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The Topological Renormalisation of the $O(3)$ Sigma Model

by

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A thesis submitted for the degree
of Doctor of Philosophy.

Department of Mathematical Sciences,
University of Durham,

September 1995

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Abstract

Like other field theories of physical interest, the moduli-space integrals of the non-linear two-dimensional $O(3)$ sigma model diverge. We show that in the one-instanton sector the imposition of a cut-off in the moduli-space leads to an unacceptable dependence of the Green's function on the way that the field is split into the quantum piece and the classical background. This dependence may be isolated in a term which may be interpreted as an anomaly to the Ward Identity of the theory.

The moduli-space divergence is associated with degeneration of the field configurations to those of another topological sector. Hence it is possible that by modifying the Green's function in, say, the zero-instanton sector will be able to cancel the divergence in the one-instanton sector. We show that the Ward Identity anomaly in the one-instanton sector may be written in the zero-instanton sector at next to leading order in powers of $\hbar$, and hence we explicitly calculate the Green's function modification. We have called the process of applying this modification "Topological Renormalisation".

A central piece of the modification term is the instanton contribution to the Green's function of the model. This is obtained by using two new methods of calculating the determinant of the fluctuation operator.

The application of Topological Renormalisation to other theories is also investigated.

To Mum and Rebecca

Preface

This thesis is based on research by the author, carried out between October 1992 and September 1995. The material presented has not been submitted previously for any degree in either this or any other University.

Chapter One and Chapter Two are review chapters. Chapter Three is also mostly review except for Section 3.4 which is original work. Chapter 4 is partly a review of part of [4], the remainder of this chapter and the whole of Chapter 5 is work done by the author in collaboration with Paul Mansfield and has been submitted for publication [3]. Chapter 6 concludes the review of [4] in Sections 6.3 and 6.4. Section 6.2 contains original ideas by the author. Chapter 7 summarises the work.

I would like to thank my supervisor Paul Mansfield for his patience and support. I would also like to thank Alex Iskandar and David Bull for some helpful comments.

Statement of Copyright

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Contents

Chapter 1

Introduction

Throughout their history field theories and other models of subatomic particle interactions have had problems with unwanted infinities. It has often been the case that integrals have diverged instead of remaining finite in the calculation of Feynman scattering amplitudes, and clever procedures have had to be devised to somehow resolve these problems. The process of cancelling these infinities, or making sure they don't occur, is known as renormalisation.

In certain field theories of physical interest perturbative expansion is often done as an expansion about classical solutions to the equations of motion. These solutions have moduli which must also be integrated out to get a final Green's Function. However these integrals over moduli space are often divergent and so some form of renormalisation needs to be devised to return the relevant model to a finite form. More explicitly, the field is split into a classical piece that solves the Euler-Lagrange equations, and a quantum piece. The classical solutions are classified into topological sectors which are parametrised by moduli. When the integral over the quantum piece is calculated in each topological sector, there remains an integral over the moduli. Typically, these integrals will diverge, and it is the purpose of this thesis to demonstrate a method of coping with this divergence.

For instance, let us consider the two dimensional non-linear $O(3)$ sigma model. Here the classical action is periodic and at the minima the solutions to the equations of motion are known as instantons, which in their most general form are given by [1]

$$v = c\prod_{j=1}^{q}\frac{(z-a_j)}{(z-b_j)} \tag{1.1}$$

The $z = z(x, y)$ are complex functions, a, b, and c are also complex and are the instanton moduli. q is known as the topological charge and is equal to the number of poles of the instanton. To introduce a quantum element into this model the fields are defined to be $w = v + \phi$ where ϕ is a quantum fluctuation about the the instanton solution continuously deformable to zero. With the fields in this form the Green's Function reduces to a functional integral over the ϕ and an integral in the moduli space. Integrating out the fields leaves us with the instanton contribution to the Green's Function [2]

$$I(\Lambda) = \langle\Lambda\rangle = \sum_q \int \Lambda(a,b,c)\, \frac{K^q}{(q!)^2}\, e^{-h_q(a,b)}\, \frac{d^2c}{\pi(1+|c|^2)^2} \prod_j d^2a_j\, d^2b_j \tag{1.2}$$

where K^q is dependent on the coupling constant and

$$h_q(a,b) = -\sum_{i<j}^{q} \ln|a_i - a_j|^2 - \sum_{i<j}^{q} \ln|b_i - b_j|^2 + \sum_{i,j}^{q} \ln|a_i - b_j|^2 \tag{1.3}$$

So we see that the problem in this case is that this is divergent as $a_i \to b_j$. We could simply apply a cut-off in the moduli space, for instance by setting $|a-b| = r$ and then later taking the limit $r \to 0$. To apply the cut-off the Green's Function needs to be written explicitly as an integral over the moduli prior to the fields being integrated out. In performing this separation the theory develops a dependence on how the field is split between the quantum piece and the classical background, i.e. it depends on our choice of co-ordinates in configuration space.

A straightforward and flexible method of separating the integral over the moduli from the integral over the fields is to use the Faddeev-Popov trick. This approach is useful as it does not make any assumptions about the size of the quantum fluctuations. However we will have to introduce an arbitrary set of constraints on the fields, corresponding to a choice of co-ordinates in configuration space. It is essential that the final version of the Green's Function for the model does not depend on our choice of these constraints. To isolate any dependence we may take a variation of the Green's Function with respect to the constraints. This results in an identity which expresses the change in the moduli-space density of an arbitrary Green's Function under a change in the constraint as a total derivative with respect to the moduli. If this identity is non-zero then the classical symmetry of the model may possibly be broken and something must be done to restore it.

However note that if the cut-off is used in the instanton solution (1.1), for instance by setting $a_i - b_i = r$, then in the limit $r \to 0$ there is a degeneracy from the q-instanton solution to the $(q-1)$-instanton solution. This suggests the possibility of modifying the action in the sector with lower instanton number in such a way that the symmetry is restored and the dependence on an arbitrary choice of quantisation procedure is removed.

In this thesis, we shall show that such a procedure is possible. We shall propose that for the $O(3)$ sigma model the symmetry may be restored by the addition of terms into the action. For instance we shall show that divergences in the one-instanton sector may be regulated by the addition of a term to the zero-instanton sector action. These new terms are a relic of the one-instanton moduli-space Jacobian.

This modification is analogous to perturbative renormalisation in that pathologies at a certain order in the expansion are cancelled by a modification of the action at a lower order. However, in perturbative renormalisation the modification is absorbed into the coupling constants to leave the Green's Functions finite. Here the modification may not be connected with the coupling constant as they have very different forms. Nevertheless, due to the nature of the regularisation procedure presented here, we shall call it "Topological Renormalisation".

This thesis is constructed as follows. In Chapter 2 we define the non-linear two dimensional $O(3)$ sigma model and look at its topology. The instanton solutions are derived and we see how they are split into homotopy classes labelled by the topological charge. This is entirely a review chapter. In Chapter 3 we present in detail the calculation of the instanton contribution to the Green's function for the model (1.2). This will largely follow [2] although Section 3.4 is entirely original work. Chapter 4 contains a study of why topological renormalisation is necessary for gauge field theories and the sigma model. We also look at the moduli divergences in terms of anomalies in the Ward Identities for the theories. In Chapter 5 the anomalous term for the $O(3)$ sigma model in the one-instanton sector is calculated. It is shown explicitly that this term may be written in terms of the zero-instanton sector fields. This chapter is entirely original work, the results of which are presented in [3]. Chapter 6 contains a review of topological renormalisation applied to other important theories [4]. Section 6.2 on the $\mathbb{CP}^{n-1}$ model is original work.

Chapter 2

The $O(3)$ Sigma Model

2.1 General Formulation

Non-linear $O(N)$ sigma models are an example of the class of field theories known as Chiral Models. In such models the existence of interaction between the fields is due solely to the geometry of the manifolds, unlike other field theories where the interaction is added into the Lagrangian by hand. In the case of the $O(3)$ sigma model the interactions are a result of the curvature of the manifolds.

Reasons for studying Chiral Models are two-fold. Firstly they are analogous to Non-Abelian gauge theories, with which they share such features as asymptotic freedom, non-trivial topological structure, the existence of instantons [5], [6], conformal invariance [7] and the presence of hidden symmetry giving, in the two dimensional case, an infinite number of conservation laws. This is an important point which we shall return to later. Secondly, in the cases where the manifolds are Kähler or hyper-Kähler, these models have supersymmetric structures in the $N = 2$ and $N = 4$ cases. This leads to the construction of superstring theories.

The general structure of the Chiral Models [8], [9] involves a scalar field taking values in an N-dimensional Riemann manifold $\mathcal{M}$ from a $(d+1)$-dimensional Minkowski spacetime. The action is

$$S = \frac{1}{4}\int g_{ij}(\phi)\ \partial_\mu\phi^i\ \partial^\mu\phi^j\ d^dxdt \tag{2.1}$$

where $\phi^i (i = 1, ..., N)$ are the co-ordinates on $\mathcal{M}$ and $g_{ij}(\phi)$ is its metric. The Greek indices label the spacetime co-ordinates. The field equations follow from the usual condition of finding the extremes of the action: $\delta S = 0$

$$\partial_\mu\partial^\mu\phi^i + \Gamma^i_{jk}\partial_\mu\phi^j\ \partial^\mu\phi^k = 0 \tag{2.2}$$

where Γ^i_{jk} are the Christoffel symbols associated with g_{ij}. Note that these field equations are generalisations of the geodesic equation.

The $O(N)$ sigma model is a specific case of this where the space is two-dimensional and parametrised by the variables x_1 and x_2 given on $\mathbb{R}^2$. Fields ϕ^i are invariant under global $O(N)$ transformations

$$\phi^i \rightarrow \phi^{i'} = O^{i'i}\phi^i \tag{2.3}$$

This can be seen as a generalisation of the $O(4)$ sigma model developed to describe pion interactions by Gell-Mann and Levý [10]. The $N = 3$ case that we are interested in is useful as a model for the isotropic ferromagnet [11] [1] [12]. Also it is equivalent to the $\mathbb{CP}^1$ model ([13] and many others), this equivalence will be investigated in further detail later as it leads to an extension of the scope of Topological Renormalisation.

So in the $O(3)$ sigma model we define the field σ^a on the two-dimensional Euclidean plane $\mathbb{R}^2$

$$\sigma^a = \sigma^a(x_1, x_2) \qquad x_1, x_2 \in \mathbb{R}^2 \qquad a = 1, 2, 3 \tag{2.4}$$

This switch to Euclidean space is done for convenience in analysing the tunneling processes that the instantons describe. Thus the action becomes

$$S = \frac{1}{2k}\int \partial_\mu\sigma^a \; \partial^\mu\sigma^a \; d^2x \tag{2.5}$$

where $\partial_\mu = \frac{\partial}{\partial x_\mu}$, $\mu = 1, 2$, $a = 1, 2, 3$ and summation over both sets of indices is assumed. This action can also be arrived at by considering the simplest $O(3)$-invariant functional with action determined by the $O(3)$-invariant metric $ds^2 = d\sigma \cdot d\sigma$.

If the σ's are defined as taking values on S^2, $(\mathcal{M} = S^2)$, then they are subject to the constraint

$$\sigma^a\sigma^a = 1 \tag{2.6}$$

which can be imposed by means of a Lagrange multiplier $\lambda(x, y)$, so

$$S = \int d^2x \left[\frac{1}{2k}\partial_\mu\sigma^a \; \partial^\mu\sigma^a \; + \lambda(x, y)\left(\sigma^a\sigma^a - 1\right)\right] \tag{2.7}$$

The field equations are then

$$\partial^2\sigma - \left(\sigma^a\partial^2\sigma^a\right)\sigma = 0 \tag{2.8}$$

The most interesting field configurations are the finite energy solutions to the field equations, i.e. soliton solutions and instantons. These configurations must approach the same limit in all directions in physical space. Thus the physical space may be compactified onto a sphere which we shall call S^2_{phys}. Also we shall now call the space of fields the internal space S^2_{int}. This natural compactification is the origin of the model's non-linearity. Mappings from sphere to sphere may be classified into homotopy classes. We shall now investigate how this comes about.

2.2 Topology of the Sigma Model

The remainder of this chapter consists of a review of the topology of the $O(3)$ sigma model and how the solutions to the model may be classed into sectors labelled by a topological index. Studies of these ideas may also be found in [14], [15], [16], [17]. Also see [8].

First we consider the group structure of the physical and target manifolds. This leads to a classification of the mappings and thus to homotopy classes. Suppose that the field ϕ is constrained to take values on a homogeneous manifold $\mathcal{M}$. The boundary conditions are that as we approach spatial infinity in any direction then ϕ tends to some limit $\phi_\infty \in \mathcal{M}$. Then by adding a point at infinity to d-dimensional physical space $\mathbb{R}^d$ it can be compactified to a sphere S^d. The fields may now be thought of as the map

$$\phi : S^d \to \mathcal{M} \tag{2.9}$$

Let us suppose that $\mathcal{M}$ is acted on transitively by a group of symmetries G. This means that for any two points on $\mathcal{M}$, say y_0 and y_1, there exists $g \in G$ such that $y_1 = g(y_0)$. Thus by using y_0 as a fixed base point and considering $g(y_0)$ we may obtain the whole of $\mathcal{M}$ as

g ranges over G. In general we may obtain it many times, otherwise we could identify $\mathcal{M}$ with G. However it may be shown that $\mathcal{M}$ may be identified with a coset space. Using these ideas, two elements $g_1, g_2 \in G$ will give the same point in $\mathcal{M}$ if and only if their action at y_0 is the same

$$g_1(y_0) = g_2(y_0) \tag{2.10}$$

this implies that

$$g_2{}^{-1} g_1(y_0) = y_0 \tag{2.11}$$

Thus $g_2{}^{-1} g_1$ is an element of the subgroup of G that leaves y_0 unchanged, known as the little group H of y_0. So

$$H = \{h \in G : h(y_0) = y_0\} \tag{2.12}$$

Hence g_1 and g_2 will give the same point in $\mathcal{M}$ if and only if $g_1 = g_2 h$ for some $h \in H$.

A left coset of G with respect to H, written gH, is the set of elements gh where g is fixed but h varies over H

$$gH = \{gh : h \in H\} \tag{2.13}$$

(Similarly right cosets Hg may be defined). Thus g_1 and g_2 will give the same point in $\mathcal{M}$ if and only if they belong to the same left coset of G with respect to H.

We shall here simply state the elementary theorem that G may be partitioned into disjoint cosets such that every element of G belongs to one and only one left coset of G with respect to H. Also transitivity implies that any point of $\mathcal{M}$ may be obtained from the action of some left coset of G on y_0. Thus we can identify $\mathcal{M}$ with the space of left cosets G/H

$$\mathcal{M} = G/H = \{gH : g \in G\} \tag{2.14}$$

In general G/H is not a group unless H is a normal subgroup (when the left and right cosets are identical). It is easy to show that G/H is independent of the choice of base point y_0 [14].

For the $O(N)$ sigma model, ϕ is a real unit vector in N-dimensional space. As ϕ is subject to the constraint $\phi \cdot \phi = 1$ then $\mathcal{M}$ is a sphere S^{N-1}. Now G can be taken to be the connected group of rotations in N-dimensional space, $SO(N)$. The rotations that leave ϕ invariant are about the direction of ϕ, so the little group is the rotations in $(N-1)$-dimensional space, i.e. $H = SO(n-1)$. Thus

$$\mathcal{M} = \frac{G}{H} = \frac{SO(N)}{S(N-1)} = S^{N-1} \tag{2.15}$$

so for $N = 3$

$$S^2 = \frac{SO(3)}{SO(2)} \tag{2.16}$$

2.3 Homotopy

It is important that the mappings $\phi : S^n \to \mathcal{M}$ are topologically stable against continuous deformations. In other words they cannot be deformed into a constant map. If this were not the case then fundamental symmetries of the physical system, such as gauge symmetries and invariance under time evolution, could be lost. Thus the division of the maps into equivalence classes is essential.

Let us formally define homotopy and continuous deformations. Let f and g be two continuous maps between the spaces X and Y such that $X \to Y$ for both maps. Then f and g are said to be homotopic if there exists a continuous map $F(x,t)$, $0 \leq t \leq 1$ such that $F : X \times I \to Y$ where I is the unit interval $[0,1]$ and

$$F(x,0) = f(x) \quad , \quad F(x,1) = g(x) \tag{2.17}$$

Homotopy is an equivalence relation, symmetry, reflexivity and transitivity are all obeyed. Thus we can partition the set of maps $X \to Y$ into disjoint classes of mutually homotopic maps - the homotopy classes.

We are interested in the maps $S^n \to \mathcal{M}$, and shall denote the set of homotopy classes for these maps by $\pi_n(\mathcal{M})$. If $\mathcal{M}$ is also a sphere S^n then two maps $f, g : S^n \to S^n$ are homotopic if and only if $f(x)$ and $g(x)$ cover S^n the same number of times that x covers it once. Then we can identify $\pi_n(\mathcal{M})$ with $\mathbb{Z}$ and say that the homotopy classes are labelled by integer winding numbers. $\pi_n(\mathcal{M})$ must have more than one number for topologically stable structures.

It still remains for us to show that $\pi_n(\mathcal{M})$ has the form of a group. For any two maps $f, g : S^n \to \mathcal{M}$ a third continuous map may be defined

$$h(x_1, x_2, \cdots, x_n) = \begin{cases} f(2x_1, x_2, \cdots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, x_2, \cdots, x_n) & \frac{1}{2} \leq x_1 \leq 1 \end{cases} \tag{2.18}$$

which may be written

$$h = f + g \tag{2.19}$$

Each of these maps are members of their own homotopy class : $[h]$, $[f]$ and $[g]$, where $[h], [f], [g] \in \pi_n(\mathcal{M})$. However, if f is varied within $[f]$ and g is varied within $[g]$ then $[f + g]$ will remain unchanged, so

$$[f + g] = [f] + [g] \tag{2.20}$$

This binary operation means that $\pi_n(\mathcal{M})$ takes on a group structure. The identity element of the group is the homotopy class of the constant map. For $n \geq 2$, $\pi_n(\mathcal{M})$ is always Abelian.

The physical significance of $\pi_n(\mathcal{M})$ having a group structure is that two solutions may be combined, by performing the group operation on their homotopy classes, to a single solution. In our case of $S^2 \to S^2$ then

$$\pi_2(S^2) = \mathbb{Z} \tag{2.21}$$

2.4 Topological Charge

In the previous section we saw how two homeomorphic mappings are members of the same homotopy class. In this section we shall show that two mappings with the same topological charge or winding number are homeomorphic and thus their homotopy class may be labelled by the said topological charge. We follow ideas given in [9] and [8] references therein.

There are several different ways of approaching an analysis of topological charge. The problem is to connect the separate strands together. We shall omit completely some of the angles of approach, such as Chern class and Pontryagin index, as they are irrelevant in the context of this work. The first approach that we shall choose is a fairly topologically rigorous one. Initially we shall make some remarks about Kähler models.

If the target manifold $\mathcal{M}$ admits a Kähler metric then the model is Kähler. This is true for $\mathcal{M} = S^2$ as the metric $g(\phi,\bar{\phi}) = (1+\bar{\phi}\phi)^{-2}$ may be written in the form $g(\phi,\bar{\phi}) = \frac{\partial}{\partial\phi}\frac{\partial}{\partial\bar{\phi}}g'(\phi,\bar{\phi})$ where $g' = \ln(1+\bar{\phi}\phi)$ is the Kähler metric density. Now suppose that $\mathcal{M}$ is an n-dimensional complex manifold parametrised by co-ordinates u^α and their complex conjugates $\bar{u}^{\bar{\alpha}}$, with $\alpha = 1, \ldots\ldots, n$. Then the sigma model action may be written

$$S = \frac{1}{4}\int d^dx \; g_{\alpha\bar{\beta}}(u) \; \partial_\mu u^\alpha \; \partial^\mu \bar{u}^{\bar{\beta}} \tag{2.22}$$

where d is the dimension of spacetime. The model is said to be Kähler if the two-form

$$\omega \equiv g_{\alpha\bar{\beta}} \; du^\alpha \times d\bar{u}^{\bar{\beta}} \tag{2.23}$$

is closed. i.e. $d\omega = 0$. This condition greatly simplifies the equation of motion. It also provides a link with cohomology.

Let us now briefly define cohomology classes. If η is a p-form, and ζ a $(p-1)$-form, on a manifold $\mathcal{M}$ then η is closed if $d\eta = 0$ and exact if $\eta = d\zeta$. If η is exact then it automatically follows that it is closed as $d^2 = 0$. However closed forms are only exact if

$\mathcal{M}$ is contractible. Actually the closed p-forms may be classified, two p-forms being in the same class if and only if they differ by an exact form. These classes are called cohomology classes.

If the set of all closed p-forms is denoted by $Z^p(\mathcal{M})$ and the set of all exact p-forms by $B^p(\mathcal{M})$, then the set of cohomology classes is defined as

$$H^p(\mathcal{M}) = Z^p(\mathcal{M})/B^p(\mathcal{M}) \tag{2.24}$$

The elements of $H^p(\mathcal{M})$ obey the rules of group operation, so $H^p(\mathcal{M})$ is called the pth cohomology group of $\mathcal{M}$.

The next step is given by the Hurewicz Theorem. In a simple form it may be stated thus

> If $\mathcal{M}$ is an $(n-1)$-connected space with $n \geq 2$ then there is a one-to-one correspondence between the homotopy classes of the maps $f : S^n \to \mathcal{M}$ and the elements of the singular homology group $H_n(\mathcal{M})$.

The proof of this theorem is outside the scope of this work, however it may be found in [18]. An equivalent statement of the Hurewicz Theorem may be made for cohomology groups, which means that we may make the identification

$$\pi_p(\mathcal{M}) \equiv H^p(\mathcal{M}) \tag{2.25}$$

which provides the link between homotopy groups and cohomology groups.

Thus we have shown that as ω is a closed two-form then it is a member of $H^2(\mathcal{M})$ and thus $\pi_2(\mathcal{M})$, and we can label the classes in each of these groups by an integer. Another quick digression is now needed to introduce the notion of pullback mapping.

For a manifold $\mathcal{M}$ there is a tangent space at a point p denoted T_pM. Linear functions ω given by $\omega : T_pM \to \mathbb{R}$ form a cotangent space of $\mathcal{M}$ at p, denoted T_p^*M. Elements of

T_p^*M are known as one-forms. the simplest example being the differential df where f is a smooth function on $\mathcal{M}$.

The action of a vector v on f is

$$v[f] = v^\mu \frac{\partial f}{\partial x^\mu} \in \mathrm{IR} \tag{2.26}$$

Then the action of $df \in T_p^*M$ on $v \in T_pM$, $df : T_pM \rightarrow \mathrm{IR}$ is defined as the inner product

$$(df, v) \equiv v[f]\mathrm{IR} \tag{2.27}$$

The map f naturally induces a map between tangent spaces called the differential map f_*

$$f_* : T_pM \rightarrow T_{f(p)}N \tag{2.28}$$

the action of which is defined as

$$(f_*v)[g] = v[gf] \tag{2.29}$$

where g is a smooth function on N. Similarly f induces the reverse mapping known as the pullback

$$f^* : T^*_{f(p)}N \rightarrow T_p^*M \tag{2.30}$$

If $h \in T^*_{f(p)}N$ then the pullback of h by f^* is defined by

$$(f^*h, v) = (h, f^*v) \tag{2.31}$$

If $k \in \Omega^r(N)$ where $\Omega^r(N)$ is the space of smooth r-forms on N then

$$d(f^*k) = f^*(dk) \tag{2.32}$$

Thus f^* maps closed forms to closed forms and exact forms to exact forms. So we may define a pullback between cohomology groups

$$f^* : H^r(N) \rightarrow H^r(M) \tag{2.33}$$

by

$$f^*[\omega] = [f^*\omega] \tag{2.34}$$

where $[\omega] \in H^r(N)$.

The relevance of pullback becomes clear when we note that if two maps f_1 and f_2 are homotopic then their pullback maps of the cohomology groups are identical. Thus each homotopy class may be assigned a pullback mapping and the topological charge may be defined as

$$Q = c^{-1} \int_N f^*[\omega] \tag{2.35}$$

where c is some normalisation factor which makes Q an integer. It makes sense to identify Q with the integer that labels the homotopy group associated with the pullback in Q. In our case of course we can take ω in Q to be the Kähler two-form given above.

We have got to the stage where we can label homotopy classes by integers known as the topological charge. However it would be convenient to be able to write Q in terms of the fields of the model. To do this consider Q as the integral of the zero component of the topological current $J_a = (J_0, J_\mu)$.

$$Q = c^{-1} \int J_0 \; d^2x \tag{2.36}$$

In the case of the $O(3)$ model

$$J_0 = \epsilon_{\mu\nu}\epsilon_{\alpha\beta\gamma} \; \phi^\alpha \; \partial_\mu\phi^\beta \; \partial_\nu\phi^\gamma \tag{2.37}$$

where $\epsilon_{\mu\nu}$ and $\epsilon_{\alpha\beta\gamma}$ are the totally antisymmetric tensors in 2 and 3 dimensions respectively.

Let us now approach topological charge from a different angle and look at it in terms of winding number. Consider the integration over the surface of the space of the fields ϕ, S^2_{int}. The winding number is the number of times a map from this space covers the target space S^2_{phys}. So the the surface area of S^2_{int} may be identified with the product of the winding number and the surface area of S^2_{phys}. So

$$\int_{S^2_{int}} dS_\phi = \int_{S^2_{phys}} dS_x \, J(x,\phi) \tag{2.38}$$

where $J(x,\phi)$ is the Jacobian of the transformation between S^2_{int} and S^2_{phys}. To calculate this Jacobian consider a small area element swept out by the vector $\boldsymbol{\phi}$ on S^2_{int}. The area of this element is the vector product $\delta_1\boldsymbol{\phi} \wedge \delta_2\boldsymbol{\phi}$ where the differential $\delta_1 = \delta/\delta x_1$. Hence $J(x,\phi)$ is

$$J(x,\phi) = \boldsymbol{\phi} \cdot \delta_1\boldsymbol{\phi} \wedge \delta_2\boldsymbol{\phi} \tag{2.39}$$

The area of a unit sphere in the physical space is 4π. Thus the winding number is given by

$$Q = \frac{1}{4\pi} \int d^2x \, \boldsymbol{\phi} \cdot \delta_1\boldsymbol{\phi} \wedge \delta_2\boldsymbol{\phi} \tag{2.40}$$

The products may be written using the anti-symmetric tensor ϵ. This creates some double counting in the space-time co-ordinates which is compensated for. Thus

$$Q = \frac{1}{8\pi} \int d^2x \, \epsilon_{\mu\nu}\epsilon_{\alpha\beta\gamma} \, \phi^\alpha \, \partial_\mu\phi^\beta \, \partial_\nu\phi^\gamma \tag{2.41}$$

as above. Also we have found the value of the normalising constant c.

Topological charge is a topological invariant, i.e. it does not change under homeomorphisms. In this way it is a conserved quantity. Similarly with the topological current.

The topological charge labels the static solutions of the model and provides a lower bound to the action. We shall now show this for the $O(3)$ sigma model. Using the fields σ from (2.5) the topological charge for the sigma model may be written as

$$Q = \frac{1}{8\pi} \int d^2x \, \epsilon_{\mu\nu} \, \sigma[\partial_\mu\sigma, \partial_\nu\sigma] \tag{2.42}$$

Now consider the inequality

$$(\partial_\mu\sigma \pm \epsilon_{\mu\nu}[\sigma, \partial_\nu\sigma])^2 \geq 0 \tag{2.43}$$

It follows from this and (2.5) that

$$S \geq 4\pi|Q| \tag{2.44}$$

so the magnitude of Q is the lower bound of the action. The equality holds when

$$\partial_\mu \sigma = \pm\epsilon_{\mu\nu}[\sigma, \partial_\nu \sigma] \tag{2.45}$$

these are called the duality conditions. It can be shown that any solution of these conditions is also a solution of the equations of motion. The solutions to the duality equations are the instanton solutions and were first derived in the form which we shall use by Belavin and Polyakov [1]. The field σ takes values on S^2

$$\sigma = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \tag{2.46}$$

If we stereographically project from the sphere onto a plane the number of components of the field can be reduced to two. If w is the field on this plane then

$$w(z) = w_1 + iw_2 = \frac{\sigma_1 + i\sigma_2}{1 - \sigma_3} = \cot\frac{\theta}{2}e^{i\varphi} \tag{2.47}$$

$$w_1 = \cot\frac{\theta}{2}\cos\varphi \tag{2.48}$$

$$w_2 = \cot\frac{\theta}{2}\sin\varphi \tag{2.49}$$

where $z = x_1 + ix_2$ are the physical co-ordinates. The antisymmetry of the duality conditions means that they reduce to

$$\partial_1 w_1 = \partial_2 w_2 \ , \ \ \partial_1 w_2 = -\partial_2 w_1 \tag{2.50}$$

which are just the Cauchy-Riemann conditions. The general solution of these may be expressed as any analytical function of z. This solution must be a continuous function within each homotopy class. Poles occur at the boundaries of topological sectors. The topological charge labels each topological sector. Thus the solutions have the form

$$w = c\prod_{j=1}^{Q}\frac{(z - a_j)}{(z - b_j)} \tag{2.51}$$

which are the instanton solutions (1.1). The parameters a_j, b_j and c define the size of the instanton.

Under the stereographic projection the action takes the form

$$S = \frac{4}{k}\int d^2x\, \frac{\partial_z w \partial_{\bar{z}} \bar{w} + \partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{2.52}$$

and the topological charge

$$Q = \frac{1}{\pi}\int d^2x\, \frac{\partial_z w \partial_{\bar{z}} \bar{w} - \partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{2.53}$$

Thus we can write the action in terms of the static solutions

$$S = \frac{4\pi Q}{k} + \frac{8}{k}\int d^2x \frac{\partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{2.54}$$

This form of the action is minimised if $\partial_z \bar{w} = 0$, which are just the Cauchy-Riemann equations again.

Chapter 3

Instanton Contribution to the Green's Function

3.1 Fluctuation Operator

A vital component of the renormalisation calculation which is our main aim in this thesis, is the instanton contribution to the Green's Function for the $O(3)$ σ-model given in (1.2) and (1.3). This was first discovered by Fateev, Frolov and Schwarz [2], [19] and is re-examined in depth in [13]. Here we shall follow their calculation in detail, but we shall also demonstrate two new methods of calculating the determinant of the fluctuation operator.

As already noted, the $O(3)$ σ-model action may be written in terms of the topological charge q, fields $w(z,\bar{z})$ and coupling constant k as

$$S = \frac{4\pi q}{k} + \frac{8}{k}\int d^2x \frac{\partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{3.1}$$

so the action is at its minimal values $S = \frac{4\pi q}{k}$ when $\partial_{\bar{z}} w = 0$. This is satisfied by

$$w(z) = \frac{P_0(z)}{P_1(z)} \tag{3.2}$$

where $P_0(z)$ and $P_1(z)$ are polynomials. This is an instanton solution where the topological charge of the instanton is equal to the maximal degree of the polynomial. Thus it is possible to write the q-instanton solution as

$$v(z) = c\frac{\prod_{i=1}^{q}(z-a_i)}{\prod_{i=1}^{q}(z-b_i)} \tag{3.3}$$

To calculate the instanton contribution to the Green's Function we shall use the steepest descent method. Suppose that w differs from v by some quantum correction $\varphi(z,\bar{z})$, then $w = v + \varphi$. As φ is small we shall approximate the action so that we only have the terms that are of the lowest order in φ. So we expand out the denominator of the second term of the action, and as $\partial_{\bar{z}} v = 0$ then to this approximation

$$S = \frac{4\pi q}{k} + \frac{8}{k}\int d^2x \frac{\partial_{\bar{z}}\varphi \partial_z \bar{\varphi}}{\rho^2} \tag{3.4}$$

where $\rho = 1 + |v|^2$. However if we consider $\partial_z(\bar{\varphi}\rho^{-2}\partial_{\bar{z}}\varphi)$ then it is easy to see that

$$\int d^2x \frac{\partial_{\bar{z}}\varphi\partial_z\bar{\varphi}}{\rho^2} = \int d^2x\partial_z(\bar{\varphi}\rho^{-2}\partial_{\bar{z}}\varphi) - \int d^2x\bar{\varphi}\partial_z\rho^{-2}\partial_{\bar{z}}\varphi \tag{3.5}$$

but the first term will disappear at the boundary when it is integrated. If we define the inner product of two complex functions ψ and χ in this space as

$$(\psi, \chi) = \int d^2x \sqrt{g}\,\rho^{-2}\,\bar{\psi}\,\chi \tag{3.6}$$

where g is the determinant of the metric on the sphere, then the action may be written in a similar form

$$(\varphi, \Delta\varphi) = -\int d^2x\bar{\varphi}\partial_z\rho^{-2}\partial_{\bar{z}}\varphi = \int d^2x\sqrt{g}\,\rho^{-2}\,\bar{\varphi}\left(-\frac{\rho^2}{\sqrt{g}}\partial_z\rho^{-2}\partial_{\bar{z}}\right)\varphi \tag{3.7}$$

So

$$\Delta = -\frac{\rho^2}{\sqrt{g}}\partial_z\rho^{-2}\partial_{\bar{z}} \tag{3.8}$$

is the fluctuation operator. This is different to the fluctuation operator used in [2], but this is just because the variables in the second term of the action are defined differently in each case. It is easy to show that the fluctuation operators are equivalent. So

$$S_0 = \frac{4\pi q}{k} + \frac{8}{k}(\varphi, \Delta\varphi) \tag{3.9}$$

The subscript indicates that this is just the leading order approximation of the action. The metric on the sphere of radius R is

$$g_{\mu\nu}(x,y) = \delta_{\mu\nu}\left[1 + \left(\frac{x^2+y^2}{4R^2}\right)\right]^{-2} \tag{3.10}$$

to return to the Euclidean metric we take $R \to 0$. The variables may be redefined as $x = 2Rx'$, $y = 2Ry'$ so that the radius R can be factored out of the metric. If the instanton parameters are also rescaled then the Green's function is unchanged and the primes can be ignored. Thus

$$g_{\mu\nu}(z,\bar{z}) = \delta_{\mu\nu}(1+|z|^2)^{-2} \tag{3.11}$$

the determinant of which is

$$detg_{\mu\nu} = (1 + |z|^2)^{-4} \tag{3.12}$$

3.2 Instanton Manifold Measure

Let the instanton contribution to the Green's function be

$$I(\eta) = \frac{\sum_q J_q(\eta)}{\sum_q J_q(1)} \tag{3.13}$$

where

$$J_q(\eta) = \int \eta(v)\, e^{-S}\, d\mu_0\, d\varphi \tag{3.14}$$

$d\mu_0$ is the measure on the manifold of instantons induced by the metric on this manifold. It is the purpose of the rest of this chapter to calculate $I(\eta)$. Let us denote the instanton parameters a, b and c by $\{t^\alpha\} = (a, b, c)$ and $\{t^{\bar{\alpha}}\} = (\bar{a}, \bar{b}, \bar{c})$. Thus the metric on the manifold of instantons can be written as

$$\sum_{\alpha,\beta} \left(\int d^2x \sqrt{g}\, \rho^{-2} \frac{\partial \bar{v}}{\partial t^{\bar{\beta}}} \frac{\partial v}{\partial t^\alpha} \right) \delta t^\alpha \delta t^{\bar{\beta}} = \sum_{\alpha,\beta} \left(\frac{\partial \bar{v}}{\partial t^{\bar{\beta}}}, \frac{\partial v}{\partial t^\alpha} \right) \delta t^\alpha \delta t^{\bar{\beta}} = \sum_{\alpha,\beta} m_{\alpha\bar{\beta}} \delta t^\alpha \delta t^{\bar{\beta}} \tag{3.15}$$

where $m_{\alpha\bar{\beta}}$ can be interpreted as the metric tensor on the manifold of instantons. We note here that this can be written in the form

$$m_{\alpha\bar{\beta}} = \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^{\bar{\beta}}} K(z, \bar{z}) \ , \quad K(z, \bar{z}) = \int d^2x \sqrt{g}\, \ln \rho \tag{3.16}$$

thus $m_{\alpha\bar{\beta}}$ has the form of a Kähler metric tensor.

The measure on the manifold of instantons $d\mu_0$ is therefore

$$d\mu_0 = (q!)^{-2} \, \det m \; dt^\alpha \; dt^{\bar{\beta}} \tag{3.17}$$

The factor $(q!)^{-2}$ is to avoid double counting. To calculate $\det m$ note that $\frac{\partial v}{\partial t^\alpha} \prod_{j=1}^{q} (z - b_j)^2$ can be represented by $\sum_{k=0}^{2q} U_{\alpha k} z^k$. Thus if we define

$$N_{kj} = \int d^2x \sqrt{g}\, \rho^{-2} \prod_i |z - b_i|^{-4} \bar{z}^k z^j \tag{3.18}$$

then $m = U^\dagger N U$, the matrix U is just dependent on the moduli a, b and c. Now $\det m = \det N |\det U|^2$, and $\det U$ can be found to be

$$\det U = c^{2q} \prod_{k>j} (a_k - a_j)(b_k - b_j) \prod_{l,m} (a_l - b_m) \tag{3.19}$$

so

$$\det m = |c|^{4q} \prod_{k>j} |a_k - a_j||b_k - b_j| \prod_{l,m} |a_l - b_m|^2 \det N \tag{3.20}$$

3.3 Regularisation

The expression $J_q(\eta)$ needs to be regularised as

$$\begin{aligned} J_q(\eta) &= k^{-2q} \int \eta(v)\, e^{-\frac{4\pi q}{k} - \frac{8}{k}(\varphi, \Delta\varphi)}\, d\mu_0\, d\varphi && (3.21) \\ &= k^{-2q} e^{-\frac{4\pi q}{k}} \int \eta(v) \det{}^{-\frac{1}{2}}(\Delta)\, d\mu_0 && (3.22) \end{aligned}$$

so that

$$I_q(\eta) = \frac{J_q(\eta)}{J_0(1)} = k^{-2q} e^{-\frac{4\pi q}{k}} \frac{\int d\mu_0\, \eta(v) \det^{-\frac{1}{2}}(\Delta)}{\int d^2c\, \det^{-\frac{1}{2}}(\Delta_0)} \tag{3.23}$$

and the determinants are divergent as they have an infinite number of dimensions. The regularisation is done by means of a proper time cut-off. Let us define the determinant to be

$$\ln \det \Delta = \operatorname{Tr} \ln \lambda_i = -\lim_{\epsilon \to 0} \sum_i \left[\int_\epsilon^\infty \frac{dt}{t}\, e^{-\lambda_i t} + \ln \epsilon \right] \tag{3.24}$$

where the λ_i are the non-zero eigenvalues of Δ. Now let us define

$$\ln \det{}_\epsilon \Delta = -\sum_i \left[\int_\epsilon^\infty e^{-\lambda_i t}\, \frac{dt}{t} + \ln \epsilon \right] \tag{3.25}$$

Δ has p zero modes, where $p = 4q + 2$. The second term in (3.25) is cancelled by a similar term from the denominator of $I_q(\eta)$. However although Δ has p zero modes, Δ_0 has 2, hence there will be an over-cancellation of $(p-2)\ln\epsilon$. Also

$$\sum_i e^{-\lambda_i t} = \operatorname{Tr}\, e^{-t\Delta} - p \tag{3.26}$$

so

$$\ln \det_\epsilon \Delta - \ln \det_\epsilon \Delta_0 = -(p-2)\ln\epsilon - \int_\epsilon^\infty \frac{dt}{t}\,(\mathrm{Tr}\ e^{-t\Delta} - p) + \int_\epsilon^\infty \frac{dt}{t}\,(\mathrm{Tr}\ e^{-t\Delta_0} - 2) \quad (3.27)$$

For $t \to 0$ the asymptotic expansion of Tr $e^{-\Delta t}$ can be calculated to be for $R = 1$

$$\mathrm{Tr}\ e^{-t\Delta} = \frac{1}{t} + 2q + O(t) \quad (3.28)$$

Thus in the limit $\epsilon \to 0$

$$\ln \det_\epsilon \Delta = -\int_\epsilon^\infty \left(\frac{1}{t} + 2q\right)\frac{dt}{t} \quad (3.29)$$

$$= \left[\frac{1}{t} - 2q\ln t\right]_\epsilon^\infty \quad (3.30)$$

We may separate the divergent part from the rest. So if $\frac{1}{\mu} > \epsilon$ but $\frac{1}{\mu}$ is still small then

$$\ln \det_\epsilon \Delta = \left[\frac{1}{t} - 2q\ln t\right]_\epsilon^{\frac{1}{\mu}} + \left[\frac{1}{t} - 2q\ln t\right]_{\frac{1}{\mu}}^\infty \quad (3.31)$$

$$= \left[\mu - \frac{1}{\epsilon} + 2q\ln\epsilon\mu\right] + \left[\frac{1}{t} - 2q\ln t\right]_{\frac{1}{\mu}}^\infty \quad (3.32)$$

Thus the divergent terms of the determinant as $\epsilon \to 0$ are the ϵ dependent pieces. Now from (3.23)

$$I_q(\eta) = k^{-2q} e^{-\frac{4\pi q}{k}} \frac{\int d\mu_0\ \eta(v)\ \exp\left[-\frac{1}{2}\left(\mu - \frac{1}{\epsilon} + 2q\ln\epsilon\mu\right)\right]}{\int d^2c\ \exp\left[-\frac{1}{2}\left(\mu - \frac{1}{\epsilon}\right)\right]} \quad (3.33)$$

so the linearly divergent pieces cancel and the logarithmic divergence can be absorbed into a renormalised coupling constant

$$I_q(\eta) = \frac{k^{-2q}}{4\pi V}\int d\mu_0\ \eta(v)\ \exp\left[-4\pi q\left(\frac{1}{k} + \frac{1}{4\pi}\ln\epsilon\mu\right)\right] \quad (3.34)$$

$$= \frac{k^{-2q}}{4\pi V}\int d\mu_0\ \eta(v)\ \exp\left[-\frac{4\pi q}{k(\mu)}\right] \quad (3.35)$$

where $V = 4\pi R^2$ is the area of the sphere. The renormalised coupling constant $k(\mu)$ given by $\frac{1}{k(\mu)} = \frac{1}{k} + \frac{1}{4\pi}\ln\epsilon\mu$ can be used to find the β-function of the model. The β-function is defined by

$$\beta(k) = \mu\frac{\partial k(\mu)}{\partial\mu} \quad (3.36)$$

Thus

$$\mu\frac{\partial}{\partial\mu}\left(\frac{1}{k(\mu)}\right) = \frac{\mu}{4\pi}\frac{\partial}{\partial\mu}\log(\epsilon\mu) \tag{3.37}$$

$$-\mu\frac{1}{k(\mu)^2}\frac{\partial k(\mu)}{\partial\mu} = \frac{1}{4\pi} \tag{3.38}$$

so

$$\beta(k) = -\frac{k(\mu)^2}{4\pi} \tag{3.39}$$

and we have obtained the negative β-function of an asymptotically free theory. We shall use this later.

The renormalised determinant is thus given by subtracting the linearly and logarithmically divergent parts from the unrenormalised determinant. i.e. if the renormalised determinant is $\det'\Delta$ then

$$\ln\ \det'\Delta = \lim_{\epsilon\to 0}\left(\ln\det{}_\epsilon\Delta - \frac{1}{\epsilon} - 2q\ln\epsilon\right) \tag{3.40}$$

3.4 Calculation of $\det'\Delta$

The determinant may be calculated by means of methods developed in [1], [2] and [20] and also used in [19]. First the variation of the determinant with respect to the instanton parameters is found. This has the effect of removing the last two terms from (3.40), $-\frac{1}{\epsilon} - 2q\ln\epsilon$, as they are independent of those parameters. If the expression found this way is then integrated, the character of the dependence of the determinant on the instanton parameters may be found.

As we are using Tr Δ it is irrelevant how the terms in Δ are ordered as long as correct permutations are used. It is convenient to use the form of Δ given in [2]

$$\Delta f = -\frac{1}{\sqrt{g}}\,\rho\,\partial_z\left[\rho^{-2}\partial_{\bar{z}}\,(\rho f)\right] \tag{3.41}$$

From (3.27) and (3.40) it follows that

$$\delta \ln \det'\Delta = \int_0^\infty (\mathrm{Tr}\ \delta\Delta e^{-t\Delta})dt \tag{3.42}$$

where

$$\delta\Delta = 2\rho^{-1}\delta\rho\Delta - 2\rho^2\partial_z\rho^{-3}\delta\rho\partial_{\bar{z}} \tag{3.43}$$

Using the properties of the trace, we may reorder Δ and $\delta\Delta$. So (neglecting g for the moment)

$$\mathrm{Tr}\ \delta\Delta e^{-t\Delta} = 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\Delta e^{-t\Delta}) - 2\mathrm{Tr}\ (\rho^2\partial_z\rho^{-3}\delta\rho\ \partial_{\bar{z}}e^{-t\Delta}) \tag{3.44}$$

$$= 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\Delta e^{-t\Delta}) - 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\ \rho^{-1}\partial_{\bar{z}}\rho^2\partial_z\rho^{-1}e^{-t\Delta}) \tag{3.45}$$

Now if we write Δ in the form $\Delta = T^\dagger T$ where

$$T^\dagger = -\frac{1}{\sqrt{g}}\rho\ \partial_z\rho^{-1}\quad,\quad T = \rho^{-1}\partial_{\bar{z}}\rho \tag{3.46}$$

we may define the conjugate operator by $\tilde{\Delta} = TT^\dagger$, hence

$$\tilde{\Delta}f = -\rho^{-1}\partial_{\bar{z}}\left[\rho^2\frac{1}{\sqrt{g}}\partial_z\left(\rho^{-1}f\right)\right] \tag{3.47}$$

so

$$\mathrm{Tr}\ \delta\Delta e^{-t\Delta} = 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\Delta e^{-t\Delta}) - 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\ \tilde{\Delta}e^{-t\Delta}) \tag{3.48}$$

but $\Delta\rho^2\partial_z\rho^{-1} = \rho^2\partial_z\rho^{-1}\tilde{\Delta}$ so $\mathrm{Tr}\Delta = \mathrm{Tr}\tilde{\Delta}$ thus we may replace Δ by $\tilde{\Delta}$ above and

$$\mathrm{Tr}\ \delta\Delta e^{-t\Delta} = 2\mathrm{Tr}\ (\rho^{-1}\delta\rho\Delta e^{-t\Delta} - \rho^{-1}\delta\rho\ \tilde{\Delta}e^{-t\tilde{\Delta}}) \tag{3.49}$$

so

$$\delta \ln \det'\Delta = 2\int_0^\infty \mathrm{Tr}\ (\rho^{-1}\delta\rho\Delta e^{-t\Delta} - \rho^{-1}\delta\rho\ \tilde{\Delta}e^{-t\tilde{\Delta}})dt \tag{3.50}$$

$$= -2\int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\ \rho^{-1}\delta\rho\ (e^{-t\Delta} - e^{-t\tilde{\Delta}})dt \tag{3.51}$$

Therefore we have to study the behaviour of the integrand for large t and for small t. The asymptotics for $t \to \infty$ are governed by the zero modes of the operator Δ (the operator $\tilde{\Delta}$ does not have any zero modes). Let

$$\Delta\psi_a = 0 \qquad a = 0, ..., 4q+1 \tag{3.52}$$

Due to the way that (3.51) has been constructed, all that needs to be calculated for the case $t \to 0$ is $\langle z|e^{-t\Delta}|z\rangle$ and $\langle z|e^{-t\tilde{\Delta}}|z\rangle$. Various methods may be used to calculate these, however we shall concentrate on two very different angles of approach to this problem. The first involves the use of Δ as an operator in the Heat Equation ([21], [22], [13])

$$\Delta\mathcal{G}(x,y;t) = \frac{\partial\mathcal{G}(x,y;t)}{\partial t} \tag{3.53}$$

where the kernel $\mathcal{G}$ satisfies $\mathcal{G}(x,y;0) = \delta(x-y)$ and $\mathcal{G} \sim e^{-t\Delta}$ thus $\langle z|e^{-t\Delta}|z\rangle \sim \mathcal{G}(x,x;t)$. Here x and y are complex numbers This method may be used to calculate the zeta function for the operator Δ

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\ \mathrm{Tr}\ \mathcal{G}(t)dt \tag{3.54}$$

which raises the question of whether $\det\Delta$ could be found directly from

$$\left.\frac{d\zeta_\Delta(s)}{ds}\right|_{s=0} = -\ln\det\Delta \tag{3.55}$$

However this is not possible even in the small s limit as Δ is too complex a function.

The heat kernel may be represented by its asymptotic expansion for small t

$$\mathcal{G}(x,y;t) = \frac{1}{4\pi t}\exp\left(-\frac{(x-y)^2}{t}\right)\sum_{n=0}^{\infty} a_n(x,y)t^n \tag{3.56}$$

where $a_0(x,x) = 1$, so for small t we need only calculate the first two or three terms of this expansion.

The other method we may employ to calculate the expectation values is to use a semi-classical expansion. Here the fluctuation operator Δ may be interpreted as the Hamiltonian of a quantum mechanical system corresponding to a classical Hamiltonian $\mathcal{H}$, so the expectation value is written as a quantum mechanical functional integral

$$\langle z|e^{-t\Delta}|z\rangle = \int \mathcal{D}(x,y,p)\exp\left(-\int dt\ \mathcal{L}(x,y,p)\right) \tag{3.57}$$

where $\mathcal{L} = \dot{x}p_x + \dot{y}p_y - \mathcal{H}$

It is necessary in both of these methods for Δ and $\tilde{\Delta}$ to be written in terms of real variables. We shall set $\sqrt{g} = 1$ for the moment as it turns out that for the calculation involving Δ, the value of the metric has no effect on the final result. However this is not true for the calculation using $\tilde{\Delta}$ and so we must be ready to replace the metric when it is needed. Also it turns out that there are no important differences between the calculations for the two operators in either method, therefore we shall only describe the calculation for Δ in detail. So if $f(x,y)$ is an arbitrary function

$$\Delta f = \partial_z\partial_{\bar{z}}f - 2\rho^{-2}(\partial_z\rho\partial_{\bar{z}}\rho)f + \rho^{-1}(\partial_z\partial_{\bar{z}}\rho)f + \rho^{-1}\partial_{\bar{z}}\rho\partial_z f - \rho^{-1}\partial_z\rho\partial_{\bar{z}}f \tag{3.58}$$

Now

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)\ ,\ \ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)\ ,\ \ \partial_z\partial_{\bar{z}} = \frac{1}{4}({\partial_x}^2 + {\partial_y}^2) = \frac{1}{4}\nabla^2 \tag{3.59}$$

where ∇ is the Laplacian operator. Thus

$$\Delta f = \left(\frac{1}{4}\nabla^2 + \frac{i}{2}\left((\partial_y \ln\rho)\,\partial_x - (\partial_x \ln\rho)\,\partial_y\right) - \frac{1}{2}\left((\partial_x \ln\rho)^2 + (\partial_y \ln\rho)^2\right) + \frac{1}{4}\rho^{-1}\nabla^2\rho\right) f \tag{3.60}$$

or

$$\Delta = \frac{1}{4}\nabla^2 + \frac{i}{2}\left((\partial_y \ln\rho)\,\partial_x - (\partial_x \ln\rho)\,\partial_y\right) + R(x,y) \tag{3.61}$$

where

$$R(x,y) = \frac{1}{4}\rho^{-1}\nabla^2\rho - \frac{1}{2}\left((\partial_x \ln\rho)^2 + (\partial_y \ln\rho)^2\right) \tag{3.62}$$

It is convenient now to see that as

$$\nabla^2 \ln\rho = \rho^{-1}\nabla^2\rho - \left((\partial_x \ln\rho)^2 + (\partial_y \ln\rho)^2\right) \tag{3.63}$$

then

$$R(x,y) = \frac{1}{4}\nabla^2 \ln\rho - \frac{1}{4}\left((\partial_x \ln\rho)^2 + (\partial_y \ln\rho)^2\right) \tag{3.64}$$

3.4.1 Heat Equation Method

We are only interested in the first couple of terms in the expansion of $\mathcal{G}(x,x;t)$ (i.e. when $n=0$ and $n=1$). As already stated, by definition $a_0(x,x)=1$ so all we really need to find is $a_1(x,x)$. Using (3.53), (3.56) and (3.61) simultaneous equations may be constructed by equating powers of t and thus $a_1(x,x)$ found.

As x and y are complex numbers, then $(x-y)$ will be treated as a vector with components along the space-time axes. Thus if the space-time co-ordinates are p and q then

$$(x-y)\cdot\partial = (x-y)_p\partial_p + (x-y)_q\partial_q \tag{3.65}$$

So we need to calculate the components of

$$\left[\frac{1}{4}\nabla^2 + \frac{i}{2}\left((\partial_y \ln\rho)\partial_x - (\partial_x \ln\rho)\partial_y\right) + R(x,y)\right]\mathcal{G} = \frac{\partial\mathcal{G}}{\partial t} \tag{3.66}$$

These are

$$\begin{aligned}\frac{1}{4}\nabla^2\mathcal{G} \;=\; & \frac{1}{4\pi t}\exp\left(-\frac{(x-y)^2}{t}\right)\left[-\sum_{n=0}^{\infty} a_n t^{n-1} - (x-y)\cdot\sum_{n=0}^{\infty}\partial a_n t^{n-1}\right.\\ & \left. + (x-y)^2\sum_{n=0}^{\infty} a_n t^{n-2} + \frac{1}{4}\sum_{n=0}^{\infty}\partial^2 a_n t^n\right]\end{aligned} \tag{3.67}$$

$$\begin{aligned}&(\partial_2\ln\rho)\partial_1 - (\partial_1\ln\rho)\partial_2\\ &= \frac{1}{4\pi t}\exp\left(-\frac{(x-y)^2}{t}\right)\left[-2\left((\partial_2\ln\rho)(x-y)_1 - (\partial_1\log\rho)(x-y)_2\right)\sum_{n=0}^{\infty} a_n t^{n-1}\right.\\ &\qquad \left. + (\partial_2\ln\rho)\sum_{n=0}^{\infty}\partial_1 a_n t^n - (\partial_1\ln\rho)\sum_{n=0}^{\infty}\partial_2 a_n t^n\right]\end{aligned} \tag{3.68}$$

and

$$\frac{\partial\mathcal{G}}{\partial t} = \frac{1}{4\pi t}\exp\left(-\frac{(x-y)^2}{t}\right)\left[-\sum_{n=0}^{\infty} a_n t^{n-1} + (x-y)^2\sum_{n=0}^{\infty} a_n t^{n-2} + \sum_{n=0}^{\infty} n a_n t^{n-1}\right] \tag{3.69}$$

where the subscripts 1 and 2 are used to indicate direction in space-time. (3.66) now becomes

$$\begin{aligned}\sum_{n=0}^{\infty} n a_n t^{n-1} &= -(x-y)\cdot\sum_{n=0}^{\infty}\partial a_n t^{n-1} + \frac{1}{4}\sum_{n=0}^{\infty}\partial^2 a_n t^n \\ &\quad - i\left((\partial_2 \ln\rho)(x-y)_1 - (\partial_1 \ln\rho)(x-y)_2\right)\sum_{n=0}^{\infty} a_n t^{n-1} \\ &\quad + \frac{i}{2}\left((\partial_2 \ln\rho)\sum_{n=0}^{\infty}\partial_1 a_n t^n - (\partial_1 \ln\rho)\sum_{n=0}^{\infty}\partial_2 a_n t^n\right) + R(x,y)\sum_{n=0}^{\infty} a_n t^n \quad (3.70)\end{aligned}$$

As each summation is independent we are free to reclassify them as we wish. Thus in all terms where the first term in the summation is a t^0 term, we make a shift $n \to n-1$. Now all the terms have t^{-1} as their first term and so this may be factored out of the equation and the summations removed leaving

$$\begin{aligned} n a_n &= -(x-y)\cdot\partial a_n + \frac{1}{4}\partial^2 a_{n-1} - i\left((\partial_2 \ln\rho)(x-y)_1 - (\partial_1 \ln\rho)(x-y)_2\right) a_n \\ &\quad + \frac{i}{2}\left((\partial_2 \ln\rho)\partial_1 a_{n-1} - (\partial_1 \ln\rho)\partial_2 a_{n-1}\right) + R a_{n-1} \end{aligned} \quad (3.71)$$

a_{-1} is zero by definition, and so for the case $n = 0$

$$-(x-y)\cdot\partial a_0 - i\left((\partial_2 \ln\rho)(x-y)_1 - (\partial_1 \ln\rho)(x-y)_2\right) a_0 = 0 \quad (3.72)$$

or

$$-\left((x-y)_1\partial_1 a_0 + (x-y)_2\partial_2 a_0\right) = i\left((\partial_2 \ln\rho)(x-y)_1 - (\partial_1 \ln\rho)(x-y)_2\right) a_0 \quad (3.73)$$

equating coefficients gives

$$-(x-y)_1\partial_1 a_0 = i(\partial_2 \ln\rho)(x-y)_1 a_0 \quad (3.74)$$

$$(x-y)_2\partial_2 a_0 = i(\partial_1 \ln\rho)(x-y)_2 a_0 \quad (3.75)$$

which we must differentiate again, with respect to the space-time co-ordinates, for use later

$$-{\partial_1}^2 a_0 = i(\partial_1\partial_2 \ln\rho)a_0 + (\partial_2 \ln\rho)^2 a_0 \quad (3.76)$$

$${\partial_2}^2 a_0 = i(\partial_2\partial_1 \ln\rho)a_0 - (\partial_1 \ln\rho)^2 a_0 \quad (3.77)$$

Here the $(x-y)_1$ and $(x-y)_2$ have been cancelled but only after the differentiation. Next we look at the case of $n=1$ in (3.71). This gives the equations

$$-(x-y)\cdot\partial a_1+\frac{1}{4}\partial^2 a_0-i\left((\partial_2\ln\rho)(x-y)_1-(\partial_1\ln\rho)(x-y)_2\right)a_1$$
$$+\frac{i}{2}\left((\partial_2\ln\rho)\partial_1 a_0-(\partial_1\ln\rho)\partial_2 a_0\right)+R(x,y)a_0=a_1 \tag{3.78}$$

Terms containing differentials of a_0 may be substituted using (3.74), (3.75), (3.76) and (3.77). A lot of the terms disappear when the trace is taken leaving us with

$$a_1(x,x)=\frac{1}{4}\partial^2\ln\rho \tag{3.79}$$

Thus

$$\mathcal{G}(x,x;t)=\frac{1}{4\pi t}+\frac{1}{16\pi}\partial^2\ln\rho+\cdots \tag{3.80}$$

An equivalent calculation may be done for $\tilde{\Delta}$. If the heat kernel for $\tilde{\Delta}$ is $\tilde{\mathcal{G}}$ then

$$\tilde{\mathcal{G}}(x,x;t)=\frac{1}{4\pi t}-\frac{1}{16\pi}\partial^2\ln\rho+\cdots \tag{3.81}$$

which is in agreement with the results found in [2].

3.4.2 Quantum Mechanical Method

We stated above that it is possible for the expectation value we are trying to calculate to be written as a functional integral with a lagrangian expressed in terms of quantum mechanical operators. Also we have said that the fluctuation operator Δ may be identified as a quantum version of a classical hamiltonian $\mathcal{H}$, hence the lagrangian may be found in the usual way

$$\mathcal{L}=\dot{x}p_x+\dot{y}p_y-\mathcal{H} \tag{3.82}$$

where the p_i's are the momenta. The standard quantum mechanical expressions of the momentum operators $p_x=-i\partial_x$ and $p_y=-i\partial_y$ (where $\hbar=1$) may be used to write Δ in terms of these momenta

$$\Delta = -\frac{1}{4}({p_x}^2 + {p_y}^2) - \frac{1}{2}\left((\partial_y \ln\rho)p_x - (\partial_x \ln\rho)p_y\right) + R(x,y) \tag{3.83}$$

where $\nabla^2 = -({p_x}^2 + {p_y}^2)$. Thus

$$\mathcal{L} = \frac{1}{4}({p_x}^2 + {p_y}^2) + \left(\dot{x} + \frac{1}{2}(\partial_y \ln\rho)\right)p_x + \left(\dot{y} + \frac{1}{2}(\partial_x \ln\rho)\right)p_y - R(x,y) \tag{3.84}$$

Let us represent the expectation value we need to find as a partition function

$$Z = \int \mathcal{D}(x,y,p)\exp\left(-\int dt\ \mathcal{L}(x,y,p)\right) \tag{3.85}$$

The momenta may be integrated out by using a standard quadratic integral, e.g. for the integration of p_x

$$\int \mathcal{D}p_x\ \exp\left[-\int dt\left(\frac{1}{4}{p_x}^2 + \left(\dot{x} + \frac{1}{2}(\partial_y \ln\rho)\right)p_x - R(x,y)\right)\right] \tag{3.86}$$

we use

$$\int_{-\infty}^{\infty} dx\ \exp\left(-ax^2 + bx + c\right) = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{4a} + c\right) \tag{3.87}$$

However care is needed as the functional measure actually represents an infinite number of measures, one for each degree of freedom of the particle

$$\mathcal{D}p_x = \lim_{n\to\infty} \prod_{i=1}^{n} d^i p_x \tag{3.88}$$

the cumulative effect of the integrals gives a coefficient of $(4\pi)^{\frac{n}{2}}$, which is infinite as $n \to \infty$. So some form of regularisation must be used. A common method of regulating infinite dimensional spaces is the zeta function method [21],[22]. Briefly this states that for a matrix operator $\mathcal{M}$ with eigenvalues λ_n and eigenfunctions ψ_n then the ζ-function is given by

$$\zeta_{\mathcal{M}}(s) = \sum_n {\lambda_n}^{-s} \tag{3.89}$$

and the regularized determinant of the matrix is given by

$$\det \mathcal{M} = e^{-\zeta'_{\mathcal{M}}(0)} \tag{3.90}$$

where $\zeta'(s) = \frac{d}{ds}\zeta(s)$. Also we can find the regularized dimension of the space $\mathcal{U}$ in which $\mathcal{M}$ is acting by

$$\dim \mathcal{U} = \zeta_{\mathcal{M}}(0) \tag{3.91}$$

This means that we need to construct the zeta function for the fluctuation operator occuring in the semi-classical expansion (3.85). Looking at (3.94) below we can see that this operator is essentially $\frac{d^2}{dt^2}$. The zeta function calculation for $\frac{d^2}{dt^2}$ is relatively straightforward (see the Appendix at the end of the chapter) and so we may this result in the regularisation of (3.88). Denoting the regularised dimension of momentum space by D, the total coefficient, including the contribution from the p_y integral, is $(4\pi)^{2D}$. As $D = -\frac{1}{2}$ then, having done the p_y integral,

$$Z = \frac{1}{4\pi}\int \mathcal{D}(x,y)\ \exp\left[\int dt\ \left(-\left(\dot{x} + \frac{1}{2}(\partial_y \log\rho)\right)^2 - \left(\dot{y} + \frac{1}{2}(\partial_x \log\rho)\right)^2 + R(x,y)\right)\right] \tag{3.92}$$

So we are left with the integral over the position variables which we shall solve by means of the saddle-point method. This involves defining the classical path of the particle. As we are only looking at the case of $t \to 0$, then the time integral may cut off at a certain time T. Thus our boundary conditions are that the particle travels from (x_1, y_1) at $t = 0$ to (x_2, y_2) at $t = T$. Using $R(x,y)$ from above the functional integral may be written

$$Z = \frac{1}{4\pi}\int_{(x_1,y_1)}^{(x_2,y_2)} \mathcal{D}(x,y)\exp(-S) \tag{3.93}$$

where now the action is

$$S = \int_0^T dt\ \left(\dot{x}^2 + \dot{y}^2 + \dot{x}(\partial_y \log\rho) + \dot{y}(\partial_x \log\rho) - \frac{1}{4}\partial^2 \log\rho\right) \tag{3.94}$$

For simplicity we define

$$f(x,y) = \partial_y \log\rho\ ,\ \ h(x,y) = \partial_x \log\rho\ ,\ \ k(x,y) = \frac{1}{4}\partial^2 \log\rho \tag{3.95}$$

Considering only small t enables us to make a change of variables which will leave a clearly dominant term in the action. If we set $t = \tau T$ where τ goes from 0 to 1, then $\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$ and the limits of the integration change as $\tau = 0$ when $t = 0$ and $\tau = 1$ when $t = T$.So

$$S = \int_0^1 d\tau \, \left(\frac{1}{T}(x'^2 + y'^2) + x'f(x,y) + y'h(x,y) - Tk(x,y)\right) \tag{3.96}$$

where $x' = \frac{dx}{d\tau}$. The action is now dominated by the first term for small T. As the first term is quadratic in the "velocity" of the particle then the classical paths may be approximated by

$$x = x_1 + \tau(x_2 - x_1) + \bar{x}\sqrt{T} \tag{3.97}$$

$$y = y_1 + \tau(y_2 - y_1) + \bar{y}\sqrt{T} \tag{3.98}$$

where x and y are constructed to obey the above boundary conditions. The variables $\bar{x}$ and $\bar{y}$, which become the integration variables, are perturbations about the classical path and are continuously deformable to zero. For simplicity we will use

$$x = \chi(\tau) + \bar{x}\sqrt{T} \tag{3.99}$$

$$y = \phi(\tau) + \bar{y}\sqrt{T} \tag{3.100}$$

and

$$x' = x_c + \bar{x}'\sqrt{T} \tag{3.101}$$

$$y' = y_c + \bar{y}'\sqrt{T} \tag{3.102}$$

where $x_c = x_2 - x_1$ and $y_c = y_2 - y_1$. The Jacobian of the change of variables $x \to \bar{x}$ is found by looking at the change in the functional measure $\mathcal{D}(x,y)$. By defining x to be $x^n = \sum_i a_i^n u_i$ where the u_i are the eigenfunctions of the action, and so $dx^n = \sum_i da_i^n u_i$, the measure becomes

$$\mathcal{D}(x) = \prod_n dx^n = \prod_n da_i^n \sqrt{\det(u_i, u_j)} \tag{3.103}$$

where the inner product is defined by

$$(u_i, u_j) = \int_0^T u^2 dt \tag{3.104}$$

under the change of variables $u_i \to \bar{u}_i\sqrt{T}$ and $t = \tau T$ then $(u_i, u_j) = T^2 \int_0^1 \bar{u}^2 d\tau$ and thus

$$\mathcal{D}(x) = \prod_n da_i^n \sqrt{\det\left(T^2(\bar{u}_i, \bar{u}_j)\right)} = T^D \mathcal{D}(\bar{x}) \tag{3.105}$$

where D is defined as above. Repeating the process for y we get the new functional measure to be

$$T^{2D}\mathcal{D}(\bar{x}, \bar{y}) = T^{-1}\mathcal{D}(\bar{x}, \bar{y}) \tag{3.106}$$

In order to write the action as an expansion in powers of T each term may be expanded in a Taylor Series, e.g.:

$$\begin{aligned} &f\left(\chi(\tau) + \bar{x}\sqrt{T}, \phi(\tau) + \bar{y}\sqrt{T}\right)\left(x_c + \bar{x}'\sqrt{T}\right) \\ &= f x_c|_{x=\chi, y=\phi} + \sqrt{T}\left(x_c\bar{x}\partial_x f + x_c\bar{y}\partial_y f + \bar{x}' f\right)_{x=\chi, y=\phi} \\ &+ T\left(\frac{x_c}{2}(\bar{x}^2\partial_x^2 f + \bar{y}^2\partial_y^2 f + \bar{x}\bar{y}\partial_x\partial_y f) + \bar{x}\bar{x}'\partial_x f + \bar{y}\bar{x}'\partial_y f\right)_{x=\chi, y=\phi} + \cdots \end{aligned} \tag{3.107}$$

All terms of order higher than T will be neglected. Also we can neglect $\bar{x}$, $\bar{y}$ and $\bar{x}\bar{y}$ terms in the action as these are meaningless when we come to calculate the expectation values. Writing

$$F(x,y)|_{x=\chi, y=\phi} = x_c f(x,y)|_{x=\chi, y=\phi} + y_c h(x,y)|_{x=\chi, y=\phi} = F(\chi, \phi) \tag{3.108}$$

gives

$$S = \int_0^1 d\tau \left(\frac{1}{T}({x_c}^2 + {y_c}^2) + ((\bar{x}')^2 + (\bar{y}')^2) + F(\chi, \phi) + \sqrt{T}\mathcal{L}_1 + T\mathcal{L}_2\right) \tag{3.109}$$

where

$$\begin{aligned} \mathcal{L}_1 &= \left(\bar{x}\partial_x F + \bar{y}\partial_y F + \bar{x}' f + \bar{y}' h\right)_{x=\chi, y=\phi} && (3.110) \\ \mathcal{L}_2 &= \left(\frac{1}{2}(\bar{x}^2\partial_x^2 F + \bar{y}^2\partial_y^2 F) + \bar{x}\bar{x}'\dot{\partial}_x f + \bar{y}\bar{y}'\partial_y h - k\right)_{x=\chi, y=\phi} && (3.111) \end{aligned}$$

Thus the functional integral is now

$$Z = \frac{1}{4\pi T}\int \mathcal{D}(\bar{x},\bar{y})\exp\left[-\int_0^1 d\tau\left(\frac{1}{T}({x_c}^2+{y_c}^2)+(\bar{x}')^2+(\bar{y}')^2+F(\chi,\phi)-\sqrt{T}S_1-TS_2\right)\right] \tag{3.112}$$

The parts of the functional integral independent of $\bar{x}$ and $\bar{y}$ we will call

$$Z_c(\chi,\phi)=\exp\left[-\frac{1}{T}({x_c}^2+{y_c}^2)-\int_0^1 d\tau F(\chi,\phi)\right] \tag{3.113}$$

The terms of order $\sqrt{T}$ and T in the action may be expanded as an exponential

$$\exp\left[-(\sqrt{T}S_1+TS_2)\right]=1-\sqrt{T}S_1+T\left(\frac{1}{2}{S_1}^2-S_2\right) \tag{3.114}$$

where again we have only gone up to order T. As $\mathcal{L}_1$ is only linear in $\bar{x}$ and $\bar{y}$ then the $\sqrt{T}$ term may be neglected. Also we have defined ${S_1}^2$ as

$$\begin{aligned}{S_1}^2 &= S_1(\tau_1)S_1(\tau_2)=\int_0^1 d\tau_1 d\tau_2\mathcal{L}_1(\tau_1)\mathcal{L}_1(\tau_2)\\ &= \int_0^1 d\tau_1 d\tau_2\,(\bar{x}_1\bar{x}_2\partial_x F_1\partial_x F_2+\bar{x}_1\bar{x}_2'\partial_x F_1 f_2+\bar{y}_1\bar{y}_2'\partial_y F_1 h_2+\bar{y}_1\bar{y}_2\partial_y F_1\partial_y F_2\\ &\quad +\bar{x}_1'\bar{x}_2\partial_x F_2 f_1+\bar{x}_1'\bar{x}_2' f_2 f_1+\bar{y}_1'\bar{y}_2\partial_y F_2 h_1+\bar{y}_1'\bar{y}_2' h_2 h_1)\end{aligned} \tag{3.115}$$

where, for convenience, we have written $\bar{x}_1' = \bar{x}(\tau_1)$ and have only included terms which will be non-zero on contraction. Also we make the change

$$\left(\frac{d\bar{x}}{d\tau}\right)^2=\frac{d}{d\tau}\left(\bar{x}\frac{d\bar{x}}{d\tau}\right)-\bar{x}\frac{d^2}{d\tau^2}\bar{x} \tag{3.116}$$

where the first term disappears as a boundary term on integration. Therefore the dominant terms in the functional integral are

$$Z=\frac{Z_c}{4\pi T}\int\mathcal{D}(\bar{x},\bar{y})\exp\left(\int_0^1 d\tau\,(\bar{x}\bar{x}''+\bar{y}\bar{y}'')\right)\left[1+T\left(\frac{1}{2}{S_1}^2-S_2\right)\right]=Z_1+Z_2 \tag{3.117}$$

The first term is simply solved by using the standard integral

$$\int dx\exp\left(-\frac{1}{2}(x,Ax)\right)=\sqrt{2}\,(\det A)^{-\frac{1}{2}} \tag{3.118}$$

but again there is the problem of an infinite coefficient $\lim_{n\to\infty} 2^n$ as there are an infinite number of integrals and two variables. However, using the same regularisation method as before we find

$$Z_1 = \frac{Z_c}{4\sqrt{2}\pi T}\left[\det\left(-2\frac{d^2}{d\tau^2}\right)\right]^{-1} \tag{3.119}$$

Note that now the operator is $\frac{d^2}{d\tau^2}$ as stated earlier. The value of the regularized determinant may also be found using the ζ-function method. The detailed zeta function calculation is given in the Appendix at the end of this chapter. We find that

$$\det\left(-2\frac{d^2}{d\tau^2}\right) = \frac{1}{\sqrt{2}} \tag{3.120}$$

Remember that we need to find only the diagonal elements of the expectation value. This is equivalent to requiring that $x_c = y_c = 0$ making $Z_c = 1$. Thus

$$Z_1 = \frac{1}{4\pi T} \tag{3.121}$$

This is the same as in the Heat Equation method.

Now to calculate Z_2 we have

$$\begin{aligned} Z_2 &= \frac{Z_c}{4\pi}\int \mathcal{D}(\bar{x},\bar{y})\exp\left(\int_0^1 d\tau\,(\bar{x}\bar{x}''+\bar{y}\bar{y}'')\right)\times \\ &\quad \left[\frac{1}{2}\int_0^1 d\tau_1 d\tau_2\,(\bar{x}_1\bar{x}_2\partial_x F_1\partial_x F_2 + \bar{x}_1\bar{x}_2'\partial_x F_1 f_2 + \bar{y}_1\bar{y}_2'\partial_y F_1 h_2 + \bar{y}_1\bar{y}_2\partial_y F_1\partial_y F_2\right. \\ &\qquad +\bar{x}_1'\bar{x}_2\partial_x F_2 f_1 + \bar{x}_1'\bar{x}_2' f_2 f_1 + \bar{y}_1'\bar{y}_2\partial_y F_2 h_1 + \bar{y}_1'\bar{y}_2' h_2 h_1) \\ &\qquad \left. -\int_0^1 d\tau\left(\frac{1}{2}(\bar{x}^2\partial_x^2 F + \bar{y}^2\partial_y^2 F) + \bar{x}\bar{x}'\partial_x f + \bar{y}\bar{y}'\partial_y h - k\right)\right] \end{aligned} \tag{3.122}$$

This can be solved term by term. Note that the first term in (3.122) may be written as (neglecting constants for the moment)

$$\int_0^1 d\tau_1 d\tau_2\;\partial_x F_1\partial_x F_2\;\int \mathcal{D}(\bar{x},\bar{y})\exp\left(\int_0^1 d\tau\,(\bar{x}\bar{x}''+\bar{y}\bar{y}'')\right)\bar{x}_1\bar{x}_2 \tag{3.123}$$

and that the Green's Function $G(\tau_1,\tau_2)$ of $\frac{d^2}{d\tau^2}$ is

$$G(\tau_1,\tau_2) = <\bar{x}_1\bar{x}_2> \;=\; \frac{\int \mathcal{D}(\bar{x},\bar{y})\exp\left(\int_0^1 d\tau\,(\bar{x}\bar{x}''+\bar{y}\bar{y}'')\right)\bar{x}_1\bar{x}_2}{\int \mathcal{D}(\bar{x},\bar{y})\exp\left(\int_0^1 d\tau\,(\bar{x}\bar{x}''+\bar{y}\bar{y}'')\right)}$$

$$= \int \mathcal{D}(\bar{x},\bar{y}) \exp\left(\int_0^1 d\tau \, (\bar{x}\bar{x}'' + \bar{y}\bar{y}'')\right) \bar{x}_1\bar{x}_2 \left[\det\left(-2\frac{d^2}{d\tau^2}\right)\right] \tag{3.124}$$

To calculate $G(\tau_1, \tau_2)$ we solve

$$\frac{d^2}{d\tau^2} G(\tau_1, \tau_2) = \delta(\tau_2 - \tau_1) \tag{3.125}$$

It is easy to see that the solution must have the general form

$$G(\tau_1, \tau_2) = |\tau_1 - \tau_2| + A\tau_1\tau_2 + B(\tau_2 + \tau_1) + C \tag{3.126}$$

where A, B and C are constants. The boundary conditions on $G(\tau_1, \tau_2)$ are the same as those for $\bar{x}$ and $\bar{y}$, so $G(\tau_1, \tau_2) = 0$ when $\tau_1 = 0$, $\tau_1 = 1$ and $\tau_2 = 1$. C can be chosen to be zero and we find

$$G(\tau_1, \tau_2) = \frac{1}{2}\left[|\tau_1 - \tau_2| + 2\tau_1\tau_2 - (\tau_2 + \tau_1)\right] \tag{3.127}$$

Comparing (3.123) and (3.124) gives the first term in Z_2 to be

$$\int_0^1 d\tau_1 d\tau_2 \, \partial_x F_1 \partial_x F_2 \left[\det\left(-2\frac{d^2}{d\tau^2}\right)\right]^{-1} G(\tau_1, \tau_2) \tag{3.128}$$

We can perform similar calculations for all the other terms in (3.122). However if we again look at only diagonal elements then $F = 0$ and f, h and k become independent of τ. This leaves us with

$$\begin{aligned} Z_2 \;=\; & \frac{1}{4\sqrt{2\pi}} \left[\det\left(-2\frac{d^2}{d\tau^2}\right)\right]^{-1} \left[\frac{1}{2}\int_0^1 d\tau_1 d\tau_2 \, (f_2 f_1 + h_2 h_1) \frac{\partial}{\partial \tau_1}\frac{\partial}{\partial \tau_2} G(\tau_1, \tau_2) \right. \\ & \left. - \int_0^1 d\tau \, (\partial_x f + \partial_y h) \frac{\partial}{\partial \tau} G(\tau, \tau) + k \right] \end{aligned} \tag{3.129}$$

The first and second terms turn out to be zero. Using results given above we are left with (using (3.95))

$$Z_2 = \frac{k}{4\pi} = \frac{1}{16\pi}\partial^2 \log \rho \tag{3.130}$$

Thus we can see that, for the first couple of terms in the expansion, Z agrees with the result obtained by solving the heat equation directly.

3.5 Assembling the Instanton Contribution

Now we are at a stage were we can return to (3.51) and find a solution for the determinant of the fluctuation operator. Using this and (3.20) and (3.17) we can find the instanton contribution by means of (3.23).

In the previous sections we found that the log det of the fluctuation operator, given by (3.51), depends upon the evaluation of the expectation values $\langle z|e^{-t\Delta}|z\rangle$ and $\langle z|e^{-t\tilde{\Delta}}|z\rangle$ for large and small t. For $t \to 0$ it is found, by two methods, that

$$\langle z|e^{-t\Delta}|z\rangle = \frac{\sqrt{g}}{4\pi t} + \frac{1}{16\pi}\partial^2 \log \rho + \cdots \tag{3.131}$$

Using the same methods, the contribution from the other expectation value in this case is found to be

$$\langle z|e^{-t\tilde{\Delta}}|z\rangle = \frac{\sqrt{g}}{4\pi t} + \frac{1}{8\pi}\partial^2 \left(\log \rho - \frac{1}{4\pi}\log g\right) + \cdots \tag{3.132}$$

Omitted terms tend to zero as $t \to 0$ in both cases. Now, with these results, and the representation of the zero modes of the fluctuation operator given in (3.52), (3.51) becomes

$$\delta \ln \, \det'\Delta = -2\int_0^\infty d^2x \; \delta \ln \rho \left[\frac{3}{16\pi}\partial^2 \log \rho - \frac{1}{32\pi^2}\partial^2 \log g\right] - 2\int_0^\infty d^2x \; \delta \ln \rho \; \mathcal{P}(x) \tag{3.133}$$

where $\mathcal{P}(x) = \sum_i \bar{\psi}_i(x)\psi_i(x)$ is the zero mode projection operator assuming $\langle \psi_i \psi_j \rangle = \delta_{ij}$. The first term may be put into a form in which it may be calculated. To do this note that

$$\partial_\mu \left(\ln \rho \; \partial_\mu(\delta \ln \rho)\right) = \partial_\mu \ln \rho \; \partial_\mu(\delta \ln \rho) + \ln \rho \; \partial_\mu \partial_\mu(\delta \ln \rho) \tag{3.134}$$

$$\partial_\mu \left(\delta \ln \rho \; \partial_\mu \ln \rho\right) = \partial_\mu(\delta \ln \rho)\partial_\mu \ln \rho + \delta \ln \rho \; \partial_\mu \partial_\mu \ln \rho \tag{3.135}$$

$$\delta \left(\ln \rho \; \partial_\mu \partial_\mu \ln \rho\right) = \delta \ln \rho \; \partial_\mu \partial_\mu \ln \rho + \ln \rho \; \partial_\mu \partial_\mu(\delta \ln \rho) \tag{3.136}$$

so that

$$\delta \ln \rho \; \partial_\mu \partial_\mu \ln \rho = \frac{1}{2}\left[\delta \left(\ln \rho \; \partial_\mu \partial_\mu \ln \rho\right) + \partial_\mu \left(\delta \ln \rho \; \partial_\mu \ln \rho\right) - \partial_\mu \left(\ln \rho \; \partial_\mu(\delta \ln \rho)\right)\right] \tag{3.137}$$

By using Stokes' Theorem the second and third terms of (3.137) may be turned into surface integrals. Therefore (3.133) may be split into four integrals

$$\delta \ln \det{}'\Delta = A_1 + A_2 + A_3 + A_4 \tag{3.138}$$

where

$$A_1 = \frac{3}{16\pi}\delta\int_0^\infty d^2x \ln\rho \, \partial_\mu\partial_\mu \ln\rho \tag{3.139}$$

$$A_2 = \frac{3}{16\pi}\oint d\sigma_\mu \left(\delta\ln\rho \, \partial_\mu \ln\rho - \ln\rho \, \partial_\mu(\delta \ln\rho)\right) \tag{3.140}$$

$$A_3 = \frac{1}{16\pi^2}\int_0^\infty d^2x \, \delta\ln\rho\partial^2 \log g \tag{3.141}$$

$$A_4 = -2\int_0^\infty d^2x \, \delta\ln\rho \, \mathcal{P}(x) \tag{3.142}$$

Now, for the remainder of these calculations we shall redefine ρ to be $\rho = (1+|v|^2)\prod_k |z-b_k|^2$ as used in [2]. This makes no difference to the final contributions to $\delta \ln \det{}'\Delta$. Note that now $\rho_0 = (1+|v|^2)$. Consequently, using these redefinitions, we are able to split A_1

$$\begin{aligned} A_1 &= \frac{3}{16\pi}\delta\int_0^\infty d^2x \ln\rho_0 \, \partial_\mu\partial_\mu \ln\rho + \frac{3}{16\pi}\delta\sum_k\int_0^\infty d^2x \, \ln|z-b_k|^2 \, \partial_\mu\partial_\mu\ln\rho \\ &= \frac{3i}{32\pi}\delta\int dz \, d\bar{z} \, \ln\rho_0 \, \frac{|\partial_z v|^2}{(1+|v|^2)^2} + \frac{3}{16\pi}\delta\sum_k\int_0^\infty d^2x \, \left(\partial^2 \ln|z-b_k|^2\right)\ln\rho \\ &+ \frac{3}{16\pi}\delta\sum_k\oint d\sigma_\mu \left((\partial_\mu \ln\rho)\ln|z-b_k|^2 - \ln\rho \, \partial_\mu \ln|z-b_k|^2\right) \end{aligned} \tag{3.143}$$

The first term in A_1 vanishes as it is just the variation of the topological number as

$$\int dz \, d\bar{z} \, \ln(1+|v|^2) \, \frac{|\partial_z v|^2}{(1+|v|^2)^2} = q\int d^2v \, \frac{\ln(1+|v|^2)}{(1+|v|^2)^2} \tag{3.144}$$

The factor of q arises because in general the z plane has q inverse images in the v plane due to the nature of the instanton solution we are using. By putting v into polar co-ordinates it is easily seen that the integral on the right-hand side of this equation is a constant.

The second integral in (3.143) is found by taking into account the fact that $\partial^2 \ln(x-a)^2 = 4\pi\delta(x-a)$. Hence

$$\frac{3}{16\pi}\delta\sum_k\int_0^\infty d^2x \, \left(\partial^2\ln|z-b_k|^2\right)\ln\rho = \frac{3}{4}\delta\sum_k\int_0^\infty d^2x \, \delta(z-b_k)\ln(1+|v|^2)$$

$$= \frac{3}{4}\delta\sum_k \ln\prod_l \left(|c|^2|b_k - a_l|^2\right) \tag{3.145}$$

The third integral in (3.143) may be calculated over an infinitely large circle. However, when the contribution from A_2 is calculated it is found to be exactly that of the third integral in A_1 up to a minus sign. Hence these cancel against each other so that the only remaining contribution to $\delta \ln \det' \Delta$ from A_1 is (3.145) and there is no overall contribution from A_2.

As $g_{\mu\nu} = \delta_{\mu\nu}\left(1 + (x^2 + y^2)/4R^2\right)^{-2}$ then A_3 may be calculated for the limit $R \to 0$. This gives a flat space and so

$$A_3 = \frac{1}{16\pi}\delta \ln(1 + |c|^2) \tag{3.146}$$

Consequently

$$\begin{aligned} A_1 + A_2 + A_3 &= 4\,\delta \ln\prod_{kl}\left(|c|^2|b_k - a_l|^2\right) + 4\,\delta\ln(1 + |c|^2) \\ &= 4\,\delta\ln\prod_{kl}\left(|b_k - a_l|^2\right) + 4\,\delta\ln\left(|c|^{2q}(1 + |c|^2)\right) \end{aligned} \tag{3.147}$$

Now to calculate A_4. Let us choose the standard basis of the zero modes of Δ to be of the form: $\chi_k = \rho^{-1}z^{\frac{k}{2}}$ if k is even, and $\chi_k = \rho^{-1}z^{\frac{k-1}{2}}$ if k is odd. This system of zero modes is not orthonormal, therefore A_4 needs to contain some form of normalisation. With the inner product defined as $(\alpha, \beta) = \int d^2x\, \sqrt{g}\, \rho^{-2}\bar{\alpha}\beta$ then

$$A_4 = -2(M_{kj})^{-1}\int d^2x\,\sqrt{g}\,\rho^{-2}\delta\ln\rho\;\bar{\chi}_j\chi_k = -2(M_{kj})^{-1}\int d^2x\,\sqrt{g}\,\rho^{-3}\delta\rho\;\bar{\chi}_j\chi_k \tag{3.148}$$

where

$$M_{kj} = \int d^2x\,\sqrt{g}\,\rho^{-2}\bar{\chi}_j\chi_k \tag{3.149}$$

but

$$\delta M_{kj} = -2\int d^2x\,\sqrt{g}\,\rho^{-3}\delta\rho\;\bar{\chi}_j\chi_k \tag{3.150}$$

Thus

$$A_4 = \mathrm{Tr}\ M^{-1}\delta M = \delta \mathrm{Tr}\ln M = \delta\ln\det M \tag{3.151}$$

However, if we refer back to (3.18) then it is easy to see that $\det M = (\det N)^2$ and so we finally obtain

$$\begin{aligned}\delta \ln \det{}'\Delta &= 4\,\delta \ln \prod_{kl}\left(|b_k - a_l|^2\right) + 4\,\delta \ln\left(|c|^{2q}(1+|c|^2)\right) + 2\delta \ln \det N \\ &= 2\,\delta \ln\left(|c|^{4q}(1+|c|^2)^2 \prod_{kl}\left(|b_k - a_l|^4\right) \det N\right) \qquad (3.152)\end{aligned}$$

So

$$(\det{}'\Delta)^{\frac{1}{2}} = |c|^{4q}(1+|c|^2)^2 \prod_{kl}\left(|b_k - a_l|^4\right) \det N \qquad (3.153)$$

and, from (3.22), (3.17) and (3.20), the instanton contribution to the Green's Function becomes

$$J_q(\eta) = k(\mu)^{-2q} e^{-\frac{4\pi q}{k(\mu)}} (q!)^{-2} \int \eta(v) \prod_{k>j} |a_k - a_j||b_k - b_j| \prod_{l,m} |a_l - b_m|^{-2} \frac{d^2c}{(1+|c|^2)^2} \prod_i d^2a_i\, d^2b_i \qquad (3.154)$$

(In [23] it was noted that an identical result is obtained for the anisotropic version of the model). This may be re-expressed as an exponential. The function $\eta(v)$ now only depends on the instanton moduli a, b and c, hence we make the change $\eta(v) = \Lambda(a,b,c)$. The factors dependent on the coupling constant may be contained in a single factor K^q [24] which will represent the coupling. Now $J_q(\Lambda)$ is the same as $I(\Lambda)$ up to some normalisation condition, so the instanton contribution to the Green's function for the $O(3)$ sigma model is

$$I(\Lambda) = \langle\Lambda\rangle = \sum_q \int \Lambda(a,b,c) \frac{K^q}{(q!)^2} e^{-h_q(a,b)} \frac{d^2c}{\pi(1+|c|^2)^2} \prod_j d^2a_j\, d^2b_j \qquad (3.155)$$

where

$$h_q(a,b) = -\sum_{i<j}^{q} \ln|a_i - a_j|^2 - \sum_{i<j}^{q} \ln|b_i - b_j|^2 + \sum_{i,j}^{q} \ln|a_i - b_j|^2 \qquad (3.156)$$

which are (1.2) and (1.3). In the one instanton sector ($q = 1$) this becomes simply

$$\begin{aligned}\langle\Lambda\rangle_1 &= \int \Lambda(a,b,c)\, K^1 \frac{1}{|a-b|^2} \frac{d^2c}{\pi(1+|c|^2)^2}\, d^2a\, d^2b \\ &= \int \Lambda(a,b,c)\, \zeta_1(a,b,c)\, d^2c\, d^2a\, d^2b \qquad (3.157)\end{aligned}$$

where $K^1 = 2^6 e^{\gamma - 2 - 4\pi/k(\mu)}$, $\gamma = 0.5772$ is the Euler number. The integrand $\zeta_1(a,b,c)$ is divergent as $a \to b$, i.e. as we approach the zero instanton sector from the one instanton sector.

Also when $q = 0$

$$\langle \Lambda \rangle_0 = \int d^2c \, \frac{\Lambda(c)}{\pi(1+|c|^2)^2} \tag{3.158}$$

3.6 Appendix: Zeta-Function Regularisation

The role of the zeta function in the evaluation of the properties of operators is described in detail in [21],[22]. Here we shall just give a basic outline of the ideas that we use, and the zeta-function calculation for the operator $\partial^2/\partial\tau^2$. More information about the properties of the zeta function may be found in [25], [26] and other works.

Thus, given a matrix operator $\mathcal{M}$ of arbitrary size with eigenvalues λ_n and eigenfunctions ψ_n then the ζ-function is given by

$$\zeta_{\mathcal{M}}(s) = \sum_n {\lambda_n}^{-s} \tag{3.159}$$

which we shall write in the form

$$\zeta_{\mathcal{M}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1}\, \mathcal{K}(\mu)\, d\mu \tag{3.160}$$

where $\mathcal{K}(\mu)$ is a kernel which is to be determined, and the Gamma Function takes its standard form

$$\Gamma(s) = \int_0^\infty t^{s-1}\, e^{-t}\, dt \tag{3.161}$$

To differentiate (3.159), each term in the summation must be differentiated separately and then the differentials summed. Thus if $z_n = {\lambda_n}^{-s}$ so that $\zeta_{\mathcal{M}}(s) = \sum_n z_n$ then

$$\left.\frac{dz_n}{ds}\right|_{s=0} = -\ln\lambda_n \tag{3.162}$$

hence

$$\left.\frac{d\zeta_{\mathcal{M}}(s)}{ds}\right|_{s=0} = -\sum_n \ln\lambda_n \tag{3.163}$$

However, as

$$\ln\det\mathcal{M} = \ln\prod_n \lambda_n = \sum_n \ln\lambda_n \tag{3.164}$$

Then the value of the determinant of $\mathcal{M}$ is given by

$$\det\mathcal{M} = \exp\left(-\zeta'_{\mathcal{M}}(0)\right) \tag{3.165}$$

where $\zeta'(s) = \frac{d}{ds}\zeta(s)$. Also we can find the regularized dimension of the space $\mathcal{U}$ in which $\mathcal{M}$ is acting by

$$\dim \mathcal{U} = \zeta_{\mathcal{M}}(0) \tag{3.166}$$

In the calculation outlined above we have

$$\mathcal{M} = -2\frac{\partial^2}{\partial \tau^2} \tag{3.167}$$

so

$$-2\partial_\tau^2 \psi_n(\tau) = \lambda_n \psi_n(\tau) \tag{3.168}$$

where $\partial_\tau^2 = \frac{\partial^2}{\partial \tau^2}$. The boundary conditions on ψ_n are the same as those on $G(\tau, \tau)$, i.e. that $\psi_n(\tau) = 0$ at $\tau = 1$ and 0. We are thus free to choose ψ_n as long as it is consistent with these conditions, so we shall simply choose $\psi_n(\tau) = A\sin(2\pi n\tau)$, where A is just some arbitrary constant. So as $\partial_\tau^2 \psi_n(\tau) = -(2\pi n)^2 \psi_n(\tau)$ then $\lambda_n = 2(2\pi n)^2 = 8\pi^2 n^2$. So from (3.159)

$$\begin{aligned}
\zeta_{\mathcal{M}}(s) &= \sum_n (8\pi^2 n^2)^{-s} && (3.169)\\
&= (8\pi^2)^{-s} \sum_n n^{-p} && (3.170)
\end{aligned}$$

where $p = 2s$, so our problem is now to find the new ζ-function

$$\zeta(p) = \sum_n n^{-p} \tag{3.171}$$

which is actually the original form of the ζ-function defined by Riemann. Comparing this with the integral form of the ζ-function (3.160)

$$\sum_n n^{-p} = \frac{1}{\Gamma(p)} \int_0^\infty \mu^{p-1} \sum_n K_n(\mu)\, d\mu \tag{3.172}$$

The kernel has been written as a summation simply to facilitate the calculation. This condition is satisfied if $K_n(\mu) = \exp(-n\mu)$. To see this, for a particular n make a change of variables $t = n\mu$ where t is the integration variable of $\Gamma(p)$. Now simply by summing it as a geometric series we find

$$\sum_{n=0}^{\infty} e^{-n\mu} = \frac{1}{e^{\mu} - 1} \tag{3.173}$$

so

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^{\infty} d\mu \frac{\mu^{p-1}}{e^{\mu} - 1} \tag{3.174}$$

We can integrate this without going into the complex plane if we let

$$\beta(\mu) = \frac{\mu}{e^{\mu} - 1} \tag{3.175}$$

so

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^{\infty} d\mu \ \mu^{p-2} \beta(\mu) \tag{3.176}$$

The behaviour of $\beta(\mu)$ at the integration limits is that $\beta(\mu) \rightarrow$a constant as $\mu \rightarrow 0$ and $\beta(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$, also $\beta'(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$, so we may integrate by parts. Note however that the integration (3.176) is only valid for $p > 2$, we shall analytically continue to the region where $\zeta(p)$ is valid for $p \geq 0$. Integrating by parts twice gives

$$\zeta(p) = \frac{1}{\Gamma(p)} \frac{1}{p(p-1)} \int_0^{\infty} d\mu \ \mu^p \beta''(\mu) \tag{3.177}$$

where the primes indicate differentiation with respect to μ. The dominant terms in (3.177) are those where p is small. In this case we may make the standard approximation $\mathcal{G}(p) \rightarrow \frac{1}{p}$, so

$$\zeta(p) = \frac{1}{(p-1)} \int_0^{\infty} d\mu \ \mu^p \beta''(\mu) \tag{3.178}$$

Thus

$$\zeta(0) = -\int_0^{\infty} d\mu \ \beta''(\mu) = -[\beta'(\mu)]_0^{\infty} \tag{3.179}$$

and as $\beta'(\mu) \rightarrow -\frac{1}{2}$ as $\mu \rightarrow 0$, and $\beta'(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$, then

$$\zeta(0) = \dim \mathcal{U} = -\frac{1}{2} \tag{3.180}$$

However, to calculate $\zeta'(0)$ we must integrate (3.174) in the complex plane. In section 2.4 of [25] it is shown that a zeta-function of the form of (3.174) may be written in a functional form

$$\zeta(p) = 2^p \pi^{p-1} \sin\left(\frac{p\pi}{2}\right) \Gamma(1-p)\, \zeta(1-p) \tag{3.181}$$

and thus that

$$\left.\frac{\zeta'(p)}{\zeta(p)}\right|_{p=0} = \log\ 2\pi \tag{3.182}$$

For details of this calculation the reader is refered to [25]. Now, from above we know that

$$\zeta_{\mathcal{M}}(s) = (8\pi^2)^{-s}\, \zeta(p) \tag{3.183}$$

and as $p = 2s$ then $\frac{d}{ds} = 2\frac{d}{dp}$ and differentiating gives

$$\zeta'_{\mathcal{M}}(s) = -(8\pi^2)^{-s} \log(8\pi^2)\zeta(p) + 2(8\pi^2)^{-s}\zeta'(p) \tag{3.184}$$

so if we divide by $\zeta_{\mathcal{M}}(s)$ we get

$$\begin{aligned} \left.\frac{\zeta'_{\mathcal{M}}(s)}{\zeta_{\mathcal{M}}(s)}\right|_{s=0} &= -\log(8\pi^2) + 2\left.\frac{\zeta'(p)}{\zeta(p)}\right|_{p=0} \\ &= \log\left(\frac{1}{2}\right) \end{aligned} \tag{3.185}$$

thus

$$\zeta'_{\mathcal{M}}(0) = -\frac{1}{2}\log\left(\frac{1}{2}\right) \tag{3.186}$$

and

$$\det \mathcal{M} = \exp\left(-\log\sqrt{2}\right) = \frac{1}{\sqrt{2}} \tag{3.187}$$

Chapter 4

General Formulation of Topological Renormalisation

4.1 Introduction

In the previous section we saw that the instanton contribution to the Green's function for the $O(3)$ sigma model is divergent at the boundaries of the instantons. Whether the divergence occurs or not is due purely to the nature of the instanton moduli. Hence, any mechanism we may devise to regulate these divergences will involve applying some sort of cut-off in the space of the instanton moduli. To do this we need to write the Greens Function as an integral over the instanton moduli to all orders in the semi-classical expansion of the functional integral. In this chapter a mechanism for doing this shall be proposed. However the act of cutting off the integral over the moduli space brings its own problems.

Separating the moduli space integral from the rest of the field variables may be done by imposing some form of constraint on the fields and the moduli. A convenient method of imposing this constraint and of generating the integral over the moduli is to use the Faddeev-Popov trick [27].

There are distinct advantages in using the Faddeev-Popov trick to introduce the integral over the moduli into the Green's Function. Firstly, in theories where we have to worry about gauge groups, such as Yang-Mills Theory, the Faddeev-Popov procedure is the accepted method of dealing with the problem of the extra multiplicative infinity arising from the integration over the gauge group measure. It does this by constraining the fields on to a gauge orbit using a delta function, the gauge group integration comes in as we have to integrate over all gauge orbits. To compensate for this, extra terms are added to the action written in terms of Grassmanian variables known as "ghost" fields and the constraints. There is nothing stopping us including the integral over the moduli in this procedure and then adding to the number of ghost fields to compensate. New fields introduced in this way will, however, not really be ghost fields, we shall thus call them "quasi-ghosts". In theories where gauge groups are not a concern, such as the $O(3)$ sigma model, the Faddeev-Popov trick as such is not needed to factor out a divergence, however the pattern of the procedure may still be used to factor out the integral over the instanton moduli, resulting in the inclusion of quasi-ghosts and constraints in the action corresponding to the separation of the field into a quantum fluctuation and a moduli dependent background.

Another advantage of the Faddeev-Popov approach is that it makes the dependence on the instanton moduli explicit without making any assumptions about the size of the fluctuations about the instanton solutions, and so it is efficient to all orders of the loop expansion. However a disadvantage of this procedure is that the constraints used remain in the action. It is important that the final theory must not have any dependence on these constraints.

First let us look in general terms at how the cut-off in moduli space my be imposed by using Stokes' Theorem [4].

Suppose that we have an (ill-defined) divergent integral over a domain M' parametrised by n variables t

$$\int_{M'} d^n t \; f(t) \tag{4.1}$$

the simplest way to regulate this is to introduce a cut-off which restricts the range of the variables. Say that only one of the variables, t_1, is restricted. Then the integration is now being taken over a domain M with boundary ∂M on which t_1 takes its cut-off values. So $\int_M d^n t \; f(t)$ is now well defined but it depends strongly on the cut-off.

However, if we were to make a reparametrisation of the variables $t^A \rightarrow \tilde{t}^A = t^A + \epsilon^A(t)$ then the cut-off changes and thus so does the value of the integral. The change in the integral is

$$\int_M d^n t \; \frac{\partial \epsilon^A(t)}{\partial t^A} f(t) + \int_M d^n t \; \epsilon^A(t) \frac{\partial f(t)}{\partial t^A} = \int_M d^n t \; \frac{\partial}{\partial t^A} \left(\epsilon^A(t) f(t) \right) \tag{4.2}$$

may be expressed using Stokes' Theorem

$$\int_M d^n t \, \frac{\partial}{\partial t^A} \left(\epsilon^A(t) f(t) \right) = \int_{\partial M} d\Sigma \, \epsilon^A(t) f(t) \tag{4.3}$$

So that now the integral is taken over the boundary of M. Now, if $\int d^n t \ f(t)$ is an amplitude, it is unacceptable for it to depend on the choice of parametrisation. Thus we wish (4.3) to be zero, if it is not then some way of cancelling it must be found.

4.2 Formalism for Gauge Field Theories

We shall now show how these ideas may be set out in terms of a field theory with instantons [4]. Such a theory may be expressed as a sum of partition functions, each partition function being integrated over a separate homotopy class $\mathcal{C}_q$ of the field configurations ϕ. The partition function Z may then be written

$$Z = \sum_q \kappa^q Z_q \ , \quad Z_q = \int_{\mathcal{C}_q} \mathcal{D}\phi \ e^{-S[\phi]} \tag{4.4}$$

where κ^q is some function of the topological coupling constant. The Greens Functions are thus also sums over the topological sectors

$$\mathcal{G}_q = \int_{\mathcal{C}_q} \mathcal{D}\phi \ e^{-S[\phi]} \ \Lambda(\phi) \tag{4.5}$$

$$\mathcal{G} = \frac{1}{Z} \sum_q \kappa^q \mathcal{G}_q \tag{4.6}$$

We will assume that the action has a gauge invariance, i.e. under the transformation $\phi \to \phi^g$ we assume that $S[\phi] \to S[\phi^g] = S[\phi]$. These symmetries will form a closed algebra. If g is close to the identity such that $g = 1 + \omega$, where ω has components ω^a, then the variation of ϕ with respect to g is $\delta_\omega \phi = \omega^a \delta_a \phi$ which forms an algebra $[\delta_a, \delta_b]\phi = f^c_{ab} \delta_c \phi$.

In each topological sector $\mathcal{C}_q$ there is a general family of solutions ϕ_0 to the classical equations of motion given by $\delta_\phi S = 0$. Let these solutions be parametrised by moduli

$\{t^A\}$ so that

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_0} = 0 \quad \text{for all t}^{\mathrm{A}} \tag{4.7}$$

Differentiating with respect to t^A shows that $\frac{\partial \phi_0}{\partial t^A}$ is a zero mode of the fluctuation operator

$$\left.\frac{\delta^2 S}{\delta \phi \delta \phi}\right|_{\phi=\phi_0} \tag{4.8}$$

which is simply the second non-vanishing term from an expansion of the action in powers of ϕ. Similarly the gauge variations of ϕ, $\delta_\omega \phi$, will be zero-modes of the fluctuation operator.

We need to consider what form of constraint may be imposed on the fields to separate out the moduli space integrals and to gauge fix the gauge symmetry. In general terms we could consider the condition that an infinite number of arbitrary functions $F_j(\phi, t)$ could be chosen such that

$$F_j(\phi^g, t) = 0 \tag{4.9}$$

The g are group elements parametrising the gauge transformations of the fields which leave the action unchanged.

More specifically, let $\phi = \phi_0(t) + \xi$ where ξ is some quantum fluctuation about ϕ_0 which is continuously deformable to zero. As a constraint we could then impose the condition that ξ is orthogonal to the zero modes. For instance in Yang-Mills Theory we would consider $\xi_{YM} = A_\mu - \mathcal{A}_\mu(t)$ where $\mathcal{A}_\mu$ is the instanton solution to the classical equations of motion and A_μ denotes the gauge potential. Thus the condition that ξ_{YM} be orthogonal to the zero modes may be expressed, on $\mathbb{R}^4$, as

$$\int d^4x \ \mathrm{tr}\left(\frac{\partial \mathcal{A}_\mu(t)}{\partial t^a}\, \xi_{YM}\right) = 0 \tag{4.10}$$

for the zero-modes which came from differentiating (4.7) with respect to the moduli, and for those which came from taking the variation of ϕ

$$[\partial_\mu + \mathcal{A}_\mu(t), \xi_{YM}] = 0 \tag{4.11}$$

The second constraint is just the background gauge condition. These constraints could be imposed by using the Faddeev-Popov construction [27], an integral over the moduli space is then naturally introduced. Using the general form of the constraints, $F_j(\phi^g, t) = 0$, we define a functional $\Delta[\phi, t]$ such that

$$\int \mathcal{D}g \; dt \; \Delta[\phi, t] \prod_j \delta\left[F_j(\phi^g, t)\right] = 1 \tag{4.12}$$

where $\mathcal{D}g$ is the Haar measure on the group of gauge transformations. Suppose that for $\phi = \hat{\phi}$ the constraints have a solution $g = \hat{g}$, $t = \hat{t}$, then $F_j(\hat{\phi}^{g}, \hat{t}) = 0$. Expanding about this solution, where $g = (1+\omega)\hat{g}$ and $t = \hat{t} + \tilde{t}$, gives

$$\begin{aligned} F_j(\phi^g, t) &= F_j(\hat{\phi}^{g}, \hat{t}) + \tilde{t}^A \frac{\partial}{\partial t^A} F_j(\phi^g, t)\Big|_{g=\hat{g}, t=\hat{t}} + \delta_\omega \phi \, \frac{\delta F_j(\phi^g, t)}{\delta \phi}\Big|_{g=\hat{g}, t=\hat{t}} + \cdots \\ &= \tilde{t}^A \frac{\partial}{\partial t^A} F_j(\phi^g, t)\Big|_{g=\hat{g}, t=\hat{t}} + \delta_\omega F_j(\phi^g, t)\Big|_{g=\hat{g}, t=\hat{t}} + \cdots \\ &\sim \left(\delta_\omega + \tilde{t}^A \frac{\partial}{\partial t^A}\right) F_j(\phi, t)\Big|_{\phi=\hat{\phi}^{g}, t=\hat{t}} \end{aligned} \tag{4.13}$$

The Haar measure is invariant under multiplication by a group element g, thus $\Delta[\phi^g, t] = \Delta[\phi, t]$ which allows Δ to be factored out of the integral and

$$\Delta^{-1} = \int \mathcal{D}g \; dt \prod_j \delta\left[F_j(\phi^g, t)\right] = \int \mathcal{D}\omega \; d\tilde{t} \prod_j \delta\left[\left(\delta_\omega + \tilde{t}^A \frac{\partial}{\partial t^A}\right) F_j(\phi, t)\Big|_{\phi=\hat{\phi}^{g}, t=\hat{t}}\right] \tag{4.14}$$

To calculate the first integral $\int \mathcal{D}\omega \prod_j \delta\left(\delta_\omega F_j(\phi, t)\right)$ remember that

$$\delta_\omega F_j(\phi, t) = \omega^a \delta_a F_j(\phi, t) \tag{4.15}$$

Now let $f^i(\phi, t)$ be the eigenfunctions of $\delta_a F_j$ and λ^i be the eigenvalues, so $(\delta_a F_j) f^i = \lambda^i f^i$. If we expand ω^a in terms of the eigenfunctions, so $\omega^a = \omega^i f^{ia}$, then $\omega^a \delta_a F_j = \omega^i f^{ia} \lambda^i$, so now we have $\int \mathcal{D}\omega \prod_j \delta\left(\omega^i f^{ia} \lambda^i\right)$. To make the integration easier we can write that delta function and the measure in terms of a single variable. Thus let $u^a = \omega^i f^{ia} \lambda^i$ and so

$$\mathcal{D}\omega = \left|\frac{\partial \omega^i}{\partial u^a}\right| \mathcal{D}u \tag{4.16}$$

To calculate the Jacobian note that $\frac{\partial u^a}{\partial \omega^i} = \lambda^i f^{ia}$ thus

$$\left|\frac{\partial u^a}{\partial \omega^i}\right| = \prod_i \lambda^i f^{ia} = (\det(\delta_a F_j))\,|f^{ia}| \tag{4.17}$$

and so

$$\int \mathcal{D}\omega \prod_j \delta(\delta_\omega F_j(\phi,t)) = \int \mathcal{D}u\,(\det(\delta_a F_j))^{-1}\,\delta(u) = (\det(\delta_a F_j))^{-1} \tag{4.18}$$

The second integral in (4.14) may be calculated in a similar manner so that

$$\int d\tilde{t} \prod_j \delta\left(\tilde{t}^A \frac{\partial}{\partial t^A} F_j(\phi,t)\right) = (\det(\partial_A F_j))^{-1} \tag{4.19}$$

Consequently

$$\Delta = \det(\delta_a F_j(\phi,t), \partial_A F_j(\phi,t))\big|_{\phi=\hat{\phi}^{\hat{g}},t=\hat{t}} \tag{4.20}$$

It is more convenient for this determinant to be written as part of the action. This may be done by using Grassmann variables in the form of ghost fields. We shall use the following notation. There is a ghost c^a for each transformation parameter ω^a, an anti-ghost b^j for each constraint F_j and a quasi-ghost τ^A for each modulus t^A. Now with (4.12) included in the partition function for the qth sector

$$Z_q = \int \mathcal{D}g\; dt \int_{\mathcal{C}_q} \mathcal{D}\phi\; e^{-S[\phi]}\;\Delta[\phi,t] \prod_j \delta\left[F_j(\phi^g,t)\right] \tag{4.21}$$

However we know that under the transformation $\phi \to \phi^g$, $S[\phi]$ and Δ are invariant. If we also assume that $\mathcal{D}\phi = \mathcal{D}\phi^g$ then the Haar measure may be factored out and the partition function may be redefined to be

$$Z_q = \int dt\; z(t) \tag{4.22}$$

$$z(t) = \int_{\mathcal{C}_q} \mathcal{D}\phi\; e^{-S[\phi]}\;\Delta[\phi,t] \prod_j \delta\left[F_j(\phi^g,t)\right] \tag{4.23}$$

In terms of the ghost fields

$$\Delta = \int \mathcal{D}(b,c)\; d\tau\;\exp\left[-\left(c^a\delta_a + \tau^A\partial_A\right)\left(b^j F_j\right)\right] \tag{4.24}$$

and as an integral the delta function is

$$\delta[F_j] = \int \prod_j d\lambda^j \; e^{i\lambda^j F_j} \tag{4.25}$$

Thus

$$\begin{aligned} z(t) &= \int_{\mathcal{C}_q} \mathcal{D}\left(\phi, c^a, b^j, \lambda^j\right) \; d\tau \; e^{-S_{tot}} & (4.26)\\ S_{tot} &= S[\phi] + \left(c^a \delta_a + \tau^A \partial_A\right)\left(b^j F_j\right) - i\lambda^j F_j & (4.27)\end{aligned}$$

This can be written more economically by using a BRST transformation [28]. The transformation is parametrised by a Grassmann number η such that $\delta_\eta \phi = \eta \varsigma \phi$ and ς operates on the fields

$$\varsigma\phi = c^a \delta_a \phi \;\;,\;\; \varsigma c^a = \frac{1}{2} c^b c^c f^a_{bc} \;\;,\;\; \varsigma b^j = i\lambda^j \;\;,\;\; \varsigma\lambda^j = 0 \;\;,\;\; \varsigma\tau^A = 0 \tag{4.28}$$

η does not act on the moduli t and ς is nilpotent, $\varsigma^2 = 0$, by construction. Using these we can write

$$S_{tot} = S[\phi] + \left(\varsigma + \tau^A \partial_A\right)\left(b^j F_j\right) \tag{4.29}$$

Now $\{\varsigma, \tau^A \partial_A\} = 0$ and $(\tau^A \partial_A)^2 = 0$ as the τ^A anti-commute but the ∂_A commute. So $\left(\varsigma + \tau^A \partial_A\right)^2 = 0$, and as $S[\phi]$ is gauge invariant and independent of the moduli then

$$\left(\varsigma + \tau^A \partial_A\right) S_{tot} = \left(\varsigma + \tau^A \partial_A\right) S[\phi] + \left(\varsigma + \tau^A \partial_A\right)^2 \left(b^j F_j\right) = 0 \tag{4.30}$$

Which means that S_{tot}, the gauge fixed action, is not BRST invariant but $\varsigma S_{tot} = -\tau^A \partial_A S_{tot}$

It can be seen from (4.22) and (4.23) that we have succeeded in our task of separating the integral over the moduli from the rest of the functional integrals. At the same time the necessary gauge fixing has been performed at the expense of introducing the constraints. The moduli space integral may now be cut off using the process described in the introduction to this chapter: restricting the integration over $\mathcal{M}$ to its boundary $\partial\mathcal{M}$ by means of Stokes' Theorem. However we need to know whether this procedure is dependent on the choice of constraint. If it is then this is unacceptable as the arbitrariness of the choice of constraint is destroyed, and some method must be found to restore it.

Thus to discern whether the Partition Function is dependent on the constraints, we make an arbitrary variation of the constraints such that $F_j \to F_j + \delta F_j$ then

$$\delta S_{tot} = \left(\varsigma + \tau^A \partial_A\right)\left(b^j\, \delta F_j\right) \tag{4.31}$$

and so

$$\delta e^{-S_{tot}} = -\left(\varsigma + \tau^A \partial_A\right)\left(b^j\, \delta F_j\right) e^{-S_{tot}} = -\left(\varsigma + \tau^A \partial_A\right)\left(b^j\, \delta F_j e^{-S_{tot}}\right) \tag{4.32}$$

giving

$$\delta z(t) = -\int_{\mathcal{C}_q} \mathcal{D}\Psi \;\left(\varsigma + \tau^A \partial_A\right)\left(b^j\, \delta F_j e^{-S_{tot}}\right) \tag{4.33}$$

where Ψ denotes the variables ϕ, c^a, b^j, λ^j and τ.

The expression (4.33) may be simplified. Consider

$$\int \mathcal{D}\Psi\; b^j\, \delta F_j e^{-S_{tot}} \tag{4.34}$$

This is zero as it is Grassmann odd. If there is a variation of the integration variables with respect to the BRST parameter, $\Psi \to \Psi + \delta_\eta \Psi = \Psi + \eta\varsigma\Psi$ then the change in the integral (4.34) is

$$\int \mathcal{D}\Psi\; \eta\varsigma\left(b^j\, \delta F_j e^{-S_{tot}}\right) \tag{4.35}$$

which is also zero as the value of the integral does not change under a change of the integration variables. This is true for all η, thus the first term in (4.33) is zero and

$$\delta z(t) = -\partial_A \int_{\mathcal{C}_q} \mathcal{D}\Psi\; \tau^A b^j\, \delta F_j e^{-S_{tot}} \tag{4.36}$$

This is the change in the moduli space density which has been generated by the constraints we have introduced. If this is non-zero then the Partition Function has gained a dependence on the constraints, which is unacceptable and some method must be found to deal with the problem. One solution is to simply cancel this anomaly with a "counter-term". A separate "counter-term" would have to be used for each topological sector and for each order in the coupling constant.

Now we are in a position to use our cut-off procedure. So applying (4.3) to (4.36) gives

$$\delta \int_M dt\ z(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi\ \tau^A(\xi^j \delta F_j) e^{-S_{tot}} \tag{4.37}$$

4.3 Formalism for Sigma Models

The difference between the Yang-Mills formulation of the problem and that of the sigma model is simply that for the sigma model we do not have to take the gauge transformations into consideration. This means that the Haar measure and any variations with respect to the gauge group never appear in the sigma model case. However, despite the fact that there is no need of gauge fixing, the Faddeev-Popov trick, or at least a simpler version of it, is still the most convenient way of separating the integrals over the moduli from the integrals over the fields.

So in this case let us consider fields w such that in the Green's Functions

$$\mathcal{G}_q = \int_{\mathcal{C}_q} \mathcal{D}w\ e^{-S[w]}\ \Lambda(w) \tag{4.38}$$

$$\mathcal{G} = \frac{1}{Z} \sum_q \kappa^q \mathcal{G}_q \tag{4.39}$$

$S[w]$ is the sigma model action. In Section 2 it was found that in each topological sector there is in general a family of solutions, v, to the classical equations of motion which are parametrized by moduli $\{t^A\}$. Thus the fields may be split into classical and quantum pieces such that $w = v + \phi$, where ϕ is a quantum fluctuation continuously deformable to zero. Our set of arbitrary constraints may thus be defined as $F_j(w, t) = 0$ which are introduced by multiplying the Green's Function by

$$\int dt\ \Delta[w,t] \prod_j \delta\left(F_j(w,t)\right) = 1 \tag{4.40}$$

Suppose that the constraints have a solution $w = \hat{w}$ and $t = \hat{t}$, then expanding the constraint about $t = \hat{t} + \tilde{t}$

$$F_j(w,t) = F_j(\hat{w},\hat{t}) + \tilde{t}^A \partial_A F_j(w,t)\big|_{w=\hat{w},t=\hat{t}} + \cdots \tag{4.41}$$

where $\partial_A = \frac{\partial}{\partial t^A}$. So

$$\int d\tilde{t}\ \Delta[w,t] \prod_j \delta\left(\tilde{t}^A \partial_A F_j(w,t)\big|_{w=\hat{w},t=\hat{t}}\right) = 1 \tag{4.42}$$

and we may factor out the operator Δ. Integrating out the delta function leaves the Jacobian, thus

$$\Delta = \det\left(\partial_A F_j(w,t)\big|_{w=\hat{w},t=\hat{t}}\right) \tag{4.43}$$

Representing this determinant using Grassmann numbers we have a quasi-anti-ghost ξ^j for each constraint F_j, and a quasi-ghost τ^A for each parameter t^A. Then

$$\Delta = \int \mathcal{D}(\xi) \exp\left[-\tau^A \partial_A \left(\xi^j F_j(w,t)\right)\right] \tag{4.44}$$

Also, if we write

$$\prod_j \delta\left(F_j(w,t)\right) = \int d\lambda\ e^{-i\lambda^j F_j} \tag{4.45}$$

then we get

$$\mathcal{G}_q = \int dt\ \mathcal{D}W\ e^{-S_{tot}}\ \Lambda(w) \tag{4.46}$$

where

$$S_{tot} = S[w] + \tau^A \partial_A \left(\xi^B F_B(w,t)\right) + i\lambda^j F_j \tag{4.47}$$

and W denotes w, ξ, λ. The equation equivalent to this one in the Gauge Field Formalism was simplified by means of a BRST transformation. Here we can go through a similar procedure as we are free to define a transformation operator as we wish. Hence we shall define a BRST-type transformation which is parametrised by a Grassmann number η, and acts on w, ξ, λ but not the moduli, then it may be written as $\delta_\eta w = \eta \varsigma w$. The transformation operator ς is thus defined by

$$\varsigma\xi^B = i\lambda^B \ , \ \varsigma\lambda = 0 \ , \ \varsigma\tau = 0 \ , \ \varsigma w = 0 \ , \ \varsigma^2 = 0 \tag{4.48}$$

So we have

$$S_{tot} = S[w] + (\varsigma + \tau^A \partial_A)(\xi^j F_j(w, t)) \tag{4.49}$$

Now $\{\varsigma, \tau^A \partial_A\} = 0$ because ∂_A acts only on the moduli whereas ς does not. Also $\left(\tau^A \partial_A\right)^2 = 0$ as derivatives with respect to the moduli commute with each other. Thus $\varsigma + \tau^A \partial_A$ is also nilpotent. Given that $S[w]$ is independent of the moduli we thus have

$$(\varsigma + \tau^A \partial_A) S_{tot} = 0 \tag{4.50}$$

So the action is not invariant under the action of the operator ς but $\varsigma S_{tot} = -\tau^A \partial_A S_{tot}$.

Again we have now achieved our purpose of being able to explicitly write the Green's Function as an integral over the moduli

$$\mathcal{G}_n = \int dt \ g(t) \ \ , \ \ g(t) = \int \mathcal{D}W \ e^{-S_{tot}} \ \Lambda(w) \tag{4.51}$$

If these integrals diverge then they may be regulated by a cut-off procedure. However we need to find out if the choice of constraint has any effect on this. If it does then this is obviously unsatisfactory and we must do something to compensate. Making an arbitrary variation of the moduli density with respect to the constraint gives

$$\begin{aligned} \delta g(t) &= -\int \mathcal{D}W \ e^{-S_{tot}} (\varsigma + \tau^A \partial_A)(\xi^j \delta F_j) \ \Lambda(w) \\ &= -\int \mathcal{D}W \ (\varsigma + \tau^A \partial_A) \ e^{-S_{tot}} \ (\xi^j \delta F_j) \ \Lambda(w) \end{aligned} \tag{4.52}$$

because of (4.50). We shall now show that the first term vanishes. Note that

$$\int \mathcal{D}W \ e^{-S_{tot}} \ (\xi^j \delta F_j) \ \Lambda(w) \tag{4.53}$$

is Grassmann odd, and under a perturbation about the field $w \to w + \eta\varsigma w$ the integration measure $\mathcal{D}W$ doesn't change but

$$\int \mathcal{D}W\ e^{-S_{tot}}\ (\xi^j \delta F_j)\ \Lambda(w) \to \int \mathcal{D}W\ \left(e^{-S_{tot}}\ (\xi^j \delta F_j) + \eta\varsigma e^{-S_{tot}}\ (\xi^j \delta F_j)\right)\ \Lambda(w) \tag{4.54}$$

However the value of the integral does not change under a change of the integration variables, hence

$$\int \mathcal{D}W\ \eta\varsigma e^{-S_{tot}}\ (\xi^j \delta F_j)\ \Lambda(w) = 0 \tag{4.55}$$

Thus

$$\delta g(t) = -\partial_A \int \mathcal{D}W\ e^{-S_{tot}} \tau^A (\xi^j \delta F_j)\ \Lambda(w) \tag{4.56}$$

If the integration region of the parameters t is M which has boundary ∂M then we may calculate the variation in the Partition Function by using Stokes' Theorem in the form

$$\int_M d^n t\ \frac{\partial}{\partial t^A} \left(f^A(t)\right) = \int_{\partial M} d\Sigma_A \left(f^A(t)\right) \tag{4.57}$$

So

$$\delta \mathcal{G}_q = \delta \int_M dt\ g(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}W\ \tau^A (\xi^j \delta F_j) e^{-S_{tot}}\ \Lambda(w) \tag{4.58}$$

If this is non-zero then the Green's Function has acqired a dependence on the arbitrary choice F. This is unacceptable and we must find some way to cure this problem. Typically the divergences are associated with a classical configuration degenerating to one of lower topology. This is true in this case as can be seen by the nature of the divergence of the instanton contribution to the Green's Function (1.2). So configurations on the boundary ∂M may be approximated by configurations in a different topological sector. Thus (4.58) may possibly be cancelled by a counterterm from another topological sector. We shall see that this kind of 'topological renormalisation' can be done in the case of the $O(3)$ σ-model, at least for the simplest kind of divergences.

4.4 Ward Identities

The existence of the anomalous term found in the previous sections means that we must reconsider the construction of the Ward Identities of the models that are being considered. This argument is also given in [4] for the case of gauge field theories. However it is directly analogous to the symmetry considerations for the $O(3)$ sigma model in [3].

For a Grassmann operator $A(\Psi)$, where Ψ represents all the fields in S_{tot}, the Ward Identities are given by

$$\langle \varsigma A(\Psi) \rangle = 0 \tag{4.59}$$

This condition is crucial to the perturbative renormalisation of Yang-Mills Theory [29] and to the decoupling of spurious states in String Theory. One approach to the analysis of (4.59) is to look at the effect of the BRST operator ς on the expectation value of the operator $A(\Psi)$. Consider the contribution to the expectation value of the operator $A(\Psi)$ from a particular topological sector $\mathcal{C}_q$

$$\int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; A(\Psi)\, e^{-S_{tot}} \tag{4.60}$$

Under the transformation $\Psi \to \Psi + \delta_\eta \Psi = \Psi + \eta\varsigma\Psi$, the invariance of the measure $\mathcal{D}\Psi$ is dependent on the Jacobian of the transformation being unitary. So under $\Psi \to \Psi + \eta\varsigma\Psi$

$$\int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; A(\Psi)\, e^{-S_{tot}} \to \int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; A(\Psi)\, e^{-S_{tot}} + \int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; \eta\varsigma \left(A(\Psi)\, e^{-S_{tot}} \right) \tag{4.61}$$

but the value of the integral does not change under a change of integration variable, so

$$\int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; \eta\varsigma \left(A(\Psi)\, e^{-S_{tot}} \right) = 0 \tag{4.62}$$

The action of the operator ς on $e^{-S_{tot}}$ is the same as its action on S_{tot}, i.e. $\varsigma e^{-S_{tot}} = -\tau^A \partial_A e^{-S_{tot}}$. Thus

$$\int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; (\varsigma A(\Psi))\; e^{-S_{tot}} + \int dt \int_{\mathcal{C}_q} \mathcal{D}\Psi \; A(\Psi)\, \tau^A \partial_A e^{-S_{tot}} = 0 \tag{4.63}$$

Hence the contribution of $\mathcal{C}_q$ to the expectation value of $\varsigma A(\Psi)$ is

$$\begin{aligned}\langle \varsigma A(\Psi)\rangle_q &= \int_M dt \int_{\mathcal{C}_q} \mathcal{D}\Psi\ (\varsigma A(\Psi))\ e^{-S_{tot}} \\ &= -\partial_A \int_M dt \int_{\mathcal{C}_q} \mathcal{D}\Psi\ A(\Psi)\ \tau^A e^{-S_{tot}} \\ &= -\int_{\partial M} d\Sigma_A \int_{\mathcal{C}_q} \mathcal{D}\Psi\ \tau^A\ A(\Psi)\ e^{-S_{tot}} \end{aligned} \tag{4.64}$$

As ς is the BRST operator in gauge theories, then this may be thought of as a BRST anomaly and must be cancelled by a counter-term if we are to retain the properties of BRST invariance.

For the $O(3)$ sigma model we could construct an analogous argument to the one just given using the BRST-like operator used in the previous section. However it is easier in this model to consider an infinitesimal transformation $w \to w + \delta_s w$ which leaves the classical action $S[w]$ and the integration element $\mathcal{D}w$ unchanged. Using this transformation to define a change of integration variables in (4.51)

$$\int \mathcal{D}W\ e^{-S_{tot}[w,\xi,\lambda]}\ \Lambda(w) = \int \mathcal{D}W\ e^{-S_{tot}[w+\delta_s w,\xi,\lambda]}\ \Lambda(w+\delta_s w) \tag{4.65}$$

expanding to first order in $\delta_s w$

$$0 = -\int \mathcal{D}W\ (\varsigma + \tau^A \partial_A)\ e^{-S_{tot}}\ (\xi^j \delta_s F_j)\ \Lambda(w) + \int \mathcal{D}W\ e^{-S_{tot}[w,\xi,\lambda]}\ \delta_s\Lambda(w) \tag{4.66}$$

Integrating over the moduli space and applying the cut-off gives

$$\langle \delta_s\Lambda(w)\rangle = \int_M dt\ \int \mathcal{D}W\ e^{-S_{tot}[w,\xi,\lambda]}\ \delta_s\Lambda(w) = \int_{\partial M} d\Sigma_A \int \mathcal{D}W\ \tau^A(\xi^j\delta_s F_j) e^{-S_{tot}}\ \Lambda(w) \tag{4.67}$$

If the right-hand side of the expression were zero, then this would represent a Ward Identity for the model expressing the effect of the symmetry of the classical action on the Green's Function. If it is non-zero then the symmetry has been broken by the cut-off, which would imply an unacceptable loss of classical symmetry.

4.5 Ghost Free Derivation

This section is a short review of Appendix C of [3]. Note that it is possible to find our basic result for the $O(3)$ sigma model (4.58) without using ghosts. Multiplying the basic Green's function (4.38) by the Faddeev-Popov-type factor (4.40), and then taking the variation with respect to the constraint gives

$$\delta\mathcal{G}_q = \delta\int_M dt \int \mathcal{D}w \; e^{-S[w]} \; \det(m) \prod_k \delta\left(F_j(w,t)\right)\Lambda \tag{4.68}$$

Here $m_{Aj} = \partial_A F_j$. Now if the ghosts are integrated out of (4.58) then we find

$$\delta\mathcal{G}_q = \int_{\partial M} d\Sigma_A \int \mathcal{D}w \; e^{-S[w]} \; \det(m)\; m_{Aj}\; \delta F_j \prod_k \delta\left(F_j(w,t)\right)\Lambda \tag{4.69}$$

Of course (4.68) and (4.69) must be identical for our result to be valid. To prove this identity we need to show that

$$\delta\left(\det(m)\prod_k \delta\left(F_k(w,t)\right)\right) = \partial_A\left(\det(m)\; m_{Aj}\; \delta F_j \prod_k \delta\left(F_k(w,t)\right)\right) \tag{4.70}$$

There is a well known formula for the variation of a determinant in terms of the variation of each element multiplied by its co-factor, which gives

$$\begin{aligned}&\delta\left(\det(m)\prod_k \delta\left(F_j(w,t)\right)\right)\\ &= \det(m)\; {m_{Aj}}^{-1}\; \delta m_{Aj} \prod_k \delta\left(F_k(w,t)\right) + \det(m)\;\delta\left(\prod_k \delta\left(F_k(w,t)\right)\right)\end{aligned} \tag{4.71}$$

As $\delta m_{Aj} = \partial_A \delta F_j$ then the first term on the right-hand side may be written as a divergence in moduli-space (which is the right-hand side of (4.70)) with a correction term. This correction term and the second term on the right-hand side of (4.71) may be shown to cancel and hence (4.70) is proved.

This version of the calculation is actually longer than the version using ghosts as the proof that the unwanted terms cancel is not trivial. Also, when we come to calculate the modification term (Section 5) the ghosts supply a straightforward method of calculation where we don't have to use determinants. Hence the ghost version is used.

Chapter 5

The Anomalous Ward Identity

5.1 Calculation of the Modification Term

We shall now calculate the 'anomaly' given in (4.58) for the case of the $O(3)$ σ-model. As the region of the moduli space that we are interested in is where $a \to b$ then the instanton solution may be written in a more convenient form. If we set $a - b = r$ the one instanton solution becomes

$$v = c\left(1 - \frac{r}{z-b}\right) \tag{5.1}$$

then as $|r|$ approaches zero, v tends to the zero instanton sector $v = c$, so we will introduce a moduli-space cut-off by taking $|r| > \epsilon$. We take r, b and c as the complex instanton parameters, defining them collectively as $\{t^\alpha\} = (r, b, c)$ and $\{t^{\bar\alpha}\} = (\bar r, \bar b, \bar c)$. Note that w is independent of $\{t^\alpha\}$. Now suppose that w differs from v by some quantum correction $\varphi(z, \bar z)$, then $w = v + \varphi$. The quantum correction is constrained to be orthogonal to a set of arbitrarily chosen functions $\{\psi\}$

$$F_{\bar\alpha} \equiv (\psi_\alpha, \varphi) = 0 = (\bar\psi_{\bar\alpha}, \bar\varphi) \equiv F_\alpha \tag{5.2}$$

The inner product is defined as

$$(\psi_\alpha, \varphi) = \int d^2x\, \sqrt{g}\, \rho^{-2}\, \bar\psi_{\bar\alpha}\, \varphi \equiv \rho^{-2}\, \bar\psi \circ \varphi \ , \quad \rho = 1 + |v|^2 \tag{5.3}$$

where $g_{\mu\nu}$ is the metric on S^2_{phys} given by (3.11). Also $g = \det g_{\mu\nu}$. The functionals F_A are the constraints which enter into the action in the manner outlined in the Section 4. Therefore the total action (4.49) is

$$S_{tot} = S_0[\varphi] + (\varsigma + \tau^A \partial_A)(\xi^j F_j) \tag{5.4}$$

where $S_0[\varphi]$ is given above in (3.9), and the 'anomaly' (4.58)

$$\delta\mathcal{G}_1 = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Phi\ e^{-S_{tot}} \tau^A (\xi^j \delta F_j)\ \Lambda(v) \tag{5.5}$$

where Φ denotes all the fields. Looking to integrate out the ghosts we expand out the action using the properties of the operator ς given in (4.48)

$$
\begin{aligned}
S_{tot} &= S_0[\varphi] + \lambda^A F_A \quad -\tau^{\alpha}\xi^{\bar{\beta}} m_{\alpha\bar{\beta}} - \tau^{\bar{\alpha}}\xi^{\beta}\bar{m}_{\bar{\alpha}\beta} \\
&\quad +\tau^{\alpha}\xi^{\beta}\partial_{\alpha}(\rho^{-2}\psi_{\beta})\circ\bar{\varphi} + \tau^{\alpha}\xi^{\bar{\beta}}\partial_{\alpha}(\rho^{-2}\bar{\psi}_{\bar{\beta}})\circ\varphi \\
&\quad +\tau^{\bar{\alpha}}\xi^{\beta}\partial_{\bar{\alpha}}(\rho^{-2}\psi_{\beta})\circ\bar{\varphi} + \tau^{\bar{\alpha}}\xi^{\bar{\beta}}\partial_{\bar{\alpha}}(\rho^{-2}\bar{\psi}_{\bar{\beta}})\circ\varphi
\end{aligned} \tag{5.6}
$$

The ghost field propagators are the inverses of the matrices

$$
m_{\alpha\bar{\beta}} = (\psi_{\bar{\beta}}, \partial_{\alpha} v) \quad , \quad \bar{m}_{\bar{\alpha}\beta} = (\partial_{\bar{\alpha}} v, \psi_{\beta}) \tag{5.7}
$$

The last four terms of (5.6) are the interaction terms between the ghost fields and φ. Now let $\tilde{S} = S_0 + \lambda^A F_A - \tau^{\alpha}\xi^{\bar{\beta}} m_{\alpha\bar{\beta}} - \tau^{\bar{\alpha}}\xi^{\beta}\bar{m}_{\bar{\alpha}\beta}$ and expand the rest of the action in a power series in φ. So if $S_{tot} = \tilde{S} + \bar{S}$ then

$$
\begin{aligned}
e^{-\bar{S}} &= 1 - \Big[\tau^{\mu}\xi^{\nu}\partial_{\mu}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} + \tau^{\bar{\mu}}\xi^{\nu}\partial_{\bar{\mu}}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} \\
&\quad + \tau^{\mu}\xi^{\bar{\nu}}\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi + \tau^{\bar{\mu}}\xi^{\bar{\nu}}\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\Big] + \cdots
\end{aligned} \tag{5.8}
$$

The unwritten terms, of higher order in φ will be neglected. The first term in the expansion disappears under contraction with the rest of $\mathcal{G}_1$ as it is linear in φ. Thus we need to find which are the non-zero terms in

$$
\begin{aligned}
\delta\mathcal{G}_1 &= -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Phi\ e^{-\tilde{S}}\ \tau^A(\xi^j\delta F_j) \\
&\quad \times \Big[\tau^{\mu}\xi^{\nu}\partial_{\mu}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} + \tau^{\bar{\mu}}\xi^{\nu}\partial_{\bar{\mu}}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} \\
&\quad +\tau^{\mu}\xi^{\bar{\nu}}\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi + \tau^{\bar{\mu}}\xi^{\bar{\nu}}\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\Big]\,\Lambda(v) \\
&= -\int_{\partial M} d\Sigma_{\alpha} \int \mathcal{D}\Phi\ e^{-\tilde{S}} \left(\tau^{\alpha}\xi^{\beta}\delta(\rho^{-2}\psi_{\beta})\circ\bar{\varphi} + \tau^{\alpha}\xi^{\bar{\beta}}\delta(\rho^{-2}\bar{\psi}_{\bar{\beta}})\circ\varphi\right) \\
&\quad \times \Big[\tau^{\mu}\xi^{\nu}\partial_{\mu}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} + \tau^{\bar{\mu}}\xi^{\nu}\partial_{\bar{\mu}}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi} \\
&\quad +\tau^{\mu}\xi^{\bar{\nu}}\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi + \tau^{\bar{\mu}}\xi^{\bar{\nu}}\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\Big]\,\Lambda(v) \\
&\quad - \int_{\partial M} d\Sigma_{\bar{\alpha}} \int \mathcal{D}\Phi\ e^{-\tilde{S}} \left(\tau^{\bar{\alpha}}\xi^{\beta}\delta(\rho^{-2}\psi_{\beta})\circ\bar{\varphi} + \tau^{\bar{\alpha}}\xi^{\bar{\beta}}\delta(\rho^{-2}\bar{\psi}_{\bar{\beta}})\circ\varphi\right)
\end{aligned}
$$

$$\times \left[\tau^{\mu}\xi^{\nu}\partial_{\mu}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi}+\tau^{\bar{\mu}}\xi^{\nu}\partial_{\bar{\mu}}(\rho^{-2}\psi_{\nu})\circ\bar{\varphi}\right.$$
$$\left.+\tau^{\mu}\xi^{\bar{\nu}}\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi+\tau^{\bar{\mu}}\xi^{\bar{\nu}}\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\right]\Lambda(v) \tag{5.9}$$

Simply by studying the action we see that to this order the non-zero terms are

$$\begin{aligned}\delta\mathcal{G}_1 &= -\int_{\partial M} d\Sigma_\alpha \int \mathcal{D}\Phi\; e^{-\tilde{S}}\left[\left(\tau^{\alpha}\xi^{\beta}\delta(\rho^{-2}\psi_\beta)\circ\bar{\varphi}\right)\left(\tau^{\bar{\mu}}\xi^{\bar{\nu}}\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\right)\ \Lambda(v)\right]\\ &\quad - \int_{\partial M} d\Sigma_{\bar{\alpha}} \int \mathcal{D}\Phi\; e^{-\tilde{S}}\left[\left(\tau^{\bar{\alpha}}\xi^{\beta}\delta(\rho^{-2}\psi_\beta)\circ\bar{\varphi}\right)\left(\tau^{\mu}\xi^{\bar{\nu}}\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\varphi\right)\ \Lambda(v)\right]\end{aligned} \tag{5.10}$$

In order to do the integration over the fields, the ghosts in the first term must be re-ordered. The ghosts are Grassmannian so they anti-commute, and due to the fact that three commutation operations need to be performed, an extra minus sign is produced in the first term. Hence

$$\begin{aligned}\delta\mathcal{G}_1 &= \int_{\partial M} d\Sigma_\alpha \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\mu}\beta}(\delta(\rho^{-2}\psi_\beta)\circ\bar{\boldsymbol{\varphi}})m^{-1}_{\alpha\bar{\nu}}(\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\boldsymbol{\varphi})\ \Lambda(v)\right)\\ &\qquad - \int_{\partial M} d\Sigma_{\bar{\alpha}} \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\alpha}\beta}(\delta(\rho^{-2}\psi_\beta)\circ\bar{\boldsymbol{\varphi}})m^{-1}_{\mu\bar{\nu}}(\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\boldsymbol{\varphi})\ \Lambda(v)\right)\\ &= \int_{\partial M} d\Sigma_\alpha \Psi^{\alpha}\ \Lambda(v) - \int_{\partial M} d\Sigma_{\bar{\alpha}}\Psi^{\bar{\alpha}}\ \Lambda(v)\end{aligned} \tag{5.11}$$

Thus

$$\Psi^{\alpha} = \zeta_1(t)\bar{m}^{-1}_{\bar{\mu}\beta}(\delta(\rho^{-2}\psi_\beta)\circ\bar{\boldsymbol{\varphi}})m^{-1}_{\alpha\bar{\nu}}(\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\boldsymbol{\varphi}) \tag{5.12}$$
$$\Psi^{\bar{\alpha}} = \zeta_1(t)\bar{m}^{-1}_{\bar{\alpha}\beta}(\delta(\rho^{-2}\psi_\beta)\circ\bar{\boldsymbol{\varphi}})m^{-1}_{\mu\bar{\nu}}(\partial_{\mu}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\boldsymbol{\varphi}) \tag{5.13}$$

Where $\zeta_1(t)$ is the one-loop partition function given above (3.157) and the boldface type indicates that the field $\boldsymbol{\varphi}$ is contracted with its conjugate field in the same term. This expression (5.11) is the lowest order contribution to $\delta\mathcal{G}_1$ from the semi-classical expansion in powers of Planck's constant. The one-loop contribution is zero as $\delta\zeta_1(t) = 0$. Our expression is one order beyond the one-loop result.

We will now show that $\Psi^{\bar{\alpha}}$ and Ψ^{α} can themselves be expressed as variations. This enables us to both simplify their evaluations and find counterterms which will cancel $\delta\int_M dt\ g(t)$.

To do this we see that the fields φ contract to introduce a two-point Green's function. The Green's function of the two-point fluctuation operator is found in Appendix A to be

$$\mathcal{I}(x,y) = <\varphi(x)\,\bar{\varphi}(y)>= \varphi(x)\bar{\varphi}(y) \tag{5.14}$$

$$= -\frac{1}{\pi^2}\int d^2z\,(1-P^\dagger\circ)\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ P) \tag{5.15}$$

here we have used a notation equivalent to the inner product, i.e.

$$P^\dagger(y,x)\circ\frac{1}{x-z} = \int d^2x'\,P^\dagger(y,x')\,\frac{1}{x'-z}\,\sqrt{g}\,\rho^{-2}(x') \tag{5.16}$$

where the operators P are projection operators which "project out" the zero modes of Δ

$$P(x,y) = \left(\rho^{-2}\psi_\alpha(x)\right)\bar{m}^{-1}_{\bar{\beta}\alpha}\bar{\mathcal{Z}}_{\bar{\beta}}(y) \tag{5.17}$$

$$P^\dagger(y,x) = \mathcal{Z}_\beta(y)m^{-1}_{\beta\bar{\alpha}}\left(\rho^{-2}\bar{\psi}_{\bar{\alpha}}(x)\right) \tag{5.18}$$

and $\mathcal{Z}(x)$ is a zero mode of Δ. Above we showed that the zero modes of the fluctuation operator are $\partial_A v$, therefore from (5.7) we see that

$$m_{\alpha\bar{\beta}} = \left(\bar{\mathcal{Z}}_{\bar{\alpha}},\bar{\psi}_{\bar{\beta}}\right) \quad , \quad \bar{m}_{\bar{\alpha}\beta} = \left(\mathcal{Z}_\alpha,\psi_\beta\right) \tag{5.19}$$

We also find in Appendix A that

$$\mathcal{I}(x,y) = (1-P^\dagger\circ)\,\mathcal{I}_0(x,y)\,(1-\circ P)\times(1+O(\epsilon)) \tag{5.20}$$

where $\mathcal{I}_0(x,y)$ is the zero instanton sector Green's Function. Thus to leading order we can use this relationship and the projection operators to evaluate $\Psi^{\bar{\alpha}}$ and Ψ^α on the boundary ∂M. To do this, in $\Psi^{\bar{\alpha}}$ and Ψ^α we may replace φ with $(1-P^\dagger\circ)\varphi_0$ and $\bar{\varphi}$ by $\bar{\varphi}_0(1-\circ P)$. Hence the one instanton sector Green's Function (5.14) may now be approximately written as

$$\varphi(x)\bar{\varphi}(y) \approx (1-P^\dagger\circ)\varphi_0\,\bar{\varphi}_0(1-\circ P) \tag{5.21}$$

and the inner products contained in $\Psi^{\bar{\alpha}}$ and Ψ^α becomes

$$(\varphi,\delta\psi_\beta) = \int d^2x\,\sqrt{g}\,\rho^{-2}\,\bar{\varphi}\,\delta\psi_\beta = \bar{\varphi}\circ(\rho^{-2}\delta\psi_\beta) \approx \bar{\varphi}_0(1-\circ P)\circ(\rho^{-2}\delta\psi_\beta) \tag{5.22}$$

Now let us consider both Ψ^{α} and $\Psi^{\bar{\alpha}}$ as the product of two terms, each containing an inner product with a ghost field propagator as a coefficient. The fields in these terms may be replaced in the manner described above, for the moment we shall consider each of these acting on an arbitrary function f which is independent of ψ. Hence for $\bar{m}^{-1}_{\bar{\mu}\beta}(\delta(\rho^{-2}\psi_{\beta}) \circ \bar{\varphi})$

$$\bar{m}^{-1}_{\bar{\mu}\beta}\left[(f(1-\circ P))\circ\delta\left(\rho^{-2}\psi_{\beta}\right)\right] = \bar{m}^{-1}_{\bar{\mu}\beta}\, f\circ\delta\left(\rho^{-2}\psi_{\beta}\right) - \bar{m}^{-1}_{\bar{\mu}\beta}\, f\circ\left(\rho^{-2}\psi_{\tau}\right)\bar{m}^{-1}_{\bar{\kappa}\tau}\bar{\mathcal{Z}}_{\bar{\kappa}}\circ\delta\left(\rho^{-2}\psi_{\beta}\right) \tag{5.23}$$

where we have used (5.17). Now $\bar{\mathcal{Z}}_{\bar{\kappa}}\circ\delta\left(\rho^{-2}\psi_{\beta}\right) = \delta\bar{m}_{\bar{\kappa}\beta}$ which can easily be seen by taking the variation with respect to ψ of the second equation in (5.19). So

$$\begin{aligned}\bar{m}^{-1}_{\bar{\mu}\beta}\left[(f(1-\circ P))\circ\delta\left(\rho^{-2}\psi_{\beta}\right)\right] &= \bar{m}^{-1}_{\bar{\mu}\beta}\, f\circ\delta\left(\rho^{-2}\psi_{\beta}\right) - \bar{m}^{-1}_{\bar{\mu}\beta}\, f\circ\left(\rho^{-2}\psi_{\tau}\right)\bar{m}^{-1}_{\bar{\kappa}\tau}\delta\bar{m}_{\bar{\kappa}\beta}\\ &= \bar{m}^{-1}_{\bar{\mu}\beta}\, f\circ\delta\left(\rho^{-2}\psi_{\beta}\right) + f\circ\left(\rho^{-2}\psi_{\tau}\right)\delta\bar{m}^{-1}_{\bar{\mu}\tau}\\ &= \delta\left(f\circ\left(\rho^{-2}\psi_{\beta}\right)\bar{m}^{-1}_{\bar{\mu}\beta}\right)\end{aligned} \tag{5.24}$$

For the second term in Ψ^{α} we may make an approximation similar to (5.22)

$$\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\boldsymbol{\varphi} \approx \partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ(1-P^{\dagger}\circ)\boldsymbol{\varphi}_0 \tag{5.25}$$

and thus

$$\begin{aligned}m^{-1}_{\alpha\bar{\nu}}\left[\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ(1-P^{\dagger}\circ)f\right] &= m^{-1}_{\alpha\bar{\nu}}\left[\partial_{\bar{\mu}}\left(\left(\rho^{-2}\bar{\psi}_{\bar{\nu}}\right)\circ(1-P^{\dagger}\circ)f\right)\right.\\ &\quad\left. -\,(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\left(\partial_{\bar{\mu}}(1-P^{\dagger}\circ)\right)f\right]\end{aligned} \tag{5.26}$$

However, one of the properties of $P^{\dagger}$ is that

$$\rho^{-2}\bar{\psi}_{\bar{\nu}} = (\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ P^{\dagger} \tag{5.27}$$

therefore the first term on the right hand side of (5.26) is zero leaving us with

$$m^{-1}_{\alpha\bar{\nu}}\left[\partial_{\bar{\mu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ(1-P^{\dagger}\circ)f\right] = m^{-1}_{\alpha\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ\partial_{\bar{\mu}}P^{\dagger}\circ f \tag{5.28}$$

Now as the zero modes are meromorphic in the moduli we have $\partial_{\bar{\mu}}\mathcal{Z}_\theta = 0$ and so using (5.18) we have

$$m^{-1}_{\alpha\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ \mathcal{Z}_\theta \partial_{\bar{\mu}}(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \circ f = \partial_{\bar{\mu}}(m^{-1}_{\alpha\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \circ f \tag{5.29}$$

where we have used (5.19). We have also assumed that $m^{-1}_{\alpha\bar{\nu}}m_{\theta\bar{\nu}} = \delta_{\alpha\theta}$. This is certainly true for the choice of constraint which we will be presenting later in this chapter, but it is not necessarily true for all choices and care must be taken in each case. A similar procedure may be used on the second term in $\Psi^{\bar{\alpha}}$, however this time there is an important modification. The derivatives of the zero modes with respect to the moduli contribute as $\partial_\mu \mathcal{Z}_\theta \neq 0$

$$\begin{aligned}
m^{-1}_{\mu\bar{\nu}}\left[\partial_\mu(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ (1 - P^\dagger \circ) f\right] &= m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ \partial_\mu(\mathcal{Z}_\theta m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \circ f \\
&= \partial_\mu(m^{-1}_{\mu\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \circ f \\
&\quad + m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ (\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \circ f
\end{aligned} \tag{5.30}$$

Now re-arranging Ψ^α and $\Psi^{\bar{\alpha}}$ using (5.24), (5.29) and (5.30) gives

$$\Psi^\alpha = \zeta_1(t)\delta\left[\bar{m}^{-1}_{\bar{\mu}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\right]\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}\alpha}\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0) \tag{5.31}$$

and as $\delta\zeta_1(t) = 0$ and $\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}\alpha}\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0)$ are independent of ψ then the variation may be put outside the whole expression. (Note that we are treating ψ and $\bar{\psi}$ as being independent of one another.

$$\Psi^\alpha = \delta\left[\zeta_1(t)\bar{m}^{-1}_{\bar{\mu}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}\alpha}\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0)\right] \tag{5.32}$$

Similarly

$$\begin{aligned}
\Psi^{\bar{\alpha}} &= \zeta_1(t)\delta\left[\bar{m}^{-1}_{\bar{\alpha}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\right]\Big(\partial_\mu(m^{-1}_{\mu\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) \\
&\quad + m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ (\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x))\Big) \circ \varphi_0 \\
&= \delta\left[\zeta_1(t)\,\bar{m}^{-1}_{\bar{\alpha}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\;\Big(\partial_\mu(m^{-1}_{\mu\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x))\right. \\
&\quad \left. + m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ (\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x))\Big) \circ \varphi_0\right]
\end{aligned} \tag{5.33}$$

Thus we have succeeded in writing Ψ^α and $\Psi^{\bar\alpha}$ on ∂M as variations, given in terms of the zero instanton sector Green's Function. Consequently the variation of the Green's Function is

$$\begin{aligned}
\delta\mathcal{G}_1 &= \int_{\partial M} d\Sigma_\alpha \delta\left[\zeta_1(t)\bar m^{-1}_{\bar\mu\beta}(\rho^{-2}\psi_\beta\circ\bar\varphi_0)\partial_{\bar\mu}(m^{-1}_{\bar\kappa\alpha}\rho^{-2}\bar\psi_{\bar\kappa}\circ\varphi_0)\right]\ \Lambda(c) \\
&\quad - \int_{\partial M} d\Sigma_{\bar\alpha}\delta\left[\zeta_1(t)\ \bar m^{-1}_{\bar\alpha\beta}(\rho^{-2}\psi_\beta\circ\bar\varphi_0)\ \left(\partial_\mu(m^{-1}_{\mu\bar\kappa}\rho^{-2}\bar\psi_{\bar\kappa}(x))\right.\right. \\
&\quad \left.\left. + m^{-1}_{\mu\bar\nu}(\rho^{-2}\bar\psi_{\bar\nu})\circ(\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar\kappa}\rho^{-2}\bar\psi_{\bar\kappa}(x))\right)\circ\varphi_0\right]\ \Lambda(c) \qquad (5.34)
\end{aligned}$$

These integrals have been restricted to a surface ∂M in the moduli space M. The one instanton moduli are b, c and $r = a - b$ and we are free to define the surface in terms of these in any manner we choose. However as the one instanton Green's Function is divergent in the region $r \to 0$ it is prudent to put the cut-off on r. Hence let us say that the magnitude of r is held fixed on ∂M such that $|r| = \epsilon$ and so $r = \epsilon e^{i\theta}$. Thus the integration measures on ∂M are

$$(d\Sigma_A) = (d\Sigma_r, d\Sigma_{\bar r}) = (\epsilon e^{-i\theta}d\theta\ d^2b\ d^2c\ ,\ \epsilon e^{i\theta}d\theta\ d^2b\ d^2c) \qquad (5.35)$$

and as

$$\zeta_1(t) = \frac{K^1}{\pi|r|^2(1+|c|^2)^2} \qquad (5.36)$$

then we can use (5.35) and (5.36) in (5.34) to give

$$\begin{aligned}
\delta\mathcal{G}_1 &= \delta\oint d\theta\ d^2b\ d^2c\ e^{-i\theta}\frac{K^1}{\pi\epsilon(1+|c|^2)^2}\bar m^{-1}_{\bar\mu\beta}(\rho^{-2}\psi_\beta\circ\bar\varphi_0)\partial_{\bar\mu}(m^{-1}_{\bar\kappa r}\rho^{-2}\bar\psi_{\bar\kappa}\circ\varphi_0)\ \Lambda(c) \\
&\quad - \delta\oint d\theta\ d^2b\ d^2c\ e^{i\theta}\frac{K^1}{\pi\epsilon(1+|c|^2)^2}\ \bar m^{-1}_{\bar r\beta}(\rho^{-2}\psi_\beta\circ\bar\varphi_0) \\
&\quad \times\left(\partial_\mu(m^{-1}_{\mu\bar\kappa}\rho^{-2}\bar\psi_{\bar\kappa}(x)) + m^{-1}_{\mu\bar\nu}(\rho^{-2}\bar\psi_{\bar\nu})\circ(\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar\kappa}\rho^{-2}\bar\psi_{\bar\kappa}(x))\right)\circ\varphi_0\ \Lambda(c) \qquad (5.37)
\end{aligned}$$

In the zero instanton sector the instanton solution is just a single complex number c, thus the two-point function in this sector is

$$\int_0 \mathcal{D}\Phi\ e^{-S_{tot}}\ \bar\varphi_0(x)\varphi_0(y) = \int \bar\varphi_0(x)\varphi_0(y)\ \frac{d^2c}{\pi(1+|c|^2)^2} \qquad (5.38)$$

hence

$$\begin{aligned}
\delta\mathcal{G}_1 &= \delta\int_0 \mathcal{D}\Phi\, e^{-S_{tot}} \oint d\theta\, d^2b\, e^{-i\theta} K^1\epsilon^{-1}\bar{m}^{-1}_{\bar{\mu}\beta}(\rho^{-2}\psi_\beta\circ\bar{\varphi}_0)\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}r}\rho^{-2}\bar{\psi}_{\bar{\kappa}}\circ\varphi_0)\,\Lambda(c) \\
&\quad -\delta\int_0 \mathcal{D}\Phi\, e^{-S_{tot}} \oint d\theta\, d^2b\, e^{i\theta} K^1\epsilon^{-1}\, \bar{m}^{-1}_{\bar{r}\beta}(\rho^{-2}\psi_\beta\circ\bar{\varphi}_0) \\
&\quad \times\Big(\partial_\mu(m^{-1}_{\mu\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) + m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ(\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x))\Big)\circ\varphi_0\,\Lambda(c) \qquad (5.39) \\
&\equiv \delta\int_0 \mathcal{D}\Phi\, e^{-S_{tot}}\, \mathcal{J}\,\Lambda(c) \qquad (5.40)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J} &= \int d^2b \oint d\theta\, K^1\epsilon^{-1}\Big[e^{-i\theta}\bar{m}^{-1}_{\bar{\mu}\beta}(\rho^{-2}\psi_\beta\circ\bar{\varphi}_0)\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}r}\rho^{-2}\bar{\psi}_{\bar{\kappa}}\circ\varphi_0) - e^{i\theta}\bar{m}^{-1}_{\bar{r}\beta}(\rho^{-2}\psi_\beta\circ\bar{\varphi}_0) \\
&\quad \times\Big(\partial_\mu(m^{-1}_{\mu\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x)) + m^{-1}_{\mu\bar{\nu}}(\rho^{-2}\bar{\psi}_{\bar{\nu}})\circ(\partial_\mu\mathcal{Z}_\theta)(m^{-1}_{\theta\bar{\kappa}}\rho^{-2}\bar{\psi}_{\bar{\kappa}}(x))\Big)\circ\varphi_0\Big] \qquad (5.41)
\end{aligned}$$

Therefore we have succeeded in writing the variation of the one instanton sector contribution as the variation of a term in the zero instanton sector. If (5.41) is not zero then the Green's Functions have gained a dependence on the arbitrary function ψ. It is important that this dependence does not occur, otherwise the Green's Functions would depend on the choice of configuration space co-ordinates, and the symmetries that enable us to renormalise may be affected. Hence we shall propose a modification of the Green's Functions designed to subtract any ψ dependent terms from the Green's Function in each instanton sector. We will thus modify (4.39) by defining

$$\tilde{\mathcal{G}} = \frac{1}{\tilde{Z}}\sum_q \kappa^q\tilde{\mathcal{G}}_q \qquad (5.42)$$

where

$$\begin{aligned}
\tilde{\mathcal{G}}_0 &= \int \mathcal{D}w\, e^{-S[w]-\kappa\mathcal{J}}\,\Lambda(w) \qquad (5.43) \\
\tilde{Z}_0 &= \int \mathcal{D}w\, e^{-S[w]-\kappa\mathcal{J}} \qquad (5.44) \\
\tilde{\mathcal{G}}_1 &= \mathcal{G}_1\ ,\ \tilde{Z}_1 = Z_1 \qquad (5.45)
\end{aligned}$$

so that to the order that we are working

$$\delta_\psi\tilde{\mathcal{G}} = 0 \qquad (5.46)$$

The modification in each sector is performed by a term from a sector of order lower by one. The leading order term is unaffected as the modification is proportional to the coupling κ. Clearly to higher orders there are further modifications.

Although this modification may be termed renormalisation, it differs from standard renormalisation in some important aspects. This is not a simple renormalisation of the action as the fields and associated constants may not be redefined to take account of the divergences. This is because the modification is not local, however, by the introduction of new fields, the modification may be localised. Note that the Faddeev-Popov procedure which enables us to extract the integration over the instanton moduli gives a non-local interaction when the ghosts are integrated out, and our modification is a new aspect of this, made necessary by the need to maintain independence of configuration space co-ordinates and internal symmetries.

The calculation of $\mathcal{J}$ may be narrowed down to the evaluation of four separate components; the inner product $(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)$ and its conjugate, and the matrix $m^{-1}_{\bar{\kappa}\alpha}$ and its conjugate. However, before that we need to decide what constraint we are going to impose on the fields, i.e. we need to choose ψ. A natural choice is to say that the quantum fluctuations ϕ are orthogonal to the zero modes of the fluctuation operator. Thus $\psi = \frac{\partial v}{\partial t}$. In this case the propagators of the ghost fields (5.7) become the Kähler metric which is the metric tensor on the manifold of the instantons, parametrized by the $\{t^\alpha\}$.

$$m_{\alpha\bar{\beta}} = \left(\partial_{\bar{\alpha}}\bar{v}, \partial_{\bar{\beta}}\bar{v}\right) = \int d^2x \,\sqrt{g(x)}\,\rho^{-2}\,\partial_\alpha v\,\partial_{\bar{\beta}}\bar{v} \tag{5.47}$$

$$\bar{m}_{\bar{\alpha}\beta} = \left(\partial_\alpha v, \partial_\beta v\right) = \int d^2x \,\sqrt{g(x)}\,\rho^{-2}\,\partial_{\bar{\alpha}}\bar{v}\,\partial_\beta v \tag{5.48}$$

$m_{\alpha\bar{\beta}}$ is Kähler as it can be written in the form [13],[15],

$$m_{\alpha\bar{\beta}} = \frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^{\bar{\beta}}}\,\mathcal{K} \;\;,\;\; \mathcal{K} = \int d^2x\,\sqrt{g(x)}\,\ln(1+|v|^2) \tag{5.49}$$

where $\mathcal{K}$ is the Kähler Potential. The inverse of the metric is found in Appendix B to be

$$\left(m_{\bar{\alpha}\beta}\right)^{-1} = \frac{\zeta^2\lambda^2}{\pi\ln|r|^2}\begin{pmatrix} |c|^{-2} & -1/\zeta & -b/(c\lambda) \\ -1/\zeta & \ln|r|^2/\zeta^2 & \bar{c}b/(\lambda\zeta) \\ -\bar{b}/(\bar{c}\lambda) & \bar{b}c/(\lambda\zeta) & \ln|r|^2/\lambda^2 \end{pmatrix} \tag{5.50}$$

in the limit $r \to 0$, where $\zeta = 1 + |c|^2$ and $\lambda = 1 + |b|^2$ when space-time is S^2_{phys} and $\lambda = 1$ when space-time is the plane.

Now for the instanton solution $v = c\left(1 - \frac{r}{z-b}\right)$ we find

$$\psi_\beta = \frac{\partial v}{\partial t^\beta} = \begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial b} \\ \frac{\partial v}{\partial c} \end{pmatrix} = \begin{pmatrix} -\frac{c}{z-b} \\ -\frac{cr}{(z-b)^2} \\ 1 - \frac{r}{z-b} \end{pmatrix} \tag{5.51}$$

and we can consider calculating the inner product

$$(\varphi_0, \psi_\beta) = (\rho^{-2}\psi_\beta \circ \bar{\varphi}_0) = \int d^2z\sqrt{g}\,\rho^{-2}\,\psi_\beta\,\bar{\varphi}_0(z,\bar{z}) \tag{5.52}$$

Earlier we defined

$$\rho^{-2} = \frac{1}{(1+|v|^2)^2} = \frac{|z-b|^4}{(|z-b|^2 + |c|^2|z-b-r|^2)^2} \tag{5.53}$$

(for $r \to 0$ note that $\rho \to 1 + |c|^2$) and the square root of the metric on S^2_{phys} is

$$\sqrt{g} = \frac{1}{(1+|z|^2)^2} \tag{5.54}$$

However, as the expression for ζ_1 from [2] that we have used is defined on the plane, this metric will presently be replaced by the flat space metric. In the units that we are using here, the flat space metric is simply 1.

Thus

$$(\varphi_0, \psi_\beta) = \int d^2z\,\frac{|z-b|^4}{(1+|z|^2)^2(|z-b|^2 + |c|^2|z-b-r|^2)^2}\begin{pmatrix} -\frac{c}{z-b} \\ -\frac{cr}{(z-b)^2} \\ 1 - \frac{r}{z-b} \end{pmatrix}\bar{\varphi}_0(z,\bar{z}) \tag{5.55}$$

For $r \neq 0$ all these integrals exist despite the singularities in ψ_β at $z \sim b$. This is due to the heavy damping factor $|z-b|^4$. Now for small r we shall look at each term individually. So for $\beta = b$

$$\begin{aligned}(\varphi_0, \psi_b) &= -\int d^2z \, \frac{cr|z-b|^4}{(1+|z|^2)^2(|z-b|^2+|c|^2|z-b-r|^2)^2(z-b)^2}\bar{\varphi}_0(z,\bar{z}) \\ &\sim -\frac{cr}{(1+|c|^2)^2}\int d^2z \, \frac{1}{(1+|z|^2)^2(z-b)^2}\bar{\varphi}_0(z,\bar{z}) \end{aligned} \tag{5.56}$$

The logarithmic divergence from the integral is regulated by the factor of r so that effectively as $r \longrightarrow 0$ this component becomes zero, i.e. $(\varphi_0, \psi_b) \sim 0$.

The $\beta = c$ component has two terms. The second of these terms is heavily damped by the factor of r and so tends to zero as $r \longrightarrow 0$. The first term, in flat space and for $r \longrightarrow 0$, becomes

$$\int d^2z \, \frac{\bar{\varphi}_0(z,\bar{z})}{(1+|c|^2)^2} = \langle 1|\bar{\varphi}_0\rangle = 0 \tag{5.57}$$

as $\bar{\varphi}_0$ is orthogonal to the constant zero mode of Δ.

Finally for $\beta = r$

$$\begin{aligned}(\varphi_0, \psi_r) &= -\int d^2z \, \frac{c|z-b|^4}{(1+|z|^2)^2(|z-b|^2+|c|^2|z-b-r|^2)^2(z-b)}\bar{\varphi}_0(z,\bar{z}) \\ &\sim -\frac{c}{(1+|c|^2)^2}\int d^2z \, \frac{1}{(1+|z|^2)^2(z-b)}\bar{\varphi}_0(z,\bar{z}) \end{aligned} \tag{5.58}$$

which is finite and is thus the dominant term in (φ_0, ψ_β) as it is the only non-zero component in the limit $r \longrightarrow 0$. Similarly for $(\bar{\varphi}_0, \bar{\psi}_{\bar{\beta}})$ the non-zero component is $(\bar{\varphi}_0, \bar{\psi}_{\bar{r}})$

We may now compute the first two terms in $\mathcal{J}$ (5.41). The first term is

$$\bar{m}^{-1}_{\bar{\mu}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\partial_{\bar{\mu}}(m^{-1}_{\bar{\kappa}r}\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0) \tag{5.59}$$

As seen above, the only non-zero components in the inner products to leading order come from $\bar{\kappa} = \bar{r}$ and $\beta = r$. As this is the case, the leading order term will be when $\bar{\mu} = \bar{r}$ and the derivative acts on the inverse matrix only. So as

$$\partial_{\bar{r}}\,(\ln|r|^2)^{-1} = -(\bar{r})^{-1}(\ln|r|^2)^{-2} \tag{5.60}$$

then the dominant contribution from (5.59) is

$$\bar{m}^{-1}_{\bar{r}r}(\rho^{-2}\psi_r \circ \bar{\varphi}_0)\partial_{\bar{r}}(m^{-1}_{\bar{r}r})(\rho^{-2}\bar{\psi}_{\bar{r}} \circ \varphi_0) \sim -\left(\frac{\zeta^2\lambda^2}{\pi|c|^2}\right)^2 \frac{1}{\bar{r}(\ln|r|^2)^3}(\rho^{-2}\psi_r \circ \bar{\varphi}_0)(\rho^{-2}\bar{\psi}_{\bar{r}} \circ \varphi_0) \tag{5.61}$$

The second term in $\mathcal{J}$ is

$$\bar{m}^{-1}_{\bar{\alpha}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)\partial_\mu(m^{-1}_{\bar{\kappa}\mu}\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0) \tag{5.62}$$

Again the dominant terms from the inner products are when $\bar{\kappa} = \bar{r}$ and $\beta = r$, leaving ∂_r to act on $m^{-1}_{\bar{r}r}$

$$\bar{m}^{-1}_{\bar{r}r}(\rho^{-2}\psi_r \circ \bar{\varphi}_0)\partial_r(m^{-1}_{\bar{r}r})(\rho^{-2}\bar{\psi}_{\bar{r}} \circ \varphi_0) \sim -\left(\frac{\zeta^2\lambda^2}{\pi|c|^2}\right)^2 \frac{1}{\bar{r}(\ln|r|^2)^3}(\rho^{-2}\psi_r \circ \bar{\varphi}_0)(\rho^{-2}\bar{\psi}_{\bar{r}} \circ \varphi_0) \tag{5.63}$$

Now (5.61) and (5.63) are identical, thus the first two terms in $\mathcal{J}$ cancel against each other and we are left with

$$\mathcal{J} = \int d^2b \oint d\theta\, K^1\epsilon^{-1}e^{i\theta}\bar{m}^{-1}_{\bar{r}\beta}(\rho^{-2}\psi_\beta \circ \bar{\varphi}_0)m^{-1}_{\mu\bar{\nu}}((\rho^{-2}\bar{\psi}_{\bar{\nu}}) \circ (\partial_\mu \mathcal{Z}_\theta))m^{-1}_{\theta\bar{\kappa}}(\rho^{-2}\bar{\psi}_{\bar{\kappa}} \circ \varphi_0) \tag{5.64}$$

Our first task is to remove the zero mode $\mathcal{Z}_\theta$ from the second inner product. Remember that as $m_{\mu\bar{\nu}} = (\rho^{-2}\psi_\mu \circ \bar{\psi}_{\bar{\nu}}) = (\rho^{-2}\mathcal{Z}_\mu \circ \bar{\psi}_{\bar{\nu}})$ then $\rho^{-2}\bar{\psi}_{\bar{\nu}} \circ \partial_\mu \mathcal{Z}_\theta = \rho^{-2}\bar{\psi}_{\bar{\nu}} \circ \partial_\mu \psi_\theta$. The dominant terms of this inner product may now be calculated

$$\rho^{-2}\bar{\psi}_{\bar{\nu}} \circ \partial_\mu\psi_\theta = \int d^2z \frac{|z-b|^4}{(|z-b|^2 + |c|^2|z-b-r|^2)^2}\bar{\psi}_{\bar{\nu}}\partial_\mu\psi_\theta \tag{5.65}$$

$\partial_\mu\psi_\theta$ is a symmetric matrix

$$(\partial_\mu\psi_\theta) = \frac{1}{z-b}\begin{pmatrix} 0 & -c/(z-b) & -1 \\ -c/(z-b) & -2cr/(z-b)^2 & -r/(z-b) \\ -1 & -r/(z-b) & 0 \end{pmatrix} \tag{5.66}$$

The term in $\bar{\psi}_{\bar{\nu}}$ of highest order in $(\bar{z}-\bar{b})^{-1}$ is $\bar{\psi}_{\bar{b}} = -\bar{c}\bar{r}/(\bar{z}-\bar{b})^2$. Thus for $r \neq 0$ all terms in $\bar{\psi}_{\bar{\nu}}\partial_\mu\psi_\theta$ which have singularities as $z \sim b$ have their divergences cancelled by the $|z-b|^4$ from ρ^{-2}. However for $r \to 0$, ρ^{-2} becomes a constant and the singularities take over.

Taking into account the explicit factors of r in ψ we conclude that $\rho^{-2}\bar{\psi}_{\bar{\nu}} \circ \partial_\mu \psi_\theta$ is of order $1/|r|$ for ν, μ, θ equal to r or b and is otherwise of lower order. Hence the dominant contribution from $m^{-1}_{\mu\bar{\nu}}\rho^{-2}\bar{\psi}_{\bar{\nu}} \circ \partial_\mu \psi_\theta$ will also be of order $1/|r|$ (as this has a higher rate of divergence than $1/|r|\ln|r|$). Since we have already restricted μ and ν to be either r or b then in this case the dominant contribution from the matrix (5.50) is the $\alpha = b$, $\beta = b$ term, and the leading order contribution from (5.65) is

$$\sum_{\theta=r,b} m^{-1}_{b\bar{b}}\rho^{-2}\bar{\psi}_{\bar{b}} \circ \partial_b \psi_\theta = \sum_{\theta=r,b} \frac{\lambda^2}{\pi}\rho^{-2}\bar{\psi}_{\bar{b}} \circ \partial_b \psi_\theta \tag{5.67}$$

The dominant contributions from the first and third inner products in (5.64) may be found by using the same considerations with which we found (5.61) and (5.63). Thus these contributions occur when $\beta = \kappa = r$. Collating all three inner products gives

$$\begin{aligned} &\sum_{\theta=r,b} \bar{m}^{-1}_{\bar{r}r}(\rho^{-2}\psi_r \circ \bar{\varphi}_0) m^{-1}_{b\bar{b}}(\rho^{-2}\bar{\psi}_{\bar{b}} \circ \partial_b\psi_\theta) m^{-1}_{\theta\bar{r}}(\rho^{-2}\bar{\psi}_{\bar{r}}(x) \circ \varphi_0) \\ \sim\; & \frac{\lambda^2}{\pi}\left(\frac{\zeta^2\lambda^2}{\pi|c|^2\ln|r|^2}\right)^2 \left(\rho^{-2}\bar{\psi}_{\bar{b}} \circ \partial_b\left[\psi_r - \frac{|c|^2}{\zeta}\psi_b\right]\right)(\rho^{-2}\psi_r \circ \bar{\varphi}_0)(\rho^{-2}\bar{\psi}_{\bar{r}}(x) \circ \varphi_0) \end{aligned} \tag{5.68}$$

The inner product $\left(\rho^{-2}\bar{\psi}_{\bar{b}} \circ \partial_b\left[\psi_r - |c|^2/\zeta\psi_b\right]\right)$ is infra-red finite and so may be computed on the plane. Using (5.51)

$$\begin{aligned} &- \int d^2z\, \frac{|z-b|^4}{(|z-b|^2 + |c|^2|z-b-r|^2)^2}\, \frac{\bar{c}\bar{r}}{(\bar{z}-\bar{b})^2}\, \partial_b\left[-\frac{c}{z-b} + \frac{rc}{(z-b)^2}\frac{|c|^2}{\zeta}\right] \\ =\; & |c|^2 \int d^2z\, \frac{|z-b|^4}{(|z-b|^2 + |c|^2|z-b-r|^2)^2}\, \frac{\bar{r}}{(\bar{z}-\bar{b})^2}\left[\frac{1}{(z-b)^2} - \frac{2r|c|^2}{(z-b)^3\zeta}\right] \\ =\; & |c|^2 \int d^2z\, \frac{\bar{r}}{(|z-b|^2 + |c|^2|z-b-r|^2)^2}\left[1 - \frac{2r|c|^2}{(z-b)\zeta}\right] \end{aligned} \tag{5.69}$$

This integral may be simplified by making a translation of z through b and then scaling by r, i.e. we define $z' = (z-b)/r$, substitute this into (5.69) and then drop the primes, leaving

$$\frac{|c|^2}{r}\int d^2z\,\frac{1}{(|z|^2+|c|^2|z-1|^2)^2}\left(1-\frac{2|c|^2}{\zeta z}\right)\equiv\frac{I(c)}{r} \tag{5.70}$$

The integral $I(c)$ is just a function of c. Consider the limiting behaviour of $I(c)$ for small c, in this case $2|c|^2/\zeta z$ will be dominated by the 1, leaving $I(c)$ as a finite integral. $I(c)$ may also be evaluated for fixed values of c.

The leading order term in $\mathcal{J}$ is now, using (5.64), (5.68) and (5.70), and remember that $r=\epsilon e^{i\theta}$

$$\begin{aligned}
\mathcal{J} &= \int d^2b \oint d\theta\, K^1\epsilon^{-1}e^{i\theta}\frac{\lambda^6}{\pi^3}\left(\frac{\zeta^2}{|c|^2\ln|r|^2}\right)^2\frac{I(c)}{r}(\rho^{-2}\psi_r\circ\bar{\varphi}_0)(\rho^{-2}\bar{\psi}_{\bar{r}}(x)\circ\varphi_0)\\
&= 2K^1\frac{\zeta^4}{\pi^2|c|^4r^2(\ln|r|^2)^2}\,I(c)\int d^2b\,\lambda^6(\rho^{-2}\psi_r\circ\bar{\varphi}_0)(\rho^{-2}\bar{\psi}_{\bar{r}}(x)\circ\varphi_0)
\end{aligned} \tag{5.71}$$

This diverges as $r\to 0$. So this is our proposed modification to the Green's function in the zero-instanton sector which will cancel the divergence in the one-instanton sector. In this form (5.71) is non-local, but it can be generated by a local action if we include additional fields [3].

(5.71) will be completely defined on the plane if we take $\lambda=1$. However if we wish to work on the sphere then the high index of λ could cause a problem. First, though, we must compactify the rest of the terms in $\mathcal{J}$ onto the sphere as ζ_1 was derived on the plane and we specifically removed the S^2 metric from the inner products. The compactification may be done by means of a conformal transformation, we shall show that this leads to a damping factor which suppresses the infra-red divergence from λ^6.

5.2 Conformal Invariance

We have the problem that there is initially an infra-red divergence in the modification term (5.71) when it is applied to a sphere. This divergence is due to the high index on the modulus $|b|$. However the remainder of the terms in $\mathcal{J}$ are defined on the plane and need to be compactified onto the sphere. As in the analogous problem in Yang-Mills Theory [4], the compactification will be done by performing a Weyl transformation on the metric. Thus the metric on the plane $ds^2 = d\bar{z}dz$ and on the sphere, $ds^2 = \Omega^2 d\bar{z}dz$ with $\Omega = 1 + \bar{z}z/h^2$, where h is the radius of the sphere, are related by a Weyl transformation, $g_{\mu\nu} \rightarrow e^{p(x)}g_{\mu\nu}$ with $p = ln\,\Omega^2$. This can be built out of infinitesimal transformations $\delta_p g_{\mu\nu} = pg_{\mu\nu}$. The field w and quasi-ghost ξ are independent of the metric, and the classical action $S[w]$ is Weyl invariant. Thus the Green's Function moduli density (4.51)

$$g(t) = \int \mathcal{D}W\; e^{-\left(S[w]+(\varsigma+\tau^A\partial_A)(\xi^j F_j)\right)}\; \Lambda(w) \tag{5.72}$$

changes under the transformation δ_p such that

$$\delta_p g(t) = \int \mathcal{D}W\; e^{-S_{tot}} \left[\left(\int p\mathcal{M} - (\varsigma + \tau^A\partial_A)(\xi^j\; \delta_p F_j)\right)\; \Lambda(w) + \delta_p\Lambda(w)\right] \tag{5.73}$$

where the first term is from the action of δ_p on the volume element and $\mathcal{M}$ is the Weyl anomaly density. The last term can be removed provided $\delta_p\Lambda(w) = 0$ and as $(\varsigma + \tau^A\partial_A)\Lambda(w) = 0$ (ς doesn't act on w and w is independent of the t) then

$$\delta_p g(t) = \int \mathcal{D}W\; e^{-S_{tot}}\Lambda(w) \int p\mathcal{M} - \int \mathcal{D}W\; (\varsigma + \tau^A\partial_A)e^{-S_{tot}}(\xi^j\; \delta_p F_j)\Lambda(w) \tag{5.74}$$

However, the second term does not contribute at the one loop level. To see this, first note that as $\int \mathcal{D}W\; e^{-S_{tot}}\; (\xi^j\delta_p F_j)\; \Lambda(w)$ is Grassmann odd, then

$$\int \mathcal{D}W\; \varsigma\; e^{-S_{tot}}\; (\xi^j\delta F_j)\; \Lambda(w) = 0 \tag{5.75}$$

Also, to first order in the expansion of $e^{-S_{tot}}$, the remaining piece in the second term is linear in φ, giving an expectation value of zero. Hence terms in (5.73) from the constraint piece in the action do not contribute at the one loop level and we can write $\delta_p g(t)$ as

$$\delta_p \int \mathcal{D}W\; e^{-S[w]} = \int \mathcal{D}W\; e^{-S[w]} \int p\mathcal{M} \tag{5.76}$$

Thus the behaviour of the Green's Function on the sphere is governed, to one loop, by the Weyl anomaly.

The calculation of the partition function involves the calculation of det Δ. However Δ has dimensions of mass so it is necessary to introduce an arbitrary parameter μ which also has dimensions of mass. To compensate for introducing μ the coupling constant becomes a function of μ, $k \rightarrow k(\mu)$. This means that the classical scale invariance of the coupling has been broken. Now an infinitesimal global scaling of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \lambda g_{\mu\nu}$, can be compensated by a shift in the mass-scale $\delta_\lambda \mu = -\frac{1}{2}\lambda\mu$ and $\delta_\lambda = \delta_\lambda \mu \, \frac{\partial}{\partial \mu} = -\frac{1}{2}\lambda\mu \, \frac{\partial}{\partial \mu}$. Thus the action $S(w)$ is no longer scale invariant either, since it contains the coupling. Using $S(w)$ from (2.52) so that now $S(w) = \frac{4}{k(\mu)} \int d^2x \hat{S}(w)$ then

$$\begin{aligned} \delta_\lambda S(w) &= -2\lambda \, \mu \frac{\partial}{\partial \mu} \left(\frac{1}{k(\mu)} \int d^2x \hat{S}(w) \right) \\ &= 2\lambda \, \mu \, \frac{1}{k(\mu)^2} \frac{\partial k(\mu)}{\partial \mu} \int d^2x \hat{S}(w) \end{aligned} \tag{5.77}$$

Here we have neglected moduli independent curvature terms which would be a relic of moving the fluctuation operator determinant from the plane to the sphere. The renormalisation group β-function for the O(3) Sigma Model is given by $\beta = \mu \frac{\partial k}{\partial \mu}$, and has been found (3.39) to be $\beta = -\frac{k^2}{4\pi}$. Considering a variation of the Green's Function with respect to this scaling

$$\delta_\lambda \int \mathcal{D}W \; e^{-S[w]} = -\int \mathcal{D}W \; e^{-S[w]} \; \delta_\lambda S[w] \equiv \int \mathcal{D}W \; e^{-S[w]} \int \lambda \mathcal{M} \tag{5.78}$$

yields

$$\int \lambda \mathcal{M} = -2\frac{\lambda}{k^2}\beta \int d^2x \hat{S}(w) \tag{5.79}$$

Hence

$$\mathcal{M} = -2\frac{\beta}{k^2}\hat{S}(w) = \frac{1}{2\pi}\hat{S}(w) \tag{5.80}$$

up to total derivatives. As we now know $\mathcal{M}$, the Green's Function can be evaluated on the sphere using the position dependent scaling p.

If $g_h(t)$ is the Green's Function moduli density and $\mathcal{D}_h W$ the functional integral volume element when the sigma model has as its space-time a sphere of radius h, then the Green's Function on that sphere in a particular topological sector is

$$\mathcal{G}_h^q = \int dt\ g_h(t) = \int dt\ \mathcal{D}_h W\ e^{-S_{tot}} \Lambda(w) \tag{5.81}$$

as $p = \ln \Omega^2$ then $\delta p = \delta h\ \frac{d}{dh} \ln \Omega^2$ and to one loop (from (5.76))

$$\begin{aligned} \delta_p \int dt\ g_h(t) &= \int dt\ \mathcal{D}_h W\ e^{-S_{tot}} \left(\int \delta p \mathcal{M} \right) \Lambda(w) \\ &= \int dt\ \mathcal{D}_h W\ e^{-S_{tot}} \left(\int \delta h \left(\frac{d}{dh} \ln \Omega^2 \right) \mathcal{M} \right) \Lambda(w) \end{aligned} \tag{5.82}$$

so

$$\frac{d}{dh} \int dt\ g_h(t) = \int dt\ \mathcal{D}_h W\ e^{-S_{tot}} \left(\int \left(\frac{d}{dh} \ln \Omega^2 \right) \mathcal{M} \right) \Lambda(w) \tag{5.83}$$

Integrating with respect to h from h to infinity we obtain

$$\begin{aligned} \int dt\ g_h(t) &= \int dt \int_h^\infty dh\ \mathcal{D}_h W\ \frac{d}{dh} \left[\exp \left(-S_{tot} + \int d^2x \ln \left(\frac{\Omega^2}{2} \right) \mathcal{M} \right) \right] \Lambda(w) \\ &= \int dt\ \mathcal{D}_\infty W\ \left[\exp \left(-S_{tot} + \int d^2x \ln \left(\frac{\Omega^2}{2} \right) \mathcal{M} \right) \right] \Lambda(w) \end{aligned} \tag{5.84}$$

The extra term involving the Weyl anomaly suppresses the divergence due to the integral over b in (5.71). Evaluating $\mathcal{M}$ at the classical solution $w = v$ gives $\hat{S}(w) = q'$ (see (3.1)) where q' is the topological charge density. For small $a - b$ the charge becomes concentrated at $z = b$ so that q' is approximately a delta-function. To see this remember that from (2.53) the classical form for q' on the sphere is

$$q' = \frac{\partial_z v \partial_{\bar{z}} \bar{v} - \partial_{\bar{z}} v \partial_z \bar{v}}{\pi (1 + |v|^2)^2} \tag{5.85}$$

The second part of this expression for q' can be expressed as a delta function as

$$\partial_{\bar{z}} v = c \partial_{\bar{z}} \left(\frac{z - a}{z - b} \right) \sim \partial_{\bar{z}} \partial \ln(z - b) \propto \delta(z - b) \tag{5.86}$$

However this is insufficient as there is still a term in q' left over. Instead we introduce a delta function regulator η such that

$$\lim_{\eta\to 0}\frac{\bar{z}-\bar{b}}{|z-b|^2+\eta^2}=\frac{1}{z-b} \tag{5.87}$$

Now with

$$v=c\left(\frac{(z-a)(\bar{z}-\bar{b})}{|z-b|^2+\eta^2}\right) \tag{5.88}$$

q' may be calculated

$$\lim_{\eta\to 0} q'=\frac{|c|^2|\epsilon|^2}{\left(|z-b|^2+|c|^2|z-b-\epsilon|^2\right)^2} \tag{5.89}$$

where $a-b=\epsilon$ and ϵ is very small, but, to start with, finite. For the case of $z\neq b$ we find that $q'\to\frac{|c|^2|\epsilon|^2}{(1+|c|^2)^2|z-b|^4}$ tends to zero as we take ϵ to zero. But if $z=b$ then $q'\to\frac{1}{|c|^2|\epsilon|^2}$ which tends to infinity as $\epsilon\to 0$. Thus $q'\propto\delta(z-b)$. Hence the additional contribution to the action due to the Weyl anomaly is

$$\begin{aligned}\int d^2x\,\ln\left(\Omega^2\right)q' &= \int d^2x\,\ln\left(\frac{2}{1+\frac{z^2}{h^2}}\right)^2\delta(z-b)\\ &= \int d^2x\,\ln\left(\frac{2}{1+\frac{b^2}{h^2}}\right)^2\end{aligned} \tag{5.90}$$

which, for large b, will supply a strong damping factor for the b-integration of Green's Functions and solves the problem of the infra-red b-integration in (5.71).

5.3 Appendix A: Fluctuation operator and Green's Function

A vital component in the calculation of the topological counter-term is the nature of the Green's Function of the fluctuation operator (3.8). In this Appendix we shall show how this Green's Function and the projection operators are constructed.

It will be useful to write the fluctuation operator

$$\Delta f = -\frac{1}{\sqrt{g}}\,\rho\,\partial_z\left[\rho^{-2}\partial_{\bar{z}}(\rho f)\right] \tag{5.91}$$

in the form $\Delta = T^\dagger T$, where $T^\dagger$ is the adjoint of T with respect to our inner product (3.6). T and $T^\dagger$ are fairly arbitrary, but a convenient construction is $T = \frac{1}{g^{\frac{1}{4}}}\partial_{\bar{z}}$ and $T^\dagger = -\frac{1}{\sqrt{g}}\rho^2\partial_z g^{\frac{1}{4}}\rho^{-2}$. The Green's Function of Δ is the two-point Green's Function for φ, and satisfies

$$\Delta\mathcal{I}(x,y) = \frac{\rho^2}{\sqrt{g}}\,\delta^2(x-y) - \rho^2 P(x,y) \tag{5.92}$$

where $P(x,y)$ is a projection operator. The form of $P(x,y)$ may be deduced by looking at the action of specific operators on (5.92). If $\mathcal{Z}(x)$ is a zero mode of T (and again we use a dot notation to indicate an inner product as in (5.16)), then $(\mathcal{Z},\Delta\mathcal{I}) = (\rho^{-2}\bar{\mathcal{Z}})(x)\circ\Delta\mathcal{I}(x,y) = (T\mathcal{Z},T\mathcal{I}(x,y)) = 0$ so $\bar{\mathcal{Z}}(x)\circ P(x,y) = \bar{\mathcal{Z}}(y)$. Also, φ is constrained by our choice of F (5.2) to be orthogonal to ψ. So the two point function of φ and $\varphi.\psi$ must vanish, hence $\mathcal{I}(x,y)\circ(\rho^{-2}\psi)(y) = 0$ which leads us to $P(x,y)\circ(\rho^{-2}\psi)(y) = (\rho^{-2}\psi)(x)$. Consequently, as $\bar{m}^{-1}_{\bar{\beta}\alpha} = \bar{\mathcal{Z}}_{\bar{\beta}}\circ(\rho^{-2}\psi_\alpha)(x)$, we can deduce that

$$P(x,y) = (\rho^{-2}\psi_\alpha)(x)\bar{m}^{-1}_{\bar{\beta}\alpha}\bar{\mathcal{Z}}_{\bar{\beta}}(y) \tag{5.93}$$

We shall show that the one instanton sector Green's Function is

$$\mathcal{I}(x,y) = -\frac{1}{\pi^2}\int d^2z\,(1-P^\dagger\circ)\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ P) \tag{5.94}$$

with $P(x,y)$ as above, and $P^{\dagger}(x,y) = \mathcal{Z}_{\beta}(x) m^{-1}_{\beta\bar{\alpha}}(\rho^{-2}\bar{\psi}_{\bar{\alpha}})(y)$. The dot notation, as above indicates an inner product. So writing this out explicitly gives

$$\begin{aligned}
\mathcal{I}(x,y) &= -\frac{1}{\pi^2}\int d^2z \left[\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}\right. \\
&\quad - \int d^2x'\; P^{\dagger}(x,x')\sqrt{g}\;\frac{1}{x'-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}} \\
&\quad - \int d^2y'\;\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}'}\sqrt{g}\;P(y',y) \\
&\quad \left. - \int d^2x'\; d^2y'\; P^{\dagger}(x,x')\sqrt{g}\;\frac{1}{x'-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}'}\sqrt{g}\;P(y',y)\right]
\end{aligned}$$

The derivatives in Δ act only on the x variable, and as

$$\frac{\partial}{\partial\bar{x}}\frac{1}{x-z} = \pi\;\delta^2(x-z) \tag{5.95}$$

and

$$\frac{\partial}{\partial\bar{x}}P^{\dagger}(x,y) = 0 \tag{5.96}$$

then

$$\begin{aligned}
\frac{\partial}{\partial\bar{x}}\mathcal{I}(x,y) &= -\frac{1}{\pi}\int d^2z \left[\delta^2(x-z)\rho^2(z)\frac{1}{\bar{z}-\bar{y}}\right. \\
&\quad \left. - \int d^2y'\;\delta^2(x-z)\rho^2(z)\frac{1}{\bar{z}-\bar{y}'}\sqrt{g}\;P(y',y)\right] \\
&= -\frac{1}{\pi}\frac{\rho^2(x)}{\bar{x}-\bar{y}} + \frac{1}{\pi}\int d^2y'\rho^2(x)\frac{\sqrt{g}}{\bar{x}-\bar{y}'}P(y',y)
\end{aligned} \tag{5.97}$$

and so

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\rho^{-2}(x)\;\frac{\partial}{\partial\bar{x}}\;\mathcal{I}(x,y)\right) &= -\delta^2(\bar{x}-\bar{y}) \\
&\quad + \int d^2y'\;\delta^2(\bar{x}-\bar{y}')\sqrt{g}\;P(y',y) \\
&= -\delta^2(\bar{x}-\bar{y}) + \sqrt{g}\;P(x,y)
\end{aligned} \tag{5.98}$$

thus

$$-\frac{\rho^2(x)}{\sqrt{g}}\frac{\partial}{\partial x}\left(\rho^{-2}(x)\;\frac{\partial}{\partial\bar{x}}\;\mathcal{I}(x,y)\right) = \Delta\;\mathcal{I}(x,y) \tag{5.99}$$

Hence (5.94) is true.

In the zero instanton sector the nature of the projection operators changes completely. The Green's Function becomes

$$\mathcal{I}_0(x,y) = -\frac{1}{\pi^2}\int d^2z\,(1-\Pi^\dagger\circ)\frac{1}{x-z}\rho_0^2(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ\Pi) \tag{5.100}$$

where Π is the zero mode projector in the zero instanton sector

$$\Pi f = \rho_0^{-2}\frac{\int\sqrt{g}\,f}{\int\sqrt{g}}\ ,\quad \rho_0 = 1+|c|^2 \tag{5.101}$$

Π and P both project constant functions onto themselves since these are zero-modes in both sectors. Hence $\Pi\circ P = \Pi$. As the divergent effects we are concerned with occur at the boundary between the one instanton sector and the zero instanton sector, we need to find the relationship between $\mathcal{I}_0(x,y)$ and $\mathcal{I}(x,y)$. First notice that

$$(1-P^\dagger\circ)\,\mathcal{I}_0(x,y)\,(1-\circ P) = -\frac{1}{\pi^2}\int d^2z\,(1-P^\dagger\circ)\frac{1}{x-z}\rho_0^2(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ P) \tag{5.102}$$

Then for ϵ small

$$\rho = \rho_0 + O(\epsilon) \tag{5.103}$$

so

$$\mathcal{I}(x,y) = (1-P^\dagger\circ)\,\mathcal{I}_0(x,y)\,(1-\circ P)\times(1+O(\epsilon)) \tag{5.104}$$

5.4 Appendix B: Kähler Metric

Here we give a review of the calculation to find the matrix $m_{\alpha\bar{\beta}} = \left(\partial_{\bar{\alpha}}\bar{v}, \partial_{\bar{\beta}}\bar{v}\right)$ and its inverse, as given in the calculation to find the modification term at (5.49) and (5.50). It is shown that the matrix may be written in the form of a Kähler metric ([13],[15]), and then the Kähler potential is calculated for the case where v is the one instanton solution

$$v = c\left(1 - \frac{r}{z-b}\right) \tag{5.105}$$

and r is small.

Given our definition of the inner product, the matrix may be written as

$$m_{\alpha\bar{\beta}} = \int d^2x\sqrt{g}\frac{1}{(1+|v|^2)^2}\left(\frac{\partial\bar{v}}{\partial t^{\bar{\beta}}}\frac{\partial v}{\partial t^{\alpha}}\right) \tag{5.106}$$

where we define $\{t^{\alpha}\} = \{r, b, c\}$ and $\{t^{\bar{\alpha}}\} = \{\bar{r}, \bar{b}, \bar{c}\}$. To show that this may be written in the form of a Kähler metric

$$m_{\alpha\bar{\beta}} = \frac{\partial}{\partial t^{\alpha}}\frac{\partial}{\partial t^{\bar{\beta}}}\mathcal{K} \tag{5.107}$$

where $\mathcal{K}$ is the Kähler potential, we exploit the fact that v is meromorphic in the moduli. Thus $\partial_{\bar{\alpha}}v = 0$ except when $\alpha = b$ in which case $\partial_{\bar{b}}v = cr\pi\delta(z-b)$. Therefore, for $\alpha, \beta = r, c$

$$\begin{aligned}\frac{\partial}{\partial t^{\alpha}}\frac{\partial}{\partial t^{\bar{\beta}}}\int d^2x\,\sqrt{g}\,\ln\left(1+|v|^2\right) &= \frac{\partial}{\partial t^{\bar{\beta}}}\int d^2x\,\sqrt{g}\,\left(\frac{1}{(1+|v|^2)}\bar{v}\frac{\partial v}{\partial t^{\alpha}}\right) \\ &= \int d^2x\sqrt{g}\left(\frac{\partial\bar{v}}{\partial t^{\bar{\beta}}}\frac{\partial v}{\partial t^{\alpha}}\right)\frac{1}{(1+|v|^2)^2}\end{aligned} \tag{5.108}$$

as above. Now as the metric $g_{\mu\nu}$ for the sphere S^2_{phys} is

$$g_{\mu\nu} = \delta_{\mu\nu}\left(1+|z|^2\right)^{-2} \tag{5.109}$$

so that

$$\sqrt{g} = \left(1+|z|^2\right)^{-2} \tag{5.110}$$

thus

$$\mathcal{K} = \int d^2x \, \left(1 + |z|^2\right)^{-2} \ln\left(1 + |v|^2\right) \tag{5.111}$$

which for the one instanton solution

$$v = c\left(\frac{z-a}{z-b}\right) \tag{5.112}$$

becomes

$$\mathcal{K} = \int dx \, dy \, (1 + |z|^2)^{-2} \left[ln(|z-b|^2 + |c|^2|z-a|^2) - ln(|z-b|^2)\right] \tag{5.113}$$

If this form of the potential is used in (5.107) then the second logarithm will generate a delta function. This is because v is not annihilated by $\partial_{\bar{b}}$ and thus

$$\frac{\partial}{\partial b}\frac{\partial}{\partial \bar{b}} ln(|z-b|^2) = \frac{1}{\pi}\delta^2(z-b) \tag{5.114}$$

However, this problem can be solved by noticing that as

$$\frac{1}{(1+|v|^2)^2}\frac{\partial v}{\partial b}\frac{\partial \bar{v}}{\partial \bar{b}} = \frac{|c|^2|z-a|^2}{(|z-b|^2 + |c|^2|z-a|^2)^2} = \frac{\partial}{\partial b}\frac{\partial}{\partial \bar{b}}\left(ln(|z-b|^2 + |c|^2|z-a|^2)\right) \tag{5.115}$$

then the second logarithm is not needed for the calculation of (5.106) and it can simply be discarded leaving us with.

$$\mathcal{K} = \int dx \, dy \, (1 + |z|^2)^{-2} \left[ln(|z-b|^2 + |c|^2|z-a|^2)\right] \tag{5.116}$$

The act of discarding the second logarithm has the effect of making the potential analytic in the moduli space.

In order to solve (5.116) we start by using two tricks to put the integrand into an exponential form

$$\int_0^\infty d\alpha \, \alpha \, \exp\left(-\alpha(1 + |z|^2)\right) = (1 + |z|^2)^{-2} \tag{5.117}$$

and

$$ln(m) = \int_0^\infty \frac{dt}{t}\left(e^{-mt} - e^{-t}\right) \tag{5.118}$$

giving

$$\begin{aligned}\mathcal{K} &= \int d\alpha\, dt\, dx\, dy \frac{\alpha}{t} \Big[\exp\Big(-(x^2+y^2)(\alpha+At)+2xtB+2ytC-Dt-\alpha\Big) \\ &\qquad - \exp\Big(-\alpha(x^2+y^2)-\alpha-t\Big)\Big]\end{aligned} \tag{5.119}$$

where the complex numbers have been split into their real and imaginary parts such that

$$|z-b|^2+|c|^2|z-a|^2 = A(x^2+y^2)+2Bx+2Cy+D \tag{5.120}$$

so

$$A=(1+|c|^2),\ B=b_1+a_1|c|^2,\ C=b_2+a_2|c|^2,\ D=|b|^2+|a|^2|c|^2 \tag{5.121}$$

The integrals over x and y can be done as simple Gaussians as

$$\int d\zeta\ e^{-m\zeta^2+n\zeta+p} = e^{\frac{n^2}{4m}+p}\left(\frac{\pi}{m}\right)^{\frac{1}{2}} \tag{5.122}$$

giving

$$\mathcal{K} = \int d\alpha\, dt\, \frac{\alpha}{t}\left[\frac{\pi}{\alpha+At}\exp\left(\frac{(B^2+C^2)t^2}{\alpha+At}-(Dt+\alpha)\right)-\frac{\pi}{\alpha}\exp(-\alpha-t)\right] \tag{5.123}$$

To progress a change of variables is needed. Let $\alpha=\lambda t$ so $d\alpha = t\, d\lambda$, this change will have no effect on the limits of the integration. However the integral over t becomes very simple

$$\begin{aligned}\mathcal{K} &= \int d\lambda\, dt\, \pi\left[\frac{\lambda}{\lambda+A}\exp\left(t\left[\frac{(B^2+C^2)}{\lambda+A}-(D+\lambda)\right]\right)-\exp\left(-t\left(\lambda+1\right)\right)\right] \\ &= \int d\lambda\, \pi\left[\frac{\lambda}{\lambda^2+(A+D)\lambda+AD-(B^2+C^2)}-\frac{1}{\lambda+1}\right]\end{aligned} \tag{5.124}$$

The first term is just a standard integral of the form

$$\begin{aligned}\int_0^\infty \frac{\tau}{\zeta+\eta\tau+\theta\tau^2}d\tau &= \frac{1}{2\theta}ln(\zeta+\eta\tau+\theta\tau^2) \\ &\quad - \frac{\eta}{2\theta(\eta^2-4\zeta\theta)^{\frac{1}{2}}}ln\left(\frac{\eta+2\theta\tau-(\eta^2-4\zeta\theta)^{\frac{1}{2}}}{\eta+2\theta\tau+(\eta^2-4\zeta\theta)^{\frac{1}{2}}}\right)\end{aligned} \tag{5.125}$$

so

$$\begin{aligned}\mathcal{K} &= \pi\left[\frac{1}{2}\ln\left(\lambda^2+(A+D)\lambda+AD-(B^2+C^2)\right)-\ln(\lambda+1)\right.\\ &\left.-\frac{1}{2(1-4F^2)^{\frac{1}{2}}}\ln\left(\frac{A+D+2\lambda-(A+D)(1-4F^2)^{\frac{1}{2}}}{A+D+2\lambda+(A+D)(1-4F^2)^{\frac{1}{2}}}\right)\right]_0^\infty \end{aligned} \tag{5.126}$$

where we have set

$$F^2=\frac{AD-B^2-C^2}{(A+D)^2} \tag{5.127}$$

The behaviour of $\mathcal{K}$ for the $\lambda\to\infty$ solution is not immediately clear. In order to study this behaviour the logarithms must be expanded for large λ. A simple piece of algebraic manipulation gives

$$\begin{aligned}\mathcal{K} &= \pi\left[\frac{1}{2}\ln\left(1+\frac{(A+D)\lambda+AD-(B^2+C^2)}{\lambda^2}\right)-\ln\left(1+\frac{1}{\lambda}\right)\right.\\ &-\frac{1}{2(1-4F^2)^{\frac{1}{2}}}\left\{\ln\left(1+\frac{(A+D)\left(1-(1-4F^2)^{\frac{1}{2}}\right)}{2\lambda}\right)\right.\\ &\left.\left.-\ln\left(1+\frac{(A+D)\left(1+(1-4F^2)^{\frac{1}{2}}\right)}{2\lambda}\right)\right\}\right]_0^\infty\end{aligned}$$

So for $\lambda\to\infty$ we may use the logarithm series expansion. To first order this is just $\ln(1+x)\sim x$ for $|x|<1$, so the result is that to this order

$$\mathcal{K}\sim\pi\left[\frac{AD-(B^2+C^2)}{2\lambda^2}+\frac{1}{2\lambda}(A+D+1)\right]^\infty\to 0 \tag{5.128}$$

Hence we are left with the $\lambda=0$ term in (5.126) which may be written as

$$\mathcal{K}=\frac{\pi}{2}\left[\frac{1}{(1-4F^2)^{\frac{1}{2}}}\ln\left(\frac{1-(1-4F^2)^{\frac{1}{2}}}{1+(1-4F^2)^{\frac{1}{2}}}\right)-\ln(AD-B^2-C^2)\right] \tag{5.129}$$

Now let us consider what happens to this Kähler Potential as the zero instanton sector is approached from the one instanton sector. Again the change of variables $a-b=r$ is made (see (5.1)), where we are interested in the limit $r\to 0$. As $AD-B^2-C^2=|c|^2|a-b|^2=$

$|c|^2|r|^2$, thus $F^2 = |c|^2|r|^2/(A+D)^2$ is small and the approximation $ln(1-F^2) \sim F^2$ is valid. Terms of order $|r|^4$ can be dropped and

$$\mathcal{K} = \frac{\pi}{2}\left[\ln F^2 + 2F^2 \ln|r|^2 - \ln|c|^2|r|^2\right] \tag{5.130}$$

We shall try to isolate terms of order r^0 and r^1. The second term in (5.130) has been kept as it converges more weakly than the other terms in $|r|^2$. Combining the first and third terms in (5.130) gives

$$\mathcal{K} = \frac{\pi}{2}\left[-2\ln(A+D) + 2F^2\ln|r|^2\right] \tag{5.131}$$

But $A+D = \zeta\lambda + |c|^2(b\bar{r} + \bar{b}r + |r|^2)$ where $\lambda = (1+|b|^2)$, $\zeta = (1+|c|^2)$. So

$$\mathcal{K} = \pi\left[-\ln\zeta\lambda - \ln\left(1 + \frac{|c|^2(b\bar{r} + \bar{b}r + |r|^2)}{S}\right) + F^2\ln|r|^2\right] \tag{5.132}$$

Now the second term is expanded and the $|r|^2$ term dropped, leaving an expression for $\mathcal{K}$ with the two terms of lowest order in r.

$$\mathcal{K} = \pi\left[-\ln\zeta\lambda - \frac{|c|^2(b\bar{r} + \bar{b}r)}{S} + F^2\ln|r|^2\right] \tag{5.133}$$

This can now be differentiated to find $m_{\alpha\bar{\beta}}$. Remember that we are now working with coordinates $\{t^\alpha\} = \{r, b, c\}$ and $\{t^{\bar{\beta}}\} = \{\bar{r}, \bar{b}, \bar{c}\}$. As we are working to leading order, only terms divergent or constant in the limit $r \to 0$ will be kept. It is thus found that

$$m \equiv (m_{\alpha,\bar{\beta}}) \approx \frac{\pi}{\zeta^2\lambda^2}\begin{pmatrix} |c|^2\ln|r|^2 & \zeta|c|^2 & \lambda\bar{c}b \\ \zeta|c|^2 & \zeta^2 & 0 \\ \lambda\bar{b}c & 0 & \lambda^2 \end{pmatrix} \tag{5.134}$$

The inverse of this is found simply by using the cofactor method

$$(m_{\bar{\alpha}\beta})^{-1} = \frac{\zeta^2\lambda^2}{\pi\ln|r|^2}\begin{pmatrix} |c|^{-2} & -1/\zeta & -b/(c\lambda) \\ -1/\zeta & \ln|r|^2/\zeta^2 & \bar{c}b/(\lambda\zeta) \\ -\bar{b}/(\bar{c}\lambda) & \bar{b}c/(\lambda\zeta) & \ln|r|^2/\lambda^2 \end{pmatrix} \tag{5.135}$$

Chapter 6

Extending Topological Renormalisation

6.1 Introduction

In this chapter we shall consider how the ideas of topological renormalisation may be applied to other models and theories. Naturally, first consideration is given to those theories which share their properties with the $O(3)$ sigma model, such as Yang-Mills Theory and Bosonic String Theory. For these theories the gauge-field version of the basic topological renormalisation theory applies (see Section 4.2). These calculations are done in [4] and we shall present a review of the results later in this chapter.

As the $O(3)$ sigma model is also the $\mathbb{CP}^1$ model then another natural extension of topological renormalisation is to apply it to the $\mathbb{CP}^{n-1}$ models. For these models the formulation of the derivation of the anomalous symmetry breaking term is identical to that for the $O(3)$ sigma model as it is independent of the nature of the action. It is also required that the instanton contribution to the Green's Function for the $\mathbb{CP}^{n-1}$ models diverge at the instanton boundaries. We would expect this to be true as the instanton contribution given in (1.2) and (1.3) would have to be a special case of the $\mathbb{CP}^{n-1}$ contribution, and the divergence displayed in the $\mathbb{CP}^1$ case would have to be a feature of the more general case. However, if the exact form of the anomaly is to be calculated, then an explicit version of the $\mathbb{CP}^{n-1}$ contribution which diverges at the instanton boundaries must be found. The instanton contribution to the Green's Function for the $\mathbb{CP}^{n-1}$ models is calculated in [20], [30] and [31]. The version of this calculation we shall concentrate on is the one given in [20] as it follows a similar pattern to the calculation of the instanton contribution for the sigma model by the same authors [2], which was reviewed in Section 3. Also the solution given in [20] is the most complete.

First, we need to show that the instanton contribution does diverge when the instanton solutions degenerate. In the $\mathbb{CP}^{n-1}$ models the most general solutions which minimise the action are

$$u_k = c_k \prod_{\alpha=1}^{q} (z - a_k^{\alpha}) \quad , \quad k = 1, ..., n \tag{6.1}$$

where q is the topological charge. The $\mathbb{CP}^{n-1}$ space is defined as the space of non-zero complex vectors $u = (u_1, ..., u_n)$ where $\sum_k |u_k|^2 \neq 0$. Vectors that differ by a complex factor are identified with each other. i.e. if $u' = \lambda u$ where λ is an arbitrary non-zero complex number, then u and u' may be identified. To see how these solutions degenerate, let us look at the two-instanton $(q = 2)$

$$u_k = c_k(z - a_k^1)(z - a_k^2) \tag{6.2}$$

For the $O(3)$ sigma model we saw that the instanton degenerated when two of the moduli became equal. Here we consider what happens when the parameters a_k^1 are identified with a single complex number b, so $a_k^1 = b$. In this case

$$u_k = (z - b)c_k(z - a_k^2) \tag{6.3}$$

but we have already stated that solutions that differ by a complex factor may be identified, hence the $(z - b)$ factor may be dropped and u_k has become the one-instanton solution.

In [20] it is found that the instanton contribution to the Green's function, in the one-instanton sector, for the $\mathbb{CP}^{n-1}$ model is

$$I_1 = \Lambda \int \Phi(c, a)\, \delta\left(\sum_j |c_j|^2 - 1\right) \left(\sum_{j<k} |c_j|^2 |c_k|^2 |a_j - a_k|^2\right)^{-\frac{n}{2}} \prod_j |c_j|^2 d^2c_j d^2a_j \tag{6.4}$$

so if *all* the $a_k = b$ then the $|a_j - a_k|^n$ term gives a strong divergence.

Now we shall look at how this fits into the basic formalism of the CP^{n-1} Models.

6.2 CP^{n-1} Models

Originally devised by Eichenherr [32] and Golo and Perelomov [33], these models are special class of sigma models as they have an $SU(n)$ symmetry. D'Adda et al [34] showed that they gave a particle model which has many similarities with Yang-Mills theory.

Summaries of the connection between $\mathbb{CP}^{n-1}$ models and $O(3)$ sigma models on the plane and on the sphere are given in [13] and [20]. The $\mathbb{CP}^{n-1}$ models are a class of two-dimensional field theories where the target space is given by the Grassmann manifold

$$G(n) = \frac{U(n)}{U(1) \times U(n-1)} \tag{6.5}$$

which is equivalent to complex projective space $\mathbb{CP}^{n-1}$. This space consists of n-component complex column vectors $\zeta = (\zeta_1, \ldots\ldots, \zeta_n)$ which are orthonormal: $\zeta_i^\dagger \zeta_k = \delta_{ik}$, and $\zeta = \zeta(x)$ where $x = (x_1, x_2) \in \mathbb{R}^2$. The action is given by

$$S = \int d^2x \ \mathrm{Tr}(D_\mu \zeta)^\dagger D_\mu \zeta \tag{6.6}$$

$$D_\mu \zeta = \partial_\mu \zeta - \zeta A_\mu \qquad A_\mu = \zeta^\dagger \partial_\mu \zeta \tag{6.7}$$

We shall use the formalism given in [20] where the action is expressed as

$$S = \frac{2}{f} \int d^2x \ H_{jk} \partial_\mu \bar{u}_j \partial_\mu u_k \tag{6.8}$$

where f is the coupling and u is an n-component complex vector $u = (u_1, \ldots\ldots, u_n)$. The metric is

$$H_{jk} = \left(\sum_i |u_i|^2 \right)^{-1} \left(\delta_{jk} - \left(\sum_i |u_i|^2 \right)^{-1} \bar{u}_j u_k \right) \tag{6.9}$$

The action in this form is at its minima when

$$u_k = c_k \prod_{\alpha=1}^{q} (z - a_k^\alpha) \tag{6.10}$$

for a topological charge q. Hence we may define the inhomogeneous co-ordinates

$$w_k = \frac{u_k}{u_n} \qquad k = 1, \ldots\ldots, n-1 \tag{6.11}$$

then (6.8) and (6.9) may be rewritten in terms of w

$$S = \frac{2}{f} \int d^2x \ \tilde{H}_{jk} \partial_\mu \bar{w}_j \partial_\mu w_k \tag{6.12}$$

$$\tilde{H}_{jk} = \left(1 + \sum_i |w_i|^2 \right)^{-1} \left(\delta_{jk} - \left(1 + \sum_i |w_i|^2 \right)^{-1} \bar{w}_j w_k \right) \tag{6.13}$$

Despite the obvious similarities between (6.13) and (2.5), there is only a direct equivalence between a $\mathbb{C}\mathbb{P}^{n-1}$ model and an $O(n+1)$ sigma model when $n = 2$. In this case there is only a single field

$$w = \frac{u_1}{u_2} = \frac{c_1 \prod_{\alpha=1}^{q}(z - a_1^\alpha)}{c_2 \prod_{\alpha=1}^{q}(z - a_2^\alpha)} \tag{6.14}$$

from which (1.1) follows, and the action (6.13) reduces to (2.52). For the original $\mathbb{C}\mathbb{P}^{n-1}$ action (6.6), this equivalence can be seen simply by setting

$$\sigma^a = z^\dagger \beta^a z \tag{6.15}$$

where β^a are the Pauli matrices.

Now the moduli-space integral (6.4) needs to be regulated, and initially the procedure is identical to that of the $O(3)$ sigma model, i.e. introduce a cut-off which creates a boundary ∂M in the moduli-space M. The divergent term resulting from taking the variation of the Green's function is (cf (4.58))

$$\delta \mathcal{G}_q = -\int_{\partial M} d\Sigma_A \int \mathcal{D}U \; \tau^A(\xi^j \delta F_j) e^{-S_{tot}} \; \Lambda(u), \tag{6.16}$$

where $U = u, \lambda, \xi$, and τ and ξ are the quasi-ghost and quasi-anti-ghost respectively. The action is (cf (4.49))

$$S_{tot} = S[u] + (\varsigma + \tau^A \partial_A)(\xi^j F_j(u,t)) \tag{6.17}$$

$\partial_A = \partial/\partial t^A$ where the t^α in the one-instanton sector are the moduli c_k and a_k, and the $t^{\bar{\alpha}}$ are their conjugates. A quantum fluctuation about the classical solution may be defined such that the field is now $w_k = u_k + \phi_k$ and the action is

$$S[\phi] = \frac{4\pi q}{f} + \frac{8}{f}(\phi, \Delta\phi) \tag{6.18}$$

The inner product is defined as

$$(\chi, \psi) = \sum_{j,k} \int d^2x \; \sqrt{g} \; H_{jk} \bar{\chi}_j \psi_k = \sum_{j,k} H_{jk} \bar{\chi}_j \circ \psi_k \tag{6.19}$$

with the fluctuation operator

$$\Delta_{kl} = -\frac{1}{\sqrt{g}}\eta^2 \partial_z H_{kl} \partial_{\bar{z}} \tag{6.20}$$

and $\eta^2 = \sum_i |u_i|^2$. We must now choose our constraint F_j. In [20] it states that the quantum fluctuations may be fixed by imposing the condition

$$\sum_k \bar{u_k}\phi_k = 0 \tag{6.21}$$

This is a gauge condition designed to gauge fix the freedom we have to multiply all the fields u_k by the same function. In addition we require that the ϕ_k are orthogonal to the zero modes of the fluctuation operator. The zero modes of Δ_{kl} are $\partial u_j/\partial t^A$ for the same reasons as outlined with (4.7) and (4.8). We shall impose this orthogonality condition in the form

$$F_\alpha = \left(\frac{\partial u}{\partial t^\alpha}, \phi\right) = 0 = \left(\phi, \frac{\partial u}{\partial t^{\bar{\alpha}}}\right) = F_{\bar{\alpha}} \tag{6.22}$$

The calculation of the modification term (6.16) to find the leading order term follows the same pattern as the $O(3)$ sigma model calculation. After expanding the exponential in powers of Planck's constant and integrating out the ghosts and the quantum fluctuation we get

$$\begin{aligned}\delta\mathcal{G}_1 \;=\; & \int_{\partial M} d\Sigma_\alpha \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\mu}\beta}(\delta(H_{ij}\partial_\beta u_i)\circ\bar{\boldsymbol{\varphi}}_j) m^{-1}_{\alpha\bar{\nu}}(\partial_{\bar{\mu}}(H_{kl}\partial_{\bar{\nu}}\bar{u}_k)\circ\boldsymbol{\varphi}_l)\ \Lambda(v)\right) \\ & - \int_{\partial M} d\Sigma_{\bar{\alpha}} \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\alpha}\beta}(\delta(H_{ij}\partial_\beta u_i)\circ\bar{\boldsymbol{\varphi}}_j) m^{-1}_{\mu\bar{\nu}}(\partial_\mu(H_{kl}\partial_{\bar{\nu}}\bar{u}_k)\circ\boldsymbol{\varphi}_l)\ \Lambda(v)\right)\end{aligned} \tag{6.23}$$

Again the boldface type indicates contraction between those fields. $\zeta_1(t)$ is the one-loop partition function and is the integrand of (6.4) without the arbitrary function $\Phi(c, a)$. The matrices $m_{\alpha\bar{\beta}}$ are given by

$$m_{\alpha\bar{\beta}} = (\partial_\alpha u_l, \partial_\beta u_k) = H_{kl}\partial_{\bar{\beta}}\bar{u}_k \circ \partial_\alpha u_l \tag{6.24}$$

We need to find out if the correction in the one-instanton sector will again come from the zero-instanton sector. To do this we may consider a similar procedure to that outlined in Appendix A of Chapter 5. Thus we define a Green's function $\mathcal{I}_{kl}(x,y) = \bar{\varphi}_k \varphi_l$ of the operator Δ_{kl} such that

$$\Delta_{kl}\mathcal{I}_{kl}(x,y) = \frac{1}{\sqrt{g}}\eta^2\, \delta^2(x-y) - \eta^2 P(x,y) \tag{6.25}$$

The form of $P(x,y)$ needs to be found. Splitting Δ_{kl} into the form $\Delta_{kl} = (T^\dagger T)_{kl}$ may be done by

$$T_{kl} = g^{-\frac{1}{4}}\left(\delta_{kl} - \eta^{-2}\bar{u}_k u_l\right)\partial_{\bar{z}} \quad , \quad T^\dagger = -g^{-\frac{1}{2}}\eta^2\partial_z\eta^{-2}g^{\frac{1}{4}} \tag{6.26}$$

Hence, if $\mathcal{Z}_j(x)$ is a zero mode of T, then $(\mathcal{Z}, \Delta\mathcal{I}) = 0$ implies that $\bar{\mathcal{Z}}_j(x) \circ P(x,y) = \bar{\mathcal{Z}}_j(y)$. Also ϕ_k is orthogonal to $\partial_A u_j$ so the two-point function of ϕ_k and $\phi_k \cdot \partial_A u_j$ must vanish. Thus $\mathcal{I}_{jk}(x,y) \circ H_{kl}\partial_A u_j(y) = 0$ which means that $P(x,y) \circ H_{kl}\partial_A u_j(y) = H_{kl}\partial_A u_j(x)$ so

$$P(x,y) = \left(H_{kl}\partial_\beta u_k\right)(x)\, \bar{m}^{-1}_{\bar{\alpha}\beta}\bar{\mathcal{Z}}_{k\bar{\alpha}} \tag{6.27}$$

Due to the similarities between this case and that of the $O(3)$ sigma model an obvious choice for $\mathcal{I}_{kl}(x,y)$ is

$$\mathcal{I}_{kl}(x,y) = -\frac{1}{\pi^2}\int d^2z\,(1 - P^\dagger\circ)\frac{1}{x-z}{H_{kl}}^{-1}(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ P) \tag{6.28}$$

It can be shown that this satisfies (6.25). Now we need to find the relationship between $\mathcal{I}_{kl}(x,y)$ and the Green's function in the zero-instanton sector $\mathcal{I}^0_{kl}(x,y)$. In this sector $w_k = c_k$ so

$$\begin{aligned} H^0_{kl} &= \left(\sum_i |c_i|^2\right)^{-1}\left(\delta_{kl} - \left(\sum_i |c_i|^2\right)^{-1} c_k\bar{c}_k\right) \\ &= \eta_0^{-2}\left(\delta_{kl} - \eta_0^{-2}c_k\bar{c}_k\right) \end{aligned} \tag{6.29}$$

thus

$$\mathcal{I}^0_{kl}(x,y) = -\frac{1}{\pi^2}\int d^2z\,(1-\Pi^\dagger\circ)\frac{1}{x-z}\left(H^0_{kl}\right)^{-1}(z)\frac{1}{\bar{z}-\bar{y}}(1-\circ\Pi) \tag{6.30}$$

where Π is the zero mode projector in the zero instanton sector

$$\Pi f = H^0_{kl}\frac{\int \sqrt{g}\, f_{kl}}{\int \sqrt{g}} \tag{6.31}$$

and $\Pi \circ P = \Pi$ as before. The one instanton (6.10) degenerates when $a_k = b + \epsilon_k$ so for small ϵ_k

$$H_{kl} \approx H^0_{kl} \times (1 + \epsilon_k) \tag{6.32}$$

which means that

$$\mathcal{I}_{kl}(x,y) = (1 - P^\dagger \circ)\, \mathcal{I}^0_{kl}(x,y)\, (1 - \circ P) \times (1 + O(\epsilon)) \tag{6.33}$$

and so (6.23) may be expressed in the zero instanton sector.

There is a lot more work to be done in this calculation and it is probable that an answer as concise as that for the $O(3)$ sigma model may be found. The transfer of (6.23) to the zero-instanton sector may be studied and the matrices (6.24) should be calculable. There is a lot of scope for further work here.

6.3 Bosonic String Theory

We shall use a formulation of String Theory where the fields g_{ab} and x^μ are functions of the world-sheet co-ordinates ξ^a, and are the co-ordinates of a surface embedded in D-dimensional space-time with metric $\eta_{\mu\nu}$. The Partition Function for closed strings is given by a sum of functional integrals over closed Riemann surfaces of genus h weighted by a power of the coupling κ [35]

$$Z = \sum_{h=0}^{\infty} \kappa^{2-2h} \int_h \mathcal{D}g_{ab}\, \mathcal{D}x^\mu\, e^{-S[g_{ab},x^\mu]} \tag{6.34}$$

where the action is

$$S[g_{ab}, \xi^\mu] = \frac{1}{2} \int d^2\xi\, \sqrt{g}\, g^{ab} \partial_a \xi^\mu \partial_b \xi^\nu\, \eta_{\mu\nu} \tag{6.35}$$

where $\partial_a = \partial / \partial \xi^a$. After integrating out the fields the one string loop partition function becomes an integral over the complex modulus τ

$$Z_1 = 4 \int_F d^2\tau\, (\mathrm{Im}\,\tau)^{-2} \left(\frac{1}{2}\mathrm{Im}\,\tau\right)^{-12} e^{4\pi \mathrm{Im}\,\tau} \left| \prod_{n=1}^{\infty} \left(1 - e^{2n\pi i \tau}\right) \right|^{-48} \tag{6.36}$$

the domain F is taken to be: $-\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2}$, $\mathrm{Im}\,\tau > 0$, $|\tau| \geq 1$. This integral diverges as $\mathrm{Im}\tau \to \infty$, and this divergence is interpreted as being infra-red. In string theory it is these divergences which we are trying to regulate. However in this case it is more convenient to look at the divergences in terms of the Ward Identities. If we define the operator A to be

$$A = \int d^2\xi\; b^{rs}\partial_r x^\mu \partial_s x^\nu\; l_{\mu\nu} e^{ik_0.x} \left(\prod_{i=1}^{n-1} \int d^2\xi\; A_i \right) \epsilon_{ab} c^a c^b A_n \tag{6.37}$$

then after a lot of calculation we get the result that to one-string-loop order

$$\langle\langle A \rangle\rangle = \kappa^{(n-6)} \langle A(1 - \kappa^6 \Lambda)\rangle_0 + \kappa^n \langle A \rangle_1 \tag{6.38}$$

where Λ is interpreted as a counter-term and can be seen as a modification of the action as $S_{tot} \to S_{tot} - \kappa^6\Lambda$. With $\langle\langle A\rangle\rangle$ now constructed to be $\langle\langle A\rangle\rangle = 0$. The topological renormalisation of String Theory is thus analogous to the sigma model in the way that the expectation values are corrected by counter-terms at each order in the expansion.

6.4 Yang-Mills Theory

Let us now look at the problem of moduli space divergences in Yang-Mills Theory. The partition function in Euclidean space-time with metric $g_{\mu\nu}$ is

$$Z = \sum_{n=-\infty}^{\infty} e^{-in\theta} \int_n \mathcal{D}A\; e^{-S_{YM}} \tag{6.39}$$

$$S_{YM} = -\frac{1}{4g^2} \int d^4x\; \sqrt{g}\; g^{\mu\rho} g_{\nu\kappa} \mathrm{tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}_{\rho\kappa}) \tag{6.40}$$

where n is the topological charge, the field strength is $\boldsymbol{F}_{\mu\nu} = [\partial_\mu + \boldsymbol{A}_\mu, \partial_\nu + \boldsymbol{A}_\nu]$. We shall take the gauge potential to be an element of the Lie algebra $su(N)$, such that $\boldsymbol{A} = A^a T_a$, $T_a^\dagger = -T_a$, $[T_a, T_b] = f_{ab}^c T_c$ and $\mathrm{tr}(T_a T_b) = -\delta_{ab}$. The theta angle plays the role of a coupling [6]. Introducing a mass scale into the quantisation of the theory breaks the conformal invariance of the classical equations of motion and produces a Weyl anomaly. This means that the measure $\mathcal{D}A$ is not conformally invariant.

The one instanton solution is [5]

$$A_\mu^a = \eta_{\mu\nu}^a \frac{2(x-y)^\nu}{(x-y)^2+\rho^2} \equiv \mathcal{A}_1(x;y,\rho) \tag{6.41}$$

$(x-y)^2 = (x^\mu - y^\mu)(x^\mu - y^\mu)$ and the $\eta_{\mu\nu}^a$, $a = 1,2,3$ form a basis for anti-self-dual tensors. In a flat space-time, and with the constraints (4.10) and (4.11), to one-loop the partition function becomes [36]

$$Z_1 = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y \, d\rho}{\rho^5} \rho^{\frac{11N}{3}} \tag{6.42}$$

which diverges for large y^μ and ρ. This divergence due to the instanton modulus implies that the Ward Identity anomaly (4.67) applies in this case and thus the BRST invariance is broken. However, instead of calculating the anomaly, let us see what happens when the space-time is compactified to S^4.

For a spherical space-time of radius a the metric is given by

$$g_{\mu\nu} = \frac{4}{\left(1+\frac{x^2}{a^2}\right)^2}\delta_{\mu\nu} \tag{6.43}$$

However for short distances R^4 and S^4 look the same, thus for small y^μ and ρ the integrand of the partition function on the sphere is approximately the same as that on the plane. Now on the sphere there is an invariance of the metric corresponding to inversion through the centre of the sphere followed by a parity transformation to reinstate the original orientation. As a co-ordinate transformation this may be represented as

$$x^\mu \rightarrow \tilde{x}^\mu = \frac{a^2 m^\mu_\nu x^\nu}{x^2} \tag{6.44}$$

where $m = \mathrm{diag}(1,-1,-1,-1)$ and

$$\frac{1}{\left(1+\frac{\tilde{x}^2}{a^2}\right)^2} d\tilde{x}^2 = \frac{1}{\left(1+\frac{x^2}{a^2}\right)^2} dx^2 \tag{6.45}$$

Now this is a conformal transformation, and as the classical equations of motion of this theory are conformally covariant, thus all the classical solutions are linked by conformal transformations. Hence this transformation applied to the instanton solution will have the effect of changing the moduli

$$\mathcal{A}_\mu(\tilde{x}; y, \rho)d\tilde{x}^\mu = \mathcal{A}_\mu(x; \tilde{y}, \tilde{\rho})dx^\mu \ , \ \tilde{\rho} = \frac{a^2\rho}{y^2+\rho^2} \ , \ \tilde{y}^\mu = \frac{a^2 m^\mu_\nu y^\nu}{y^2+\rho^2} \tag{6.46}$$

The partition function is invariant under a simple change of co-ordinates. Thus, as argued above, for small $\tilde{y}^\mu$ and $\tilde{\rho}$ we may use

$$Z_1 = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4\tilde{y}\ d\tilde{\rho}}{\tilde{\rho}^5} \tilde{\rho}^{\frac{11N}{3}} \tag{6.47}$$

Furthermore the measure $d^4y\ d\rho/\rho^5$ of conformal transformations on the moduli. Thus $d^4y\ d\rho/\rho^5 = d^4\tilde{y}\ d\tilde{\rho}/\tilde{\rho}^5$ and so

$$Z_1 = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y\ d\rho}{\rho^5} \tilde{\rho}^{\frac{11N}{3}} = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y\ d\rho}{\rho^5} \left(\frac{a^2\rho}{y^2+\rho^2}\right)^{\frac{11N}{3}} \tag{6.48}$$

which, being true for small $\tilde{y}^\mu$ and $\tilde{\rho}$, is true for *large* y^μ and ρ. Consequently, when we are working on the sphere, the moduli space integration converges in the one instanton sector and there is no BRST anomaly.

So there is no moduli-space divergence in Yang-Mills theory in the one-instanton sector. However we do not know whether this is a special case or whether it is true to all orders of the expansion. Let us look at the two-instanton contribution to the partition function. In this sector the instanton solutions are

$$Z_2 = e^{-\frac{16\pi^2}{g^2(\mu)}} \left(\frac{4\pi}{g^2(\mu)}\right)^8 e^{-2\alpha(1)}\frac{4\pi}{3} \int \frac{d\lambda_1}{\lambda_1}\frac{d\lambda_2}{\lambda_2} d^4y_0 d^4y_1 d^4y_2\ W^4 \frac{N_A^{\frac{4}{3}}}{N_S^{\frac{1}{3}}} \sqrt{\Gamma} \tag{6.49}$$

$$\begin{aligned}
W &= z^3\lambda_1\lambda_2 \ , \ z^2 = \frac{1}{1+{\lambda_1}^2+{\lambda_2}^2} \\
N_A &= z^2\left({\lambda_2}^2(y_0-y_1)^2 + {\lambda_0}^2(y_1-y_2)^2 + {\lambda_1}^2(y_2-y_0)^2\right) \\
N_S &= W^2(y_0-y_1)^2(y_1-y_2)^2(y_2-y_0)^2 \\
\Gamma &= \Gamma\left((y_0-y_1)^2, (y_1-y_2)^2, (y_2-y_0)^2\right) \\
\Gamma(a,b,c) &= 2(ab+bc+ca) - a^2 - b^2 - c^2
\end{aligned}$$

There is a degeneracy to configurations of lower instanton number when $(y_i - y_j)^2 \rightarrow 0$ and the integrand does diverge in this limit. For instance as $y_2 \rightarrow y_1$ it behaves as $((y_1 - y_2)^2)^{-\frac{1}{3}}$. Also in this case the integrals are now taken over $y_0 - y_1$ and $y_0 + y_1/2$, and the integration over the latter gives an infinite volume factor. However we still need to see what effect compactifying the theory onto S^4 has.

We may compactify the theory onto a sphere by means of a Weyl transformation, as we did for the sigma model in Section 5.2. The classical action $S[\phi]$ is Weyl invariant but the regulated volume element $\mathcal{D}A$ is not. Hence when making the Weyl transformation of the partition function there is a Weyl anomaly from the measure as well as the term due to the symmetry breaking given in (4.37). So under the transformation $\delta_p g_{\mu\nu} = \delta p g_{\mu\nu}$ the partition function becomes

$$\delta_p \int_M dt\ z(t) = \int_M dt \int \mathcal{D}\Psi \left(\int d^4x\ pW(x)\right) e^{-S_{tot}} - \int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi\ \tau^A(\xi^j \delta_p F_j) e^{-S_{tot}} \quad (6.50)$$

To one loop the second term does not contribute as our choice of F_j (given in (4.10) and (4.11)) is linear in $\xi_{YM} = A_\mu - \mathcal{A}_\mu(t)$, which means that this term contributes at higher order in the expansion in powers of Planck's constant. Let $z_a(t)$ and $\mathcal{D}_a\Psi$ denote the partition function moduli density and the volume element on a sphere of radius a. Now taking $\delta p = \delta a \frac{d}{da} \ln \Omega^2$, where $\Omega = 2/(1 + x^2/a^2)$, gives to one loop

$$\delta_p \int_M dt\ z_a(t) = \int_M dt \int \mathcal{D}_a\Psi \left(\int d^4x\ \frac{d}{da} \ln \Omega^2\ W\right) e^{-S_{tot}} \equiv \frac{d}{da} \int_M dt\ z_a(t) \quad (6.51)$$

Integrating with respect to a from ∞ to a gives

$$\int_M dt\ z_a(t) = \int_M dt \int \mathcal{D}_\infty \Psi\ e^{-S_{tot} + \int d^4x\ \ln(\Omega^2/4) W} \quad (6.52)$$

The anomaly density W should contain a contribution from the Euler density for the sphere. We shall ignore this as it is independent of the moduli. At this stage we need to evaluate W. To this order it turns out to be analogous to the sigma model case (5.80), i.e. it is a product of the beta-function and the Lagrangian density of the theory

$$\int d^4x \, \ln\left(\frac{\Omega^2}{4}\right) \, W = \frac{\beta(g^2)}{4g^2} \int d^4x \, \ln\left(\frac{\Omega^2}{4}\right) \, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \tag{6.53}$$

where the one-loop beta-function is

$$\beta = \mu \frac{\partial}{\partial \mu} g^2(\mu) = -\frac{g^4 11N}{24\pi^2} \tag{6.54}$$

For the two-instanton solution

$$\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) = 2\partial^2\partial^2 \ln \sigma \tag{6.55}$$

$$\sigma \equiv \left(\lambda_0^2(x-y_1)^2(x-y_2)^2\lambda_1^2(x-y_2)^2(x-y_0)^2\lambda_2^2(x-y_0)^2(x-y_1)^2\right) \tag{6.56}$$

where ∂^2 is the flat four-dimensional Laplacian. The case that we are interested in is when the two-instanton sector solution degenerates, i.e. when $y_2 = y_1$. In this case the function σ develops an overall factor of $(x-y_1)^2$, but $\partial^2\partial^2 \ln (x-y_1)^2 = -16\pi^2\delta(x-y_1)$ so

$$\begin{aligned} \int d^4x \, \ln\left(\frac{\Omega^2}{4}\right) W &= \frac{11N}{3} \int d^4x \, \ln\left(\frac{\Omega^2}{4}\right) \delta(x-y_1) \\ &= -\frac{11N}{3} \ln\left(1+\frac{x^2}{a^2}\right)^2 \end{aligned} \tag{6.57}$$

The presence of this factor in the partition function density $z_a(t)$ makes the integral over $(y_0 + y_1)/2$ converge. Thus in Yang-Mills theory we can say that the moduli-space divergences are removed by compactifying the theory onto S^4. In this way Yang-Mills theory is very different from the sigma model and Bosonic String Theory.

Chapter 7

Conclusions

In order to perform the semi-classical expansion in certain important field theories, the field is split into a quantum piece and a classical piece. The classical piece is a solution of the Euler-Lagrange equations and is parametrised by moduli. The functional integral over the quantum fluctuation reduces to an integral over the moduli which, typically, will diverge. To regulate this divergence we could introduce a cut-off in moduli-space. However this has the effect of breaking the rotational symmetry of the action and making the model dependent on the configuration space co-ordinates. In order to apply the cut-off, the integral over the moduli needs to be separated out from the rest of the functional integral. We do this by means of the Faddeev-Popov trick, which also involves the introduction into the action of an arbitrary constraint on the quantum fluctuation. We use this constraint to set the configuration space co-ordinates. However it is essential that the final Green's Function is independent of this choice.

For the case of the $O(3)$ sigma model we see that the problem is that the instanton contribution to the Green's function is divergent at the instanton boundaries, but applying a cut-off introduces an unacceptable constraint on the fields.

To solve this problem we make two observations. Firstly that the divergence and the constraint may be isolated in an anomaly to the Ward Identities of the model. Secondly that the divergence is associated with a degeneracy from a particular instanton sector to one of lower order. This suggests the possibility that the anomaly in the one-instanton sector may be cancelled by adding a term, written in terms of the zero-instanton sector fields, into the action.

We show that the Ward Identity anomaly for the $O(3)$ sigma model in the one-instanton sector, may be written in terms of the zero-instanton sector fields. We also show how this term may be used to cancel the moduli-space integral divergences. This term is then calculated explicitly for a natural choice of the constraint.

A start has been made in looking at how Topological Renormalisation affects other important models. However, there is scope for much more work, even within sigma models themselves. The CP^{n-1} calculation looks like it will yield a positive result, but it would be interesting to study the effect Topological Renormalisation has on theories with fermions, with WZW terms or supersymmetric theories.

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