On Breathers in Affine Toda Theories

by

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To my Mother and in memory of my Father

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Abstract

Oscillating solitonic solutions, the *breathers*, of affine Toda theory are studied. These breather solutions are constructed from two solitons of the same mass with velocity opposite of each other; by analytically continuing its velocity or rapidity to a complex value, the resulting solution becomes a periodic solution. Generally, the parameters in the soliton solutions are restricted to a certain range of definition. In particular, it is shown for $a_n^{(1)}$ and $d_4^{(1)}$ cases, these restrictions can be calculated explicitly.

To some cases of $a_n^{(1)}$ theories, one can show that there are sine-Gordon embedded solitons which give rise to a sine-Gordon breather.

Furthermore, these breather solutions carry topological charges. These topological charges are calculated and it is found that they are exactly the same as the topological charges of some single soliton cases. Moreover, for the non-zero topological charges, one can show they belong to the irreducible fundamental representation component of the tensor product of two fundamental representations associated with the constituent solitons. This Clebsch-Gordan decomposition property is in agreement with the fusing rule of soliton which in turn is similar to the fusing rule of the fundamental Toda particles. One can also make a conjecture that the zero topological charge is always carried by a breather whose constituent solitons are associated with either conjugate or self-conjugate fundamental representations. Although it is not possible to know the individual topological charge carried by the constituent solitons in a breather, nevertheless using the crossing symmetry similar to that of the crossing symmetry of the S-matrix, one can perform a superficial calculation to determine the constituent soliton's topological charges.

Attempts to understand the exact scattering matrices of the sine-Gordon solitons and breathers from a root space point of view is also discussed. This study tries to mimic the

exact S-matrix construction of the real coupling regime affine Toda theory from the root space by Dorey. In this study, one replaces the ordinary Coxeter element, which plays an important rôle in the real coupling regime, with other transformations to incorporate the infinite product nature of the sine-Gordon soliton scattering matrix. However, the desired consistent construction seems to elude the author in this study.

Preface

This thesis is based on my research, carried out between April 1992 and March 1995. The material presented has not been submitted previously for any degree in either this or any other University.

No claim of originality is made for the work contained in the review parts of this thesis (*Chapter Two* and *Four*) unless stated otherwise. *Section One, Two* and *Three* of *Chapter Three* are based on a paper in collaboration with Uli Harder and William A. McGhee [47]. While the rest of *Chapter Three* and *Chapter Five* are my own work.

I am indebted to my supervisor Ed Corrigan for his continuing support and helpful suggestions and comments. I would also like to thank all of those with whom I have had discussions over the past three years: Peter Bowcock, Patrick Dorey, Uli Harder, Richard Hall, Niall Mackay, William McGhee and Gérard Watts.

My stay in Durham has been most enjoyable, with the company of my friends Adrian Askew, Adnan Bashir, Haluk Bayrakdar, David Bull, Richard Costambeys, Antonis Giolas, Ayşe Kizilersü, Marlene Lopez, Nick Myers and Eugenia Petropoulou and many others.

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Statement of Copyright

Contents

Chapter 1

Introduction

Prelude to a Kiss

Edward Kennedy Ellington

This thesis is a study of some aspects of the *complex* Affine Toda theories. A particular attention is given to the $a_n^{(1)}$ series of the theory.

The affine Toda theories became an interest of the author because of the various algebraic properties which have been discovered recently. The introduction of affine Toda theories to the author was the series of classic papers of Olive-Turok [3, 4] and the more recent papers on its S-matrices by Braden-Corrigan-Dorey-Sasaki [8, 9]. It was the work of Dorey [21, 23] which motivated the author to the study of the algebraic techniques in affine Toda theories. In these works, the ordinary Coxeter element plays an important rôle in the real coupling regime. This has led the author to the study of a general Coxeter element of the Weyl group, the *affine* Coxeter element. It was hoped that this natural generalization of the ordinary Coxeter element will shed a new light to the imaginary coupling regime of the theories. In particular, on the relation of its soliton solutions and the root space. This study, in someway, is a speculative study, as the complex nature of the Hamiltonian is not well understood, especially the question of the existence of its quantum theory is not settled (in the sense of the soliton spectrum and its mass corrections [33, 34]).

The known classical soliton spectrum [11, 12, 13, 15, 35] carries topological charges which lie in the fundamental representation associated with each soliton. Generally, the soliton spectrum does not fill the whole space of its fundamental representation space. On the other hand, the quantum states are conjectured to fill the whole fundamental representation space [14]. Thus, one is led to find additional soliton solutions. One posibility is oscillating soliton solutions, the *breathers*. In a collaboration with U. Harder and W. McGhee, the author has found the breather solutions made up of two solitons in the $a_n^{(1)}$ theories [47]. It has been shown by the author that to have a non singular breather solution, one has to restrict the *displacement* parameter and the oscillating velocity (rapidity). Furthermore, from the two possible ways to build this breather, the author has shown that the breather topological charges always fall in the irreducible component of the tensor product of the fundamental representations associated with the constituent solitons. Moreover, it is found that some of the zero topological charge breathers are a sine-Gordon embedded breather. The non-zero topological charges were found to be exactly the same as topological charges of a single soliton. Thus, these breathers do not create any new topological charges such that the fundamental representation space is filled up. Calculations on other theories (in particular the $d_4^{(1)}$ theory) show a similar pattern of results. That is, the non zero topological charges carried by a breather is exactly the same as the topological charge of a certain single soliton case. The understanding of the discrepancy between the classical and quantum states of this imaginary coupling regime of the affine Toda theories is far from complete.

In general, this thesis results from these somewhat parallel studies of the relation of the sine-Gordon soliton and breather spectrum with root space and the breather spectrum of the affine Toda theories.

Organisation of this thesis is as follows.

Chapter Two provides a review of the Affine Toda field theories. A brief account of two-dimensional field theory will be given. Integrability of a two-dimensional theory enables one to solve the theory exactly. Selection rules of processes obtained from the integrability lead to the factorization of an N-body process into a product of 2-body processes. In the quantum theory of two-dimensional field theories, the S-matrix factorises. An example of an integrable two-dimensional field theory is the affine Toda field theory. The classical aspect of affine Toda theories will be presented followed by its quantum theory and the S-matrices. In the *imaginary* coupling parameter regime, the affine Toda theories will give rise to soliton solutions which carry topological charges. Thus they come in a degenerate state. Because of this degeneracy, to calculate the S-matrix, one has to solve the Yang-Baxter equation. This chapter is closed with a detailed discussion on several methods to obtain the soliton solutions.

Having shown how to construct the soliton solutions of affine Toda theories, *Chapter Three* discusses a special type of soliton solution in the affine Toda theories. These soliton solutions are periodic in time and it is customary to call them *breathers.* The sine-Gordon case will be presented as a guiding prescription in the construction of the breather solutions. Taking the sine-Gordon prescription and applying to the $a_n^{(1)}$ series, one obtains breather solutions of the associated theory. The breather solutions are shown to be of two types. Type A breathers have constituent solitons of the same species whereas type B breathers have constituent solitons of opposite species. The topological charges of these breathers are then determined and shown to lie in an irreducible component of the tensor product of fundamental representation associated with the constituent solitons. Further examination shows that some of the type A breathers are sine-Gordon embedded breathers. A crossing of one of the constituent solitons in a breather results in the crossing of a type A breather into a type B breather and vice versa. In particular, it will be shown that this crossing symmetry can be used to superficially calculate the topological charge of the constituent solitons. Breathers in other theories are also discussed, in particular breathers of $d_4^{(1)}$ theory are calculated in detail using the algebraic τ-function. Examination of the representation where the topological charges of these breathers lie shows the same pattern of result, i.e. the non zero topological charges always lie in the irreducible fundamental representation component of the tensor product of the fundamental representations associated to the constituent solitons. This Clebsch-Gordan decomposition property is in agreement with and gives a further support to the fusing of solitons. The zero topological charges are also carried by a breather with constituent solitons which come from conjugate (or self-conjugate) fundamental representations, such that in the Clebsch-Gordan decomposition there is a trivial singlet component.

Chapter Four reviews the interesting relation between root space of the underlying simple Lie algebra of the affine Toda theories and the S-matrix structure of the real coupling parameter regime of the theory [21, 23]. This relation hinges on the structure of the Coxeter element of the Weyl group. The distinct Coxeter orbits of roots leads to the *fusing rule* of the Toda fundamental particles. Using data obtained from these Coxeter orbits, the S-matrices of fundamental Toda particles can be written down from an S-matrix *building block*. These S-matrices are finite products of periodic functions. The main part of this chapter is devoted to the Coxeter element and some of its properties. An account of the Coxeter element in the simple Lie algebras and the affine Kac-Moody algebras will open the chapter. It is then followed with a review of the S-matrix construction via the Coxeter element for the real coupling parameter Toda theories.

From the established result of the S-matrices for the sine-Gordon theory solitons and its breathers [1], one tries to follow Dorey's algebraic construction of the S-matrices in *Chapter Five*. Several observations lead to the idea behind this work. First, there is a natural extension of the Coxeter element, which has a finite order, into the *affine* Coxeter element which has an infinite order. Secondly, root triangles in the affine root space can also be constructed from the affine Coxeter orbits. Finally, as discussed in *Chapter Three*, the imaginary coupling parameter regime has a richer solution. So, one is led to speculate that the affine Coxeter element can play the same rôle in the complex coupling regime as the ordinary Coxeter element will play in the real coupling regime. However, subsequent calculations show that associating the soliton and breather spectrum with the orbits of the affine Coxeter transformation is too naïve.

Conclusion and an outlook is given in *Chapter Six*.

Having in mind a self-contained thesis, some algebraic materials are provided in the *Appendices.* In particular, *Appendix* **A** contains an exposition on several basis of the Kac-Moody algebra and its root space. It is complemented with a short review on Quantum groups. *Appendix* **B** explains the reciprocal polynomial which is used in the discussion on the order of the affine Coxeter transformation.

Chapter 2

Aspects of Affine Toda Field Theory

String of Pearls

Glenn Miller

Two dimensional field theory has been the subject of a vast amount of research interest. This is due to the richness of the two dimensional theory, such as integrability and conformal invariance. Conformal invariance results from a symmetry under the general coordinate transformation. Due to its integrability, a two dimensional theory has an exact solution [1]. Toda theories [2] are examples of integrable two dimensional field theories.

The Conformal Toda theories [3, 4] are a class of massless scalar field theories with exponential interaction between the scalar fields. This interaction term is specified by the simple roots of the underlying Lie algebra for the Toda theory. The addition of a mass term to the Lagrangian will break the conformal invariance.

A perturbation of conformal Toda theories which retains their integrability will result in a massive scalar field theory [5]. This massive field theory is achieved by introducing a perturbation term in the Lagrangian in the form of the exponential interaction term. The perturbation term depends on the extended root which together with the simple roots of the associated Lie algebra constitute the simple roots system of an *affine* Kac-Moody algebra. In particular, for the *untwisted* Kac-Moody algebras, the extended root is the negative of highest root of the associated Lie algebra [6, 7]. Thus, the resulting field theories are called the Affine Toda theories [8, 9].

Conformal invariance of the affine Toda theories can be restored by the addition of an auxiliary field [10, 12, 13]. These integrable and conformally invariant massive field theories are known as the Conformal Affine Toda theories.

In the affine and the conformal affine Toda theories, one can take a *purely* imaginary coupling parameter [11, 12]. The potentials of these *complex* theories have multiple vacua. There exist complex soliton solutions which have real energy and momentum which interpolate between these vacua. Further examination reveals that these complex theories also have oscillating soliton solutions, the *breathers*, which will be discussed in *Chapter Three.*

Although most parts of this chapter are applicable to the *simply-laced* affine Toda theories, only the results of $a_n^{(1)}$ series will be presented. This chapter provides a review on some of the structure of a two dimensional field theory [1]. It is then followed by a review of classical affine Toda theories and a brief account of the quantum theories in the real coupling regime

[8, 9]. The quantum theory of the *imaginary* coupling regime is different compared with the real coupling regime. In this imaginary coupling regime, the classical Toda solitons are degenerate in mass, hence scattering processes can be *non-diagonal* [1, 14]. Several methods to the construction of the soliton solutions namely the Hirota's method [11, 15], the algebraic method [12, 13] and the Bäcklund transformation [16] for the $a_n^{(1)}$ theories will be presented.

2.1 Two-dimensional Quantum Field Theory

The momentum of a particle a in two-dimensional space-time is denoted in terms of the rapidity variable, θ_a, as

$$\mathbf{p}_a = m_a(\cosh\theta_a, \sinh\theta_a),$$

such that the velocity of a particle is given by $v_a = \tanh\theta_a$. Let Q_s denote a locally conserved operator. Under the Lorentz transformation L, Q_s transforms as a rank s tensor,

$$[L, Q_s] = sQ_s.$$

Q_s is called a *spin s* conserved operator. Integrability of a field theory implies an infinite number of locally conserved commuting operators. In an integrable theory, a single particle state must be represented by the simultaneous eigenvectors of these operators with eigenvalues $q_s^a(\theta)$, called the *spin s* conserved quantity. Thus, for a particle a with momentum p,

$$Q_s|p_a(\theta)> = q_s^a(\theta)|p_a(\theta)> .$$

Under a Lorentz transformation the rapidity θ of the one particle state, $|p_a(\theta)>$, changes by the parameter ε of the Lorentz transformation,

$$e^{-\varepsilon.L}|p_a(\theta)> = |p_a(\theta+\varepsilon)> .$$

Thus, Lorentz covariance of the one particle state fixes the following dependence on rapidity θ and spin s of the conserved quantities $q_s^a(\theta)$,

$$q_s^a(\theta) = q_s^a e^{s\theta}. \tag{2.1.1}$$

Consider a multiparticle in-state denoted by $|p_1, p_2, \ldots, p_n>$ and an out-state denoted by $|p'_1, p'_2, \ldots, p'_{n'}>$. By the locality nature of the conserved operators, it acts additively on a multiparticle state. Hence conservation laws for conserved quantities of *spin s* are given as,

$$\sum_{i=1,\text{ in}}^{n} q_s^i(p_i) = \sum_{i=1,\text{ out}}^{n'} q_s^i(p'_i).$$

The existence of infinite conservation laws such as above, leads to a strict selection rule [1],

- The number of particles with mass m_a in the in-state and out-state are the same.
- The set of momenta of incoming and outgoing particles are the same.

Thus, in a scattering process there is no particle production. In fact, these selection rules lead to the *factorization* of the S-matrix . There are several arguments for this, one of them will be presented below.

In a massive theory, the interaction between particles happen in a finite range, R. Consider the position configuration phase space of a three-body scattering. A typical coordinate in this configuration space will be (x_1, x_2, x_3), where x_i corresponds to the position of particle i. If the particles are far apart, the wave function of this state in the configuration space will be a linear combination of three-dimensional plane wave functions permitted by the selection rule above. The in-state and out-state of scattered particles are represented by an extrapolation of these plane waves. The task to extrapolate within the interaction range is very complicated. But, one can consider an extrapolation of wave functions when only two of the particles are within the interaction range. For example, suppose particles 1 and 2 come close together while particle 3 is far. In the configuration space, scattering of particles 1 and 2 happen on the plane $x_1 = x_2$. The extrapolation of wave functions can be done quantum mechanically by inserting the interaction potential which starts at the distance $\frac{R}{2}$ normal to the plane $x_1 = x_2$. In the interpolation, the x_3-component of the wave function is fixed. The extrapolation of the in-state and out-state wave functions of an N-body scattering can be done by considering interactions of two particles at a time. This means that any N-body scattering can be written as a product of two-body scattering processes. This is the *wave function* argument of the Zamolodchikovs, [1]. Besides this argument, there is also the *wave packet* (also known as the *particle displacement*) argument by Shankar and Witten, [17]. A rigorous axiomatic argument can be found in the works of Iagolnitzer, [18]. In S-matrix language, it is said that the S-matrix factorizes into a product of two-particle S-matrices.

Furthermore, since the S-matrix factorizes, the two-particle S-matrix has to satisfy a cubic equation. This cubic equation results from examining the processes depicted in the following figure.

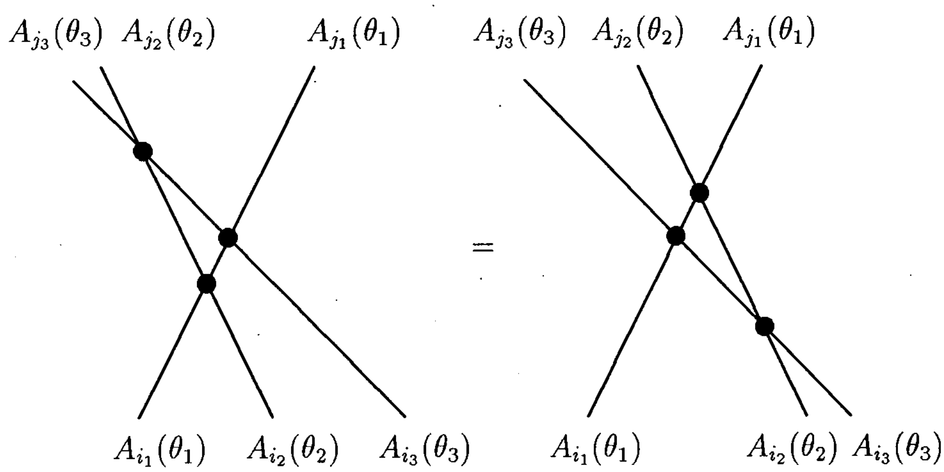

Figure 2.1: Factorization of S-matrix.

These two processes must describe an identical scattering process. It is easy to see this using the *particle displacement* argument, see [17, 19]. This argument states that, when acting on a localized wave packet, the conserved operator will only shift the position of the wave packet peak. Since the conserved operator commutes with the Hamiltonian, the action of a conserved operator will not change the scattering process. Hence one can shift any wave packet without changing the S-matrix. From the figure, we can read off the *Yang-Baxter equation* or also known as the *Factorization equation* of the S-matrices as follows,

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 i_3}^{j_1 k_3}(\theta_{13}) S_{k_2 k_3}^{j_2 j_3}(\theta_{23}) = S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) S_{i_1 k_3}^{k_1 j_3}(\theta_{13}) S_{i_2 i_3}^{k_2 k_3}(\theta_{23}) \tag{2.1.2}$$

where, $\theta_{ab} = \theta_a - \theta_b$. Thus only two-body scattering is needed. Write the two-particle S-matrix with the particles labelled A_i and parametrised by its rapidity θ as,

$$|A_i(\theta_1), A_j(\theta_2) >_{in} = S_{ij}^{kl}(\theta_1 - \theta_2)|A_k(\theta_1), A_l(\theta_2) >_{out} . \tag{2.1.3}$$

In the above expression, summation runs through all possible states allowed by the selection rules.

Note that a theory in which the particle spectrum contains no mass degenerate multiplets, the Yang-Baxter equation becomes trivial. Moreover, there is no summation in equation (2.1.3) above, and the S-matrix is just a phase factor. In this case, one does not need to solve the Yang-Baxter equation, and the form of the S-matrices are guessed. However, this S-matrices have to satisfy the following constraints: unitarity, crossing symmetry properties, and the bootstrap principle (see below). This is the case of affine Toda field theory with real coupling parameter.

The rapidity θ can be exchanged with the Mandelstam variable s,

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta_{ab},$$

or,

$$\theta_{ab} = \cosh^{-1}\frac{s - m_a^2 - m_b^2}{2m_a m_b}.$$

Analytic continuation of the S-matrix to the complex s-plane, yields two square root branch cuts at $(m_a \pm m_b)^2$. As a function of the rapidity difference θ_{ab}, the S-matrix admits several poles. Only those poles which lie in the physical strip $0 \leq \theta_{ab} \leq i\pi$ correspond to physical processes.

The S-matrix as a function of (complex) rapidity has the following unitarity and crossing symmetry properties.

- *Unitarity*

  The total probability that an initial state evolves into a final state is, of course, unity. But, since there can only be $2 \to 2$ processes, then unitarity of the S-matrix is given as,

$$S_{ab}^{kl}(\theta) S_{kl}^{cd}(-\theta) = \delta_a^c \delta_b^d. \tag{2.1.4}$$

- *Crossing symmetry*

  In a scattering of $a+b \to c+d$ (see *figure 2.2*), an s-channel or t-channel process can take place. So the S-matrix has to be symmetric under the interchange of $s \leftrightarrow t$, or $\theta \leftrightarrow (i\pi - \theta)$,

$$S_{ab}^{cd}(\theta) = S_{a\bar{d}}^{c\bar{b}}(i\pi - \theta), \tag{2.1.5}$$

  where antiparticles are denoted by *bars*.

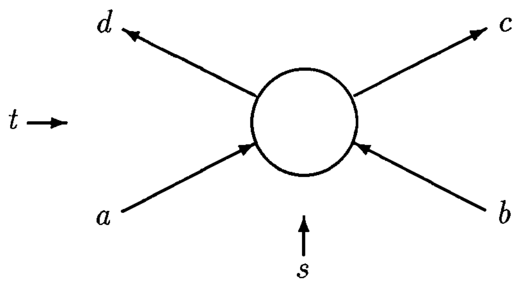

Figure 2.2: s and t channel.

Suppose the energy of the incoming particles, $s = (p_a + p_b)^2$, equals the $mass^2$ of a third particle with mass less than $(m_a + m_b)$, i.e.

$$s = m_{\bar{c}}^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh \Theta_{ab}^c. \tag{2.1.6}$$

Thus, it is implied that the angle $\Theta_{ab}^c \equiv i\theta_{ab}^c$ is purely imaginary so that $m_c \leq m_a + m_b$.

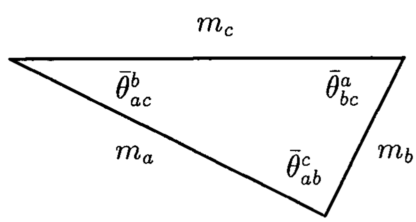

Figure 2.3: Mass triangle.

It is said that, particles a and b may *fuse* into particle $\bar{c}$, $a + b \to \bar{c}$. And the coupling of these three particles is non-vanishing. The S-matrix for particles a and b will have a pole at $\theta_{ab} = i\theta^c_{ab}$. It is also possible for particles b and c to fuse into $\bar{a}$, $b + c \to \bar{a}$, and particles c and a to fuse into $\bar{b}$, $c + a \to \bar{b}$. From this, the *mass triangle* relation is obtained by interchanging the labels a, b and c in equation (2.1.6) in a cyclic manner.

In *figure 2.3*, $\bar{\theta}^c_{ab} = \pi - \theta^c_{ab}$, thus

$$\theta^c_{ab} + \theta^b_{ac} + \theta^a_{bc} = 2\pi.$$

From conservation of energy and momentum in a fusing process, one gets the following relations,

$$\begin{aligned} \theta_a &= \theta_c + i\bar{\theta}^b_{ac}, \\ \theta_b &= \theta_c - i\bar{\theta}^a_{bc}. \end{aligned} \tag{2.1.7}$$

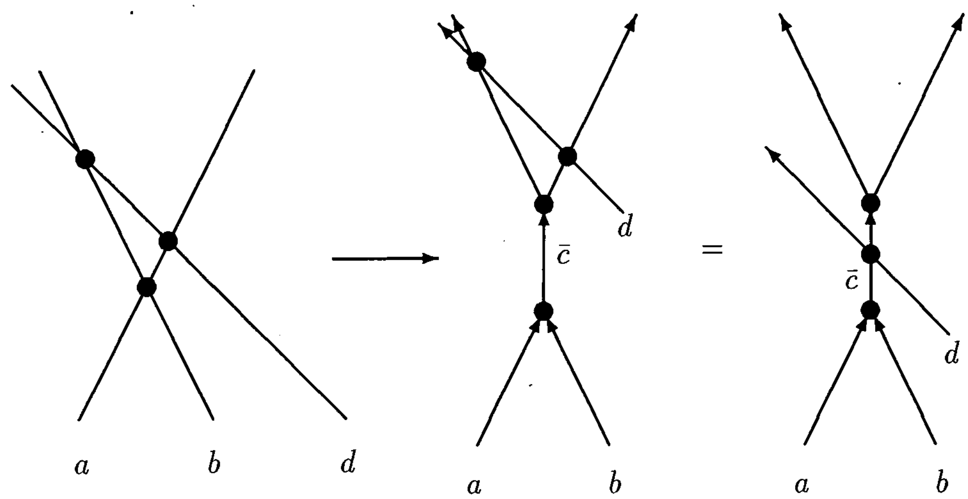

Figure 2.4: Bootstrap Relation.

In a *bootstrap* principle, the possibility of a particle *species* appearing as a pole in the S-matrix, is restricted to some set of previously determined particles which are contained in the theory. Consider the *purely* elastic scattering process of particles a and b. If their S-matrix has a pole at position θ^c_{ab}, then a bound state particle $\bar{c}$ can appear for some time during the process. If preceding the scattering of particles a and b with particle d, a bound state of particle $\bar{c}$ appears for a short time, see *figure 2.4*, then by the *particle displacement* argument, the product of scatterings before the apperance of the bound state $\bar{c}$ has to be the same as the S-matrix of particles $\bar{c}$ and d,

$$S_{d\bar{c}}(\theta) = S_{da}(\theta - i\bar{\theta}^b_{ac})S_{db}(\theta + i\bar{\theta}^a_{bc}), \tag{2.1.8}$$

this is called the *S-matrix bootstrap.*

If one writes the fusing of $a + b \to \bar{c}$ as, $|A_a A_b> \simeq |A_{\bar{c}}>$. Then applying the conserved charge operator, and remembering its additive property and the form of equation (2.1.1), one has the *conserved charge bootstrap*,

$$q^{\bar{c}}_s = q^a_s e^{is\bar{\theta}^b_{ac}} + q^b_s e^{-is\bar{\theta}^a_{bc}}. \tag{2.1.9}$$

On the other hand, the fusing $\bar{a} + \bar{c} \to b$ yields

$$q^b_s = q^{\bar{a}}_s e^{-is\bar{\theta}^c_{ab}} + q^{\bar{c}}_s e^{is\bar{\theta}^a_{bc}}.$$

Substituting q^b_s and remembering that

$$\bar{\theta}^c_{ab} + \bar{\theta}^b_{ac} + \bar{\theta}^a_{bc} = \pi,$$

one obtains a relation of the conserved quantities of particle a with its conjugate $\bar{a}$, $q^{\bar{a}}_s = (-1)^{s+1} q^a_s$. Thus, equation (2.1.9) can be written as,

$$q^a_s + q^b_s e^{is\theta^c_{ab}} + q^c_s e^{is(\theta^c_{ab} + \theta^a_{bc})} = 0. \tag{2.1.10}$$

For *spin* $s = 1$, $q^a_1 = m_a$ and the conserved charge bootstrap yields the mass triangle.

2.2 Real Coupling Affine Toda Theories

The affine Toda theory is an integrable two dimensional field theory which is described by the following Lagrangian density [2, 5, 8, 9],

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{\tilde{\beta}^2} \sum_{j=0}^{n} n_j (e^{\tilde{\beta}\alpha_j \cdot \phi} - 1), \tag{2.2.1}$$

where ϕ is an n-dimensional scalar field, m and $\tilde{\beta}$ are the mass and coupling parameters respectively. The vectors α_j with $j = 1, 2, \ldots, n$ are the simple roots of the Lie algebra g of rank n associated with the theory. And α_0 is chosen such that the set $\{\alpha_0, \alpha_j\}$ represents the affine Dynkin diagram of an affine Kac-Moody algebras. For the *simply-laced* Kac-Moody algebras, α_0 is the negative of the highest root $\psi = \sum_{j=1}^{n} n_j \alpha_j$; in particular, for the $a_n^{(1)}$ series, $n_j = 1$ for all j. The constants n_j are inserted into the potential term so that $\phi = 0$ is a solution to the minima of the potential.

Expanding the potential term yields,

$$V(\phi) = \frac{m^2}{\tilde{\beta}^2} \sum_{j=0}^{n} n_j \left[\tilde{\beta}(\alpha_j \cdot \phi) + \frac{\tilde{\beta}^2}{2}(\alpha_j \cdot \phi)^2 + \frac{\tilde{\beta}^3}{6}(\alpha_j \cdot \phi)^3 + \dots \right].$$

The order $\tilde{\beta}^{-1}$ term vanishes by the linear dependence relation between α_0 and the simple roots α_j for $j = 1, 2, \dots, n$. The second and the third term are the mass-squared and the three particle interaction terms respectively,

$$(M^2)^{ab} = m^2 \sum_{j=0}^{n} n_j \alpha_j^a \alpha_j^b, \tag{2.2.2}$$

$$C^{abc} = \frac{m^2 \tilde{\beta}}{6} \sum_{j=0}^{n} n_j \alpha_j^a \alpha_j^b \alpha_j^c. \tag{2.2.3}$$

Higher terms are N particle interaction terms. Coupling of interacting particles in the higher interaction terms can be obtained from the three particle coupling [20]. It was noted independently by Dorey [21] and Freeman [22] that non-vanishing three-point couplings are related to orbits of the Coxeter element of the associated Weyl group. But it was Dorey [21, 23] who provided the rigorous exposition of this fusing phenomena. Other related results are also due to Fring and Olive [24]. It was realised by Dorey that the non-vanishing three-point couplings are related to a root triangle made from elements of three Coxeter orbits. Upon projection of this root triangle to a two-dimensional eigenplane of the Coxeter element, the resulting equation resembles the conserved quantity boostrap (2.1.10), (this subject is discussed in more detail in *Chapter Four*).

The mass-squared matrix (2.2.2) can be diagonalized to obtain the masses of the fundamental Toda particles. A curious fact was noted in [9] that, if these masses are arranged as the n-component of a vector $\mathbf{m}$, then $\mathbf{m}$ is an eigenvector of the Cartan matrix with eigenvalue $2 - 2\cos(\frac{\pi}{h})$. In fact, a further examination [21, 25] shows that the conserved charges

q_s^a can be arranged into the eigenvectors of the Cartan matrix defined by $C_{jk} = \frac{2\alpha_j\cdot\alpha_k}{\alpha_j^2}$,

$$C\mathbf{q}_s = (2 - 2\cos(\theta_s))\mathbf{q}_s, \tag{2.2.4}$$

where $\theta_s = \frac{s\pi}{h}$ and s is an exponent of the underlying Lie algebra.

When a three particle coupling is non-vanishing it was noted [9] that the absolute value of this coupling is proportional to the area of triangle whose sides have lengths equal to the masses of the interacting particles,

$$|C^{abc}| = \frac{2\tilde{\beta}}{\sqrt{h}} m_a m_b \sin\theta_{ab}^c, \tag{2.2.5}$$

where h is the Coxeter number of the underlying Lie algebra and θ_{ab}^c is the fusing angle which give the mass triangle relation of (2.1.6).

These curious facts, lead to the assignment of the particles to each node of the Dynkin diagram of the underlying Lie algebra. An algebraic proof of (2.2.4) for $s = 1$ (the mass eigenvector) and (2.2.5) has been provided by several authors [22, 26].

In particular, for the $a_n^{(1)}$ series the masses of the fundamental particles are [9],

$$\tilde{m}_a = 2m\sin(\frac{a\pi}{h}) \qquad a = 1, 2, \ldots, n, \tag{2.2.6}$$

in the above, the Coxeter number of $a_n^{(1)}$ is $h = n + 1$. And the three-particle coupling is non-vanishing when a mass triangle (2.1.6) can be formed from particles a, b and c which satisfies $a + b + c = 0 \bmod h$ and the fusing angles are given by,

$$\theta_{ab}^c = \begin{cases} \frac{a+b}{h}\pi & a + b + c = h, \\ (2 - \frac{a+b}{h})\pi & a + b + c = 2h. \end{cases} \tag{2.2.7}$$

The S-matrix for affine Toda theories are obtained in the following way. First note that although there are mass degeneracies between particles associated with automorphisms of the Dynkin diagram, each particle is uniquely labelled by the higher conserved charges. Thus in this real coupling parameter regime, the fundamental Toda particles do not develop a particle multiplet with mass degeneracy. And, as a consequence of this fact, the *Yang-Baxter equation* (2.1.2) is trivial. The S-matrices can be 'determined' using the unitarity and crossing symmetry conditions derived from (2.1.4) and (2.1.5) respectively as follows,

$$S_{ab}(\theta)S_{ab}(-\theta) = 1, \tag{2.2.8}$$

$$S_{ab}(i\pi - \theta) = S_{b\bar{a}}(\theta). \tag{2.2.9}$$

Secondly, as there are many solutions to the above constraint. These S-matrices are fixed by the requirement that it develops a set of poles coresponding to the set of fusing angles given from the mass triangles or the non-vanishing three particle coupling. Finally, one has make sure that these *exact* S-matrices agree with perturbation theory.

One starts with the S-matrix of the *lightest* particle and then using the boostrap relation (2.1.8) to obtain the complete set of S-matrix components.

An algebraic construction of the S-matrices of the *simply-laced* theories and its relation with *Dorey's Rule* will be illuminated in the review section of the second part of this thesis (*Chapter Four*).

In particular, the S-matrix element of the $a_n^{(1)}$ series are given as follows [9],

$$S_{ab} = \prod_{|a-b|+1}^{a+b-1} \{p\}, \tag{2.2.10}$$

in the above expression, the product are taken in *step* of two, and the notation for the S-matrix building block is the following,

$$\{x\} = \frac{(x+1)(x-1)}{(x+1-B)(x-1+B)}, \tag{2.2.11}$$

with,

$$(x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})} \qquad \text{and} \qquad B(\tilde{\beta}) = \frac{1}{2\pi}\frac{\tilde{\beta}^2}{1+\frac{\tilde{\beta}^2}{4\pi}}. \tag{2.2.12}$$

Having obtained the expression for the S-matrices, one can then examine the pole structures of the S-matrix. The existence of *multiple* poles of the *simply-laced cases* has been explained using perturbation theory [8, 27].

Renormalising the *simply-laced* theory yields the quantum mass correction which has been calculated to the lowest order. It turns out that all the masses of the fundamental particles are renormalised by a universal factor [8, 9, 27]. And hence, the classical mass ratio of the fundamental particles are preserved in the quantum theory. A quantum calculation of higher conserved charges have been provided by Niedermaier [28].

More recently, results on the *non-simply-laced* affine Toda theories show a fascinating feature of *duality* between dual-pairs of the underlying algebra [29].

2.3 Imaginary Coupling Affine Toda Theories

In the lagrangian (2.2.1), one can change the coupling parameter $\tilde{\beta}$ into a *purely* imaginary coupling parameter $i\beta$ [11]. The resulting theory will have the following potential term,

$$V = -\frac{m^2}{\beta^2}\sum_{j=0}^{n} n_j \left(e^{i\beta\alpha_j\cdot\phi} - 1\right).$$

This potential term now has a multiple vacua given by the solution,

$$\phi = \frac{2\pi}{\beta}\lambda^v,$$

where λ^v is an element of the *coweight lattice.* The existence of these vacua signals a possibility of topological solitonic solutions which interpolate between them.

Although the hamiltonian of this theory is complex, nevertheless the energy and momentum of soliton solutions are real as shown by Hollowood [11] for $a_n^{(1)}$ series and generally by Olive et.al. [12]. Continuing the results of [12, 13], Freeman [30] has provided the calculation of higher conserved charges for the soliton solutions. It is shown that these conserved charges are also real.

The equation of motion can be derived easily from (2.2.1) upon substitution of the *imaginary* coupling parameter,

$$\partial^2\phi = -\frac{m^2}{i\beta}\sum_{j=0}^{n} n_j\alpha_j \exp(i\beta\alpha_j\cdot\phi). \tag{2.3.1}$$

Soliton solutions to the above equation of motion can be constructed using several methods [11, 12, 13, 15, 16]; a comparison of these will be discussed in detail in the following section. Each soliton solution is associated with a node on the Dynkin diagram of the underlying Lie algebra. Masses of these soliton solutions have been calculated [11, 12, 15]. For the *simply-laced* cases the single soliton of *species a* has a mass,

$$M_a = \frac{2mh}{\beta^2}\sqrt{\lambda_a}, \tag{2.3.2}$$

where h is the Coxeter number of the underlying Lie algebra and λ_a is the eigenvalue of the matrix $N\hat{C}$ where $\hat{C}$ is the affine Cartan matrix and $N = \mathrm{diag}(n_0, n_1, \ldots, n_n)$. In particular the mass of $a_n^{(1)}$ single soliton of species a is,

$$M_a = \frac{4hm}{\beta^2}\sin\left(\frac{\pi a}{h}\right) = \frac{2h}{\beta^2}\tilde{m}_a, \tag{2.3.3}$$

where $\tilde{m}_a$ is the mass of the fundamental affine Toda particle, c.f. (2.2.6). The quantum correction of the soliton mass is calculated in a different fashion compared with the quantum correction of the mass of the fundamental particles. Following the semiclassical approach of Dashen et.al. [31], Hollowood [32] has calculated the mass correction for the $a_n^{(1)}$ solitons. This mass correction was calculated via a small perturbation around the classical solution. It was found that for the $a_n^{(1)}$ solitons the quantum correction is again a universal factor, and hence the classical mass ratios of the solitons are preserved in the quantum theory. The corrections for other theories are still not completely determined [33, 34].

These soliton solutions are topological solitons, their topological charges are calculated from the following definition,

$$q = \frac{\beta}{2\pi}\int_{-\infty}^{\infty}(\partial_x\phi)\,\mathrm{d}x. \tag{2.3.4}$$

It has been found that each species of soliton associated with a node of the Dynkin diagram of the underlying Lie algebra can have several topological charges [11, 35, 36]. These topological charges lie in the *highest weight representation space* of the fundamental weight associated with each single soliton species. Thus, there is a mass degenerate multiplet for each species of solitons.

Due to this mass degenerate multiplet associated with each fundamental representation, scattering of solitons can be *non-diagonal.* Hence, in the calculation of scattering matrix the Yang-Baxter equation becomes non-trivial. The earliest result in solving the Yang-Baxter equation for the complex affine Toda theories were the S-matrices of the sine-Gordon solitons [1]. Although the solutions to the Yang-Baxter equation for various algebraic cases have long been known [39, 40], one still has to impose additional restrictions to extract the S-matrix. Namely unitarity, crossing symmetry and bootstrap. Several methods has been suggested for these calculations. Bernard and Leclair [41] noted that the symmetry of the non-local conserved currents is a *quantum symmetry* of a quantum group. This symmetry leads to a possible way of extracting the S-matrix from the solution of the Yang-Baxter equation. However, there exist very few explicit example of these S-matrix calculations. Hollowood [14], using the relation between the representation of the q-Hecke algebra with the representation of the quantum group $Sl_q(n)$, provides an alternative way of calculating the S-matrix. The $a_n^{(1)}$ series, for which the sine-Gordon theory is the simplest case, has been provided in [14]. This will be disscused in detail in the following. Recently, the quantum bound states exact S-matrices of the $a_2^{(1)}$ theory have been calculated [42].

Let a particle multiplet with degenerate mass M_a be denoted by the irreducible representation vector space V_a of the Lie algebra [14, 40]. Representations of the in-state and out-state of a scattering of two particles are then written as tensor product representations, and the Yang-Baxter equation is evaluated as a triple tensor product space [37, 40]. Solutions to the Yang-Baxter equation, denoted by R-matrices, are found without imposing the unitarity and crossing symmetry conditions [14, 41]. These conditions are imposed only when extracting the S-matrix from the R-matrix solution of the Yang-Baxter equation.

The Yang-Baxter equation in terms of these R-matrices is given in the following form [37],

$$\begin{aligned}(\check{R}_{V_2V_3}(x) \otimes 1)(1 \otimes \check{R}_{V_1V_3}(xy))(\check{R}_{V_1V_2}(y) \otimes 1) \\ = (1 \otimes \check{R}_{V_1V_2}(y))(\check{R}_{V_1V_3}(xy) \otimes 1)(1 \otimes \check{R}_{V_2V_3}(x)).\end{aligned} \tag{2.3.5}$$

In the above, $\check{R}_{V_1V_2}(x)$ is the invertible element of a *quasitriangular Hopf algebra* [37, 40, 43]. It acts on the tensor product vector spaces $V_1 \otimes V_2$, i.e. $\check{R} = R^i_{V_1} \otimes R^i_{V_2}$. Where $R^i_{V_1}$ and $R^i_{V_2}$ is an element of the first and the second copy of the *Hopf* algebra in a *quasitriangular Hopf algebra* respectively (see *Appendix* **A**). By definition $\check{R}_{V_1V_2}$ maps $V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$. Hence, the left hand side and right hand side of equation (2.3.5) maps $V_1 \otimes V_2 \otimes V_3 \rightarrow V_3 \otimes V_2 \otimes V_1$.

These R-matrices can be determined up to an overall scaling [14, 40], this freedom of scale is used to convert an R-matrix into an S-matrix. For this reason, it is convenient to use,

$$\check{R}_{V_iV_j}(x) = \sigma \cdot R_{V_iV_j}(x), \tag{2.3.6}$$

where $\sigma(v_i \otimes v_j) = (v_j \otimes v_i)$ and $v_i \in V_i$. Hence $R_{V_1V_2}$ maps $V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$, and the Yang-Baxter equation becomes,

$$R_{V_2V_3}(x)R_{V_1V_3}(xy)R_{V_1V_2}(y) = R_{V_1V_2}(y)R_{V_1V_3}(xy)R_{V_2V_3}(x), \tag{2.3.7}$$

where it is understood that $R_{V_iV_j}$ acts on the vector spaces V_i and V_j of $V^{\otimes 3}$. Now, the right and left hand sides is an $End(V_1 \otimes V_2 \otimes V_3)$. This last relation is similar to the *factorization* equation for the S-matrices, c.f. equation (2.1.2).

Jimbo [38] proposed the solution of the Yang-Baxter equation (2.3.5) for the $U_q(Sl(n))$ quantum group using the *q-deformed permutation group* called the *Hecke algebra*, $\mathcal{H}_m$. Elements of the Hecke algebra, $\{T_a,\ a = 1, 2, \ldots, m-1\}$ with $T_a \in End(V^{\otimes m})$, satisfy

$$\begin{aligned} &(T_a - q^{-1})(T_a + q) = 0, \\ &T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1}, \\ &[T_a, T_b] = 0, \qquad |a - b| \geq 2. \end{aligned} \tag{2.3.8}$$

In the above, q is the deformation parameter. T_a permutes the a^{th} and $(a+1)^{th}$ space in the tensor product space $V^{\otimes m}$. In general, one can label the elements of the Hecke algebra by $T_\omega \in \mathcal{H}_m$ with $\omega \in \mathcal{S}_m$ where $\mathcal{S}_m$ is the permutation group of m objects. Such that $T_{\omega\omega'} = T_\omega T_{\omega'}$ if $l(\omega\omega') = l(\omega) + l(\omega')$, where $l(\omega)$ is the length of ω.

With $\{e_i,\ i = 1, 2, \ldots, n\}$ as basis of V, let $T \in End(V^{\otimes 2})$ be defined as [38, 14],

$$T(e_i \otimes e_j) \doteq \begin{cases} q^{-1}(e_i \otimes e_i) & \text{if } i = j, \\ (q^{-1} - q)(e_i \otimes e_j) + (e_j \otimes e_i) & \text{if } i > j, \\ (e_j \otimes e_i) & \text{if } i < j. \end{cases} \tag{2.3.9}$$

Then the solution to equation (2.3.5) is given by

$$\check{R}(x) \doteq xT^{-1} - x^{-1}T. \tag{2.3.10}$$

To prove this solution, one can simplify the notation by dropping the tensor product sign since T already signifies which vector space it acts on. Using the Hecke algebra properties one can write,

$$T^2 - q^{-1}T + qT - 1 = 0 \quad \text{or} \quad T - q^{-1} + q - T^{-1} = 0,$$

and one can use these relations to change for example $(T_1 T_2^{-1} T_1 - T_1^{-1} T_2 T_1^{-1})$ into $(T_2 T_1^{-1} T_2 - T_2^{-1} T_1 T_2^{-1})$. Then equation (2.3.10) can be proved to satisfy relation (2.3.5).

Furthermore, these R-matrices satisfy the following *fusion procedure* [38, 14],

$$\check{R}^{a,b+c}(x) = \{I \otimes \check{R}^{a,b}[x(-q)^{c/2}]\}\{\check{R}^{a,c}[x(-q)^{-b/2}] \otimes I\}. \tag{2.3.11}$$

In particular one can easily show the following

$$\check{R}^{a,b}(x)\check{R}^{b,a}(x^{-1}) = \prod_{i,j=1}^{a,b} g[x(-q)^{-(a+b-2i-2j+2)/2}] I \otimes I, \tag{2.3.12}$$

with the scalar function $g(x) = (xq^{-1} - x^{-1}q)(x^{-1}q^{-1} - xq)$.

The generators of Hecke algebra can also be used in the q-analogue of the full symmetriser and antisymmetriser [38],

$$s_m^{\pm} = \frac{q^{\pm m(m-1)/2}}{[m]!} \sum_{\omega \in S_m} (\pm q)^{\mp l(w)} T_w, \tag{2.3.13}$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]! = [m][m-1]\dots[1]$. For example,

$$s_2^+ = \frac{1}{[2]}(q + T), \qquad s_2^- = \frac{1}{[2]}(q^{-1} - T).$$

Thus, (2.3.10) can also be written as,

$$\check{R}(x) = (xq - x^{-1}q^{-1})s_2^+ + (x^{-1}q - xq^{-1})s_2^-. \tag{2.3.14}$$

As in the representation theory of $Sl(n)$, one can project out the representation space V_a from the a-fold tensor product $V_1^{\otimes a}$ using the full antisymmetriser above,

$$V_a \cong s_a^-(V_1^{\otimes a}).$$

In the following discussion, V_1 will be written as V with basis $\{e_i,\ i = 1, 2, \dots, n\}$.

The S-matrix can be constructed from the R-matrix by the following [14],

$$S^{a,b}(\theta) = S^{a,b}_{min}(\theta) v(\theta) \check{R}^{a,b}(x(\theta), q). \tag{2.3.15}$$

Where $S^{a,b}_{min}(\theta)$ is the minimal (coupling parameter independent) solution to the unitarity condition (2.2.8), crossing symmetry (2.2.9) and the bootstrap relation (2.1.8). The scalar function $v(\theta)$ is inserted to ensure unitarity and crossing symmetry. Thus, the full S-matrix is a mapping of tensor product spaces due to the R-matrix,

$$S^{a,b} : V_a \otimes V_b \to V_b \otimes V_a.$$

Furthermore it satisfies the following conditions,

- *Unitarity*

$$S^{a,b}(\theta) S^{b,a}(-\theta) = I_a \otimes I_b.$$

- *Crossing symmetry*

$$S^{\bar{a},b}(\theta) = (I \otimes C_a) \cdot \left[\sigma \cdot S^{b,a}(i\pi - \theta)\right]^{t_2} \cdot \sigma \cdot (C_{\bar{a}} \otimes I). \tag{2.3.16}$$

- *S-matrix Bootstrap*

$$S^{d,c}(\theta) = \left[I \otimes S^{d,a}(\theta - i\bar{\theta}^{\bar{b}}_{a\bar{c}})\right]\left[S^{d,b}(\theta + i\bar{\theta}^{\bar{a}}_{b\bar{c}}) \otimes I\right].$$

While the Yang-Baxter equation is now given as,

$$\begin{aligned}&\left[I \otimes S^{a,b}(\theta_1)\right]\left[S^{a,c}(\theta_1+\theta_2) \otimes I\right]\left[I \otimes S^{b,c}(\theta_2)\right]\\ &\quad = \left[S^{b,c}(\theta_2) \otimes I\right]\left[I \otimes S^{a,c}(\theta_1+\theta_2)\right]\left[S^{a,b}(\theta_1) \otimes I\right].\end{aligned} \tag{2.3.17}$$

Explanations of the notations used above are as follows. I_a is the identity of the vector space V_a. The *charge conjugation operator* C_a is defined as a mapping $C_a : V_a \to V_{n-a}$, thus one would expect that the action of C_a on V_a would be proportional to $s^-_{n-a}(V^{\otimes n-a})$. So, defining the dual basis of V as $\{e^*_j\}$ where $e_i \cdot e^*_j = \delta_{ij}$, yields C_a explicitly as

$$C_a = k_a \sum_{\{i_j\}\in P_n} (-q)^{l(\{i_j\})}(e_{i_{a+1}} \otimes e_{i_{a+2}} \otimes \ldots \otimes e_{i_n})(e^*_{i_1} \otimes e^*_{i_2} \otimes \ldots \otimes e^*_{i_a}),$$

where P_n is the set of permutation of $(1,2,\ldots,n)$ and k_a some constant. For a fixed set of labels $(i_1,i_2,\ldots,i_a)$, C_a can be written as

$$C^{(i_a)}_a = k_a \sum_{\{i_a\}\in P_a} (-q)^{l(\{(i_a)(i_{\bar{a}})\})_{lowest}} \left(\sum_{\{i_{\bar{a}}\}\in P_{\bar{a}}} (-q)^{l(\{i_{\bar{a}}\})}(e_{i_{a+1}} \otimes \ldots \otimes e_{i_n})\right)(e^*_{i_1} \otimes \ldots \otimes e^*_{i_a}), \tag{2.3.18}$$

where $l(\{(i_a)(i_{\bar{a}})\})_{lowest}$ is the *lowest* length of permutation of two cycle $\{(i_a)(i_{\bar{a}})\}$ for a fixed set of (i_a). The constant k_a is determined by requiring that $C_{\bar{a}}C_a = I_a$. As these charge conjugation operators are always used in pairs in the crossing symmetry condition of the S-matrix, one only needs to know the value of $k_{\bar{a}}k_a$. For any choice of the fixed labels (i_a) one obtains the following relation,

$$k_{\bar{a}}k_a \left(\sum_{\{i_a\}\in P_a} (-q)^{l(\{(i_a)(i_{\bar{a}})\})_{lowest}+l(\{i_a\})}\right)\left(\sum_{\{i_{\bar{a}}\}\in P_{\bar{a}}} (-q)^{l(\{(i_{\bar{a}})(i_a)\})_{lowest}+l(\{i_{\bar{a}}\})}\right) = 1. \tag{2.3.19}$$

The crossing symmetry equation (2.3.16) can be viewed as follows. The left hand side is a mapping from $V_{\bar{a}} \otimes V_b$ into $V_b \otimes V_{\bar{a}}$. Thus, reading off from the right, the right hand side of the same equation give $\sigma \cdot (C_{\bar{a}} \otimes I)(V_{\bar{a}} \otimes V_b) = V_b \otimes V_a$. Applying $\sigma \cdot S^{b,a}$ on this tensor product space results in $V_b \otimes V_a$. But note, that the conjugate operator will always give the basis vector of the conjugate vector space in a reverse ordering. If $e_i^{(a)}$ for $i = 1, 2, \ldots, (\dim V_a)$ is the basis of the vector space V_a and $e_i^{(\bar{a})}$ for $i = 1, 2, \ldots, (\dim V_a)$ is the basis of the vector space $V_{\bar{a}}$, then $C_a(e_j^{(a)}) \cong e^{(\bar{a})}_{(\dim V_a)+1-j}$. Thus, one needs to *transpose* the second space of $\sigma \cdot S^{b,a}$, denoted by $[\sigma \cdot S^{b,a}]^{t_2}$, before applying it to $V_b \otimes V_a$. Finally, one needs to take a conjugate of the vector space V_a one more time.

Comparing the Yang-Baxter equation of the S-matrices (2.3.17) with that of the R-matrices, one immediately see that $x = e^{\lambda\theta+\lambda'}$. By the unitarity condition and equation (2.3.12), one can deduce that $\lambda' = 0$.

Generally, the S-matrix $S^{a,b}(\theta)$ will have a pole at $\theta = i\theta_{ab}^c$ (from the minimal part) corresponding to particle $c = a + b \bmod h$ at the direct channel. Thus, $S^{a,b}(\theta)$ must be proportional to the full antisymmetriser s^-_{a+b} at the pole. For the complex $a_{n-1}^{(1)}$ affine Toda theories, the minimal S-matrix is given by [9],

$$S^{a,b}_{min}(\theta) = \prod_{p=|a-b|}^{(a+b)} (p)$$

where the product above is taken in step 2 and (p) is the S-matrix building block (2.2.12) with Coxeter number $h = n$. With the pole of $S^{1,1}_{min}(\theta)$ at $\theta = \frac{2i\pi}{h}$ one can deduce from (2.3.14) that $x = (-q)^{-\frac{h\theta}{2i\pi}}$.

The simplest example of the $a_n^{(1)}$ complex affine Toda theories is the Sine-Gordon theory which is based on the affine algebra $a_1^{(1)}$. Thus one has to consider the quantum group $Sl_q(2)$ with the vector space V is considered to be a doublet space with basis $\{e_1, e_2\}$. Hence T can be written as the following matrix,

$$T \begin{pmatrix} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{pmatrix} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (q^{-1} - q) & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \begin{pmatrix} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{pmatrix}. \qquad (2.3.20)$$

Unitarity condition yields the following condition on the scalar function $v(x)$

$$v(x)v(x^{-1}) = \frac{1}{(xq^{-1} - x^{-1}q)(x^{-1}q^{-1} - xq)},$$

or, with $q = -e^{-i\pi\lambda}$ and $x = e^{\theta\lambda}$ one can write this condition in terms of Γ-functions

$$v(x)v(x^{-1}) = -\frac{1}{4\pi^2}\Gamma(-\frac{i\theta\lambda}{\pi}+\lambda)\Gamma(1+\frac{i\theta\lambda}{\pi}-\lambda)\Gamma(\frac{i\theta\lambda}{\pi}+\lambda)\Gamma(1-\frac{i\theta\lambda}{\pi}-\lambda). \quad (2.3.21)$$

One can easily see that the $\check{R}$-matrix for $Sl_q(2)$ is crossing symmetric. Thus crossing symmetry restricts the function scalar $v(x)$ by the following condition,

$$v(x) = v(-\frac{1}{xq}). \quad (2.3.22)$$

The solution which satisfies the unitarity condition (2.3.21) and crossing property (2.3.22) can be found by an iterative method. One starts from the simplest solution to the unitarity condition, by adding extra terms on this solution one can satisfy the crossing symmetry. But the resulting solution will not satisfy unitarity, so one adds some more extra terms in order to satisfy the unitarity condition. Now the crossing symmetry is again spoiled. Thus, this iterative process can be continued infinitely many times to produce an infinite product of some function. There are several solutions which satisfies these conditions. One has to fix these solutions using the known particle spectrum and non-zero three-point coupling, i.e. from the mass triangle one obtains the value of fusing angles which are the poles of the S-matrices. For the sine-Gordon theory, one finds that $v(x)$ is given by the following infinite product of Γ-functions (note that this is different from Bernard-Leclair solution [41], their solution seems to have incorrect pole positions),

$$v(x) = \frac{1}{2i\pi}\prod_{k=1}^{\infty} \frac{\Gamma(\frac{i\theta\lambda}{\pi}+2k\lambda)\Gamma(\frac{i\theta\lambda}{\pi}+1+2(k-1)\lambda)}{\Gamma(-\frac{i\theta\lambda}{\pi}+2k\lambda)\Gamma(-\frac{i\theta\lambda}{\pi}+1+2(k-1)\lambda)} \frac{\Gamma(-\frac{i\theta\lambda}{\pi}+(2k-1)\lambda)\Gamma(-\frac{i\theta\lambda}{\pi}+1+(2k-3)\lambda)}{\Gamma(\frac{i\theta\lambda}{\pi}+(2k+1)\lambda)\Gamma(\frac{i\theta\lambda}{\pi}+1+(2k-1)\lambda)}, \quad (2.3.23)$$

and, together with the minimal S-matrix solution and equations (2.3.10) and (2.3.20), one has all the doublet state S-matrices of the Sine-Gordon theory. In this solution, the basis e_1 can be interpreted as a soliton with e_2 as an antisoliton, or vice-versa.

In the sine-Gordon soliton and anti-soliton scattering, one can have a transmision or reflection, thus one can write $S_{e_1\otimes e_2, e_2\otimes e_1}$ as the transmision element of the S-matrix, S_t, and $S_{e_1\otimes e_2, e_1\otimes e_2}$ as the reflection element of the scattering matrix, S_r. Scattering element of identical soliton $S_{e_1\otimes e_1, e_1\otimes e_1}$ or $S_{e_2\otimes e_2, e_2\otimes e_2}$ is written as S. After some Γ-function algebra, one can reproduce the matrix elements of Zamolodchikovs [1] as follows,

$$S_t = \prod_{k=1}^{\infty} \frac{\Gamma(\frac{i\theta}{2\pi}+1+\frac{k}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+1+\frac{k-1}{2\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{k}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{k-1}{2\lambda})} \frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{2}+\frac{k-1}{2\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{3}{2}+\frac{k}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k-1}{2\lambda})}, \tag{2.3.24}$$

$$S_r = -\prod_{k=1}^{\infty} \frac{\Gamma(\frac{3}{2}+\frac{k}{2\lambda})\Gamma(\frac{1}{2}+\frac{k-1}{2\lambda})}{\Gamma(\frac{1}{2}+\frac{k}{2\lambda})\Gamma(-\frac{1}{2}+\frac{k-1}{2\lambda})} \frac{\Gamma(\frac{i\theta}{2\pi}+1+\frac{k-1}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k}{2\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+1+\frac{k-1}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{3}{2}+\frac{k}{2\lambda})} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{k}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{2}+\frac{k-1}{2\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{k}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k-1}{2\lambda})}, \tag{2.3.25}$$

$$S = \prod_{k=1}^{\infty} \frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k-1}{2\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{k-1}{2\lambda})} \frac{\Gamma(\frac{i\theta}{2\pi}+1+\frac{k}{2\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{k-1}{2\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+1+\frac{k}{2\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{k-1}{2\lambda})}. \tag{2.3.26}$$

The matrix element S_t has physical poles at the following rapidity values,

$$\theta = i\pi - \frac{in\pi}{\lambda}, \tag{2.3.27}$$

for $n = 1, 2, \ldots, [\lambda]$ where $[\lambda]$ means the largest integer less than λ. These poles correspond to the *direct channel* production of the soliton - anti-soliton bound states, the *breathers*, where the mass of the n^{th} breather is given by

$$M_n = 2M\sin(\frac{n\pi}{2\lambda}),$$

and M is the soliton mass given with the first quantum correction as [32],

$$M = \frac{2m}{\pi}\lambda.$$

Comparing these results with the known facts from a semi-classical calculation of Dashen et.al. [31], one finds the relations between the coupling parameter β and the parameter λ as follows,

$$\frac{1}{\lambda} = \frac{1}{4\pi}\frac{\beta^2}{1-\frac{\beta^2}{4\pi}}.$$

The physical poles of the matrix element S correspond to the *cross channel* process of soliton-antisoliton scattering.

Note that contrary to the S-matrices of the real coupling regime affine Toda theories, these S-matrix elements are not products of periodic functions.

To calculate the soliton-breather and breather-breather scatterings, one uses the S-matrix bootstrap principle. In writing down the S-matrix bootstrap, one has to keep in mind that the allowed S-matrix in this equation has to yield a *purely elastic* process. This means, that only S and S_t will be used, since S_r is not purely elastic.

Denoting the soliton as A, the anti-soliton as $\bar{A}$ and the n^{th} breather as B_n, then the fusing angle of $A\bar{A}B_n$ is given from equation (2.3.27),

$$i\bar{\theta}^A_{n\bar{A}} = \frac{i\pi}{2} - \frac{in\pi}{2\lambda}.$$

This value of fusing angle will be used in the following boostrap relations. The soliton-breather S-matrix is obtained from

$$S_{An}(\theta) = S(\theta - i\bar{\theta}^A_{n\bar{A}})S_t(\theta + i\bar{\theta}^{\bar{A}}_{An}),$$

and the breather-breather S-matrix is obtained from

$$S_{nm}(\theta) = S_{nA}(\theta - i\bar{\theta}^{\bar{A}}_{Am})S_{n\bar{A}}(\theta + i\bar{\theta}^A_{\bar{A}m}).$$

After some Γ-function algebra, one obtains the soliton-breather S-matrix element as follows,

$$S_{An}(\theta) = \prod_{k=1}^{n} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2-n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}-\frac{1}{4}+\frac{2k-2-n}{4\lambda})} \cdot \frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-2-n}{4\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2-n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{5}{4}+\frac{2k-n}{4\lambda})}. \tag{2.3.28}$$

Multiplying with a unity term, for example the following

$$\prod_{k=n+1}^{\infty} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2-n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}-\frac{1}{4}+\frac{2k-2-n}{4\lambda})} \cdot$$
$$\prod_{k=1}^{\infty} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k+n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}-\frac{1}{4}+\frac{2k-2+n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k+n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2+n}{4\lambda})},$$

the finite product in equation (2.3.28) can easily be rewritten as an infinite product,

$$S_{An}(\theta) = \prod_{k=1}^{\infty} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-2-n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}-\frac{1}{4}+\frac{2k-2-n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{5}{4}+\frac{2k-n}{4\lambda})}$$
$$\frac{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2-n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-n}{4\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2-n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-n}{4\lambda})}$$
$$\frac{\Gamma(\frac{i\theta}{2\pi}-\frac{1}{4}+\frac{2k-2+n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{5}{4}+\frac{2k+n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k+n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k-2+n}{4\lambda})}$$
$$\frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2+n}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k+n}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{4}+\frac{2k-2+n}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{3}{4}+\frac{2k+n}{4\lambda})}. \tag{2.3.29}$$

The breather-breather scattering matrix element is as follows,

$$S_{nm}(\theta) = \prod_{k=1}^{m} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{2k+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+1-\frac{2k+n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}-\frac{2k+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+1+\frac{2k+n-m}{4\lambda})}$$
$$\frac{\Gamma(\frac{i\theta}{2\pi}+\frac{2k-2+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+1-\frac{2k-2+n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}-\frac{2k-2+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+1+\frac{2k-2+n-m}{4\lambda})}$$
$$\frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}-\frac{2k-n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-n-m}{4\lambda})}$$
$$\frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}-\frac{2k-2-n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2-n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}-\frac{2k-2-n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2-n-m}{4\lambda})}, \tag{2.3.30}$$

or a more symmetrical form as follows,

$$S_{nm}(\theta) = \prod_{k=1}^{m} \frac{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2-n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+1+\frac{2k-n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{2k-2-n-m}{4\lambda})}$$
$$\frac{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2+n-m}{4\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{2k-2+n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+1+\frac{2k+n-m}{4\lambda})}$$
$$\frac{\Gamma(\frac{i\theta}{2\pi}+1+\frac{2k+n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{2k-2+n-m}{4\lambda})}{\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}-\frac{2k+n-m}{4\lambda})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2+n-m}{4\lambda})}$$
$$\frac{\Gamma(-\frac{i\theta}{2\pi}+1+\frac{2k-n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{2k-2-n-m}{4\lambda})}{\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-n-m}{4\lambda})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{2}+\frac{2k-2-n-m}{4\lambda})}. \tag{2.3.31}$$

These S-matrices, S_{An} and S_{nm}, can also be written as a finite product of the sinh-function [1]. Thus, in contrast to the S-matrices for soliton-antisoliton scattering, the S-matrices involving the breathers have a $2i\pi$-periodicity. In the physical strip, the only meaningful poles of $S_{An}(\theta)$ are

$$\theta = \frac{i\pi}{2} \pm \frac{in\pi}{2\lambda} \tag{2.3.32}$$

where the positive and negative sign correspond to the antisoliton poles in the s- and t-channel respectively. While in $S_{nm}(\theta)$, the only physical poles are

$$\theta = \frac{i(n+m)\pi}{2\lambda},\ i\pi - \frac{i(n+m)\pi}{2\lambda}, \tag{2.3.33}$$

which correspond to the $(n+m)^{th}$ breather in the s- and t-channel respectively. Other poles in the physical strip are double poles which do not correspond to any physical bound states [1]. The existence of these double poles may be explained by the *Coleman-Thun mechanism* [44].

2.4 Soliton Solutions

As mentioned in the previous section, soliton solutions to the complex affine Toda theories have been calculated by several authors. Namely using the Hirota's method [11, 15], construction by algebraic methods [12, 13] and for the $a_n^{(1)}$ series using the Bäcklund transformation [16]. This section provides a comparison study of these constructions. A particular attention is given to the two soliton solutions as they will be explored further in the following chapter.

2.4.1 Hirota's Method

Soliton solutions to the affine Toda theory equation of motion, (2.3.1), can be derived using Hirota's method [45]. An ansatz for the soliton solutions of the affine Toda theories [11, 15] is the following,

$$\phi = \frac{i}{\beta}\sum_{j=0}^{n} \eta_j \alpha_j \ln \tau_j, \tag{2.4.1}$$

where, $\eta_j = \frac{2}{\alpha_j^2}$. Thus for $a_n^{(1)}$ series, $\eta_j = 1$ for all j.

In Hirota's method, the τ-functions are a power series expansion in an arbitrary parameter ε which will be set to 1 at the end of the construction,

$$\tau_j = 1 + \varepsilon t_j^{(1)} + \varepsilon^2 t_j^{(2)} + \dots. \tag{2.4.2}$$

In the above expression, t_j is a function of (x,t). The equation of motion is solved order by order in the arbitrary parameter ε. Inserting the ansatz (2.4.1) into the equation of motion written as a Hirota bilinear form [15, 46], one can solve the soliton solution by restricting ϕ to have a finite value at its asymptote. In particular for the $a_n^{(1)}$ series, an N soliton solution is calculated by setting $t_j^{(a)} = 0$ for $a > N$ in the τ-function.

For $a_n^{(1)}$ series, inserting (2.4.1) into the equation of motion (2.3.1), and assuming that the set of $h = n + 1$ equations of motion decouple, results in the following,

$$\ddot{\tau}_j \tau_j - \dot{\tau}_j^{\,2} - \tau_j'' \tau_j + \tau_j'^2 = m^2(\tau_{j-1}\tau_{j+1} - \tau_j^2) \qquad j = 0, 1, \dots, n, \tag{2.4.3}$$

where the subscript of the τ-functions are labelled modulo the Coxeter number $h = n + 1$. The single soliton solution is obtained as

$$\tau_j^{(a)} = 1 + \exp[\Omega_a + \rho_a + ij\theta_a] \tag{2.4.4}$$

where,

$$\Omega_a = \sigma_a(x - u_a t), \qquad \rho_a = \eta_a + i\xi_a, \qquad \theta_a = \frac{2\pi a}{h}, \tag{2.4.5}$$

$\sigma_a, u_a, \eta_a, \xi_a \in \mathbb{R}$. The parameter σ_a and the velocity u_a are related by,

$$\sigma_a^2(1 - u_a^2) = 4m^2 \sin^2 \frac{\pi a}{h}. \tag{2.4.6}$$

In the rest of this thesis, all σ is taken to have positive real values. The superscript of the τ-function (2.4.4) indicates the *species* of the soliton. One can also write the τ-functions with an explicit dependence on the *lightcone* coordinates, $x_\pm = \frac{1}{\sqrt{2}}(t \pm x)$, thus Ω_a becomes,

$$\Omega_a = \delta_a^{(-)} x_+ - \delta_a^{(+)} x_-,$$

with,

$$\delta_a^{(\pm)} = \frac{1}{\sqrt{2}} \sigma_a (1 \pm u_a).$$

A two soliton solution is obtained by solving the equation of motion (2.4.3) after inserting the expansion of the τ-function up to second order in ε. The two soliton solution for *species* a and b is,

$$\begin{aligned} \tau_j^{(ab)} &= 1+\exp[\Omega_a+\rho_a+ij\theta_a]+\exp[\Omega_b+\rho_b+ij\theta_b] \\ &\quad +A\exp[\Omega_a+\Omega_b+\rho_a+\rho_b+ij(\theta_a+\theta_b)], \end{aligned} \tag{2.4.7}$$

which can be compactly written as,

$$\tau_j^{(ab)} = 1+(\tau_j^{(a)}-1)+(\tau_j^{(b)}-1)+A(\tau_j^{(a)}-1)(\tau_j^{(b)}-1). \tag{2.4.8}$$

The interaction coefficient in the above expression is given by,

$$A = -\frac{(\sigma_a-\sigma_b)^2-(\sigma_a u_a-\sigma_b u_b)^2-4m^2\sin^2(\frac{\pi}{h}(a-b))}{(\sigma_a+\sigma_b)^2-(\sigma_a u_a+\sigma_b u_b)^2-4m^2\sin^2(\frac{\pi}{h}(a+b))}. \tag{2.4.9}$$

If the rapidity variable Θ is defined through the velocity u as, $u_a = \tanh\Theta_a$, then the interaction coefficient can also be written in terms of rapidity difference $\Theta = \Theta_a - \Theta_b$,

$$A = \frac{\sin\left(\frac{\Theta}{2i}+\frac{\pi(a-b)}{2h}\right)\sin\left(\frac{\Theta}{2i}-\frac{\pi(a-b)}{2h}\right)}{\sin\left(\frac{\Theta}{2i}+\frac{\pi(a+b)}{2h}\right)\sin\left(\frac{\Theta}{2i}-\frac{\pi(a+b)}{2h}\right)}. \tag{2.4.10}$$

As the case for two soliton solutions, a multi-soliton solution can be constructed from a collection of single soliton solutions [11]. The interaction between the solitons is pair-wise, with the interaction coefficient as given above.

Following [12], the energy-momentum tensor of soliton solutions can be written as,

$$T_{\mu\nu} = (\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)C. \tag{2.4.11}$$

Or in terms of the lightcone components,

$$T_{+-} = \partial_+\partial_- C, \tag{2.4.12}$$

$$T_{\pm\pm} = -\partial_\pm^2 C. \tag{2.4.13}$$

Notice that T_{+-} is actually the trace of the energy-momentum tensor, then (2.4.12) becomes,

$$\partial^2 C = -\frac{2m^2}{\beta^2}\sum_{j=0}^{n}\left(e^{i\beta\alpha_j\cdot\phi}-1\right). \tag{2.4.14}$$

Using the soliton solutions in the energy-momentum tensor, leads to the following solution for C up to linear functions of only x_+ or x_- variables,

$$C = -\frac{2}{\beta^2}\sum_{j=0}^{n} \ln \tau_j. \tag{2.4.15}$$

The mass of the soliton solution can be calculated as follows. The energy and momentum density is given in terms of components of the energy-momentum density as, $\mathcal{E} = T_{00}$ and $\mathcal{P} = T_{10}$. Consider instead, the *lightcone* energy-momentum density,

$$\mathcal{P}^{\pm} = \frac{\mathcal{E} \pm \mathcal{P}}{\sqrt{2}}. \tag{2.4.16}$$

Thus, using (2.4.11), the components of the lightcone energy-momentum are given by,

$$P^{+} = (-\partial_+ C)_{x=-\infty}^{\infty}, \qquad P^{-} = (\partial_- C)_{x=-\infty}^{\infty}. \tag{2.4.17}$$

For the single soliton solution (2.4.4), one finds that

$$\partial_{\pm} C = \mp\frac{2}{\beta^2}\sum_{j=0}^{n} \frac{\delta_a^{\mp}(\tau_j^{(a)} - 1)}{\tau_j^{(a)}}. \tag{2.4.18}$$

In the limit as $x \to -\infty$ the ratios $\frac{(\tau_j^{(a)}-1)}{\tau_j^{(a)}}$ vanish, and in the limit $x \to \infty$ all the ratios tend to 1. Recall the constraint (2.4.6) and write the rapidity of the soliton as $\Theta_a = \frac{1}{2}\ln\left(\frac{1+u_a}{1-u_a}\right)$, then the lightcone energy-momentum of a single soliton is given by

$$P^{\pm} = \frac{4hm}{\sqrt{2}\beta^2}\sin\left(\frac{\theta_a}{2}\right)e^{\mp\Theta_a}, \tag{2.4.19}$$

and its mass is calculated to be

$$M_a^2 = 2P^+P^- = \left(\frac{4hm}{\beta^2}\sin\left(\frac{\theta_a}{2}\right)\right)^2. \tag{2.4.20}$$

Note, that the mass of the species a soliton is proportional to the mass of the fundamental particle of the $a_n^{(1)}$ affine Toda field theory, $\tilde{m}_a = 2m\sin(\frac{\theta_a}{2})$.

For a multi-soliton solution the calculation above can be performed in the same way. In particular a two soliton solution of species a and b yields,

$$\sqrt{2}P^{\pm} = M_a e^{\mp\Theta_a} + M_b e^{\mp\Theta_b}. \tag{2.4.21}$$

As seen from the example of the $a_n^{(1)}$ series, the Hirota's method is useful in performing explicit calculations such as in determining the topological charges [35] or in the construction of oscillating soliton solutions, the breathers [47] (see *Chapter Three*).

2.4.2 Bäcklund Transformation

Solitons of the $a_n^{(1)}$ series can also be constructed using the Bäcklund transformation [16]. The essence of a Bäcklund transformation is to relate different solutions of the equation of motion one to the other by two coupled first order differential equations. These first order differential equations are relatively easier to solve. One starts with a vacuum solution and using the Bäcklund transformation one obtains a single soliton solution. To construct a two soliton solution, one has to do four steps transformations. First from a vacuum solution using two different parameters, one can create the single soliton solutions of species a and b. Secondly, transforming these single soliton solutions with the opposite parameter will give two solutions for a double soliton. Finally, one has to *match* the parameters of these double soliton solutions such that it is compatible with the single soliton solutions.

One can write the extended simple roots system of $a_n^{(1)}$ Kac-Moody algebra as $\alpha_j = e_j - e_{j+1}$, where $\{e_1, e_2, \ldots, e_n, e_{n+1} \equiv e_0\}$ are orthonormal basis of $\mathbb{R}^{n+1}$ vector space. The field ϕ is an n-dimensional scalar field in the vector space span by the simple roots of the Lie algebra A_n. Thus it can be written as $\phi = \sum_{j=0}^{n} e_j \phi_j$ such that $\phi_j = e_j \cdot \phi$ and $\sum_{j=0}^{n} \phi_j = 0$. Using these extended simple roots representation the component of the equation of motion (2.3.1) becomes,

$$F_j(\phi) \equiv \partial^2 \phi_j - \frac{im^2}{\beta}\{\exp[i\beta(\phi_j - \phi_{j+1})] - \exp[i\beta(\phi_{j-1} - \phi_j)]\} = 0. \tag{2.4.22}$$

Let ϕ and $\tilde{\phi}$ be vector fields orthogonal to $\sum_{j=0}^{n} e_j$ which satisfy the following Bäcklund transformation in the form of a coupled first order differential equation,

$$\begin{aligned} \partial_+(\phi_j - \tilde{\phi}_j) &= \frac{m}{\sqrt{2}\beta} A\{\exp[i\beta(\tilde{\phi}_j - \phi_{j+1})] - \exp[i\beta(\tilde{\phi}_{j-1} - \phi_j)]\}, & (2.4.23) \\ \partial_-(\phi_j - \tilde{\phi}_{j-1}) &= \frac{m}{\sqrt{2}\beta} A^{-1}\{\exp[i\beta(\phi_j - \tilde{\phi}_j)] - \exp[i\beta(\phi_{j-1} - \tilde{\phi}_{j-1})]\}, & (2.4.24) \end{aligned}$$

where A is the parameter of the Bäcklund transformation. Taking the derivative of (2.4.23) with respect to x_- and the derivative of (2.4.24) with respect to x_+ results in,

$$F_j(\phi) = F_j(\tilde{\phi}), \qquad F_j(\phi) = F_{j-1}(\tilde{\phi}), \tag{2.4.25}$$

which are true for all j and hence all the functions F are equal. Furthermore,

$$F_j(\phi) = F_j(\tilde{\phi}) = \frac{1}{n+1} \sum_{j=0}^{n} F_j = 0,$$

these are the equations of motion of $a_n^{(1)}$ series for fields ϕ and $\tilde{\phi}$. This statement means that the $n+1$ equations of motion decouple. Thus, the Bäcklund transformations (2.4.23) and (2.4.24) map one solution, $\tilde{\phi}$, into another, ϕ.

One starts with the trivial solution $\tilde{\phi} = 0$ to obtain the single soliton solution by solving (2.4.23) and (2.4.24). With $\tilde{\phi} = 0$, write

$$B_j = A^{-1} \exp(i\beta\phi_j), \tag{2.4.26}$$

from (2.4.23) and (2.4.24) one obtains,

$$\partial_+ \phi_j = \frac{m}{\sqrt{2}\beta}[B_{j+1}^{-1} - B_j^{-1}], \tag{2.4.27}$$

$$\partial_- \phi_j = \frac{m}{\sqrt{2}\beta}[B_j - B_{j-1}]. \tag{2.4.28}$$

A further substitution yield the equations (2.4.23) and (2.4.24) in terms of B,

$$\partial_+ B_j = \frac{im}{\sqrt{2}}(B_{j+1}^{-1} B_j - 1) = \partial_- B_{j+1}^{-1}. \tag{2.4.29}$$

Assume that the soliton is at rest, i.e. $\partial_t \phi = 0$, one obtains the relation,

$$B_{j+1}^{-1} + B_j = c, \tag{2.4.30}$$

which is true for all $j = 0, 1, \ldots, n$ for some constant c. This relation is useful as it relates different B_j. Hence knowing one B for a particular j all other Bs are also known. The quantities Bs has the following property which can be proved by induction,

$$B_1 B_2 \ldots B_k = a_k B_k - a_{k-1}.$$

In the above equation, a_k are solved from the recurrence relation: $a_{k+1} = c a_k - a_{k-1}$, with initial condition $a_1 = 1$ and $a_2 = c$. A further observation shows that the consistency condition $a_{n+1} = 0$ is also needed, this yields the solution $c = 2\cos(\frac{\theta}{2})$ where $\theta = \frac{2\pi a}{n+1}$.

Using (2.4.30) in (2.4.29) yields,

$$\frac{\mathrm{d}B_j}{\mathrm{d}x} = -im[B_j - \exp(\frac{i\theta}{2})][B_j - \exp(-\frac{i\theta}{2})], \tag{2.4.31}$$

which can be integrated directly to give,

$$B_j = \exp(\frac{i\theta}{2}) \frac{Q\exp[2mx\sin(\frac{\theta}{2}) + i(j-1)\theta] - 1}{Q\exp[2mx\sin(\frac{\theta}{2}) + ij\theta] - 1}. \tag{2.4.32}$$

In the above Q is the complex integration constant which will determine the topological charges of the soliton solution.

To obtain a moving single soliton solution, one only has to Lorentz boost (2.4.32). The rapidity variable is defined as $u = \tanh\Theta$ or $\Theta = \frac{1}{2}\ln(\frac{1+u}{1-u})$, such that coordinates transform as $x'_\pm = e^{\mp\Theta}x_\pm$. Define the elementary T-function as,

$$T_j = \exp(-j\Theta - ij\frac{\theta}{2})\{\exp(ij\theta)Q\exp[2m\frac{x-ut}{\sqrt{1-u^2}}\sin(\frac{\theta}{2})] - 1\}, \tag{2.4.33}$$

then B can be written,

$$B_j = \frac{T_{j-1}}{T_j}. \tag{2.4.34}$$

The above single soliton solution is easily compared with the solution obtained from Hirota's method. Recall the ansatz (2.4.1) for $a_n^{(1)}$ series, one can write $\phi = \sum_{j=0}^n e_j\phi_j$ with

$$\phi_j^{(a)} = \frac{i}{\beta}\ln\frac{\tau_j^{(a)}}{\tau_{j-1}^{(a)}}. \tag{2.4.35}$$

On the other hand, choosing $Q = \exp(i\pi + \rho)$ in (2.4.33) gives,

$$T_j^{(a)} = -e^{-j\Theta_a}\omega_a^{-\frac{j}{2}}\{1 + \omega_a^j\exp[\sigma_a(x - u_a t) + \rho]\}, \tag{2.4.36}$$

where,

$$\omega_a = \exp(\frac{2i\pi a}{n+1}), \qquad \sigma_a = \frac{2m\sin(\frac{\pi a}{n+1})}{\sqrt{1-u_a^2}}.$$

Then (2.4.26) gives,

$$\phi_j^{(a)} = \frac{i}{\beta}\ln A_a^{-1}e^{-\Theta_a}\omega_a^{-\frac{1}{2}}\frac{\tau_j^{(a)}}{\tau_{j-1}^{(a)}},$$

which is (2.4.35) provided that

$$A_a = e^{-\Theta_a}\omega_a^{-\frac{1}{2}} = \left(\frac{1+u_a}{1-u_a}\right)^{-\frac{1}{2}}\omega_a^{-\frac{1}{2}},$$

and $\tau_j^{(a)}$ is given as (2.4.4).

As stated earlier, a two soliton solution is obtained by successive applications of Bäcklund transformations. One starts with a vacuum solution and uses two different transformation parameters in (2.4.23) and (2.4.24) to obtain two single soliton solutions. Inserting again this single soliton solution into (2.4.23) and (2.4.24) with the opposite transformation parameter will give a two soliton solution. This procedure can be depicted in the following diagram.

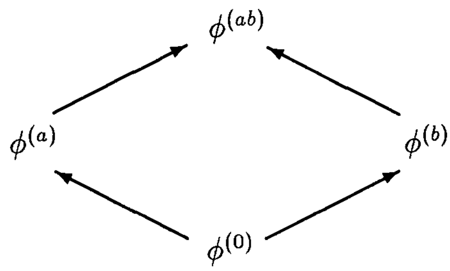

Figure 2.5: Bäcklund Transformation for two soliton solution.

Equating the four Bäcklund transformations corresponding with ∂_+ yields,

$$f_{j-1}\exp[-i\beta\phi_j^{(ab)}] - f_j\exp[-i\beta\phi_{j+1}^{(ab)})] = g_{j+1} - g_j. \tag{2.4.37}$$

And transformations corresponding with ∂_- yields,

$$g_j\exp[i\beta\phi_j^{(ab)}] - g_{j+1}\exp[i\beta\phi_{j+1}^{(ab)})] = f_j - f_{j-1}. \tag{2.4.38}$$

Where the functions f and g are defined as follows,

$$\begin{aligned} f_j &\equiv A_b\exp[i\beta\phi_j^{(a)}] - A_a\exp[i\beta\phi_j^{(b)}], \\ g_j &\equiv A_a\exp[-i\beta\phi_j^{(a)}] - A_b\exp[-i\beta\phi_j^{(b)}]. \end{aligned}$$

Eliminating $\phi_{j+1}^{(ab)}$ terms from (2.4.37) and (2.4.38) results in a quadratic equation of $\phi_j^{(ab)}$,

$$\left\{\exp[i\beta\phi_j^{(ab)}] + \frac{f_{j-1}}{g_j}\right\}\left\{\exp[i\beta\phi_j^{(ab)}] + \frac{f_{j-1} - f_j}{g_j - g_{j+1}}\right\} = 0. \tag{2.4.39}$$

The first solution of the above quadratic equation was given in [16],

$$\begin{aligned} B_j^{(ab)} &\equiv A_a^{-1}A_b^{-1}\exp(i\beta\phi_j^{(ab)}) \\ &= \frac{T_{j-1}^{(a)}T_{j-2}^{(b)} - T_{j-1}^{(b)}T_{j-2}^{(a)}}{T_j^{(a)}T_{j-1}^{(b)} - T_j^{(b)}T_{j-1}^{(a)}}. \end{aligned} \tag{2.4.40}$$

With (2.4.36) and recalling (2.4.4) one obtains the following,

$$\phi_j^{(ab)} = \frac{i}{\beta}\ln\frac{e^{\Theta_b}\omega_b^{\frac{1}{2}}\tau_j^{(a)}\tau_{j-1}^{(b)} - e^{\Theta_a}\omega_a^{\frac{1}{2}}\tau_{j-1}^{(a)}\tau_j^{(b)}}{e^{\Theta_b}\omega_b^{\frac{1}{2}}\tau_{j-1}^{(a)}\tau_{j-2}^{(b)} - e^{\Theta_a}\omega_a^{\frac{1}{2}}\tau_{j-2}^{(a)}\tau_{j-1}^{(b)}}. \tag{2.4.41}$$

Write the τ-functions (2.4.4) as,

$$\tau_j^{(a)} = 1 + \omega_a^j t_a. \tag{2.4.42}$$

Then rescaling t_a and t_b by the following,

$$\bar{t}_a = \frac{e^{\Theta_b}\omega_b^{\frac{1}{2}} - e^{\Theta_a}\omega_a^{-\frac{1}{2}}}{e^{\Theta_b}\omega_b^{\frac{1}{2}} - e^{\Theta_a}\omega_a^{\frac{1}{2}}} t_a,$$

$$\bar{t}_b = \frac{e^{\Theta_b}\omega_b^{-\frac{1}{2}} - e^{\Theta_a}\omega_a^{\frac{1}{2}}}{e^{\Theta_b}\omega_b^{\frac{1}{2}} - e^{\Theta_a}\omega_a^{\frac{1}{2}}} t_b,$$

one can write (2.4.41) as

$$\phi_j^{(ab)} = \frac{i}{\beta} \ln \frac{\tau_j^{(ab)}}{\tau_{j-1}^{(ab)}},$$

where,

$$\tau_j^{(ab)} = 1 + \omega_a^j \bar{t}_a + \omega_b^j \bar{t}_b + A(\omega_a\omega_b)^j \bar{t}_a\bar{t}_b, \tag{2.4.43}$$

with A given in (2.4.10). Thus, contrary to the Hirota's methods, the single soliton τ-function produced by a Bäcklund transformation cannot be directly used to construct the two soliton τ-function (compare (2.4.42) and (2.4.43) with (2.4.4) and (2.4.8)).

Using the Bäcklund transformations (2.4.27) one can show that the energy and momentum densities are surface terms which can be integrated over all space yielding real energy and momentum. However, subtleties arises if one tries to use the same methods in calculating the energy and momentum of multi-soliton solutions. The methods used in [16] do not generalize easily to multi-soliton solutions.

One can conclude that the construction of soliton solutions and the calculations of their energies and momenta using the Bäcklund transformations is not practical. Moreover, this method does not generalize and so far only the Bäcklund transformation of the $a_n^{(1)}$ series is known.

2.4.3 Algebraic Method

A more elegant constructions of soliton solutions is the group-algebraic constructions provided by Olive et.al [12, 13]. This algebraic approach is a generalization of the Leznov-Saveliev solution of the conformal Toda theory [48] where the simplest affine case, i.e. the sine-Gordon theory, has been discussed in [49]. The Leznov-Saveliev solutions are given in term of the fundamental weight expectation value of a certain group element. The key generalization of Olive et.al. is to use an *alternative basis* for the affine Kac-Moody algebra which ad-diagonalises, and replacing the fundamental weights of the Lie algebras with the fundamental weight of the associated affine Kac-Moody algebras. As a result of these, the reality of energy and momentum was proved in an algebra-independent way.

One starts with the generalization of the Leznov-Saveliev solution for the *conformal affine Toda theories*,

$$e^{-\beta\Lambda_j\cdot\Phi} = e^{-\beta\Lambda_j\cdot\Phi_0} < \Lambda_j|U(x_+)V(x_-)|\Lambda_j >, \qquad (2.4.44)$$

where Λ_j is the affine fundamental weight defined by $\frac{2\Lambda_j\cdot a_k}{a_k^2} = \delta_{jk}$. For the untwisted cases, the affine simple roots are related to the ordinary simple roots of the rank-r Lie algebra as $a_0 = (-\psi, 0, 1)$ and $a_j = (\alpha_j, 0, 0)$ with $j = 1, 2, \ldots, r$. Thus, $\Lambda_0 = (0, \frac{1}{2}\psi^2, 0)$ and $\Lambda_j = (\lambda_j, \frac{1}{2}m_j\psi^2, 0)$, and the constant m_j's are given by $m_j = n_j\frac{\alpha_j^2}{\psi^2}$. The field Φ has components (ϕ, η, ξ) where the extra fields η and ξ correspond to the additional generators of the Cartan subalgebra, the generators d and k respectively. The affine Toda theories are obtained from the *conformal* affine Toda theories by setting the the field $\eta = 0$, this is called the conformal gauge [10, 12]. The quantities $U(x_+)$ and $V(x_-)$ satisfy the following differential equations [12],

$$\partial_+ U = -\mu\left\{e^{\beta\phi_0^+\cdot H_0}\left(\sum_{j=1}^{r}\sqrt{m_j}E_0^{\alpha_j} + E_1^{\alpha_0}\right)e^{-\beta\phi_0^+\cdot H_0}\right\}U, \qquad (2.4.45)$$

$$\partial_- V = -\mu V\left\{e^{-\beta\phi_0^-\cdot H_0}\left(\sum_{j=1}^{r}\sqrt{m_j}E_0^{-\alpha_j} + E_{-1}^{-\alpha_0}\right)e^{\beta\phi_0^-\cdot H_0}\right\}. \qquad (2.4.46)$$

Here, the generators are written in the modified Cartan-Weyl basis (actually in the following discussion, one will freely change from this basis to the Chevalley basis or the *alternative basis* as explained in the *Appendix* **A**) normalized to the Chevalley basis, i.e.

$$[E_0^{\alpha_i}, E_0^{-\alpha_j}] = [E^{a_i}, E^{-a_j}] = \frac{2a_j\cdot H_0}{a_j^2}\delta_{ij}, \qquad i, j = 1, 2, \ldots, r.$$

The parameter μ is related to the mass parameter m as $m^2 = \frac{4\mu^2}{\psi^2}$, and the field components $\phi_0 = \phi_0^+ + \phi_0^-$ of Φ_0 provide a free field solution. The auxiliary field ξ can be eliminated from equation (2.4.44) by dividing through by $e^{-\beta\Lambda_0\cdot\Phi}$ to obtain,

$$e^{-\beta\lambda_j\cdot\phi} = e^{-\beta\lambda_j\cdot\phi_0}\frac{<\Lambda_j|U(x_+)V(x_-)|\Lambda_j>}{<\Lambda_0|U(x_+)V(x_-)|\Lambda_0>^{m_j}}. \qquad (2.4.47)$$

Furthermore, given two highest weight states $|\Lambda_1>$ and $|\Lambda_2>$, one can construct a third highest weight state defined as,

$$|\Lambda_1+\Lambda_2> \equiv |\Lambda_1>\otimes|\Lambda_2> .$$

Where action of any group element factorises,

$$<\Lambda_1+\Lambda_2|g|\Lambda_1+\Lambda_2> = <\Lambda_1|g|\Lambda_1><\Lambda_2|g|\Lambda_2> .$$

Therefore, for any dominant weight $\Lambda = (\lambda, 0, ?)$ with level p one has the same relation as (2.4.47),

$$e^{-\beta\lambda\cdot\phi} = e^{-\beta\lambda\cdot\phi_0}\frac{<\Lambda|U(x_+)V(x_-)|\Lambda>}{<\Lambda_0|U(x_+)V(x_-)|\Lambda_0>^{p}}. \qquad (2.4.48)$$

Recall that the action of E^{-a_j} on the highest weight state $|\Lambda_k>$ is given by,

$$E^{-a_j}|\Lambda_k> = \delta_{jk}|\Lambda_k - a_k> .$$

Then, using (2.4.47) one obtain,

$$\begin{aligned} -\beta\partial_+\partial_-(\lambda_j\cdot\phi) &= -\mu^2 e^{\beta\phi_0\cdot\alpha_j}m_j\frac{<2\Lambda_j-a_j|UV|2\Lambda_j-a_j>}{<\Lambda_j|UV|\Lambda_j>^2} \\ &\quad -\mu^2 e^{\beta\phi_0\cdot\alpha_0}m_j\frac{<2\Lambda_0-a_0|UV|2\Lambda_0-a_0>}{<\Lambda_0|UV|\Lambda_0>^2}, \end{aligned}$$

where $|2\Lambda_j - a_j>$ is defined as

$$|2\Lambda_j - a_j> \equiv \frac{1}{\sqrt{2}}(|\Lambda_j>\otimes|\Lambda_j-a_j> - |\Lambda_j-a_j>\otimes|\Lambda_j>),$$

and it is a dominant weight of level $2m_j$. Then, upon a further substitution of (2.4.48) and setting $\psi^2 = 2$, one finally see that indeed (2.4.47) solves the affine Toda equation of motion. Note, that up until this stage one has not specified the value of the coupling parameter, i.e. it can be real or imaginary.

The generators,

$$\hat{E}_{\pm1} = \sum_{j=1}^{r}\sqrt{m_j}E_0^{\pm\alpha_j} + E_{\pm1}^{\pm\alpha_0} = \sum_{j=0}^{r}\sqrt{m_j}E^{\pm a_j}, \qquad (2.4.49)$$

are known as elements of the Heisenberg subalgebra of $\hat{g}$, which plays the rôle of a new Cartan subalgebra of the alternative basis. The complete alternative basis of $\hat{g}$ consist of the generators $\hat{E}_{\pm M}$ where $M = [M] + mh$ is the exponent of $\hat{g}$ ($m \in \mathbb{Z}$), i.e. $[M]$ and h are the exponent and Coxeter number of g respectively; and the step operators $\hat{F}^j_N$ where $N = [N] + nh$, $n \in \mathbb{Z}$. The formal power series expansion of $\hat{F}^j_N$,

$$\hat{F}^j(z) = \sum_{N=-\infty}^{\infty} z^{-N} \hat{F}^j_N \tag{2.4.50}$$

will ad-diagonalised $\hat{E}_M$. Their commutation relations are given by the following,

$$[\hat{E}_M, \hat{E}_N] = M\delta_{M+N,0}\, x, \tag{2.4.51}$$

$$[\hat{E}_M, \hat{F}^j(z)] = \gamma_j \cdot q([M]) z^M \hat{F}^j(z), \tag{2.4.52}$$

where x (or alternatively also denoted by k) is the central element of $\hat{g}$, $q([M])$ is an eigenvector of the Coxeter element ω of the Weyl group of g, i.e. $\omega(q([M])) = e^{\frac{2i\pi[M]}{h}} q([M])$ with the following orthogonality and completeness properties

$$q([M]) \cdot q([M'])^* = h\delta_{[M],[M']}, \qquad \sum_{[M]} q([M])\, q([M])^* = hI,$$

where $q(h - [M]) = q([M])^*$ and $\gamma_j = c(j)\alpha_j$ where $c(j) = 1$ or -1 depending on the simple roots α_j of g is *black* or *white* according to the bicolouring of the Dynkin diagram (see *Chapter Four*). A short review on the construction of the alternative basis of $\hat{g}$ is given in *Appendix* **A**.

As already noted in the previous discussions, taking an imaginary coupling parameter results in the existence of degenerate vacua, $\phi \in \frac{2i\pi}{\beta}\Lambda_w(g^v)$ where $\Lambda_w(g^v)$ is the coweight lattice. Thus, the soliton solution is obtained from (2.4.47) by taking $\phi_0^\pm = \frac{2i\pi}{\beta}\Lambda_w(g^v)$ and assuming that β is imaginary. One readily obtains a very simple version of (2.4.45) and (2.4.46),

$$\partial_+ U = -\mu \hat{E}_1 U, \qquad \partial_- V = -\mu V \hat{E}_{-1}.$$

Which can be integrated directly to yield,

$$U = e^{-\mu \hat{E}_1 x_+} g_+(0), \qquad V = g_-(0) e^{-\mu \hat{E}_{-1} x_-}.$$

Thus,

$$UV = e^{-\mu\hat{E}_1 x_+} g(0) e^{-\mu\hat{E}_{-1} x_-}, \tag{2.4.53}$$

where $g(0) = g_+(0)g_-(0)$ is a group valued integration constant. The *time development operators* in (2.4.53) can be eliminated by defining a *normal ordered time development operator*,

$$V(t) \equiv \exp(\mu\hat{E}_{-1}x_-)\ \exp(-\mu\hat{E}_1 x_+), \tag{2.4.54}$$

and,

$$g(t) \equiv V(t)g(0)V(t)^{-1}. \tag{2.4.55}$$

Such that,

$$< \Lambda_j|g(t)|\Lambda_j >= e^{-\mu^2 x_+ x_- m_j} < \Lambda_j|e^{-\mu\hat{E}_1 x_+} g(0) e^{-\mu\hat{E}_{-1} x_-}|\Lambda_j > .$$

Suppose that $g(0)$ is a group element generated by $\hat{E}_N$, these expression can be *normal ordered* such that for $N \geq 0$, $\hat{E}_N$ will annihilate $|\Lambda_j >$. Thus, the soliton solution can be written as,

$$e^{-\beta\lambda_j\cdot\phi} = \frac{< \Lambda_j|g(t)|\Lambda_j >}{< \Lambda_0|g(t)|\Lambda_0 >^{m_j}}, \tag{2.4.56}$$

and the group valued integration constant will only depend on the step operators $\hat{F}^j(z_j)$,

$$g(0) = e^{Q_j\hat{F}^j(z_j)} e^{Q_k\hat{F}^k(z_k)} \ldots. \tag{2.4.57}$$

From (2.4.55) one realises that $g(t)$ is given by $g(0)$ with each step operator $\hat{F}^j(z_j)$ replaced with the following expression,

$$\hat{F}^j(z_j) \longrightarrow \exp(-\mu x_+ \gamma_j \cdot q([1])z_j + \mu x_- \gamma_j \cdot q([1])^* z_j^{-1})\hat{F}^j(z_j). \tag{2.4.58}$$

Furthermore, to this solution one can perform a Lorentz boost with rapidity Θ to a rest frame, $x_\pm \rightarrow x_\pm e^{\pm\Theta}$, such that the parameter z takes the value,

$$z = \pm\sqrt{\frac{\gamma_j \cdot q([M])^*}{\gamma_j \cdot q([M])}}.$$

Hence one can choose the parameter z to be,

$$z = \mp\frac{|\gamma_j \cdot q([M])|}{\gamma_j \cdot q([M])} e^{\Theta}. \tag{2.4.59}$$

Using this value of z, one finally obtains

$$g(t) = e^{W_j(\Theta_j)Q_j\hat{F}^j(z_j)}\, e^{W_k(\Theta_k)Q_k\hat{F}^k(z_k)} \ldots, \tag{2.4.60}$$

where

$$W_j(\Theta_j) = \exp[\pm\mu|\gamma_j \cdot q([M])|(x_+e^{\Theta_j} - x_-e^{-\Theta_j})]. \tag{2.4.61}$$

It turns out that the element $\hat{F}^k(z_k)$ is nilpotent. Hence, in the expansion of $g(t)$, the product of $\hat{F}^k(z_k)$ terminates at a certain finite value which depends on the level of the fundamental representation. To calculate the soliton solutions explicitly, one needs to know the expectation value of $\hat{F}^k(z_k)$,

$$F_{jk} = <\Lambda_j|\hat{F}^k(z_k)|\Lambda_j> . \tag{2.4.62}$$

Recall that the positive and negative grading of $\hat{F}^k(z_k)$ are linear combinations of positive and negative grading of the modified Cartan-Weyl step operators, respectively. Thus, only the zero grade of $\hat{F}^k(z_k)$ will contribute in (2.4.62). Then one can write,

$$\hat{F}^k_0 = \sum_{j=0}^{r} h_j F_{jk}.$$

The matrix F_{jk} can be made into a square matrix by adding a zeroth column given by the central element x, $\hat{F}^0(z) = x$ such that $F_{j0} = m_j$.

Taking the expectation value of the commutation relation $[\hat{E}_1, [\hat{E}_{-1}, \hat{F}^k_0]]$ one obtains,

$$\sum_i m_j\hat{C}_{ji}F_{ik} = \begin{cases} |\gamma_k \cdot q(1)|^2 F_{jk} & k \neq 0, \\ 0 & k = 0. \end{cases} \tag{2.4.63}$$

Thus, the columns of F_{jk} are eigenvectors of the matrix $m_j\hat{C}_{ji}$, where $\hat{C}_{ji} = \frac{2a_j \cdot a_i}{a_i^2}$ is the affine Cartan matrix, with eigenvalues proportional to the mass^2 of the fundamental particle,

$$\tilde{m}_a = \sqrt{2}\mu|\gamma_a \cdot q(1)|.$$

For the $a_n^{(1)}$ series, the fundamental mass is given by (2.2.6). One can easily solve for the matrix F_{jk} of $a_n^{(1)}$ series to obtain,

$$F_{jk} = e^{\frac{2i\pi}{h}jk}. \tag{2.4.64}$$

The next task is to know the expectation value of a product of several step operators $\hat{F}^k(z_k)$. Confining oneself to the level one representation of simply-laced cases, one can represent $\hat{F}^k(z_k)$ with the vertex operator on the irreducible representation with highest weight Λ_j as,

$$\rho_j(\hat{F}^k(z_k)) = F_{jk}\exp\left(\sum_{N>0}\frac{\gamma_k\cdot q([N])}{N}z^N\hat{E}_{-N}\right)\exp\left(-\sum_{N>0}\frac{\gamma_k\cdot q([N])^*}{N}z^{-N}\hat{E}_N\right). \quad (2.4.65)$$

This vertex operator has the same commutation relation with $\hat{E}_M$ as (2.4.52) and the right expectation value with respect to $|\Lambda_j>$ as (2.4.62).

The operator product of two vertex operators are normal ordered, i.e. moving all positive grade operators to the right of negative grade operators. The resulting normal ordering gives,

$$\rho(\hat{F}^j(z_j))\rho(\hat{F}^k(z_k)) = X_{jk}(z_j,z_k) :\rho(\hat{F}^j(z_j))\rho(\hat{F}^k(z_k)):, \quad (2.4.66)$$

and the interaction coefficient is given by,

$$X_{jk}(z_j,z_k) = \prod_{p=1}^{h}(z_j-\zeta^{-p}z_k)^{\omega^p(\gamma_j)\cdot\gamma_k}, \quad (2.4.67)$$

where ζ is a primitive h-root of unity and ω is the Coxeter element. The interaction coefficient is symmetric under the interchange of its indices and its arguments, $X_{jk}(z_j,z_k) = X_{kj}(z_k,z_j)$. Recall that the simple roots of g can be bicoloured (*black* and *white*) into two sets of orthogonal simple roots, the Coxeter element can written as $\omega = \omega_{\{\bullet\}}\omega_{\{\circ\}}$ where $\omega_{\{\bullet\}}$ and $\omega_{\{\circ\}}$ is product of reflections with respect to the black and white simple roots respectively. Then the symmetry property can be shown using the following identities,

$$\gamma_j = \omega^{-\frac{1}{2}[1-c(j)]}(\lambda_j) - \omega^{\frac{1}{2}[1+c(j)]}(\lambda_j), \qquad \text{and} \qquad \sum_{p=1}^{h}p\omega^p(\gamma_j) = -h\omega^{\frac{1}{2}[1+c(j)]}(\lambda_j),$$

such that $\gamma_k\cdot\sum_{p=1}^{h}p\omega^p(\gamma_j)\in h\mathbb{Z}$.

From this interaction coefficient, one can immediately see the nilpotency of the level one vertex operator,

$$\hat{F}^j(z_j)\hat{F}^j(z_j) = 0. \quad (2.4.68)$$

Generalization to products of more than two vertex operators can be done in the same way resulting in the following,

$$\rho(\hat{F}^{i_1}(z_{i_1}))\ldots\rho(\hat{F}^{i_k}(z_{i_k})) = \prod_{1\le p<q\le k} X_{i_p i_q}(z_{i_p}, z_{i_q}) : \rho(\hat{F}^{i_1}(z_{i_1}))\ldots\rho(\hat{F}^{i_k}(z_{i_k})) :,$$

and the expectation value with respect to $|\Lambda_j>$ gives,

$$<\Lambda_j|\hat{F}^{i_1}(z_{i_1})\ldots\hat{F}^{i_k}(z_{i_k})|\Lambda_j> = \prod_{1\le p<q\le k} X_{i_p i_q}(z_{i_p}, z_{i_q}) \prod_{p=1}^{k} F_{j i_p}.$$

Thus it is readily seen that the interaction coefficient factorises.

Generalization to vertex operators of higher level is done by recalling that the irreducible representations of simply-laced $\hat{g}$ of level one are labelled as follows,

$$|\Lambda_0>=|1,0> \quad ; \quad |\Lambda_j>=|1,\lambda_j>.$$

Thus, higher level will also be labelled by the same fundamental weight λ_j of g. It is expected that the level x irreducible representation of $\hat{g}$ occur in the decomposition of representation as [13],

$$D^{(1,\lambda_j)} \otimes D^{(1,0)} \otimes \ldots \otimes D^{(1,0)},$$

where in the above, the irreducible representation $D^{(1,0)}$ appears $(x-1)$ times. Hence, a vertex operator on the level x can be defined using the vertex operator of the level one as,

$$\hat{\mathcal{F}}^j(z) = \hat{F}^j(z)\otimes 1\otimes\ldots\otimes 1 + 1\otimes\hat{F}^j(z)\otimes\ldots\otimes 1 + \ldots + 1\otimes 1\otimes\ldots\otimes\hat{F}^j(z). \quad (2.4.69)$$

Then,

$$\hat{\mathcal{F}}^j(z)^x = x!\hat{F}^j(z)\otimes\hat{F}^j(z)\otimes\ldots\otimes\hat{F}^j(z),$$

such that in virtue of (2.4.68) one obtains the nilpotency of a level x vertex operator as follows,

$$\hat{\mathcal{F}}^j(z)^{x+1} = 0. \quad (2.4.70)$$

Using the above informations one can now construct the soliton solution explicitly. Writing the expectation value of the group element $g(t)$ as $\hat{\tau}_j \equiv <\Lambda_j|g(t)|\Lambda_j>$, one has

$$e^{-\beta\lambda_j\cdot\phi} = \frac{\hat{\tau}_j}{(\hat{\tau}_0)^{m_j}}.$$

Or, to put it in the familiar Hirota's τ-function solution, one has

$$\phi = -\frac{1}{\beta}\sum_{j=1}^{r}\eta_j\alpha_j \ln\frac{\hat{\tau}_j}{(\hat{\tau}_0)^{m_j}} = -\frac{1}{\beta}\sum_{j=0}^{r}\eta_j\alpha_j \ln\hat{\tau}_j,$$

where $\eta_j = \frac{2}{\alpha_j^2}$, $\psi^2 = 2$ and $n_0 = m_0 = 1$.

Specializing on the $a_n^{(1)}$ series, one obtain the following results. The single soliton τ-function which is *created* by $g(t) = e^{WQ\hat{F}^a}$,

$$\begin{aligned}\hat{\tau}_j^{(a)} &= <\Lambda_j|(1+WQ\hat{F}^a)|\Lambda_j> \\ &= 1 + e^{\frac{2i\pi}{h}aj}QW.\end{aligned}$$

A two soliton solution τ-function *created* by $e^{W_aQ_a\hat{F}^a}$ and $e^{W_bQ_b\hat{F}^b}$ is given by,

$$\begin{aligned}\hat{\tau}_j^{(ab)} &= <\Lambda_j|(1+W_aQ_b\hat{F}^a)(1+W_bQ_b\hat{F}^b)|\Lambda_j> \\ &= 1 + e^{\frac{2i\pi}{h}aj}Q_aW_a + e^{\frac{2i\pi}{h}bj}Q_bW_b + X_{a,b}(z_a, z_b)e^{\frac{2i\pi}{h}(a+b)j}Q_aQ_bW_aW_b.\end{aligned}$$

Where Ws are given by (2.4.61) and the complex quantity Q will determine the topological charges of these soliton solutions.

In [12], it is also shown that the energy-momentum tensor of the general solution to the affine Toda theories (2.4.44) splits into two parts,

$$T_{\mu\nu} = \Theta_{\mu\nu} + C_{\mu\nu}, \tag{2.4.71}$$

where the improvement term is given by,

$$C_{\mu\nu} = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)C, \tag{2.4.72}$$

and the *improved* energy-momentum tensor is forced to be traceless, $\Theta^\mu_\mu = 2\Theta_{+-} = 0$. Defining a dominant weight S as,

$$S = \sum_{i=0}^{r}\frac{a_0^2\Lambda_i}{a_i^2},$$

the non-zero *improved* energy-momentum tensor components are,

$$\Theta_{\pm\pm} = (\partial_\pm\Phi, \partial_\pm\Phi) - \frac{2}{\beta}\partial_\pm^2(S\cdot\Phi),$$

where (,) is defined as,

$$(\partial_\pm\Phi, \partial_\pm\Phi) = \sum_{i,j=0}^{r} \frac{2a_i \cdot a_j}{a_i^2 a_j^2} \partial_\pm\phi\partial_\pm\phi.$$

The explanation of [12] is straightforward; inserting the general solution (2.4.44) to the above equation one obtains,

$$\Theta_{\pm\pm} = (\partial_\pm\Phi_0, \partial_\pm\Phi_0) - \frac{2}{\beta}\partial_\pm^2(S \cdot \Phi_0),$$

i.e. the improved part only depends on the free solution. For the soliton solutions, one chooses $\phi_0 \in \frac{2i\pi}{\beta}\Lambda_w(g^v)$, $\xi_0 = 0$ and $\eta_0 = 0$, which results in the vanishing of the improved energy-momentum tensor, $\Theta_{\pm\pm} = 0$. Thus the energy-momentum tensor is given by the function C which is found from the following equation,

$$\partial^2 C = \frac{4\mu^2}{\beta^2} \sum_{i=0}^{r} n_i [e^{\beta a_i \cdot \Phi} - 1].$$

The solution for C is easily found to be

$$C = -\frac{2}{\beta} S \cdot \Phi - \frac{2h\mu^2}{\beta^2} x_+ x_-. \tag{2.4.73}$$

Inserting the soliton solution (2.4.44) into (2.4.73) yields

$$C = \frac{2}{\beta} \ln < S|g(t)|S > . \tag{2.4.74}$$

Then the lightcone energy-momentum component is given as (2.4.17).

The mass of a single soliton can be determined from the lightcone energy-momentum component. As an example consider a two soliton solution, where $g(t)$ is given by,

$$g(t) = e^{W_a Q_a \hat{F}^a} e^{W_b Q_b \hat{F}^b}.$$

It is already known that the expansion of the argument of $g(t)$ terminates at a finite number. Introduce a short hand notation $\mu_a = \sqrt{2}\mu|\gamma_a \cdot q(1)|$ and $\varepsilon_a = \pm 1$ depending on the choice of sign of W in (2.4.61). Then the expectation value can be written as,

$$\begin{aligned} < S|g(t)|S > &= < S| \left(\sum_{p_a=0} \frac{(W_a Q_a \hat{F}^a)^{p_a}}{p_a!} \right) \left(\sum_{p_b=0} \frac{(W_b Q_b \hat{F}^b)^{p_b}}{p_b!} \right) |S > \\ &= < S| \left(\sum_{p_a, p_b} e^{(\varepsilon_a p_a \mu_a [t \sinh(\Theta_a) + x \cosh(\Theta_a)])} \right. \\ & \qquad \left. e^{(\varepsilon_b p_b \mu_b [t \sinh(\Theta_b) + x \cosh(\Theta_b)])} \frac{Q_a^{p_a} Q_b^{p_b}}{p_a! p_b!} (\hat{F}^a)^{p_a} (\hat{F}^b)^{p_b} \right) |S > . \end{aligned} \tag{2.4.75}$$

It follows directly that in the limit $x \to \pm\infty$, the value of $\frac{\partial_\pm <S|g(t)|S>}{<S|g(t)|S>}$ is a constant and does not depend on the choice of ε. Thus one obtains

$$P^\pm = -\frac{4h}{\beta^2\gamma_a^2}\frac{\mu_a}{\sqrt{2}}e^{\mp\Theta_a} - \frac{4h}{\beta^2\gamma_b^2}\frac{\mu_b}{\sqrt{2}}e^{\mp\Theta_b}.$$

Further development using this algebraic method has also been done [50, 51]. Finally, one can conclude that although this algebraic method is elegant, in fact to perform an explicit calculation one has to calculate the same τ-functions as in Hirota's method.

Chapter 3

Breather Solutions of Affine Toda Theories

Body and Soul

Charlie Parker

As it has been discussed in detail in the previous chapter, the imaginary coupling regime of the affine Toda theories admits soliton solutions interpolating the degenerate vacua of the affine Toda potential. The sine-Gordon theory is the simplest example of the $a_n^{(1)}$ series, i.e. the $a_1^{(1)}$ Toda field theory. Besides soliton and antisoliton solutions of the sine-Gordon theory, there also exist oscillating solitonic solutions, the *breathers*. These breathers are bound states of the sine-Gordon soliton and antisoliton pair. Since the $a_n^{(1)}$ affine Toda theories are a generalization of the sine-Gordon theory, it is natural to ask if such breather solutions also exist for these theories, and indeed if such breather solutions exist for all affine Toda theories. Much has been conjectured about the breathers of the affine Toda field theory [11, 12, 13, 51]. Calculation of the scattering processes of the $a_n^{(1)}$ affine Toda solitons [14] also points to the existence of these breathers, since there are poles in the soliton S-matrix which correspond to bound states of soliton pairs.

Furthermore, generally the spectrum of topological charges carried by the single soliton solutions do not fill up the coresponding fundamental representation space [35, 36]. Thus, it is hope that the *missing* topological charges are carried by these breathers.

For the $a_1^{(1)}$ Toda field theory, which is the sine-Gordon theory, the single soliton solution is given by,

$$\phi = \frac{i\sqrt{2}}{\beta} \ln\left(\frac{1 - e^{\sigma(x-vt)+\rho}}{1 + e^{\sigma(x-vt)+\rho}}\right), \tag{3.0.1}$$

with the constraint $\sigma^2(1 - v^2) = 4m^2$. The parameter $\rho = \eta + i\xi$ is complex, and its imaginary part determines the topological charge of the soliton. In this case, there are two possibilities, the soliton and the antisoliton, which have the same mass equal to $\frac{8m}{\beta^2}$. It is well known (see for instance Rajaraman [52]) that the oscillating solitonic solutions, or *breathers*, of the sine-Gordon theory can be constructed from two approaching soliton solutions by changing the velocity v into iv,

$$\phi_{breather} = \frac{i\sqrt{2}}{\beta} \ln\left(\frac{1 - e^{\sigma(x-ivt)+\rho_1} - e^{\sigma(x+ivt)+\rho_2} - v^2 e^{2\sigma x+(\rho_1+\rho_2)}}{1 + e^{\sigma(x-ivt)+\rho_1} + e^{\sigma(x+ivt)+\rho_2} - v^2 e^{2\sigma x+(\rho_1+\rho_2)}}\right). \tag{3.0.2}$$

Taking $\xi_1 = -\xi_2 = -\frac{\pi}{2}$ and $\eta_1 = \eta_2 = \eta$, yields a soliton solution oscillating about the point $\frac{\ln(v^2)+2\eta}{2\sigma}$. As it is constructed from a soliton-antisoliton pair, this sine-Gordon breather has zero topological charge; its mass is equal to $\frac{16m}{\beta^2\sqrt{1+v^2}}$.

Following the prescription of analytically continuing the real velocity into *imaginary* velocity, the breathers created from two solitons of the $a_n^{(1)}$ affine Toda theories can be constructed [47]. It turns out that in order for the energy of the breathers built from two solitons to be real, the constituent solitons must be of the same mass and moving with opposite imaginary velocity, resulting in a stationary breather. To obtain a moving breather, one can apply the usual Lorentz boost to the breather solution. The condition for the reality of the energy also produces an expression for the masses of these breathers which is less than the sum of the constituent solitons. One type of breather can carry topological charge which coincides with the topological charge of a certain single soliton, while the other type has zero topological charge. It will be shown also that these topological charges lie in a fundamental representation which is a subset of a tensor product representation of the fundamental representations which are associated to the topological charges of the constituent solitons. Moreover, these topological charges are analogous to the single soliton case [35, 36]. Thus, only particular combinations of the constituent solitons are permitted, such that the sum of their topological charges is the appropriate topological charge of the breather. However, the topological charges of the constituent solitons cannot be calculated explicitly. As the complex $a_n^{(1)}$ Toda theories is a direct generalization of the sine-Gordon theory, it is possible that some linear combination of the components of the solution will remain invariant under a certain automorphism of the Dynkin diagram. This led to the embedding of the sine-Gordon solution in some cases of the $a_n^{(1)}$ family. The crossing of one of the constituent soliton in the breather solution can be used to superficially calculate the topological charges of the constituent solitons. Finally, breathers in the $d_4^{(1)}$ theory will be discussed and the breathers of the other theories will be briefly commented on.

One has to point out that a recent result on the exact S-matrices for bound states in the $a_2^{(1)}$ theory [42] agrees with the classical breathers considered in this chapter.

3.1 Breathers of $a_n^{(1)}$ Theories

As discussed in the previous chapter, in performing explicit calculations one has to consider the τ-function expressions. Thus, in what follows, Hirota's τ-function of the $a_n^{(1)}$ solitons will be used.

To obtain an oscillating solitonic solution which is constructed from two solitons, one follows the sine-Gordon prescription of changing the velocity into an imaginary velocity, i.e. changing u into iv in the τ-functions (2.4.7). Care must be taken in the analytic continuation of $u \to iv$ such that the energy and momentum are real, although the energy and momentum densities generally are complex.

Further, one assumes that in order to have a stable classical bound state of solitons (*breather*), there must exist a *rest frame* in which the constituent solitons of the breather oscillate about a fixed point in space. In this rest frame the energy of the breather is localised and time independent, although the energy density is time dependent.

Changing u into iv also means changing a real rapidity into an imaginary rapidity, with a relation between velocity v and rapidity $\bar{\Theta}$ becomes $v = \tan(\bar{\Theta})$, $\bar{\Theta} = -i\Theta$. From the lightcone energy-momentum of the two soliton solution (2.4.21), a real energy and momentum can be achieved provided that the two solitons forming a breather are of the same mass and moving towards to each other with the same velocity giving a *stationary* breather. One can make an oscillating solution from solitons of two different masses, but the energy and momentum of this solution will not be real. Generally, one can add a real rapidity Θ_0 as a phase in the energy-momentum tensor, which acts as a Lorentz boost to the breather solution. Thus, from (2.4.21) one obtains,

$$P^{\pm}_{breather} = \frac{4h\tilde{m}_a}{\sqrt{2}\beta^2}\cos(\bar{\Theta}_a)e^{\mp\Theta_0}, \tag{3.1.1}$$

$\tilde{m}_a$ above is the mass of the fundamental particles of the $a_n^{(1)}$ affine Toda theory. For simplicity, in what follows only stationary breathers are considered. Hence (3.1.1) becomes,

$$P^{\pm}_{breather} = \frac{4h}{\sqrt{2}\beta^2}\frac{\tilde{m}_a}{\sqrt{1+v^2}}, \tag{3.1.2}$$

then the mass of a breather is calculated to be

$$M_{breather} = \frac{2M_a}{\sqrt{1+v^2}} = \frac{4h\tilde{m}_a}{\beta^2\sqrt{1+v^2}}. \tag{3.1.3}$$

It is obvious that the mass of a breather is less than the sum of its constituent solitons. This result generalizes the sine-Gordon case, i.e. taking $h = 2$ gives the mass of the sine-Gordon breather. This result was also noted in [13].

The masses m_a of the fundamental particles of $a_n^{(1)}$ series are degenerate with respect to the Z_2 symmetry of the A_n Dynkin diagram, i.e. $\tilde{m}_a = \tilde{m}_{h-a}$. Hence, there are two possibilities of forming a breather. Either the two constituent solitons are of the same species, these breathers will be called *type* A breathers or, the two constituent solitons are of opposite species, *type* B breathers. Exceptions to this classification are the breathers constructed from solitons of species $(n+1)$ of the $a_{2n+1}^{(1)}$ theories. These breathers are sine-Gordon embedded breathers which belong to both type A and B as will be explained in *Section Three* of this chapter.

Looking back at the τ-function of a two soliton solution of the same constituent mass, choosing $u_a = -u_b = iv$ yields the breather τ-function,

$$\begin{aligned} \tau_j^{(ab)} &= 1 + \exp[\sigma_a(x - ivt) + \rho_a + ij\theta_a] + \exp[\sigma_b(x + ivt) + \rho_b + ij\theta_b] \\ &\quad + \exp[\sigma_+ x + \lambda + \rho_+ + ij\theta_+], \end{aligned} \tag{3.1.4}$$

the interaction coefficient is written as $A = e^\lambda$ with $\lambda = \zeta + i\delta$, where $\zeta, \delta \in \mathbb{R}$ and

$$\sigma_\pm = \sigma_a \pm \sigma_b, \qquad \rho_\pm = \rho_a \pm \rho_b, \qquad \theta_\pm = \theta_a \pm \theta_b,$$

$$\eta_\pm = \eta_a \pm \eta_b, \qquad \xi_\pm = \xi_a \pm \xi_b,$$

recalling (2.4.5). Note that for solitons of the same mass, $\sigma_a = \sigma_b$. By the ansatz (2.4.1), it is clear that in order to have a well-defined solution, each α_j component of the solution ϕ must be well defined. Thus, for each j, the ratio $\frac{\tau_j}{\tau_0}$ must not become zero or infinite. Evaluation of the behaviour of the τ-function can be done easily by writing the real and imaginary part of (3.1.4) explicitly. It turns out that to avoid the real and imaginary part of (3.1.4) becoming zero simultaneously at the same point, the parameters ξ_+ and η_- are restricted to a certain range of definition.

Moreover, for type A and B breathers, the interaction coefficient can have either positive or negative value. The critical velocity when the interaction coefficient changes sign is, for a type A breather

$$v_c^{(A)} = \tan(\frac{\theta_a}{2}), \tag{3.1.5}$$

and for a type B breather,

$$v_c^{(B)} = \frac{1}{v_c^{(A)}}. \tag{3.1.6}$$

For breathers with constituent solitons of species $a = \frac{h}{2}$, the interaction coefficient never changes sign.

With $\sigma_a = \sigma_b = \sigma$, when the interaction coefficient is negative, i.e. $\delta = \pi$, the breather τ-function (3.1.4) can be written as,

$$\begin{aligned}
\tau_j^{(ab)} &= 2\exp\frac{1}{2}[2\sigma x + \zeta + \eta_+ + i(\delta + \xi_+ + j\theta_+)] \\
&\times \left\{ \left(-\cosh\frac{1}{2}[2\sigma x + \zeta + \eta_+] \sin\frac{1}{2}[\xi_+ + j\theta_+] \right.\right. \\
&\left. -e^{-\frac{\zeta}{2}} \sinh\frac{1}{2}[\eta_-] \sin\frac{1}{2}[2\sigma v t - (\xi_- + j\theta_-)] \right) \\
&+ i\left(\sinh\frac{1}{2}[2\sigma x + \zeta + \eta_+] \cos\frac{1}{2}[\xi_+ + j\theta_+] \right. \\
&\left.\left. -e^{-\frac{\zeta}{2}} \cosh\frac{1}{2}[\eta_-] \cos\frac{1}{2}[2\sigma v t - (\xi_- + j\theta_-)] \right) \right\}. \qquad (3.1.7)
\end{aligned}$$

While for positive interaction coefficient, i.e. $\delta = 0$, the breather τ-function (3.1.4) becomes,

$$\begin{aligned}
\tau_j^{(ab)} &= 2\exp\frac{1}{2}[2\sigma x + \zeta + \eta_+ + i(\delta + \xi_+ + j\theta_+)] \\
&\times \left\{ \left(\cosh\frac{1}{2}[2\sigma x + \zeta + \eta_+] \cos\frac{1}{2}[\xi_+ + j\theta_+] \right.\right. \\
&\left. +e^{-\frac{\zeta}{2}} \cosh\frac{1}{2}[\eta_-] \cos\frac{1}{2}[2\sigma v t - (\xi_- + j\theta_-)] \right) \\
&+ i\left(\sinh\frac{1}{2}[2\sigma x + \zeta + \eta_+] \sin\frac{1}{2}[\xi_+ + j\theta_+] \right. \\
&\left.\left. -e^{-\frac{\zeta}{2}} \sinh\frac{1}{2}[\eta_-] \sin\frac{1}{2}[2\sigma v t - (\xi_- + j\theta_-)] \right) \right\}. \qquad (3.1.8)
\end{aligned}$$

Thus, one only needs to examine the behaviour of the real and imaginary part of the curly bracket of (3.1.7) or (3.1.8) since in the τ-function ratio $\frac{\tau_j}{\tau_0}$ the prefactor of the curly bracket is just a phase.

3.1.1 Type A Breathers

The breathers with constituent solitons of the same species will have a negative interaction coefficient, $A < 0$, when $v^2 < v_c^{(A)2}$ (see *figure 3.1*). In the following one will deduce the restriction on the parameters η and ξ such that the breather solution is well behaved.

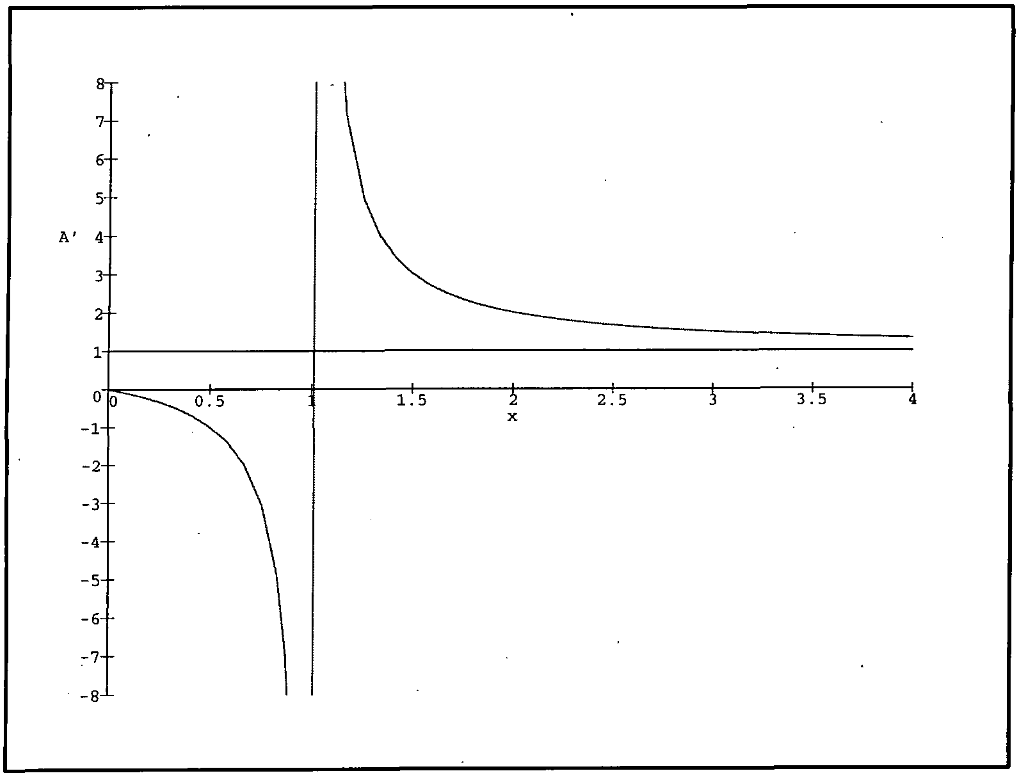

Figure 3.1: The graph shows the behaviour of the interaction coefficient for the type A breather solution. The interaction A' is given by $A' = A\cos(\theta_a/2)$, the velocity as $x = v^2/v_c^2$. As mentioned in *Subsection 3.1.1* there is an upper bound for the velocity. This has not been marked in the graph.

Suppose at coordinate (t_0, x_0) the real and imaginary part of the curly bracket of (3.1.7) becomes zero simultaneously, i.e.

$$\cosh\Gamma_0 = -c_1 \sin\Delta_0 \qquad \text{and} \qquad \sinh\Gamma_0 = c_2\cos\Delta_0, \tag{3.1.9}$$

where,

$$\Gamma_0 = \sigma x_0 + \frac{1}{2}(\zeta + \eta_-), \qquad \Delta_0 = \sigma v t_0 - \frac{\xi_-}{2},$$
$$c_1 = \frac{e^{\frac{-\zeta}{2}}\sinh\frac{\eta_-}{2}}{\sin\frac{1}{2}(\xi_+ + j\theta_+)}, \qquad c_2 = \frac{e^{\frac{-\zeta}{2}}\cosh\frac{\eta_-}{2}}{\cos\frac{1}{2}(\xi_+ + j\theta_+)},$$

provided $\sin\frac{1}{2}(\xi_+ + j\theta_+)$ is not zero. When $\cos\frac{1}{2}(\xi_+ + j\theta_+)$ is zero, one can readily avoid the singularity at (t_0, x_0) by setting $\eta_- = 0$. Squaring and subtracting (3.1.9) yields the conditions

$$c_1^2 \geq 1 \qquad \text{and} \qquad c_2^2 \geq -1.$$

Obviously, in order that in the whole space-time there are *no* sets of points (t_0, x_0), i.e. to have a well behaved solution, one has to demand that $c_1^2 < 1$ or

$$\sinh(\frac{\eta_-}{2})^2 < |A|\sin(\frac{1}{2}(\xi_+ + j\theta_+))^2. \tag{3.1.10}$$

Thus, to have a well behaved solution the parameter ξ_+ must *not* take the following values,

$$\xi_+ = (2\pi - j\theta_+) \bmod 2\pi, \qquad j = 0, \ldots, n. \tag{3.1.11}$$

This divides the range of ξ_+ into several regions which will determine the topological charge of the breather.

Furthermore, the upper bound of (3.1.10) will give the maximum distance of separation of the parameter η between the two constituent solitons which is restricted as follows,

$$-\min_j(\eta_c^j) < \eta_- < \min_j(\eta_c^j), \tag{3.1.12}$$

where,

$$\eta_c^j = \operatorname{arcosh}\left[2|A|\sin^2\frac{1}{2}(\xi_+ + j\theta_+) + 1\right],$$

with $j = 0, \ldots, n$. Each η_c^j above will restrict τ_j such that it will never be zero. Hence, for all τ-functions to avoid zero, the minimum value of η_c^j is taken as the limit on the range of η_-.

For breathers of type A with positive interaction coefficient (as well as type B breathers) one can perform a similar evaluation as above. In the case of type A breather with positive interaction coefficient, from the τ-function (3.1.7) one finds that the parameter ξ_+ must *not* have the following values,

$$\xi_+ = (\pi - j\theta_+) \bmod 2\pi, \qquad j = 0, \ldots, n. \tag{3.1.13}$$

The separation distance of parameter η is restricted as above with,

$$\eta_c^j = \operatorname{arcosh}\left[2A\cos^2\frac{1}{2}(\xi_+ + j\theta_+) - 1\right].$$

As η_c^j are defined through an arcosh-function, one has to satisfy the following restriction

$$A\cos^2\frac{1}{2}(\xi_+ + j\theta_+) \geq 1. \tag{3.1.14}$$

Thus η_c^j in turn will restrict the allowed velocity of the constituent solitons. Generally, not all $v^2 > v_c^{(A)2}$ are allowed. In fact all velocities with absolute value greater than the absolute value of the critical velocity are allowed if for all j the following is true,

$$\left[1 - \frac{\cos^2\frac{1}{2}(\xi_+ + j\theta_+)}{\cos^2(\frac{\theta_+}{4})}\right] < 0.$$

Otherwise, the velocity is bounded from above by,

$$v_c^{(A)^2} < v^2 \leq \frac{v_c^{(A)^2}}{\max\limits_j\left[1 - \frac{\cos^2\frac{1}{2}(\xi_+ + j\theta_+)}{\cos^2(\frac{\theta_+}{4})}\right]}. \tag{3.1.15}$$

3.1.2 Type B Breathers

The constituent solitons of type B breathers are of opposite species. The negative interaction coefficient regime is accomplished with $v^2 > v_c^{(B)2}$. In this case, to have a well behaved solution the parameter ξ_+ must *not* take the following values

$$\xi_+ = 0 \bmod 2\pi, \tag{3.1.16}$$

i.e. the parameter ξ_+ must not be an integer multiple of 2π. For each τ_j to avoid zero, the separation of parameter ηs is limited to take values between,

$$-\min_j(\eta_c^j) < \eta_- < \min_j(\eta_c^j),$$

where,

$$\eta_c^j = \operatorname{arcosh}\left[2|A|\sin^2\frac{1}{2}(\xi_+ + j\theta_+) + 1\right],$$

with $j = 0, \ldots, n$. As for type A breathers, each τ_j has its own η_c^j, and the smallest of these is taken as the limit.

For positive interaction coefficient, one obtains the restriction that ξ_+ must *not* take the following values,

$$\xi_+ = \pi \bmod 2\pi.$$

Furthermore, if one considers the restriction (3.1.14) then the following has to be fulfilled

$$v^2 \leq v_c^{(B)^2}\left[1 - \frac{1}{\cos^2(\frac{\theta_a}{2})\cos^2\frac{1}{2}(\xi_+ + j\theta_+)}\right],$$

which can never be satisfied. Thus, contrary to the type A breathers, it is not possible to have type B breathers with positive interaction coefficient, or to have velocity $v^2 < v_c^{(B)2}$. Since the τ-functions would necessarily pass the origin of the complex plane and this would lead to a solution which is not well-defined.

3.2 Properties of the $a_n^{(1)}$ Breather Solutions

Having found the breather solutions of $a_n^{(1)}$ series, one can proceed further by examining some of their properties. Namely, the interaction coefficient and their topological charges.

3.2.1 The Interaction Coefficient

The interaction coefficient A has properties similar to the properties of the S-matrix of the fundamental Toda particles. Recall the expressions for A from (2.4.9) and (2.4.10). The interaction term of the type A breathers is,

$$A_{aa} = \frac{v^2}{(1+v^2)\cos^2(\frac{\theta_a}{2}) - 1}, \tag{3.2.1}$$

or in terms of rapidity difference $\bar{\Theta} = -i\Theta$,

$$A_{aa} = \frac{\sin^2(\frac{\bar{\Theta}}{2})}{\sin(\frac{\bar{\Theta}}{2} + \frac{\theta_a}{2})\sin(\frac{\bar{\Theta}}{2} - \frac{\theta_a}{2})}. \tag{3.2.2}$$

And for type B breathers,

$$A_{a\bar{a}} = (1 + v^2)\cos^2(\frac{\theta_a}{2}) - v^2, \tag{3.2.3}$$

in terms of rapidity difference $\bar{\Theta} = -i\Theta$,

$$A_{a\bar{a}} = \frac{\cos(\frac{\bar{\Theta}}{2} + \frac{\theta_a}{2})\cos(\frac{\bar{\Theta}}{2} - \frac{\theta_a}{2})}{\cos^2(\frac{\bar{\Theta}}{2})}. \tag{3.2.4}$$

Similarly to the case of S-matrices of fundamental Toda particles, these interaction coefficients admit a pole. For type A breathers,

$$v = v_c^{(A)} \qquad \text{or,} \qquad \Theta = i\frac{2\pi a}{h} \tag{3.2.5}$$

and, for type B breathers,

$$v \longrightarrow \infty \qquad \text{or,} \qquad \Theta = i\pi. \tag{3.2.6}$$

It is readily seen that the pole of A_{aa} is exactly the fusing angle related to the process $a + a \longrightarrow \overline{(h - 2a)}$ of the fundamental particles [9]. Hollowood noted that the same fusing rule also applies to soliton fusings in $a_n^{(1)}$ theories [11]. In fact, the fusing rule of fundamental particles applies also to all *simply laced* affine Toda solitons [12, 46]. The fusing of two solitons of species a into $\overline{(h - 2a)}$ hinted that the topological charge of type A breathers has to be found in the same representation as the topological charges of $\overline{(h - 2a)}$ single solitons.

It is not surprising that at the pole of the interaction coefficient, the breathers fail to exist. Using the breather τ-function, (3.1.4), with the interaction coefficient approaching its pole, the interaction term dominates. Hence, the solution falls into one of the vacuum solutions of the complex affine Toda potential,

$$\phi = -\frac{1}{\beta}\sum_{j=1}^{n} j\alpha_j\theta_+. \tag{3.2.7}$$

On the other hand, if the positions of the constituent solitons are simultaneously shifted by $-\frac{\zeta}{2}$, i.e. changing $\eta \longrightarrow \eta - \frac{\zeta}{2}$, then from (3.1.4), as the interaction coefficient approaches its pole one obtains,

$$\tau_j^{(ab)} = 1 + \exp[\sigma_+ x + \eta_+ + i(\delta + \xi_+ + j\theta_+)].$$

Thus the breather turns into a static solution as the interaction coefficient approaches its pole.

When the interaction coefficient approches its zero, there is no interaction between the constituent solitons. In this case, the τ-functions do not give a well defined solution, as the τ-functions will vanish at a particular point in space-time. This is quite obvious, since the breathers are bound states of soliton pairs and therefore a breather interaction coefficient cannot be zero.

Furthermore, the interaction coefficient A has the following general properties some of which are similar to but not the same as the properties of the S-matrix,

- *Crossing symmetry*

$$A_{aa}(v) = A^{-1}_{a\bar{a}}(\frac{1}{v}) \qquad \text{or,} \qquad A_{aa}(\bar{\Theta}) = A^{-1}_{a\bar{a}}(\bar{\Theta} - \pi). \tag{3.2.8}$$

- *Evenness*

$$A(v) = A(-v) \qquad \text{or,} \qquad A(\bar{\Theta}) = A(-\bar{\Theta}). \tag{3.2.9}$$

- *Symmetry*

$$A_{a\bar{a}}(v) = A_{\bar{a}a}(v) \qquad \text{or,} \qquad A_{a\bar{a}}(\bar{\Theta}) = A_{\bar{a}a}(\bar{\Theta}). \tag{3.2.10}$$

- *Periodicity*

$$A(\bar{\Theta}) = A(\bar{\Theta} + 2\pi). \tag{3.2.11}$$

3.2.2 The Topological Charges

The topological charges of a soliton solution is a conserved quantity of zero spin. For the $a_n^{(1)}$ series, topological charges of the single and multi-soliton solutions have been calculated [35, 36]. This calculation can be outlined as follows.

The topological charge q of a solution ϕ is defined by (c.f. (2.3.4))

$$q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \partial_x \phi \mathrm{d}x = \frac{\beta}{2\pi} \lim_{x \to \infty} (\phi(x,t) - \phi(-x,t)). \tag{3.2.12}$$

Writing the solution ϕ in terms of the breather τ-functions, defining $f_j = \frac{\tau_j}{\tau_0}$ for $j = 1 \ldots n$ and making use of the definition of the logarithm of a complex number, (3.2.12) can be recast as

$$q = -\frac{1}{2\pi i} \sum_{j=1}^{n} \alpha_j \lim_{x\to\infty} \{\ln |f_j(x,t)| - \ln |f_j(-x,t)| \\ + i \arg(f_j(x,t)) + 2i\pi k' - i \arg(f_j(-x,t)) - 2i\pi k''\}, \quad (3.2.13)$$

where $k', k'' \in \mathbb{Z}$. A simplification of (3.2.13) results from the fact that $\lim_{|x|\to\infty} |f_j(x,t)| = 1$, thus

$$q = -\frac{1}{2\pi} \sum_{j=1}^{n} \alpha_j \lim_{x\to\infty} \{\arg(f_j(x,t)) - \arg(f_j(-x,t)) + 2\pi k\} \quad (3.2.14)$$

with $k = k' - k''$. The number k determines the curve f_j in the complex plane, and in particular, how often and in what direction it winds around the origin. The topological charge is therefore determined by the change in the argument of f_j as $|x|$ goes to infinity.

In the soliton solutions cases, the parameter ξ is also divided into several allowed sectors which constitutes the number of distinct topological charges. For the single soliton, the change of argument above can be easily evaluated by examining the behaviour of the τ-function near to the excluded values of the parameter ξ. This will also give the direction traced by the curve of the ratio of the τ-function as x goes from $-\infty$ to ∞, hence the change of argument. In the multi-soliton case, one can confine each of these solitons by sending the other solitons far away from it such that the soliton in question receives a negligible interaction. Then, the topological charge of this multi-soliton system is just the sum of the topological charges of its constituent solitons.

As seen from the construction of the breather solutions in the previous section, the two solitons constituting the breather can only separate out to a finite 'distance' from each other. Recall that η_- is restricted. Thus, the line of calculation for the topological charges of a multi-soliton solution, as explained above, is not applicable for the breather case.

The topological charges of the breather solutions can also be determined in a similar way, but instead of sending the constituent solitons away from each other, one can evaluate the τ-function near the pole of the interaction coefficent. This will significantly simplify the calculations.

For the type A breather it is not too difficult to deduce the number of distinct topological charges in the fashion of [35]. The number of the topological charges is determined by the number of sectors of allowed values of ξ_+. From the expression for forbidden ξ_+'s, (3.1.11) or (3.1.13), it follows that one looks for the smallest number p for which

$$\frac{2ap}{h} = k \,\text{with}\, p, k \in \mathbb{N}. \tag{3.2.15}$$

Here a is the species of the constituent solitons and h is the Coxeter number.

With $2\tilde{a} = \frac{2a}{\gcd(2a,h)}$, $\tilde{h} = \frac{h}{\gcd(2a,h)}$ then (3.2.15) can be rewritten as

$$2\tilde{a}p = \tilde{h}k. \tag{3.2.16}$$

Because $2\tilde{a}$ and $\tilde{h}$ are coprime it follows that $p = \tilde{h}$ and $k = 2\tilde{a}$. Thus the range of the allowed values for ξ_+ is divided into $\tilde{h}$ sectors. This leads to the following formula for the maximum number of topological charges of type A breather with constituent solitons of species a

$$\tilde{h} = \frac{h}{\gcd(2a,h)}. \tag{3.2.17}$$

This argument holds independently of the sign of the interaction coefficient. The argument of $f_j(x,t)$ can only change when $f_j(x,t)$ is undefined or zero, hence the topological charge within each sector is constant. It turns out that in each of these sectors, the topological charges take a different unique value. These topological charges are related by permutation of the roots α_j for $j = 0, \ldots, n$. The topological charge of a specific sector will be determined first, and is called the *highest charge* [35]. Then it will be shown that in all other sectors, the topological charges will be different. This means that $\tilde{h}$ is indeed the number of topological charges associated with a breather.

In the following, a calculation of the highest charge will be performed for a type A breather with negative interaction coefficient. The calculation for positive interaction coefficient is the same and will not be presented here. To calculate the highest topological charge, one has to employ a little trick first to simplify the breather τ-function. The type A breather τ-function is given by

$$\begin{aligned} \tau_j^{(aa)} &= 1 + \exp[\sigma_a(x + ivt) + \rho + ij\theta_a] + \exp[\sigma_a(x - ivt) + \rho' + ij\theta_a] \\ &\quad - \exp[\zeta + 2\sigma_a x + \rho_+ + 2ij\theta_a]. \end{aligned}$$

First choose $t = 0$ because the topological charge does not depend on the time. Secondly choose $\rho = -\zeta/2 + \hat{\rho}$ and $\rho' = -\zeta/2 + \hat{\rho}'$. This corresponds to a simultaneous shift of the constituent solitons to the left. By this shifting, the last term in the breather τ-function will not depend on the interaction coefficient A,

$$\tau_j^{(aa)} = 1 + \exp(\sigma_a x + ij\theta_a - \zeta/2)(e^{\hat{\rho}} + e^{\hat{\rho}'}) - \exp(2\sigma_a x + \hat{\rho}_+ + 2ij\theta_a).$$

With $\mu_a(2j) = \frac{4\pi a j}{h} \bmod 2\pi$, the limits $|x| \to \infty$ of f_j will give

$$\begin{aligned} \lim_{x\to\infty} f_j &= e^{i\mu_a(2j)}, \\ \lim_{x\to-\infty} f_j &= 1. \end{aligned} \tag{3.2.18}$$

Moreover one can take the limit ζ approaching $+\infty$, this corresponds to choosing the velocity v very near to $v_c^{(A)}$. As long as v is not equal to $v_c^{(A)}$ the breather solution is well-defined by construction. Write $y = e^{2\sigma_a x}$, then provided one does not take the limit $x \to \infty$, the $y^{1/2}$ term can be dropped,

$$\begin{aligned} \tau_j^{(aa)} &= 1 + y^{1/2}\exp(ij\theta_a - \zeta/2)(e^{\hat{\rho}} + e^{\hat{\rho}'}) - y\exp(\hat{\rho}_+ + 2ij\theta_a) \\ &= 1 - y\exp(\hat{\rho}_+ + i\mu_a(2j)). \end{aligned} \tag{3.2.19}$$

By splitting the ratio f_j into its real and imaginary part one can now easily show that f_j traces out a clockwise curve in the complex plane, i.e. the winding number k is zero. To see this take $\hat{\rho} = \hat{\rho}' = i(\pi - \frac{\varepsilon}{2})$ where ε is a real, positive and infinitesimal parameter,

$$\tau_j = 1 - y\exp(i(\mu_a(2j) - \varepsilon)).$$

Then the ratio f_j can be written as

$$\begin{aligned} f_j = & \frac{1}{|1 - ye^{\hat{\rho}_+}|^2}[1 - y(\cos(\mu_a(2j) - \varepsilon) + \cos(\varepsilon)) + y^2\cos(\mu_a(2j)) \\ & + i\{-y(\sin(\mu_a(2j) - \varepsilon) + \sin(\varepsilon)) + y^2\sin(\mu_a(2j))\}]. \end{aligned}$$

The only zeros for the imaginary part occur when $y = 0$ and

$$y = \frac{\sin(\mu_a(2j) - \varepsilon) + \sin(\varepsilon)}{\sin(\mu_a(2j))},$$

with $\mu_a(2j) \neq 0$ or π. For small ε this is

$$y = 1 + \varepsilon \frac{1 - \cos(\mu_a(2j))}{\sin(\mu_a(2j))} + O(\varepsilon^2). \tag{3.2.20}$$

Now inserting (3.2.20) into the real part of f_j results in,

$$\Re e(f_j|1 - ye^{\hat{\rho}_+}|^2) = -2\varepsilon \frac{1 - \cos(\mu_a(2j))}{\sin(\mu_a(2j))} + O(\varepsilon^2).$$

One should also observe that in the small ε regime the imaginary part behaves for small y like

$$\Im m(f_j|1 - ye^{\hat{\rho}_+}|^2) = -y\sin(\mu_a(2j)) + \dots.$$

So, for $0 < \mu_a(2j) < \pi$, the curve starts at $(1,0)$ with a negative imaginary part and crosses the negative part of the real line. For $\pi < \mu_a(2j) < 2\pi$, it starts at $(1,0)$ with a positive imaginary part and crosses the positive part of the real line. When $\mu_a(2j) = 0$ then $f_j = 1$, this does not contribute to the topological charge. Whereas when $\mu_a(2j) = \pi$ the change of argument is π. In any case, it winds around the origin in the clockwise sense. Thus, the change of argument of f_j is given by $\mu_a(2j) - 2\pi$. The explicit formula for the *highest topological charge* is therefore determined by (3.2.14)

$$q_a^{(1)} = \sum_{j=0}^{n} \frac{2a(h-j) \bmod h}{h} \alpha_j. \tag{3.2.21}$$

In the summation above, the extended root α_0 is included for convenience in the permutation of the simple roots and α_0. As mentioned previously, this result does not depend on the sign of the interaction coefficient.

From this highest charge, one can obtain all the other charges as follows. Suppose initially the value of ξ_+ is chosen. Then, making a shift of $\frac{4\pi a}{h}$ on this ξ_+ amounts to sending the breather solution to a different sector of ξ_+. Successive applications of this shift will bring the breather solution to every allowed sector of ξ_+. With the $\tilde{h}^{th}$ application it will return to the original sector. However, note that the resulting sectors in successive shift are not necessarily adjacent to each other. Recall that with (3.2.19) the breather solution is given by,

$$\phi = \frac{i}{\beta} \sum_{j=0}^{n} \alpha_j \ln(1 - \omega_a^{2j} y e^{\hat{\rho}_+}).$$

Making the shift $\xi_+ \to \xi_+ - \frac{4\pi a}{h}$ in the above solution results in lowering the power of ω_a by one, i.e.

$$\phi = \frac{i}{\beta}\sum_{j=0}^{n}\alpha_j \ln(1-\omega_a^{2j} y e^{\hat{\rho}_+ - \frac{4i\pi a}{h}}) = \frac{i}{\beta}\sum_{j=0}^{n}\alpha_{j+1}\ln(1-\omega_a^{2j} y e^{\hat{\rho}_+}).$$

Thus, this shifting is the same as cyclically permuting the roots α_j for all $j = 0,\ldots,n$. And hence, each shifting results to a different topological charge. Since the maximum number one can shift ξ_+ is $\tilde{h}$ times, then $\tilde{h}$ is exactly the number of topological charges of the breather solution. The expression for all the topological charges is

$$q_a^{(k)} = \sum_{j=0}^{n} \frac{2a(h-j) \bmod h}{h}\alpha_{(j+k-1)}, \qquad k = 1,2,\ldots,\tilde{h}, \tag{3.2.22}$$

where the roots α_j are labelled modulo h.

This is analogous to the one soliton case [35]. Furthermore, all these topological charges lie in the same representation because they are related by a Weyl transformation as will be shown in the next subsection. Interestingly, the fundamental representation space in which these topological charges lie is the irreducible component of the Clebsch-Gordan decomposition of a tensor product representation.

For the type B breather it has been determined in a preceding calculation (subsect. 3.1.2) that there is only one sector of allowed values for ξ_+. The only possible way for the topological charge to change is whenever the ratio f_j is not well-defined, i.e ξ_+ changes from one sector to another. So, in this case there cannot be a change in the topological charge. The only open question now is what value the topological charge takes. To determine this one simply follows the previous prescription [35]. The τ-functions for the type B breather are given by

$$\begin{aligned}\tau_j^{(a\bar{a})} &= 1+\exp[\sigma_a(x+ivt)+\rho+ij\theta_a]+\exp[\sigma_a(x-ivt)+\rho'-ij\theta_a] \\ &\quad -\exp[\zeta+2\sigma_a x+\rho_+],\end{aligned}$$

where $\bar{a} = h-a$. Because the topological charge is time independent, one can set $t=0$. Also, one can substitute $\exp(\sigma_a x) = z$, $\rho = \rho' = i(\pi+\frac{\epsilon}{2})$ with $\epsilon \in \mathbb{R}$ and infinitesimal. The τ-function is then given in the compact form

$$\tau_j^{(a\bar{a})} = 1 - 2z\cos(j\theta_a)e^{i\frac{\epsilon}{2}} - z^2 e^{\zeta+i\epsilon}.$$

Let f_j be defined as before. The start and end point of the curve traced out by f_j as x goes from $-\infty$ to ∞ are in this case the same, $f_j(x = \pm\infty) = 1$. Solving an equation for the imaginary part of f_j one finds that these are also the only points for which the imaginary part vanishes. Therefore the winding number k is zero, because the curve cannot wrap around the origin. Moreover, since the change of arguments of f_j as x goes from $-\infty$ to ∞ is zero, the topological charge of any type B breather is deduced to be zero. In a sense, type B breathers are *sine-Gordon like* breathers. The constituent solitons are of opposite topological charges such that the resulting breather has zero topological charge. In fact, as will be discussed in the next section, type B breathers do not come from a sine-Gordon embedding in the theory.

Note however, that individual topological charges of the constituent solitons cannot be calculated explicitly since this topological charges is determined by ξ_+, i.e. individual ξ loses its importance. Nevertheless, using crossing symmetry, one can perform a superficial calculation for these individual topological charges.

3.2.3 Topological Charge and Representation Space

It is natural to expect that the topological charges which have been derived in the previous calculation lie in the tensor product representation of the fundamental representation associated with the topological charges of the constituent solitons. In fact, for the type A breather, with the exception for breathers built from species $(n+1)$ in the $a_{2n+1}^{(1)}$ cases, the topological charges lie in the fundamental representation which is a component of the Clebsch-Gordan decomposition of the tensor product representation. For type B breathers and the exceptional cases above, the topological charge (which is zero) lies in the singlet representation component of the Clebsch-Gordan decomposition of the tensor product representation.

For the non-zero highest topological charge, the first step is to show that it lies in the $\mathcal{R}_{\lambda_{2a \bmod h}}$ fundamental representation. This will be shown using a combination of Weyl transformations [35]. Then, the second step is to show the other topological charges are related to the highest charge by a special Coxeter element of the Weyl group.

It is convenient to write the highest charge (3.2.21) as

$$q_a^{(1)} = \sum_{j=0}^{n} \frac{bj \bmod h}{h} \alpha_j, \tag{3.2.23}$$

where $b = h - (2a \bmod h)$. Because of the Z_2 symmetry of the simple roots, it is necessary to consider only the case $b \leq \left[\frac{h}{2}\right]$. The notation $[x]$ means the largest integer less or equal to x. Hence, for h even, $\left[\frac{h}{2}\right] = \frac{h}{2}$, and for h odd, $\left[\frac{h}{2}\right] = \frac{h-1}{2}$. Furthermore, (3.2.23) can be rewritten in terms of the fundamental weights λ_j defined by $\frac{2\lambda_j \cdot \alpha_k}{\alpha_k^2} = \delta_{jk}$ as follows,

$$\begin{aligned} q_a^{(1)} &= \frac{1}{h}\{2[b \bmod h] - [2b \bmod h]\}\lambda_1 + \frac{1}{h}\{2[bn \bmod h] - [b(n-1) \bmod h]\}\lambda_n \\ &+ \sum_{j=1}^{n-1} \frac{1}{h}\{2[bj \bmod h] - [b(j-1) \bmod h] - [b(j+1) \bmod h]\}\lambda_j. \end{aligned} \tag{3.2.24}$$

Then the following can be demonstrated easily,

$$q_a^{(1)} \cdot \alpha_j = \begin{cases} 1 & j = n, \\ 0 \text{ or } -1 & j = n-1, \\ 0 \text{ or } -1 \text{ or } 1 & 1 \leq j < n-1. \end{cases} \tag{3.2.25}$$

The part $q_a^{(1)} \cdot \alpha_j = -1$ for $j < n-1$ will be demonstrated in the following. Let, $bj = ch + d$ where $d < b$ and $c \geq 0$, thus $j = 1$ is excluded. Then with (3.2.24) one finds that,

$$\begin{aligned} q_a^{(1)} \cdot \alpha_j &= \frac{1}{h}\{2[bj \bmod h] - [b(j-1) \bmod h] - [b(j+1) \bmod h]\} \\ &= \frac{1}{h}\{2d - (d - b + h) - (d + b)\} = -1. \end{aligned}$$

There are $(b-1)$ terms of $q_a^{(1)} \cdot \alpha_j = -1$ for $j < n-1$, since this happens only when $d < b$. Furthermore, using a similar procedure as above, it is straightforward to see that for $1 < j < n-1$

$$q_a^{(1)} \cdot \alpha_j = -1 \Longrightarrow q_a^{(1)} \cdot \alpha_{j-1} = 1. \tag{3.2.26}$$

Thus, if the scalar products of $q_a^{(1)}$ with the simple roots $\{\alpha_j\}$ is written as a row vector, it has the entry 1 at n^{th} position and there are $(b-1)$ pairs of $(1,-1)$ to the left of it,

$$q_a^{(1)} \cdot \{\alpha_j\} = (0, \ldots, 0, 1, -1, 0, \ldots, 1, -1, 1, -1, \ldots, 0, 1),$$

the j^{th} entry of the row vector on the right-hand side is $q_a^{(1)} \cdot \alpha_j$. With this presentation, action of a certain element of the Weyl group becomes clear as can be seen in what follows.

It is elementary to see the following. Suppose a weight γ_1 has a scalar product with the simple roots as $\gamma_1 \cdot \{\alpha_j\} = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$. Consider the Weyl reflection r with respect to the simple root α_k where $\gamma_1 \cdot \alpha_k = -1$. The action of r on γ_1 will shift the pair $(1, -1)$ in $\gamma_1 \cdot \{\alpha_j\}$ one step to the right, i.e. $r : \gamma_1 \longrightarrow \gamma_1'$ with

$$\gamma_1' \cdot \{\alpha_j\} = (0, \ldots, 0, 0, 1, -1, \ldots, 0).$$

For a weight γ_2 which has $\gamma_2 \cdot \{\alpha_j\} = (0, \ldots, 0, 1, -1, 1, \ldots, 0)$, consider the Weyl reflection r' with respect to the simple root α_k where $\gamma_2 \cdot \alpha_k = -1$. The action of r' on γ_2 will give γ_2' where,

$$\gamma_2' \cdot \{\alpha_j\} = (0, \ldots, 0, 0, 1, 0, \ldots, 0).$$

So, using a combination of these Weyl transformations, $q_a^{(1)}$ can be transformed into a fundamental weight, $q_a^{(1)} \longrightarrow \lambda$, where

$$\lambda \cdot \{\alpha_j\} = (0, \ldots, 0, 1, 0, \ldots, 0).$$

Since there are $(b-1)$ pairs of $(1, -1)$ in $q_a^{(1)} \cdot \{\alpha_j\}$ row vector, then after these combination of Weyl transformations the entry 1 will appear at the position $n - (b-1) = 2a \bmod h$. Hence the highest topological charge $q_a^{(1)}$ lies in the fundamental representation $\mathcal{R}_{\lambda_{2a \bmod h}}$.

Recall that the rest of the topological charges are obtained by cyclically permuting the simple roots and α_0, (3.2.22). This cyclic permutation is the same as the action of the following Coxeter element of the Weyl group on $q_a^{(1)}$,

$$\omega_{tc} = r_1 r_2 \ldots r_n, \tag{3.2.27}$$

where r_j is a Weyl reflection with respect to the simple root α_j. Then, the topological charges are related to the highest charge by,

$$q_a^{(k)} = \omega_{tc}^{k-1}(q_a^{(1)}). \tag{3.2.28}$$

Note that the ordering of Weyl reflections above is special, other orderings do not necessarily relate one topological charge to another. The relation (3.2.28) is straightforward to see using the fact that,

$$\omega_{tc}(\alpha_j) = \alpha_{j+1} \qquad \text{for } j = 0, 1, \ldots, n \tag{3.2.29}$$

the simple roots and α_0 are labelled modulo h. Further examination of (3.2.22) shows that the set of topological charges $\{q_a^{(k)}\}$ coincides with the topological charges of the species $2a$ mod h single solitons.

The next task is to show that $\mathcal{R}_{\lambda_{2a\bmod h}}$ is a component of the Clebsch-Gordan decomposition of $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$. This will be shown using a conjecture attributed to Parthasarathy, Ranga Rao and Varadarajan [53]. The PRV conjecture may be stated as follows: let $\bar{\gamma}$ be a unique dominant weight of the Weyl orbit of $\gamma = \lambda + \omega\mu$ for any ω in the Weyl group and λ, μ are highest weights, then $\mathcal{R}_{\bar{\gamma}}$ appears with multiplicity of at least one in the decomposition of $\mathcal{R}_\lambda \otimes \mathcal{R}_\mu$, where $\mathcal{R}_\lambda$ and $\mathcal{R}_\mu$ are finite dimensional irreducible representations with highest weights λ and μ respectively. This conjecture has been proved recently [54]; it was first used in the context of affine Toda theories by Braden [55].

For convenience of calculation, one can write the fundamental weights of the Lie algebra A_n as follows,

$$\lambda_a = \sum_{j=0}^{a} \frac{(h-a)j}{h}\alpha_j + \sum_{j=a+1}^{n} \frac{a(h-j)}{h}\alpha_j. \tag{3.2.30}$$

By the Z_2 symmetry of the simple roots of A_n, one has to consider only the case $a \leq [\frac{h}{2}]$. Choose ω to be the Coxeter element defined in (3.2.27). Then, remembering the action of this Coxeter element on the simple roots, c.f. (3.2.29), it is easy to show that

$$\lambda_a + \omega_{tc}^a \lambda_a = \lambda_{2a}. \tag{3.2.31}$$

It is obvious that λ_{2a} is a unique dominant weight of the Weyl orbit. Thus by PRV conjecture $\mathcal{R}_{\lambda_{2a\bmod h}} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$.

This completes the claim that all the topological charges lie in the same fundamental representation $\mathcal{R}_{\lambda_{2a\bmod h}}$ which is an irreducible component of $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$; i.e.

$$\{q_a^{(k)}\} \in \mathcal{R}_{\lambda_{2a\bmod h}} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}. \tag{3.2.32}$$

Note that $2a$ mod $h = \overline{(h-2a)}$. Hence, (3.2.32) suggest that the *fusing* rule of the constituent solitons [12, 46] and the representation space to which the topogical charge of the breather belongs, as shown by the Clebsch-Gordan component, are related. Furthermore, as noted in the previous calculation, the number of topological charges is $\tilde{h} = \frac{h}{\gcd(2a,h)}$

which is generally less than the dimension of $\mathcal{R}_{\lambda_{2a \bmod h}}$. So, the topological charges of type A breathers, normally do not fill the fundamental representation $\mathcal{R}_{\lambda_{2a \bmod h}}$. Only particular combinations of the topological charges of the constituent solitons can make up a breather. A special case of the type A breather is when the constituent solitons come from the fundamental representation $\mathcal{R}_{\lambda_a}$ which is self-conjugate, this happens for $\mathcal{R}_{\lambda_{n+1}}$ in the representation of A_{2n+1}. This breather belongs to both type A and B.

For the type B breathers and the exceptional case above, the fundamental representations of their constituent solitons are conjugates of each other (or self-conjugate). Thus, the topological charges of these breathers will lie in the tensor product $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_{h-a}}$. Using the PRV conjecture as before, it can be shown that

$$\lambda_a + \omega_{tc}^a \lambda_{h-a} = 0. \tag{3.2.33}$$

Hence, the trivial *singlet* representation appears in the Clebsch-Gordan decomposition of this tensor product. It is in this singlet representation that the topological charge lies.

Examples of the calculation of the topological charges for $a_3^{(1)}$ and $a_4^{(1)}$ cases can be found in [47].

3.3 Sine-Gordon Embedding

Automorphisms of the Dynkin diagram can be used to reduce an affine Toda theory to another affine Toda theory with fewer scalar fields [3]. Using this reduction method, Sasaki noted that in the $a_n^{(1)}$ affine Toda theories with a real coupling parameter, there are ways to reduce some members of the $a_n^{(1)}$ family to the $a_1^{(1)}$ theory, i.e. the sinh-Gordon theory [56]. The same procedure can be applied in the case of complex Toda theories. Define the solution to the equation of motion (2.3.1) as,

$$\phi = \mu\psi, \tag{3.3.1}$$

where μ is some vector to be determined. Then (2.3.1) becomes,

$$\mu\partial^2(\beta\psi) = im^2 \sum_{j=1}^{n} \alpha_j \left(e^{i\beta\alpha_j\cdot\mu\psi} - e^{i\beta\alpha_0\cdot\mu\psi}\right). \tag{3.3.2}$$

The aim is to reduce (3.3.2) above into the sine-Gordon equation of motion by choosing a suitable μ,

$$\mu\partial^2(\beta\psi) = im^2\mu\left(e^{i\beta\psi} - e^{-i\beta\psi}\right) = -2m^2\mu\sin(\beta\psi). \tag{3.3.3}$$

There are two kinds of reductions. A *direct reduction* results when several nodes of the affine Dynkin diagram which do not have a direct link are identified. When linked nodes are transposed, this results in a *non-direct reduction.*

One can reduce the $a_{2n+1}^{(1)}$ theories to $a_1^{(1)}$ theory using a direct reduction by choosing μ as follows [56],

$$\mu_1 = \alpha_1 + \alpha_3 + \ldots + \alpha_{2n-1} + \alpha_{2n+1}. \tag{3.3.4}$$

The vector μ_1 is an invariant vector under the Z_{n+1} symmetry which identifies $\alpha_j \rightarrow \alpha_{j+2}$. Projecting the simple roots of $a_{2n+1}^{(1)}$ to μ_1 subspace gives the simple roots of $a_1^{(1)}$ with multiplicity $(n+1)$,

$$\alpha_j \cdot \mu_1 = 2 \text{ or } -2,$$

for j odd or even respectively. A *non-direct reduction* is a two step reduction: first $a_{4n-1}^{(1)}$ can be reduced to the three dimensional subspace of $a_3^{(1)}$ then $a_3^{(1)}$ can be reduced to $a_1^{(1)}$ [56]. There are two choices of μ for this non-direct reduction,

$$\mu_2 = \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \ldots + \alpha_{4n-3} + \alpha_{4n-2}, \tag{3.3.5}$$

$$\mu_3 = \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \ldots + \alpha_{4n-2} + \alpha_{4n-1}, \tag{3.3.6}$$

in the above, μ_3 is obtained from μ_2 by cyclically permuting the simple roots of $a_{4n-1}^{(1)}$ once. Together with the vector μ_1, these three vectors are invariant under the Z_n symmetry which identifies $\alpha_j \rightarrow \alpha_{j+4}$. The simple roots of $a_{4n-1}^{(1)}$ can be projected to μ_2 or μ_3 giving the simple roots of $a_1^{(1)}$ with multiplicity $2n$.

In terms of the single soliton τ-functions (2.4.4), direct reduction forces some τ-functions to be equal leaving only two different τ-functions,

$$\tau_0^{(a)} = \tau_2^{(a)} = \ldots = \tau_{2n}^{(a)} \qquad \text{and} \qquad \tau_1^{(a)} = \tau_3^{(a)} = \ldots = \tau_{2n-1}^{(a)}, \tag{3.3.7}$$

with,

$$\tau_j^{(a)} = 1 + \omega_a^j e^{(\Omega_a+\rho)}.$$

Since $\omega_a^j = \exp(\frac{2i\pi a}{h}j)$, it is clear that for the $a_{2n+1}^{(1)}$ theories, only solitons of species $a = (n+1)$ are the *true* sine-Gordon solitons embedded in the theory. For the non-direct reductions, one has to have the following conditions for the τ-functions. Using μ_2 yields,

$$\tau_0^{(a)} = \tau_3^{(a)} = \tau_4^{(a)} = \ldots = \tau_{4n-4}^{(a)} = \tau_{4n-1}^{(a)},$$
$$\tau_1^{(a)} = \tau_2^{(a)} = \tau_5^{(a)} = \ldots = \tau_{4n-3}^{(a)} = \tau_{4n-2}^{(a)},$$

and for the choice μ_3,

$$\tau_0^{(a)} = \tau_1^{(a)} = \tau_4^{(a)} = \ldots = \tau_{4n-4}^{(a)} = \tau_{4n-3}^{(a)},$$
$$\tau_2^{(a)} = \tau_3^{(a)} = \tau_6^{(a)} = \ldots = \tau_{4n-2}^{(a)} = \tau_{4n-1}^{(a)}.$$

These conditions on the τ-functions of $a_{4n-1}^{(1)}$ will never be satisfied. This is because for $h = 4n$, the factor ω_a^j cannot be equal to ω_a^{j+1} since j and $(j+1)$ are coprime.

Thus, the solitons associated with middle spot of the A_{2n+1} Dynkin diagram are the only sine-Gordon solitons embedded in the $a_{2n+1}^{(1)}$ affine Toda theories. Hence, these solitons can bind together resulting in sine-Gordon breathers, i.e. type A breathers with zero topological charge. Note also that type B breathers by the above definitions are not formed from any sine-Gordon embedded solitons.

3.4 Soliton Crossing and Breather

Recall that in previous construction of multi-soliton and breather solutions, one always takes the parameter σ to be positive. Replacing σ by $-\sigma$ turns a single soliton of species a into its anti-species $\bar{a}$; in particular for the $a_n^{(1)}$ series, $\bar{a} = h - a$. Kneipp and Olive [50] have shown that this crossing can be viewed in an almost similar manner to S-matrix crossing, i.e. analytic continuation of rapidity Θ into $\Theta - i\pi$.

Under the crossing transformation, $\sigma \to -\sigma$, the parameters η and ξ transform in a simple way, while the interaction coefficient A is inverted as already noted in the interaction coefficient of the breather (3.2.8). Since the topological charges of the soliton solutions are determined from the parameter ξ, one will see that crossing will necessary invert the sign of the original topological charge.

In the $a_n^{(1)}$ breather case, crossing of one of its constituent solitons will result in the crossing from type A breathers into type B breathers and vice versa as one would expect. One already noted from the construction of the breather solutions, that the topological charge of a breather is determined by ξ_+. Thus, individual ξ of the constituent solitons loses its importance. From previous calculation one can only determine the total topological charge of the breather solution without knowing the constituent topological charges. However, by examining the crossing of breathers with negative interaction coefficient and assuming that each parameter ξ of the constituent solitons will determine the topological charges of constituent solitons, one can find an exact relation between the constituent topological charges in a breather solution. In other words, given a constituent topological charge of a breather, one can directly know the second constituent topological charge.

3.4.1 Crossing Transformation

As stated earlier, there are two ways of making a crossing from a soliton into an anti-soliton. The first way is to replace the species index a with $\bar{a} = h - a$. If one takes the positive square root of (2.4.6),

$$\sigma_a\sqrt{1-u_a^2} = 2m\sin(\frac{\theta_a}{2}),$$

making the mentioned replacement one obtains,

$$\sigma_{\bar{a}}\sqrt{1-u_{\bar{a}}^2} = 2m\sin(\frac{\theta_{\bar{a}}}{2}) = -2m\sin(\frac{\theta_a}{2}).$$

With $u_a = u_{\bar{a}}$, one has $\sigma_{\bar{a}} = -\sigma_a$. The second alternative is an analytic continuation of the soliton rapidity from Θ_a into $\Theta_a + i\pi$. One can see this crossing by writing Ω_a of (2.4.5) in terms of rapidity using the relation $\Theta_a = \frac{1}{2}\ln\left(\frac{1+u_a}{1-u_a}\right)$ to have,

$$\Omega_a = \sqrt{2}m\sigma_a\sin(\frac{\theta_a}{2})(x_+e^{\Theta_a} - x_-e^{-\Theta_a}).$$

Replacing Θ_a with $\Theta_a + i\pi$ will give an overall negative sign. Thus, crossing is achieved by replacing σ with $-\sigma$. In what follows σ is always taken to be positive.

To derive the crossing transformations of parameters η and ξ, one starts from the *crossed version* of the τ-function and rewrites it in terms of positive σ.

The single soliton case is quite trivial, recall that $\theta_{\bar{a}} = 2\pi - \theta_a$, then

$$\begin{aligned} \bar{\tau}_j^{(a)} &= 1 + e^{-\sigma_a(x-ut)+\bar{\eta}+i(\bar{\xi}+j\theta_a)} \\ &= e^{-\sigma_a(x-ut)+\bar{\eta}+i(\bar{\xi}+j\theta_a)}\{1 + e^{\sigma_a(x-ut)+\eta+i(\xi+j\theta_{\bar{a}})}\} = (\bar{\tau}_j^{(a)} - 1)\tau_j^{(\bar{a})}, \end{aligned} \tag{3.4.1}$$

where the parameters $\bar{\eta}$ and $\bar{\xi}$ transform into,

$$\eta = -\bar{\eta} \qquad \text{and} \qquad \xi = -\bar{\xi}.$$

For the double-soliton case, one performs a crossing of the species b constituent soliton,

$$\begin{aligned} \bar{\tau}_j^{(ab)} &= 1 + e^{\sigma_a(x-u_at)+\bar{\eta}_a+i(\bar{\xi}_a+j\theta_a)} + e^{-\sigma_b(x-u_bt)+\bar{\eta}_b+i(\bar{\xi}_b+j\theta_b)} \\ &\quad + \bar{A}_{ab}e^{(\sigma_a-\sigma_b)x-(\sigma_au_a-\sigma_bu_b)t+\bar{\eta}_++i(\bar{\xi}_++j\theta_+)} \\ &= e^{-\sigma_b(x-u_bt)+\bar{\eta}_b+i(\bar{\xi}_b+j\theta_b)}\{1 + e^{\sigma_a(x-u_at)+\eta_a+i(\xi_a+j\theta_a)} + e^{\sigma_b(x-u_bt)+\eta_{\bar{b}}+i(\xi_{\bar{b}}+j\theta_{\bar{b}})} \\ &\quad + A_{a\bar{b}}e^{(\sigma_a+\sigma_b)x-(\sigma_au_a+\sigma_bu_b)t+\eta_++i(\xi_++j(\theta_a+\theta_{\bar{b}}))}\} \\ &= e^{-\sigma_b(x-u_bt)+\bar{\eta}_b+i(\bar{\xi}_b+j\theta_b)}\tau_j^{(a\bar{b})}. \end{aligned} \tag{3.4.2}$$

It is straightforward to see by direct replacement of σ_b with $-\sigma_b$ in (2.4.9) that,

$$\bar{A}_{ab} = (A_{a\bar{b}})^{-1} \equiv e^{\zeta+i\delta}. \tag{3.4.3}$$

Moreover, note that the interaction coefficient of a multi-soliton, A_{ab}, is always positive [35], i.e. $\delta = 0$. Then from the above, one sees that the parameters transform as follows,

$$\eta_a = \bar{\eta}_a + \zeta \qquad \text{and} \qquad \xi_a = \bar{\xi}_a, \tag{3.4.4}$$

$$\eta_{\bar{b}} = -\bar{\eta}_b \qquad \text{and} \qquad \xi_{\bar{b}} = -\bar{\xi}_b. \tag{3.4.5}$$

From this last result, one can generalize the crossing transformation (3.4.4) and (3.4.5) into the N-soliton solution as was shown in [50].

In the breather case, the interaction coefficient for type A breathers can have positive or negative value while for type B breathers only negative values are allowed. Crossing of one of its constituent soliton yield the following transformation of the parameters,

$$\eta_a = \bar{\eta}_a + \zeta \qquad \text{and} \qquad \xi_a = \bar{\xi}_a + \delta, \tag{3.4.6}$$

$$\eta_{\bar{b}} = -\bar{\eta}_b \qquad \text{and} \qquad \xi_{\bar{b}} = -\bar{\xi}_b. \tag{3.4.7}$$

As is visible from (3.4.2), a type A breather will be crossed into a type B breather and vice-versa. From the crossed version τ-function of the breathers (replacing $u_a = -u_b = iv$ in (3.4.2)), one can evaluate the restrictions on parameters $\bar{\eta}$ and $\bar{\xi}$ and the allowed velocity v in the same manner as shown in *Section 3.1.*

From the above mentioned evaluation, one obtains the restrictions on the parameters $(\bar{\xi}_- + \delta)$ and $(\bar{\eta}_+ + \zeta)$ as oppose to the original *uncrossed* case with restrictions on the parameters ξ_+ and η_-, respectively. For example, in the case of the crossed version of type A breather with negative interaction coefficient, $v^2 > \cot(\frac{\theta_a}{2})^2$, $\bar{\tau}_j^{(aa)}$ will be non-singular provided $(\bar{\xi}_- + \pi)$ never takes the values of (3.1.16) and $(\bar{\eta}_+ + \zeta)$ is bounded by a similar expression as (3.1.12) with an appropriate $\bar{\eta}_c^j$. The positive interaction case is always found to be singular. A non-singular solution for the crossed version of type B breather yields the same restriction as the ordinary type A breather, eqns. (3.1.11), (3.1.13) and (3.1.12), with its appropriate $\bar{\eta}_c^j$ and bounds on the allowed velocity (3.1.15).

Furthermore, one can calculate the topological charge carried by these crossed version breathers. It is indeed found that the crossed version of type A breathers always have zero topological charge, hence it correspond to the ordinary type B breathers. Whereas the crossed version of type B breathers have $\tilde{h}$ different topological charges, depending on the value of $\bar{\xi}_-$, and it is given by (3.2.22), thus these are exactly the ordinary type A breathers.

Finally, one can also imagine crossing both constituent solitons in a breather solution. The type of breathers is not changed by this crossing, but the constituent solitons now belongs to their anti-species, i.e. $\bar{\tau}_j^{(ab)} \sim \tau_j^{(\bar{a}\bar{b})}$. As a result, the topological charges of the *double crossed* version of type A breather calculated from $\bar{\tau}_j^{(aa)}$ is given by (3.2.22) with a replaced by $h - a$. Hence the topological charge of the *double crossed* version of type A breather of species a will now lie in the fundamental representation $\mathcal{R}_{\lambda_{2\bar{a} \bmod h}}$, instead of $\mathcal{R}_{\lambda_{2a \bmod h}}$. While the topological charge of the *double crossed* version of type B breather remains in the trivial representation space, i.e. the singlet component of the tensor product of conjugate fundamental representations. However, a recent result from considering the breather S-matrices [42] shows a different *double crossing* behaviour for the quantum bound states corresponding to the type B breather.

3.4.2 Crossing and Topological Charges

The crossing transformation $\sigma \to -\sigma$ can be viewed as a space inversion, $x \to -x$. Thus, since the asymptotic values of the solution are now interchanged, the sign of the topological charge calculated from this asymptotic value will be inverted. An alternative way of looking at this inversion of sign is through the transformation of parameter ξ, which determines the topological charges. It can be shown that this transformation of parameter ξ yields the desired inversion of sign on the topological charge.

Single Soliton and Multi-soliton

Consider first the crossing of a single soliton solution (3.4.1). Writing,

$$\bar{f}_j^{(a)} \equiv \frac{\bar{\tau}_j^{(a)}}{\bar{\tau}_0^{(a)}} = \omega_a^j \frac{\tau_j^{(\bar{a})}}{\tau_0^{(\bar{a})}} = \omega_a^j f_j^{(\bar{a})},$$

then a crossed single soliton solution is given by,

$$\bar{\phi}^{(a)} = \frac{i}{\beta} \sum_{j=1}^{n} \alpha_j \ln \bar{f}_j^{(a)} = \frac{i}{\beta} \sum_{j=1}^{n} (\ln \omega_a^j + \ln f_j^{(\bar{a})}).$$

Thus, it is readily seen that the set of topological charges of the crossed species a single soliton is equal to the set of topological charges of the species $\bar{a}$ single soliton,

$$\bar{q}_a = -\frac{1}{2\pi} \sum_{j=1}^{n} \alpha_j \Delta(\arg(f_j^{(\bar{a})})) = q_{\bar{a}}. \tag{3.4.8}$$

This analysis can be made precise when one considers the result of replacing $\xi_{\bar{a}}$ by $-\xi_{\bar{a}}$ in $\tau_j^{\bar{a}}$, as will be discussed shortly, to show the claim that the mentioned transformation of $\xi_{\bar{a}}$ also inverts the sign of the topological charge of the *crossed version* of the single soliton of species a.

Note that the asymptotic behaviour of $\bar{f}_j^{(a)}$ are as follows,

$$\bar{f}_j^{(a)} = \begin{cases} x \to \infty : & |\bar{f}_j^{(a)}| = 1 \quad \text{and} \quad \arg(\bar{f}_j^{(a)}) = 0, \\ x \to -\infty : & |\bar{f}_j^{(a)}| = 1 \quad \text{and} \quad \arg(\bar{f}_j^{(a)}) = \mu_j^{(a)}, \end{cases}$$

where $\mu_j^{(a)} = j\theta_a \bmod 2\pi$. Performing the same analysis of [35] on $\bar{f}_j^{(a)}$ with $\bar{\xi}_a = \pi - \varepsilon$, where ε is a positive infinitesimal parameter, one finds that the complex function $\bar{f}_j^{(a)}$ traces out a counter-clockwise curve in the complex plane of $\Re e(\bar{f}_j^{(a)})$ and $\Im m(\bar{f}_j^{(a)})$, and it only crosses the real axis once. The change of argument of $\bar{f}_j^{(a)}$ as x evolve from $-\infty$ to ∞ is,

$$\Delta(\arg(\bar{f}_j^{(a)})) = (2\pi - \frac{2\pi a j}{h}) \bmod 2\pi = 2\pi \left(\frac{a(h-j) \bmod h}{h} \right).$$

Hence, the *highest* topological charge of the crossed species a single soliton is given by,

$$\bar{q}_a^{(1)} = -\sum_{j=0}^{n} \frac{a(h-j) \bmod h}{h} \alpha_j = -q_a^{(1)}, \tag{3.4.9}$$

and by cyclically permuting the simple roots and α_0 one obtains $\bar{q}_a^{(k)} = -q_a^{(k)}$ with $k = 1, 2, \ldots, \tilde{h}_a$. Thus, crossing inverts the sign of the single soliton's topological charge.

Although it is obvious from (3.4.8) that the set of topological charges $\{\bar{q}_a^{(k)}\}$ is the same as the set of topological charges $\{q_{\bar{a}}^{(k)}\}$, one can make this claim more precise by showing that with $\xi_{\bar{a}} = -\bar{\xi}_a$, the solution $\bar{\phi}^a \sim \phi^{\bar{a}}$ does indeed have a topological charge which is a member of the set of topological charges $\{q_{\bar{a}}^{(k)}\}$ calculated from *uncrossed* $\phi^{\bar{a}}$. From [35] one knows that the set of species $\bar{a}$ single soliton's topological charges is,

$$q_{\bar{a}}^{(k)} = \sum_{j=0}^{n} \frac{aj \bmod h}{h} \alpha_{j-1+k}. \tag{3.4.10}$$

If one replaces $\xi_a = \pi - \epsilon$ with $-\xi_a$ in the analysis of $\bar{f}_j^{(a)}$ in determining the *highest charge* as performed in [35], one finds that the complex function $\bar{f}_j^{(a)}$ traces out a counter-clockwise curve in the complex plane of $\Re e(\bar{f}_j^{(a)})$ and $\Im m(\bar{f}_j^{(a)})$ which only crosses the real axis once. The resulting set of topological charges of the species $\bar{a}$ single soliton (denoted by $\tilde{q}_{\bar{a}}$ to differentiate with $q_{\bar{a}}$ in (3.4.10)) is

$$\tilde{q}_{\bar{a}}^{(k')} = -\sum_{j=0}^{n} \frac{(h-a)j \bmod h}{h} \alpha_{j-1+k'}. \tag{3.4.11}$$

With $p = \gcd(h, a)$ there exists a $\tilde{j}$, $1 \leq \tilde{j} \leq (h-1)$, such that $(h-a)\tilde{j} \bmod h = (h-p)$. Then setting $k' = h + 1 - \tilde{j}$, by rewriting (3.4.11), one can show that

$$\tilde{q}_{\bar{a}}^{(k')} = \sum_{j=0}^{n} \frac{aj \bmod h}{h} \alpha_j = q_{\bar{a}}^{(1)}, \tag{3.4.12}$$

where $q_{\bar{a}}^{(1)}$ is given by (3.4.10). Thus the set $\{\tilde{q}_{\bar{a}}^{(k')}\}$, derived above, is equal to the set $\{q_{\bar{a}}^{(k)}\}$, derived in [35], with $k' = h + k - \tilde{j}$.

To summarize, in this single soliton case, one has shown that crossing of $\sigma \to -\sigma$ inverts the sign of the topological charge, i.e. $\bar{q}_a^{(k)} = -q_a^{(k)}$. Further, this inversion of sign can also be seen as a result of taking $-\xi_{\bar{a}}$ instead of $\xi_{\bar{a}}$ in the evaluation of the topological charge of $\phi^{\bar{a}}$. This is because one has the above exact relation between the two derivations resulting from the two choices of $\xi_{\bar{a}}$.

In the multi-soliton case, the generalization is straightforward. In the crossed version of a multi-soliton solution, for each *uncrossed* constituent soliton, the topological charge remains the same because the parameters transform as $\xi_a = \bar{\xi}_a$. For the *crossed* constituent soliton, its topological charge changes sign since the parameter $\xi_{\bar{a}}$ transforms to $-\bar{\xi}_a$. Because there is no static multi-soliton solution (if the constituent solitons are of the same species), one can imagine taking a *snap-shot* of this system of multi-solitons when each constituent soliton is at some distance from the others, such that each constituent soliton does not feel the presence of the others, then the total topological charge of this system is just the sum of the topological charges of the constituent solitons. Hence after crossing, the topological charge of the system changes.

Breather

Contrary to the multi-soliton case, in the breather case the parameter ξ of each constituent soliton loses its importance in determining the topological charge of the breather. Instead, it is the parameter $\xi_+ = \xi_a + \xi_b$ which holds the information of the topological charge. And, as seen from *Section 3.2*, a consequence of this is that one can only know the total topological charge of a breather without knowing what are the topological charge of its constituent solitons. Thus, a similar crossing evaluation of the topological charges, as done in the multi-soliton case above, cannot be performed. However, some information can still be extracted from the breather with negative interaction coefficient if one assumes that the individual ξ carry the information of constituent charges and ignore the importance of ξ_+.

One begins with the examination of crossed type B breather, $\bar{\tau}_j^{(a\bar{a})} \sim \tau_j^{(a_1 a_2)}$ with $a = a_1 = a_2$, which has the topological charge given by,

$$\bar{q}_{a\bar{a}}^{(k)} = \sum_{j=0}^{n} \frac{2a(h-j) \bmod h}{h} \alpha_{j+k-1} = q_{aa}^{(k)}. \tag{3.4.13}$$

Recall that in a single soliton case of species a, for each topological charge $q_a^{(k)}$ the parameter ξ_a lie in the k^{th} sector [35],

$$\xi_a \in I_a^{(k)} = \left(\pi - \frac{2\pi}{\tilde{h}_a}\{[\tilde{a}(k-1) \bmod \tilde{h}_a] + 1\}, \pi - \frac{2\pi}{\tilde{h}_a}[\tilde{a}(k-1) \bmod \tilde{h}_a]\right), \qquad (3.4.14)$$

where $\tilde{a} = \frac{a}{\gcd(h,a)}$.

Under the crossing of $\sigma_{\bar{a}} \to -\sigma_{\bar{a}}$, the parameter $\bar{\xi}_{\bar{a}}$ transforms as,

$$\xi_{a_2} = -\bar{\xi}_{\bar{a}}.$$

From the discussion of previous subsection this transformation yields an inversion of the sign of the topological charge of constituent soliton $\bar{a}$, $q_{a_2} = -q_{\bar{a}}$. Let $s \equiv \left[\frac{h}{2a}\right] = \frac{h}{2a} - \delta_s$, where $\delta_s = 0$ if h is even and $\delta_s = \frac{1}{2a}$ if h is odd and assume that $a \leq \left[\frac{h}{2}\right]$. Then the transformation of $\bar{\xi}_a$ can be seen as a shifting of sector from where $\bar{\xi}_a$ originally lay,

$$\xi_{a_1} = \bar{\xi}_a - \pi = \bar{\xi}_a - \frac{2\pi a}{h}\left(\frac{h}{2a}\right) = \bar{\xi}_a - \frac{2\pi a}{h}s - \frac{2\pi a}{h}\delta_s.$$

If $\delta_s \neq 0$, then the extra shift of $\frac{2\pi a}{h}\delta_s = \frac{\pi}{\tilde{h}_a \gcd(h,a)}$ will be maximum when $\gcd(h,a) = 1$.

When the range $(-\pi, \pi)$ of parameter ξ is divided into even number of sectors, $\tilde{h}_a$ is even, then $\bar{\xi}_a \in I_a^{(k)}$ is shifted into $\xi_{a_1} = \bar{\xi}_a - \pi \in I_a^{(k+s)}$. On the otherhand, when there are odd possibilities of the constituent soliton's topological charges, then the final sector after shifting will depend on where originally $\bar{\xi}_a$ lay in $I_a^{(k)}$. Suppose $\bar{\xi}_a$ is in the upper-half of $I_a^{(k)}$, i.e.

$$\bar{\xi}_a \in \left(\pi - \frac{\pi}{\tilde{h}_a}\{2[\tilde{a}(k-1) \bmod \tilde{h}_a] + 1\}, \pi - \frac{2\pi}{\tilde{h}_a}[\tilde{a}(k-1) \bmod \tilde{h}_a]\right).$$

Then the shift of $-\pi$ on $\bar{\xi}_a$ amounts to shifting the sector from $I_a^{(k)}$ into $I_a^{(k+s)}$. This is because the extra shift cause by $\frac{2\pi a}{h}\delta_s$ will only result in placing $\xi_{a_1} = \bar{\xi}_a - \pi$ in the lower-half of sector $I_a^{(k+s)}$,

$$\xi_{a_1} \in \left(\pi - \frac{2\pi}{\tilde{h}_a}\{[\tilde{a}(k+s-1) \bmod \tilde{h}_a] + 1\}, \pi - \frac{\pi}{\tilde{h}_a}\{2[\tilde{a}(k+s-1) \bmod \tilde{h}_a] + 1\}\right).$$

Whereas, if originally $\bar{\xi}_a$ is in the lower-half of sector $I_a^{(k)}$, then one can view the transformation of $\bar{\xi}_a$ as a shift of π instead, and this will bring $\xi_{a_1} = \bar{\xi}_a + \pi$ into the upper-half of sector $I_a^{(k-s)}$.

Thus one can summarize the transformation of the constituent topological charges of type B breather under the crossing, i.e. $\tau^{(a\bar{a})} \to \bar{\tau}^{(a\bar{a})} \sim \tau^{(a_1 a_2)}$,

$$\begin{aligned}
q_a^{(k_1)} &\longrightarrow q_{a_1}^{(k_1')} = q_a^{(k_1+s)} \quad \text{or} \quad q_a^{(k_1-s)}, \\
q_{\bar{a}}^{(k_2)} &\longrightarrow q_{a_2}^{(k_2')} = -q_{\bar{a}}^{(k_2)} = q_a^{(k_1)},
\end{aligned}$$

since $q_a^{(k_1)} + q_{\bar{a}}^{(k_2)} = 0$. Thus one of the topological charges changes sign while the other is shifted such that the total topological charge of the system takes one of the values given by (3.4.13).

For cases with $a > \left[\frac{h}{2}\right]$, the transformation of constituent topological charges are as above with s being replaced by $\bar{s} \equiv -\left[\frac{h}{2\bar{a}}\right]$.

From this result, one can show explicitly that $q_{a_1}^{(k_1')} + q_{a_2}^{(k_2')}$ is the topological charge of type A breather. Suppose that ξ_a is in the upper-half of sector $I_a^{(k)}$ and $a \leq \left[\frac{h}{2}\right]$, then after crossing one has (remember that $a = a_1 = a_2$)

$$\begin{aligned}
q_{a_2}^{(k)} &= \sum_j^n \frac{a(h-j) \bmod h}{h} \alpha_{j+k-1} \\
q_{a_1}^{(k+s)} &= \sum_{j=s}^{n+s} \frac{a(h-j+s) \bmod h}{h} \alpha_{j+k-1}.
\end{aligned}$$

Write $a(h-j)$ as follows $a(h-j) = c_j + hd_j$ where $d_j \in \mathbb{N}$, $c_j < h$ and $c_0 = 0$. Then $a(h-j) \bmod h = c_j$ and

$$a(h-j+s) \bmod h = \tilde{c}_j + \left[\frac{h}{2}\right],$$

where,

$$\tilde{c}_j = \begin{cases} c_j & \text{if } c_j \leq \left[\frac{h}{2}\right], \\ c_j - h & \text{if } c_j > \left[\frac{h}{2}\right]. \end{cases}$$

Hence, $2a(h-j) \bmod h = c_j + \tilde{c}_j$. Then one can write,

$$\begin{aligned}
q_{a_1}^{(k+s)} &= \frac{[h/2]}{h}\alpha_0 + \sum_{j=s}^{n} \frac{\tilde{c}_j + [h/2]}{h}\alpha_{j+k-1} + \sum_{j=n+2}^{n+s} \frac{\tilde{c}_j + [h/2]}{h}\alpha_{j+k-1} \\
&= \sum_{j=s}^{n} \frac{\tilde{c}_j}{h}\alpha_{j+k-1} + \sum_{j=h+1}^{h-1+s} \frac{\tilde{c}_j}{h}\alpha_{j+k-1} \\
&= \sum_{j=1}^{n} \frac{\tilde{c}_j}{h}\alpha_{j+k-1}.
\end{aligned}$$

Summing with $q_{a_2}^{(k)}$ one obtains

$$q_{a_1}^{(k+s)} + q_{a_2}^{(k)} = \sum_{j=1}^{n} \frac{c_j + \tilde{c}_j}{h} \alpha_{j+k-1} = \sum_{j=0}^{n} \frac{2a(h-j) \bmod h}{h} \alpha_{j+k-1} = q_{aa}^{(k)}.$$

Thus, now one knows what are the constituent topological charges of a type A breather individually, namely $q_a^{(k+s)}$ and $q_a^{(k)}$.

The evaluation of the topological charge transformation of the crossed type A breather with negative interaction coefficient, $\tau^{(a_1a_2)} \rightarrow \bar{\tau}^{(a_1a_2)} \sim \tau^{(a\bar{a})}$ with $a = a_1 = a_2$, can be performed in a similar manner to give

$$\begin{aligned} q_{a_1}^{(k)} &\longrightarrow q_a^{(k_1')} = q_{a_1}^{(k+s)}, \\ q_{\bar{a}}^{(k+s)} &\longrightarrow q_{\bar{a}}^{(k_2')} = -q_{a_2}^{(k+s)}, \end{aligned}$$

which give a total of zero topological charge.

However a similar evaluation is not possible when the interaction coefficient is positive. This is as would be expected, because the shifting of one of the topological charges is due precisely to the presence of the extra π in the transformation of ξ arising from the crossing. In the positive interaction case there is no extra π.

To this end, one has to bear in mind that ξ of constituent solitons do not play a rôle in determing the topological charges of a breather. Instead, it is the value of ξ_+ which will determine the topological charge of a breather. The crossing treatment of the topological charges of a breather, just discussed, is just a superficial way of evaluating the constituent topological charges.

3.5 Breathers in other Theories

In this section the breathers in the $d_4^{(1)}$ affine Toda theory will be determined. This calculation provide an example for the general calculation of breathers in other Toda theories. One will use Olive et.al.'s algebraic method to construct the τ-functions. As expected, these τ-functions are the same as τ-functions derived using the Hirota's method.

Having obtained the breather solutions, one proceeds further to determine its topological charges. It turns out that for breathers with constituent solitons coming from the heavy soliton, i.e. the middle spot of $d_4^{(1)}$ Dynkin diagram, the topological charges of certain breather solutions do not come from the sum of known topological charges of its constituent solitons. Furthermore, it is found that the topological charges of all possible $d_4^{(1)}$ breathers are exactly the same as the topological charges of certain single soliton solutions. Thus, these breather solutions do not produce new topological charges.

Finally, a short remark on the breathers of other theories will also be given.

3.5.1 Algebraic τ-function of $d_4^{(1)}$ Theory

To construct the algebraic τ-function, one has to be able to evaluate all expectation values of the terms in the expansion of (2.4.57) in (2.4.56). For a given fundamental representation $|\Lambda_j>$, one has to calculate the expectation value of the operator $\hat{F}^a$,

$$F_{ja} \equiv <\Lambda_j|\hat{F}^a|\Lambda_j> .$$

When $|\Lambda_j>$ is of level 1, F_{jk} can be calculated using the fact that the set of automorphism of the *affine* Dynkin diagram which do not have fixed nodes, $W_0(g)$, is isomorphic with the *centre*, $Z(g)$ (for details see [13, 57]). It is found that

$$F_{ja} \equiv \varepsilon(a,j) = e^{-2i\pi\lambda_a\cdot\lambda_j}. \tag{3.5.1}$$

If $|\Lambda_j>$ is of level higher than 1, one can write a representation of this fundamental weight as a tensor product of level 1 representations.

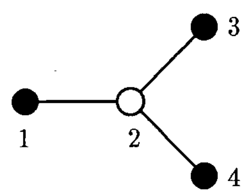

Figure 3.2: Numbering and Colouring of the Dynkin diagram of D_4.

An example of this is the level 2 fundamental weight of $d_4^{(1)}$ given in [13],

$$|\Lambda_2> \; = \; \frac{1}{\sqrt{2}}(f_j|\Lambda_j>\otimes|\Lambda_j> - \; |\Lambda_j>\otimes f_j|\Lambda_j>)$$

$$= \frac{1}{\sqrt{2}}\left(\hat{E}_{-1}|\Lambda_j> \otimes|\Lambda_j> - \ |\Lambda_j> \otimes\hat{E}_{-1}|\Lambda_j>\right), \qquad (3.5.2)$$

where $|\Lambda_j>$ is the level 1 representation for the outer nodes of $d_4^{(1)}$ Dynkin diagram (see *figure 3.2*). Using this representation, one obtains the following expectation values,

$$F_{2a} \ = \ \begin{cases} 0 & \text{for } a = 1,3,4 \\ -4 & \text{for } a = 2. \end{cases} \qquad (3.5.3)$$

With this information, one can write down the τ-function defined by,

$$\tau_j = < \Lambda_j|g(t)|\Lambda_j >, \qquad (3.5.4)$$

with $g(t)$ is given by (2.4.60). The algebraic τ-functions for single solitons of the $d_4^{(1)}$ theory have been written down in [13] and coincide with Hirota's τ-function [15]. In writing down these τ-functions, one uses the fact that for the representation $|\Lambda_j>$ of level m_j, the highest non-vanishing power of step-operators $\hat{F}^a$ can be written as a vertex operator [13, 50],

$$\frac{1}{m_j!}\left(\hat{F}^a(z_a)\right)^{m_j} = e^{-2i\pi\lambda_j\cdot\lambda_a}Y_-^a Y_+^a, \qquad (3.5.5)$$

where,

$$Y_\pm^a = \exp\left(\mp\sum_{N>0}\frac{\gamma_a\cdot q[\mp N]}{N}z_a^{\mp N}\hat{E}_{\pm N}\right), \qquad (3.5.6)$$

with $\gamma_a, q[\pm N]$ as discussed in *Subsection 2.4.3* and z_a is given in (2.4.59).

The two soliton τ-function can be calculated in the same manner. For $|\Lambda_j>$ representation of level 1, one obtains a general expression for the τ-function as follows,

$$\begin{aligned} \tau_j^{(ab)} &= < \Lambda_j|\exp(Q_aW_a\hat{F}^a)\exp(Q_bW_b\hat{F}^b)|\Lambda_j> \\ &= < \Lambda_j|(1+Q_aW_a\hat{F}^a)(1+Q_bW_b\hat{F}^b)|\Lambda_j> \\ &= < \Lambda_j|(1+Q_aW_a\hat{F}^a+Q_bW_b\hat{F}^b+X_{a,b}Q_aQ_bW_aW_b : \hat{F}^a\hat{F}^b :)|\Lambda_j> \\ &= 1+\varepsilon(a,j)Q_aW_a+\varepsilon(b,j)Q_bW_b+\varepsilon(a,j)\varepsilon(b,j)X_{a,b}Q_aQ_bW_aW_b. \end{aligned} \qquad (3.5.7)$$

In the above, the parameter $Q_a = \exp(\eta_a + i\xi_a)$ determines the *position* of the soliton and its topological charge. Further with $\tilde{m}_a$ being the mass of the a^{th} fundamental particle, W_a is given by

$$W_a(\Theta_a) = \exp[\tilde{m}_a(x\cosh\Theta_a - t\sinh\Theta_a)]. \qquad (3.5.8)$$

The general interaction coefficient $X_{a,b}$ is given as in (2.4.67), or after some algebra

$$X_{a,b}(\Theta_a,\Theta_b)=\prod_{p=1}^{h}\left(\frac{\sinh\frac{1}{2}[\Theta+\frac{i\pi}{h}(2p+\frac{c_a+c_b}{2}-1)]}{\sinh\frac{1}{2}[\Theta+\frac{i\pi}{h}(2p+\frac{c_a+c_b}{2}+1)]}\right)^{\lambda_a\cdot\omega^p(\gamma_b)}, \qquad (3.5.9)$$

with $\Theta=\Theta_a-\Theta_b$ is the rapidity difference and c_a is the colour factor of the simple roots.

For $|\Lambda_2>$, which is a level 2 representation, one has the following τ-function,

$$\begin{aligned}\tau_2^{(ab)} &= <\Lambda_2|(1+Q_aW_a\hat{F}^a+\frac{1}{2}Q_a^2W_a^2(\hat{F}^a)^2)(1+Q_bW_b\hat{F}^b+\frac{1}{2}Q_b^2W_b^2(\hat{F}^b)^2)|\Lambda_2> \\ &= 1+F_{2a}Q_aW_a+F_{2b}Q_bW_b+\varepsilon(a,2)Q_a^2W_a^2+\varepsilon(b,2)Q_b^2W_b^2 \\ &\quad+\varepsilon(a,2)F_{2a}X_{a,b}Q_a^2Q_bW_a^2W_b+\varepsilon(b,2)F_{2b}X_{a,b}Q_aQ_b^2W_aW_b^2 \\ &\quad+Z_{a,b}Q_aQ_bW_aW_b+\ \varepsilon(a,2)\varepsilon(b,2)(X_{a,b})^2Q_a^2Q_b^2W_a^2W_b^2. \end{aligned} \qquad (3.5.10)$$

In the derivation above, one has used the following identity relation [51]

$$Y_+^a\hat{F}^b=X_{a,b}\hat{F}^bY_+^a, \qquad \hat{F}^bY_-^a=X_{a,b}Y_-^a\hat{F}^b.$$

The interaction coefficient $Z_{a,b}$ above is calculated to be

$$\begin{aligned}Z_{a,b}(\Theta) &\equiv <\Lambda_2|\hat{F}^a\hat{F}^b|\Lambda_2> \\ &= (Z_-X_{a,b}+Z_+)\varepsilon(a,j)\varepsilon(b,j),\end{aligned} \qquad (3.5.11)$$

where

$$Z_\pm=2-|\gamma_a\cdot q[1]|^2-|\gamma_b\cdot q[1]|^2\pm 2|\gamma_a\cdot q[1]||\gamma_b\cdot q[1]|\cosh\Theta.$$

These results are exactly the same as results derived from Hirota's method [46].

3.5.2 Breathers in $d_4^{(1)}$ Theory

For this $d_4^{(1)}$ theory, the fundamental particles associated to the outer nodes of the Dynkin diagram are degenerate in mass, $\tilde{m}_a=m\sqrt{2}$. And the fundamental particle associated with the middle node is heavier, $\tilde{m}_2=m\sqrt{6}$. So, there are 3 possible combinations of two soliton solutions which can make a breather: 22-breathers, aa-breathers and ab-breathers where $a,b=1,3$ or 4. These breather solutions are obtained by analytic continuation into complex rapidity, setting $\Theta_a=-\Theta_b=i\bar{\Theta}$. Then, the masses of these breathers are given as in (3.1.3), which is always smaller than the sum of the mass of its constituent solitons. For each possible two soliton combination mentioned above, one would like to find the *magic recipe* of all the parameters involved in the breather solution. However, subtleties arises in the 22-breather τ-function, as one has to solve two quartic equations of transcendental functions. For the other cases, one can derive the complete restrictions of parameters η, ξ and the rapidity $\bar{\Theta}$.

22-breathers

In this case, the X-interaction coefficient and the Z-interaction coefficient are given by the following

$$X_{2,2} = \frac{[\cos(2\bar{\Theta}) - 1][\cos(2\bar{\Theta}) - \frac{1}{2}]}{[\cos(2\bar{\Theta}) + 1][\cos(2\bar{\Theta}) + \frac{1}{2}]}, \tag{3.5.12}$$

$$Z_{2,2} = 16\frac{[\cos(2\bar{\Theta}) + \frac{\sqrt{10}}{4}][\cos(2\bar{\Theta}) - \frac{\sqrt{10}}{4}]}{[\cos(2\bar{\Theta}) + 1][\cos(2\bar{\Theta}) + \frac{1}{2}]}. \tag{3.5.13}$$

Although the range of definition of the rapidity is from 0 to $\frac{\pi}{2}$ (recall that the velocity and rapidity are related by $v = \tan\bar{\Theta}$). In *figure 3.3*, only the rapidity range from 0 to $\frac{\pi}{3}$ is drawn.

The behaviour of the τ-function on level 1 representation can be evaluated in a similar manner as the $a_n^{(1)}$ breather cases done in *Section 3.1.* Separating the real and imaginary part of the τ-function, and evaluating the behaviour of these parts such that there is no point in space time which will give a simultaneous zero for the real and imaginary part yields a restriction on the parameter η and the rapidity. With the notation,

$$X = e^{\zeta_x + i\delta_x}, \qquad Z = e^{\zeta_z + i\delta_z}, \qquad \text{where} \qquad \zeta_x, \zeta_z, \delta_x, \delta_z \in \mathbb{R},$$

without difficulties one finds that

$$-\eta_c' < \eta_- < \eta_c', \tag{3.5.14}$$

where,

$$\eta_c' = \operatorname{arcosh}\{2e^{\zeta_x}\cos\frac{1}{2}(\xi_+ + \delta_x)^2 - e^{i\delta_x}\}. \tag{3.5.15}$$

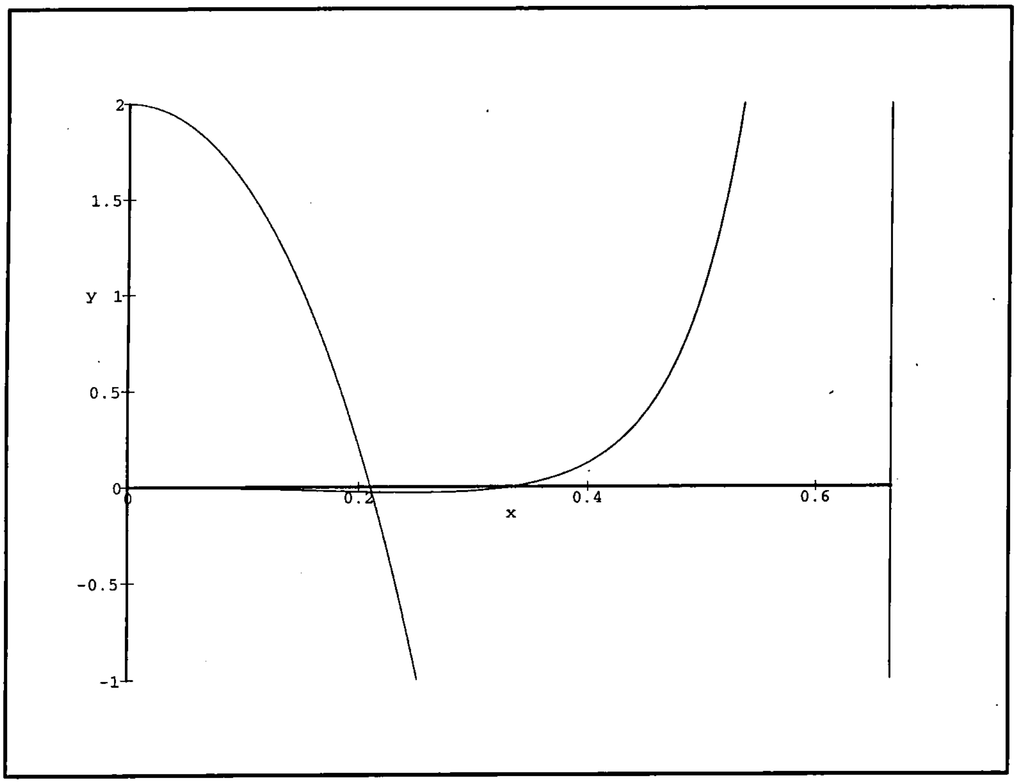

Figure 3.3: The graph shows the behaviour of the interaction coefficients $X(2\bar{\Theta})$ (full line) and $Z(2\bar{\Theta})$ (dotted line) for the 22-breather of $d_4^{(1)}$ in the rapidity range 0 to $\frac{\pi}{3}$. The horizontal axis is chosen in terms of the parameter $x = \frac{2\bar{\Theta}}{\pi}$. Note that both interaction coefficients become singular at $x = \frac{2}{3}$ or $\bar{\Theta} = \frac{\pi}{3}$.

Furthermore, from the above definition of η_c' it is clear that all values of rapidity for which $X < 0$ are allowed, while in the region where $X > 0$ the rapidity is restricted by,

$$\bar{\Theta}_0 \leq \bar{\Theta} < \frac{\pi}{3},$$

where

$$\bar{\Theta}_0 = \frac{1}{2}\arccos\left(\frac{-3 - 3\cos(\frac{\xi_+}{2})^2 + \sqrt{\cos(\frac{\xi_+}{2})^4 + 34\cos(\frac{\xi_+}{2})^2 + 1}}{4[\cos(\frac{\xi_+}{2}) - 1]}\right). \tag{3.5.16}$$

Also, the parameters ξ are restricted *not* to take the following values,

$$\xi_+ = 0 \bmod 2\pi \qquad \text{for} \quad X < 0, \tag{3.5.17}$$

$$\xi_+ = \pi \bmod 2\pi \qquad \text{for} \quad X > 0. \tag{3.5.18}$$

Up to a complex phase, the τ-function on the level 2 representation can be written as follows

$$\begin{aligned}\tau_2 \;\sim\; & e^{\zeta_z - \zeta_x + i(\delta_z - \delta_x)} + 2\cosh[\Gamma^{(2)} + i(\xi_+ + \delta_x)] + 2e^{-\zeta_x - i\delta_x}\cosh[i\Delta^{(2)} - \eta_-] \\ & -16e^{-\frac{1}{2}(\zeta_x + i\delta_x)}\cosh\frac{1}{2}[i\Delta^{(2)} - \eta_-]\cosh\frac{1}{2}[\Gamma^{(2)} + i(\xi_+ + \delta_x)],\end{aligned} \tag{3.5.19}$$

where,

$$\Gamma^{(a)} = 2\tilde{m}_a x\cos\bar{\Theta} + \eta_+ + \zeta_x, \qquad \Delta^{(a)} = 2\tilde{m}_a t\sin\bar{\Theta} - \xi_-. \tag{3.5.20}$$

Suppose one examines the case when $X > 0$ and $Z < 0$, and assumes that there exist (x_0, t_0) such that the real and imaginary parts of (3.5.19) are zero simultaneously, i.e.

$$\begin{aligned}\Re e(\tau_2) \;\sim\; & e^{\zeta_z} - 2e^{\zeta_x}\left[\cos\left(\frac{\xi_+}{2}\right)^2 - \sin\left(\frac{\xi_+}{2}\right)^2\right]\left[\cosh\left(\frac{\Gamma_0^{(2)}}{2}\right)^2 + \sinh\left(\frac{\Gamma_0^{(2)}}{2}\right)^2\right] \\ & -2\left[\cosh\left(\frac{\eta_-}{2}\right)^2 + \sinh\left(\frac{\eta_-}{2}\right)^2\right]\left[\cos\left(\frac{\Delta_0^{(2)}}{2}\right)^2 - \sin\left(\frac{\Delta_0^{(2)}}{2}\right)^2\right] \\ & +16e^{\frac{\zeta_x}{2}}\left[\cos\left(\frac{\xi_+}{2}\right)\cosh\left(\frac{\eta_-}{2}\right)\cos\left(\frac{\Delta_0^{(2)}}{2}\right)\cosh\left(\frac{\Gamma_0^{(2)}}{2}\right)\right. \\ & \left. + \sin\left(\frac{\xi_+}{2}\right)\sinh\left(\frac{\eta_-}{2}\right)\sin\left(\frac{\Delta_0^{(2)}}{2}\right)\sinh\left(\frac{\Gamma_0^{(2)}}{2}\right)\right] = 0,\end{aligned}$$

and,

$$
\begin{aligned}
\Im m(\tau_2) \quad \sim \quad & 8e^{\zeta_x} \sin\left(\frac{\xi_+}{2}\right) \cos\left(\frac{\xi_+}{2}\right) \sinh\left(\frac{\Gamma_0^{(2)}}{2}\right) \cosh\left(\frac{\Gamma_0^{(2)}}{2}\right) \\
& -8e^{\zeta_x} \sinh\left(\frac{\eta_-}{2}\right) \cosh\left(\frac{\eta_-}{2}\right) \sin\left(\frac{\Delta_0^{(2)}}{2}\right) \cos\left(\frac{\Delta_0^{(2)}}{2}\right) \\
& -16e^{\frac{\zeta_x}{2}} \left[\sin\left(\frac{\xi_+}{2}\right) \cosh\left(\frac{\eta_-}{2}\right) \cos\left(\frac{\Delta_0^{(2)}}{2}\right) \sinh\left(\frac{\Gamma_0^{(2)}}{2}\right)\right. \\
& \left. - \cos\left(\frac{\xi_+}{2}\right) \sinh\left(\frac{\eta_-}{2}\right) \sin\left(\frac{\Delta_0^{(2)}}{2}\right) \cosh\left(\frac{\Gamma_0^{(2)}}{2}\right)\right] \quad = \ 0.
\end{aligned}
$$

So, the idea is to eliminate $\Delta_0^{(2)}$ or $\Gamma_0^{(2)}$ using the above relations, and from there one tries to find a further restriction for the parameters η. Unfortunately, as seen from the above, these involve manipulation of a transcendental function which in turn gives rise to subtleties in the evaluation. Thus, one resorts to a simplified restriction and further shows that indeed with this simplified restriction on η there exist solutions.

Setting $\eta_- = 0$, from the imaginary relation above one obtains the following relations

$$
\cos\frac{1}{2}(\Delta_0^{(2)}) = \frac{1}{2}e^{\frac{\zeta_x}{2}} \cos\frac{1}{2}(\xi_+) \cosh\frac{1}{2}(\Gamma_0^{(2)}). \tag{3.5.21}
$$

Inserting this into the real part, one sees that in order that there exist non-singular solutions, i.e. *no* (x_0, t_0) exist such that the real and imaginary parts are zero simultaneously, the following must hold,

$$
g(\bar{\Theta}, \xi_+) \equiv \left(2e^{\zeta_x}[1 - 2\cos\frac{1}{2}(\xi_-)^2] - e^{\zeta_x} - 2\right) < 0 \qquad \text{for } \frac{\pi}{6} < \bar{\Theta} < \frac{\pi}{3}.
$$

It is straightforward to see that this inequality indeed is fulfilled. Furthermore, in deriving (3.5.21) one has assumed that $\sinh\frac{1}{2}(\Gamma_0^{(2)}) \neq 0$ and $\sin\frac{1}{2}(\xi_+) \neq 0$. But, one can show that even taking $\sinh\frac{1}{2}(\Gamma_0^{(2)}) = 0$, the real part will never vanish. Further, choosing $\sin\frac{1}{2}(\xi_+) = 0$ will always give a singular solution, thus the parameters ξ are chosen such that ξ_+ never takes the values,

$$
\xi_+ = 0 \bmod 2\pi. \tag{3.5.22}
$$

Thus, combining with the results of the τ-function of the level 1 representations, one has succeeded in proving that with $\eta_- = 0$, there exist non-singular solutions with $X > 0$ for rapidities $\bar{\Theta}_0 \leq \bar{\Theta} < \frac{\pi}{3}$ where $\bar{\Theta}_0$ is given in (3.5.16). Moreover from (3.5.18) and (3.5.22), the parameters ξ are chosen so that ξ_+ will not take the following values,

$$\xi_+ = 0, \pi \bmod 2\pi. \tag{3.5.23}$$

One can perform the same evaluation for cases of $X < 0$. However, due to the behaviour of a similar function as the function $g(\bar{\Theta}, \xi_+)$ above, one cannot say quite clearly that there exist or does not exist non-singular solutions for rapidity value between $\frac{\pi}{3} < \bar{\Theta} < \frac{\pi}{2}$. This is because there are regions of $(\bar{\Theta}, \xi_+)$ where this function has negative values while in other regions it has positive values. Nevertheless, for rapidity values $0 < \bar{\Theta} < \frac{\pi}{6}$ one can show that there exist non-singular solutions, with parameters ξ are chosen such that ξ_+ never takes the values,

$$\xi_+ = 0 \bmod 2\pi. \tag{3.5.24}$$

At special values of rapidities, these breather solutions fail to exist. In particular at rapidity $\bar{\Theta} = 0$ it becomes an obvious static solution, at $\bar{\Theta} = \frac{\pi}{6}$ singularity always occur, and rapidity of $\bar{\Theta} = \frac{\pi}{3}$ or $\frac{\pi}{2}$ will lead to a static single soliton solution or a vacuum solution. This last phenomena was also noted in the two soliton case in [46].

aa-breathers

These are breathers with constituent solitons coming from 2 of the lighter solitons of $d_4^{(1)}$ associated with the same species (node of the Dynkin diagram). The interaction coefficients are calculated to be (see *figure 3.4*),

$$X_{a,a} = \frac{[\cos(2\bar{\Theta}) - 1][\cos(2\bar{\Theta}) + \frac{1}{2}]}{[\cos(2\bar{\Theta}) + 1][\cos(2\bar{\Theta}) - \frac{1}{2}]}, \tag{3.5.25}$$

$$Z_{a,a} = \frac{2}{[\cos(2\bar{\Theta}) + 1][\cos(2\bar{\Theta}) + \frac{1}{2}]}. \tag{3.5.26}$$

The τ-functions of level 1 representations has the same restriction as that of the 22-breathers, relations (3.5.14) and (3.5.15). And as before, the case of $X > 0$ yields further restriction on the rapidity,

$$\frac{\pi}{6} < \bar{\Theta} \leq \bar{\Theta}_0,$$

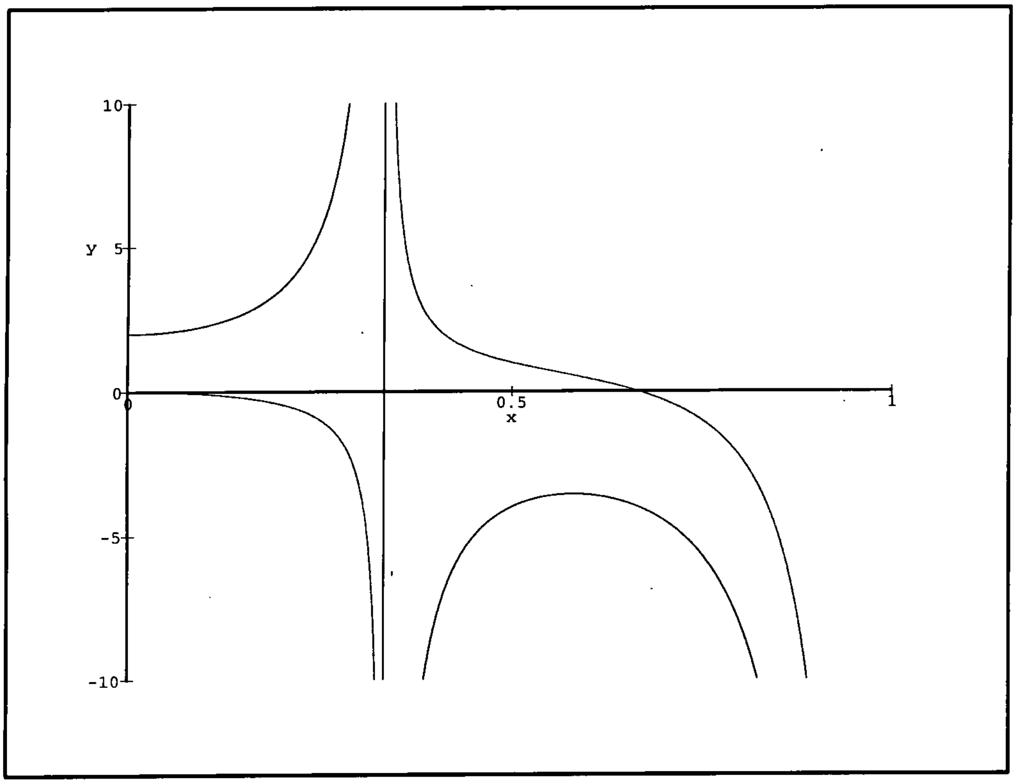

Figure 3.4: The graph shows the behaviour of the interaction coefficients $X(2\bar{\Theta})$ (dotted line) and $Z(2\bar{\Theta})$ (full line) for the aa-breather of $d_4^{(1)}$ in the rapidity range 0 to $\frac{\pi}{2}$. The horizontal axis is chosen in terms of the parameter $x = \frac{2\bar{\Theta}}{\pi}$. Note that both interaction coefficients become singular at $x = \frac{1}{3}$ or $\bar{\Theta} = \frac{\pi}{6}$ and at $x = 1$ or $\bar{\Theta} = \frac{\pi}{2}$.

where

$$\bar{\Theta}_0 = \frac{1}{2}\arccos\left(\frac{-1-\cos(\frac{\xi_+}{2})^2+\sqrt{9\cos(\frac{\xi_+}{2})^4-14\cos(\frac{\xi_+}{2})^2+9}}{4[1-\cos(\frac{\xi_+}{2})]}\right). \tag{3.5.27}$$

And, the parameters ξ are restricted as (3.5.17) and (3.5.18).

Furthermore, for this case one has a much simplified τ-function for the level 2 representation, since $F_{2a} = 0$. And as a result, a detailed restriction on the parameter η can be found. The evaluation method for this τ-function is similar to the evaluation done in the $a_n^{(1)}$ cases. The results can be cited here as follows. In order that τ_2 never vanishes, the parameters η have to be chosen such that its difference is bounded as in (3.5.14), i.e.

$$-\eta_c < \eta_- < \eta_c,$$

with

$$\eta_c = \operatorname{arcosh}\{2e^{\zeta_z} + e^{\zeta_x}\cos(\xi_+ + \delta_z)\}. \tag{3.5.28}$$

Also, the parameters ξ have to be chosen such that for rapidity values between $0 < \bar{\Theta} < \frac{\pi}{6}$ and $\frac{\pi}{6} < \bar{\Theta} \leq \bar{\Theta}_0$, ξ_+ never takes the values

$$\xi_+ = 0, \pi \bmod 2\pi, \tag{3.5.29}$$

and for rapidity values between $\frac{\pi}{3} < \bar{\Theta} < \frac{\pi}{2}$, ξ_+ may not take the following values

$$\xi_+ = 0 \bmod 2\pi. \tag{3.5.30}$$

Thus to summarise the results, an aa-breather can be constructed provided one choses the parameter η_- to between the bounds of the smallest of η_c' or η_c and parameters ξ are chosen such that (3.5.29) is not fulfilled for rapidity values between $0 < \bar{\Theta} < \frac{\pi}{6}$ and $\frac{\pi}{6} < \bar{\Theta} \leq \bar{\Theta}_0$ or (3.5.30) is not fulfilled for rapidity values between $\frac{\pi}{3} < \bar{\Theta} < \frac{\pi}{2}$.

At special values of rapidities, these breather solutions fail to exist. At rapidity $\bar{\Theta} = 0$, the solution become a static solution, i.e. not a breather. For rapidity $\bar{\Theta} = \frac{\pi}{3}$, there always exist (x_0, t_0) such that the real and imaginary parts of τ_j, with $j = 1, 3$ or 4, becomes zero simultaneously. While at the pole rapidities, the breather solution becomes a static single soliton solution at $\bar{\Theta} = \frac{\pi}{6}$ or a vacuum solution at $\bar{\Theta} = \frac{\pi}{2}$. These last results are similar to the two soliton fusing results of Hall [46].

ab-breathers

From all $d_4^{(1)}$ breathers, this type of breather are the easiest to evaluate, since its interaction coefficients are as follows,

$$X_{a,b} = \frac{[\cos(2\bar{\Theta}) - \frac{1}{2}]}{[\cos(2\bar{\Theta}) + \frac{1}{2}]}, \tag{3.5.31}$$

$$Z_{a,b} = 0. \tag{3.5.32}$$

And with $F_{ab} = 0$ for $a, b = 1, 3$ or 4, all the τ-functions are similar to that of the $a_n^{(1)}$ series. Thus, evaluation procedure of these τ-functions are the same as in type B of $a_n^{(1)}$ cases. It turns out that it is not possible to construct a breather solution for $X > 0$, $0 < \bar{\Theta} < \frac{\pi}{6}$ and $\frac{\pi}{3} < \bar{\Theta} < \frac{\pi}{2}$, since zeros always appear simultaneously in the real and imaginary parts of the τ-function. However, for the $X < 0$ case, the rapidities between $\bar{\Theta}_0 \leq \bar{\Theta} < \frac{\pi}{3}$ will give a valid breather solution, where

$$\bar{\Theta}_0 = \frac{1}{2} \arccos\left(\frac{1}{2}\frac{|\cos(\xi_+)| - 1}{|\cos(\xi_+)| + 1}\right). \tag{3.5.33}$$

Further, the parameter η has to be chosen such that

$$-\eta_c < \eta_- < \eta_c,$$

where η_c is the smallest of the following,

$$\eta_c = \begin{cases} \operatorname{arcosh}\{2e^{\zeta_x} \sin \frac{1}{2}(\xi_+)^2 + 1\}, \\ \operatorname{arcosh}\{2e^{\zeta_x} \cos \frac{1}{2}(\xi_+)^2 + 1\}, \\ \operatorname{arcosh}\{e^{\zeta_x} |\cos(\xi_+)|\}. \end{cases} \tag{3.5.34}$$

And, the parameters ξ are chosen such that their sum do not take the following values,

$$\xi_+ = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \bmod 2\pi. \tag{3.5.35}$$

3.5.3 The Topological Charges of $d_4^{(1)}$ Breathers

The topological charges are calculated from the asymptotic value of the breather solutions ϕ at $x \longrightarrow \pm\infty$. However, since the solution involves a logarithm of a complex function, the solution traces a curve in the complex plane. The asymptotic values depends on the argument difference at $x \longrightarrow \pm\infty$, winding number and the direction of winding. Examination of the topological charges of the $d_4^{(1)}$ breathers follows the same steps as in *Section 3.2.*

As an example, the topological charges of 22-breather will be determined in what follows. The α_j-component of the topological charge for $j = 1, 3$ and 4, is determined from the ratio $f_j = \frac{\tau_j}{\tau_0} = 1$ for all region of definition. Thus, the α_j-component of the topological charge is zero. For the α_2-component, one notes from the previous calculation that there are two different cases.

The first case is when $X < 0$, from (3.5.23) one sees that there is only one type of topological charge for this case. Note that the asymptotic value of the ratio $f_2 = \frac{\tau_2}{(\tau_0)^2}$ is again equal to 1 at $x \longrightarrow \pm\infty$. Set all ηs equal to zero, and choose all ξs equal to $\frac{\pi}{2}$ and further choose $\bar{\Theta} = \frac{1}{2}\arccos\left(\frac{\sqrt{10}}{4}\right)$ such that $Z = 0$. Explicit numerical evaluation of the real and imaginary parts of the ratio $f_2 = \frac{\tau_2}{(\tau_0)^2}$ shows that f_2 never winds around the origin of the complex plane. Thus, the α_2-component for this case is zero.

The second case when $X < 0$, from (3.5.24) one sees that there can be 2 type of topological charge for this case. The asymptotic value of the ratio $f_2 = \frac{\tau_2}{(\tau_0)^2}$ is again equal to 1 at $x \longrightarrow \pm\infty$. Performing a similar trick as in *Section 3.2*, i.e. shifting the position of the breather such that at rapidity near the pole several terms drops out as they are negligible. Choose both ξ to be $\frac{\varepsilon}{2}$ where $\varepsilon > 0$ and infinitesimal. Shift $\eta = \bar{\eta} - \frac{\zeta_x}{2}$, and set $\bar{\Theta} \longrightarrow (\frac{\pi}{3})_-$ such that,

$$\lim_{\bar{\Theta}\longrightarrow(\frac{\pi}{3})_-} e^{\zeta_z - \zeta_x} = 4, \qquad \lim_{\bar{\Theta}\longrightarrow(\frac{\pi}{3})_-} e^{-\zeta_x} = 0.$$

Then, one can easily see that f_2 starts off from (1,0) of the complex plane with negative imaginary part at $x \longrightarrow -\infty$, and winds around the origin in a clockwise manner back to (1,0) as $x \longrightarrow \infty$. While choosing $\varepsilon < 0$ yields the opposite winding, i.e. a counter-clockwise manner. Thus, the α_2-component for this case is ± 1.

To summarize the 22-breather topological charges, one has

$$q^{(22)} = \quad 0 \qquad \text{for } \bar{\Theta}_0 \leq \bar{\Theta} < \frac{\pi}{3}, \tag{3.5.36}$$

$$q^{(22)} = \quad \pm\alpha_2 \qquad \text{for } 0 < \bar{\Theta} < \frac{\pi}{6}, \tag{3.5.37}$$

where $\bar{\Theta}_0$ is given in (3.5.16).

Topological charges of the aa-breathers and ab-breathers are calculated in a similar way and one finds the following,

$$q^{(aa)} = \pm\alpha_2 \qquad \text{for } 0 < \bar{\Theta} < \frac{\pi}{6}, \frac{\pi}{6} < \bar{\Theta} \leq \bar{\Theta}_0, \tag{3.5.38}$$

$$q^{(aa)} = 0 \qquad \text{for } \frac{\pi}{3} < \bar{\Theta} < \frac{\pi}{2}, \tag{3.5.39}$$

where $\bar{\Theta}_0$ is given in (3.5.27), and

$$q^{(ab)} = \pm\frac{1}{2}(\alpha_a + \alpha_b),\ \pm(\alpha_2 + \frac{1}{2}(\alpha_a + \alpha_b)). \tag{3.5.40}$$

Next, as in the $a_n^{(1)}$ series, one can check that the non-zero topological charges lie in the irreducible component of the Clebsch-Gordan decomposition of the tensor product of two fundamental representations associated with the constituent solitons. In order to do this, one uses the PRV conjecture. Consider the Coxeter element of the Weyl group given as follows,

$$\omega = r_1 r_3 r_4 r_2, \tag{3.5.41}$$

where the Weyl reflections with respect to the simple roots has been ordered in bicolour manner. A direct check of the Coxeter orbit of λ_2 shows that $\pm\alpha_2$ are elements of this orbit. Also, the weights $\{\pm\frac{1}{2}(\alpha_a + \alpha_b), \pm(\alpha_2 + \frac{1}{2}(\alpha_a + \alpha_b))\}$ are elements of the Coxeter orbit of λ_c with a, b and c being different labels chosen from 1,3 or 4. This means that the non-zero topological charges of $q^{(22)}, q^{(aa)}$ and $q^{(ab)}$ above lie in the following fundamental representations,

$$\begin{aligned} \{q^{(22)}\}, \{q^{(aa)}\} &\in \mathcal{R}_{\lambda_2}, \\ \{q^{(ab)}\} &\in \mathcal{R}_{\lambda_c}. \end{aligned}$$

Moreover, using the Coxeter element above, one can show for example

$$\lambda_2 + \omega^2 \lambda_2 = \alpha_2,$$

and since α_2 lies in the Coxeter orbit of λ_2 then by PRV conjecture $\mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_2} \otimes \mathcal{R}_{\lambda_2}$. Also,

$$\lambda_a + \omega\lambda_a = \lambda_2,$$

thus directly one sees that $\mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$. Finally,

$$\lambda_a + \omega^4 \lambda_b = \frac{1}{2}(\alpha_a + \alpha_b),$$

but it is already found that $\frac{1}{2}(\alpha_a + \alpha_b)$ is an element of the Coxeter orbit of λ_c, thus $\mathcal{R}_{\lambda_c} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_b}$. Furthermore, the zero topological charges are in fact belonging to the singlet component of this Clebsch-Gordan decomposition, since

$$\lambda_j + \omega^3 \lambda_j = 0 \qquad \text{for } j = 1, 2, 3, 4.$$

However, note that the choice of Coxeter element (3.5.41) is not unique, for example one can also use the natural ordering of the Coxeter element and still be able to prove the Clebsch-Gordan decomposition property.

To summarize, the representation space where these topological charges lie is as follows,

$$\begin{aligned} \{q^{(22)}\} &\in \mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_2} \otimes \mathcal{R}_{\lambda_2}, \\ \{q^{(aa)}\} &\in \mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}, \\ \{q^{(ab)}\} &\in \mathcal{R}_{\lambda_c} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_b}. \end{aligned} \qquad (3.5.42)$$

In fact, the non-zero topological charges are exactly same as those of the topological charges of single solitons associated with the fundamental representation in the irreducible component above. Furthermore, these Clebsch-Gordan decomposition property are in agreement with the fusing rule of solitons [13, 46].

The fact that these non-zero topological charges and its fundamental representation agrees with the soliton fusing, explains the results that the non-zero topological charges of the 22-breathers are not a simple sum of the topological charges of its constituent solitons. To be precise, these non-zero topological charges are actually *the* topological charges of the single soliton associated with the middle node of the Dynkin diagram. Furthermore, it was noted that the soliton associated with the middle node of the Dynkin diagram can coupled to itself. Thus, this phenomena leads one to think of the non-zero topological charge 22-breather as an excited single soliton with mass (energy) higher than the ordinary single soliton. All other non-zero topological charge breathers can be thought as an excited single soliton of the appropriate species determined by the Clebsch-Gordan decomposition or fusing rule above.

Finally, one also finds that there are no sine-Gordon embedded solitons in the $d_4^{(1)}$ theory. This agrees with the fact that there is no reduction of $d_4^{(1)}$ to $a_1^{(1)}$ [56]. On the other hand, the zero topological charge breathers can be thought as a sine-Gordon *like* breathers as the type B breathers of $a_n^{(1)}$ series.

3.5.4 Remarks on Other Theories

As seen in previous calculations of $a_n^{(1)}$ and $d_4^{(1)}$ breathers, one can only construct breather solutions provided the parameters η, ξ and rapidity $\bar{\Theta}$ are restricted to certain ranges of definition. To obtain these restrictions, one has to evaluate the complete τ-function and not only looks at its asymptotic behaviour. Thus, one needs to know the two soliton τ-functions, these can be derived using Hirota's methods or alternatively using Olive et.al.'s algebraic construction.

In the algebraic construction of the τ-functions, the crucial step is in writing down the higher level representation as a tensor product of level 1 representations or its descendent. The example of (3.5.2) will be sufficient for all fundamental representation of level 2 associated with the nodes in the Dynkin diagram which are next to the end nodes. In particular, the $d_5^{(1)}$ case can also be calculated using (3.5.2). For other theories, these tensor product representations of level 1 fundamental representations may not easily be found.

From the two soliton τ-function, one makes an analytic continuation of the real rapidity into a complex rapidity (or from real velocity into a complex velocity) to obtain the breather τ-function. The evaluation of the behaviour of the τ-functions is performed by separating out the real part and imaginary part explicitly. However, as the treatment of $d_4^{(1)}$ breathers has shown, these evaluations are not readily calculable. Thus, one may make a concession by opting to a simplified restriction such as setting $\eta_- = 0$. With this choice, one can find the range of definition for the allowed rapidities. Although the restriction on individual parameter ξ of the constituent solitons are not known, the topological charge parameters ξ_+ are restricted into a range of definition which is divided into several sectors, hence giving the number of different topological charge of the breather solution.

Finally, the topological charges carried by these breather solutions can be calculated using the same procedure as already shown for the previous $a_n^{(1)}$ and $d_4^{(1)}$ cases. It is expected that the Clebsch-Gordan decomposition or, specifically, the fusing rule will agree with the fundamental representation space where these topological charges lie. And thus, will give a further support to the soliton fusing and the existence of an excited soliton, i.e. single soliton with higher mass (energy) such that its starts to *vibrate*, some call this the *breathing solitons.* It can be conjectured also that the breathers whose constituent solitons are associated with conjugate (or self conjugate) fundamental representation space always carry zero topological charge as well as the possible non-zero topological charges permitted by the fusing rule. Examples of this conjecture are the type A breathers of $a_{2n+1}^{(1)}$ series with constituent solitons coming from the $(n+1)$ fundamental representations, the type B breathers of $a_n^{(1)}$ series and jj-breathers of $d_4^{(1)}$ theory (where $j = 1, 2, 3, 4$). Since the *longest* Weyl word which sends a positive weight to a negative weight always exists, then this can be used to show the singlet component of the Clebsch-Gordan decomposition. Thus, the above conjecture is in line with the PRV conjecture which is used in previous calculations.

Chapter 4

Coxeter Transformation, Root Space and S Matrix

Take Five

Paul Desmond

Dorey in [21, 23] has shown that, all the proposed S-matrices of the simply-laced affine Toda theories of [8, 9] can be constructed using information extracted from the root space of the underlying Lie algebra. Some related results can also be found in [24]. In this algebraic construction of the S-matrix, a special element of the Weyl group plays a major rôle, namely the Coxeter element (sometimes also called Coxeter transformation). This chapter is a review of the algebraic S-matrix of Dorey, preceded by a review of the general Coxeter transformation which constitutes the main part of this chapter.

A Coxeter transformation is a product of all reflections which form the basis of a Weyl group. In the Lie algebra cases, the basis of the associated Weyl Group is just the set of Weyl reflections with respect to the simple roots. As a linear transformation, the Coxeter element has a *finite* order h, where h is commonly named the Coxeter number. Acting on roots, the Coxeter transformation yields rank-r distinct orbits which consist of h elements [58]. In the case of affine algebras, one can also construct an *affine* Coxeter transformation as the product of all reflections with respect to the simple roots of the associated affine Kac-Moody algebra. Thus, the affine Coxeter transformation is just the ordinary Coxeter transformation for the simple Lie algebra g, augmented by a reflection with respect to the extended simple root of the *affine* Kac-Moody algebra $\hat{g}$. But the resulting transformation has very different properties compared with those of the ordinary Coxeter transformation. Some of these properties will be discussed in this chapter. As an affine Coxeter transformation is more general than the *ordinary* Coxeter transformation, the main discussion of this chapter will concentrate on the affine cases and almost all results are also applicable to the ordinary Coxeter transformation of the Weyl group of a simple Lie algebra unless stated otherwise.

After the introduction of the relation between the Coxeter transformation with the Killing matrix, the eigenvalue spectrum of the affine Coxeter transformation and its exponents is discussed. It will also be shown that the eigenvectors of the affine Coxeter transformation are related to the eigenvectors of the affine Cartan matrix. The order of affine Coxeter transformation is found to be infinite. Restricting oneself on the algebras with bicolouring of its affine Dynkin diagram, orbits of the affine Coxeter transformation are explored.

The discussion of the affine Coxeter transformation follows the work of Coleman [59] which is applicable to any Kac-Moody algebra, while the work of Berman et.al. [60], is restricted to algebras which have bicolouring of their simple roots, i.e. excluding the $a_{2n}^{(1)}$ series.

4.1 Coxeter Transformation and Killing Matrix

For an arbitrary Coxeter transformation one can define a *Killing Matrix* through components of the associated Cartan matrix. As a result of this relation, the spectrum of the Coxeter transformation will be given by the characteristic equation of the Killing matrix.

4.1.1 Preliminaries

This preliminary section explains the terms used in the following discussions. Let $\hat{C}$ be an $(r+1) \times (r+1)$ matrix of zero or positive determinant with integer matrix elements c_{ij} and with diagonal entries 2. A non-zero off-diagonal entry implies that its transpose entry is also non-zero.

$$\begin{array}{l} c_{ij} \in \mathbb{Z},\ c_{ii} = 2, \\ \text{If } i \neq j,\ c_{ij} \neq 0 \Leftrightarrow c_{ji} \neq 0,\ a_{ij} \leq 0, \\ |\hat{C}| \geq 0. \end{array} \tag{4.1.1}$$

Thus, the affine Cartan matrix defined by

$$\hat{C}_{jk} = \frac{2a_j \cdot a_k}{a_j^2}, \tag{4.1.2}$$

is a subset of the above general matrix $\hat{C}$. Note that this definiton of Cartan matrix is the transpose of the Cartan matrix used in the algebraic soliton solutions of *Section 2.4*. Henceforth, in the remaining of the thesis, definition (4.1.2) will be used.

Let the matrix $\hat{C}$ be partitioned as,

$$\hat{C} = C_- + C_0 + C_+, \tag{4.1.3}$$

where $C_0 = 2I$ is the diagonal and C_- and C_+ are the lower and the upper triangular parts, respectively.

For a non-symmetric affine Cartan matrix, there is a diagonal matrix D, with non-zero diagonal entries given by

$$(D)_{ij} = a_i^2 \delta_{ij},$$

such that $D\hat{C} = B$ where B is a *symmetric* matrix.

For obvious reasons, a Dynkin diagram is called a *cycle* or a *tree* if it does or does not contain a cycle respectively. Thus, Dynkin diagram of a simple Lie algebra is always a *tree*. The ordering of the labels on the nodes in any of the Dynkin diagrams can be arbitrary, i.e. labels of the n nodes of a Dynkin diagram can be taken from any permutation of the integers $1, 2, \ldots, n$.

To a node i in a *tree* Dynkin diagram, one can assign two integers u_i and v_i such that

$$u_i + v_i = 1. \tag{4.1.4}$$

And if i and j are adjacent nodes, then

$$u_i + v_j = 0 \quad \text{or} \quad 2 \tag{4.1.5}$$

when the labels going from i to j is falling or rising respectively. Examples of the labelling for the *tree* Dynkin diagram are given in *figure 4.1*.

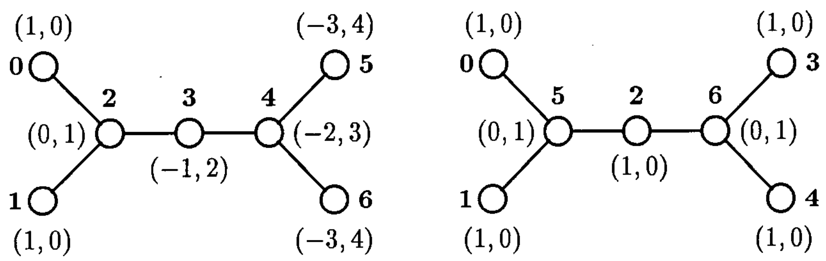

Figure 4.1: Affine Dynkin diagrams of $d_6^{(1)}$ with two different labellings denoted by the bold numbers. The pairs of numbers denote (u_i, v_i).

In a *cycle* Dynkin diagram, i.e. the $a_n^{(1)}$ affine Kac-Moody algebra cases, one can chose any two adjacent nodes, say i and j, and call the link between them the *special link*. Removing this *special link* yields a *tree* Dynkin diagram of the A_{n+1} Lie algebra. Assign to this *tree*

Dynkin diagram, the integers u_j and v_j starting from the node j. Count the rises and falls of the labels starting from the label j going around the *cycle* towards the label i and back to label j again. Let ν denote the difference between these falls and rises, then

$$u_i + v_j = \nu \quad \text{or} \quad \nu + 2 \tag{4.1.6}$$

according to a fall or rise of the labels i to j respectively. Examples are given in *figure 4.2*.

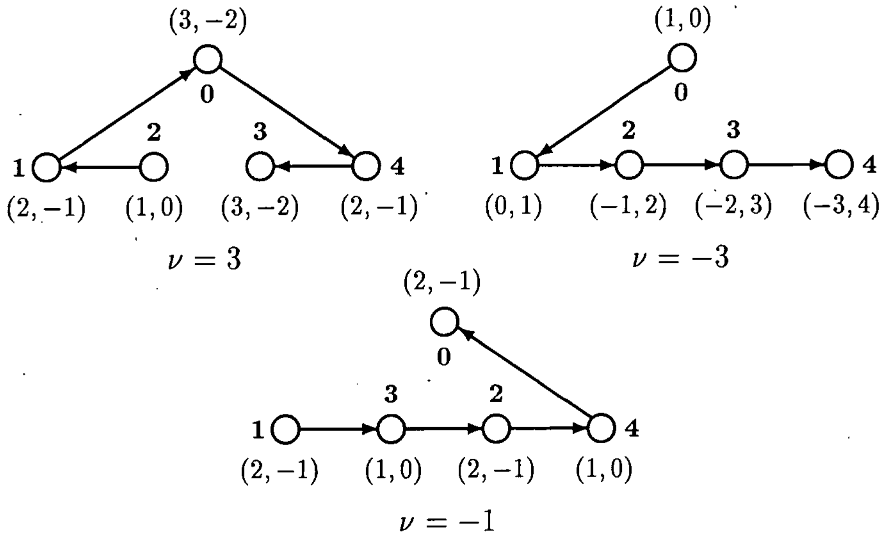

Figure 4.2: Affine Dynkin diagrams of $a_4^{(1)}$ with two different labellings denoted by the bold numbers. For clarity, the *special link* has been removed. The pairs of numbers denote (u_i, v_i) and are calculated in the direction of the arrows.

4.1.2 Relation of Coxeter Transformation and Killing Matrix

Consider an $(r+1)$-dimensional root vector space with basis a_i. A *Weyl* reflection of a_j with respect to the basis a_i is defined by,

$$r_i a_j = a_j - c_{ij} a_i, \tag{4.1.7}$$

with c_{ij} is the Cartan matrix element. For an arbitrary vector $\mathbf{x} = \sum_j x^j a_j$, one obtains

$$r_i \mathbf{x} = \mathbf{x} - c_{ij} x^j a_i,$$

where summation over j is implied. Thus, only the a_i component of $\mathbf{x}$ changes by $\sum_j c_{ij}x^j$. From the affine Cartan matrix one can define the matrix $(C_i)_{kj} = \delta_{ik}c_{ij}$, i.e. everywhere is zero except the i-th row. Then by writing $\mathbf{x}$ as a column vector, one can write the Weyl reflection as a matrix transformation operator,

$$r_i\mathbf{x} = (I - C_i)\mathbf{x}. \tag{4.1.8}$$

An affine Coxeter transformation is defined as the product of all *Weyl* reflections with respect to the simple roots of the associated affine Kac-Moody algebra $\hat{g}$,

$$\hat{\omega} = \prod_{all\ simple\ roots} r_i. \tag{4.1.9}$$

The ordering sequence of this product can be arbitrary, just like the arbitrariness in the labels of an affine Dynkin diagram. In the affine cases, this ordering may lead to different spectrum of eigenvalues. Consider an ordering which corresponds to the *natural* ordering of the affine Dynkin diagram, i.e. labelling the nodes from 0 to r in consecutive manner. Thus, the affine Coxeter transformation is given as,

$$\hat{\omega} = \prod_{i=0}^{r} r_i = r_0 r_1 \dots r_r. \tag{4.1.10}$$

In finding the spectrum of an affine Coxeter transformation, one needs to solve the *characteristic equation* of $\hat{\omega}$,

$$\hat{\omega}(\mu) = \mu I - \hat{\omega}. \tag{4.1.11}$$

which is obtained by setting the determinant of $\hat{\omega}(\mu)$ equal to zero, $|\hat{\omega}(\mu)| = 0$. Writing equation (4.1.10) in matrix form yields,

$$\begin{aligned} \hat{\omega} &= (I - C_0)(I - C_1)\dots(I - C_r) \\ &= I - \sum_{i=0}^{r} C_i + \sum_{i_1<i_2} C_{i_1}C_{i_2} - \sum_{i_1<i_2<i_3} C_{i_1}C_{i_2}C_{i_3} \\ &\quad + \dots + (-1)^r C_1 C_2 \dots C_r. \end{aligned} \tag{4.1.12}$$

Let the matrix $C_{i_1}C_{i_2}\dots C_{i_p}$ be written as $B_{i_1 i_2 \dots i_p}$. Then it is not difficult to see that,

$$\begin{aligned} C_+ B_{i_1 i_2 \dots i_p} &= B_{(i_1-1)i_1 i_2 \dots i_p} + B_{(i_1-2)i_1 i_2 \dots i_p} + \dots + B_{0 i_1 i_2 \dots i_p}, \\ C_+ B_{012\dots r} &= 0, \end{aligned}$$

where C_+ is the upper triangular matrix of the affine Cartan matrix. Define the *Killing* matrix $K(\mu)$, and the T matrix by,

$$K(\mu) = \mu C_+ + (\mu+1)I + C_- \tag{4.1.13}$$

$$T = I + C_+. \tag{4.1.14}$$

Then the relation between the eigenvalue problem of the Coxeter transformation (4.1.11) and the Killing matrix is as folllows,

$$T\hat{\omega}(\mu) = \mu(I + C_+) + I + C_- = K(\mu). \tag{4.1.15}$$

By definition, the determinant of the matrix T is equal to one, $|T| = 1$. Thus, setting the characteristic equation of the Killing matrix equals to zero is the same as setting the characteristic equation of the affine Coxeter transformation equals to zero,

$$|K(\mu)| = 0 \Longrightarrow |\hat{\omega}(\mu)| = 0. \tag{4.1.16}$$

Hence, solving the characteristic equation of the Killing matrix, which is much easier to construct by the definition (4.1.13), gives the spectrum of the affine Coxeter transformation.

If the affine Cartan is symmetric, i.e. $\hat{C} = \hat{C}^t$ then $C_-^t = C_+$. From equation (4.1.13) one has,

$$\mu K(\mu^{-1}) = C_+ + (1+\mu)I + \mu C_-.$$

And taking the determinant of the above equation yields,

$$\mu^{r+1}|K(\mu^{-1})| = |C_+ + (1+\mu)I + \mu C_-| = |K(\mu)|, \tag{4.1.17}$$

where the last equality is evaluated by taking a transpose. This means that the characteristic function $|K(\mu)|$ is reciprocal (see *Appendix* **B**). If $\hat{C}$ is symmetrisable by D, i.e. $D\hat{C} = B$, then

$$DK(\mu) = \mu B_+ + (1+\mu)D + B_-.$$

Using the same argument as above, one sees that $|DK(\mu)|$ is also a reciprocal polynomial. And since $|D| \neq 0$ then $|K(\mu)| = 0$ is a reciprocal equation. Thus,

for all Lie algebras and affine Kac-Moody algebras, $|K(\mu)|$ is reciprocal.

The fact that $|K(\mu)|$ is reciprocal will be used to show that the order of the corresponding affine Coxeter transformation is *infinite*.

4.2 The Spectrum and Exponents

The eigenvalues of an ordinary Coxeter transformation and of an affine Coxeter transformation are found to be some power of a root of unity; these powers are called the *exponents* of the Coxeter transformation. Further, it is easily shown that the eigenvectors of the Coxeter transformation (at least for the *tree cases*) are related to the eigenvectors of the corresponding Cartan matrix. But, as mentioned in the previous section, the ordering sequence of Weyl reflections in an affine Coxeter transformation can be arbitrary. This arbitrariness may lead to several different spectra of the affine Coxeter transformation.

Starting from the natural ordering of the affine Dynkin diagram, consider a permutation σ^{-1} of the labels. And from equation (4.1.9), $\hat{\omega}^\sigma$ is meant to denote the following affine Coxeter transformation

$$\hat{\omega}^\sigma = \prod_{i=0}^{r} r_{\sigma^{-1}(i)}. \tag{4.2.1}$$

Define, $c_{ij}^\sigma = c_{\sigma^{-1}(i)\sigma^{-1}(j)}$ such that $\hat{C}^\sigma = (c_{ij}^\sigma) = C_-^\sigma + 2I + C_+^\sigma$. Then, equation (4.1.15) becomes: $T^\sigma \hat{\omega}^\sigma(\mu) = K^\sigma(\mu)$.

Permutation of the natural ordering labels of the affine Dynkin diagram by σ^{-1} results in the following assignment for the pairs of numbers (u_i, v_j). Let $i = \sigma(a)$, $j = \sigma(b)$, then for the *tree* Dynkin diagram,

$$\begin{array}{ll} u_b + v_b = 1 & \text{for all b} \\ u_a + v_b = 0 & \text{if } \sigma(a) > \sigma(b) \\ u_a + v_b = 2 & \text{if } \sigma(a) < \sigma(b) \end{array} \tag{4.2.2}$$

and in a cycle with difference of rises and falls equals to ν, one also has for the nodes link by the *special link*

$$\begin{array}{ll} u_a + v_b = \nu & \text{if } \sigma(a) > \sigma(b) \\ u_a + v_b = \nu + 2 & \text{if } \sigma(a) < \sigma(b). \end{array} \tag{4.2.3}$$

Write the Killing matrix as $[K(\mu)]_{ij} = k_{ij}$, so that

$$\begin{array}{ll} k_{ij} = \mu c_{ij} & \text{for } i < j \\ k_{ij} = c_{ij} & \text{for } i > j \\ k_{ii} = \mu + 1. & \end{array}$$

Set $\zeta^2\mu = 1$, and define diagonal matrices $U = (u_{ab})$ and $V = (v_{ab})$ as

$$u_{ab} = \zeta^{u_a}\delta_{ab}, \qquad v_{ab} = \zeta^{v_a}\delta_{ab}. \tag{4.2.4}$$

To examine the possibility of several spectral classes, one utilizes the following $F^\sigma(\zeta)$ matrix,

$$[F^\sigma(\zeta)]_{ij} = f^\sigma_{ij} = [U^\sigma K^\sigma(s) V^\sigma]_{ij} = u_{aa}k_{ab}v_{bb}. \tag{4.2.5}$$

As will be seen shortly, the advantage of using the $F^\sigma(\zeta)$ matrix is that for a *tree* Dynkin diagram the matrix $F^\sigma(\zeta)$ has a dependence on ζ only in the diagonal.

4.2.1 The Trees

Consider first the cases when the affine Dynkin diagram is a *tree*, i.e. all cases of affine Kac-Moody algebras excluding $a_n^{(1)}$ cases. With equation (4.2.2), $F^\sigma(\zeta)$ in terms of its components is given as,

$$\begin{array}{lll} i < j & \text{or} & \sigma(a) < \sigma(b) \Rightarrow f^\sigma_{ij} = \zeta^{u_i+v_j}\mu c^\sigma_{ij} = c^\sigma_{ij} \\ i > j & \text{or} & \sigma(a) > \sigma(b) \Rightarrow f^\sigma_{ij} = c^\sigma_{ij} \\ i = j & \text{or} & \sigma(a) = \sigma(b) \Rightarrow f^\sigma_{ii} = \zeta + \zeta^{-1}. \end{array} \tag{4.2.6}$$

It is clear that changing $K^\sigma(\mu)$ into $F^\sigma(\zeta)$ results to a dependence on μ or ζ only in the diagonal.

Note that from equations (4.2.2) and (4.2.4) one has $UV = U^\sigma V^\sigma = \zeta I$. By definition of the Killing matrix, equation (4.1.13), and the characteristic equation $|K(\mu)| = 0$, one notice that μ cannot be equal to zero, hence ζ is finite and $\zeta \neq 0$. Thus, the characteristic equations of $F^\sigma(\zeta)$ and $K^\sigma(\mu)$ imply each other,

$$|F^\sigma(\zeta)| = 0 \Leftrightarrow |K^\sigma(\mu)| = 0. \tag{4.2.7}$$

But, $F^\sigma(\zeta)$ can be obtained from $F(\zeta)$ by permuting the rows and columns using σ. Therefore there is a permutation matrix P such that $F^\sigma(\zeta) = PF(\zeta)P^{-1}$, i.e. $F^\sigma(\zeta)$ is similar to $F(\zeta)$, thus the zeros of $|F^\sigma(\zeta)|$ and $|F(\zeta)|$ are the same. Hence,

> *all permutation of reflections in Coxeter transformation of a tree Dynkin diagram have the same spectrum.*

Since for a *tree*, $\hat{C}$ is symmetrisable, then $\hat{C}$ and $\hat{C}^t$ will give the same characteristic equation of $K(\mu)$. Thus, dual-pairs of affine Kac-Moody algebras (and simple Lie algebras) will have the same characteristic equation of $K(\mu)$. For the case $a_{2n}^{(2)}$, the affine Cartan matrix is obtained from the affine Cartan matrix of $c_n^{(1)}$ case by interchanging two rows and two columns. Hence the characteristic equation of $K(\mu)$ for $a_{2n}^{(2)}$ cases is the same as for $c_n^{(1)}$ cases, since the negative factor multiplying the determinant comes in twice.

The characteristic functions $K(\mu)$ for *tree* cases of the Kac-Moody algebras are listed in *Table 4.1*, which is reproduced from Coleman [59]. And for the simple Lie algebras are listed in *Table 4.2*.

Algebras	Characteristic functions	Exponents
$b_n^{(1)}, a_{2n-1}^{(2)}$	$(\mu^2-1)(\mu^{n-1}-1)$	$0,2,4,\ldots,2(n-1),(n-1)$ $0,1,2,\ldots,(n-1),\frac{n-1}{2}$
$c_n^{(1)}, d_{n+1}^{(2)}, a_{2n}^{(2)}$	$(\mu-1)(\mu^n-1)$	$0,1,2,\ldots,n$
$d_n^{(1)}$	$(\mu+1)(\mu^2-1)(\mu^{n-2}-1)$	$0,1,2,\ldots,(n-2),\frac{n-2}{2},\frac{n-2}{2}$ $0,2,4,\ldots,2(n-2),(n-2),(n-2)$
$e_6^{(1)}$	$(\mu+1)(\mu^3-1)^2$	$0,2,2,3,4,4,6$
$e_7^{(1)}$	$(\mu+1)(\mu^3-1)(\mu^4-1)$	$0,3,4,6,6,8,9,12$
$e_8^{(1)}$	$(\mu+1)(\mu^3-1)(\mu^5-1)$	$0,6,10,12,15,18,20,24,30$
$f_4^{(1)}, e_6^{(2)}$	$(\mu^2-1)(\mu^3-1)$	$0,2,3,4,6$
$g_2^{(1)}, d_4^{(3)}$	$(\mu-1)(\mu^2-1)$	$0,1,2$

Table 4.1: Characteristic functions of $K(\mu)$ and exponents for *tree* cases of affine Kac-Moody algebras (in $b_n^{(1)}$ and $d_n^{(1)}$ series the first sets of exponents are for n even and the second set are for n odd).

Looking at the tables above, one sees that μ is a root of unity. This means that it can be written as,

$$\mu_s = \exp\left[2i\pi\frac{s}{m}\right] \tag{4.2.8}$$

where $s, m \in \mathbb{Z}$ (for the simple Lie algebra cases m is denoted differently by h, the Coxeter number). The integers s are called the *exponents* of the Coxeter transformation. For the affine Kac-moody algebra cases, m is the largest number in the list of exponents in *Table 4.1*. But contrary to the affine cases, in the simple Lie algebra cases $\mu = 1$ is not an eigenvalue of the Coxeter transfromation, thus 0 is not an exponent. Further, h is always one more than the highest exponent.

Algebras	Characteristic functions	Exponents	h
A_n	$(\mu^n + \mu^{n-1} + \ldots + \mu + 1)$	$1, 2, 3, \ldots, n$	$(n+1)$
B_n, C_n	$(\mu^n + 1)$	$1, 3, 5, \ldots, (2n-1)$	$2n$
D_n	$(\mu^n + \mu^{n-1} + \mu + 1)$	$1, 3, 5, \ldots, (2n-3), (n-1)$	$2n-2$
E_6	$(\mu^6 + \mu^5 - \mu^3 + \mu + 1)$	$1, 4, 5, 7, 8, 11$	12
E_7	$(\mu^7 + \mu^6 - \mu^4 - \mu^3 + \mu + 1)$	$1, 5, 7, 9, 11, 13, 17$	18
E_8	$(\mu^8 + \mu^7 - \mu^5 - \mu^4 - \mu^3 + \mu + 1)$	$1, 7, 11, 13, 17, 19, 23, 29$	30
F_4	$(\mu^4 - \mu^2 + 1)$	$1, 5, 7, 11$	12
G_2	$(\mu^2 - \mu + 1)$	$1, 5$	6

Table 4.2: Characteristic functions of $K(\mu)$, exponents and Coxeter number for simple Lie algebras.

A further examination shows that, there is no straightforward relation between the exponents of the affine Coxeter transformation and the exponents of the Coxeter transformation. This relation may come from the relation of the characteristic equation of the Killing matrix for the simple Lie algebra g, $K^g(\mu)$, and the characteristic equation of the Killing matrix for the associated affine algebra $\hat{g}$, $K^{\hat{g}}(\mu)$. Generally one has,

$$|K^{\hat{g}}(\mu)| = (\mu+1)|K^g(\mu)| + \sum_{i=1}^{r}(-1)^{i+1}\mu c_{i0}|\widetilde{K^{\hat{g}}_{i0}}|, \tag{4.2.9}$$

where the extended root is assigned to α_0, c_{i0} is the entry of the affine Cartan matrix at row i and column 0; $|\widetilde{K^{\hat{g}}_{i0}}|$ denotes the *minor* of $K^{\hat{g}}_{i0}$.

A relation between the eigenvalues and eigenvectors of the affine Coxeter transformation and the eigenvalues and eigenvectors of the affine Cartan matrix can be established using the following *incidence* matrix,

$$M^\sigma = 2I - \hat{C}^\sigma = -C^\sigma_- - C^\sigma_+. \tag{4.2.10}$$

Hence with $X = \zeta + \zeta^{-1}$, equation (4.2.6) becomes

$$F^\sigma(\zeta) = XI - M^\sigma. \tag{4.2.11}$$

From elementary matrix algebra, one knows that $|K^\sigma(\mu)| = 0$ implies that there are non-trivial solutions for the following,

$$K^\sigma(\mu_s)\eta_s = 0, \tag{4.2.12}$$

for all μ satisfying $|K^\sigma(\mu)| = 0$. Let q_s be the null-vector for $F^\sigma(\zeta)$, $F^\sigma(\zeta)q_s = 0$, then with equation (4.2.5) and using the relation of the Killing matrix and the eigenvalue problem of the Coxeter transformation, one has $V^\sigma q_s$ as an eigenvector of $\hat{\omega}^\sigma$ with eigenvalue μ,

$$\hat{\omega}^\sigma(V^\sigma q_s) = \mu(V^\sigma q_s). \tag{4.2.13}$$

Let, $\eta_s = V^\sigma q_s$, from (4.2.11) with $\zeta + \zeta^{-1} = 2a$ one has,

$$M^\sigma q_s = 2aq_s$$

or,

$$\hat{C}^\sigma q_s = (2 - 2a)q_s. \tag{4.2.14}$$

For each μ there are two possibilities ζ and $-\zeta$, by the relation $\mu = \zeta^{-2}$. This means that, for each μ there are a pair of eigenvalues, $2a$ and $-2a$, of M^σ. Thus eigenvalues of $\hat{C}^\sigma$ come in pairs,

$$\begin{aligned} &\hat{C}^\sigma q_s = 2(1 - \cos(\pi \tfrac{s}{m}))q_s \\ \text{and}\quad & \\ &\hat{C}^\sigma \tilde{q}_s = 2(1 + \cos(\pi \tfrac{s}{m}))\tilde{q}_s, \end{aligned} \tag{4.2.15}$$

with $q_s = V^\sigma(\zeta)^{-1}\eta_s$ and $\tilde{q}_s = V^\sigma(-\zeta)^{-1}\eta_s$.

Since the values of s are always less than m, then the angles in equation (4.2.15) lie in the first and second quadrant. Thus, one can write equation (4.2.15) compactly as,

$$\hat{C}^\sigma q_s = 4\sin^2(\frac{\pi s}{2m})q_s. \tag{4.2.16}$$

As already noted, the eigenvectors q_s of $\hat{C}^\sigma$, equation (4.2.14), and the eigenvectors η_s of $\hat{\omega}^\sigma$, equation (4.2.13), are related by the matrix V^σ,

$$\eta_s = V^\sigma q_s. \tag{4.2.17}$$

In the simple Lie algebra cases, inserting $\mu = 1$ in equation (4.1.13) yields the Cartan matrix, $K^g(1) = C$. But it is well known that $|C| > 0$, thus $\mu = 1$ cannot be an eigenvalue of the Coxeter transformation. In other words, there is no vector invariant under the Coxeter transformation. Furthermore, the relations between eigenvalues and eigenvectors of the Coxeter transformation ω^σ, and the eigenvalues and eigenvectors of the Cartan matrix C^σ, are the same as in the affine cases above.

In these *tree* cases, the affine Dynkin diagram can always be bicoloured, i.e. the nodes of affine Dynkin diagram can be coloured black and white such that the set of simple roots corresponding to black and white nodes are orthogonal in each sets. Berman et.al. [60], use this bicolouring to obtain the sets of exponents of affine Coxeter transformations and the relation between eigenvalues of an affine Cartan matrix and an affine Coxeter transformation.

4.2.2 The Cycle

Next, one considers the $a_n^{(1)}$ case which is a *cycle*. Labelling the nodes of the affine Dynkin diagram in a consecutive manner (*natural* ordering), the difference between falls and rises of the labels is $\nu = -(n-1)$. Reversing the order one gets $\nu = (n-1)$, while changing the order of two adjacent nodes will result in ν changing by 2 (see for example *figure 4.2*). So the value of ν lies between $-(n-1) \leq \nu \leq (n-1)$ although not all the integers in this range are valid possibilities for ν.

As before, set $i = \sigma(a)$, $j = \sigma(b)$ and $\zeta^2\mu = 1$, for the *special nodes* i and j, $i < j$; using equation (4.2.3) one has

$$f_{ij}^\sigma = \zeta^{\nu+2}\mu c_{ij}^\sigma = \zeta^\nu c_{ij}^\sigma. \tag{4.2.18}$$

Since $u_a + v_b = 2 + \nu$ and for all b, $u_b + v_b = 1$, then $u_b + v_a = -\nu$, hence,

$$f_{ji}^\sigma = \zeta^{-\nu} c_{ji}^\sigma. \tag{4.2.19}$$

If the special nodes are $i > j$, all the signs of ν are inverted (since the ordering is reversed).

So one gets the dependence of $F^\sigma(\zeta)$ on ζ only in the diagonal and the special nodes,

$$\begin{array}{ll} \text{diagonal} & : \zeta + \zeta^{-1} \\ (i,j) \text{ position} & : \zeta^{\nu} c^{\sigma}_{ij} \text{ or } \zeta^{-\nu} c^{\sigma}_{ji} \end{array}$$

For a certain permutation σ, changing into permutation σ^{-1} is the same as relabelling the nodes of affine Dynkin diagram backwards, so $\nu(\sigma) = -\nu(\sigma^{-1})$, further $\hat{\omega}^\sigma$ changes into $(\hat{\omega}^\sigma)^{-1}$. Due to the fact that for this cases $\hat{C}$ is symmetric then, $|K(\mu)|$ is reciprocal and hence ζ and ζ^{-1} can be interchange. Under the interchange of ζ to ζ^{-1}, $F^\sigma(\zeta)$ becomes $F^\sigma(\zeta^{-1}) = F^{\sigma^{-1}}(\zeta)$. Thus,

- $(\hat{\omega}^\sigma)$ and $(\hat{\omega}^\sigma)^{-1}$ have the same spectrum (by (4.1.16) and (4.2.7)), and
- $(\hat{\omega}^\sigma)$ corresponding to ν and $-\nu$ also have the same spectrum.

However, in general, $F^\sigma(\zeta)$ with different value of ν will lead to distinct spectrum.

For an arbitrary ν, one starts with natural ordering from node 0 to $(k-1)$ and relabels the node $k = (n - \nu + 1)/2$ with n and the following nodes in decreasing manner from n. Calculating the resulting determinant of the Killing matrix one obtains the following characteristic function [59],

$$(\mu^j - 1)(\mu^{n+1-j} - 1) \quad \text{where } 2j = \nu + n + 1. \tag{4.2.20}$$

In the following, two most obvious cases will be examined.

Bicolouring, $\nu = 0$

This case can only occur when n is odd. The characteristic function of the Killing matrix becomes,

$$(\mu^{\frac{n+1}{2}} - 1)^2. \tag{4.2.21}$$

And the exponents are,

$$0, 1, 1, 2, 2, \ldots, (\frac{n+1}{2}).$$

Since $\nu = 0$, then there is no dependence on ζ in the off-diagonal entry of the matrix $F^\sigma(\zeta)$. Hence this is exactly like the *tree* cases, thus the relation between eigenvalues and eigenvectors of the affine Coxeter transformation and the affine Cartan matrix are also given by equations (4.2.13), (4.2.15-4.2.17). These are the cases considered by Berman et.al. [60].

Natural Ordering, $\nu = n - 1$

The characteristic function of the Killing matrix for this case is,

$$(\mu - 1)(\mu^n - 1), \tag{4.2.22}$$

with the following exponents,

$$0, 1, 2, \ldots, n.$$

The dependence on $\zeta^{\pm\nu}$ in the off-diagonal entry of $F^\sigma(\zeta)$ gives rise to subtleties in relating the eigenvalues and eigenvectors of the affine Coxeter transformation with eigenvalues and eigenvectors of the affine Cartan matrix. However, the affine Cartan matrices $\hat{C}$ and $\hat{C}^\sigma$ are related by some permutation matrix P, $\hat{C}^\sigma = P\hat{C}P^{-1}$; hence the eigenvalues of $\hat{A}$ are the same with any other ordering with different ν.

4.2.3 Special cases $a_1^{(1)}$ and $a_2^{(2)}$

The characteristic function of the Killing matrix for these two very simple cases can be directly calculated from their affine Cartan matrix. These characteristic equations happen to be the same,

$$(\mu - 1)^2, \tag{4.2.23}$$

and the exponents are $\widehat{0}, 1$.

4.3 Order of Affine Coxeter Transformation

From the last discussion in *Section 4.1*, one knows that for all affine Kac-Moody algebras, the characteristic function of the Killing matrix, $|K(\mu)|$, is reciprocal (see also *Appendix* **B**). Moreover, setting μ equals to one in equation (4.1.13), one obtains $K(\mu = 1) = \hat{C}$. It is also known that the affine Cartan matrix of an affine Kac-Moody algebra has nullity one [6, 7], i.e. it has one null-vector, $\hat{C}\mathbf{x} = 0$. Multiplying the vector $\mathbf{x}$ to the Killing matrix at $\mu = 1$ one has $k_{ij}(1)x^j = 0$. Since x^j are coefficients of a linearly independent basis for $\mathbf{x}$, one can write for each i,

$$k_{ij}(\mu)x^j = (\mu - 1)h_i,$$

for some function h_i. Thus, $(\mu - 1)$ must divide the reciprocal polynomial $|K(\mu)|$. From *Appendix* **B**, one knows that the zeros of a reciprocal polynomial which are equal to 1 must have even multiplicity. Hence, taking the lowest even multiplicity, one can write $|K(\mu)|$ as,

$$|K(\mu)| = (\mu - 1)^2\phi(\mu), \tag{4.3.24}$$

where $\phi(\mu)$ is a reciprocal polynomial of order $(n - 1)$, as $(\mu - 1)^2$ is reciprocal.

Inserting $\mu = 1$ to the defining relation between an affine Coxeter transformation and the Killing matrix, equation (4.1.15), one sees that the affine Coxeter transformation $\hat{\omega}$, has only one eigenvector corresponding to eigenvalue 1. This eigenvector is the null-vector of the affine Cartan matrix. But, by equation (4.3.24), $\mu = 1$ has to have at least multiplicity of two. Hence, one does not have enough eigenvectors to diagonalize $\hat{\omega}$. Or in other words, in the Jordan canonical form of $\hat{\omega}$ there is a block with dimension greater than one. Thus, there is no $m \in \mathbb{Z}$ such that $\hat{\omega}^m = I$.

An affine Coxeter transformation, $\hat{\omega}$, has an infinite order.

For the *bicoloured* cases, one can show explicitly using the eigenvectors of the affine Cartan matrix that, the eigenvalue 1 of $\hat{\omega}$ has only one eigenvector.

In contrast to the affine cases, in the simple Lie algebra cases the Coxeter transformation has a *finite* order, h. To see this, one notes that the only possible degenerate eigenvalues of ω^σ are in the D_{2n} series which correspond to the exponents equal to $(2n - 1)$, these are doublet degeneracies. Examining the Killing matrix, equation (4.1.13), with $\mu = -1$ corresponding to these degenerate eigenvalues, yields the *nullity* of $K^g(-1)$ exactly two. Thus, there are enough eigenvectors to diagonalize ω^σ.

4.4 Orbits of Affine Coxeter Transformation

Only the cases of bicoloured affine Dynkin diagram will be considered in this section. Split the simple roots of $\hat{g}$ into two sets. Elements in each sets are simple roots which are orthogonal to each other. This is the bicolouring procedure of the affine Dynkin diagram, i.e. nodes are coloured *black* and *white* alternatingly. Label the simple roots as follows, $\bullet \doteq \{0,1,2,\ldots,k\}$ and $\circ = \{k+1,\ldots,r\}$. With an abuse of notation, $\bullet$, $\bullet'$ and $\circ$, $\circ'$ will also be used as indices of the simple roots of black or white type. Thus,

$$a_\bullet \cdot a_{\bullet'} = 0 = a_\circ \cdot a_{\circ'}.$$

Let the affine Coxeter transformation be written as,

$$\hat{\omega} = \omega_{\{\bullet\}}\omega_{\{\circ\}} \tag{4.4.25}$$

where $\omega_{\{\bullet\}} = \prod_{i\in\bullet} r_i$ and $\omega_{\{\circ\}} = \prod_{i\in\circ} r_i$ are products of reflections with respect to *black* and *white* simple roots respectively. Note that the ordering of reflections in $\omega_{\{\bullet\}}$ or $\omega_{\{\circ\}}$ can be arbitrary since within each set these reflections commute with each other.

Because of the orthogonality of the simple roots in each of the sets, $\{a_\bullet\}$ and $\{a_\circ\}$, one has the following

$$\begin{aligned} \omega_{\{\bullet\}}a_\bullet &= -a_\bullet, & \omega_{\{\circ\}}a_\circ &= -a_\circ, \\ \omega_{\{\bullet\}}a_\circ &= a_\circ - \sum_\bullet \hat{C}_{\bullet\circ}a_\bullet, & \omega_{\{\circ\}}a_\bullet &= a_\bullet - \sum_\circ \hat{C}_{\circ\bullet}a_\circ, \end{aligned}$$

and,

$$i \in \bullet \qquad \hat{\omega}a_i = -a_i - \sum_{j\in\circ}\hat{C}_{ji}a_j + \sum_{j\in\circ,k\in\bullet}\hat{C}_{kj}\hat{C}_{ji}a_k, \tag{4.4.26}$$

$$i \in \circ \qquad \hat{\omega}a_i = -a_i + \sum_{j\in\bullet}\hat{C}_{ji}a_j. \tag{4.4.27}$$

These relations are also valid for the *ordinary* Coxeter transformation of *simple* Lie algebra. Since in the *simple* Lie algebra cases, the simple roots can always be bicoloured.

Remember that $c_{ij} \leq 0$ for $i \neq j$. Suppose that after $p > 0$ applications of affine Coxeter transformations on a affine simple root a_i produce the following,

$$\hat{\omega}^p a_i = b = (\beta, 0, k) \tag{4.4.28}$$

where b is some arbitrary affine root, β is a member of the root system of the associated simple Lie algebra and $k \in \mathrm{IR}$ (see *Appendix* **A** for notation of the affine simple roots in terms of the ordinary simple roots and the *imaginary* roots). Then using (4.4.26) and (4.4.27), one sees that

$$k > 0 \text{ for } i \in \bullet \qquad \text{and} \qquad k < 0 \text{ for } i \in \circ,$$

i.e. the *imaginary* root-component of a black (white) simple root always goes to positive (negative) direction.

As in the simple Lie algebra cases, one can introduce a generalization of Kostant's representation of simple roots [58],

$$\phi_i = r_r r_{r-1} \dots r_{i+1} a_i. \tag{4.4.29}$$

Although ϕ_i is not always a simple root, nevertheless the $r+1$ vectors ϕ_i are linearly independent, thus it can be thought of as basis of the root space. Then one can see that the orbits of these representation of simple roots are *distinct* from each other. To see this, one can utilize the trick by Dorey [21, 23]. Recall that for the fundamental weight Λ_j defined by $\frac{2a_j \cdot \Lambda_k}{a_j^2} = \delta_{jk}$, one has

$$r_j \Lambda_k = \Lambda_k - \delta_{jk} a_j,$$

then,

$$\phi_j = (1 - \hat{\omega}^{-1}) \Lambda_j. \tag{4.4.30}$$

Now suppose that $\hat{\omega}^p \phi_j = \phi_k$ then one has,

$$\hat{\omega}^p \Lambda_j = \Lambda_k,$$

which is not true since all fundamental weights are dominant highest weight which are not related by any Weyl element to each other [61].

Looking back at equation (4.2.8), one suspects that after m *rotations*, the simple root will come back to itself. But actually, it is only the *euclidean* part of the root which returns to itself, i.e. m applications of affine Coxeter transformations on the affine simple root a_i (see *Appendix* **A** for notation of the affine simple roots) yields

$$\hat{\omega}^m a_i = (\alpha_i, 0, k), \qquad k \in \mathrm{IR}. \tag{4.4.31}$$

Because the *imaginary*-direction always grows, one obtains an infinite orbit, although the *euclidean*-part has a finite order, m. The finite order of the euclidean-part of the affine root was proved by Steinberg [62], which can be stated as the following.

The euclidean part of $\hat{\omega}$ is,

$$\bar{\omega} = r_\psi r_0 \hat{\omega}. \tag{4.4.32}$$

Let β be the fork node in an affine Dynkin diagram, i.e. the node correspond to the long simple root which has multiple links or the $(n+1)$ node in the $a_{2n+1}^{(1)}$ cases. Further, if $D(g)$ is the Dynkin diagram of the simple Lie algebra g associated with the Kac-Moody algebra $\hat{g}$, then deleting the node β from $D(g)$ will yield a disconnected Dynkin diagram of $D_\beta(g)$. The Coxeter transformation $\bar{\omega}_\beta$ associated with the Weyl group of $D_\beta(g)$ can be obtained from $\bar{\omega}$,

$$\bar{\omega}_\beta = \sigma \bar{\omega} \sigma^{-1}, \tag{4.4.33}$$

where σ is an element of the Weyl group which sends the longest root of g to $-\beta$,

$$\sigma \psi = -\beta. \tag{4.4.34}$$

Obviously, $\bar{\omega}_\beta$ has a finite order since it is a Coxeter element of disconnected A_n type Dynkin diagrams. Moreover, following Steinberg [62], one can show that the order of $\bar{\omega}_\beta$ is exactly m as listed in *Table 4.1.* Thus, the euclidean part of the affine Coxeter transformation $\bar{\omega}$, has a finite order of m.

Next, one can also show that the action of $t_\psi \equiv r_0 r_\psi$ is a translation in the imaginary direction $\delta = (0,0,1)$,

$$t_\psi x \;=\; r_0 r_\psi x \;=\; x - \frac{2a_0 \cdot x}{a_0^2} a_0 - \frac{2\psi \cdot x}{\psi^2}\psi + \frac{2\psi \cdot x}{\psi^2}\frac{2a_0 \cdot \psi}{a_0^2} a_0. \tag{4.4.35}$$

With $\frac{2\psi \cdot x}{\psi^2} = -\frac{2a_0 \cdot x}{a_0^2}$ and $\frac{2a_0 \cdot \psi}{a_0^2} = -2$, one obtains

$$t_\psi x = x - \frac{2\psi \cdot x}{\psi^2}\delta. \tag{4.4.36}$$

Thus with $x = (\gamma, 0, k)$ after m applications of affine Coxeter transformation $\hat{\omega}$ one has,

$$\hat{\omega}^m x = (\gamma, 0, k - \frac{2\psi \cdot x}{\psi^2} m). \tag{4.4.37}$$

As an example, one looks at the $a_1^{(1)}$ case which has only two simple roots $\{a_0, a_1\} = \{(-\alpha, 0, 1), (\alpha, 0, 0)\}$. The *imaginary* root δ is given by $\delta = a_0 + a_1$, and the affine Coxeter transformation constructed from these simple roots is,

$$\hat{\omega} = r_0 r_1,$$

and its action on the simple roots and the *imaginary* root are,

$$\hat{\omega}^p a_0 = (-\alpha, 0, 1 + 2p), \qquad \hat{\omega}^p a_1 = (\alpha, 0, -2p), \qquad \hat{\omega}^p n\delta = n\delta, \tag{4.4.38}$$

for any $p \in \mathbb{Z}$.

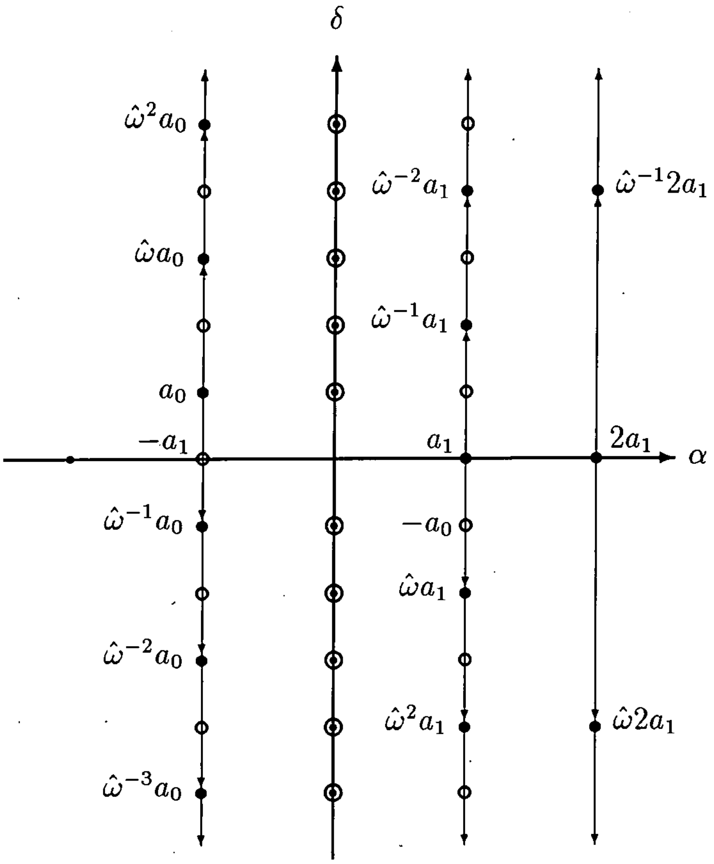

Figure 4.3: Orbits of the roots a_0, a_1 and $2a_1$. The *imaginary* roots $n\delta$ are invariant, denoted by a circle on each $n\delta$.

Thus, as mentioned above, it is clear that the orbit of the simple roots are infinite in the *imaginary* direction, while the *euclidean* part does not change, i.e. the *finite order* for the *euclidean* part is one. The jump in *imaginary* direction for each application of $\hat{\omega}$ is two steps. The *imaginary* roots themselves are invariant under the applications of the affine Coxeter transformation. Furthermore, to cover the whole root subspaces of a_0 and a_1, one needs two orbits for each subspace. For the root subspace of a_0, the orbit of the root $-a_1$ is also needed. And by changing the sign one gets the root subspace of a_1. One can visualize these orbits as in *figure 4.3.*

4.5 Affine Toda S-matrices and Root Space

It was found that, for a simply-laced affine Toda field theory, a *root triangle* constructed from elements of three *Coxeter orbits* of a representation of the simple roots, i.e. a triangle in $\mathbb{R}^r$, can be projected to another triangle in a two-dimensional plane of the conserved charge, q_s. This *conserved charges triangle* is exactly the conserved charge bootstrap relation, equation (2.1.10). This leads to *Dorey's Fusing Rule* [21] which states that, *a three-point coupling is non-vanishing if and only if there exists a root triangle associated with the three particles.* Information of the *fusing angle* can be obtained from the root triangle and the Coxeter orbit. Inserting the value of these fusing angles into an S-matrix *building block* yields the correct S-matrix.

There is more information which can be extracted from the root space and two of these features will be explained briefly in this paragraph. Firstly, in a *purely elastic* process, the selection rule forbids a non-diagonal scattering. This selection rule is related to the *flipping* of the mass quadrilateral, see [19] and [8]. The existence of the flipping of the mass quadrilateral can also be explained in terms of two root triangles having one common side, see [21]. Secondly, for the simply-laced affine Toda theories, the forward-channel (s-channel) poles are always an odd-order pole with residue of $+i$ times a positive factor, [8]. These S-matrices are built by *stacking* the S-matrix building block with an appropriate power which depend on the root representation of the particles. The possible ways of *stacking* these building blocks, in order to obtain an odd-order pole, can be explained in terms of the root representations, see [23, 24]. Some related results can also be found in [24].

In what follows, a short review of the algebraic construction of the simply-laced S-matrices [21, 23] will be considered.

It is well known that the conserved charges of the simply-laced affine Toda field theory are eigenvectors of the Cartan matrix A of the associated Lie algebra g [8, 9, 22, 26]. If one writes the conserved charge vector as $\mathbf{q}_s = \{q_s^1, q_s^2, \dots, q_s^r\}$, then the following eigenvalue relation holds, c.f. equation (4.2.16),

$$C\mathbf{q}_s = 4\sin^2(\frac{\pi s}{2h})\mathbf{q}_s, \tag{4.5.1}$$

where s and h are the exponents and Coxeter number of the Lie algebra, respectively. This fact leads one to assign to the a^{th} node in the Dynkin diagram of the Lie algebra, the a^{th} particle of the affine Toda field theory. However, from [21, 22] one realizes that actually one should associate a fundamental particle of the affine Toda theory with a Coxeter orbit, as discussed below.

Let r_i represent a Weyl reflection corresponding to the simple root α_i. It is convenient to use the equivalent representation of the simple roots as in (4.4.29), i.e.

$$\phi_i = r_r r_{r-1} \dots r_{i+1}\alpha_i.$$

The Coxeter transformation is defined as (4.4.25), i.e.

$$\omega = \omega_{\{\bullet\}}\omega_{\{\circ\}},$$

where $\bullet = 1, 2, \dots, k$ and $\circ = (k+1), \dots, r$. Thus,

$$\begin{aligned} \phi_\bullet &= \omega_{\{\circ\}}(\alpha_\bullet) = \omega_{\{\circ\}}\omega_{\{\bullet\}}(-\alpha_\bullet) = \omega^{-1}(-\alpha_\bullet), && (4.5.2)\\ \phi_\circ &= \alpha_\circ. && (4.5.3) \end{aligned}$$

The Coxeter orbit of the roots ϕ_i denoted by Γ_i are disjoint set. Since the Coxeter transformation has a finite order h, then each orbit Γ_i has r elements, see [58, 63]. This means that the union of all orbits, $\bigcup_{i=1}^r \Gamma_i$, yields the set of all hr elements of the roots of g.

Consider the eigenspace of the Coxeter transformation ω, i.e. the space spanned by η_s, where η_s is an eigenvector of ω with eigenvalue $\exp(\frac{2i\pi s}{h})$, (c.f. equation (4.2.13)) i.e.

$$\omega\eta_s = e^{\frac{2i\pi s}{h}}\eta_s, \tag{4.5.4}$$

where η_s and q_s are related by, (c.f. equation (4.2.17)),

$$\eta_s = V\mathbf{q}_s. \tag{4.5.5}$$

Here the diagonal matrix V, (c.f. equation (4.2.4)), is obtained from the assigment of integers (u_i, v_i) to the nodes of the Dynkin diagram. Setting all black nodes on the Dynkin diagram to have $v_\bullet = -1$, one has

$$V_{ij} = \begin{cases} e^{\frac{i\pi s}{h}}\delta_{\bullet\bullet'}, \\ \delta_{\circ\circ'}. \end{cases}$$

An example for the above assignment of the integers (u_i, v_i) in a Dynkin diagram can be found in *figure 4.4* in which the bicolouring of the A_4 Dynkin diagram is considered.

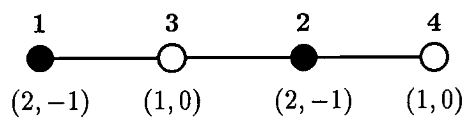

Figure 4.4: Bicolouring of the A_4 Dynkin diagram. The pairs of numbers denote (u_i, v_i).

Thus, from the above eigenvector relation, it seems that it is more accurate to associate a particle of the affine Toda theory with a Coxeter orbit instead of associating a particle with a simple root.

Consider the two-dimensional eigenspace of ω for each s with the basis,

$$a_s = \sum_\bullet \eta_s^\bullet \hat{\alpha}_\bullet,$$
$$b_s = \sum_\circ \eta_s^\circ \hat{\alpha}_\circ.$$

In the above, η_s^i with $i \in \bullet$ or $i \in \circ$ is the i^{th} component of η_s and $\hat{\alpha}$ are the dual roots of the simple roots α defined as

$$\hat{\alpha}_i \cdot \alpha_j = \delta_{ij}.$$

Using the complex coordinate representation of equations (4.5.5) and (4.5.6), it is readily seen that the angle between a_s and b_s is $\frac{i\pi s}{h} \equiv \theta_s = s\theta_1$ with $\theta_1 = \frac{i\pi}{h}$. Also, from equation

(4.5.4), one sees that the Coxeter transformation ω rotates the basis a_s and b_s by the angle $\frac{2i\pi s}{h} = 2s\theta_1$ (see also [21] and [63]).

Define a linear mapping from the $\mathbb{R}^r$ root space into this two-dimensional eigenspace as,

$$P_s(\alpha_\bullet) \equiv -\eta_s^\bullet = -e^{is\theta_1} q_s^\bullet, \quad P_s(\alpha_\circ) \equiv \eta_s^\circ = q_s^\circ, \tag{4.5.6}$$

with linear property,

$$P_s[\omega^k(\alpha_i)] = e^{\frac{2i\pi sk}{h}} \eta_s^i. \tag{4.5.7}$$

Let α, β and γ be any three roots which satisfy,

$$\alpha + \beta + \gamma = 0.$$

Write each of α, β and γ as a Coxeter transformation of a simple root. Then the mapping P_s from $\mathbb{R}^r$ root space into the two-dimensional eigenspace of s yields,

$$q_s^\alpha e^{\theta_s^\alpha} + q_s^\beta e^{\theta_s^\beta} + q_s^\gamma e^{\theta_s^\gamma} = 0,$$

which has a form of the *conserved charge bootstrap* relation, c.f. equation (2.1.10).

Let C_{ijk} be the three-point coupling of particles i, j and k. Then the following is always true for simply-laced affine Toda field theory. *Dorey's Fusing Rule*, [21]

> $C_{ijk} \neq 0$ iff $\exists$ roots $\alpha_{(i)} \in \Gamma_i$, $\alpha_{(j)} \in \Gamma_j$ and $\alpha_{(k)} \in \Gamma_k$ such that,
>
> $$\alpha_{(i)} + \alpha_{(j)} + \alpha_{(k)} = 0.$$

In the above, $\alpha_{(i)}$ is any element of the Coxeter orbit Γ_i. For example, suppose one has a root triangle from three white orbits as

$$\alpha_{(i)} + \alpha_{(j)} + \alpha_{(k)} = \omega^{p_i}(\alpha_i) + \omega^{p_j}(\alpha_j) + \omega^{p_k}(\alpha_k) = 0,$$

where α_i, α_j and α_k are white simple roots. Then, upon projection to the two-dimensional eigenplane one obtains,

$$q_s^i + q_s^j e^{i2s(p_j - p_i)\theta_1} + q_s^k e^{i2s(p_k - p_i)\theta_1} = 0. \tag{4.5.8}$$

Which is a triangle of three coplanar vectors, written in complex coordinates. Comparison with the *conserved charge bootstrap* of particles i, j and k, equation (2.1.10), leads to the identification of the fusing angles as follows,

$$\theta_{ij}^{k} = 2(p_j - p_i)\theta_1, \qquad \theta_{jk}^{i} = 2(p_k - p_j)\theta_1, \qquad \theta_{ki}^{j} = 2(p_i - p_k)\theta_1.$$

From the above relations, one sees that fusing of particles belonging to the same type (in terms of the type of their simple roots representation, black or white), always has a fusing angle which is an even multiple of θ_1. If one had started with two white simple roots and a black simple root, one would have found that the fusing of particles corresponding to different type of simple roots has a fusing angle which is an odd multiple of θ_1.

Introduce the following notation. Define a non-commutative $*$ product of two unit vectors of the projection P_s (written in complex coordinate) as follows,

$$\begin{aligned} P_s[\omega^p(\alpha)] * P_s[\omega^q(\beta)] &= e^{2isp\theta_1} e^{-2isq\theta_1} \\ &= \left(e^{2isq\theta_1} e^{-2isp\theta_1}\right)^{-1} \\ &\equiv \left(P_s[\omega^q(\beta)] * P_s[\omega^p(\alpha)]\right)^{-1}, \end{aligned} \tag{4.5.9}$$

with identity is given as,

$$P_s(\alpha) * P_s(\alpha) \equiv id.$$

Let the bracket $< f_s * g_s >$ be defined on a $*$ product of two functions as,

$$< f_s * g_s > \equiv \frac{-i}{\theta_s} \ln(f_s * g_s). \tag{4.5.10}$$

Define the integers $u(\alpha, \beta)$ mod $2h$ as

$$u(\alpha, \beta) \equiv < P_s(\alpha) * P_s(\beta) > . \tag{4.5.11}$$

Thus, it is obvious that $u(\alpha, \beta)$ counts the angle, in multiple of θ_1, between the projection of the roots α and β, in the eigenspace s of ω. Moreover, using equation (4.5.11) one has the following properties,

$$\begin{aligned} &u(\alpha, \beta) = -u(\beta, \alpha), \\ &u(\omega^p \alpha, \beta) = 2p + u(\alpha, \beta), \\ &u(\alpha, \beta) + u(\beta, \gamma) + u(\gamma, \alpha) = 0 \quad \text{for any roots } \alpha, \beta, \gamma. \end{aligned} \tag{4.5.12}$$

For the black and white roots one has,

$$\begin{aligned} &u(\alpha_\bullet, \alpha_{\bullet'}) = 0 = u(\alpha_\circ, \alpha_{\circ'}), \\ &u(\phi_\bullet, \phi_{\bullet'}) = 0 = u(\phi_\circ, \phi_{\circ'}), \\ &u(\phi_\bullet, \phi_\circ) = -1. \end{aligned} \tag{4.5.13}$$

And, since $u(\alpha_i, \alpha_j)$ is a *signed* integer, one has

$$\theta_{ij}^k \equiv \frac{\pi}{h}|u(\alpha_{(i)}, \alpha_{(j)})|.$$

With these integers as data obtained *purely* from *group-algebraic* consideration, one can write down the S-matrices of the simply-laced affine Toda field theory as follows (either case of building blocks yields the same results) [23],

$$S_{ij} = \prod_{\alpha_{(j)}\epsilon\Gamma_j} \{u(\phi_i, \alpha_{(j)}) + 1\}_{\pm}^{\lambda_i\cdot\alpha_{(j)}}, \tag{4.5.14}$$

where λ_i is the i^{th} fundamental weight and the building blocks $\{x\}_\pm$ and its properties are given as follows,

$$\begin{aligned} &(x)_+ = \sinh(\tfrac{\theta}{2} + \tfrac{i\pi x}{2h}), \\ &\{x\}_+ = \tfrac{(x-1)_+(x+1)_+}{(x-1+B)_+(x+1-B)_+}, \\ &\{x\}_- = \{x\}_+^{-1}, \\ &\{x\} = \tfrac{\{x\}_+}{\{-x\}_+} = \tfrac{\{x\}_-}{\{-x\}_-}, \\ &\{x\}_\pm = \{x \pm 2h\}_\pm, \end{aligned} \tag{4.5.15}$$

where,

$$B(\tilde{\beta}) = \frac{1}{2\pi}\frac{\tilde{\beta}^2}{1 + \frac{\tilde{\beta}^2}{4\pi}}. \tag{4.5.16}$$

This S-matrix expression of equation (4.5.14) is consistent with the S-matrix bootstrap principle. From equation (2.1.8) one has the S-matrix bootstrap relation as follows,

$$S_{l\bar{i}}(\theta) = S_{lj}(\theta - i\bar{\theta}_{ij}^k) S_{lk}(\theta + i\bar{\theta}_{ik}^j).$$

Applying the unitarity and crossing symmetry conditions, and remembering that the affine Toda S-matrices are $2i\pi$-periodic functions, yields

$$S_{li}(\theta) S_{lj}(\theta + i\theta_{ij}^k) S_{lk}(\theta - i\theta_{ik}^j) = 1. \tag{4.5.17}$$

The above S-matrix bootstrap relation has a very compelling similarity with the conserved charge bootstrap. Define the matrix T [9, 19] as,

$$T_{ab} = \frac{d}{d\theta} \ln S_{ab},$$

then equation (4.5.17) gives,

$$T_{li}(\theta) + T_{lj}(\theta + i\theta_{ij}^{k}) + T_{lk}(\theta - i\theta_{ik}^{j}) = 0. \tag{4.5.18}$$

Since $S_{ab}(\theta)$ is a $2i\pi$-periodic function, then $T_{ab}(\theta)$ is also a $2i\pi$-periodic function. Hence, $T_{ab}(\theta)$ can be expanded as a Fourier series in $i\theta$,

$$T_{ab}(\theta) = \sum_{-\infty}^{\infty} t_s^{ab} e^{is\theta}.$$

Thus a consequence of equation (4.5.18), is that for each s one has

$$t_s^{li} + t_s^{lj} e^{is\theta_{ij}^{k}} + t_s^{lk} e^{-is\theta_{ik}^{j}} = 0,$$

which resembles the conserved charge bootstrap, equation (2.1.10). Although in the above s is not necessary be an exponent of the Lie algebra.

To follow on the bootstrap, introduce the shift operator $\mathcal{T}_y$ [19], which acts distributively on products of functions, as

$$\mathcal{T}_y f(\theta) = f(\theta + \frac{i\pi y}{h}).$$

Then it is easily seen that,

$$\mathcal{T}_y \{x\}_+ = \{x + y\}_+$$
$$\mathcal{T}_y \{x\}_- = \{x - y\}_-.$$

Hence, equation (4.5.17) can be written as,

$$(S_{li})(\mathcal{T}_{u(\alpha_{(i)},\alpha_{(j)})} S_{lj})(\mathcal{T}_{u(\alpha_{(i)},\alpha_{(k)})} S_{lk})(\theta) = 1.$$

Applying $\mathcal{T}_{u(\phi_l,\alpha_{(i)})}$ and remembering that $\mathcal{T}$ acts distributively yields,

$$(\mathcal{T}_{u(\phi_l,\alpha_{(i)})} S_{li})(\mathcal{T}_{u(\phi_l,\alpha_{(j)})} S_{lj})(\mathcal{T}_{u(\phi_l,\alpha_{(k)})} S_{lk})(\theta) = 1, \tag{4.5.19}$$

where one has use equation (4.5.12) and $\alpha_{(i)}+\alpha_{(j)}+\alpha_{(k)}=0$. It is now easy to see that the building blocks of equation (4.5.14) is consistent with the bootstrap. Insert the S-matrix in terms of the $\{x\}_-$ building blocks (the positive building blocks yield the same result) to equation (4.5.19), one obtains

$$\begin{aligned}
&\prod_{p=0}^{h-1} \{u(\phi_l,\omega^{-p}\alpha_{(i)}) - u(\phi_l,\alpha_{(i)}) + 1\}_-^{\lambda_l\cdot\omega^{-p}\alpha_{(i)}} \\
&\quad \times\{u(\phi_l,\omega^{-p}\alpha_{(j)}) - u(\phi_l,\alpha_{(j)}) + 1\}_-^{\lambda_l\cdot\omega^{-p}\alpha_{(j)}} \\
&\quad \times\{u(\phi_l,\omega^{-p}\alpha_{(k)}) - u(\phi_l,\alpha_{(k)}) + 1\}_-^{\lambda_l\cdot\omega^{-p}\alpha_{(k)}} \\
&\quad = \prod_{p=0}^{h-1}\{2p+1\}_-^{\lambda_l\cdot\omega^{-p}(\alpha_{(i)}+\alpha_{(j)}+\alpha_{(k)})} \equiv 1.
\end{aligned}$$

In the above derivation, the properties in equation (4.5.12) and $\alpha_{(i)}+\alpha_{(j)}+\alpha_{(k)}=0$ have been used again.

Chapter 5

Sine-Gordon S Matrix and Root Space

It Don't Mean a Thing (If It Ain't Got That Swing)
Edward Kennedy Ellington

This chapter reports on attempts to construct the sine-Gordon soliton and breather S-matrices from the root space which mimics Dorey's S-matrix construction for real coupling affine Toda theories. As in the real coupling regime, one wants to associate the soliton and breather spectrum of the sine-Gordon theory with an orbit of a Weyl transformation in the root space. However, after several calculations, it seems that the naïve thought of associating the soliton and breather spectrum of the sine-Gordon model to the affine roots of $a_1^{(1)}$ Kac-Moody algebra does not yield a consistent way of extracting the necessary data from the root space to construct the S-matrix.

5.1 Preliminary Observations

As seen in *Chapter Four*, the S-matrices of the simply-laced affine Toda theories in the real coupling regime can be constructed by taking a product of the building block factor to some power which depends on the orbit of the Coxeter transformation of a black or white simple roots representation. Furthermore, since the orbit of a Coxeter transformation is finite, then the S-matrices are written as a finite product of the building block.

For the sine-Gordon theory, as well as the $a_n^{(1)}$ affine Toda field theory with complex coupling parameter [14], the soliton scattering matrices involve an *infinite product* of Γ-functions. Thus, one wants to relate the sine-Gordon spectrum with a root orbit under a certain transformation in root space. Furthermore, this linear transformation has to have an infinite order to be able to produce the infinite product in the S-matrix expression. In other words one needs to replace the Coxeter transformation with a new transformation which will naturally have an infinite orbit.

As the affine root system has more feature by the inclusion of the imaginary root δ, and as already seen in the previous chapter that an affine Coxeter transformation is a natural generalization of the Coxeter transformation, it is worthwhile to examine the possibility of using these affine systems for the algebraic construction of the soliton and breather S-matrices.

By considering the affine root system instead of the Euclidean root system one might speculate the following,

$C_{ijk} \neq 0$ iff $\exists$ affine roots $a_{(i)} \in \hat{\Gamma}_i$, $a_{(j)} \in \hat{\Gamma}_j$ and $a_{(k)} \in \hat{\Gamma}_k$ such that,

$$a_{(i)} + a_{(j)} + a_{(k)} = 0, \qquad a_{(i)}, a_{(j)}, a_{(k)} \text{ is non-zero}$$

where $\hat{\Gamma}_i$ is some orbit of root $a_{(i)}$ with respect to a certain *linear* transformation. One will then try to find an appropriate linear mapping from the root space, i.e. linear mapping of the orbits, to the conserved charge space. Using this mapping one wants to reproduce the conserved charge bootstrap, i.e. conserved charge triangle, from a root triangle.

In the sine-Gordon theory, the only non-vanishing three-point couplings are $C_{A\bar{A}n}$ and $C_{nm(n+m)}$. And the conserved charge bootstraps which one will compare with, are the mass triangles:

$$M + Me^{(i\pi - \frac{in\pi}{\lambda})} + 2M\sin(\frac{n\pi}{2\lambda})e^{(\frac{3i\pi}{2} - \frac{in\pi}{2\lambda})} = 0, \qquad (5.1.1)$$

for soliton–antisoliton–n^{th}-breather coupling, and

$$M_{n+m} + M_n e^{(i\pi - \frac{im\pi}{2\lambda})} + M_m e^{(i\pi + \frac{in\pi}{2\lambda})} = 0. \qquad (5.1.2)$$

for three-breathers coupling. Both equations can be rescaled by dividing with M, the mass of the soliton. The fusing angles which one wants to recover are given in equations (2.3.27), (2.3.32) and (2.3.33),

$$\begin{aligned} i\theta^n_{A\bar{A}} &= i\pi - \frac{in\pi}{\lambda} \\ i\theta^A_{n\bar{A}} &= \frac{i\pi}{2} + \frac{in\pi}{2\lambda} \\ i\theta^{n+m}_{nm} &= \frac{i(n+m)\pi}{2\lambda}. \end{aligned}$$

5.2 Attempt Using Affine Coxeter Transformation

Looking at the affine Coxeter orbits of the simple roots of the $a_1^{(1)}$ affine algebra, one sees that a root triangle can also be constructed, although not in the *Euclidean* sense since the affine root space is actually of a *Minkowskian* type (this will be explained in the next section). One visualizes an example of the root triangle in *figure 5.1*.

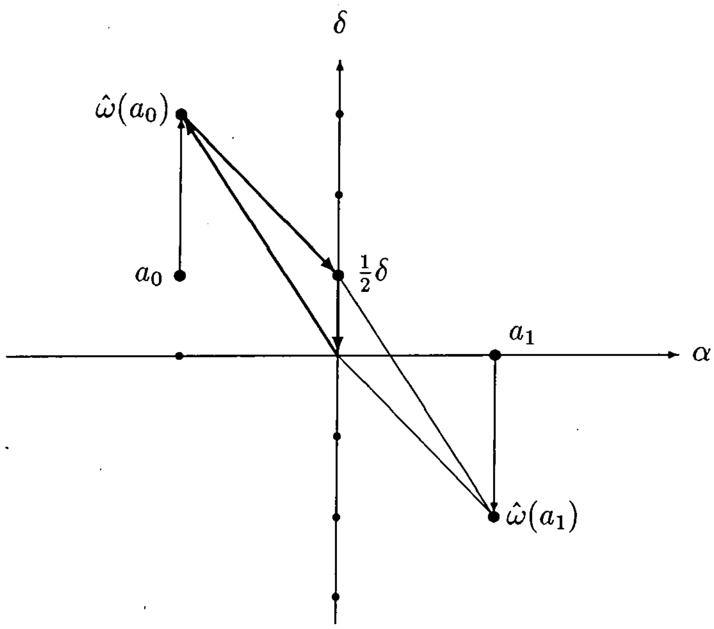

Figure 5.1: Root triangle of $\hat{\omega}(a_0) + \hat{\omega}(a_1) - \frac{1}{2}\delta = 0$.

Let $\hat{\omega}$ be the affine Coxeter transformation defined as the product of Weyl reflections with respect to the simple roots of $a_1^{(1)}$: $\{a_0, a_1\}$. In this section, the imaginary root δ will be set to

$$\delta = (0, 0, 2).$$

Then for the example in *figure 5.1*, one has a the root triangle given as

$$\hat{\omega}(a_0) + \hat{\omega}(a_1) - \frac{1}{2}\delta = 0.$$

Applying any number of affine Coxeter transformations to the root triangle does not alter the nature of the root triangle since an imaginary root is invariant under $\hat{\omega}$,

$$\hat{\omega}^p(a_0) + \hat{\omega}^p(a_1) - \frac{1}{2}\delta = 0.$$

This suggest that, from the affine Coxeter orbit an infinite number of of triangles with the same $\frac{1}{2}\delta$ as one of its side can be constructed. Other root triangles with common imaginary side can also be constructed using two different orbits. Thus, one is lead to take the affine Coxeter orbit of the simple root a_1 as a representation of a soliton, the affine Coxeter orbit of $-a_1$ as a representation of an antisoliton and the n^{th}-breather is proportional to the imaginary root $n\delta$.

With the affine fundamental weight $\Lambda_1 = (\frac{\alpha}{2}, 1, 0)$, see [7], define the vectors,

$$u = q\Lambda_1, \qquad \hat{u} = q^{-1}a_1, \tag{5.2.1}$$

such that,

$$\hat{u}\cdot u = 1, \qquad a_1 \cdot u = q, \tag{5.2.2}$$

where q is a coupling parameter dependent factor. The vector $\hat{u}$ is a special case of the vector $\hat{a}$ of [21]. It is straightforward to see that under $\hat{\omega}$, $\hat{u}$ is rotated by the angle 2π and the imaginary component increases by $2q^{-1}$.

Following [21] one defines the mapping from the root space into the u-space as,

$$\mathcal{P}[a_1] = q\hat{u}, \tag{5.2.3}$$

such that, the image of this mapping has the same behaviour as the root a_1, i.e. $\mathcal{P}[a_1]\cdot u = q$. Note that the projection space is actually the root space again, this is because the simplicity of the the simple roots of $a_1^{(1)}$ algebra.

One defines the mapping under repeated applications of $\hat{\omega}$ as

$$\begin{aligned} \mathcal{P}[\hat{\omega}^k(a_1)] &\equiv qe^{-2ikq^{-1}}\hat{u} \\ \mathcal{P}[\hat{\omega}^k(-a_1)] &\equiv -qe^{2ikq^{-1}}\hat{u} \\ &= qe^{i\pi+2ikq^{-1}}\hat{u}. \end{aligned} \tag{5.2.4}$$

Further, define the mapping of the imaginary root δ as

$$\mathcal{P}[n\delta] \equiv q^{\delta}e^{\frac{i\pi}{2}+inq^{-1}}\hat{u}, \tag{5.2.5}$$

for some quantity q^{δ}. In the above, one can drop the direction vector, $\hat{u}$, and using $q = \frac{2\lambda}{\pi}$, one can write down the conserved charge bootstrap relations of the soliton, antisoliton and the n^{th}-breather. But, as it is directly seen from the definition above, equations (5.2.4-5.2.5), although an imaginary root can be written as a combination of two root orbits, the mapping of these two root orbits does not satisfy the definition given in equation (5.2.5). Hence, the mapping $\mathcal{P}$ is not *well defined* and not linear. Thus, this is misconstruction.

Using the complex coordinate representation for the projection as done in *Section 4.5*, only one η will be obtained, since the eigenvalues of $\hat{\omega}$ are degenerate with only one eigenvector. Proceeding as in *Section 4.5*, one finds that the projection P_s is again not well defined, since the projection of the imaginary roots is

$$P_s(n\delta) \neq nP_s(\delta).$$

A digression of what has been learned so far is as follows. One needs a coupling parameter dependent factor with increasing power as more and more *rotations* are applied. A coupling parameter dependence on the power is needed to reproduce a mass triangle, written in complex coordinate, c.f. equations (5.1.1) and (5.1.2). This dependence cannot be achieved by $\hat{\omega}$, although one has a freedom in adding a free parameter in the eigenvector, $\eta = \eta(\lambda)$. In fact, compared with *ordinary* Coxeter transformation which is a coplanar rotation, the *affine* Coxeter transformation causes a *spiral* rotation. Furthermore, it is desirable that this coupling parameter dependent transformation to act on the root space, instead of a coupling parameter dependent mapping from the root space to the conserved quantities space. Thus, to overcome this one may have to define an additional transformation.

5.3 Attempts Using the Imaginary Roots δ and δ'

Since the length of an imaginary root δ is zero, associating the breather with this imaginary root alone seems to be incorrect. Because one cannot make a mass triangle in an Euclidean sense. But one can take into account the second imaginary root, namely

$$\delta' = (0, 1, 0).$$

From this section onwards, the imaginary root δ is set to be $(0, 0, 1)$.

Using these imaginary roots one can construct two combinations of imaginary roots which have non-zero length,

$$\Delta_1 = \delta + \delta' \qquad \text{and} \qquad \Delta_2 = \delta - \delta'$$

with $(\Delta_1)^2 > 0$ and $(\Delta_2)^2 < 0$, i.e. Δ_1 is a *space-like* vector while Δ_2 is a *time-like* vector, borrowing the terminology in the Minkowski space.

5.3.1 Combining $\hat{\omega}$ with δ and δ'

Write the affine Coxeter transformations on the simple roots a_0 and a_1 as follows, c.f. equation (4.4.38),

$$\hat{\omega}^p(a_0) = (2p+1)a_0 + 2pa_1$$

$$\hat{\omega}^p(a_1) = -2pa_0 - (2p-1)a_1.$$

Moreover, the affine Coxeter transformation on an arbitrary root of the following form,

$$B = a\alpha + b\delta' + c\delta,$$

is given by

$$\hat{\omega}^p(B) = B - p(pb+2a)a_0 - p((p-1)b+2a)a_1. \qquad (5.3.1)$$

Next, one tries to construct a root triangle using the roots $A = \alpha + k\delta' + d\delta$, $\bar{A} = -A$ and B, having in mind to associate A with a soliton, $\bar{A}$ with an antisoliton and B with a breather,

$$\hat{\omega}^p(A) + \hat{\omega}^q(\bar{A}) + \hat{\omega}^r(B) \equiv 0.$$

Equating the coefficients, one finds that the coefficients a will depend on p and q, b is equal to zero, and c will also depend on r,

$$a = (q-p)k, \qquad b = 0, \qquad c = 2ra + (p^2 - q^2)k.$$

If p and q are both odd or even, then c has to be an integer multiple of $2(q-p)k$ since r has to be an integer. If, p and q are of different type, then c has to be an integer multiple of $(q-p)k$.

On the other hand, if one wants to associate the breathers with the *space-like* imaginary root Δ_1, one finds that the coefficient k and d will depend on p and q. Which is undesirable, since one is free to choose A without the knowledge of the root triangle relation above.

As seen from the discussion above, the affine Coxeter transformation used on these roots failed to produce a coupling parameter dependent transformation. Even if one takes the coefficients a and c to be dependent to the coupling parameter, one still lacks the exponent behaviour.

5.3.2 Defining New Transformation 1

From the results above, it seems that one needs a new operation in the root space. On the other hand, one wants to keep the picture that a soliton is associated to an affine root which has a non-zero Euclidean component and breathers are associated with purely imaginary roots. So that a root triangle similar to *figure 5.1* can be drawn. Then a possibility of a new transformation on the root space can be achieved when the transformation has a finite order in the Euclidean part of an affine root a. Furthermore it has to be defined such that when acting on an imaginary root it will give rise to a factor dependent on the coupling parameter, λ.

Suppose one defines a new transformation, $\mathcal{R}$, on the root space as follows,

$$\begin{aligned} \mathcal{R}^k(\alpha) &= \omega^k(\alpha) \\ \mathcal{R}^k(\delta') &= q^k\delta' \\ \mathcal{R}^k(\delta) &= q^{-k}\delta, \end{aligned} \tag{5.3.2}$$

where ω is just the ordinary Weyl reflection with respect to the root α which in this case is also the Coxeter transformation. Consider the following combinations of roots,

$$\begin{aligned} A &= \alpha + \delta' \\ \bar{A} &= -\alpha + \delta \\ B_{m+n} &= -q^{\frac{m+n}{2}}(\delta + \delta'). \end{aligned} \tag{5.3.3}$$

The most general root triangle constructed from the above roots is,

$$\mathcal{R}^m(A) + \mathcal{R}^{-n}(\bar{A}) + \mathcal{R}^{\frac{m-n}{2}}(B_{m+n}) = 0, \tag{5.3.4}$$

with m and n are both odd or even integers. This restriction on n and m comes from the last term, since it does not make any sense to have a *fractional* application of $\mathcal{R}$.

Next one needs a linear mapping $\mathcal{P}$, defined on the components root, which brings this root triangle into a mass triangle. But such linear map will automatically cancel contributions from the imaginary roots. And one is left with,

$$\mathcal{P}[\omega^m(\alpha)] - \mathcal{P}[\omega^{-n}(\alpha)] = 0.$$

This will not give the mass triangle.

So, instead of defining a mapping which act on components root, one define a mapping, from the root space into a q-space, which acts on A, $\bar{A}$ and B_{m+n} as,

$$\begin{aligned} \mathcal{P}[\mathcal{R}^n(A)] &= q^{-n} \\ \mathcal{P}[\mathcal{R}^n(\bar{A})] &= e^{i\pi}q^n, \end{aligned} \tag{5.3.5}$$

and by definition (5.3.3), one has

$$B_n = -q^{-\frac{n}{2}}(\delta' + \delta) = -\mathcal{R}^{\frac{n}{2}}(A) - \mathcal{R}^{\frac{n}{2}}(\bar{A}).$$

One immediately sees that the above relation is only true for n even. Furthermore, if one continues to take a mapping of B_n, using equations (5.3.5), a triviallity will again be obtained. Moreover one sees that, although $\mathcal{R}^n(A) \neq -\mathcal{R}^{-n}(\bar{A})$, its mapping is the same, $\mathcal{P}[\mathcal{R}^n(A)] = \mathcal{P}[-\mathcal{R}^{-n}(\bar{A})]$.

Restricting to the B_{even} breathers, one can revise the transformation given in equation (5.3.2) to be

$$\begin{aligned} \mathcal{R}^k(\alpha) &= \omega^k(\alpha) \\ \mathcal{R}^k(\delta') &= q^{|k|}\delta' \\ \mathcal{R}^k(\delta) &= q^{-|k|}\delta, \end{aligned} \tag{5.3.6}$$

then the mappings, equations (5.3.5), become

$$\begin{aligned} \mathcal{P}[\mathcal{R}^k(A)] &= q^{-|k|} \\ \mathcal{P}[\mathcal{R}^k(\bar{A})] &= e^{i\pi}q^{|k|}, \end{aligned} \tag{5.3.7}$$

with the restriction that $k \neq 0$.

If $m, n > 0$, from equation (5.3.4) one has the following root triangle,

$$\mathcal{R}^m(A) + \mathcal{R}^{-n}(\bar{A}) + \mathcal{R}^{\frac{m-n}{2}}(-\mathcal{R}^{\frac{m+n}{2}}(A) - \mathcal{R}^{-\frac{m+n}{2}}(\bar{A})) = 0, \tag{5.3.8}$$

which upon the projections (5.3.7) yields,

$$1 + e^{i\pi}q^{m+n} + e^{i\pi} + q^{m+n} = 0, \tag{5.3.9}$$

this is the scaled *mass* triangle of soliton, antisoliton and the $(m+n)^{th}$-breathers if one identifies $q = e^{-\frac{i\pi}{\lambda}}$. But, somehow this does not seem to be right, since there is a sense of triviality in equation (5.3.8). Moreover one is restricted to B_{even} breathers. Changing the combinations for the roots A and $\bar{A}$ will still give a restriction to B_{even} or B_{odd} only.

As mentioned in the previous section, the affine Coxeter transformation cannot give an increasing power to coupling dependent factor which one is free to introduce. In fact, if one rescaled the imaginary root with the coupling dependent factor q, $q\delta$, each application of the affine Coxeter transformation will on increase or decrease the imaginary component δ by $2q\delta$. On the other hand, one wants to keep the affine Coxeter transformation such that a nice root triangle can be constructed as in *figure 5.1*.

Thus, one tries to combine the affine Coxeter transformation with a *scaling* transformation on the imaginary root δ. Define the scaling transformation as,

$$\mathcal{T}^k(\delta) \equiv q^k\delta \tag{5.3.10}$$

$$\mathcal{T}^k(\alpha) \equiv \alpha. \tag{5.3.11}$$

A combined transformation on the root space is defined as,

$$\mathcal{R} \equiv \mathcal{T}\hat{\omega}, \tag{5.3.12}$$

such that,

$$\begin{aligned} \mathcal{R}^k(\alpha) &= \alpha - 2kq^k\delta \\ \mathcal{R}^k(\delta) &= q^k\delta. \end{aligned}$$

Thus, basically the transformation $\mathcal{R}$ above is the same as in equation (5.3.6), with ω replaced by $\hat{\omega}$.

Let $A = a_1$, $\bar{A} = -a_1$ and $B_n = 4n\delta$, then a root triangle can be constructed as follows,

$$\mathcal{R}^n(A) + \mathcal{R}^{-n}(\bar{A}) + \mathcal{R}^n(B_n) = 0.$$

But, there is no non-trivial mapping from this root triangle which can reproduce the soliton–antisoliton–n^{th}-breather mass triangle.

5.3.3 Defining New Transformation 2

Looking carefully at the affine root space generated by the simple roots a_0 and a_1, one sees that this basis can be exchanged with the basis α and δ. One will try to mimic the affine Coxeter transformation using this new basis.

Thus, define a reflection with respect to the imaginary root δ as,

$$\omega_\delta(x) = x - 2\frac{x \cdot \delta'}{\delta \cdot \delta'} q\delta. \tag{5.3.13}$$

Such that,

$$\begin{aligned} \omega_\delta(\alpha) &= \alpha \\ \omega_\delta(q^k\delta) &= q^k\delta - 2q^{k+1}\delta. \end{aligned}$$

Note, when $q \to 1$, ω_δ is a Weyl reflection, as $\omega_\delta(\delta) = -\delta$. Now, define the transformation $\mathcal{R}$ as,

$$\mathcal{R} \equiv \omega_\delta \omega, \tag{5.3.14}$$

where ω is the Weyl reflection with respect to α. The result of repeated application of the transformation $\mathcal{R}$ has a polynomial of q as the coefficient of the imaginary root δ. This is undesirable, since one cannot construct a root triangle.

5.4 Restriction from Root and Mass Triangles

If one had demanded that the root triangle and the mass triangle must be related by a mapping, then these triangles will give a restriction on the representation of solitons and breathers in terms of roots. Furthermore, this restriction will give constraint on the orbit transformation.

One wants a transformation $\mathcal{R}$ such that the following root triangle can be satisfied,

$$\mathcal{R}^a(A) + \mathcal{R}^b(\bar{A}) + \mathcal{R}^c(B_n) = 0. \tag{5.4.1}$$

This transformation must give the dependence on q but still be a linear transformation. On the other hand, one also wants a mapping $\mathcal{P}$ which will give the correct mass triangle,

$$\mathcal{P}[\mathcal{R}^a(A)] + \mathcal{P}[\mathcal{R}^b(\bar{A})] + \mathcal{P}[\mathcal{R}^c(B_n)]$$

$$\begin{aligned} &\equiv 1 + e^{i\pi} q^n + e^{i\pi} + q^n \\ &= q^a + e^{i\pi} q^{-b} + (e^{i\pi} q^a + q^{-b}), \end{aligned} \tag{5.4.2}$$

where $q = e^{-\frac{i\pi}{\lambda}}$, and suppose that $n = -a - b$. Hence, equation (5.4.2) implies that,

$$\mathcal{R}^c(B_n) = \mathcal{R}^{-b}(A) + \mathcal{R}^a(\bar{A}). \tag{5.4.3}$$

This relation together with equation (5.4.1) leaves two possibilities,

$$\mathcal{R}^a(A) = -\mathcal{R}^{-b}(A) \quad \text{and} \quad \mathcal{R}^{-b}(\bar{A}) = -\mathcal{R}^a(\bar{A}) \tag{5.4.4}$$

or

$$\mathcal{R}^a(A) = -\mathcal{R}^a(\bar{A}) \quad \text{and} \quad \mathcal{R}^{-b}(\bar{A}) = -\mathcal{R}^{-b}(A). \tag{5.4.5}$$

Thus, one has to find such a transformation $\mathcal{R}$ which satisfies either of the above restriction. And then one can define a suitable mapping $\mathcal{P}$ which will give the mass triangle.

So far, the following transformations have been examined and do not yield the desired results:

- $\mathcal{R} = \hat{\omega}$ and $\mathcal{P}[\mathcal{R}^{2k}(a_1)] = q^{-k}$, this mapping is not linear.
- $\mathcal{R} = \mathcal{T}\hat{\omega}$ where $\mathcal{T}(\alpha) = \alpha$ and $\mathcal{T}^k(\delta) = q^k\delta$, root triangle can not be constructed.
- $\mathcal{R} = \mathcal{T}\omega$ where $\mathcal{T}$ is defined as above, and ω is the ordinary Coxeter transformation, i.e. reflection with respect to the simple root α, root triangle cannot be constructed.
- All the transformation discussed in *Section 5.3.*

Chapter 6

Conclusion and Outlook

So What

Miles Davis

This chapter concludes the report presented in this thesis and an outlook of future research is also given.

The first part of this thesis reports on the classical oscillating solitonic solutions of the affine Toda theories.

As a well established fact, the (topological) spectrum of classical soliton solutions of the affine Toda theories do not fill up the associated representation space. This is contrary to the quantum states, in which the associated representation spaces are conjectured to be filled up. Thus one is led to find new classical solitonic solutions to fill up the associated representation space. A possible new solution is the oscillating two soliton solution, the *breather*.

One has seen that a classical oscillating solitonic solution of the affine Toda theories can be constructed. By analytically changing the real velocity u (rapidity Θ) into purely imaginary velocity iv (imaginary rapidity $i\bar{\Theta}$), one obtains a periodic two soliton solution. In order to have a real energy and momentum for this periodic two soliton solution, the constituent solitons have to be degenerate in mass. Further, to have a well defined solution, one has to examine the behaviour of the breather solution which is given as a logarithm of complex functions. Due to this, the 'displacement' parameter η, the topological charge parameter ξ and the velocity v (rapidity $\bar{\Theta}$) are restricted to a certain range of definition. These restrictions are obtained by examining the real and imaginary parts of the solutions. Examples which have been presented in this thesis are the cases of $a_n^{(1)}$ and $d_4^{(1)}$ breathers. The exercise of constructing the breather solutions in other theories can be performed in a similar fashion.

In the $a_n^{(1)}$ series, breathers constructed from two solitons of the same species (type A breathers) carry non-zero topological charge; breathers constructed from two solitons of anti-species of each other carry zero topological charge. In the $d_4^{(1)}$ case, the above arrangement yields different results. In particular, breathers constructed with constituent solitons associated to the same fundamental representation can carry zero and non-zero topological charges. While the breathers constructed from the solitons associated with two different outer nodes of the $d_4^{(1)}$ case only carry non-zero topological charges.

These topological charges lie in the fundamental representation which is the irreducible component (or the trivial singlet representation) of the Clebsch-Gordan decomposition of the tensor product of the fundamental representations associated with the constituent solitons. It turns out that these topological charges are the same as the topological charges of the single soliton cases. Thus it seems that these breather solutions do not give additional states in the representation space. Furthermore, the topological charges of the constituent solitons cannot be determined individually. Due to the fact that this topological charge quantum number differentiates the solitons of the same species (and for some cases, the solitons in a mass degenerate multiplets), then one does not know which solitons have been taken to make this oscillating solution.

Furthermore, the Clebsch-Gordan decomposition property of the representation space of the topological charges and the fusing rule of solitons [13, 46], which are similar the same as the fusing rule of the fundamental Toda particle [9, 21], are in agreement with each other. This gives a further support to the soliton fusing and for the existence of an excited soliton, a *breathing soliton*, which is a single soliton with higher mass (energy) because it has started to *vibrate*. Moreover, one can make a conjecture that the breathers with constituent solitons associated with conjugate (or self conjugate) fundamental representation spaces carry zero topological charge as well as the possible non-zero topological charges permitted by the fusing rule. Examples of this conjecture are the type A breathers of $a_{2n+1}^{(1)}$ series with constituent solitons coming from the $(n+1)$ fundamental representations, the type B breathers of $a_n^{(1)}$ series and jj-breathers of $d_4^{(1)}$ theory (where $j = 1, 2, 3, 4$). This conjecture is in line with the PRV conjecture used in determining the Clebsch-Gordan property of the tensor product of the fundamental representations associated with the constituent solitons.

One can examine these breather solutions more deeply. First, one can show that not all zero-topological charge breathers result from an embedded sine-Gordon breather. Secondly, it has been shown for the $a_n^{(1)}$ series that crossing transformation of one of the constituent soliton can be achieved by replacing the parameter σ by $-\sigma$ or replacing the rapidity $\bar{\Theta}$ by $\bar{\Theta} + i\pi$. This crossing transformation sends a type A breather into a type B breather and vice-versa. As a by product of this crossing symmetry, one can superficially calculate the topological charges of the constituent solitons in the breather solution.

Several future lines of research can be mentioned at this stage. First, it might be possible to calculate the quantum corrections of the breather masses in the same spirit as the quantum corrections of the soliton masses [32, 33, 34]. Secondly, one might be able to calculate the classical non-zero three-point coupling (*fusing*) of the soliton and breather solutions in a similar prescription as done in [13, 46]. As a possible guiding principle, the non-zero three-point coupling of the quantum solitons and soliton bound states of the $a_2^{(1)}$ theory has been given in [42]. Finally, one can further ask if multi-soliton breathers can also be constructed. Of course, the original problem of filling up the representation space associated to the classical solitons is still unanswered.

The second part of this thesis reports on attempts to associate the sine-Gordon solitons and breathers to a root space. This is done by associating the spectrum of the sine-Gordon soliton and breather with the orbits of an affine Coxeter transformation. However, it seems that the naïve thought of generalizing the Coxeter transformation, which plays a central rôle in the real coupling regime of affine Toda theories, into an affine Coxeter transformation does not lead to a consistent result.

On this point one may need to reflect more carefully on the symmetry group associated to the imaginary coupling regime of the affine Toda theories. One has seen that at classical level of the soliton solutions, the symmetry group is the simple Lie group, as hinted by the Coxeter transformation of the associated Weyl group which plays a significant rôle. However, for the quantum theory as seen by the construction of its exact S-matrices [14, 41, 42], the symmetry group is provided by a *quantum* group. These different symmetry groups may be the reason for the discrepancy between the classical and quantum states of the soliton solutions. Furthermore, the determination of the exact scattering matrix elements of all affine Toda solitons are still an open problem at this moment. Two most recent attempts can be mentioned. The first is the realization of exchange operator which has as its symmetry group the *quantum affine Heisenberg algebra* [67], this work continues previous results of Corrigan and Dorey [66]. When exchanged these operators give rise to the scattering matrix, thus it is hoped that at least in the diagonal processes of soliton scattering can be calculated using this exchange operators. The second recent result is the calculation of a new R-matrix solution to the Yang-Baxter equation for the quantum affine algebra cases. These new R-matrices possess non-rigid pole structure [68]. Since the S-matrices can be extracted from the R-matrices, this result may provide the S-matrices for the complex affine Toda theory which has *floating* poles, mirroring the floating poles of the exact S-matrix of the *non-simply-laced* affine Toda theories in the real coupling regime.

Appendix A

Affine Kac-Moody Algebras and Quantum Group

A.1 Affine Kac-Moody Algebras and Root System

This is a brief review of the untwisted affine Kac-Moody algebras and the root space related to it [6, 7, 64].

Let $\{H^i, E^\alpha\}$ be the *modified Cartan-Weyl* basis of a Lie algebra g of rank r. That is, H^i with $i = 1, 2, \ldots, r$ are the maximal set of commuting Hermitian generators, and E^α is the step operator corresponding to the root α. From this algebra, one can make a *loop algebra*, i.e. an infinite dimensional complex Lie algebra, by adding a spectral parameter λ to the generators. For any $\gamma \in g$ define $\gamma_m \equiv \lambda^m \otimes \gamma$, with commutation relation between γ_m and γ'_n is given by,

$$[\lambda^m \otimes \gamma, \lambda^n \otimes \gamma'] = \lambda^{m+n} \otimes [\gamma, \gamma']. \tag{A.1.1}$$

The Kac-Moody algebra can be viewed as a central extension of this loop algebra with the central element k, and the commutation relation (A.1.1) is amended to be,

$$[\lambda^m \otimes \gamma, \lambda^n \otimes \gamma'] = \lambda^{m+n} \otimes [\gamma, \gamma'] + \delta_{m+n,0}(\gamma, \gamma')mk, \tag{A.1.2}$$

where (,) is the Killing form on g. Further, the elements of the Kac-Moody algebra are *graded* by the *derivation operator* d which has the property,

$$[d, \lambda^m \otimes \gamma] = m\lambda^m \otimes \gamma. \tag{A.1.3}$$

This gradation of the Kac-Moody algebra is called the *homogeneous gradation.* With the modified Cartan Weyl basis, the untwisted affine Kac-Moody algebra $\hat{g}$ based on a simple Lie algebra g of rank r is given by the following commutation relations:

$$\begin{aligned}
[H^i_m, H^j_n] &= km\delta^{ij}\delta_{m,-n} \\
[H^i_m, E^\alpha_n] &= \alpha^i E^\alpha_{m+n} \\
[E^\alpha_m, E^\beta_n] &= \begin{cases} \varepsilon(\alpha,\beta)E^{\alpha+\beta}_{m+n} & \text{if } (\alpha+\beta) \text{ is a root} \\ \frac{2}{\alpha^2}\{\alpha \cdot H_{m+n} + km\delta_{m,-n}\} & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \\
[k, E^\alpha_m] &= [k, H^i_m] = [k, d] = 0 \\
[d, E^\alpha_n] &= nE^\alpha_n \\
[d, H^i_n] &= nH^i_n
\end{aligned} \tag{A.1.4}$$

The index i and j runs from 1 to r and $m, n \in \mathbb{Z}$. The vectors α and β are roots of g. The constant $\varepsilon(\alpha, \beta)$ is antisymmetric in α and β, and can be calculated from the normalization of the roots α and β. Let $\psi = \sum_{j=1}^{r} n_j \alpha_j$ be the *highest root* of the simple Lie algebra g. Then, the central element which commutes with all generators is given by

$$k = \sum_{j=0}^{r} m_j \frac{2\alpha_j \cdot H_0}{\alpha_j^2},$$

where m_j is related to n_j as,

$$m_j = n_j \frac{\alpha_j^2}{\psi^2}.$$

To construct the root system, one needs to look for a Cartan Subalgebra (CSA) of the Kac-Moody algebra. One might take the CSA to consist of H_0^i and k. Then the commutation relations of the CSA with the step operators yields,

$$\begin{aligned} [H_0^i, E_n^\alpha] &= \alpha^i E_n^\alpha, \\ [k, E_n^\alpha] &= 0, \end{aligned} \tag{A.1.5}$$

with $(r+1)$-dimensional roots $(\alpha, 0)$, where $\alpha \in \Phi$, and Φ is the root system of g. These roots are infinitely degenerate. Moreover, the CSA is not a maximal abelian subalgebra since for any n

$$[H_0^i, H_n^j] = 0.$$

To overcome this, one has to include the derivation operator d in the CSA. Thus, the CSA consists of H_0^i, k and d with $1 \leq i \leq r$. The step operators and their corresponding roots are as follows,

E_n^α corresponds to the root $a = (\alpha, 0, n)$,

H_n^i, $n \in \mathbb{Z}$, $n \neq 0$, corresponds to the root $\delta = (0, 0, n)$.

Thus, the basis roots of the untwisted Kac-Moody algebra $\hat{g}$ can be taken as

$$\begin{aligned} a_i &= (\alpha_i, 0, 0), \qquad 1 \leq i \leq r, \\ a_0 &= (\alpha_0, 0, 1), \end{aligned} \tag{A.1.6}$$

where α_i are the *simple roots* of g and $\alpha_0 = -\psi$. The root lattice is generated by these simple roots. An arbitrary root can be written as a linear combination of these simple roots,

$$a = \sum_{i=0}^{r} k_i a_i \tag{A.1.7}$$

with $k_i \in \mathbb{Z}$ and all $k_i \leq 0$ or $k_i \geq 0$. The scalar product of two roots $a = (\alpha, k, d)$ and $b = (\beta, k', d')$ is defined as,

$$a \cdot b = \alpha \cdot \beta + kd' + dk'. \tag{A.1.8}$$

With this definition, one can calculate the length of a root. Since $\delta^2 = 0$, henceforth the root δ will be called the *imaginary root.*

The *affine* Cartan matrix becomes

$$\hat{C}_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}, \qquad 0 \leq i, j \leq r \tag{A.1.9}$$

i.e. it is the Cartan matrix of g with an additional row and column. Furthermore, defining $m_0 = 1$, then the $(r+1)$-vector with components m_j is the right null-vector of the affine Cartan matrix, i.e.

$$\sum_{k=0}^{r} \hat{C}_{jk} m_k = 0.$$

From this affine Cartan matrix, we can draw the corresponding affine Dynkin diagrams, which are the ordinary Dynkin diagrams augmented with a node corresponding to the root α_0. For a list of affine Dynkin diagrams see [6, 7, 64].

The Kac-Moody algebras can equivalently be represented in the *Chevalley basis* by the generators $\{h_j, e_j, f_j\}$ where j runs from 0 to r with the following commutation relations [6],

$$\begin{aligned} [h_j, h_k] &= 0, \\ [h_j, e_k] &= \hat{C}_{kj} e_k, \\ [h_j, f_k] &= -\hat{C}_{kj} f_k, \\ [e_j, f_k] &= \delta_{jk} h_k, \end{aligned} \tag{A.1.10}$$

together with the Serre relations $(j \neq k)$,

$$(\text{ad } e_j)^{1-\hat{C}_{kj}} e_k = 0,$$
$$(\text{ad } f_j)^{1-\hat{C}_{kj}} f_k = 0,$$

This basis is supplemented by a gradation which is provided by the element $d' = hd + T_3^0$,

$$[d', e_j] = e_j, \qquad [d', h_j] = 0, \qquad [d', f_j] = -f_j. \tag{A.1.11}$$

This is called the *principal gradation*, and T_3^0 is defined to be $\lambda^0 \otimes T_3$ where T_3 generates the maximal subalgebra of g,

$$T_3 = \frac{1}{2}\sum_{\beta>0}\frac{2\beta\cdot H}{\beta^2}, \tag{A.1.12}$$

such that,

$$[T_3, E^\alpha] = \text{height}(\alpha)E^\alpha. \tag{A.1.13}$$

One can see that the last commutation relation comes from the following [61, 65]. An arbitrary root β of g can expressed as a linear combination of the simple roots $\alpha \in \Delta$,

$$\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha,$$

with k_α are all positive or all negative. Thus, the root space of g can be decomposed into positive roots and negative roots, $\Phi = \Phi_+ + \Phi_-$. Further, the height of a root is defined as

$$\text{height}(\alpha) = \sum_{\alpha\in\Delta} k_\alpha.$$

Let $\alpha \in \Delta$ and $\beta \in \Phi_+ - \{\alpha\}$ so that $\beta = \sum_{\gamma\in\Delta} k_\gamma\gamma$ $(k_\gamma \in \mathbb{Z}_+)$. A Weyl reflection of β with respect to the simple root α, $r_\alpha\beta = \beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha$, changes the coefficient of α in β. Since β is not proportional to α (the only roots proportional to α are $\pm\alpha$), then after reflection r_α, some k_γ of β will remain positive and hence all k_γ will remain positive. Thus, $r_\alpha\beta \in \Phi_+ - \{\alpha\}$, i.e. r_α permutes the positive roots other than α.

Set $\delta = \frac{1}{2}\sum_{\beta>0}\beta$, then since r_α leaves $\delta - \frac{1}{2}\alpha$ invariant, one has $r_\alpha\delta = \delta - \alpha$. From this one notices that with $\alpha \in \Delta$,

$$\delta - \frac{2\alpha\cdot\delta}{\alpha^2}\alpha = \delta - \alpha \Longleftrightarrow \frac{2\alpha\cdot\delta}{\alpha^2} = 1.$$

Hence, for the simply-laced cases one can calculate the height of any root γ from the following,

$$\frac{1}{2}\sum_{\beta>0}\frac{2\beta\cdot\gamma}{\beta^2} = \text{height}(\gamma). \tag{A.1.14}$$

Thus, using (A.1.14) and (A.1.4) one can show that the commutation relation (A.1.13) holds.

Further, the fundamental weights of a Kac-Moody algebra are defined by the relation,

$$\frac{2a_j\cdot\Lambda_k}{a_j^2} = \delta_{jk}. \tag{A.1.15}$$

Thus, with the simple roots given by (A.1.6), the fundamental weights are calculated to be

$$\Lambda_j = (\lambda_j, \frac{1}{2}m_j\psi^2, 0), \qquad j = 1, 2, \ldots r, \tag{A.1.16}$$

$$\Lambda_0 = (0, \frac{1}{2}\psi^2, 0), \tag{A.1.17}$$

where λ_j are the fundamental weights of the corresponding simple Lie algebra. The representation of a Kac-Moody algebra is built up from these fundamental weights. In particular, the action of the generators H_0 and h on a *weight* vector $|\Lambda>$ are as follows,

$$H_0^j|\Lambda> = \Lambda^{(j)}|\Lambda> \tag{A.1.18}$$

$$h_j|\Lambda> = \frac{2a_j \cdot \Lambda}{a_j^2}|\Lambda>. \tag{A.1.19}$$

A.2 Alternative Basis of the Affine Kac-Moody Algebra

In the algebraic construction of the solution to the affine Toda equation, one uses a special basis of the Kac-Moody algebra. The step operators of this basis *create* the soliton solutions. One starts with the alternative basis of the simple Lie algebra g, and by affinization this basis becomes the alternative basis of the Kac-Moody algebra $\hat{g}$.

A.2.1 Alternative basis of g

Let the generators of the simple Lie algebra g be graded by $S = e^{\frac{2i\pi}{h}T_3}$ as follows,

$$Sg_\nu S^{-1} = e^{\frac{2i\pi}{h}\nu}g_\nu, \tag{A.2.1}$$

where ν is an exponent of g, such that

$$g = g_0 \oplus g_1 \oplus \ldots \oplus g_{h-1}.$$

In particular,

$$g_0 = \mathcal{H} = \text{CSA of } g \text{ in Cartan-Weyl basis},$$

$$g_1 = \{E^{\alpha_j}, E^{-\psi}\}, \qquad \alpha_j \in \Delta.$$

It is claimed in [4] that the following generators provides g with a new Cartan subalgebra in *apposition* (see also [58]),

$$E_\nu = \sum_j G^j_\nu + \sum_{j'} G^{j'}_{\nu-h} \equiv q(\nu) \cdot T \tag{A.2.2}$$

where $G^j_\nu \in g_\nu$ and $q(\nu)$ is the eigenvector of the Coxeter element ω, i.e

$$\omega(q(\nu)) = e^{\frac{2i\pi}{h}\nu} q(\nu). \tag{A.2.3}$$

Their orthogonality and completeness relations are as follows,

$$q(\nu) \cdot q(\nu') = h\delta_{\nu,\nu'} \tag{A.2.4}$$

$$\sum_\nu q(\nu) q(\nu)^* = hI, \tag{A.2.5}$$

where $q(h-\nu) = q(\nu)^*$.

One can evaluate the commutation relations of these generators using the known commutation relations of the *old* CSA. In particular,

$$E_1 = \sum_{\alpha_j \in \Delta} \sqrt{m_j} E^{\alpha_j} + E^{-\psi}, \tag{A.2.6}$$

has the following commutation relation,

$$[E_1, E_{-1}] = 0.$$

The step operators F^β corresponding to the root β in the *old* basis are defined through the following commutation relation,

$$[E_\nu, F^\beta] = q(\nu) \cdot \beta F^\beta. \tag{A.2.7}$$

These step operators can be calculated using the known commutation relations in the *old* basis, since one can expand F^{γ_j} into its graded components [12],

$$F^{\gamma_j} = \sum_{\nu=0}^{h-1} F^j_\nu, \tag{A.2.8}$$

with $SF^j_\nu S^{-1} = e^{\frac{2i\pi}{h}\nu} F^j_\nu$, and $\gamma_j = c(j)\alpha_j$ with $c(j) = 1$ if α_j is a *black* simple root or $c(j) = -1$ if α_j is a *white* simple root. Alternatively, one notes that the following commutation relation,

$$[E_\nu, SF^\beta S^{-1}] = q(\nu) \cdot \omega(\beta) SF^\beta S^{-1}, \tag{A.2.9}$$

yields $SF^{\beta}S^{-1} = F^{\omega(\beta)}$. Thus, one can consider the following operators [26],

$$A_j = \frac{1}{\sqrt{h}} \sum_{k=1}^{h} F^{\omega^k(\gamma_j)}, \tag{A.2.10}$$

this means that one is considering the combination of step operators in a certain Coxeter orbit. Then,

$$SA_jS^{-1} = A_j$$

i.e. $A_j \in g_0 = \mathcal{H}$. With this relation and the appropriate commutation relations one can find the expression for F^{β}.

The alternative CSA are normalized and they are orthogonal to step operators,

$$(E_\nu, E_{\nu'}) = h\delta_{\nu,h-\nu'}, \tag{A.2.11}$$

$$(E_\nu, F^j_\mu) = 0. \tag{A.2.12}$$

Next, taking the conjugation of the commutation relation $[E_\nu, F^{\gamma_j}]$ with respect to S and comparing coefficient of graded components one obtains,

$$[E_\nu, F^j_\mu] = q(\nu) \cdot \gamma_j F^j_{(\nu+\mu)\mathrm{mod}h}. \tag{A.2.13}$$

From conjugation of F^{β} with respect to S, one notes that all F^{β} of the same Coxeter orbit are linear combinations of the same set $\{F^j_0, \dots, F^j_{h-1}\}$. Thus, as the set $\{E_\nu, F^{\beta}\}$ span the simple Lie algebra of g, so is the following set

$$\{E_\nu, F^j_\mu;\ j = 1, 2, \dots, r,\ 0 \le \mu \le (h-1),\ \nu \text{ is an exponent}\}.$$

A.2.2 Alternative basis of $\hat{g}$

Affinization of the alternative basis of g just discussed is done by considering the loop algebra extension of this basis [12].

The affinization of operator E_ν with principal grade $M = \nu + mh$ is

$$\hat{E}_M = \lambda^m \otimes E_\nu, \tag{A.2.14}$$

such that,

$$[d', \hat{E}_M] = M\hat{E}_M.$$

Further, contrary to the commutation relation of the alternative CSA of g, i.e. $[E_\nu, E_\mu] = 0$, one would expect that the commutation relation of the alternative CSA of $\hat{g}$ to be proportional to the central element k. Recall that the scalar product of two elements of g provided by the Killing form has to be invariant under the action of the Lie group G, i.e. it has the following property [7],

$$(X, [Y, Z]) = ([X, Y], Z) \qquad X, Y, Z \in g \tag{A.2.15}$$

Furthermore, the scalar product of the derivation operator of a Kac-Moody algebra is defined to be,

$$(d, k) = 1, \qquad (d, \text{anything}) = 0, \tag{A.2.16}$$

and the scalar product of T_3^0 with k is zero. The alternative CSA of $\hat{g}$ is normalised by its scalar product,

$$(\hat{E}_M, \hat{E}_N) = h\delta_{M+N,0}. \tag{A.2.17}$$

Using this information, one can evaluate the commutation relation of two elements of the alternative basis to find its proportionality to the central element in the following way,

$$[\hat{E}_M, \hat{E}_N] = Mk\delta_{M+N,0}. \tag{A.2.18}$$

The affinization of operator F_ν^j with principal grade $M = \nu + mh$ is

$$\hat{F}_M^j = \lambda^m \otimes F_\nu^j, \tag{A.2.19}$$

such that,

$$[d', \hat{F}_M^j] = M\hat{F}_M^j.$$

And, this affinization changes the commutation relation (A.2.13) into,

$$[\hat{E}_M, \hat{F}_N^j] = q([M]) \cdot \gamma_j \hat{F}_{M+N}^j, \tag{A.2.20}$$

where $M = [M] + mh$ and $N = [N] + nh$ with $[M]$ and $[N]$ are exponents of g. However, taking the scalar product of the derivative operator with the commutation relation $[\hat{E}_M, \hat{F}_{-M}^j]$ forces one to amend the affinization of F_0^j to be,

$$\hat{F}_0^j = \lambda^0 \otimes F_0^j - \frac{1}{h}(T_3, F_0^j)k, \tag{A.2.21}$$

recall that,

$$(\lambda^m \otimes \gamma, \lambda^n \otimes \gamma') = \delta_{m+n,0}(\gamma, \gamma').$$

Consider the following formal power series expansion in a complex variable z,

$$\hat{F}^j(z) = \sum_{N=-\infty}^{\infty} z^{-N} \hat{F}^j_N, \tag{A.2.22}$$

where $\hat{F}^j_N$ is recovered by the usual contour integral,

$$\hat{F}^j_N = \oint_{z=0} \frac{z^{N+1}}{2i\pi} \hat{F}^j(z) \mathrm{d}z. \tag{A.2.23}$$

Then the commutation relation of the alternative CSA with $\hat{F}^j(z)$ is given by,

$$[\hat{E}_M, \hat{F}^j(z)] = q([M]) \cdot \gamma_j z^M \hat{F}^j(z), \tag{A.2.24}$$

i.e. the alternative CSA ad-diagonalises $\hat{F}^j(z)$.

A.3 Quantum Group

This section contains a brief discussion on *Quantum Groups.* A detailed introduction can be found in [43, 69, 70]. There are two algebraic structures which will be introduce, the first one is the *deformation* of a *universal enveloping algebra* and the second is the *quasitriangular Hopf algebra* structure. A solution to the *quantum Yang-Baxter equation* can be constructed from the elements of this *quasitriangular Hopf algebra.*

A.3.1 Quantized Universal Enveloping Algebra

Let g be a Lie algebra of rank r with elements given by the Chevalley basis $\{H_i, E_i, F_i\}$, where $i = 1, 2, \ldots, r$. Let c_{ij} be the entries of the Cartan matrix. Consider the *Universal Enveloping algebra* of g, $U(g)$. A *deformation* of this algebraic structure called a *quantized Universal Enveloping Algebra*, $U_q(g)$, or also known as *Quantum Groups*, is given by the following relations,

$$\left.\begin{array}{rcl} [H_i, E_j] & = & c_{ij} E_j, \\ [H_i, F_j] & = & -c_{ij} F_j, \\ [E_i, F_j] & = & \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \end{array}\right\} \tag{A.3.1}$$

$$\left.\begin{array}{rcl} \Delta(H_i) &=& H_i \otimes 1 + 1 \otimes H_i, \\ \Delta(E_i) &=& E_i \otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}} \otimes E_i, \\ \Delta(F_i) &=& F_i \otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}} \otimes F_i, \end{array}\right\} \tag{A.3.2}$$

$$\varepsilon(H_i) = \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(1) = 1 \tag{A.3.3}$$

$$\left.\begin{array}{rcl} S(H_i) &=& -H_i, \\ S(E_i) &=& -qE_i, \\ S(F_i) &=& -q^{-1}F_i. \end{array}\right\} \tag{A.3.4}$$

The parameter q is called the *deformation parameter.* The operation Δ is called a *coproduct,* which is a mapping $\Delta : U \to U \otimes U$, compare this with the product in a Lie algebra which is a mapping $m : U \otimes U \to U$. The inverse operation is given by the *antipode* mapping $S : U \to U$. And, ε is called the *counit,* which is a mapping $\varepsilon : U \to \mathbb{C}$, compare this with the unit mapping $\eta : \mathbb{C} \to U$. Thes structures satisfy the *Hopf* algebra axioms,

$$\begin{array}{rclll} m(m \otimes 1) &=& m(1 \otimes m), & & \text{associativity} \\ m(1 \otimes \eta) &=& m(\eta \otimes 1) &= id, & \\ (\Delta \otimes id)\Delta &=& (id \otimes \Delta)\Delta, & & \text{coassociativity} \\ (1 \otimes \varepsilon)\Delta &=& (\varepsilon \otimes 1)\Delta, & & \\ m(S \otimes id)\Delta &=& m(id \otimes S)\Delta &= \eta \cdot \varepsilon. & \end{array} \tag{A.3.5}$$

It is the fact that Δ is an algebra homomorphism,

$$\Delta(ab) = \Delta(a)\Delta(b),$$

allows one to calculate the commutation relation equation (A.3.1), i.e. one uses $\Delta([a, b]) = [\Delta(a), \Delta(b)]$. To calcutate Δ is more tedious, the form of Δ is restricted by the constraint that Δ has to become a coproduct structure of a *Universal Enveloping Algebra,* $U(g)$, as $q \to 1$. Also, Δ has to be *coassociative*, and Δ has to satisfy a further requirement relating it with a *co-Poisson structure*, see Tjin [70] for details.

If one takes the deformation parameter $q \to 1$, then $U_q(g) \to U(g)$. Also, the commutation relations (A.3.1) become the usual commutation relation of g in Chevalley basis, i.e. the only one that changes is

$$[E_i, F_j] = 2\delta_{ij} H_i.$$

A.3.2 Quasitriangular Hopf Algebra

Let A be a Hopf algebra, namely an algebra with *coproduct, counit* and *antipode* structures which satisfy axioms (A.3.5). Let R be an invertible element of $A \otimes A$, then the pair (A, R) is called a *quasitriangular Hopf algebra* if the following are satisfied,

$$\tau \cdot \Delta(a) = R\Delta(a)R^{-1}, \tag{A.3.6}$$

$$(\Delta \otimes 1)R = R_{13}R_{23}, \tag{A.3.7}$$

$$(1 \otimes \Delta)R = R_{13}R_{12}. \tag{A.3.8}$$

Here $\tau(a \otimes b) = (b \otimes a)$ and

$$R = R_i^{(1)} \otimes R_i^{(2)}, \tag{A.3.9}$$

(summation is implied), where $R_i^{(1)}$ is an element of the first copy of the Hopf algebra A and $R_i^{(2)}$ is an element of the second copy. Thus, for example in the equations (A.3.7) and (A.3.8), R_{13} is given as $R_i^{(1)} \otimes 1 \otimes R_i^{(2)}$.

This quasitriangular Hopf algebra has an important property. Namely, if V_1 and V_2 are representation spaces, then $V_1 \otimes V_2$ is isomorphic with $V_2 \otimes V_1$. This isomorphism is provided by R, i.e. equation (A.3.6). This follows because Δ relates to the tensor product space $V_1 \otimes V_2$, while $\tau \cdot \Delta$ relates to $V_2 \otimes V_1$.

The elements R of a quasitriangular Hopf algebra satisfy the *quantum Yang-Baxter equation,*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{A.3.10}$$

To see this, one needs to calculate $(1 \otimes \Delta')R$ in 2 different ways, where $\Delta' = \tau \cdot \Delta$. First, one has

$$\begin{aligned}(1 \otimes \Delta')R &= R_i^{(1)} \otimes \Delta' R_i^{(2)} \\ &= R_i^{(1)} \otimes R\Delta(R_i^{(2)})R^{-1} \\ &= (1 \otimes R)(R_i^{(1)} \otimes \Delta(R_i^{(2)}))(1 \otimes R^{-1}) \\ &= R_{23}(1 \otimes \Delta)RR_{23}^{-1} \\ &= R_{23}R_{13}R_{12}R_{23}^{-1},\end{aligned} \tag{A.3.11}$$

and secondly, one can also calculate $(1 \otimes \Delta')R$ as

$$\begin{aligned}(1 \otimes \Delta')R &= (1 \otimes \tau)(1 \otimes \Delta)R \\ &= (1 \otimes \tau)R_{13}R_{12} \\ &= R_{12}R_{13}. \end{aligned} \qquad \text{(A.3.12)}$$

Combining equations (A.3.11) and (A.3.12) yields (A.3.10).

Appendix B

Reciprocal Polynomial

A polynomial $P(z)$ of degree p is called a reciprocal polynomial if the following holds

$$P(z) = z^p P(z^{-1}).$$

Or, writing $P(z)$ as

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_p z^p, \tag{B.0.1}$$

it is a reciprocal polynomial if $a_i = a_{p-i}$ for all i and $a_i \in \mathrm{IR}$. A first degree reciprocal polynomial is always of the form $(1 + z)$. Any *even* degree reciprocal polynomial, $p = 2n$, can be written as,

$$P(z) = \prod_i^n (z^2 + b_i z + 1)$$

with,

$$\begin{aligned}
\frac{a_{2k}}{a_0} &= C_{(n-2k,0)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k}} b_{i_1} \ldots b_{i_{2k}} + C_{(n-2k+2,1)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k-2}} b_{i_1} \ldots b_{i_{2k-2}} \\
&+ C_{(n-2k+4,2)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k-4}} b_{i_1} \ldots b_{i_{2k-4}} + \ldots + C_{(n-2,k-1)} \sum_{i_1 \neq i_2} b_{i_1} b_{i_2} \\
&+ C_{(n,k)}, \qquad k = 1, 2, \ldots, [\frac{n}{2}], \\
\frac{a_{2k+1}}{a_0} &= C_{(n-2k-1,0)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k+1}} b_{i_1} \ldots b_{i_{2k+1}} + C_{(n-2k+1,1)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k-1}} b_{i_1} \ldots b_{i_{2k-1}} \\
&+ C_{(n-2k+3,2)} \sum_{i_1 \neq i_2 \neq \ldots i_{2k-3}} b_{i_1} \ldots b_{i_{2k-3}} + \ldots + C_{(n-3,k-1)} \sum_{i_1 \neq i_2 \neq i_3} b_{i_1} b_{i_2} b_{i_3} \\
&+ C_{(n-1,k)} \sum_{i_1} b_{i_1}, \qquad k = 0, 1, \ldots, [\frac{n-1}{2}].
\end{aligned}$$

In the above equations, $[x]$ means zero or largest integers less than or equal to x, and the coefficients $C_{(n,r)}$ are combinatorial coefficients, i.e.

$$C_{(n,r)} = \frac{n!}{(n-r)!r!}.$$

An *odd* degree reciprocal polynomial, $p = 2n + 1$, can be written as,

$$\begin{aligned}
P(z) &= a_0(1 + z^{2n+1}) + a_1 z(1 + z^{2n-1}) + a_2 z^2 (1 + z^{2n-3}) \\
&+ \ldots + a_n z^n (1 + z).
\end{aligned}$$

Since $(1+z)$ divide any term of the form $(1+z^{2n+1})$, then

$$P(z) = a_0(1+z)[z^{2n} + c_1 z^{2n-1} + c_2 z^{2n-2} + \ldots + c_2 z^2 + c_1 z + 1]$$

where,

$$\begin{aligned} c_{2k} &= \frac{1}{a_0}\sum_{i=1}^{2k}(-1)^{i+1}a_i - 1\,, \qquad k = 1,2,\ldots,[\frac{n}{2}], \\ c_{2k-1} &= \frac{1}{a_0}\sum_{i=1}^{2k-1}(-1)^{i}a_i + 1\,, \qquad k = 1,2,\ldots,[\frac{n+1}{2}] \end{aligned}$$

Thus any reciprocal polynomial can be written as,

$$P(z) = a(z+1)^d \prod_i (z^2 + 2b_i z + 1) \tag{B.0.2}$$

where $a, b_i \in \mathrm{IR}$ and the integer d is odd if the degree of $P(z)$ is odd. Note that if one or more of the b_is are equal to -1, then $z = 1$ is a root of the reciprocal equation with even multiplicity.

If $P(z)$ is a reciprocal polynomial, then the equation $P(z) = 0$ is called a reciprocal equation.

A specific type of reciprocal equation used in the discussions of *Chapter Four* is the reciprocal equation with roots in the form of m^{th} *root of unity.* Consider a reciprocal equation of degree two, $(z^2 + 2b_i z + 1) = 0$. Suppose that this reciprocal equation has roots in the form of $e^{i\theta}$, then the coefficient b_i has to lie in the range of $-1 \leq b_i \leq 1$. Now consider the general reciprocal equation (B.0.1), and suppose that all the coefficients a_i equal to ± 1 or 0. Then, when written in the form of equation (B.0.2), all the coefficients b_i lie in the range of $-1 \leq b_i \leq 1$. Hence, the roots of this reciprocal equation are *roots of unity.*

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