Constructing vertices in QED

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Abstract

We study the Dyson Schwinger Equation for the fermion propagator in the quenched approximation. We construct a non-perturbative fermion-boson vertex that ensures the fermion propagator satisfies the Ward-Takahashi identity, is multiplicatively renormalizable, agrees with the lowest order perturbation theory for weak couplings and has a critical coupling for dynamical mass generation that is strictly gauge independent. This is in marked contrast to the *rainbow* approximation in which the critical coupling changes by 50% just between the Landau and Feynman gauges. We also show how to construct a vertex which not only has the aforementioned properties but also agrees with the results obtained from the CJT effective potential for the critical exponent of the mass function. These vertices are expressed in terms of two functions which satisfy an integral and a derivative condition. By considering the perturbative expansion for the transverse vertex, we have performed numerical evaluation of the first of these functions which will hopefully guide their non-perturbative structure. The use of vertices satisfying these properties should lead to a more believable study of mass generation.

Acknowledgments

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Declaration

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

The research in this thesis has been carried out in collaboration with J.C.R. Bloch and A. Kızılersü (Section 2.4), and with M.R. Pennington (Chapter 3). Parts of this thesis have been summarised in the following publications :

1. "Gauge-independent chiral symmetry breaking in quenched QED", A. Bashir and M.R. Pennington, Phys. Rev. **D50** 7679 (1994).

2. "Chiral symmetry breaking for fundamental fermions", A. Bashir, to be published in the proceedings volume for the summer school "Frontiers in Particle Physics - Cargèse 94".

Contents

Chapter 1

Introduction

"One should never underestimate the pleasure we feel from hearing something we already know."

Enrico Fermi

What determines the mass of the fermions has long been a problem in gauge theories. Unification of electromagnetic and weak forces was once hindered by the fact that the introduction of mass terms broke the gauge invariance of the theory. This problem was solved in the Standard Model (SM) by the introduction of the Higgs field. This causes the spontaneous breakdown of the $SU_L(2) \times U_Y(1)$ symmetry. The gauge bosons gain mass and the masses for the fermions are generated through their Yukawa interaction with this Higgs field. However, there has been a widespread dissatisfaction with this mechanism since the masses are not predictable. Rather, they must be fixed by experiment. This leaves an unsatisfactory number of parameters in the SM free. In more ambitious attempts to embed $SU_L(2) \times U_Y(1)$ in a bigger gauge group, e.g. $SU(5)$, in order to include the strong interactions in the unification scheme, a more serious problem of fine tuning the parameters arises.

All this serves as a motivation to study non-perturbative aspects of gauge theories through Dyson-Schwinger Equations (DSEs). Such a study suggests that if the interactions are strong enough, they are capable of generating masses for the particles dynamically even if they start with zero bare mass. There are also indications that models based upon dynamical mass generation are capable of overcoming the fine tuning problem. This is in the context of Quantum Chromodynamics (QCD) and some suggested 4-fermion interaction models. As QCD is not very simple, Quantum Electrodynamics (QED) provides a starting point for such investigation to be studied in detail. Once this has been achieved, the next step to extend the work to QCD will be less forbidding. Moreover, the non-perturbative study of QED is interesting in its own right especially because heavy-ion collision experiments suggest the possibility that QED has a non-perturbative phase. In the vicinity of intensely high electromagnetic fields, this non-trivial phase of QED is triggered. e^+ and e^- are able to add to their masses dynamically on the breakdown of chiral symmetry, and can produce a temporary bound state.

In the following sections, we shall give a brief review of the SM, the fine tuning problem, *technicolor*, $t\bar{t}$ condensate models and the heavy-ion collision experiments. We shall then introduce the DSEs with an emphasis on the importance of the vertex function in the development of their study so far.

1.1 The Standard Model

The SM has been extremely successful in all areas of its applicability. However, it has drawbacks which we discuss in this section. We start by recalling the SM Lagrangian for the electroweak interactions, excluding the Higgs sector for the time being :

$$\begin{aligned} \mathcal{L}_{SM-H} &= \sum_f \left[\overline{f}_L \gamma^\mu \left(i\partial_\mu - \frac{g}{2}\tau_i W^i_\mu - \frac{g'}{2} Y B_\mu \right) f_L + \overline{f}_R \gamma^\mu \left(i\partial_\mu - \frac{g'}{2} Y B_\mu \right) f_R \right] \\ &\quad - \frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \quad , \qquad (1.1) \end{aligned}$$

where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\,\epsilon_{ijk} W^j_\mu W^k_\nu \quad . \qquad (1.2) \end{aligned}$$

For simplicity, quarks and gluons have not been included. f stands for all lepton fields and subscripts L and R for their handedness. B_μ and W_μ^i are the gauge fields corresponding to the groups $U_Y(1)$ and $SU_L(2)$ respectively. g' is the coupling constant for B_μ interacting with both f_L and f_R, g is the coupling constant for W_μ^i interacting with f_L, and Y is the hypercharge operator. The left and right handed fields are defined by

$$\begin{aligned} f_L &= \frac{1}{2}(1-\gamma_5)f \\ f_R &= \frac{1}{2}(1+\gamma_5)f \quad . \end{aligned}$$

Under the local transformation $SU_L(2) \times U_Y(1)$, the fields transform as follows

$$\begin{aligned} f_R &\rightarrow e^{i\beta Y} f_R \\ f_L &\rightarrow e^{i\vec{\alpha}\cdot\frac{\vec{\tau}}{2}+i\beta Y} f_L \\ B_\mu &\rightarrow B_\mu - \frac{1}{g'}\partial_\mu \beta \\ \vec{W}_\mu &\rightarrow \vec{W}_\mu - \frac{1}{g}\partial_\mu \vec{\alpha} - \vec{\alpha} \times \vec{W}_\mu \end{aligned}$$

which leaves the Lagrangian invariant. Mass terms, e.g. $m\overline{f}f$, cannot be included as the Lagrangian would lose its gauge invariance under $SU_L(2) \times U_Y(1)$ transformations. It is at this stage that the scalar Higgs field ϕ is introduced with the following term added to the Lagrangian :

$$\mathcal{L}_H = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \quad , \tag{1.3}$$

where $\lambda > 0$ and $\mu^2 < 0$. The field ϕ is an $SU(2)$ doublet defined as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad . \tag{1.4}$$

Because of the wrong sign for μ^2, spontaneous breakdown of $SU_L(2) \times U_Y(1)$ symmetry takes place and the minimum of the field ϕ is shifted to a non-zero value $<\phi>$:

$$<\phi> \equiv <0|\phi|0> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad ,$$

where $v = \sqrt{-\mu^2/\lambda}$. The gauge bosons acquire masses by interacting with the Higgs field through the following $SU_L(2)\times U_Y(1)$ invariant piece of the Lagrangian :

$$
\begin{aligned}
\mathcal{L}_{BH} &= \left|\left(-ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu - i\frac{g'}{2}B_\mu\right)\phi\right|^2 \\
&= \frac{1}{4}g^2v^2W_\mu^+W^{-\mu} + \frac{1}{2}\left[\frac{1}{4}v^2(g^2+g'^2)\right]Z_\mu Z^\mu + 0A_\mu A^\mu + \cdots \quad , \qquad (1.5)
\end{aligned}
$$

where the fields $W_\mu^\pm$, Z_μ and A_μ are defined as follows :

$$
\begin{aligned}
W_\mu^\pm &= \frac{1}{\sqrt{2}}\left[W_\mu^1 \mp iW_\mu^2\right] \\
Z_\mu &= \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}} \\
A_\mu &= \frac{g'W_\mu^3 + gB_\mu}{\sqrt{g^2+g'^2}} \quad . \qquad (1.6)
\end{aligned}
$$

From Eq. (1.5), we can read off the masses of the gauge bosons :

$$
M_W = \frac{1}{2}gv\,, \qquad M_Z = \frac{1}{2}v\left[g^2+g'^2\right]^{\frac{1}{2}}\,, \qquad M_A = 0 \quad .
$$

Defining

$$
\begin{aligned}
Z_\mu &= W_\mu^3\cos\theta_w - B_\mu\sin\theta_w \\
A_\mu &= W_\mu^3\sin\theta_w + B_\mu\cos\theta_w \quad ,
\end{aligned}
$$

where θ_w is called the Weinberg angle, we find

$$
M_W = M_Z\cos\theta_w \quad .
$$

Fermion masses are obtained by assuming that they couple with the Higgs field through Yukawa-type interactions :

$$
\begin{aligned}
\mathcal{L}_{FH} &= -G_F\left[\bar{f}_L\phi f_R + \bar{f}_R\phi^\dagger f_L\right] \\
&= -\frac{1}{\sqrt{2}}G_F\,v\bar{f}f + \cdots \quad , \qquad (1.7)
\end{aligned}
$$

which implies that the mass for the fermions is

$$m_f = \frac{1}{\sqrt{2}} G_F \, v \quad .$$

The Model does not determine the values of v, θ_w and G_F, and hence fails to predict M_W, M_Z and m_f. These parameters are determined experimentally and are fed into the SM. It would be more acceptable if we could find a theory capable of predicting the masses of the particles, without referring to experiment. Moreover, the introduction of the Higgs field is *ad hoc* and is not completely satisfactory as we have no other experience of a fundamental scalar.

The SM is not the end of the story. In order to stretch the attempts of unification to include QCD, one has to look for a bigger symmetry group. We then face the problem of artificial tuning of parameters as we discuss in the next section.

1.2 The Fine Tuning Problem

At low energies, the world of elementary particles is symmetric under the group $SU(3) \times U_Q(1)$, where subscript Q suggests that this group corresponds to electromagnetism. The SM predicts that at higher energies the symmetry group expands to $SU(3) \times SU_L(2) \times U_Y(1)$. This is a step forward in the direction of unification of forces. There are speculations that at yet higher energies, the symmetry group is larger, $SU(5)$ being the simplest. Despite the fact that theories based on this group have many nice features, there are some unsatisfactory implications as we now discuss.

The breakdown of a symmetry is triggered by the non-vanishing vacuum expectation value of a local field. This value sets a scale for the masses resulting from the symmetry breakdown. In the SM, $< \phi > \simeq 250\,\mathrm{GeV}$. Consequently, the masses of all the fermions and bosons in the model are of the order of or smaller than the symmetry breaking scale $\simeq 250\,\mathrm{GeV}$. In an attempt to unify the three forces in the symmetry group $SU(5)$, we encounter twelve new gauge bosons. Some of these bosons (X) are able to mediate the process

$$\mathrm{ud} \rightarrow \mathrm{X} \rightarrow \mathrm{e}^{+}\,\overline{\mathrm{u}}$$

which can cause a proton to decay. In order to be consistent with the limit on the life-time of a proton, these bosons should be as heavy as 10^{15}GeV. This means that a new completely different energy scale is required for $SU(5)$ symmetry to break down to $SU(3) \times SU_L(2) \times U_Y(1)$. The picture now looks like

$$SU(5) \xrightarrow{<\Phi>\sim 10^{15}\mathrm{GeV}} SU(3) \times SU_L(2) \times U_Y(1) \xrightarrow{<\phi>\sim 10^{2}\mathrm{GeV}} SU(3) \times U(1)$$

where Φ corresponds to the heavy Higgs and ϕ is the SM Higgs. The problem with having two Higgs multiplets is that they communicate with each other through the exchange of heavy bosons and hence it is very difficult to keep the two mass scales separate from each other. In order to achieve this, the parameters in the expression for their potential terms have to be adjusted to an accuracy of about 24 significant figures. Moreover, to retain the balance, a fine tuning at each order in perturbation theory is required. This is highly unsatisfactory. There are indications that if asymptotically free theories are involved in the symmetry breakdown, the problem of fine tuning the parameters can be solved.

The drawbacks discussed above in the conventional way of generating masses through Higgs mechanism led to the introduction of *technicolor.*

1.3 Technicolor

Technicolor [1, 2] was invented in order to circumvent the introduction of a scalar Higgs to generate masses for bosons and fermions. The basic underlying idea can be understood just by considering a massless doublet of quarks u and d, interacting through ordinary QCD. Such a theory has $SU_L(2) \times SU_R(2)$ symmetry. Now the following non-zero vacuum expectation value of the quark fields

$$< \overline{\mathrm{u}}\mathrm{u} + \overline{\mathrm{d}}\mathrm{d} > \neq 0 \tag{1.8}$$

triggers the breakdown of chiral $SU_L(2) \times SU_R(2)$ symmetry to $SU(2)_{\text{isospin}}$ symmetry. The breakdown of this global symmetry will produce three Goldstone bosons (pions) out of the vacuum leading to the following equation :

$$< 0|\, J^{\mu}_{5a}\, |\pi^a> \;=\; f_\pi\, q^\mu \quad , \tag{1.9}$$

where f_π is a constant which we can identify with the pion decay constant and the axial current

$$J^{\mu}_{5a} \sim \overline{\text{q}}\, \gamma^\mu\, \gamma^5\, \frac{\tau_a}{2}\, \text{q} \quad .$$

Here $q \equiv (u, d)$. Let us now turn on $SU_L(2) \times U_Y(1)$ electroweak interactions but without any fundamental scalar. The four gauge fields $W^{\pm,0}_\mu$ and B_μ will couple to the Goldstone bosons through the above current and generate masses. A proper combination Z_μ and A_μ of these fields can prevent the photon from acquiring mass and give

$$M_{W^\pm} = \frac{1}{2} g f_\pi^2 \qquad m_\gamma = 0 \qquad M_Z = \frac{1}{2}(g^2 + g'^2)^{1/2} f_\pi^2 \quad ,$$

where the symbols have their usual meaning.

This is of course just a toy model to explain the idea of mass generation without referring to the fundamental Higgs. This model cannot represent the real world for two simple reasons. We know that $f_\pi \approx 93$MeV which implies the wrong result $M_{W^\pm,Z} \approx 100$MeV. Moreover, we do observe the pions which, in this model, are supposed to be eaten up by $W^\pm$ and Z. However, it is not difficult to construct a realistic model incorporating this underlying idea. Assume that a force stronger than the strong force exists whose dynamics is just a scaled up version of QCD. It is named Quantum Technidynamics (QTD). In order to meet the empirical requirements, the following comparison between QCD and QTD can serve as a useful guide

QCD	$\longrightarrow$	QTD
$\begin{bmatrix} \mathrm{u} \\ \mathrm{d} \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} \mathrm{A} \\ \mathrm{B} \end{bmatrix}$
Colour	$\longrightarrow$	*Technicolor*
$f_\pi \approx 93\,\mathrm{MeV}$	$\longrightarrow$	$F_\pi \approx 246\,\mathrm{GeV}$
$M_{W^\pm} = \frac{1}{2} g f_\pi^2$	$\longrightarrow$	$M_{W^\pm} = \frac{1}{2} g F_\pi^2$
$M_Z = \frac{1}{2}(g^2 + g'^2)^{1/2} f_\pi^2$	$\longrightarrow$	$M_Z = \frac{1}{2}(g^2 + g'^2)^{1/2} F_\pi^2$
$\Lambda_{QCD} \approx 200\,\mathrm{MeV}$	$\longrightarrow$	$\Lambda_{QTD} \approx 500\,\mathrm{GeV}$

Now consider a world consisting of both the fermions and the technifermions interacting with the weak gauge bosons of $SU_L(2) \times U_Y(1)$. In order to ensure that the physical pions are the QCD pions and the technipions are the ones responsible for generating masses for the weak bosons, we require

$$< 0 \mid J_\mu^5 \mid \text{physical pion} > \; = \; 0 \quad , \tag{1.10}$$

where

$$J_\mu^5 \;\sim\; \overline{\mathrm{q}}\gamma_\mu\gamma_5\mathrm{q} \;+\; \overline{\mathrm{q_t}}\gamma_\mu\gamma_5\mathrm{q_t} \tag{1.11}$$

which satisfies

$$< 0 | \; J_5^\mu \; |\text{QCD pion} > \; = \; f_\pi \, q^\mu$$
$$< 0 | \; J_5^\mu \; |\text{QTD pion} > \; = \; F_\pi \, q^\mu \quad .$$

As before, $q \equiv (u, d)$ and $q_t \equiv (A, B)$. F_π is the technipion decay constant. Eq. (1.10) can be satisfied if we define

$$|\text{pion absorbed} > \ = \ \frac{F_\pi |\,\text{QTD pion} > + f_\pi |\,\text{QCD pion} >}{\sqrt{F_\pi^2 + f_\pi^2}} \quad , \tag{1.12}$$

$$|\text{physical pion} > \ = \ \frac{F_\pi |\,\text{QCD pion} > - f_\pi |\,\text{QTD pion} >}{\sqrt{F_\pi^2 + f_\pi^2}} \quad . \tag{1.13}$$

Since $F_\pi >> f_\pi$, the physical pion is mostly the QCD pion, while the absorbed pion is mostly the QTD pion. The energy scale for QTD can be estimated from

$$\frac{\Lambda_{QTD}}{\Lambda_{QCD}} \sim \frac{F_\pi}{f_\pi} \sim 2600 \quad . \tag{1.14}$$

If we take $\Lambda_{QCD} \sim 200\,\text{MeV}$, then we obtain $\Lambda_{QTD} \sim 500\,\text{GeV}$. Therefore, it is obvious that in order to produce technihadrons, one would have to go to much higher energies than the energies required to produce ordinary hadrons.

So far, what we have been able to achieve with the above model is to generate masses for the weak gauge bosons. What is still not clear is how the fermions acquire masses as the term proportional to $\overline{\psi}\psi$ is not allowed by $SU(2)$ invariance. It is here that another ingredient is required. It is assumed that there exist interactions between the fermions and the technifermions. Such interactions are referred to as *extended technicolor interactions.* It is found, as discussed below, that in order to generate quarks of masses of the order $\sim 1\,\text{GeV}$, the mass scale Λ_E of the *extended technicolor interactions* should be $\sim 20\,\text{TeV}$. It is, therefore, reasonable to assume that these interactions, at low energies, take the form of non-renormalizable vertices with dimensionful couplings, whereas, at much higher energies, these may arise due to the exchange of heavy bosons. It is easy to see that for an n-fermion interaction, the coupling has the dimensions $M^{(8-3n)/2}$, where M is the scale of interaction. Then the lowest dimensional operator representing fermionic interactions, permissible by the gauge covariance requirements, is $(\overline{\psi}\psi)^2/\Lambda_E^2$. This leads to the relation

$$m_q \ \approx \ \frac{< \overline{\text{A}}\text{A} + \overline{\text{B}}\text{B} >}{\Lambda_E^2} \quad , \tag{1.15}$$

where Λ_E can be estimated from the approximate relation

$$\sqrt{\frac{N}{3}}\ \frac{<\overline{\mathrm{A}}\mathrm{A}+\overline{\mathrm{B}}\mathrm{B}>/2}{<\overline{\mathrm{q}}\mathrm{q}>} \quad\approx\quad \left[\frac{F_\pi}{f_\pi}\right]^3 \quad . \tag{1.16}$$

where N is the number of *technicolors*. The QCD analysis gives $<\overline{\mathrm{q}}\mathrm{q}> \approx\ 17\, f_\pi^3$. Then for example with four *technicolors*, we have

$$\frac{<\overline{\mathrm{A}}\mathrm{A}+\overline{\mathrm{B}}\mathrm{B}>}{2} \quad\approx\quad (600\,\mathrm{GeV})^3 \quad . \tag{1.17}$$

Using Eq. (1.15), we can see that in order to achieve $m_{\mathrm{q}} \approx 1\,\mathrm{GeV}$, we should have

$$\Lambda_E \quad\approx\quad 20\,\mathrm{TeV} \quad .$$

However, we immediately encounter a serious problem. The existence of extended technibosons allows flavour changing neutral currents through diagrams of the type

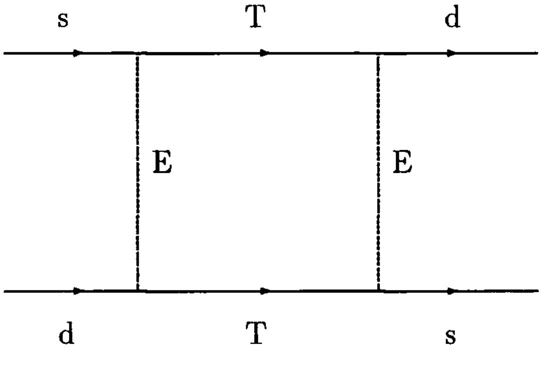

Fig 1.1 : Flavour changing neutral currents mediated by extended technibosons E between quarks and techniquarks T.

Such diagrams contribute to the $K_L - K_S$ mass difference. Experimental limits on this mass difference suggest that $\Lambda_E > 20\,\mathrm{TeV}$ which is barely consistent with the number quoted before. There have been attempts made within the domain of *technicolor* models to solve this problem. However, difficulties seem to persist especially for quarks as heavy as the top. At this stage, it was realised that the problem of large top quark mass could be dealt with in a more direct way by discarding *technicolor* altogether and postulating four-fermion operators of a different kind. This was the birth of $t\bar{t}$ condensates which we discuss in the next section.

1.4 $t\bar{t}$ Condensates

The idea of $t\bar{t}$ condensation [3, 4, 5, 6] is only a step forward from *technicolor* models. It assumes new forces act on the top quark. One of the reasons to treat the top quark specially is because experiments tell us that the top quark is very heavy and so in the ordinary SM, the Yukawa coupling g_t for top-Higgs interaction is $\mathcal{O}(1)$. One then naturally expects that non-perturbative effects become important. $t\bar{t}$ models suggest that the top quark may acquire mass non-perturbatively through four-fermion interactions, and the Higgs can then be viewed as the condensate of the top and the antitop. In these models, the Higgs sector of the SM is omitted just as in *technicolor* models and is replaced by a new gauge-invariant four-fermion vertex as follows :

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + G(\overline{L}^{ia} t_{Ra})(\bar{t}_R^b L_{ib}) \quad , \tag{1.18}$$

where $L = (t_L, b_L)$ and the index i is summed over $SU_L(2)$ indices and a, b over colour indices. $\mathcal{L}_{\text{kinetic}}$ consists of the kinetic terms for massless fermions and gauge bosons. G is the dimensionful coupling which can be expressed in terms of the dimensionless coupling g and the cut-off Λ_t as $G \equiv g^2/\Lambda_t^2$. The cut-off Λ_t can be regarded as some high mass scale beyond which the new force is presumably mediated by the exchange of some heavy bosons. G and Λ_t are the fundamental parameters of the theory. We can now look at this model in the fermion bubble approximation which can best be described by the following diagram representing the self interaction of the top quark :

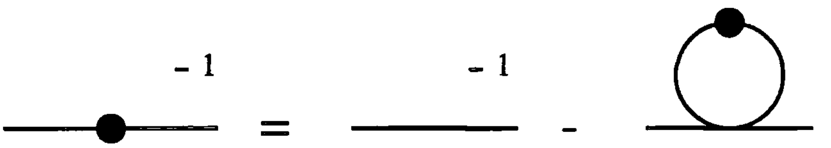

Fig. 1.2 : The gap equation.

The blob represents the full fermion propagator which includes in itself a sum of all the fermion bubble diagrams. Refer to Section 1.6 for more discussion on the meaning of blobs on the Greens functions. If the momentum flowing through the fermion propagator is p, the full fermion propagator can be defined in the most general way as follows :

$$S_F(p) \;=\; \frac{F(p^2)}{\not p - \mathcal{M}(p^2)} \quad . \tag{1.19}$$

Fig [1.2] is an example of the DSE we shall investigate in this thesis : here the interactions are only 4-fermion. Mathematically, the gap equation can be written as

$$\frac{\not p - \mathcal{M}(p^2)}{F(p^2)} \;=\; \not p - 2iGN_c \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2)}{\not k - \mathcal{M}(k^2)} \quad . \tag{1.20}$$

where N_c is the number of colours. This is a matrix equation consisting of two equations in $F(p^2)$ and $\mathcal{M}(p^2)$. These two equations are straightforwardly separated by a method we shall use many times. Multiplying this equation by $\not p$, and taking the trace, we obtain one of the two equations :

$$\frac{1}{F(p^2)} \;=\; 1 - 2GN_c \frac{1}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2) k \cdot p}{k^2 + \mathcal{M}^2(k^2)} \quad . \tag{1.21}$$

where Wick rotation from Minkowski to Euclidean space has been performed. One can easily see that the angular integration in the above equation gives zero. So we get $F(p^2) = 1$.

The other equation can be obtained by taking the trace of Eq. (1.20) and Wick rotating to Euclidean space :

$$\mathcal{M}(p^2) \;=\; 2GN_c \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k^2)}{k^2 + \mathcal{M}^2(k^2)} \quad , \tag{1.22}$$

where we have used the fact that $F(p^2) = 1$. Note that $\mathcal{M}(p^2) = 0$ is a trivial solution. Let us try

$$\mathcal{M}(p^2) \;=\; m_t \qquad\qquad m_t \neq 0 \tag{1.23}$$

as a non-trivial solution. We then get

$$1 \;=\; 2GN_c \int \frac{d^4k}{(2\pi)^4} \frac{m_t}{k^2 + m_t^2} \quad . \tag{1.24}$$

Now carrying out the angular integration and doing a bit of straightforward algebra gives

$$m_t^2 \ln \frac{\Lambda_t^2}{m_t^2} \;=\; \frac{8\pi^2}{N_c} \left[\frac{1}{G_c} - \frac{1}{G} \right] \quad , \tag{1.25}$$

where $G_c = 8\pi^2/N_c\Lambda^2$. G_c is the critical value of the coupling below which no solution exists for m_t. For $\Lambda_t \approx 10^{15}$ GeV, we find $m_t \approx 230$ GeV. It is surprising that with such a crude approximation, one finds a result which is not very far from reality. As a natural next step, we can turn on QCD, and the gap equation will now be modified to

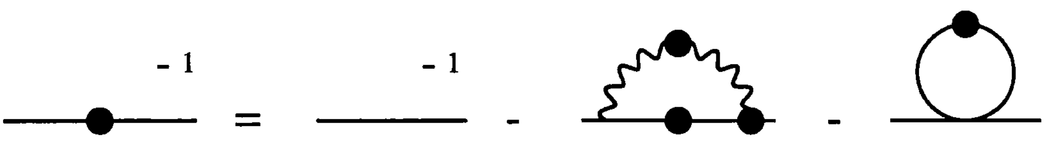

Fig. 1.3 : The gap equation with the gauge boson exchange terms included.

where the blobs represent the full quantities. In an attempt to solve this equation, approximations made may mean one loses gauge invariance of the physical quantities. Of course, physical quantities must be gauge independent. This motivates the study of how to achieve this in non-perturbative calculations. As QCD is a complicated theory, QED can serve as a starting point to investigate dynamical mass generation within gauge theories. Moreover, there are indications that QED itself has a non-perturbative phase observed in the heavy ion collisions. This observation makes the non-perturbative study of QED interesting in its own right.

1.5 Why Non-perturbative QED?

It is now established that QED is the best tested of all physical theories. As a usual example, theory and experiment agree for the magnetic moment of the electron to many significant figures. QED is also regarded as the best understood quantum field theory. But all its success is in the domain of perturbation theory. In the past few years, there has been a great deal of interest in the possibility that QED has a non-perturbative phase which is very different from the perturbative one. Work in this direction seems to conclude that there is another phase of QED as the coupling is theoretically increased to the order of unity. This phase is strikingly different from the one encountered in perturbative analysis. Chiral symmetry is broken in this phase and masses are generated dynamically. Intriguingly data on heavy ion collisions has generated much interest due to the appearance of at least three narrow peaks in the e^+e^- coincidence spectra in the mass range 1.6-1.8 MeV. The theoretical proposal which has gained most attention is that the peaks provide evidence that the strong phase of QED is actually realised in the laboratory [7, 8].

1.5.1 Heavy Ion Collisions

A few years ago, experiments started at Gesellschaft für Schwerionenforschung (GSI) in Darmstadt, Germany to look at heavy ion collisions at energies around 6 MeV per nucleon. The initial interest was in detecting positrons. It was expected that spontaneous positron emission should take place if one had a nucleus with $Z > Z_{critical}(\simeq 173)$ and an empty K-shell with binding energy E_0 greater than $2m_ec^2$. Under these circumstances, it may be energetically favourable to produce an e^+e^- pair from the vacuum. The electron will then be captured by the K-shell and be bound with energy E_0. On the other hand, the positron will be emitted with kinetic energy equal to $E_0 - 2m_ec^2$. No such nuclei are known to exist naturally. Therefore, it was suggested that such a nucleus could be created for a short time by the collision of heavy nuclei forming some sort of compound nucleus with $Z > Z_{critical}$.

Two groups mainly worked on carrying out the experiments : the Electron Positron Spectrometer group (EPOS) and the Orange Spectrometer group. Two of the colliding systems used were U + Cm (Combined Z=188) and Th + Th (Combined Z=180). Peaks of positronium production were observed but it was realised that these peaks did not have the expected characteristics. They were too narrow and were observed even below $Z_{critical}$, e.g. in the case of Th + Ta (combined Z=163). The data suggested that the source of e^+ was most likely to be a neutral object decaying, almost at rest in the centre of mass, to e^+e^-. The detector was then modified in order to see coincidental e^+ and e^- events. At least a few such events have been reported by both the groups. The main characteristics of these events were that their positions in mass did not depend on Z and that the electron and positrons had equal energies and were back-to-back.

There have been many attempts to explain the existence of positron peaks, though most of them have already been ruled out on phenomenological grounds. As nuclei are involved in the collision, many suggestions were based on effects from nuclear physics. However, they were all deficient in explaining all the exotic features of the peaks. Similar attempts in terms of atomic physics also faded into insignificance. In the beginning, when there was only one peak, there was some excitement about the discovery of a new elementary particle, possibly the axion. However, when many peaks appeared, the interest in this explanation died away as it was difficult to believe in so many new particles.

After the failure of conventional explanations, the best possibility was that the aforementioned events could be attributed to the decay of some composite particle. A composite particle has many energy levels. This fact helps in interpreting the data as some of the levels decaying to e^+ and e^-. The best known composite system of e^+ and e^- is positronium. However, all its levels are lower than the threshold of $2m_e$. The next natural and simplest assumption is that the decaying states are composites of e^+ and e^- in a non-perturbative phase of QED.

The new-phase explanation allows all the pieces of the puzzle to fit together.

- It is easily understandable why the composite states are found only in collisions of very heavy ions and are difficult, if not impossible, to be seen in other systems where the background electromagnetic fields are weak. The idea is that when the two nuclei merge to form, temporarily, a nucleus with large value of Z, a phase transition is induced to the new QED vacuum as the effective coupling $(\alpha Z) \sim 1$. The non-perturbative effects come into play which result in the production of a bound state of e^+ and e^-. After the ions go apart again, the strength of the field diminishes. The new vacuum becomes metastable and finally decays into the original vacuum. Consequently, the composite system liberates an e^+ and an e^- which come out with equal energies and back-to-back, as the experiment reveals.

- As we are dealing with the bound state of e^+e^-, it is self-explanatory why the bound state prefers to decay into e^+ and e^- as compared to photons.

- The decay of the bound state takes place after the restoration of the original vacuum, i.e. after the electromagnetic configuration with the combined value of Z has died off. This explains the Z-independence of the states.

Theoretical work in non-perturbative QED, both through lattice and continuum studies, has shown that the phase transition does take place as the coupling is increased to something of order unity. Quantitative work is underway to demonstrate that in the collision of heavy ions, the intense background electromagnetic field can trigger such a phase transition. The present situation is not yet conclusive. However, there is indirect evidence from the spectrum calculation of the observed states using various models incorporating the dynamical breakdown of chiral symmetry. It causes the electron to have an additional contribution to its mass, and hence the bound states can lie in the neighbourhood of 1.7 MeV.

The continuum studies of non-perturbative QED are carried out through the DSEs. It is here that it becomes impossible to proceed any further without introducing these equations.

1.6 Dyson-Schwinger Equations

The Dyson Schwinger Equations (DSEs) relate Green's functions to each other. There is a one to one correspondence between the number of Green's functions and the number of DSEs. We know that a field theory is completely defined when all of its Green's functions are known. Therefore, solving these coupled integral equations, we can extract all the possible information about a field theory. Their derivation is independent of any recourse to perturbation theory. Hence, they provide a suitable framework to explore non-perturbative characteristics of field theories. Being an infinite tower of equations, it is impossible to carry out there study without any truncation. The approximations needed in this regard call into question their validity as a reliable mathematical tool. However, efforts are being made to improve the simplifying assumptions and find sound physical grounds for them.

We shall not go into the derivation of DSEs which is carried out through path integral formulation. We shall write down the mathematical expressions for the first two DSEs, namely, the ones for the fermion propagator and the photon propagator along with their diagrammatic representation and then discuss the attempts made so far to solve them.

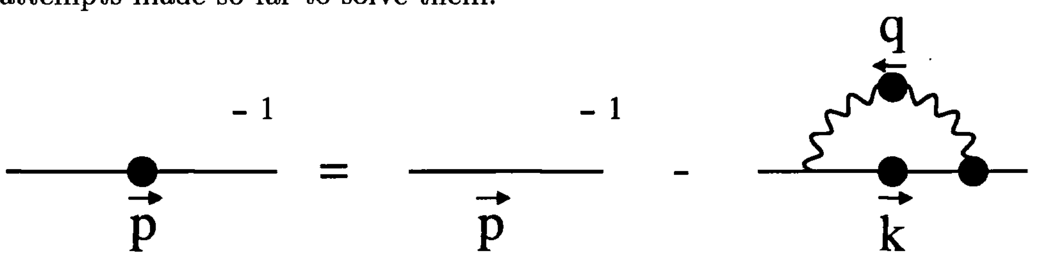

Fig. 1.4 : Dyson-Schwinger equation for the fermion propagator.

$$iS_F^{-1}(p) = iS_F^{0^{-1}}(p) - e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \, S_F(k) \, \Gamma^\nu(k,p) \, \Delta_{\mu\nu}(q) \quad . \tag{1.26}$$

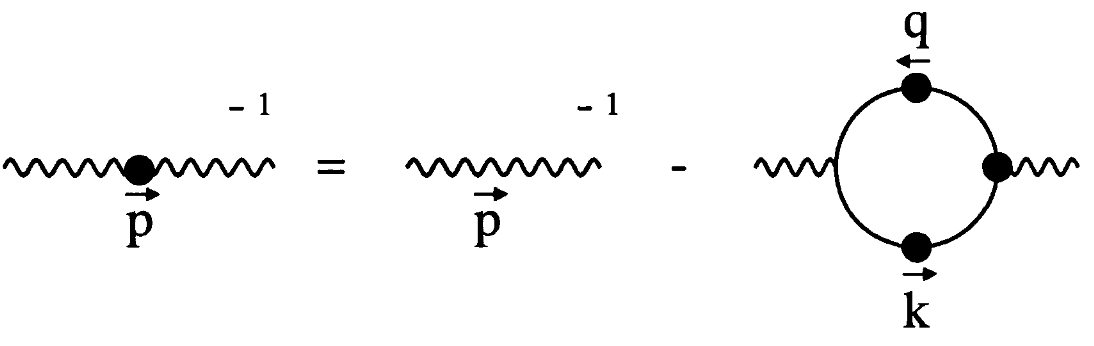

Fig. 1.5 : Dyson-Schwinger equation for the photon propagator.

$$i\Delta_{\mu\nu}^{-1}(p) = i\Delta_{\mu\nu}^{0^{-1}}(p) - e^2 N_f \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \, S_F(k) \, \Gamma^\nu(k,p) \, S_F(q) \quad . \tag{1.27}$$

The blobs on the propagators and the vertex represent the full quantities. Probably the simplest way to understand this is to look at the perturbative expansion of one of the propagators, e.g. the fermion propagator to order e^4 :

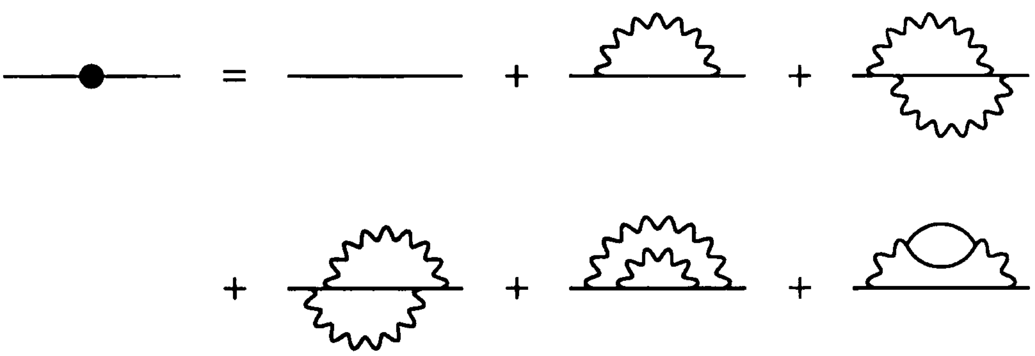

Fig. 1.6 : Perturbative expansion of the fermion propagator.

This diagram shows that the blob on the Fermion propagator can be thought of as a summation over all the activities that a bare propagator can undergo by emitting virtual photons and then recapturing them in infinitely many ways. Some of the parts in these diagrams can be regarded as a correction to the bare photon and fermion propagators and some to the bare vertex which results in the appearance of blobs on these quantities. Collectively, they can be summed up as shown in Fig. [1.4]. Note the absence of the blob on one of the vertices in Fig. [1.4]. The reason is to avoid double counting of the diagrams. In Eqs. (1.26,1.27), symbols with the superscript 0 represent bare quantities, while the ones without a superscript correspond to the full quantities. $S_F^0(p)$ and $\Delta^0_{\mu\nu}(q)$ are the bare fermion and photon propagators carrying momenta p and q respectively. They are defined by

$$
\begin{aligned}
S_F^0(p) &= \frac{1}{\not p - m_0} \\
\Delta^0_{\mu\nu}(q) &= \frac{1}{q^2}\left(g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right) \qquad ,
\end{aligned}
$$

where m_0 is the bare mass of the fermion. $S_F(p)$ is the full fermion propagator defined by Eq. (1.19) and $\Delta_{\mu\nu}(q)$ is the full photon propagator which can be defined in its most general form in a covariant gauge by :

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2}\left[\frac{1}{\mathcal{G}(q^2)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \xi\frac{q_\mu q_\nu}{q^2}\right].$$

where ξ is the covariant gauge parameter. Similarly, $\Gamma(k,p)$ is the full vertex. N_f corresponds to the number of flavours.

There is an infinite set of DSEs all coupled to each other. The structure is such that the two-point function is related to the three-point function, the three-point function is related to the four-point function and so on. The two equations exemplified above inter-relate the fermion and photon propagators to the fermion-photon vertex. This vertex is then related to a 4-point function through the following equation

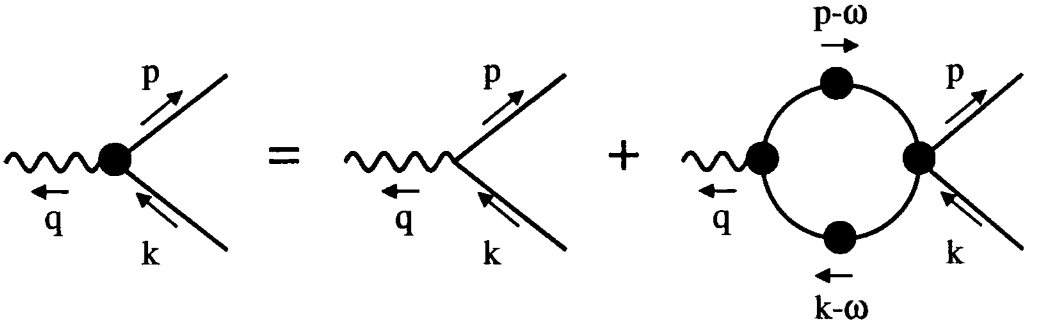

Fig. 1.7 : Dyson-Schwinger equation for the fermion-photon vertex.

In order to solve this infinite tower of equations, a truncation is inevitable. In the region of $\alpha << 1$, perturbation theory is one of the ways to achieve this. However, when $\alpha \simeq 1$, a non-perturbative way has to be sought. In order to solve the DSEs for the fermion and photon propagators, we have to substitute expressions for the vertex function. As it is a prohibitively difficult task to solve the DSE for the vertex, the most economic way is to look for a clever *ansatz* for the vertex. This enables us to decouple the first two DSEs from the rest of the tower. However, some important features of DSEs can be understood even by making a few more simplifying assumptions.

The **quenched approximation** corresponds to neglecting the fermion loop contribution to the vacuum polarization, which enables us to replace the full 2-

point photon function by its bare counterpart. The mathematical justification for this comes from the fact that all the fermion loops carry a factor N_f (number of flavours) with them. Regarding this factor as a mathematical parameter, we set it zero. This situation does not, of course, represent the real world. However, the simplicity that the quenched theory brings provides us with a useful framework within which we can gain insight into solving DSEs and it serves as a natural starting point for more realistic problems to be attacked later on. Quenched QED ([12] - [20]) is also an interesting theory in its own right. In the quenched approximation, the equation for the fermion propagator can be written as

$$iS_F^{-1}(p) \;=\; iS_F^{0^{-1}}(p) \;-\; e^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu\, S_F(k)\,\Gamma^\nu(k,p)\,\Delta^0_{\mu\nu}(q) \quad . \tag{1.28}$$

We cannot proceed any further unless we make an *ansatz* for the 3-point vertex function. The crudest of assumptions is to write

$$\Gamma_\mu(k,p) = \gamma_\mu \quad . \tag{1.29}$$

This is commonly referred to as the *ladder* or the *rainbow* approximation. Combined with the quenched approximation, it enables the equation for the fermion propagator to be decoupled from the rest of the infinite tower of DSEs. This is a great advantage of using this vertex. Despite the fact that there are significant problems associated with this approximation, such as the loss of gauge covariance and multiplicative renormalizability (MR), it has proved to be sufficiently interesting in studying dynamical generation of mass as we discuss in the next section.

1.7 Dynamical Mass Generation

As discussed before, one of the more important motivations for studying the non-perturbative behaviour of gauge theories is the fact that if the interactions are strong enough, they are capable of generating masses for the particles dynamically even if they start with zero bare mass. In order to study this feature of the DSEs, let us look at the DSE for the fermion propagator in the quenched approximation with the bare vertex :

$$iS_F^{-1}(p) \;=\; iS_F^{0^{-1}}(p) \;-\; e^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu\, S_F(k)\, \gamma^\nu\, \Delta^0_{\mu\nu}(q) \quad . \tag{1.30}$$

This is a matrix equation consisting of two equations, one each for the mass function $\mathcal{M}(p^2)$ and the wavefunction renormalization $F(p^2)$. Substituting the expressions for the full fermion propagator and the bare photon propagator in this equation and taking its trace after multiplying it with $\not{p}$ and 1 respectively, we obtain

$$\begin{aligned} \frac{1}{F(p^2)} &= 1 - \frac{\alpha}{4\pi^3}\frac{1}{p^2}\int d^4k\, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\,\frac{1}{q^2} \\ &\qquad \left\{ -2k\cdot p - \frac{(\xi-1)}{q^2}\left[2k^2p^2-(k^2+p^2)k\cdot p\right]\right\} && (1.31) \\ \frac{\mathcal{M}(p^2)}{F(p^2)} &= m_0 - \frac{\alpha}{4\pi^3}(3+\xi)\int d^4k\, \frac{F(k^2)\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\,\frac{1}{q^2} \quad , && (1.32) \end{aligned}$$

where a Wick rotation has been performed from Minkowski to Euclidean space. In the simple rainbow approximation, none of the unknown functions F and $\mathcal{M}$ is a function of the variable q^2. Consequently, we can perform the angular integrations to arrive at the following result :

$$\begin{aligned} \frac{1}{F(p^2)} &= 1 + \frac{\alpha\xi}{4\pi}\int_0^{\Lambda^2} dk^2\, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{k^4}{p^4}\theta(p^2-k^2)+\theta(k^2-p^2)\right] && (1.33) \\ \frac{\mathcal{M}(p^2)}{F(p^2)} &= m_0 + \frac{\alpha(3+\xi)}{4\pi}\int_0^{\Lambda^2} dk^2\, \frac{F(k^2)\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\theta(k^2-p^2)\right] . && (1.34) \end{aligned}$$

It becomes clear at this stage that the Landau gauge $\xi = 0$ is a preferred gauge as $F(p^2)$ is obviously equal to 1 in this gauge and, hence, Eq. (1.33) and (1.34) decouple from each other. We then only have to worry about the equation for the mass function $\mathcal{M}(p^2)$:

$$\mathcal{M}(p^2) = m_0 + \frac{3\alpha}{4\pi}\int_0^{p^2} dk^2\, \frac{k^2}{p^2}\,\frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} + \frac{3\alpha}{4\pi}\int_{p^2}^{\infty} dk^2\, \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} \tag{1.35}$$

It is here that we can start talking about dynamical mass generation. Note that for $m_0 = 0$, $\mathcal{M}(p^2) = 0$ is the trivial solution of the above equation which corresponds to the fact that in perturbation theory, a particle with bare mass equal to zero is incapable of acquiring mass at any level of truncation, a point we amplify a little later. However, we find that a non-trivial solution for the mass function also exists indicating the generation of mass even when the bare mass of the particle is zero. It is interesting to see that even without formally solving the above equation, we can learn some important features of the solution.

Let us start by trying to find the large p^2 behaviour of $\mathcal{M}(p^2)$. We assume that it is of the form

$$\mathcal{M}(p^2) = B\left(p^2\right)^{-s} \qquad \text{for} \qquad p^2 \to \infty \tag{1.36}$$

where B is a constant and we expect s to be positive. Let us define momentum p_c^2 such that above this momentum, $\mathcal{M}^2(p^2)$ is so small that it can be neglected in comparison with p^2. We can then write Eq. (1.35) as

$$\begin{aligned}\mathcal{M}(p^2) &= m_0 + \frac{3\alpha}{4\pi}\int_0^{p_c^2} dk^2\,\frac{k^2}{p^2}\,\frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} + \frac{3\alpha}{4\pi}\int_{p_c^2}^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\\ &\quad + \frac{3\alpha}{4\pi}\int_{p^2}^{\infty}\frac{dk^2}{k^2}\,\mathcal{M}(k^2) \quad .\end{aligned} \tag{1.37}$$

Using Eq. (1.36), and carrying out the radial integration, we obtain

$$B(p^2)^{-s} = m_0 + \frac{\alpha}{4p^2}\left[\frac{3}{\pi}\,I(p_c^2) - \frac{B}{1-s}\,(p_c^2)^{1-s}\right] + \frac{3\alpha B}{4\pi s(1-s)}\,(p^2)^{-s} \quad , \tag{1.38}$$

where

$$I(p_c^2) = \int_0^{p_c^2} dk^2\,\frac{k^2\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} \quad . \tag{1.39}$$

Comparing the coefficients of $(p^2)^{-s}$, we get a quadratic equation in s which gives

$$s = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{\alpha}{\alpha_c}} \quad , \tag{1.40}$$

where $\alpha_c = \pi/3$. Therefore, the exponent s is completely determined by the large p^2 behaviour of the mass function. For $\alpha > \alpha_c$, the solution of the mass function enters the complex plane. As we shall see in Chapter 2, this is an indication that a phase transition has taken place from perturbative to non-perturbative solution corresponding to the dynamical generation of mass. Comparing the constant terms on both sides of Eq. (1.38), we deduce $m_0 = 0$. We shall see that this is a consequence of not introducing an ultraviolet cut-off on the momentum integral. Equating the coefficients of $1/p^2$, the constant B is given by

$$B = \frac{3(1-s)}{\pi}(p_c^2)^{s-1} I(p_c^2) \quad . \tag{1.41}$$

Hence, we find that in order to evaluate B, we have to know the behaviour of $\mathcal{M}(p^2)$ in all ranges of p^2 including the infrared region. This is unlike the exponent s which is completely fixed by the knowledge of $\mathcal{M}(p^2)$ when $p^2 \to \infty$.

Had we introduced an ultraviolet cut-off Λ in Eq. (1.37) then, instead of Eq. (1.38), we would have arrived at the following equation.

$$\begin{aligned} B(p^2)^{-s} &= m_0 - \frac{3\alpha B}{4\pi s}(\Lambda)^{-s} + \frac{\alpha}{4p^2}\left[\frac{3}{\pi} I(p_c^2) - \frac{B}{1-s}(p_c^2)^{1-s}\right] \\ &+ \frac{3\alpha B}{4\pi s(1-s)}(p^2)^{-s} \quad . \end{aligned} \tag{1.42}$$

Then equating the constant terms gives

$$m_0 = \frac{3\alpha B}{2\pi\left[1 \pm \sqrt{1-\alpha/\alpha_c}\right]}(\Lambda^2)^{-\frac{1}{2}\left[1\pm\sqrt{1-\alpha/\alpha_c}\right]} \quad . \tag{1.43}$$

This expression tells us how fast the bare mass falls to zero with the cut-off. The rate of fall is different for the two solutions. As expected, the faster the mass function drops off, the faster the bare mass approaches zero.

We shall now investigate the equation for the mass function $\mathcal{M}(p^2)$ in the limit when $p^2 \to 0$. We expect $\mathcal{M}(p^2)$ to be a constant. Let ${p_c'}^2$ be the momentum below which the mass function behaves as a constant. Then, on assuming the solution

$$\mathcal{M}(p^2) = A \qquad \text{for} \qquad p^2 \to 0 \quad , \tag{1.44}$$

we obtain

$$A = m_0 + \frac{3\alpha A}{4\pi p^2}\int_0^{p^2} dk^2 \frac{k^2}{k^2+A^2} + \frac{3\alpha A}{4\pi}\int_{p^2}^{{p_c'}^2} dk^2 \frac{1}{k^2+A^2} + \frac{3\alpha}{4\pi} I({p_c'}^2) \quad ,$$

where

$$I({p_c'}^2) = \int_{{p_c'}^2}^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} \quad . \tag{1.45}$$

On carrying out the radial integration, and neglecting terms of the order $\mathcal{O}\left(p^2/A^2\right)$ in comparison with 1, we find

$$A = m_0 + \frac{3\alpha A}{4\pi}\left[1 + \ln\left({p_c'}^2 + A^2\right)\right] + \frac{3\alpha}{4\pi}\, I({p_c'}^2) \quad . \tag{1.46}$$

This equation determines A. But in order to find its exact value, we again have to know $\mathcal{M}(p^2)$ at all p^2.

From the discussion in this section, we come to conclude that for $p^2 \to 0$, the mass function has a constant value, whereas, for asymptotic values of p^2, it drops off as $(p^2)^{-s}$, where s is given by Eq. (1.40). This behaviour is worth emphasizing because we shall later discover that these features are retained by the mass function even for more sophisticated vertices such as the Ball-Chiu (BC) [22] or the Curtis-Pennington (CP) vertex [24].

The fact that the mass function $\mathcal{M}(p^2)$ can have a non-zero solution despite the bare mass m_0 being zero is an indication that we shall encounter features which are alien to perturbation theory. We therefore discuss this a little more in the next section.

Eq. (1.35) was formally solved by Miransky et al. [14]. Defining $M = \mathcal{M}(M^2)$, they obtained the following solution for the Euclidean mass $\mathcal{M}(M^2) = M$:

$$\begin{aligned} M &= 4\,\Lambda\,\exp\left(-\frac{\pi}{\tau}\right) \qquad && \text{for} \quad \alpha > \alpha_c \\ M &= 0 && \text{for} \quad \alpha < \alpha_c \quad , \end{aligned} \tag{1.47}$$

where

$$\tau = \sqrt{\frac{\alpha}{\alpha_c} - 1}$$

with $\alpha_c = \pi/3$. Below $\alpha = \alpha_c$, this solution coincides with that of the perturbation theory. However. beyond the critical value of the coupling, the non-zero solution bifurcates away from the trivial solution. This is best illustrated in Fig. [1.8]. Such a behaviour of the mass function is in complete contrast with the perturbation theory, where, even if we perform an all orders resummation using the Renormalization Group Equation, we end up with a result of the following form,

$$
\begin{aligned}
\mathcal{M}(p^2) &= m_0 X(p^2) \\
X(p^2) &= \sum_n^{\infty} \sum_m^{n} \alpha^n A_n B_{m,n} \ln^m(p^2/\Lambda^2)
\end{aligned}
$$

and the field remains massless to all orders if we start with a zero bare mass, $m_0 = 0$.

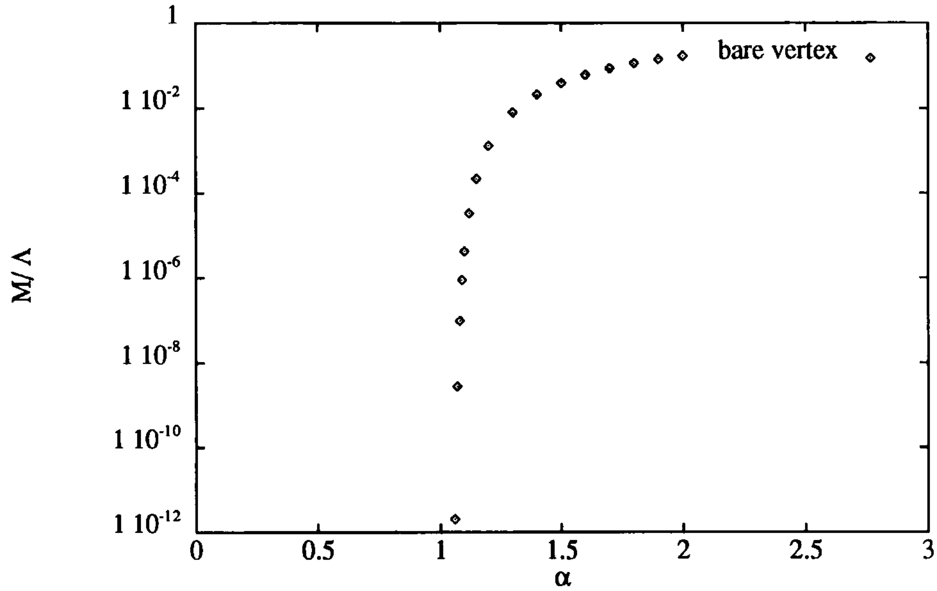

Fig. 1.8 : The Euclidean mass plotted against the coupling. At a critical value of the coupling, the non-zero solution bifurcates away from the perturbative solution.

The idea of DCSB and mass generation can have a very important role to play in physical theories. However, attempts to achieve this cannot be reliable unless the assumptions made have a sound physical basis. In the following sections, we outline the progress made so far in that direction.

1.8 Gauge Covariance and the Vertex Ansatz

An immediate objection to the choice of the bare vertex as the full *ansatz* is that the bare vertex does not respect one of the key features of a gauge theory, i.e. its gauge covariance as we shall see shortly. This fact led Ball and Chiu [22] to construct a non-perturbative vertex which ensures the gauge covariance of the fermion propagator by satisfying the corresponding Ward Takahashi Identity (WTI). This is the minimal requirement. The complete 3-point function complying with the requirements of local gauge covariance will, however, be the one that fulfills the so called LKF transformations discussed in Chapter 3 in detail. Below, we discuss the role of the WTI in restricting the vertex function.

1.8.1 WTI and the Bare Vertex

The Ward Takahashi Identity (WTI) that relates the fermion propagator with the fermion boson vertex is :

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p) \quad . \tag{1.48}$$

This identity is true non-non-perturbatively as well as at every level of truncation in perturbation theory. If we substitute Eq. (1.19) in the above equation and replace $\Gamma^\mu(k,p)$ by the bare vertex γ^μ, we can write

$$\not{q} = \frac{\not{k}}{F(k^2)} - \frac{\not{p}}{F(p^2)} - \frac{\mathcal{M}(k^2)}{F(k^2)} + \frac{\mathcal{M}(p^2)}{F(p^2)} \quad . \tag{1.49}$$

Obviously, the WTI cannot be satisfied in all gauges. In particular, in the Landau gauge, where $F(k^2) = F(p^2) = 1$ in this rainbow approximation, the above equation gives

$$\mathcal{M}(p^2) = \mathcal{M}(k^2) \quad .$$

This equation does not hold true unless both k^2 and $p^2 \to 0$, the region where $\mathcal{M}$ is roughly constant. Therefore, we conclude that *the bare vertex fails to satisfy the WTI except in the Landau gauge where it is true only for values of k^2 and p^2 less than M^2.*

1.8.2 WTI and The Longitudinal Vertex

Apparently, the straightforward conclusion from the WTI is that

$$\Gamma^\mu(k,p) = \frac{S_F^{-1}(k) - S_F^{-1}(p)}{k^2 - p^2} \, (k+p)^\mu \quad .$$

However, it is easy to see that this *ansatz* is plagued with the presence of a kinematic singularity when $k^2 \to p^2$. This can be seen by using Eq. (1.19) in the above equation which gives

$$\Gamma^\mu(k,p) = \frac{1}{k^2-p^2}\left[\frac{\not k}{F(k^2)} - \frac{\not p}{F(p^2)} - \frac{\mathcal{M}(k^2)}{F(k^2)} + \frac{\mathcal{M}(p^2)}{F(p^2)}\right] (k+p)^\mu \quad .$$

Let $k^2 \to p^2$ without demanding $k \to p$. Then the term

$$\frac{1}{k^2-p^2}\left[\frac{\not k}{F(k^2)} - \frac{\not p}{F(p^2)}\right]$$

has the above mentioned unacceptable kinematic singularity. However, it is not very difficult to get rid of this undesirable feature. The form of the WTI suggests, as noticed by Ball and Chiu [22], that we can decompose the full non-perturbative vertex into two components— longitudinal and transverse :

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p) \tag{1.50}$$

where the transverse component is defined by

$$q_\mu \Gamma^\mu_T(k,p) = 0 \quad . \tag{1.51}$$

It is trivial to realise that the transverse part of the vertex is, by definition, completely unspecified by the WTI.

Ball and Chiu [22] made the crucial assumption that the vertex be free of kinematic singularities. It led them to a unique form for the longitudinal part of the vertex. This assumption is automatically taken care of if we start from the limit $k \to p$ of the WTI, known as the Ward Identity (WI)

$$\frac{\partial S_F^{-1}(p)}{\partial p_\mu} = \Gamma^\mu(p,p) \tag{1.52}$$

and proceed systematically as described below. The expression for the full fermion propagator permits us to write

$$\begin{aligned}\Gamma^\mu(p,p) &= \frac{\partial}{\partial p_\mu}\left[\frac{\not p}{F(p^2)}\right] - \frac{\partial}{\partial p_\mu}\left[\frac{\mathcal{M}(p^2)}{F(p^2)}\right] \\ &= \frac{\gamma^\mu}{F(p^2)} + 2p^\mu \not p \frac{\partial}{\partial p^2}\left[\frac{1}{F(p^2)}\right] - 2p^\mu \frac{\partial}{\partial p^2}\left[\frac{\mathcal{M}(p^2)}{F(p^2)}\right]\end{aligned} \tag{1.53}$$

where we have used

$$\frac{\partial}{\partial p_\mu} = 2p^\mu \frac{\partial}{\partial p^2} \quad .$$

Now keeping in mind the correct symmetry of the vertex under the interchange of k and p, we can write the longitudinal part of the vertex as follows :

$$\Gamma^{\mu}_{L}(k,p) = \frac{1}{2}\left[\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right]\gamma^{\mu}+\frac{1}{2}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]\frac{(k+p)^{\mu}(\not k+\not p)}{k^2-p^2}$$
$$-\left[\frac{\mathcal{M}(k^2)}{F(k^2)}-\frac{\mathcal{M}(p^2)}{F(p^2)}\right]\frac{(k+p)^{\mu}}{k^2-p^2} \quad . \tag{1.54}$$

It is obviously free of kinematic singularities. This expression for the longitudinal part of the vertex is usually referred to as the Ball-Chiu vertex. The *BC* vertex has the nice feature of preserving both the WI and WTI. These identities, however, leave the transverse part of the vertex undefined. We shall see in the next section that the transverse part is constrained by the requirement of the MR of the fermion propagator.

1.9 The Transverse Vertex

Although the WTI completely fixes the longitudinal vertex but any *ansatz* can serve as the transverse vertex as long as it satisfies Eq. (1.51) and

$$\Gamma_T(p,p) = 0 \quad . \tag{1.55}$$

This is because the *BC* longitudinal part alone satisfies the WI.

What more can be said about the transverse vertex? A systematic approach towards answering this question is to try to find basis vectors in terms of which the transverse part of the vertex can be written in its most general form. This was again accomplished by Ball and Chiu [22]. In case of a spin-$\frac{1}{2}$ Dirac particle interacting with the photon, we have three 4-vectors

$$\gamma^{\mu}, \quad k^{\mu}, \quad p^{\mu}$$

and four Lorentz scalars

$$1, \quad \not k, \quad \not p, \quad \not k\,\not p$$

to play with, from which, 12 independent vectors can be constructed. These can be

$$\begin{array}{cccc} \gamma^\mu, & \not k\gamma^\mu, & \not p\gamma^\mu, & \not k \not p \gamma^\mu \\ k^\mu, & \not k k^\mu, & \not p k^\mu, & \not k \not p k^\mu \\ p^\mu, & \not k p^\mu, & \not p p^\mu, & \not k \not p p^\mu \end{array}$$

or any linear combination of these. Then all vectors involving momenta k and p can be expressed in terms of the chosen set of basis vectors. Eq. (1.54) suggests that three of the basis vectors are γ^μ, $(k+p)^\mu$ and $(\not k + \not p)(k+p)^\mu$. The only scalar absent is $\not k \not p$ which indicates that the coefficient of the corresponding basis vector is identically zero. The remaining eight tensors serve as a basis to express the transverse vertex. They must satisfy

$$q_\mu T_i^\mu = 0 \qquad \text{for} \qquad i = 1, 2, \cdots, 8 \quad .$$

A set of such independent T's is [22] :

$$\begin{aligned} T_1^\mu(k,p) &= p^\mu(k.q) - k^\mu(p.q) \\ T_2^\mu(k,p) &= T_1^\mu(\not k + \not p) \\ T_3^\mu(k,p) &= q^2\gamma^\mu - q^\mu \not q \\ T_4^\mu(k,p) &= T_1^\mu p^\nu k^\rho \sigma_{\nu\rho} \\ T_5^\mu(k,p) &= \sigma^{\mu\nu} q_\nu \\ T_6^\mu(k,p) &= \gamma^\mu(k^2 - p^2) - (k+p)^\mu(\not k - \not p) \\ T_7^\mu(k,p) &= \frac{1}{2}(k^2 - p^2)[\gamma^\mu(\not k + \not p) - p^\mu - k^\mu] + (k+p)^\mu p^\nu k^\rho \sigma_{\nu\rho} \\ T_8^\mu(k,p) &= -\gamma^\mu p^\nu k^\rho \sigma_{\nu\rho} + p^\mu \not k - k^\mu \not p \quad . \end{aligned} \tag{1.56}$$

More discussion on these basis vectors can be found in Chapter 3. The transverse vertex can now be expressed as

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau_i(k^2, p^2, q^2) T_i^\mu(k,p) \quad . \tag{1.57}$$

We shall see that it is the coefficients $\tau_i(k^2, p^2, q^2)$ which are constrained by the requirements of MR. In the next section, we present a brief reminder of its importance. We shall then check each vertex *ansatz* against the demands of MR.

1.10 MR and the Vertex Ansatz

It is due to the presence of ultraviolet and/or infrared divergences that we usually have to to renormalize a theory to obtain physical predictions. Regularization is needed to separate the divergent parts from the finite ones. For example, Λ serves as a regulator to cut-off the ultraviolet divergences. MR means that by rescaling the fields, masses and couplings of a theory, we can make Green's functions finite as the regularization that we introduced is lifted. QED is multiplicatively renormalizable. In general, although the proof exists only within perturbation theory, it is believed to be true for the complete theory even outside the domain of the perturbative expansion. Multiplicative renormalizability is of vital importance as it leads to the derivation of the renormalization group equations for the Green's functions, revealing how these functions evolve with various mass scales in the theory. Therefore, it sounds reasonable to start by demanding the MR of the quenched QED.

Let us pick up massless QED for simplicity. The WTI can then be written as

$$q_\mu \Gamma^\mu = \frac{\not k}{F(k^2)} - \frac{\not p}{F(p^2)} \quad . \tag{1.58}$$

MR of the fermion propagator requires that there exists a factor $Z_2^{-1}(\mu^2/\Lambda^2)$ that makes $F(p^2/\Lambda^2)$ independent of Λ^2 to give the renormalized fermion function $F_R(p^2/\mu^2)$, μ being the renormalization scale :

$$F_R(p^2/\mu^2) \;=\; Z_2^{-1}(\mu^2/\Lambda^2)F(p^2/\Lambda^2) \quad . \tag{1.59}$$

This implies

$$\frac{F(k^2/\Lambda^2)}{F(p^2/\Lambda^2)} \;=\; \left(\frac{k^2}{p^2}\right)^\nu \tag{1.60}$$

where $\nu = f(\alpha, \xi)$ is a constant as α does not run in quenched QED. Eq. (1.58) shows that the function $F(p^2/\Lambda^2)$ depends on the choice of the full vertex. Brown and Dorey [25] have argued that an arbitrary *ansatz* for the vertex does not satisfy the requirement of MR. It was realised that neither the bare vertex nor the BC vertex were good enough to fulfill the demands of MR. Curtis and Pennington [24] showed that this requirement restricts the form of the transverse vertex. They put forward an *ansatz* for the 3-point function which not only satisfies the WTI but also guarantees the MR of the fermion propagator.

In the following three sub-sections, we shall discuss the bare vertex, the *BC* vertex and the *CP* vertex in the context of MR of the fermion propagator. In order to clarify some of the ideas, a few mathematical results will be presented whose derivation will be postponed till Chapter 2.

1.10.1 The Bare Vertex

We start by recalling the equation for the wavefunction renormalization $F(p^2)$, Eq. (1.33), in the case of the bare vertex. For simplicity, we analyse it only in the massless limit :

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha\xi}{4\pi} \int_0^{\Lambda^2} \frac{dk^2}{k^2} F(k^2) \left[\frac{k^4}{p^4} \theta(p^2 - k^2) + \theta(k^2 - p^2) \right] \quad .$$

where Λ is the ultraviolet cut-off. Let

$$F(p^2) = A(p^2)^\nu \quad . \tag{1.61}$$

Only if we find a consistent solution for A and ν, shall we be able to say that the bare vertex leads to a multiplicatively renormalizable solution for $F(p^2)$. Substituting this in the above equation, we obtain :

$$\frac{1}{F(p^2)} = 1 - \frac{\alpha\xi}{4\pi\nu} \left[\frac{2F(p^2)}{\nu + 2} - F(\Lambda^2) \right] \quad . \tag{1.62}$$

This equation does not have any solution for A and ν, except in the Landau gauge, where $A = 1$ and $\nu = 0$ solve the equation. Therefore, for the bare vertex, $F(p^2)$ has a multiplicatively renormalizable solution only in the Landau gauge.

1.10.2 BC Vertex

If instead of the bare vertex, we had used the BC vertex, an analogous calculation would have led us to the following equation for $F(p^2)$ in the massless limit.

$$\begin{aligned}\frac{1}{F(p^2)} &= 1-\frac{\alpha\xi}{4\pi}\frac{1}{F(p^2)}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)\\ &-\frac{3\alpha}{16\pi}\int_0^{p^2}\frac{dk^2}{p^2}\frac{k^2}{p^2}\frac{k^2+p^2}{k^2-p^2}\left[1-\frac{F(k^2)}{F(p^2)}\right]\\ &-\frac{3\alpha}{16\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{k^2+p^2}{k^2-p^2}\left[1-\frac{F(k^2)}{F(p^2)}\right] \quad . \end{aligned} \tag{1.63}$$

Again assuming a multiplicatively renormalizable solution $F(p^2)=A(p^2)^\nu$ and carrying out the integration, we arrive at

$$F(p^2)=1-\frac{\alpha\xi}{4\pi\nu}F(\Lambda^2)+\frac{3\alpha}{16\pi}F(p^2)\left[\frac{5}{2}+2\pi\cot\pi\nu-\frac{1}{\nu}-\frac{2}{\nu+1}-\frac{1}{\nu+2}+\ln\frac{\Lambda^2}{p^2}\right] \quad .$$

The explicit presence of the term $\ln(\Lambda^2/p^2)$ prevents a multiplicatively renormalizable solution.

1.10.3 CP Vertex

Curtis and Pennington [24] looked for a simple transverse vertex that could restore the MR of the fermion propagator. They noticed that probably the simplest way to achieve this is by choosing all the coefficients τ_i in Eq. (1.57) to be equal to zero except τ_6. We can then define

$$\Gamma^\mu_T(k,p)=\tau_6(k^2,p^2)T^\mu_6(k,p) \quad , \tag{1.64}$$

where we have assumed that τ_6 does not depend on q^2. Now repeating the same exercise as was carried out for the bare and the BC vertices, gives

$$\begin{aligned}\frac{1}{F(p^2)} &= 1+\frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)}-\frac{\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)\\ &\left\{\frac{k^4}{p^4}\left[\frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)+\frac{3}{2}(k^2-p^2)\tau_6(k^2,p^2)\right]\theta(p^2-k^2)\right.\\ &\left.\left[\frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)+\frac{3}{2}(k^2-p^2)\tau_6(k^2,p^2)\right]\theta(k^2-p^2)\right\} \quad .\end{aligned}$$

One can easily see that the required cancellation of the divergent term that spoiled the MR of $F(p^2)$ in case of the *BC* vertex, takes place with the choice

$$\tau_6(k^2,p^2) = -\frac{1}{2}\frac{k^2+p^2}{(k^2-p^2)^2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \quad . \tag{1.65}$$

We then get

$$F(p^2) = \left[\frac{p^2}{\Lambda^2}\right]^{(\alpha\xi/4\pi)} \quad . \tag{1.66}$$

Hence the *CP* vertex proved to be a success in restoring the MR of the fermion propagator, a key feature of a gauge theory. We have seen in this chapter that there has been a gradual progress in finding an increasingly refined *ansatz* for the vertex, the guideline being provided by the WTI which is a consequence of gauge covariance, and the MR of the fermion propagator. The natural next step is to use the *CP* vertex in solving the DSE for the fermion propagator and to compare the results with the solution obtained from using the bare vertex. This may serve as a guide to improve further on the *CP* vertex.

Chapter 2

Looking for the Vertex that Does It All

In the last chapter, we have discussed the importance of the fermion-photon vertex function in the study of Dyson Schwinger Equations (DSE) for the fermion propagator and presented a brief review of the advantages and disadvantages of a few vertex *ansatze* in this regard. All this discussion was carried out in the context of the Ward Takahashi Identity (WTI) and the multiplicative renormalizability (MR) of the fermion propagator. WTI is a consequence of gauge covariance. However, gauge covariance requires more. Various quantities have to have the correct gauge parameter dependence. The issue of gauge covariance in the DSE approach to the solution of gauge field theories is very important. Lack of gauge covariance in much work in this field has hindered its acceptance as a perfectly satisfactory non-perturbative tool. There are many contemporary works that fail to address this issue. If the fermion-boson vertex is known, the DSEs for the propagators decouple from the rest of the infinite tower. The structure of the fermion-boson vertex is crucial in obtaining the correct gauge parameter dependence of the quantities one calculates in the DSE approach. This indicates the importance of this vertex. In this chapter, we shall see how well or badly various vertex *ansatze* perform under this requirement. We shall then try to construct a vertex with the aim of improving on the gauge dependence of the physical observables in comparison with that obtained from any of the previous vertex *ansatze.*

2.1 Critical Coupling and Vertex Ansatze

Recall Fig. [1.8]. It beautifully illustrates the dynamical generation of mass beyond a critical value of the coupling, α_c, using the bare vertex as the full *ansatz*. The calculation was performed in the Landau gauge. Despite the fact that this is a very interesting result, one can readily discover that there are problems. As the critical coupling corresponds to a change of phase, it is expected to be independent of the gauge parameter. But when one solves the Eqs. (1.33) and (1.34) for different gauges, one finds that this is not the case, as depicted in Fig. [2.1].

However, it is not difficult to trace the root of this problem. The full vertex has to satisfy the Ward-Takahashi Identity (WTI) for the fermion propagator to ensure its gauge covariance. The bare vertex that was used in Eqs. (1.33) and (1.34) does not obey this identity. Therefore, one should not expect physical outputs to be gauge independent when the input is not gauge covariant.

The *CP* vertex incorporates both the features which the bare vertex lacks, i.e. it obeys WTI in all gauges, and it provides a multiplicatively renormalizable solution for $F(p^2)$. Curtis and Pennington [23] solved the coupled equations for F and M, using this *ansatz*. They found that the gauge dependence of the critical coupling at which the non-perturbative behaviour bifurcates away from the perturbative one reduces considerably, as seen by comparing Figs. [2.1] and [2.2].

The reduced gauge dependence has not been imposed explicitly, but is a consequence of satisfying the WTI and requiring MR of the fermion propagator. Fig. [2.2] was an outcome of the numerical evaluation of the Euclidean mass. It does not tell us the exact value for the critical coupling. Atkinson et al., [26, 27, 28], have recently proposed a method using bifurcation analysis to locate the critical coupling precisely. We discuss this method in the next section.

2.2 Bifurcation Analysis

Bifurcation analysis is the study of the critical point where the non-perturbative solution bifurcates away from the perturbative solution, and mass is generated. A solution for the mass function is a power of the momentum that has to satisfy a transcendental equation. The onset of criticality is governed by the coming together of two solutions of the transcendental equation. It indicates that oscillatory behaviour takes over from the non-oscillatory one. To investigate this critical point, one has to take the Frechet derivative of the nonlinear operators with respect to $\mathcal{M}(p^2)$ and evaluate it at the trivial point, $\mathcal{M}(p^2) = 0$. This amounts in fact simply to throwing away all terms that are quadratic or higher in the mass function. It must be emphasized that this is not an approximation : it is a precise way to locate the critical point by applying bifurcation theory. In this section, we shall apply the bifurcation analysis to various choices of the vertex, in particular the *CP* vertex, and study the precise dependence of the critical coupling on the gauge parameter. The critical coupling is potentially a physically measurable quantity, since it signals a change of phase, and so it should be gauge invariant. We shall see that although this is not exactly true with the use of the *CP* vertex, it is approximately so. Indeed, the requirement that α_c be gauge invariant will, in next sections, be used to constrain the vertex function further.

2.2.1 Bare Vertex

Let us start with the bare vertex. Here we can readily reproduce the results of Section (1.7). Dropping the terms quadratic and higher in mass, the mass function satisfies the following equation :

$$\mathcal{M}(p^2) \;=\; \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)+\frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\mathcal{M}(k^2)\,. \qquad (2.1)$$

This equation has the multiplicatively renormalizable solution,

$$\mathcal{M}(p^2) = B\left(p^2\right)^{-s} \qquad (2.2)$$

It is only at the bifurcation point that this simple behaviour of the mass function holds at all momenta. There, only when the mass is still effectively zero is there just one scale, Λ, for the momentum dependence of $\mathcal{M}(p^2)$. MR then forces a simple power behaviour. The above equation requires

$$\frac{4\pi}{3\alpha} \;=\; \frac{1}{s}+\frac{1}{1-s}\quad , \qquad (2.3)$$

where it has been assumed that $0 < s \leq 1$. This quadratic equation in s has the solution (1.40). Bifurcation occurs when the two roots of s merge with each other. The value of the coupling at this point corresponds to the critical coupling. Similar analytical steps cannot be carried out in an arbitrary gauge as there is no analytic expression for $F(p^2)$. Note that for the bare vertex, $F(p^2)$ does not have a multiplicatively renormalizable solution.

2.2.2 CP Vertex

Consider the *CP* vertex in the Landau gauge where it is identical to the *BC* vertex. If the terms quadratic and higher in mass are dropped, the mass function satisfies the following equation :

$$\begin{aligned} \mathcal{M}(p^2) &= \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)-\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\right)\right] \\ &+ \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)-\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\right)\right] . \qquad (2.4) \end{aligned}$$

As before, using the multiplicatively renormalizable solution $\mathcal{M}(p^2) \sim (p^2)^{-s}$, we arrive at the following equation,

$$\frac{8\pi}{3\alpha} = 1+\frac{3}{s}+\frac{1}{1-s}-\pi\cot\pi s \equiv f(s) \qquad (2.5)$$

There are two roots for s between 0 and 1. Bifurcation occurs when the two roots for s merge at $s = s_c$, specified by $f'(s_c) = 0$. This point defines the critical coupling [26, 27, 28], $\alpha_c = 8\pi/3f(s_c)$. Numerically, $\alpha_c = 0.933667$ and $s_c = 0.470966$.

In contrast to the situation with the bare vertex, it is now possible to go ahead and employ the same procedure for an arbitrary gauge, as the *CP* vertex ensures a multiplicatively renormalizable solution for $F(p^2)$, i.e. $F(p^2) \sim (p^2)^\nu$. One then arrives at the following equations in an arbitrary gauge :

$$\nu = \frac{\alpha\xi}{4\pi} \qquad (2.6)$$

$$\begin{aligned} \xi &= \frac{3\nu(\nu-s+1)}{2(1-s)}\Big[3\pi\cot\pi(\nu-s)+2\pi\cot\pi s-\pi\cot\pi\nu \\ &+ \frac{1}{\nu}+\frac{1}{\nu+1}+\frac{2}{1-s}+\frac{3}{s-\nu}+\frac{1}{s-\nu-1}\Big] . \qquad (2.7) \end{aligned}$$

In other than the Landau gauge, particularly when ξ is large, Eq. (2.7) has more than two roots for s between 0 and 1, but one is interested in those that are continuously connected to the two that are present in the Landau gauge. Bifurcation occurs when the two roots for s merge at a point specified by the necessary condition $\partial\xi/\partial s = 0$. This condition leads to the following equation :

$$\xi = \frac{-3(\nu - s + 1)^2}{2}\left[3\pi^2 \csc^2 \pi(\nu - s) - 2\pi^2 \csc^2 \pi s + \frac{2}{(1-s)^2} - \frac{3}{(s-\nu)^2} - \frac{1}{(s-\nu-1)^2}\right] . \quad (2.8)$$

Simultaneous solution of Eqs. (2.6, 2.7, 2.8) gives α_c as a function of ξ. The results have been illustrated in Fig. [2.3].

One clearly sees that the gauge dependence of α_c is much less severe with the use of the *CP* vertex than with the bare vertex. However, all physical observables should be strictly gauge independent. Therefore, there will always be room for improvement however mild the gauge dependence might be. We shall now aim to look for an *ansatz* which can further reduce the gauge dependence of α_c. It is here that it seems essential to summarise all the requirements of the vertex function in order to be equipped with the necessary information to construct a vertex.

2.3 Requirements of the Vertex Function

We expect that any reasonable *ansatz* for the vertex should fulfill the following requirements which extend the list of Burden et al [34]:

- It must satisfy the WTI in all gauges.

$$q^\mu \Gamma_\mu = S_F^{-1}(k) - S_F^{-1}(p)$$

- It must ensure that the fermion propagator is multiplicatively renormalizable.

- It must result in a critical coupling, at which mass is generated dynamically, that is gauge independent.

- It must be free of any kinematic singularities, i.e. it should have a unique limit when $k^2 \to p^2$.

- It must have the same transformation properties as the bare vertex γ^μ under the operation of charge conjugation and parity.
- It must reduce to the bare vertex in the free field limit in the manner prescribed by perturbation theory.
- It should ensure local gauge covariance of the propagators and the vertex.

Although the first condition is a consequence of gauge invariance, it only restricts the longitudinal part of the vertex, and says nothing about the transverse part. By itself, it is insufficient to ensure the last condition. A well defined set of transformation rules in quantum electrodynamics, which relate a Green's function in one gauge to the same Green's function in another gauge, has been given by Landau and Khalatnikov (1956) and Fradkin (1956) [32]— and henceforward is known as the LKF transformations. These rules leave the DSEs and WTI form-invariant. One can in principle ensure the last condition by choosing an *ansatz* for the vertex which is covariant under the action of the LKF transformations. Unfortunately, it has so far been practically impossible to implement this procedure, as the transformation rule for the vertex is far from simple, and, moreover, these rules are expressed in coordinate space, which makes their use more complicated. However, the LKF transformation rule for the fermion propagator is relatively straightforward. One can use this transformation to check whether the solution for the fermion propagator using a particular vertex *ansatz* transforms appropriately. Although such a procedure has played an important role in constructing an improved vertex, it does not seem to be sufficient to ensure that the physical observables calculated are gauge invariant. The aim of the rest of this chapter will be to see how we can proceed to reduce this gauge dependence.

2.4 Gauge Invariance and the Vertex Function

As has clearly been seen in the last section, in dramatic contrast to the *rainbow* approximation, the critical coupling found with the *CP* vertex is only weakly gauge dependent in the neighbourhood of the Landau gauge. In this section, we shall try to improve on the *CP* vertex, hoping to reduce further the gauge dependence of the critical coupling in a broader range of the gauge parameter.

We start by using the full transverse vertex with its eight unknown components $\tau_i(k^2, p^2, q^2)$, Eqs. (1.56,1.57), in the DSE for the fermion propagator, Eq. (1.26). Now multiplying Eq. (1.26) by $\not p$ and taking the trace of the equation, we arrive at :

$$
\begin{aligned}
\frac{1}{F(p^2)} = 1 - \quad & \frac{\alpha}{4\pi^3}\frac{1}{p^2}\int d^4k \, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)} \, \frac{1}{q^2} \\
\Bigg\{ \quad & \qquad a(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2 - 3k\cdot pq^2\right] \\
+ \quad & \qquad b(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2(k^2+p^2)\right] \\
+ \quad & \mathcal{M}(k^2)c(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2\right] \\
- \quad & \frac{\xi}{q^2F(p^2)} \left[p^2(k^2-k\cdot p) + \mathcal{M}(k^2)\mathcal{M}(p^2)(k\cdot p - p^2)\right] \\
+ \quad & \mathcal{M}(k^2)\tau_1(k^2,p^2,q^2)\left[\Delta^2\right] \\
+ \quad & \qquad \tau_2(k^2,p^2,q^2)\left[-\Delta^2(k^2+p^2)\right] \\
+ \quad & \qquad \tau_3(k^2,p^2,q^2)\left[2\Delta^2 + 3q^2k\cdot p\right] \\
+ \quad & \mathcal{M}(k^2)\tau_4(k^2,p^2,q^2)\left[\Delta^2(k\cdot p - p^2)\right] \\
+ \quad & \mathcal{M}(k^2)\tau_5(k^2,p^2,q^2)\left[3p^2 - 3k\cdot p\right] \\
+ \quad & \qquad \tau_6(k^2,p^2,q^2)\left[(k^2-p^2)3k\cdot p\right] \\
+ \quad & \mathcal{M}(k^2)\tau_7(k^2,p^2,q^2)\left[\Delta^2 + \frac{3}{2}(k^2-p^2)(k\cdot p + p^2)\right] \\
+ \quad & \qquad \tau_8(k^2,p^2,q^2)\left[2\Delta^2\right]\Bigg\} \quad ,
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
a(k^2,p^2) &= \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \quad , \\
b(k^2,p^2) &= \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{k^2-p^2} \quad , \\
c(k^2,p^2) &= \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right)\frac{1}{k^2-p^2} \quad , \\
\Delta^2 &= (k\cdot p)^2 - k^2p^2 \quad .
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
\frac{\mathcal{M}(p^2)}{F(p^2)} = m_0 - \quad & \frac{\alpha}{4\pi^3}\int d^4k\,\frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\,\frac{1}{q^2} \\
& \Big\{ \mathcal{M}(k^2)\,a(k^2,p^2)\,\left[-4+(1-\xi)\right] \\
& -\mathcal{M}(k^2)\,b(k^2,p^2)\,\left[(k+p)^2-\frac{(1-\xi)}{q^2}(k^2-p^2)^2\right] \\
& + \qquad c(k^2,p^2)\,\left[(k^2+k\cdot p)-\frac{(1-\xi)}{q^2}(k^2-p^2)(k^2-k\cdot p)\right] \\
& + \qquad \tau_1(k^2,p^2,q^2)\,\left[\Delta^2\right] \\
& +\mathcal{M}(k^2)\,\tau_2(k^2,p^2,q^2)\,\left[2\Delta^2\right] \\
& +\mathcal{M}(k^2)\,\tau_3(k^2,p^2,q^2)\,\left[3q^2\right] \\
& + \qquad \tau_4(k^2,p^2,q^2)\,\left[\Delta^2(k\cdot p-k^2)\right] \\
& + \qquad \tau_5(k^2,p^2,q^2)\,\left[3\left(k^2-k\cdot p\right)\right] \\
& +\mathcal{M}(k^2)\,\tau_6(k^2,p^2,q^2)\,\left[3(k^2-p^2)\right] \\
& + \qquad \tau_7(k^2,p^2,q^2)\,\left[\Delta^2-\frac{3}{2}(k^2-p^2)(k\cdot p+k^2)\right] \\
& +\mathcal{M}(k^2)\,\tau_8(k^2,p^2,q^2)\,[0]\Big\} \quad . \qquad (2.11)
\end{aligned}
$$

In order to keep track of the signs of various terms while switching over from Minkowski to Euclidean space, it is emphasized that as a convention, $\mathcal{M}(k^2)\mathcal{M}(p^2)$ does not change sign although it has dimensions of m^2. τ_i also remain unchanged. However, $a(k^2,p^2)$, $b(k^2,p^2)$ and $c(k^2,p^2)$ change sign. All the momenta in their definition given above lie in Euclidean space. Now, let us assume that the τ_i do not depend on q^2. This enables us to carry out the integration over the angular variable. On doing so, we arrive at the the following equations :

$$
\begin{aligned}
\frac{1}{F(p^2)} \;=\; & 1 - \frac{\alpha}{4\pi}\int_0^{\Lambda^2} dk^2 \, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)} \\
& \Bigg[\; \frac{k^4}{p^4} \Bigg\{ \; b(k^2,p^2)\left[\frac{3}{2}(k^2+p^2)\right] \\
& \quad + \; c(k^2,p^2)\left[\frac{3}{2}\mathcal{M}(k^2)\right] + \; \frac{\xi}{F(p^2)} \, \frac{\mathcal{M}(k^2)\mathcal{M}(p^2)}{k^2} \\
& \quad + \; \mathcal{M}(k^2)\tau_1(k^2,p^2)\left[\frac{1}{4}(k^2-3p^2)\right] \\
& \quad + \; \tau_2(k^2,p^2)\left[-\frac{1}{4}(k^2+p^2)(k^2-3p^2)\right] \\
& \quad + \; \tau_3(k^2,p^2)\left[\frac{1}{2}(k^2-3p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_4(k^2,p^2)\left[\frac{1}{8}(6p^4-4k^2p^2+k^4)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_5(k^2,p^2)\left[\frac{3}{2k^2}(2p^2-k^2)\right] \\
& \quad + \; \tau_6(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_7(k^2,p^2)\left[\frac{1}{2k^2}(2k^4-3p^4)\right] \\
& \quad + \; \tau_8(k^2,p^2)\left[\frac{1}{2}(k^2-3p^2)\right] \Bigg\}\theta(p^2-k^2) \\
& \; + \Bigg\{ \; b(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
& \quad + \; c(k^2,p^2)\left[\frac{3}{2}\mathcal{M}(k^2)\right] - \; \frac{\xi}{F(p^2)} \\
& \quad + \; \mathcal{M}(k^2)\tau_1(k^2,p^2)\left[\frac{1}{4}(p^2-3k^2)\right] \\
& \quad + \; \tau_2(k^2,p^2)\left[-\frac{1}{4}(k^2+p^2)(p^2-3k^2)\right] \\
& \quad + \; \tau_3(k^2,p^2)\left[\frac{1}{2}(p^2-3k^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_4(k^2,p^2)\left[\frac{1}{8}p^2(4k^2-p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_5(k^2,p^2)\left[\frac{3}{2}\right] \\
& \quad + \; \tau_6(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_7(k^2,p^2)\left[\frac{1}{2}(3k^2-4p^2)\right] \\
& \quad + \; \tau_8(k^2,p^2)\left[\frac{1}{2}(p^2-3k^2)\right] \Bigg\}\theta(k^2-p^2)\Bigg] \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{M}(p^2)}{F(p^2)} \;=\;& m_0 - \frac{\alpha}{4\pi}\int_0^{\Lambda^2} dk^2\, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)} \\
& \Bigg[\; \frac{k^2}{p^2}\, \Bigg\{ \mathcal{M}(k^2)\left(\frac{-3}{F(k^2)} - \frac{\xi}{F(p^2)}\right) \\
& \quad - \; b(k^2,p^2)\left[3p^2\mathcal{M}(k^2)\right] + c(k^2,p^2)\left[\frac{3}{2}k^2\right] \\
& \quad + \; \tau_1(k^2,p^2)\left[\frac{1}{4}k^2(k^2-3p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_2(k^2,p^2)\left[\frac{1}{2}k^2(k^2-3p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_3(k^2,p^2)\left[3p^2\right] \\
& \quad + \; \tau_4(k^2,p^2)\left[\frac{1}{8}k^4(4p^2-k^2)\right] \\
& \quad + \; \tau_5(k^2,p^2)\left[\frac{3}{2}k^2\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_6(k^2,p^2)\left[3(k^2-p^2)\right] \\
& \quad + \; \tau_7(k^2,p^2)\left[\frac{1}{2}k^2(3p^2-4k^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_8(k^2,p^2)\left[\,0\,\right] \Bigg\}\theta(p^2-k^2) \\
& + \Bigg\{ \frac{\mathcal{M}(k^2)}{F(p^2)}\left(-3 - \xi\frac{\mathcal{M}(p^2)}{\mathcal{M}(k^2)}\right) \\
& \quad -b(k^2,p^2)\left[3k^2\mathcal{M}(k^2)\right] + c(k^2,p^2)\left[\frac{3}{2}p^2\right] \\
& \quad + \; \tau_1(k^2,p^2)\left[\frac{1}{4}p^2(p^2-3k^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_2(k^2,p^2)\left[\frac{1}{2}p^2(p^2-3k^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_3(k^2,p^2)\left[3k^2\right] \\
& \quad + \; \tau_4(k^2,p^2)\left[\frac{1}{8}p^2(6k^4-4k^2p^2+p^4)\right] \\
& \quad + \; \tau_5(k^2,p^2)\left[\frac{3}{2}(2k^2-p^2)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_6(k^2,p^2)\left[3(k^2-p^2)\right] \\
& \quad + \; \tau_7(k^2,p^2)\left[\frac{1}{2}(-3k^4+2p^4)\right] \\
& \quad + \; \mathcal{M}(k^2)\tau_8(k^2,p^2)\left[\,0\,\right] \Bigg\}\theta(k^2-p^2)\Bigg]\,. \qquad (2.13)
\end{aligned}
$$

At this stage, an *ansatz* for the vertex is needed. In order to be well-equipped to provide this, let us recall some of the **properties** of the τ_i.

- Under the operation of charge conjugation, the vertex transforms as follows :

$$C\,\Gamma_\mu(k,p)\,C^{-1} = -\,\Gamma_\mu^T(-p,-k) \quad . \tag{2.14}$$

Using the identities,

$$C = -C^T \qquad\qquad C\,\gamma_\mu\,C^{-1} = -\,\gamma_\mu^T \tag{2.15}$$

it is easy to see that

$$\begin{aligned} C\,T_i^\mu(k,p)\,C^{-1} &= -\,T_i^{\mu\,T}(-p,-k) \qquad \text{for} \qquad i \neq 6 \\ C\,T_6^\mu(k,p)\,C^{-1} &= T_6^{\mu\,T}(-p,-k) \quad . \end{aligned} \tag{2.16}$$

This implies that all the τ_i are symmetric, except for τ_6 which is antisymmetric under $k^2 \leftrightarrow p^2$.

- In order that the vertex be dimensionless, we must have,

$$\begin{aligned} \tau_2 &\sim \frac{1}{M^4}, \quad \tau_3 \sim \frac{1}{M^2}, \quad \tau_6 \sim \frac{1}{M^2}, \quad \tau_8 \sim \frac{1}{M^2}, \\ \tau_1 &\sim \frac{1}{M^3}, \quad \tau_4 \sim \frac{1}{M^5}, \quad \tau_5 \sim \frac{1}{M}, \quad \tau_7 \sim \frac{1}{M^3}. \end{aligned} \tag{2.17}$$

- We learn from perturbation theory that when momentum in one of the fermion legs is much greater than that in the other, e.g. $k^2 \gg p^2$, the vertex behaves as follows for the leading logarithmic terms [24] :

$$\Gamma_T^\mu(k,p) \simeq -\frac{\alpha\xi}{8\pi}\,\ln\frac{k^2}{p^2}\left[\gamma^\mu - \frac{k^\mu \not{k}}{k^2}\right] \quad , \tag{2.18}$$

where, as usual, $\alpha = e^2/4\pi$. One can easily see that only T_3 and T_6 have this large momentum behaviour. Therefore, for $k^2 \gg p^2$, the following must hold true :

$$\tau_3 + \tau_6 \simeq \frac{\alpha\xi}{8\pi}\frac{1}{k^2}\ln\frac{k^2}{p^2} \tag{2.19}$$

Moreover, in the same limit $k^2 \gg p^2$:

$$\begin{aligned} \tau_1 &< \frac{1}{k^2}, \quad \tau_2 < \frac{1}{k^3}, \quad \tau_4 < \frac{1}{k^3}, \\ \tau_5 &< \frac{1}{k^3}, \quad \tau_7 < \frac{1}{k^3}, \quad \tau_8 < \frac{1}{k}. \end{aligned} \tag{2.20}$$

- MR of the fermion propagator suggests that the transverse vertex must contain information about the function $F(p^2)$. It would, in general, depend on all the Green's functions of the theory, but the correct dependence on $F(p^2)$ is necessary to ensure the MR of the fermion propagator. Therefore, the τ_i should be functions of $F(p^2)$.

Impressed with the success of the *CP* vertex, our natural starting point will be to guess the simplest form for the rest of the seven τ_i in close analogy with the τ_6 suggested by Curtis and Pennington [24]. It is here that we should list the **assumptions** that we start with :

- As mentioned before, it is assumed that the τ_i do not depend on q^2. This enables us to carry out integration over the angular variable. At this stage, it seems to be essential to do so. However, we shall see in chapter 4 how we could proceed without this simplifying assumption.

- We demand that a chirally-symmetric solution should be possible when the bare mass is zero, just as in perturbation theory. Looking at the equation for the mass function, one can see that this is most easily accomplished if only those transverse vectors with odd numbers of gamma matrices contribute to $\Gamma^\mu_T(k,p)$. Then the sum in Eq. (1.57) involves just $i = 2, 3, 6$ and 8. An added advantage of assuming this is that at the bifurcation point, the equations for F and $\mathcal{M}$ decouple from each other.

- We assume that, in the Landau gauge, the transverse component of the vertex vanishes. This is motivated by its large momentum behaviour in perturbation theory, Eq. (2.18).

We can then start with the following form for the τ_i :

$$\begin{aligned}
\tau_2(k^2,p^2) &= a_2 \frac{(k^2+p^2)^{m_2}}{(k^2-p^2)^{m_2+2}} \left(\frac{1}{F(k^2)} + (-1)^{m_2}\frac{1}{F(p^2)}\right) \\
\tau_3(k^2,p^2) &= a_3 \frac{(k^2+p^2)^{m_3}}{(k^2-p^2)^{m_3+1}} \left(\frac{1}{F(k^2)} + (-1)^{m_3+1}\frac{1}{F(p^2)}\right) \\
\tau_6(k^2,p^2) &= a_6 \frac{(k^2+p^2)^{m_6}}{(k^2-p^2)^{m_6+1}} \left(\frac{1}{F(k^2)} + (-1)^{m_6}\frac{1}{F(p^2)}\right) \\
\tau_8(k^2,p^2) &= a_8 \frac{(k^2+p^2)^{m_8}}{(k^2-p^2)^{m_8+1}} \left(\frac{1}{F(k^2)} + (-1)^{m_8+1}\frac{1}{F(p^2)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tau_1(k^2,p^2) &= a_1 \frac{(k^2+p^2)^{m_1}}{(k^2-p^2)^{m_1+2}} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} + (-1)^{m_1}\frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_4(k^2,p^2) &= a_4 \frac{(k^2+p^2)^{m_4}}{(k^2-p^2)^{m_4+3}} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} + (-1)^{m_4}\frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_5(k^2,p^2) &= a_5 \frac{(k^2+p^2)^{m_5}}{(k^2-p^2)^{m_5+1}} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} + (-1)^{m_5}\frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_7(k^2,p^2) &= a_7 \frac{(k^2+p^2)^{m_7}}{(k^2-p^2)^{m_7+2}} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} + (-1)^{m_7+1}\frac{\mathcal{M}(p^2)}{F(p^2)}\right) .
\end{aligned}$$

Probably the simplest choice is

$$\begin{aligned}
\tau_2(k^2,p^2) &= a_2 \frac{1}{(k^2+p^2)(k^2-p^2)} \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \\
\tau_3(k^2,p^2) &= a_3 \frac{1}{(k^2-p^2)} \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \\
\tau_6(k^2,p^2) &= a_6 \frac{(k^2+p^2)}{(k^2-p^2)^2} \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \\
\tau_8(k^2,p^2) &= a_8 \frac{1}{(k^2-p^2)} \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tau_1(k^2,p^2) &= a_1 \frac{1}{(k^2+p^2)(k^2-p^2)} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_4(k^2,p^2) &= a_4 \frac{1}{(k^2+p^2)^2(k^2-p^2)} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_5(k^2,p^2) &= a_5 \frac{1}{(k^2-p^2)} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right) \\
\tau_7(k^2,p^2) &= a_7 \frac{1}{(k^2+p^2)(k^2-p^2)} \left(\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right) .
\end{aligned}$$

Note that the CP vertex corresponds to choosing $a_i = 0$ $(i = 1,2,3,4,5,7,8)$ and $a_6 = 1/2$. We are then left with the following τ_6 in the Euclidean space :

$$\tau_6(k^2,p^2) = \frac{1}{2}\frac{(k^2+p^2)}{(k^2-p^2)^2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \quad .$$

Our aim is to keep four τ_i, and try to find out their coefficients a_i $(i = 2,3,6,8)$, such that the gauge dependence of the critical coupling is the least. Plugging the above τ_i in Eqs. (2.12) and (2.13), and denoting $p^2 = x$, $k^2 = y$, we obtain, for the fermion wavefunction renormalization :

$$\begin{aligned}\frac{1}{F(x)} = 1 &+ \frac{\alpha\xi}{4\pi}\int_x^{\Lambda^2}\frac{dy}{y}\frac{F(y)}{F(x)} + \frac{3\alpha}{16\pi}\int_0^x \frac{dy}{y}\frac{1}{F(x)}\left(\frac{F(y)-F(x)}{y-x}\right)\\ &\left[\frac{y^2}{x^2}\left\{(1-2a_6)(y+x) - \frac{1}{3}(a_2+2a_3+2a_8)(y-3x)\right\}\theta(x-y)\right.\\ &\left. + \left\{(1-2a_6)(y+x) - \frac{1}{3}(a_2+2a_3+2a_8)(x-3y)\right\}\theta(y-x)\right] .\end{aligned} \tag{2.21}$$

As expected, this equation has no dependence on $\mathcal{M}(p^2)$. The choice of the CP vertex is equivalent to setting $a_6 = 1/2$ and $a_2 = a_3 = a_8 = 0$. A great deal of simplification is achieved with this choice, and the relations $F(p^2) = A(p^2)^\nu$ and $\nu = \alpha\xi/4\pi$ are recovered.

A similar expression for the mass function is :

$$\begin{aligned}\frac{\mathcal{M}(x)}{F(x)} = m_0 &+ \frac{3\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dy}{y}F(y)\\ &\left[\frac{y}{x}\left\{\frac{\mathcal{M}(y)}{F(y)} + \frac{\xi}{3}\frac{\mathcal{M}(y)}{F(x)} - \frac{1}{2}\frac{y}{y-x}\left(\frac{\mathcal{M}(y)}{F(y)} - \frac{\mathcal{M}(x)}{F(x)}\right)\right\}\theta(x-y)\right.\\ &\left.\left\{\frac{\mathcal{M}(y)}{F(x)} + \frac{\xi}{3}\frac{\mathcal{M}(x)}{F(x)} - \frac{1}{2}\frac{x}{y-x}\left(\frac{\mathcal{M}(y)}{F(y)} - \frac{\mathcal{M}(x)}{F(x)}\right)\right\}\theta(y-x)\right]\\ &+ \frac{3\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dy}{y}\frac{\mathcal{M}(y)}{F(x)}\left(\frac{F(y)-F(x)}{y-x}\right)\\ &\left[\frac{y}{x}\left\{-\frac{1}{2}x + \frac{a_2}{6}\frac{y(y-3x)}{y+x} - a_3 x - a_6(y+x)\right\}\theta(x-y)\right.\\ &\left.\left\{-\frac{1}{2}y + \frac{a_2}{6}\frac{x(x-3y)}{y+x} - a_3 y - a_6(y+x)\right\}\theta(y-x)\right] .\end{aligned} \tag{2.22}$$

At the critical coupling, MR of the fermion propagator restricts the form of the solution for $F(p^2)$ and $\mathcal{M}(p^2)$ to be :

$$F(p^2) = A(p^2)^{\nu} \quad , \quad \mathcal{M}(p^2) = B(p^2)^{-s} \quad , \tag{2.23}$$

with A and B constants. One can find a detailed discussion of how to solve various integrals that arise in the above two equations in the appendix at the end of this chapter. As shown below, all the integrals can be brought to the form discussed in the appendix :

$$\begin{aligned}
F(y)\left(\frac{\mathcal{M}(y)}{F(y)} - \frac{\mathcal{M}(x)}{F(x)}\right) &= B(y^{-s} - x^{-s}) - Bx^{-(s+\nu)}(y^{\nu} - x^{\nu}) \\
F(y)\mathcal{M}(y)\left(\frac{1}{F(y)} - \frac{1}{F(x)}\right) &= B(y^{-s} - x^{-s}) - Bx^{-\nu}(y^{\nu-s} - x^{\nu-s}) \\
\frac{1}{y^2 - x^2} &= \frac{1}{2x}\left(\frac{1}{y-x} - \frac{1}{y+x}\right) \quad .
\end{aligned}$$

On carrying out the radial integration, the equation for the wavefunction renormalization gives :

$$\begin{aligned}
Ax^{\nu} &= 1 - \frac{\alpha\xi}{4\pi\nu} A\Lambda^{2\nu} + \frac{\alpha\xi}{4\pi\nu} Ax^{\nu} + \frac{\alpha}{16\pi} Ax^{\nu} \\
&\times \left[(3 - (a_2 + 2a_3 + 2a_8) - 6a_6) \left\{ \frac{3}{2} + \pi \cot \pi\nu - \frac{1}{\nu+2} - \frac{1}{\nu+1} - \frac{1}{\nu} \right\} \right. \\
&\quad \left. +3(1 + (a_2 + 2a_3 + 2a_8) - 2a_6) \left\{ 1 + \pi \cot \pi\nu - \frac{1}{\nu+1} + \ln\frac{\Lambda^2}{x} \right\} \right]
\end{aligned}$$

Comparing the coefficients of x^0 on both sides of the equation, we get :

$$\begin{aligned}
A &= \frac{4\pi\nu}{\alpha\xi}\Lambda^{-2\nu} \\
F(p^2) &= \frac{4\pi\nu}{\alpha\xi}\left[\frac{p^2}{\Lambda^2}\right]^{\nu} \quad .
\end{aligned} \tag{2.24}$$

MR of the fermion propagator puts the following constraint on the coefficients a_i :

$$1 \,+\, (a_2 + 2a_3 + 2a_8) \,-\, 2a_6 \,=\, 0 \quad . \tag{2.25}$$

Now, on comparing the coefficients of x^{ν} on both sides of the equation, we arrive at :

$$1 = \frac{\alpha\xi}{4\pi\nu} + \frac{\alpha}{4\pi}\left[(1-2a_6)\left\{\frac{3}{2} + \pi\cot\pi\nu - \frac{1}{\nu+2} - \frac{1}{\nu+1} - \frac{1}{\nu}\right\}\right]$$
$$-3 < \nu < 1 \qquad (2.26)$$

We can similarly proceed with the equation for the mass function. Condition (1.32) leaves 3 of the 4 a_i independent of each other. The fourth one is fixed once we have chosen the other three. We choose the independent coefficients to be a_2, a_3 and a_6 for no other reason than that the analytical expressions for the two equations appear simpler. On carrying out radial integration, we get :

$$\begin{aligned}\frac{8\pi}{3\alpha A x^{\nu}} &= \frac{1}{\nu} + \frac{1}{\nu+1} - \pi\cot\pi\nu + \frac{1}{s} + \frac{1}{1-s} + \frac{2}{s-\nu} - \pi\cot\pi(s-\nu) \\ &- \frac{2}{3}\xi\left[\frac{1}{\nu} - \frac{1}{\nu}\left(\frac{\Lambda^2}{x}\right)^{\nu} + \frac{1}{s-\nu-1}\right] \\ &- \frac{1}{3}a_2\left[\frac{1}{s} + \frac{1}{1-s} - \pi\cot\pi s - \frac{1}{s-\nu} + \frac{1}{s-\nu-1}\right. \\ &\qquad \left. -2\left\{\overline{\psi}\left(\frac{s}{2}\right) - \overline{\psi}(s) - \overline{\psi}\left(\frac{s-\nu}{2}\right) + \overline{\psi}(s-\nu)\right\}\right] \\ &+ 2a_3\left[\pi\cot\pi s - \pi\cot\pi(s-\nu)\right] \\ &- 2a_6\left[\frac{1}{s} - \frac{1}{1-s} - 2\pi\cot\pi s - \frac{1}{s-\nu} - \frac{1}{s-\nu-1} + 2\pi\cot\pi(s-\nu)\right],\end{aligned} \qquad (2.27)$$

where

$$\overline{\psi}(x) = \psi(x) + \psi(-x) \quad . \qquad (2.28)$$

$\psi(x)$ is the Euler's psi function [39] :

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) \quad ,$$
$$\Gamma(x) = \int_0^{\infty} dt e^{-t} t^{x-1} \quad .$$

Comparing the coefficients of $x^{-\nu}$ does not give us any new information. It only confirms the expression for A. Comparing the coefficients of x^0 on both sides of the equation, we get :

$$
\begin{aligned}
\xi \;=\;& \frac{3\nu(\nu-s+1)}{2(1-s)}\Bigg[\Bigg\{\frac{1}{\nu}+\frac{1}{\nu+1}-\pi\cot\pi\nu+\frac{1}{s}+\frac{1}{1-s}+\frac{2}{s-\nu}-\pi\cot\pi(s-\nu)\Bigg\} \\
&+\left(2a_3+\frac{a_2}{3}+4a_6\right)\Bigg\{\pi\cot\pi s-\pi\cot\pi(s-\nu)\Bigg\} \\
&-\left(\frac{a_2}{3}-2a_6\right)\Bigg\{-\frac{1}{s}+\frac{1}{1-s}+\frac{1}{s-\nu}+\frac{1}{s-\nu-1}\Bigg\} \\
&-\frac{2a_2}{3}\Bigg\{\frac{1}{s}-\frac{1}{s-\nu}-\overline{\psi}\left(\frac{s}{2}\right)+\overline{\psi}(s)+\overline{\psi}\left(\frac{s-\nu}{2}\right)-\overline{\psi}(s-\nu)\Bigg\}\Bigg] \\
&\qquad\qquad -1<s<1 \qquad -1<s-\nu<1\,. \qquad (2.29)
\end{aligned}
$$

As already discussed in the section on the bifurcation analysis, the necessary condition for the two roots of s to merge with each other is $\partial\xi/\partial s = 0$. This leads us to the following equation :

$$
\begin{aligned}
\xi \;=\;& -\frac{3(\nu-s+1)^2}{2}\Bigg[\Bigg\{-\frac{1}{s^2}+\frac{1}{(1-s)^2}-\frac{2}{(s-\nu)^2}+\pi^2\csc^2\pi(s-\nu)\Bigg\} \\
&+\left(2a_3+\frac{a_2}{3}+4a_6\right)\Bigg\{-\pi^2\csc^2\pi s+\pi^2\csc^2\pi(s-\nu)\Bigg\} \\
&-\frac{2a_2}{3}\Bigg\{-\frac{1}{s^2}+\frac{1}{(s-\nu)^2}-\overline{\psi}'\left(\frac{s}{2}\right)+\overline{\psi}'(s)+\overline{\psi}'\left(\frac{s-\nu}{2}\right)-\overline{\psi}'(s-\nu)\Bigg\}\Bigg] \\
&\qquad\qquad (2.30)
\end{aligned}
$$

We have Eqs. (2.26), (2.29) and (2.30) to be solved simultaneously in the three variables ν, α and s for various values of the gauge parameter ξ. The aim is to choose the coefficients a_2, a_3 and a_6 in such a way that the corresponding values of the coupling α depend least on the gauge parameter ξ. The large momentum behaviour of the vertex puts the following constraint :

$$
a_3+a_6 = \frac{1}{2} \quad . \qquad (2.31)
$$

Therefore, the task of mapping the multi-dimensional space of the coefficients is simplified as we are left with only 2 independent coefficients, say, a_2 and a_6. It is found that there do exist values of a_2 and a_6 for which the gauge dependence of the critical coupling is much less than that obtained from the *CP* vertex. It can clearly be seen in Figs. [2.4] and [2.5] for the choice $a_6 = -0.5$ and $a_2 = 2.75$.

For a comparison, the curve for the *CP* vertex has also been plotted. Not only does the new choice of the coefficients improve the situation in the neighbourhood of the Landau gauge but keeps the curve much flatter around $\alpha_c = 0.93$ even upto quite large values of the gauge parameter. Changing ξ from $\xi = 0$ to $\xi = 10$ reduces the gauge dependence by about 15% in comparison with that using the *CP* vertex. The improvement becomes more significant when we are further away from the Landau gauge. For example, in going upto $\xi = 70$, the change in α_c is improved by more than 60%. These results are encouraging in the sense that we have managed to find a vertex which serves our aims better than the ones constructed before. But, however weak the variation of α_c with ξ may be, any gauge dependence shows that the new vertex cannot be the exact choice. *Therefore, even if it does a lot, it does not do it all.*

2.5 Appendix

This appendix deals with evaluating the integrals that have been used to simplify the equations for the wavefunction renormalization and the mass function. An obvious but important thing to note is that the nature of these integrals puts strict constraints on the ranges of ν, s and $s-\nu$. As a consequence, it seems hard to expect that the analysis will work perfectly over an infinitely broad range of the gauge parameter. However, our aim is less ambitious. We are content, at least in this chapter, to find an improvement on the previous vertex *ansatze* irrespective of how little the improvement is. As each integral is convergent or divergent depending on the range of the parameter λ, therefore, for every integral, the results have been stated for increasing range of the parameter λ. We start with the following integral:

$$
\begin{aligned}
\int_0^x dy\,\frac{y^\lambda - x^\lambda}{y-x} &= x^\lambda\left[\psi(\lambda+1)-\psi(1)\right] && \lambda > -1 \\
\int_\epsilon^x dy\,\frac{y^\lambda - x^\lambda}{y-x} &= x^\lambda\left[\psi(\lambda+1)-\psi(1)\right] + x^\lambda\,\frac{\epsilon^{\lambda+1}}{\lambda+1} && \lambda > -2 \\
\int_\epsilon^x dy\,\frac{y^\lambda - x^\lambda}{y-x} &= x^\lambda\left[\psi(\lambda+1)-\psi(1)\right] + x^\lambda\left[\frac{\epsilon^{\lambda+1}}{\lambda+1} + \frac{\epsilon^{\lambda+2}}{\lambda+2}\right] && \lambda > -3 \\
\int_\epsilon^x dy\,\frac{y^\lambda - x^\lambda}{y-x} &= x^\lambda\left[\psi(\lambda+1)-\psi(1)\right] + \sum_{\beta=1}^{-(n+1)}\frac{\epsilon^{\lambda+\beta}}{\lambda+\beta} && \lambda > n\,.
\end{aligned}
\tag{2.32}
$$

An example of how such integrals are evaluated has been given below. Consider the second of integrals Eq. (2.32). Denote the integral by I. Using the substitution $y = xz$:

$$I \;=\; x^\lambda \int_\epsilon^1 dz \, \frac{z^\lambda - 1}{z - 1} \qquad .$$

As $z < 1$, one can use the expansion $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$ to arrive at the following expression :

$$I \;=\; -x^\lambda \sum_{k=0}^{\infty} \int_\epsilon^1 dz \, \left(z^{k+\lambda} - z^k\right) \qquad .$$

As $\lambda > -2$, the first term in the above equation has a pole corresponding to $k = 0$. Separating out this divergent term, and carrying out the integration after changing the order of summation and integration, we get :

$$I \;=\; x^\lambda \frac{\epsilon^{\lambda+1}}{\lambda+1} - \frac{x^\lambda}{\lambda+1} - x^\lambda \sum_{k=0}^{\infty} \left[\frac{1}{k+\lambda+2} - \frac{1}{k+1}\right] \quad . \qquad (2.33)$$

The series representation of Euler's psi function ψ is [39] :

$$\psi(\lambda) \;=\; -C - \sum_{k=0}^{\infty} \left(\frac{1}{\lambda+k} - \frac{1}{k+1}\right) \qquad , \qquad (2.34)$$

where $C = -\psi(1) = 0.57721566490\ldots$. This enables one to write Eqn. (2.33) as :

$$I \;=\; x^\lambda \frac{\epsilon^{\lambda+1}}{\lambda+1} - \frac{x^\lambda}{\lambda+1} - x^\lambda \left[\psi(1) - \psi(\lambda+2)\right] \quad .$$

Now, using the identity

$$\psi(x+1) \;=\; \psi(x) + \frac{1}{x} \, , \qquad (2.35)$$

it is trivial to arrive at the required result.

Below are listed some more integrals :

$$\begin{aligned}
\int_0^x dy \, \frac{y}{x} \frac{y^\lambda - x^\lambda}{y - x} &= x^\lambda \left[\psi(\lambda+2) - \psi(2)\right] & \lambda > -2 \\
\int_\epsilon^x dy \, \frac{y}{x} \frac{y^\lambda - x^\lambda}{y - x} &= x^\lambda \left[\psi(\lambda+2) - \psi(2)\right] + x^\lambda \frac{1}{\lambda+2} \epsilon^{\lambda+2} & \lambda > -3 \\
\int_\epsilon^x dy \, \frac{y}{x} \frac{y^\lambda - x^\lambda}{y - x} &= x^\lambda \left[\psi(\lambda+2) - \psi(2)\right] + x^\lambda \sum_{\beta=2}^{-(n+1)} \frac{1}{\lambda+\beta} \epsilon^{\lambda+\beta} & \lambda > n
\end{aligned} \qquad (2.36)$$

For $\lambda > 1$, the integral is infra-red divergent. Similar divergent terms are present in all other integrals discussed below in slightly different regions of λ.

$$\int_0^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(\lambda+3)-\psi(3)\,] \qquad \lambda > -3$$
$$\int_\epsilon^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(\lambda+3)-\psi(3)\,] + x^\lambda \frac{1}{\lambda+3}\epsilon^{\lambda+3} \qquad \lambda > -4$$
$$\int_\epsilon^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(\lambda+3)-\psi(3)\,] + x^\lambda \sum_{\beta=3}^{-(n+1)} \frac{1}{\lambda+\beta}\epsilon^{\lambda+\beta} \qquad \lambda > n \tag{2.37}$$

When the variable of integration, y, runs from $x \to \Lambda^2$, as is the case for the next two sets of integrals, the only difference in evaluating them is to start with the substitution $x = yz$ instead of $y = xz$:

$$\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] - x^\lambda\left[\ln\frac{\Lambda^2}{x} + \frac{1}{\lambda} - \frac{1}{\lambda}\left(\frac{\Lambda^2}{x}\right)^\lambda\right] \qquad \lambda < 1$$
$$\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] - x^\lambda\left[\ln\frac{\Lambda^2}{x} + \frac{1}{\lambda} - \frac{1}{\lambda}\left(\frac{\Lambda^2}{x}\right)^\lambda - \frac{1}{\lambda-1}\left(\frac{\Lambda^2}{x}\right)^{\lambda-1}\right] \qquad \lambda < 2$$
$$\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] - x^\lambda\left[\ln\frac{\Lambda^2}{x} + \frac{1}{\lambda} - \sum_{\beta=0}^{n-1}\frac{1}{\lambda-\beta}\left(\frac{\Lambda^2}{x}\right)^{\lambda-\beta}\right] \qquad \lambda < n \tag{2.38}$$

Divergence of the type $\ln(\Lambda^2/x)$ cannot be dealt with by choosing a particular range of λ. It puts a constraint on the coefficients a_i.

$$\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] \qquad \lambda < 1$$
$$\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] + x^\lambda\frac{1}{\lambda-1}\left(\frac{\Lambda^2}{x}\right)^{\lambda-1} \qquad \lambda < 2$$
$$\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y-x} = x^\lambda\,[\,\psi(1)-\psi(1-\lambda)\,] + x^\lambda\sum_{\beta=1}^{n-1}\frac{1}{\lambda-\beta}\left(\frac{\Lambda^2}{x}\right)^{\lambda-\beta} \qquad \lambda < n \tag{2.39}$$

For the term involving τ_2, integrals of the following type are encountered in which a factor of $(x+y)$ appears in the denominator :

$$\begin{aligned}
\int_0^x dy\, \frac{y}{x}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+3}{2}\right) - \psi\left(\frac{\lambda+2}{2}\right) - \psi\left(\frac{3}{2}\right) + \psi(1)\right] \quad \lambda > -2 \\
\int_\epsilon^x dy\, \frac{y}{x}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+3}{2}\right) - \psi\left(\frac{\lambda+2}{2}\right) - \psi\left(\frac{3}{2}\right) + \psi(1)\right] \\
&\quad - x^\lambda \frac{\epsilon^{\lambda+2}}{\lambda+2} \quad \lambda > -3 \\
\int_\epsilon^x dy\, \frac{y}{x}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+3}{2}\right) - \psi\left(\frac{\lambda+2}{2}\right) - \psi\left(\frac{3}{2}\right) + \psi(1)\right] \\
&\quad - x^\lambda \sum_{\beta=2}^{-(n+1)} (-1)^\beta \frac{\epsilon^{\lambda+\beta}}{\lambda+\beta} \quad \lambda > n
\end{aligned} \tag{2.40}$$

It is only for these type of integrals that the final result cannot be written just in terms of the more familiar function cot x. All the following sets of integrals fall in the same category:

$$\begin{aligned}
\int_0^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+4}{2}\right) - \psi\left(\frac{\lambda+3}{2}\right) - \psi(2) + \psi\left(\frac{3}{2}\right)\right] \quad \lambda > -3 \\
\int_\epsilon^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+4}{2}\right) - \psi\left(\frac{\lambda+3}{2}\right) - \psi(2) + \psi\left(\frac{3}{2}\right)\right] \\
&\quad - x^\lambda \frac{\epsilon^{\lambda+3}}{\lambda+3} \quad \lambda > -4 \\
\int_\epsilon^x dy\, \frac{y^2}{x^2}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{\lambda+4}{2}\right) - \psi\left(\frac{\lambda+3}{2}\right) - \psi(1) + \psi\left(\frac{3}{2}\right)\right] \\
&\quad + x^\lambda \sum_{\beta=3}^{-(n+1)} (-1)^\beta \frac{\epsilon^{\lambda+\beta}}{\lambda+\beta} \quad \lambda > n
\end{aligned} \tag{2.41}$$

Also,

$$\begin{aligned}
\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{1-\lambda}{2}\right) - \psi\left(\frac{-\lambda}{2}\right) + \psi(1) - \psi\left(\frac{1}{2}\right)\right] \\
&\quad x^\lambda\left[-\ln\frac{\Lambda^2}{x} + \frac{1}{\lambda}\left(\frac{\Lambda^2}{x}\right)^\lambda\right] \quad \lambda < 1
\end{aligned}$$

$$
\begin{aligned}
\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{1-\lambda}{2}\right) - \psi\left(\frac{-\lambda}{2}\right) + \psi(1) - \psi\left(\frac{1}{2}\right)\right] \\
& \quad x^\lambda\left[-\ln\frac{\Lambda^2}{x} + \frac{1}{\lambda}\left(\frac{\Lambda^2}{x}\right)^\lambda + \frac{1}{\lambda-1}\left(\frac{\Lambda^2}{x}\right)^{\lambda-1}\right] \qquad \lambda < 2 \\
\int_x^{\Lambda^2} dy\, \frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{1-\lambda}{2}\right) - \psi\left(\frac{-\lambda}{2}\right) + \psi(1) - \psi\left(\frac{1}{2}\right)\right] \\
& \quad x^\lambda\left[-\ln\frac{\Lambda^2}{x} + \sum_{\beta=0}^{n-1}(-1)^\beta\frac{1}{\lambda-\beta}\left(\frac{\Lambda^2}{x}\right)^{\lambda-\beta}\right] \qquad \lambda < n
\end{aligned}
\tag{2.42}
$$

and finally,

$$
\begin{aligned}
\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{2-\lambda}{2}\right) - \psi\left(\frac{1-\lambda}{2}\right) - \psi(1) + \psi\left(\frac{1}{2}\right)\right] \quad \lambda < 1 \\
\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{2-\lambda}{2}\right) - \psi\left(\frac{1-\lambda}{2}\right) - \psi(1) + \psi\left(\frac{1}{2}\right)\right] \\
& \quad + x^\lambda\frac{1}{\lambda-1}\left(\frac{\Lambda^2}{x}\right)^{\lambda-1} \qquad \lambda > -4 \\
\int_x^{\Lambda^2} dy\, \frac{x}{y}\frac{y^\lambda - x^\lambda}{y+x} &= \frac{x^\lambda}{2}\left[\psi\left(\frac{2-\lambda}{2}\right) - \psi\left(\frac{1-\lambda}{2}\right) - \psi(1) + \psi\left(\frac{1}{2}\right)\right] \\
& \quad + x^\lambda\sum_{\beta=1}^{n-1}(-1)^\beta\frac{1}{\lambda-\beta}\left(\frac{\Lambda^2}{x}\right)^{\lambda-\beta} \qquad \lambda > n
\end{aligned}
\tag{2.43}
$$

Following is an identity which is used after carrying out the radial integration to write the result in terms of more commonly used function $\cot x$:

$$
\psi(1-\lambda) - \psi(\lambda) = \pi\cot\pi\lambda
$$

The integrals and identities stated so far are sufficient to arrive at the results derived in this chapter.

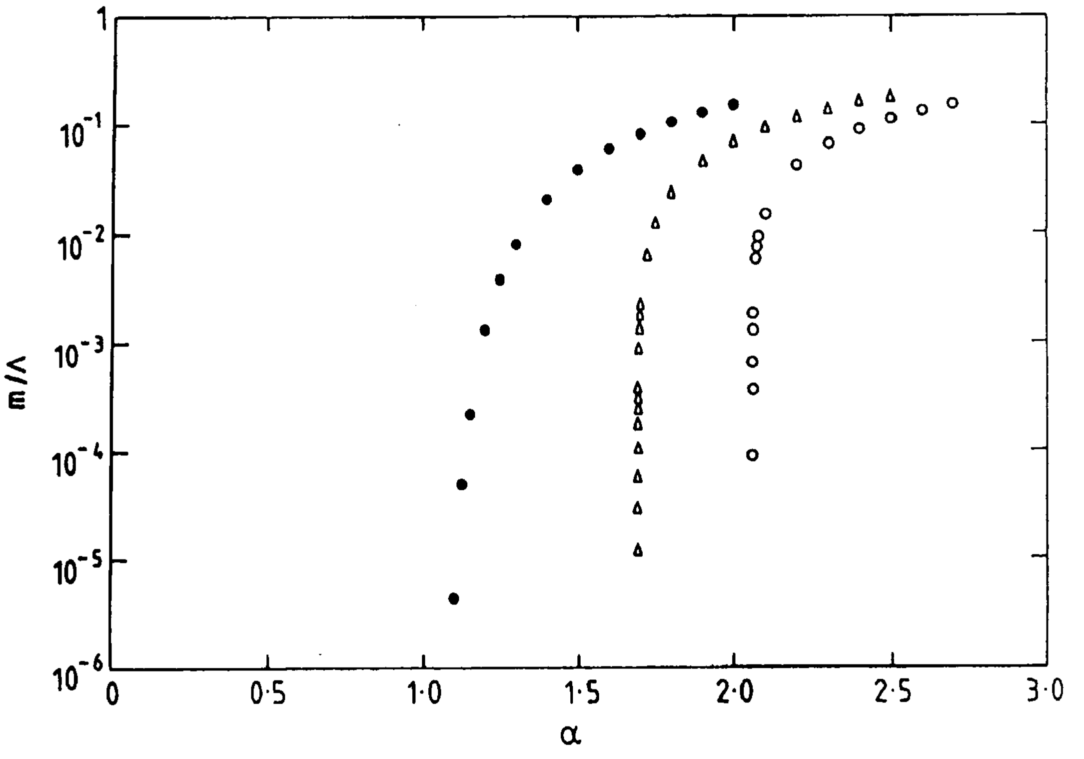

Figure 2.1: Euclidean mass, $M = \mathcal{M}(M^2)$ dynamically generated in the rainbow approximation as a function of the coupling α in three different gauges: Landau ($\xi = 0$) $\bullet$, Feynman ($\xi = 1$) $\vartriangle$, and Yennie ($\xi = 3$) $\circ$ gauges.

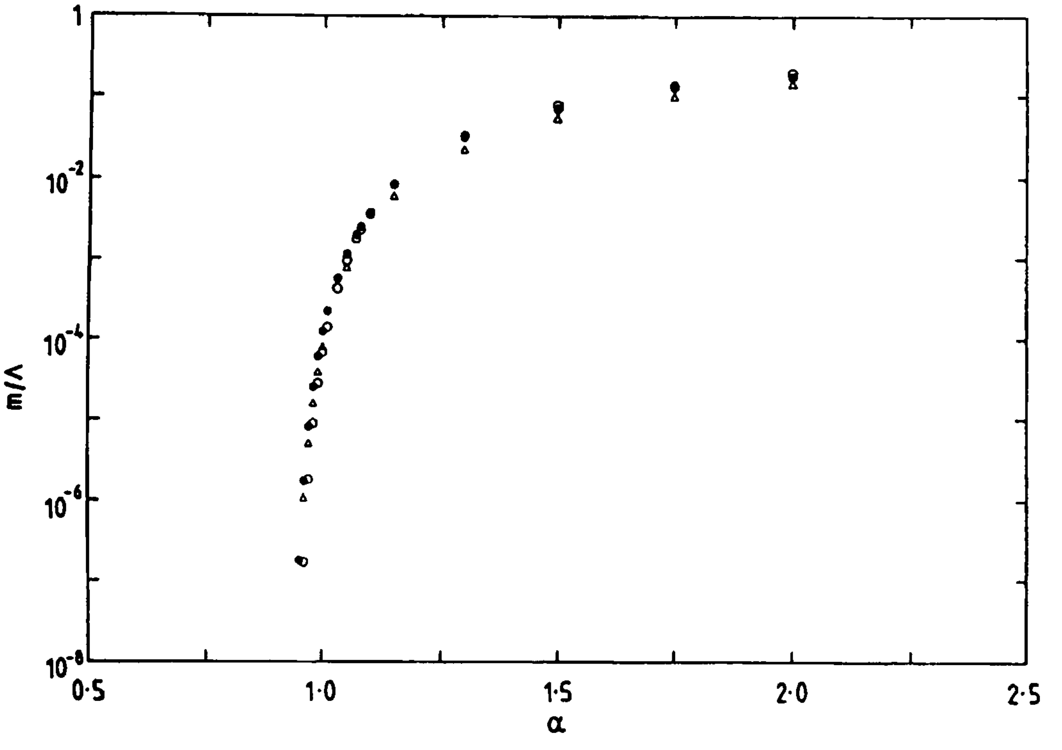

Figure 2.2: Euclidean mass, $M = \mathcal{M}(M^2)$ dynamically generated with the *CP* vertex as a function of the coupling α in three different gauges: Landau ($\xi = 0$) $\bullet$, Feynman ($\xi = 1$) $\triangle$, and Yennie ($\xi = 3$) $\circ$ gauges.

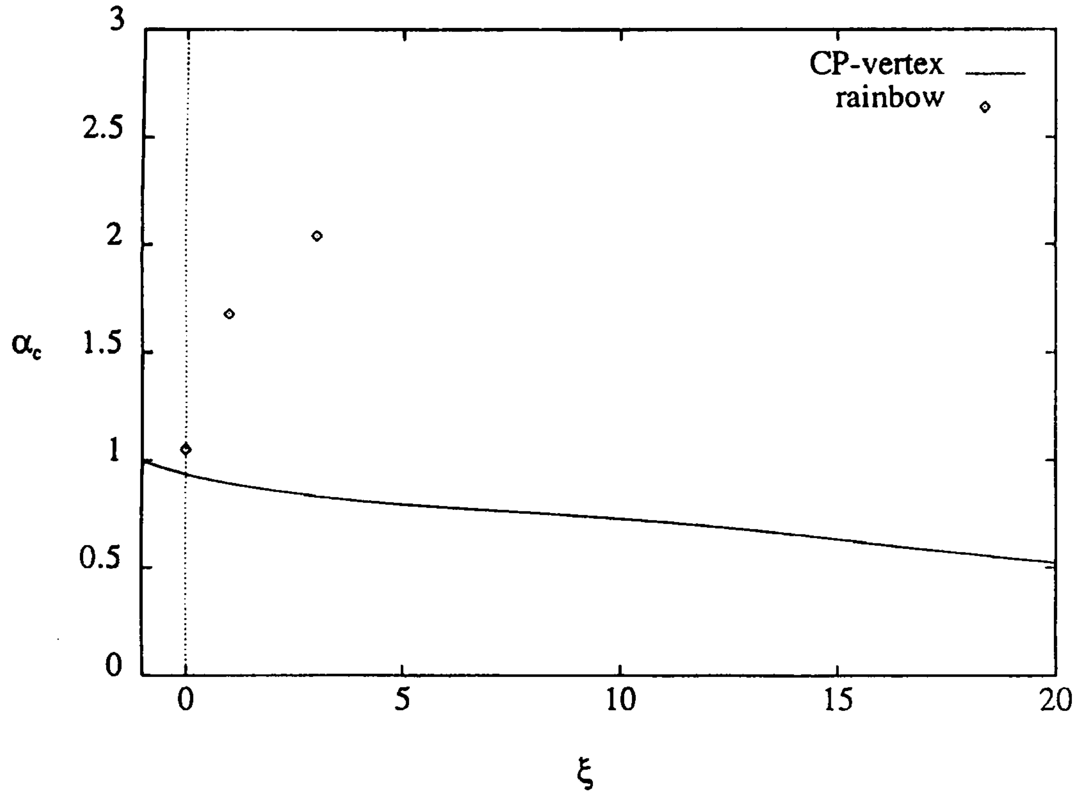

Figure 2.3: Critical coupling, α_c as a function of the gauge parameter, ξ (solid line). For a comparison, the corresponding values for the rainbow approximation have also been shown.

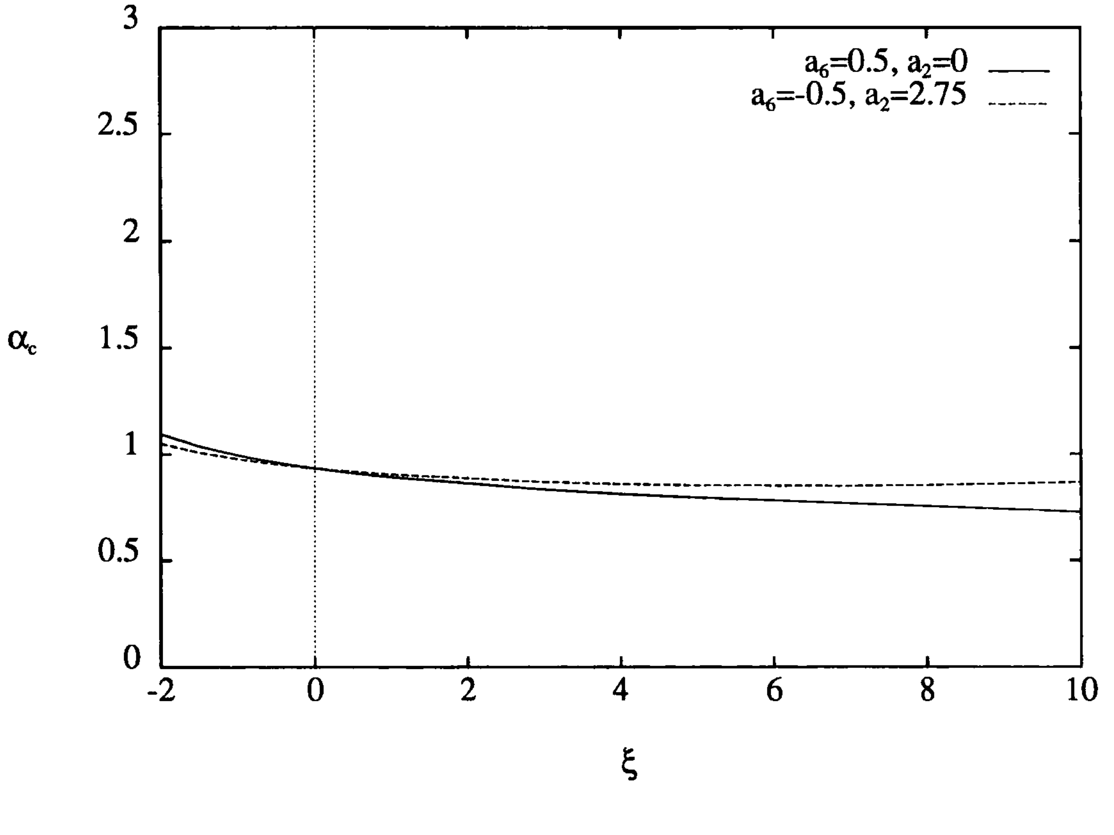

Figure 2.4: Critical coupling, α_c as a function of the gauge parameter, ξ (dashed line), for the vertex with $a_6 = -0.5$ and $a_2 = 2.75$, in the neighbourhood of the Landau gauge. For a comparison, the corresponding values for the *CP* vertex have also been shown.

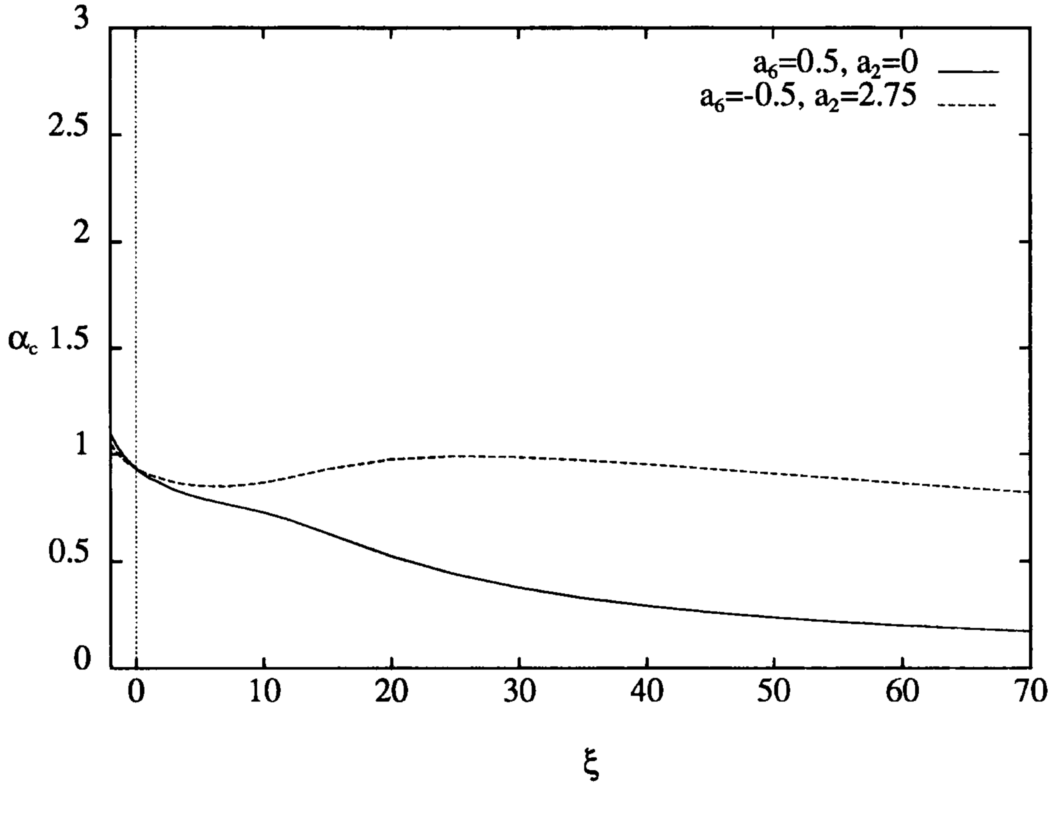

Figure 2.5: Critical coupling, α_c as a function of the gauge parameter, ξ (dashed line), for the vertex with $a_6 = -0.5$ and $a_2 = 2.75$, in a broad range of the gauge parameter. For a comparison, the corresponding values for the *CP* vertex have also been shown.

Chapter 3

Gauge Independent Chiral Symmetry Breaking

"Perhaps it is just what you want to hear from my lips. Well, then, listen."

Ivan, *The Brothers Karamazov*

Despite all the efforts made so far, the need remains to construct a non-perturbative fermion-boson vertex that embodies in it all the necessary requirements of a gauge theory simultaneously, i.e. it ensures that the fermion propagator satisfies the Ward Takahashi Identity (WTI), is multiplicatively renormalizable (MR), agrees with perturbation theory for weak couplings and has a critical coupling for dynamical mass generation that is strictly gauge independent. In Chapter 2, we have discussed in detail the gradual progress in looking for such a vertex *ansatz.* WTI rules out the use of the bare vertex which is then replaced by the *BC* vertex. Later on, the constraint from MR of the fermion propagator indicates the necessity of adding an appropriate transverse piece to the *BC* vertex. This results in the introduction of the *CP* vertex. However, this *ansatz* still fails to resolve the issue of gauge independent chiral symmetry breaking. In Chapter 2, we made attempts to find an improvement to the *CP* vertex. We managed to construct a new vertex, the use of which reduces the gauge dependence of the critical coupling considerably over a much broader range of the gauge parameter. However weak this variation, any gauge dependence shows that the use of such a vertex cannot lead to a believable study of mass generation.

In this chapter, we determine the constraints on the full fermion-boson vertex that ensures gauge covariance for the fermion propagator and exact gauge independence for the critical coupling. This extends the work of Dong et al. [35]. In general, only the position of the pole in a propagator has to be gauge independent. At that value of the momentum, when $p^2 = m^2$ in Minkowski space, (or equivalently at $p^2 = -m^2$ in the Euclidean space in which we work) the fermion mass function has to be independent of the gauge. Atkinson and Fry [29] proved this independence follows from the Ward-Takahashi identities. However, at the critical coupling for dynamical mass generation, MR imposes such a simple form on the mass function that this whole function becomes gauge independent. This is embodied in our construction.

3.1 Wavefunction Renormalization $F(p^2)$

One of our basic starting points for finding the constraints on the vertex will be the knowledge of the fermion wavefunction renormalization $F(p^2)$. Therefore, it seems appropriate to have a detailed discussion of this function before we proceed any further. In the following two subsections, we shall attempt to see what we can learn about $F(p^2)$ through perturbation theory and LKF-transformations respectively.

3.1.1 Perturbation Theory

In the leading logarithm approximation in perturbation theory, fermion wavefunction renormalization $F(p^2)$ can be written as [24]

$$F(p^2/\Lambda^2) \quad = \quad 1 + \alpha A_1 \ln\frac{p^2}{\Lambda^2} + \alpha^2 A_2 \ln^2\frac{p^2}{\Lambda^2} + \cdots \qquad , \tag{3.1}$$

where Λ is the ultraviolet cut-off and $\alpha = e^2/4\pi$. MR demands that $A_2 = A_1^2/2!$, $A_3 = A_1^3/3!$ and so on, so that we can write $F(p^2)$ as follows :

$$F(p^2/\Lambda^2) = \exp\left[\alpha A_1 \ln\frac{p^2}{\Lambda^2}\right] = \left[\frac{p^2}{\Lambda^2}\right]^{\alpha A_1} \quad . \tag{3.2}$$

It is only now that we can find a function $Z_2^{-1}(\mu^2/\Lambda^2)$ such that its product with $F(p^2/\Lambda^2)$ will be independent of Λ^2 :

$$Z_2^{-1}(\mu^2/\Lambda^2) = \exp\left[\alpha A_1 \ln\frac{\Lambda^2}{\mu^2}\right] = \left[\frac{\Lambda^2}{\mu^2}\right]^{\alpha A_1} \quad . \tag{3.3}$$

The renormalized $F_R(p^2)$ can now be written as :

$$F_R(p^2/\mu^2) = F_R(1)\exp\left[\alpha A_1 \ln\frac{p^2}{\mu^2}\right] = F_R(1)\left[\frac{p^2}{\mu^2}\right]^{\alpha A_1} \quad , \tag{3.4}$$

which is independent of Λ^2, as required by MR. Conventionally, the renormalization scale μ is chosen so that $F_R(p^2 = \mu^2) = 1$. We also know that in the leading logarithm approximation in perturbation theory

$$F_R(p^2/\mu^2) = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\mu^2} + \cdots \quad . \tag{3.5}$$

Comparing expressions (3.4) and (3.5), we have $A_1 = \xi/4\pi$. Therefore,

$$F_R(p^2/\mu^2) = \left[\frac{p^2}{\mu^2}\right]^{\alpha\xi/4\pi} \quad . \tag{3.6}$$

3.1.2 LKF Transformations

LKF transformations are a set of rules which formulate the dependence of a Green's function on the gauge parameter ξ [32]. In general, these rules are far from simple. The fact that they are written in coordinate space adds to their complexity. As a consequence, these transformations have hardly played any significant and practical role in the study of DSEs, especially when 4-dimensional QED is under discussion. However, recently, they have shed light on the fermion wavefunction renormalization $F(p^2)$, as discussed below.

Recall Eq. (2.12). Let us consider its massless version in the Landau gauge alone. Under the assumption that the transverse part of the vertex vanishes in this gauge, i.e,

$$\Gamma_T^{\mu\nu}(k,p,\xi=0) \;\; = 0 \quad , \tag{3.7}$$

$F(p^2) = 1$ is a solution to this equation. We shall see that this assumption is sufficient to ensure that the solution of the DSE is LKF covariant. Another way to achieve the same thing is as follows. Recall Eq. (1.28) and assume that we are working in d-dimensions. Let us write $\Delta^0_{\mu\nu}(q)$ as

$$\Delta^0_{\mu\nu}(q) \;\; = \;\; \Delta^T_{\mu\nu}(q) + \xi\frac{q_\mu q_\nu}{q^4} \quad ,$$

where

$$\Delta^T_{\mu\nu}(q) \;=\; \frac{1}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \quad .$$

Eq. (1.28) can now be written as follows :

$$\begin{aligned} S_F^{-1}(p) = S_F^{0^{-1}}(p) \;&+\; ie^2 \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{q^2}\,\gamma^\mu\, S_F(k)\,\Gamma^\nu(k,p)\,\Delta^T_{\mu\nu}(q) \\ &+\; ie^2\xi \int \frac{d^dk}{(2\pi)^d}\,\frac{\not q}{q^4}\,S_F(k)\left(S_F^{-1}(k) - S_F^{-1}(p)\right) \quad . \end{aligned} \tag{3.8}$$

It is easy to see that the condition

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{q^2}\,\gamma^\mu\, S_F(k)\,\Gamma^\nu(k,p)\,\Delta^T_{\mu\nu}(q) \;=\; 0 \tag{3.9}$$

is sufficient to ensure that $F(p^2) = 1$ in the Landau gauge. Condition (3.9) is assumed to hold true by Burden et al. [34, 36]. The thing which needs to be emphasized is that $F(p^2;\xi=0)=1$ *is only an assumption that stems either from the assumption (3.7) or (3.9) which may or may not be realised in perturbation theory.*

Knowing $F(p^2)$ in one gauge, we should, in principle, be able to evaluate it in any other gauge under the operation of LKF transformations. Prior to carrying out this task in 4-dimensional QED, it will be a useful exercise to go through the details in the case of 3-dimensional massless QED for the reasons of simplicity [34]. It will make the extension to higher dimensions clear. Note that all the following discussion in this section will be carried out in the Euclidean space. The following sequence of steps is to be carried out :

- **Step 1:** *We know what the fermion propagator is in one gauge in the momentum space. We carry out the Fourier transformation to find the corresponding expression in coordinate space:*

We can write the massless fermion propagator in its most general form in the momentum and coordinate spaces, respectively, as follows :

$$\begin{aligned} S_F(p;\xi) &= -\frac{i}{\not p}\,F(p;\xi) \quad , \\ S_F(x;\xi) &= \not x\,\mathrm{X}(x;\xi) \quad . \end{aligned}$$

The Fourier transformation rule

$$S_F(x;\xi) = \frac{1}{(2\pi)^3}\int d^3p e^{-ip\cdot x} S_F(p;\xi)$$

allows us to write

$$\not{x}X(x;\xi) = -\frac{i}{(2\pi)^3}\int \frac{d^3p}{p^2} e^{-ip\cdot x}\ \not{p}F(p;\xi) \quad .$$

On multiplying this equation with $\not{x}$ and taking the trace, we obtain :

$$X(x;\xi) = -\frac{i}{(2\pi)^3}\frac{1}{x^2}\int \frac{d^3p}{p^2} e^{-ip\cdot x} p\cdot x F(p;\xi) \quad .$$

We can now carry out the angular integration and arrive at the following equation :

$$X(x;\xi) = -\frac{1}{2\pi^2 x^3}\int \frac{dp}{p}\left(\sin px - px\cos px\right) F(p;\xi) \quad .$$

As $F(p,0) = 1$, we have

$$X(x;0) = -\frac{1}{4\pi x^3} \qquad \forall \quad x \neq 0 \quad , \tag{3.10}$$

where we have used the standard integration formulae

$$\int_0^\infty dp\frac{\sin p}{p} = \frac{\pi}{2} \qquad \text{and} \qquad \int_0^\infty dp \cos px = \delta(x) \quad .$$

- **Step 2:** *Now using the LKF transformation rule, we obtain the expression for the fermion propagator in an arbitrary gauge in the coordinate space:*

The LKF transformation law for the fermion propagator is [32, 33]

$$S_F(x;\Delta) \quad = \quad S_F(x;0)e^{-i[\Delta(0)-\Delta(x)]} \quad , \tag{3.11}$$

where

$$\Delta(x) \quad = \quad -i\xi e^2\int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip\cdot x}}{p^4} \quad . \tag{3.12}$$

Now using $d^3p = p^2 dp \sin\theta d\theta d\phi$, we can write

$$\Delta(0) - \Delta(x) \;=\; -i\frac{\alpha\xi}{\pi}\int_0^\infty \frac{dp}{p^2}\int_0^\pi d\theta\left[1 - e^{-ipxcos\theta}\right] \quad .$$

On carrying out the angular integration,

$$\Delta(0) - \Delta(x) \;=\; -2i\frac{\alpha\xi}{\pi x}\int_0^\infty \frac{dp}{p^3}\left[px - \sin px\right] \quad .$$

We can now perform the radial integration by parts and arrive at the following result :

$$\Delta(0) - \Delta(x) = -i\frac{\alpha\xi}{\pi}x\int_0^\infty dp\frac{\sin px}{p} = -i\frac{\alpha\xi}{2}x \quad .$$

Inserting this in Eq. (3.11), we obtain

$$S_F(x;\xi) = -\frac{1}{4\pi x^3}\, \not{x} e^{-(\alpha\xi/2)x} \tag{3.13}$$

and

$$X(x;\xi) = X(x;0)e^{-(\alpha\xi/2)x} \quad . \tag{3.14}$$

- **Step 3:** *We then Fourier transform the result back into momentum space:*

The inverse Fourier transform

$$S_F(p;\xi) = \int d^3x\, e^{ip\cdot x} S_F(x;\xi)$$

permits us to write

$$\frac{\not{p}}{p^2}F(p;\xi) = -\frac{i}{4\pi}\int \frac{d^3x}{x^3}e^{ip\cdot x}\; \not{x} e^{-(\alpha\xi/2)x} \quad .$$

Multiplying both sides of the equation by $\not{p}$ and taking the trace, we obtain

$$F(p;\xi) = -\frac{ip}{2}\int_0^\infty dx e^{-(\alpha\xi/2)x}\int_0^\pi d\theta \sin\theta\cos\theta e^{ipx\cos\theta} \quad .$$

On carrying out the angular integration, we readily obtain

$$F(p;\xi) = -\frac{1}{p}\int_o^\infty \frac{dx}{x^2}\mathrm{e}^{-(\alpha\xi/2)x}\,(px\cos px - \sin px) \quad .$$

Evaluation of this radial integration has been discussed in the appendix. It leads us to write

$$F(p;\xi) = 1 - \frac{\alpha\xi}{2p}\tan^{-1}\left[\frac{2p}{\alpha\xi}\right] \quad . \tag{3.15}$$

Therefore,

$$S_F(p;\xi) = -\frac{i}{\not p}\left[1 - \frac{\alpha\xi}{2p}\tan^{-1}\left(\frac{2p}{\alpha\xi}\right)\right] \quad . \tag{3.16}$$

This equation describes how the fermion propagator in 3-dimensional QED evolves as a function of the gauge parameter.

The whole procedure outlined above can be repeated for 4-dimensional QED. The difference is that the integrals involved are a bit harder to evaluate. Moreover, $\Delta(0)$ is divergent and, therefore, has to be regulated. Defining the most general fermion propagator as before, the Fourier transform

$$S_F(x;\xi) = \frac{1}{(2\pi)^4}\int d^4p\,\mathrm{e}^{-ip\cdot x}S_F(p;\xi)$$

allows us to write

$$\not x \mathrm{X}(x;\xi) = -\frac{i}{(2\pi)^4}\int \frac{d^4p}{p^2}\mathrm{e}^{-ip\cdot x}\;\not p F(p;\xi) \quad .$$

On multiplying with $\not x$ and taking the trace, we obtain

$$\mathrm{X}(x;\xi) = -\frac{i}{(2\pi)^3}\frac{1}{x}\int dp^2 pF(p;\xi)\int_0^\pi d\psi\sin^2\psi\cos\psi\mathrm{e}^{-ipx\cos\psi} \quad .$$

Evaluation of the angular integration has been detailed in the appendix, Eq. (3.76). Substituting the result in the above equation, we get

$$\mathrm{X}(x;\xi) = -\frac{1}{4\pi^2x^2}\int_0^\infty dp\,pJ_2(px)F(p;\xi) \quad , \tag{3.17}$$

where J_2 is the Bessel function of order 2. Recall that $F(p;0) = 1$. If we use the damping factor $\exp(-k^2/\Lambda^2)$ to serve as the cut-off, we can show that

$$\lim_{\Lambda^2\to\infty}\int_0^\infty dp\,p\exp(-k^2/\Lambda^2)\,J_2(px) = \frac{2}{x^2} \quad .$$

We, therefore, arrive at

$$X(x;0) = -\frac{1}{2\pi^2 x^4} \quad .$$

Now we want to calculate the quantity $[\Delta(0) - \Delta(x)]$. We shall see that this is divergent in 4-dimensions, unlike the case in 3-dimensions where the difference was finite despite the fact that both $\Delta(0)$ and $\Delta(x)$ were divergent separately. Moreover, an interesting thing to note is that in 4-dimensions, $\Delta(x)$ is dimensionless. Therefore, we have to introduce an external distance or momentum scale in order to form a dimensionless quantity and to make the integral convergent. For clarity, it is better to work in d-dimensions and let $d \to 4$ at the end. In other than four dimensions the coupling e^2 is dimensionful. As is usual we introduce a scale μ to maintain e^2 dimensionless. Hence, we start with

$$\Delta_d(x) \quad = \quad -i\xi e^2 \mu^{4-d} \int_0^\infty \frac{d^d p}{(2\pi)^d} \frac{e^{-ip\cdot x}}{p^4} \quad .$$

In d-dimensional Euclidean space, we have $d^d p = dp\, p^{d-1} \sin^{d-2}\psi\, d\psi\, \Omega_{d-2}$, where $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma(\frac{d-1}{2})$. Therefore,

$$\Delta_d(x) = -i\xi e^2 \mu^{4-d} f(d) \int_0^\infty dp\, p^{d-5} \int_0^\pi d\psi \sin^{d-2}\psi e^{-ipx\cos\psi} \quad ,$$

where $f(d) = \Omega_{d-2}/(2\pi)^d$. Making use of the integral formulae (3.75) and (3.76), listed in the appendix, in succession, and then letting $d = 4 + \epsilon$, we arrive at the following equation :

$$\Delta(x) = -i\frac{\xi e^2}{16\pi^{2+\frac{\epsilon}{2}}} (\mu x)^{-\epsilon}\, \Gamma\left(\frac{\epsilon}{2}\right) \quad .$$

Using the expansions

$$\begin{aligned} \Gamma\left(\frac{\epsilon}{2}\right) &= \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \\ x^{-\epsilon} &= 1 - \epsilon \ln x + \mathcal{O}(\epsilon^2) \quad , \end{aligned}$$

we obtain

$$\Delta(x) = -i\frac{\xi e^2}{16\pi^{2+\frac{\epsilon}{2}}} \left[\frac{2}{\epsilon} - \gamma - 2\ln(\mu x) + \mathcal{O}(\epsilon)\right] \quad .$$

Note that we cannot write a similar expression for $\Delta(0)$ because of the presence of the term proportional to $\ln x$. Therefore, we introduce a cut-off scale x_{min}. Now

$$\Delta(x_{min}) - \Delta(x) = -i \ln \left(\frac{x^2}{x_{min}^2} \right)^{\nu} \quad . \tag{3.18}$$

where $\nu = \alpha\xi/4\pi$. Hence,

$$S(x;\xi) = -\frac{\not{x}}{2\pi^2 x^4} \left(\frac{x^2}{x_{min}^2} \right)^{-\nu} \tag{3.19}$$

and

$$X(x;\xi) = -\frac{1}{2\pi^2 x^4} \left(\frac{x^2}{x_{min}^2} \right)^{-\nu} \quad . \tag{3.20}$$

As before, using the inverse Fourier transform

$$S_F(p;\xi) = \int d^4x e^{ip\cdot x} S_F(x;\xi) \quad ,$$

we can write

$$F(p;\xi) = -i\frac{2p}{\pi} \int_0^\infty dx \left(\frac{x^2}{x_{min}^2} \right)^{-\nu} \int_0^\pi d\psi \cos\psi \sin^2\psi e^{ipx\cos\psi}$$

which, on angular integration, yields

$$F(p;\xi) = 2 \left(x_{min}^2 \right)^{\nu} \int_0^\infty dx x^{-2\nu-1} J_2(px) \quad .$$

We reach the following final result after performing radial integration :

$$F(p;\xi) = \frac{1}{2^{2\nu}} \frac{\Gamma(1-\nu)}{\Gamma(2+\nu)} \left(p^2 x_{min}^2 \right)^{\nu} \quad .$$

The requirement of MR of $F(p^2)$ is

$$\frac{F_R(p^2/\mu^2)}{F_R(k^2/\mu^2)} = \frac{F(p^2/\Lambda^2)}{F(k^2/\Lambda^2)}$$

Choosing $F_R(k^2/\mu^2)|_{k^2=\mu^2} = 1$, we get

$$F_R(p^2/\mu^2) = \frac{F(p^2/\Lambda^2)}{F(\mu^2/\Lambda^2)}$$

which then permits us to write

$$F(p^2/\mu^2;\xi) = \left(\frac{p^2}{\mu^2} \right)^{\nu} \quad . \tag{3.21}$$

This result reaffirms that $F(p;\xi)$ has a power structure and that the exponent $\nu = \alpha\xi/4\pi$ *provided* $F(p;\xi=0) = 1$.

3.2 Constraint from $F(p^2)$

In this section, we aim to deduce constraints on the fermion-photon vertex arising from the MR of $F(p^2)$. We start by rewriting Eq. (3.8) in 4-dimensions as follows :

$$\begin{aligned} S_F^{-1}(p) = S_F^{0^{-1}}(p) &+ ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{q^2} \gamma^\mu S_F(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q) \\ &+ ie^2\xi \int \frac{d^4k}{(2\pi)^4} \frac{\not q}{q^4} \\ &- ie^2\xi \int \frac{d^4k}{(2\pi)^4} \frac{\not q}{q^4} S_F(k) S_F^{-1}(p) \quad . \end{aligned} \tag{3.22}$$

The third term on the right vanishes, as it is an odd integral, and we are left with

$$\begin{aligned} S_F^{-1}(p) = S_F^{0^{-1}}(p) &+ ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{q^2} \gamma^\mu S_F(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q) \\ &- ie^2\xi \int \frac{d^4k}{(2\pi)^4} \frac{\not q}{q^4} S_F(k) S_F^{-1}(p) \quad . \end{aligned} \tag{3.23}$$

To solve this equation, we must make an *ansatz* for the full vertex, $\Gamma^\mu(k,p)$. Our aim is to construct a vertex that automatically embodies as much of the physics of the interaction as possible. Exactly as discussed in Chapter 2, we divide the vertex into longitudinal and transverse components, following Ball and Chiu, and make the same assumptions about the transverse part as outlined there.

The fermion propagator is determined by the two functions $F(p^2)$ and $\mathcal{M}(p^2)$. We can project out equations for these by taking the trace of Eq. (3.23), having multiplied by $\not p$ and 1 in turn. On Wick rotating to Euclidean space,

$$\begin{aligned} \frac{1}{F(p^2)} = 1 - \quad & \frac{\alpha}{4\pi^3} \frac{1}{p^2} \int d^4k \frac{F(k^2)}{k^2 + \mathcal{M}^2(k^2)} \frac{1}{q^2} \\ & \Big\{ \quad a(k^2,p^2) \left[-2k\cdot p - \frac{1}{q^2} \left(-2k^2p^2 + (k^2+p^2) k\cdot p \right) \right] \\ & + \quad b(k^2,p^2) \left[2k^2p^2 + (k^2+p^2)k\cdot p - \frac{1}{q^2} (k^2-p^2)^2 k\cdot p \right] \\ & + \quad \mathcal{M}(k^2) c(k^2,p^2) \left[p^2 + k\cdot p - \frac{1}{q^2} (k^2-p^2)(k.p - p^2) \right] \\ & - \frac{\xi}{q^2 F(p^2)} \left[p^2(k^2 - k\cdot p) + \mathcal{M}(k^2)\mathcal{M}(p^2)(k\cdot p - p^2) \right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \tau_2(k^2,p^2)\left[(k^2+p^2)(k^2p^2-(k\cdot p)^2)\right] \\
&+ \quad \tau_3(k^2,p^2)\left[-2k^2p^2+3(k^2+p^2)k\cdot p-4(k\cdot p)^2\right] \\
&+ \quad \tau_6(k^2,p^2)\left[(k^2-p^2)3k\cdot p\right] \\
&+ \quad \tau_8(k^2,p^2)\left[-2k^2p^2+2(k\cdot p)^2\right]\Big\}
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
\frac{\mathcal{M}(p^2)}{F(p^2)} = m_0 - \quad & \frac{\alpha}{4\pi^3}\int d^4k\,\frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\,\frac{1}{q^2} \\
& \Big\{ -\, a(k^2,p^2)\mathcal{M}(k^2)\,[3] \\
- \quad & b(k^2,p^2)\mathcal{M}(k^2)\left[(k+p)^2-\frac{1}{q^2}\,(k^2-p^2)^2\right] \\
+ \quad & c(k^2,p^2)\left[(k^2+k\cdot p)-\frac{1}{q^2}\,(k^2-p^2)(k^2-k\cdot p)\right] \\
- \quad & \frac{\xi}{q^2F(p^2)}\,\left[\mathcal{M}(p^2)(k^2-k\cdot p)-\mathcal{M}(k^2)(p\cdot k-p^2)\right] \\
+ \quad & \tau_2(k^2,p^2)\mathcal{M}(k^2)\left[-2k^2p^2+2(k\cdot p)^2\right] + 3\,q^2\tau_3(k^2,p^2)\mathcal{M}(k^2) \\
+ \quad & \tau_6(k^2,p^2)\mathcal{M}(k^2)\left[3(k^2-p^2)\right]\Big\} \quad .
\end{aligned}
\tag{3.25}
$$

We are only interested in solving this equation when the bare mass, m_0 is zero. One solution of the mass equation, Eq. (3.25), is, as anticipated, $\mathcal{M}(p^2)=0$. We first consider the wavefunction renormalization, $F(p^2)$, in this case. Carrying out the angular integrations in Euclidean space gives :

$$
\begin{aligned}
\frac{1}{F(p^2)} = 1 \quad & + \quad \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\frac{F(k^2)}{F(p^2)} \\
& - \quad \frac{3\alpha}{16\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\frac{k^2}{p^2}\,\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right) \\
& - \quad \frac{3\alpha}{16\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right) \\
& - \quad \frac{\alpha}{8\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\frac{k^2}{p^2}\,F(k^2)K_1(k^2,p^2) \\
& - \quad \frac{\alpha}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,F(k^2)K_2(k^2,p^2) \quad ,
\end{aligned}
\tag{3.26}
$$

where

$$K_1(k^2,p^2) = (k^2-3p^2)\left[\tau_3(k^2,p^2)+\tau_8(k^2,p^2)-\frac{1}{2}(k^2+p^2)\tau_2(k^2,p^2)\right] + 3(k^2-p^2)\tau_6(k^2,p^2) \quad (3.27)$$

$$K_2(k^2,p^2) = (p^2-3k^2)\left[\tau_3(k^2,p^2)+\tau_8(k^2,p^2)-\frac{1}{2}(k^2+p^2)\tau_2(k^2,p^2)\right] + 3(k^2-p^2)\tau_6(k^2,p^2) \quad . \quad (3.28)$$

As noted by Dong et al. [35], it is convenient to define the combination $\overline{\tau}$ of τ_2, τ_3 and τ_8,

$$\overline{\tau}(k^2,p^2) = \tau_3(k^2,p^2)+\tau_8(k^2,p^2)-\frac{1}{2}(k^2+p^2)\tau_2(k^2,p^2) \quad . \quad (3.29)$$

Then,

$$K_1(k^2,p^2) = (k^2-3p^2)\overline{\tau}(k^2,p^2)+3(k^2-p^2)\tau_6(k^2,p^2) \quad (3.30)$$

$$K_2(k^2,p^2) = (p^2-3k^2)\overline{\tau}(k^2,p^2)+3(k^2-p^2)\tau_6(k^2,p^2) \quad , \quad (3.31)$$

which can be re-expressed in terms of functions with definite symmetry properties when $k \leftrightarrow p$. Thus,

$$K_1(k^2,p^2) = h_s(k^2,p^2)+h_a(k^2,p^2) \quad (3.32)$$

$$K_2(k^2,p^2) = h_s(k^2,p^2)-h_a(k^2,p^2) \quad , \quad (3.33)$$

where $h_s(k^2,p^2)$ and $h_a(k^2,p^2)$ are symmetric and antisymmetric respectively under the interchange of k and p,

$$h_s(k^2,p^2) = -(k^2+p^2)\overline{\tau}(k^2,p^2)+3(k^2-p^2)\tau_6(k^2,p^2) \quad (3.34)$$

$$h_a(k^2,p^2) = 2(k^2-p^2)\overline{\tau}(k^2,p^2) \quad . \quad (3.35)$$

As discussed in detail in the previous section, MR requires that the solution of this integral equation for the wavefunction renormalization, $F(p^2)$, must be of the form,

$$F(p^2) = A\left(p^2\right)^{\nu} \quad . \quad (3.36)$$

Perturbation theory to $\mathcal{O}(\alpha)$, as well as LKF transformations, suggest that $\nu = \alpha\xi/4\pi$. This simple power behaviour is generated by the 1 and the first integral on the right hand side of Eq. (3.26). This requires, as noted in Refs. [37, 35], a cancellation among the remaining integrals. Thus MR imposes the following constraint :

$$\begin{aligned}
&\frac{3}{2}\int_0^{p^2}\frac{dk^2}{p^2}\,\frac{k^2}{p^2}\,\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right)\\
&+\frac{3}{2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\frac{k^2+p^2}{k^2-p^2}\left(1-\frac{F(k^2)}{F(p^2)}\right)\\
&+\int_0^{p^2}\frac{dk^2}{p^2}\,\frac{k^2}{p^2}\,F(k^2)\left(h_s(k^2,p^2)+h_a(k^2,p^2)\right)\\
&+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,F(k^2)\left(h_s(k^2,p^2)-h_a(k^2,p^2)\right)=0 \quad , \qquad (3.37)
\end{aligned}$$

where $F(p^2) = A(p^2)^\nu$ and the artificial cut-off, Λ, can be taken to infinity with impunity. The scale invariance of the integrals makes it convenient to introduce the variable x, where for $0 \leq k^2 < p^2$, $x = k^2/p^2$, and for $p^2 \leq k^2 < \infty$, $x = p^2/k^2$ [27, 28]. Then,

$$\begin{aligned}
&\frac{3}{2}\int_0^1 dx\,\frac{x+1}{x-1}\,r_1(x)\\
&+\int_0^1 dx\,x^{\nu+1}F(p^2)\left(h_s(xp^2,p^2)+h_a(xp^2,p^2)\right)\\
&+\int_0^1 dx\,x^{-\nu-1}F(p^2)\left(h_s\left(p^2/x,p^2\right)-h_a\left(p^2/x,p^2\right)\right)=0 \quad , \qquad (3.38)
\end{aligned}$$

where

$$\begin{aligned}
r_1(x) &= x(1-x^\nu)-x^{-1}(1-x^{-\nu})\\
r_1(1/x) &= -r_1(x) \quad .
\end{aligned}$$

Since this equation must hold true at all p^2, the integrands cannot be functions of p^2 but only of x. Thus,

$$\begin{aligned}
F(p^2)\,h_s(xp^2,p^2) &\equiv h_1(x)\\
F(p^2)\,h_a(xp^2,p^2) &\equiv h_2(x)
\end{aligned}$$

defines h_1, h_2. Then, Eq. (3.38) becomes

$$\frac{3}{2}\int_0^1 dx\,\frac{x+1}{x-1}\,r_1(x) \;+\; \int_0^1 dx\,x^{\nu+1}\left(h_1(x)+h_2(x)\right) \\ +\; \int_0^1 dx\,x^{-\nu-1}\left(h_1(1/x)-h_2(1/x)\right)=0 \quad . \tag{3.39}$$

The original symmetry of the τ's under the exchange of k^2 and p^2 translates as follows in terms of the x-variable [35] :

$$\begin{aligned} h_1(1/x) &= x^\nu h_1(x) \\ h_2(1/x) &= -x^\nu h_2(x) \quad . \end{aligned}$$

In the most compact way, Eq. (3.39) can be written as :

$$\int_0^1 dx\; W_1(x)=0 \qquad , \tag{3.40}$$

where

$$W_1(x)=\frac{3}{2}\,\frac{x+1}{x-1}\,r_1(x)+\left(x^{\nu+1}+x^{-1}\right)\left(h_1(x)+h_2(x)\right) \quad . \tag{3.41}$$

Thus, this function $W_1(x)$ fixes $\tau_6(k^2,p^2)$ and the combination $\overline{\tau}(k^2,p^2)$, so that

$$\overline{\tau}(k^2,p^2) = \frac{1}{4}\,\frac{1}{k^2-p^2}\,\frac{1}{s_1(k^2,p^2)}\left[W_1\left(\frac{k^2}{p^2}\right)-W_1\left(\frac{p^2}{k^2}\right)\right] \tag{3.42}$$

$$\begin{aligned} \tau_6(k^2,p^2) = {} & -\frac{1}{2}\,\frac{k^2+p^2}{(k^2-p^2)^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)+\frac{1}{3}\,\frac{k^2+p^2}{k^2-p^2}\,\overline{\tau}(k^2,p^2) \\ & +\frac{1}{6}\,\frac{1}{k^2-p^2}\,\frac{1}{s_1(k^2,p^2)}\left[W_1\left(\frac{k^2}{p^2}\right)+W_1\left(\frac{p^2}{k^2}\right)\right] , \end{aligned} \tag{3.43}$$

where

$$s_1(k^2,p^2)=\frac{k^2}{p^2}\,F(k^2)+\frac{p^2}{k^2}\,F(p^2) \quad .$$

It is the first term in Eq. (3.43) that is essentially the CP vertex in the massless theory. Note the automatic appearance of the difference $\left(F(k^2)^{-1}-F(p^2)^{-1}\right)$, which Curtis et al. [24] conjectured was the non-perturbative generalization of the leading logarithm behaviour in lowest order perturbation theory, Eq. (2.18). Indeed, agreement with this behaviour is naturally achieved if $W_1 \to 0$ in this limit.

3.3 Constraint on W_1 from Avoiding Kinematic Singularities

The vertex can only have singularities for good dynamical reasons. It cannot have kinematic singularities. A sufficient condition for this is to assume that each of the τ_i (i=1,8) is free of kinematic singularities. Ball and Chiu [22] found that, with their choice of basis vectors T_i^μ, this is indeed true at one loop order in perturbation theory in the Feynman gauge. However, more recently Kizilersü, Reenders and Pennington [38] have shown this does not hold in arbitrary covariant gauges at this order. However, by a simple redefinition of one of the basis vectors, T_7^μ of Eq. (1.56), the τ_i are free of singularities. In the present non-perturbative analysis, we assume that this freedom from kinematic singularities continues to hold with these new basis vectors. (Note that the redefinition of T_7^μ from that given in Eq. (1.56) does not, in fact, affect the present analysis.) Thus,

$$\lim_{k^2 \to p^2} (k^2 - p^2)\, \tau_6(k^2, p^2) = 0 \quad , \tag{3.44}$$

which requires

$$W_1(1) + W_1'(1) = -6\nu \qquad , \tag{3.45}$$

as found by Dong et al. [35]. Perturbation theory demands $W_1(x)$ be $\mathcal{O}(\alpha)$. While the form of the coefficient function τ_6 is determined by the constrained function $W_1(x)$, it is only the combination $\overline{\tau}$ of τ_2, τ_3, τ_8 that is so specified. By imposing the gauge independence of the critical coupling for mass generation, we will be able to separate these functions as we shall show in the next section.

3.4 Constraint from Gauge Invariance

While for $\alpha < \alpha_c$, there is only one solution $\mathcal{M}(p^2) = 0$, as $\alpha \to \alpha_c$, a second non-zero solution becomes possible. This solution bifurcates away from the other solution. Bifurcation analysis allows us to investigate precisely when this happens. In the neighbourhood of the critical coupling, terms quadratic in the mass function can be rigorously neglected. Thus, the wavefunction renormalization, $F(p^2)$, is that of the massless theory, and the equation for the mass function, $\mathcal{M}(p^2)$, Eq. (3.25) with $m_0 \equiv 0$, linearizes :

$$
\begin{aligned}
\frac{\mathcal{M}(p^2)}{F(p^2)} &= \frac{\alpha\xi}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} + \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)} \\
&+ \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2) + \frac{p^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2) - \mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&+ \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} + \frac{k^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2) - \mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&- \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\tau_2(k^2,p^2)\right. \\
&\qquad\left. + p^2\,\tau_3(k^2,p^2) + (k^2-p^2)\,\tau_6(k^2,p^2)\right] \\
&- \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\tau_2(k^2,p^2)\right. \\
&\qquad\left. + k^2\,\tau_3(k^2,p^2) + (k^2-p^2)\,\tau_6(k^2,p^2)\right] . \qquad (3.46)
\end{aligned}
$$

If this equation is to be multiplicatively renormalizable with a gauge independent bifurcation, then this imposes further constraints on the transverse vertex, τ_i ($i = 2,3,6$). We first work in the Landau gauge, where we assume the transverse vertex vanishes. This is motivated by the perturbative result of Eq. (2.18), but as we shall stress later is only true when $k^2 \gg p^3$ or $p^2 \gg k^2$ as shown by Kizilersü et al. [38]. Then we have simply :

$$
\begin{aligned}
\mathcal{M}(p^2) &= \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2) - \frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\right)\right] \\
&+ \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2) - \frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\right)\right] . \qquad (3.47)
\end{aligned}
$$

This equation has the multiplicatively renormalizable solution,

$$\mathcal{M}(k^2) = B\left(k^2\right)^{-s} \quad , \qquad (3.48)$$

where Eq. (3.47) requires,

$$\frac{8\pi}{3\alpha} \;=\; 1+\frac{3}{s}+\frac{1}{1-s}-\pi\cot\pi s \;\equiv\; f(s)\,. \tag{3.49}$$

There are two roots for s between 0 and 1. Bifurcation occurs when the two roots for s merge at $s = s_c$, specified by $f'(s_c) = 0$. This point defines the critical coupling [26, 27, 28], $\alpha_c = 8\pi/3f(s_c)$. Numerically, $\alpha_c = 0.933667$ and $s_c = 0.470966$. A little away from this critical point, the exponent s is given by

$$s \;=\; s_c \pm \sqrt{\frac{2f(s_c)}{f''(s_c)}}\sqrt{1-\frac{\alpha}{\alpha_c}}\,. \tag{3.50}$$

It is only at the bifurcation point that the simple behaviour of Eq. (3.48) holds at all momenta. There, only when the mass is still effectively zero is there just one scale, Λ, for the momentum dependence of $\mathcal{M}(k^2)$. MR then forces a simple power behaviour. Such a multiplicatively renormalizable mass function must exist in all gauges. Consequently, the exponent, s_c, must be gauge independent. Moreover, dynamical mass generation marks a physical phase change and so the critical coupling, α_c, must also be gauge independent. Thus, the critical values, α_c, s_c, found in the Landau gauge must hold in all gauges. This is achieved as follows. We recall Eqs. (3.26,3.37) :

$$\frac{1}{F(p^2)} = 1+\frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\frac{F(k^2)}{F(p^2)} \quad . \tag{3.51}$$

Multiplying this equation by $\mathcal{M}(p^2)$ and subtracting it from Eq. (3.46), we obtain :

$$\begin{aligned}
\mathcal{M}(p^2) \;=\;& \frac{\alpha\xi}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} \\
&+ \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)+\frac{p^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&+ \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)}+\frac{k^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
- & \frac{3\alpha}{4\pi} \int_0^{p^2} \frac{dk^2}{p^2} \mathcal{M}(k^2) F(k^2) \Bigg[\frac{k^2}{6} (k^2 - 3p^2) \tau_2(k^2,p^2) \\
& \qquad + p^2 \, \tau_3(k^2,p^2) \, + \, (k^2 - p^2) \, \tau_6(k^2,p^2) \Bigg] \\
- & \frac{3\alpha}{4\pi} \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \mathcal{M}(k^2) F(k^2) \Bigg[\frac{p^2}{6} (p^2 - 3k^2) \, \tau_2(k^2,p^2) \\
& \qquad + k^2 \, \tau_3(k^2,p^2) \, + \, (k^2 - p^2) \, \tau_6(k^2,p^2) \Bigg] \, . \qquad (3.52)
\end{aligned}
$$

In order for the above equation to reduce to Eq. (3.47), it must be true that :

$$
\begin{aligned}
\frac{\xi}{3} \int_0^{p^2} \frac{dk^2}{p^2} \, \mathcal{M}(k^2) \frac{F(k^2)}{F(p^2)} \; = & - \int_0^{p^2} dk^2 \, \frac{\mathcal{M}(k^2)}{2(k^2 - p^2)} \left(1 - \frac{F(k^2)}{F(p^2)} \right) \\
& - \int_{p^2}^{\Lambda^2} dk^2 \, \frac{\mathcal{M}(k^2)}{2(k^2 - p^2)} \left(1 - \frac{F(k^2)}{F(p^2)} \right) \\
& + \int_0^{p^2} \frac{dk^2}{p^2} \mathcal{M}(k^2) F(k^2) \Bigg[\frac{k^2}{6} (k^2 - 3p^2) \, \tau_2(k^2,p^2) \\
& \qquad + p^2 \, \tau_3(k^2,p^2) \, + \, (k^2 - p^2) \, \tau_6(k^2,p^2) \Bigg] \\
& + \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \mathcal{M}(k^2) F(k^2) \Bigg[\frac{p^2}{6} (p^2 - 3k^2) \, \tau_2(k^2,p^2) \\
& \qquad + k^2 \, \tau_3(k^2,p^2) \, + \, (k^2 - p^2) \, \tau_6(k^2,p^2) \Bigg]
\end{aligned}
\qquad (3.53)
$$

at all momentum p and in all gauges ξ. This equation can be written as follows :

$$
\begin{aligned}
\xi \int_0^{p^2} \frac{dk^2}{p^2} \, \mathcal{M}(k^2) \frac{F(k^2)}{F(p^2)} \; = & - \int_0^{p^2} dk^2 \, \frac{3\mathcal{M}(k^2)}{2(k^2 - p^2)} \left(1 - \frac{F(k^2)}{F(p^2)} \right) \\
& - \int_{p^2}^{\Lambda^2} dk^2 \, \frac{3\mathcal{M}(k^2)}{2(k^2 - p^2)} \left(1 - \frac{F(k^2)}{F(p^2)} \right) \\
& + \int_0^{p^2} \frac{dk^2}{p^2} \, \mathcal{M}(k^2) F(k^2) K_3(k^2,p^2) \\
& + \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \, \mathcal{M}(k^2) F(k^2) K_4(k^2,p^2) \quad , \qquad (3.54)
\end{aligned}
$$

where $K_3(k^2,p^2)$ and $K_4(k^2,p^2)$ can, like $K_1(k^2,p^2)$ and $K_2(k^2,p^2)$, be expressed in terms of functions with definite symmetry properties under the interchange of k and p :

$$
\begin{aligned}
g_s(k^2,p^2) &= \frac{1}{4}\left[(k^2-p^2)^2-4k^2p^2\right]\tau_2(k^2,p^2)+\frac{3}{2}(k^2+p^2)\,\tau_3(k^2,p^2) \\
&\qquad +3(k^2-p^2)\,\tau_6(k^2,p^2) \\
g_a(k^2,p^2) &= \frac{1}{4}(k^2-p^2)\left[(k^2+p^2)\tau_2(k^2,p^2)-6\tau_3(k^2,p^2)\right] \quad , \qquad (3.55)
\end{aligned}
$$

so that

$$
\begin{aligned}
K_3(k^2,p^2) &= g_s(k^2,p^2)+g_a(k^2,p^2) \\
K_4(k^2,p^2) &= g_s(k^2,p^2)-g_a(k^2,p^2) \quad .
\end{aligned}
$$

Introducing the variable x as before and knowing that $\mathcal{M}(k^2) \sim (k^2)^{-s_c}$ and $F(k^2) \sim (k^2)^{\nu}$, Eq. (3.54) becomes,

$$
\begin{aligned}
&\xi\int_0^1 dx\, x^{\nu-s_c}+\frac{3}{2}\int_0^1\frac{dx}{x-1}\left[x^{-s_c}-x^{\nu-s_c}-x^{s_c-1}+x^{s_c-\nu-1}\right] \\
&-\int_0^1 dx\, x^{\nu-s_c}F(p^2)\left[g_s(xp^2,p^2)+g_a(xp^2,p^2)\right] \\
&-\int_0^1 dx\, x^{s_c-\nu-1}F(p^2)\left[g_s\left(p^2/x,p^2\right)-g_a\left(p^2/x,p^2\right]\right) = 0 \quad . \qquad (3.56)
\end{aligned}
$$

Once again, this equation must hold true for all p^2, and so the integrands cannot be functions of p^2 but solely of x. Thus, we conveniently define,

$$
\begin{aligned}
F(p^2)\,g_s(xp^2,p^2) &\equiv g_1(x) \\
F(p^2)\,g_a(xp^2,p^2) &\equiv g_2(x) \quad .
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\xi\int_0^1 dx\, x^{\nu-s_c}+\frac{3}{2}\int_0^1\frac{dx}{x-1}\left[x^{-s_c}-x^{\nu-s_c}-x^{s_c-1}+x^{s_c-\nu-1}\right] \\
&-\int_0^1 dx\, x^{\nu-s_c}\left[g_1(x)+g_2(x)\right]-\int_0^1 dx\, x^{s_c-\nu-1}\left[g_1\left(1/x\right)-g_2\left(1/x\right)\right] = 0 \quad .
\end{aligned}
$$

(3.57)

The symmetry of the vertex [35] under $k \leftrightarrow p$ means that,

$$
\begin{aligned}
g_1\left(1/x\right) &= x^{\nu}g_1(x) \\
g_2\left(1/x\right) &= -x^{\nu}g_2(x) \quad .
\end{aligned}
$$

In contrast to our discussion in section 3.2, when the equations for the wavefunction renormalization, $F(p^2)$, apply for all values of the coupling, Eq. (3.57) only hold when $\alpha = \alpha_c$.

Eq. (3.57) can be written in a compact way as

$$\int_0^1 \frac{dx}{\sqrt{x}} W_2(x) = 0 \quad , \tag{3.58}$$

where

$$\begin{aligned} W_2(x) &= \xi\, x^{\nu - s_c + \frac{1}{2}} + \frac{3}{2}\frac{r_2(x)}{x-1} - x^{\nu - s_c + \frac{1}{2}} \left[g_1(x) + g_2(x)\right] \\ &- x^{-\nu + s_c - \frac{1}{2}} \left[g_1(1/x) - g_2(1/x)\right] , \end{aligned} \tag{3.59}$$

with

$$r_2(x) = x^{\frac{1}{2} - s_c} \left(1 - x^{\nu}\right) - x^{s_c - \frac{1}{2}} \left(1 - x^{-\nu}\right) \quad , \tag{3.60}$$

which has the property, $r_2(1/x) = -r_2(x)$. Conveniently defining the combination,

$$s_2(k^2, p^2) = \frac{k}{p}\frac{\mathcal{M}(k^2)}{\mathcal{M}(p^2)} F(k^2) + \frac{p}{k}\frac{\mathcal{M}(p^2)}{\mathcal{M}(k^2)} F(p^2) \quad , \tag{3.61}$$

we have

$$\begin{aligned} g_s(k^2, p^2) &= \frac{\xi}{2 s_2(k^2,p^2)}\left[\frac{k}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} + \frac{p}{k}\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] \\ &+ \frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\frac{1}{s_2(k^2,p^2)}\, r_2\left(\frac{k^2}{p^2}\right) \\ &- \frac{1}{2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right] \end{aligned} \tag{3.62}$$

$$\begin{aligned} g_a(k^2, p^2) &= \frac{\xi}{2 s_2(k^2,p^2)}\left[\frac{k}{p}\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - \frac{p}{k}\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] \\ &- \frac{3}{4}\frac{1}{s_2(k^2,p^2)}\, r_2\left(\frac{k^2}{p^2}\right) \\ &- \frac{1}{2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] . \end{aligned} \tag{3.63}$$

Solving the last two equations for τ_2 and τ_3 in terms of τ_6 and W_2, we obtain :

$$\begin{aligned}
\tau_2(k^2,p^2) &= \frac{2\xi}{(k^2-p^2)^2}\frac{q_2(k^2,p^2)}{s_2(k^2,p^2)} - 6\,\frac{\tau_6(k^2,p^2)}{(k^2-p^2)} \\
&\quad -\frac{1}{(k^2-p^2)^2}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)+W_2\left(\frac{p^2}{k^2}\right)\right] \\
&\quad -\frac{k^2+p^2}{(k^2-p^2)^3}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)-W_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned} \tag{3.64}$$

where

$$q_2(k^2,p^2) = \frac{1}{k^2-p^2}\left[\frac{k^3}{p}\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - \frac{p^3}{k}\,\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right], \tag{3.65}$$

where $q_2(k^2,p^2)$ is obviously a symmetric function of k and p, and

$$\begin{aligned}
\tau_3(k^2,p^2) &= -\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2) \\
&\quad +\frac{1}{k^2-p^2}\,\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}\,r_2\left(\frac{k^2}{p^2}\right)-\frac{\xi}{3}\,q_3(k^2,p^2)\right] \\
&\quad -\frac{1}{6}\frac{k^2+p^2}{(k^2-p^2)^2}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)+W_2\left(\frac{p^2}{k^2}\right)\right] \\
&\quad +\frac{1}{6}\frac{k^4+p^4-6k^2p^2}{(k^2-p^2)^3}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)-W_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned} \tag{3.66}$$

where

$$q_3(k^2,p^2) = \frac{kp}{(k^2-p^2)^2}\left[(p^2-3k^2)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} - (k^2-3p^2)\,\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right], \tag{3.67}$$

where $q_3(k^2,p^2)$ is antisymmetric in k and p. The relation, Eq. (3.29),

$$\overline{\tau}(k^2,p^2) = \tau_3(k^2,p^2)+\tau_8(k^2,p^2)-\frac{1}{2}\,(k^2+p^2)\tau_2(k^2,p^2)$$

then fixes $\tau_8(k^2,p^2)$.

$$\begin{aligned}
\tau_8(k^2,p^2) &= -2\,\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2)+\overline{\tau}(k^2,p^2) \qquad (3.68)\\
&\quad -\frac{1}{k^2-p^2}\,\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}\,r_2\left(\frac{k^2}{p^2}\right)-\frac{\xi}{3}\,q_8(k^2,p^2)\right]
\end{aligned}$$

$$
-\frac{1}{3}\,\frac{k^2+p^2}{(k^2-p^2)^2}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)+W_2\left(\frac{p^2}{k^2}\right)\right]
$$
$$
-\frac{2}{3}\,\frac{k^4+p^4}{(k^2-p^2)^3}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)-W_2\left(\frac{p^2}{k^2}\right)\right], \quad (3.69)
$$

where

$$
q_8(k^2,p^2)=\frac{1}{(k^2-p^2)^2}\left[\frac{k}{p}(3k^4+p^4)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}-\frac{p}{k}(k^4+3p^4)\,\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right], \quad (3.70)
$$

which is clearly antisymmetric in k and p.

3.5 Constraint on W_2 from Avoiding Kinematic Singularities

Imposing the condition that the vertex and its components should be free of kinematic singularities means that,

$$
\lim_{k^2\to p^2}(k^2-p^2)\,\tau_i(k^2,p^2)=0 \qquad i=2,3,8 \quad ,
$$

noting that the antisymmetry of τ_6 means $\tau_6(p^2,p^2)=0$. Thus,

$$
W_2(1)+2W_2'(1)=2\xi(\nu-s+1) \qquad , \qquad (3.71)
$$

where $s=s_c$ at the critical point.

3.6 An Example

We have now constructed a vertex that ensures the fermion propagator is multiplicatively renormalizable and that the critical coupling above which mass can be dynamically generated is gauge independent. The resulting vertex involves two unknown functions W_1 and W_2. Each of these satisfies a sum rule, Eqs. (3.40,3.58), and a constraint on their derivatives, Eqs. (3.45,3.71), Any choice of these fulfills our fundamental constraints as long as it correctly matches onto perturbation theory. Here we give too very simple examples that satisfy the necessary constraints, merely as illustrations :

- $\mathbf{W_1}$: The simplest example for W_1 satisfying Eqs. (3.40,3.45) is perhaps

$$W_1\left(\frac{k^2}{p^2}\right) = 2\nu\left(1 - 2\frac{k^2}{p^2}\right) \quad . \tag{3.72}$$

There are, of course, an infinity of possible guesses. In practice, we expect that W_1 should be expressible solely in terms of the ratio $F(k^2)/F(p^2)$. However, we have not been able to find simple examples that achieve this.

- $\mathbf{W_2}$: The transverse vertex has the correct lowest order perturbative limit, viz. $\Gamma^\mu_T = \mathcal{O}(\alpha)$, provided,

$$W_2\left(k^2/p^2\right) = \xi\,\frac{k}{p}\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)} + \mathcal{O}(\alpha) \quad . \tag{3.73}$$

Since at large momenta we expect the power behaviour of Eqs. (3.36,3.48) even away from criticality, Eq. (3.71) will hold for all values of the coupling, α. In contrast, Eq. (3.58) is only true at the bifurcation point. Its exact form for all α is not known, but Eq. (3.50) might suggest

$$\int_0^1 \frac{dx}{\sqrt{x}}\,W_2(x) \approx \xi\sqrt{1 - \frac{\alpha}{\alpha_c}} \quad , \tag{3.74}$$

to agree with both the $\alpha = 0$ and $\alpha = \alpha_c$ limits, Eqs. (3.73,3.58). We expect that W_2 should surely also involve $\mathcal{M}(k^2)/\mathcal{M}(p^2)$ as well in addition to $F(k^2)/F(p^2)$.

The exact form of the full vertex would, of course, determine these functions $W_1(x), W_2(x)$ precisely. Thus, solving the Schwinger-Dyson equation for the three point function would specify the unknowns. However, that has not been our aim. Our aim is to construct a vertex that ensures the fermion propagator is gauge covariant, multiplicatively renormalizable and has a gauge independent chiral symmetry breaking phase transition. One does not need to know the exact form of the full vertex to achieve these properties, only the *effective* vertex for the fermion equation, Eq. (1.26). However, we believe that this effective vertex should nevertheless satisfy the appropriate WTI and agree with perturbation theory at least in the leading logarithmic limit of the weak coupling regime. This is the construction we have achieved for any functions $W_i(x)$ $(i = 1, 2)$. This *effective* vertex is thus given by Eqs. (1.50,1.54,1.56,1.57, 3.43,3.64–3.70).

3.7 Appendix

Following is the list of integrals used in the section on LKF transformations. Each of them is followed by the standard integral formulae marked alphabetically which were needed in their derivation :

$$\frac{1}{p}\int_0^\infty \frac{dx}{x^2}\,\mathrm{e}^{-ax}\left[px\cos px - \sin px\right] = -1 + \frac{a}{p}\tan^{-1}\frac{p}{a} \tag{3.75}$$

$$\int_0^\infty dx\,\mathrm{e}^{-ax}\sin px = \frac{p}{a^2+p^2} \tag{3.75a}$$

$$\int_0^\infty dx\,\mathrm{e}^{-ax}\frac{\sin px}{x} = \tan^{-1}\frac{p}{a} \tag{3.75b}$$

$$\int_0^\pi d\psi\,\sin^2\psi\cos\psi\,\mathrm{e}^{-ipx\cos\psi} = -\frac{i\pi}{px}J_2(px)$$

$$\int_0^\pi d\psi\,\sin^{d-2}\psi\,\mathrm{e}^{-ipx\cos\psi} = \sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{px}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(px)$$

$$\int_0^\pi d\psi\,\sin^{d-2}\psi\cos\psi\,\mathrm{e}^{-ipx\cos\psi} = -i\sqrt{\pi}\left(\frac{px}{2}\right)^{1-\frac{d}{2}}\Gamma\left(\frac{d-1}{2}\right)J_{\frac{d}{2}}(px) \tag{3.76}$$

$$\int_0^\pi dx\,\cos^{2m+1}x = 0 \tag{3.76a}$$

$$\int_0^\pi dx\,\cos^{2m}x = \frac{\pi}{2^{2m}}\binom{2m}{m} \tag{3.76b}$$

$$\int_0^{\frac{\pi}{2}} dx\,\sin^{\mu-1}x\cos^{\nu-1}x = \frac{1}{2}B\left(\frac{\mu}{2},\frac{\nu}{2}\right) \qquad \mu>0,\nu>0 \tag{3.76c}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.76d}$$

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \tag{3.76e}$$

$$\Gamma(n+1) = n! \tag{3.76f}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k} \tag{3.76g}$$

$$\lim_{\Lambda^2\to\infty}\int_0^\infty dx\,x\exp(-x^2/\Lambda^2)J_2(ax) = \frac{2}{a^2}$$

$$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}}dx = \frac{1}{2^{\nu-q}a^{q-\nu+1}}\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\nu+\frac{1-q}{2}\right)} \qquad -1<q<\nu-\frac{1}{2} \tag{3.77}$$

Example 1:

$$\text{Let} \qquad I = \frac{1}{p} \int_0^\infty \frac{dx}{x^2}\, \mathrm{e}^{-ax} \left[px \cos px - \sin px\right] \quad .$$

Integrate this by parts, regarding $\mathrm{e}^{-ax}\left[px \cos px - \sin px\right]$ and $1/x^2$ as two functions. This gives

$$I = -\frac{1}{p} \int_0^\infty dx\, \mathrm{e}^{-ax} \left[ap \cos px - a\frac{\sin px}{x} + p^2 \sin px\right] \quad .$$

Integrating the first term by parts gives

$$I = -\frac{a^2 + p^2}{p} \int_0^\infty dx\, \mathrm{e}^{-ax} \sin px + \frac{a}{p} \int_0^\infty dx\, \mathrm{e}^{-ax} \frac{\sin px}{x} \quad .$$

Now, making use of the standard integral formulae (3.74a) and (3.74b), we arrive at the required result (3.74).

Example 2:

$$\text{Let} \qquad I = \int_0^\pi d\psi \sin^2\psi \cos\psi \mathrm{e}^{-ipx\cos\psi} \quad .$$

Using the power expansion of the exponential function, we obtain

$$I = \sum_{n=0}^{\infty} \frac{(-ipx)^n}{n!} \left[\int_0^\pi d\psi \cos^{n+1}\psi - \int_0^\pi d\psi \cos^{n+3}\psi\right] \quad .$$

The standard integral (3.75a) permits us to disregard all the terms for which n is an even number. A bit of re-arrangement, in order that the summation variable runs over all non-negative integers, gives

$$I = \sum_{n=0}^{\infty} \frac{(-ipx)^{2n+1}}{(2n+1)!} \left[\int_0^\pi d\psi \cos^{2(n+1)}\psi - \int_0^\pi d\psi \cos^{2(n+2)}\psi\right] .$$

This will help later in identifying the sum with the series representation of the Bessel function of order 2. For the even powers of the cosine function, integral expression 3.75b enables us to arrive at the following desired form with a little bit of algebraic manipulation :

$$I = -\frac{i\pi}{px}\left(\frac{px}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(2+n+1)} \left(\frac{px}{2}\right)^{2n} \quad .$$

We can now compare it with the series representation of the Bessel function, (3.75g), to arrive at the required result.

Example 3:

$$\text{Let} \qquad I = \int_0^{\pi} d\psi \sin^{d-2}\psi e^{-ipx\cos\psi} \quad .$$

Now, we make use of the power expansion of the exponential function to write the above equation as follows :

$$I = \sum_{n=0}^{\infty} \frac{(-ipx)^n}{n!} \left[\int_0^{\frac{\pi}{2}} d\psi \sin^{d-2}\psi \cos^n\psi - \int_{\frac{\pi}{2}}^{\pi} d\psi \sin^{d-2}\psi \cos^n\psi \right] \quad .$$

Splitting the integral into two terms enables us to identify them with the integral representation of the B-function, by making the change of variables $\theta = \pi - \psi$ in the second term. We then get

$$I = \sum_{n=0}^{\infty} \frac{(-ipx)^n}{n!} \left[1 + (-1)^n\right] \int_0^{\frac{\pi}{2}} d\psi \sin^{d-2}\psi \cos^n\psi \quad .$$

With the use of Eq. (3.76c), we are able to write

$$I = \sum_{n=0}^{\infty} \frac{(-ipx)^{2n}}{(2n)!} B\left(\frac{d-1}{2}, \frac{2n+1}{2}\right) \quad .$$

We now use the formulae (3.76d) and (3.76e) in succession, and tidy up the result in the form

$$I = \sqrt{\pi}\, \Gamma\left(\frac{d-1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\frac{d}{2}+n\right)} \left(\frac{px}{2}\right)^{2n} \quad ,$$

which can be readily identified with the series representation of the Bessel function, (3.76g), to arrive at the result (3.76).

Chapter 4

The Mass Function and The Vertex

In the last chapter, we presented the construction of an effective vertex that ensures gauge independent chiral symmetry breaking. The vertex is written in terms of two unknown functions W_1 and W_2 which obey certain conditions, Eqs. (3.40,3.45,3.58,3.71). The function W_1 corresponds to the equation for $F(p^2)$, while W_2 to that for the mass function $\mathcal{M}(p^2)$. The assumption that the transverse vertex vanishes in the Landau gauge does not enter the discussion of W_1. However, the conditions for W_2, Eq. (3.58,3.71), crucially depend on the aforementioned assumption. The discussion on this issue is intimately related to the value of the exponent, s, of the mass function($\mathcal{M}(p^2) = (p^2)^{-s}$), at criticality. If the assumption holds true, then $s_c = 0.47$. However, Holdom [40] uses the arguments based on Cornwall-Jackiw-Tomboulis (CJT) effective potential technique [41] to show that s is strictly equal to $1/2$ regardless of the choice of the vertex. This would suggest that there is a piece in the transverse part of the vertex which does not vanish in the Landau gauge and has the property that it restores the result obtained by the use of the bare vertex, Eq. (1.40). The study of DSE seems to suggest that, although $s = 1/2$ is a possibility, it does not have to be $1/2$. In fact, only a particular family of vertices will ensure $s = 1/2$. In this chapter, we shall attempt to find constraints on such a group of vertices through the same method as the one employed in Chapter 3.

An added motivation to carry out this work comes from the recent perturbative calculation of the transverse vertex in an arbitrary covariant gauge, performed by

Kizilersü et al. [38]. Their work will be discussed in detail in the next chapter. However, it is important to mention that this calculation reveals, for the first time, that the transverse part of the vertex does not vanish in the Landau gauge, an assumption which has been made frequently in various works, including the one discussed in the last chapter. It may well be that the non-zero transverse piece in the Landau gauge restores the simplicity of the result which is the characteristic of the bare vertex, spoiled by an additional term introduced in the longitudinal vertex constructed by Ball and Chiu.

4.1 Is $s = 1/2$?

Using the arguments based on the CJT effective potential, Bob Holdom claims that, regardless of the choice of the transverse vertex, the mass function $\mathcal{M}(p^2)$ could be proved to obey the equation

$$p\mathcal{M}(p^2) = \frac{1}{2}\int_{\mathcal{M}(0)}^{\Lambda} dk G(k,p)\mathcal{M}(k^2) \quad . \tag{4.1}$$

where $G(k,p)$ is a function independent of $\mathcal{M}$. Holdom goes on to deduce [40] from this equation that $s = 1/2$ is not merely an artifact of the bare vertex, but that it is a "universal consequence of quenched theories".

It seems natural to believe that the arguments using CJT effective potential should not be in contradiction with those using the DSEs. Let us recall the DSE for the fermion propagator Eq. (3.23) :

$$\begin{aligned} S_F^{-1}(p^2) \; = \; S_F^{0^{-1}}(p^2) \; &+ \; ie^2 \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{q^2} \, \gamma^\mu \, S_F(k) \, \Gamma^\nu(k,p) \; \Delta^T_{\mu\nu}(q) \\ &- \; ie^2\xi \int \frac{d^4k}{(2\pi)^4} \, \frac{\not q}{q^4} \, S_F(k) S_F^{-1}(p) \quad . \end{aligned}$$

Using the definition of the full fermion propagator in this equation and once more taking the trace, we obtain the following linearized equation in $\mathcal{M}$:

$$\begin{aligned} \frac{\mathcal{M}(p^2)}{F(p^2)} \; = \; \frac{-ie^2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2)}{q^2k^2} \Bigg\{ &\mathrm{Tr}\left[\gamma^\mu \left(\not k + \mathcal{M}(k^2)\right)\Gamma_\mu - \frac{\not q}{q^2}\left(\not k + \mathcal{M}(k^2)\right) q^\mu \Gamma_\mu\right] \\ &- \frac{4}{q^2}\frac{\xi}{F(p^2)}\left[\mathcal{M}(k^2) p\cdot q - \mathcal{M}(p^2) k\cdot q\right] \Bigg\} . \end{aligned}$$

As a consequence of the WTI, $\mathcal{M}(p^2)$, as well as $\mathcal{M}(k^2)$, appears on the right hand side of the above equation. Therefore, unless a miraculous cancellation occurs, it is not possible to write this equation in the form Eq. (4.1) where $G(k,p)$ is independent of $\mathcal{M}$, as proposed by Holdom. Such a cancellation does not occur in the case of the CP vertex although the value of s, for the CP vertex, comes out to be very close to 1/2. It is around 0.47 in the Landau gauge.

So far, there has been no *ansatz* for the transverse vertex that is not based upon the assumption that the transverse part of the vertex vanishes in the Landau gauge. Our aim in the rest of this chapter is to find constraints on the vertex such that the aforementioned miracle does indeed take place.

4.2 Constraint on the Vertex from $s = 1/2$

Recall the linearized equation for the mass function $\mathcal{M}(p^2)$ from Chapter 3, Eq.(3.46).

$$
\begin{aligned}
\mathcal{M}(p^2) &= \frac{\alpha\xi}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} \\
&+ \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\left[\mathcal{M}(k^2)+\frac{p^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{k^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&+ \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\left[\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)}+\frac{k^2}{2(k^2-p^2)}\,\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\qquad\left. -\frac{p^2}{2(k^2-p^2)}\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&- \frac{3\alpha}{4\pi}\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\tau_2(k^2,p^2)\right. \\
&\qquad\left. +p^2\,\tau_3(k^2,p^2)\,+\,(k^2-p^2)\,\tau_6(k^2,p^2)\right] \\
&- \frac{3\alpha}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\,\tau_2(k^2,p^2)\right. \\
&\qquad\left. +k^2\,\tau_3(k^2,p^2)\,+\,(k^2-p^2)\,\tau_6(k^2,p^2)\right].
\end{aligned}
$$

In the case of the bare vertex, the mass function obeys the following equation in the Landau gauge :

$$\mathcal{M}(p^2) = \frac{3\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\theta(k^2-p^2)\right] \quad . \tag{4.2}$$

In order that Eq. (3.46) is identical to Eq. (4.2) for all values of the gauge parameter, the following must hold true :

$$\begin{aligned}
\frac{\xi}{3}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} = &-\int_0^{p^2}\frac{dk^2}{p^2}\frac{1}{2(k^2-p^2)} \\
&\left[p^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\left.\qquad -k^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&-\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{1}{2(k^2-p^2)} \\
&\left[k^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
&\left.\qquad -p^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
&+\int_0^{p^2}\frac{dk^2}{p^2}\mathcal{M}(k^2)F(k^2)\left[\frac{k^2}{6}(k^2-3p^2)\,\tau_2(k^2,p^2)\right. \\
&\left.\qquad +p^2\,\tau_3(k^2,p^2)\;+\;(k^2-p^2)\,\tau_6(k^2,p^2)\right] \\
&+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\mathcal{M}(k^2)F(k^2)\left[\frac{p^2}{6}(p^2-3k^2)\,\tau_2(k^2,p^2)\right. \\
&\left.\qquad +k^2\,\tau_3(k^2,p^2)\;+\;(k^2-p^2)\,\tau_6(k^2,p^2)\right]
\end{aligned} \tag{4.3}$$

at all momenta p. As before, we write the above equation in terms of functions K_3 and K_4. They are, respectively, the sum and difference of a symmetric function $g_s(k^2,p^2)$ and an antisymmetric function $g_a(k^2,p^2)$,

$$\begin{aligned}
g_s(k^2,p^2) &= \frac{1}{4}\left[(k^2-p^2)^2-4k^2p^2\right]\tau_2(k^2,p^2)+\frac{3}{2}\,(k^2+p^2)\,\tau_3(k^2,p^2) \\
&\quad +3(k^2-p^2)\,\tau_6(k^2,p^2) \\
g_a(k^2,p^2) &= \frac{1}{4}(k^2-p^2)\left[(k^2+p^2)\tau_2(k^2,p^2)-6\tau_3(k^2,p^2)\right] \quad ,
\end{aligned} \tag{4.4}$$

so that

$$\begin{aligned}
K_3(k^2,p^2) &= g_s(k^2,p^2)+g_a(k^2,p^2) \\
K_4(k^2,p^2) &= g_s(k^2,p^2)-g_a(k^2,p^2) \quad .
\end{aligned}$$

Eq. (4.3) can then be written as

$$
\begin{aligned}
\frac{\xi}{3}\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)\frac{F(k^2)}{F(p^2)} \;=\; & -\int_0^{p^2}\frac{dk^2}{p^2}\frac{1}{2(k^2-p^2)} \\
& \left[p^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
& \qquad \left.-k^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
& -\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{1}{2(k^2-p^2)} \\
& \left[k^2\mathcal{M}(k^2)\left(1-\frac{F(k^2)}{F(p^2)}\right)\right. \\
& \qquad \left.-p^2\left(\mathcal{M}(k^2)-\mathcal{M}(p^2)\frac{F(k^2)}{F(p^2)}\right)\right] \\
& +\int_0^{p^2}\frac{dk^2}{p^2}\,\mathcal{M}(k^2)F(k^2)K_3(k^2,p^2) \\
& +\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\,\mathcal{M}(k^2)F(k^2)K_4(k^2,p^2) \quad . \qquad (4.5)
\end{aligned}
$$

Introducing the variable x, where, for $0 \le k^2 < p^2$, $x = k^2/p^2$, and for $p^2 \le k^2 < \infty$, $x = p^2/k^2$, we get

$$
\begin{aligned}
& \xi\int_0^1\frac{dx}{\sqrt{x}}x^\nu - \frac{3}{2}\int_0^1\frac{dx}{\sqrt{x}}\left[\frac{x^\nu - x^{-\nu}}{x-1}\right] + \frac{3}{2}\int_0^1\frac{dx}{\sqrt{x}}\left[\frac{x\left(x^{\nu+\frac{1}{2}} - x^{-(\nu+\frac{1}{2})}\right)}{x-1}\right] \\
& -\int_0^1\frac{dx}{\sqrt{x}}\;x^\nu\;F(p^2)\left[g_s(xp^2,p^2)+g_a(xp^2,p^2)\right] \\
& -\int_0^1\frac{dx}{\sqrt{x}}x^{-\nu}F(p^2)\left[g_s\left(p^2/x,p^2\right)-g_a\left(p^2/x,p^2\right]\right) \;=\; 0 \quad . \qquad (4.6)
\end{aligned}
$$

Again, this equation must hold true for all p^2, and so the integrands cannot be functions of p^2 but only of x. Thus, we define,

$$
\begin{aligned}
F(p^2)\,g_s(xp^2,p^2) &\equiv g_1(x) \\
F(p^2)\,g_a(xp^2,p^2) &\equiv g_2(x)
\end{aligned}
$$

to arrive at

$$
\begin{aligned}
& \xi\int_0^1\frac{dx}{\sqrt{x}}x^\nu - \frac{3}{2}\int_0^1\frac{dx}{\sqrt{x}}\left[\frac{x^\nu - x^{-\nu}}{x-1}\right] + \frac{3}{2}\int_0^1\frac{dx}{\sqrt{x}}\left[\frac{x\left(x^{\nu+\frac{1}{2}} - x^{-(\nu+\frac{1}{2})}\right)}{x-1}\right] \\
& -\int_0^1\frac{dx}{\sqrt{x}}\;x^\nu\;\left[g_1(x)+g_2(x)\right] - \int_0^1\frac{dx}{\sqrt{x}}x^{-\nu}\left[g_1\left(1/x\right)-g_2\left(1/x\right)\right] \;=\; 0 \quad . \qquad (4.7)
\end{aligned}
$$

The symmetry of the vertex under $k \leftrightarrow p$ translates as,

$$
\begin{aligned}
g_1(1/x) &= x^\nu g_1(x) \\
g_2(1/x) &= -x^\nu g_2(x) \quad .
\end{aligned}
$$

Eq. (4.7) can now be written in a compact way as

$$
\int_0^1 \frac{dx}{\sqrt{x}}\, V_2(x) = 0 \quad , \tag{4.8}
$$

where

$$
\begin{aligned}
V_2(x) &= \xi\, x^\nu + \frac{3}{2}\left[\frac{x^{-\nu}-x^\nu}{x-1}\right] - \frac{3x}{2}\left[\frac{x^{-(\nu+\frac{1}{2})}-x^{(\nu+\frac{1}{2})}}{x-1}\right] \\
&\quad -x^\nu\left[g_1(x)+g_2(x)\right] - x^{-\nu}\left[g_1(1/x)-g_2(1/x)\right] \ . \qquad (4.9)
\end{aligned}
$$

Unlike the case for W_2, V_2 does not, in general, vanish in the Landau gauge. Instead, it is

$$
V_2(x) = \frac{3\sqrt{x}}{2} \ - \ 2\left[g_1(x)+g_2(x)\right] \quad . \tag{4.10}
$$

In terms of variables k^2 and p^2,

$$
V_2\left(\frac{k^2}{p^2}\right) = \frac{3}{2}\frac{k}{p} - \left[k^2(k^2-3p^2)\tau_2(k^2,p^2) + 6p^2\tau_3(k^2,p^2) + 6(k^2-p^2)\tau_6(k^2,p^2)\right] \ . \tag{4.11}
$$

Coming back to the discussion in an arbitrary gauge, we would like to invert Eq. (4.11) to evaluate the expressions for τ_i in terms of $V_1(k^2/p^2)$ and $V_2(k^2/p^2)$. As an intermediate step, we have

$$
\begin{aligned}
g_s(k^2,p^2) &= \frac{1}{2\left[F(k^2)+F(p^2)\right]}\Bigg[\xi\left(\frac{F(k^2)}{F(p^2)}+\frac{F(p^2)}{F(k^2)}\right) - \frac{3}{2}\frac{k^2+p^2}{k^2-p^2} \\
&\quad \left\{\left(\frac{F(k^2)}{F(p^2)}-\frac{F(p^2)}{F(k^2)}\right) - \left(\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)} - \frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right)\right\} \\
&\quad -\left\{V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right\}\Bigg] \\
g_a(k^2,p^2) &= \frac{1}{2\left[F(k^2)-F(p^2)\right]}\Bigg[\xi\left(\frac{F(k^2)}{F(p^2)}+\frac{F(p^2)}{F(k^2)}\right) + \frac{3}{2} \qquad (4.12) \\
&\quad \left\{\left(\frac{F(k^2)}{F(p^2)}-\frac{F(p^2)}{F(k^2)}\right) + \left(\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)} - \frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right)\right\} \\
&\quad -\left\{V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right\}\Bigg] \ .
\end{aligned}
$$

Solving the last two equations for τ_2 and τ_3 in terms of τ_6 and V_2, we obtain :

$$\begin{aligned}
\tau_2(k^2,p^2) &= -6\,\frac{\tau_6(k^2,p^2)}{(k^2-p^2)} \\
&+\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]}\left\{2\xi q_2(k^2,p^2)+3(k^2+p^2)Q_2(k^2,p^2)\right\} \\
&-\frac{1}{(k^2-p^2)^2}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&-\frac{k^2+p^2}{(k^2-p^2)^3}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right], \qquad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
q_2(k^2,p^2) &= \frac{1}{k^2-p^2}\left[k^2\,\frac{F(k^2)}{F(p^2)}-p^2\,\frac{F(p^2)}{F(k^2)}\right] \\
Q_2(k^2,p^2) &= \frac{1}{k^2-p^2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)}-\frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right].
\end{aligned}$$

Both $q_2(k^2,p^2)$ and $Q_2(k^2,p^2)$ are symmetric functions of k and p. Also,

$$\begin{aligned}
\tau_3(k^2,p^2) &= -\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2) \\
&+\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]}\left\{-\frac{\xi}{3}q_3(k^2,p^2)+2k^2p^2Q_3(k^2,p^2)\right\} \\
&-\frac{1}{6}\frac{k^2+p^2}{(k^2-p^2)^2}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&+\frac{1}{6}\frac{k^4+p^4-6k^2p^2}{(k^2-p^2)^3}\,\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right], \qquad (4.14)
\end{aligned}$$

where

$$\begin{aligned}
q_3(k^2,p^2) &= \frac{1}{k^2-p^2}\left[p^2(p^2-3k^2)\frac{F(k^2)}{F(p^2)}-k^2(k^2-3p^2)\frac{F(p^2)}{F(k^2)}\right] \\
Q_3(k^2,p^2) &= Q_2(k^2,p^2) \quad .
\end{aligned}$$

As before, $q_3(k^2,p^2)$ is a symmetric function of k and p. Now, using the relation

$$\overline{\tau}(k^2,p^2) \;=\; \tau_3(k^2,p^2)+\tau_8(k^2,p^2)-\frac{1}{2}\,(k^2+p^2)\tau_2(k^2,p^2)\,,$$

we find $\tau_8(k^2,p^2)$:

$$
\begin{aligned}
\tau_8(k^2,p^2) &= -2\frac{k^2+p^2}{k^2-p^2}\tau_6(k^2,p^2)+\overline{\tau}(k^2,p^2) \\
&+\frac{1}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]} \\
&\quad\left\{\frac{\xi}{3}q_8(k^2,p^2)+\frac{1}{2}(3k^4+3p^4+2k^2p^2)Q_8(k^2,p^2)\right\} \\
&-\frac{1}{3}\frac{k^2+p^2}{(k^2-p^2)^2}\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)+V_2\left(\frac{p^2}{k^2}\right)\right] \\
&-\frac{2}{3}\frac{k^4+p^4}{(k^2-p^2)^3}\frac{1}{[F(k^2)+F(p^2)]}\left[V_2\left(\frac{k^2}{p^2}\right)-V_2\left(\frac{p^2}{k^2}\right)\right],
\end{aligned}
\tag{4.15}
$$

where

$$
\begin{aligned}
q_8(k^2,p^2) &= \frac{1}{(k^2-p^2)}\left[(3k^4+p^4)\frac{F(k^2)}{F(p^2)}-(k^4+3p^4)\frac{F(p^2)}{F(k^2)}\right] \\
Q_8(k^2,p^2) &= Q_2(k^2,p^2) = Q_3(k^2,p^2)\,.
\end{aligned}
$$

It is here that we note the restoration of simplicity. The explicit appearance of the mass term in τ_2, τ_3 and τ_8 is through the same factor

$$
\frac{1}{k^2-p^2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)}-\frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right],
$$

unlike the case $s=0.47$, where $q_2(k^2,p^2)$, $q_3(k^2,p^2)$ and $q_8(k^2,p^2)$ are all different from each other and more complicated.

4.3 Constraint on V_2 from Avoiding Kinematic Singularities

Imposing the condition that the vertex and its components should be free of kinematic singularities means that,

$$
\lim_{k^2\to p^2}(k^2-p^2)\,\tau_i(k^2,p^2)=0 \qquad i=2,3,8 \quad ,
$$

which implies

$$
V_2(1)+2V_2'(1)=\xi(2\nu+1)\;+\;6(\nu+s) \quad , \tag{4.16}
$$

where we have used the fact that the antisymmetry of τ_6 means $\tau_6(p^2, p^2) = 0$.

We have been able to see that the whole procedure of constructing the transverse vertex for the case $s = 1/2$ is identical to the one for $s = 0.47$ detailed in Chapter 3. The function V_2 here is the counterpart of W_2. On comparing Eq. (3.59) and Eq. (4.9), we can see that the main difference between them is that V_2 has additional piece coming from the longitudinal part of the vertex. As a result of this difference, V_2 *does not vanish in the Landau gauge in contrast to* W_2. We have felt no need to mention the function V_1 because it is exactly the same as W_1 as pointed out in the beginning of this chapter.

The transverse vertex has the correct lowest order perturbative limit, viz. $\Gamma^\mu_T = \mathcal{O}(\alpha)$, provided

$$V_2\left(k^2/p^2\right) = \xi\,\frac{F(k^2)}{F(p^2)} + \frac{3}{2}\left[\frac{F(k^2)\mathcal{M}(p^2)}{F(p^2)\mathcal{M}(k^2)} - \frac{F(p^2)\mathcal{M}(k^2)}{F(k^2)\mathcal{M}(p^2)}\right] + \mathcal{O}(\alpha) \quad . \tag{4.17}$$

Since at large momenta we expect the power behaviour of Eqs. (3.36,3.48) even away from criticality, Eq. (4.16) will hold for all values of the coupling, α. In contrast, Eq. (4.8) is only true at the bifurcation point. Its exact form for all α is not known, but Eq. (1.40) might suggest

$$\int_0^1 \frac{dx}{\sqrt{x}}\,V_2(x) \approx 2\xi\sqrt{1-\frac{\alpha}{\alpha_c}} \quad . \tag{4.18}$$

to agree with both the $\alpha = 0$ and $\alpha = \alpha_c$ limits, Eqs. (4.17,4.8). Apart from the greater simplicity of the expressions, there is no technical difference between the calculations for $s = 1/2$ and $s = 0.47$.

We hope that the perturbative calculation of the transverse vertex in an arbitrary gauge may provide some clue to resolve the issue as to whether $s = 1/2$ or not. As the derivative condition only represents the fact that the τ_i do not have any kinematic singularity, it is the integral condition, Eq. (4.8), which is more likely to shed light on what s might be. However, as the difference between the two values of s is very small, it may not be very simple to lift the degeneracy between the two cases.

Chapter 5

Effective Vertex In Perturbation Theory

We have repeatedly emphasized in previous chapters the importance of perturbation theory as a guide to determining the non-perturbative aspects of a gauge theory. Perturbation theory is the only known truncation of the full set of DSEs that maintains the two key features, namely, gauge invariance (GI) and multiplicative renormalizability (MR) of a gauge theory at every level of approximation. A solution of the DSEs can be acceptable and physically meaningful only if it agrees with perturbative results in the weak coupling regime. As an example of how perturbation theory can hint at the allowed non-perturbative structure of a gauge theory, let us concentrate on the fermion-photon vertex. In principle, the non-perturbative form of the vertex would contain information about all the other Green's functions of the theory. The Ward Takahashi Identity (WTI) indicates that at least some of these must involve the fermion functions $F(p^2)$ and $\mathcal{M}(p^2)$. Perturbation theory can be a guide to the form in which these functions may appear. This is true only if we calculate the vertex in an arbitrary gauge. As an example, the form $[F^{-1}(k^2) - F^{-1}(p^2)]$ for the transverse vertex was hinted at by the perturbative result being proportional to $(\alpha\xi/4\pi)\ln(k^2/p^2)$ for $k^2 >> p^2$ as mentioned in Chapter 2. We shall discuss this a bit more in the next section. Had this calculation been performed in the Landau gauge alone, it would not have been possible to deduce the aforementioned non-perturbative form. Therefore, in this final chapter, we aim to gain information about functions W_1 and W_2, which appear in the non-perturbative construction of the vertex in Chapter 3, through the perturbative expansion of the transverse vertex.

5.1 Perturbative Expansion of the Vertex

We start by recalling that what remains undefined in the vertex after the application of the WTI is the transverse piece. However, perturbative expansion of the transverse vertex can be obtained by the subtraction of a similar expansion for the longitudinal part from that of the full vertex. In the following three sub-sections, we shall briefly review the perturbative transverse vertex in :

- the Feynman gauge at all momenta,
- an arbitrary covariant gauge in the limit when momentum in one of the fermion legs is much greater than that in the other, e.g., $k^2 \gg p^2$,
- an arbitrary covariant gauge at all momenta.

5.1.1 In Feynman Gauge at All Momenta

Employing the procedure outlined above, Ball and Chiu [22] evaluated the transverse part of the vertex in the Feynman Gauge at all momenta. They calculated analytic expressions for the coefficients τ_i of each of these tensors $T_i(k,p)$, Eq. (1.56). Employing a tensor method permitted each coefficient to be expressed in terms of a single scalar integral plus elementary functions. Other than simplicity, the only criterion that was used for choosing a particular set of basis vectors rather than some linear combinations of these was that, with their choice, each of the coefficients, in itself, was free of kinematical singularities. It was found, by Ball and Chiu, that if instead of T_3 given in Chapter 2, Eq. (1.56), they used $T_1^\mu \not{q}$, which is a linear combination of T_2, T_3 and T_6, a kinematical singularity appeared in τ_6, while for their choice of T_i, all the τ_i were separately analytic. We list τ_2, τ_3, τ_6 and τ_8 in the massless fermion case :

$$
\begin{aligned}
\tau_2(k^2,p^2,q^2) &= -\frac{3}{4}\frac{k\cdot p}{\Delta^2}\tau_8 + \frac{\alpha}{8\pi\Delta^2}\left\{\frac{q^2}{4}J_0 - \frac{1}{2}\frac{(k+p)^2}{k^2-p^2}\ln\frac{k^2}{p^2} - 1\right\} \\
\tau_3(k^2,p^2,q^2) &= -\frac{\alpha}{8\pi\Delta^2}\left\{ \quad J_0 \quad \left[\frac{3}{8}\frac{(k^2-p^2)^2(k\cdot p)^2}{\Delta^2} - \frac{1}{8}(k^2+p^2)^2 - \Delta^2\right]\right. \\
&\qquad + \ln\frac{k^2}{p^2} \quad \left[\frac{1}{4}(k^2-p^2)\left(1-\frac{3}{2}\frac{(k\cdot p)(k+p)^2}{\Delta^2}\right)\right] \\
&\qquad + \ln\frac{q^4}{k^2p^2}\left[\frac{k\cdot p}{2}\left(\frac{3}{4\Delta^2}(k^2-p^2)^2 - 1\right)\right] \\
&\qquad \left. + \qquad (k+p)^2\right\} \\
\tau_6(k^2,p^2,q^2) &= -\frac{k^2-p^2}{2}\tau_2 \\
\tau_8(k^2,p^2,q^2) &= \frac{\alpha}{8\pi\Delta^2}\left\{q^2\left[k\cdot pJ_0 + \ln\frac{q^4}{k^2p^2}\right] - (k^2-p^2)\ln\frac{k^2}{p^2}\right\} \quad , \qquad (5.1)
\end{aligned}
$$

where

$$
\begin{aligned}
q &= k-p \\
\Delta^2 &= (k\cdot p)^2 - k^2p^2 \\
J_0 &= \frac{2}{i\pi^2}\int d^4\omega \,\frac{1}{\omega^2(\omega-p)^2(\omega-k)^2} \\
J_0 &= \frac{2}{\Delta}\left[f\left(\frac{k\cdot p-\Delta}{p^2}\right) - f\left(\frac{k\cdot p+\Delta}{p^2}\right) + \frac{1}{2}\ln\frac{q^2}{p^2}\ln\left(\frac{k\cdot p-\Delta}{k\cdot p+\Delta}\right)\right] \\
f(x) &= Sp(1-x) \\
Sp(x) &= -\int_0^x dy\frac{\ln(1-y)}{y} \quad . \qquad (5.2)
\end{aligned}
$$

Although the Eqs. (5.1) appear a bit complicated, the nice thing is that all the τ_i are expressed in terms of elementary functions and a single scalar integral J_0. Ball and Chiu proved that the τ_i individually are free of kinematic singularities at $\Delta^2 = 0$ and $k^2 = p^2$. However, at $q^2 = 0$, τ_3 has logarithmic divergence. This singularity is allowed for good dynamical reasons. If we now take the limit $q_\mu \to 0$, T_3 vanishes and the vertex has a finite limit. This is in accordance with the Ward Identity. It was also found that, after taking the limits $q_\mu \to 0$ and $q^2 \to 0$, the transverse part is finite and the longitudinal piece is logarithmically divergent in the mass-shell limit $k^2 \to 0$ (recall here we only consider the massless fermion case).

The above calculation has been an important step towards a better understanding of the analytic behaviour of the transverse vertex and possibly provides us with a sound basis of tensors in terms of which one can attempt to construct a non-perturbative vertex. However, the only major draw-back in this calculation is that it has only been performed in the Feynman gauge. Therefore, the perturbative expression for the τ_i cannot serve as a guide to their non-perturbative form.

5.1.2 In an Arbitrary Gauge at Large Momenta

Curtis and Pennington [24] realised that, unless the vertex is calculated in an arbitrary gauge, it would not be possible to put forward an educated guess for its non-perturbative form. However, as the task seemed formidable, they carried out this calculation only in a particular range of the external momenta, i.e. when $k^2 \gg p^2$. As mentioned in Chapter 3, they showed that in the leading logarithm approximation,

$$\Gamma^\mu_T(k,p) \simeq -\frac{\alpha\xi}{8\pi} \ln\frac{k^2}{p^2}\left[\gamma^\mu - \frac{k^\mu \not{k}}{k^2}\right] \quad .$$

Since

$$F(p^2) = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2} + \cdots \quad , \tag{5.3}$$

probably the simplest way to achieve the factor $(-\alpha\xi/4\pi)$, in the large k^2 approximation of the vertex, is the non-perturbative factor $[F^{-1}(k^2) - F^{-1}(p^2)]$. Therefore, they were guided by the perturbative expansion of the vertex to put forward the following *ansatz* for the non-perturbative transverse vertex :

$$\Gamma^\mu_T(k,p) = \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{d(k^2,p^2)} T^\mu_6(k,p) \; , \tag{5.4}$$

where

$$\begin{aligned} d(k^2,p^2) &= \frac{(k^2-p^2)^2 + [\mathcal{M}^2(k^2) + \mathcal{M}^2(p^2)]^2}{k^2+p^2} \\ T^\mu_6(k,p) &= \gamma^\mu(k^2-p^2) - (k+p)^\mu(\not{k} - \not{p}) \quad . \end{aligned}$$

Their choice of the factor $d(k^2,p^2)$ guarantees the multiplicative renormalizability of the fermion propagator and for massive fermions avoids kinematic singularities.

It seems needless to mention that this *ansatz* should be regarded as a minimal vertex which uses only a little amount of information available from the perturbation theory to solve the problem of multiplicative renormalizability. The form for $d(k^2, p^2)$ is a guess as it is only determined when $k^2 >> p^2$ or $p^2 >> k^2$. The success of this vertex *ansatz* has already been discussed in Chapter 2. There we saw that the *CP* vertex corresponds to $W_1 = 0$ in Eqs. (3.42,3.43), but in massless QED, W_1 must be non-zero to avoid a kinematic singularity, Eq. (3.45).

5.1.3 In an Arbitrary Gauge at All Momenta

Recently, Kizilersü et al. [38] have performed the complete one loop calculation of the fermion-boson vertex in QED in an arbitrary covariant gauge. What makes this calculation significantly longer and more complicated than that of Ball and Chiu in the Feynman gauge is the additional term $q^\mu q^\nu / q^4$ of the photon propagator. This term brings greater complexity because of the potential appearance of infra-red divergences. Although this calculation has been performed for all τ_i in massive QED, we shall list only τ_2, τ_3, τ_6 and τ_8 in the massless case :

$$\begin{aligned}\tau_2(k^2,p^2,q^2) \;=\; \frac{\alpha}{8\pi\Delta^2}\Bigg\{ & J_0 \left[\xi'\left(-\frac{3}{2\Delta^2}q^2k^2p^2 - (k^2+p^2)\right) + k\cdot p\right] \\ & + \ln\frac{k^2}{p^2}\left[\xi'\frac{3}{2\Delta^2}(k^2-p^2)k\cdot p - \frac{\xi}{2}\frac{(k+p)^2}{k^2-p^2}\right] \\ & + \ln\frac{q^4}{k^2p^2}\left[\xi'\frac{-3}{2\Delta^2}\,q^2\,k\cdot p - (1-\xi)\right] \\ & - \;2\;\xi'\Bigg\} \end{aligned} \tag{5.5}$$

$$\begin{aligned}\tau_3(k^2,p^2,q^2) \;=\; \frac{\alpha}{8\pi\Delta^2}\Bigg\{ & J_0 \left[\frac{\xi'}{4}\left(-\frac{3}{\Delta^2}(k^2-p^2)^2(k\cdot p)^2 + (k^2+p^2)^2\right) + \Delta^2\right] \\ & + \ln\frac{k^2}{p^2}\left[\xi'\frac{k^2-p^2}{2}\left(-1+\frac{3}{2\Delta^2}(k+p)^2k\cdot p\right)\right] \\ & + \ln\frac{q^4}{k^2p^2}\left[\xi'\,k\cdot p\left(\frac{-3}{4\Delta^2}(k^2-p^2)^2+1\right)\right] \\ & - \;\xi'\,(k+p)^2\Bigg\} \end{aligned} \tag{5.6}$$

$$
\begin{aligned}
\frac{\tau_6(k^2,p^2,q^2)}{k^2-p^2} &= \frac{-\alpha}{16\pi\Delta^2}\Bigg\{ J_0 \quad \left[\xi'\left(\frac{q^2}{2}\left(1-\frac{3}{\Delta^2}(k\cdot p)^2\right)+\Delta^2\right)\right] \\
&\quad +\ln\frac{k^2}{p^2}\quad\left[\xi'\left(\frac{3}{2\Delta^2}(k^2-p^2)-\frac{(k+p)^2}{k^2-p^2}\right)\right] \\
&\quad +\ln\frac{q^4}{k^2p^2}\left[\xi'\,\frac{-3}{2\Delta^2}\,q^2\,k\cdot p\right] \\
&\quad -\quad 2\quad \xi'\Bigg\}
\end{aligned} \tag{5.7}
$$

$$
\tau_8(k^2,p^2,q^2) = \frac{\alpha}{8\pi\Delta^2}\left\{q^2\left[k\cdot pJ_0+\ln\frac{q^4}{k^2p^2}\right]-(k^2-p^2)\ln\frac{k^2}{p^2}\right\}, \tag{5.8}
$$

where $\xi' = 1-\xi/2$. Despite the extra complication involved, the final result can still be written in terms of elementary functions and a single scalar integral J_0. Checking the singularity structure of these four components of the vertex, it is found that they continue to be free of any kinematic singularities even in an arbitrary covariant gauge. Therefore, Ball and Chiu's choice of corresponding basis tensors remains unaltered in the case of these four $T_2^\mu, T_3^\mu, T_6^\mu, T_8^\mu$. The main advantage of this work is that it would serve as a guide to the construction of a non-perturbative *ansatz* for the 3-point vertex which must agree with perturbation theory in the weak coupling limit. All this will be discussed in the rest of this chapter.

5.2 Exact Vertex and Effective Vertex

As discussed in detail in Chapter 2, in order to solve the DSE for the fermion propagator, some assumptions about the structure of the fermion-boson vertex $\Gamma^\mu(k,p,q)$ have to be made. One of them has been that it does not depend upon q^2. It seems impossible to proceed without this assumption because, otherwise, we cannot even carry out the integration over the angular variable. As mentioned earlier, a motivation for this simplifying assumption comes from the large momentum behaviour of the vertex, where it does, indeed, only depend on the variables k^2 and p^2, and **not on** q^2, Eq. (2.18). However, it is clear from the perturbative calculation of Kizilersü, Reenders and Pennington [38] that the same does not hold true for all the ranges of k^2 and p^2. Instead, the q^2-dependence occurs in almost every term of each of the τ_i. Therefore, we should keep in mind that whenever we are neglecting q^2-dependence, we are not talking about the exact but only the effective vertex. In order to find a connection between the two, recall the equation for $F(p^2)$:

$$
\begin{aligned}
\frac{1}{F(p^2)} = 1 - & \frac{\alpha}{4\pi^3}\frac{1}{p^2}\int d^4k \, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)}\,\frac{1}{q^2} \\
& \Big\{ \quad a(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2 - 3k\cdot pq^2\right] \\
+ & \quad b(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2(k^2+p^2)\right] \\
+ & \ \mathcal{M}(k^2)c(k^2,p^2)\frac{1}{q^2}\left[-2\Delta^2\right] \\
- & \ \frac{\xi}{q^2F(p^2)}\left[p^2(k^2-k\cdot p)+\mathcal{M}(k^2)\mathcal{M}(p^2)(k\cdot p-p^2)\right] \\
+ & \quad \tau_2(k^2,p^2,q^2)\left[-\Delta^2(k^2+p^2)\right] \\
+ & \quad \tau_3(k^2,p^2,q^2)\left[2\Delta^2+3q^2k\cdot p\right] \\
+ & \quad \tau_6(k^2,p^2,q^2)\left[(k^2-p^2)3k\cdot p\right] \\
+ & \quad \tau_8(k^2,p^2,q^2)\left[2\Delta^2\right]\Big\} \quad .
\end{aligned} \tag{5.9}
$$

τ_i in the above equation are so far exact. The equation for $F(p^2)$, after the angular integration has been carried out, contains the effective τ_i :

$$
\begin{aligned}
\frac{1}{F(p^2)} = \ & 1 - \frac{\alpha}{4\pi}\int_0^{\Lambda^2} dk^2 \, \frac{F(k^2)}{k^2+\mathcal{M}^2(k^2)} \\
& \left[\frac{k^4}{p^4} \Big\{ b(k^2,p^2)\left[\frac{3}{2}(k^2+p^2)\right]\right. \\
& + c(k^2,p^2)\left[\frac{3}{2}\mathcal{M}(k^2)\right] + \frac{\xi}{F(p^2)}\,\frac{\mathcal{M}(k^2)\mathcal{M}(p^2)}{k^2} \\
& + \quad \tau_2^{\text{eff}}(k^2,p^2)\left[-\frac{1}{4}(k^2+p^2)(k^2-3p^2)\right] \\
& + \quad \tau_3^{\text{eff}}(k^2,p^2)\left[\frac{1}{2}(k^2-3p^2)\right] \\
& + \quad \tau_6^{\text{eff}}(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
& + \quad \tau_8^{\text{eff}}(k^2,p^2)\left[\frac{1}{2}(k^2-3p^2)\right]\Big\}\theta(p^2-k^2)
\end{aligned}
$$

$$
\begin{aligned}
+ \Bigg\{ & b(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
+ & c(k^2,p^2)\left[\frac{3}{2}\mathcal{M}(k^2)\right] - \frac{\xi}{F(p^2)} \\
+ & \tau_2^{\text{eff}}(k^2,p^2)\left[-\frac{1}{4}(k^2+p^2)(p^2-3k^2)\right] \\
+ & \tau_3^{\text{eff}}(k^2,p^2)\left[\frac{1}{2}(p^2-3k^2)\right] \\
+ & \tau_6^{\text{eff}}(k^2,p^2)\left[\frac{3}{2}(k^2-p^2)\right] \\
+ & \tau_8^{\text{eff}}(k^2,p^2)\left[\frac{1}{2}(p^2-3k^2)\right] \Bigg\}\theta(k^2-p^2)\Bigg] .
\end{aligned}
\tag{5.10}
$$

On comparing the last two expressions, we find the following exact relation between the exact and the effective τ_i :

$$
\begin{aligned}
\tau_2^{\text{eff}}(k^2,p^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\,\frac{sin^2\theta}{q^2}\tau_2(k^2,p^2,q^2)\,\Delta^2 \\
\tau_3^{\text{eff}}(k^2,p^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\frac{sin^2\theta}{q^2}\tau_3(k^2,p^2,q^2)\left(\Delta^2+\frac{3}{2}q^2k\cdot p\right) \\
\tau_6^{\text{eff}}(k^2,p^2) &= \frac{1}{f_6(k^2,p^2)}\int_0^\pi d\theta\frac{sin^2\theta}{q^2}\tau_6(k^2,p^2,q^2)k\cdot p \\
\tau_8^{\text{eff}}(k^2,p^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\theta\frac{sin^2\theta}{q^2}\tau_8(k^2,p^2,q^2)\Delta^2 \quad ,
\end{aligned}
\tag{5.11}
$$

where

$$
\begin{aligned}
f(k^2,p^2) &= \frac{\pi}{8}\left[\frac{k^2}{p^2}(k^2-3p^2)\,\theta(p^2-k^2)+\frac{p^2}{k^2}(p^2-3k^2)\,\theta(k^2-p^2)\right] \\
f_6(k^2,p^2) &= \frac{\pi}{4}\left[\frac{k^2}{p^2}\,\theta(p^2-k^2)+\frac{p^2}{k^2}\,\theta(k^2-p^2)\right] \quad .
\end{aligned}
$$

In Fig. [5.7] are shown the integrands of Eq. (5.11) illustrating their smoothness that allows the integrals to be computed accurately provided k^2 is not very much bigger or very much smaller than p^2. In this asymptotic limit, the integrals can be evaluated analytically if perfect accuracy is required.

Our aim is to find the perturbative expansion for the functions W_1 and W_2 defined in Chapter 3. As it is non-trivial to carry out the angular integration required in Eqs. (5.11) analytically, we shall first restrict our aim to the limit $k^2 \gg p^2$. Recall the expression for W_1 :

$$W_1\left(\frac{k^2}{p^2}\right) = s_1(k^2,p^2)\Bigg[(k^2-3p^2)\overline{\tau}(k^2,p^2)+\frac{3}{2}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)$$
$$+3(k^2-p^2)\tau_6(k^2,p^2)\Bigg], \qquad (5.12)$$

where

$$s_1(k^2,p^2) = \frac{k^2}{p^2}F(k^2)+\frac{p^2}{k^2}F(p^2).$$

All the terms within the square brackets in Eq. (5.12) are proportional to α. Therefore, the perturbative expansion of W_1, to the lowest order in α, corresponds to :

$$s_1(k^2,p^2) = \frac{k^4+p^4}{k^2p^2} \quad .$$

We learnt from Chapter 3 that if the solution to the DSE for $F(p^2)$, Eq. (3.26), is $(p^2/\Lambda^2)^\nu$ where $\nu = \alpha\xi/4\pi$, then the integral $\int_0^1 dx\, W_1(x) = 0$. We want to see how, (and if) this is satisfied in perturbation theory. To differentiate between the non-perturbative $W_1(x)$ of Sect. (3.2) and its perturbative approximation, we call the latter $\omega_1(x)$. Since the behaviour for $x \to 0$ is critical to the possible convergence of the integral of $\omega(x)$, we first consider the behaviour in this region, which corresponds to $k^2 \gg p^2$, or equally $p^2 \gg k^2$, if we take into account the symmetry properties of the τ_i appropriately.

In order to have a perturbative expansion for ω_1, we have to go up to $\mathcal{O}(1/k^4)$ in $\tau_3^{\rm eff}$, $\tau_6^{\rm eff}$ and $\tau_8^{\rm eff}$, and $\mathcal{O}(1/k^6)$ in $\tau_2^{\rm eff}$, instead of just keeping the terms of order $\mathcal{O}(1/k^2)$ and $\mathcal{O}(1/k^4)$ respectively. Consequently, for the real τ_i, the above statement means that τ_3 and τ_6 would have to be evaluated to $\mathcal{O}(1/k^5)$, τ_8 to $\mathcal{O}(1/k^4)$ and τ_2 to $\mathcal{O}(1/k^6)$. The difference for τ_3 and τ_6 arises due to the fact that some of the angular integrals in Eqs. (5.11) are odd, and we have to be careful in collecting terms of the same order.

The above discussion implies that, in an arbitrary gauge, we have to go up to $\mathcal{O}(1/k^7)$ in evaluating J_0 for k^2 large. The expansions of J_0 and $\ln(q^4/k^2p^2)$ to the required order in the limit when $k^2 \gg p^2$ are :

$$
\begin{aligned}
\ln\frac{q^4}{k^2p^2} &= \ln\frac{k^2}{p^2} - 4\frac{k\cdot p}{k^2} + 2\frac{p^2}{k^2} - 4\frac{(k\cdot p)^2}{k^4} + 4\frac{p^2 k\cdot p}{k^4} - \frac{16}{3}\frac{(k\cdot p)^3}{k^6} - \frac{p^4}{k^4} \\
&+ 8\frac{p^2(k\cdot p)^2}{k^6} - 8\frac{(k\cdot p)^4}{k^8} - 4\frac{p^4 k\cdot p}{k^6} + 16\frac{p^2(k\cdot p)^3}{k^8} - \frac{64}{5}\frac{(k\cdot p)^5}{k^{10}} \\
&+ \frac{2}{3}\frac{p^6}{k^6} - 12\frac{p^4(k\cdot p)^2}{k^8} + 32\frac{p^2(k\cdot p)^4}{k^{10}} - \frac{64}{3}\frac{(k\cdot p)^6}{k^{12}}
\end{aligned}
$$

and

$$
\begin{aligned}
J_0 &= \frac{2}{k^2}\left[\left(1 + \frac{k\cdot p}{k^2} - \frac{1}{3}\frac{p^2}{k^2} + \frac{4}{3}\frac{(k\cdot p)^2}{k^4} - \frac{p^2 k\cdot p}{k^4} + 2\frac{(k\cdot p)^3}{k^6} + \frac{1}{5}\frac{p^4}{k^4}\right.\right. \\
&\quad -\frac{12}{5}\frac{p^2(k\cdot p)^2}{k^6} + \frac{16}{5}\frac{(k\cdot p)^4}{k^8} + \frac{p^4 k\cdot p}{k^6} - \frac{16}{3}\frac{p^2(k\cdot p)^3}{k^8} \\
&\quad \left. +\frac{16}{3}\frac{(k\cdot p)^5}{k^{10}}\right) \ln\frac{k^2}{p^2} \\
&\quad +\left(2 + \frac{k\cdot p}{k^2} - \frac{2}{9}\frac{p^2}{k^2} + \frac{8}{9}\frac{(k\cdot p)^2}{k^4} - \frac{1}{2}\frac{p^2 k\cdot p}{k^4} + \frac{(k\cdot p)^3}{k^6}\right. \\
&\quad +\frac{2}{25}\frac{p^4}{k^4} - \frac{24}{25}\frac{p^2(k\cdot p)^2}{k^6} + \frac{32}{25}\frac{(k\cdot p)^4}{k^8} + \frac{1}{3}\frac{p^4 k\cdot p}{k^6} \\
&\quad \left.\left. -\frac{16}{9}\frac{p^2(k\cdot p)^3}{k^8} + \frac{16}{9}\frac{(k\cdot p)^5}{k^{10}}\right)\right] \quad . \qquad (5.13)
\end{aligned}
$$

A detailed evaluation of $J_0(k^2 \gg p^2)$ has been given in the appendix. A tedious but straightforward calculation leads us to the following perturbative expansions for the τ_i :

$$
\begin{aligned}
\tau_2(k^2,p^2,q^2) &= -\frac{\alpha}{12\pi k^4}\Bigg\{ 1+2\,\frac{k\cdot p}{k^2}+\frac{1}{5k^4}\Big(18(k\cdot p)^2-k^2p^2\Big) \\
&\quad -\xi\Bigg[2+3\,\frac{k\cdot p}{k^2}+\frac{1}{5k^4}\Big(24(k\cdot p)^2+7k^2p^2\Big)\Bigg]\Bigg\}\ln\frac{k^2}{p^2} \\
&\quad +\frac{\alpha}{36\pi k^4}\Bigg\{ 1+5\,\frac{k\cdot p}{k^2}+\frac{1}{50k^4}\Big(594(k\cdot p)^2-193k^2p^2\Big) \\
&\quad -\xi\Bigg[5+\frac{21}{2}\frac{k\cdot p}{k^2}+\frac{1}{50k^4}\Big(972(k\cdot p)^2-209k^2p^2\Big)\Bigg]\Bigg\} \\
\tau_3(k^2,p^2,q^2) &= \frac{\alpha}{12\pi k^2}\Bigg\{ 1+0\,\frac{k\cdot p}{k^2}-\frac{1}{5k^4}\Big(4(k\cdot p)^2+7k^2p^2\Big) \\
&\quad -\frac{k\cdot p}{k^6}\Big(2(k\cdot p)^2+3k^2p^2\Big) \\
&\quad +\xi\Bigg[1+\frac{3}{2}\frac{k\cdot p}{k^2}+\frac{1}{5k^4}\Big(12(k\cdot p)^2+k^2p^2\Big) \\
&\quad +4\frac{(k\cdot p)^3}{k^6}\Bigg]\Bigg\}\ln\frac{k^2}{p^2} \\
&\quad +\frac{7\alpha}{18\pi k^2}\Bigg\{ 1+\frac{27}{28}\frac{k\cdot p}{k^2}+\frac{1}{175k^4}\Big(268(k\cdot p)^2-71k^2p^2\Big) \\
&\quad +\frac{1}{28}\frac{k\cdot p}{k^6}\Big(74(k\cdot p)^2-33k^2p^2\Big) \\
&\quad +\xi\Bigg[\frac{1}{7}-\frac{9}{56}\frac{k\cdot p}{k^2}+\frac{1}{175k^4}\Big(-84(k\cdot p)^2+23k^2p^2\Big) \\
&\quad +\frac{1}{7}\frac{k\cdot p}{k^6}\Big(-7(k\cdot p)^2+3k^2p^2\Big)\Bigg]\Bigg\} \\
\frac{\tau_6(k^2,p^2,q^2)}{(1-\xi/2)} &= -\frac{\alpha}{12\pi k^2}\Bigg\{ 1+\frac{k\cdot p}{k^2}+\frac{3}{5k^4}\Big(2(k\cdot p)^2+k^2p^2\Big) \\
&\quad +\frac{4k\cdot p}{5k^6}\Big(2(k\cdot p)^2+k^2p^2\Big)\Bigg\}\ln\frac{k^2}{p^2} \\
&\quad +\frac{\alpha}{9\pi k^2}\Bigg\{ 1+\frac{11}{8}\frac{k\cdot p}{k^2}-\frac{27}{100k^4}\Big(-7(k\cdot p)^2+4k^2p^2\Big) \\
&\quad -\frac{2k\cdot p}{25k^6}\Big(-34(k\cdot p)^2+23k^2p^2\Big)\Bigg\} \\
\tau_8(k^2,p^2,q^2) &= -\frac{\alpha}{4\pi k^2}\Bigg\{ 1+\frac{2}{3}\frac{k\cdot p}{k^2}+\frac{2}{3k^4}(k\cdot p)^2\Bigg\}\ln\frac{k^2}{p^2} \\
&\quad -\frac{\alpha}{4\pi k^2}\Bigg\{ 1-\frac{2}{9}\frac{k\cdot p}{k^2}-\frac{1}{18k^4}\Big(10(k\cdot p)^2-9k^2p^2\Big)\Bigg\}.
\end{aligned}
\tag{5.14}
$$

We learn the following points from the above calculation :

- To the lowest order in $1/k^2$, all the four τ_i are independent of the angular variable.
- Substituting these τ_i in the expression for the full transverse vertex, we retrieve the perturbative result of Eq. (2.18), which was derived by Curtis and Pennington. This serves as one of the checks of the calculation. However, it does not, by any means, rule out the possibility of errors in the the expressions (5.5,5.6,5.7,5.8) for τ_i.
- Comparing Eqs. (5.5–5.8) and (5.14), one can see that all the Δ^2 have disappeared from the denominator. Hence, for large k^2, the τ_i are explicitly finite for all values of the angular variable.

We can now use Eq. (5.11) to find out the large k^2 expansion of the effective τ_i, which comes out to be :

$$\begin{aligned}
\tau_2^{\text{eff}}(k^2,p^2) &= -\frac{\alpha}{12\pi k^4}\left\{1 - 2\xi + \frac{16}{5}\left(\frac{1}{3} - \xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2} \\
&\quad + \frac{\alpha}{36\pi k^4}\left\{1 - 5\xi - \frac{16}{25}\left(\frac{1}{3} + 4\xi\right)\frac{p^2}{k^2}\right\}, \qquad (5.15) \\
\tau_3^{\text{eff}}(k^2,p^2) &= +\frac{\alpha}{12\pi k^2}\left\{1 + \frac{1}{4}\xi + \frac{1}{5}\left(\frac{7}{3} - \frac{3}{4}\xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2} \\
&\quad + \frac{\alpha}{144\pi k^2}\left\{29 + \frac{25}{2}\xi - \frac{1}{25}\left(\frac{257}{3} - \frac{169}{2}\xi\right)\frac{p^2}{k^2}\right\}, \qquad (5.16) \\
\frac{\tau_6^{\text{eff}}(k^2,p^2)}{(1-\xi/2)} &= -\frac{\alpha}{24\pi k^2}\left\{3 + 5\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2} \\
&\quad + \frac{\alpha}{16\pi k^2}\left\{3 + \frac{5}{9}\frac{p^2}{k^2}\right\}, \qquad (5.17) \\
\tau_8^{\text{eff}}(k^2,p^2) &= -\frac{\alpha}{4\pi k^2}\left\{1 + \frac{1}{3}\frac{p^2}{k^2}\right\}\left(\ln\frac{k^2}{p^2} + 1\right), \qquad (5.18) \\
\overline{\tau}(k^2,p^2) &= -\frac{\alpha}{8\pi k^2}\left\{1 + \frac{1}{2}\xi - \frac{1}{3}\left(1 - \frac{11}{2}\xi\right)\frac{p^2}{k^2}\right\}\ln\frac{k^2}{p^2} \\
&\quad - \frac{\alpha}{16\pi k^2}\left\{1 - \frac{5}{2}\xi + \frac{1}{9}\left(17 - \frac{37}{2}\xi\right)\frac{p^2}{k^2}\right\}. \qquad (5.19)
\end{aligned}$$

Eq. (5.12) now permits us to evaluate $\omega_1(k^2/p^2)$ from Eqs. (5.15–5.19) :

$$\omega_1\left(\frac{k^2}{p^2}\right) \overset{p^2>>k^2}{=} \frac{3\alpha}{4\pi}(1-\xi)\frac{p^2}{k^2}+\frac{\alpha}{2\pi}\left[\ln\frac{k^2}{p^2}-\frac{1}{3}\right] \quad . \tag{5.20}$$

It is readily seen that despite the complicated coefficients arising in the asymptotic limits of τ_i, the expression for ω_1 is incredibly simple. However, the presence of the first term means that the integral $\int_0^1 dx\omega_1(x)$ is not a convergent integral except in the Feynman gauge. Consequently $\int_0^1 dx\omega_1(x)$ does not vanish. The only assumption that had gone into the vanishing of the aforementioned integral was that $F(p^2) = (p^2/\Lambda^2)^\nu$ where $\nu = \alpha\xi/4\pi$, following the perturbative calculation of $F(p^2)$ in leading logarithms and the non-perturbative result obtained from the application of LKF transormations. However, we see that there are the reasons to believe that this assumption may not necessarily be correct :

- The perturbative calculation for $F(p^2)$ to order α gives $F(p^2) = 1$ in the Landau gauge. However, there is nothing to prevent the possibility that to the next order, there may be terms which do not vanish in the Landau gauge, while still preserving the power behaviour of $F(p^2)$.

- Under the generally made assumption that the transverse part of the vertex vanishes in the Landau gauge, it is trivial to see that $F(p^2) = 1$ satisfies Eq. (3.26) for $\xi = 0$. However, the perturbative calculation of the vertex carried out by Kizilersü et. al. clearly shows that this assumption is not true.

- In order to make use of the LKF transformations, we have to input the value of $F(p^2)$ in one gauge. The aforementioned exponent for $F(p^2)$ is generated through the LKF transformations only when we input $F(p^2) = 1$ for $\xi = 0$. If this input were incorrect, we expect LKF transformations to yield a different result.

The knowledge of ω_1 to the lowest order in α can shed light on the 2nd order perturbative behaviour of $F(p^2)$. This is what we aim to do now. The most general multiplicatively renormalizable form for $F(p^2)$ is indeed proportional to $(p^2/\Lambda^2)^\gamma$ where γ is a constant power for α fixed as in the quenched theory. This result can be expanded perturbatively as :

$$F\left(\frac{p^2}{\Lambda^2}\right) = (1+\alpha C_1+\alpha^2 C_2+\cdots)\left(\frac{p^2}{\Lambda^2}\right)^{\alpha A_1+\alpha^2 A_2+\cdots} \quad . \tag{5.21}$$

We insert this expression in Eq. (3.26) and obtain

$$\frac{1}{F(p^2)} = 1+\frac{\alpha\xi}{4\pi\gamma}\left[\left(\frac{\Lambda^2}{p^2}\right)^{\gamma}-1\right]-\frac{\alpha}{8\pi}\left[\int_0^1 dx\,\frac{x^2\omega_1(x)}{x^2+x^{-\gamma}}+\int_{p^2/\Lambda^2}^1 dx\,\frac{\omega_1(x)}{1+x^{2+\gamma}}\right] \tag{5.22}$$

By expanding both sides of this equation in powers of α and comparing the coefficients of α^0, α^1 and α^2, we obtain :

$$C_1 = 0 \tag{5.23}$$

$$A_1 = \frac{\xi}{4\pi} \tag{5.24}$$

$$\alpha\left[C_2+A_2\ln\frac{p^2}{\Lambda^2}\right] = \frac{1}{8\pi}\,[I_1+I_2] \quad . \tag{5.25}$$

where

$$I_1 = \int_0^1 dx\,\frac{x^2\omega_1(x)}{1+x^2}$$

$$I_2 = \int_{p^2/\Lambda^2}^1 dx\,\frac{\omega_1(x)}{1+x^2}$$

with

$$\frac{x\omega_1(x)}{1+x^2} = \frac{3\alpha\xi}{8\pi}\frac{1+x}{1-x}\ln x-(3-x)\,\overline{\tau}(x)-3(1-x)\,\tau_6(x) \quad .$$

The coefficient A_2 can be found just by using the asymptotic analytic expansion of $\omega_1(x)$. In order to see how this can be achieved, we rewrite the integral I_2 as follows :

$$I_2 = \int_{p^2/\Lambda^2}^{y} dx\,\frac{\omega_1(x)}{1+x^2}+\int_y^1 \frac{\omega_1(x)}{1+x^2} \quad ,$$

where y is the maximum value of x such that we can still use the following expansion of $\omega_1(x)$, Eq. (5.20) :

$$\omega_1(x) \overset{x\to 0}{=} \frac{3\alpha}{4\pi x}(1-\xi)+\frac{\alpha}{2\pi}\left(\ln x-\frac{1}{3}\right) \quad .$$

Substituting the expressions for I_1 and I_2 in Eq. (5.22), and comparing the coefficients of $\ln(p^2/\Lambda^2)$ and the constant terms, we find

$$\begin{aligned} A_2 &= -\frac{3(1-\xi)}{2(4\pi)^2} \qquad (5.26) \\ C_2 &= \frac{1}{8\pi}\left[\int_0^1 dx\,\frac{x^2\omega_1(x)}{1+x^2} + \frac{3}{4\pi}(1-\xi)\ln y + \frac{y}{2\pi}\left(\ln y - \frac{4}{3}\right) + \int_y^1 dx\,\frac{\omega_1(x)}{1+x^2}\right] . \end{aligned}$$

In the numerical evaluation of C_2, y should be chosen such that C_2 is insensitive to small variations in y. For a large value of y, the analytic part of the integral will not be reliable, and for small values, the numerical evaluation will not be exact as each of the τ_i are logarithmically divergent at the lower end of the integration range. In an attempt to evaluate C_2, we realise that its value is of the same order as the error in numerical evaluation unless we are in the Feynman gauge where all the logarithmic singularities cancel. In this gauge, we find that $C_2 = 0.00949$, a number which can also be written as

$$C_2(\,\xi = 1\,) = -\frac{3}{2(4\pi)^2} \quad .$$

With the numerical inaccuracy that we have in the evaluation of the τ_i, it could well be that C_2 is independent of the gauge parameter ξ.

It should be stressed that non-leading logarithms are calculation-scheme dependent. Here, we have used an ultraviolet cut-off in momentum as a regulator, whereas the perturbative calculation of the transverse vertex by Kizilersü et al. [38] uses dimensional regularization. It is possible that the fact that $A_2 \neq 0$, in Eqs. (5.21,5.26), is a result of this difference. As an example, the integral

$$I_d(\mathrm{cut-off}) \;=\; (d-4)\int_0^{\Lambda^2} \frac{dk^2}{k^2+p^2}$$

is zero in four dimensions, while

$$\begin{aligned} I_d(\mathrm{dim.\ reg.}) &= (d-4)\int_0^{\infty} \frac{dk^2}{k^2+p^2}\left(\frac{k^2}{\mu^2}\right)^{d/2-2} \\ &= (d-4)\left(\frac{p^2}{\mu^2}\right)^{d/2-2} \Gamma(d/2-1)\Gamma(2-d/2) \\ &\to -2 \end{aligned}$$

when $d \to 4$. However, without redoing the horrendously long calculation of Kizilersü et al. [38], we have not been able to identify such an ultraviolet divergent term proportional to $\alpha^2(1-\xi)$.

The other condition on $W_1(x)$ that we can check in perturbation theory results from the requirement that the τ_i are free of kinematic singularities. This gives

$$W_1(1) \;+\; W_1'(1) = -6\nu \quad .$$

from Eq. (3.45). To $\mathcal{O}(\alpha)$ we can check whether

$$\omega_\Delta \equiv \omega_1(1) \;+\; \omega_1'(1) = -\frac{3\alpha\xi}{2\pi} \tag{5.27}$$

numerically. In Fig. [5.6], we have plotted ω_Δ/α versus the gauge parameter ξ. The numerical and analytical results are in excellent agreement with each other. This serves as a reassuring check of our angular integration routines.

We have been able to find the numerical perturbative expansion of the effective vertex in terms of τ_i and of the function $\omega_1(x)$. In Figs. [5.4,5.8,5.9,5.10], we show the effective τ_i for $\xi = 0, 1, 3$. In Fig. [5.11], we show the corresponding results for $u_1(x)$, where

$$u_1(x) \;\;\equiv\;\; \omega_1(x) - \frac{3\alpha}{4\pi x}(1-\xi) \quad , \tag{5.28}$$

with $\omega_1(x)$ given by Eq. (5.26). As seen from this definition of $\omega_1(x)$, the function $u_1(x)$ is integrable for $0 \le x \le 1$. It is these functions that will hopefully aid the construction of non-perturbative forms for the transverse vertex. We have also been able to evaluate these results analytically in the region where $k^2 \gg p^2$ or $p^2 \gg k^2$. Every non-perturbative construction of the vertex must agree with these perturbative results. This calculation has also been of vital importance in improving our understanding of the fermion propagator, in particular the wave function renormalization $F(p^2)$. It has provided us with the 2nd order perturbative expansion of $F(p^2)$. We expect that a similar calculation for the function $\omega_2(x)$ will reveal important facts about the mass function $\mathcal{M}(p^2)$ and may be able to shed light on some of the issues which are still unresolved. This is for the future.

5.3 Appendix

We start from the expression

$$J_0 = \frac{2}{i\pi^2} \int d^4\omega \, \frac{1}{\omega^2(\omega-k)^2(\omega-p)^2} \quad . \tag{5.29}$$

In order to solve this integral with the use of Feynman parametrization, recall the following identity :

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \, \frac{1}{[\,bx + cy + (1-x-y)a\,]^3} \, . \tag{5.30}$$

Now identify

$$\begin{aligned} a &= \omega^2 \\ b &= (\omega-k)^2 \\ c &= (\omega-p)^2 \quad . \end{aligned}$$

This permits us to write

$$\frac{1}{\omega^2(\omega-k)^2(\omega-p)^2} = \int_0^1 dx \int_0^{1-x} dy \, \frac{2}{[(\omega-k)^2x + (\omega-p)^2y + (1-x-y)\omega^2]^3} \, .$$

Now we write the denominator as follows :

$$(\omega-k)^2x + (\omega-p)^2y + (1-x-y)\omega^2 = \omega'^2 + L \, ,$$

where

$$\begin{aligned} \omega' &= \omega - xk - yp \quad , \\ L &= xk^2(1-x) + yp^2(1-y) - 2xyk\cdot p \quad . \end{aligned}$$

Eq. (5.29) can then be written as

$$J_0 = \frac{4}{i\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^4\omega \, \frac{1}{\left(\omega'^2 + L\right)^3} \quad .$$

Changing the variable of integration from ω to ω', and making use of the standard integral,

$$\int d^4\omega' \, \frac{1}{\left(\omega'^2 + L\right)^3} = \frac{i\pi^2}{2L} \quad ,$$

we obtain

$$J_0 \;=\; 2\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{L} \quad .$$

We now make another change of variables, $y = z(1-x)$, to arrive at the following expression :

$$J_0 \;=\; 2\int_0^1 dx \int_0^1 dz \, \frac{1}{z(1-z)p^2 + x(k^2 + z^2p^2 - 2zk\cdot p)} \quad .$$

It is trivial to carry out integration over the variable x. On performing this integration, we get :

$$J_0 \;=\; 2\int_0^1 dz \, \frac{1}{k^2 + z^2p^2 - 2zk\cdot p} \ln \left[\frac{k^2 + zp^2 - 2zk\cdot p}{z(1-z)p^2}\right] \quad .$$

We are interested in solving this integral only in the limit when $k^2 \gg p^2$. In that case, it is more convenient to write the above expression as follows :

$$\begin{aligned} J_0 \;=\; & 2\int_0^1 dz \, \frac{1}{k^2 + z^2p^2 - 2zk\cdot p} \\ & \left[\ln\frac{k^2}{p^2} + \ln\left\{1 + \frac{zp^2 - 2zk\cdot p}{k^2}\right\} - \ln\left\{z(1-z)\right\}\right] . \end{aligned} \tag{5.31}$$

In order to have a perturbative expansion for W_1, we have to go up to $\mathcal{O}(1/k^5)$ in τ_3 and τ_6, $\mathcal{O}(1/k^6)$ in τ_2 and $\mathcal{O}(1/k^4)$ in τ_8. Therefore, in an arbitrary gauge, we have to go up to $\mathcal{O}(1/k^7)$ in evaluating J_0. We list here the perturbative expansions of the quantities involved up to the required order :

$$\begin{aligned} \frac{1}{k^2 + z^2p^2 - 2zk\cdot p} \;=\; & \frac{1}{k^2} + 2z\frac{k\cdot p}{k^4} - z^2\frac{p^2}{k^4} + 4z^2\frac{(k\cdot p)^2}{k^6} - 4z^3\frac{p^2k\cdot p}{k^6} \\ + \; & 8z^3\frac{(k\cdot p)^3}{k^8} + z^4\frac{p^4}{k^6} - 12z^4\frac{p^2(k\cdot p)^2}{k^8} + 16z^4\frac{(k\cdot p)^4}{k^{10}} \\ + \; & 6z^5\frac{p^4(k\cdot p)}{k^8} - 32z^5\frac{p^2(k\cdot p)^3}{k^{10}} + 32z^5\frac{(k\cdot p)^5}{k^{12}} \end{aligned} \tag{5.32}$$

$$
\begin{aligned}
\ln\left\{1+\frac{zp^2-2zk\cdot p}{k^2}\right\} &= -2z\frac{k\cdot p}{k^2}+z\frac{p^2}{k^2}-2z^2\frac{(k\cdot p)^2}{k^4}+2z^2\frac{p^2k\cdot p}{k^4} \\
&- \frac{8}{3}z^3\frac{(k\cdot p)^3}{k^6}-\frac{1}{2}z^2\frac{p^4}{k^4}+4z^3\frac{p^2(k\cdot p)^2}{k^6}-4z^4\frac{(k\cdot p)^4}{k^8} \\
&- 2z^3\frac{p^4k\cdot p}{k^6}+8z^4\frac{p^2(k\cdot p)^3}{k^8}-\frac{32}{5}z^5\frac{(k\cdot p)^5}{k^{10}}\,. \qquad (5.33)
\end{aligned}
$$

Now we make use of the following standard integrals to carry out integration over the variable z :

$$
\begin{aligned}
\int_0^1 dz\ln\left[z(1-z)\right] &= -2 \\
\int_0^1 dz\, z\ln\left[z(1-z)\right] &= -1 \\
\int_0^1 dz\, z^2\ln\left[z(1-z)\right] &= -\frac{13}{18} \\
\int_0^1 dz\, z^3\ln\left[z(1-z)\right] &= -\frac{7}{12} \\
\int_0^1 dz\, z^4\ln\left[z(1-z)\right] &= -\frac{149}{300} \\
\int_0^1 dz\, z^5\ln\left[z(1-z)\right] &= -\frac{157}{360}\quad . \qquad (5.34)
\end{aligned}
$$

Using Eqs. (5.32–5.34) in Eq. (5.31), it is quite straightforward to arrive at the result of Eq. (5.13).

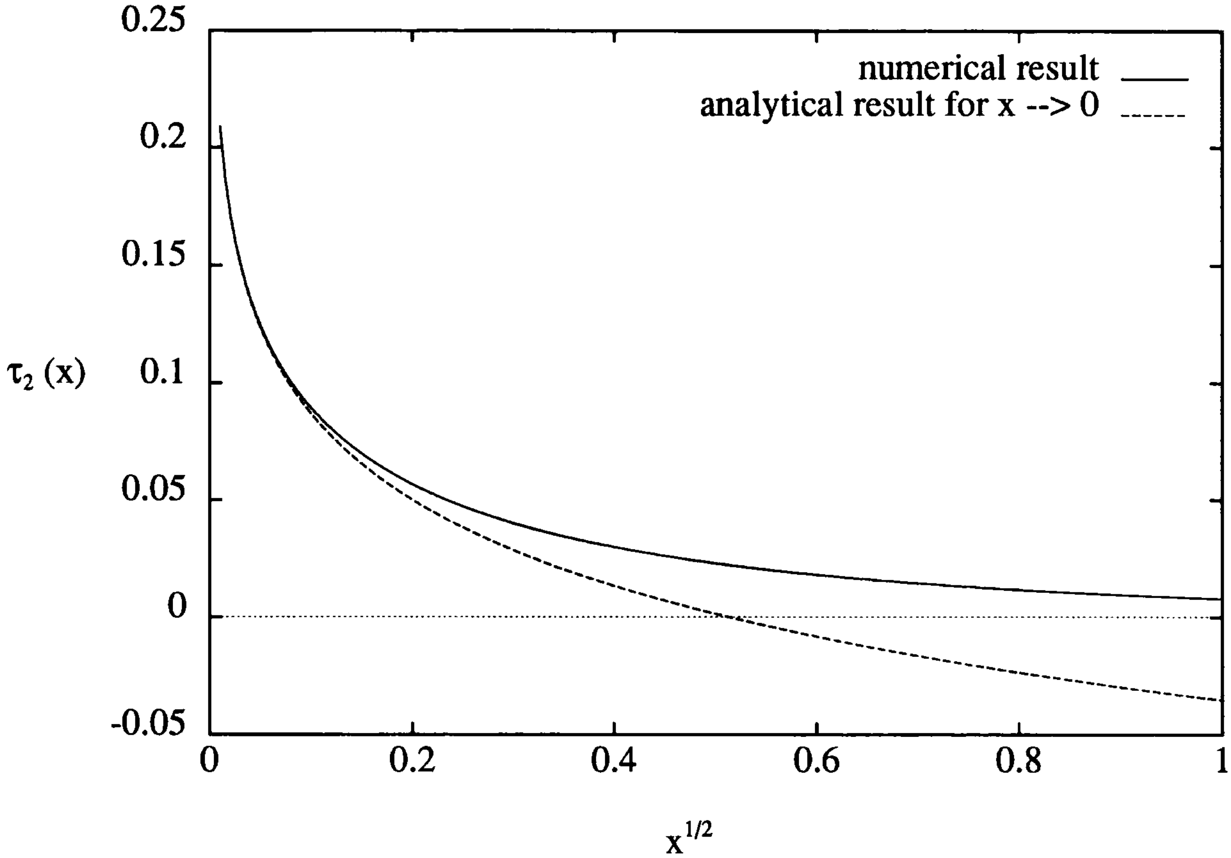

Figure 5.1: Dimensionless quantity $\tau_2(x)$ plotted as a function of $x^{1/2}$ in the Feynman ($\xi = 1$) gauge. The solid line represents the numerical evaluation. The dashed line is the analytical result which is true only in the limit when $x \to 0$.

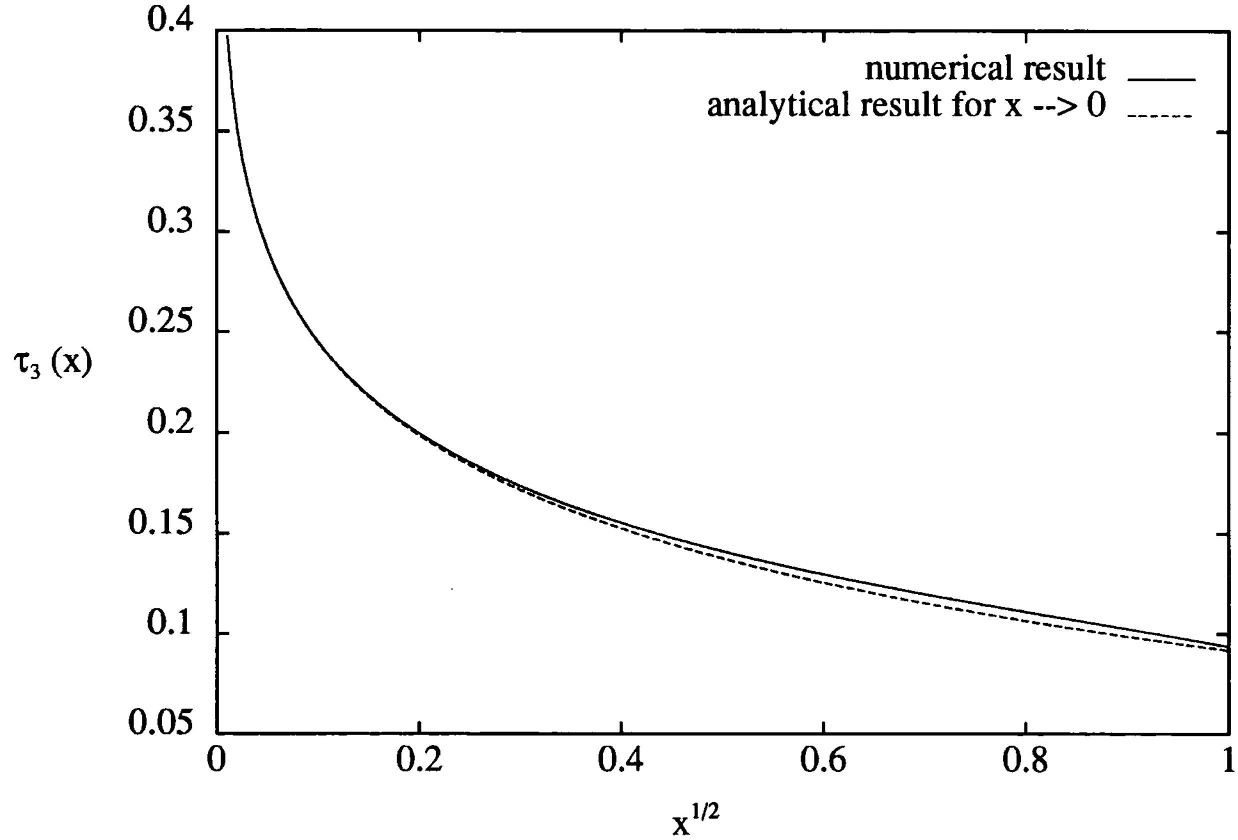

Figure 5.2: Dimensionless quantity $\tau_3(x)$ plotted as a function of $x^{1/2}$ in the Feynman ($\xi = 1$) gauge. The solid line represents the numerical evaluation. The dashed line is the analytical result which is true only in the limit when $x \to 0$.

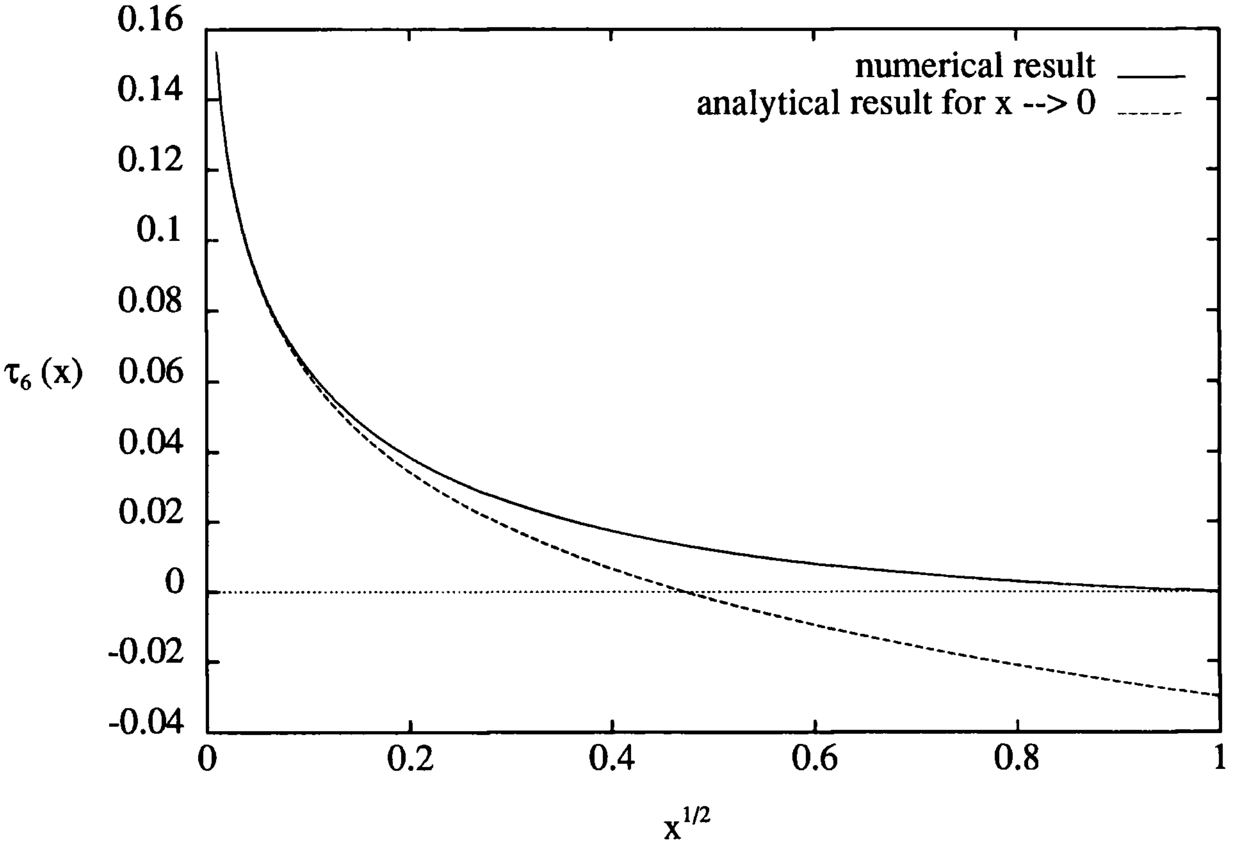

Figure 5.3: Dimensionless quantity $\tau_6(x)$ plotted as a function of $x^{1/2}$ in the Feynman ($\xi = 1$) gauge. The solid line represents the numerical evaluation. The dashed line is the analytical result which is true only in the limit when $x \to 0$.

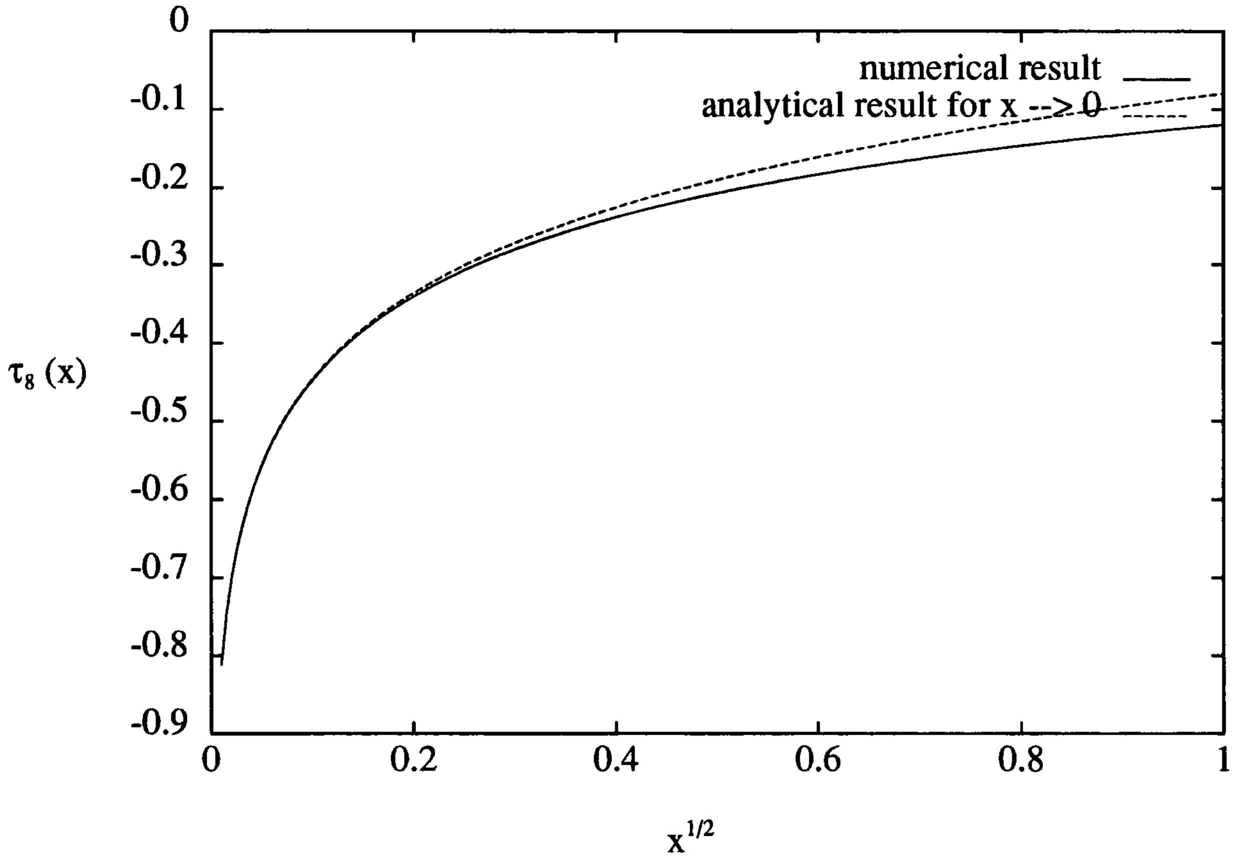

Figure 5.4: Dimensionless quantity $\tau_8(x)$ plotted as a function of $x^{1/2}$. The solid line represents the numerical evaluation. The dashed line is the analytical result which is true only in the limit when $x \to 0$.

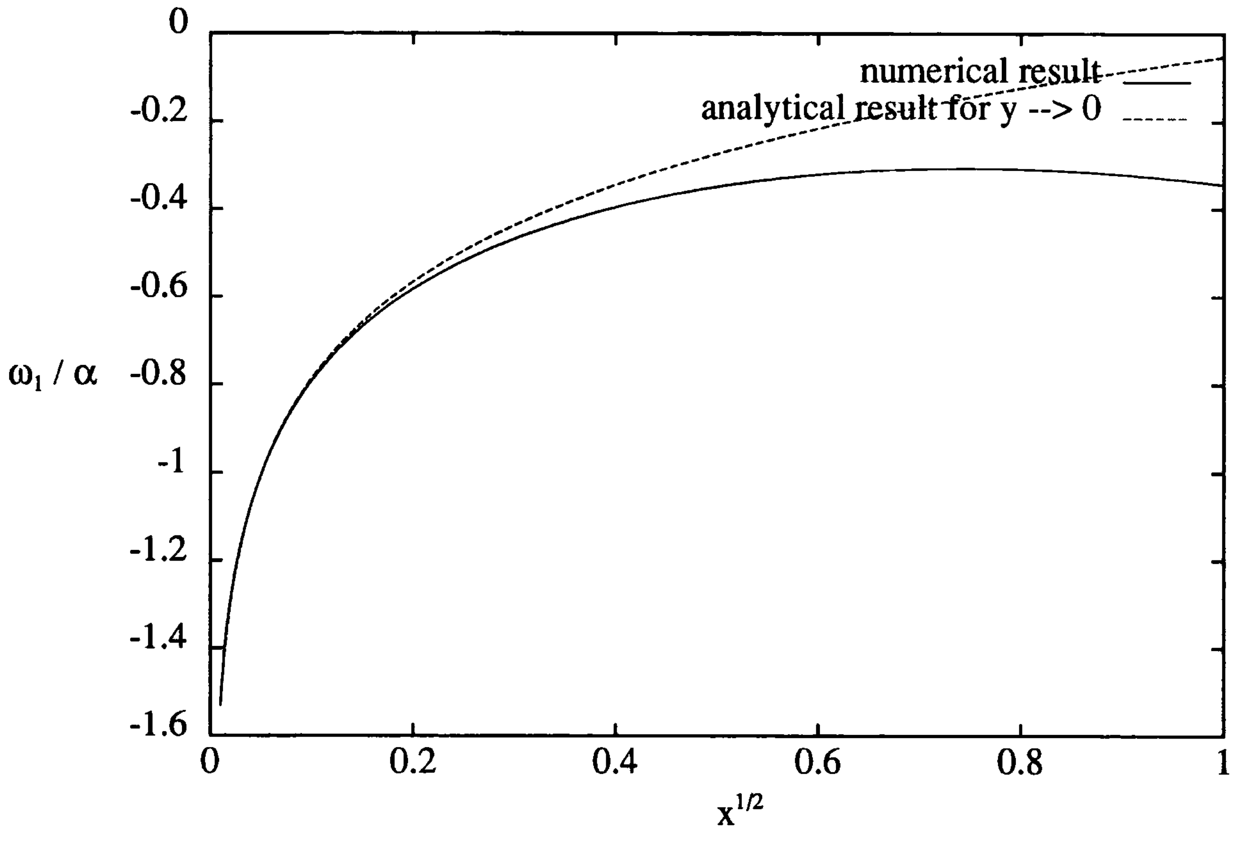

Figure 5.5: $\omega_1(x)/\alpha$ plotted as a function of $x^{1/2}$ in the Feynman ($\xi = 1$) gauge. The solid line represents the numerical evaluation. The dashed line is the analytical result which is true only in the limit when $x \to 0$.

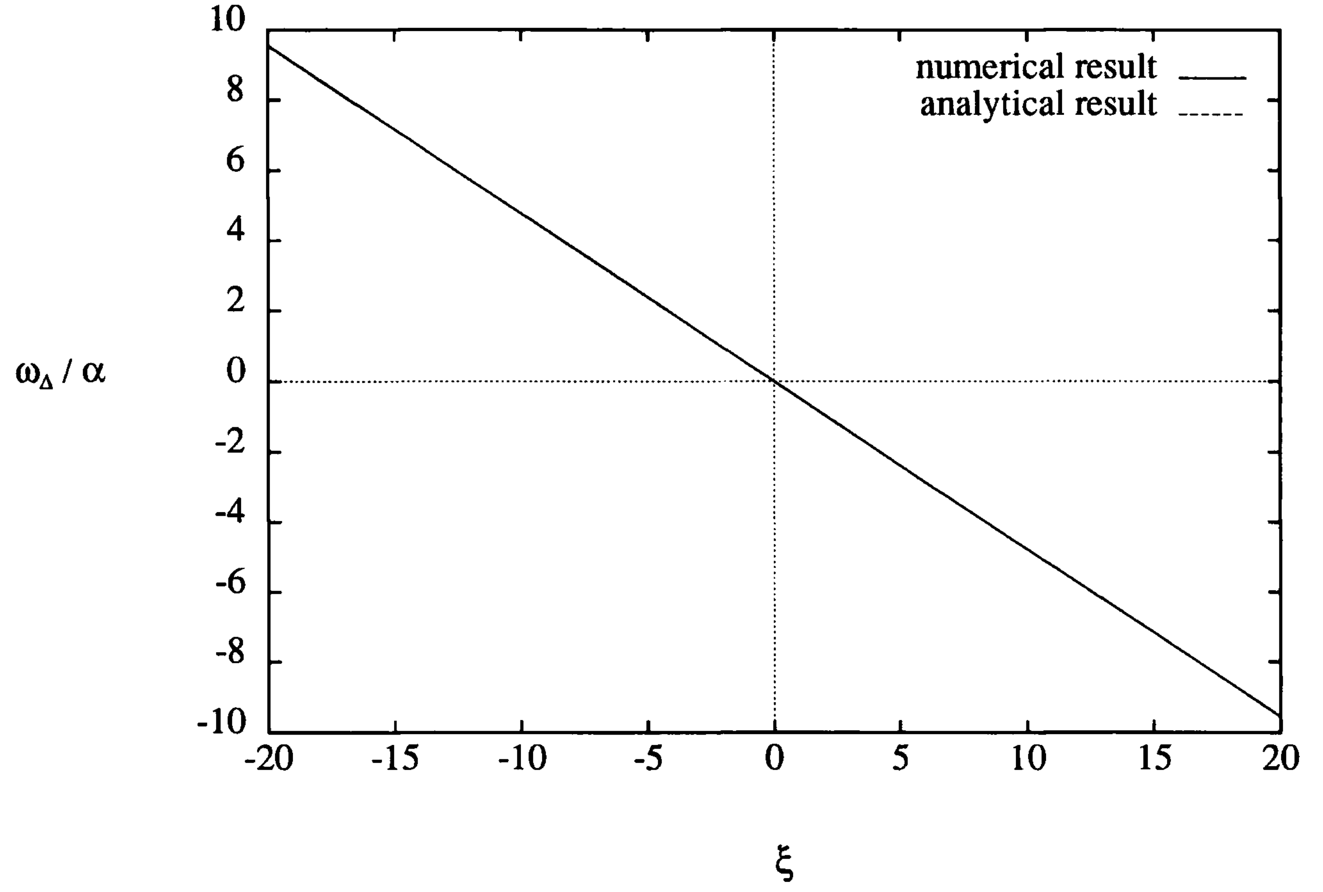

Figure 5.6: ω_Δ/α, which is the notation used for $[\omega_1(1) + \omega_1'(1)]/\alpha$ plotted as a function of the gauge parameter ξ. The solid line which represents the numerical result lies completely on top of the dashed analytical result in boringly perfect agreement.

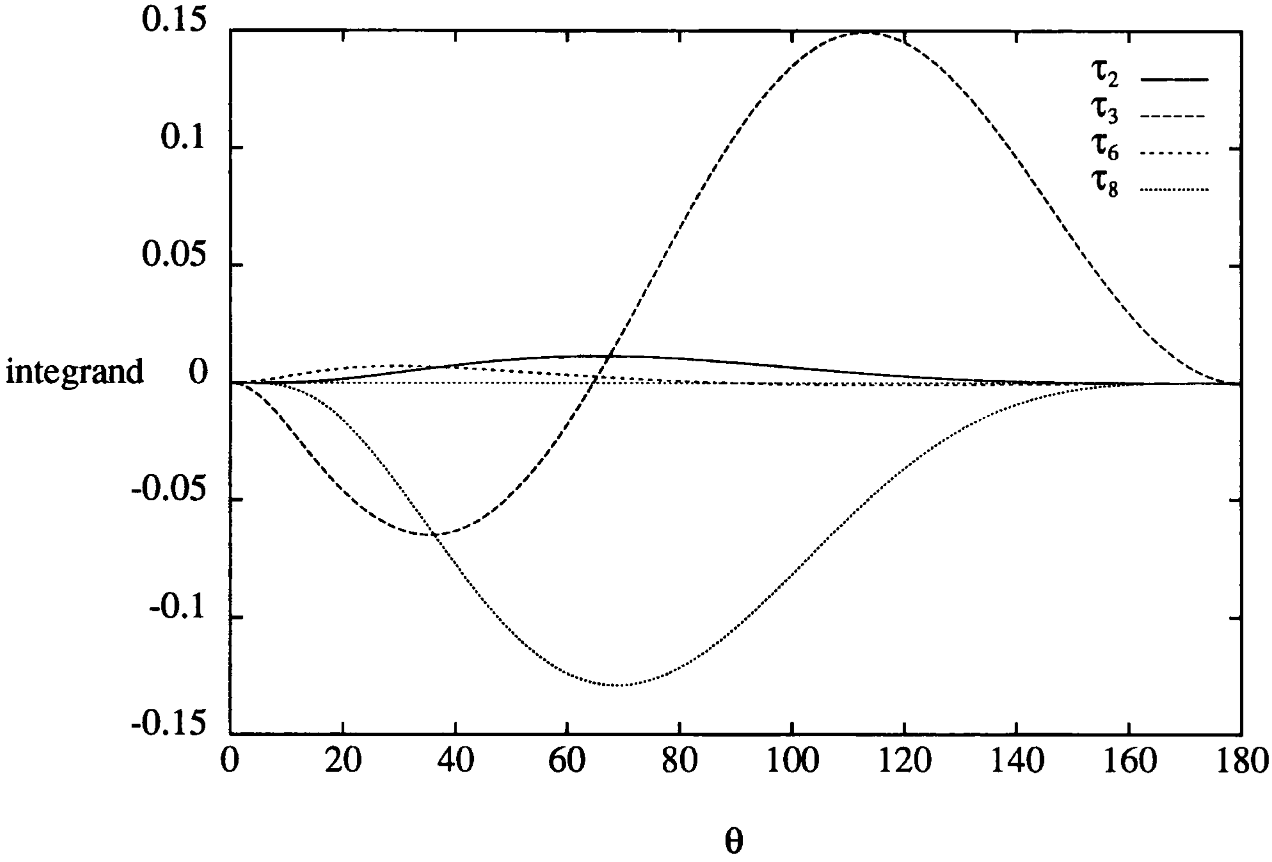

Figure 5.7: The integrand for each of the four τ_i plotted as a function of the angle θ at $x = 0.5$ in the Feynman ($\xi = 1$) gauge.

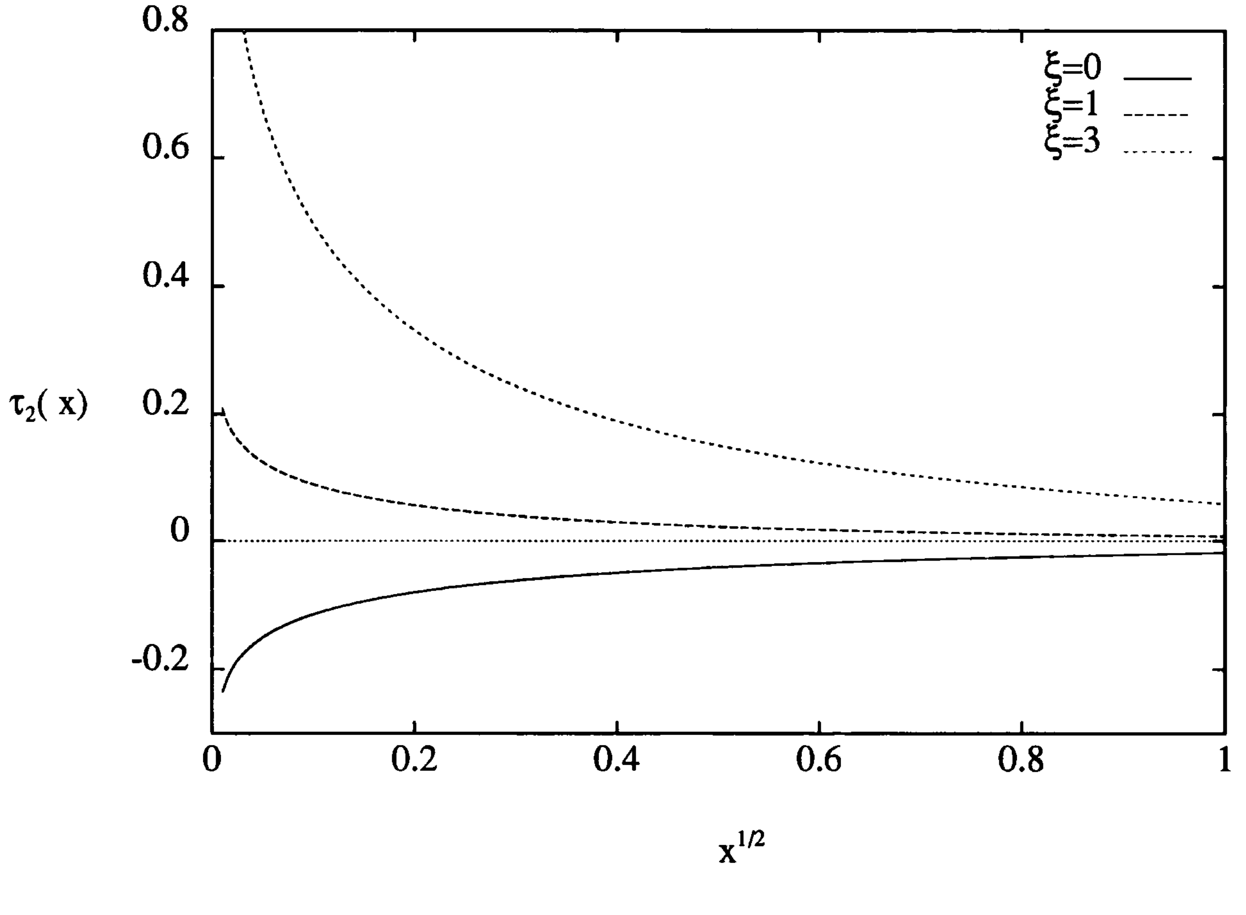

Figure 5.8: Dimensionless quantity $\tau_2(x)$ plotted as a function of $x^{1/2}$ for gauges $\xi = 0, 1$ and 3.

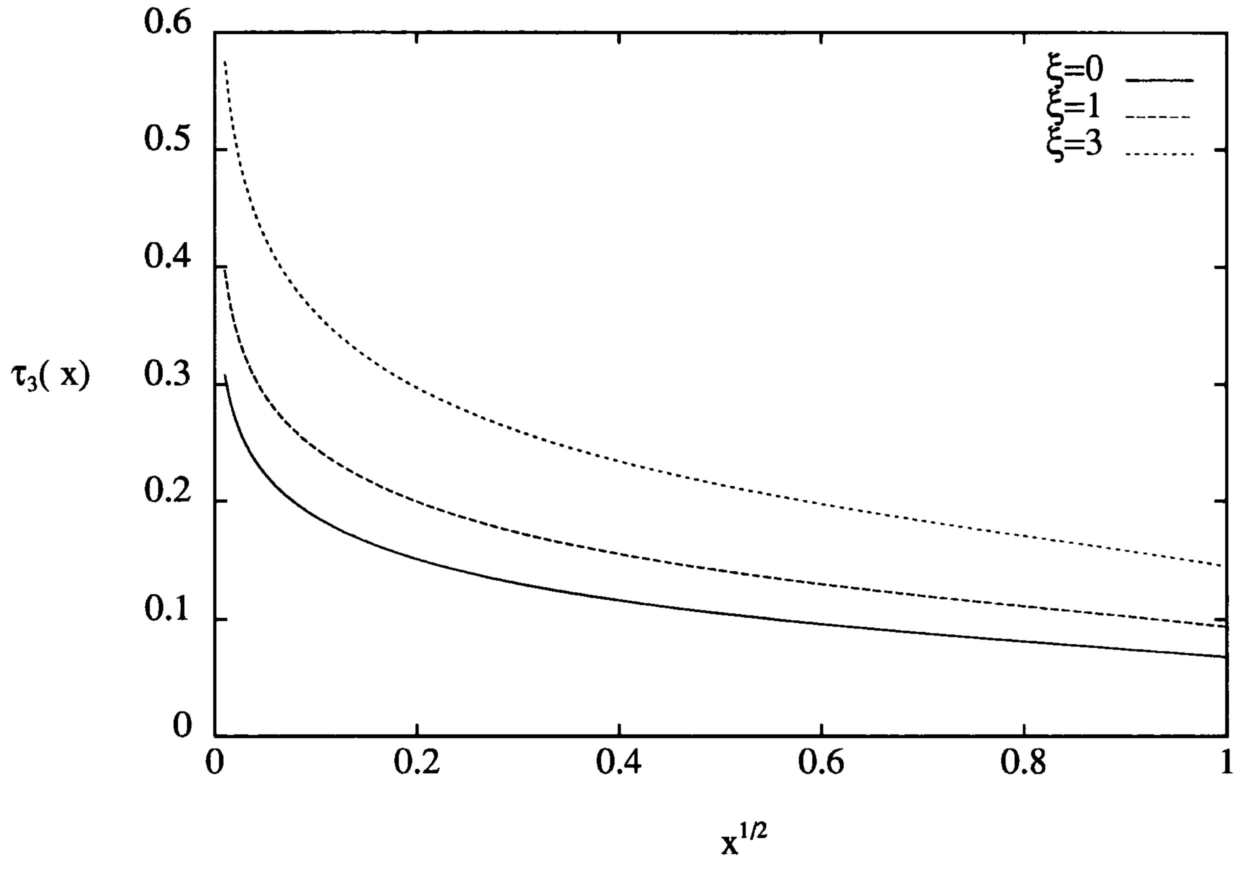

Figure 5.9: Dimensionless quantity $\tau_3(x)$ plotted as a function of $x^{1/2}$ for gauges $\xi = 0, 1$ and 3.

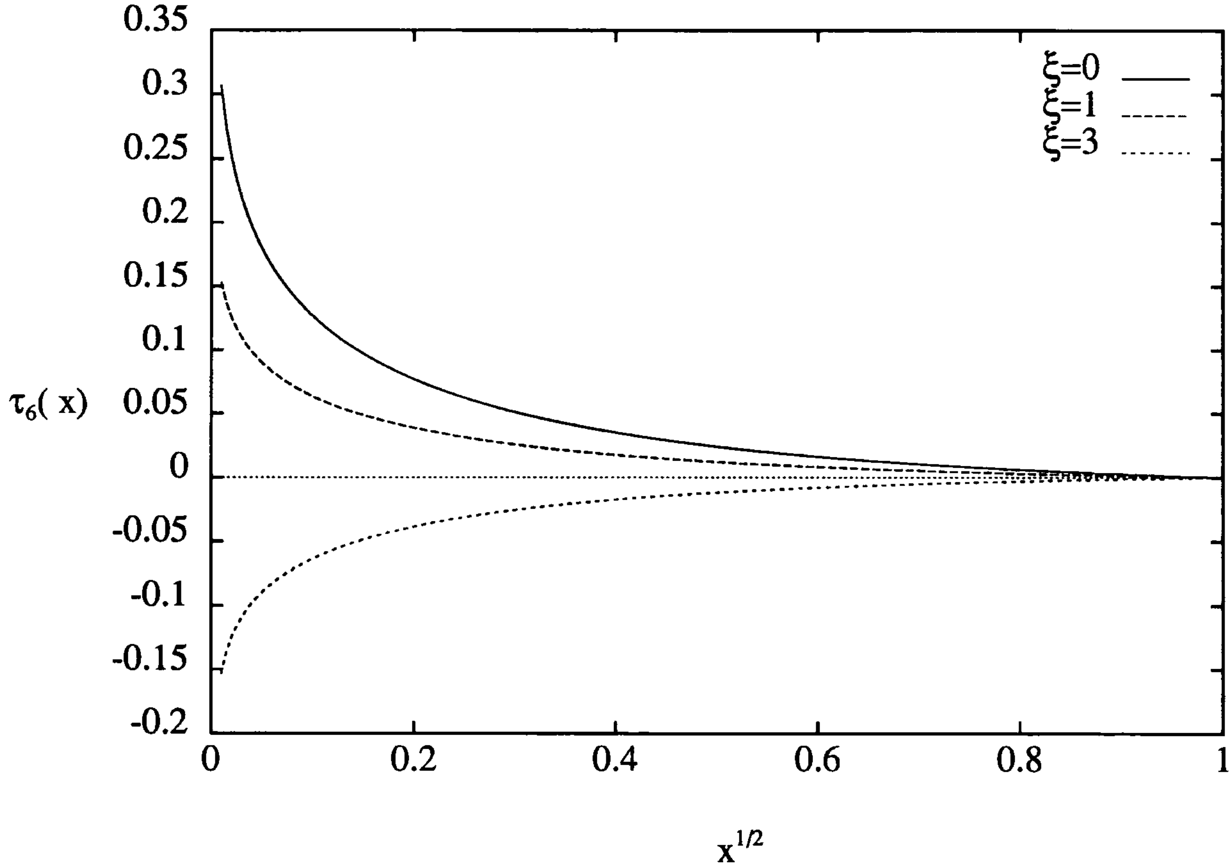

Figure 5.10: Dimensionless quantity $\tau_6(x)$ plotted as a function of $x^{1/2}$ for gauges $\xi = 0, 1$ and 3.

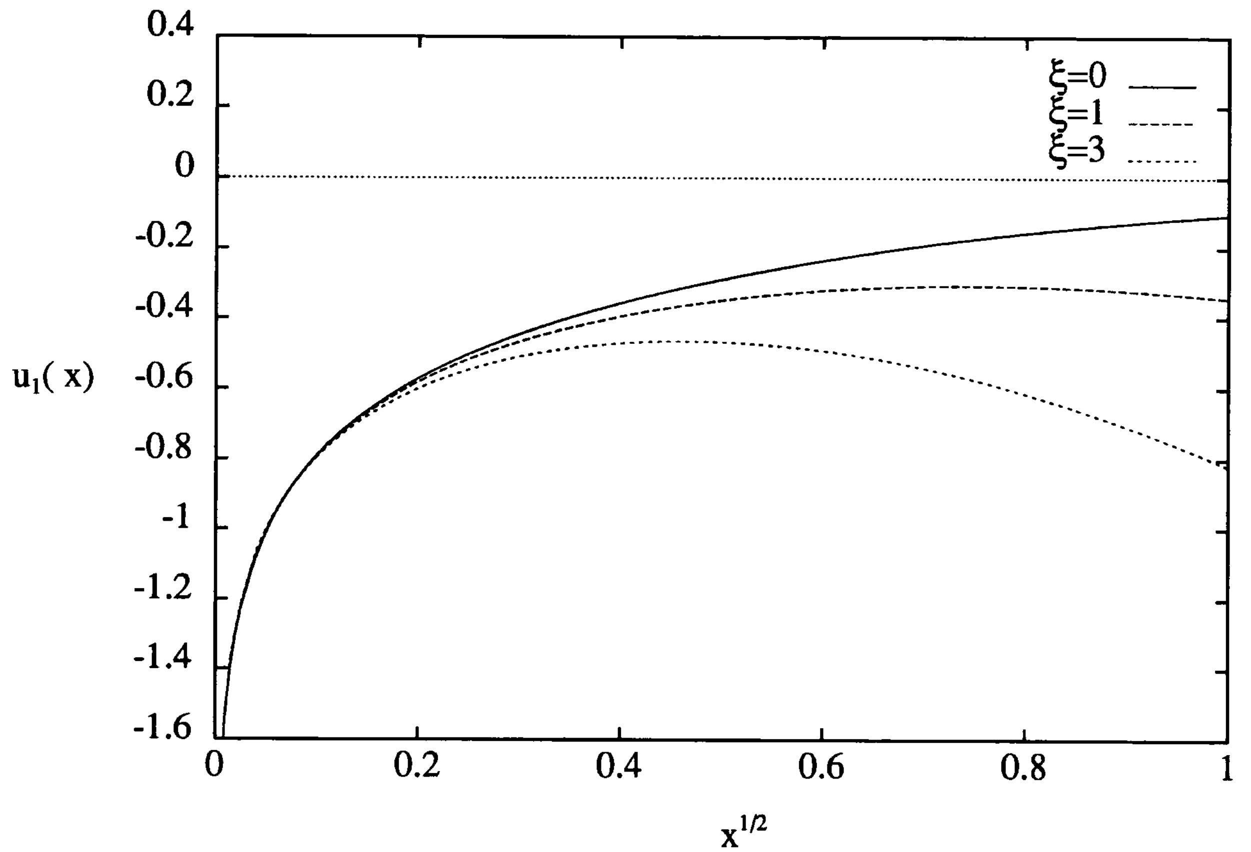

Figure 5.11: Dimensionless quantity $u_1(x)$ of Eq. (5.28) plotted as a function of $x^{1/2}$ for gauges $\xi = 0, 1$ and 3.

Chapter 6

Conclusions

In Chapter 2, we first review the fact that in terms of the gauge invariance of the critical coupling, α_c, the *CP* vertex is a much better *ansatz* to study dynamical chiral symmetry breaking than the bare vertex. However, following the realisation by Atkinson et al. [28] that the *CP* vertex, after all, performs this task only approximately, we succeeded in finding an *ansatz* that could do better than the *CP* vertex and achieve less gauge dependence of α_c over a much broader range of the gauge parameter ξ.

However, the fact remains that any gauge dependence of α_c is a cause of alarm. In Chapter 3, we constructed an effective transverse vertex that not only ensures that the fermion propagator satisfies the Ward-Takahashi Identity, is multiplicatively renormalizable and agrees with perturbation theory for weak couplings but also makes sure that the critical coupling for dynamical mass generation is strictly gauge independent. We write the transverse vertex in terms of two unknown functions W_1 and W_2 each satisfying an integral and a derivative condition. We gave a simple example for each of these functions. The results obtained in this chapter have to be compared with earlier work. For example, Rembiesa [30] and Haeri [18] construct fermion-boson vertices that make the fermion propagator itself gauge independent. This is, of course, at variance with its behaviour in perturbation theory and consequently with the renormalization group in the weak coupling limit. Rembiesa [30] then went on to find that the critical coupling for mass generation with such a vertex is strongly gauge dependent, being given by $\alpha_c = \pi/(3+\xi)$. In complete contrast, Kondo [31] finds a gauge independent coupling as here, but at the expense of using a vertex that has singularities. The construction presented in Chapter 3 overcomes these deficiencies.

One of the underlying assumptions for the constraints on W_2 is that the transverse part of the vertex vanishes in the Landau gauge. This fixes the critical exponent s_c of the mass function to be 0.47. However, using the arguments based on the effective potential, Bob Holdom [40] claims that $s = 1/2$ is a universal fact of the quenched theories which would imply that there is non-vanishing piece in the transverse vertex for $\xi = 0$. In Chapter 4, we showed that a similar analysis to that given in Chapter 3 can be carried out and an effective vertex can be constructed which ensures that $s = 1/2$.

In Chapter 5, we attempted to find the perturbative expansion $\omega_1(x)$ of the function $W_1(x)$ to $\mathcal{O}(\alpha)$. This was possible after the perturbative calculation of the transverse vertex by Kizilersü et al. [38] in an arbitrary covariant gauge. We related the exact coefficients $\tau_i(k^2, p^2, q^2)$, Eq. (1.56), obtained from their calculation to the effective $\tau_i^{\rm eff}(k^2, p^2)$ through the equation for $F(p^2)$, Eqs. (5.9,5.10). In the momentum region $k^2 \gg p^2$ or $p^2 \gg k^2$, such an evaluation was performed analytically. However, in other regions, because of the complicated angular integrals involved, this task was carried out numerically. It was found that the perturbative analogue of the Eq. (3.45) for $\omega_1(x)$ is satisfied. However, the condition $\int_0^1 dx\, \omega_1(x) = 0$ is violated. This could be because of a mismatch between an ultraviolet cut-off and dimensional regularization, requiring an investigation beyond the scope of this thesis.

We have seen in Chapter 5 how to construct effective τ_i to order α from the real τ_i calculated by Kizilersü et al. Although the motivation for their calculation has been to have a guide to construct non-perturbative τ_i, it is quite obvious to note that it is a prohibitively difficult task to find a non perturbative set of τ_i which can analytically reduce to complicated functions involving Spence functions in the perturbative regime. However, the numerical evaluation of the effective τ_i presented here suggests the possibility of proposing some simple functions which could fit the numerical results of Chapter 5 for weak coupling. These functions would then serve as the effective non-perturbative set of τ_i with the correct perturbative limit.

The perturbative evaluation of the function W_1 presented here shows that the implementation of the Landau-Khalatnikov-Fradkin transformations in 4-dimensions is not yet fully understood and requires further work to be able to understand the complete gauge dependence of the fermion wavefunction renormalization, even in quenched QED. We hope that a parallel calculation for the function W_2 would be of even more importance. We already know that the transverse vertex does not vanish in the Landau gauge. The aforementioned calculation would tell us what part in the transverse vertex, if any, influences the equation for the mass function. This may provide us a clue to the dilemma of the critical exponent s. Moreover, recall that the integral condition, Eq. (3.58), on the function W_2 is true only at criticality. And, therefore, we are unable to say what this condition might be in the perturbative region. The above mentioned calculation is expected to shed light on this issue as well.

The study presented in this thesis also motivates the need for a realistic investigation of $t\bar{t}$ condensates as the source of the electroweak symmetry breaking. The need is to solve the DSE of Fig. [1.3] in a gauge invariant way. The study of quenched QED presented here suggests that a proper choice of the vertex can guarantee the gauge independence of the physical observables. However, a realistic calculation, of course, requires the unquenching of the theory which complicates the problem significantly. The fermion-boson vertex (in particular its transverse part) will intimately depend on the photon renormalization function in a non-perturbative way not yet understood. The discussion for quenched QED presented here provides the starting point for such an investigation of full QED.

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