## *Explicit wavefunction collapse and quantum measurement*

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## Abstract

In this thesis, we are concerned with models of explicit wavefunction collapse as a possible solution to the measurement problem of quantum mechanics. We examine the models where collapse is to near-position eigenstates, originally introduced by Ghirardi, Rimini and Weber in 1986, where the wavefunction is spontaneously localized at random times. Subsequent models where some of the problems of the GRW model are solved, are discussed, for both sudden localization and continuous localization processes. We comment briefly on the possible origins of collapse. The consequences of possible wavefunction collapse on the operation of quantum computers are described. Finally, we look at an attempt to describe the collapse process in a Lorentz-invariant manner.

# Preface

This thesis is based on research carried out by the author between Oct 1993 and Aug 1996 at the University of Durham in the Department of Mathematical Science. No part of this thesis has been previously submitted for any degree in either this or any other university.

For the most part, this research was carried out under the supervision of Euan Squires, who sadly died in June 1996, before the completion of this thesis. I am very grateful for all the helpful comments and encouragement that Euan gave to me.

Of the material in the thesis, Chapter 2 is basically a review of [11]. Chapter 3 contains a review of [13] with Sections 3.5 and 3.6 also dealing with material in [20, 21], and Chapter 5 has a brief description of [15]. The material in Chapter 4 is original work, some of which is contained in an article written jointly with Euan Squires, [14]. Chapter 6 is original work, and was published in [16]. Finally, Chapter 7 is again based on joint work with Euan Squires [17].

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# Chapter 1

# Introduction

In quantum mechanics, in constrast to classical mechanics, the particle is not everything. Sometimes it behaves as a wave. Thus in quantum mechanics, the particle is described by a wavefunction $|\Psi>$. The theory is probabilistic in nature, although the evolution equation is deterministic. We replace the exact values of quantities with operators, including

$$\mathbf{p} \rightarrow -i\hbar\nabla \qquad\qquad E \rightarrow i\hbar\frac{\partial}{\partial t}, \tag{1.1}$$

such that the wavefunction satisfies the Schrödinger equation

$$\frac{d}{dt}|\Psi> = -\frac{i}{\hbar}H|\Psi>, \tag{1.2}$$

where the Hamiltonian $H$ can be written as

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V, \tag{1.3}$$

for a single particle. The Schrödinger equation is linear and deterministic. The linearity enables the superposition of states in a solution of the equation. The total wavefunction can be written as the sum of a number of eigenstates (of a particular operator)

$$|\Psi> = \sum_j c_j|\psi_j>. \tag{1.4}$$

If the states $|\psi_j>$ are eigenfunctions of a particular operator $\hat{O}$ with corresponding eigenvalues $o_j$ then the probability of a measurement of the dynamical variable represented by $\hat{O}$ producing the result $o_j$ is given by $|c_j|^2$. The question is what exactly constitutes the measurement.

This leads us to the fundamental difficulty of the theory. Taking into account that the measuring apparatus has quantum constituents, we can describe the measuring process as the evolution of the closed physical system $S + A$, where $S$ and $A$ denote the measured system and measuring apparatus respectively. The measurement may be of any dynamical variable, and the eigenstates of the measuring apparatus may be different positions of a pointer perhaps. Thus the wavefunction resulting from the interaction between the microscopic system and the macroscopic measuring apparatus can be written as

$$\Psi = \sum_i c_i |\phi_i> |\psi_i>, \tag{1.5}$$

where $|\phi_i>$ represent the eigenfunctions of the microscopic system of the dynamical variable that is being measured, and $|\psi_i>$ are the eigenstates of the measuring apparatus.

From this last equation, we can see the basis of the measurement problem of quantum mechanics. The result of the measuring process is a superposition of macroscopically distinguishable states. This contradicts the projection postulate, which requires a statistical mixture of the states $|\phi_i>|\psi_i>$, as "A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured". [1]

Since the result of a measurement is always indicated by the different eigenstates of a macroscopic system, the problem can be traced back to the behaviour of macroscopic objects and their interactions with microscopic ones. The linearity of the quantum theory causes the inevitability of such problems as it allows the

superposition of macroscopically distinguishable states, i.e. pointer positions.

A breakdown in quantum mechanics under certain conditions does not seem possible.[2, 3, 4, 5] We must believe that a macroscopic object is not a truly quantum mechanical system, but behaves classically. We then necessarily have dualism in nature, requiring a different evolution for microscopic and macroscopic objects. But what are the criteria for distinguishing between them? There can be no neat definition of the boundary between the two types of physical object or system. According to Bell, it "compels the physicist to disregard from time to time the exact equations of the theory and to supplement them with vague verbal assertions."[6]

What are the possible solutions of this quantum measurement problem? If we leave aside the theories in which we divide systems and objects into two distinct types, we have three possible solutions:(See also [7])

1) The de Broglie - Bohm model

2) The many worlds interpretation

3) Models of wavefunction collapse

In (1), the de Broglie - Bohm model[8, 9], instead of the answer 'wave' or 'particle' to the question of wave-particle duality, this question is answered by 'particle' *and* 'wave'. We still have a wavefunction $\psi(t, \mathbf{r})$, but it is an objective field, not simply a wave of information. The associated particle then rides along the wave at some well-defined position $\mathbf{x}(t)$ with velocity

$$\dot{\mathbf{x}}(t) = \frac{1}{m}\frac{\partial}{\partial \mathbf{r}}\Im \log \psi(t, \mathbf{r})|_{\mathbf{r}=\mathbf{x}}. \tag{1.6}$$

For a double-slit experiment, the wave in this picture goes through both slits, whilst the particle only goes through one, but is guided by the wave toward

positions where $|\psi|^2$ is large, and away from positions with small values of $|\psi|^2$.

The de Broglie - Bohm picture is quite deterministic. The initial configuration of the system of wave and particle fixes completely any subsequent development.

If the wavefunction is in a superposition of two macroscopically distinguishable states, then the trajectory calculated from eq. (1.6) will be for the particle in the first state *or* in the second state, giving a statistical mixture.

Possibility (2) advocates the existence of many worlds. [10] The number of actual worlds present corresponds to the number of possibilities present. Whenever there is any doubt as to the outcome of any event because of quantum uncertainty, the world multiplies so that all possibilities are realized, but in different worlds. As the observers will also multiply with the world, those in any particular world will experience only what happens in that world.

In this thesis, we shall concentrate on possibility (3). The wavefunction must be reduced at some stage from the superposition of states, as in eq. (1.5) to individual eigenstates, resulting in a particular eigenvalue being measured. At what point in the measurement the wavefunction is reduced is not clear, only that it occurs. In models of wavefunction collapse, extra terms are added to the evolution equations of the system such that the wave-packet reduction is described by precise dynamical processes and equations. We may maintain the linearity of the evolution equation for the wavefunction, but at the level of a normalized wavefunction, this reduction is nonlinear.

The evolution obtained from these extra terms must solve two basic problems. There is required to be a choice as to which states the reduction process leads, and the process must become more and more effective as the system becomes more macroscopic in nature.

In Chapter 2, we introduce the pioneering hitting model of Ghirardi, Rimini

and Weber(GRW)[11]. In this model, each constituent particle of any system is 'hit' at randomly distributed times and positions, such that the wavefunction is suddenly localized to a near-position eigenstate. These collapses of the wavefunction are instantaneous and are independent of whether a measurement has taken place. This model gives answers to both the problems above; the final states are localized in position, and the effectiveness of the process for macroscopic objects follows from the fact that a single collapse in a macroscopic object is sufficient to localize the whole object. However, one defect of this model was that the collapse process destroyed any symmetry that the wavefunction may possess, so the model needed development if the required symmetry properties of systems of identical particles were to be maintained.

In Chapter 3, we review the continuous spontaneous localization model(CSL) which differs from the GRW hitting model in that the wavefunction evolves continuously from the initial state to the final state, not instantaneously[12, 13]. This is achieved by the introduction of a stochastic field into the evolution equations. We discuss the consequences of the model, and examine how certain theoretical and experimental constraints can influence the choice of parameters in the model, and suggest a connection to gravity.

Discrete hitting models which, similarly to CSL, preserve the symmetry properties of the wavefunction, and have a more-or-less greater conceptual simplicity in not needing a CSL stochastic field, are described in Chapter 4, and the relationship between the hitting and continuous models is examined[14].

One possible relationship to gravity is discussed in Chapter 5[15], with the postulate of planckons appearing peripatetically to explain the origin of the collapse process. The energy increase caused by these planckons is compared to the energy increase which accompanies collapse, and the correlation of the planckon

potential related to the stochastic fields introduced in CSL.

In the following section, Chapter 6, we look at the effects of possible wavefunction collapse on the operations of quantum computers[16]. We calculate the sizes of possible errors arising as a result of the wavefunction being localized and whether they would be sufficiently large to be significant.

Chapter 7 introduces the first attempts at describing a hitting model that respects Lorentz-invariance[17]. In the GRW model, the collapse is instantaneous over all space, and here we postulate that the collapse is felt only inside of the forward light-cone of the point of origin. The consequences for the Born probability rule are examined.

Finally, Chapter 8 contains a summary of the collapse models and conclusions.

# Chapter 2

# Spontaneous Localization

In this chapter, we review the model originally proposed by Ghirardi, Rimini and Weber in 1986[11] (hereafter to be referred to as the GRW model), which modifies the dynamical evolution of a quantum system by postulating that in addition to the normal Hamiltonian evolution, there is also a stochastic evolution which localizes the wavefunction.

The model is defined in such a way as to leave the dynamics of a microscopic system essentially unaltered from the standard Hamiltonian quantum dynamics. The dynamics of a macroscopic system can be derived from that of its microscopic constituents, and the evolution is found to be compatible with classical mechanics. In the model, linear superpositions of states corresponding to a macroscopic object being localized in markedly different spatial regions are suppressed. The reduction of a pointer, initially in a superposition, into a definite final position can be deduced from the modified equations.

The basic postulate is that these collapse processes occur spontaneously; we do not enquire here as to their origin. (For one attempt, see Chapter 5.)

We shall define the model, and examine some of its consequences for the dynamics of microscopic and macroscopic objects, including the deviation from

the normal quantum-mechanical behaviour, and the rate of collapse of a pointer. We note that the collapse described here necessarily implies a non-conservation of energy.

## 2.1 Definition of Model

Consider a system consisting of $N$ distinguishable particles with wavefunction given by $\Psi = \Psi(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \equiv \Psi(\mathbf{q})$. It was proposed that at random times, the wavefunction will be 'hit' such that it changes instantaneously according to

$$\Psi \rightarrow \Psi^H_{\mathbf{x},j} = \frac{f(\mathbf{x} - \mathbf{q}_j)\Psi}{R_j(\mathbf{x})}, \tag{2.1}$$

where the function $f$ is localized around the zero value of its argument. GRW chose this function such that $f^2$ is a normalized gaussian ($\int f^2(\mathbf{z})\, d^3\mathbf{z} = 1$):

$$f(\mathbf{z}) = \left(\frac{\beta}{\pi}\right)^{\frac{3}{4}} \exp\left(-\frac{\beta}{2}z^2\right), \tag{2.2}$$

where the parameter $\beta$ gives the order of magnitude of the radius of the collapsed wavefunction

$$a = \sqrt{\frac{1}{\beta}}. \tag{2.3}$$

The function $R$ is chosen so that this hitting process preserves the normalization of the wavefunction. Hence

$$|R_j(\mathbf{x})|^2 = \int d^3\mathbf{q}_1 \ldots d^3\mathbf{q}_N |f(\mathbf{x} - \mathbf{q}_j)|^2 |\Psi|^2. \tag{2.4}$$

It follows from these equations that the $R$ functions themselves are normalized,

$$\int |R_j(\mathbf{x})|^2 d^3\mathbf{x} = 1. \tag{2.5}$$

Now we suppose that the probability of hitting particle $j$ in time $dt$ is given by $\lambda_j dt$, and that the probability distribution of the hitting position $\mathbf{x}$ is given

by the function $P_j(\mathbf{x})$. Then the density matrix at time $t + dt$ is given by

$$\rho(t+dt) = \left(1 - \sum \lambda_j dt\right)\left(\rho(t) - \frac{i}{\hbar}\left[H, \rho(t)\right] dt\right) + \sum \lambda_j \rho_j^H dt, \tag{2.6}$$

where $\rho_j^H$ is the density matrix after particle $j$ has been hit. This equation leads to a differential equation governing the evolution of $\rho$:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] - \sum \lambda_j (\rho - \rho_j). \tag{2.7}$$

The model is completely determined when we impose the requirement that *position* averages are conserved, i.e. that the diagonal elements of the density matrix are unchanged by the hitting process. Using a position representation, we have from eq. (2.1)

$$<\mathbf{q}'|\rho_j^H|\mathbf{q}> = \int d^3\mathbf{x} \frac{P_j(\mathbf{x}) f(\mathbf{x} - \mathbf{q}_j') \Psi(\mathbf{q}')^* f(\mathbf{x} - \mathbf{q}_j) \Psi(\mathbf{q})}{|R_j(\mathbf{x})|^2}, \tag{2.8}$$

and hence we require

$$\sum_j \lambda_j \int d^3\mathbf{x} \frac{P_j(\mathbf{x}) \left(f(\mathbf{x} - \mathbf{q}_j)\right)^2}{|R_j(\mathbf{x})|^2} = \sum_j \lambda_j. \tag{2.9}$$

Since this equation must be true for all $\mathbf{q}$, it follows that $P$ is proportional to $|R|^2$, and so, because eq. (2.5) shows that $|R|^2$ is already normalized as a probability, that

$$P_j(\mathbf{x}) = |R_j(\mathbf{x})|^2. \tag{2.10}$$

This probability that the occurrence of collapses occurs most often where the wavefunction is largest, is analogous to the dependence on probabilities of the outcomes of a measurement in standard quantum mechanics. However, these hits occur spontaneously and randomly, regardless of whether any measurement has been performed.

Now, the elements of the density matrix after a hit can be written

$$<\mathbf{q}'|\rho^H|\mathbf{q}''> = \exp\left(-\frac{\beta}{4}(\mathbf{q}' - \mathbf{q}'')^2\right) <\mathbf{q}'|\rho|\mathbf{q}''>, \tag{2.11}$$

where we are dealing with a single particle, and the evolution equation, eq. (2.7) becomes

$$\frac{d}{dt}<\mathbf{q}'|\rho|\mathbf{q}''> = -\frac{i}{\hbar}<\mathbf{q}'|[H,\rho]|\mathbf{q}''> - \lambda\left(1-\exp\left(-\frac{\beta}{4}(\mathbf{q}'-\mathbf{q}'')^2\right)\right)<\mathbf{q}'|\rho|\mathbf{q}''>. \tag{2.12}$$

We can easily see that this last equation is trace-preserving, and we have that

$$\frac{d}{dt}(\mathrm{tr}\rho^2) < 0, \tag{2.13}$$

from eq. (2.7). Thus originally pure states will be transformed into statistical mixtures.

## 2.2 Change of dynamics for a free particle

It is necessary to examine the consequences of the collapse mechanism for the dynamics of a single particle, and to compare with the standard Schrödinger evolution. We consider a single, free particle with Hamiltonian $H$. Let us calculate the mean values of position and momentum and their spreads. For a simple dynamical variable $S$ we have that $<S> = \mathrm{tr}(S\rho)$ as the mean value. Thus, from eq. (2.7), the time derivative of this mean value is given by

$$\begin{aligned}\frac{d}{dt}<S> &= -\frac{i}{\hbar}\mathrm{tr}([S,H]\rho) - \lambda\,\mathrm{tr}\left[\rho\left(S-\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}\right.\right. \\ &\left.\left.\times\int d^3\mathbf{x}\exp\left(-\frac{\beta}{2}(\mathbf{q}-\mathbf{x})^2\right)S\exp\left(-\frac{\beta}{2}(\mathbf{q}-\mathbf{x})^2\right)\right)\right].\end{aligned} \tag{2.14}$$

It is easily seen from this equation, that for any dynamical variable $X$ which is a function of position only, we have

$$\frac{d}{dt}<X(\mathbf{q})> = -\frac{i}{\hbar}\,\mathrm{tr}\left(X(\mathbf{q})[H,\rho]\right), \tag{2.15}$$

as holds for the Schrödinger evolution. There follows immediately

$$\frac{d}{dt}<q_i> = \frac{1}{m}<p_i>, \tag{2.16}$$

and

$$\begin{aligned} \frac{d}{dt}<q_i^2> &= -\frac{i}{2m\hbar}\mathrm{tr}([q_i^2, p_i^2]\rho) \\ &= <q_i p_i + p_i q_i>. \end{aligned} \tag{2.17}$$

One can find the other values of interest by substituting the relevant operators into eq. (2.14). We have immediately

$$\frac{d}{dt}<p_i> = 0, \tag{2.18}$$

and since $<q_i p_i + p_i q_i> = <2q_i p_i - i\hbar>$, we have

$$\begin{aligned} \frac{d}{dt}<q_i p_i + p_i q_i> &= -\frac{i}{2m\hbar}\mathrm{tr}\left([2q_i p_i, p_i^2]\rho\right) \\ &= \frac{2}{m}<p_i^2>. \end{aligned} \tag{2.19}$$

Finally, to find the momentum spreading, and hence the energy increase;

$$\begin{aligned} \frac{d}{dt}<p_i^2> &= -\lambda\hbar^2\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}\mathrm{tr}\left(\int d^3\mathbf{x}[-\beta + \beta^2(q_i - x_i)^2]\exp\left(-\beta(\mathbf{q}-\mathbf{x})^2\right)\rho\right) \\ &= \frac{\beta\lambda\hbar^2}{2}. \end{aligned} \tag{2.20}$$

The expressions given in eqs. (2.16) and (2.18) correspond with the pure Schrödinger evolution,

$$<q_i> = <q_i>_0, \qquad <p_i> = <p_i>_0, \tag{2.21}$$

but the other values differ from the pure Schrödinger evolution because of the extra term on the right-hand side of eq. (2.20). We find that

$$<q_i^2> = <q_i^2>_0 + \frac{\beta\lambda\hbar^2}{6m^2}t^3 \tag{2.22}$$

$$<q_i p_i + p_i q_i> = <q_i p_i + p_i q_i>_0 + \frac{\beta\lambda\hbar^2}{2m}t^2 \tag{2.23}$$

$$<p_i^2> = <p_i^2>_0 + \frac{\beta\lambda\hbar^2}{2}t, \tag{2.24}$$

where the subscript 0 indicates the pure Schrödinger evolution.

For reasonable initial values of the position and momentum spreads, $\Delta q = (<(q - <q>)^2>)^{\frac{1}{2}}$, and $\Delta p = (<(p - <p>)^2>)^{\frac{1}{2}}$, the standard Schrödinger evolution causes the increase in $\Delta q$ with time to be insignificant for all interesting times, due to the smallness of the Planck constant, $\hbar$.

In the case of the dynamics resulting from the collapse process however, the spreads now contain other time-dependent terms. We can calculate, in terms of the parameters $\beta$ and $\lambda$, the interval $T$ for which these terms are smaller than the pure Schrödinger terms. From eqs. (2.22) and (2.24), we have

$$T_1 = \left[\frac{6m^2(\Delta q_s)^2}{\beta\lambda\hbar^2}\right]^{\frac{1}{3}}, \qquad T_2 = \frac{2(\Delta p_s)^2}{\beta\lambda\hbar^2}. \tag{2.25}$$

For times less than these values, the position and momentum spreads are lower than the Schrödinger values, and hence the overall spreads are negligible. The smallness of these terms is related to the small influence of the collapse term on the off-diagonal elements of the density matrix $<\mathbf{q}'|\rho|\mathbf{q}''>$ in the case where $|\mathbf{q}' - \mathbf{q}''| \ll a$.

For the off-diagonal elements, if we ignore the Hamiltonian term in the evolution (as we wish to concentrate on just the collapse behaviour), we can write a solution to eq. (2.12) in the form

$$<\mathbf{q}'|\rho(t)|\mathbf{q}''> = \exp\left[-\lambda t\left(1 - \exp\left(-\frac{\beta}{4}(\mathbf{q}' - \mathbf{q}'')^2\right)\right)\right] <\mathbf{q}'|\rho(0)|\mathbf{q}''>, \tag{2.26}$$

and if we assume that $|\mathbf{q}' - \mathbf{q}''| \gg a$, then we have approximately

$$<\mathbf{q}'|\rho|\mathbf{q}''> \approx \exp(-\lambda t) <\mathbf{q}'|\rho|\mathbf{q}''>. \tag{2.27}$$

This last equation shows that the off-diagonal elements of the density matrix will decay exponentially, with lifetime $T = \frac{1}{\lambda}$. This means that in a time interval of order $T$, a linear superposition of states separated by a distance much larger than $a = \frac{1}{\sqrt{\beta}}$ will be transformed into a statistical mixture.

## 2.3 Rate of collapse of a pointer

We wish to examine the rate of collapse of a macroscopic system of $N$ distinguishable particles, relative to that of a single particle. In GRW, each particle is assumed to experience collapse independently of all the other particles. It will be noted that this assumption is not valid for the case of identical particles, as the collapse process will not then preserve the necessary symmetries of the wavefunction.

We take the system to be pointer of $N$ particles, in a linear superposition of states, separated by a distance $|\mathbf{Q}' - \mathbf{Q}''| \gg a$.

Here, each particle will be localized at a rate $\lambda$, and the evolution of the total density matrix is given by eq. (2.7). We can replace the absolute coordinates of all the particles in the system with the center of mass and relative coordinates,

$$\mathbf{q}_i = \mathbf{Q} + \mathbf{r}_i, \tag{2.28}$$

and assume that the Hamiltonian separates into terms for the center of mass and internal motion, $H = H_{\mathbf{Q}} + H_{\mathbf{r}}$. The dynamical evolution of the center of mass of the system can be found by tracing over the internal variables in the evolution equation, eq. (2.7). If we assume that the internal variable spread is negligible, i.e. $|\mathbf{r}'_j - \mathbf{r}''_j| \ll a$ for all constituents, then we can write $\rho = \rho_Q \rho_r$ and hence

$$\frac{d}{dt}\rho_Q = -\frac{i}{\hbar}[H_Q, \rho_Q] - \sum_{i=1}^{N} \lambda_i \left[ \rho_Q - \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{x} \, \exp\left(-\frac{\beta}{2}(\hat{\mathbf{Q}} - \mathbf{x})^2\right) \rho_Q \times \exp\left(-\frac{\beta}{2}(\hat{\mathbf{Q}} - \mathbf{x})^2\right) \right]. \tag{2.29}$$

In the position representation, we can evaluate elements $<\mathbf{Q}'|\rho|\mathbf{Q}''>$, to find

$$\begin{aligned} \frac{d}{dt}<\mathbf{Q}'|\rho|\mathbf{Q}''> &= -\frac{i}{\hbar}<\mathbf{Q}'|[H_Q, \rho_Q]|\mathbf{Q}''> \\ &\quad - \sum_{i=1}^{N} \lambda_i \left[1 - \exp\left(-\frac{\beta}{4}(\mathbf{Q}' - \mathbf{Q}'')^2\right)\right] <\mathbf{Q}'|\rho|\mathbf{Q}''>. \end{aligned} \tag{2.30}$$

Thus a localization of a single constituent of the pointer is sufficient to localize the whole pointer. To calculate the rate at which this occurs, we can approximate the solution to this last equation as

$$<\mathbf{Q}'|\rho|\mathbf{Q}''> = \exp\left[-\left(\sum_{i=1}^{N}\lambda_i\right)t\left(1-\exp\left(-\frac{\beta}{4}(\mathbf{Q}'-\mathbf{Q}'')^2\right)\right)\right]<\mathbf{Q}'|\rho|\mathbf{Q}''>. \tag{2.31}$$

If we take each constituent particle to localize at the same rate, $\lambda$, then we can write

$$<\mathbf{Q}'|\rho|\mathbf{Q}''> \approx \exp(-N\lambda t)<\mathbf{Q}'|\rho|\mathbf{Q}''>, \tag{2.32}$$

and a pointer of $N$ particles collapse at a rate $N$ times greater than a single particle.

## 2.4 Excitation rate of a bound state

The localization process can cause an atomic bound state to be excited from its ground state. The rate of excitation can be calculated from the expression $R(\phi) = \frac{d}{dt}<\phi|\rho|\phi>$, where the excitation is to the state $|\phi>$. We evaluate this by multiplying eq. (2.7) by $<\phi|\mathbf{q}'><\mathbf{q}''|\phi>$ and integrating over $\mathbf{q}'$ and $\mathbf{q}''$.

For simplicity, we take an atom of hydrogen, initially in its ground state $|\psi_0>$. We make the assumption that the spatial extension of the atom is less than $a$ so that we may expand the exponential in eq. (2.11) to first order in $\beta(\mathbf{q}'-\mathbf{q}'')^2$ to give

$$R(\phi) = \frac{\lambda_e\beta}{2}\left|<\phi|\mathbf{q}|\psi_0>\right|^2, \tag{2.33}$$

where $\mathbf{q}$ is the coordinate of the electron, and the collapse is assumed to affect only the electron. We can replace the absolute coordinate by the relative coordinate, as $\mathbf{q} = \mathbf{Q} + \frac{m_p}{m_e+m_p}\mathbf{r}$, the center of mass operator will not excite the internal atomic states and $\frac{m_p}{m_e+m_p} \approx 1$.

To this order, the $p$-states only will be excited. For example, the rate of exitation to each sublevel of the $2p$-level in hydrogen is $0.277\lambda\beta a_0^2$.

We can also calculate the total rate of excitation, from the expression

$$R_{tot} = \lambda\left(1 - \int d^3\mathbf{x}\left|<\psi_{\mathbf{x}}^H|\psi_0>\right|^2\right). \tag{2.34}$$

Working again to lowest order, using the hydrogen ground state wavefunction,

$$\psi_0 = \frac{1}{(\pi a_0^3)^{\frac{1}{2}}}\exp\left(-\frac{r}{a_0}\right), \tag{2.35}$$

we find that

$$R_{tot} = \frac{3}{2}\lambda\beta a_0^2. \tag{2.36}$$

Using the values of the parameters chosen by GRW, which we discuss in the next section, we find that this formula gives a rate of photoemission of about 25 photons/g Hydrogen/sec.[18] This article also suggests that a possible place in which to search for these photons from collapse is in the deep underground masses of water also used for the detection of possible proton decays and for the observation of neutrinos from supernova explosions.

## 2.5 Choice of parameters

The localization model described in this chapter is all very well, but we need to be able to choose values for the parameters $\beta$ and $\lambda$ such that both the quantum-mechanical predictions for a microscopic system are valid for long time intervals, and a macroscopic object does not persist in a superposition of spatially distinguishable states for sufficient time for it to be perceived.

We must follow some important criteria in the choice of each parameter. For the parameter $\lambda$, we require that the rate of localization for a single particle, or a microscopic system to be very small, such that the dynamics of such a system be

essentially unaltered by the collapse process. For a macroscopic object, however, we require that the mean time, $T_N = \frac{1}{\lambda_N}$ between successive collapses to be small enough that the transition from the superposition to a statistical mixture for states with spatial extent larger than the localization distance $a$ occurs in a small fraction of a second.

For the parameter $\beta$, we need to choose this such that the collapse radius, $a = \frac{1}{\sqrt{\beta}}$ is large compared with an atomic radius, and with mean spreads around equilibrium positions of crystal lattice points. This is so that when a collapse does occur, it does not significantly alter the internal structure of the system. However, as $a$ represents the distance after which a linear superposition would be transformed into a statistical mixture, it must be made sufficiently small that we avoid the occurrence of linear superpositions of markedly different locations of a macroscopic object.

The GRW choice of these parameters was

$$\lambda = 10^{-16}\,\text{sec}^{-1}, \tag{2.37}$$

and

$$a = \frac{1}{\sqrt{\beta}} = 10^{-5}\,\text{cm}. \tag{2.38}$$

The value of $\lambda$ means that a single particle will be localized once every $10^8 - 10^9$ years, practically not at all, and for a macroscopic object containing a number of particles of order Avogadro's number, the mean time between localizations will be $\sim 10^{-7}$ sec.

Note that when we consider the smallest object that can be observed through an optical microscope, it will not be possible to simultaneously satisfy both of the above criteria for $\lambda$, but this is mainly due to the fact that the GRW model in its original form fails to respect the symmetry of the wavefunction. We will see

in later chapters that some of the versions of the collapse models which respect the symmetry have a pointer collapse rate with a higher proportionality to the particle number than the GRW model.

We can evaluate the times in eq. (2.25) for a macroscopic object of mass $M$, as this equation still holds for a macroscopic object with the mass appearing now the total mass $M$ and the collapse rate, $\lambda$ replaces by the collapse rate of the object, $N\lambda$. If we take the object to have a mass of $\sim 1\,\mathrm{g}$, and have initial spread given by $\Delta q_0 \sim 10^{-5}\,\mathrm{cm}$, then the additional term appearing in $\Delta q^2$ will equal $\Delta q_0^2$ at a time of order 100 years, and the time for the corresponding equality in $\Delta p^2$ will be appreciably longer. These are very long times to keep a macroscopic object isolated.

## 2.6 Energy non-conservation

The non-Hamiltonian terms appearing in the evolution equation give rise to a non-conservation of energy. From eq. (2.20), we have that the mean value of the energy is given by

$$<E> = <E>_0 + \frac{3\hbar^2\lambda\beta}{4m}t, \tag{2.39}$$

for each particle, and hence

$$\frac{d}{dt}<E> = \frac{3\hbar^2\lambda\beta}{4m}. \tag{2.40}$$

For a single electron, this result in a rate of energy increase of about

$$\frac{d}{dt}<E> \approx 10^{-21}\,\mathrm{eV\ sec}^{-1}, \tag{2.41}$$

giving a 1 eV increase in $10^{18}$ years.

Whether this energy increase can be accounted for by the possible origins of the collapse processes will be discussed later (see Chapter 5).

## 2.7 Concluding remarks

The GRW model causes the wavefunction to undergo spontaneous localization to near position eigenstates. A superposition of macroscopically distinguishable states is transformed into a statistical mixture as required by the projection postulate of the theory of quantum measurement, and thus is a possible solution of the measurement problem. The modification of the Schrödinger evolution by an additional stochastic behaviour leaves essentially unchanged the dynamics of microscopic objects, whilst macroscopic objects are quickly localized. The effect of the stochastic terms is quite small, but the process necessarily causes an apparent non-conservation of energy. Perhaps all one can say is that this energy increase may be explained by the origin of the collapse process.

# Chapter 3

# The CSL model

The GRW model described in the previous chapter was a discret hitting model which causes the wavefunction to undergo sudden spontaneous localization to near position eigenstates.

In this chapter we describe a localization model which does not cause the wavefunction to change suddenly, but rather evolves continuously from the initial state to the collapsed state. This is known as the continuous spontaneous localization model, or CSL for short.[12, 19, 13]

The CSL theory is in the class of Markov processes in Hilbert space and has a stochastic evolution equation of the form

$$d|\psi> = \left[-\frac{i}{\hbar}Hdt + dh - \frac{1}{2}\overline{(dh)^2}\right]|\psi>, \tag{3.1}$$

where $dh$ is a random, self-adjoint, linear operator. This process is not itself norm-conserving, so if we wish, we can deal instead with the process

$$d|\phi> = \left[-\frac{i}{\hbar}Hdt + dh_\phi - \frac{1}{2}\overline{(dh_\phi)^2}\right]|\phi>, \tag{3.2}$$

which, whilst now norm-conserving, is also non-linear, as now the operator $dh_\phi$ is a function of the state $\phi$. This last equation also embodies an assumption as to the probabilities assigned to the state vectors $|\phi>$.

In section §3.1, we introduce the basic formalism required to describe the CSL model, and show that with a suitable choice of operators, state vector reduction can occur. The actual CSL model is described in subsequent sections, and attention is paid to the values of the parameters in the model in §3.6 .

## 3.1 Markov processes

The CSL model is derived from assuming that the wavefunction in Hilbert space satisfies an Ito stochastic differential equation. In this section we define the necessary formalism for deriving the model. We take the Markov process $|\psi_B(t)>$ to satisfy the Ito equation

$$d|\psi> = (Cdt + \mathbf{A}\cdot d\mathbf{B})|\psi>, \tag{3.3}$$

where $C$ is an operator, $\mathbf{A} \equiv \{A_i\}$ is a set of operators, and $\mathbf{B} \equiv \{B_i\}$ is a real Wiener process such that

$$\overline{dB_i} = 0, \tag{3.4}$$

$$\overline{dB_i dB_j} = \delta_{ij}\gamma dt, \tag{3.5}$$

where $\gamma$ is a real constant. The index $i$ can be continuous, in which case the Kronecker $\delta$ becomes a Dirac $\delta$. The Ito equation, eq. (3.3) generates an ensemble of state vectors $|\psi_B(t)>$, and each state vector has the same probability as the process $\mathbf{B}$ from which it originated. We should note that the evolution equation, eq. (3.3) is not norm-conserving. In fact,

$$\begin{aligned} d\|\psi\|^2 &= <\psi|d\psi> + <d\psi|\psi> + \overline{<d\psi|d\psi>} \\ &= <\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi>\cdot d\mathbf{B} + <\psi|(C+C^\dagger)|\psi>dt \\ &\quad + <\psi|\mathbf{A}^\dagger\cdot\mathbf{A}|\psi>\gamma dt, \end{aligned} \tag{3.6}$$

where the notation $d|\psi> = |d\psi>$ has been used.

As the vectors $|\psi_B(t)>$ do not have unit norm, we consider the ensemble of normalized vectors given by

$$|\chi_B(t)> = \frac{|\psi_B(t)>}{\|\psi_B(t)\|}, \tag{3.7}$$

having the same probabilities as the corresponding vectors $|\psi_B(t)>$, and also the ensemble of vectors

$$|\phi_{B'}(t)> = \frac{|\psi_{B'}(t)>}{\|\psi_{B'}(t)\|}, \tag{3.8}$$

whose probabilities are given by the probabilities of the vectors $|\psi_B(t)>$ multiplied by their squared norms $\|\psi_B(t)\|^2$. The ensemble of vectors $|\phi_B(t)>$ will be called the physical ensemble and the corresponding stochastic process the physical process, in contrast to the raw ensemble and process described above.

To describe the relationship between the raw and physical processes, we denote by $p(\mathbf{B}(t,t_0))$ the probability of the occurrence of the state vector $|\psi_B(t)>$ or the Wiener process $\mathbf{B}(t,t_0)$, and by $q(\mathbf{B}(t,t_0))$ the probability of the state vector $|\phi_B(t)>$. According to definition, we have

$$q(\mathbf{B}(t,t_0)) = \|\psi_B(t,t_0)\|^2 p(\mathbf{B}(t,t_0)). \tag{3.9}$$

Due to the linearity of eq. (3.3) and the Markov nature of the Wiener process $\mathbf{B}$, the procedure from the raw to physical ensemble can be performed at any number of times between the initial and final times. Thus we may consider an infinitesimal time interval $(t_0, t_0 + dt)$, giving

$$q(d\mathbf{B}) = \left(1 + d\|\psi\|^2\right) p(d\mathbf{B}). \tag{3.10}$$

For the ensemble to be deemed physical, we require that the total probability associated with the distribution $q$ is unity. This amounts to requiring that the

average of the weighting factor $\|\psi\|^2$ relative to the distribution $p$ is unity, i.e., $d\overline{\|\psi\|^2} = \overline{d\|\psi\|^2} = 0$. Thus we require, from eq. (3.6). that

$$C + C^\dagger = -\gamma \mathbf{A}^\dagger \cdot \mathbf{A}. \tag{3.11}$$

We can use this condition to rewrite eq. (3.3) in the form

$$d|\psi> = \left(-\frac{i}{\hbar}Hdt + \mathbf{A} \cdot d\mathbf{B} - \frac{\gamma}{2}\mathbf{A}^\dagger \cdot \mathbf{A}dt\right)|\psi>, \tag{3.12}$$

where we have written the anti-Hermitian part of $C$ as $-\frac{i}{\hbar}H$. For $\mathbf{A}$ self-adjoint, this is of the form in eq. (3.1). Also we now have

$$d\|\psi\|^2 = <\psi|(\mathbf{A} + \mathbf{A}^\dagger)|\psi> \cdot d\mathbf{B}, \tag{3.13}$$

and from eq. (3.10),

$$q(d\mathbf{B}) = (1 + 2\mathbf{R} \cdot d\mathbf{B})p(d\mathbf{B}), \tag{3.14}$$

where

$$\mathbf{R} = \frac{1}{2}<\psi|(\mathbf{A} + \mathbf{A}^\dagger)|\psi>, \tag{3.15}$$

and the probability distribution for $q$ is normalized. The process that generates the ensemble, with probability distribution $q$ is denoted by $d\mathbf{B}'$, and one has

$$\overline{dB_i'} = 2R_i\gamma dt, \tag{3.16}$$

$$\overline{dB_i'dB_j'} = \delta_{ij}\gamma dt, \tag{3.17}$$

so that

$$d\mathbf{B}' = d\mathbf{B} + 2\mathbf{R}\gamma dt, \tag{3.18}$$

and $\mathbf{B}'$ is a diffusion process having the same diffusion as $\mathbf{B}$ and drift $2\mathbf{R}\gamma$. It should be noted that the physical process for the un-normalized vectors $|\psi>$ with probabilities equal to the corresponding vector $|\phi>$, can be written, analogously to eq. (3.12), as

$$d|\psi_{B'}> = \left(-\frac{i}{\hbar}Hdt + \mathbf{A} \cdot d\mathbf{B}' - \frac{\gamma}{2}\mathbf{A}^\dagger \cdot \mathbf{A}dt\right)|\psi_{B'}>.$$

To write the stochastic differential equation for the physical process, we must first write it down for the vectors $|\chi>$. From eqs. (3.12) and (3.13) it is found that (See Appendix A for derivation)

$$d|\chi> = \left[\left(-\frac{i}{\hbar}H - \frac{\gamma}{2}\mathbf{A}^\dagger \cdot \mathbf{A} - \gamma \mathbf{A} \cdot \mathbf{R} + \frac{3\gamma}{2}\mathbf{R} \cdot \mathbf{R}\right) dt + (\mathbf{A} - \mathbf{R}) \cdot d\mathbf{B}\right] |\chi>, \tag{3.19}$$

where now we have

$$\mathbf{R} = \frac{1}{2}<\chi|(\mathbf{A} + \mathbf{A}^\dagger)|\chi>. \tag{3.20}$$

The physical process is now obtained by replacing the process $\mathbf{B}(t)$ with the process having the correct probability distribution, namely $\mathbf{B}'(t)$, so that

$$d|\phi> = \left[\left(-\frac{i}{\hbar}H - \frac{\gamma}{2}\mathbf{A}^\dagger \cdot \mathbf{A} - \gamma \mathbf{A} \cdot \mathbf{R} + \frac{3\gamma}{2}\mathbf{R} \cdot \mathbf{R}\right) dt + (\mathbf{A} - \mathbf{R}) \cdot d\mathbf{B}'\right] |\phi>, \tag{3.21}$$

with

$$\mathbf{R} = \frac{1}{2}<\phi|(\mathbf{A} + \mathbf{A}^\dagger)|\phi>. \tag{3.22}$$

We can rewrite this eq. (3.21) in terms of the original Wiener process $\mathbf{B}$, to give

$$d|\phi> = \left[\left(-\frac{i}{\hbar}H - \frac{\gamma}{2}(\mathbf{A}^\dagger - \mathbf{R}) \cdot \mathbf{A} + \frac{\gamma}{2}(\mathbf{A} - \mathbf{R}) \cdot \mathbf{R}\right) dt + (\mathbf{A} - \mathbf{R}) \cdot d\mathbf{B}\right] |\phi>. \tag{3.23}$$

It should be noted that the equations for the norm-conserving processes, eqs. (3.19) and (3.21/3.23), are nonlinear, in contrast to eq. (3.12). The case in which $\mathbf{A}$ is a set of self-adjoint operators is important for our model. In this instance, eq. (3.23) becomes

$$d|\phi> = \left[\left(-\frac{i}{\hbar}H - \frac{\gamma}{2}(\mathbf{A} - \mathbf{R})^2\right) dt + (\mathbf{A} - \mathbf{R}) \cdot d\mathbf{B}\right] |\phi>, \tag{3.24}$$

with now

$$\mathbf{R} = <\phi|\mathbf{A}|\phi>, \tag{3.25}$$

which is of the form of eq. (3.2). We can easily obtain the equation for the evolution of the density matrix corresponding to the physical ensemble, from the definition

$$\rho = \overline{|\phi><\phi|}, \tag{3.26}$$

and by using eq. (3.24). Again using the theory of stochastic differentials, we obtain

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \gamma \mathbf{A}\rho \cdot \mathbf{A}^\dagger - \frac{\gamma}{2}\left\{\mathbf{A}^\dagger \cdot \mathbf{A}, \rho\right\}, \tag{3.27}$$

where the symbols [ , ] and { , } denote the commutator and anticommutator repsectively.

### 3.1.1 Reduction from the Markov process

We shall now see under what conditions the previously desribed process causes the state vector to be reduced onto eigenspaces of the operator **A**. This will cause localization of the wavefunction when **A** is dependent on position.

We are interested in the case when **A** is a set of commuting, self-adjoint operators, and we ignore the Hamiltonian part of the evolution as we are concerned with the new terms. Thus we have

$$d|\phi> = \left[-\frac{\gamma}{2}(\mathbf{A}-\mathbf{R})^2 dt + (\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\phi>, \tag{3.28}$$

with **R** as given in eq. (3.25). We rewrite **A** in terms of orthogonal projections $P_\sigma$;

$$\mathbf{A} = \sum_\sigma \mathbf{a}_\sigma P_\sigma, \tag{3.29}$$

where $\mathbf{a}_\sigma \neq \mathbf{a}_\tau$ for $\sigma \neq \tau$, and the projections sum to the identity. We consider the real, non-negative variables

$$<\phi|P_\sigma|\phi> = z_\sigma, \tag{3.30}$$

with the $z_\sigma$ satisfying

$$\sum_\sigma z_\sigma = 1. \tag{3.31}$$

In terms of these new variables, we can write

$$\mathbf{R} = \sum_\sigma \mathbf{a}_\sigma z_\sigma, \tag{3.32}$$

$$(\mathbf{A} - \mathbf{R})|\phi> = \sum_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau) P_\sigma|\phi>, \tag{3.33}$$

$$(\mathbf{A} - \mathbf{R})^2|\phi> = \sum_\sigma \left[\sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau)\right]^2 P_\sigma|\phi>. \tag{3.34}$$

Hence we can rewrite the evolution equation, eq. (3.28) as

$$dP_\sigma|\phi> = \left(-\frac{\gamma}{2}\left[\sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau)\right]^2 dt + \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau)\cdot d\mathbf{B}\right) P_\sigma|\phi>. \tag{3.35}$$

We use this last equation in the relation

$$d<\phi|P_\sigma|\phi> = <\phi|P_\sigma(dP_\sigma|\phi>) + (d<\phi|P_\sigma)P_\sigma|\phi> + \overline{(d<\phi|P_\sigma)(dP_\sigma|\phi>)}, \tag{3.36}$$

which follows from $P_\sigma^2 = P_\sigma$ and the stochastic differentials. Thus

$$dz_\sigma = 2z_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau)\cdot d\mathbf{B}. \tag{3.37}$$

This last equation shows that when $\{z_\sigma\}$ approaches the solutions to the set of equations

$$z_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau) = 0, \tag{3.38}$$

then the diffusion of $\{z_\sigma\}$ vanishes, and the values of $\{z_\sigma\}$ will eventually converge to such solutions. To prove this, we simply find the differential of $z_\sigma^2$ from eq. (3.37):

$$dz_\sigma^2 = 2z_\sigma dz_\sigma + [2z_\sigma(\mathbf{a}_\sigma - \mathbf{a}_\tau)]^2 \gamma dt, \tag{3.39}$$

and subsequently

$$d\overline{z_\sigma^2} = \overline{dz_\sigma^2} = \left[2z_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau)\right]^2 \gamma dt. \tag{3.40}$$

It then follows that

$$\frac{d}{dt}\overline{z_\sigma^2} \geq 0. \tag{3.41}$$

Since we must have the boundedness property

$$\overline{z_\sigma^2} \leq 1, \tag{3.42}$$

then as $t \rightarrow \infty$, we have

$$\frac{d}{dt}\overline{z_\sigma^2} \rightarrow 0. \tag{3.43}$$

Hence from eq. (3.40),

$$z_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau) \rightarrow 0. \tag{3.44}$$

The only solutions to eq. (3.41) are given by $z_i = \delta_{ij}$ for some $j$. It is immediately apparent that $z_1 = 0, \ldots, z_\sigma = 1, \ldots$ is an acceptable solution, but to show that this type is the only type of solution, suppose that, say, $z_1 \neq 0$ and $z_2 \neq 0$. Then we have

$$\sum_\tau z_\tau(\mathbf{a}_1 - \mathbf{a}_\tau) = 0 \qquad \text{and} \qquad \sum_\tau z_\tau(\mathbf{a}_2 - \mathbf{a}_\tau) = 0, \tag{3.45}$$

and hence

$$\sum_\tau z_\tau(\mathbf{a}_1 - \mathbf{a}_2) = 0, \tag{3.46}$$

giving $\mathbf{a}_1 = \mathbf{a}_2$, contrary to the statement that $\mathbf{a}_\sigma \neq \mathbf{a}_\tau$ unless $\sigma = \tau$. Thus we have that $|\phi>$ reduces to one of the components $P_\sigma|\phi(0)>$ times a real normalizing factor, under the process $B$.

## 3.2 The model

With the general formalism defined in the previous sections, we now must define the set of operators $A_i$ required to cause the continuous spontaneous localization

of a system of particles. At this stage we take the system to consist of identical particles. A locally averaged density operator $N(\mathbf{x})$ is defined according to

$$N(\mathbf{x}) = \sum_s \int d^3\mathbf{y} g(\mathbf{y}-\mathbf{x}) a^\dagger(\mathbf{y},s) a(\mathbf{y},s), \tag{3.47}$$

where $a(\mathbf{x},s)$ and $a^\dagger(\mathbf{x},s)$ are the annihilation and creation operators for a particle at point $\mathbf{x}$ with spin component $s$ satisfying suitable commutation relations. The function $g(\mathbf{x})$ is required to localize the operator, and needs to be a spherically symmetric, positive real function. Following the GRW model, this is taken to be a normalized gaussian,

$$g(\mathbf{x}) = \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta}{2}\mathbf{x}^2\right), \tag{3.48}$$

where $\beta$ is the length parameter as in GRW such that $\beta^{-\frac{3}{2}}$ represents the volume over which the average is taken for $N(\mathbf{x})$.

The operators $N(\mathbf{x})$ are self-adjoint and commute with each other, and with the definition of $g$ we have

$$\int d^3\mathbf{x} N(\mathbf{x}) = N, \tag{3.49}$$

where N is the total number operator.

The eigenvectors of these operators $N(\mathbf{x})$ are given by

$$|\mathbf{q},s> = \mathcal{N} a^\dagger(\mathbf{q}_1,s_1)\ldots a^\dagger(\mathbf{q}_n,s_n)|0>, \tag{3.50}$$

with corresponding eigenvalues

$$n(\mathbf{x}) = \sum_{i=1}^{n} g(\mathbf{q}_i - \mathbf{x}). \tag{3.51}$$

These density operators $N(\mathbf{x})$ can now be identified as the operators $A_i$ in the previous sections, with the index $i$ being replaced by the position $\mathbf{x}$. With this replacement, the evolution equation can be written

$$d|\psi> = \left[-\frac{i}{\hbar}Hdt + \int d^3\mathbf{x} N(\mathbf{x}) dB(\mathbf{x}) - \frac{1}{2}\gamma \int d^3\mathbf{x} N^2(\mathbf{x}) dt\right]|\psi>, \tag{3.52}$$

where the quantities $B$ satisfy the equations

$$\overline{dB(\mathbf{x})} = 0, \tag{3.53}$$

and

$$\overline{dB(\mathbf{x})dB(\mathbf{y})} = \gamma\delta^3(\mathbf{x}-\mathbf{y})dt. \tag{3.54}$$

The density matrix in this case, follows directly by replacing $A$ by $N(\mathbf{x})$ in eq. (3.27) to give

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + \gamma\int d^3\mathbf{x}N(\mathbf{x})\rho N(\mathbf{x}) - \frac{1}{2}\gamma\left\{\int d^3\mathbf{x}N^2(\mathbf{x}),\rho\right\}. \tag{3.55}$$

In the position representation, we can use eq. (3.47) to find

$$\begin{aligned}\frac{\partial}{\partial t}<\mathbf{q}',s'|\rho|\mathbf{q}'',s''> &= -\frac{i}{\hbar}<\mathbf{q}',s'|[H,\rho]|\mathbf{q}'',s''> - \frac{\gamma}{2}\left(\frac{\beta}{4\pi}\right)^{\frac{3}{2}}<\mathbf{q}',s'|\rho|\mathbf{q}'',s''> \\ &\quad\times\sum_{j,k}\left[\Phi(\mathbf{q}'_j-\mathbf{q}'_k)+\Phi(\mathbf{q}''_j-\mathbf{q}''_k)-2\Phi(\mathbf{q}'_j-\mathbf{q}''_k)\right]\end{aligned} \tag{3.56}$$

where we have defined

$$\begin{aligned}\Phi(\mathbf{y}'-\mathbf{y}'') &= \left(\frac{4\pi}{\beta}\right)^{\frac{3}{2}}\int d^3\mathbf{x}\, g(\mathbf{y}'-\mathbf{x})g(\mathbf{y}''-\mathbf{x}) \\ &= \exp\left(-\frac{\beta}{4}(\mathbf{y}'-\mathbf{y}'')^2\right).\end{aligned} \tag{3.57}$$

For a single particle, we can see that this reduces to

$$\begin{aligned}\frac{\partial}{\partial t}<\mathbf{q}'|\rho|\mathbf{q}''> &= -\frac{i}{\hbar}<\mathbf{q}'|[H,\rho]|\mathbf{q}''> \\ &\quad+\gamma\left(\frac{\beta}{4\pi}\right)^{\frac{3}{2}}\left[\exp\left(-\frac{\beta}{4}(\mathbf{q}'-\mathbf{q}'')^2\right)-1\right]<\mathbf{q}'|\rho|\mathbf{q}''>.\end{aligned} \tag{3.58}$$

With the parameter choice

$$\lambda = \gamma\left(\frac{\beta}{4\pi}\right)^{\frac{3}{2}}, \tag{3.59}$$

this coincides exactly with the equation obtained for a single particle in GRW described in §2.1 .

## 3.3 Rate of collapse of a pointer

We want to be able to calculate the relative collapse rate of a macroscopic body to that of a single particle. We assume that the internal wavefunction of the macroscopic body is sharply localized with respect to the length parameter $a = \beta^{-\frac{1}{2}}$, and that it is the center of mass of the body that is to be localized by the collapse process. The macroscopic body, or pointer, is taken to be a system of $n$ identical particles, with the center of mass given by

$$\mathbf{Q} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i. \tag{3.60}$$

We can also write

$$\mathbf{r}_i = \mathbf{q}_i - \mathbf{Q}, \tag{3.61}$$

for the internal coordinates of the particles in the pointer. These coordinates $\mathbf{r}_i$ with respect to the center of mass, sum up to zero, so that they are functions of $3n - 3$ independent internal variables, which will be indicated by $r$.

Thus we wish to consider the wavefunction

$$\psi(\mathbf{q}, s) = \Psi(\mathbf{Q})\chi(r, s), \tag{3.62}$$

where the internal wavefunction $\chi$ is understood to be correctly symmetrized (or antisymmetrized), and both $\chi$ and the center of mass wavefunction $\Psi$ are separately normalized. The wavefunction $\chi(r, s)$ will be taken to be sharply (with respect to $\beta^{-\frac{1}{2}}$ ) peaked around the value $r_0$ of $r$.

We can easily calculate the action of the operator $N(\mathbf{x})$ on this wavefunction,

$$N(\mathbf{x})\Psi(\mathbf{Q})\chi(r, s) = \Psi(\mathbf{Q})\sum_i \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{2}(\mathbf{Q} + \mathbf{r}_i - \mathbf{x})^2\right)\chi(r, s). \tag{3.63}$$

As the function $\chi(r, s)$ is sharply peaked around $r = r_0$, the factor in front of $\chi$ will vary more slowly than $\chi$ itself, and we may take $r = r_0$ in this factor. In

other words, we treat $\chi$ as if it were of the form $\delta^{3n-3}(r-r_0)\xi(s)$. Then we have

$$N(\mathbf{x})\Psi(\mathbf{Q})\chi(r,s) = F(\mathbf{Q}-\mathbf{x})\Psi(\mathbf{Q})\chi(r,s), \tag{3.64}$$

where we have written

$$F(\mathbf{Q}-\mathbf{x}) = \sum_i \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta}{2}(\mathbf{Q}+\mathbf{r}_i(r_0)-\mathbf{x})^2\right). \tag{3.65}$$

Hence the operator $N(\mathbf{x})$ acts solely on the factor $\Psi$ of $\psi$. Thus, if we have that $H = H_Q + H_r$ then $\Psi$ and $\chi$ satisfy the equations

$$d|\Psi> = \left[-\frac{i}{\hbar}H_Q dt + \int d^3\mathbf{x}F(\mathbf{Q}-\mathbf{x})dB(\mathbf{x}) - \frac{\gamma}{2}\int d^3\mathbf{x}F^2(\mathbf{Q}-\mathbf{x})dt\right]|\Psi>, \tag{3.66}$$

and

$$d\chi = -iH_r dt\,\chi, \tag{3.67}$$

respectively. Hence the motion of the center of mass of the pointer and the internal motions decouple in this case. Furthermore, the internal structure is not affected by the stochastic terms under this approximation, whilst the center of mass wavefunction obeys a stochastic differential equation, again of the type given by eq. (3.52). The operators $F(\mathbf{Q}-\mathbf{x})$ which appear in this equation are real functions of the center of mass position operator $\mathbf{Q}$, and correspond to the operators $A_i$ of §3.1 Thus, these non-Schrödinger terms induce the collapse of the wavefunction onto approximate eigenstates of the position $\mathbf{Q}$.

In order to calculate the rate at which this reduction occurs, we can study the off-diagonal elements of the density matrix, $<\mathbf{Q}'|\rho|\mathbf{Q}''>$, disregarding the Schrödinger term as this will not be significant for a process which turns out to be very fast. The evolution equation, eq. (3.55) becomes in this case

$$\frac{\partial}{\partial t}<\mathbf{Q}'|\rho|\mathbf{Q}''> = -\Gamma(\mathbf{Q}',\mathbf{Q}'')<\mathbf{Q}'|\rho|\mathbf{Q}''>, \tag{3.68}$$

where

$$\Gamma(\mathbf{Q}', \mathbf{Q}'') = \frac{\gamma}{2} \int d^3\mathbf{x} \left[ F^2(\mathbf{Q}' - \mathbf{x}) + F^2(\mathbf{Q}'' - \mathbf{x}) - 2F(\mathbf{Q}' - \mathbf{x})F(\mathbf{Q}'' - \mathbf{x}) \right]. \tag{3.69}$$

Equation (3.68) of course gives

$$<\mathbf{Q}'|\rho(t)|\mathbf{Q}''> = \exp(-\Gamma t) <\mathbf{Q}'|\rho(0)|\mathbf{Q}''>. \tag{3.70}$$

We can evaluate the integral in eq. (3.69) exactly in terms of the $\mathbf{r}_i$, but we can gain a clearer physical insight if we use the macroscopic body approximation by replacing the sum by an integral, since we are assuming that a volume $\beta^{-\frac{3}{2}}$ contains a large number of particles. Thus we write

$$F(\mathbf{Q} - \mathbf{x}) = \int d^3\mathbf{x} D(\tilde{\mathbf{y}}) \left( \frac{\beta}{2\pi} \right)^{\frac{3}{2}} \exp\left( -\frac{\beta}{2} (\mathbf{Q} - \tilde{\mathbf{y}} - \mathbf{x})^2 \right), \tag{3.71}$$

where $D(\tilde{\mathbf{y}})$ is the density of particles in the region of the point $\mathbf{y} = \mathbf{Q} + \tilde{\mathbf{y}}$.

Since we are not interested in the details of $\Gamma$ for $\mathbf{Q}' - \mathbf{Q}'' \to 0$, we can make a further approximation by replacing the normalized Gaussian function in eq. (3.71) by the corresponding $\delta$ function, which means

$$F(\mathbf{Q} - \mathbf{x}) = D(\mathbf{x} - \mathbf{Q}), \tag{3.72}$$

resulting in

$$\Gamma(\mathbf{Q}' - \mathbf{Q}'') = \gamma \int d^3\mathbf{x} \left[ D^2(\mathbf{x}) - D(\mathbf{x}) D(\mathbf{x} - \mathbf{Q}' - \mathbf{Q}'') \right]. \tag{3.73}$$

If we take the simple case when the pointer is a homogeneous body of constant density $D_0$ then we have

$$\Gamma = \gamma D_0 n_{out}, \tag{3.74}$$

where $n_{out}$ is the number of particles that are not contained in both the volume occupied when the pointer is in the center of mass position $\mathbf{Q}'$ and when the

pointer has center of mass $\mathbf{Q}''$, i.e. $n_{out} = n - n_{overlap}$. Thus the ratio of the collapse rate of a pointer to that of a single particle is given by

$$\frac{\lambda_p}{\lambda} = n_{out} D_0 \left(\frac{4\pi}{\beta}\right)^{\frac{3}{2}}, \tag{3.75}$$

valid for when $|\mathbf{Q}' - \mathbf{Q}''| \gg \beta^{-\frac{1}{2}}$.

We note here the presence of the extra factor $\sim D_0 a^3$, the number of paritcles in the 'collapse volume'. This was not present in the collapse rate of a pointer in GRW, and gives more scope for the choice of the parameters when constraints are imposed.(§3.6)

## 3.4 Position and momentum spreads

If we substitute $\mathbf{Q}' = \mathbf{Q}''$ into eq. (3.69), then we can see that the diagonal elements of the density matrix, $<\mathbf{Q}|\rho|\mathbf{Q}>$ are unaffected by the localization process. As in the case of the GRW model, we wish to ensure that the time dependence of both position and momentum spreads does not result in unacceptable behaviour.

The equation for the density matrix in operator form is written

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \gamma \int d^3\mathbf{x} \left[F(\mathbf{Q} - \mathbf{x})\rho F(\mathbf{Q} - \mathbf{x}) - \frac{1}{2}\left\{F^2(\mathbf{Q} - \mathbf{x}), \rho\right\}\right], \tag{3.76}$$

where we now retain the Schrödinger term. For the case of a free macroscopic body, we have $H = \frac{1}{2M}P^2$, and for a dynamical variable $S$ the mean value is defined to be

$$<S> = \mathrm{tr}(S\rho). \tag{3.77}$$

The time derivative of $S$ according to eq. (3.76) is given by

$$\begin{aligned} \frac{d}{dt}<S> &= -\frac{i}{\hbar}\mathrm{tr}([S, H]\rho) \\ &+\gamma \int d^3\mathbf{x}\, \mathrm{tr}\left(\left[F(\mathbf{Q} - \mathbf{x})SF(\mathbf{Q} - \mathbf{x}) - \frac{1}{2}\left\{S, F^2(\mathbf{Q} - \mathbf{x})\right\}\right]\rho\right). \end{aligned} \tag{3.78}$$

Using this equation, we find that the position and momentum spreads are

$$\frac{d}{dt}<Q_i> = \frac{1}{M}<P_i>, \tag{3.79}$$

$$\frac{d}{dt}<P_i> = 0, \tag{3.80}$$

and

$$\frac{d}{dt}<Q_i^2> = <Q_iP_i + P_iQ_i>, \tag{3.81}$$

$$\frac{d}{dt}<Q_iP_i + P_iQ_i> = \frac{2}{M}<P_i^2>, \tag{3.82}$$

$$\frac{d}{dt}<P_i^2> = \frac{1}{2}\gamma\delta_i\hbar^2, \tag{3.83}$$

where

$$\delta_i = \int d^3\mathbf{y}\left[\frac{\partial F(\mathbf{y})}{\partial y_i}\right]^2. \tag{3.84}$$

Eqs. (3.79) and (3.80) are the same as in the case of pure Schrödinger evolution, so that

$$<Q_i> = <Q_i>_0 \qquad \text{and} \qquad <P_i> = <P_i>_0, \tag{3.85}$$

where the suffix 0 indicates the pure Schrödinger solution satisfying the same initial conditions.

The system of equations, eqs. (3.81), (3.82) and (3.83) differ from the normal Schrödinger evolution, because of the non-zero term on the right-hand side of eq. (3.83). We can rewrite them as

$$<Q_i^2> = <Q_i^2>_0 + \gamma\delta_i\frac{\hbar^2}{6M^2}t^3, \tag{3.86}$$

$$<Q_iP_i + P_iQ_i> = <Q_iP_i + P_iQ_i>_0 + \gamma\delta_i\frac{\hbar^2}{2M}t^2, \tag{3.87}$$

$$<P_i^2> = <P_i>_0 + \gamma\delta_i\frac{\hbar^2}{2}t. \tag{3.88}$$

The momentum diffusion $\frac{\gamma}{2}\delta_i\hbar^2$ appearing in the third of these equations causes the extra terms in the first two equations. These differ from GRW in

that the factor $n\lambda\beta$ is here replaced with $\gamma\delta_i$. To evaluate the quantities $\delta_i$, we need the details of the macroscopic body in question, and we need to make the macroscopic body approximation, as in §3.3. For simplicity, we take the body to be a box of side $L$, with constant density $D_0$. The macroscopic body approximation gives

$$\begin{aligned} F(\mathbf{y}) &= D_0 \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \int_V d^3\tilde{\mathbf{y}} \exp\left(-\frac{\beta}{2}(\tilde{\mathbf{y}}+\mathbf{y})^2\right) \\ &= \frac{D_0}{8} \prod_{i=1}^{3} \left\{ \mathrm{erf}\left[\left(\frac{\beta}{2}^{\frac{1}{2}}\right)(y_i+L)\right] - \mathrm{erf}\left[\left(\frac{\beta}{2}\right)^{\frac{1}{2}} y_i\right]\right\}. \end{aligned} \tag{3.89}$$

We then substitute this expression into the equation for $\delta_i$, eq. (3.84), to find

$$\delta_i = \frac{D_0^2}{8}\left(\frac{1}{\pi\beta}\right)^{\frac{1}{2}} \left[1-\exp\left(\frac{\beta}{4}\mathrm{L}^2\right)\right] E^2\left[\left(\frac{\beta}{2}\right)^{\frac{1}{2}} L\right], \tag{3.90}$$

where

$$E(x) = \int_{-\infty}^{+\infty} dz \left[\mathrm{erf}(z+x) - \mathrm{erf}(z)\right]^2. \tag{3.91}$$

Since we are dealing with a macroscopic body, we have $x = \left(\frac{\beta}{2}\right)^{\frac{1}{2}} L \gg 1$, and we can take $E(x) = 4x$ so that

$$\delta_i = \left(\frac{\beta}{\pi}\right)^{\frac{1}{2}} D_0^2 L^2. \tag{3.92}$$

For an idea on the order of magnitude of this term, we can choose the GRW values for the parameters $\beta$ and $\lambda$, and a density of, say, $D_0 \approx 10^{24}\,\mathrm{cm}^{-3}$, the minimum value of $L \approx 10^{-4}\,\mathrm{cm}$ for which the approximations are valid, to give

$$\frac{\gamma}{2}\delta_i\hbar^2 \approx 10^{-40}(\mathrm{g\,cm\,s^{-1}})^2\,\mathrm{s}^{-1}. \tag{3.93}$$

This means that the term on the right-hand side of eq. (3.86) will become of the order of $10^{-10}\,\mathrm{cm}^2$ after a time of the order of $10^2\,\mathrm{s}$.

These position and momentum spreads were specifically for the center of mass of the macroscopic body in question. It is perhaps useful to calculate the average

rate of increase of energy per particle. This can be calculated from a similar expression to eq. (3.78),

$$\frac{d}{dt}<H> = \gamma \int d^3\mathbf{x} \mathrm{tr}\left(\left[n(\mathbf{x})Hn(\mathbf{x}) - \frac{1}{2}\left\{H, n^2(\mathbf{x})\right\}\right]\rho\right), \tag{3.94}$$

where $n(\mathbf{x})$ is as given in eq. (3.51). We treat the particles in the system to be non-interacting, so that we can take

$$H = \frac{1}{2m}\sum_{j=1}^{N}\sum_{i=1}^{3}(p_{ji})^2. \tag{3.95}$$

The majority of the terms from eq. (3.95) will either cancel or sum to zero when all the particles are taken into account, leaving just one term

$$\frac{d}{dt}<H> = \frac{\hbar^2\gamma}{2m}\sum_i\sum_j \int d^3\mathbf{x}\ \mathrm{tr}\Bigg(\sum_{\sigma,\tau}\left(\frac{\beta}{2\pi}\right)^3\left[\beta\delta_{\sigma j}(q_\sigma - x)_i \exp\left(-\frac{\beta}{2}(\mathbf{q}_\sigma - \mathbf{x})^2\right)\right] \times\left[\beta\delta_{\tau j}(q_\tau - x)_i \exp\left(-\frac{\beta}{2}(\mathbf{q}_\tau - \mathbf{x})^2\right)\right]\rho\Bigg). \tag{3.96}$$

This expression can be evaluated easily, giving

$$\frac{d}{dt}<H> = \frac{3\hbar^2\gamma\beta N}{32m}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}. \tag{3.97}$$

When the expression for $\gamma$, eq. (3.59) is substituted into this equation, we can see that the average rate of energy increase for a system of $N$ identical particles is given by the same formula as for GRW,

$$\frac{d}{dt}<H> = \frac{3\hbar^2}{4ma^2}\frac{N}{T}, \tag{3.98}$$

for any state of the system.

## 3.5 Bound State Excitation

If we have a system of $N$ particles which belong to a bound state, then the collapse process can cause the system to be excited from the ground state. We assume

that the system has spatial extension much less than $a = \beta^{-\frac{1}{2}}$, and calculate the rate of excitation from the ground state to an orthogonal excited state $|\phi>$.

The rate of excitation is given by

$$R(\phi) = \frac{\partial}{\partial t}<\phi|\rho|\phi>, \tag{3.99}$$

and in order to calculate this, we take the evolution equation for the density matrix, eq. (3.56), multiply through by $<\phi|\mathbf{q}><\mathbf{q}'|\phi>$ and integrate over $\mathbf{q}$ and $\mathbf{q}'$. We take the initial wavefunction to be the ground state so that $\rho = \rho(0) \equiv |\psi_0><\psi_0|$. Due to the orthogonality of the ground state and the excited state, all terms excepting $\Phi(\mathbf{q}_j - \mathbf{q}'_k)$ will vanish. As the spatial extension of the state is much less than $a$, we may expand the function $\Phi$ as

$$\Phi(\mathbf{q}_j - \mathbf{q}'_k) \approx 1 - \frac{q_j^2 + q_k'^2 - 2\mathbf{q}_j \cdot \mathbf{q}'_k}{4a^2}, \tag{3.100}$$

which leads to

$$\begin{aligned} R(\phi)_{t=0} &= \frac{1}{2Ta^2}\sum_{j=1}^{N}\sum_{k=1}^{N}<\phi|\mathbf{q}_j|\psi_0>\cdot<\psi_0|\mathbf{q}_k|\phi>, \\ &= \frac{N^2}{2Ta^2}\left|<\phi|\mathbf{Q}|\psi_0>\right|^2, \end{aligned} \tag{3.101}$$

where $\mathbf{Q}$ is the center of mass position operator for the $N$-particles, which may or may not be the center of mass for the entire system.

If only the electrons in the system were affected by the collapse process, then this equation gives a rate of photoemission similar to GRW. If, however, the process affects both electrons and nucleons, this rate can change considerably. The most economical way to accommodate both electrons and nucleons in the collapse process is to allow the coupling of each type of particle to be different, whilst keeping the parameters $\lambda$ and $\beta$ the same for all particles. If we take the coupling of particle $j$ to be $\mu_j$, then this introduces a factor $\mu_j\mu_k$ into the double

sums in both the equation for the elements of the density matrix, eq. (3.56) and that for the rate of excitation, eq. (3.101).[20]

We can eliminate the excitation to this order in CSL entirely if we choose the couplings to be proportional to the mass, $\mu_j = \frac{m_j}{m_0}$, where $m_0$ is a constant mass-scaling factor. Then the rate of excitation reads

$$R(\phi)_{t=0} = \frac{M^2}{2Ta^2m_0^2} |<\phi|\mathbf{Q}|\psi_0>|^2, \tag{3.102}$$

$$= 0, \tag{3.103}$$

where $\mathbf{Q}$ is now the center of mass position operator for all the particles in the system. The expression is zero, as the center of mass operator does not excite the internal atomice states. With this coupling, the average energy increase for the system can now be written

$$\frac{d}{dt}<H> = \frac{3\hbar^2}{4m_0a^2}\frac{M}{m_0T}. \tag{3.104}$$

The first non-vanishing contribution to the rate of excitation is of order $\frac{1}{T}\left(\frac{m_e}{m_0}\right)^2\left(\frac{a_0}{a}\right)^4$, where $a_0$ is the atomic radius, and will be evaluated for a hydrogen atom in §4.5.

We can make the simplification that atoms consist of just two different types of particles; electrons (mass $m_e$) and nucleons (mass $m_p$), ignoring the nuclear binding energy and the neutron-proton mass-difference. Then we can take the coupling to be $\alpha$ for the electrons, and $1-\alpha$ for the nucleons, so that the excitation rate for an atom $(Z, A)$ becomes

$$R(\phi) = \frac{1}{2Ta^2}\left|<\phi|\alpha\sum_{j=1}^{Z}\mathbf{r}_j + (1-\alpha)\sum_{j=1}^{A}\mathbf{R}_j|\psi_0>\right|^2, \tag{3.105}$$

where we have used the relative coordinates $\mathbf{r}_j$ and $\underline{\mathrm{R}}_j$ for the electrons and nucleons respectively, as we can ignore any occurrence of the atom center of mass

position operator. We can eliminate the position variables $\mathbf{R}_j$ by noting that $m_e \sum_{j=1}^{Z} \mathbf{r}_j + m_p \sum_{j=1}^{A} \mathbf{R}_j \equiv 0$ and thus

$$<\phi| \sum_{j=1}^{A} \mathbf{R}_j |\psi_0> = -\frac{m_e}{m_p} <\phi| \sum_{j=1}^{Z} \mathbf{r}_j |\psi_0>. \tag{3.106}$$

Hence the rate becomes

$$R(\phi) = \frac{1}{2Ta^2} \left(1 + \frac{m_e}{m_p}\right)^2 \left(\alpha - \frac{m_e}{m_e + m_p}\right)^2 \left| <\phi| \sum_{j=1}^{Z} \mathbf{r}_j |\psi_0> \right|^2 . \tag{3.107}$$

## 3.6 Constraint on Parameters

The window for parameters in the GRW model is relatively small. It is necessary that a macroscopic object collapses faster than the time of human perception (at least), whereas there should be no obvious difference in the behaviour of a macroscopic object. However, there were no experimental constraints to be imposed on the parameters, so the initially chosen values could not be ruled out.

We can now put a constraint on the value of $Ta^2$ appearing in the excitation rate by comparing the theory to an experiment measuring the rate of spontaneous X-ray emission in Germanium[21]. In this experiment, we examine the data for the ionization of the $1s$ electrons. Double-beta decay experiments have long been performed to set upper bounds on the rate of spontaneous X-ray emission. If the collapse process causes an electron to be knocked out of the $1s$ shell in Ge, an X-ray pulse will immediately follow. The total energy of the photons emitted will be 11.1keV (the ionization energy), and also the free electron will lose several keV in collisions.

All that is needed from this experiment is the count rate for total energy detecton just over 11.1keV. The experimental upper bound on the X-ray flux is found to be 0.049counts/keV/kg/day. We now need to calculate the rate predicted by CSL.

Initially we will assume that only electrons are affected by the collapse, ($\alpha = 1$), and take as a starting point, eq. (3.101) for the excitation, where we can replace $\mathbf{q}_i$ by the relative coordinates, $\mathbf{r}_i$. We can write $<\phi|\mathbf{r}_1+\mathbf{r}_2+\ldots+\mathbf{r}_n|\psi> = Z<\phi|\mathbf{r}_1|\psi>$ because of the symmetry of the electrons. $|\psi_0>$ will denote the $1s$ electron state and $|\psi_i>$ the other electron states. $|\phi_0, k, m>$ will denote the normalized $l = 1$ ionized electron state with spin index $m$ and momentum $k$. Then using eq. (3.101), the probability that one of the two $1s$ electrons will be ionized to a state with momentum between $k$ and $k + dk$ is

$$dR = \frac{2dk}{2Ta^2}\sum_m |<\phi_0, k, m|\mathbf{r}_1|\psi_0>|^2 \prod_{i=1}^{Z-1} |<\phi_i|\psi_i>|^2 . \tag{3.108}$$

If we let $<\mathbf{r}|\psi_0> = R_0(r)Y_{00}$ and $<\mathbf{r}|\phi_0, k, m> = R_k(r)Y_{1m}$ then we have

$$dR = \frac{dk}{Ta^2}\left[\int_0^\infty r^3 dr R_k(r)R_o(r)\right]^2 \prod_{i=1}^{Z-1}\left[\int_0^\infty r^2 dr R_i'(r)R_i(r)\right]^2 , \tag{3.109}$$

where $R_i'(r)$ and $R_i(r)$ are the wavefunctions of the other, non-$1s$ electrons in the ionized state and ground state respectively.

The integrals here can be approximated by assuming that the electrons in the state are non-interacting, but this will underestimate the count rate. More accurately we can apply Hartree's method, in which each electron is taken to move in a potential due to the nucleus and the average potential due to the other electrons in the atom. For each electron, we solve a differential equation of the form

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(rR_\psi) + V_n R_\psi + V_e R_\psi + V_l R_\psi = E R_\psi, \tag{3.110}$$

where $R_\psi$ is the radial part of the wavefunction, $V_n$ the nuclear potential, $V_e$ is the electron potential from the other electrons, $V_l$ is the potential from the angular momentum and $E$ is the electron energy.

This equation can be solved numerically, by treating the atom as if it were placed in a box. The wavefunctions and transition probabilities can be computed,

and we have a theoretical count rate of

$$\frac{dR}{dE} = \frac{\tilde{a}^2}{2Ta^2}\Gamma, \tag{3.111}$$

where $\tilde{a} = \frac{a_0}{Z}$ is the Bohr radius, and $\Gamma$ is a function of the energy which, according to the numerical calculation, is $\Gamma = 0.548\,\mathrm{keV}^{-1}$ when $E = 11.1\mathrm{keV}$, the ionization energy of the electron. As there are $8.64 \times 10^4\,\mathrm{sec/day}$ and $8.29 \times 10^{24}\,\mathrm{atoms/kg}$ in Ge, the theoretical count rate can by written

$$\mathrm{Counts/keV/kg/day} = 7.17 \times 10^{29}\frac{\tilde{a}^2}{2Ta^2}\Gamma = 5.37 \times 10^9\frac{1}{Ta^2}, \tag{3.112}$$

where $T$ is measured in seconds, and $a$ in cm.

We can substitute the experimental limit into this equation to give the inequality

$$Ta^2 \geq 1.1 \times 10^{11}\,\mathrm{sec\,cm}^2. \tag{3.113}$$

With the GRW choice of parameters, we only have $Ta^2 = 10^6\,\mathrm{sec\,cm}^2$, and clearly we cannot choose these parameters if only the electrons were affected by collapse.

If however both electrons and nucleons are involved in collapse, with coupling $\alpha$ and $1-\alpha$, then the excitation rate changes according to eq. (3.107), and this inequality now becomes

$$Ta^2 \geq 1.1 \times 10^{11}\left(1 + \frac{m_e}{m_p}\right)^2\left(\alpha - \frac{m_e}{m_e + m_p}\right)^2\,\mathrm{sec\,cm}^2. \tag{3.114}$$

For the GRW values of the parameters to be consistent with this inequality, we require that $\alpha \leq 3.5 \times 10^{-3}$. Thus, if we want to maintain these values, then nucleons must be mainly responsible for the collapse. The right-hand side of this inequality vanishes if we choose the couplings to be mass-dependent, $\alpha = \frac{m_e}{m_e+m_p} \approx 0.54 \times 10^{-3}$, and the order of magnitude of the excitation rate is $\frac{1}{T}\left(\frac{m_e}{m_0}\right)^2\left(\frac{a_0}{a}\right)^4$. The experimental data then leads to a new, very weak constraint

$$Ta^4 \geq 10^{-15}\,\mathrm{sec\,cm}^4. \tag{3.115}$$

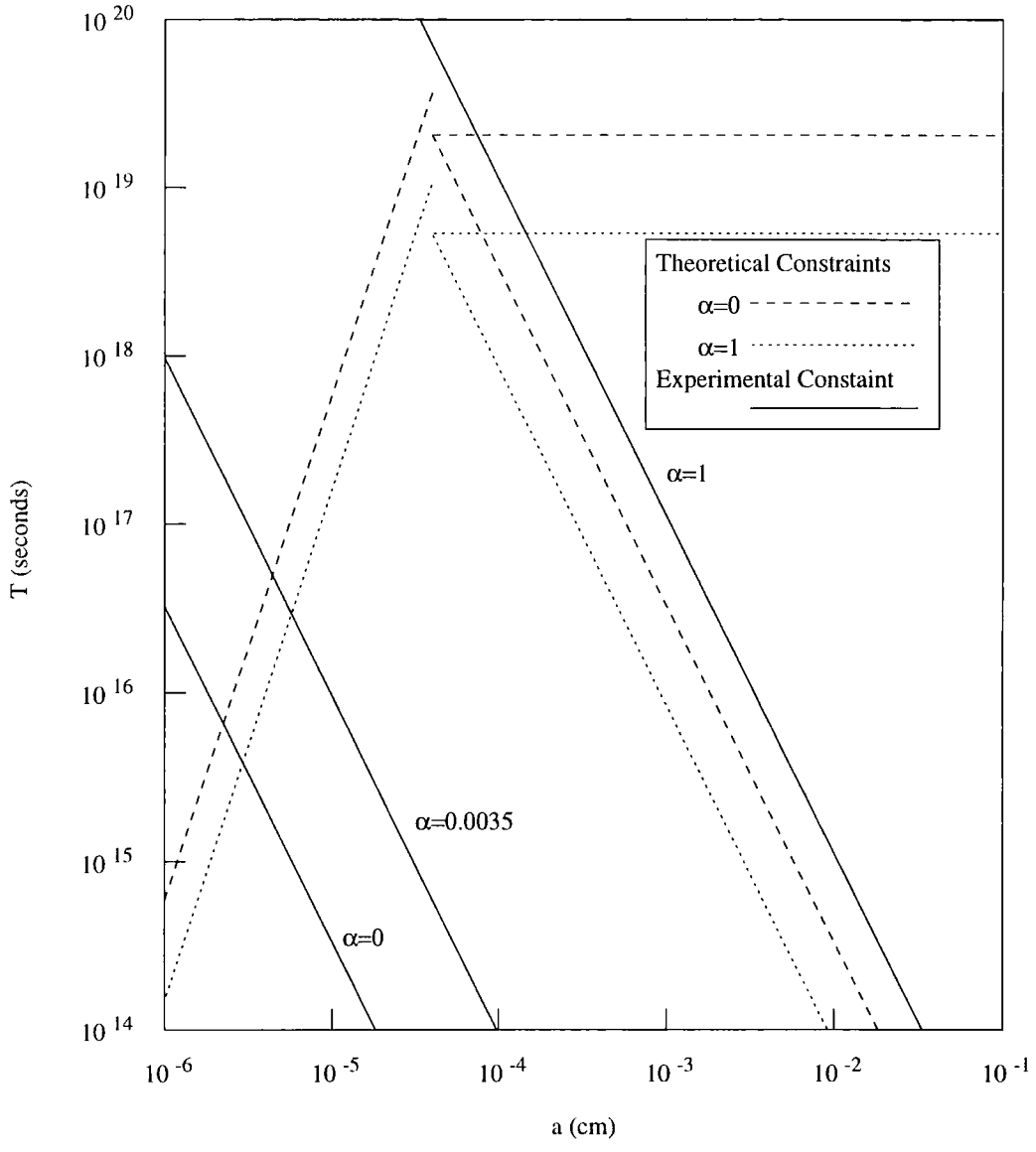



Figure 3.1: The boundaries imposed by the theoretical constraints, eqs. (3.119), (3.120) and (3.121) in the $\log T(\text{sec}) - \log a(\text{cm})$ plane for the cases $\alpha = 0, 1$. The allowed region lies below each boundary. Also shown is the constraint imposed by the Germanium experimental data, eq. (3.115), with the allowed region lying above the boundary

We can also find some theoretical constraints for the model by requiring that macroscopic objects do not persist in a superposition longer than the human perception time, which we conservatively take to be of order 0.01 sec.

We will take the macroscopic object to the smallest object that can be seen through an optical microscope, a sphere of about $4 \times 10^{-5}$ cm diameter. We assume that the sphere is initially in a superposition of states separated by a distance $l$. The collapse times of such an object are then given by

$$\tau \approx \frac{T}{Da^3 N[\alpha Z + (1-\alpha)A]^2}, \qquad a < 4 \times 10^{-5}\,\text{cm} \tag{3.116}$$

$$\tau \approx \frac{T}{N^2[\alpha Z + (1-\alpha)A]^2}, \qquad a > 4 \times 10^{-5}\,\text{cm}, \qquad l \geq a \tag{3.117}$$

$$\tau \approx \frac{4Ta^2}{N^2 l^2[\alpha Z + (1-\alpha)A]^2}, \qquad a > 4 \times 10^{-5}\,\text{cm}, \qquad l \ll a. \tag{3.118}$$

The first two of these equations come from the pointer collapse rate, of §3.3, whereas the third can be derived easily from eq. (3.56), by expanding to first order in $l^2$. Here, $D$ is the particle number density and $N$ is the number of atoms in the sphere. If we take the sphere to be made of carbon, then $D \approx 1.1 \times 10^{23}\,\text{atoms/cm}^3$, $N \approx 3.8 \times 10^9$ atoms, $Z = 6$ and $A = 12$. We take the two states to be on the point of touching, such that $l \approx 4 \times 10^{-5}$ cm, and hence we have three theoretical constraints,

$$Ta^{-3} \leq 1.5 \times 10^{32}(2-\alpha)^2\,\text{sec cm}^{-3}, \qquad a < 4 \times 10^{-5}\,\text{cm} \tag{3.119}$$

$$T \leq 5.2 \times 10^{18}(2-\alpha)^2\,\text{sec}, \qquad a > 4 \times 10^{-5}\,\text{cm}, \qquad l \geq a \tag{3.120}$$

$$Ta^2 \leq 8.3 \times 10^9(2-\alpha)^2\,\text{sec cm}^2, \qquad a > 4 \times 10^{-5}\,\text{cm} = l. \tag{3.121}$$

These theoretical constraints are shown in Fig. (3.1), together with the experimental constraints from the Germanium data. It should be noted that eqs. (3.114) and (3.121) are in fact inconsistent unless the value of $\alpha \leq 0.43$, indicating that the collapse process should certainly affect the nucleons, and suggesting

there is some sort of mass-dependence of the couplings.

## 3.7 Conclusion

The continuous spontaneous localization model introduces a new stochastic field which causes the wavefunction to collapse continuously to near position eigenstates, as opposed to the sudden collapses of the GRW model. The requirements of a negligible effect on microscopic systems and a rapid collapse of the wavefunctions of macroscopice objects are maintained. Moreover, the CSL process can describe all the particles in the system, be they distinguishable or identical. The constraints that can be placed on the parameters of the model suggest that the collapse mechanism, and the new stochastic field may in some manner be related to gravity, and it is there that we should look for the source of collapse.

# Chapter 4

# Symmetric Collapse Models

Although the pioneering GRW 'hitting' model [11] described in the first part of Chapter 2 can cause collapse at a rate consistent with whether the system under consideration is microscopic or macroscopic in nature (with a suitable choice of parameters), the collapse process, eq. (2.1) does not respect any symmetry that the initial wavefunction may possess. Whereas this may not be a problem when all of the particles are distinguishable, we require the symmetry to be preserved in the case of identical particles (Whether the wavefunction is totally symmetric or antisymmetric is not an issue).

This is not a problem in the case of the continuous collapse models that were subsequently developed (described in Chapter 3)[12, 13], and here we will describe possible versions of the explicit collapse models which do maintain the previous symmetry of the wavefunction[14].

We will then introduce a hitting process which will affect distinguishable particles in a different manner, by modifying the equations to include a coupling which is mass-dependent.

It will be observed that there is no unique model to achieve these ends. One degree of freedom will be the choice of the power taken of a particular sum in the

collapse function, and another will be the coupling coefficients. We will compare the theoretical predictions of the models here with the GRW model and CSL for the amount of excitation of bound states and the rate of collapse of a pointer, and using experimental constraints to try and reduce the freedom of choice.

It is found that these comparisons strongly suggest that if collapse does actually occur, it has gravitational origin[23] (See §3.6).

Finally we will discuss the connection between the family of explicit collapse models discussed in this chapter with the continuous CSL-type models.

## 4.1 A hitting process for identical particles

Given that we wish to maintain the wavefunction symmetry, the collapses experienced by individual particles cannot be totally independent as in the GRW model. We require the *system* to undergo the collapse process.

Initially, we will suppose that all of the $N$ particles in this system are identical, and that the wavefunction is correctly symmetrized (or antisymmetrized). An obvious generalization of eq. (2.1) that preserves this property is

$$\Psi \rightarrow \Psi^H_{\mathbf{x}} = \frac{F(\mathbf{x}, \{\mathbf{q}_i\})\Psi}{R(\mathbf{x})}, \tag{4.1}$$

where the function $R(\mathbf{x})$ is required to normalize the wavefunction,

$$R(\mathbf{x}) = \int d^3\mathbf{q}_1 \ldots d^3\mathbf{q}_N F^2(\mathbf{x}, \{\mathbf{q}_i\})\Psi^2 = 1.$$

We must now ask exactly what form the new 'hitting' function $F$ should take. An early attempt by Ghirardi et al.[22] suggested that to maintain the symmetry, the collapse should involve as many collapse centers as there are particles involved, and then the collapse function should be a sum over all permutations, i.e.

$$F(\{\mathbf{x}_i\}, \{\mathbf{q}_i\}) = \sum_{\pi} \left[ \exp\left( -\frac{\beta}{2} \sum_{i=1}^{N} \left( \mathbf{q}_i - \mathbf{x}_{\pi(i)} \right)^2 \right) \right], \tag{4.2}$$

where the $\pi(i)$ are permutations of the numbers 1 to $N$ and the sum is over all such permutations.

However, this does not appear to have nice features when translated into the language of second quantization.

It is much more preferable for a single collapse to only one collapse center, even if the collapse does affect all particles. A single collapse on a macroscopic body will cause its wavefunction to collapse. We can take the system collapsed wavefunction to be an equal superposition of the wavefunctions produced when each individual particle experiences the collapse, i.e. the collapse function becomes

$$F(\mathbf{x}, \{\mathbf{q}_i\}) = \frac{C(\{\mathbf{q}_i\})}{N} \sum_j \left(\frac{\beta}{\pi}\right)^{\frac{3}{4}} \exp\left(-\frac{\beta}{2}(\mathbf{q}_j - \mathbf{x})^2\right). \tag{4.3}$$

The necessity of the factor $C(\{\mathbf{q}_i\})$ will be made clear below.

To ascertain the probability distribution of the collapse centers, and the function $C$, we must again look at the diagonal elements of the density matrix. Since any individual hit may affect all of the particles in the system, we shall denote the probability of a single hit in the system in a time $dt$ by $\lambda_s dt$, where $\lambda_s$ could now be very large. It should be emphasized at this point that later we will require $N = N_u$, where $N_u$ is the number of particles in the universe, and each hit can occur anywhere in the universe. Here we shall assume that there is nothing except the system.

The density matrix evolves almost the same as before, with Hamiltonian evolution when there is no hit in time $dt$,

$$\rho(t + dt) = (1 - \lambda_s dt)\left(\rho(t) - \frac{i}{\hbar}[H, \rho(t)]dt\right) + \lambda_s \rho^H dt, \tag{4.4}$$

which leads to

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] - \lambda_s(\rho - \rho^H). \tag{4.5}$$

If we take the probability of a single hit occurring at $\mathbf{x}$ to be $P(\mathbf{x})$, then we can write $\rho^H$, the density matrix after the hit as (in a position representation):

$$<\mathbf{q}'|\rho^H|\mathbf{q}> = \int d^3\mathbf{x} \frac{P(\mathbf{x})F(\mathbf{x},\{\mathbf{q}'_j\})\Psi(\mathbf{q}')^* F(\mathbf{x},\{\mathbf{q}_j\})\Psi(\mathbf{q})}{|R(\mathbf{x})|^2}, \tag{4.6}$$

with $F$ as given in eq. (4.3).

The requirement that the probability predictions of quantum theory are preserved again means that the diagonal elements of the density matrix must be unchanged by the collapse. This implies that

$$\int d^3\mathbf{x} \frac{P(\mathbf{x})F^2(\mathbf{x},\{\mathbf{q}_j\})}{|R(\mathbf{x})|^2} = 1. \tag{4.7}$$

Substituting in the form of $F$ from eq. (4.3) and making use of the identity

$$(\mathbf{q}_j - \mathbf{x})^2 + (\mathbf{q}_k - \mathbf{x})^2 = 2\left(\mathbf{x} - \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_k)\right)^2 + \frac{1}{2}(\mathbf{q}_j - \mathbf{q}_k)^2, \tag{4.8}$$

we find

$$\frac{C^2}{N^2}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \sum_{j,k} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right) \times \int d^3\mathbf{x} \frac{P(\mathbf{x})}{|R(\mathbf{x})|^2} \exp\left(-\beta\left[\mathbf{x} - \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_k)\right]^2\right) = 1. \tag{4.9}$$

In order for this last equation to be satisfied, the function $C$ must depend on the position variables, $\{\mathbf{q}_j\}$. It seems a reasonable physical requirement that it should depend only on the differences between any two of the position variables, and thus the integrals must be independent of $(\mathbf{q}_j + \mathbf{q}_k)$, and hence we must again choose

$$P(\mathbf{x}) = |R(\mathbf{x})|^2, \tag{4.10}$$

analogous to eq. (2.10). Thus we can now write down the explicit form of the function $C$ from eq. (4.9):

$$\frac{C(\{\mathbf{q}_i\})}{N} = \left[\sum_{j,k} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right)\right]^{-\frac{1}{2}}. \tag{4.11}$$

Note that is the absence of the factor $C(\{\mathbf{q}_i\})$ that causes an earlier attempt at a symmetric version of GRW to fail[24].

### 4.1.1 An alternative hitting model which preserves the symmetry

We can see that an extra complication of the model in the previous section which manifests itself as cross-terms, for example in eq. (4.6), is caused by the factor $C(\{\mathbf{q}_i\})$. It is possible to eliminate this extra factor in the collapse function if we replace eqs. (4.1) and (4.3) with

$$\Psi \to \Psi_{\mathbf{x}}^{H} = \frac{G(\mathbf{x}, \{\mathbf{q}_i\})\Psi}{R(\mathbf{x})}, \tag{4.12}$$

where

$$G(\mathbf{x}, \{\mathbf{q}_i\}) = \frac{1}{N^{\frac{1}{2}}} \left(\frac{\beta}{\pi}\right)^{\frac{3}{4}} \left[\sum_j \exp\left(-\beta(\mathbf{x} - \mathbf{q}_j)^2\right)\right]^{\frac{1}{2}}, \tag{4.13}$$

and the square root removes the need for the position-dependent $C$ factor. Note that the coeffeicient of the square position deviation $(\mathbf{q}_j - \mathbf{x})^2$ in the gaussian is $\beta$, a factor of 2 different to before to ensure that all the models are in agreement when there is only one particle.

In eq. (4.13), we have taken the collapse function to be the square root of the original function of eq. (4.3). In principle, we could take an arbitrary power of this sum to be the collapse function, but the square root is the only one for which the function required to separately normalize the collapse function is in fact constant.

The equation governing the evolution of the density matrix for this model is the same as for the first model, eq. (4.5), with the elements of the hit density

matrix given by

$$<\mathbf{q}'|\rho^H|\mathbf{q}> = \frac{1}{N_u}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{x} \left[\sum_j \exp\left(-\beta(\mathbf{x}-\mathbf{q}'_j)^2\right)\right]^{\frac{1}{2}} <\mathbf{q}'|\rho|\mathbf{q}> \times \left[\sum_k \exp\left(-\beta(\mathbf{x}-\mathbf{q}_k)^2\right)\right]^{\frac{1}{2}}. \quad (4.14)$$

## 4.2 A mass-dependent hitting model

As we shall see below there are reasons for believing that the collapse model should treat all 'matter' particles (fermions) in a symmetrical way, apart from a dependence on the mass. In order to permit this dependence, we can replace eq. (4.3) with

$$F(\mathbf{x},\{\mathbf{q}_i\}) = \frac{C(\{\mathbf{q}_i\})}{N}\sum_j \alpha_j \left(\frac{\beta}{\pi}\right)^{\frac{3}{4}} \exp\left(-\frac{\beta}{2}(\mathbf{q}_j-\mathbf{x})^2\right), \quad (4.15)$$

where $\alpha_j$ is the "coupling" of the $j^{th}$ particles, and the sum is now over all particles. If particles $j_1$ and $j_2$ are identical, then of course the couplings $\alpha_{j_1}$ and $\alpha_{j_2}$ will be equal, thus maintaining the symmetry of the wavefunction. The simplest mass dependence, which would be expected for a gravitational effect, comes from taking $\alpha_j = \frac{m_j}{m_0}$, where $m_0$ is some arbitrary mass scaling factor. We may choose this arbitrarily as $\lambda$ may still be adjusted to satisfy any constraints. Later we will consider atomic systems where we have electrons that couple with strength $\alpha$ and nucleons with strength $1-\alpha$. With this change in the hitting function, the diagonal elements of the density matrix are unchanged if we have

$$\frac{C(\{\mathbf{q}_i\})}{N} = \left[\sum_{j,k}\alpha_j\alpha_k \exp\left(-\frac{\beta}{4}(\mathbf{q}_j-\mathbf{q}_k)^2\right)\right]^{-\frac{1}{2}}. \quad (4.16)$$

We can of course make a similar replacement for the alternative model, with now

$$G(\mathbf{x},\{\mathbf{q}_i\}) = \frac{1}{N^{\frac{1}{2}}}\left(\frac{\beta}{\pi}\right)^{\frac{3}{4}}\left[\sum_j \alpha_j \exp\left(-\beta(\mathbf{q}_j-\mathbf{x})^2\right)\right]^{\frac{1}{2}}. \quad (4.17)$$

## 4.3 The pointer collapse rate

We now wish to compare the rate of collapse of a macroscopic system, which we shall call the pointer, to the rate of collapse of a microscopic system, consisting of a single particle. In particular we shall compare the relative rates for the original GRW model, the continuous localization model (CSL), with mass-dependent collapse, the symmetrical hitting model of §4.1 (which we shall call SHM1) and the alternative model of §4.1.1 (Henceforth to be known as SHM2), together with their modified versions for mass-dependent collapse of §4.2.

Initially, we assume that the pointer is in a state of superposition of two macroscopically distinguishable states, and these states are isolated from the rest of the universe, i.e. the wavefunction can be written

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)\chi_u, \tag{4.18}$$

where $\psi_1$ and $\psi_2$ are the pointer states and $\chi_u$ is the wavefunction for the rest of the universe.

### 1. GRW

In this model each particle is hit independently of all of the others, so the rate of collapse of the pointer is given by

$$\lambda_p = N_p\lambda, \tag{4.19}$$

where $N_p$ is the number of particles in the pointer.

### 2. CSL

As noted in §3.3, the collapse rate has an extra factor causing the pointer to collapse to either $\psi_1$ or $\psi_2$ much faster:

$$\lambda_p = (N_p D_N a^3)\lambda, \tag{4.20}$$

where $D_N$ is the particles-number density of the pointer.

If we consider this model with mass-dependence, then we must include the $\alpha$ factors. In particular, if we take electrons to have a coupling $\alpha$ and nucleons a coupling $1-\alpha$, then for a pointer made of one element (Z, A), the collapse rate becomes

$$\lambda_p = \left(N_p D_n a^3[\alpha Z + (1-\alpha)A]^2\right)\lambda, \tag{4.21}$$

where $N_p$ is now the number of atoms in the pointer, and $D_N$ is the density of atoms in the pointer.

### 3. SHM1

Since we earlier defined $\lambda_s$ to be the collapse rate of the system in question, we must either consider a subset of this system or indeed that the system is a subset of the entire universe.

It is perhaps enlightening to examine further the extra factor $C(\{\mathbf{q}_i\})$ present in the collapse function. We can use a similar method to that of CSL to approximate this factor, by replacing the sum in eq. (4.11) by an integral with extra density factors:

$$\left[\sum_{j,k}\exp\left(-\frac{\beta}{4}(\mathbf{q}_j-\mathbf{q}_k)^2\right)\right] \rightarrow \int\int d^3\mathbf{q}_j d^3\mathbf{q}_k D(\mathbf{q}_j)D(\mathbf{q}_k)\exp\left(-\frac{\beta}{4}(\mathbf{q}_j-\mathbf{q}_k)^2\right). \tag{4.22}$$

This should be a reasonable replacement if we are dealing with a macroscopic object. We are not able to perform the integration without knowing the exact form of the density functions in the expression, but we can make a further approximation by replacing the gaussian by a suitable delta function, which enables us to calculate $C$.

$$C(\{\mathbf{q}_i\}) = \left[N\tilde{D}\left(\frac{4\pi}{\beta}\right)^{\frac{3}{2}}\right]^{-\frac{1}{2}}, \tag{4.23}$$

where $\tilde{D}$ is the average particle density of the macroscopic system. Now this factor is in fact identical to that factor present in the pointer collapse rate in CSL.

The similarity is in fact even more marked if in the original sum over particles in the collapse function, we consider all of the particles in the universe. As the pointer states are taken to be isolated from the rest of the universe, we may rewrite the $C$ factor in the form

$$\frac{C(\{\mathbf{q}_i\})}{N} = \left[\sum_{j,k\in u-p} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right) + \sum_{j,k\in p} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right)\right]^{-\frac{1}{2}}, \tag{4.24}$$

where the first sum is over all the universe apart from the pointer and the second is over the particles in the pointer. As the number of particles in the universe is clearly much larger than the number of particles in the pointer, $N_u \gg N_p$, we can clearly approximate this expression as:

$$\frac{C(\{\mathbf{q}_i\})}{N} = \left(\frac{1}{\Sigma_u}\right)^{\frac{1}{2}} \left[1 - \frac{1}{2\Sigma_u} \sum_{j,k\in p} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right)\right], \tag{4.25}$$

where we have written

$$\Sigma_u = \sum_{j,k\in u} \exp\left(-\frac{\beta}{4}(\mathbf{q}_j - \mathbf{q}_k)^2\right). \tag{4.26}$$

Clearly this number is not going to change significantly by incorporating the particles in the pointer into the sum. The number is very large, and approximately constant.

We may also make these approximations in the equation for the elements of the density matrix, eq. (4.6). Since we are only interested in how the pointer is affected by the collapse, we may take $\mathbf{q}_j = \mathbf{q}'_j$ for particle $j \notin p$. Thus we have

$$\begin{aligned} <\mathbf{q}'|\rho^H|\mathbf{q}> \;=\; & \frac{1}{\Sigma_u}\left[1 - \frac{1}{2\Sigma_u}\sum_{j,k\in p}\Phi(\mathbf{q}_j - \mathbf{q}_k)\right]\left[1 - \frac{1}{2\Sigma_u}\sum_{j,k\in p}\Phi(\mathbf{q}'_j - \mathbf{q}'_k)\right] \\ & \times \left(\Sigma_u + \sum_{j,k\in s}\Phi(\mathbf{q}_j - \mathbf{q}'_k)\right) <\mathbf{q}'|\rho|\mathbf{q}>, \end{aligned} \tag{4.27}$$

where $\Phi(\mathbf{z}) = \exp\left(-\frac{\beta}{4}z^2\right)$.

Taking into account the fact that $\Sigma_u$ is very large, we can substitute the above expression into the equation for the evolution of the density matrix, eq. (4.5) to give

$$\begin{aligned}\frac{\partial<\mathbf{q}'|\rho|\mathbf{q}>}{\partial t} &= -\frac{i}{\hbar}<\mathbf{q}'|[H,\rho]|\mathbf{q}> - \frac{\lambda}{2}<\mathbf{q}'|\rho|\mathbf{q}> \\ &\times \sum_{j,k\in s}\left[\Phi(\mathbf{q}_j-\mathbf{q}_k)+\Phi(\mathbf{q}'_j-\mathbf{q}'_k)-2\Phi(\mathbf{q}_j-\mathbf{q}'_k)\right], \quad (4.28)\end{aligned}$$

where $\Phi$ is given as above and we have written

$$\lambda = \frac{\lambda_u}{\Sigma_u}. \quad (4.29)$$

The above equation for the evolution of the density matrix is in fact identical to that obtained from CSL, and hence the rate of collapse for a pointer in the model SHM1 will be as given for CSL in eqs. (3.70), (3.74) and (3.75).

It is perhaps worth noting that the SHM1 model does allow the possibility of a novel type of superluminal signalling. If we take the overall rate of collapse for the universe to be constant($\lambda_u$), then the effective collapse rate, proportional to $\lambda$, depends on the quantity $\Sigma_u$, which in principle can be altered by changing the density anywhere in the universe (at least when the universe is finite).

### 4. SHM2

As the pointer is isolated from the rest of the universe, we may split the sum over particles into two separate sums; one for the particles in the pointer and the other for the rest of the universe. If the collapse center $\mathbf{x}$ is located on the pointer, then the sum over particles in the rest of the universe will be essentially zero, and vice versa. Thus we can write

$$\left[\sum_j \exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right)\right]^{\frac{1}{2}} = \left[\sum_{j\in u-p} \exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right)\right]^{\frac{1}{2}}$$

$$+\left[\sum_{j\in p}\exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right)\right]^{\frac{1}{2}}, \qquad (4.30)$$

as for all values of $\mathbf{x}$ we may take one of these terms to be zero.

As we are only interested in the pointer, we may again take $\mathbf{q}_i = \mathbf{q}'_i$ for particle $i \notin p$, and we can write the hit density matrix, $\int d^3\mathbf{x} P(\mathbf{x}) |\Psi^H_{\mathbf{x}}><\Psi^H_{\mathbf{x}}|$, as

$$\begin{aligned} <\mathbf{q}'|\rho^H|\mathbf{q}> \;=\; & \left(\frac{1}{N_u}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{x}\left[\sum_{j\in p}\exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right)\right]^{\frac{1}{2}}\right. \\ & \times\left[\sum_{j\in p}\exp\left(-\beta(\mathbf{x}-\mathbf{q}'_j)^2\right)\right]^{\frac{1}{2}} \\ & \left.+\frac{1}{N_u}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}\int d^3\mathbf{x}\left[\sum_{j\in u-p}\exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right)\right]<\mathbf{q}'|\rho|\mathbf{q}>\right) \end{aligned} \qquad (4.31)$$

We may of course evaluate the second term in this expression exactly, giving a value of $\frac{N_u-N_p}{N_u}$, where $N_u$ is the number of particles in the universe, and $N_p$ the number of particles in the pointer.

As the pointer is in a superposition of two macroscopically distinguishable states, the first term in eq. (4.31) is essentially zero, since one of the factors will vanish for each value of $\mathbf{x}$.[1] Substituting this equation into the evolution equation, eq. (4.5) and ignoring the Hamiltonian term, we have

$$\frac{\partial<\mathbf{q}'|\rho|\mathbf{q}>}{\partial t} = -\lambda N_p<\mathbf{q}'|\rho|\mathbf{q}>, \qquad (4.32)$$

where we have written

$$\lambda = \frac{\lambda_u}{N_u}, \qquad (4.33)$$

analogously to eq. (4.29). Therefore the rate of collapse of the pointer is given by

$$\lambda_p = N_p\lambda. \qquad (4.34)$$

[1]More precisely, with the macroscopic body and sharp-scanning approximations, the term is approximately $n_{overlap}$.

Again, we can modify these equations for the case when the couplings are mass-dependent, leading to a similar expression for the collapse rate:

$$\lambda_p = \sum_{j \in p} \alpha_j \lambda, \tag{4.35}$$

where now we have

$$\lambda = \frac{\lambda_u}{\sum_{j \in u} \alpha_j}, \tag{4.36}$$

replacing eq. (4.33). In particular, for a pointer made solely of atoms (Z, A) with the coupling $\alpha$ for electrons and $1 - \alpha$ for nucleons, then

$$\lambda_p = \left(N_p[Z\alpha + (1-\alpha)A]\right)\lambda, \tag{4.37}$$

where $N_p$ is the number of atoms in the pointer. We can see that this model has the same pointer collapse rate as the GRW model; it is linear in the number of particles, lacking the extra factor present in the SHM1 and CSL models.

Exactly how the pointer collapse rate varies with $N_p$ when the overlap between $\psi_1$ and $\psi_2$ does not vanish, will be discussed later.

## 4.4 Position and Momentum Spreads

As in the case of the GRW model and CSL, we will calculate the effects of the stochastic terms in the evolution equation on the mean values and spreads of position and momentum.

For SHM1, the values calculated will, in the limit as the number of particles in the universe tends to infinity, be the same as for CSL, eqs. (3.85) -(3.88).

For SHM2, we calculate the values as before, using the equation

$$\frac{\partial}{\partial t}<S> = -\frac{i}{\hbar}\mathrm{tr}([S,H]\rho) - \lambda_u \mathrm{tr}(S\rho) + \frac{\lambda_u}{N_u}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \mathrm{tr}\left(\int d^3\mathbf{x} F^{\frac{1}{2}}\, S\, F^{\frac{1}{2}}\rho\right), \tag{4.38}$$

which follows from the evolution equation, eq. (4.5). We have also written

$$F = \sum_j \exp\left(-\beta(\mathbf{q}_j - \mathbf{x})^2\right). \tag{4.39}$$

As in previous cases, if the operator $S$ is solely a function of position, then the values are given by the standard Schödinger evolution. Similarly, when $S$ contains only one power of momentum, this will also hold. The problem case is where $S = p_l^2$, in which case we have

$$\begin{aligned}
\frac{\partial}{\partial t}<p_{l(i)}^2> &= -\lambda\hbar^2\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \mathrm{tr}\left(\int d^3\mathbf{x} F^{\frac{1}{2}}\left[\frac{\partial^2}{\partial q_{l(i)}^2} F^{\frac{1}{2}}\right]\rho\right) \\
&= \lambda\hbar^2\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \mathrm{tr}\left(\int d^3\mathbf{x}\left[\beta - 2\beta^2(q_l - x)_i^2\right]\exp\left(-\beta(\mathbf{q}_l - \mathbf{x})^2\right)\rho\right) \\
&\quad +\lambda\hbar^2\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \mathrm{tr}\left(\int d^3\mathbf{x}\frac{1}{F}\beta^2(q_l - x)_i^2\exp\left(-2\beta(\mathbf{q}_l - \mathbf{x})^2\right)\rho\right)
\end{aligned} \tag{4.40}$$

The first term on the right-hand side of this last equation vanishes when an integration by parts is performed. The second term cannot be integrated exactly because of the presence of the factor $\frac{1}{F}$ in the integrand. However, we may estimate the magnitude of the term by condsidering a couple of special cases.

Firstly, let us examine this integral when the $l^{th}$ particle is isolated from all the other particles. (Isolated in this case means with a separation greater than the collapse radius $a$.) In this case, the function $F$ may esssentially only contains one term over the relevant range of $\mathbf{x}$, since the numerator will ensure the integrand is small for other values. Thus, in this case,

$$\begin{aligned}
\frac{\partial}{\partial t}<p_{l(i)}^2> &\approx \lambda\hbar^2\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \mathrm{tr}\left(\int d^3\mathbf{x}\beta^2(q_l - x)_i^2\exp\left(-\beta(\mathbf{q}_l - \mathbf{x})^2\right)\rho\right) \\
&\approx \frac{1}{2}\lambda\beta\hbar^2.
\end{aligned} \tag{4.41}$$

So we recover eq. (2.40) for the average rate of energy increase per particle in this instance. However, if we assume that the particles are packed more densely

around the particle in question, then more particles will contribute in the sum in $F$. If we make the approximation of a constant particle density around $\mathbf{q}_l$, then the number of particles that will contribute to the sum is approximately $Da^3$ where $D$ is the particle number density. Thus the momentum spread will be as in eq. (4.41), but reduced by about a factor of $Da^3$, and the average rate of energy increase will differ from that of the GRW and CSL models.

To summarize, the mean values and spreads are given by

$$<q_i> = <q_i>_0, \qquad <p_i> = <p_i>_0, \tag{4.42}$$

as with the pure Schrödinger evolution, and

$$<q_i^2> = <q_i^2>_0 + \frac{\beta\lambda\hbar^2}{6m^2}\frac{1}{f(D)}t^3 \tag{4.43}$$

$$<q_ip_i + p_iq_i> = <q_ip_i + p_iq_i>_0 + \frac{\beta\lambda\hbar^2}{2m}\frac{1}{f(D)}t^2 \tag{4.44}$$

$$<p_i^2> = <p_i^2>_0 + \frac{\beta\lambda\hbar^2}{2}\frac{1}{f(D)}t, \tag{4.45}$$

where the function $f(D)$ indicates the approximate effect of the particle density on the spreads, and is approximately equal to the number of particles in the collapse volume, c.f. eqs. (2.22)-(2.24).

## 4.5 Bound State Excitation

In section §4.3, we were concerned with the collapse rate of a pointer, giving a lower limit to the collapse rate. Now we switch our attention to an effect which has not been observed thus far, and therefore will give an upper limit to the collapse rate.

We suppose that the particles in the system belong to an isolated bound state, with spatial separation much less than $a = \frac{1}{\sqrt{\beta}}$, as for example in an atom.

Initially we take the system to be in its ground state $\psi_0$. We will consider the rate of excitation to an orthogonal excited state $\phi$ in the various models.

As the spatial separation of the particles are small, so are the $\mathbf{q}_i$, and we can make an expansion to lowest order in $\beta q_i^2$.

### 1. GRW

The expression for the rate of excitation is given by

$$R(\phi) = \frac{\lambda\beta}{2} \sum_{k=1}^{N_s} |<\phi|\mathbf{q}_k|\psi_0>|^2 , \tag{4.46}$$

where $N_s$ is the number of particles in the system. This has order of magnitude $\frac{\lambda}{2}\left(\frac{a_0}{a}\right)^2$, where $a_0$ is the radius of the bound state.

### 2. CSL

For the continuous model, the corresponding expression to eq. (4.46) is

$$R(\phi) = \frac{\lambda\beta}{2} \left| <\phi| \sum_{k=1}^{N_s} \alpha_k \mathbf{q}_k |\psi_0> \right|^2 , \tag{4.47}$$

where the sum is over all the particles in the bound state. If the $\alpha_k$ are all equal, the effect is dominated by the electrons, and the excitation rate is similar to that given above for GRW.

A very different situation arises if we choose mass-dependent couplings, $\alpha_i = \frac{m_i}{m_0}$. In particular if we have $\alpha_e = \frac{m_e}{m_e+m_p}$ and $\alpha_p = 1 - \alpha_e$ (ignoring the neutron-proton mass-difference), then

$$R(\phi) = \left(\frac{M_a}{m_0}\right)^2 \frac{\lambda\beta}{2} |<\phi|\mathbf{Q}|\psi_0>|^2 , \tag{4.48}$$

where $\mathbf{Q}$ is the center of mass operator and $M_a$ is the mass of the bound atom (Here, it is actually the sum of masses of its constituent particles). This last expression is equal to zero, as the center of mass operator cannot excite the

internal atomic states. Thus there is no excitation to lowest order in the mass-dependent case. The first non-vanishing term gives an excitation proportional to $\lambda\left(\frac{a_0}{a}\right)^4$.

### 3. SHM1

As we noted above, the equation for the evolution of the off-diagonal elements of the density matrix is identical to that obtained in CSL, and hence the expressions for the rate of excitation of a bound state will be as in eqs. (4.47) and (4.48) above.

### 4. SHM2

To calculate the rate of excitation in SHM2, we need to work with the equation for the elements of the density matrix. Whereas the first term in eq. (4.31) is exact, an approximation is needed to proceed further with the second term. We write

$$\Sigma_s = \sum_{j\in s} \exp\left(-\beta(\mathbf{x}-\mathbf{q}_j)^2\right), \tag{4.49}$$

where the sum is over the particles in the system, and hence

$$\Sigma_s^{\frac{1}{2}} = \exp\left(-\frac{\beta}{2}x^2\right)\left[\sum_{j\in s}\exp\left(\beta(2\mathbf{q}_j\cdot\mathbf{x}-q_j^2)\right)\right]^{\frac{1}{2}}. \tag{4.50}$$

The gaussian at the front of this expression, $\exp\left(-\frac{\beta}{2}x^2\right)$, ensures that only terms with $x < a$ will be significant. Since we are concerned with a system with spatial separation much less than $a$, i.e. $q \ll a$, we have $(xq) \ll a^2$ and hence $\beta(xq) \ll 1$ so we can make a power series expansion of the exponentials and just keep the first few terms.

$$\Sigma_s^{\frac{1}{2}} = \exp\left(-\frac{\beta}{2}x^2\right)\left[\sum_{j\in s}\left(1+\beta(2\mathbf{q}_j\cdot\mathbf{x}-q_j^2)+2\beta^2(\mathbf{x}\cdot\mathbf{q}_j)^2\right)\right]^{\frac{1}{2}}, \tag{4.51}$$

as we are only interested in terms that give a contribution to first order in $q^2$. We can similarly expand the square root to give

$$\begin{aligned}\Sigma_s^{\frac{1}{2}} &= N_s^{\frac{1}{2}} \exp\left(-\frac{\beta}{2}x^2\right) \\ &\times \left[1 + \beta \mathbf{Q}_s \cdot \mathbf{x} - \frac{1}{N_s}\sum_{j\in s}\left(\frac{\beta}{2}q_j^2 - \beta^2(\mathbf{x}\cdot\mathbf{q}_j)^2\right) - \frac{\beta^2}{2}(\mathbf{x}\cdot\mathbf{Q}_s)^2\right] \quad (4.52)\end{aligned}$$

where $\mathbf{Q}_s$ is the center of mass operator for the system. We can then substitute this expression into the equation for the evolution of the density matrix, eq. (4.5), and by making use of

$$\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{x}(\mathbf{x}\cdot\mathbf{q}_j)^2 \exp\left(-\beta x^2\right) = \frac{1}{2\beta}q_j^2, \quad (4.53)$$

we find that

$$\begin{aligned}<\mathbf{q}'|\rho^H|\mathbf{q}> &= <\mathbf{q}'|\rho|\mathbf{q}>\left[\frac{N_u - N_s}{N_u}\right. \\ &+ \left.\frac{N_s}{N_u}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{x} \exp\left(-\beta x^2\right)\left(1 - \frac{\beta^2}{2}[\mathbf{x}\cdot(\mathbf{Q}_s - \mathbf{Q}_s')]^2\right)\right] \\ &= \left[1 - \left(\frac{N_s}{N_u}\right)\frac{\beta}{4}(\mathbf{Q}_s - \mathbf{Q}_s')^2\right] <\mathbf{q}'|\rho|\mathbf{q}>. \quad (4.54)\end{aligned}$$

We can substitute this last expression back into the evolution equation to get

$$\frac{\partial <\mathbf{q}'|\rho|\mathbf{q}>}{\partial t} = -\frac{i}{\hbar}<\mathbf{q}'|[H,\rho]|\mathbf{q}> - \frac{N_s\lambda\beta}{4}(\mathbf{Q}_s - \mathbf{Q}_s')^2 <\mathbf{q}'|\rho|\mathbf{q}>. \quad (4.55)$$

To calculate the rate of excitation, $R(\phi) = \frac{\partial <\phi|\rho|\phi>}{\partial t}$, we must multiply both sides of eq. (4.55) by $<\phi|\mathbf{q}'><\mathbf{q}|\phi>$, and integrate over $\mathbf{q}$ and $\mathbf{q}'$, leading to

$$R(\phi) = \frac{N_s\lambda\beta}{2}\left|<\phi|\mathbf{Q}|\psi_0>\right|^2, \quad (4.56)$$

and there is again no excitation to this order, as this equation only involves the center of mass operator. Including the $\alpha_j$ factors yields

$$R(\phi) = \frac{\lambda\beta}{2N_s}\left|<\phi|\sum_{k=1}^{N_s}\alpha_k\mathbf{q}_k|\psi_0>\right|^2. \quad (4.57)$$

Apart from the factor of $N_s$, which is 2 for hydrogen for example, this is the same as the rate for CSL and SHM1.

As before, in the case where the $\alpha_j$ are linearly proportional to the mass of particle $j$, then the the contribution to the excitation rate $R(\phi)$ is zero to this order.

When this is not the case, we can denote the ratio of the excitation rate to the pointer collapse rate by $\Gamma_{SHM2}$ and similarly by $\Gamma_{SHM1}$ for these two models, and we have that

$$\frac{\Gamma_{SHM1}}{\Gamma_{SHM2}} = \frac{N_s}{D_n a^3[\alpha Z + (1-\alpha)A]}, \tag{4.58}$$

which will be very small, of order $10^{-8}$ when we consider a carbon pointer and take $a = 10^{-5}$cm. This is one reason why the constraints on the parameters are much tighter than those for CSL or SHM1.

### 4.5.1 Excitation with mass-dependent couplings

For the case when the couplings $\alpha_j$ are proportional to the mass of the $j^{\text{th}}$ particle, we will evaluate the first non-vanishing contribution to the excitation rate of the bound state.

#### 1. CSL/SHM1

The calculation of the next order term is straightforward. We can calculate the excitation rate directly from the evolution equation, eq. (4.28). Only terms which contain both primed and unprimed coordinates will give a contribution to the excitation rate. Furthermore, we may neglect any occurrence of the center of mass operator, as it does not excite the internal atomic states. The rate of

excitation becomes

$$R(\phi) = \frac{\lambda\beta^2}{16}\int d^3\mathbf{q}\, d^3\mathbf{q}' \left[\sum_{j,k}\frac{m_j m_k}{m_0^2}\left(q_j^2 q_k'^2 + 2(\mathbf{q}_j\cdot\mathbf{q}_k')^2\right)\right] \times <\phi|\mathbf{q}'><\mathbf{q}'|\psi_0><\psi_0|\mathbf{q}><\mathbf{q}|\phi>. \tag{4.59}$$

To get any sort of numerical result, we need to consider a particular atomic system and substitute explicit forms for the wavefunctions in eq. (4.59). For simplicity, we will take the simplest atomic system, that of a solitary hydrogen atom, and consider the collapse affecting both the proton and the electron. Initially, we take the atom to be in its ground state(1s), and will be calculate the rate of excitation to the lowest excited states, the 2s and 2p levels. The $m = 0$ state of the 2p level is not excited here, unlike the case of GRW where it is excited in first order, even in the mass-dependent case.

In order to evaluate eq. (4.59), we can change the coordinate basis, and make the replacements $\mathbf{q}_1 = \mathbf{Q} - \frac{m_e}{M}\mathbf{r}$ and $\mathbf{q}_2 = \mathbf{Q} + \frac{m_p}{M}\mathbf{r}$ for the proton and electron respectively. Any appearances of the center of mass operators will cancel out, leaving the integral over this coordinate as trivial.

### 2. SHM2

In this case, we have already expanded $\Sigma_s^{\frac{1}{2}}$ as far as we require. The same terms will contribute as in the calculation for CSL/SHM1. This means that the relevant part of $\Sigma_2^{\frac{1}{2}}$ can be written as

$$\Sigma_s^{\frac{1}{2}} = \left(\frac{M_s}{m_0}\right)\exp\left(-\frac{\beta}{2}x^2\right)\left[1 - \frac{m_0}{M_s}\sum_{j\in s}\left(\frac{\beta}{2}q_j^2 - \beta^2(\mathbf{x}\cdot\mathbf{q}_j)^2\right)\right]. \tag{4.60}$$

In order to simplify the expression for the excitation rate obtained using this expansion, we make further use of eq. (4.53) together with

$$\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}\int d^3\mathbf{x}(\mathbf{x}\cdot\mathbf{q}_j)^2(\mathbf{x}\cdot\mathbf{q}_k')^2\exp\left(-\beta x^2\right) = \frac{1}{4\beta^2}\left(q_j^2 q_k'^2 + 2(\mathbf{q}_j\cdot\mathbf{q}_k')^2\right), \tag{4.61}$$

| Excitation | GRW | CSL/SHM1 | SHM2 |
|---|---|---|---|
| 2s | 0 | $0.231\lambda\beta^2\left(\frac{m_e}{m_0}\right)^2 a_0^4$ | $0.370\lambda\beta^2\frac{m_0}{M}\left(\frac{m_e}{m_0}\right)^2 a_0^4$ |
| 2p(m=0) | $0.277\lambda\beta\left(\frac{m_e}{m_0}\right) a_0^2$ | 0 | 0 |
| 2p(m=±1) | $0.277\lambda\beta\left(\frac{m_e}{m_0}\right) a_0^2$ | $0.267\lambda\beta^2\left(\frac{m_e}{m_0}\right)^2 a_0^4$ | $1.070\lambda\beta^2\frac{m_0}{M}\left(\frac{m_e}{m_0}\right)^2 a_0^4$ |

Table 4.1: Rates of Excitation for a Hydrogen Atom

which leads to the expression

$$R(\phi) = \frac{\lambda\beta^2}{2}\left(\frac{m_0}{M_s}\right)\int d^3\mathbf{q}\, d^3\mathbf{q}' \sum_{j,k}\left(\frac{m_j m_k}{m_0^2}(\mathbf{q}_j\cdot\mathbf{q}'_k)^2\right) \times <\phi|\mathbf{q}'><\mathbf{q}'|\psi_0><\psi_0|\mathbf{q}><\mathbf{q}|\phi>. \tag{4.62}$$

To compare the rates here with those obtained in CSL/SHM1, we use the same atomic system as before, the hydrogen atom, and we replace the absolute coordinates with the center of mass and relative coordinates as before.

Table 3.1 shows the excitation rates for all the models under discussion for the case of mass-dependent couplings.

If we take $m_0 = M = m_e + m_p$ as suggested before, then the extra factor $\frac{m_0}{M}$ in the rates for SHM2 is unity, and the excitation rates have a dependence on $\lambda\beta^2\left(\frac{m_e}{M}\right)^2 a_0^4$ for both CSL/SHM1 and SHM2.

## 4.6 Constraints on Parameters

We can find similar constraints for the SHM2 model to those of CSL/SHM1, which are described in §3.6.[21] There are two theoretical constraints, formed in the same way as for CSL, on what we need the model to achieve. (Only two as there is no density factor here in the rate of collapse of a pointer.)

Again, we take the pointer to consist of a sphere with diameter $\approx 4\times10^{-5}$ cm, in a superposition with separation $l$ between the two states.

When the separation $l$ is greater than the collapse parameter $a$, the collapse rate from eq. (4.32) is

$$\tau \approx \frac{T}{N[\alpha Z + (1-\alpha)A]} \qquad l \geq a, \tag{4.63}$$

where $T = \frac{1}{\lambda}$. Note that if the two pointer positions were to overlap, then $N$ here would be reduced to the number in those positions that are not in the overlap region.

When the separation is less than $a$, we must approximate the first term in eq. (4.31). If we assume that the two states in the superposition are identical in configuration, with a shift in the center of mass only, then we can write

$$\begin{aligned} \exp\left(-\beta(\mathbf{x}-\mathbf{q}'_k)^2\right) = \; & \exp\left(-\beta(\mathbf{x}-\mathbf{q}_k)^2\right)\exp\left(-\beta l^2\right) \\ & \times \exp\left(2\beta(\mathbf{Q}'-\mathbf{Q})\cdot(\mathbf{x}-\mathbf{q}_k)\right). \end{aligned} \tag{4.64}$$

To enable us to evaluate the integral in eq. (4.31), we must approximate the final term. Since $|\mathbf{Q}'-\mathbf{Q}| = l \ll a$, we can expand the exponential and take just the first two terms. A futher approximation enables us to write:

$$\begin{aligned} \left[\sum_k \exp\left(-\beta(\mathbf{x}-\mathbf{q}'_k)^2\right)\right]^{\frac{1}{2}} = \; & \left[\sum_k \exp\left(-\beta(\mathbf{x}-\mathbf{q}_k)^2\right)\right]^{\frac{1}{2}} \exp\left(-\frac{\beta}{2}l^2\right) \\ & \times \left[1+\beta\frac{\sum_k \exp\left(-\beta(\mathbf{x}-\mathbf{q}_k)^2\right)(\mathbf{Q}'-\mathbf{Q})\cdot(\mathbf{x}-\mathbf{q}_k)}{\sum_k \exp\left(-\beta(\mathbf{x}-\mathbf{q}_k)^2\right)}\right]. \end{aligned} \tag{4.65}$$

If we substitute this last expression into eq. (4.31), we find that the last term in eq. (4.65) will cause this part of the integral to vanish through symmetry, leaving us with

$$<\mathbf{q}'|\rho^H|\mathbf{q}> = \left(1-\frac{N_s}{N_u}\left(\exp\left(-\frac{l^2}{2a^2}\right)-1\right)\right)<\mathbf{q}'|\rho|\mathbf{q}>. \tag{4.66}$$

We of course can modify this expression to take into account the mass-dependent couplings, $\alpha$, and we have a collapse time given by

$$\tau \approx \frac{2Ta^2}{Nl^2[\alpha Z + (1-\alpha)A]}, \qquad l < a. \tag{4.67}$$

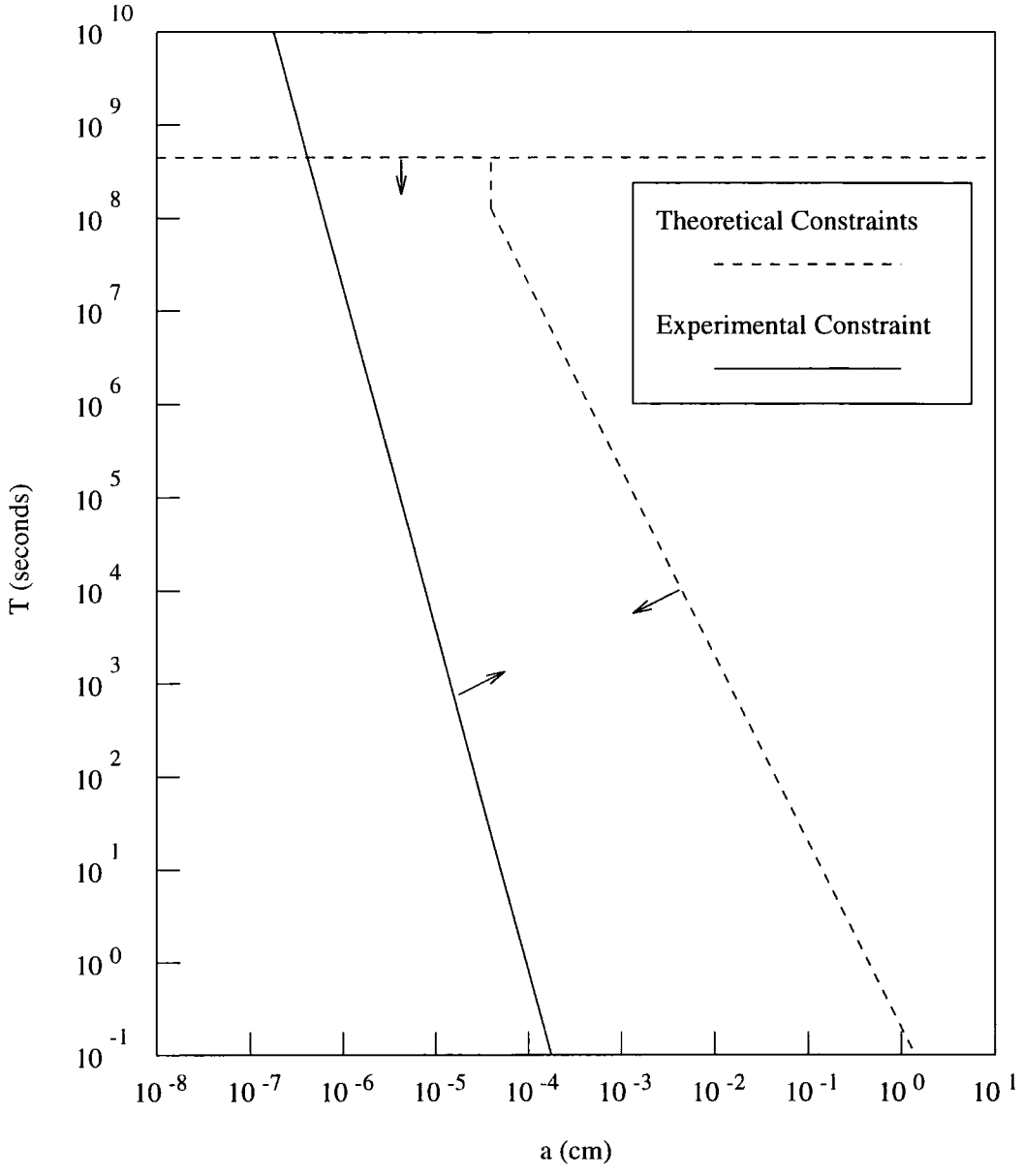



Figure 4.1: The boundaries imposed by the theoretical constraints, eqs. (4.68) and (4.69) in the $\log T(\text{sec}) - \log a(\text{cm})$ plane for the case $\alpha = \frac{m_e}{m_e+m_p}$. The allowed region lies below each boundary. Also shown is the constraint imposed by the Germanium experimental data, eq. (4.72), with the allowed region lying above the boundary.

With the sphere made of carbon atoms, $N \approx 3.8 \times 10^9$ atoms, $Z = 6$ and $A = 12$, the two constraints, eqs. (4.63) and (4.67) become

$$T \leq 2.3 \times 10^8 (2 - \alpha) \text{ sec} \qquad l \geq a, \tag{4.68}$$

and

$$Ta^2 \leq 0.1(2 - \alpha) \text{ sec cm}^2 \qquad a > 4 \times 10^{-5} \text{cm} = l. \tag{4.69}$$

It should be noted that the GRW value of $T = 10^{16}$ sec is forbidden by some 8 orders of magnitude by eq. (4.68).

Here, we may use the same experimental data obtained from the germanium experiments used for the constraint of the parameters. Since the first order rate of excitation in SHM2 only differs from that of CSL/SHM1 by a factor $N[\alpha Z + (1 - \alpha)A]$, we may immediately write down the constraint which is appropriate here.

$$Ta^2 \geq 1.1 \times 10^{11} \left(1 + \frac{m_e}{m_p}\right)^2 \left(\alpha - \frac{m_e}{m_e + m_p}\right)^2 (74 - 42\alpha)^{-1} \text{ sec cm}^2 \tag{4.70}$$

The inequalities in eqs. (4.69) and (4.70) impose very strong limits on the value of $\alpha$. In fact, we have

$$\left| \frac{\alpha}{m_e/(m_e + m_p)} - 1 \right| \leq 0.03. \tag{4.71}$$

Given this result, it would be surprising were $\alpha$ not to be equal to $\frac{m_e}{m_e + m_p}$, i.e. mass-proportional coupling. In other words in the model SHM2, which appears to be one of the more natural hitting models consistent with preserving wavefunction symmetry, the experimental requirements force a mass dependence suggestive of the effect being ultimately connected with gravity.

With this choice of coupling, the excitation vanishes at this order, and we have a new, much weaker constraint from at second order (to order of magnitude):

$$Ta^4 \geq 10^{-16} \text{ sec cm}^4. \tag{4.72}$$

The allowed parameter space formed from the inequalities eqs. (4.68), (4.69) and (4.72) is shown in Fig. (4.1), which may be compared with the corresponding diagram for CSL/SHM1, Fig. (3.1).

## 4.7 Connection between discrete and continuous localization models

We saw earlier that the elements of the density matrix in the model SHM1 reduce to those of CSL in the limit when the system is isolated from the rest of the universe. One subtle difference is the coupling constant $\lambda$. As defined in SHM1, the coupling for a single particle is not in fact totally constant, the constant instead being the total 'hit rate' of the universe, $\lambda_u$. In CSL, the rate of hitting of a single particle is defined to be the constant.

Here we investigate the connection between the second model discussed above and CSL. As the difference between SHM1 and SHM2 is simply a square root, it seems logical to examine the CSL equations with an added square root.

Since the eigenvectors $n(\mathbf{x})$ of the operators $N(\mathbf{x})$, defined in §3.2. are basically the collapse functions of SHM1, we need operators whose eigenvalues are the square roots of those for CSL, i.e. we simply need to take the square root of the original operator, i.e.

$$N(\mathbf{x}) = \left[\sum_s \int d^3\mathbf{y} g(\mathbf{y}-\mathbf{x}) a^\dagger(\mathbf{y},s) a(\mathbf{y},s)\right]^{\frac{1}{2}}, \tag{4.73}$$

and thus the eigenvalues are

$$n'(\mathbf{x}) = \left[\sum_{i=1}^{n} g(\mathbf{q}_i - \mathbf{x})\right]^{\frac{1}{2}}. \tag{4.74}$$

These eigenvalues are now basically the collapse function of SHM2, subject to an insignificant difference of a factor of 2 in the expoentials. To check this, we

substitute eq. (4.73) into the equation for the evolution of the density matrix in CSL, eq. (3.55) and using eq. (4.74) we find

$$\frac{\partial}{\partial t}<\mathbf{q}'|\rho|\mathbf{q}> = -\gamma\left(N_u - \int d^3\mathbf{x}\left[\sum_{i=1}^{n} g(\mathbf{q}_i - \mathbf{x})\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n} g(\mathbf{q}'_i - \mathbf{x})\right]^{\frac{1}{2}}\right)<\mathbf{q}'|\rho|\mathbf{q}>, \tag{4.75}$$

ignoring the Hamiltonian evolution.

This is of course precisely the same form as the evolution equation for SHM2, from eqs. (4.05) and (4.14), for the parameter choice $\gamma = \lambda$, the single particle 'hit rate'.

Subject to manipulation of the parameters in the models, and the odd factor of 2, the hitting model formed by taking an arbitrary power of the symmetric sum in eq. (4.3) as the collapse function, can be related directly to CSL when we take the same power of the operator $N(\mathbf{x})$ in the CSL equations.

## 4.8 Concluding Remarks

The models described in this chapter have shown that it is possible to have a discrete hitting model which preserves the symmetry of the wavefunction, whilst maintaining the necessary properties for the collapse models to be a possible solution of the measurement problem of quantum theory. In the limit when the number of particles tends to infinity, the disrete hitting models can be approximated to become equivalent to CSL or a variant. The parameters in the models can again be constrained, and the original choice of GRW is found not to be compatible with the constraints for SHM2. Moreover the suggestion of a link to gravity is evident from the severity of the constraints.

# Chapter 5

# Modelling the origin of collapse

In the collapse models discussed in earlier chapters, the Schrödinger equation is modified by the addition of a stochastic term causing the localization of the wavefunction[11, 13]. However, there have been no attempts to explain the origin of these extra terms causing wavefunction collapse, other than to say that the experimental data relating to Germanium excitation suggest that the models are in some way have a gravitational basis[21, 23]

In CSL, the stochastic process $d\mathbf{B}$ which causes the collapse is not specified, only its mean value and drift. The parameters in the models, $\lambda$ and $\beta$ are ad hoc, with the choice being so as to fit certain theoretical and experimental constraints (see §2.5 §3.6 and §4.6).If collapse is a fundamental physical process, then we should expect to be able to write these parameters in terms of other physical constants. A third criticism is that the narrowing of the wavefunction after collapse produces an associated energy increase (see §2.6 and §3.6), requiring us to ask whether this is really a non-conservation of energy or simply has an as yet unspecified source.

In this chapter, we will examine a model which produces a fluctuating gravitational source. The gravitational potential arising will be taken to determine the

stochastic process $d\mathbf{B}$ in CSL, and also as a possible source of the accompanying energy increase. This model postulates that a point particle has an effect on the potential as if its mass were smeared over the GRW scale $a$[25, 15] The values of the parameters calcualted in the models will be compared to the GRW values.

## 5.1 A fluctuating gravitational potential

In CSL, the quantum system is influenced by a field $w = d\mathbf{B}'$ which can be written as

$$w(\mathbf{x}, t) = w_0(\mathbf{x}, t) + <\mathbf{A}(\mathbf{x}, t)>, \tag{5.1}$$

where $<\mathbf{A}>$ is the quantum expectation value of the mass density operator when we use mass dependent couplings, and $w_0$ is a field with zero drift(See §3.1). The average value of $w(\mathbf{x}, t)$ is $<\mathbf{A}>$, so proceeding by analogy with classical gravity, we write

$$w(\mathbf{x}, t) \equiv \frac{1}{4\pi G}\nabla^2\phi(\mathbf{x}, t). \tag{5.2}$$

The presence of the randomly fluctuating field $w_0$ in eq. (5.1) indicates that the gravitational potental also fluctuates. We shall describe one naive classical model of the source of the fluctuations.

We assume that the source of the fluctuations is the appearance at random times and positions of particles(monopoles) of mass $\mu$. Each persists for a fixed short time interval $\mathcal{T}$ in a cell of volume $\mathcal{L}^3$. Each 'real' particle feels a gravitational force due to these peripatetic particles, causing it to undergo undamped Brownian motion, with its kinetic energy increasing linearly with time as a result. The energy increase is precisely what is predicted by the collapse models to accompany the collapse. We can equate the expressions calculated here and from the collapse models to gain a relationship between the two sets of parameters.

We take the whole of space to be partioned into these cells of volume $\mathcal{L}^3$ and time into intervals $\mathcal{T}$. We define a characteristic function $u$ according to $u_{\alpha,\tau}(\mathbf{z},t) = 1$ if a monopole is in the $\alpha^{th}$ cell during the $\tau^{th}$ interval for $\mathbf{z}$ contained in this cell and $t$ in the interval, and $u_{\alpha,\tau}(\mathbf{z},t) = 0$ otherwise. So that there is on average zero mass in each cell, we take there to be a constant mass density of $-\frac{\mu\mathcal{P}}{\mathcal{L}^3}$ throughout space, where $\mathcal{P}$ is the probability of a monopole appearing in a cell during an interval $\mathcal{T}$.

The potential at $(\mathbf{x},t)$ can be written

$$\phi(\mathbf{x},t) = -G\mu \sum_{\alpha} \frac{1}{\mathcal{L}^3} \int d^3\mathbf{z} \frac{[u_{\alpha,\tau}(\mathbf{z},t) - \mathcal{P}]}{|\mathbf{x}-\mathbf{z}|}, \tag{5.3}$$

leading to $<\phi(\mathbf{x},t)> = 0$ since $<u_{\alpha,\tau}(\mathbf{z},t)> = \mathcal{P}$.

From this last equation, we can find the correlation function of the potential

$$\begin{aligned} <\phi(\mathbf{x},t)\,\phi(\mathbf{x}',t)> &= \delta_{\tau\tau'}\Theta_\tau(t)\Theta_\tau(t')(G\mu)^2 \frac{\mathcal{P}(1-\mathcal{P})}{\mathcal{L}^6} \\ &\quad \times \sum_{\alpha} \int\int d^3\mathbf{z}\, d^3\mathbf{z}' \frac{\Theta_\alpha(\mathbf{z})\Theta_\alpha(\mathbf{z}')}{|\mathbf{x}-\mathbf{z}|\,|\mathbf{x}'-\mathbf{z}'|} \\ &\approx \delta(t-t')(G\mu)^2\tilde{\mathcal{P}} \int d^3\mathbf{z} \frac{1}{|\mathbf{x}-\mathbf{z}|\,|\mathbf{x}'-\mathbf{z}|}, \end{aligned} \tag{5.4}$$

where we have defined $\tilde{\mathcal{P}} = \frac{\mathcal{P}\mathcal{T}}{\mathcal{L}^3}$. Also, the functions $\Theta_\alpha(\mathbf{z})$ is defined to be 1 if $\mathbf{z}$ lies in the $\alpha^{th}$ cell and 0 otherwise. Similarly, $\Theta_\tau(t)$ is unity if $t$ lies in the $\tau^{th}$ interval. In eq. (5.4) we have made the approximations $\mathcal{P} \ll 1$, $\delta_{\tau\tau'}\Theta_\tau(t)\Theta_\tau(t') \approx \mathcal{T}\delta(t-t')$ together with $\Theta_\alpha(\mathbf{z})\Theta_\alpha(\mathbf{z}') \approx \mathcal{L}^3\delta(\mathbf{z}-\mathbf{z}')$.

From the potential correlation, we can calculate the correlation of the field $w_0$:

$$<w_0(\mathbf{x},t)\,w_0(\mathbf{x}',t)> = \mu^2\tilde{\mathcal{P}}\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'). \tag{5.5}$$

This is the same as the correlation function used in CSL provided we take $\gamma = \frac{1}{\mu^2\tilde{\mathcal{P}}}$. Thus we can obtain an expression for the collapse rate of a single particle of mass $m$:

$$\lambda_m = \frac{m^2}{8\pi^{\frac{3}{2}}\mu^2 a^3 \mathcal{P}}. \tag{5.6}$$

We now evaluate the rate of energy increase of a particle of mass $m$ due to the fluctuating potential. We take the mass of the particle to be smeared over the GRW distance $a$ such that it has a mass distribution $m\left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}}\exp\left(-\frac{\beta}{2}x^2\right)$. The force correlation follows from eq. (5.4)

$$\begin{aligned} <F^i(t)\,F^j(t')> &= \delta(t-t')(G\mu m)^2\tilde{\mathcal{P}}\left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \\ &\quad\times\int d^3\mathbf{x}\,d^3\mathbf{x}'\,d^3\mathbf{z}\,\exp\left(-\frac{\beta}{2}(x^2+x'^2)\right)\partial_i\partial'_j\frac{1}{|\mathbf{x}-\mathbf{z}|\,|\mathbf{x}'-\mathbf{z}|} \\ &= \delta(t-t')\delta_{ij}\frac{4\pi^{\frac{1}{2}}}{3}\beta(G\mu m)^2\tilde{\mathcal{P}}. \end{aligned} \tag{5.7}$$

This last equation follows from the symmetry of the integrand enabling us to make the replacement $\partial_i\partial'_j \rightarrow -\frac{1}{3}\delta^{ij}\nabla_z^2$. This enables us to identify $F_i(t) = K\frac{dB_i(t)}{dt}$ where $K^2$ is the constant factor in eq. (5.7) and $B_i(t)$ is Brownian motion, $<B_i(t)\,B_j(t)> = \delta_{ij}t$. The momentum is hence $KB_i(t)$ and the energy $E = \frac{K^2}{2m}\mathbf{B}(t)\cdot\mathbf{B}(t)$. Thus

$$\frac{d}{dt}<E> = 2\pi^{\frac{1}{2}}m(G\mu)^2\beta\tilde{\mathcal{P}}. \tag{5.8}$$

The effect of the fluctuating potential is an increase in energy, which we equate to the energy increase which we know accompanies collapse, leading to a second relationship involving $\lambda_m$ and $a = \frac{1}{\sqrt{\beta}}$:

$$\frac{3\hbar^2}{4ma^2}\lambda_m = \frac{2\pi^{\frac{1}{2}}m(G\mu)^2}{a}\tilde{\mathcal{P}}. \tag{5.9}$$

We can solve eqs. (5.8) and (5.9) to find

$$a = \left(\frac{3}{64\pi^2}\right)^{\frac{1}{4}}\left[\frac{c\hbar}{G\mu^2}\right]^{\frac{1}{2}}\left(\frac{1}{\tilde{\mathcal{P}}c}\right)^{\frac{1}{2}}, \tag{5.10}$$

and

$$\lambda_m = \frac{1}{(3\pi)^{\frac{1}{2}}}\frac{Gm^2}{a\hbar}. \tag{5.11}$$

We note that $a$ is independent of $m$, and $\lambda_m$ is thus proportional to $m^2$, as the collapse models to date have assumed. It is also interesting that $\lambda_m a$ is independent of the parameters in the model.

If we wish to keep the GRW value for $a$, then from eq. (5.10) we have $\lambda_m \approx 10^{-24}\,\mathrm{sec}^{-1}$ for a nucleon, which is some 8 orders of magnitude smaller than the value suggested by GRW. It is not possible to retain the GRW value of this parameter, as then $a$ is required to be of order $10^{-13}\,\mathrm{cm}$, too small as it is less than atomic radii.

The model only determines $a$ if we specify the mass of the particles $\mu$ and the probability density $\tilde{\mathcal{P}}$. There are no obvious choices for these parameters, but let us see the results if we choose the Planck mass for $\mu = \left(\frac{c\hbar}{G}\right)^{\frac{1}{2}}$, and also the Planck time for $\mathcal{T} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}}$. Maintaining the GRW parameter $a$, we require that $\frac{\mathcal{P}}{\mathcal{L}^3} \approx 4 \times 10^{41}\,\mathrm{cm}^{-3}$. We note that $\left(\frac{\hbar}{Mc}\right)^{-3} \approx 10^{41}\,\mathrm{cm}^{-3}$, where $M$ is the nucleon mass. Using the nucleon Compton wavelength for $\mathcal{L}$ we find that

$$a \approx 1.4 \times 10^{-5}\,\mathrm{cm}, \qquad \text{and} \qquad \lambda_m \approx 2 \times 10^{-24}\,\mathrm{sec}^{-1} \quad \text{for the nucleon.} \tag{5.12}$$

Thus we obtain the GRW value for $a$ if the probability density for these monopole 'Planckons' is about 1 per proton volume.

If we decide we do not require CSL, only a discrete hitting model, then we remove the necessity of equating the field correlation to that of the CSL stochastic field, and eq. (5.6) need no longer hold. Thus the only equation relating the two fundamental parameters of the collapse model is then eq. (5.9).

In this instance, we have no problem in maintaining the GRW values for the parameters $\lambda_m$ and $a$. To do this we would then require that $\tilde{\mathcal{P}} \approx 4 \times 10^{11}\,\mathrm{sec\ m}^{-3}$. This value is about 8 orders of magnitude larger than before. If we use the Planck time again for $\mathcal{T}$, it is difficult to relate the value of $\frac{\mathcal{P}}{\mathcal{L}^3}$ to any fundamental quantities.

It should be noted that the additional negative mass background included in the potential is necessary to prevent the planckon distribution from being sufficient to overclose the universe by many orders of magnitude. If we ignore this negative mass background, the correlations and energy relation are unchanged from before, but we now have an average mass density of $\rho = \frac{\mathcal{P}_\mu}{\mathcal{L}^3}$ over all space. The closure condition for the universe is given by

$$\rho \approx 2 \times 10^{-26}\,\mathrm{kg\ m^{-3}}, \tag{5.13}$$

so if we were to maintain the values of the parameters we had before, we would have $\rho_{model} \approx 10^{40}\,\mathrm{kg\ m^{-3}}$, thus overclosing the universe by some 66 orders of magnitude.

If on the other hand, we use the closure relation to define the value of $\rho$ and hence $\frac{\mathcal{P}}{\mathcal{L}^3}$, we would find it impossible to use the Planck value for $\mathcal{T}$ if we wished to maintain sensible values for the collapse parameters.

# Chapter 6

# Collapse and Quantum Computers

One of the main features that is required in quantum computing is the placing of the system in a superposition of a large number of states. How quantum computers perform, and limitations on the number of states involved, will be influenced by how straightforward it is to maintain these superpositions over the course of the calculation. Unruh[26] has discussed interactions with the environment, and the constraints decohering noise may put on the length of time the computation may last. In this chapter, we will assess the effects of possible wavefunction collapse on the superposition.[16] In particular, we are concerned with the explicit collapse to (near) position eigenstates as postulated in the GRW model in Chapter 2.

Firstly, we need to know how the bits of memory are to be 'stored'. If they are to be stored as superpositions of $|+>$ and $|->$ in a suitable spin system, then we do not have to worry about collapse, as the collapse process (as it stands) does not act on the space of spin states.

However, any quantum two-level system could feasibly be used in the memory of the computer (exactly which system will of course also influence the length of the computation). Collapse processes could be significant in any system which uses spatial separation to distinguish between the two states. (This is true to a certain extent if the states are two different energy eigenstates, as the spatial wavefunctions will of course differ.)

Here we shall examine the case when the $|0>$ and $|1>$ states are distinguished solely by spatial separation. It is not important that this should be the case, but we are not concerned here with effects arising from other sources.

## 6.1 Effect of collapse on a single-bit calculation

The most general state of the system with a single 'input' and single 'output' bit is

$$|\psi> = \frac{1}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} [\alpha|0, f(0)> + \beta|1, f(1)>], \tag{6.1}$$

where $f$ is the output memory state associated with the input state at any stage of the calculation. We will assume that the $|0>$ and $|1>$ states of any one bit are differentiated solely by a spatial separation $d$. If any collapse process occurs, it has a certain probability of being centered on either the $|0>$ or $|1>$ states. (Probabilities proportional to $\alpha^2$ and $\beta^2$ respectively). In what follows, we shall take the collapse center to be located close to the $|0>$ state of the input. In this case, a collapse will cause the wavefunction to become

$$|\psi> \rightarrow |\psi'> = \frac{1}{\left(\alpha^2 + \beta^2 \exp\left[-\frac{d^2}{a^2}\right]\right)^{\frac{1}{2}}} \left[\alpha|0, f(0)> + \beta \exp\left(-\frac{d^2}{2a^2}\right)|1, f(1)>\right], \tag{6.2}$$

where $a$ is the collapse radius. It is immediately apparent that the relationship between the spatial separation $d$ of the two states and the collapse parameter $a$

will be the major factor determining the significance of collapse.

We will first assess the effect of possible wavefunction collapse on the simplest calculation, due to Deutsch[27], where we have $f : \mathbf{Z}_2 \rightarrow \mathbf{Z}_2$, and we would like to utilise quantum parallelism to compute the Boolean sum $G = f(0) \oplus f(1)$. In what follows, we take $\alpha = \beta$, to give four possible final states of the system before measurement, namely

$$|u_0> = \frac{1}{\sqrt{2}}[|0,0> + |1,0>] \tag{6.3}$$

$$|u_{01}> = \frac{1}{\sqrt{2}}[|0,1> + |1,0>] \tag{6.4}$$

$$|u_{10}> = \frac{1}{\sqrt{2}}[|0,0> + |1,1>] \tag{6.5}$$

$$|u_1> = \frac{1}{\sqrt{2}}[|0,1> + |1,1>] \tag{6.6}$$

In order to extract information about $G$ by the use of one measurement only, we construct an observable $\mathcal{G}$ with eigenstates

$$|E_0> = \frac{1}{2}(|0,0> + |1,0> - |0,1> - |1,1>) = \frac{1}{\sqrt{2}}(|u_0> - |u_1>) \tag{6.7}$$

$$|E_1> = \frac{1}{2}(|0,0> - |1,0> - |0,1> + |1,1>) = \frac{1}{\sqrt{2}}(|u_{01}> - |u_{10}>) \tag{6.8}$$

$$|E_f> = \frac{1}{2}(|0,0> + |1,0> + |0,1> + |1,1>) = \frac{1}{\sqrt{2}}(|u_0> + |u_{01}>) \tag{6.9}$$

$$|E_e> = \frac{1}{2}(|0,0> - |1,0> + |0,1> - |1,1>) \tag{6.10}$$

Note that $|E_f>$ can also be written as $\frac{1}{\sqrt{2}}(|u_1> + |u_{10}>)$ and that $|E_e>$ is orthogonal to all the states $|u>$.

Upon the performance of a measurement of $\mathcal{G}$ on any state there is a probability of $\frac{1}{2}$ that the eigenvalue corresponding to $|E_f>$ will be seen; the observation has failed to determine the value of $G$. Also with probability $\frac{1}{2}$, the eigenvalue

seen will correspond to the eigenstate $|E_0>$ (respectively $|E_1>$),and the value of $G$ is then definitely 0 (respectively 1). A false value for G will never be obtained.

This is all very well when we have an equal (or at least known) superposition of the two states, but what happens if a collapse occurs, disturbing this balance?

When the state separation $d \ll a$, we can write the collapsed state as

$$|\psi'> = \frac{1}{\left(2 - \frac{d^2}{a^2}\right)^{\frac{1}{2}}} \left[|0, f(0)> + \left(1 - \frac{d^2}{2a^2}\right)|1, f(1)>\right]. \tag{6.11}$$

We will take the function $f$ to be the constant function, $f \equiv 0$. In this instance, the eigenvalue corresponding to $|E_1>$ should never be seen, but it will be with probability

$$P_{er} = |<\psi'|E_1>|^2 \tag{6.12}$$

$$\approx \frac{1}{16} \frac{\left(\frac{d^2}{a^2}\right)^2}{2 - \frac{d^2}{a^2}}. \tag{6.13}$$

If for example, we take the spatial separation $d$ to be about an order of magnitude smaller than the collapse parameter $a$, we would find an error of about $3 \times 10^{-6}$. It may very well be that inaccuracies of this order could be tolerated, as the system itself will not be completely precise.

With the spatial separation $d$ equal in magnitude to the collapse parameter $a$, then the resultant error becomes much more significant. Of all the supposedly 'definite' values of $G$ produced, about 5.6% of the results will actually be false.

It should be remarked that if the separation $d$ is greater than the value of $a$, then effectively the collapse has destroyed the superposition, and any hopes of information being obtained from the system have gone.

## 6.2 Effects of collapse on many bits

The chance of any individual bit being affected by a collapse during the course of a computation is relatively small, but when the superpositions of states involve a large number of bits, then the expected number of collapse processes occurring will rise by a factor at least linear in the number of bits involved. The number of bits required to deal with a number is given by $\log_2 N$, and the advantage of a quantum computer is that in a superposition of a large number of states representing a sequence of numbers, the number of bits required is still only logarithmic in the number of states.

As a first example, we shall still take the 'output' state, described by $f$, to be a single bit. However, we will restrict the possible functions $f$ to two types: (a) those where $f$ is a constant ($f \equiv 0$ or $f \equiv 1$), and (b) those where $f$ takes the value 1 for exactly half of all the possible input states[28]. We are to determine the type of function $f$ involved. If there are $2N$ different input states, then classically we would need $N+1$ computations to be sure of the type of function $f$ that we have. However, by utilising quantum parallelism, we can determine $f$ with only two computations (described by unitary transformations $U_f$, together with an intermediate operation $S|i,j> = (-1)^j|i,j>$. The implementation of the operations $U_f$,$S$,$U_f^\dagger$ to an initial state

$$|\phi> = \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i,0>, \tag{6.14}$$

causes the system to finish in the state

$$|\psi> = \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} (-1)^{f(i)}|i,0>. \tag{6.15}$$

It should be noted that $|\phi>$ and $|\psi>$ are orthogonal when the function $f$ is of type (b). Upon measuring the projection observable $|\phi><\phi|$, an outcome of 1

means that we can be sure that $f$ is of type (a), whereas if the outcome is 0 then $f$ must have been of type (b).

In assessing the effect of collapse, we will take $f(i) \equiv 0$. In this case, the only difference between the initial state $|\phi>$ and the final state $|\psi>$ will be that the coefficients of half of the states will have been reduced by a factor $\alpha = \exp\left(-\frac{d^2}{2a^2}\right)$, together with a slightly different normalization. This means that

$$|<\phi|\psi>| = \frac{1}{2}\left(\frac{2}{1+\alpha^2}\right)^{\frac{1}{2}}(1+\alpha). \tag{6.16}$$

Whereas before, when this inner product was unity, we could be sure that the outcome of a measurement of $|\phi><\phi|$ would be 1, it is now possible that an outcome of 0 would occur, with probability

$$\begin{aligned} P_{er} &= 1 - |<\phi|\psi>|^2 \\ &= \frac{(1-\alpha)^2}{2(1+\alpha^2)}. \end{aligned} \tag{6.17}$$

As before, we can expand this expression in terms of the spatial separation $d$, provided that $d \ll a$. To lowest order, we have

$$P_{er} = \frac{1}{16}\frac{\frac{d^4}{a^4}}{(1-\frac{d^2}{2a^2})}. \tag{6.18}$$

This is the same expression, for a single collapse (apart from a factor of 2) as that in the previous section, as should be expected.

On average, for each additional collapse occurring, the magnitude of the inner product between $|\phi>$ and $|\psi>$ decreases by the same factor. Given $n$ collapses, the probability of a given state being affected by $m$ of these is $2^{-n}\binom{n}{m}$. The system will finish in the state

$$|\psi> = \frac{1}{\sqrt{(2N)}}\left(\frac{2}{1+\alpha^2}\right)^{\frac{n}{2}}\sum_{i=0}^{2N-1}\alpha^{m(i)}|i,0>, \tag{6.19}$$

leading to

$$\begin{aligned} |<\phi|\psi>| &= \frac{1}{2N}\left(\frac{2}{1+\alpha^2}\right)^{\frac{n}{2}} \sum_{i=0}^{2N-1} \alpha^{m(i)} \\ &= \left(\frac{2}{1+\alpha^2}\right)^{\frac{n}{2}} \sum_{m=0}^{n} 2^{-n}\binom{n}{m}\alpha^m \\ &= \frac{(1+\alpha)^n}{[2(1+\alpha^2)]^{\frac{n}{2}}}. \end{aligned} \tag{6.20}$$

In deriving this expression, it has been assumed that no two collapses are centered on any individual bit. Since $\lambda \approx 10^{-16}\mathrm{s}^{-1}$ for a nucleon, and lower by perhaps a factor of $\left(\frac{m_e}{m_p}\right)^2$ for an electron if the couplings in the model are mass-dependent (see §3.5 and §3.6), this does not seem an unreasonable assumption. Also, for ease of calculation, we take $2N = 2^L$, so that the superposition uses all $L$ bits equally.

This of course leads to a probability of an incorrect measurement of

$$P_{er} = 1 - \frac{(1+\alpha)^{2n}}{[2(1+\alpha^2)]^n}. \tag{6.21}$$

Again, we would like to expand this in terms of the spatial separation, $d$ of the states. To do this, we now require that

$$n\frac{d^2}{a^2} \ll 1. \tag{6.22}$$

It is very unlikely that the computation will involve enough bits and run for a long enough time for there to be significantly many collapse processes occurring, so this is effectively the same condition as before. When this condition is satisfied, $P_{er}$ becomes, to leading order

$$P_{er} = \frac{n}{16}\frac{\frac{d^4}{a^4}}{(1-\frac{n}{2}\frac{d^2}{a^2})}. \tag{6.23}$$

## 6.3 Collapse and Shor's Algorithms

It will be shown that collapse has effects of the same magnitude when the states in the superposition have different phases.

Although the algorithm for factoring is no doubt the most useful, in assessing the errors arising from collapse, it is easier to examine the algorithm for evaluating discrete logarithms, as it is clearer to see where the errors occur[29].

The algorithms rely on destructive interference between states which represent unwanted results to leave only those states which may in some way be useful. If any collapse process occurs, it will disturb the amplitides of the phase factors in the superposition, the interference will not be totally destructive, and some unwanted states will persist (i.e. those that will give an incorrect result). We wish to calculate the probability that the state of the system will be found to be in one of these states upon measurement.

The algorithm for discrete logarithms relies on the superposition of states representing numbers $a$,$b$,$c$,$d$. Collapses centered on the bits of different numbers will have differing effects. It should be noted that collapses centered on the bits of the numbers $c$ and $d$ will not cause unwanted states to persist. These collapses will diminish the probability of obtaining certain states, but will not give unwanted results.

We consider the effect of one collapse on an arbitrary bit of the superposition of $b$;

$$|b_i\prime> = \frac{1}{(1+\alpha^2)^{\frac{1}{2}}}[|0> + \alpha|1>]. \tag{6.24}$$

As before, each individual collapse will affect half of the states in the superposition. The probability of a state $|c, d, y>$ (where $y \equiv g^k \pmod p$) being

observed becomes

$$\left|\frac{1}{(p-1)^2}\left(\frac{2}{1+\alpha^2}\right)^{\frac{1}{2}}\sum_{b=0}^{p-2}\left[1\times(b_j=0)+\alpha\times(b_j=1)\right]\times\exp\left(\frac{2\pi i(kc+b(d+rc))}{(p-1)}\right)\right|^2, \tag{6.25}$$

where we have taken the collapse to be centered on the $|0>$ state of the $j^{\text{th}}$ bit of $b$. It is straightforward to calculate this probability when $d+rc\equiv 0 \pmod{p-1}$. We have

$$P_{d+rc\equiv 0}=\frac{1}{(p-1)^4}\left[\frac{p-1}{2}(1+\alpha)^2\right]^2\frac{2}{1+\alpha^2}. \tag{6.26}$$

Noting that there are $(p-1)^2$ possible states satisfying the equivalence condition, we have the probability of a non-useful state being observed

$$P_{non-u}=\frac{(1-\alpha)^2}{2(1+\alpha^2)}, \tag{6.27}$$

which is precisely the same probability as was obtained before. It can easily be shown that for the case when we have a number of collapses occuring during the computation, the expression for the probability of error is given by eq. (6.21), and eq. (6.23) when $d \ll a$.

## 6.4 Sizes of possible errors

We have assessed the errors occurring when collapses occur, but of course as the collapses are random, we can only estimate the number that are likely to occur during the course of any computation. If the number of bits that are used in the calculation is $L$, then the expected number of collapse processes occurring is

$$N_c=\lambda L T_c, \tag{6.28}$$

where $T_c$ is the length of time for which the computation proceeds. If the spatial separation of the states $d$, is small compared with the collapse parameter $a$, we

can approximate the total error in the results as

$$P_{er} \approx \frac{1}{16}\lambda L T_c \frac{d^4}{a^4}. \tag{6.29}$$

As an idea of the order of magnitude of this error, again using an electron state, we can let the spatial separation be almost as large as possible, only one order of magnitude smaller than $a$, using $L = 10^4$ bits, and $T_c = 10^2$ s, giving $P_{er} \approx 10^{-15}$ for a nucleon, and $P_{er} \approx 10^{-22}$ for an electron, which should be a very insignificant chance of producing an incorrect result. Taking a maximum permissible error to be of order $10^{-6}$, say, and using a state described by a single electron, then we could employ $L = 10^{19}$ bits in the computation, and be dealing with numbers of order $10^{3\times 10^{18}}$. We could hold this number in superposition for the length of the calculation, or let the state evolve in some unitary way, but the final error would be of the same order of magnitude. Having a computational state represented by a number of superpositions (as in Shor's algorithms for example) may bring down the number of permissible bits by an order of magnitude, but the maximum number with which we could deal would still be extremely large, showing that for most situations that could be envisaged, wavefunction collapse is a very minor effect.

All of these results have assumed that each bit is represented by a single particle. It is most probable that we would at the very least be dealing with single atoms. Using a large number of particles to describe each bit would be cumbersome, but it should be asked whether the errors would be significantly different in this situation.

The presence of many particles would increase the minimum separation $d$ between $|0>$ and $|1>$ that could be used, as the two states must still be orthogonal, yet this distance must not approach the collapse parameter $a \approx 10^{-5}$ cm. If we take the particle separation to be of order $10^{-8}$ cm, then we could be permitted

$N \sim 10^6$ particles describing the state.

An increase in the number $N$ of particles involved would also increase the number of collapses centered on the bit in any given time. Depending on the model used for many-particle collapse, this increase could be by a factor $N$ if we take SHM2 to be correct for describing many particles, or even $N^2$ if we assume SHM1 or CSL to be the appropriate model. Taking the most severe case, when the number of collapses is quadratic in the number of particles present, and $N = 10^6$, then the probability of an error becomes $P_{er} = 10^{-3}$ for nucleons (for the same values of $T_c$,$L$ as before.) This error is no longer very small, and it is apparent that using this number of particles in any one bit is impractical if we want an error-free computation. The error is still quite small, however, if we consider electron states, of order $P_{er} = 10^{-10}$.

It should be remarked that all of this assumes that wavefunction collapse does occur. This is of course, yet to be confirmed. It may be possible that a quantum computer, working on a calculation with a known result, could be a test of the existence of explicit wavefunction collapse. This is of course if other sources of decoherence, such as interactions with the environment do not swamp the effects due to collapse.

# Chapter 7

# A Local Model of Collapse

As described in Chapter 2, the GRW model [11] showed the possibililty of constructing a realistic model describing explicit wavefunction collapse in such a way that, in many situations, the correct predictions of quantum theory were maintained but real experiments actually had results. The work has since been developed in a number of ways, some of which are described in Chapters 3 and 4[13, 14], and it is generally agreed that it provides a satisfactory resolution of the measurement problem of quantum theory, at least in the non-relativistic domain. As originally presented, however, the model was clearly non-local and not Lorentz invariant. Both the symmetric versions of the hitting model and CSL share these attributes. Recently, attempts have been made to develop versions of the collapse models which, whilst retaining the non-locality, are nevertheless Lorentz invariant[30, 31, 32] Perhaps the best one can say of these models is that they are partially successful. They certainly raise several interesting issues.

We shall take a different approach in this chapter and endeavour to construct a *local*, and Lorentz invariant version of the collapse model[17]. We know of course that this cannot agree in all respects with the predictions of orthodox quantum theory, and one object here is to see where the disagreement lies and whether it

is detectable. Note that even the original GRW model does not completely agree with quantum theory, and this requires severe constraints to be placed on the parameters[20, 14, 21] (See §2.5 ,§3.6 and §4.6). We are concerned here with a different type of departure from quantum theory, which is caused by our insistence on the theory being local.

## 7.1 A local model of collapse

In the original GRW model(see chapter 2), it was proposed that 'hits' occurred in a random fashion, at certain space-time points. The effect of a given hit spread throughout all space instantaneously. Thus, if we have a single particle wavefunction $\psi(\mathbf{x}, t)$, a hit at the point $\mathbf{x}_1$, would cause this to change according to:

$$\psi \rightarrow \psi' = N \exp\left(-\frac{\beta}{2}(\mathbf{x} - \mathbf{x}_1)^2\right)\psi. \tag{7.1}$$

In order to make this into something that is both local and Lorentz invariant, we propose instead that a hit at the space-time point $\mathrm{X}_1 \equiv (\mathbf{x}_1, t_1)$ only has an effect inside the forward light-cone from that point. To ensure Lorentz invariance of the hitting function, we must replace the 3-dimensional distance in eq. (7.1) by a four-dimensional distance. We cannot use the distance from the hitting point to the point on the light-cone since this is identically zero. Instead, we propose the perpendicular distance from the point on the light-cone to a four-momentum vector $\mathrm{P}_\mu$ originating from $\mathrm{X}_1$, where perpendicular is meant in the sense of a Minkowski metric. For a state with a single momentum component (this of course can only be true approximately unless we have a plane wave), this $\mathrm{P}_\mu$ will just be the four-momentum of the particle. For a more general wavefunction it is probably necessary to consider each component of momentum in a separate

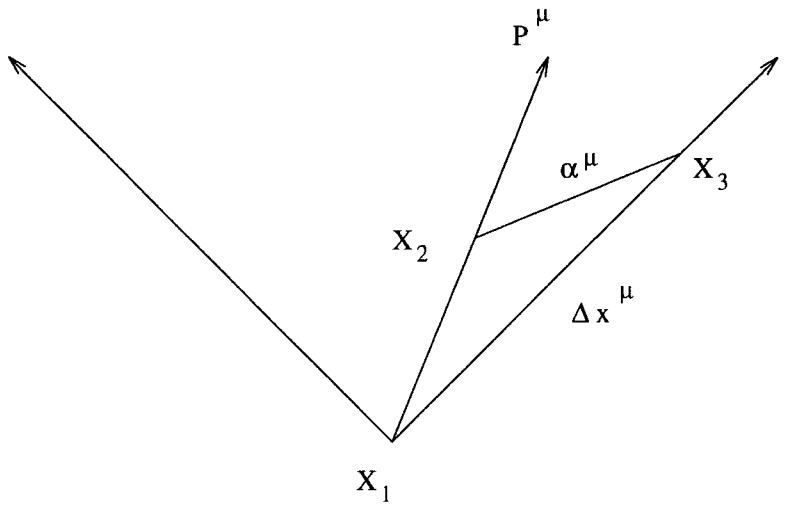



Figure 7.1: Constructing a Lorentz invariant distance

manner. Alternatively, it may be possible to define a 4-momentum vector at a point in analogy to the Bohm hidden-variable model (i.e. from a 4-dimensional analogue of eq. (1.6) ), although for a complicated wavefunction, this may not lead to a momentum vector lying within the forward light-cone.[33]

If we denote the vector from the light-cone to $P^\mu$ by $\alpha^\mu$ (see Fig.7.1) then the condition that it is perpendicular to $P^\mu$ is

$$P_\mu \alpha^\mu = 0. \tag{7.2}$$

The path from $X_1$ to $X_3$, where $X_3$ is a point on the forward light-cone from $X_1$ that is at one end of a perpendicular vector to $P^\mu$, can be traversed in two ways, giving another condition

$$kP^\mu + \alpha^\mu = \Delta x^\mu, \tag{7.3}$$

for some $k$. These two equations enable us to find the value of $\alpha_\mu \alpha^\mu$. From eq. (7.3), and using eq. (7.2), we have

$$kP_\mu P^\mu = P_\mu \Delta x^\mu, \tag{7.4}$$

and

$$\alpha_\mu \alpha^\mu = \alpha_\mu \Delta \mathrm{x}^\mu. \tag{7.5}$$

Also, since $\Delta \mathrm{x}^\mu$ is a null vector, eq. (7.3) gives

$$k \Delta \mathrm{x}_\mu \mathrm{P}^\mu + \Delta \mathrm{x}_\mu \alpha^\mu = 0. \tag{7.6}$$

We can rearrange these three equations to eliminate $k$ and, putting $\mathrm{P}_\mu \mathrm{P}^\mu = m^2c^2$, we have

$$\alpha^\mu = \Delta \mathrm{x}^\mu - \frac{1}{m^2c^2} \mathrm{P}^\mu \left( \mathrm{P}_\nu \Delta \mathrm{x}^\nu \right), \tag{7.7}$$

and

$$\alpha_\mu \alpha^\mu = -\frac{1}{m^2c^2} \left( \mathrm{P}_\mu \Delta \mathrm{x}^\mu \right)^2. \tag{7.8}$$

This reduces to $\alpha_\mu \alpha^\mu = -(\Delta \mathbf{x})^2$ in the rest frame of the particle, $\mathrm{P}_\mu = (mc, \mathbf{0})$.

We therefore postulate that the collapse takes effect along the forward light cone from $\mathrm{X}_1$, according to

$$\psi_{H_1}(\mathrm{X}) = \exp\left( -\frac{\beta}{2m^2c^2} \left( \mathrm{P}^{(1)}_\mu (\mathrm{X}^\mu - \mathrm{X}^\mu_1) \right)^2 \right) \psi(\mathrm{X}), \tag{7.9}$$

and use this as a boundary condition to finding the effect of the collapse inside the forward light-cone. This is our local analogue of eq. (7.1).

In what follows, we shall simplify the discussion by constraining the particle to a single spatial dimension (z). Ideally, we should take the wavefunction to be a solution of the Dirac equation. However, we wish not to be concerned with any Dirac bispinor, as the collapse process does not act on the space of spins. For a free particle, we can instead take the wavefunction to be a solution of the Klein-Gordon equation. We shall work with a single momentum for which the initial wavefunction is

$$\psi_0(t, z) = N \exp\left( -iEt + ipz \right), \tag{7.10}$$

where $N$ is some normalization factor.

Given that the forward light-cone is the boundary under consideration, it is sensible to use light-cone coordinates, $x_+ = ct+z$, $x_- = ct-z$. The Klein-Gordon equation in this coordinate system reads

$$\frac{\partial^2}{\partial x_+ \partial x_-}\psi = -\frac{1}{4}\left(\frac{mc}{\hbar}\right)^2 \psi. \tag{7.11}$$

Then, if we choose the origin to be at the point of collapse, the boundary conditions in a general frame of reference are

$$\psi(x_+,0) = N\exp\left(-\frac{i}{2}(E-p)x_+\right)\exp\left(-\frac{\beta}{8m^2}(E-p)^2x_+^2\right) \tag{7.12}$$

$$\psi(0,x_-) = N\exp\left(-\frac{i}{2}(E+p)x_-\right)\exp\left(-\frac{\beta}{8m^2}(E+p)^2x_-^2\right). \tag{7.13}$$

The solution of the Klein-Gordon equation inside the forward light-cone from the point of collapse is uniquely defined by these boundary conditions. In order to be able to write this down in a simple form, we shall ignore the quantum evolution, i.e. assume $\frac{\hbar}{mc}$ is very small. We will later specialize to the rest frame, in which $p = 0$, but we will use a more general frame at the moment. We can write

$$\psi(x_+,x_-) = N(p,E)\exp\left(-\frac{i}{2}\left[(E-p)x_+ + (E+p)x_-\right]\right)w(x_+,x_-). \tag{7.14}$$

Substituting this expression into the Klein-Gordon equation, we have

$$\frac{(E+p)}{m}\frac{\partial w}{\partial x_+} + \frac{(E-p)}{m}\frac{\partial w}{\partial x_-} = -\frac{2i\hbar}{mc}\frac{\partial^2 w}{\partial x_+ \partial x_-}, \tag{7.15}$$

where we have included all constants which had been previously set to unity. The right-hand-side is responsible for the quantum evolution. It can be treated as a perturbation. First we calculate the zeroth order solution which is $w(x_+,x_-) = h(x_+ - x_-)$. Substituting in the boundary conditions leads to

$$w(x_+,x_-) = \exp\left(-\frac{\beta}{8m^2}\left[(E-p)x_+ - (E+p)x_-\right]^2\right), \tag{7.16}$$

within the forward light-cone of X. Outside of this region, the original free-particle solution holds. This of course can be written as

$$w(x_+, x_-) \equiv W(z,t) = \exp\left(-\frac{\beta}{2}\left[\gamma\left(z - \frac{p}{E}t\right)\right]^2\right), \tag{7.17}$$

with $\gamma = \left(1 - \left(\frac{p}{E}\right)^2\right)^{-\frac{1}{2}}$. Hence the zeroth order solution to the Klein-Gordon equation may we written as

$$\psi(t,z) = N(p,E)\exp(-iEt + ipz)\exp\left(-\frac{\beta}{2}\left[\gamma\left(z - \frac{p}{E}t\right)\right]^2\right), \tag{7.18}$$

which in the rest frame reads

$$\psi(t,z) = N\exp(-imt)\exp\left(-\frac{\beta}{2}z^2\right). \tag{7.19}$$

For completeness, we will evaluate the contribution of the first order 'correction' to this result, which follows from treating the right-hand side of eq. (7.15) as a perturbation.

Rewriting the boundary conditions as

$$w(x_+, 0) = f(x_+) \tag{7.20}$$

$$w(0, x_-) = f(x_-), \tag{7.21}$$

we have an equation for the correction term (in the rest frame) of

$$\frac{\partial w_1}{\partial x_+} + \frac{\partial w_1}{\partial x_-} = \frac{2i\hbar}{mc} f''(x_+ - x_-). \tag{7.22}$$

This equation is most easily solved by working in normal space-time coordinates and then transforming back into the light-cone system, leading to a solution

$$w_1(x_+, x_-) = \frac{2i\hbar}{mc}(x_+ + x_-)f''(x_+ - x_-). \tag{7.23}$$

We now substitute the boundary conditions, eqs. (7.20) and (7.21) to find the solution to eq. (7.15) to first order

$$w(x_+, x_-) = f(x_+ - x_-) + \frac{2i\hbar}{mc} f''(x_+ - x_-)\left[(x_+ + x_-) - |x_+ - x_-|\right]. \tag{7.24}$$

Substituting in the function $f$ for the boundary conditions and transforming back into Minkowski space, gives us a wavefunction (in the rest frame) within the forward light-cone, to first order

$$\psi(t,z) = N \exp\left(-i\frac{mc^2}{\hbar}t\right) \exp\left(-\frac{\beta}{2}z^2\right) \left[1 - \frac{i\beta\hbar}{mc}(ct - |z|)(1 - \beta z^2)\right]. \quad (7.25)$$

We must ask the question as to the conditions under which this 'correction' term is smaller than the first term. If we take the GRW value for the parameter $\beta = 10^{14}\,\mathrm{m}^{-2}$, then this quantum spreading of the wavefunction could become important over distances of order $10^{-7}\,\mathrm{m}$ and times of order $10^{-7}\,\mathrm{sec}$. However, as the quantum spreading is ignored in the non-relativistic collapse models, we shall continue to do so here.

If we take the initial wavefunction to be a gaussian with a large spread, $\psi_0 \sim \exp\left(-\frac{z^2}{a^2}\right)$, with $a \gg \frac{\hbar}{mc}$, then the momentum states contributing will have $p < \frac{\hbar}{a} \ll mc$. We should note that using the collapse radius for $a$ here gives $p < 2 \times 10^{-9}mc$, and we are justified in taking this to have a single momentum component.

To summarise this section, the effect of a single collapse on a single particle is the same as in the non-relativistic case, except that the effect is only felt within the forward light-cone of the point of origin of the collapse, $\mathrm{X}_1$.

## 7.2 Single particle affected by two collapses

A major difference between our local collapse model and that of GRW is that two independent collapses can occur at space-like separations so that neither collapse 'knows about' the other. There is no problem with consistency until we arrive at the intersection of the light-cones arising from the two collapse centers. In the region formed by the forward light-cone originating from the point of intersection both collapses will be felt, and we need to define precisely how this happens.

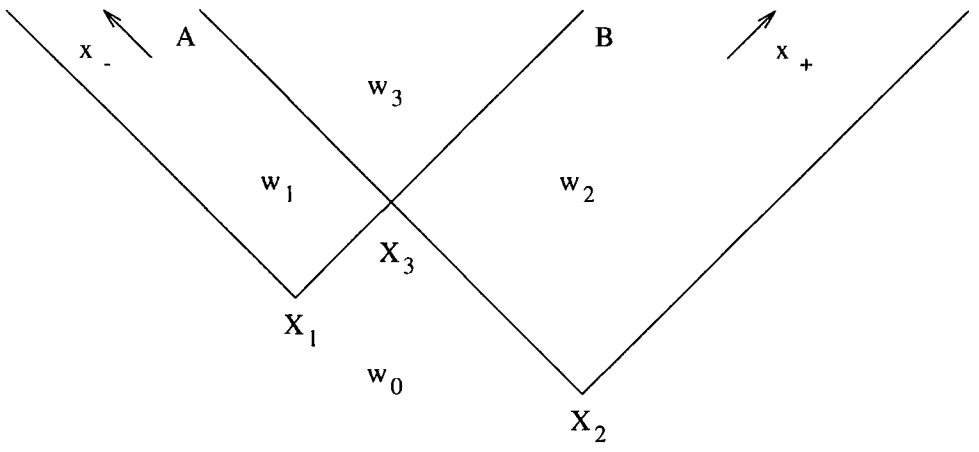



Figure 7.2: Wavefunction affected by two collapses

We shall take the two collapses to occur at the points $\mathrm{X}_1 \equiv (z_1, t_1) \equiv (x_{1+}, x_{1-})$ and $\mathrm{X}_2 \equiv (z_2, t_2) \equiv (x_{2+}, x_{2-})$, see Fig. 7.2. The wavefunction in the regions $w_1$ and $w_2$ will be calculated as in the previous section. On reaching the intersection point, $\mathrm{X}_3 \equiv (x_{2+}, x_{1-})$, we shall solve the differential equation again with new boundary conditions along this third light-cone.

The boundary conditions are formed, as indeed they were before, by taking the wavefunction outside of the light-cone, and multiplying by the collapse factor which arises from the collapse along that particular light-cone. Thus, for the boundary condition along $\mathrm{X}_3 A$ we take the hit wavefunction in the region $w_1$, which is $\psi_{w_1}(z,t)$, and multiply by the collapse factor arising from the collapse at $\mathrm{X}_2$ along $\mathrm{X}_2 A$. Hence

$$\psi_{\mathrm{X}_1\mathrm{X}_2}(z,t) = \exp\left(-\frac{\beta}{2}(\alpha_{\mathrm{X}_2})^2\right)\psi_{w_1}(z,t), \tag{7.26}$$

and similarly along $\mathrm{X}_3 B$

$$\psi_{\mathrm{X}_2\mathrm{X}_1}(z,t) = \exp\left(-\frac{\beta}{2}(\alpha_{\mathrm{X}_1})^2\right)\psi_{w_2}(z,t), \tag{7.27}$$

where $\alpha_{\mathrm{X}_1}$ and $\alpha_{\mathrm{X}_2}$ are the perpendicular four-distances from the momentum

vectors arising from the collapses at $X_1$ and $X_2$ respectively.

In general, for an arbitrary initial wavefunction the momentum vectors arising from each point will be different. Even ignoring the quantum evolution, this could lead to solutions of the differential equation which are quite complicated. For simplicity, we shall deal with the case when the momenta arising from each collapse center are equal. In this situation, we can again work in the frame where $\mathbf{p} = 0$. This means that the boundary conditions are (having extracted the plane-wave term and normalization as before):

$$w(x_{2+}, x_-) = \exp\left(-\frac{\beta}{2}(z - z_1)^2\right) \exp\left(-\frac{\beta}{8}(x_- - x_{2-})^2\right), \tag{7.28}$$

along $X_3A$, and

$$w(x_+, x_{1-}) = \exp\left(-\frac{\beta}{2}(z - z_2)^2\right) \exp\left(-\frac{\beta}{8}(x_+ - x_{1+})^2\right), \tag{7.29}$$

along $X_3B$.

We are only interested in the zeroth order solution to the differential equation, eq. (7.15), so of course we have $w(x_+, x_-) = h(x_+ - x_-)$, as before. Substituting the boundary conditions, we find that

$$h(x_+ - x_{1-}) = \exp\left(-\frac{\beta}{8}\left[(x_+ - x_{1-} - x_{2+} + x_{2-})^2 + (x_+ - x_{1+})^2\right]\right), \tag{7.30}$$

which may be rewritten as

$$h(\lambda) = \exp\left(-\frac{\beta}{8}\left[(\lambda - x_{2+} + x_{2-})^2 + (\lambda - x_{1+} + x_{1-})^2\right]\right), \tag{7.31}$$

leading to a wavefunction

$$\psi(t, z) = N(0, m) \exp\left(-i\frac{mc^2}{\hbar}t\right) \exp\left(-\frac{\beta}{2}(z - z_1)^2\right) \exp\left(-\frac{\beta}{2}(z - z_2)^2\right). \tag{7.32}$$

Here we find that, in this case, the two collapses are equivalent to a single collapse at the point $\frac{1}{2}(X_1 + X_2)$, but with twice the 'strength'. This is certainly what would be expected in the non-relativistic limit if we were to have

two collapses, although this solution only holds in the forward light-cone of the intersection point $X_3$. It should be noted that in the non-relativistic situation, the wavefunction at the point of the second collapse would have already been reduced by the first, so the probability of the second collapse occurring would be very small.

We now briefly consider the question of the order of the two collapses. Take the situation shown in Fig. 7.3, where the collapse at $X_1$ happens later than the collapse at $X_2$, in the frame in which $p = 0$. The relative size of the two peaks depends on the distances from $X_1$ to $A_2$ and $X_2$ to $A_1$. As can be seen from the diagram, these distances are of course equal, so the exponentials by which we multiply the two wavefunctions will be equal, at the two peaks, and thus as the wavefunction is not time-dependent, the two peaks will have the same size. The order of the collapses is immaterial when we have a single momentum component.

### 7.2.1 Superposition of Two Wavepackets

We now want to examine a typical measurement situation, where the initial wavefunction is a sum of two well-separated peaks, e.g.

$$\psi_0(z) = N\left[\exp\left(-\alpha(z-z_1)^2\right) + \exp\left(-\alpha(z-z_2)^2\right)\right], \tag{7.33}$$

with $|z_1 - z_2| \gg \frac{1}{\sqrt{\alpha}}$. Any collapse which occurs will, with probability essentially one, be centered around one of the two peaks of the wavefunction, as a result of an important property of the GRW-type models which we wish to retain in the rest frame; that the probability of a collapse occurring at a point $\mathbf{x}$ is proportional to $|\psi^H(\mathbf{x})|^2$. With a single collapse, the wavefunction will be reduced to a single peak in a time $t \approx \frac{|z_1 - z_2|}{c}$.

However, as before, the relativistic model allows the possibility of there being two collapse events, one centered on each peak, providing that each collapse event

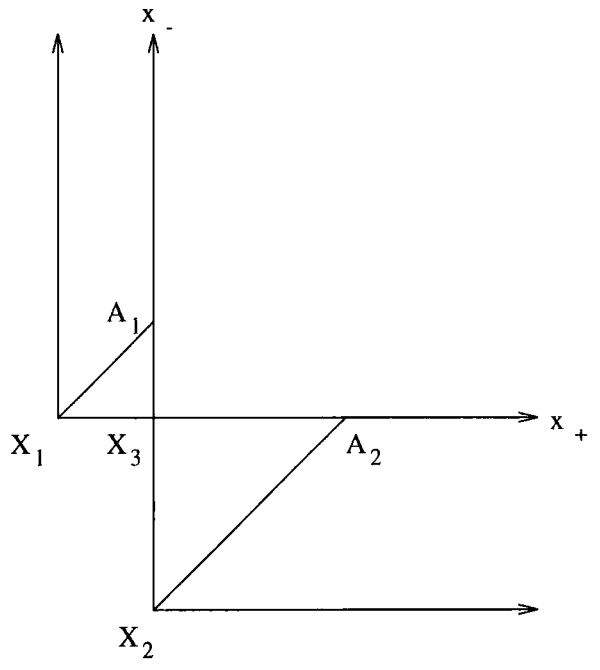



Figure 7.3: The effect of the relative times of the collapses

is outside of the forward light-cone of the other. On a constant time slice, there may persist peaks in each region, but we are predominately concerned with the shape of the wavefunction at later times, i.e. after the intersection of the two forward light-cones.

We assume that the momentum vectors defined at the two collapse points are (to a sufficiently good approximation) the same, so that we can again work in the frame for which $\mathbf{p} = 0$. Then, with the same approximations as before, the final state wavefunction will just be the initial state wavefunction multiplied by the collapse functions arising from the spatially separated collapses. This final state wavefunction can be written as

$$\psi_f = N\left[\exp\left(-(\alpha+\beta)(z-z_1')^2\right) + \exp\left(-(\alpha+\beta)(z-z_2')^2\right)\right], \qquad (7.34)$$

and we again have the two peaks, only now their centers have been shifted,

according to

$$z_1' = z_1 + \frac{\frac{1}{2}\beta}{\alpha+\beta}(z_2 - z_1) \tag{7.35}$$

$$z_2' = z_2 + \frac{\frac{1}{2}\beta}{\alpha+\beta}(z_1 - z_2). \tag{7.36}$$

Obviously if the peaks were very sharp in the initial wavefunction, then the shift will be quite small. However, it is certainly possible that the shift will be sufficient for the new peak to lie well into the tail of the initial peak, where it would have been extremely unlikely that the particle could be found. This will be the case if $\exp\left(-\frac{\alpha\beta^2}{4(\alpha+\beta)^2}(z_2 - z_1)^2\right) \ll 1$. For a pair of sharp initial peaks, and $\alpha \gg \beta$, this reads $|z_2 - z_1| \gg \frac{2\sqrt{\alpha}}{\beta}$.

In general we might expect the localization of the two peaks to be less than, but of the order of, the GRW collapse size, i.e. $\beta < \alpha$ but of the same order. This means that the peaks are shifted by something around $\frac{1}{4}$ of their separation.

## 7.3 The Born Probability Rule

In orthodox quantum theory, the probability that a measurement outcome will correspond to one of 2 peaks is proportional to the square integral of the weight of each peak. The same result holds in GRW, where it is a consequence of the probability rule for a hit occurring at a particular point (see §2.1). Here we shall again guarantee this result, *for a single collapse*, by postulating that the probability of this collapse occurring at one of the peaks is proportional to the integral of the square of the wavefunction over the peak.

It should be noted that in the non-relativistic GRW model, the probability of a collapse in an infinitesimal element $d\mathbf{x}dt$ is given by $P = \lambda|\psi_I^H(\mathbf{x},t)|^2 d\mathbf{x}dt$, where $\psi_I^H(\mathbf{x},t)$ is the integrated square of the wavefunction after a hit, equivalent to eqs. (2.10) and (2.4) of the GRW model. We would like to retain some version of

this rule. However, the integral in the non-relativistic case is over the hyperplane with constant time. In this local version, the hit is not felt instantaneously at all points, so the integral must be taken over the hypersurface that is the forward light-cone of the point in question, assuming no further hits.

Thus with an initial state (in the rest frame):

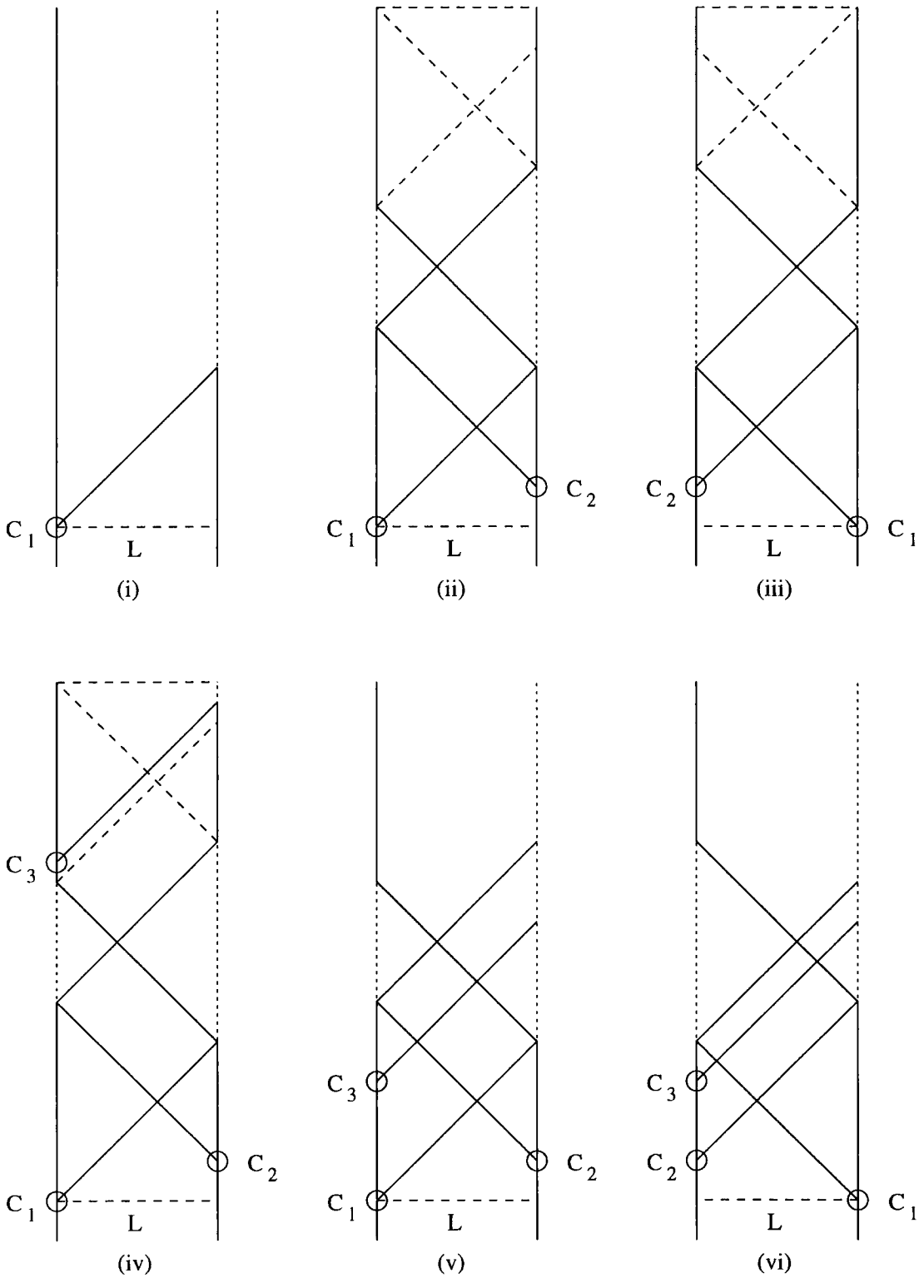
$$\psi_0(z) = N\left[a\exp\left(-\alpha(z-z_1)^2\right) + b\exp\left(-\alpha(z-z_2)^2\right)\right], \qquad (7.37)$$

with $|a|^2 + |b|^2 = 1$, the collapse will occur near $z_1$ or $z_2$ in the ratio of $|a|^2$ to $|b|^2$.

However, we now have to consider carefully the possibility of more than one collapse occurring. This means of course that both peaks can change their magnitudes. We take account of this by allowing $a$ and $b$ in eq. (7.38) to be functions of time. Consistent with the requirement of a *local* model we postulate that the probability of a collapse at $z_1$ at time $t_1$ is proportional to $\frac{|a(t)|^2}{|a(t)|^2+|b(t_R)|^2}$, where $t_R = t_1 - \frac{|z_1-z_2|}{c}$, the retarded time.

The probability of a particular peak persisting depends of course on the number of collapses that occur, and the time taken for the signal of a collapse to reach the other peak($T$). Here we shall evaluate the probability of peak 1 dominating. There will be contributions to this probability from all possible numbers of collapses. We shall assume that $\lambda T \ll 1$, and so make an expansion in this parameter. We shall calculate the first three terms in this series, i.e. work to order $(\lambda T)^2$.

The collapse processes which contribute to this order are illustrated in Fig. 7.4. We shall take the first collapse to occur at time $t = 0$. The separation of the two peaks is $z_1 - z_2 \equiv L \equiv Tc$. In order to assess the probability of a collapse occuring on a particular peak, we look at the relative sizes of the peaks along the backward light-cone. In the figure, the solid vertical lines indicate where a collapse on the peak is possible, whereas the dotted lines indicate that a collapse


C 1
L
(i)
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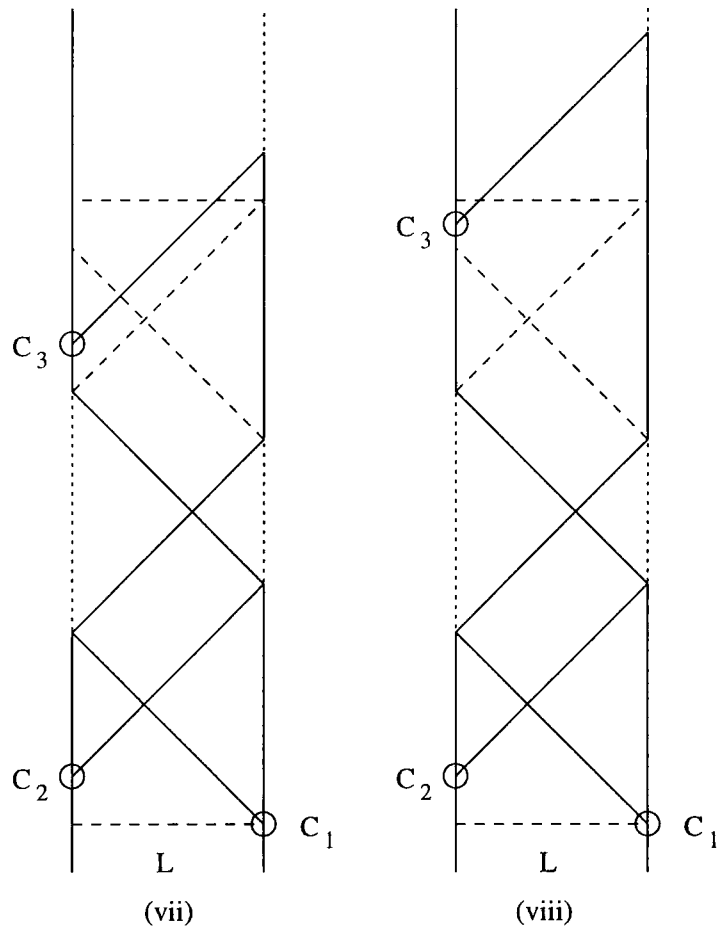


Figure 7.4: Collapse processes which contribute to second order in $\lambda T$.

is not possible. When a collapse is deemed possible, the probabilities for each side will be $|a|^2$ and $|b|^2$ respectively, if both can occur, or 1 and 0 if only one of these is possible. We need to calculate the probability contributions from each diagram separately.

**Diagram (i)** In this case a single collapse is successful. There are no collapses on the other peak before it has received the signal from the first collapse. The

probability of this occurring is

$$P_i = |a|^2 \exp\left(-\lambda T|b|^2\right) = |a|^2\left(1 - \lambda T|b|^2 + \frac{1}{2}(\lambda T)^2|b|^4 + O([\lambda T]^3)\right). \quad (7.38)$$

**Diagram (ii)** Here we have two specified collapses, $C_1$ and $C_2$, one centered on each peak, with that on the peak number 1 occurring first, and with no further collapses before the time indicated by the dashed line. The probability can be written as

$$\begin{aligned} P_{ii} &= |a|^2 \int_0^T \exp\left(-\lambda T|b|^2\right) \lambda dt|b|^2 \exp\left(-\lambda(T+t)|a|^2\right) \\ &\qquad \times \exp\left(-2\lambda T\right) \exp\left(-\lambda t|b|^2\right) P \\ &= |a|^2|b|^2(\lambda T)[1 - \frac{7}{2}(\lambda T)]\, P + O([\lambda T]^3), \end{aligned} \quad (7.39)$$

where $P$ is the overall probability of peak 1 dominating.

**Diagram (iii)** This diagram is a mirror image of diagram (ii), and the probability contribution is in fact the same.

$$P_{iii} = P_{ii}. \quad (7.40)$$

**Diagram (iv)** The initial collapses are the same as in diagram (ii), only we have a further collapse occurring before the dashed line. (n.b. if there are no further collapses before this time, then we essentially have the situation with which we started.) This added collapse gives rise to another integral in the calculation of the probability

$$\begin{aligned} P_{iv} &= |a|^2 \int_0^T \exp\left(-\lambda T|b|^2\right) \lambda dt|b|^2 \exp\left(-\lambda(T+t)|a|^2\right) \\ &\qquad \times \int_0^{T+t} \exp\left(-2\lambda T\right) \lambda dt' \exp\left(-\lambda t'|b|^2\right) \\ &= \frac{3}{2}(\lambda T)^2|a|^2|b|^2 + O([\lambda T]^3). \end{aligned} \quad (7.41)$$

**Diagram (v)** This time we have two collapses centered on peak 1 before the collapse on the second peak has taken full effect, with the first collapse centered

on peak 1.

$$\begin{aligned} P_v &= |a|^2 \int_0^T \exp\left(-\lambda T|b|^2\right) \lambda dt |b|^2 \lambda |a|^2 (T+t) \exp\left(-\lambda(T+t)|a|^2\right) \\ &= \frac{3}{2}(\lambda T)^2 |a|^4 |b|^2 + O([\lambda T]^3). \end{aligned} \tag{7.42}$$

**Diagram (vi)** This is similar to the previous diagram, with the solitary collapse on peak number 2 occurring first.

$$\begin{aligned} P_{vi} &= |b|^2 \int_0^T \exp\left(-\lambda T|a|^2\right) \lambda dt |a|^2 \exp\left(-\lambda(T+t)|b|^2\right) \int_t^T \lambda dt' |a|^2 \\ &= \frac{1}{2}(\lambda T)^2 |a|^4 |b|^2 + O([\lambda T]^3). \end{aligned} \tag{7.43}$$

**Diagram (vii)** This diagram is similar to diagram (iii) in the same way as diagram (iv) is related to diagram (ii).

$$\begin{aligned} P_{vii} &= |b|^2 \int_0^T \exp\left(-\lambda T|a|^2\right) \lambda dt |a|^2 \exp\left(-\lambda(T+t)|b|^2\right) \\ &\qquad \int_0^{T-t} \lambda dt' \exp\left(-2\lambda T\right) \exp\left(-\lambda t'|b|^2\right) \\ &= \frac{1}{2}|a|^2 |b|^4 (\lambda T)^2 + O([\lambda T]^3). \end{aligned} \tag{7.44}$$

**Diagram (viii)** This is almost the same as the last diagram, except that the time of the last collapse gives it a different probability of occurring.

$$\begin{aligned} P_{viii} &= |b|^2 \int_0^T \exp\left(-\lambda T|a|^2\right) \lambda dt |a|^2 \exp\left(-\lambda(T+t)|b|^2\right) \\ &\qquad \int_0^t \lambda dt' |a|^2 \exp\left(-2\lambda T\right) \exp\left(-\lambda t|a|^2\right) \exp\left(-\lambda(T-t+t')|b|^2\right) \\ &= \frac{1}{2}|a|^4 |b|^2 (\lambda T)^2 + O([\lambda T]^3). \end{aligned} \tag{7.45}$$

Adding all these calculated probabilities together and rearranging, leads to

$$P = |a|^2 + \lambda T |a|^2 |b|^2 (|a|^2 - |b|^2) - \frac{1}{2}(\lambda T)^2 |a|^2 |b|^2 (|a|^2 - |b|^2)(5 - 4|a|^2|b|^2). \tag{7.46}$$

If the initial superposition is equally weighted, then as expected the probabilities for each peak to dominate are equal. However, if we start with an unequal superposition of the two gaussian peaks, then in this model the probability of obtaining the initially higher peak increases with the peak separation.

## 7.4 Two-particle correlated wavefunction

We will now deal with the case where we have two particles with a correlated wavefunction, for instance an EPR-type situation analogous to the measurement of the spins of two fermions in a correlated state

The initial wavefunction can be written in the form:

$$\begin{aligned}\psi(z_1,z_2) &= N\left[a\phi_1(z_1)\phi_2(z_2)+b\chi_1(z_1)\chi_2(z_2)\right] \\ &= N\Big[a\exp\left(-\alpha(z_1-z_{11})^2\right)\exp\left(-\alpha(z_2-z_{21})^2\right) \\ &\qquad +b\exp\left(-\alpha(z_1-z_{12})^2\right)\exp\left(-\alpha(z_2-z_{22})^2\right)\Big],\end{aligned} \tag{7.47}$$

where $z_{11}, z_{12}$ refer to the center of the peaks corresponding to particle 1, and similarly for particle 2. We assume that the two peaks for each particle do not overlap significantly, so that a collapse centered on one will kill the other peak, i.e. $\alpha(z_{11}-z_{12})^2 \gg 1$ and $\beta(z_{11}-z_{12})^2 \gg 1$. Also, for simplicity, we will consider only the case when this peak separation itself is negligible compared with the separation of the two particles, for instance, $|z_{11}-z_{21}| \gg |z_{11}-z_{12}|$. This of course corresponds to the actual experimental situations in tests of the Bell inequality.

Collapse processes centered on each particle will be taken to be independent, and they will have the same effect on the wavefunction as before. However, in the case where we have two 'incompatible' collapses, the situation will have changed in that as each collapse acts on a different part of the wavefunction, they will not 'interfere' at any point in space. At a time after signals from both collapses have reached the other, the wavefunction in the intermediate region will just be multiplied by the two independent collapse factors irrespective of the momentum states from which they were constructed, i.e. for collapses centered at $z_{11}$ and

$z_{22}$,

$$\psi' = \psi \exp\left(-\frac{\beta}{2}(z_1 - z_{11})^2\right) \exp\left(-\frac{\beta}{2}(z_2 - z_{22})^2\right). \tag{7.48}$$

As before, we should again ask which part of the wavefunction will dominate. We can do a similar calculation to before, with the probability of a collapse occurring on a particular peak being either $|a|^2, |b|^2$, 1 or zero. However, as the peak separation of either particle is considered to be negligible compared to the separation between the two particles, we shall take the signal of a collapse to travel instantaneously to the other peak connected to that particle. For a single particle, the probability of another collapse in the actual time taken is very small.

It turns out that the probability for a particular peak to dominate is actually the same as when we only have one particle.

Fig. 7.5 illustrates the diagrams which contribute to second order in $\lambda T$, ignoring the separation between the two peaks on one side. As before the solid lines indicate that a collapse is deemed possible, whereas a collapse cannot occur where the line is dotted. The points where a collapse occurs are circled. The peaks for particle 1 are those to the left, with the $\phi$ peak the right one of these and the $\chi$ peak on the left. We calculate the probability of the $\phi$ peaks dominating.

**Diagram (i)** The simplest case where we have no incompatible collapses.

$$\begin{aligned} P_{2i} &= |a|^2 \left[\exp(-\lambda T) + \int_0^T \lambda |a|^2 dt \exp(-\lambda t)\right] \\ &= |a|^2 \left(1 - \lambda T |b|^2 + (\lambda T)^2 |b|^2\right) + O([\lambda T]^3). \end{aligned} \tag{7.49}$$

**Diagram (ii)** As in the one-particle case, we have two specified collapses, one affecting each particle, but incompatible.

$$\begin{aligned} P_{2ii} &= |a|^2 \int_0^T \lambda |b|^2 dt \exp(-2\lambda(T + t))\, P_2 \\ &= |a|^2 |b|^2 (\lambda T)\,(1 - 3(\lambda T))\, P_2 + O([\lambda T]^3), \end{aligned} \tag{7.50}$$

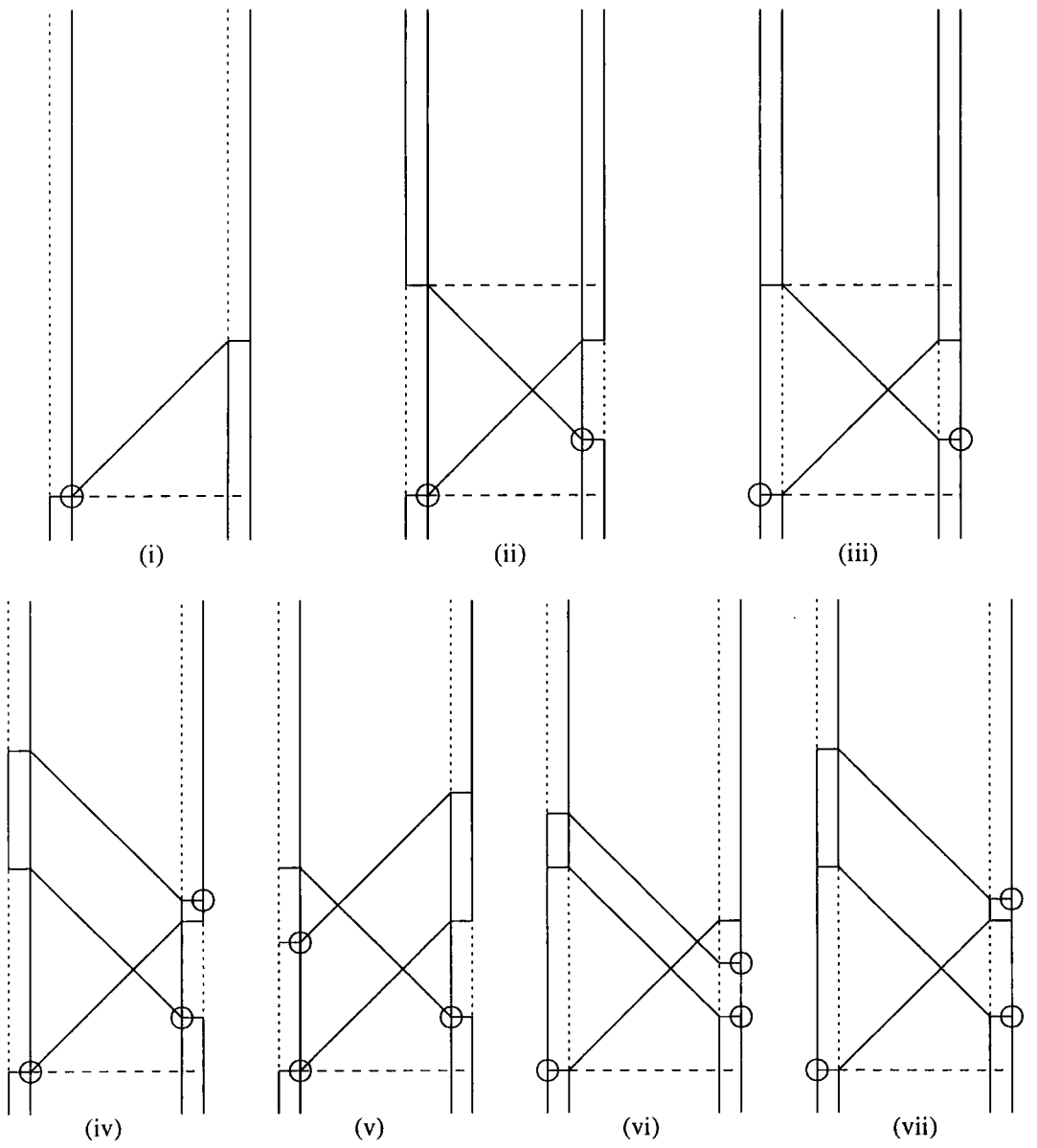


Figure 7.5: The collapse processes which contribute to second order in $\lambda T$ for two correlated particles.

where $P_2$ is the overall probability of the $\phi$ peaks dominating for this two-particle wavefunction.

**Diagram (iii)** This is just the mirror image of the previous diagram, with identical contribution to the probability.

$$P_{2iii} = P_{2ii}. \tag{7.51}$$

**Diagram (iv)** As in diagram (ii) but with an additional collapse prior to both particles having knowledge of both previous collapses.

$$\begin{aligned} P_{2iv} &= |a|^2 \int_0^T \lambda|b|^2 dt \int_0^t \lambda|a|^2 dt' \exp\left(-\lambda(T+t')\right) \exp\left(-\lambda(T-t+t')\right) \\ &= \frac{1}{2}|a|^4|b|^2(\lambda T)^2. \end{aligned} \tag{7.52}$$

**Diagram (v)** One particle has two compatible collapses dominating the collapse on the other particle.

$$\begin{aligned} P_{2v} &= |a|^2 \int_0^T \lambda|b|^2 dt \exp\left(-\lambda T\right) \exp\left(-\lambda(T+t)\right) \int_0^{T+t} \lambda dt' \exp\left(-\lambda t'\right) \\ &= \frac{3}{2}|a|^2|b|^2(\lambda T)^2. \end{aligned} \tag{7.53}$$

**Diagram (vi)** As in the last diagram, but the order of the first two collapses is reversed.

$$\begin{aligned} P_{2vi} &= |b|^2 \int_0^T \lambda|a|^2 dt \exp\left(-\lambda T\right) \exp\left(-\lambda(T+t)\right) \int_0^{T-t} \lambda dt' \exp\left(-\lambda t'\right) \\ &= \frac{1}{2}|a|^2|b|^2(\lambda T)^2. \end{aligned} \tag{7.54}$$

**Diagram (vii)** Similar to diagram (iii) with the same difference as between diagrams (ii) and (iv).

$$\begin{aligned} P_{2vii} &= |b|^2 \int_0^T \lambda|a|^2 dt \exp\left(-\lambda T\right) \exp\left(-\lambda(T+t)\right) \\ &\qquad \times \int_0^t \lambda|a|^2 dt' \exp\left(-\lambda(T-t+t')\right) \\ &= \frac{1}{2}|a|^4|b|^2(\lambda T)^2. \end{aligned} \tag{7.55}$$

Adding these probabilities together, gives

$$P_2 = |a|^2 + \lambda T|a|^2|b|^2(|a|^2 - |b|^2) - \frac{1}{2}(\lambda T)^2|a|^2|b|^2(|a|^2 - |b|^2)(5 - 4|a|^2|b|^2), \quad (7.56)$$

which is the same probability as that obtained for the one-particle, two-peak wavefunction.

## 7.5 Problems

The crucial parameter in the above discussion is $\lambda T$. For a single particle, or two particle system, this will be small, even with the separation $L \approx 10\,\mathrm{m}$ corresponding to the largest separation in the Aspect experiments[34]. However, for examining the quantum probability rule, we need to interact the two-particle system with a measuring apparatus. It is extremely unlikely that the two-particle system will have collapsed, so the result of any measurements will be dependent on the collapses of the measuring pointers. If the interactions on the opposite sides of the system occur almost simultaneously (separated by less than time $T$), then the collapses which occur on the pointers on each side will be independent (as the pointers themselves are not correlated), and thus the correlations between the two particles will have disappeared. Thus our local collapse process does not seem to be able to explain the results of the Aspect experiments.

## 7.6 Summary

An important contribution of the original GRW model was that it showed the *possibility* of defining a precise model in which collapse happens as a physical process in such a way that the tested predictions of quantum theory still held (This is independent of the issue of whether nature actually *chooses* this particular

solution of the measurement problem).

In the same spirit we have attempted to find here a precise definition of a collapse model which is *local* and *Lorentz-invariant.* The issue of whether it can be made consistent with all experiments is somewhat less clear, especially involving correlated wavefunctions, may well render the condition on locality impossible to retain.

# Chapter 8

# Summary and Conclusions

The models of wavefunction collapse postulate that the quantum jumps that take us from a superposition to a statistical mixture do occur, but not in the rather vague sort of way originally intimated. The collapses are built into the evolution equations of the quantum system in a stochastic manner, in such a way that although the wavefunction experiences sudden, random localizations, and hence is discontinuous at the point of a collapse, the evolution of the density matrix of the system is completely deeterministic. It should be noted that the narrowing of the wave-packet is necessarily accompanied by an increase of energy.

The models possess the two features to enable them to be considered as possible solutions of the measurement problem of quantum theory. As stated above, they cause the reduction of an initial superposition of states into a statistical mixture. However, this process is only rapid for macroscopic objects, and the effect is negligible for a microscopic system. This difference can be seen explicitly from the equations of the model, without the need for *a priori* splitting of objects into two classes. Another crucial feature is that the collapse is to spatially localized states, thus accounting for the world we actually see.

A simple extension of the original hitting model renders it consistent for a sys-

tem of identical particles, and this model can be related to a continuous version in which the wavefunction is localized continuously as opposed to changing suddenly. The discrete models introduce two new parameters into nature, $\lambda$ which gives the average rate of collapse of a single particle (or the whole universe, depending on exactly what you require to be constant.), and $a$, the localization scale to which the wavefunctions are reduced. In addition, the CSL model introduces a new stochastic field, $d\mathbf{B}$ to be the source of the collapse.

Theoretical constraints can be imposed on the models by considering the collapse rate of a macroscopic object, whilst experimental constraints can be derived from examining the maximum rates of atomic excitation not already ruled out. These constraints taken together give a strong indication that the couplings for the particles are related to their mass, and hence the collapse mechanism has a gravitional source.

An early attempt at explaining the relationship between gravity and collapse postulates a random fluctuating gravitational potential arising from the appearance of planckons at random times and places. This model can give the required rate of energy increase accompanying collapse, but does not give the previously taken values for both parameters simultaneously. It is doubtful whether any parameter values could be found to be consistent with the constraints for the models formulated here.

These collapse models exist in the non-relativistic domain, and are non-local in that the collapse has an immediate effect on the wavefunction over all space. We postulate that the effect of the collapse is felt only inside of the forward light-cone of the collapse center. However, this does lead to a slight deviation from the Born probability rule for a simple state of one particle or two correlated particles. If the effect of the collapses were to be completely additive for many particles,

then the deviation could become large, although it is negligible at microscopic level. Observation or non-observation of this effect could be a test as to whether the collapse mechanism exists.

To conclude, models of wavefunction collapse, whether discrete hitting versions or continuous models can be seen to be a logical consistent solution to the measurement problem of quantum mechanics. They introduce two new parameters into nature at this level, and indications are that they have a gravitational origin. To date there have been no experiments performed that can prove the existence of the collapse mechanism, but none can disprove it either.

# Appendix A

# Derivation of eq. (3.19)

The derivation of eq. (3.19) from eqs. (3.12) and (3.13) is not immediately obvious. This is because the differential of $\|\psi\|$ does not follow immediately given $d\|\psi\|^2$ when using Ito calculus. To derive the evolution of $|\chi>$, we must find the differential $d\left(\frac{|\psi>}{\|\psi\|}\right)$. Using the Ito calculus, we find

$$d\left(\frac{|\psi>}{\|\psi\|}\right) = \frac{d|\psi>}{\|\psi\|} + |\psi> d\left(\frac{1}{\|\psi\|}\right) + \overline{(d|\psi>)\, d\left(\frac{1}{\|\psi\|}\right)}. \tag{A.1}$$

To calculate the differential of $\frac{1}{\|\psi\|}$, we will first evaluate $d\|\psi\|$.

We know all of the terms that can occur in the differential, so we can write

$$d\|\psi\| = c_1 \mathbf{R}\cdot d\mathbf{B} + c_2(\mathbf{A}-\mathbf{A}^\dagger)\cdot d\mathbf{B} + \left(c_3\mathbf{A}\cdot\mathbf{R} + c_4\mathbf{R}\cdot\mathbf{R} + c_5\mathbf{A}^\dagger\cdot\mathbf{A} + c_6\right)dt, \tag{A.2}$$

where the $c_i$ are to be determined. To do this, we substitute this expression into the formula

$$2\mathbf{R}\cdot d\mathbf{B} = d\|\psi\|^2 = 2\|\psi\| d\|\psi\| + \overline{d\|\psi\| d\|\psi\|}. \tag{A.3}$$

Using the correlations of $d\mathbf{B}$ given by eq2. (3.4) and (3.5), we find that $1 = c_1\|\psi\|$ and $2\|\psi\|c_4 + c_1^2\gamma = 0$ with all the other $c_i$ vanishing. Thus

$$d\|\psi\| = \frac{1}{\|\psi\|}\mathbf{R}\cdot d\mathbf{B} - \frac{1}{2\|\psi\|^3}\gamma dt\mathbf{R}\cdot\mathbf{R}. \tag{A.4}$$

To extract the required differential, we make use of the identity

$$0 \equiv d\left(\|\psi\|\frac{1}{\|\psi\|}\right) = d\|\psi\|\frac{1}{\|\psi\|} + \|\psi\| d\left(\frac{1}{\|\psi\|}\right) + \overline{d\|\psi\| d\left(\frac{1}{\|\psi\|}\right)}. \tag{A.5}$$

We write the differential in the same manner as $d\|\psi\|$:

$$d\left(\frac{1}{\|\psi\|}\right) = c_1 \mathbf{R}\cdot d\mathbf{B} + c_2 \gamma dt \mathbf{R}\cdot\mathbf{R}, \tag{A.6}$$

in the knowledge that these will be the only terms present. We then substitute this expression and that for $d\|\psi\|$ of eq. (A.4) into eq. (A.5) to find that $\frac{1}{\|\psi\|^2} + \|\psi\| c_1 = 0$ and $-\frac{1}{2\|\psi\|^4} + c_2\|\psi\| + c_1\frac{1}{\|\psi\|} = 0$, which leads to

$$d\left(\frac{1}{\|\psi\|}\right) = -\frac{1}{\|\psi\|^3}\mathbf{R}\cdot d\mathbf{B} + \frac{3}{2\|\psi\|^5}\gamma dt \mathbf{R}\cdot\mathbf{R}. \tag{A.7}$$

Using eq. (3.12) for $d|\psi>$, we find that

$$\overline{(d|\psi>) d\left(\frac{1}{\|\psi\|}\right)} = -\frac{1}{\|\psi\|^3}\gamma dt \mathbf{A}\cdot\mathbf{R}, \tag{A.8}$$

and hence

$$\begin{aligned} d\left(\frac{|\psi>}{\|\psi\|}\right) &= \left(-\frac{i}{\hbar}H dt + \mathbf{A}\cdot d\mathbf{B} - \frac{\gamma}{2}\mathbf{A}^\dagger\cdot\mathbf{A} dt\right)\frac{|\psi>}{\|\psi\|} \\ &+ \left(-\frac{1}{\|\psi\|^2}\mathbf{R}\cdot d\mathbf{B} + \frac{3}{2\|\psi\|^4}\gamma dt \mathbf{R}\cdot\mathbf{R}\right)\frac{|\psi>}{\|\psi\|} - \frac{1}{\|\psi\|^2}\gamma dt \mathbf{A}\cdot\mathbf{R}\frac{|\psi>}{\|\psi\|}, \end{aligned} \tag{A.9}$$

where $\mathbf{R} = \frac{1}{2}<\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi>$. We can of course rewrite this expression in terms of the new state $|\chi>$,

$$d|\chi> = \left[\left(-\frac{i}{\hbar}H - \frac{\gamma}{2}\mathbf{A}^\dagger\cdot\mathbf{A} - \gamma\mathbf{A}\cdot\mathbf{R} + \frac{3\gamma}{2}\mathbf{R}\cdot\mathbf{R}\right)dt + (\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\chi>, \tag{A.10}$$

with $\mathbf{R}$ now given by

$$\mathbf{R} = \frac{1}{2}<\chi|(\mathbf{A}+\mathbf{A}^\dagger)|\chi>. \tag{A.11}$$

We can see that we have recovered eq. (3.19).

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