed and extended.

DECLARATION

The work presented in this thesis was carried out in the Department of Mathematical Sciences at the University of Durham between October 1993 and September 1996. This material has not been submitted previously for any degree in this or any other university.

No claim of originality is made for the second and third chapters; the work in Chapters 4, 5 and 6 is claimed as original, except where the authors have been specifically acknowledged in the text. Part of Chapter 4 has been published as [43].

ACKNOWLEDGEMENT

I would like to thank my supervisor Dr. David Fairlie for all his help over the last three years. I would also like to thank everyone at Marseille University for making my visit there so enjoyable and profitable –in particular Thomas Schücker, Bruno Iochum, Daniel Kastler, Robert Coquereaux and Lionel Carminati. I am also very grateful to Peter Bongaarts and Jacek Brodzki for some very useful discussions and for their support. Finally I'd like to thank my family Cath, Nor, Lem and Phil.

To Cath ♡♡♡

Contents

Chapter 1

Introduction

Non-commutative geometry can be loosely described as the study of spaces whose algebra of functions is non-commutative. This has been an area of interest to both mathematicians (see for example Gelfand [68]) and physicists (see for example Dirac [69]) throughout this century. However compared with 'classical' geometry (the geometry of spaces whose algebra of functions is commuting) non-commutative geometry was very under developed. Recently this has begun to change with the introduction by Connes [1][2] and Dubois-Violette [32] of (independent) generalised de Rham differential algebras on the "non-commutative manifold"[1]. Since then, non-commutative geometry has been enhanced and refined until, in its present state [7][9] it is highly developed and contains many of the tools of classical geometry.

Non-commutative geometry was first applied to physics in 1990 by Dubois-Violette et al [33]. The use of non-commutative geometry in physics and in particular for constructing gauge theories [2] has become something of a growth industry in the last five years. Work in this field can be roughly split into three main groups. The first one, based around the southern Paris group [33]-[35][41] uses the differential algebra first constructed in [32]. The second, the Marseille-Mainz group [36]-[40] works within a framework first introduced by Coquereaux et al [36]. The third grouping [10]-[26] takes as its starting point the Connes-Lott model [5] (later refined [8],[9]). This thesis falls into the third category –it is an exploration of the Connes-Lott standard model.

The aim of this thesis is to help elucidate and develop the Connes-Lott standard model, to answer the questions

- What are non-commutative gauge theories?
- How do the intricacies of the standard model follow from non-commutative geometry?

[1] terms such as this will be explained in the following chapter

[2] non-commutative geometry is also being used to describe phenomena in solid state physics

- What are the strong and weak points of the Connes-Lott standard model and can any of the weak points be improved upon?

Throughout this thesis, unless specified otherwise 'non-commutative geometry' refers to Connes formulation of non-commutative geometry [7][8]. A list of definitions and conventions is provided at the end of this thesis in Appendix A.

Chapter 2

The Basic Mathematical Concepts

2.1 Summary of this Chapter

In this thesis non-commutative geometry is used essentially as a tool for building Yang-Mills models and the mathematics is largely taken on trust. As explained in the introduction non-commutative geometry can be viewed as a rewriting of classical geometry so that a much larger class of manifolds can be described. This chapter outlines the non-commutative formulation of some of the tools you would expect in a mathematical system calling itself a geometry. The tools that are dealt with are those necessary for applying non-commutative geometry to physics –namely a notion of manifold, metric, differential and integral calculus. A far more detailed explanation can be found in [7] or [23].

Classical differential geometry can be reformulated in algebraic rather than 'spatial terms', switching the emphasis from the local properties of the compact manifold to a (unital, involutive) algebra $\mathcal{A}$. Gelfand showed [62] (see section 2.2 below) that a manifold X can be dealt with algebraically by considering a commutative algebra $\mathcal{A}$ such that X is in one to one correspondence with the spectrum of $\mathcal{A}$. The generalisation of this concept to a non-commutative algebra is the starting point of non-commutative geometry.

Section 2.3 deals with Connes' 'quantised calculus' [7], the calculus of non-commutative geometry. The quantised calculus is a new, purely algebraic calculus that replaces the usual classical differential and integral calculus. The basic information needed for this is a pair $(\mathcal{H}, \mathrm{F})$, $\mathcal{H}$ a Hilbert space and F an operator on $\mathcal{H}$.

To use this quantised calculus on a space (described at this level by an algebra $\mathcal{A}$) it is necessary to find a pair $(\mathcal{H}, \mathrm{F})$ and a representation of $\mathcal{A}$ on $\mathcal{H}$ that satisfy certain criteria. It transpires that these criteria are exactly the definition of a Fredholm module over $\mathcal{A}$ (section 2.4). Sections 2.3 and 2.4 are very brief and are only included to give a feeling for how the notion of a Fredholm module arises in non-commutative geometry. They are by no means rigorous or complete.

In section 2.5 a metric is then defined on this space. This is done most naturally using a K cycle –a Fredholm module with additional structure. Section 2.6 describes a differential algebra on the non-commutative manifold, the generalisation of the differential algebra of de Rham forms.

No information is lost in this reformulation and standard Riemannian differential geometry can be recovered by taking the Dirac K-cycle ($\mathcal{A}$,$\mathcal{H}$,D) where $\mathcal{A} = C^\infty(M)$ the algebra of infinitely differentiable functions on a Riemannian manifold M, $\mathcal{H} = L^2(S)$ the Hilbert space of square integrable spinors and D is the ordinary Dirac operator. However Connes' approach is much more powerful than this because it can be extended to a much wider class of spaces (that is not just Riemmanian) simply by taking an algebra $\mathcal{A}$ other than $\mathcal{A} = C^\infty(M)$ or D other than the Dirac operator. If $\mathcal{A}$ is taken to be a non-commutative algebra then a 'non-commutative geometry' will be derived. In section 2.7 two manifolds are described using non-commutative geometry to illustrate some of the points of this chapter. The first example is the flat Euclidean manifold and the second is a discrete two point space.

2.2 The Non-Commutative Manifold

Classically, given a compact Hausdorff space X, a commutative C^* algebra $\mathcal{A}$ can be associated to it. This algebra is $\mathcal{A}$ =C(X), the algebra of complex valued functions on X with the involution given by complex conjugation in $\mathbb{C}$ ie

$$a^*(x) := \overline{a(x)} \quad a \in \mathcal{A} \quad x \in X$$

and the norm given by the supremum norm

$$\|a\| := \sup_{x \in X} |a(x)| \quad a \in \mathcal{A}\ .$$

This algebra contains all the information necessary to reconstruct the space X.

For a general Banach Algebra $\mathcal{A}$ (the algebra $\mathcal{A}$ =C(X) discussed above being a C* algebra –a special case of a Banach algebra) the Gelfand Transform $\wedge$ is a map between $\mathcal{A}$ and C(Sp($\mathcal{A}$)) [62]. Where Sp($\mathcal{A}$) is the spectrum or character space of $\mathcal{A}$ –the set of complex homomorphisms on $\mathcal{A}$

$$Sp(\mathcal{A}) = \{\chi \mid \chi : \mathcal{A} \xrightarrow{hom} \mathbb{C}\}.$$

The Gelfand transform is given by

$$\begin{array}{rcl} \wedge : \mathcal{A} & \longrightarrow & C(Sp(\mathcal{A})) \\ a & \mapsto & \hat{a} \end{array}$$

where

$$\begin{array}{rcll} \hat{a} : Sp(\mathcal{A}\,) & \longrightarrow & \mathbb{C} & \\ \chi & \mapsto & \hat{a}(\chi) = \chi(a) & a \in \mathcal{A}\,. \end{array}$$

It can be shown that in general the Gelfand transform is a surjective homomorphism, if $\mathcal{A}$ is semi-simple then it is an isomorphism and if $\mathcal{A}$ is a B* algebra then it is a *-isomorphism (ie the isomorphism respects the involution).

Returning to the specific case of interest $\mathcal{A}$ =C(X), $\mathcal{A}$ is a semi-simple C* algebra so the Gelfand transform is a *-isomorphism between $\mathcal{A}$ =C(X) and C(Sp($\mathcal{A}$)). In fact there is a one-to-one correspondence between X and Sp($\mathcal{A}$):

$$\begin{array}{rcl} X & \longleftrightarrow & Sp(\mathcal{A}\,) \\ x & \longleftrightarrow & \chi_x \end{array}$$

where the homomorphism χ_x is defined on $\mathcal{A}$ as

$$\begin{array}{rcl} \chi_x : \mathcal{A}\, = C(X) & \longrightarrow & \mathbb{C} \\ a & \mapsto & a(x). \end{array}$$

It can be shown that all elements of Sp($\mathcal{A}$) are of this form, so the one-to-one correspondence holds. The situation is summarised in the following diagram:

$$
\begin{tikzcd}
\begin{array}{c}\text{compact space}\\ X \leftrightarrow Sp(\mathcal{A})\end{array} \arrow[r, bend left] & \begin{array}{c}\text{commutative } C^* \text{ algebra}\\ \mathcal{A} = C(X) \cong C(Sp(\mathcal{A}))\end{array} \arrow[l, bend left]
\end{tikzcd}
$$

Two commutative C^* algebras are isomorphic if and only if their spectra are homeomorphic. So, it can be seen that no information is lost if the algebra $\mathcal{A}$ rather than the space Sp($\mathcal{A}$) is worked with.

Non-commutative geometry rests on the heuristic generalisation of the above argument to a non-commutative algebra. A non-commutative manifold is <u>defined</u> to be the 'manifold' associated to a non-commutative C* algebra in exactly the same way as a classical manifold is associated to a commutative C* algebra.

2.3 Quantised Calculus

As explained in the introduction to this chapter Connes' quantised or spectral calculus is an algebraic reformulation of the usual differential and integral calculus. It is the next logical step down the road to a completely algebraic geometry after the description of a manifold in terms of a C^* algebra as described in section 2.2. The quantised calculus is based on a pair ($\mathcal{H}$,F), where $\mathcal{H}$ is a Hilbert space and F is an operator on $\mathcal{H}$ such that F=F* and $F^2 = 1$. Connes [7] then gives the following 'dictionary' -translating the familiar concepts of classical calculus into the corresponding concepts in quantum calculus.

CLASSICAL	QUANTUM
topological space	C* algebra
complex variable	operator in $\mathcal{H}$
real variable	self-adjoint operator in $\mathcal{H}$
differential of variable	df=[F,f]
infinitesimal	compact operator in $\mathcal{H}$
integral	Dixmier Trace

The first entry in the above table has already been explained in the preceeding section. The next three entries go towards explaining why Connes uses the name 'quantised' calculus. A quantum mechanical description associates an operator on $\mathcal{H}$ to a variable and in particular associates a self-adjoint operator to an observable (real variable). Similarly the substitution of df=[F,f] for the classical definition of a differential is considered by Connes [7] to be analagous to the quantisation process in which the Poisson bracket {f,g} of classical mechanics is replaced by the commutator [f,g]. This explains at least part of the origin of the name quantised calculus. Note that since $[F, fg] = [F, f]g + f[F, g]$ the Leibniz rule holds for this new differential. As summarised in the table the role of infinitesimals is played by compact operators. An infinitesimal is said to be of order α if the eigenvalues μ_n of the corresponding compact operator satisfy $\mu_n = O(n^{-\alpha})$ as $n \to \infty$ (the μ_n are ordered by decreasing size). In Connes' scheme the role of the integral is taken by the Dixmier trace. The Dixmier trace is defined on all operators $T \in \mathcal{L}^{1\infty}(\mathcal{H})$ in terms of a generalised limiting process ω.

$$Tr_\omega(|T|) = lim_\omega \frac{1}{logN} \sum_{i=0}^{N} \mu_i(T)$$

where T is a positive element of $\mathcal{L}^{1\infty}(\mathcal{H})$, $\mu_n(T)$ are the eigenvalues of T

$\mu_0 \geq \mu_1 \geq \cdots$ and $\mathcal{L}^{1\infty}(\mathcal{H})$ is the ideal of order one infinitesimals:

$$\mathcal{L}^{1\infty}(\mathcal{H}) = \{T; \text{T compact operator on } \mathcal{H}\ , \mu_n(T) = O(n^{-1})\}.$$

$\mathcal{L}^{1\infty}$ is sometimes referred to as the Dixmier ideal in the literature. A proper explanation of the Dixmier trace is beyond the scope of this thesis but can be found in [7][23] and the references within. The important equalities needed for calculating non-commutative integrals are quoted below without proof.

The Dixmier trace has the following properties (for T$\geq 0, T \in \mathcal{L}^{1\infty}(\mathcal{H})$) that would be expected of an integral

1. Positivity: $Tr_\omega(T) \geq 0$
2. Finiteness: $Tr_\omega(T) \leq \infty$
3. Unitary Invariance: $Tr_\omega(UTU^*) = Tr_\omega(T)$ for every unitary U
4. Linearity: $Tr_\omega(S+T) = Tr_\omega(S) + Tr_\omega(T)$ for S$\geq 0, S \in \mathcal{L}^{1+}(\mathcal{H})$
5. The Dixmier trace is zero on infinitesimals of order greater than 1

Clearly in general the value of the Dixmier trace will depend on the limiting process ω. However, there is a certain class of operators known as measurable operators for which it can be shown [3] that their Dixmier trace is independent of ω. In all the applications of non-commutative geometry to physics that will be dealt with in the following chapters T will be measurable.

In fact, the only non-commutative manifolds that will be dealt with in this thesis are the Euclidean four-space, discrete point spaces and the product of these two manifolds [1]. The Dixmier trace on such manifolds reduces to an extremely simple

[1] A description of the K cycles associated to these non-commutative manifolds can be found in section 2.7. An explanation of how K cycles relate to non-commutative manifolds can be found in section 2.5.

form. For the Euclidean four-space (described by the Dirac K cycle) the Dixmier trace reduces to the usual integral over Euclidean space

$$Tr_\omega(T|\partial\!\!\!/|^{-4}) = \frac{1}{32\pi^2}\int Tr_\gamma(T)d^4x \tag{2.1}$$

where Tr_γ denotes the trace over the Clifford algebra. For a finite (zero dimensional) K cycle associated to a discrete space the Dixmier trace reduces to the ordinary trace

$$Tr_\omega(T) = Tr(T).$$

For the product of two K cycles $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ of dimension p_1 and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ of dimension p_2 the Dixmier trace can be written as a product of Dixmier traces:

$$Tr_\omega[(T_1 \otimes T_2)|D|^{-(p_1+p_2)}] \propto Tr_\omega(T_1|D_1|^{-p_1})Tr_\omega(T_2|D_2|^{-p_2}) \tag{2.2}$$

where $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$.

2.4 Fredholm Modules

To apply the above quantised calculus given by ($\mathcal{H}$,F) to a (possibly non-commutative) manifold X it is necessary to use a Fredholm module over the algebra $\mathcal{A}$ associated to X (as outlined in section 2.2).

Definition Fredholm Module

A Fredholm Module ($\mathcal{H}$,F) over an algebra $\mathcal{A}$ consists of

1. $\mathcal{H}$ a Hilbert space
2. F a self-adjoint operator on $\mathcal{H}$ with $F^2 = 1$
3. $\mathcal{A}$ a unitary, involutive algebra
4. λ an involutive, injective representation of $\mathcal{A}$ into B($\mathcal{H}$) (the bounded operators on $\mathcal{H}$) such that da is an infinitesimal for all $a \in \mathcal{A}$, that is, such that the operator $[F, \lambda(a)]$ is compact for all $a \in \mathcal{A}$.

2.5 Metric Space

Next Connes defined [6] a metric on this space this is done using a K-cycle or spectral triple ($\mathcal{A}$,$\mathcal{H}$,D), a Fredholm module with extra structure.

<u>Definition</u> Spectral Triple

A Spectral Triple or K cycle ($\mathcal{A}$,$\mathcal{H}$,D) consists of

1. $\mathcal{A}$ a unitary, involutive algebra
2. $\mathcal{H}$ a Hilbert space
3. D a self adjoint operator on $\mathcal{H}$, $(D^2+1)^{-1}$ compact
4. λ a faithful, involutive representation of $\mathcal{A}$ into B($\mathcal{H}$) such that $[D, \lambda(a)]$ is bounded for all a $\in \mathcal{A}$.

<u>Definition</u> Graded Spectral Triple

A graded spectral triple is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with grading Γ written $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ such that

1. Γ is a grading operator on the Hilbert space, $\Gamma^2 = 1$
2. $\mathcal{H}$ is $\mathbb{Z}_2$ graded. That is $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, $\mathcal{H}^0$ and $\mathcal{H}^1$ closed, mutually orthogonal subspaces. $\Gamma\mathcal{H}^0 = \mathcal{H}^0$, $\Gamma\mathcal{H}^1 = -\mathcal{H}^1$.
3. $\Pi(a)$ is even for all $a \in \mathcal{A}$ $\quad \Gamma\Pi(a) - \Pi(a)\Gamma = 0$
4. D is odd $\quad \Gamma D + D\Gamma = 0$

Given a K cycle a metric can then be defined on the manifold (corresponding to the algebra $\mathcal{A}$). The geodesic distance between two 'points' χ and ξ; $\chi, \xi \in Sp(\mathcal{A})$ is given by

$$d(\chi, \xi) = sup\{\mid \chi(a) - \xi(a) \mid : a \in \mathcal{A}; \| [D, a] \| \leq 1\} \tag{2.3}$$

where the norm $\|\cdot\|$ is the Hilbert space norm. Note that unlike the Riemmanian geodesic this definition does not rely on the notion of a path between the two points. That is the space does not have to be arcwise connected for a meaningful and consistent definition of distance.

2.6 Graded Differential Algebra

A classical manifold has a differential algebra –the algebra of de Rham forms associated to it. In this section the generalisation of the de Rham algebra to a non-commutative manifold is discussed. The properties that are required of this generalised algebra are that

1. it is $\mathbb{Z}$ graded

$$\Omega(\mathcal{A}) = \bigoplus_{p=0}^{\infty} \Omega^p(\mathcal{A})$$

$$\text{if } \phi \in \Omega^p(\mathcal{A}),\ \omega \in \Omega^q(\mathcal{A}) \quad \text{then } \phi\omega \in \Omega^{p+q}(\mathcal{A})$$

2. there exists a linear map d

$$d : \Omega^p(\mathcal{A}) \longrightarrow \Omega^{p+1}(\mathcal{A})$$

such that $d^2 = 0$ and d obeys the graded Leibniz rule

$$d(\phi\omega) = (d\phi)\omega + (-1)^p \phi(d\omega) \quad \phi \in \Omega^p(\mathcal{A}), \omega \in \Omega^q(\mathcal{A})$$

3. $\Omega^0(\mathcal{A}) = \mathcal{A}$.

For every algebra $\mathcal{A}$ there exists at least one such system of differential forms the so called universal algebra $\Omega_u\mathcal{A}$. The universality of $\Omega_u\mathcal{A}$ means that there exists a unique degree preserving homomorphism ρ between $\Omega_u\mathcal{A}$ and any other differential algebra admitted by $\mathcal{A}$.

$$\rho : \Omega_u\mathcal{A} \longrightarrow \Omega\mathcal{A}$$

such that $\rho d_u = d\rho$ where d_u is the exterior derivative associated to the universal algebra $\Omega_u\mathcal{A}$ and d is the exterior derivative associated to $\Omega\mathcal{A}$. This means that all the differential algebras associated to $\mathcal{A}$ can be obtained as quotients of $\Omega_u\mathcal{A}$.

The Universal Differential Algebra

The universal differential algebra $(\Omega_u\mathcal{A}, d_u)$ can be written in the following way: The space of p forms $\Omega^p_u\mathcal{A}$ is generated by symbols $a, d_u a \quad a \in \mathcal{A}$ with

$$\begin{aligned} d_u(ab) &= (d_u a)b + a(d_u b) \quad a, b \in \mathcal{A} \\ d_u 1 &= 0 \\ d_u^2 &= 0. \end{aligned}$$

$\Omega^p_u\mathcal{A}$ consists of a finite sum of terms of the form $a_0 d_u a_1 ... d_u a_p$

$$\Omega^p_u\mathcal{A} = \{\sum_j a_0^j d_u a_1^j ... d_u a_p^j \mid a_0, ... a_p \in \mathcal{A}\}.$$

It is easily checked that d_u obeys the graded Leibniz rule and that $\Omega^0_u\mathcal{A} = \mathcal{A}$. The involution $*$ on $\mathcal{A}$ is extended to $\Omega_u\mathcal{A}$ by putting $(d_u a)^* := d_u(a^*) := d_u a^*$. Given this identification it follows simply that $(d_u\phi)^* = (-1)^n d_u(\phi^*)$ for $\phi \in \Omega^n_u\mathcal{A}$.

The universal differential algebra is represented on the Hilbert space by a homomorphism Π obtained by extending the representation λ of $\mathcal{A}$ on $\mathcal{H}$.

$$\begin{aligned} \Pi : \Omega_u\mathcal{A} &\longrightarrow B(\mathcal{H}) \\ a_0 d_u a_1 ... d_u a_p &\mapsto (-i)^p \lambda(a_0)[D, \lambda(a_1)]...[D, \lambda(a_p)]. \end{aligned}$$

However the representation Π is ambiguous. There exist forms $\phi_u \in \Omega_u\mathcal{A}$ such that $\Pi(\phi_u) = 0$ but $\Pi(d_u\phi_u)$ is not necessarily zero, such forms need to be quotiented out. Such a differential algebra can be constructed by quotienting out the graded differential ideal J

$$J = \bigoplus_k J^k$$

$$J^k = (ker\Pi)^k + d_u(ker\Pi)^{k-1}.$$

In doing this we are moving from the space of 'formal differential forms' to one of genuine differential forms so it is the elements of the differential algebra $\Omega\mathcal{A} = \Omega_u\mathcal{A}/J$ that are of physical interest, ie that are the genuine connections and curvatures.

Consider obtaining the space of one forms $\Omega^1\mathcal{A}$ from the space of universal one forms $\Omega^1_u\mathcal{A}$

$$\Omega^1_u\mathcal{A} = \{\sum_j a_0^j d_u a_1^j \mid a_i^j \in \mathcal{A}\}$$

$$\Omega^1\mathcal{A} = \frac{\Omega^1_u\mathcal{A}}{J^1}$$

$$\Omega^1\mathcal{A} \cong \Pi(\Omega^1\mathcal{A}) = \frac{\Pi(\Omega^1_u\mathcal{A})}{\Pi(J^1)}$$

$J^1 = (ker\Pi)^1$ so $\Pi(J^1) = \{0\}$, therefore $\Pi(\Omega^1\mathcal{A}) = \Pi(\Omega^1_u\mathcal{A})$.

Similarly for the space of two forms

$$\Omega^1_u\mathcal{A} = \{\sum_j a_0^j d_u a_1^j d_u a_2^j \mid a_i^j \in \mathcal{A}\}$$

$$\Omega^2\mathcal{A} = \frac{\Omega^2_u\mathcal{A}}{J^2}$$

$$\Omega^2\mathcal{A} \cong \Pi(\Omega^2\mathcal{A}) = \frac{\Pi(\Omega^2_u\mathcal{A})}{\Pi(J^2)}$$

$$\begin{aligned}\Pi(J^2) &= \Pi((ker\Pi)^2 + d_u(ker\Pi)^1) \\ &= \Pi(d_u(ker\Pi)^1).\end{aligned}$$

So

$$\Pi(\Omega^2\mathcal{A}) = \frac{\Pi(\Omega^2_u\mathcal{A})}{\Pi((d_u(ker\Pi)^1)}$$

and in general

$$\Omega^k\mathcal{A} \cong \Pi(\Omega^k\mathcal{A}) = \frac{\Pi(\Omega^k_u\mathcal{A})}{\Pi((d_u(ker\Pi)^{k-1})}. \tag{2.4}$$

The forms of the differential algebra $\Pi(\Omega^*\mathcal{A})$ constructed by quotienting are equivalence classes of operators on $\mathcal{H}$. A method of selecting a unique representative from any given equivalence class is needed so that a form is a unique operator rather than a class of operators. This is done via an inner product $(\cdot,\cdot)$ on $\mathrm{B}(\mathcal{H})$

$$(x, y) := Tr_\omega(x^\dagger y).$$

Once this inner product has been defined $\Pi(\Omega_u\mathcal{A})$ can be written as the direct sum of two orthogonal vector spaces J and V, where J is the differential graded ideal defined above and

$$V := \{v \in \Pi(\Omega_u\mathcal{A}) \mid (v,j) = 0 \ \ \forall j \in J\}.$$

Let P be the orthogonal projection from $\Pi(\Omega_u\mathcal{A})$ onto V

$$\begin{aligned} P: \ \Pi(\Omega_u\mathcal{A}) &\longrightarrow V \\ v + j &\mapsto v. \end{aligned}$$

Using P a map $\tilde{\mathrm{P}}$ can be constructed

$$\begin{aligned} \tilde{P}: \ \frac{\Pi(\Omega_u\mathcal{A})}{\Pi(J)} &\longrightarrow V \\ [v] &\mapsto P(v). \end{aligned} \tag{2.5}$$

It can be shown that $\tilde{\mathrm{P}}$ is an isomorphism so the algebras $\Pi(\Omega_u\mathcal{A})/\Pi(J)$ and V can be identified and P(v) can be selected as the unique representative of the equivalence class [v].

As mentioned in the introduction, the non-commutative generalisation of the de Rham algebra is not unique, Dubois-Violette [32] has constructed a different generalisation based on Der $\mathcal{A}$ the space of derivatives of $\mathcal{A}$.

2.7 Examples

1) Euclidean Manifold

For non-commutative geometry to be consistent with classical geometry it would be expected that the non-commutative description of a (compact, flat) Euclidean manifold yields the same result as the classical description (though of course via different methods), this is indeed the case.

The algebra $\mathcal{A}$ associated to the Euclidean manifold X is the commutative C^* algebra $C^\infty(X)$. This is represented on the Hilbert space $\mathcal{H} = L^2(S)$, the space of square integrable spinors. The generalised Dirac operator D is the genuine Dirac operator $i\not{\partial}$. Given this K cycle the generalised differential algebra of section 2.6 is isomorphic to the de Rham algebra and the metric 2.3 reproduces the geodesic separation.

<u>Metric Structure</u>

The geodesic separation $d_g(p,q)$ of two point p and q is reproduced by the metric formula 2.3. It can be shown that $\| da \|_\infty^2 = \| [D,a] \|_{L^2}^2$ so 2.3 can be rewritten as

$$d(p,q) = sup\{\mid a(p) - a(q) \mid : a \in \mathcal{A}; \| da \|_\infty \leq 1\}.$$

Note that the one-to-one correspondence between X and Sp($\mathcal{A}$) for $\mathcal{A} = C^\infty(X)$ has been used. Now

$$\begin{aligned} |a(p) - a(q)| &= \int_p^q |\nabla a \cdot ds| \\ &\leq \| da \|_\infty \, d_g(p,q), \end{aligned}$$

so $d(p,q) \leq d_g(p,q)$. Conversely if $a(q) := d_g(p,q)$ (a valid choice since for this choice $\| da \|_\infty = 1$) then

$$d(p,q) = sup \;\; d_g(p,q).$$

Therefore it can be seen that $d(p,q)$ is equal to the geodesic separation of p and q.

<u>The Differential Algebra</u>

The differential algebra formed (after quotienting) from the Dirac K cycle can be identified with the de Rham algebra of differential forms. This is done via the isomorphism γ (first noticed by Kähaler) between differential forms with the vee ($\vee$) or Clifford product and the Clifford algebra [64]

$$\begin{array}{rccc} \gamma : & \text{basis of differential forms} & \longrightarrow & \text{matrix representation of the} \\ & \text{multiplication given by } \vee & & \text{basis of the Clifford Algebra} \\ & dx^\mu & \mapsto & \gamma^\mu \\ & dx^\mu \vee dx^\nu & \mapsto & \gamma^\mu\gamma^\nu \\ & dx^\mu \vee dx^\nu \vee dx^\sigma & \mapsto & \gamma^\mu\gamma^\nu\gamma^\sigma \end{array}$$

where

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}$$

and

$$dx^\mu \vee dx^\nu \vee dx^\sigma = dx^\mu \wedge dx^\nu \wedge dx^\sigma + g^{\mu\nu} dx^\sigma - g^{\mu\sigma} dx^\nu + g^{\nu\sigma} dx^\mu.$$

It is worth spelling out exactly how the identification between de Rham forms and the non-commutative forms of the Dirac K cycle works. Consider the two form $\sigma_u = da_0 da_1$ in the universal differential algebra constructed from the Dirac K cycle, it is represented on the Hilbert space $L^2(S)$ by

$$\begin{aligned}\Pi(\sigma_u) &= (i\not{\partial} a_0)(i\not{\partial} a_1) \\ &= -1/2\gamma^\mu\gamma^\nu(\partial_\mu a_0 \partial_\nu a_1 - \partial_\nu a_0 \partial_\mu a_1) - \partial_\mu a_0 \partial^\mu a_1\end{aligned}$$

On quotienting (see section 2.6) the scalar term is eliminated and the two form in the genuine differential algebra is

$$\Pi(\sigma) = -1/2\gamma^\mu\gamma^\nu(\partial_\mu a_0 \partial_\nu a_1 - \partial_\nu a_0 \partial_\mu a_1).$$

Equally using the map γ above we can write

$$\begin{aligned}\Pi(\sigma_u) &= \gamma(da_0)\gamma(da_1) \\ &= \gamma(\partial_\mu a_0 dx^\mu)\gamma(\partial_\nu a_1 dx^\nu) \\ &= \gamma(\partial_\mu a_0 \partial_\nu a_1 dx^\mu \vee dx^\nu) \\ &= \gamma(\partial_\mu a_0 \partial_\nu a_1 dx^\mu \wedge dx^\nu) + \gamma(\partial_\mu a_0 \partial^\mu a_1)\end{aligned}$$

which, on quotienting yields the two form realised by the Clifford algebra

$$\Pi(\sigma) = 1/2\gamma((\partial_\mu a_0 \partial_\nu a_1 - \partial_\nu a_0 \partial_\mu a_1) dx^\mu \wedge dx^\nu).$$

So it can be seen that the de Rham two form $(\partial_\mu a_0 \partial_\nu a_1 - \partial_\nu a_0 \partial_\mu a_1) dx^\mu \wedge dx^\nu$ can be identified with the non-commutative two form $-\gamma^\mu\gamma^\nu(\partial_\mu a_0 \partial_\nu a_1 - \partial_\nu a_0 \partial_\mu a_1)$.

2) Discrete Two Point Space

The two point space X is described in non-commutative geometry by a zero-dimensional K cycle $(\mathcal{A}, \mathcal{H}, D)$:

$$\begin{aligned} \mathcal{A} &= \mathbb{C} \oplus \mathbb{C} \\ \mathcal{H} &= \mathbb{C} \oplus \mathbb{C} \\ D &= \begin{bmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{bmatrix} \quad \mu \in \mathbb{C} \end{aligned}$$

$$\begin{array}{cccc} \lambda : & \mathcal{A} & \longrightarrow & B(\mathcal{H}) \\ & (a_1, a_2) & \mapsto & \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}. \end{array}$$

The space X is in one-to-one correspondence with $Sp(\mathcal{A}) = \{(p_1, p_2)\}$

$$\begin{array}{ccccccccc} \text{where} & p_1 : & \mathcal{A} & \xrightarrow{hom} & \mathbb{C} & p_2 : & \mathcal{A} & \xrightarrow{hom} & \mathbb{C} \\ & & (a_1, a_2) & \mapsto & a_1 & & (a_1, a_2) & \mapsto & a_2. \end{array}$$

Metric Structure

The separation of two "points" of Sp($\mathcal{A}$) is given by the metric formula 2.3. Let $a = (a_1, a_2)$ then

$$[D, a] = (a_1 - a_2) \begin{bmatrix} 0 & -\mu \\ \overline{\mu} & 0 \end{bmatrix}$$

and $\|[D, a]\| = |a_1 - a_2|(\mu\overline{\mu})^{1/2}$ so $d(p_1, p_2) = 1/(\mu\overline{\mu})^{1/2}$.

The Differential Algebra

Calculating the Hilbert space representation of forms in the universal differential algebra is just a matter of matrix multiplication. For instance, a general one form $\rho_u = a d_u b$ is represented explicitly on the Hilbert space $\mathbb{C} \oplus \mathbb{C}$ as

$$\begin{aligned} \Pi(\rho_u) &= -i\lambda(a)[D, \lambda(b)] \\ &= -i \begin{bmatrix} 0 & \mu a_1(b_2 - b_1) \\ \overline{\mu} a_2(b_1 - b_2) & 0 \end{bmatrix}. \end{aligned}$$

Similarly a general element $\alpha_u \in \Omega^2_u\mathcal{A}$, $\alpha_u = ad_u b d_u c$ is represented as

$$\begin{aligned}\Pi(\alpha_u) &= -\lambda(a)[D,\lambda(b)][D,\lambda(c)] \\ &= -\begin{bmatrix} \mu\overline{\mu}a_1(b_2-b_1)(c_1-c_2) & 0 \\ 0 & \overline{\mu}\mu a_2(b_1-b_2)(c_2-c_1) \end{bmatrix}.\end{aligned}$$

Transfer to genuine differential forms is achieved by quotienting by the graded differential ideal J as described in section 2.6. For concreteness consider $\Omega^2(\mathcal{A})$

$$\Pi(\Omega^2\mathcal{A}) = \frac{\Pi(\Omega^2_u\mathcal{A})}{\Pi(J^2)}.$$

A general element σ in $(Ker\Pi)^1$ is of the form $\sum_j a^j d_u b^j$ subject to the conditions

$$\sum_j \mu a^j_1(b^j_2 - b^j_1) = 0 \quad \text{and} \quad \sum_j \overline{\mu} a^j_2(b^j_1 - b^j_2) = 0 \tag{2.6}$$

$$\begin{aligned}\Pi(d_u\sigma) &= \Pi(\textstyle\sum_j d_u a^j d_u b^j) \ \text{ subject to constraints 2.6} \\ &= \textstyle\sum_j \begin{bmatrix} \mu\overline{\mu}(a^j_1 - a^j_2)(b^j_2 - b^j_1) & 0 \\ 0 & \overline{\mu}\mu(a^j_1 - a^j_2)(b^j_2 - b^j_1)\end{bmatrix} \begin{array}{l}\text{subject to} \\ \text{constraints 2.6}\end{array} \\ &= 0\end{aligned}$$

so $\Pi(d_u(ker\Pi)^1) = 0$ and (in this case) $\Pi(\Omega^2_u\mathcal{A}) = \Pi(\Omega^2\mathcal{A})$.

The product of two non-commutative manifolds is found by multiplying the associated K cycles using the theorem[18][7] below.

Theorem

Given two manifolds X_1 and X_2, described by the K cycles $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ respectively and with $\mathcal{H}_1$ having a $\mathbb{Z}_2$ grading Γ_1, then the product manifold $X_1 \cdot X_2$ is associated to a triple $(\mathcal{A}, \mathcal{H}, D)$ with

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \\ \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ D &= D_1 \otimes 1 + \Gamma_1 \otimes D_2.\end{aligned}$$

In fact all the non-commutative manifolds discussed in this thesis will have this product form. They will all be the product of a flat Euclidean space associated to a K cycle with an infinite, commuting algebra henceforth denoted $(\mathcal{A}_I, \mathcal{H}_I, D_I)$ and a discrete space associated to a K cycle henceforth denoted $(\mathcal{A}_F, \mathcal{H}_F, D_F)$. For example the product space obtained by multiplying the Euclidean manifold (example 1 above) by the discrete two point space (example 2 above) will be associated with a K cycle $(\mathcal{A}, \mathcal{H}, D)$ where

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_I \otimes \mathcal{A}_F \\ \mathcal{H} &= \mathcal{H}_I \otimes \mathcal{H}_F \\ D &= D_I \otimes 1 + \Gamma_I \otimes D_F \end{aligned}$$

with

$$\begin{array}{rclrcl} \mathcal{A}_I &=& C^\infty(M_4) & \mathcal{A}_F &=& \mathbb{C} \oplus \mathbb{C} \\ \mathcal{H}_I &=& L^2(S) & \mathcal{H}_F &=& \mathbb{C} \oplus \mathbb{C} \\ D_I &=& i\partial\!\!\!/ & D_F &=& \begin{bmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{bmatrix} \\ \Gamma_I &=& \gamma_5. & & & \end{array}$$

This product space can be visualised as two copies of a Euclidean manifold separated by a small distance $(1/(\mu\overline{\mu})^{1/2})$ and described by the K cycle $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A} = (\mathbb{C} \oplus \mathbb{C}) \otimes C^\infty(M_4)$, $\mathcal{H} = (\mathbb{C} \oplus \mathbb{C}) \otimes L^2(S)$, $D = i\partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$.

Chapter 3

Non-Commutative Geometry and Physics

3.1 Summary of this Chapter

Ultimately, the main aim of applying non-commutative geometry to physics is to reformulate quantum field theory in terms of non-commutative geometry. At the moment this goal is a long way off.

In the short term however Connes' non-commutative geometry has provided some very interesting developments in the area of classical particle physics in particular when applied to the problem of the standard model [5][11][13].

As already discussed (section 2.6) it is possible to develop a non-commutative analogue of de Rham cohomology. And so, via, as usual the curvature of a Lie algebra valued one form a Yang-Mills action can be defined (section 3.2). As non-commutative geometry is able to describe many more spaces than classical geometry it is possible to construct Yang-Mills actions over previously untreatable spaces.

One such space, the product of a continuous Euclidean 4-manifold and a discrete two point space is of particular interest to particle physicists. This is because when a pure Yang-Mills action with gauge group $SU(2) \times U(1)$ is constructed over this space the Glashow-Weinberg-Salam Lagrangian [60] (with leptons as the only fermionic matter) is retrieved but this time with the Higgs terms (that is the Higgs-gauge, kinetic Higgs and quartic potential terms) arising naturally: the complete bosonic sector of the Lagrangian can be derived as a pure Yang-Mills theory. This is clearly a great improvement on the usual formulation of the standard model. Details of the Connes-Lott formulation of the 'non-commutative Glashow-Weinberg-Salam model' are given in section 3.3. Section 3.4 discusses the full Connes-Lott non-commutative standard model (including the strong force) and 3.5 outlines the advantages and problems of this formulation compared to the usual formulation of the standard model. In the last section of this chapter (section 3.6) the most recent development in the application of non-commutative geometry to particle physics is outlined.

3.2 Construction Of Yang-Mills Models Over A Non-Commutative Manifold

Classically, a Yang-Mills Lagrangian is constructed by squaring the curvature of an anti-hermitian one form that is valued in the Lie algebra of the gauge group of the model. This is exactly the method that is used for the construction of a Yang-Mills model over a non-commutative manifold.

Consider constructing a Yang-Mills model on a non-commutative manifold X specified by a C* algebra $\mathcal{A}$ as outlined in section 2.2. It is necessary to know the K cycle $(\mathcal{A}, \mathcal{H}, D)$ associated to X. The Hilbert space $\mathcal{H}$ is the Hilbert space of Euclidean fermions and so must be chosen to match the desired fermionic content of the model. The 'generalised Dirac operator' D, contains information about the masses of the fermions and of course the metric structure of the manifold X.

Given the above inputs a Yang-Mills model can then be constructed. From $\mathcal{A}$ the graded differential algebra $\Omega\mathcal{A}$ is formed as outlined in section 2.6. An anti-hermitian one form $\rho \in \Omega(\mathcal{A})$ will be valued in u the Lie algebra of the gauge group U and can be considered as a vector potential. The curvature of ρ is defined as usual to be $\theta = \rho^2 + d\rho$.

The Yang-Mills action is then defined to be

$$A_{YM} = Tr_\omega(\Pi(\theta)^2 D^{-4}).$$

The gauge group U of the Yang-Mills action is the group of unitary elements of $\mathcal{A}$

$$U = \{u \mid uu^* = u^*u = 1; u \in \mathcal{A}\ \}.$$

As expected the curvature θ transforms homogeneously and the Yang-Mills action is invariant under the gauge transformation

$$\rho \rightarrow udu^\dagger + u\rho u^\dagger.$$

By imposing algebraic conditions the gauge group of the model can be reduced to a subgroup of the group of unitaries of $\mathcal{A}$. The representation λ of $\mathcal{A}$ on $\mathcal{H}$ determines the representation of the gauge group U. The requirement that the representation of the gauge group on $\mathcal{H}$ is a restriction of the representation of $\mathcal{A}$ greatly reduces the number of group representations that are available for model building. This is to be compared with the usual formulation of the standard model where any irreducible group representation is allowable. This point will be expanded on in section 3.5.

3.3 Construction of a Non-Commutative Glashow-Weinberg-Salam Model

The simplest physically interesting model to illustrate the construction of a non-commutative Yang-Mills is the non-commutative Glashow-Weinberg-Salam (GWS) model with leptons as the only fermionic matter.

The non-commutative GWS model is constructed over the non-commutative manifold given by the product of a Euclidean manifold (described, -see example 1 section 2.7, by the infinite commuting algebra $\mathcal{A}_I = C^\infty(M_4)$) by the space of the internal degrees of freedom of the model. In this case the internal degrees of freedom are SU(2) weak isospin and U(1) hypercharge, the (finite, non-commutative) algebra which must therefore be used to describe this internal space is $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C}$. Therefore the algebra associated with the product manifold is

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_I \otimes \mathcal{A}_F \\ &= C^\infty(M_4, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{C}). \end{aligned}$$

The Hilbert space is the space of Euclidean fermions

$$\mathcal{H} = L^2(S) \otimes [(\mathbb{C}^2 \otimes 1_N) \oplus ((\mathbb{C} \oplus \mathbb{C}) \otimes 1_N))]$$

corresponding to a fermionic content of $\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_R^e \\ e_R \end{pmatrix}$ for $N = 1$;

$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_R^e \\ e_R \end{pmatrix}, \begin{pmatrix} \nu_R^\mu \\ \mu_R \end{pmatrix}$ for $N = 2$, etc.

The representation λ of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\begin{array}{cccc} \lambda : & \mathcal{A} & \longrightarrow & B(\mathcal{H}) \\ & f \otimes a & \mapsto & \begin{bmatrix} f \otimes q \otimes 1_N & \\ & f \otimes C \otimes 1_N \end{bmatrix} \end{array}$$

where $C = \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$, $f \in C^\infty(M_4, \mathbb{R}), q \in \mathbb{H}, c \in \mathbb{C}$.

Note that a right handed neutrino has been included. This is so that $\mathbb{C}$ can be represented as a quaternion, it will be projected out at a later stage.

The generalised Dirac operator D is taken to be

$$D = i\partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$$

where the Euclidean gamma matrices (γ^μ, $\mu = 0, \cdots, 3$) are taken self-adjoint and where D_F is the leptonic mass matrix

$$D_F = \begin{bmatrix} 0 & M \\ M^\dagger & 0 \end{bmatrix}$$

or more explicitly for one generation

$$D_F = \begin{array}{c} \begin{array}{cccc} \nu_L & e_L & \nu_R & e_R \end{array} \\ \begin{pmatrix} 0 & 0 & m_\nu & 0 \\ 0 & 0 & 0 & m_e \\ m_\nu^\dagger & 0 & 0 & 0 \\ 0 & m_e^\dagger & 0 & 0 \end{pmatrix} \end{array} \quad \text{that is } M = \begin{pmatrix} m_\nu & 0 \\ 0 & m_e \end{pmatrix}.$$

The Yang-Mills action can then be calculated explicitly. This example is outlined in some detail to establish notations and conventions. Similar calculations can be found in [23][10][16].

Consider $\rho_u \in \Omega_u^1\mathcal{A}$, a general ρ_u will be of the form $\rho_u = \sum_j a_0^j d_u a_1^j$ then, dropping the j summation to ease the notation (though this will always be implied)

$$\begin{aligned}
\Pi(\rho) &= \Pi(\rho_u) \\
&= -i\lambda(a_0)[D, \lambda(a_1)] \\
&= -i\begin{bmatrix} f_0(i\partial\!\!\!/ f_1)\otimes q_0q_1\otimes 1_N & \gamma_5 f_0 f_1 \otimes [q_0(C_1 - q_1)\otimes 1_N]M \\ \gamma_5 f_0 f_1 \otimes M^\dagger[C_0(q_1 - C_1)\otimes 1_N] & f_0(i\partial\!\!\!/ f_1)\otimes C_0C_1\otimes 1_N \end{bmatrix} \\
&=: -i\begin{bmatrix} A_1\otimes 1_N & \gamma_5(h\otimes 1_N)M \\ \gamma_5 M^\dagger(g\otimes 1_N) & A_2\otimes 1_N \end{bmatrix}.
\end{aligned}$$

We wish $\Pi(\rho)$ to be Lie algebra valued so impose $\Pi(\rho)$ anti-hermitian ie impose $A_1^\dagger = A_1$, $A_2^\dagger = A_2$ and $g = h^\dagger$. The curvature of ρ is $\theta = d\rho + \rho^2$ so it is necessary to calculate $\Pi(d\rho)$ and $\Pi(\rho^2)$:

$$\Pi(d_u\rho_u) = -[D, \lambda(a_0)][D, \lambda(a_1)]$$

$\Pi(\rho^2) = \Pi(\rho)^2$ (since Π is a homomorphism) so

$$\Pi(\rho^2) = -\lambda(a_0)[D, \lambda(a_1)]\lambda(a_0)[D, \lambda(a_1)].$$

Both these terms can easily be calculated using matrix multiplication. For instance $\Pi(d_u\rho_u)$ is found to be

$$
\begin{aligned}
\Pi(d_u\rho_u)_{11} &= +(\partial\!\!\!/ f_0)(\partial\!\!\!/ f_1)\otimes q_0q_1\otimes 1_N+\\
&\quad -f_0f_1\otimes[(C_0-q_0)\otimes 1_N]MM^\dagger[(q_1-C_1)\otimes 1_N]\\
\Pi(d_u\rho_u)_{12} &= -(i\partial\!\!\!/ f_0)\gamma_5 f_1\otimes[q_0(C_1-q_1)\otimes 1_N]M+\\
&\quad -\gamma_5 f_0(i\partial\!\!\!/ f_1)\otimes[(C_0-q_0)C_1\otimes 1_N]M\\
\Pi(d_u\rho_u)_{21} &= -\gamma_5 f_0(i\partial\!\!\!/ f_1)\otimes M^\dagger[(q_0-C_0)q_1\otimes 1_N]+\\
&\quad -(i\partial\!\!\!/ f_0)\gamma_5 f_1\otimes M^\dagger[C_0(q_1-C_1)\otimes 1_N]\\
\Pi(d_u\rho_u)_{22} &= +(\partial\!\!\!/ f_0)(\partial\!\!\!/ f_1)\otimes C_0C_1\otimes 1_N+\\
&\quad -f_0f_1\otimes M^\dagger[(q_0-C_0)(C_1-q_1)\otimes 1_N]M
\end{aligned}
\tag{3.1}
$$

It is then necessary to pass to the space of 'genuine forms' by quotienting by $\Pi(J^2)$.

Quotienting

As discussed in section 2.6 this is done using a map P

$$
\begin{aligned}
P:\Pi(\Omega_u\mathcal{A})=J\oplus V &\longrightarrow V\cong\Pi(\Omega\mathcal{A})\\
j+v &\mapsto v
\end{aligned}
$$

where $V=\{v\in\Pi(\Omega_u\mathcal{A})\mid (v,j)=0\ \ \forall j\in J\}$. So an explicit description of the map P is needed. Consider a generic element $t=\Pi(a_0d_ua_1d_ua_2)$ of $\Pi(\Omega_u^2\mathcal{A})$ then

$$
\begin{aligned}
t_{11} &= +1/2\gamma^\mu\gamma^\nu[f_0(\partial_\mu f_1)(\partial_\nu f_2)-f_0(\partial_\nu f_1)(\partial_\mu f_2)]\otimes q_0q_1q_2\otimes 1_N+\\
&\quad +f_0(\partial_\mu f_1)(\partial^\mu f_2)\otimes q_0q_1q_2\otimes 1_N-f_0f_1f_2\otimes q_0(C_1-q_1)(q_2-C_2)\otimes\Sigma+\\
&\quad -f_0f_1f_2\otimes q_0(C_1-q_1)\sigma_3(q_2-C_2)\otimes\Delta\\
t_{12} &= -[\gamma_5 f_0(i\partial\!\!\!/ f_1)f_2\otimes q_0q_1(q_2-C_2)\otimes 1_N+\gamma_5 f_0f_1(i\partial\!\!\!/ f_2)\otimes q_0(C_1-q_1)C_2\otimes 1_N]M\\
t_{21} &= -\gamma_5 M^\dagger[f_0f_1(i\partial\!\!\!/ f_2)\otimes C_0(q_1-C_1)q_2\otimes 1_N-f_0(i\partial\!\!\!/ f_1)f_2\otimes C_0C_1(q_2-C_2)\otimes 1_N]\\
t_{22} &= +1/2\gamma^\mu\gamma^\nu[f_0(\partial_\mu f_1)(\partial_\nu f_2)-f_0(\partial_\nu f_1)(\partial_\mu f_2)]\otimes C_0C_1C_2\otimes 1_N+\\
&\quad +f_0(\partial_\mu f_1)(\partial^\mu f_2)\otimes C_0C_1C_2\otimes 1_N-f_0f_1f_2\otimes M^\dagger[C_0(q_1-C_1)(C_2-q_2)\otimes 1_N]M
\end{aligned}
$$

where the identity $MM^\dagger=I\otimes\Sigma+\sigma_3\otimes\Delta$ has been used. With Σ and Δ defined to be $\Sigma = 1/2(M_l M_l^\dagger + M_\nu M\nu^\dagger)$ and $\Delta = 1/2(M_l M_l^\dagger - M_\nu M\nu^\dagger)$ where

$$M_l := \begin{bmatrix} m_e & & & \\ & m_\mu & & \\ & & m_\tau & \\ & & & \ddots \end{bmatrix} \text{ and } M_\nu := \begin{bmatrix} m_{\nu e} & & & \\ & m_{\nu\mu} & & \\ & & m_{\nu\tau} & \\ & & & \ddots \end{bmatrix}.$$

So a generic element of $\Pi(\Omega_u^2\mathcal{A})$ has the form

$$\begin{bmatrix} A\otimes 1_N + B\otimes 1_N + C\otimes\Sigma + iD\otimes\Delta & (\gamma_5 F\otimes 1_N)M \\ M^\dagger(\gamma_5 E\otimes 1_N) & G\otimes 1_N + H\otimes 1_N + M^\dagger(K\otimes 1_N)M \end{bmatrix}$$

where $A \in \Omega_2(M)\otimes\mathbb{H}$; $B, C, D \in C^\infty(M, \mathbb{R})\otimes\mathbb{H}$; $E, F \in \Omega_1(M)\otimes\mathbb{H}$
$G \in \Omega_2(M)\otimes\mathbb{H}_{diag}$; $H \in C^\infty(M,\mathbb{R})\otimes\mathbb{H}_{diag}$ and $K \in C^\infty(M,\mathbb{R})\otimes\mathbb{H}$.

Similarly a generic element $j \in \Pi(J^2)$, $j = \Pi(d_u a_0 d_u a_1)$ subject to $\Pi(a_0 d_u a_1) = 0$ has the form

$$\begin{aligned}
j_{11} &= +(\not{\partial} f_0)(\not{\partial} f_1)\otimes q_0 q_1\otimes 1_N + \\
&\quad -f_0 f_1\otimes[(C_0 - q_0)\otimes 1_N]MM^\dagger[(q_1 - C_1)\otimes 1_N] \\
j_{12} &= -(i\not{\partial} f_0)\gamma_5 f_1\otimes[q_0(C_1 - q_1)\otimes 1_N]M + \\
&\quad -\gamma_5 f_0(i\not{\partial} f_1)\otimes[(C_0 - q_0)C_1\otimes 1_N]M \\
j_{21} &= -\gamma_5 f_0(i\not{\partial} f_1)\otimes M^\dagger[(q_0 - C_0)q_1\otimes 1_N] + \\
&\quad -(i\not{\partial} f_0)\gamma_5 f_1\otimes M^\dagger[C_0(q_1 - C_1)\otimes 1_N] \\
j_{22} &= +(\not{\partial} f_0)(\not{\partial} f_1)\otimes C_0 C_1\otimes 1_N + \\
&\quad -f_0 f_1\otimes M^\dagger[(q_0 - C_0)(C_1 - q_1)\otimes 1_N]M
\end{aligned}$$

subject to the constraints

$$\begin{aligned}
f_0(\not{\partial} f_1) &= 0 \\
C_0(q_1 - C_1) &= 0 \\
q_0(C_1 - q_1) &= 0
\end{aligned}$$

ie

$$
\begin{aligned}
j_{11} &= -f_0\Box f_1 \otimes q_0 q_1 \otimes 1_N - f_0 f_1 \otimes (C_0 - q_0)\sigma_3(q_1 - C_1) \otimes \Delta \\
j_{12} &= 0 \\
j_{21} &= 0 \\
j_{22} &= -f_0\Box f_1 \otimes C_0 C_1 \otimes 1_N .
\end{aligned}
\tag{3.2}
$$

So every element of $\Pi(J^2)$ is of the form

$$
\begin{bmatrix} J_1 \otimes 1_N + iJ_2 \otimes \Delta & 0 \\ 0 & J_3 \otimes 1_N \end{bmatrix}
$$

where $J_1, J_2 \in C^\infty(M, \mathbb{R}) \otimes \mathbb{H}$, $\quad J_3 \in C^\infty(M, \mathbb{R}) \otimes \mathbb{H}_{diag}$.

Then, imposing

$$
\begin{aligned}
(j,t) &= \int d^4x \; Tr_\gamma \otimes Tr_2 \otimes Tr_2 \otimes Tr_N \;\; (j^\dagger t) \\
&= 0 \qquad\qquad\qquad\qquad \forall j \in \Pi(J^2)
\end{aligned}
$$

it immediately follows that j and t are orthogonal if

$$
\begin{aligned}
B &= -\frac{Tr_N(\Sigma)}{N} C & \qquad D &= 0 \\
H &= -\frac{Tr_N(MM^\dagger)}{N} \begin{bmatrix} \alpha & \\ & \overline{\alpha} \end{bmatrix} & \qquad \text{where } \alpha &= [K]_{11}.
\end{aligned}
$$

So P the map projecting from the universal two forms onto the two forms of interest is given by

$$
P : \begin{bmatrix} A \otimes 1_N + B \otimes 1_N + C \otimes \Sigma + iD \otimes \Delta & (\gamma_5 F \otimes 1_N)M \\ M^\dagger(\gamma_5 E \otimes 1_N) & G \otimes 1_N + H \otimes 1_N + M^\dagger(K \otimes 1_N)M \end{bmatrix}
$$

$$
\longrightarrow
$$

$$
\begin{bmatrix} A \otimes 1_N - \frac{Tr_N(\Sigma)C}{N} \otimes 1_N + C \otimes \Sigma & (\gamma_5 F \otimes 1_N)M \\ M^\dagger(\gamma_5 E \otimes 1_N) & G \otimes 1_N - \frac{Tr_N(MM^\dagger)}{N} \begin{bmatrix} \alpha & \\ & \overline{\alpha} \end{bmatrix} \otimes 1_N + M^\dagger(K \otimes 1_N)M \end{bmatrix}
\tag{3.3}
$$

$\alpha = [K]_{11}$.

Applying this map to 3.1 and to the equivalent expression for $\Pi(\rho)$ yields

$$\begin{aligned}
\Pi(\theta)_{11} &= -1/2\gamma^\mu\gamma^\nu iF^1_{\mu\nu}\otimes 1_N - (\Phi^\dagger\Phi - 1)\otimes\Sigma + \tfrac{Tr_N(\Sigma)}{N}(\Phi^\dagger\Phi - 1)\otimes 1_N \\
\Pi(\theta)_{12} &= -(D\Phi\gamma_5\otimes 1_N)M \\
\Pi(\theta)_{21} &= M^\dagger((D\Phi)^\dagger\gamma_5\otimes 1_N) \\
\Pi(\theta)_{22} &= -1/2\gamma^\mu\gamma^\nu iF^2_{\mu\nu}\otimes 1_N - M^\dagger(\Phi^\dagger\Phi - 1)M + \tfrac{Tr_N(MM^\dagger)}{N}[\Phi^\dagger\Phi - 1]_{\substack{diag\\ only}}\otimes 1_N
\end{aligned} \tag{3.4}$$

with

$$\begin{aligned}
F^i_{\mu\nu} &:= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - i[A^i_\mu, A^i_\nu] \\
\Phi &:= h + 1 \\
D\Phi &:= i\partial\!\!\!/\Phi - \Phi A^2 + A^1\Phi.
\end{aligned}$$

The Yang-Mills action can then be calculated

$$\begin{aligned}
A_{YM} &= (\Pi(\theta), \Pi(\theta)) \\
&= \tfrac{1}{32\pi^2}\int d^4x\; Tr_\gamma\otimes Tr_2\otimes Tr_2\otimes Tr_N\;\; [\Pi(\theta)^\dagger\Pi(\theta)]
\end{aligned}$$

with the physical identifications

$$\begin{aligned}
A^\mu_1 &= -1/2 g\sigma\cdot W^\mu \\
A^\mu_2 &= g'\begin{pmatrix} B^\mu & \\ & -B^\mu \end{pmatrix} \\
\Phi &= \begin{pmatrix} \overline{\phi}_2 & \phi_1 \\ -\overline{\phi}_1 & \phi_2 \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ the genuine Higgs doublet.}
\end{aligned}$$

This yields the Yang-Mills Lagrangian (after projecting out the right handed neutrino)

$$\begin{aligned}
\mathcal{L}_{YM} &= Ng^2 W_{\mu\nu}\cdot W^{\mu\nu} + 2Ng'^2 B_{\mu\nu}B^{\mu\nu} + 16tr(M_L)\,(D_\mu\phi)^\dagger(D^\mu\phi) + \\
&\quad +6[tr_N(\Sigma^2) - \tfrac{tr_N(\Sigma)^2}{N}][(\phi^\dagger\phi)^2 - 2\phi^\dagger\phi - 1]
\end{aligned}$$

where

$$\begin{aligned}
W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\mathcal{E}_{ijk}W^j_\mu W^k_\nu \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\
D_\mu\phi &= i\partial_\mu\phi + g'B_\mu\phi - 1/2 g\sigma\cdot W_\mu\phi
\end{aligned}$$

and Σ (after removing the right handed neutrino) is

$$\Sigma = 1/2 \begin{bmatrix} m_e m_e^\dagger & & & \\ & m_\mu m_\mu^\dagger & & \\ & & m_\tau m_\tau^\dagger & \\ & & & \ddots \end{bmatrix}.$$

The fermionic action is defined to be

$$A_F = (\psi, (D + i\Pi(\rho))\psi)$$

giving (for N=1)

$$\begin{aligned} \mathcal{L}_F &= f_L^\dagger \gamma^\mu (i\partial_\mu - 1/2 g\sigma \cdot W_\mu) f_L + e_R^\dagger \gamma^\mu (i\partial_\mu - g' B_\mu) e_R \\ &\quad + m_e f_L^\dagger \gamma_5 \phi e_R + m_e e_R^\dagger \gamma_5 \phi^\dagger f_L \end{aligned}$$

where $f_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$.

The Lagrangian $\mathcal{L}_{YM} + \mathcal{L}_F$ is of roughly the correct form. However it should be noted that the left handed leptons have the incorrect hypercharge (zero instead of –1/2) and that the Higgs-gauge boson interaction is incorrect ($D_\mu \phi = i\partial_\mu \phi + g' B_\mu \phi - 1/2 g\sigma \cdot W_\mu \phi$ instead of $D_\mu \phi = i\partial_\mu \phi - 1/2 g' B_\mu \phi - 1/2 g\sigma \cdot W_\mu \phi$).
These problems are solved by introducing quarks and the strong force (-please see the next section). Interestingly the coefficient of the Higgs potential is

$$tr_N(\Sigma^2) - \frac{1}{N} tr_N(\Sigma)^2$$

which is clearly zero for the case $N = 1$. So the non-commutative standard model gives a reason for why (at least if we require massive particles) there should be more than one generation of fermions, it answers I I Rabi's question "who ordered the muon?". It should also be noted that the coefficient of the Higgs potential is positive for $m_\tau >> m_\mu >> m_e$ as would be expected in a Euclidean Lagrangian. A further comment should also be made on the subject of the non-commutative GWS

Lagrangian: if the fermionic Lagrangian is Wick rotated then it can be seen that half of the fermionic mass terms will have the incorrect sign. This is an, as yet, unresolved problem in all Connes-Lott models.

One subtlety that has been ignored in the above calculation is the question of the choice of scalar product. The scalar product that has been used on the differential algebra associated to the finite algebra $\mathcal{A}_F$ is

$$(\omega, \eta) = Tr(\omega^\dagger \eta) \quad \omega, \eta \in \Omega^k(\mathcal{A}_F).$$

Whilst this is a very natural choice it is not the most general one and since the Hilbert space of fermions $\mathcal{H}_F$ is not irreducible its use artificially imposes relationships between the different parts of the representation. A more general scalar product has been proposed [6]

$$(\omega, \eta) = Tr(z\omega^\dagger \eta) \quad \omega, \eta \in \Omega^k(\mathcal{A}_F) \tag{3.5}$$

where z, 'the non-commutative coupling constant' has the following properties

- $[z, \lambda(a)] = 0 \quad a \in \mathcal{A}_F$
- $[z, J\lambda(a)J^{-1}] = 0$
- $[z, D_F] = 0.$

It has been shown [42] that the above properties are necessary to insure that if the scalar product 3.5 is used then the map $\tilde{P}$ (equation 2.5) is still an isomorphism of involutive algebras.

To summarise, the input to the non-commutative GWS model is

1. a double sheeted space
2. the gauge group of the model SU(2) $\times$ U(1)
3. the fermionic content of the model

4. the fermion masses, Yukawa coupling constants and the Cabbibo-Kobayashi-Maskawa constants.

Given this input and using the Connes-Lott recipe for building non-commutative Yang-Mills models, the unique output is (upto Higgs hypercharge) the Glashow-Weinberg-Salam Lagrangian.

3.4 Real Structure -Incorporating the Strong Force into the Connes-Lott Model

In this section the revised Connes-Lott non-commutative standard model [8] which includes quarks and the SU(3) strong force is outlined.

Introducing quarks and the strong force into the non-commutative standard model is a non-trivial step. There are two problems, both have their origin in the fact that the gauge group of the model and the representation of this gauge group that acts on the fermions is derived from the algebra (as the group of unitaries and as a restriction of the algebra representation respectively). This is a construction that is particular to non-commutative geometry and whilst it is in general a strength (see section 3.5 on advantages of the non-commutative standard model) it does make the extension to SU(3) quarks rather difficult. The first of the two problems is that SU(3) is the group of unitaries of no algebra and so it does not fit naturally into the non-commutative framework. This can be got around by choosing $\mathcal{A} = M_3(\mathbb{C})$ whose group of unitaries is U(3), this is then broken down to SU(3) by imposing what Connes calls the unimodularity condition –essentially a tracelessness condition. The second problem is much harder to solve but throws up some very rich and interesting structure in the non-commutative standard model. Consider the left handed up quark u_L. It sits in an SU(2) weak isospin doublet $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ and in an

SU(3) colour triplet $\begin{pmatrix} u_R \\ u_G \\ u_B \end{pmatrix}$ ie the representation of $SU(2) \times SU(3)$ that acts on it is $\underline{2}_{SU(2)} \times \underline{3}_{SU(3)}$ –a product of two group representations. This presents no problem in the usual formulation of the standard model since any irreducible unitary group representation can be used and the product of two group representations is indeed a group representation. It does however present a problem in the non-commutative standard model where the group representation used is a restriction of the algebra representation λ used:

$$\begin{aligned} \lambda: \quad & \mathcal{A} \longrightarrow \quad B(\mathcal{H}) \\ \lambda|_U: \quad & U \mapsto \quad B(\mathcal{H}). \end{aligned}$$

It is easy to check that the product of two algebra representations is <u>not</u> in general an algebra representation (it doesn't preserve the linear structure of the algebra). In fact this is also the reason that the hypercharges of the left handed leptons in the non-commutative GWS model are zero. The left handed leptons in the standard model are acted upon by a product representation of SU(2)×U(1). For the reasons explained above it is not possible to realise this within the simple Connes-Lott model introduced in section 3.3 and the leptons are taken to be in an SU(2) doublet only -that is their hypercharge is zero. To accommodate quarks and the left handed leptons correctly a more complicated algebra bimodule structure [7][6], a Poincaré dual spectral triple needs to be introduced.

<u>Definition</u> Poincaré dual spectral triple

A Poincaré dual spectral triple $(\hat{\mathcal{B}} \otimes \hat{\mathcal{B}}', \hat{\mathcal{H}}, \hat{D})$ is defined to be a spectral triple with $\hat{\mathcal{B}}$ and $\hat{\mathcal{B}}'$ in Poincaré duality that is they satisfy the algebraic Poincaré duality conditions

1. $[\hat{\lambda}(b), \hat{\lambda}'(b')] = 0 \qquad b \in \hat{\mathcal{B}}\ , b' \in \hat{\mathcal{B}}'$

2. $[[\hat{D}, \hat{\lambda}(b)], \hat{\lambda}'(b')] = 0 \qquad b \in \hat{\mathcal{B}}\ , b' \in \hat{\mathcal{B}}'$

where $\hat{\lambda}$ and $\hat{\lambda}'$ are representations of $\hat{\mathcal{B}}$ and $\hat{\mathcal{B}}'$ respectively on a common Hilbert space:

$$\begin{array}{rccc} \hat{\lambda}: & \hat{\mathcal{B}} & \longrightarrow & B(\hat{\mathcal{H}}) \\ & b & \mapsto & b \otimes 1' \\ \hat{\lambda}': & \hat{\mathcal{B}}' & \longrightarrow & B(\hat{\mathcal{H}}) \\ & b' & \mapsto & 1 \otimes b'. \end{array}$$

To reproduce the standard model a Poincaré dual spectral triple over the algebras $\hat{\mathcal{B}} = \mathbb{C} \oplus \mathbb{H}$ and $\hat{\mathcal{B}}' = \mathbb{C} \oplus M_3(\mathbb{C})$ is taken. $\hat{\mathcal{B}} = \mathbb{C} \oplus \mathbb{H}$ reflects the electroweak structure of the model and $\hat{\mathcal{B}}' = \mathbb{C} \oplus M_3(\mathbb{C})$ the strong structure. A non-commutative Yang-Mills model built using this algebra will have gauge group $U_{\hat{\mathcal{B}}} \times U_{\hat{\mathcal{B}}'} = U(1) \times SU(2) \times U(1) \times U(3)$. This is broken down to $SU(2) \times U(1)_Y \times SU(3)$ by two unimodularity conditions which essentially identify (upto scalar multiples) the three U(1) factors.

This rather clumsy Poincaré dual structure consisting of two separate algebras can be reduced to a spectral triple over one Poincaré self-dual algebra with an interesting extra structure that reflects physics. This is done using the theorem [18] below

Theorem

A real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ can be obtained from a Poincaré dual spectral triple $(\hat{\mathcal{B}} \otimes \hat{\mathcal{B}}', \hat{\mathcal{H}}, \hat{D})$ if $\hat{\mathcal{B}}$ is of the form $\hat{\mathcal{B}} = \hat{\mathcal{A}} \oplus \hat{\mathcal{C}}$ and if $\hat{\mathcal{B}}'$ is of the form $\hat{\mathcal{B}}' = \hat{\mathcal{A}} \oplus \hat{\mathcal{C}}'$ by setting

$$\begin{aligned} \mathcal{A} &= \hat{\mathcal{A}} \oplus \hat{\mathcal{C}} \oplus \hat{\mathcal{C}}' \\ \mathcal{H} &= \hat{\mathcal{H}} \oplus \bar{\hat{\mathcal{H}}} \\ D &= \hat{D} \,\bar{\oplus}\, \hat{D} \\ \lambda(\mathcal{A}) &= \hat{\lambda}(\hat{\mathcal{A}} \oplus \hat{\mathcal{C}}\,) \,\bar{\oplus}\, \hat{\lambda}'(\hat{\mathcal{A}} \oplus \hat{\mathcal{C}}') \end{aligned}$$

where $\bar{\hat{\mathcal{H}}}$ denotes the conjugate Hilbert space of $\hat{\mathcal{H}}$ and $S \,\bar{\oplus}\, T$ is shorthand for $S \oplus 0 + J(T \oplus 0)J$.

Definition Real Spectral Triple

A real spectral triple is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with real structure J. Where J is an operator on the Hilbert space

$$\begin{aligned} J : \hat{\mathcal{H}} \oplus \bar{\hat{\mathcal{H}}} &\longrightarrow \hat{\mathcal{H}} \oplus \bar{\hat{\mathcal{H}}} \\ (\psi, \bar{\eta}) &\mapsto (\eta, \bar{\psi}) \end{aligned}$$

which satisfies the following conditions

1. $JD = DJ$
2. $J^2 = \pm 1$
3. $[\lambda(a), J\lambda(a')J^{-1}] = 0 \quad a, a' \in \mathcal{A}$
4. $[[D, \lambda(a)], J\lambda(a')J^{-1}] = 0 \quad a, a' \in \mathcal{A}$

Definition Real Graded Spectral Triple

A real graded spectral triple is a graded spectral triple $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ with real structure J which satisfies the above conditions as well as the additional condition

5. $J\Gamma = \pm \Gamma J \qquad \Gamma$ the K cycle grading.

If the real spectral triple corresponding to Riemannian space is considered then the real structure can be seen to be charge conjugation J=$\mathcal{C}$. The real structure on a generalised non-commutative manifold is therefore the non-commutative generalisation of charge conjugation. So in trying to incorporate quarks into the non-commutative standard model deeper links between non-commutative geometry and physics have been uncovered. Other features of the physics of the non-commutative standard model that are revealed by incorporating quarks using a real spectral triple are discussed in chapter 4.

The non-commutative standard model (including quarks and the strong force) is then obtained by building a non-commutative Yang-Mills model over the real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$

$$\mathcal{A} = C^\infty(M) \otimes [\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})] \tag{3.6}$$

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c$$

with

$$\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}),$$

$$\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}),$$

where superscript c denotes charge conjugation. Corresponding to the following basis of $\mathcal{H}$

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, u_R, d_R, e_R, \begin{pmatrix} u_L \\ d_L \end{pmatrix}^c, \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}^c, u_R^c, d_R^c, e_R^c$$

for $N = 1$. The generalised Dirac operator D is

$$D = i\partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F \tag{3.7}$$

with

$$D_F = \begin{bmatrix} 0 & M & 0 & 0 \\ M^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & M^\dagger \\ 0 & 0 & M & 0 \end{bmatrix} \text{ where } M = \begin{bmatrix} M_q \otimes 1_3 & \\ & M_l \end{bmatrix}.$$

For N=1

$$M_q = \begin{bmatrix} m_u & \\ & m_d \end{bmatrix}, \quad M_l = \begin{bmatrix} 0 \\ m_e \end{bmatrix}.$$

And the real structure J is

$$J = \mathcal{C} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} *$$

where $\mathcal{C}$ denotes charge conjugation on the spinor space, that is multiplication by the charge conjugation matrix C followed by complex conjugation. $\mathcal{A}$ is represented on $\mathcal{H}$ by the faithful homomorphism λ

$$\lambda(a) = \lambda_w(a) \oplus \lambda_s(a)^*$$

$$\lambda_w(a) = f \otimes \begin{bmatrix} q \otimes 1_N \otimes 1_3 & & & \\ & q \otimes 1_N & & \\ & & \begin{bmatrix} c & \\ & \overline{c} \end{bmatrix} \otimes 1_N \otimes 1_3 & \\ & & & \overline{c} 1_N \end{bmatrix}$$

$$\lambda_s(a) = f \otimes \begin{bmatrix} 1_2 \otimes 1_N \otimes m & & & \\ & \overline{c} 1_2 \otimes 1_N & & \\ & & 1_2 \otimes 1_N \otimes m & \\ & & & \overline{c} 1_N \end{bmatrix}.$$

The Yang-Mills action is calculated as before, the fermionic action is now defined to be

$$A_F = (\psi, (D + A + JAJ^{-1})\psi).$$

Using the properties of J it is easy to check that this new action is gauge invariant. Given

$$\begin{aligned} \psi \longrightarrow \psi^u &= u\psi u^* = uJuJ^\dagger \\ A \longrightarrow A^u &= uAu^* + u[D, u^*] \end{aligned}$$

since

$$\begin{aligned} Ju^*J^\dagger u^* DuJuJ^\dagger &= u^*[D, u] + Ju^*[D, u]J^\dagger + D \\ Ju^*J^\dagger u^*(uAu^* + u[D, u^*])uJuJ^\dagger &= A + [D, u^*]u \\ Ju^*J^\dagger u^*(JuAu^*J^\dagger + Ju[D, u^*]J^\dagger)uJuJ^\dagger &= JAJ^\dagger + J[D, u^*]uJ^\dagger \end{aligned}$$

then

$$(\psi^u, (D + A^u + JA^uJ^{-1})\psi) = (\psi, (D + A + JAJ^{-1})\psi).$$

This model yields the following predictions [20]

- Higgs Mass $m_H^2 \approx \frac{3(m_t/m_W)^4 + 2(m_t/m_W)^2 - 1}{(m_t/m_W)^2 + 3} m_W^2$ giving $m_H = 288 \pm 22 GeV$

- Weinberg Angle $sin^2(\theta_W) < \frac{2}{3(1 + (m_W/m_t)^2 + (g_2/3g_3)^2)}$ giving $sin^2(\theta_W) < 0.54$

- W Boson mass $m_e < M_W < m_t/\sqrt{3}$ giving $0.5 < M_W < 103 \times 10^3$MeV

where the following notation has been used

m_e	electron mass	θ_w	Weinberg angle
m_t	top quark mass	g_2	weak coupling constant
M_W	W boson mass	g_3	strong coupling constant
m_H	Higgs boson mass.		

Comparing these 'predictions' with experimental results [67]

$$\begin{aligned} \sin^2(\theta_W) &= 0.2319 \\ M_W &= 80.22\ GeV \\ m_H &> 58.4\ GeV \end{aligned}$$

it can be seen that there is no conflict between the predictions and the experimental results. However the range of the predictions for the W mass and the Weinberg angle is so wide as to be virtually meaningless. The best judge of the quality of these predictions will be made when the Higgs mass is known.

Of course these constraints are classical and therefore subject to quantum corrections. Quantisation of non-commutative Lagrangians is still an open question, it is felt by some that a new quantisation procedure that reflects non-commutative geometry needs to be developed. If the non-commutative Lagrangian is treated as a normal Lagrangian and is quantised in the usual way then the above constraints can be shown [22][26] to vary weakly under the renormalisation flow.

3.5 Advantages and Problems of the Non-Commutative Standard Model

3.5.1 Advantages

To better illustrate the advantages of the non-commutative standard model a very brief outline of the usual formulation of the standard model and its problems is given.

Usual Formulation Of The Standard Model

The standard model Lagrangian consists of the sum of five pieces: the Yang-Mills Lagrangian, the Dirac Lagrangian, the Higgs potential, the Klein-Gordon Lagrangian and the Yukawa terms.

To obtain the Yang-Mills Lagrangian a gauge group (out of the infinite number of finite dimensional compact Lie groups) must be selected. There is no a priori theoretical reason for choosing $SU(2)\times U(1)\times SU(3)$. Given this gauge group the Yang-Mills Lagrangian is constructed, it is well motivated geometrically. To construct the Dirac Lagrangian a representation of the gauge group $SU(2) \times U(1) \times SU(3)$ must be chosen -out of the infinite number of unitary, irreducible representations that are available to build a model with. Nature, as shown by experiment, selects the fundamental representation, again there is no a priori reason for this choice.

So far, the Lagrangian constructed, that is the sum of the Yang-Mills and Dirac Lagrangians results in massless gauge bosons and fermions. To break the group symmetry and introduce mass terms the Higgs potential, Klein-Gordon Lagrangian and Yukawa terms need to be added. This is a totally ad hoc procedure with no theoretical motivation.

Even given the basic form of the standard model Lagrangian as described above it is still necessary to fine tune so that its predictions agree with experimental results. It is necessary to input that the weak force is parity violating; that the strong force is vectorial, that its carriers (gluons) are massless and that there are three generations of fermions. It is also necessary to input eighteen parameters –the three gauge couplings, the W mass, the Higgs mass, nine fermion masses (assuming the neutrinos to be massless) and four Cabbibo-Kobayashi-Maskawa parameters. So to summarise, the arbitrary features of the standard model that, in many peoples minds debar it from being a fundamental theory are

1. arbitrary gauge group
2. arbitrary group representation
3. no theoretical motivation for the introduction of the Higgs
4. arbitrary force structure (weak non-vectorial, strong vectorial)
5. arbitrary masslessness of the gluons
6. arbitrary choice of three generations of fermions
7. 18 free parameters

Having said all this the standard model does agree with experiment to a high degree of accuracy and at least part of it, the Yang-Mills Lagrangian, is well motivated. It would be foolish to just abandon it, especially given the lack of alternative theories.

The main achievement of the non-commutative standard model is in solving problem (3) –it gives a very natural, geometric explanation for the existence of the Higgs particle. It also explains (4) and (5):- given that the weak force is maximally parity violating it asserts that the strong force is vectorial, that the SU(2) gauge group is broken (so its gauge bosons $W^{\pm}$ and Z are massive) and that the SU(3) gauge group is unbroken (so the gluons remain massless). The non-commutative standard model helps to a certain extent with (2) and (6) but is essentially no improvement when it comes to (1) and (7).

Existence of the Higgs

The main advantage of the non-commutative formulation of the standard model is conceptual -it provides a geometric interpretation of the Higgs. The Higgs boson arises naturally as an extra gauge boson associated with the discreteness of the space. It is unified with the other usual gauge bosons of the model (photons, $W^{\pm}$ and Z) and appears on exactly the same footing as them. This is precisely because in non-commutative geometry, unlike in classical geometry the discrete space is treated on the same footing as the continuous space.

Structure of the Strong Force and Masslessness of the Gluons

It can be shown [43] that, because of the requirement of Poincaré duality in the non-commutative standard model, given the parity violating structure of the weak force the strong force must be vectorial (see chapter 4 for more details). Furthermore it follows from the non-commutative standard model that the gauge group associated to a vectorial force remains unbroken (and its bosons therefore remain massless). This is because the Higgs boson arises as a one form in the differential algebra $\Omega\mathcal{A}_F$ constructed from the finite algebra, but in the case of a vectorial force we have $\lambda_L(\mathcal{A}_F) = \lambda_R(\mathcal{A}_F)$ and $[\lambda_L(\mathcal{A}_F), M] = 0$ so the differential algebra is trivial

$$\begin{aligned} \Omega^0\mathcal{A}_F &= \mathcal{A}_F \\ \Omega^p\mathcal{A}_F &= 0 \quad p \geq 1 \end{aligned}$$

therefore there are no Higgs terms and vectorial forces remain unbroken. That is, in the case of the standard model, non-commutative geometry explains why the $W^{\pm}$ and Z bosons are massive and the gluons are massless.

Gauge Group Representation

In the usual formulation of the standard model the fermions can be placed in any of the infinite number of unitary irreducible representations of the gauge group. In the non-commutative formulation of the standard model the representation of the gauge group is a restriction of the representation of the algebra. This is a very limiting condition –typically an algebra has only one or two possible representations. This point was analysed by Schücker and Iochum [13], their results are summarised below

Gauge Group U	Possible Representations of U
O(n,$\mathbb{R}$)	fundamental representation
U(n)	fundamental or conjugate fundamental representation
Sp(2n)	fundamental representation.

For the case of the standard model gauge group $SU(2) \times U(1) \times SU(3)$ this compels us to work in the fundamental (or conjugate fundamental representation). It can be seen that the possibility of constructing non-commutative grand unified theories based on SU(5) or SO(10) is ruled out as both these schemes utilise representations which are neither fundamental nor conjugate fundamental.

Number of Generations of Fermions

As already discussed (section 3.3) the existence of the Higgs potential requires at least two generations of fermions. It has also been noted [27] that, since the mass of the top quark is thought to be 174GeV, the non-commutative constraint $m_t > \sqrt{N} m_w$ constrains the number of fermions to be less than five.

Choice of Gauge Group

Here the non-commutative standard model has little advantage over the usual formulation, almost any compact Lie group can be used though the exceptionals can be ruled out as they are not the group of unitaries of any semi-simple algebra.

Number of Free Parameters

The non-commutative standard model has a marginally improved free parameter count as the Higgs mass and the Weinberg angle are both constrained.

3.5.2 Problems

There are three main problems associated with the non-commutative standard model (apart from the fact that non-commutative geometry doesn't uniquely select the standard model). Firstly, as mentioned earlier the problem of quantisation. Secondly, the fact that the non-commutative standard model Lagrangian is in Euclidean space. And thirdly the requirement of the unimodularity condition.

Quantisation

It is not known how to quantise non-commutative Lagrangians in a 'non-commutative way'. All the renormalisation analysis that has been applied to the non-commutative standard model [22][26] is based on the conventional quantisation process. There is no reason to believe that this is the method that should be applied in the non-commutative case.

Euclidean Space

All the non-commutative standard model Lagrangians that have been constructed are essentially in Euclidean rather than Minkowski space. This is because non-commutative geometry is firmly rooted in a Hilbert space setting: the fundamental building block of non-commutative geometry, the K cycle is a Hilbert space notion. If we consider the space of spinors in Minkowski space where the inner product is

$$(\psi_1, \psi_2) = \int \psi_1(x)^\dagger \gamma^0 \psi_2(x) d^4x$$

then, since this inner product is not positive definite the vector space is not a Hilbert space. Unlike the Euclidean case where the inner product

$$(\psi_1, \psi_2) = \int \psi_1(x)^\dagger \psi_2(x) d^4x$$

is positive definite so we do have a Hilbert space –the space $L^2(S)$. For this reason we are compelled to work in Euclidean rather that Minkowski space.

Furthermore, the operator D in a K cycle ($\mathcal{A}$,$\mathcal{H}$,D) is required to be elliptic. However the Dirac operator $i\partial\!\!\!/$, necessary for building non-commutative models over space-time is not elliptic for Minkowski space-time (although it is for Euclidean space-time). This is another reason why constructing a Minkowski space formulation of non-commutative geometry will be extremely difficult.

The most commonly used method to get around this problem is to Wick rotate the final Lagrangian. Alternatively it has been argued [26] that since, in the calculation of the non-commutative standard model after the one form has been calculated and integration defined (in this case just the usual Euclidean integration), the calculation of the Lagrangian just proceeds in the usual way (without any further reference to non-commutative geometry) it is possible to introduce integration over a Minkowski space and a Minkowski rather than Euclidean Dirac operator. Essentially amounting to 'Wick rotation' at an earlier stage. Neither method is particularly satisfactory.

Unimodularity

As mentioned earlier it is necessary to reduce the 'natural' gauge group of the standard model to the correct gauge group via a unimodularity condition. This is a rather ugly and ad hoc process. This subject will be covered in more depth in chapter 5.

3.6 Gravity and the Non-Commutative Standard Model

Non-commutative geometry has recently [9] been extended in such a way that the Dirac action and the Yang-Mills action can be naturally unified with the Einstein-Hilbert action [30][31]. This unified action can be written as

$$Tr\mathcal{F}\left[\frac{(D+A+JAJ^{\dagger})^2}{\Lambda^2}\right]+(\psi,(D+A+JAJ^{\dagger})\psi)$$

where $\mathcal{F}$ is the characteristic function of the unit interval [0,1], Λ is a cut off and $D+A+JAJ^{\dagger}$ is as defined in section 3.4 with D being the generalised Dirac operator, $iA = i\sum_j a^j[D,b^j]$ an anti-hermitian one form ($a, b \in \mathcal{A}$) and J the real structure. The standard model action unified with the Einstein-Hilbert action is obtained by defining $\mathcal{A}$ and D as in section 3.4 (equations 3.6 and 3.7 respectively).

This unification occurs at high energies, Λ in the range $10^{15} - 10^{19}GeV$. At low energies the universal action just reduces to the usual Connes-Lott standard model action. For this reason these new developments do not, on the whole, impinge on the topic of this thesis. There is one exception to this –the formulation of the universal action could perhaps give an explanation for the unimodularity condition, this is explained in section 5.3 of chapter 5.

Chapter 4

Poincaré Duality and the Chiral Structure of the Non-Commutative Standard Model

4.1 Summary of this Chapter

The most convincing argument for non-commutative geometry being the natural setting for the standard model is undoubtedly the geometric explanation for the Higgs that it provides. But it also has many other interesting features, some of which are summarised in section 3.5, which are surprisingly consistent with the standard model. This chapter deals with one of these features, namely how non-commutative Poincaré duality dictates the chiral structure of the standard model. To be more precise it can be shown that given the structure of the weak force Poincaré duality asserts that the strong force must be vectorial, and conversely given the form of the strong force the weak force is constrained to be parity violating. Section 4.2 introduces the notion of Poincaré duality both in the classical and non-commutative setting. Section 4.3 explains the calculations that constrain the strong force to be vectorial. Section 4.4 briefly examines the converse statement namely that given the structure of the strong force in the non-commutative standard model Poincaré duality constrains the weak force to be parity violating. In section 4.5 the chiral structure of more general non-commutative Yang-Mills models is examined. Section 4.6 is a short conclusion.

The aim of this chapter is not to explain the non-commutative formulation of Poincaré duality which is mathematically complicated and beyond the scope of this thesis (please see [7] for an in depth discussion); but rather to examine what, assuming Connes' formulation of non-commutative Poincaré duality, its implications are.

4.2 Poincaré Duality in Classical and Non-Commutative Geometry

For all classical compact orientated manifolds there is a duality [61] between homology and cohomology known as Poincaré duality. That is there exists an isomorphism γ

$$\gamma : H^i(M) \longrightarrow H_{n-i}(M)$$

for M an n dimensional compact orientated manifold. Poincaré duality can alternatively be expressed as the requirement that the map

$$\begin{array}{ccc} H^i(M) \times H^{n-i}(M) & \longrightarrow & \mathbb{R} \\ (\omega, \eta) & \mapsto & \int \omega \wedge \eta \end{array}$$

is nondegenerate.

Connes argues [7] that for a non-commutative space, described by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be a smooth manifold it is necessary that Poincaré duality is satisfied by the triple. The conditions required for the existence of the Poincaré duality isomorphism are

$$[\lambda(a), J\lambda(a')J^{-1}] = 0 \quad \forall a, a' \in \mathcal{A}' \tag{4.1}$$

$$[[D, \lambda(a)], J\lambda(a')J^{-1}] = 0 \quad \forall a, a' \in \mathcal{A}' \tag{4.2}$$

$$Tr_\omega(\Gamma[D, \lambda(a^0)][D, \lambda(a^1)]...[D, \lambda(a^n)]\,|D|^{-n}) = 0 \quad \forall a^j \in \mathcal{A}'. \tag{4.3}$$

From these conditions it can be clearly seen that not every spectral triple is equivalent to a non-commutative manifold (as has been implicitly assumed until now). Indeed whether or not the algebra $\mathcal{A}$ is Poincaré self-dual depends not only on the algebra but also on the representation λ of the algebra.

At this point it should be noted that the requirement that a non-commutative space be a non-commutative manifold (that is the requirement of non-commutative Poincaré duality) is precisely the requirement necessary to incorporate the strong force into the non-commutative standard model as discussed in section 3.4. So in the non-commutative standard model the strong and weak algebras are Poincaré dual to one another. That is there can be considered to be a geometric relationship between the strong and the weak force.

4.3 Poincaré Duality and the Strong Force

Claim: If, within the framework of the non-commutative standard model, the form of the electroweak sector is assumed then the condition that the algebra $\mathcal{A}$ must be Poincaré self-dual (that is there exists the Poincaré duality isomorphism on the non-commutative space) constrains

1. the strong force to be be vectorial.

Additionally, it forces

2. the strong force to be blind to isospin

3. the action of the strong force on each generation of quarks to be the same.

Proof of (1) and (2)

(1) and (2) will be shown first, for convenience (3) will be assumed at first but proved in a later section.

Notation and Assumptions: In these calculations only the finite part $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ of the full K-cycle is worked with. The full model is then obtained by tensoring with the infinite sector. $\mathcal{A}_F$ is taken to be

$$\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}).$$

For the present it is assumed that the action of the strong force is the same on every generation of quarks so

$$\lambda(a) = \begin{bmatrix} \lambda_w(a) \otimes 1_N & \\ & \lambda_s(a) \otimes 1_N \end{bmatrix} \quad a \in \mathcal{A}_F$$

where $a = (q, c, x) \in \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. The electroweak sector is assumed so $\lambda_w(a)$ is taken to be

$$\lambda_w(q,c,x) = \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} \lambda_1(q) & \\ & \lambda_2(c) \end{pmatrix} \end{array}$$

with

$$\lambda_1(q) = \begin{array}{c} \begin{array}{cc} (u,d)_L & (\nu,e)_L \end{array} \\ \begin{pmatrix} q \otimes 1_3 & \\ & q \end{pmatrix} \end{array} \text{ and } \lambda_2(c) = \begin{array}{c} \begin{array}{cc} (u,d)_R & e_R \end{array} \\ \begin{pmatrix} C \otimes 1_3 & \\ & \bar{c} \end{pmatrix} \end{array}, \quad C = \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}.$$

(The basis for the first generation of fermions is given as an example). From experimental evidence [63] it is known that quarks exist in 'threes' (ie what we call colour triplets) of identical mass so the form of the fermionic mass matrix is known. The form of the mass matrix in the above basis is awkward due to (Cabbibo-Kobayashi-Maskawa) quark mass mixing. The following notation is used

$$D_F = \begin{array}{c} \begin{array}{cc} \textit{particles} & \textit{antiparticles} \end{array} \\ \begin{pmatrix} \mathcal{M} & \\ & \bar{\mathcal{M}} \end{pmatrix} \end{array} \quad \mathcal{M} = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1N} \\ \vdots & & & \vdots \\ D_{N1} & \cdots & \cdots & D_{NN} \end{bmatrix}$$

$$\text{with} \quad D_{ii} = \begin{bmatrix} 0 & M_i \\ M_i^\dagger & 0 \end{bmatrix} \quad \text{and} \quad M_i = \begin{bmatrix} M_q^i \otimes 1_3 & \\ & M_l^i \end{bmatrix} \tag{4.4}$$

where i=1,...N denotes generation number. J_F, the non-commutative generalisation of charge conjugation (on the finite algebra) is taken to be $J_F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * .$

The aim is then to prove statements (1) and (2) given the conditions 4.1 and 4.2.

<u>Calculations:</u>

Firstly the constraints imposed by 4.1 are examined. From 4.1 it follows that

$$[\lambda_w(a), \lambda_s(a')] = 0. \tag{4.5}$$

This means that $\lambda_s(a)$ is block diagonal,

$$\lambda_s(a) = \begin{bmatrix} \lambda_3(a) \otimes 1_N & \\ & \lambda_4(a) \otimes 1_N \end{bmatrix}$$

with

$$[\lambda_1(a), \lambda_3(a')] = 0 \tag{4.6}$$

and

$$[\lambda_2(a), \lambda_4(a')] = 0. \tag{4.7}$$

Inserting $\lambda_1(a) = \lambda_1(q) = (q \otimes 1_3) \oplus q$ into 4.6 and using Schur's first lemma leads to

$$\begin{aligned} \lambda_3(a) &= (Z 1_2 \otimes Y') \oplus 1_2 X \qquad && Z, X \in \mathbb{C},\ Y' \in M_3(\mathbb{C}) \\ &= (1_2 \otimes Y) \oplus 1_2 X && Y := ZY'. \end{aligned}$$

Using the additional fact that $\lambda(a)$ is an algebra representation of $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ it follows that the possible choices for Y and X are

$Y = x, \bar{x}$ or any 3×3 diagonal matrix whose entries are either c or $\bar{c}$ (denoted $M_3(c, \bar{c})$) or any 3×3 block diagonal matrix whose entries are [q,c] (plus permutations)(denoted $M_3(q, c)$)

$X = c$ or $\bar{c}$.

Similarly, inserting $\lambda_2(a) = \lambda_2(c) = (C \otimes 1_3) \oplus \bar{c}$ into 4.7 and using Schur's lemma leads to

$$\lambda_4(a) = (W \otimes V) \oplus U \qquad W \in M_2(\mathbb{C})_{diag},\ V \in M_3(\mathbb{C}),\ U \in \mathbb{C}.$$

Again, using the representation properties of $\lambda(a)$, it follows that the possible choices for W,V,U are

$W = c1_2, \bar{c}1_2, \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$ or $\begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$ and $V = 1_3$

or

$W = 1_2$ and $V = x,\ \bar{x},\ M_3(c, \bar{c})$ or $M_3(q, c)$

$U = c$ or $\bar{c}$.

So possible combinations of values for

$$\begin{aligned}\lambda_s(a) &= \lambda_3(a) \oplus \lambda_4(a) \\ &= [(1_2 \otimes Y) \oplus 1_2 X] \oplus [(W \otimes V) \oplus U]\end{aligned}$$

are

	Y	X	W	V	U
I	$x,\bar{x}$	c,$\bar{c}$	$c1_2, \bar{c}1_2, \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$	1_3	c,$\bar{c}$
II	$M_3(c,\bar{c})$	c,$\bar{c}$	$c1_2, \bar{c}1_2, \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$	1_3	c,$\bar{c}$
III	$M_3(q,c)$	c,$\bar{c}$	$c1_2, \bar{c}1_2, \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$	1_3	c,$\bar{c}$
IV	$x,\bar{x}$	c,$\bar{c}$	1_2	$x,\bar{x}$	c,$\bar{c}$
V	$M_3(c,\bar{c})$	c,$\bar{c}$	1_2	$x,\bar{x}$	c,$\bar{c}$
VI	$M_3(q,c)$	c,$\bar{c}$	1_2	$x,\bar{x}$	c,$\bar{c}$
VII	$x,\bar{x}$	c,$\bar{c}$	1_2	$M_3(q,c)$	c,$\bar{c}$
VIII	$M_3(c,\bar{c})$	c,$\bar{c}$	1_2	$M_3(q,c)$	c,$\bar{c}$
IX	$M_3(q,c)$	c,$\bar{c}$	1_2	$M_3(q,c)$	c,$\bar{c}$
X	$x,\bar{x}$	c,$\bar{c}$	1_2	$M_3(c,\bar{c})$	c,$\bar{c}$
XI	$M_3(c,\bar{c})$	c,$\bar{c}$	1_2	$M_3(c,\bar{c})$	c,$\bar{c}$
XII	$M_3(q,c)$	c,$\bar{c}$	1_2	$M_3(c,\bar{c})$	c,$\bar{c}$

Options II[1], III, VIII, IX, XI and XII can be ruled out immediately because $\lambda(a)$ must represent $M_3(\mathbb{C})$ and these choices do not.

Next the constraints imposed by 4.2 are examined to see if this rules out any of the other options tabulated above. Inserting D_F and $\lambda(a)$ into 4.2 and using the

[1] alternatively, as remarked by Martín et al [26] this option is also ruled out because it violates the third Poincaré duality constraint.

shorthand $\lambda' := \lambda(a')$ gives

$$[[\mathcal{M}, \lambda_w \otimes 1_N], \lambda'_s \otimes 1_N] = 0 \tag{4.8}$$

$$[[\bar{\mathcal{M}}, \lambda_s \otimes 1_N], \lambda'_w \otimes 1_N] = 0. \tag{4.9}$$

If 4.8 is expanded in the generation index it can be written as

$$\begin{bmatrix} [[D_{11}, \lambda_w], \lambda'_s] & [[D_{12}, \lambda_w], \lambda'_s] & \cdots & [[D_{1N}, \lambda_w], \lambda'_s] \\ \vdots & & & \vdots \\ [[D_{NN}, \lambda_w], \lambda'_s] & \cdots & \cdots & [[D_{NN}, \lambda_w], \lambda'_s] \end{bmatrix} = 0.$$

Consider the diagonal entries

$$[[D_{ii}, \lambda_w], \lambda'_s] = 0 \quad i = 1, \cdots N. \tag{4.10}$$

The N equations 4.10 represent only one condition since all the D_{ii} have the same structure 4.4 just with different positive entries. Substituting 4.4 into 4.10 and dropping the i index gives

$$(M\lambda_2 - \lambda_1 M)\lambda'_4 - \lambda'_3(M\lambda_2 - \lambda_1 M) = 0 \tag{4.11}$$

and

$$(M^\dagger\lambda_1 - \lambda_2 M^\dagger)\lambda'_3 - \lambda'_4(M^\dagger\lambda_1 - \lambda_2 M^\dagger) = 0. \tag{4.12}$$

The λ are faithful representations so setting q=0 and c=1 gives $\lambda_1 = 0$ and $\lambda_2 = 1$. Evaluating 4.11 at q=0 and c=1 gives

$$M\lambda'_4 = \lambda'_3 M$$

inserting $M = (M_q \otimes 1_3) \oplus M_l$, $\lambda_3(a) = (1_2 \otimes Y) \oplus 1_2 X$ and $\lambda_4(a) = (W \otimes V) \oplus U$ then gives

$$(M_q W \otimes V) \oplus M_l U = (M_q \otimes Y) \oplus X M_l$$

this condition yields

$$U = X \quad W = T1_2 \text{ and } TV = Y \quad T \in \mathbb{C}.$$

Comparing this with the tabulated options it can be seen that options I, V, VI, VII and X are ruled out leaving only IV. The possible choices for $\lambda_s(c,x)$ are then either

$$\begin{bmatrix} 1_2 \otimes x & & & \\ & c1_2 & & \\ & & 1_2 \otimes x & \\ & & & c \end{bmatrix} \text{ or } \begin{bmatrix} 1_2 \otimes x & & & \\ & \bar{c}1_2 & & \\ & & 1_2 \otimes x & \\ & & & \bar{c} \end{bmatrix}.$$

The relative sign of x (that is whether x or $\bar{\text{x}}$ is chosen) is irrelevant as λ_w does not represent $M_3(\mathbb{C})$. It can be immediately seen that the action of $\mathcal{A}$ associated with the strong force on the left handed fermions is the same as that on the right handed fermions ($\lambda_3 = \lambda_4$, upto the fact that there is no right handed neutrino assumed in this calculation[2]) and that it commutes with the mass matrix ($[\mathcal{M}, \lambda_s] = 0$) –that is it has been shown that the strong force is constrained to be vectorial. Additionally the quark doublet $\begin{bmatrix} u \\ d \end{bmatrix}$ is acted upon by $1_2 \otimes x$. So it follows that the strong force does not see flavour.

This concludes the proof of statements (1) and (2).

Proof of (3)

Aim: to show that the strong force acts in the same way on quarks of all generations.

Notation: The same notation for $\lambda_w(a)$ and D_F will be used in this proof of (3) as in the proof of (1) and (2). Here the generational structure of $\lambda_s(a)$ is not assumed so $\lambda(a)$ will be taken to be

$$\lambda(a) = \begin{bmatrix} \lambda_w(a) \otimes 1_N & \\ & R_s(a) \end{bmatrix} \tag{4.13}$$

[2]a right handed neutrino could easily be included in this calculation [43] leading to an exact equality but since it has been shown [19] that this would violate the third Poincaré duality condition it has been omitted.

where R_s is a general matrix, its elements, expanded by generation will be labelled

$$R_s(a) = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ \vdots & & & \vdots \\ R_{N1} & \cdots & \cdots & R_{NN} \end{bmatrix}$$

where each R_{ij} will have the same matrix dimensions as $\lambda_w(a)$.

Calculations: Substituting 4.13 into 4.1 yields N^2 equations

$$[\lambda_w(a), R'_{ij}] = 0$$

$\lambda_w(a) = (q \otimes 1_3) \oplus q \oplus (C \otimes 1_3) \oplus \bar{c}$ so each R_{ij} is of the form

$$R_{ij} = (1_2 \otimes Z_{ij}) \oplus Y_{ij} 1_2 \oplus (X_{ij} \otimes W_{ij}) \oplus V_{ij} \tag{4.14}$$

where $Z_{ij}, W_{ij} \in M_3(\mathbb{C})$; $Y_{ij}, V_{ij} \in \mathbb{C}$ and $X_{ij} \in M_2(\mathbb{C})_{diag}$.

Similarly, substituting 4.13 into equation 4.2 yields N^2 equations, the ith–jth one being

$$\sum_{k=1}^{N} [D_{ik}, \lambda_w(a)] R'_{kj} - R'_{ik} [D_{kj}, \lambda_w] = 0. \tag{4.15}$$

Consider first the off diagonal equations ($i \neq j$). Each i–j equation will consist of the sum of 2N terms, 2N-2 of these terms will depend on a different D_{ik} multiplied by R'_{kj} $(k \neq j)$, each of the D_{ik} are independent so for this sum to be zero for all a and a' $\in \mathcal{A}$ each term must vanish independently ie $R_{ij} = 0 \quad i \neq j$. So, the off diagonal equations reduce to

$$[D_{ij}, \lambda_w(a)] R'_{jj} - R'_{ii} [D_{ij}, \lambda_w] = 0 \quad i \neq j. \tag{4.16}$$

The D_{ij} are the elements of the mass matrix which contain the quark mixing terms, they are therefore of the form

$$
D_{ij} = \begin{array}{cc}
\overbrace{\overbrace{q \quad\; l}^{L} \quad \overbrace{q \quad\; l}^{R}}^{\text{generation j}} & \\
\begin{bmatrix} 0 & 0 & G_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ H_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \left.\begin{array}{l} \left.\begin{array}{l} \text{q} \\ \text{l} \end{array}\right\} \text{L} \\ \left.\begin{array}{l} \text{q} \\ \text{l} \end{array}\right\} \text{R} \end{array}\right\} \text{generation i}
\end{array}
$$

$$
\text{with} \quad G_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & g_{ij} \end{bmatrix} \otimes 1_3 \quad \text{and} \quad H_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & h_{ij} \end{bmatrix} \otimes 1_3.
$$

So, substituting the above, $\lambda_w(a)$ and R_{ij} (equation 4.14) into 4.16 yields

$$
[G_{ij}(C \otimes 1_3) - (q \otimes 1_3)G_{ij}][X_{jj} \otimes W_{jj} - 1_2 \otimes Z_{ii}] = 0
$$

and

$$
[1_2 \otimes Z_{jj} - X_{ii} \otimes W_{ii}][H_{ij}(q \otimes 1_3) - (C \otimes 1_3)H_{ij}] = 0,
$$

implying that $X_{jj} \otimes W_{jj} = 1_2 \otimes Z_{ii}$ $i \neq j$.

Similarly expanding the diagonal terms of the matrix equation 4.15 gives $X_{ii} \otimes W_{ii} = 1_2 \otimes Z_{ii}$ putting these two results together gives

$$
1_2 \otimes Z_{ii} = 1_2 \otimes Z_{jj} \quad \text{and} \quad X_{ii} \otimes W_{ii} = X_{jj} \otimes W_{jj} \quad i, j = 1 \cdots N.
$$

By comparing with equation 4.14, and bearing in mind that the $1_2 \otimes Z_{ii}$ and the $X_{ii} \otimes W_{ii}$ terms act on the quarks, (the Y_{ii} and V_{ii} terms act on the leptons) and that the subscript i is a generational index, it can easily be seen that the strong force acts in the same way on each generation of quarks. Hence the proof of statement (3).

4.4 Poincaré Duality and the Weak Force

So, it has been shown that given the weak force the Poincaré duality condition constrains the strong force to be vectorial. An interesting and obvious question of course is does the converse statement hold? That is, given the form of the strong force that occurs in the standard model is the weak force constrained to be parity violating? At first glance the answer to this question appears to be no since for a vectorial strong force the second Poincaré duality condition 4.2 is trivially satisfied for any representation associated to the weak force and the remaining constraint 4.1 is not restrictive enough to constrain the weak force to be parity violating.

However there is a third Poincaré duality condition which was not exploited in the previous calculations of this chapter. This constraint has been examined by Testard [19] and found to rule out the possibility of a right handed neutrino in the standard model. (This proof holds for any number of generations of fermions and assumes, apart from the right handed neutrino, the usual particle spectrum and the usual form of the weak and strong forces). If Poincaré duality is expressed in terms of K theory the third Poincaré duality constraint for a finite algebra can be written [26] as the requirement that the map

$$(p_i, p_j) \mapsto Tr(\Gamma\lambda(p_i)J\lambda(p_j)J^\dagger) \tag{4.17}$$

is nondegenerate, where p_i and p_j are generators of the K theory group $K_0(\mathcal{A}_F)$ of $\mathcal{A}_F$. For the algebra of interest in the standard model, namely $\mathcal{A}_F = \mathbb{H}\oplus\mathbb{C}\oplus M_3(\mathbb{C})$, the K theory group is $K_0(\mathcal{A}_F) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Let the generators of $K_0(\mathcal{A}_F)$ be [3]

$$p_1 = 1_{\mathbb{C}} \quad p_2 = 1_{\mathbb{H}} \quad p_3 = e$$

[3] in [26] a different basis for the generators is used namely

$$p_1 = (-1_{\mathbb{C}}) \oplus e \quad p_2 = 1_{\mathbb{C}} \oplus 1_{\mathbb{H}} \quad p_3 = 1_{\mathbb{C}}$$

this will not affect the calculations of whether or not the map 4.17 is invertible.

$$\text{with } e = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

For example, consider the representation $\lambda(q,c,m) = \lambda_w(q,c,m) \oplus \lambda_s(q,c,m)$ of $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ with $q \in \mathbb{H}, c \in \mathbb{C}$ and $m \in M_3(\mathbb{C})$.

$$\lambda_w(q,c,m) = \begin{bmatrix} q \otimes 1_3 & & & \\ & q & & \\ & & c1_6 & \\ & & & c \end{bmatrix} \tag{4.18}$$

$$\lambda_s(q,c,m) = \begin{bmatrix} 1_2 \otimes m & & & \\ & 1_2 c & & \\ & & 1_2 \otimes m & \\ & & & c \end{bmatrix}.$$

In this representation

$$\begin{aligned} \lambda(p_1) &= diag(0,0,1_6,1) && \oplus\ diag(0,1_2,0,1) \\ \lambda(p_2) &= diag(1_6,1_2,0,0) && \oplus\ diag(0,0,0,0) \\ \lambda(p_3) &= diag(0,0,0,0) && \oplus\ diag(1_2 \otimes e, 0, 1_2 \otimes e, 0) \\ \Gamma &= diag(-1_6,-1_2,1_6,1) && \oplus\ diag(-1_6,-1_2,1_6,1) \end{aligned}$$

then, by 4.17

$$
\begin{aligned}
(p_1, p_1) &= Tr(0,0,0,1) \oplus (0,0,0,1) \\
&= 2 \\
(p_2, p_2) &= Tr(0,0,0,0) \oplus (0,0,0,0) \\
&= 0 \\
(p_3, p_3) &= Tr(0,0,0,0) \oplus (0,0,0,0) \\
&= 0 \\
(p_1, p_2) &= Tr(0,0,0,0) \oplus (0,-1_2,0,0) \\
&= -2 \\
(p_1, p_3) &= Tr(0,0,1_2 \otimes e,0) \oplus (0,0,0,0) \\
&= 2 \\
(p_2, p_3) &= Tr(-1_2 \otimes e,0,0,0) \oplus (0,0,0,0) \\
&= -2
\end{aligned}
$$

where (a,b,c,d) denotes the matrix with diagonal entries (a,b,c,d).

So the matrix whose $i^{th} - j^{th}$ entry is (p_i, p_j) is

$$
\begin{bmatrix} 2 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}.
$$

The determinant of this matrix is non-zero so the representation 4.18 in this example satisfies the third Poincaré duality condition 4.17.

The aim then of the following calculation is to see, given the algebra representation associated to the strong force, what constraints the three Poincaré duality conditions place on the form of the weak force representation. Firstly a particle spectrum with no right hand neutrino is considered and then a particle spectrum including a right hand neutrino is considered.

Calculations (no right handed neutrinos)

The strong force representation (no right handed neutrino) is therefore taken to be the usual strong force representation of the non-commutative standard model

$$\lambda_s(q,c,m) = \begin{bmatrix} 1_2 \otimes m & & & \\ & \bar{c}1_2 & & \\ & & 1_2 \otimes m & \\ & & & \bar{c} \end{bmatrix} \quad q \in \mathbb{H},\ c \in \mathbb{C}, m \in M_3(\mathbb{C}). \tag{4.19}$$

The first Poincaré duality condition 4.1 immediately constrains the weak force representation to be block diagonal

$$\lambda_w(q,c,m) = \begin{bmatrix} A & & & \\ & B & & \\ & & C & \\ & & & D \end{bmatrix}.$$

The possible options for A, B, C and D are listed below, [q,c] denotes the block diagonal matrix $\begin{bmatrix} q & \\ & c \end{bmatrix}$.

	A	B	C	D
I	$1_2 \otimes m$	1_2c	$1_2 \otimes m$	c
II	$1_2 \otimes m$	1_2c	$q \otimes 1_3$	c
III	$1_2 \otimes m$	1_2c	$c1_6$	c
IV	$1_2 \otimes m$	q	$1_2 \otimes m$	c
V	$1_2 \otimes m$	q	$q \otimes 1_3$	c
VI	$1_2 \otimes m$	q	$c1_6$	c
VII	$q \otimes 1_3$	1_2c	$1_2 \otimes m$	c
VIII	$q \otimes 1_3$	1_2c	$q \otimes 1_3$	c
IX	$q \otimes 1_3$	1_2c	$c1_6$	c
X	$q \otimes 1_3$	q	$1_2 \otimes m$	c
XI	$q \otimes 1_3$	q	$q \otimes 1_3$	c
XII	$q \otimes 1_3$	q	$c1_6$	c
XIII	$c1_6$	1_2c	$1_2 \otimes m$	c
XIV	$c1_6$	1_2c	$q \otimes 1_3$	c
XV	$c1_6$	1_2c	$c1_6$	c
XVI	$c1_6$	q	$1_2 \otimes m$	c

	A	B	C	D
XVII	$c1_6$	q	$q \otimes 1_3$	c
XVIII	$c1_6$	q	$c1_6$	c
XIX	$1_2 \otimes [q,c]$	1_2c	$1_2 \otimes m$	c
XX	$1_2 \otimes m$	1_2c	$1_2 \otimes [q,c]$	c
XXI	$1_2 \otimes [q,c]$	1_2c	$1_2 \otimes [q,c]$	c
XXII	$1_2 \otimes [q,c]$	1_2c	$q \otimes 1_3$	c
XXIII	$1_2 \otimes [q,c]$	1_2c	$c1_6$	c
XXIV	$1_2 \otimes [q,c]$	q	$1_2 \otimes [q,c]$	c
XXV	$1_2 \otimes [q,c]$	q	$1_2 \otimes m$	c
XXVI	$1_2 \otimes m$	q	$1_2 \otimes [q,c]$	c
XXVII	$1_2 \otimes [q,c]$	q	$q \otimes 1_3$	c
XXVIII	$1_2 \otimes [q,c]$	q	$c1_6$	c
XXIX	$q \otimes 1_3$	1_2c	$1_2 \otimes [q,c]$	c
XXX	$q \otimes 1_3$	q	$1_2 \otimes [q,c]$	c
XXXI	$c1_6$	1_2c	$1_2 \otimes [q,c]$	c
XXXII	$c1_6$	q	$1_2 \otimes [q,c]$	c

<u>Note:</u> A shorthand has been employed in the above list of weak representations. 1_2c (in the B column) denotes any one of the four possible algebra representations of $\mathbb{C}$

$$\begin{bmatrix} c & \\ & c \end{bmatrix}, \begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} \bar{c} & \\ & \bar{c} \end{bmatrix} \text{ or } \begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}.$$

Similarly c (in the D column) denotes c or $\bar{c}$ and 1_6c (in the A and C column) denotes any one of the twelve matrices of the form $k \otimes 1_3$ or $1_2 \otimes h$ where k (resp. h) is a 2×2 (resp. 3×3) matrix with diagonal entries consisting of either c or $\bar{c}$. $1_2 \otimes m$, q, $q \otimes 1_3$ and [q,c] (columns A,B and C) similarly denote $1_2 \otimes m$ or $1_2 \otimes \overline{m}$; q or $\bar{q}$; $q \otimes 1_3$ or $\bar{q} \otimes 1_3$ and $\begin{bmatrix} q & \\ & c \end{bmatrix}, \begin{bmatrix} q & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} \bar{q} & \\ & c \end{bmatrix}, \begin{bmatrix} \bar{q} & \\ & \bar{c} \end{bmatrix}, \begin{bmatrix} c & \\ & q \end{bmatrix}, \begin{bmatrix} c & \\ & \bar{q} \end{bmatrix},$

$\begin{bmatrix} \bar{c} & \\ & q \end{bmatrix}$, or $\begin{bmatrix} \bar{c} & \\ & \bar{q} \end{bmatrix}$.

It is possible to use this shorthand because we are only interested in whether or not a representation is ruled out by the Poincaré duality conditions. Consider the four 2×2 matrices written above as $1_2 c$. If a weak representation containing $\begin{bmatrix} c & \\ & c \end{bmatrix}$ fails to commute with the strong representation (that is violates the first Poincaré duality condition 4.1) then replacing $\begin{bmatrix} c & \\ & c \end{bmatrix}$ with $\begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$, $\begin{bmatrix} \bar{c} & \\ & \bar{c} \end{bmatrix}$ or $\begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$ will not produce a weak representation that commutes with the strong representation. Similarly whether or not an algebra representation violates the third Poincaré duality condition 4.17 will not be affected by replacing $\begin{bmatrix} c & \\ & c \end{bmatrix}$ by $\begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$, $\begin{bmatrix} \bar{c} & \\ & \bar{c} \end{bmatrix}$ or $\begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$ since all the generators calculated are real. The second Poincaré duality condition 4.2 does not play a role in these calculations since it is trivially satisfied. So, 'approximating' the four matrices $\begin{bmatrix} c & \\ & c \end{bmatrix}$, $\begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$, $\begin{bmatrix} \bar{c} & \\ & \bar{c} \end{bmatrix}$ and $\begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$ by $\begin{bmatrix} c & \\ & c \end{bmatrix}$ in the following calculations will not result in a representation that does satisfy Poincaré duality being ruled out as not satisfying the Poincaré duality conditions. However if a representation containing $c1_2$ does satisfy all three Poincaré duality conditions it is not necessarily the case that replacing $\begin{bmatrix} c & \\ & c \end{bmatrix}$ by $\begin{bmatrix} c & \\ & \bar{c} \end{bmatrix}$, $\begin{bmatrix} \bar{c} & \\ & \bar{c} \end{bmatrix}$ or $\begin{bmatrix} \bar{c} & \\ & c \end{bmatrix}$ will result in a representation that satisfies Poincaré duality (at this stage the three other options will have to be checked by hand).

Of the thirty-two (69312 when the shorthand is expanded) options listed all but four (3×2^{10} when the shorthand is expanded) (IX, XII, XIV and XVII) are ruled out by one or more of the Poincaré duality conditions. Note that all of the allowed weak representations are vectorial. So, given the strong force representation 4.19 the conditions necessary for Poincaré duality constrain the weak force to be parity violating.

Calculations (right handed neutrino included)

Next the analysis is repeated but this time including a right handed neutrino in the particle spectrum so that the algebra representation associated to the strong force is

$$\lambda_s(q,c,m) = \begin{bmatrix} 1_2 \otimes m & & & \\ & \bar{c}1_2 & & \\ & & 1_2 \otimes m & \\ & & & \bar{c}1_2 \end{bmatrix} \quad q \in \mathbb{H},\ c \in \mathbb{C}, m \in M_3(\mathbb{C}). \tag{4.20}$$

There are sixty-four (207936 if the shorthand detailed above is fully expanded) possible weak representations acting on a particle spectrum with a right handed neutrino. Everyone of these is ruled out by at least one of the three Poincaré duality conditions. So, the rather strong conclusion can be reached that if the strong force representation is of the form 4.20 (as in the standard model) allowing for a right handed neutrino in the particle spectrum then a weak representation of any form cannot be constructed that will satisfy the Poincaré duality conditions.

4.5 Poincaré Duality in a 'General' Standard Model

In section 4.3 (resp. 4.4) the representation associated to the strong (resp. weak) force in the non-commutative standard model was assumed. In this section neither representation is assumed to see if it is still possible in this more general model to say anything about the relationship between the chiral structure of the two forces. This work is done as a precursor to research on the much harder question of whether or not any statements can be made about the chiral structure imposed by Poincaré duality in a general non-commutative Yang-Mills model. If nothing can be said about the generalised standard model then it follows that attempts at making statements about a general Yang-Mills model will be futile.

The questions that are addressed in this section are

1. Can any general conclusion be drawn about the necessity of having one force vectorial and one force parity violating?

2. Is a lepton-quark asymmetry, as conjectured by Martín et al [26], necessary for the Poincaré duality conditions to be satisfied?

3. Is a left-right fermion asymmetry necessary for the Poincaré duality conditions to be satisfied?

Assumptions and Notation

The most recent framework for the non-commutative standard model [8] as detailed in section 3.4 is employed. The gauge group of the general standard model is taken to be, as usual, $SU(2) \times U(1) \times SU(3)$ so $\mathcal{A}_F$ is taken to be $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. The particle spectrum (unless specified otherwise) is the usual one of $u_L, d_L, \nu_L, e_L, u_R, d_R, e_R$. However nothing is assumed about the weak and strong representations –that is nothing is assumed about the interaction between the electroweak and strong forces and the fermions of the model. Though it is assumed that λ_w is associated to a different force to λ_s (as is the case in the standard model). This is what is meant by the general standard model in this context.

Calculations

The three questions listed above are answered in turn. All details of the calculations are omitted as they are very similar to those in section 4.3 and 4.4

Question 1:

A force is said to be parity violating if its interaction with left handed particles is different from its interaction with right handed particles. Because the particle spectrum considered is asymmetric in left and right handed leptons (that is there is no right handed neutrino) it is difficult to define what it means for a force to be vectorial on the leptons. So, in answering question 1, only the quark interactions are considered.

It is found that, in order to satisfy all three Poincaré duality conditions, it is necessary to have one force vectorial and one force parity violating. That is all combinations of the weak and strong force representations that correspond to both forces being vectorial or both forces being parity violating are ruled out by at least one of the three conditions.

<u>Question 2:</u>

In answering this and the following question the particle spectrum is altered from the usual one to see if their are any 'essential features' of the usual standard model particle spectrum that enable it to satisfy Poincaré duality.

It is found that lepton-quark asymmetry is not a necessary condition for the Poincaré duality conditions to be satisfied. For instance the algebra representation

$$\lambda_w(q,c,m) = \begin{array}{c} \begin{matrix} u_L d_L & \nu_R & e_R \end{matrix} \\ \begin{pmatrix} q \otimes 1_3 & & \\ & \bar{c} & \\ & & c \end{pmatrix} \end{array}$$

$$\lambda_s(q,c,m) = \begin{array}{c} \begin{matrix} u_L d_L & \nu_R & e_R \end{matrix} \\ \begin{pmatrix} 1_2 \otimes m & & \\ & \bar{c} & \\ & & \bar{c} \end{pmatrix} \end{array}$$

satisfies all three Poincaré duality conditions and acts on a particle spectrum of two leptons and two quarks.

Question 3:

It is found that left-right fermion asymmetry is not necessary for Poincaré duality to be satisfied. For instance the representation

$$\lambda_w(q,c,m) = \begin{array}{c} \begin{array}{ccc} u_L d_L & u_R & e_R \end{array} \\ \begin{pmatrix} q\otimes 1_3 & & \\ & c\otimes 1_3 & \\ & & c \end{pmatrix} \end{array}$$

$$\lambda_s(q,c,m) = \begin{array}{c} \begin{array}{ccc} u_L d_L & u_R & e_R \end{array} \\ \begin{pmatrix} 1_2\otimes m & & \\ & m & \\ & & \bar{c} \end{pmatrix} \end{array}$$

on the Hilbert space with basis corresponding to two left handed fermions and two right handed fermions satisfies all three Poincaré duality conditions.

4.6 Conclusions

It can be concluded that, (assuming the particle spectrum $u_L, d_L, \nu_L, e_L, u_R, d_R, e_R$), a non-commutative Yang-Mills model with gauge group $SU(2)\times U(1)\times SU(3)$ is constrained to have one force vectorial and one force parity violating. It can also be seen that it is impossible to build any model that includes the right handed neutrino in the particle spectrum (assuming the usual form of the strong force).

From these calculations it can be seen that there are deep and intriguing links between the geometric structure of the non-commutative standard model and its chiral structure. This is in accordance with other observations about the non-commutative standard model which place its chiral structure 'centre stage'. That is the chiral structure is an integral part of the non-commutative standard model –in complete contrast to the usual formulation of the standard model where it is something of a curious anomaly. It can be argued that the discrete structure of the non-commutative manifold used in building the non-commutative standard model (which is directly linked to the existence of the Higgs sector) is related to the chiral structure of the standard model. This is perhaps most clearly seen in the non-commutative Glashow-Weinberg-Salam model where the left handed fermions can be interpreted as being on one sheet of the manifold (fibred over by SU(2)) and the right handed fermions on the other (fibred over by U(1)). The link between the chiral structure of a force and whether or not its gauge group is broken also supports the notion that chirality is a fundamental feature of the non-commutative standard model.

It would be interesting, given more time, to consider what constraints Poincaré duality places on the chiral structure of a general non-commutative Yang-Mills model.

Chapter 5

Non-Commutative Geometry and the Unimodularity Condition

5.1 Summary of this Chapter

In a non-commutative Yang-Mills model the gauge group G is obtained as the group of unitary elements of the algebra $\mathcal{A}$ describing the non-commutative manifold:

$$G = U(\mathcal{A}\ ) = \{a \mid a^*a = aa^* = 1\}.$$

For the standard model we require this gauge group to be $SU(2) \times U(1) \times SU(3)$. The group SU(2) can be obtained as the group of unitaries (unitary elements) of the algebra of the quaternions $\mathbb{H}$, the group U(1) as the group of unitaries of the algebra of the complex numbers $\mathbb{C}$ but SU(3) is the the group of unitaries of no algebra. The nearest group that can be obtained is U(3) (from the algebra $M_3(\mathbb{C})$). So, building a Yang-Mills model over the non-commutative manifold described by the algebra $\mathcal{A} = C^\infty(M) \otimes (\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}))$ yields a Lagrangian with gauge group $G = SU(2) \times U(1) \times U(3)$. Traditionally in the non-commutative standard model [7] the group U(3) is broken down to SU(3) essentially by identifying the U(1) component within the U(3)[1] with the U(1) of hypercharge. To be more specific, the subgroup U of G^o (the connected component in G containing the identity) is defined by

$$U := \{g = e^x \in G^o,\ tr[\Lambda(x)] = 0\}, \tag{5.1}$$

where $\Lambda(x)$ is the restriction of $\lambda_w(x) \oplus \lambda_s(x)$ to the particles. This, when applied to the standard model gauge group gives $U = SU(2) \times U(1) \times SU(3)$ due to the resulting condition $trA_3 = A_2$ where A_3 is the U(3) gauge boson and A_2 is the U(1) gauge boson. This is known as the unimodularity condition.

The unimodularity condition whilst being perhaps the most natural and simplest method of reducing the gauge group is not the only one. The more general definition of U as

$$U := \{g = e^x \in G^o,\ tr[\Lambda(xT)] = 0\} \tag{5.2}$$

[1] at the level of Lie Algebras $u(3) \cong su(3) \oplus u(1)$

$$\text{where } T = \begin{bmatrix} T_w & \\ & T_s \end{bmatrix} \quad T_w = T_s = \begin{bmatrix} 1_6 & & & \\ & 1_2\alpha & & \\ & & 1_6 & \\ & & & 1_2\alpha \end{bmatrix} \quad \alpha \in \mathbb{C}$$

also yields the required gauge group $U = SU(2) \times U(1) \times SU(3)$ when applied to the standard model but this time via the constraint $TrA_3 = \alpha A_2$. And, crucially, while 5.1 gives the correct hypercharges namely

u_L	d_L	ν_l	e_L	u_R	d_R	e_R
1/3	1/3	-1	-1	4/3	-2/3	-2

5.2 gives the more general (and for the quarks incorrect) hypercharges

u_L	d_L	ν_l	e_L	u_R	d_R	e_R
$\alpha/3$	$\alpha/3$	-1	-1	$1+\alpha/3$	$-1+\alpha/3$	-2

Such an approach is not ideal and could be considered to be rather ad hoc. It would be preferable for the gauge group of the model to be $SU(2) \times U(1) \times SU(3)$ from the beginning or at least that there was a unique method of reducing the gauge group from $SU(2) \times U(1) \times U(3)$ to $SU(2) \times U(1) \times SU(3)$ that also gave the required hypercharges.

E Alvarez et al [24] have argued that the unimodularity condition is equivalent to anomaly cancellation. They have shown that given the fermions in the SU(2) and SU(3) representations of the standard model (that is the left handed quarks in $\underline{2}_{SU(2)} \times \underline{3}_{SU(3)}$ the right handed electron in $\underline{1}_{SU(2)} \times \underline{1}_{SU(3)}$ etc.) then the requirement of anomaly cancellation exactly leads to the desired fermion hypercharges.

In this chapter an alternative approach to the unimodularity condition is considered. This approach (discussed in section 5.2) is based on an idea of Schücker and leads to an extra gauge boson. The calculations in this section were done with Schücker and Carminati.

Section 5.3 examines the unimodularity condition within the recent Chamseddine-Connes model. Section 5.4 is a short conclusion.

5.2 Fifth Force

One possible way round the problem of the unimodularity condition is to choose not to break down $G = SU(2) \times U(1) \times U(3)$ but instead to postulate a 'fifth force' associated with the extra U(1) factor. The purpose of this section is to examine the properties that such a fifth force would have and to see whether or not they are compatible with experimental data.

Consider the subspace of $su(2) \oplus u(1) \oplus u(3)$ associated to colourless, neutral gauge bosons, this is spanned by three generators (iB, iW^3, i$Z^\perp$)

$$\begin{aligned}
iB &= g_1\left[\ 0,\ \tfrac{i}{2},\ \tfrac{i}{6}1_3\ \right] \\
iW^3 &= g_2\left[\begin{pmatrix} i/2 & \\ & -i/2\end{pmatrix}, 0, 0\right] \\
iZ^\perp &= g_{z^\perp}\left[\ 0,\ (1+w)\tfrac{i}{2},\ -v\tfrac{i}{6}1_3\ \right]
\end{aligned}$$

where

$$v = \frac{9(Nx + \frac{3}{4}try + \frac{3}{4}tr\tilde{y})}{N\tilde{x}}$$

and

$$w = \frac{Nx + try}{Nx + \frac{1}{2}try + \frac{3}{2}tr\tilde{y}}.$$

The positive real constant x and the $N \times N$ diagonal matrices y and $\tilde{y}$ (with positive real entries) have arisen from z, the non-commutative coupling constant (described in section 3.3)

$$z = z_w \oplus z_s$$

$$z_w = \begin{bmatrix} x/3 1_2 \otimes 1_3 & & & \\ & 1_2 \otimes y & & \\ & & x/3 1_2 \otimes 1_3 & \\ & & & y \end{bmatrix}$$

$$z_s = \begin{bmatrix} \tilde{x}/3 1_2 \otimes 1_3 & & & \\ & 1_2 \otimes \tilde{y} & & \\ & & \tilde{x}/3 1_2 \otimes 1_3 & \\ & & & \tilde{y} \end{bmatrix}.$$

iB and iW^3 are the hypercharge and isospin generators respectively (iB is a linear combination of the two *u(1)* factors). The third generator i$Z^\perp$ is associated with the additional U(1) gauge boson that is projected out by the unimodularity condition in the usual formulation of the non-commutative standard model. It too is a linear combination of the two *u(1)* factors. Rotating the basis of the (iB, iW^3) vectors by the Weinberg angle θ_W results in the basis $(iQ, iZ, iZ^\perp)$

$$\begin{aligned} iQ &= \left[g_2 sin\theta_w \begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, \tfrac{i}{2} g_1 cos\theta_w, \tfrac{i}{6} 1_3 cos\theta_w \right] \\ &=: e \left[\begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, \tfrac{i}{2}, \tfrac{i}{6} 1_3 \right] \\ iZ &= \left[g_2 cos\theta_w \begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, -\tfrac{i}{2} g_1 sin\theta_w, -\tfrac{i}{6} 1_3 sin\theta_w \right] \\ &=: g_z \left[cos^2\theta_w \begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, -\tfrac{i}{2} sin^2\theta_w, -\tfrac{i}{6} 1_3 sin^2\theta_w \right] \end{aligned}$$

and $iZ^\perp$ as given above.

In the usual formulation (iQ, iZ) diagonalises the mass matrix and is thus normally associated with the physical bosons the photon and the Z boson respectively. However, since the $iZ^{\perp}$ boson couples to the Z boson this basis does not diagonalise the mass matrix if the $iZ^{\perp}$ boson is included. The basis that does diagonalise the mass matrix is (iQ, iV, iX).

$$iV = g_v \left[\begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, \tfrac{i}{2}, -\tfrac{i}{6} v 1_3 \right]$$

$$iX = g_x \left[\begin{pmatrix} i/2 & \\ & -i/2 \end{pmatrix}, -w\tfrac{i}{2}, 0 \right]$$

$$v = \frac{9(Nx + 3/4try + 3/4tr\tilde{y})}{N\tilde{x}} \qquad w = \frac{Nx + try}{Nx + 1/2try + 3/2tr\tilde{y}}$$

and iQ as given above.

Where the rotation matrix that has been used to rotate from the (iZ, $iZ^{\perp}$) to the (iV,iX) basis is

$$R = \begin{pmatrix} -cos\Gamma & -sin\Gamma \\ -sin\Gamma & cos\Gamma \end{pmatrix} \quad cos\Gamma = \frac{g_v}{[g_x^2 + g_v^2]^{1/2}}, \; sin\Gamma = \frac{g_x}{[g_x^2 + g_v^2]^{1/2}}.$$

iQ is the usual generator of U(1) charge, it is vectorial and therefore (as expected) $U(1)_Q$ remains unbroken and the photon remains massless. Instead of the usual massive Z boson there are now two new bosons the V and the X. From the vectorial form of iV it can be seen that the V boson is massless, similarly because of the non-vectorial form of iX X is massive. To see if such a scheme is compatible with experiment it is necessary to calculate the mass of the X boson.

Mass of the X Boson:

Gauge boson masses come from the $tr(D\Theta^{\dagger}D\Theta z)$ term in the (non-commutative) Yang-Mills Lagrangian, $D\Theta = i\partial\!\!\!/\Theta + A\Theta - \Theta A$ (equation (72) [11])

$$1/2M_x^2 = tr([\lambda(iX), \Theta]^{\dagger}[\lambda(iX), \Theta]z_w). \tag{5.3}$$

$\lambda(iX)$,Θ and z_w are defined as follows

$$\lambda(iX) = g_x \begin{bmatrix} a \otimes 1_N \otimes 1_3 & & & \\ & a \otimes 1_N & & \\ & & B \otimes 1_N \otimes 1_3 & \\ & & & B \otimes 1_N \end{bmatrix}$$

$$\text{with } a = \begin{bmatrix} i/2 & \\ & -i/2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -wi/2 & \\ & wi/2 \end{bmatrix}.$$

$$\Theta = \begin{bmatrix} 0 & 0 & (\Phi \otimes 1_N)M_q \otimes 1_3 & 0 \\ 0 & 0 & 0 & (\phi \otimes 1_N)M_l \\ M_q^\dagger(\Phi^\dagger \otimes 1_N) \otimes 1_3 & 0 & 0 & 0 \\ 0 & M_l^\dagger(\phi \otimes 1_N) \otimes 1_3 & 0 & 0 \end{bmatrix}$$

$$\text{with } \Phi = \begin{pmatrix} \overline{\phi}_2 & \phi_1 \\ -\overline{\phi}_1 & \phi_2 \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ the genuine Higgs doublet}$$

and the fermionic mass matrices M_q and M_l given by

$$M_q = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes M_u + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes M_d \right) \otimes 1_3$$

$$M_l = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes M_e,$$

where for N=3

$$M_u = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{bmatrix} \qquad M_d = C_{KM} \begin{bmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{bmatrix}$$

C_{KM} is the Cabbibo-Kobayashi-Maskawa matrix

$$C_{KM} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix}$$

and

$$M_e = \begin{bmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{bmatrix}.$$

$$z_w = \begin{bmatrix} x/3 1_2 \otimes 1_N \otimes 1_3 & & & \\ & 1_2 \otimes y & & \\ & & x/3 1_2 \otimes 1_N \otimes 1_3 & \\ & & & \otimes y \end{bmatrix} \text{ with } y = \begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}.$$

Substituting these into 5.3 gives

$$M_x^2 = k^2(w+1)^2 L g_x^2$$

where $L = (m_u^2 + m_c^2 + m_t^2)x + [|V_{ud}M_d|^2 + |V_{us}M_s|^2 + |V_{ub}M_b|^2 + |V_{cd}M_d|^2 + |V_{cs}M_s|^2 + |V_{cb}M_b|^2 + |V_{td}M_d|^2 + |V_{ts}M_s|^2 + |V_{tb}M_b|^2]x + m_e^2 y_1 + m_\mu^2 y_2 + m_\tau^2 y_3$ and k is the vacuum expectation value of the Higgs doublet. Similarly the mass of the W boson is found to be

$$M_w^2 = k^2 L g_2^2$$

so

$$M_x = (1+w)\frac{g_x}{g_2} M_w.$$

The coupling constants g_2 and g_x are chosen so that the field strength terms in the Yang-Mills Lagrangian are normalised to $1/4F_{\mu\nu}F^{\mu\nu}$, this yields

$$\begin{aligned} g_2^{-2} &= Nx + try \\ g_x^{-2} &= Nx + try + \frac{w^2}{2}(2Nx + try + 3tr\tilde{y}) \end{aligned}$$

so

$$M_x^2 = \frac{2Nx + 3/2try + 3/2tr\tilde{y}}{Nx + 1/2try + 3/2tr\tilde{y}} M_w^2.$$

$$\text{Now} \quad g_1^{-2} = Nx + 2/9N\tilde{x} + 1/2try + 3/2tr\tilde{y}$$

$$\text{and} \quad g_3^{-2} = 4/3N\tilde{x}$$

so the mass of the X boson can be rewritten as

$$M_x^2 = \frac{6g_2^2g_3^2 + 6g_1^2g_3^2 - g_1^2g_2^2}{(6g_3^2 - g_1^2)g_2^2}M_w^2. \tag{5.4}$$

Experimentally the values of these coupling constants and the mass of the W boson are known [67]

$$g_1 = 0.3575 \pm 0.0001$$

$$g_2 = 0.6505 \pm 0.0007$$

$$g_3 = 1.207 \pm 0.026$$

$$M_w = 80.22 \pm 0.26GeV$$

substituting these into 5.4 yields $M_x = 91.69 \pm 0.3GeV$.

X and V Coupling Strengths:

The X and V boson couplings are

$$g_x^{-2} = Nx + try + w^2/2(2Nx + try + 3tr\tilde{y})$$

and

$$g_v^{-2} = 2Nx + 3/2try + 3/2tr\tilde{y} + 2v^2N\tilde{x}/9.$$

Rewriting these in terms of g_1, g_2 and g_3 yields

$$g_x^2 = \frac{g_2^4(g_3^2 - 1/6g_1^2)}{g_2^2(g_3^2 - 1/6g_1^2) + g_1^2g_3^2}$$

and

$$g_v^2 = \frac{g_3^{-2}}{6(g_1^{-2} + g_2^{-2})(g_1^{-2} + g_2^{-2} - 1/6g_3^{-2})},$$

substituting in the experimental values for g_1, g_2 and g_3 gives

$$g_x \sim 0.57$$

$$g_v \sim 0.03$$

So, if the unimodularity condition is not imposed the gauge boson spectrum consists of a neutral massless gauge boson with very weak coupling ($g_v \sim 0.03$), a massive gauge boson (mass=$91.69 \pm 0.3 GeV$) with moderately weak coupling ($g_x \sim 0.57$) and the usual photon, $W^{\pm}$ bosons and gluons.

<u>**Comparison with Experiment:**</u>

The boson spectrum calculated above is phenomenologically unacceptable.

5.3 The Unimodularity Condition in the Universal Chamseddine-Connes Action

Chamseddine and Connes argue [30][31] that the physical gauge fields arise as fluctuations of the metric where the metric is now defined as

$$d(\chi, \xi) = sup\{\mid \chi(a) - \xi(a) \mid : a \in \mathcal{A}; \| [\tilde{D}, \lambda(a)] \| \leq 1\} \tag{5.5}$$

where $\tilde{D} = D + A + JAJ^{\dagger}$, A the gauge potential of the theory. Note the change from the previous definition 2.3. If the standard model is considered then the diagonal elements of the gauge potential restricted to the particles (before the unimodularity condition has been imposed) are

$$A_{diag} = \begin{bmatrix} A_1 \otimes 1_3 & & & \\ & A_1 & & \\ & & \begin{bmatrix} A_2 & \\ & -A_2 \end{bmatrix} \otimes 1_3 & \\ & & & -A_2 \end{bmatrix} \oplus \begin{bmatrix} 1_2 \otimes A_3 & & & \\ & -A_2 1_2 & & \\ & & 1_2 \otimes A_3 & \\ & & & -A_2 \end{bmatrix}$$

where iA_1 is an *su(2)* valued gauge field, iA_2 is a *u(1)* valued gauge field and iA_3 is a *u(3)* valued gauge field. So

$$\begin{aligned}
[\tilde{D}_{diag}|_{particles}]_{11} &= i\partial\!\!\!/ \otimes 1_2 \otimes 1_3 + A_1 \otimes 1_3 + 1_2 \otimes A_3 \\
[\tilde{D}_{diag}|_{particles}]_{22} &= i\partial\!\!\!/ \otimes 1_2 + A_1 - A_2 1_2 \\
[\tilde{D}_{diag}|_{particles}]_{33} &= i\partial\!\!\!/ \otimes 1_2 \otimes 1_3 + \begin{bmatrix} A_2 & \\ & -A_2 \end{bmatrix} \otimes 1_3 + 1_2 \otimes A_3 \\
[\tilde{D}_{diag}|_{particles}]_{44} &= i\partial\!\!\!/ - 2A_2
\end{aligned}$$

Since the metric is determined by 5.5 it can be seen that removing any elements from $\tilde{D}_{diag}$ which commute with $\lambda(a)$ will not effect the metric of the theory as $\tilde{D}$ enters as a commutator $[\tilde{D}, \lambda(a)]$. So, replacing A_3 (the *u(3)* valued gauge field) by $\tilde{A}_3$ (an *su(3)* valued gauge field) will not effect the metric as the *u(1)* component is proportional to the identity matrix.

Chamseddine and Connes use this argument to give the unimodularity condition a more natural, less ad hoc footing. However it does not seem clear why such an argument would not also lead to the (phenomenologically unacceptable) removal of the *u(1)* field A_2.

5.4 Conclusions

To date there is no satisfactory method for reducing the gauge group of the non-commutative standard model. In previous incarnations of the non-commutative standard model [7] there were two unimodularity conditions (and a clumsy algebraic structure involving two algebras $\mathcal{A}$ and $\mathcal{B}$ that were Poincaré dual to one another). This was later refined [8] to the current situation where there is just a single unimodularity condition (and a more economic algebraic structure with just one algebra $\mathcal{A}$). It must be hoped that another fundamental revision of the non-commutative standard model removes this final unimodularity condition at some point in the future.

Chapter 6

SU(2|1) 'Weak Unification'

6.1 Summary of this Chapter

This chapter explores the possibility of unifying weak SU(2) and electromagnetic U(1) in a single graded gauge group SU(2|1). The first section, section 6.2, surveys attempts at grand unification within the programme of non-commutative geometry. Section 6.3 contains introductory material about SU(2|1), or more precisely about its graded Lie algebra *su(2/1)*, and explains the motivation of this chapter. Section 6.4 discusses the construction of Hermitian representations. Section 6.5 contains the calculations aimed at unifying the weak and electromagnetic forces and the final section, section 6.6, is a conclusion.

6.2 Grand Unification within Non-Commutative Geometry

One very popular way to reduce the apparent arbitrariness of the standard model is to embed its gauge group $SU(2) \times U(1) \times SU(3)$ into a larger simple gauge group with a single coupling constant –the concept of grand unification. A natural question to ask is "Can any of the Grand Unified models be realised within the Connes-Lott non-commutative scheme?" The answer to this question [44][12] appears to be no. Below, the popular grand unified models [65][66] are listed and reasons why they are not compatible with non-commutative geometry are outlined. It should be noted that this refers to a strict interpretation of the Connes-Lott Yang-Mills model building scheme. Other methods of constructing non-commutative models [28][29] do permit grand unification.

- SU(5)

  Minimal SU(5) is unobtainable as a Connes-Lott model since the fermions are required to sit in a $\bar{\mathbf{5}} + \mathbf{10}$ representation. Whilst the $\bar{\mathbf{5}}$ is fundamental the

10 is not and, as already explained, in the Connes-Lott model all fermions appear in the fundamental representation.

- **SO(10)**

  SO(10) unification is ruled out by non-commutative geometry for two reasons. Firstly, [29][12] it is not possible to break the initial left-right symmetry (as required) in the non-commutative framework. Secondly as for SU(5) the fermions are accommodated in a non-fundamental representation.

- **E(6), E(7) and E(8)**

  Grand unification schemes have been proposed [66] based on the exceptional Lie groups E(6), E(7) and E(8). As explained in section 3.5 none of these are obtainable within the Connes-Lott scheme [13] as the exceptionals are the group of unitaries of no semi-simple algebra.

- **Unification by a Semi-Simple Group**
  $SU(4)_{PS}\times SU(2)_L\times SU(2)_R$

  This unification scheme is ruled out by non-commutative geometry [44] since it requires the generalised Dirac operator to contain Majorana mass terms which connect the particle and anti-particle sector. Such operators would violate the second Poincaré duality condition. Furthermore the $SU(4)_{PS}\times SU(2)_L\times SU(2)_R$ unified model is a left-right symmetric model and therefore not realisable within the Connes-Lott scheme.

So, it can be seen that all popular grand unified theories based on simple gauge groups, and indeed the 'halfway house' of unification based on a semi-simple group, are ruled out by Connes-Lott non-commutative geometry. The aim of the chapter is to explore the possibility of a weaker model in which just the electromagnetic and the weak force are unified in a single group (the strong force is excluded) and to see if this is compatible with non-commutative geometry.

6.3 The Graded Lie Algebra *Su(2/1)*

The Lie algebra *su(2/1)* is $\mathbb{Z}_2$ graded. That is it has even generators (which close into themselves under commutation and therefore generate an ordinary Lie algebra) and odd generators (which close into the whole algebra under anti-commutation). In the case of *su(2/1)* there are eight generators. Four even ones, denoted Y, I_i, $i = 1 \cdots 3$, which generate the underlying Lie algebra *su(2)*$\times$*u(1)* and four odd ones denoted Ω_a, Ω'_a $a = 1, 2$. The commutation/anticommutation relations can be written as

$$
\begin{array}{lcllcl}
[I_i, I_j] & = & i\mathcal{E}_{ijk} I_k & [I_3, \Omega'_1] & = & 1/2\Omega_1 \\
[I_i, Y] & = & 0 & [Y, \Omega_a] & = & -\Omega_a \\
[I_i, \Omega_1] & = & +1/2(\sigma_i)_{1b}\Omega_b & [Y, \Omega'_a] & = & \Omega'_a \\
[I_i, \Omega'_2] & = & -1/2(\sigma_i)_{b2}\Omega'_b & \{\Omega_a, \Omega_a\} & = & 0 \\
[I_1, \Omega_2] & = & 1/2\Omega_1 & \{\Omega_a, \Omega_b\} & = & 0 \\
[I_2, \Omega_2] & = & -1/2i\Omega_1 & \{\Omega'_a, \Omega'_a\} & = & 0 \\
[I_3, \Omega_2] & = & -1/2\Omega_2 & \{\Omega'_a, \Omega'_b\} & = & 0 \\
[I_1, \Omega'_1] & = & -1/2\Omega'_2 & \{\Omega_a, \Omega'_a\} & = & I_1 - i(-1)^a I_2 \\
[I_2, \Omega'_1] & = & -1/2i\Omega'_2 & \{\Omega_a, \Omega'_b\} & = & 1/2Y + (-1)^b I_3.
\end{array}
$$

An irreducible representation (irrep) of *su(2/1)* has at most four *su(2)* $\times$ *u(1)* multiplets [55]. All finite dimensional irreps of *su(2/1)* contain a multiplet with isospin i and hypercharge y, at most one multiplet with isospin $i - 1/2$ and hypercharge $y - 1$, at most one multiplet with isospin $i - 1/2$ and hypercharge $y + 1$ and at most one multiplet with isospin $i - 1$ and hypercharge y. They split into five cases:

1. the trivial one dimensional representation of *su(2/1)*.
2. a 4i+1 dimensional representation with $-y/2 = i \geq 0$ containing the multiplets $|y, i, i_3>$ and $|y - 1, i - 1/2, i_3>$.
3. a 4i+1 dimensional representation with $y/2 = i \geq 0$ containing the multiplets $|y, i, i_3>$ and $|y + 1, i - 1/2, i_3>$.

4. a four dimensional representation containing the multiplets $|y, 1/2, \pm 1/2>$, $|y-1, 0, 0>$ and $|y+1, 0, 0>$.

5. an 8i dimensional representation with $-y/2 \neq \pm i$ containing all four possible *su(2)* $\times$ *u(1)* multiplets ie $|y, i, i_3>$, $|y-1, i-1/2, i_3>$, $|y+1, i-1/2, i_3>$ and $|y, i-1, i_3>$.

The special graded Lie algebras *su(n/m)* $n > m \geq 1$ (also denoted spl(n,m) in the literature) obey the condition of supertracelessness (the counterpart to tracelessness in the special Lie algebras *su(n)*). The algebra *su(n/m)* can be represented as the set of $(n+m) \times (n+m)$ matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{array}{l} \text{a an n} \times \text{ n matrix, d an m} \times \text{ m matrix} \\ \text{b an n} \times \text{ m matrix, c an m} \times \text{ n matrix.} \end{array}$$

The Lie algebra consists of the diagonal block matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and the odd subspace consists of off diagonal block matrices $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. The supertrace (or graded trace) is defined to be

$$str(A) = tr(a) - tr(d).$$

More details of graded Lie algebras can be found in [51][52][53][55].

$SU(2|1)$ was first considered as a possible gauge group in the late 70's [45][46]. However gauging the odd sector of the group [50] leads to bosons with anti-commutation properties –a clear violation of the spin-statistics theorem. Attempts [47][48] at avoiding this problem by introducing anticommuting supergroup parameters also led to the introduction of ghosts which could not be removed. So work on *su(2/1)*, at least by physicists, was largely abandoned [1]until recently when it was reintroduced by Coquereaux [39]. Interestingly Coquereaux et al and Ne'eman et al have

[1]with the exception of Ne'eman who continued to work in this field, see for example [56]-[59]

both studied *su(2/1)* in the context of non-commutative geometry. However the role that *su(2/1)* plays in their work is completely different from each other and completely different from the role that it will take here.

Su(2/1) is of interest to particle physicists in general and non-commutative geometrers in particular for several reasons listed below.

1. The Irreducible Representations

   The irreps of *su(2/1)* (as listed above) can accommodate all the particles of the standard model. If we take i=1/2 in case 2 a 3 dimensional representation with a doublet of hypercharge –1 (identifiable with the left lepton doublet (ν_L,e_L)) and a singlet of hypercharge –2 (identifiable with e_R) is obtained. If we take y=1/3 in case 4 an irrep containing a y=1/3 doublet, a y=4/3 singlet and a y=–2/3 singlet is obtained. Which is suitable for describing the left quark doublet (u_L,d_L) and the two right singlets u_R and d_R. Further generations of leptons and quarks can be accommodated by taking direct sums of these irreps. The gauge particles can be described by the supermultiplet of case 4. If y=0 and i=1 then this multiplet contains an *su(2)* triplet of zero hypercharge (which can be identified with the gauge fields W_1, W_2 and W_3); a singlet of zero hypercharge (identifiable with the gauge field B) and two doublets of hypercharge +1/2 and –1/2 (identifiable with the Higgs and Higgs conjugate fields respectively). The appearance of the Higgs boson in the same multiplet as the traditional gauge bosons is of course very reminiscent of the situation in non-commutative geometry. So, in summary all the particles of the standard model fit into three basic representations of one group

$$\begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix} \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} \begin{pmatrix} W^i \\ B \\ \phi \\ \phi^c \end{pmatrix}$$

as opposed to the usual inelegant formulation which fits the particles into nine representations of two groups

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \begin{pmatrix} e_R \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \begin{pmatrix} u_R \end{pmatrix} \begin{pmatrix} d_R \end{pmatrix}$$

$$\begin{pmatrix} W^i \end{pmatrix} \begin{pmatrix} B \end{pmatrix} \begin{pmatrix} \phi \end{pmatrix} \begin{pmatrix} \phi^c \end{pmatrix}$$

(Here W^i i=1...3 is an *su(2)* triplet and ϕ and ϕ^c are *su(2)* doublets).

2. The Indecomposable Representations

In most Yang-Mills models extra generations of fermions are entered trivially simply by tensoring by extra identical representations. This leads to Lagrangians in which the fermions just interact with other fermions of the same generation –mixing between generation (as observed experimentally) has to be added in by hand. Graded Lie algebras have some very interesting representations that might yield more sophisticated methods of introducing extra fermionic generations that automatically lead to mixing between the generations [39]. These representations are called reducible indecomposable representations. Unlike normal Lie groups not all reducible [2] representations of a graded Lie group are decomposable [3]. In particular there exists reducible indecomposable representations that would be ideal for two (or three) generations

[2] A representation is said to be reducible if it is equivalent to a representation of the form

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \quad \forall g \in G.$$

[3] A representation is said to be decomposable if it is equivalent to a representation of the form

$$D(g) = \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix} \quad \forall g \in G.$$

of quarks. These representations are obtained by taking semi-direct sums of two (or three) copies of the 4 dimensional quark irrep previously described. Interestingly if we assume that there is no right handed neutrino then there is no similar indecomposable representation for the leptons that is this model cannot describe lepton mixing (unless there is a right neutrino) –in agreement with experiment.

3. Chirality

   The standard model fails to explain why the right handed particles are singlets under SU(2) whilst the left handed particles transform as doublets. In *su(2/1)* the chirality of matter is given by the grading of the Lie algebra and hence their different transformation laws are entirely natural. For example consider the lepton multiplet

$$\psi = \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix}.$$

   Its transformation as a fundamental representation $\psi \to U\psi$ automatically leads to the correct (different) transformations for the left handed and the right handed particles. It should be noted that parity invariance is a fundamental feature of Yang-Mills models with *su(2/1)* gauge groups just as it is a fundamental feature of non-commutative Yang-Mills-Higgs models.

For all these reasons $SU(2|1)$ is interesting as a possible gauge group. In particular its use leads naturally to work [45] which it can be argued foreshadowed one of the major claims of non-commutative Yang-Mills models –the proposition that the Higgs field be regarded as a gauge boson. For this reason and because of the failure to incorporate any other unification scheme into non-commutative geometry I believe that it is worthwhile question to ask whether or not some form of unification can be achieved using *su(2/1)* within the context of non-commutative geometry.

6.4 Hermitian Representations and the Quark Multiplet

Given the normal adjoint operation on the even elements of a graded Lie algebra there are two possible definitions of the generalisation of the adjoint operation to the odd elements [54] –the adjoint operation (denoted $\dagger$) and the graded adjoint operation (denoted $\ddagger$).

Definition Adjoint Operation

An adjoint operation in a graded Lie algebra L is a mapping

$$\begin{array}{ccc} L & \longrightarrow & L \\ A & \mapsto & A^{\dagger} \end{array}$$

such that

1. the adjoint of an even (odd) operator is even (odd)

2. $(aA+bB)^{\dagger} = \overline{a}A^{\dagger} + \overline{b}B^{\dagger}$

3. $(A,B)^{\dagger} = (B^{\dagger}, A^{\dagger})$

4. $(A^{\dagger})^{\dagger} = A$

where A and B are elements of L and $a, b \in \mathbb{C}$.

Definition Grade Adjoint Operation

A grade adjoint operation in a graded Lie algebra L is a mapping

$$\begin{array}{ccc} L & \longrightarrow & L \\ A & \mapsto & A^{\ddagger} \end{array}$$

such that

1. the adjoint of an even (odd) operator is even (odd)

2. $(aA+bB)^{\ddagger} = \overline{a}A^{\ddagger} + \overline{b}B^{\ddagger}$

3. $(A,B)^\ddagger = (-1)^{\delta_A\delta_B}(B^\ddagger, A^\ddagger)$

4. $(A^\ddagger)^\ddagger = (-1)^{\delta_A} A$

where A and B are homogeneous elements of L of degree δ_A and δ_B respectively and $a, b \in \mathbb{C}$.

This area is being explored because it is necessary to have hermitian operators in order that the physical transformations under SU(2|1) be unitary.

The adjoint operator can be defined on the fundamental representation of *su(2/1)* in the usual way

$$I_i^\dagger = I_i \quad Y^\dagger = Y \quad \Omega_1^\dagger = -\Omega_2' \quad \Omega_2^\dagger = -\Omega_1 \quad \Omega_1'^\dagger = -\Omega_2 \quad \Omega_2'^\dagger = -\Omega_1$$

and linear combinations of the odd operators taken so that all the generators are hermitian.

However for the 4 dimensional representation (used in model building for accommodating the quarks) the situation is not so simple. To see why this is the case consider the odd generators in the 4d representation

$$\Omega_1 = \begin{bmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \qquad \Omega_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_1' = \begin{bmatrix} 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Omega_2' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha\gamma = 1/2 + y/2 \quad \beta\epsilon = 1/2 - y/2.$$

Consider first the construction of the adjoint operator. The closure of the adjoint operation can be achieved in an infinite number of ways, or (upto positive real scalars) in two ways. The first of these two ways, such that

$$\Omega_1^\dagger = \Omega_2' \quad \Omega_2'^\dagger = \Omega_1 \quad \Omega_2^\dagger = \Omega_1' \quad \Omega_1'^\dagger = \Omega_2$$

is obtained by imposing $\overline{\gamma} = \alpha$ and $\overline{\beta} = -\epsilon$ which clearly leads to the requirement $y > 1$. The second way, such that

$$\Omega_1^\dagger = -\Omega_2' \quad \Omega_2'^\dagger = -\Omega_1 \quad \Omega_2^\dagger = -\Omega_1' \quad \Omega_1'^\dagger = -\Omega_2$$

is obtained by imposing $\overline{\gamma} = -\alpha$ and $\overline{\beta} = \epsilon$ which leads to the requirement $y < -1$. Neither of these choices of hermitian representation is suitable for describing the quarks since, as explained in section 6.3 it is required that $y = 1/3$ in order that the decomposition of the 4 dimensional *su(2/1)* representation under $su(2) \times su(1)$ is such that the quarks can be accommodated.

As it is not possible to construct a suitable hermitian representation via the adjoint operation consider the possibility of constructing a grade hermitian representation using the grade adjoint operation. If we write a general operator in L in block diagonal matrix form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix representation of $A^\ddagger$ is

$$A^\ddagger = \begin{bmatrix} a^\dagger & -c^\dagger \\ b^\dagger & d^\dagger \end{bmatrix}$$

where $\dagger$ within the bracket denotes normal hermitian conjugation of the matrix. There are two possible choices of the parameters $(\alpha,\ \beta,\ \gamma,\ \epsilon)$ in the 4d irrep to make the grade adjoint operation close into the generators. The first choice $\overline{\alpha} = \gamma$, $\overline{\epsilon} = \beta$ leads to the following relations

$$\Omega_1^\ddagger = \Omega_2' \quad \Omega_2'^\ddagger = -\Omega_1 \quad \Omega_2^\ddagger = \Omega_1' \quad \Omega_1'^\ddagger = -\Omega_2$$

and to the restriction $-1 < y < 1$. The second choice $\overline{\alpha} = -\gamma$, $\overline{\epsilon} = -\beta$ leads to the relations

$$\Omega_1^\ddagger = -\Omega_2' \quad \Omega_2'^\ddagger = \Omega_1 \quad \Omega_2^\ddagger = -\Omega_1' \quad \Omega_1'^\ddagger = \Omega_2$$

and to the restriction $y < -1$, $1 < y$. The first of these choices is able to accommodate the quarks (y=1/3) however due to the grading it is not possible to construct a grade hermitian representation. Therefore it is not possible to chose a basis (in this example or in general) such that the odd generators are represented by self-grade-adjoint operators that is such that $A^\ddagger = A$ or such that $A^\ddagger = (-1)^{\delta_A} A$. It can be seen from condition (4) of the definition of the grade adjoint that it is impossible to construct self-grade-adjoint odd operators. So for this reason the generalised adjoint (the grade adjoint) is not suitable for constructing physical models.

This results of this section effectively rule out the 4 dimensional irrep as a suitable representation for the quarks. This result, though initially disappointing, is in fact in complete agreement with non-commutative geometry which asserts that fermions must only be accommodated in the fundamental representation. And perhaps makes physical sense as an *su(2/1)* theory with quarks would mean quarks with no strong force. It would be interesting to see if any of the graded Lie algebras (such as *su(5/2)* or *su(7/1)*) which contain *su(3)*$\times$ *su(2)* $\times$ *u(1)* in their underlying Lie algebra have a fundamental representation that can be made hermitian and which can accommodate all the leptons and quarks.

6.5 Calculations

Due to the unresolved spin-statistic problems incurred on gauging the full *su(2/1)* algebra an alternative approach is proposed here. Only the even part (*su(2)* $\times$ *u(1)*) of the graded Lie algebra will be gauged, invariance under the whole group will be global. This is achieved within the context of a non-commutative model by tensoring the finite algebra that is associated with *su(2)* $\times$ *u(1)* by the infinite space-time dependent algebra $\mathcal{A}_I = C^\infty(M)$ but not tensoring the finite algebra associated with the odd sector of *su(2/1)* by $\mathcal{A}_I$. So the K cycle with which the model is built is $(\mathcal{A}, \overline{\mathcal{H}}, \overline{D})$. The algebra $\mathcal{A}$ is as described above

$$\mathcal{A} = [C^\infty(M) \otimes (\mathbb{H} \oplus \mathbb{C})] \oplus M_3(\mathbb{C}).$$

The Hilbert space $\overline{\mathcal{H}}$ is two copies of the leptonic Hilbert space (for convenience only one generation of leptons is considered) so

$$\overline{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \quad \mathcal{H} = L^2(M) \otimes (\mathbb{C}^2 \oplus \mathbb{C}).$$

The basis of this Hilbert space is

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, e_R, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, e_R.$$

The generalised Dirac operator $\overline{D}$ is given by two copies of the usual generalised Dirac operator:

$$\overline{D} = D \oplus D$$

$$D = \begin{bmatrix} i\partial\!\!\!/ \otimes 1_2 & \gamma_5 \otimes M \\ \gamma_5 \otimes M^\dagger & i\partial\!\!\!/ \end{bmatrix} \quad M = \begin{pmatrix} 0 \\ m_e \end{pmatrix}.$$

$\mathcal{A}$ is faithfully represented on $\overline{\mathcal{H}}$ by λ

$$\lambda(a) = \begin{bmatrix} f \otimes q & & \\ & f \otimes c & \\ & & 1_4 \otimes m \end{bmatrix} \quad f \in C^\infty(M, \mathbb{R}),\ q \in \mathbb{H},\ c \in \mathbb{C},\ m \in M_3(\mathbb{C}).$$

The global invariance group is reduced from U(3) to $SU(2|1)$ by imposing supertracelessness. The fundamental representation of *su(2/1)* is worked with. Note that the basis of the odd generators has been changed from that in section 6.3 so that all the generators are self-adjoint:

$$I_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad I_2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad I_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \Omega_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Omega_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Omega_1' = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \Omega_2' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$$

The Fermionic Lagrangian

A one form in $\Omega\mathcal{A}$ has, as usual, the form $\rho = a_0 d_u a_1$

$$\begin{aligned}
\Pi(\rho) &= -i\lambda(a_0)[\overline{D}, \lambda(a_1)] \\
&= -i \begin{bmatrix} f_0(i\partial\!\!\!/ f_1) \otimes q_0 q_1 & \gamma_5 f_0 f_1 \otimes q_0(c_1 - q_1)M & 0 \\ \gamma_5 f_0 f_1 \otimes M^\dagger c_0(q_1 - c_1) & f_0(i\partial\!\!\!/ f_1) \otimes c_0 c_1 & 0 \\ 0 & 0 & \gamma_5 \otimes m_0[\mu, m_1] \end{bmatrix} \\
&=: -i \begin{bmatrix} A_1 & \gamma_5 h M & 0 \\ \gamma_5 M^\dagger g & A_2 & 0 \\ 0 & 0 & \gamma_5 \otimes C \end{bmatrix}
\end{aligned}$$

where $\mu = \begin{bmatrix} 0 & M \\ M^\dagger & 0 \end{bmatrix}$. We wish $\Pi(\rho)$ to be valued in the graded Lie algebra *su(2/1)* so we impose anti-hermiticity and supertracelessness. That is impose $A_1^\dagger = A_1$, $A_2^\dagger = A_2$, $C^\dagger = C$, $g = h^\dagger$ and $Str(C) = 0$. A_1 and A_2 are genuine gauge fields (they are space-time dependent) but C is not. Constructing the fermionic Lagrangian as usual via

$$A_F = (\psi, (D + i\Pi(\rho))\psi)$$

yields

$$\mathcal{L}_F = f_L^\dagger(i\partial\!\!\!/\otimes 1_2 + A_1)f_L + e_R^\dagger(i\partial\!\!\!/ + A_2)e_R + f_L^\dagger(\gamma_5\Phi M)e_R + e_R^\dagger(\gamma_5 M^\dagger\Phi^\dagger)f_L + \psi^\dagger(\gamma_5\otimes C)\psi.$$

Since C is space-time independent it is an non-dynamical field (it has no kinetic term). Differentiating $\mathcal{L}_F$ with respect to C yields the constraints

$$\begin{aligned}
\nu_L^\dagger\gamma_5\nu_L + e_R^\dagger\gamma_5 e_R &= 0\\
e_L^\dagger\gamma_5 e_L + e_R^\dagger\gamma_5 e_R &= 0\\
\nu_L^\dagger\gamma_5 e_L &= 0\\
e_L^\dagger\gamma_5\nu_L &= 0\\
\nu_L^\dagger\gamma_5 e_R &= 0\\
e_R^\dagger\gamma_5\nu_L &= 0\\
e_L^\dagger\gamma_5 e_R &= 0\\
e_R^\dagger\gamma_5 e_L &= 0
\end{aligned} \tag{6.1}$$

These constraints are automatically satisfied once the fermionic Lagrangian has been Wick rotated into Minkowski space. With the physical field assignments

$$\begin{aligned}
A_1^\mu &= -1/2 g\sigma\cdot W^\mu\\
A_2^\mu &= -g' B^\mu\\
\Phi &= \begin{pmatrix} \bar{\phi}_2 & \phi_1 \\ -\bar{\phi}_1 & \phi_2 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{the genuine Higgs doublet,}
\end{aligned}$$

this leads to the fermionic Lagrangian

$$\begin{aligned}
\mathcal{L}_F &= f_L^\dagger\gamma^\mu(i\partial_\mu - 1/2 g\sigma\cdot W_\mu)f_L + e_R^\dagger\gamma^\mu(i\partial_\mu - g' B_\mu)e_R\\
&\quad + m_e f_L^\dagger\gamma_5\phi e_R + m_e e_R^\dagger\gamma_5\phi^\dagger f_L.
\end{aligned}$$

The Yang-Mills Lagrangian

To calculate the Yang-Mills Lagrangian it is necessary to calculate the curvature θ of the one form ρ using $\Pi(\theta) = \Pi(\rho^2) + \Pi(d\rho)$:

$$\Pi(d_u \rho_u) = -[D, \lambda(a_0)][D, \lambda(a_1)]$$

giving

$$
\begin{aligned}
\Pi(d_u\rho)_{11} &= +(\not{\partial} f_0)(\not{\partial} f_1) \otimes q_0 q_1 + \\
&\quad -f_0 f_1 \otimes (c_0 - q_0) M M^\dagger (q_1 - c_1) \\
\Pi(d_u\rho)_{12} &= -(i\not{\partial} f_0)\gamma_5 f_1 \otimes q_0(c_1 - q_1) M + \\
&\quad -\gamma_5 f_0 (i\not{\partial} f_1) \otimes (c_0 - q_0) c_1 M \\
\Pi(d_u\rho)_{13} &= 0 \\
\Pi(d_u\rho)_{21} &= -\gamma_5 f_0 (i\not{\partial} f_1) \otimes M^\dagger (q_0 - c_0) q_1 + \\
&\quad -(i\not{\partial} f_0)\gamma_5 f_1 \otimes M^\dagger c_0 (q_1 - c_1) \\
\Pi(d_u\rho)_{22} &= +(\not{\partial} f_0)(\not{\partial} f_1) \otimes c_0 c_1 + \\
&\quad -f_0 f_1 \otimes M^\dagger (q_0 - c_0)(c_1 - q_1) M \\
\Pi(d_u\rho)_{23} &= 0 \\
\Pi(d_u\rho)_{31} &= 0 \\
\Pi(d_u\rho)_{32} &= 0 \\
\Pi(d_u\rho)_{33} &= -1_4 \otimes [\mu, m_0][\mu, m_1]
\end{aligned}
\tag{6.2}
$$

and $\Pi(\rho^2) = -\lambda(a_0)[D,\lambda(a_1)]\lambda(a_0)[D,\lambda(a_1)]$ giving

$$\begin{aligned}
\Pi(\rho^2)_{11} &= +f_0(\partial\!\!\!/ f_1)f_0(\partial\!\!\!/ f_1)\otimes q_0q_1q_0q_1 + \\
&\quad -f_0f_1f_0f_1\otimes q_0(c_1-q_1)MM^\dagger c_0(q_1-c_1) \\
\Pi(\rho^2)_{12} &= -f_0(i\partial\!\!\!/ f_1)\gamma_5f_0f_1\otimes q_0q_1q_0(c_1-q_1)M + \\
&\quad -\gamma_5f_0f_1f_0(i\partial\!\!\!/ f_1)\otimes q_0(c_1-q_1)c_0c_1M \\
\Pi(\rho^2)_{13} &= 0 \\
\Pi(\rho^2)_{21} &= -\gamma_5f_0f_1f_0(i\partial\!\!\!/ f_1)\otimes M^\dagger c_0(q_1-c_1)q_0q_1 + \\
&\quad -f_0(i\partial\!\!\!/ f_1)\gamma_5f_0f_1\otimes M^\dagger c_0c_1c_0(q_1-c_1) \\
\Pi(\rho^2)_{22} &= +f_0(\partial\!\!\!/ f_1)f_0(\partial\!\!\!/ f_1)\otimes c_0c_1c_0c_1 + \\
&\quad -f_0f_1f_0f_1\otimes M^\dagger c_0(q_1-c_1)q_0(c_1-q_1)M \\
\Pi(\rho^2)_{23} &= 0 \\
\Pi(\rho^2)_{31} &= 0 \\
\Pi(\rho^2)_{32} &= 0 \\
\Pi(\rho^2)_{33} &= -1_4\otimes C^2.
\end{aligned} \tag{6.3}$$

Now it is necessary to quotient by the graded differential ideal $\Pi(J^2)$. $\Pi(J^2)_{ij}$ $i,j=1\cdots 2$ is as given in Section 3 equation 3.2 (so it is possible to use an extension of the map P (eqn. 3.3) calculated explicitly there), $\Pi(J^2)_{33}$ is of the form

$$\Pi(J^2)_{33} = 1_4\otimes m \quad m\in M_3(\mathbb{C})$$

and all the other $\Pi(J^2)_{ij}$ are zero. So, applying the quotient map P to $\Pi(\theta)$ yields

$$
\begin{aligned}
\Pi(\theta)_{11} &= -1/2\gamma^\mu\gamma^\nu i F^1_{\mu\nu} - (\Phi^\dagger\Phi - 1)M_L + \tfrac{Tr_N(M_L)}{N}(\Phi^\dagger\Phi - 1) \\
\Pi(\theta)_{12} &= -D\Phi\gamma_5 M \\
\Pi(\theta)_{13} &= 0 \\
\Pi(\theta)_{21} &= M^\dagger(D\Phi)^\dagger\gamma_5 \\
\Pi(\theta)_{22} &= -1/2\gamma^\mu\gamma^\nu i F^2_{\mu\nu} - M^\dagger(\Phi^\dagger\Phi - 1)M + 2\tfrac{Tr_N(M_L)}{N}[\Phi^\dagger\Phi - 1]_{22} \\
\Pi(\theta)_{23} &= 0 \\
\Pi(\theta)_{31} &= 0 \\
\Pi(\theta)_{32} &= 0 \\
\Pi(\theta)_{33} &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
F^i_{\mu\nu} &:= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - i[A^i_\mu, A^i_\nu] \\
\Phi &:= h + 1 \\
D\Phi &:= i\partial\!\!\!/\Phi - \Phi A^2 + A^1\Phi \\
M_L &:= 1/2 \begin{bmatrix} m_e m_e^\dagger & & & \\ & m_\mu m_\mu^\dagger & & \\ & & m_\tau m_\tau^\dagger & \\ & & & \ddots \end{bmatrix},
\end{aligned}
$$

and the identity $MM^\dagger = [I - \sigma_3] \otimes M_L$ has been used. It can be seen that there is no contribution from the odd sector of *su(2/1)* to the Yang-Mills Lagrangian. The Lagrangian is calculated to be

$$
\begin{aligned}
\mathcal{L}_{YM} &= Ng^2 W_{\mu\nu}\cdot W^{\mu\nu} + 2Ng'^2 B_{\mu\nu}B^{\mu\nu} + 16tr(M_L)\,(D_\mu\phi)^\dagger(D^\mu\phi) + \\
&\quad +6[tr_N(M_L^2) - \tfrac{tr_N(M_L)^2}{N}][(\phi^\dagger\phi)^2 - 2\phi^\dagger\phi - 1]
\end{aligned}
$$

where

$$
\begin{aligned}
W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\mathcal{E}_{ijk}W^j_\mu W^k_\nu \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\
D_\mu\phi &= i\partial_\mu\phi + 1/2g' B_\mu\phi - 1/2g\sigma\cdot W_\mu\phi.
\end{aligned}
$$

Gauge Invariance

The calculated Lagrangian $\mathcal{L}_{YM} + \mathcal{L}_F$ is invariant under an *su(2/1)* transformation where the even part is local and the odd part is global that is it is invariant under

$$\begin{aligned} \psi &\mapsto u\psi \\ A &\mapsto uAu^\dagger + u[D, u^\dagger] \end{aligned}$$

where $u = e^{i\alpha(x)\cdot I + i\beta(x)Y + i\delta\cdot\Omega}$ (note that α and β are space-time dependent and that the δ are not). However this transformation needs to be examined more carefully. Consider the infinitesimal transformation of A. Denote the generators of *su(2/1)* by K_i and the parameters of the transformation by t^i then

$$\begin{array}{lcllcll} K^i &=& I^i \quad i = 1\cdots 3 & t^i &=& \alpha^i(x) & i = 1\cdots 3 \\ K^4 &=& Y & t^4 &=& \beta(x) & \\ K^{5,6} &=& \Omega_{1,2} & t^j &=& \delta^j & j = 5\cdots 8 \\ K^{7,8} &=& \Omega'_{1,2} & & & & \end{array}$$

where in this abbreviated notation u is written $u = e^{iK\cdot t}$. Then, to first order in t,

$$A\cdot K \mapsto A\cdot K + [iK\cdot t, A\cdot K] + [iK\cdot t, D]$$

which will not close into the graded Lie algebra for all K^i since odd generators close under anticommutation not commutation.

Generalising slightly and replacing the commutator [,] by a generalised bracket $[\![\, ,\,]\!]$

$$\begin{aligned} [\![\text{even}, \text{even}]\!] &:= [\text{even}, \text{even}] \\ [\![\text{even}, \text{odd}]\!] &:= [\text{even}, \text{odd}] \\ [\![\text{odd}, \text{odd}]\!] &:= \{\text{odd}, \text{odd}\} \end{aligned}$$

leads to a transformation

$$A\cdot K \mapsto A\cdot K + [\![\, iK\cdot t, A\cdot K]\!] + [\![\, iK\cdot t, D]\!]$$

which closes under the algebra. It can be shown, (using the constraints 6.1), that the fermionic Lagrangian is invariant under this transformation. However it does not appear to be possible to show that such a transformation forms a representation of the graded Lie algebra.

6.6 Conclusions

It does not appear to be possible to construct a non-commutative Yang-Mills model based on *su(2/1)*. The problems met are associated with using a graded gauge group rather than with non-commutative geometry.

Chapter 7

Conclusions

At low energies non-commutative geometry appears to be an extremely satisfactory tool for Yang-Mills model building. It provides a beautiful, geometric explanation for many features of the standard model. The Higgs particles are described geometrically as the gauge bosons associated with gauging the discrete structure of space-time. Charge conjugation (and its non-commutative generalisation) appears naturally, in fact is essential, for a complete description of smooth manifolds via Poincaré duality. This in turn forces the algebra to have a bimodule structure - perfect for accommodating the strong force. It has also been shown that the chiral structure of the standard model has a deeply geometric origin and that this in turn is vital to the process of spontaneous symmetry breaking since the existence of a Higgs sector is a consequence of parity violation in the model.

At higher energies non-commutative physical models run into many problems. It is widely believed that the standard model is only a low energy approximation to a 'true' description of particle behaviour. However the very restrictive nature of non-commutative geometry makes it strangely incompatible with most higher energies theories. No grand unified theory seems to be realisable within the non-commutative geometric program. Similarly the high energy unification of the standard model and the Einstein-Hilbert action [30][31] contains many problems. Whilst conceptually the unification of gravity and gauge theories on a geometric footing is very pleasing the model does have its faults. First and foremost the theory is non-unitary [31], its numerical prediction are untenable [21] and of course the action is a Euclidean action.

So non-commutative geometry is in quite an unusual position vis-a-vis physics. It provides a highly convincing description of the standard model and is a great improvement on the usual formulation. However extending the model in the usual ways familiar to physicists appear very difficult. Perhaps this is a strong point of non-commutative geometry, that it will only admit a very small number of theories, that the correct high energy description has yet to be formulated and that non-commutative geometry will indicate the way forward to such a theory. Chamseddine and Connes suggest that one interpretation of the problems at higher energies could be that the concept of space-time as a manifold may be inadequate at small distances and that this would also need to be described by a non-commutative algebra.

Appendix A

Definitions and Conventions

A.1 Definitions

The following definitions are taken from [62], [61] and [70]

- Banach Algebra

  A Banach algebra is an algebra A that is also a Banach space (completely normed space) with respect to the norm $\|\cdot\|$ that satisfies

  i)the multiplicative inequality $\| xy \| \leq \| x \| \| y \|$ for all x,y in A

  ii)if A contains a unit e then $\| e \| = 1$

- B* Algebra

  A B* algebra is a Banach algebra A with an involution *

$$\begin{array}{cccc} * : & A & \longrightarrow & A \\ & x & \longmapsto & x^* \end{array}$$

  that satisfies $\| xx^* \| = \| x \|^2$ for all x in A

- C* Algebra

A C* algebra A is a B* algebra in which the involution is the adjoint of a matrix or an operator on Hilbert space

- Semi-Simple Algebra

A Banach algebra A is said to be semi-simple if the intersection of the kernels of all irreps of A is null. All B* algebras are semi-simple [71]

- Symbol of a Differential Operator

Consider D a differential operator mapping between sections of vector bundles (E and F) over a manifold M of dimension d

$$D : \Gamma(M, E) \to \Gamma(M, F).$$

Let U be a chart of M whose local co-ordinates are denoted x^μ over which E and F are trivial. Adopting the notation of [61] we write

$$\begin{aligned} T &= (\mu_1, \mu_2, ..., \mu_d) \quad \mu_j \in \mathbb{Z},\ \mu_j \geq 0 \\ |T| &= \mu_1 + \mu_2 + ... + \mu_d \\ D_T &= \frac{\partial^{|T|}}{\partial x^T} = \frac{\partial^{\mu_1+\mu_2+\dots+\mu_d}}{\partial(x^1)^\mu_1 ... \partial(x^d)^\mu_d}. \end{aligned}$$

Then, if the dimension of E is k and the dimension of F is k' the most general form of D is

$$[Ds(x)]^\alpha = \sum_{1\leq a\leq k} \sum_{|T|\leq N} A_a^{T\,\alpha} D_T s^a(x) \quad 1 \leq \alpha \leq k'$$

where $s(x) \in \Gamma(M, E)$ and N is the order of D. The symbol of D is then defined to be the k×k' matrix

$$\sigma(D, \xi) = \sum_{|T|=N} A_a^{T\,\alpha}(x)\, \xi_T$$

where ξ is a real d-tuple $\xi = (\xi_1, \xi_2, ..., \xi_d)$ and ξ_T is defined to be

$$\xi_T := \xi_1^{\mu_1} + \xi_2^{\mu_2} + \cdots + \xi_d^{\mu_d}.$$

○ Elliptic Operator

A differential operator D is said to be elliptic if the symbol of D is invertible for each $x \in M$ and each $\xi \in \mathbb{R}^d - \{0\}$.

○ Kernel and Cokernel of an Elliptic Operator

The kernel and cokernel of an elliptic operator D

$$D : \Gamma(M, E) \rightarrow \Gamma(M, F)$$

are defined as follows

$$\begin{aligned} kerD &:= \{s \in \Gamma(M, E) | Ds = 0\} \\ cokerD &:= \tfrac{\Gamma(M,F)}{imD}. \end{aligned}$$

○ Fredholm Operator

An elliptic operator D is said to be Fredholm if ker D and coker D are finite dimensional.

A.2 Conventions

A.2.1 Gamma Matrices

Throughout this thesis the following representation of the Euclidean gamma matrices has been used:

$$\begin{aligned} \gamma^0 &= -I \otimes \sigma_1 \\ \gamma^i &= \sigma_i \otimes \sigma_2 \\ \gamma^5 &= \gamma^0\gamma^1\gamma^2\gamma^3. \end{aligned}$$

With this notation the γ^μ and γ^5 are self-adjoint and C the charge conjugation matrix is $C = \sigma_2 \otimes \sigma_3$.

A.2.2 Involutions

The following notation for involutions has been adopted

$\overline{a}$	denotes the complex conjugate of a
$A^{\dagger}$	denotes the hermitian conjugate of A
x^{*}	denotes a general involution on x.

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