Dimensional regularisation and gauge theories

Shaban, Neil Tamim

Dimensional Regularisation and Gauge Theories

Neil Tamim Shaban
University of Durham

September 1994

Abstract

Dimensional regularisation is formulated without using the assumption that $\int d^D k (k^2)^n = 0$. Alternative definitions of $\varepsilon_{\kappa\lambda\mu\nu}$ and γ^5 are also considered. In the reformulated scheme, quadratic divergences are present, in general, in the scalar and gauge boson self-energies, and remain unregularised. The possible cancellation of such divergences is investigated. Phenomenological aspects of unified gauge theories are studied.

Acknowledgements

Remarks made by John Collins and Gerhard 't Hooft were helpful in the early stages of this work. Discussions and correspondence with Tim Jones played an essential part in my developing an understanding of the problems in dimensional regularisation. An earlier project, conceived by Colin Wilkin, gave me valuable experience in dealing with special functions, which has been pertinent in this more recent research. I am also grateful to Mark Homewood (Fred) for TEX-nical support.

Thanks are due to many people at the Centre for Particle Theory in Durham. Discussions with Mike Pennington, Chris Maxwell, Alan Martin, Nigel Glover, Mark Hindmarsh, Philip Tuckey and Ramon Munoz-Tapia directly benefitted the course of this work, and suggestions made by David Summers contributed to the numerical analysis of the unified gauge theories.

My fellow students, - in particular Foster Langbein, David Barclay, Peter Sutton, Duncan Curtis and Francisco Astorga-Saenz, - were a great source of support and companionship.

Above all, thanks go to James Stirling, who supervised this work. James set the direction for much of my research. I benefitted from numerous discussions with James, and his sound advice and strong encouragement enabled me to develop some of my own ideas.

I consider myself fortunate to have studied at Durham, to have been supervised by James Stirling and to have worked in a very interesting area of research. For me at this time, each of these three aspects could not have been bettered anywhere else.

I am indebted to my family for their constant and unwavering support - my wife Mandy, my children Sandy and Lawrie, my mother Bessie and my late father Abdulhayy.

Declaration

I declare that no material presented in this thesis has previously been submitted for a degree at this or any other University. Contributions from this work have been published elsewhere in the form of the following papers;

Threshold Effects at the GUT Scale in Minimal SU(5) using LEP Coupling Constant Measurements;

N.T. Shaban and W. J. Stirling, Mod. Phys. Lett. A7 (1992) 2028.

Minimal Left-Right Symmetry and SO(10) Grand Unification using LEP Coupling Constant Measurements;

N.T. Shaban and W. J. Stirling, Phys. Lett. B291 (1992) 281.

Contents

Chapter 1

Dimensional Regularisation

Regularisation is the means by which the divergent integrals occuring in field theories are controlled. There are many different techniques of regularisation, although probably the most widely used is dimensional regularisation. In this scheme the regularisation is achieved by using extended definitions of integration and vector algebra to define a field theory in a non-integer number of dimensions. Here the main concern will be with the way of extending the definition of integration.

Consider the integral of a function $f(x)$ in D dimensional Euclidean space $(D = 1, 2, 3, \ldots)$ with $-\infty < x_i < \infty$

$$I = \int d^D x f(x) \tag{1.1}$$

In polar coordinates this becomes

$$I = \int f(x) r^{D-1} dr \sin^{D-2}\theta_{D-1} d\theta_{D-1} \sin^{D-3}\theta_{D-2} d\theta_{D-2} \ldots d\theta_1 \tag{1.2}$$

where $r = \mid x \mid$ and $0 \leq \theta_i \leq \pi$ except for $0 \leq \theta_1 \leq 2\pi$.
If the function f(x) is spherically symmetric, then $f(x) = f(r)$ and the relation (Wallis' formula)

$$\int_0^\pi \sin^m \theta d\theta = \pi^{(1/2)} \frac{\Gamma(\frac{1}{2}(m+1))}{\Gamma(\frac{1}{2}(m+2))} \tag{1.3}$$

$$m \geq 0$$

may be used to carry out the angular integrations

$$I = \int_0^\infty f(r) r^{D-1} dr \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(D-1))}{\Gamma(\frac{1}{2}D)} \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(D-2))}{\Gamma(\frac{1}{2}(D-1))} \ldots \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(2))}{\Gamma(\frac{1}{2}3)} \times 2\pi \tag{1.4}$$

which implies that

$$I = \int_0^\infty f(r) r^{D-1} dr 2\pi (\pi^{\frac{1}{2}})^{D-2} \frac{1}{\Gamma(\frac{1}{2}D)} \tag{1.5}$$

$$\Rightarrow I = \frac{2\pi^{(D/2)}}{\Gamma(D/2)} \int_0^\infty f(r) r^{D-1} dr \tag{1.6}$$

$$\Rightarrow I = DV(D) \int_0^\infty f(r) r^{D-1} dr \tag{1.7}$$

$$D \geq 2 \ (D = 2, 3, 4, \ldots)$$

where

$$V(D) = \frac{\pi^{(D/2)}}{\Gamma((D/2)+1)} \tag{1.8}$$

is the volume of the D-dimensional unit sphere, for $D = 2, 3, 4, \ldots$). The formula is also correct for D=1. There are no angular integrations here so the limit $D \geq 2$ in (1.8) which comes from $m \geq 0$ in (1.3) does not apply. The factor $V(1) = 2$ comes from

$$\int_{-\infty}^\infty f(r) dr = 2 \int_0^\infty f(r) dr \tag{1.9}$$

Hence (1.5) and (1.7) hold for $D = 1, 2, 3, 4, \ldots$

An expression for I has now been worked out which is an explicit function of D, the number of dimensions. Therefore this expression can be used to define a function, or rather a functional of the function f, which is also a function of a continuous variable D, and which corresponds to standard integration when $D = 1, 2, 3, 4, \ldots$ This is how integration in a non-integer number of dimensions can be defined in this rather limited context. This is not an analytic continuation in D, as the function was previously only defined at a set of points $D = 1, 2, 3, 4, \ldots$ and not in an analytic region. Therefore this definition is not unique. For example, anything of the form

$$\int d^D x f(x) = g(D) DV(D) \int_0^\infty f(r) r^{D-1} dr \tag{1.10}$$

where $g(D) = 1$ for $D = 1, 2, 3, 4, \ldots$ would serve equally well. The analytic properties of

$$I = \int d^D x f(x) = DV(D) \int_0^\infty f(r) r^{D-1} dr \tag{1.11}$$

will be determined by

$$\int_0^\infty f(r)r^{D-1}dr \qquad (1.12)$$

since $DV(D)$ is an analytic function.
Only $\Re D \geq 0$ will be considered here, so as a basic definition

$$I(D) = \frac{2\pi^{(D/2)}}{\Gamma(d/2)} \int_0^\infty f(r)r^{D-1}dr \qquad (1.13)$$

$$\Re D \geq 0$$

In general the following axioms, due to Wilson [4], will define integration in non-integer dimensions

$$\int d^D x(f_1(x) + f_2(x)) = \int d^D x f_1(x) + \int d^D x f_2(x) \qquad (1.14)$$

$$\int d^D x f(\mu x) = \mu^{-D} \int d^D x f(x) \qquad (1.15)$$

$$\int d^D x f(x+a) = \int d^D x f(x) \qquad (1.16)$$

When D is an integer these properties are satisfied. For non-integer D the basic integral (1.13) automatically satisfies the linearity (1.14) and scaling (1.15) conditions. If translational invariance (1.16) is imposed on (1.13), then this can be used to derive other non-spherically symmetric integrals.
A specific function in $D = 2\omega$ dimensional momentum space will now be considered, namely

$$f(k) = \frac{1}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)^n} \qquad (1.17)$$

where m is a real constant (hence $m^2 \geq 0$), but D is not necessarily integer, or real, but may be complex. The factor of $1/(2\pi)^{2\omega}$ is included for convenience.
The integral of this function in 2ω dimensions is

$$I(\omega, m) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)^n} \qquad (1.18)$$

From the definition (1.13) this is equal to

$$I(\omega, m) = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty \frac{2k^{2\omega-1}dk}{(k^2+m^2)^n} \qquad (1.19)$$

since $I(\omega, m)$ is spherically symmetric. Making the substitution $k^2 = m^2 x$ gives

$$I(\omega, m) = \frac{(m^2)^{\omega-n}}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty \frac{x^{\omega-1}}{(x+1)^n} dx \qquad (1.20)$$

From (B.13) the integral in (1.20) is the integral representation of the beta function provided that $\Re\omega > 0$ and $\Re(n-\omega) > 0$, so that

$$I(\omega, m) = \frac{(m^2)^{\omega-n}}{(4\pi)^{\omega}\Gamma(\omega)}\beta(\omega, n-\omega) \tag{1.21}$$

$$\Re\omega > 0 \; , \; \Re(n-\omega) > 0$$

$$\Rightarrow I = \frac{(m^2)^{\omega-n}}{(4\pi)^{\omega}}\frac{\Gamma(n-\omega)}{\Gamma(n)} \tag{1.22}$$

$$\Re\omega > 0 \; , \; \Re(n-\omega) > 0$$

The first condition, $\Re\omega > 0$ is not a problem, since as in (1.13) only $\Re D \geq 0$ is being considered (and $D = 2\omega$), but $\Re(n-\omega) > 0$ is more restrictive. It is important to realise that for $\Re(n-\omega) < 0$ the integral in (1.20) is not the integral representation of the beta function, which instead will be given by an integral of the form of (B.26).

In dimensional regularisation, the main concern is with integrals in a dimension $D = 2\omega$ close to $D = 4$. It will be useful to also look at dimensions close to $D = 2$. Consider

$$D = 2 - \epsilon \Rightarrow \omega = 1 - \epsilon/2 \tag{1.23}$$

where ϵ is small and positive i.é. $|\epsilon| \ll 1$, $\Re\,\epsilon \geq 0$. For this case the integral $I(\omega, m)$ will be given by (1.21) for $\Re\, n > 1 - \Re\,\epsilon/2$. Specifically, this is true for $n = 1, 2, 3 \ldots$ For the case $D = 4 - \epsilon$ ($\omega = 2 - \epsilon/2$), $I(\omega, m)$ is given by (1.21) provided that $\Re n > 2 - \Re\,\epsilon/2$ and (1.21) will be correct for $n = 2, 3, 4, \ldots$, but not for $n = 1$.

For $D = 2 - \epsilon$ with real n

$$I^n(\omega, m) = \frac{(m^2)^{\omega-n}}{(4\pi)^{\omega}}\frac{\Gamma(n-\omega)}{\Gamma(n)} \tag{1.24}$$

$$0 < 1 - \Re\,\epsilon/2 < n$$

If $n > 1$, (1.24) is regular as $\epsilon \to 0$ ($D \to 2$ from below). If $n = 1$, then

$$I^1(\omega, m) = \frac{(m^2)^{\omega-1}}{(4\pi)^{\omega}}\Gamma(1-\omega) \tag{1.25}$$

has a simple pole at $\epsilon = 0$ ($\omega = 1$). This is an ultraviolet logarithmic divergence, since in 2 dimensions

$$I^1_M(1, m) = \int^M \frac{d^2k}{(2\pi)^2}\frac{1}{k^2 + m^2} \tag{1.26}$$

$$= \frac{1}{4\pi} \int_0^M \frac{2k\,dk}{(k^2 + m^2)} \tag{1.27}$$

$$= \frac{1}{4\pi} [\ln(k^2 + m^2)]_0^M \tag{1.28}$$

$$= \frac{1}{4\pi} \ln \left(\frac{M^2 + m^2}{m^2} \right) \tag{1.29}$$

and therefore

$$I_M^1(1, m) \sim \ln M \qquad \text{as} \qquad M \to \infty \tag{1.30}$$

For $D = 4 - \epsilon$

$$I^n(\omega, m) = \frac{(m^2)^{\omega - n}}{(4\pi)^\omega} \frac{\Gamma(n - \omega)}{\Gamma(n)} \tag{1.31}$$

$$0 < 2 - \Re\,\epsilon/2 < n$$

If $n > 2$, $I^n(\omega, m)$ is regular as $\epsilon \to 0$, and for $n = 2$

$$I^2(\omega, m) = \frac{(m^2)^{\omega - 2}}{(4\pi)^\omega} \Gamma(2 - \omega) \tag{1.32}$$

has a simple pole at $\epsilon = 0$ ($\omega = 2$). This again is an ultraviolet (UV) logarithmic divergence, which can be compared with

$$I_M^2(2, m) = \int^M \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^2} \tag{1.33}$$

$$= \frac{1}{16\pi^2} \int_0^M \frac{2k^3\,dk}{(k^2 + m^2)^2} \tag{1.34}$$

$$= \frac{1}{16\pi^2} \left\{ \ln \left(\frac{M^2 + m^2}{m^2} \right) - \frac{M^2}{M^2 + m^2} \right\} \tag{1.35}$$

and as with $I_M^1(1, m)$

$$I_M^2(2, m) \sim \ln M \qquad \text{as} \qquad M \to \infty \tag{1.36}$$

Now consider $I^1(\omega, m)$ for $D = 4 - \epsilon$. In 4 dimensions

$$I_M^1(2, m) = \int^M \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)} \tag{1.37}$$

$$= \frac{1}{16\pi^2} \int_0^M \frac{2k^3\,dk}{(k^2 + m^2)} \tag{1.38}$$

$$= \frac{1}{16\pi^2}\left\{M^2 - m^2 \ln\left(\frac{M^2+m^2}{m^2}\right)\right\} \tag{1.39}$$

Now the dominant behaviour of $I^1_M(2,m)$ as $M \to \infty$ is

$$I^1_M(2,m) \sim M^2 \qquad \text{as} \qquad M \to \infty \tag{1.40}$$

This is a (UV) quadratic divergence. It is apparent that $I^1_M(2,m)$ has a log. divergent part, and also a part that is a 'pure' quadratic divergence i.e. which is independent of m. The pure quadratic divergence dominates in the large M limit.
In $D = 4 - \epsilon$ dimensions

$$I^1(\omega,m) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)} \tag{1.41}$$

$I^1(\omega,m)$ is spherically symmetric, so

$$I(\omega,m) = \frac{1}{(4\pi)^\omega\Gamma(\omega)} \int_0^\infty \frac{(k^2)^{\omega-1}d(k^2)}{(k^2+m^2)} \tag{1.42}$$

When $1 < \Re\,\omega < 2$ the integral in (1.42) is not the integral representation of the beta function. Rearranging the integrand

$$I(\omega,m) = \frac{1}{(4\pi)^\omega\Gamma(\omega)} \int_0^\infty \frac{(k^2)^{\omega-2}(k^2+m^2-m^2)d(k^2)}{(k^2+m^2)} \tag{1.43}$$

$$= \frac{1}{(4\pi)^\omega\Gamma(\omega)} \int_0^\infty \left((k^2)^{\omega-2} - m^2\frac{(k^2)^{\omega-2}}{(k^2+m^2)}\right) d(k^2) \tag{1.44}$$

The integral in the second part of (1.44) is the integral representation of the beta function in the domain $\Re\,(\omega - 1) > 0$ and $\Re\,(2-\omega) > 0$, therefore

$$-\frac{m^2}{(4\pi)^\omega\Gamma(\omega)} \int \frac{(k^2)^{\omega-2}}{(k^2+m^2)} d(k^2) = -\frac{(m^2)^{\omega-1}}{(4\pi)^\omega\Gamma(\omega)}\beta(\omega-1, 2-\omega) \tag{1.45}$$

$$1 < \Re\,\omega < 2$$

$$= +\frac{(m^2)^{\omega-1}}{(4\pi)^\omega\Gamma(\omega)}\beta(\omega, 1-\omega) \tag{1.46}$$

$$1 < \Re\,\omega < 2$$

since $\beta(\omega-1, 2-\omega) = -\beta(\omega, 1-\omega)$. (1.46) is logarithmically divergent, and corresponds to the second part of (1.39). The first part of (1.44) is the 'pure' quadratic divergence corresponding to the first part of (1.39).

$$\frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty (k^2)^{\omega-2} d(k^2) \tag{1.47}$$

is divergent, even for $\omega \neq 2$ ($\epsilon \neq 0$), and therefore remains unregularised. Strictly, this means that the whole of (1.44) is unregularised, and the regularisation procedure has failed. However, since the second part of (1.44) is regular for $\epsilon \neq 0$, it is possible that the scheme will still work if the pure quadratic terms from different parts of a calculation ultimately cancel each other out. To determine this, a linear separation of $I^1(\omega, m)$ into two integrals must be carried out.

$$\begin{aligned} I^1(\omega, m) &= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2 + m^2} \\ &= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} - \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{m^2}{k^2(k^2+m^2)} \\ &= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} + \frac{(m^2)^{\omega-1}}{(4\pi)^\omega \Gamma(\omega)} \beta(\omega, 1-\omega) \end{aligned} \tag{1.48}$$

$$1 < \Re\omega < 2$$

The success of this approach depends on the consistent manipulation of such expressions, even though they contain an unregularised quadratic divergence. To carry out such manipulations, the use of linearity (1.14) will be the principal tool. In certain circumstances, translational invariance (1.16) may also be used, as will be shown later. The scaling property (1.15) must be avoided, however, since e.g. for

$$f(k) = \frac{1}{k^2} \tag{1.49}$$

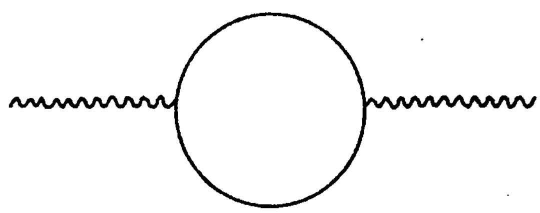
(1.15) would imply that

$$\mu^{-2} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} = \mu^{-2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \tag{1.50}$$

Chapter 2

The Photon Self-Energy

2.1 Feynman Parametrisation



Consider the photon self-energy $\Pi_{\mu\nu}(p)$ in QED to 1-loop order. In $D = 2\omega = 4 - \epsilon$ dimensions (see e.g. [5, 6]).

$$\Pi_{\mu\nu}(p) = -e^2 \mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \gamma_\mu \frac{1}{\not{k} - \mathrm{m} + \mathrm{i}\epsilon} \gamma_\nu \frac{1}{\not{k} + \not{p} - \mathrm{m} + \mathrm{i}\epsilon}\right] \tag{2.1}$$

In Euclidean space, after Wick rotation

$$\Pi_{\mu\nu}(p) = -ie^2 \mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \gamma_\mu \frac{1}{\not{k} + \mathrm{m}} \gamma_\nu \frac{1}{\not{k} + \not{p} + \mathrm{m}}\right] \tag{2.2}$$

$$= -ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{Tr}[\gamma_\mu(\not{k} - \mathrm{m})\gamma_\nu(\not{k} + \not{p} - \mathrm{m})]}{(\mathrm{k}^2 + \mathrm{m}^2)((\mathrm{k} + \mathrm{p})^2 + \mathrm{m}^2)} \tag{2.3}$$

Evaluating the trace using gamma matrix formulae

$$\Pi_{\mu\nu}(p) = -4ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu - \delta_{\mu\nu}(k^2 + k.p + m^2)}{(k^2+m^2)((k+p)^2+m^2)} \tag{2.4}$$

Carrying out Feynman parametrisation leads to

$$\Pi_{\mu\nu}(p) = -4ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^1 dx$$

$$\left\{ \frac{2k_\mu k_\nu}{(k^2+a^2)^2} - \frac{2x(1-x)[p_\mu p_\nu - \delta_{\mu\nu}p^2]}{(k^2+a^2)^2} - \frac{\delta_{\mu\nu}}{(k^2+a^2)} \right\} \tag{2.5}$$

where

$$a^2 = m^2 + p^2 x(1-x) \tag{2.6}$$

For spherically symmetric integrals

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} f(k^2) = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2)(k^2)^{\omega-1} f(k^2) \tag{2.7}$$

It will also be assumed that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2+a^2)^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\left(\frac{\delta_{\mu\nu}}{2\omega}\right) k^2}{(k^2+a^2)^2} \tag{2.8}$$

Therefore interchanging integrations in (2.5) gives

$$\Pi_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega \Gamma(\omega)} \int_0^1 dx \left\{ A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} \right\} \tag{2.9}$$

where

$$A_{\mu\nu} = \frac{\delta_{\mu\nu}}{\omega} \int_0^\infty d(k^2)(k^2)^{\omega-1} \frac{k^2}{(k^2+a^2)^2} \tag{2.10}$$

$$B_{\mu\nu} = -2x(1-x)[p_\mu p_\nu - \delta_{\mu\nu}p^2] \int_0^\infty d(k^2)(k^2)^{\omega-1} \frac{1}{(k^2+a^2)^2} \tag{2.11}$$

$$C_{\mu\nu} = -\delta_{\mu\nu} \int_0^\infty d(k^2)(k^2)^{\omega-1} \frac{1}{(k^2+a^2)} \tag{2.12}$$

For $A_{\mu\nu}$ and $C_{\mu\nu}$ the integral representation of the beta function (B.26) is required, with $n=0$. In (B.26) $n=0$ implies that

$$F(1/t, x+y) = 1 - (1+1/t)^{x+y} \tag{2.13}$$

$$\Rightarrow \beta(x,y) = \int_0^\infty dt\{t^{y-1}(1+t)^{-x-y} - t^{-x-1}\} \tag{2.14}$$

$$\Re(x+y) > 0,\ -1 < \Re x < 0$$

Letting $t = k^2/a^2$

$$\beta(x,y) = (a^2)^x \int_0^\infty d(k^2)\left\{\frac{(k^2)^{y-1}}{(k^2+a^2)^{x+y}} - \frac{1}{(k^2)^{x+1}}\right\} \tag{2.15}$$

$$\Re(x+y) > 0,\ -1 < \Re x < 0$$

For (2.10) $x + y = 2$ and $y - 1 = \omega$ are required, implying that $x = 1 - \omega$ $(-1 < x < 0)$ and so

$$\beta(1-\omega,\omega+1) = (a^2)^{1-\omega}\int_0^\infty d(k^2)\left\{\frac{(k^2)^\omega}{(k^2+a^2)^2} - \frac{1}{(k^2)^{2-\omega}}\right\} \tag{2.16}$$

This gives

$$A_{\mu\nu} = \frac{\delta_{\mu\nu}}{\omega}\left((a^2)^{\omega-1}\beta(1-\omega,\omega+1) + \int_0^\infty d(k^2)\frac{1}{(k^2)^{2-\omega}}\right) \tag{2.17}$$

Similarly, for (2.12) the appropriate values are $x + y = 1$ and $y = \omega$ which again imply $x = 1 - \omega$ $(-1 < x < 0)$ and so

$$\beta(1-\omega,\omega) = (a^2)^{1-\omega}\int_0^\infty d(k^2)\left\{\frac{(k^2)^{\omega-1}}{(k^2+a^2)} - \frac{1}{(k^2)^{2-\omega}}\right\} \tag{2.18}$$

(2.12) therefore becomes

$$C_{\mu\nu} = -\delta_{\mu\nu}\left((a^2)^{\omega-1}\beta(1-\omega,\omega) + \int_0^\infty d(k^2)\frac{1}{(k^2)^{2-\omega}}\right) \tag{2.19}$$

For $B_{\mu\nu}$ the usual integral representation (B.13) must be used. Making the substitution $v = k^2/a^2$ in (B.13)

$$\beta(x,y) = (a^2)^y \int_0^\infty d(k^2)\frac{(k^2)^{x-1}}{(k^2+a^2)^{x+y}} \tag{2.20}$$

$$\Re x > 0,\ \Re y > 0$$

For (2.11) $x + y = 2$ and $x = \omega$ and so $y = 2 - \omega$ and therefore

$$\beta(\omega,2-\omega) = (a^2)^{2-\omega}\int_0^\infty d(k^2)\frac{(k^2)^{\omega-1}}{(k^2+a^2)^2} \tag{2.21}$$

This implies that (2.11) is

$$B_{\mu\nu} = -2x(1-x)[p_\mu p_\nu - \delta_{\mu\nu}p^2](a^2)^{\omega-2}\beta(\omega, 2-\omega) \tag{2.22}$$

Using (B.12) and the Γ-function recurrence relation

$$z\Gamma(z) = \Gamma(z+1) \tag{2.23}$$

it can be shown that

$$\frac{1}{\omega}\beta(1-\omega, \omega+1) = \beta(1-\omega, \omega) \tag{2.24}$$

From this it follows that

$$\begin{aligned} A_{\mu\nu} + C_{\mu\nu} &= \\ \delta_{\mu\nu}(a^2)^{\omega-1}\beta(1-\omega,\omega) &+ \frac{\delta_{\mu\nu}}{\omega}\int_0^\infty d(k^2)(k^2)^{\omega-2} \\ -\delta_{\mu\nu}(a^2)^{\omega-1}\beta(1-\omega,\omega) &- \delta_{\mu\nu}\int_0^\infty d(k^2)(k^2)^{\omega-2} \\ &= \delta_{\mu\nu}\left(\frac{1}{\omega}-1\right)\int_0^\infty d(k^2)(k^2)^{\omega-2} \end{aligned} \tag{2.25}$$

and so $A_{\mu\nu} + C_{\mu\nu} = 0$ for $\omega = 1$ ($D = 2$) only.
The regularised photon self-energy is

$$\begin{aligned} \Pi_{\mu\nu}(p) &= \frac{-4ie^2}{(4\pi)^\omega\Gamma(\omega)} \\ &\Big\{[\delta_{\mu\nu}p^2 - p_\mu p_\nu]\beta(\omega, 2-\omega)\int_0^1 dx 2x(1-x)(a^2)^{\omega-2} \\ &+ \delta_{\mu\nu}\left(\frac{1}{\omega}-1\right)\int_0^\infty d(k^2)(k^2)^{\omega-2}\Big\} \end{aligned} \tag{2.26}$$

For the case $m^2 = 0$, $a^2 = p^2x(1-x)$ and

$$\begin{aligned} \Pi_{\mu\nu}(p) &= \frac{-4ie^2}{(4\pi)^\omega\Gamma(\omega)} \\ &\Big\{[\delta_{\mu\nu}p^2 - p_\mu p_\nu]\beta(\omega, 2-\omega)2(p^2)^{\omega-2}\beta(\omega,\omega) \\ &+ \delta_{\mu\nu}\left(\frac{1}{\omega}-1\right)\int_0^\infty d(k^2)(k^2)^{\omega-2}\Big\} \end{aligned} \tag{2.27}$$

2.2 Expansion of the integrand

The photon self-energy $\Pi_{\mu\nu}(p)$ can be obtained without using translational invariance on the quadratic divergence. This is possible by using identities to expand the integrand of the 1-loop $\Pi_{\mu\nu}(p)$ in (2.4). From the identity

$$\frac{(k+p)^2 - 2k.p - p^2 + m^2}{k^2+m^2} = 1 \tag{2.28}$$

it follows that

$$\frac{1}{(k^2+m^2)((k+p)^2+m^2)} =$$
$$\frac{1}{(k^2+m^2)^2} - \frac{2p.k+p^2}{(k^2+m^2)((k+p)^2+m^2)} \tag{2.29}$$

and from

$$\frac{(k^2+m^2-m^2)}{k^2} = 1 \tag{2.30}$$

it follows that

$$\frac{1}{(k^2+m^2)^2} = \frac{1}{k^4} - \frac{2m^2}{k^4(k^2+m^2)^2} + \frac{m^4}{k^4(k^2+m^2)^2} \tag{2.31}$$

Therefore the photon self-energy $\Pi_{\mu\nu}(p)$ can be written as

$$\Pi_{\mu\nu}(p) = -4ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{2k_\mu k_\nu - \delta_{\mu\nu}k^2}{k^4} \right.$$
$$- \frac{2m^2(2k_\mu k_\nu - \delta_{\mu\nu}k^2)}{k^4(k^2+m^2)^2} + \frac{m^4(2k_\mu k_\nu - \delta_{\mu\nu}k^2)}{k^4(k^2+m^2)^2}$$
$$+ \frac{(2k_\mu k_\nu - \delta_{\mu\nu}k^2)(-p^2-2p.k)}{(k^2+m^2)^2((k+p)^2+m^2)}$$
$$\left. + \frac{k_\mu p_\nu + p_\mu k_\nu - \delta_{\mu\nu}k.p - m^2\delta_{\mu\nu}}{(k^2+m^2)((k+p)^2+m^2)} \right\} \tag{2.32}$$

For $m = 0$ this becomes

$$\Pi_{\mu\nu}(p) = \Pi^A_{\mu\nu} + \Pi^B_{\mu\nu} + \Pi^C_{\mu\nu}$$
$$= -4ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{2k_\mu k_\nu - \delta_{\mu\nu}k^2}{k^4} \right.$$

$$+\frac{(2k_\mu k_\nu - \delta_{\mu\nu}k^2)(-p^2 - 2p.k)}{(k^2)^2(k+p)^2}$$
$$+ \left.\frac{k_\mu p_\nu + p_\mu k_\nu - \delta_{\mu\nu}k.p}{k^2(k+p)^2}\right\} \qquad (2.33)$$

The first line of (2.32) and in the massless case (2.33) contains the pure quadratic divergences. These are the dominant UV divergences. All of the remaining terms are either linearly or logarithmically divergent. After Feynman parametrisation of these remaining terms, only logarithmic divergences will survive.

Assuming that the pure quadratic terms satisfy (C.8)

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{2k_\mu k_\nu - \delta_{\mu\nu}k^2}{k^4} = \delta_{\mu\nu}(\frac{1}{\omega} - 1) \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \qquad (2.34)$$

these terms will only cancel in 2 dimensions ($\omega = 1$). For $\omega \neq 1$ the remaining logarithmic and linear divergences will not be sufficient to cancel this dominant divergence.

This argument is based on a simple algebraic rearrangement of the integrand, which does not confuse infrared and ultraviolet divergences (in the domain $1 < \omega < 2$). It can therefore be applied within other regularisation schemes. This leads to the conclusion that quadratic divergences are present in the photon self-energy whatever the regularisation scheme, and that the only way to remove them is to introduce additional fields which contribute quadratic divergences that exactly cancel with those of $\Pi_{\mu\nu}(p)$.

Returning to the dimensional regularisation of $\Pi_{\mu\nu}(p)$, it should be confirmed that the remaining logarithmic and linear terms produce the expected result. For the massless case ($m = 0$) this means that (2.33) should agree with (2.27). All of these terms can be evaluated using the standard form of the β-function (B.2). To demonstrate this the following integrals, which are of the form (C.29) - (C.31), are required;

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{k^2(k+p)^2} = \frac{1}{(4\pi)^\omega}(p^2)^{\omega-2}(p_\mu)\Gamma(2-\omega)\beta(\omega, \omega-1) \qquad (2.35)$$
$$1 < \Re\omega \, , 1 < \Re(\omega+1), 0 < \Re\omega < 2 \quad \Rightarrow \quad 1 < \Re\omega < 2$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{k^4(k+p)^2} = \frac{1}{(4\pi)^\omega}(p^2)^{\omega-3}\left[p_\mu p_\nu \Gamma(3-\omega)\beta(\omega-1, \omega)\right.$$
$$\left.+\frac{1}{2}\delta_{\mu\nu}p^2\Gamma(2-\omega)\beta(\omega, \omega-1)\right] \qquad (2.36)$$
$$1 < \Re\omega \, , 2 < \Re(\omega+1), 0 < \Re\omega < 2 \quad \Rightarrow \quad 1 < \Re\omega < 2$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu k_\lambda}{k^4(k+p)^2} =$$

$$\frac{1}{(4\pi)^\omega}(p^2)^{\omega-3}\Big[-p_\mu p_\nu p_\lambda \Gamma(3-\omega)\beta(\omega+1,\omega-1)$$

$$-\frac{1}{2}p^2(\delta_{\mu\nu}p_\lambda + \delta_{\mu\lambda}p_\mu + \delta_{\lambda\mu}p_\nu)\Gamma(2-\omega)\beta(\omega,\omega)\Big] \tag{2.37}$$

$$1 < \Re\omega\ ,\ 2 < \Re(\omega+2),\ 0 < \Re\omega < 2 \quad \Rightarrow \quad 1 < \Re\omega < 2$$

Using these formulae, which apply strictly for $1 < \Re\omega < 2$

$$\Pi^C_{\mu\nu} = \frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}(\delta_{\mu\nu}p^2 - 2p_\mu p_\nu)\Gamma(2-\omega)\beta(\omega,\omega-1) \tag{2.38}$$

$$\Pi^B_{\mu\nu} = \frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Big[(\delta_{\mu\nu}p^2 - 2p_\mu p_\nu)\Gamma(3-\omega)\beta(\omega,\omega-1)$$

$$+\frac{1}{2}p^2\delta_{\mu\nu}(2\omega-2)\Gamma(2-\omega)\beta(\omega,\omega-1)\Big]$$

$$+\frac{(-4ie^2)(p^2)^{\omega-3}}{(4\pi)^\omega}2p_\lambda\delta_{\mu\nu}\Big[-p^2p_\lambda\Gamma(3-\omega)\beta(\omega+1,\omega-1)$$

$$-\frac{1}{2}p^2p_\lambda(2+2\omega)\Gamma(2-\omega)\beta(\omega,\omega)\Big]$$

$$-\frac{(-4ie^2)(p^2)^{\omega-3}}{(4\pi)^\omega}4p_\lambda\Big[-p_\mu p_\nu p_\lambda\Gamma(3-\omega)\beta(\omega+1,\omega-1)$$

$$-\frac{1}{2}p^2(\delta_{\mu\nu}p_\lambda + \delta_{\mu\lambda}p_\mu + \delta_{\lambda\mu}p_\nu)\Gamma(2-\omega)\beta(\omega,\omega)\Big] \tag{2.39}$$

It is important to note that all of these terms are free of infrared divergences. Collecting similar terms gives

$$\Pi^B_{\mu\nu} = \frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Big[(\delta_{\mu\nu}p^2 - 2p_\mu p_\nu)\Gamma(3-\omega)\beta(\omega,\omega-1)$$

$$+\frac{1}{2}p^2\delta_{\mu\nu}(2\omega-2)\Gamma(2-\omega)\beta(\omega,\omega-1)\Big]$$

$$+\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Big[(-2\delta_{\mu\nu}p^2 + 4p_\mu p_\nu)\Gamma(3-\omega)\beta(\omega+1,\omega-1)$$

$$+\left(-\delta_{\mu\nu}p^2 2(1+\omega)+2(2p_\mu p_\nu+\delta_{\mu\nu}p^2)\right)\Gamma(2-\omega)\beta(\omega,\omega)\Big] \tag{2.40}$$

Using the Γ-function recurrence relation $z\Gamma(z)=\Gamma(z+1)$ this becomes

$$\Pi^B_{\mu\nu}=\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)$$

$$\Big[(\delta_{\mu\nu}p^2-2p_\mu p_\nu)(2-\omega)\frac{(2\omega-1)}{(\omega-1)}+p^2\delta_{\mu\nu}(\omega-1)\frac{(2\omega-1)}{(\omega-1)}$$

$$+(-2\delta_{\mu\nu}p^2+4p_\mu p_\nu)(2-\omega)\frac{\omega}{(\omega-1)}+4p_\mu p_\nu+\delta_{\mu\nu}p^2(-2\omega)\Big] \tag{2.41}$$

and so

$$\Pi^B_{\mu\nu}=\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)$$

$$\left[(\delta_{\mu\nu}p^2-2p_\mu p_\nu)\frac{(\omega-2)}{(\omega-1)}-p^2\delta_{\mu\nu}+4p_\mu p_\nu\right] \tag{2.42}$$

Similarly

$$\Pi^C_{\mu\nu}=\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)\left[(\delta_{\mu\nu}p^2-2p_\mu p_\nu)\frac{(2\omega-1)}{(\omega-1)}\right] \tag{2.43}$$

The sum of (2.42) and (2.43) is

$$\Pi^B_{\mu\nu}+\Pi^C_{\mu\nu}=\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)$$

$$\left[(\delta_{\mu\nu}p^2-2p_\mu p_\nu)\frac{(3\omega-3)}{(\omega-1)}-p^2\delta_{\mu\nu}+4p_\mu p_\nu\right] \tag{2.44}$$

$$=\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)\left[2\delta_{\mu\nu}p^2-2p_\mu p_\nu\right] \tag{2.45}$$

Therefore the photon self-energy is

$$\Pi_{\mu\nu}(p)=-4ie^2\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left(\frac{2k_\mu k_\nu-\delta_{\mu\nu}k^2}{k^4}\right)$$

$$+\frac{(-4ie^2)(p^2)^{\omega-2}}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega,\omega)2\left[\delta_{\mu\nu}p^2-p_\mu p_\nu\right] \tag{2.46}$$

which agrees with (2.27).

The quadratic term is again present explicitly, and remains unregularised. If the standard integral representation of the β-function $\beta(x,y)$ (B.2) is used for all values of x and y, this leads to the quadratic divergences being ignored.

2.3 Schwinger Parametrisation

Instead of Feynman parametrisation, it is also possible to use exponential, or Schwinger parametrisation, whereby

$$\frac{1}{k^2+m^2} = \int_0^\infty dx \exp[-x(k^2+m^2)] \tag{2.47}$$

Using this approach, the photon self-energy can be written

$$\Pi_{\mu\nu}(p) = -4ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu - \delta_{\mu\nu}(k^2+k.p+m^2)}{(k^2+m^2)((k+p)^2+m^2)} \tag{2.48}$$

$$\begin{aligned} \Rightarrow \Pi_{\mu\nu}(p) = -4ie^2 \int_0^\infty dx_1 \int_0^\infty dx_2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \\ [2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu - \delta_{\mu\nu}(k^2+k.p+m^2)] \\ \exp[-x_1(k^2+m^2) - x_2((k+p)^2+m^2)] \end{aligned} \tag{2.49}$$

Changing variables to

$$k' = k + \frac{px_2}{x_1+x_2} = k + p - \frac{px_1}{x_1+x_2} \tag{2.50}$$

$$\begin{aligned} \Pi_{\mu\nu}(p) = -4ie^2 \int_0^\infty dx_1 \int_0^\infty dx_2 \int \frac{d^{2\omega}k'}{(2\pi)^{2\omega}} \\ \left[2k'_\mu k'_\nu + \frac{(x_1-x_2)}{(x_1+x_2)}(k'_\mu p_\nu + p_\mu k'_\nu) - 2p_\mu p_\nu \frac{x_1x_2}{(x_1+x_2)^2} \right. \\ \left. -\delta_{\mu\nu}\left((k')^2 + k'.p\frac{x_1-x_2}{x_1+x_2} - p^2\frac{x_1x_2}{(x_1+x_2)^2} + m^2\right)\right] \\ \exp\left[-(x_1+x_2)(k')^2 - \frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \end{aligned} \tag{2.51}$$

Using the Gaussian type integrals

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2 + 2k.px) = \frac{1}{(4\pi x)^\omega} \exp(xp^2) \tag{2.52}$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu \exp(-xk^2 + 2k.px) = \frac{1}{(4\pi x)^\omega} p_\mu \exp(xp^2) \tag{2.53}$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu k_\nu \exp(-xk^2 + 2k.px) = \frac{1}{(4\pi x)^\omega}\left(p_\mu p_\nu + \frac{\delta_{\mu\nu}}{2x}\right)\exp(xp^2) \quad (2.54)$$

$$\Pi_{\mu\nu}(p) = -4ie^2 \int_0^\infty dx_1 \int_0^\infty dx_2$$
$$\frac{1}{(4\pi)^\omega (x_1+x_2)} \exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right]$$
$$\left[(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\frac{x_1x_2}{(x_1+x_2)^2} + \delta_{\mu\nu}\left(\frac{1-\omega}{x_1+x_2} - p^2\frac{x_1x_2}{(x_1+x_2)^2} - m^2\right)\right] \quad (2.55)$$

Writing

$$\Pi_{\mu\nu}(p) = \Pi^T_{\mu\nu}(p) + \Pi^L_{\mu\nu}(p) \quad (2.56)$$

where $\Pi^T_{\mu\nu}(p)$ is proportional to $(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)$, then using the identity

$$\int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{x_1+x_2}{\lambda}\right) = 1 \quad (2.57)$$

$\Pi^T_{\mu\nu}(p)$ becomes

$$\Pi^T_{\mu\nu}(p) = -4ie^2 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{x_1+x_2}{\lambda}\right)$$
$$\frac{1}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\frac{x_1x_2}{(x_1+x_2)^{2+\omega}}\exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \quad (2.58)$$

$$= \frac{-4ie^2}{(4\pi)^\omega}\int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty d\lambda\delta\left(1 - \frac{x_1+x_2}{\lambda}\right)$$
$$(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\frac{x_1x_2}{\lambda^{\omega+3}}\exp\left[-\frac{x_1x_2}{\lambda}p^2 - m^2\lambda\right] \quad (2.59)$$

Letting $x_1 = \lambda x_1'$ and $x_2 = \lambda x_2'$

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\int_0^\infty dx_1' \int_0^\infty dx_2' \int_0^\infty d\lambda\delta(1 - (x_1' + x_2'))$$
$$x_1'x_2'\lambda^{1-\omega}\exp\left[\lambda(-x_1'x_2'p^2 - m^2)\right] \quad (2.60)$$

Now the integral over λ gives a Γ-function

$$\int_0^\infty d\lambda\lambda^{1-\omega}\exp[-\lambda(x_1'x_2'p^2 + m^2)] = \Gamma(2-\omega)(x_1'x_2'p^2 + m^2)^{\omega-2} \quad (2.61)$$

$$\Re(2-\omega) > 0$$

Therefore after integrating over x_2'

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\Gamma(2-\omega)$$

$$\times \int_0^\infty dx_1' x_1'(1-x_1')[x_1'(1-x_1')p^2 + m^2]^{\omega-2} \tag{2.62}$$

$$\Re(2-\omega) > 0$$

This is equal to the transverse part of (2.26).
Taking part of the longitudinal term, and using the same identity

$$\Pi^{L_1}_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{1}{(x_1+x_2)^\omega}$$

$$(-p^2\frac{x_1x_2}{(x_1+x_2)^2} - m^2)\exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \tag{2.63}$$

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1' \int_0^\infty dx_2' \int_0^\infty d\lambda\delta\left(1-(x_1'+x_2')\right)$$

$$\lambda^{1-\omega}(-x_1'x_2'p^2 - m^2)\exp[-\lambda(x_1'x_2'p^2 + m^2)] \tag{2.64}$$

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\Gamma(2-\omega)\int_0^\infty dx_1'(p^2x_1'(1-x_1') + m^2)^{\omega-1} \tag{2.65}$$

$$\Re(2-\omega) > 0$$

The remainder of the longitudinal part is given by

$$\Pi^{L_2}_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{1}{(x_1+x_2)^\omega}$$

$$\frac{1-\omega}{(x_1+x_2)}\exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \tag{2.66}$$

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1' \int_0^\infty dx_2' \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1-(x_1'+x_2')\right)$$

$$\times (1-\omega)\lambda^{-\omega}\exp[-\lambda(x_1'x_2'p^2 + m^2)] \tag{2.67}$$

For $1 < \Re\omega < 2$ the integral over λ is divergent, and can be related to the Cauchy-Saalschutz form (A.13). for $n = 0$ in (A.13)

$$\Gamma(z) = \int_0^\infty (e^{-t} - 1)t^{z-1} \tag{2.68}$$

$$-1 < \Re z < 0$$

$$\Rightarrow \Gamma(z) = a^z \int_0^\infty (e^{-at} - 1)t^{z-1} \tag{2.69}$$

$a > 0,\ -1 < \Re z < 0$

Therefore (2.67) becomes

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}(1-\omega)\int_0^\infty dx_1' \int_0^\infty dx_2' \delta\left(1 - (x_1' + x_2')\right)$$

$$\left\{(x_1'x_2'p^2 + m^2)^{\omega-1}\Gamma(1-\omega) + \int_0^\infty \lambda^{-\omega} d\lambda\right\} \tag{2.70}$$

$1 < \Re\omega < 2$

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\left\{\Gamma(2-\omega)\int_0^\infty dx_1'(x_1'(1-x_1')p^2 + m^2)^{\omega-1}\right.$$

$$\left. +(1-\omega)\int_0^\infty \lambda^{-\omega} d\lambda\right\} \tag{2.71}$$

$1 < \Re\omega < 2$

The complete longitudinal part is (2.71) + (2.65)

$$\Pi^L_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}(1-\omega)\int_0^\infty \lambda^{-\omega} d\lambda \tag{2.72}$$

$1 < \Re\omega < 2$

This is not quite the same as the longitudinal part of (2.26). $\Pi^L_{\mu\nu}(p)$ in (2.72) has been scaled by a factor $\omega\Gamma(\omega)$. This happens because of the illegal interchange of integrations over x_1', x_2', and k^2 which occurs when using exponential parametrisation and the Gaussian formulae. A similar rescaling can be seen comparing (3.31) with (3.29), although here the scaling factor is only $\Gamma(\omega)$. The extra factor of ω in (2.72) is presumably associated with the additional scaling $x_1, x_2 \to x_1', x_2'$.

This contrasts with the results obtained using Feynman parametrisation (2.26) and (2.27). Here it is not actually neccessary to interchange integrations for the pure quadratic term - this remains in (2.26) explicitly as an integral over k^2. Translational invariance of the quadratic term was used to derive (2.26), although this will be justified to some extent in (3.70).

Scaling such as has arisen in (2.72) can be tolerated, as long as the same procedure is applied throughout a calculation, and the pure quadratic divergences ultimately cancel between terms. In general, of course, scaling destroys any consistent cancellation, since with $\lambda = \lambda'\mu$

$$\int_0^\infty \lambda^{-\omega} d\lambda = \mu^{1-\omega} \int_0^\infty \lambda^{-\omega} d\lambda \tag{2.73}$$

Therefore, treating the integral as though it were finite, leads to

$$\int_0^\infty \lambda^{-\omega} d\lambda = 0 \tag{2.74}$$

2.4 Pauli-Villars Regularisation

The longitudinal part of the photon self-energy $\Pi^L_{\mu\nu}(p)$ can be made to vanish, if finiteness is assumed a few steps earlier. Taking

$$\Pi^L_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{1}{(x_1+x_2)^\omega}$$

$$\left(\frac{1-\omega}{(x_1+x_2)} - p^2\frac{x_1x_2}{(x_1+x_2)^2} - m^2\right)\exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \tag{2.75}$$

and rescaling $x_1 = \lambda x_1'$ and $x_2 = \lambda x_2'$

$$\Pi^L_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1' \int_0^\infty dx_2' \frac{1}{(x_1'+x_2')^\omega}\lambda^{2-\omega}$$

$$\left(\frac{1-\omega}{\lambda(x_1'+x_2')} - p^2\frac{x_1'x_2'}{(x_1'+x_2')^2} - m^2\right)\exp\left[-\lambda\left(\frac{x_1'x_2'}{(x_1'+x_2')}p^2 + m^2(x_1'+x_2')\right)\right] \tag{2.76}$$

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\lambda\frac{\partial}{\partial\lambda}\int_0^\infty dx_1' \int_0^\infty dx_2' \frac{1}{(x_1'+x_2')^{\omega+1}}\lambda^{1-\omega}$$

$$\exp\left[-\lambda\left(\frac{x_1'x_2'}{(x_1'+x_2')}p^2 - m^2(x_1'+x_2')\right)\right] \tag{2.77}$$

and rescaling back again

$$\Pi^L_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\lambda\frac{\partial}{\partial\lambda}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{1}{(x_1+x_2)^{\omega+1}}$$

$$\times \exp\left[-\frac{x_1x_2}{(x_1+x_2)}p^2 - m^2(x_1+x_2)\right] \tag{2.78}$$

$$\Rightarrow \Pi^L_{\mu\nu}(p) = 0 \tag{2.79}$$

This is exactly the argument used in [10] for $D = 2\omega = 4$, to suggest that $\Pi_{\mu\nu}(p)$ is only logarithmically divergent. In [10] Pauli-Villars regularisation is used [11]. In this regularisation scheme all integrations are 4-dimensional, and the divergences are regularised by a careful cancellation between similar terms. In Pauli- Villars the transverse $\Pi^T_{\mu\nu}(p)$ becomes

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\int_0^\infty dx_1'\int_0^\infty dx_2'\int_0^\infty \frac{d\lambda}{\lambda}\delta(1-(x_1'+x_2'))$$

$$\lambda^{-1}\sum_i c_i x_1' x_2' \exp\left[\lambda(-x_1' x_2' p^2 - m_i^2)\right] \tag{2.80}$$

Clearly $c_0 = 1$, and $\Pi^T_{\mu\nu}(p)$ is regularised if $c_1 = -1$, $c_i = 0$ $(i > 1)$. The relation

$$\int_0^\infty dt \frac{e^{-\nu t} - e^{-\mu t}}{t^{\rho+1}} = \frac{\mu^\rho - \nu^\rho}{\rho}\Gamma(1-\rho) \tag{2.81}$$

$$\mu > 0, \qquad \nu > 0; \qquad \rho < 1$$

implies that

$$\int_0^\infty \frac{dt}{t}\left(e^{-at} - e^{-bt}\right) = \ln\left(\frac{b}{a}\right) \tag{2.82}$$

$$a > 0, \qquad b > 0$$

Using this, $\Pi^T_{\mu\nu}(p)$ becomes

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)$$

$$\int_0^1 dx_1' x_1'(1-x_1')\ln\left[\frac{x_1'(1-x_1')p^2 + M^2}{x_1'(1-x_1')p^2 + m^2}\right] \tag{2.83}$$

for large M this is

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)$$

$$\left\{\frac{1}{6}\ln\frac{M^2}{m^2} - \int_0^1 dx x(1-x)\ln\left[1 + \frac{p^2}{m^2}x(1-x)\right]\right\} \tag{2.84}$$

The longitudinal part $\Pi^L_{\mu\nu}(p)$ is now

$$= \frac{-4ie^2}{(4\pi)^\omega}\delta_{\mu\nu}\int_0^\infty dx_1'\int_0^\infty dx_2'\int_0^\infty d\lambda\delta\left(1-(x_1'+x_2')\right)$$

$$\times \lambda^{-1} \sum_i c_i \left(-\frac{1}{\lambda} - x_1' x_2' p^2 - m_i^2 \right) \exp[-\lambda(x_1' x_2' p^2 + m_i^2)] \qquad (2.85)$$

With $c_1 = -1$, $c_i = 0$ $(i > 1)$, the middle term is regularised in the same way as $\Pi^T_{\mu\nu}(p)$, but both the $-1/\lambda$ term and the m_i^2 term still appear to be divergent. A double subtraction would be sufficient to regularise both of these terms, but if the expression (2.72) is used for $\Pi^L_{\mu\nu}(p)$ at $D = 4$ the two contributions to $\Pi^L_{\mu\nu}(p)$ from the sum in (2.72) will cancel each other out, and only one subtraction will be enough. However, to show that $\Pi^L_{\mu\nu}(p)$ reduces to (2.72) it is neccessary to carry out the integrations in (2.70) and (2.65). These are divergent at $D = 4$ so it is not clear that this can be done. Using both one Pauli-Villars subtraction, and dimensional regularisation

$$\Pi_{\mu\nu}(p) = \Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)\Gamma(2-\omega)$$

$$\times \sum_i \int_0^1 dx x(1-x)[x(1-x)p^2 + m_i^2]^{\omega-2} \qquad (2.86)$$

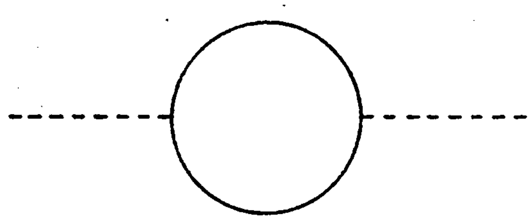
The Pauli-Villars regularisation has immediately cancelled the quadratic part. In the limit $\omega \to 2$

$$\Pi^T_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega}(2p^2\delta_{\mu\nu} - 2p_\mu p_\nu)$$

$$\int_0^1 dx x(1-x) \ln\left[\frac{x(1-x)p^2 + M^2}{x(1-x)p^2 + m^2}\right] \qquad (2.87)$$

which agrees with (2.83). This implies that one Pauli-Villars subtraction is sufficient to cancel both quadratic and logarithmic divergences, and fully regularise $\Pi_{\mu\nu}(p)$.

A similar treatment applied to the scalar boson self-energy gives



$$= -iG^2 \mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \frac{1}{(\not{k} + \mathrm{m})} \frac{1}{(\not{k} + \not{p} + \mathrm{m})}\right] \qquad (2.88)$$

$$= \frac{-4iG^2}{(4\pi)^\omega} \left\{ \int_0^1 dx(x(1-x)p^2 + m^2)^{\omega-1} \frac{1}{1-\omega} \Gamma(2-\omega) \right.$$

$$\left. + \omega \int_0^\infty \lambda^{-\omega} d\lambda \right\} \tag{2.89}$$

and clearly now one Pauli-Villars subtraction is not sufficient to fully regularise (2.89). It is also interesting to note that the quadratic divergence does not now cancel in 2-dimensions ($\omega = 1$).

Chapter 3

Domain of Convergence

3.1 The 't Hooft-Veltman Conjecture

In ref. [7] 't Hooft and Veltman suggest that the domain of convergence of integrals in D dimensions may be extended by means of analytic continuation. Their argument starts with the integral (in the notation of [7])

$$I = \int d^\kappa p \frac{p_a^{\lambda_1} p_b^{\lambda_2} \dots p_c^{\lambda_j}}{((p+k_1)^2+m_1^2)^{\alpha_1}((p+k_2)^2+m_2^2)^{\alpha_2}\dots((p+k_l)^2+m_l^2)^{\alpha_l}} \tag{3.1}$$

The integral will be convergent if

$$\lambda_1 > -1, \qquad \lambda_2 > -1, \qquad \dots \qquad \lambda_j > -1$$

$$\kappa + \lambda_1 + \lambda_2 + \dots \lambda_j - 2(\alpha_1 + \alpha_2 \dots + \alpha_l) > 0 \tag{3.2}$$

The identity

$$\frac{1}{\kappa}\sum_{i=1}^{\kappa}\frac{\partial p_i}{\partial p_i} = 1 \tag{3.3}$$

can be inserted into (3.1) and partial integration performed to give

$$I = \left(\frac{-\lambda_1 - \lambda_2 - \dots \lambda_j + 2(\alpha_1 + \alpha_2 \dots + \alpha_l)}{\kappa}\right) I - \frac{1}{\kappa} I' \tag{3.4}$$

or

$$I = \frac{-1}{(\kappa + \lambda_1 + \lambda_2 + \dots \lambda_j - 2(\alpha_1 + \alpha_2 \dots + \alpha_l))} I' \tag{3.5}$$

where I' is

$$I' = \int d^\kappa p p_a^{\lambda_1} \dots p_c^{\lambda_j}$$

$$\left[\frac{2\alpha_1(m_1^2+k_1^2+p.k_1)}{((p+k_1)^2+m_1^2)^{\alpha_1+1}((p+k_2)^2+m_2^2)^{\alpha_2}\dots((p+k_l)^2+m_l^2)^{\alpha_l}}\right.$$
$$+\frac{2\alpha_2(m_2^2+k_2^2+p.k_2)}{((p+k_1)^2+m_1^2)^{\alpha_1}((p+k_2)^2+m_2^2)^{\alpha_2+1}\dots((p+k_l)^2+m_l^2)^{\alpha_l}}$$
$$\left.+\frac{2\alpha_l(m_l^2+k_l^2+p.k_l)}{((p+k_1)^2+m_1^2)^{\alpha_1}((p+k_2)^2+m_2^2)^{\alpha_2}\dots((p+k_l)^2+m_l^2)^{\alpha_l+1}}\right] \tag{3.6}$$

The integral I' converges if

$$\lambda_1 > -1, \qquad \lambda_2 > -1, \qquad \dots \qquad \lambda_j > -1$$
$$\kappa+\lambda_1+\lambda_2+\dots\lambda_j-2(\alpha_1+\alpha_2\dots+\alpha_l) > 1 \tag{3.7}$$

In [7] it is suggested that because this is a larger domain than (3.2) the RHS of (3.5) is the explicit representation of the analytic continuation of I into this domain. Then by repeating the process of partial integration (and for sufficiently large λ) an explicit representation can be obtained which is valid in an arbitrary large domain in the complex D-plane.
Consider the case $\lambda_1=\lambda_2=\dots\lambda_j=0$, $\alpha_2\dots=\alpha_l=0$ and $k_1=0$. Then

$$I=\int d^\kappa p\frac{1}{(p^2+m_1^2)^{\alpha_1}} \tag{3.8}$$

This is identical to the integral (1.18), with $\alpha_1=n$, $m_1^2=m^2$, $\kappa=2\omega$ and $p=k$ (plus a factor $\frac{1}{(2\pi)^{2\omega}}$). The convergence condition for I (3.7) becomes

$$\kappa-2\alpha_1<0 \tag{3.9}$$

and (3.5) becomes

$$I=\frac{-1}{(\kappa-2\alpha_1)}I' \tag{3.10}$$

with

$$I'=\int d^\kappa p\frac{2\alpha_1 m_1^2}{(p^2+m_1^2)^{\alpha_1+1}} \tag{3.11}$$

The convergence condition for I' is

$$\kappa-2\alpha_1<1 \tag{3.12}$$

Consider the integral (3.8) in the notation of (1.18)

$$I=\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(k^2+m^2)^n} \tag{3.13}$$

From (1.13)

$$I = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2)(k^2)^{\omega-1} \frac{1}{(k^2+m^2)^n} \tag{3.14}$$

Integrating by parts

$$I = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \left\{ \left[\frac{1}{(k^2+m^2)^n} \frac{(k^2)^\omega}{\omega} \right]_0^\infty - \int_0^\infty d(k^2) \frac{(k^2)^\omega}{\omega} \frac{(-n)}{(k^2+m^2)^{n+1}} \right\} \tag{3.15}$$

which implies that

$$I = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \left[\frac{1}{(k^2+m^2)^n} \frac{(k^2)^\omega}{\omega} \right]_0^\infty + \frac{n}{\omega} I - \frac{n}{\omega} I' \tag{3.16}$$

where

$$I' = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2) \frac{m^2 (k^2)^{\omega-1}}{(k^2+m^2)^{n+1}} \tag{3.17}$$

and so

$$I = \frac{-n}{(\omega-n)} I' + \frac{\omega}{(\omega-n)} \frac{1}{(4\pi)^\omega \Gamma(\omega)} \left[\frac{1}{(k^2+m^2)^n} \frac{(k^2)^\omega}{\omega} \right]_0^\infty \tag{3.18}$$

I is convergent if $\omega - n < 0$ (3.9) and
I' is convergent if $\omega - n < 1$ (3.12), however

$$\lim_{k^2 \to 0} \left[\frac{1}{(k^2+m^2)^n} \frac{(k^2)^\omega}{\omega} \right] = 0 \qquad (\omega > 0)$$

$$\lim_{k^2 \to \infty} \left[\frac{1}{(k^2+m^2)^n} \frac{(k^2)^\omega}{\omega} \right] = 0 \qquad (n - \omega > 0) \tag{3.19}$$

Therefore the expression

$$I = \frac{-n}{\omega - n} I' \tag{3.20}$$

is only valid for $n - \omega > 0$, and so cannot be used to extend the definition of I into the region $n - \omega > -1$. Note that it is the UV divergence that prevents this. To conclude, as in (1.21)

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)^n} = \frac{(m^2)^{\omega-n}}{(4\pi)^\omega \Gamma(\omega)} \beta(\omega, n-\omega) \tag{3.21}$$

strictly within the domain

$$\Re\omega > 0 \qquad \Re(n-\omega) > 0 \tag{3.22}$$

3.2 The Leibbrandt-Capper Analysis

Leibbrandt and Capper [8, 9] also try to justify the use of the formula (3.21) outside the limits of (3.22), by appealing to the principle of analytic continuation. They begin by stating the basic theorem;

Let an analytic function $g_1(z)$ be defined in a region $\mathcal{D}_1$ and let $\mathcal{D}_2$ be another region which has a certain subregion $\mathcal{R}$, but only this one, in common with $\mathcal{D}_1$. Then if a function $g_2(z)$ exists which is analytic in $\mathcal{D}_2$ and coincides with $g_1(z)$ in $\mathcal{R}$, there can only be one such function.
This theorem asserts that $g_2(z)$ is unique provided $\mathcal{R}$ is not the empty set, $\mathcal{R} = \mathcal{D}_1 \cap \mathcal{D}_2 \neq 0$ ($\mathcal{R}$ contains infinitely many points). It further implies that the representations of $g_1(z)$ and $g_2(z)$ are equal in the subregion $\mathcal{R}$. Outside $\mathcal{R}$, the functions g_1 and g_2 possess, of course, different representations.

The Euler integral (B.1) is, in the notation of [8]

$$\Gamma_E(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \tag{3.23}$$

$$\Re z > 0$$

Weierstrass' partial fraction expansion is (see (B.5) and (B.6))

$$\Gamma_W(z) = \sum_{n=0}^{\infty} (-1)^n [n!(z+n)]^{-1} + \int_1^\infty dt\, t^{z-1} e^{-t} \tag{3.24}$$

which is analytic in the entire z-plane, except at the points $z = 0, -1, -2, \ldots$. Therefore, the representation Γ_W is an analytic continuation of Γ_E, since its domain of analatycity clearly overlaps that of Γ_E.
This does not, however, prove that the integral $\Gamma_E(z)$ (3.23) is analytic for $\Re z < 0$, which it clearly is not. In this region the appropriate representation, or rather, representations, are those of the Cauchy-Saalshutz form (B.13)

$$\Gamma_n = \int_0^\infty [e^{-t} - \sum_{m=0}^{n} (-t)^m/m!] t^{z-1} dt \tag{3.25}$$

$$-(n+1) < \Re z < -n$$

In a given region $-(n_0+1) < \Re z < -n_0$, the Cauchy -Saalschutz integral $\Gamma_{n_0}(z)$ is equal to $\Gamma_W(z)$. $\Gamma_{n_0}(z)$ is therefore analytic in this region. The difference between $\Gamma_{n_0}(z)$ and $\Gamma_E(z)$ in this region

$$\Gamma_E(z) - \Gamma_{n_0}(z) = \int_0^\infty [\sum_{m=0}^{n_0} (-t)^m/m!] t^{z-1} dt \tag{3.26}$$

$$-(n_0+1) < \Re z < -n_0$$

is clearly not an analytic function. So the integral $\Gamma_E(z)$ is not analytic for any $\Re z < 0$. it is important to distinguish between the integral $\Gamma_E(z)$ and the gamma-function $\Gamma(z)$ in this region.

In [8] it is proposed that integration leads, in the region where the integrals exist, to Γ-functions. The analytic continuation is then implemented by using for the Γ-functions the Weierstrass representation.
The response to this is implicit in the statement itself; in regions where the integrals do not exist, they are not equal to Γ-functions, no matter what representation is used.
This is discussed further in [8] in relation to massless integrals. The integral

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \tag{3.27}$$

is evaluated in two ways. Firstly using parametrisation

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^\infty e^{-xk^2}\, dx = \int_0^\infty \frac{dx\, x^{-\omega}}{(4\pi)^\omega} \tag{3.28}$$

$$= \frac{1}{(4\pi)^\omega} \left[\frac{x^{1-\omega}}{1-\omega} \right]_0^\infty \tag{3.29}$$

which diverges at $x = 0$ for $1 - \omega < 0$. This is equivalent to

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2)(k^2)^{\omega-1} \frac{1}{k^2} \tag{3.30}$$

$$= \frac{1}{(4\pi)^\omega \Gamma(\omega)} \left[\frac{(k^2)^{\omega-1}}{\omega - 1} \right]_0^\infty \tag{3.31}$$

which diverges at $k^2 = \infty$ for $\omega - 1 > 0$.
Secondly (3.27) is expressed as

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \frac{(k-p)^2}{(k-p)^2} \tag{3.32}$$

$$= p^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2(k-p)^2} - 2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{p.k}{k^2(k-p)^2} + \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k^2}{k^2(k-p)^2} \tag{3.33}$$

Evaluating the three integrals separately [8] finds

$$I = (4\pi)^{-\omega}\left\{(p^2)^{\omega-1}\Gamma(1-\omega)\left[(1-\omega)\beta(\omega-1,\omega-1)\right.\right.$$
$$\left.\left.-2(1-\omega)\beta(\omega-1,\omega)+(1-\omega)\beta(\omega-1,\omega+1)\right]\right\}$$
$$+\left[\omega(4\pi)^{-\omega}(p^2)^{\omega-1}\Gamma(1-\omega)\beta(\omega,\omega)\right] \quad (3.34)$$

In deriving (3.34) it has been assumed that the Euler integral $\Gamma_E(z)$ is valid for all z. Using (B.12) and the recurrence relation $z\Gamma(z) = \Gamma(z+1)$ (3.34) reduces to

$$I = 0 \quad (3.35)$$

Leibbrandt [8] notes that (3.35) may not be the case since each of the terms in the bracket $\{\ldots\}$ in (3.34) is analytic in the finite strip $\mathcal{D}_1 : 1 < \Re\omega < 2$, whereas the last expression involving $\Gamma(1-\omega)\beta(\omega,\omega)$ is only defined in the domain $\mathcal{D}_1 : 0 < \Re\omega < 1$. Since the domains of definition $\mathcal{D}_1$ and $\mathcal{D}_2$ do not overlap ($\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$), Leibbrandt comments that cancellations between the analytic continuations of the corresponding functions in (3.34) may not be justified.

Actually (3.34) is analytic for all ω (except integer values) because the Γ-function and β-function are so defined. The argument of [8] above is entirely correct, however, when applied to the integrals arising from (3.33).
Leibbrandt attempts to justify (3.35) using a redefinition of integration in D dimensions. The original definition used in [8] is based on a generalisation of the Gaussian integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2 + 2k.b) = \frac{1}{(4\pi x)^{\omega}} \exp(\frac{b^2}{x}) \quad (3.36)$$

$$x > 0$$

This reduces to the standard Gaussian formula for $2\omega = 1, 2, 3\ldots$ but for complex ω, the RHS of (3.36) is taken as the definition of the integral on the LHS. This can be compared to the earlier definition based on spherical symmetry (C.1)

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)^n} = \frac{(m^2)^{\omega-n}}{(4\pi)^{\omega}} \frac{\Gamma(n-\omega)}{\Gamma(n)} \quad (3.37)$$

$$0 < \Re\omega < \Re n$$

For $b = 0$ (3.36) becomes

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2) = \frac{1}{(4\pi x)^\omega} \tag{3.38}$$

$$x > 0$$

The LHS of (3.38) is spherically symmetric, and so

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2) = \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2)(k^2)^{\omega-1} \exp(-xk^2) \tag{3.39}$$

$$= \frac{1}{(4\pi)^\omega \Gamma(\omega)} x^{-\omega} \int_0^\infty d(k^2)(k^2)^{\omega-1} e^{-k^2} \tag{3.40}$$

$$= \frac{1}{(4\pi)^\omega} x^{-\omega} \tag{3.41}$$

$$\Re\,\omega > 0,\ \Re\,x > 0$$

which agrees with (3.38). Using translational invariance in (3.39) gives

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-x(k - b/x)^2) = \frac{1}{(4\pi x)^\omega} \tag{3.42}$$

which implies

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2 + 2k.b) = \frac{1}{(4\pi x)^\omega} \exp(b^2/x) \tag{3.43}$$

$$\Re\,\omega > 0,\ \Re\,x > 0$$

which agrees with (3.36). Hence the two definitions are equivalent. The Gaussian approach has the same difficulties. Consider

$$I^1(\omega, m) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2 + m^2} \tag{3.44}$$

Using an exponential parametrisation

$$I^1(\omega, m) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^\infty d\alpha \exp(-\alpha(k^2 + m^2)) \tag{3.45}$$

$$m^2 \neq 0$$

Interchanging integrations, and using (3.38)

$$I' = \int_0^\infty d\alpha e^{-\alpha m^2} \frac{1}{(4\pi\alpha)^\omega} \tag{3.46}$$

$$\Rightarrow I' = \frac{1}{(4\pi)^{\omega}}(m^2)^{\omega-1}\Gamma(1-\omega) \tag{3.47}$$

$$\Re(1-\omega) > 0$$

exactly as in (1.25). Again I' is only defined for $0 < \Re\omega < 1$, and not for $1 < \Re\omega < 2$.

Ref. [8] attempts to justify (3.35) using a redefinition of the generalised Gaussian integral. (3.36) is replaced by

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp[-xk^2 + 2b.k] = \frac{1}{(4\pi x)^{\omega}} \exp[\frac{b^2}{x} - xf(\omega)] \tag{3.48}$$

$$x > 0$$

where the function $f(\omega)$ is called the continuity function, and has the following properties;

1. $f(\omega)$ is a non-zero analytic function of the complex variable ω.
2. $f(\omega) = 0$ for $2\omega = 0, \pm1, \pm2, \pm3, \ldots$.
3. $f^l(\omega) = 0$ for $2\omega = 0, \pm1, \pm2, \pm3, \ldots$ and $l \leq l_0$ where l_0 is finite; l denotes the number of ordinary derivatives with respect to ω.
4. $\Re[f(\omega)] > 0$ for any $\Re(2\omega) = 0, \pm1, \pm2, \pm3, \ldots$

This corresponds to (1.10) with $g(2\omega) = \exp[-xf(\omega)]$. Repeating the steps (3.44) to (3.46) using this definition leads to

$$I' = \frac{1}{(4\pi)^{\omega}} \int_0^{\infty} d\alpha \alpha^{-o} e^{-\alpha(m^2+f(\omega))} \tag{3.49}$$

which after rescaling (for real $f(\omega)$) becomes

$$I' = \frac{1}{(4\pi)^{\omega}}(m^2 + f(\omega))^{\omega-1} \int_0^{\infty} d\alpha'(\alpha')^{-o} e^{-\alpha'} \tag{3.50}$$

which is equal to

$$I' = \frac{1}{(4\pi)^{\omega}}(m^2 + f(\omega))^{\omega-1}\Gamma(1-\omega) \tag{3.51}$$

$$\text{for } \Re(1-\omega) > 0,\ (m^2 + f(\omega)) > 0$$

just as in (3.47). I' is again only well defined for $0 < \Re\omega < 1$, and not for $1 < \Re\omega < 2$. The condition $\Re(f(\omega))$ is not sufficient to allow the domain of analyticity to be extended.

3.3 The Bjorken-Drell Test

In [10] ,there is a very simple and compelling theorem concerning the photon self-energy $\Pi_{\mu\nu}(p)$. The Ward identity requires that

$$p^{\mu}\Pi_{\mu\nu}(p) = 0 \tag{3.52}$$

After Wick rotating to Euclidean space $\Pi_{\mu\nu}(p)$ is given by (2.1). Contracting with p^{μ}

$$p^{\mu}\Pi_{\mu\nu}(p) = -ie^2\mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \not{p}\frac{1}{(\not{k}+\mathrm{m})}\gamma_{\nu}\frac{1}{(\not{k}+\not{p}+\mathrm{m})}\right] \tag{3.53}$$

Using the identity $\not{p} = \not{p} + \not{k} + m - \not{k} - m$ and the cyclic property of the trace

$$p^{\mu}\Pi_{\mu\nu}(p) = -ie^2\mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \frac{1}{(\not{k}+\not{p}+\mathrm{m})}(\not{p}+\not{k}+\mathrm{m}-\not{k}-\mathrm{m})\frac{1}{(\not{k}+\mathrm{m})}\gamma_{\nu}\right] \tag{3.54}$$

which implies that

$$p^{\mu}\Pi_{\mu\nu}(p) = -ie^2\mathrm{Tr}\left[\int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \left(\frac{1}{(\not{k}+\mathrm{m})} - \frac{1}{(\not{k}+\not{p}+\mathrm{m})}\right)\gamma_{\nu}\right] \tag{3.55}$$

If the integral in (3.55) were finite, translational invariance could be used in the second term ($k \to k-p$) to give $\Pi_{\mu\nu}(p) = 0$. With regularisation procedures such as momentum cut-off, this is obviously not possible because the cut-off precludes translational invariance. In dimensional regularisation, translational invariance is generally preserved, implying that the ward identity can be satisfied.

However, explicit calculation of $\Pi_{\mu\nu}(p)$, using dimensional regularisation, yielded for the case $m = 0$ ((2.27) and (2.46))

$$\begin{aligned}\Pi_{\mu\nu}(p) = {} & \frac{-4ie^2}{(4\pi)^{\omega}\Gamma(\omega)}2[\delta_{\mu\nu}p^2 - p_{\mu}p_{\nu}]\beta(\omega, 2-\omega)\beta(\omega,\omega)(p^2)^{\omega-2} \\ & + \frac{-4ie^2}{(4\pi)^{\omega}\Gamma(\omega)}\delta_{\mu\nu}(\frac{1}{\omega}-1)\int_0^{\infty} d(k^2)(k^2)^{\omega-2}\end{aligned} \tag{3.56}$$

which implies that

$$p^{\mu}\Pi_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^{\omega}\Gamma(\omega)}p_{\nu}(\frac{1}{\omega}-1)\int_0^{\infty} d(k^2)(k^2)^{\omega-2} \tag{3.57}$$

The unregularised quadratic divergence present in $\Pi_{\mu\nu}(p)$ prevents (3.52) from being satisfied. If (3.57) is true, then the unregularised (even in 2ω dimensions) divergences present in the integrals in (3.55) must forbid the use of translational invariance.

In deriving (2.27), translational invariance was employed, after Feynman parametrisation, and before the pure quadratic divergences could be clearly identified. In the derivation of (2.46), which agrees with (2.27), the pure quadratic divergences were first separated out using only linearity. This would appear to support the use of translational invariance in the derivation of (2.27).

The problem has arisen that using translational invariance to derive (3.57) leads to one result

$$p^{\mu}\Pi_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^{\omega}\Gamma(\omega)}p_{\nu}(\frac{1}{\omega} - 1)\int_0^{\infty} d(k^2)(k^2)^{\omega-2} \tag{3.58}$$

whereas using translational invariance in (3.55) gives another

$$p^{\mu}\Pi_{\mu\nu}(p) = 0 \tag{3.59}$$

Therefore the issue of translational invariance must be looked at more closely. From (C.4) it follows that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_{\mu}}{(k^2 + m^2)^n} = 0 \tag{3.60}$$

for $0 < \Re\omega < \Re(n-1)$

It is tempting to assume that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_{\mu} f(k^2) = 0 \tag{3.61}$$

is true for all $f(k^2)$. This is certainly the case if $f(k^2) \to 0$ sufficiently quickly as $k^2 \to \infty$, but consider the simple example $f(k^2) = 1$. For $2\omega = 4$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_{\mu} = \int_{-\infty}^{\infty} \frac{dk_1}{(2\pi)} \int_{-\infty}^{\infty} \frac{dk_2}{(2\pi)} \int_{-\infty}^{\infty} \frac{dk_3}{(2\pi)} \int_{-\infty}^{\infty} \frac{dk_4}{(2\pi)} k_{\mu} \tag{3.62}$$

For e.g. $\mu = 1$, the integral over k_1 is zero, whilst the integrals over k_2,k_3,k_4 diverge. Hence the 4-dimensional integral is not well-defined. In 2ω dimensions, if it is assumed that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_{\mu} = 0 \tag{3.63}$$

then since this is finite, translational invariance must be permitted, implying that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} (k_\mu + p_\mu) \tag{3.64}$$

and from (3.63)

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} (k_\mu + p_\mu) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} p_\mu \tag{3.65}$$

but this integral is spherically symmetric, so

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} p_\mu = p_\mu \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty d(k^2)(k^2)^{\omega-1} \tag{3.66}$$

which is divergent. Therefore (3.63) can not be true. Even if (3.63) does not hold (3.66) implies that (3.64) is not true. It is therefore important to consider other integrals of the form

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left(f(k_\mu + p_\mu) - f(k_\mu)\right) \tag{3.67}$$

This is done in appendix D, where it is demonstrated that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k)^4} \tag{3.68}$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu + p_\mu}{(k+p)^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k)^4} \tag{3.69}$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k)^2} \tag{3.70}$$

and finally that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu + p_\mu}{(k+p)^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{k_\mu}{(k)^2} + \frac{p_\mu}{(k)^2} - \frac{2p.kk_\mu}{(k)^4} \right\} \tag{3.71}$$

To obtain these expressions, linearity is assumed at all times, and translational invariance is used only for integrals known to be well-defined. It is also assumed at all times that $2\omega = 4 - \epsilon$ ($1 < \Re\omega < 2$).
For $m = 0$ (3.55) becomes

$$p^\mu \Pi_{\mu\nu}(p) = -ie^2 \mathrm{Tr}\left[\int \frac{d^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \left(\frac{\mathrm{k}^\mu}{\mathrm{k}^2} - \frac{(\mathrm{k}^\mu + \mathrm{p}^\mu)}{(\mathrm{k}+\mathrm{p})^2} \right) \gamma_\mu \gamma_\nu \right] \tag{3.72}$$

and from (3.71) this is

$$p^\mu \Pi_{\mu\nu}(p) = -ie^2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left(\frac{p^\mu}{k^2} - \frac{(2p.kk^\mu)}{(k)^4} \right) \mathrm{Tr}\,[\gamma_\mu \gamma_\nu] \tag{3.73}$$

which implies that

$$p^\mu \Pi_{\mu\nu}(p) = \frac{-4ie^2}{(4\pi)^\omega \Gamma(\omega)} p_\nu (\frac{1}{\omega} - 1) \int_0^\infty d(k^2)(k^2)^{\omega-2} \tag{3.74}$$

This agrees with (3.57).
The identity used in (3.54) has expressed the quadratically divergent (3.53) as the difference between two cubic divergences (3.55). As shown by (3.71) this difference is a non-vanishing unregularised integral, even for $2\omega \neq 4$. Therefore translational invariance fails for divergences that are cubic, or higher. No inconsistencies arise, though, if translational invariance is used in the quadratically divergent integrals, as shown by (3.70). This provides some justification for the use of translational invariance in the derivation of (2.27).

Chapter 4

$\varepsilon_{\kappa\lambda\mu\nu}$ and γ_5

4.1 Dirac γ-matrices in D dimensions

In $D = 2\omega$ dimensions the γ matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbf{1} \tag{4.1}$$

where $\mathbf{1}$ is the 2ω dimensional unit matrix. The trace operation is defined such that

$$\mathrm{Tr}[\mathbf{1}] = \mathrm{f}(2\omega) \tag{4.2}$$

In an even integer dimension, the standard representation of the γ^μ's has dimension $2^{D/2}$, although for most calculations it is usually assumed that $f(2\omega) = 4$.
The trace operation is linear

$$\mathrm{Tr}[\mathrm{aA} + \mathrm{bB}] = \mathrm{aTr}[\mathrm{A}] + \mathrm{bTr}[\mathrm{B}] \tag{4.3}$$

and cyclic

$$\mathrm{Tr}[\mathrm{ABC}] = \mathrm{Tr}[\mathrm{BCA}] \tag{4.4}$$

These properties define the trace of any linear combination of products of γ's. For example, from cyclicity (4.4)

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \mathrm{Tr}[\gamma^\nu\gamma^\mu] \tag{4.5}$$

then anticommutation (4.1)

$$= \mathrm{Tr}[-\gamma^\mu\gamma^\nu + 2\delta^{\mu\nu}\mathbf{1}] \tag{4.6}$$

and linearity (4.3)

$$= -\mathrm{Tr}[\gamma^\mu \gamma^\nu] + 2\delta^{\mu\nu}\mathrm{Tr}[\mathbf{1}]$$
$$\Rightarrow \mathrm{Tr}[\gamma^\mu \gamma^\nu] = \delta^{\mu\nu}\mathrm{Tr}[\mathbf{1}] \tag{4.7}$$

Similarly

$$\mathrm{Tr}[\gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu] = (\delta^{\kappa\lambda}\delta^{\mu\nu} - \delta^{\kappa\mu}\delta^{\lambda\nu} + \delta^{\kappa\nu}\delta^{\lambda\mu})\mathrm{Tr}[\mathbf{1}] \tag{4.8}$$

Now consider $D\mathrm{Tr}[\gamma^\lambda]$

$$\begin{aligned} D\mathrm{Tr}[\gamma^\lambda] &= \mathrm{Tr}[\gamma^\kappa \gamma_\kappa \gamma^\lambda] \\ &= -\mathrm{Tr}[\gamma^\kappa \gamma^\lambda \gamma_\kappa] + 2\mathrm{Tr}[\gamma^\lambda] \\ &= -\mathrm{Tr}[\gamma_\kappa \gamma^\kappa \gamma^\lambda] + 2\mathrm{Tr}[\gamma^\lambda] \end{aligned} \tag{4.9}$$

$$\Rightarrow (D-1)\mathrm{Tr}[\gamma^\lambda] = 0 \tag{4.10}$$

So $\mathrm{Tr}[\gamma^\lambda] = 0$ except at $D = 1$. For the trace of three γ matrices the same treatment leads to

$$(D-3)\mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\nu] = 2\delta^{\lambda\mu}\mathrm{Tr}[\gamma^\nu] \tag{4.11}$$

so $\mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\nu] = 0$ except for $D = 1, 3$.
In 4 dimensions it is possible to define a fifth matrix γ^5 by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.12}$$

or more formally

$$\gamma^5 = \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \tag{4.13}$$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is the totally antisymmetric Levi-Cevita tensor, with

$$\varepsilon_{0123} = +1 \tag{4.14}$$

With this definition of γ^5, it follows that

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{4.15}$$

$$(\gamma^5)^2 = \mathbf{1} \tag{4.16}$$

$$\mathrm{Tr}[\mathbf{1}]\varepsilon_{\kappa\lambda\mu\nu} = \mathrm{Tr}[\gamma^5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu] \tag{4.17}$$

These definitions of both γ^5 and $\varepsilon_{\kappa\lambda\mu\nu}$ are unique to 4 dimensions. This is why difficulties arise when trying to use these objects within the scheme of dimensional regularisation. If it is assumed that γ^5 and γ^μ anticommute for arbitrary D,

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{4.18}$$

for all D

then starting with $\mathrm{Tr}[\gamma^5]$ and using the properties (4.1) - (4.4)

$$\begin{aligned} D\mathrm{Tr}[\gamma^5] &= \mathrm{Tr}[\gamma^5\gamma^\mu\gamma_\mu] \\ &= \mathrm{Tr}[\gamma_\mu\gamma^5\gamma^\mu] \\ &= -\mathrm{Tr}[\gamma^5\gamma_\mu\gamma^\mu] \\ &= -D\mathrm{Tr}[\gamma^5] \end{aligned} \tag{4.19}$$

$$\Rightarrow D\mathrm{Tr}[\gamma^5] = 0 \tag{4.20}$$

So $\mathrm{Tr}[\gamma^5] = 0$, except at $D = 0$. In the first and last lines

$$\gamma_\mu\gamma^\mu = \frac{1}{2}\{\gamma_\mu, \gamma^\mu\} = \delta^\mu_\mu \mathbf{1} = D\mathbf{1} \tag{4.21}$$

was used. In a similar fashion

$$\begin{aligned} D\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\lambda] \\ &= \mathrm{Tr}[\gamma_\lambda\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda] \\ &= -\mathrm{Tr}[\gamma^5\gamma_\lambda\gamma^\mu\gamma^\nu\gamma^\lambda] \end{aligned}$$

$$\begin{aligned} &= -2\delta^\mu_\lambda\mathrm{Tr}[\gamma^5\gamma^\lambda\gamma^\mu] + 2\delta^\nu_\lambda\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] - \mathrm{D}\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] \\ &= -2\mathrm{Tr}[\gamma^5\{\gamma^\mu, \gamma^\nu\}] + (4 - \mathrm{D})\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] \end{aligned} \tag{4.22}$$

$$\Rightarrow (2 - D)\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 2\delta^{\mu\nu}\mathrm{Tr}[\gamma^5] \tag{4.23}$$

which implies that $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0$, except at $D = 0, 2$. For $\mathrm{Tr}[\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ the algorithm gives

$$(8 - 2D)\mathrm{Tr}[\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu] = 4\delta^{\kappa\lambda}\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] + 4\delta^{\mu\nu}\mathrm{Tr}[\gamma^5\gamma^\kappa\gamma^\lambda] \tag{4.24}$$

$$\Rightarrow \mathrm{Tr}[\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu] = 0 \tag{4.25}$$

except at $D = 0, 2, 4$

Therefore a γ^5 that anticommutes with all γ^μ is not possible outside $D = 4$. The usual way around this problem is to define

$$\gamma^5 = \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \tag{4.26}$$

with

$$\varepsilon_{\kappa\lambda\mu\nu} = \begin{cases} 1 & \text{if } (\kappa\lambda\mu\nu) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\kappa\lambda\mu\nu) \text{ is an odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases} \tag{4.27}$$

This definition is not Lorentz invariant on the full space, but only on the first four dimensions. The definition implies that

$$\begin{aligned} \{\gamma^5, \gamma^\mu\} = 0 \qquad & \text{if } \mu = 0,1,2,3 \\ [\gamma^5, \gamma^\mu] = 0 \qquad & \text{otherwise} \\ (\gamma^5)^2 = 1 & \\ (\gamma^5)^\dagger = \gamma^5 & \end{aligned} \tag{4.28}$$

and so γ^5 does not anticommute with all γ^μ.
Another possibility is to use (4.13), and take $\varepsilon_{\kappa\lambda\mu\nu}$ outside the expression to be generalised to non-integer D. However, then neither (4.15) nor (4.17) will hold.

4.2 Definition of γ_5

The definition that will be investigated here is to again take

$$\gamma^5 = \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \tag{4.29}$$

but with $\varepsilon_{\kappa\lambda\mu\nu}$ defined to be totally antisymmetric for all $(\kappa\lambda\mu\nu)$ and

$$\varepsilon_{\kappa\lambda\mu\nu} = 1 \qquad \text{for} \qquad \kappa < \lambda < \mu < \nu \tag{4.30}$$

Now γ^5 will, in general, not anticommute with all γ^ρ. In $D = 4$ dimensions the set of indices $(\kappa\lambda\mu\nu)$ must be the same as the set (1234), so ρ will be equal to one of them, and distinct from the other three. Therefore γ^ρ will be equal to one of the γ matrices in (4.29)(and so commute with it) and will anticommute with the other three. So γ^ρ will anticommute with γ^5. In $D \neq$ integer dimensions, the values of the indices $(\kappa\lambda\mu\nu)$ will no longer be restricted to 4 integers, and so γ^ρ will not anticommute with γ^5.
Some properties of γ^5 will remain unchanged. From (4.8) it follows that

$$\mathrm{Tr}[\gamma^5] = \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\mathrm{Tr}[\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$$

$$= \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}(\delta^{\kappa\lambda}\delta^{\mu\nu} - \delta^{\kappa\mu}\delta^{\lambda\nu} + \delta^{\kappa\nu}\delta^{\lambda\mu})\mathrm{Tr}[\mathbf{1}] \tag{4.31}$$

$$\Rightarrow \mathrm{Tr}[\gamma^5] = 0 \tag{4.32}$$

using the antisymmetry of $\varepsilon_{\kappa\lambda\mu\nu}$.
In the same way i.e. by successive reduction of the trace of $2n$ γ matrices to traces of $2n-2$ γ matrices, it follows that

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0 \tag{4.33}$$

and also that

$$\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}] = \varepsilon^{\mu_1\mu_2\mu_3\mu_4}\mathrm{Tr}[\mathbf{1}] \tag{4.34}$$

Note that within this particular trace γ^5 does anticommute with the other γ's. This can be seen explicitly, since by virtue of (4.1) and (4.33), γ^{μ_1} can be anticommuted to the right

$$\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}] = -\mathrm{Tr}[\gamma^5\gamma^{\mu_2}\mathrm{g}^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_1}] \tag{4.35}$$

and as the trace is cyclic

$$\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}] = -\mathrm{Tr}[\gamma^{\mu_1}\gamma^5\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}] \tag{4.36}$$

This does not mean that γ^5 will anticommute within any other trace, as will now be demonstrated. The trace of γ^5 with 6 γ matrices is rather more lengthy, and yields

$$\begin{aligned}
&\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}] = \\
&\mathrm{Tr}[\mathbf{1}] \times \\
&[\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\delta^{\mu_5\mu_6} - \varepsilon^{\mu_1\mu_2\mu_3\mu_5}\delta^{\mu_4\mu_6} + \varepsilon^{\mu_1\mu_2\mu_3\mu_6}\delta^{\mu_4\mu_5} \\
&+\varepsilon^{\mu_1\mu_2\mu_4\mu_5}\delta^{\mu_3\mu_6} - \varepsilon^{\mu_1\mu_2\mu_4\mu_6}\delta^{\mu_3\mu_5} + \varepsilon^{\mu_1\mu_2\mu_5\mu_6}\delta^{\mu_3\mu_4} \\
&-\varepsilon^{\mu_1\mu_3\mu_4\mu_5}\delta^{\mu_2\mu_6} + \varepsilon^{\mu_1\mu_3\mu_4\mu_6}\delta^{\mu_2\mu_5} - \varepsilon^{\mu_1\mu_3\mu_5\mu_6}\delta^{\mu_2\mu_4} \\
&+\varepsilon^{\mu_1\mu_4\mu_5\mu_6}\delta^{\mu_2\mu_3} + \varepsilon^{\mu_2\mu_3\mu_4\mu_5}\delta^{\mu_1\mu_6} - \varepsilon^{\mu_2\mu_3\mu_4\mu_6}\delta^{\mu_1\mu_5} \\
&+\varepsilon^{\mu_2\mu_3\mu_5\mu_6}\delta^{\mu_1\mu_4} - \varepsilon^{\mu_2\mu_4\mu_5\mu_6}\delta^{\mu_1\mu_3} + \varepsilon^{\mu_3\mu_4\mu_5\mu_6}\delta^{\mu_1\mu_2}]
\end{aligned} \tag{4.37}$$

Using the same method as in (4.35), but with (4.34) instead of (4.33), it is apparent that

$$\begin{aligned}
&\mathrm{Tr}[\{\gamma^5,\gamma^{\mu_1}\}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}] \\
&= 2\mathrm{Tr}[\mathbf{1}]\,(\delta^{\mu_1\mu_2}\varepsilon^{\mu_3\mu_4\mu_5\mu_6} - \delta^{\mu_1\mu_3}\varepsilon^{\mu_2\mu_4\mu_5\mu_6} \\
&+\delta^{\mu_1\mu_4}\varepsilon^{\mu_2\mu_3\mu_5\mu_6} - \delta^{\mu_1\mu_5}\varepsilon^{\mu_2\mu_3\mu_4\mu_6} + \delta^{\mu_1\mu_6}\varepsilon^{\mu_2\mu_3\mu_4\mu_5})
\end{aligned} \tag{4.38}$$

Clearly now $(\gamma^5)^2 \neq \mathbf{1}$, although from (4.34)

$$\mathrm{Tr}[\gamma^5\gamma^5] = \frac{1}{4!}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\mathrm{Tr}[\mathbf{1}] \tag{4.39}$$

$$\Rightarrow \mathrm{Tr}[\gamma^5\gamma^5] = \frac{1}{4!}\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)(\mathrm{D}-3)\mathrm{Tr}[\mathbf{1}] \tag{4.40}$$

It is also important to note that the identity

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4} = \begin{vmatrix} \delta^{\mu_1\nu_1} & \dots & \delta^{\mu_4\nu_1} \\ \vdots & & \vdots \\ \delta^{\mu_1\nu_4} & \dots & \delta^{\mu_4\nu_4} \end{vmatrix} \tag{4.41}$$

is only true in 4 dimensions. It is only for $D = 4$ that the two sets of indices $(\mu_1\mu_2\mu_3\mu_4)$ and $(\nu_1\nu_2\nu_3\nu_4)$ are both equal to the set (1234). For $D \neq 4$, this is not true for the definition of $\varepsilon_{\kappa\lambda\mu\nu}$ given in (4.30). Hence, in general $(\gamma^5)^2 \neq \mathbf{1}$.

4.3 Anomalies

Consider the axial extension of the QED Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eJ_\mu A^\mu - e' J_{5\mu}A_5^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{5\mu\nu}F^{5\mu\nu} \tag{4.42}$$

where the vector and axial vector currents are

$$J_\mu = \overline{\psi}\gamma_\mu\psi \tag{4.43}$$

$$J_\mu^5 = \overline{\psi}\frac{1}{2}\left[\gamma_\mu, \gamma^5\right]\psi \tag{4.44}$$

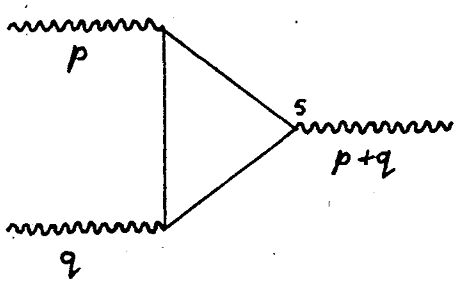
The vector current is conserved

$$\partial^\mu J_\mu = 0 \tag{4.45}$$

whereas for the axial current, the Dirac eqation gives

$$\partial^\mu J_{5\mu} = 2im\overline{\psi}\gamma^5\psi \tag{4.46}$$

and, as expected, $J_{5\mu}$ is conserved only when $m = 0$, i.e. when (4.42) is invariant under axial gauge transformations. The ABJ anomaly arises in the triangle diagram. The amplitude from this diagram



and the crossed diagram with $p \leftrightarrow q$ and $\mu \leftrightarrow \nu$ is

$$T^{(3)}_{\lambda\mu\nu}(p,q) = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\gamma_\nu\right] + (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.47}$$

In dimensional regularisation, γ^5 will no longer anticommute with γ^μ's. Consider $p^\mu T^{(3)}_{\lambda\mu\nu}$

$$p^\mu T^{(3)}_{\lambda\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{\mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\ \not p \frac{1}{\not k - \not p + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\gamma_\nu\right] + \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\gamma_\nu \frac{1}{\not k - \not q + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not p + \mathrm{m}}\ \not p\right]\right\} \tag{4.48}$$

Using the identity

$$\not p = (\not k + m) - (\not k - \not p + m) = (\not k + \not p + m) - (\not k + m) \tag{4.49}$$

in the first and third traces respectively, gives

$$p^\mu T^{(3)}_{\lambda\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}$$

$$\left\{\mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k - \not p + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\right] - \mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\right]\right.$$

$$+\mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k - \not q + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \mathrm{m}}\right] - \mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k - \not q + \mathrm{m}}\gamma_\lambda\gamma^5 \frac{1}{\not k + \not p + \mathrm{m}}\right]\Bigg\} \tag{4.50}$$

where the linear and cyclic properties of the trace operation have been used but γ^5 has not been anticommuted. Now a translation of the third term $k \rightarrow k + q$ shows that it cancels identically with the second term, whilst the translation $k \rightarrow k + q - p$ in the last term implies that it cancels with the first. The result is that

$$p^\mu T^{(3)}_{\lambda\mu\nu} = 0 \tag{4.51}$$

Similarly, it can be shown that

$$q^\nu T^{(3)}_{\lambda\mu\nu} = 0 \tag{4.52}$$

in agreement with the Ward identities. In using the identity (4.49) the linearly divergent terms in (4.48) have been expressed as the difference of quadratically divergent terms in (4.50). Even though these quadratic divergences remain unregularised by dimensional regularisation, the differences between them are regular. Therefore, as was argued in chapter 2, translational invariance can legitimately be used.
The remaining identity should imply that

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = 2mT_{\mu\nu} \tag{4.53}$$

where

$$T_{\mu\nu} = -ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}}\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\gamma_\nu\right] + (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.54}$$

However from (4.47)

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}}(\not p + \not q)\gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\gamma_\nu\right] + (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.55}$$

Using the identity

$$(\not p + \not q)\gamma^5 = (\not k + \not q + m)\gamma^5 - (\not k - \not p + m)\gamma^5$$
$$= -\gamma^5(\not k + \not q + m) - (\not k - \not p + m)\gamma^5 + 2m\gamma^5 + \{\gamma^5, \not k + \not q\} \tag{4.56}$$

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = 2mT_{\mu\nu}$$

$$+ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ -\mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k + \mathrm{m}} \gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}} \gamma^5\right] \right.$$

$$-\mathrm{Tr}\left[\gamma_\nu \frac{1}{\not k + \mathrm{m}} \gamma_\mu \gamma^5 \frac{1}{\not k + \not q + \mathrm{m}}\right]$$

$$\left. + \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}} \gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}} \{\gamma^5, \not k + \not q\} \frac{1}{\not k + \not q + \mathrm{m}} \gamma_\nu\right] \right\}$$

$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.57}$$

Using translational invariance

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = 2mT_{\mu\nu}$$

$$+ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ -\mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}} \gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}} \{\gamma^5, \gamma_\nu\}\right] \right.$$

$$\left. -\mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}} \{\gamma^5, \gamma_\mu\} \frac{1}{\not k + \not q + \mathrm{m}} \gamma_\nu\right] \right\}$$

$$+ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}} \gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}} \{\gamma^5, \not k + \not q\} \frac{1}{\not k + \not q + \mathrm{m}} \gamma_\nu\right] \right.$$

$$\left. + (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \right\} \tag{4.58}$$

Using the definition of γ^5 given in section 4.2 for $2\omega = D \neq 4$

$$\gamma^5 = \frac{1}{4!} \varepsilon_{\kappa\lambda\mu\nu} \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu \tag{4.59}$$

it follows that

$$\mathrm{Tr}[\gamma^5 \gamma_\mu] = 0 \tag{4.60}$$

except for $2\omega = 5$

$$\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu] = 0 \tag{4.61}$$

$$\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu \gamma_\rho] = 0 \tag{4.62}$$

except for $2\omega = 5, 7$

$$\mathrm{Tr}[\{\gamma^5, \gamma_\mu\} \gamma_\nu \gamma_\rho \gamma_\sigma] = 0 \tag{4.63}$$

Therefore the middle two terms in (4.58) vanish identically, leaving

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = 2mT_{\mu\nu} + A^{ABJ}_{\mu\nu} \tag{4.64}$$

where

$$A^{ABJ}_{\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \mathrm{Tr}\left[\frac{1}{\not k + \mathrm{m}}\gamma_\mu \frac{1}{\not k - \not p + \mathrm{m}}\{\gamma^5, \not k + \not q\}\frac{1}{\not k + \not q + \mathrm{m}}\gamma_\nu\right]$$
$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.65}$$

$$\Rightarrow A^{ABJ}_{\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}$$
$$\mathrm{Tr}\left[\frac{(\not k - \mathrm{m})}{(\mathrm{k}^2 + \mathrm{m}^2)}\gamma_\mu \frac{(\not k - \not p - \mathrm{m})}{((\mathrm{k}-\mathrm{p})^2 + \mathrm{m}^2)}\{\gamma^5, \not k + \not q\}\frac{(\not k + \not q - \mathrm{m})}{((\mathrm{k}+\mathrm{q})^2 + \mathrm{m}^2)}\gamma_\nu\right]$$
$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.66}$$

and again from (4.60) to (4.63), $A^{ABJ}_{\mu\nu}$ reduces to

$$A^{ABJ}_{\mu\nu} = ie^2e' \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{Tr}\left[\not k\gamma_\mu(\not k - \not p)\{\gamma^5, \not k + \not q\}(\not k + \not q)\gamma_\nu\right]}{(\mathrm{k}^2 + \mathrm{m}^2)((\mathrm{k}-\mathrm{p})^2 + \mathrm{m}^2)((\mathrm{k}+\mathrm{q})^2 + \mathrm{m}^2)}$$
$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.67}$$

Using (4.38) for the γ matrix trace

$$A^{ABJ}_{\mu\nu} = ie^2e' 2\mathrm{Tr}[\mathbf{1}]\,(\delta^{\mu_1\mu_2}\varepsilon^{\nu\mu_4\mu\mu_6} - \delta^{\mu_1\nu}\varepsilon^{\mu_2\mu_4\mu\mu_6}$$
$$+\delta^{\mu_1\mu_4}\varepsilon^{\mu_2\nu\mu\mu_6} - \delta^{\mu_1\mu}\varepsilon^{\mu_2\nu\mu_4\mu_6} + \delta^{\mu_1\mu_6}\varepsilon^{\mu_2\nu\mu_4\mu})$$
$$\times \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{(k+q)^{\mu_1}(k+q)^{\mu_2}k^{\mu_4}(k-p)^{\mu_6}}{(k^2+m^2)((k-p)^2+m^2)((k+q)^2+m^2)}$$
$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.68}$$

Using Feynman parametrisation, the integral becomes

$$\Gamma(3)\int \frac{d^{2\omega}k'}{(2\pi)^{2\omega}} \int_0^1 dx_1 \int_0^{x_1} dx_2$$
$$\frac{(k+q)^{\mu_1}(k+q)^{\mu_2}k^{\mu_4}(k-p)^{\mu_6}}{[(k')^2 + m^2 + q^2x_2 - p^2(x_1 - x_2) - (qx_1 - p(x_1 - x_2))^2]^3}$$
$$+ (p \leftrightarrow q,\ \mu \leftrightarrow \nu) \tag{4.69}$$

with $k = k' - qx_2 + p(x_1 - x_2)$. The whole expression simplifies to

$$A^{ABJ}_{\mu\nu} = 2ie^2e'4\mathrm{Tr}[\mathbf{1}]\left(1 - \frac{2}{\omega}\right)\mathrm{q}^{\mu_4}\mathrm{p}^{\mu_6}\varepsilon^{\nu\mu_4\mu\mu_6}$$

$$\int \frac{d^{2\omega}k'}{(2\pi)^{2\omega}}\int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{(k')^2}{[(k')^2 + a^2]^3} \tag{4.70}$$

where

$$a^2 = m^2 + q^2x_2 - p^2(x_1 - x_2) - (qx_2 - p(x_1 - x_2))^2 \tag{4.71}$$

and the inital factor of 2 comes from the crossed term. Integrating over k' gives

$$A^{ABJ}_{\mu\nu} = 2ie^2e'4\mathrm{Tr}[\mathbf{1}]\left(1 - \frac{2}{\omega}\right)\mathrm{q}^{\mu_4}\mathrm{p}^{\mu_6}\varepsilon^{\nu\mu_4\mu\mu_6}$$

$$\int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{(a^2)^{\omega-2}}{(4\pi)^\omega}\frac{\Gamma(2-\omega)}{\Gamma(3)}\omega \tag{4.72}$$

for $0 < \Re\omega < 2$

$$\Rightarrow A^{ABJ}_{\mu\nu} = -ie^2e'4\mathrm{Tr}[\mathbf{1}]\Gamma(3-\omega)\frac{1}{(4\pi)^\omega}\mathrm{q}^{\mu_4}\mathrm{p}^{\mu_6}\varepsilon^{\nu\mu_4\mu\mu_6}$$

$$\times \int_0^1 dx_1 \int_0^{x_1} dx_2 (a^2)^{\omega-2} \tag{4.73}$$

$$\Rightarrow A^{ABJ}_{\mu\nu} = -ie^2e'\frac{2\mathrm{Tr}[\mathbf{1}]}{(4\pi)^\omega}q^{\mu_4}p^{\mu_6}\varepsilon^{\nu\mu_4\mu\mu_6} + O(2-\omega) \tag{4.74}$$

In $2\omega = 4$ dimensions it follows that

$$(p^\lambda + q^\lambda)T^{(3)}_{\lambda\mu\nu} = 2mT_{\mu\nu} - ie^2e'\frac{1}{2\pi^2}\varepsilon^{\mu\nu\rho\sigma}q^\rho p^\sigma \tag{4.75}$$

which is the standard ABJ anomaly.

Chapter 5

General Gauge Theory

5.1 Full Quadratic Term

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}\overline{\psi}^{i}\gamma^{\mu}(D_{\mu}\psi)^{i}$$

$$+ (D_{\mu}\phi)^{a}(D_{\mu}\phi)^{a\dagger} - \frac{1}{2}\overline{\psi}^{i}\psi^{i}m_{\psi} - \frac{1}{2}m^{2}_{\phi}\phi^{a}\phi^{a\star}$$

$$+ \frac{1}{2}\overline{\psi}^{i}(G^{a})^{ij}\phi^{a}\psi^{j} - \frac{1}{4}\lambda_{abcd}\phi^{a}\phi^{b}\phi^{c}\phi^{d} \tag{5.1}$$

where

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} - gf^{abc}A^{b}_{\mu}A^{c}_{\nu} \tag{5.2}$$

$$(D_{\mu}\phi)^{a} = \partial_{\mu}\phi^{a} + ig(\Theta^{A})^{ab}A^{A}_{\mu}\phi^{b} \tag{5.3}$$

$$(D_{\mu}\psi)^{i} = \partial_{\mu}\psi^{i} + ig(T^{A})^{ij}A^{A}_{\mu}\psi^{j} \tag{5.4}$$

The scalar fields are taken to be real, and the spinor fields Majorana. The representation matrices T^{A} and Θ^{A} obey the Lie algebra of the gauge group;

$$[T^{A}, T^{B}] = if^{ABC}T^{C} \tag{5.5}$$

$$[\Theta^{A}, \Theta^{B}] = if^{ABC}\Theta^{C} \tag{5.6}$$

The corresponding 1-loop 2-point functions are shown in figs. (5.1), (5.2) and (5.3). In general both the scalar boson self-energy, and the gauge boson self-energy contain quadratic divergences. Using the usual Feynman rules, and the same techniques as in chapter 2, or simply by setting masses and external momenta to zero, the pure quadratic parts can be identified.

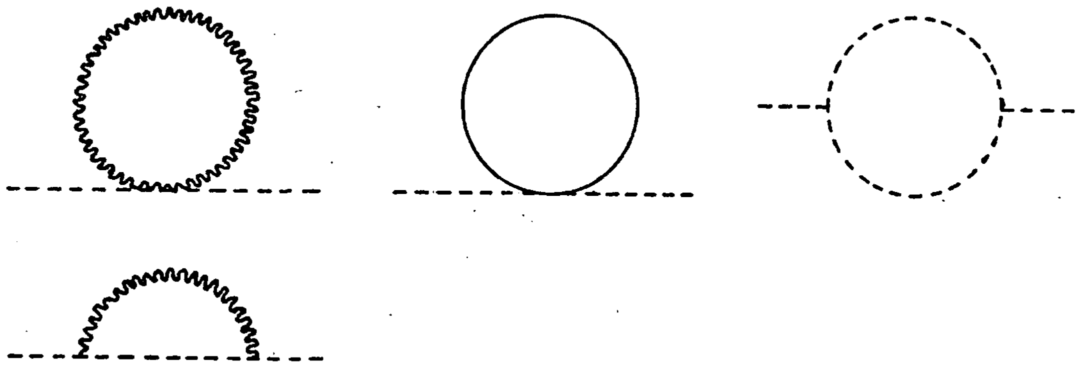


Figure 5.1: Scalar 2-point graphs

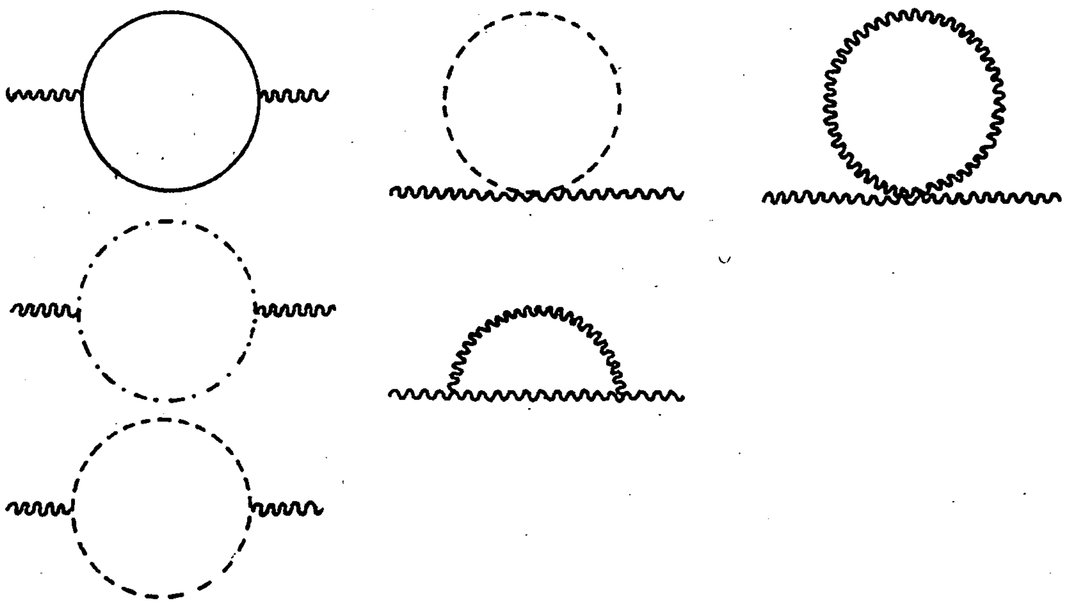
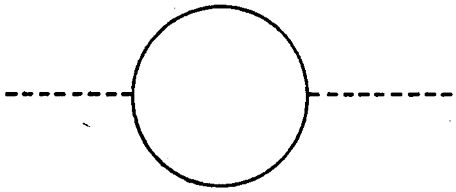


Figure 5.2: Gauge boson 2-point graphs

Figure 5.3: Fermion 2-point graphs

To begin with, consider the scalar self-energy. The fermion loop contribution is

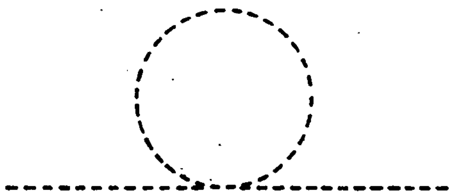


$$= \frac{1}{2} i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} i(G^a)^{ij} i(G^b)^{ji} \mathrm{Tr}\left[\frac{\mathrm{i}}{\not{k} + \mathrm{m}_\psi} \frac{\mathrm{i}}{\not{k} + \not{p} + \mathrm{m}_\psi}\right] \tag{5.7}$$

The pure quadratic part is

$$-\frac{1}{2} i (G^a)^{ij} (G^b)^{ji} \mathrm{Tr}[\mathbf{1}] \int \frac{\mathrm{d}^{2\omega}\mathrm{k}}{(2\pi)^{2\omega}} \frac{1}{\mathrm{k}^2} \tag{5.8}$$

The scalar loop contribution is

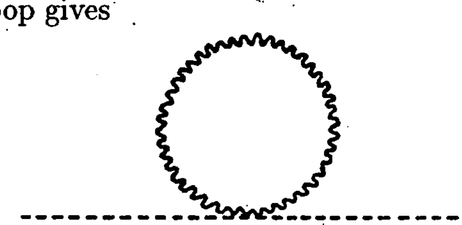


$$= i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} (-i\lambda_{abcd}) \delta_{cd} \frac{i}{k^2 + m_\phi^2} \tag{5.9}$$

The quadratic part is

$$i\lambda_{abcd}\delta_{cd} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \tag{5.10}$$

The gauge boson loop gives

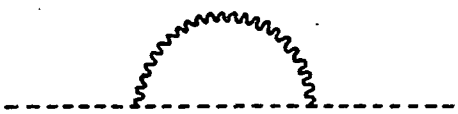


$$= -\frac{1}{2} i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{i\delta_{cd}}{k^2} \left(-\delta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) ig^2 \delta^{\mu\nu} \left((\Theta^c\Theta^d)^{ba} + (\Theta^d\Theta^c)^{ba}\right) \tag{5.11}$$

with quadratic part (the whole expression)

$$ig^2(2\omega - (1-\xi))(\Theta^c\Theta^c)^{ba}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.12}$$

and also



$$= i\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i}{k^2}\left(-\delta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{i}{(k+p)^2}$$
$$\times(-ig)(k^\mu + 2p^\mu)(\Theta^c)^{da}(-ig)(k^\nu + 2p^\nu)(\Theta^c)^{bd} \tag{5.13}$$
$$= ig^2(\Theta^c\Theta^c)^{ba}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2(k+p)^2}\left(-(k+2p)^2 + (1-\xi)\frac{(k^2+2p.k)^2}{k^2}\right) \tag{5.14}$$

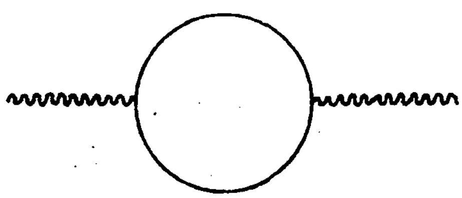
The quadratic part is

$$= ig^2(\Theta^c\Theta^c)^{ba}(-\xi)\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.15}$$

Combining (5.8), (5.10), (5.12), and (5.15), the coefficient of the pure quadratic divergence is

$$-\frac{1}{2}i\mathrm{Tr}[\mathbf{1}](\mathrm{G^a})^{\mathrm{ij}}\mathrm{i}(\mathrm{G^b})^{\mathrm{ji}} + \mathrm{i}\lambda_{\mathrm{abcc}} + \mathrm{i}g^2(2\omega - 1)(\Theta^c\Theta^c)^{\mathrm{ba}} \tag{5.16}$$

Turning now to the gauge boson self-energy, the fermion loop contribution is

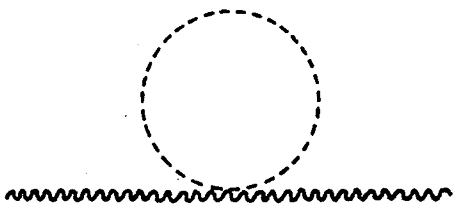


$$= -\frac{1}{2}ig^2\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\mathrm{Tr}[\mathrm{T^aT^b}]\mathrm{Tr}\left[\gamma_\mu\frac{1}{\not{k}+\mathrm{m}_\psi}\gamma_\nu\frac{1}{\not{k}+\not{p}+\mathrm{m}_\psi}\right] \tag{5.17}$$

The quadratic part is

$$-\frac{1}{2}ig^2\mathrm{Tr}[\mathrm{T^a T^b}]\mathrm{Tr}[\mathbf{1}]\delta_{\mu\nu}\left(\frac{1}{\omega}-1\right)\int\frac{\mathrm{d^{2\omega}k}}{(2\pi)^{2\omega}}\frac{1}{\mathrm{k}^2} \tag{5.18}$$

(There is a symmetry factor of $\frac{1}{2}$ because the fermion is Majorana.) There are two contributing scalar loops. The first is

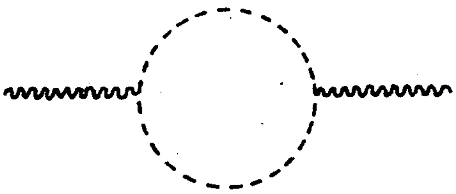


$$=\frac{1}{2}i\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i}{k^2+m_\phi^2}ig^2\delta_{\mu\nu}\left((\Theta^a\Theta^b)^{cc}+(\Theta^b\Theta^a)^{cc}\right) \tag{5.19}$$

The quadratic part is

$$-ig^2\delta_{\mu\nu}\mathrm{Tr}[\Theta^\mathrm{a}\Theta^\mathrm{b}]\int\frac{\mathrm{d^{2\omega}k}}{(2\pi)^{2\omega}}\frac{1}{\mathrm{k}^2} \tag{5.20}$$

The second diagram is



$$=\frac{1}{2}i\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i}{k^2+m_\phi^2}(-ig(\Theta^a)^{dc})(2k_\mu+p_\mu)$$

$$\frac{i}{(k+p)^2+m_\phi^2}(-ig(\Theta^b)^{cd})(2k_\nu+p_\nu) \tag{5.21}$$

$$=\frac{1}{2}ig^2\mathrm{Tr}[\Theta^\mathrm{a}\Theta^\mathrm{b}]\int\frac{\mathrm{d^{2\omega}k}}{(2\pi)^{2\omega}}\frac{(2\mathrm{k}_\mu+\mathrm{p}_\mu)(2\mathrm{k}_\nu+\mathrm{p}_\nu)}{(\mathrm{k}^2+\mathrm{m}_\phi^2)((\mathrm{k}+\mathrm{p})^2+\mathrm{m}_\phi^2)} \tag{5.22}$$

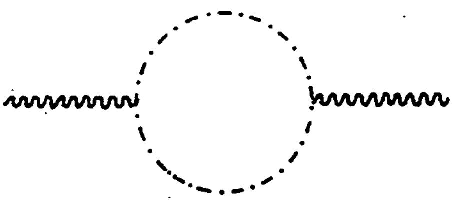
The quadratic part is

$$ig^2\mathrm{Tr}[\Theta^\mathrm{a}\Theta^\mathrm{b}]\delta_{\mu\nu}\frac{1}{\omega}\int\frac{\mathrm{d^{2\omega}k}}{(2\pi)^{2\omega}}\frac{1}{\mathrm{k}^2} \tag{5.23}$$

The complete scalar quadratic part is (5.20) plus (5.23)

$$- ig^2\delta_{\mu\nu}\mathrm{Tr}[\Theta^a\Theta^b]\left(1-\frac{1}{\omega}\right) \tag{5.24}$$

In a general covariant gauge the ghost loop is



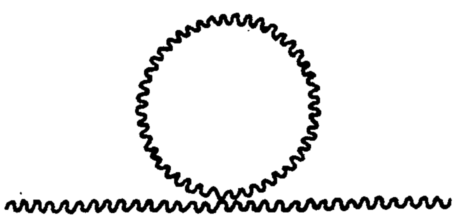
$$= -i\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i\delta_{ce}}{(k+p)^2}(-gf_{cda})(k_\mu+p_\mu)\frac{i\delta_{df}}{(k)^2}(-gf_{feb})k_\nu \tag{5.25}$$

$$= ig^2 f_{cda}f_{dcb}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{(k_\mu+p_\mu)k_\nu}{(k+p)^2k^2} \tag{5.26}$$

The quadratic part is

$$- ig^2 f_{acd}f_{bcd}\frac{\delta_{\mu\nu}}{2\omega}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.27}$$

There are two gauge boson loop diagrams. The first is

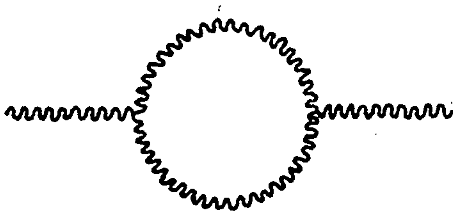


$$\begin{aligned}= \frac{1}{2}i\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{i\delta_{cd}}{k^2}&\left(-\delta_{\lambda\rho}+(1-\xi)\frac{k_\lambda k_\rho}{k^2}\right)\\ &\times(-ig^2)[f_{eca}f_{ebd}(\delta_{\lambda\nu}\delta_{\mu\rho}-\delta_{\mu\nu}\delta_{\lambda\rho})\\ &\quad+f_{eba}f_{ecd}(\delta_{\nu\lambda}\delta_{\mu\rho}-\delta_{\mu\lambda}\delta_{\nu\rho})\\ &\quad f_{ecb}f_{ead}(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\nu\mu}\delta_{\lambda\rho})]\end{aligned} \tag{5.28}$$

The quadratic part is

$$- ig^2 f_{aec}f_{bec}\delta_{\mu\nu}\left((2\omega-1)-(1-\xi)\left(1-\frac{1}{2\omega}\right)\right)\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.29}$$

The final diagram to consider is



$$= \frac{1}{2}i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} g f_{adc}[-(2k_\mu + p_\mu)\delta_{\rho\lambda} + (k_\lambda - p_\lambda)\delta_{\mu\rho} + (2p_\rho + k_\rho)\delta_{\lambda\mu}]$$

$$\times \frac{i}{k^2}\left(-\delta^{\rho\sigma} + (1-\xi)\frac{k^\rho k^\sigma}{k^2}\right) \frac{i}{(k+p)^2}\left(-\delta^{\lambda\kappa} + (1-\xi)\frac{(k^\lambda + p^\lambda)(k^\sigma + p^\sigma)}{(k+p)^2}\right)$$

$$\times g f_{bcd}[(k_\sigma + 2p_\sigma)\delta_{\nu\kappa} + (k_\kappa - p_\kappa)\delta_{\sigma\nu} - (2k_\nu + p_\nu)\delta_{\kappa\sigma}] \quad (5.30)$$

The quadratic part is contained in

$$-\frac{1}{2}ig^2 f_{adc} f_{bcd} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2(k+p)^2}$$

$$\Big[4k_\mu k_\nu(2\omega - 1) + \delta_{\mu\nu} 2k^2 - 2k_\mu k_\nu$$

$$+ (1-\xi)\frac{1}{k^2}(-\delta_{\mu\nu}k^4 + k^2 k_\mu k_\nu)$$

$$+ (1-\xi)\frac{1}{(k+p)^2}(-\delta_{\mu\nu}k^4 + k^2 k_\mu k_\nu)\Big] \quad (5.31)$$

The quadratic part is therefore

$$\frac{1}{2}ig^2 f_{acd} f_{bcd} \delta_{\mu\nu}\left[4 - \frac{4}{2\omega} + 2 - \frac{2}{2\omega} + 2(1-\xi)\left(-1 + \frac{1}{2\omega}\right)\right] \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \quad (5.32)$$

Combining (5.27), (5.29) and (5.32), the gauge boson loop contribution is

$$ig^2 f_{acd} f_{bcd} \delta_{\mu\nu}\Big[3 - \frac{3}{2\omega} + (1-\xi)\left(-1 + \frac{1}{2\omega}\right)$$

$$-(2\omega - 1) + (1-\xi)\left(1 - \frac{1}{2\omega}\right) - \frac{1}{2\omega}\Big] \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \quad (5.33)$$

$$= ig^2 f_{acd} f_{bcd} \delta_{\mu\nu}\left[3 - \frac{3}{2\omega} - (2\omega - 1) - \frac{1}{2\omega}\right] \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \quad (5.34)$$

$$= ig^2 f_{acd} f_{bcd} \delta_{\mu\nu} \left[4 - 2\omega - \frac{4}{2\omega}\right] \tag{5.35}$$

$$= ig^2 f_{acd} f_{bcd} \delta_{\mu\nu} 2 \left(1 - \frac{1}{\omega}\right)(1-\omega) \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{k^2} \tag{5.36}$$

The coefficient of the whole thing, combining the quadratic parts of gauge, fermion and scalar sectors is

$$\begin{aligned} & ig^2 f_{acd} f_{bcd} \delta_{\mu\nu} 2 \left(1 - \frac{1}{\omega}\right)(1-\omega) \\ & - ig^2 \delta_{\mu\nu} \mathrm{Tr}[\Theta^a \Theta^b] \left(1 - \frac{1}{\omega}\right) \\ & - ig^2 \delta_{\mu\nu} \frac{1}{2} \mathrm{Tr}[\mathbf{1}] \mathrm{Tr}[T^a T^b] \left(\frac{1}{\omega} - 1\right) \end{aligned} \tag{5.37}$$

$$= ig^2 \delta_{\mu\nu} \left(1 - \frac{1}{\omega}\right) \left[-2(\omega - 1) f_{acd} f_{bcd} - \mathrm{Tr}[\Theta^a \Theta^b] + \frac{1}{2} \mathrm{Tr}[\mathbf{1}] \mathrm{Tr}[T^a T^b]\right] \tag{5.38}$$

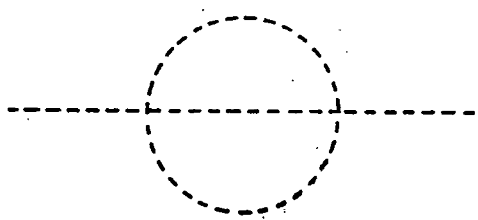
Taking $\mathrm{Tr}[\mathbf{1}] = 4$ and $2\omega = 4$, this becomes

$$= ig^2 \frac{1}{2} \delta_{\mu\nu} \left[-2 f_{acd} f_{bcd} - \mathrm{Tr}[\Theta^a \Theta^b] + 2\mathrm{Tr}[T^a T^b]\right] \tag{5.39}$$

For the regularised self-energy to satisfy the Ward identity, the quadratic divergences must cancel, i.e. (5.39) must equal zero. This is rather difficult to achieve, since the only way to vary (5.39) is by choice of representation. It is much more restrictive than the corresponding scalar condition (5.16) which contains 3 distinct dimensionless couplings.
The only obvious solution to(5.39), and the most natural is supersymmetry. In a supersymmetric gauge theory, not only does (5.39) vanish, but (5.16) also.

5.2 Two-Loop Example

Consider the two-loop diagram



$$I = -\frac{1}{6}i\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}i\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{1}{k_1^2}\frac{1}{k_2^2}\frac{1}{(k_1+k_2+p)^2} \tag{5.40}$$

taking the scalar mass to be zero. Carrying out the k_2 integration using Feynman parametrisation leads to

$$I = \frac{1}{6}i\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{1}{k_1^2}\frac{1}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega-1,\omega-1)((k_1+p)^2)^{\omega-2} \tag{5.41}$$

$$1 < \Re\omega < 2$$

Feynman parametrising the k_1 integral implies that

$$I = \frac{1}{6}\frac{1}{(4\pi)^\omega}\Gamma(2-\omega)\beta(\omega-1,\omega-1)(2-\omega)\frac{1}{(4\pi)^\omega\Gamma(\omega)}$$

$$\int_0^\infty d(k_1^2)\int_0^1 dx x^{1-\omega}\frac{(k_1^2)^{\omega-1}}{[(k_1+xp)^2+p^2x(1-x)]^{3-\omega}} \tag{5.42}$$

The integral over k_1 diverges quadratically. It is not possible to use the simple identites of section 2.2 to identify the pure quadratic divergence, because the exponent of the denominator is non-integer, but the identification can be made using the formula for the beta function worked out in appendix B.
For $1 < \Re\omega < 2$ (B.26) implies that

$$\beta(3-2\omega,\omega) = \int_0^\infty dt\left(\frac{t^{\omega-1}}{(1+t)^{3-\omega}} - t^{2\omega-4}\right) \tag{5.43}$$

which implies that

$$\beta(3-2\omega,\omega) = a^{3-2\omega}\int_0^\infty dt'\left(\frac{(t')^{\omega-1}}{(a+t')^{3-\omega}} - (t')^{2\omega-4}\right) \tag{5.44}$$

and so

$$I = \frac{1}{6}\frac{1}{(4\pi)^\omega}\Gamma(3-\omega)\beta(\omega-1,\omega-1)\frac{1}{(4\pi)^\omega\Gamma(\omega)}$$

$$\int_0^1 dx x^{1-\omega}\left\{(p^2x(1-x))^{2\omega-3}\beta(3-2\omega,\omega)+\int_0^\infty dt'(t')^{2\omega-4}\right\} \tag{5.45}$$

$$\Rightarrow I = \frac{1}{6}\frac{1}{(4\pi)^{2\omega}\Gamma(\omega)}\Gamma(3-\omega)\beta(\omega-1,\omega-1)$$

$$\left\{(p^2)^{2\omega-3}\beta(\omega-1,2\omega-2)\beta(3-2\omega,\omega)+\frac{1}{2-\omega}\int_0^\infty d(k^2)(k^2)^{2\omega-4}\right\} \tag{5.46}$$

$$1 < \Re\omega < 2$$

5.3 Dimensional Reduction

In section (5.1) it was shown that the condition for the cancellation of quadratic divergences in the gauge boson self-energy is

$$ig^2\delta_{\mu\nu}\left(1-\frac{1}{\omega}\right)\left[-2(\omega-1)f_{acd}f_{bcd}-\mathrm{Tr}[\Theta^a\Theta^b]+\frac{1}{2}\mathrm{Tr}[\mathbf{1}]\mathrm{Tr}[\mathrm{T^aT^b}]\right]=0 \tag{5.47}$$

where a sum over representations is assumed.
Even with an appropriate choice of representations, the cancellation is only exact for $2\omega = D = 4$ (barring a particularly contrived choice of $\mathrm{Tr}[\mathbf{1}]$). Therefore dimensional regularisation is still not 'manifestly' gauge invariant in 2ω dimensions, although if an exact cancellation were to be obtained for $D = 4$, the model would be expected to be well behaved, and free of quadratic divergences when other types of regularisation scheme were used.
It has been established that dimensional regularisation does not manifestly preserve supersymmetry, and to solve this problem, a modification was proposed by Siegel [12]. In the modified scheme, known as dimensional reduction, the spacetime dimension $D = 2\omega$ is non-integer, whilst the dimension of all fields is fixed at $D' = 4$. Symbolically

$$D' = D \oplus (D' - D) \tag{5.48}$$

4-dimensional space is decomposed into a sum of D- and $(4-D)$- dimensional subspaces, and in the $(4 - D) = \epsilon$ dimensional subspace all derivatives are required to vanish.
Consider the pure gauge Lagrangian, including gauge-fixing and ghost terms

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \eta^{a*}\partial_\mu D^{\mu ab}\eta^b \tag{5.49}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{5.50}$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - gf^{abc}A^c_\mu \tag{5.51}$$

In dimensional reduction the range of the index μ of A_μ is split into $0 \leq i \leq D$ and $D \leq \sigma \leq 4$ [13], leading to

$$(\mathcal{L}^G)^4 = (\mathcal{L}^G)^D + (\mathcal{L}^G)^\epsilon \tag{5.52}$$

where

$$(\mathcal{L}^G)^D = -\frac{1}{4}F^a_{ij}F^{ija} - \frac{1}{2\xi}(\partial_i A^{ia})^2 + \eta^{a*}\partial_i D^{iab}\eta^b \tag{5.53}$$

$$(\mathcal{L}^G)^\epsilon = -\frac{1}{2}(\partial_i A^a_\sigma)^2 - gf^{abc}A^b_i A^c_\sigma \partial^i A^{\sigma a}$$
$$-\frac{1}{2}g^2 f^{abc}f^{ade}A^b_i A^c_\sigma A^{id}A^{\sigma e}$$
$$-\frac{1}{4}g^2 f^{abc}f^{ade}A^b_\sigma A^c_{\sigma'}A^{\sigma d}A^{\sigma' e} \quad (5.54)$$

The D-dimensional part $(\mathcal{L}^G)^D$ is the same as in dimensional regularisation. The ϵ scalars A^a_σ couple to the gauge fields A^a_i as though they were scalar bosons in the adjoint representation. The gauge transformations are

$$\delta A^a_i = \partial_i \Lambda^a + gf^{abc}A^b_i \Lambda^c \quad (5.55)$$

$$\delta A^a_\sigma = gf^{abc}A^b_\sigma \Lambda^c \quad (5.56)$$

The fermionic lagrangian

$$(\mathcal{L}^F)^4 = \frac{1}{2}\overline{\psi}^a \gamma^\mu (D_\mu \psi)^a \quad (5.57)$$

becomes $(\mathcal{L}^F)^D + (\mathcal{L}^F)^\epsilon$ where

$$(\mathcal{L}^F)^D = \frac{1}{2}\overline{\psi}^a \gamma^i (D_i \psi)^a \quad (5.58)$$

$$(\mathcal{L}^F)^\epsilon = \frac{1}{2}ig\overline{\psi}^a (T^A)^{ba}\gamma^\sigma A^A_\sigma \psi^b \quad (5.59)$$

The scalar lagrangian

$$(\mathcal{L}^S)^4 = \frac{1}{2}(D_\mu \phi)^{a\dagger}(D^\mu \phi)^a \quad (5.60)$$

becomes $(\mathcal{L}^S)^D + (\mathcal{L}^S)^\epsilon$ where

$$(\mathcal{L}^S)^D = \frac{1}{2}(D_i \phi)^{a\dagger}(D^i \phi)^a \quad (5.61)$$

$$(\mathcal{L}^S)^\epsilon = \frac{1}{2}g^2(\Theta^B \Theta^A)^{cb}A^A_\sigma A^{\sigma b}\phi^b \phi^c \quad (5.62)$$

In deriving the terms of the ϵ-scalar lagrangian it has only been assumed that a sum over 4-dimensional indices could be separated into a sum over D and ϵ dimensional indices, i.e.

$$A_\mu B^\mu = A_i B^i + A_\sigma B^\sigma \quad (5.63)$$

It must also be assumed in the gamma matrix algebra that

$$\{\gamma_\sigma, \gamma_i\} = 0 \tag{5.64}$$

In the original formulation [12] it was also assumed that the D and ϵ dimensional vectors could be projected out of the 4 dimensional ones.
In 4 dimensions

$$\delta_{\mu\nu} = \delta_{\nu\mu} \qquad \delta_{\mu\alpha}\delta_{\alpha\nu} = \delta_{\mu\nu} \qquad \delta_{\mu\mu} = 4 \tag{5.65}$$

In D dimensions

$$\hat{\delta}_{\mu\nu} = \hat{\delta}_{\nu\mu} \qquad \hat{\delta}_{\mu\alpha}\hat{\delta}_{\alpha\nu} = \hat{\delta}_{\mu\nu} \qquad \hat{\delta}_{\mu\mu} = D \tag{5.66}$$

In ϵ dimensions

$$\hat{\hat{\delta}}_{\mu\nu} = \hat{\hat{\delta}}_{\nu\mu} \qquad \hat{\hat{\delta}}_{\mu\alpha}\hat{\hat{\delta}}_{\alpha\nu} = \hat{\hat{\delta}}_{\mu\nu} \qquad \hat{\hat{\delta}}_{\mu\mu} = \epsilon \tag{5.67}$$

If $\hat{\delta}_{\mu\nu}$ is regarded as an orthogonal projection operator from 4 dimensional space onto its D dimensional subspace, then

$$\delta_{\mu\alpha}\hat{\delta}_{\alpha\nu} = \hat{\delta}_{\mu\nu} \tag{5.68}$$

whilst

$$\hat{\hat{\delta}}_{\mu\nu} = \delta_{\mu\nu} - \hat{\delta}_{\mu\nu} \tag{5.69}$$

projects to ϵ dimensions.
In fact, as has been shown [14], these projections can not be carried out consistently. Siegel first showed that (5.68) is inconsistent when applied to the 4 dimensional identity

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = \begin{vmatrix} \delta_{\mu_1\nu_1} & \cdots & \delta_{\mu_4\nu_1} \\ \vdots & & \vdots \\ \delta_{\mu_1\nu_4} & \cdots & \delta_{\mu_4\nu_4} \end{vmatrix} \tag{5.70}$$

Projecting to D dimensions gives

$$\varepsilon_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}\varepsilon_{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4} = \begin{vmatrix} \hat{\delta}_{\mu_1\nu_1} & \cdots & \hat{\delta}_{\mu_4\nu_1} \\ \vdots & & \vdots \\ \hat{\delta}_{\mu_1\nu_4} & \cdots & \hat{\delta}_{\mu_4\nu_4} \end{vmatrix} \tag{5.71}$$

$$\varepsilon_{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4}\varepsilon_{\hat\nu_1\hat\nu_2\hat\nu_3\hat\nu_4} = 0 \tag{5.72}$$

From (5.71)

$$\varepsilon_{\hat a\hat b\hat c\hat d}\varepsilon_{\hat a\hat b\hat c\hat d} = (D-3)(D-2)(D-1)D \tag{5.73}$$

$$\varepsilon_{\hat{\hat a}\hat{\hat b}\hat{\hat c}\hat{\hat d}}\varepsilon_{\hat{\hat a}\hat{\hat b}\hat{\hat c}\hat{\hat d}} = (1-D)(2-D)(3-D)(4-D) \tag{5.74}$$

It follows that

$$0 = \varepsilon_{\hat a\hat b\hat c\hat d}\varepsilon_{\hat{\hat e}\hat{\hat f}\hat{\hat g}\hat{\hat h}}\varepsilon_{\hat a\hat b\hat c\hat d}\varepsilon_{\hat{\hat e}\hat{\hat f}\hat{\hat g}\hat{\hat h}} = (D-4)(D-3)^2(D-2)^2(D-1)^2 D \tag{5.75}$$

$$\Rightarrow D = 0,1,2,3,4 \tag{5.76}$$

which is Siegel's result. It can be seen that (5.71) is incompatible with the defiinition of ε_{abcd} given in chapter 4.
Another proof of the inconsistency is due to Avdeev and Vladimirov [15]. They note that in 4 dimensions

$$\begin{vmatrix} \delta_{\mu_1\nu_1} & \dots & \delta_{\mu_5\nu_1} \\ \vdots & & \vdots \\ \delta_{\mu_1\nu_5} & \dots & \delta_{\mu_5\nu_5} \end{vmatrix} = 0 \tag{5.77}$$

but using (5.68)

$$\hat\delta_{\mu_1\nu_1}\dots\hat\delta_{\mu_5\nu_5}\begin{vmatrix} \delta_{\mu_1\nu_1} & \dots & \delta_{\mu_5\nu_1} \\ \vdots & & \vdots \\ \delta_{\mu_1\nu_5} & \dots & \delta_{\mu_5\nu_5} \end{vmatrix} = \hat\delta_{\mu_1\nu_1}\dots\hat\delta_{\mu_5\nu_5}\begin{vmatrix} \hat\delta_{\mu_1\nu_1} & \dots & \hat\delta_{\mu_5\nu_1} \\ \vdots & & \vdots \\ \hat\delta_{\mu_1\nu_5} & \dots & \hat\delta_{\mu_5\nu_5} \end{vmatrix} \tag{5.78}$$

$$= D(D-1)(D-2)(D-3)(D-4) \tag{5.79}$$

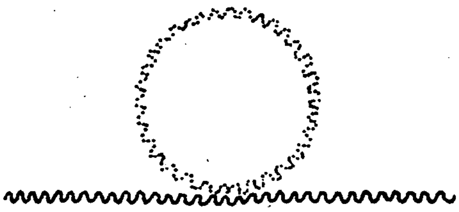
which again yields (5.76). This proof shows that the inconsistency is due to the projection itself, and not to the problems of defining ε_{abcd} in dimensions other than 4. This lends support to the view taken in chapter 4 that ε_{abcd} can be defined consistently in non-integer dimensions.
The ϵ scalar terms in the Lagrangian were worked out using only (5.63) and without using $\hat\delta_{\mu\nu}$ as a projection operator. Any calculation carried out using dimensional reduction will be consistent as long as the projection

$$\delta_{\mu\alpha}\hat\delta_{\alpha\nu} = \hat\delta_{\mu\nu} \tag{5.80}$$

is not required.

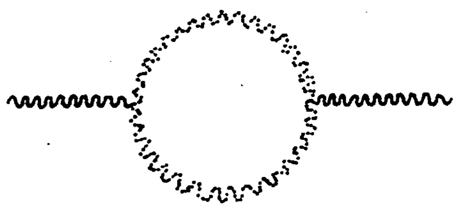
5.4 ϵ Scalar Contributions

With dimensional reduction, there will be an additional contribution to the gauge boson self-energy arising from the ϵ scalar loops.



$$= \frac{1}{2} i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} i g^2 \delta_{\mu\nu}(f_{ace}f_{bde} + f_{bce}f_{ade})\delta_{\sigma_1\sigma_2}\frac{i\delta_{\sigma_1\sigma_2}\delta cd}{k^2} \tag{5.81}$$

$$= -ig^2 f_{ace}f_{bce}\delta_{\mu\nu}\epsilon \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.82}$$



$$= \frac{1}{2} i \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} (-g f_{acd}(2k_\mu + p_\mu)\delta_{\sigma_1\sigma_3})(-g f_{bfe}(2k_\nu + p_\nu)\delta_{\sigma_4\sigma_2})$$

$$\times \frac{i\delta_{\sigma_1\sigma_2}\delta_{ce}}{(k+p)^2}\frac{i\delta_{\sigma_3\sigma_4}\delta_{df}}{k^2} \tag{5.83}$$

$$= \frac{1}{2} i g^2 f_{acd}f_{bcd}\epsilon \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{(2k_\mu + p_\mu)(2k_\nu + p_\nu)}{k^2(k+p)^2} \tag{5.84}$$

The quadratic part is

$$= ig^2 f_{acd}f_{bcd}\epsilon\delta_{\mu\nu}\frac{1}{\omega}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.85}$$

Combining (5.82) and (5.85) gives a contribution to the coefficient of the pure quadratic divergence of

$$= ig^2 \delta_{\mu\nu} f_{acd}f_{bcd}\left(1 - \frac{1}{\omega}\right)(-\epsilon) \tag{5.86}$$

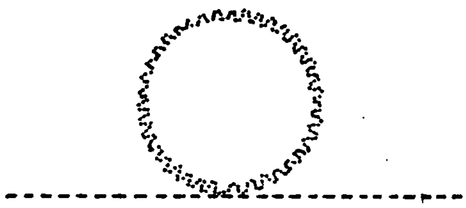
When added to (5.38) the total coefficient becomes

$$= ig^2\delta_{\mu\nu}\left(1-\frac{1}{\omega}\right)\left[-2f_{acd}f_{bcd}-\mathrm{Tr}[\Theta^a\Theta^b]+\frac{1}{2}\mathrm{Tr}[\mathbf{1}_D]\mathrm{Tr}[\mathrm{T^aT^b}]\right] \tag{5.87}$$

As long as it is assumed that inside the trace

$$\mathbf{1}_D = \mathbf{1}_\epsilon = \mathbf{1}_4 \tag{5.88}$$

then (5.87) can exactly vanish, even for $D \neq 4$, with appropriate choices of representations.
The additional contribution to the scalar self-energy is from



$$= \frac{1}{2}i\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(-ig^2)((\Theta^c\Theta^d)^{ba}+(\Theta^d\Theta^c)^{ba})\delta_{\sigma_1\sigma_2}\frac{i\delta_{\sigma_1\sigma_2}\delta cd}{k^2} \tag{5.89}$$

$$= ig^2\epsilon(\Theta^c\Theta^c)^{ba}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2} \tag{5.90}$$

Adding this to (5.14) gives the total coefficient of the quadratic divergence in the scalar self-energy, which now becomes

$$-\frac{1}{2}i\mathrm{Tr}[\mathbf{1}_D](\mathrm{G^a})^{\mathrm{ij}}\mathrm{i}(\mathrm{G^b})^{\mathrm{ji}}+\mathrm{i}\lambda_{\mathrm{abcc}}+3\mathrm{ig}^2(\Theta^\mathrm{c}\Theta^\mathrm{c})^{\mathrm{ba}} \tag{5.91}$$

Again this is now independent of ϵ.
As was mentioned in section 2.2, if the standard integral representation of the β- function $\beta(x,y)$ is assumed throughout for all values of x and y, then the 'pure' quadratic divergences are ignored, or effectively set to zero, i.e. $\int d^{2\omega}k\,(k^2)^{-1} = 0$.
However, a function that diverges quadratically in 4 dimensions will also have a pole at $D = 4 - 2/L$ for an L-loop diagram. Veltman [16] used this connection to identify one-loop quadratic divergences by considering $D = 2$ poles, and effectively suggested that dimensional reduction should be used for this identification. More recently, this idea has been investigated extensively by Jack, Jones and Al-Sarhi [17, 18, 19].

Veltman showed that imposing a cancellation of the quadratic divergences in the self-energy of the standard model Higgs boson leads to a relation between Higgs, top and Z-boson masses. In [17], though, it was demonstrated that the corresponding two-loop condition is not compatible with the Veltman condition. In fact, the much more general result was obtained that if the one-loop condition in a general renormalisable gauge theory is renormalisation scale invariant, then the two-loop condition follows automatically provided one further condition is met. It was pointed out in [19] that this condition could be obtained by cancelling the quadratic divergences in the ϵ-scalar self-energy. This condition also corresponds to the vanishing of (5.87), the quadratic divergence in the gauge boson self-energy.

The cancellation of quadratic divergences in the scalar self-energy can also be called a 'naturalness' condition. The concept of naturalness is related to the heirarchy problem in grand unified theories. In such models, the need to maintain two vastly different scales of symmetry breaking, the electroweak scale and the grand unification scale, requires extreme fine tuning of the model parameters, which must be adjusted at each successive order of perturbation theory.

In supersymmetric theories, quadratic divergences are completely absent (or at any rate, completely cancel) and although an initial fine tuning of vacuum expectation values is required in supersymmetric grand unified theories, no further adjustment is necessary.

By reformulating dimensional regularisation another difficulty has been identified here. Quadratic divergences appear in the gauge boson self-energy, as well as in the scalar self-energy. Consequently the regularised gauge boson self-energy does not satisfy the Ward identity (3.52), and manifest gauge invariance is lost.

If manifest gauge invariance is not obtained, in general, with dimensional regularisation (and Pauli-Villars regularisation is difficult to extend to non-abelian theories) the question arises as to whether it is possible with any regulator. This is not an anomaly as discussed in section 4.3, since the Ward identity can still be imposed on the renormalised function, but it does appear to indicate a shortcoming in the current formulation of (non-supersymmetric) gauge theories.

It would seem that manifest gauge invariance requires supersymmetry.

Chapter 6

Unified Gauge Theories

6.1 The Standard Model Couplings at M_Z

The couplings corresponding to the gauge group of the Standard Model, $SU(3) \times SU(2)_L \times U(1)_Y$ are usually denoted by g_s, g and g'. The symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_Q$; effectively at the scale of the Z^0, M_Z. The residual symmetry, the little group $U(1)_Q$, is the gauge group of QED, with associated coupling e. The couplings g and g' are related to e by the electroweak mixing angle $\sin^2\theta$. There are different definitions of $\sin^2\theta$ corresponding to different possible renormalisation schemes. With the couplings defined in the $\overline{\text{MS}}$ scheme

$$g'^2 = \frac{e^2}{\cos^2\theta_{\overline{\text{MS}}}}, \quad g^2 = \frac{e^2}{\sin^2\theta_{\overline{\text{MS}}}}. \tag{6.1}$$

Another possible definition, denoted by $\sin^2\theta_W$, is given in terms of the W and Z masses

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}. \tag{6.2}$$

This corresponds to the 'on-mass-shell' scheme. At lowest order $M_W = \frac{1}{2}vg'$, $M_Z = \frac{1}{2}v\sqrt{g'^2 + g^2}$ where v is the vacuum expectation value of the Higgs, and so $\sin^2\theta_W = \sin^2\theta_{\overline{\text{MS}}}$. However, this is no longer the case if higher order effects are included.
The parameter $\sin^2\theta_W$ can also be determined using the Sirlin relation [20]

$$\sin^2\theta_W = \frac{\pi\alpha}{\sqrt{2}G_F M_W^2(1-\Delta r)}, \tag{6.3}$$

where α is the Thompson charge, $\alpha = e^2(m_e)/4\pi$ which has the value

$$\alpha = 1/137.0359895(61), \tag{6.4}$$

G_F is the Fermi constant, given by

$$G_F = 1.166389(22) \times 10^{-5}\text{GeV}^{-2}, \tag{6.5}$$

and Δr is a radiative correction term, which contains all the higher order effects. Δr can be calculated in terms of the two unknowns M_{top} and M_H, the top mass and the Higgs mass, assuming no dependence on any non-Standard Model particles. Using a value of $\sin^2\theta_W$ determined from measurements of M_W and M_Z, this can then be used to constrain M_{top} (the dependence on M_H is weak).

There is a similar expression for $\sin^2\theta_{\overline{\text{MS}}}$ defined at the scale M_Z [21], namely

$$\sin^2\theta_{\overline{\text{MS}}}(M_Z) = \frac{\pi\alpha}{\sqrt{2}G_F M_W^2(1-\Delta\hat{r}_W)}, \tag{6.6}$$

but now also

$$\sin^2\theta_{\overline{\text{MS}}}(M_Z)\cos^2\theta_{\overline{\text{MS}}}(M_Z) = \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta\hat{r})}, \tag{6.7}$$

with $\Delta\hat{r} \neq \Delta\hat{r}_W$. In addition

$$\alpha_{\overline{\text{MS}}}(M_Z) = \frac{\alpha}{1-\Delta\hat{r}_W}. \tag{6.8}$$

$\Delta\hat{r}_W$ and $\Delta\hat{r}$ also depend on M_{top} and M_H and can be related to Δr. Similarly, $\sin^2\theta_{\overline{\text{MS}}}$ can be determined from $\sin^2\theta_W$.

The value of $\sin^2\theta_W$ has been measured in deep inelastic neutrino scattering. From the CDHS and CHARM collaborations, [22]

$$\sin^2\theta_W = 0.2300 \pm 0.0064, \tag{6.9}$$

and from measurements of the ratio M_W/M_Z at the UA1, UA2 and CDF detectors, [23]

$$\sin^2\theta_W = 0.2275 \pm 0.0052. \tag{6.10}$$

Combining these two values gives

$$\sin^2\theta_W = 0.2285 \pm 0.0040. \tag{6.11}$$

	$M_H = 51$ GeV	$M_H = 1000$ GeV	average
$\sin^2\theta_{\overline{\rm MS}}(M_Z)$	0.2326 ± 0.0008	0.2334 ± 0.0010	0.2331 ± 0.0013
$\Delta\hat{r}_W \times 10^{-2}$	6.82 ± 0.16	7.15 ± 0.09	6.96 ± 0.29
M_{top} GeV	112 ± 35	144 ± 32	127 ± 50

Table 6.1: Values of $\sin^2\theta_{\overline{\rm MS}}(M_Z)$, $\Delta\hat{r}_W \times 10^{-2}$ and M_{top} for $M_H = 51$ Gev, 1000 GeV and the average of the two.

This value can be used, along with the tables in ref.[21], to determine a value for $\sin^2\theta_{\overline{\rm MS}}(M_Z)$ and corresponding values of $\Delta\hat{r}_W$ and M_{top} for different values of M_H.

The results for $M_H = 51$ GeV and $M_H = 1000$ GeV are shown in Table (6.1). This represents the largest possible range of M_H. From direct observation $M_H \geq 51$ GeV and conservatively the upper limit can be taken to be $M_H \leq 1000$ GeV. Also shown in the table are the averages of the two sets. The average value of $\sin^2\theta_{\overline{\rm MS}}(M_Z)$ is

$$\sin^2\theta_{\overline{\rm MS}}(M_Z) = 0.2331 \pm 0.0013. \tag{6.12}$$

From the corresponding value of $\Delta\hat{r}_W = (6.96 \pm 0.29) \times 10^{-2}$, and eqn.(14)

$$\alpha_{\overline{\rm MS}}(M_Z) = \frac{1}{127.5 \pm 0.4}. \tag{6.13}$$

For the strong coupling α_s $(= g_s^2/4\pi)$ the average of the results obtained by the four LEP experiments, as reported by Hebbeker is used[24]:

$$\alpha_s(M_Z) = 0.120 \pm 0.007. \tag{6.14}$$

Generally $\alpha = g^2/4\pi$, but for the coupling of $U(1)_Y$

$$\alpha_1 = \alpha_Y = \frac{5}{3}\frac{g'^2}{4\pi} \tag{6.15}$$

where the factor of 5/3 is to standardise the normalisation of the $U(1)_Y$ generator to be the same as that of the other generators (*i.e.* we rescale g'). This is necessary if the standard model gauge group is to be embedded in a larger group. So with

$$\begin{aligned} \alpha_2 &= \alpha_L = g^2/4\pi \\ \alpha_3 &= \alpha_s = g_s^2/4\pi \end{aligned} \tag{6.16}$$

it follows that

$$\begin{aligned}\alpha_1(M_Z) &= 0.017045 \pm 0.000036 \\ \alpha_2(M_Z) &= 0.03365 \pm 0.00022 \\ \alpha_3(M_Z) &= 0.120 \pm 0.007.\end{aligned} \tag{6.17}$$

With the couplings at M_Z determined to this degree of accuracy, the couplings can be evolved to large scales to see if they are consistent with the unified gauge models.

6.2 Threshold Effects in SU(5)

In Recent years there has been renewed interest in experimental tests of Grand Unified Theories (GUT's), stemming from the more accurate determinations of the Standard Model couplings that have been achieved at LEP and elsewhere. Speculation has centred on the SU(5) model of Georgi and Glashow [25, 26]. Amaldi *et al.* [27] (see also refs.[28, 29]) have shown that the minimal SU(5) model is inconsistent with the values of the couplings measured at M_Z, $\alpha_i(M_Z)$, but that if supersymmetry is introduced at an intermediate scale, then supersymmetric SU(5) *is* consistent and also satisfies the lower bound on the proton lifetime.
In a unified model, the symmetry of the grand unifying group is broken at some scale M_X. Above this scale there is a single coupling, below it the different couplings evolve separately. So the theory predicts that the couplings, if evolved up from M_Z should meet at a single point. Amaldi *et al.* showed that this does not happen in minimal SU(5) but that if supersymmetry is included above $M_{SUSY} \approx 1$ TeV then there is a single unification point.
This is consistent with theoretical expectations of the SUSY breaking scale. Supersymmetric Grand Unified Theories (if softly broken) can be made almost entirely free of fine tuning problems. This is related to the cancellation of quadratic divergences which occurs in these models. This property only holds as long as $M_W^2 \approx \alpha M_{SUSY}^2$ and so $M_{SUSY} \approx 1$ TeV, in agreement with the value suggested by the data.
The running of the couplings, *i.e.* their variation with respect to the renormalisation scale, is determined by their β-functions. Up to second order,

$$\mu\frac{\partial \alpha_i}{\partial \mu} = \frac{b_i}{2\pi}\alpha_i^2 + \sum_j \frac{b_{ij}}{8\pi^2}\alpha_i^2\alpha_j + \dots \quad . \tag{6.18}$$

This has the approximate solution

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(\mu_\prime)} + \frac{b_i}{2\pi}\ln\left(\frac{\mu}{\mu_\prime}\right) + \sum_j \frac{b_{ij}}{4\pi b_j}\ln\left(\frac{\alpha_j(\mu_\prime)}{\alpha_j(\mu)}\right) \tag{6.19}$$

The coefficients b_i, b_{ij} can be calculated perturbatively in a given renormalisation scheme. The $\overline{\mathrm{MS}}$ scheme will be assumed throughout. For the Standard Model, the coefficients are [30]

$$b_i = \begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix} + N_{fam}\begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + N_{Higgs}\begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix} \tag{6.20}$$

$$b_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -136/3 & 0 \\ 0 & 0 & -102 \end{pmatrix} + N_{fam}\begin{pmatrix} 19/15 & 3/5 & 44/15 \\ 1/5 & 49/3 & 4 \\ 11/30 & 3/2 & 76/3 \end{pmatrix} \tag{6.21}$$
$$+N_{Higgs}\begin{pmatrix} 9/50 & 9/10 & 0 \\ 3/10 & 13/6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first order solution to the renormalisation group equation is

$$\frac{1}{\alpha_i(\mu')} = \frac{b_i}{2\pi}\ln\left(\frac{\mu'}{\mu}\right) + \frac{1}{\alpha_i(\mu)}. \tag{6.22}$$

Using this equation with $N_{fam} = 3$ and $N_{Higgs} = 1$, Amaldi *et al.* [27] found that α_3 misses the meeting point of α_1 and α_2 by more than 7 standard deviations. A value of $\alpha_3(M_Z) = 0.07$ is required to force a single meeting point.

As it stands, there can only be limited confidence in a first order result. This uncertainty can easily be removed by carrying out a second order analysis. However, a full two-loop treatment should include threshold effects around the unification scale, where the superheavy particles may have a significant effect. In the $\overline{\mathrm{MS}}$ renormalisation scheme this is achieved by 'matching functions' that relate the three effective couplings $\alpha_i(\mu)$ to the unified coupling $\alpha_G(\mu)$ at any scale close to the unification scale. To estimate these effects α_1 and α_2 are used to define the GUT, and then the implied values of M_X and $\alpha_3(M_Z)$ computed.

In the minimal SU(5) model there are $5^2 - 1 = 24$ gauge bosons. Of these, 12 correspond to the gauge bosons of the standard model. The remaining 12 are the superheavy X and Y vector bosons responsible for proton decay.

There is one fundamental representation (5 dimensional) scalar H, and one adjoint representation (24 dimensional) scalar Φ. Of the 24 components of Φ, 12 are 'eaten' by the X and Y gauge bosons to give them their mass M_X. The remaining 12 divide into an SU(3) octet Φ_8, an SU(2) triplet Φ_3 and a singlet Φ_0. The masses of Φ_8 and Φ_3 are automatically constrained so that

$$M^2_{\Phi_8} = \frac{1}{4} M^2_{\Phi_3}. \tag{6.23}$$

The H field splits into a colour triplet H_3 that becomes superheavy, and a doublet H_2 that becomes the usual Higgs of the Standard Model.
Including the effects of these particles, the approximate second order solution to the renormalisation group equation is [31, 32]

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_G(M_X)} + \frac{b_i}{2\pi} \ln\left(\frac{M_X}{\mu}\right) + \sum_j \frac{b_{ij}}{4\pi b_j} \ln(X_j) - 4\pi\lambda_i(M_X), \tag{6.24}$$

where

$$X_j = 1 + \frac{1}{2\pi} b_j \alpha_G^{(1)}(M_X^{(1)}) \ln\left(\frac{M_X^{(1)}}{\mu}\right). \tag{6.25}$$

X_j approximates $\alpha_G(M_X)/\alpha_j(\mu)$ ($\alpha_G^{(1)}$ and $M_X^{(1)}$ are the first order values). The $\lambda_i(M_X)$ are the matching functions. For the SU(5) model with the minimal scalar content discussed above, they are given in the $\overline{\mathrm{MS}}$ scheme by

$$\lambda_1(M_X) = \frac{1}{16\pi^2}\left(\frac{2}{15}\ln\left(\frac{M_H}{M_X}\right) + \frac{5}{3}\right) \tag{6.26}$$

$$\lambda_2(M_X) = \frac{1}{16\pi^2}\left(\frac{2}{3}\ln\left(\frac{M_S}{M_X}\right) + 1\right) \tag{6.27}$$

$$\lambda_3(M_X) = \frac{1}{16\pi^2}\left(\frac{1}{3}\ln\left(\frac{M_H}{M_X}\right) + \ln\left(\frac{\frac{1}{2}M_S}{M_X}\right) + \frac{2}{3}\right). \tag{6.28}$$

Here M_H is the mass of the triplet H_3, and M_S is the mass of the triplet Φ_3. In eqn.(6.28) $\frac{1}{2}M_S$ is the mass of the octet Φ_8 (see eqn.(6.23)).
The matching functions evaluated at M_X, $\lambda_i(M_X)$ only depend on the ratios M_H/M_X and M_S/M_X and not explicitly on M_X. Therefore, these have been taken as input parameters, varying in the range 10^{-3} — 10^3, and then the difference $1/\alpha_1 - 1/\alpha_2$ used to calculate M_X (2$^{\rm nd}$ order) in terms of these mass ratios. $M_{H,S}$ would not be expected to be very different from M_X, so $10^{-3} - 10^3$ is 'reasonable' – anything much outside this range would be hard

M_H/M_X	M_S/M_X	$M_X \times 10^{13}$ GeV	$\alpha_3(M_Z)$
LO	LO	0.7 ± 0.2	0.069 ± 0.002
1	1	0.7 ± 0.2	0.072 ± 0.002
10^3	10^3	0.6 ± 0.2	0.073 ± 0.002
10^3	10^{-3}	1.1 ± 0.3	0.073 ± 0.002
10^{-3}	10^3	0.5 ± 0.2	0.070 ± 0.002
10^{-3}	10^{-3}	1.0 ± 0.3	0.070 ± 0.002

Table 6.2: predicted values of M_X and $\alpha_3(M_Z)$

to motivate or understand. Only the case $N_{fam} = 3$, $N_{Higgs} = 1$ is considered *i.e.* three generations of fermions and one Higgs doublet .
The results obtained at the corners of the parameter space for the scale M_X and for $\alpha_3(M_Z)$ are shown in Table (6.2). The value of $\alpha_3(M_Z)$ is strongly correlated to M_H/M_X whereas M_X depends mainly on M_S/M_X. The most favourable case is $M_H/M_X = 10^3$, $M_S/M_X = 10^{-3}$. Even this does not predict a value of $\alpha_3(M_Z)$ that comes near to the measured value of 0.120 ± 0.007. This is illustrated in fig.(6.1), where the measured $\alpha_3(M_Z)$ and its evolution are indicated. The proton lifetime τ_p depends on $(M_X)^4$. For this case it is $\sim 8 \times 10^{24}$ years, with the mass splittings contributing only a factor of 6, as compared to the case when the masses are all equal. This is incompatible with the experimental lower limit on τ_p, which is $\tau_p \geq 1.9 \times 10^{32}$ years [33] (assuming a $p \rightarrow e^+\pi^0$ branching ratio of 35% [34]). As stated above, the range of input values of the mass ratios was so chosen because the masses M_H, M_S, and M_X are all assumed to be of comparable magnitude, so that they can be considered as part of the same threshold, and not split into two or three separate thresholds. There is a natural upper limit to the size of any GUT masses set by the Planck mass. In fact the upper limits on these masses are actually much lower, because if either M_H/M_X or M_S/M_X are greater than ~ 10 the model is pushed towards the strongly interacting regime, where perturbation theory does not apply. However, even with the limits relaxed to M_{Planck} and M_Z, there is still no hope of obtaining results consistent with the measured value of $\alpha_s(M_Z)$ and the lower limit on τ_p.
So it must be concluded that even with superheavy threshold effects, the values of the Standard Model coupling constants are inconsistent with the minimal SU(5) model, by at least 6 standard deviations.

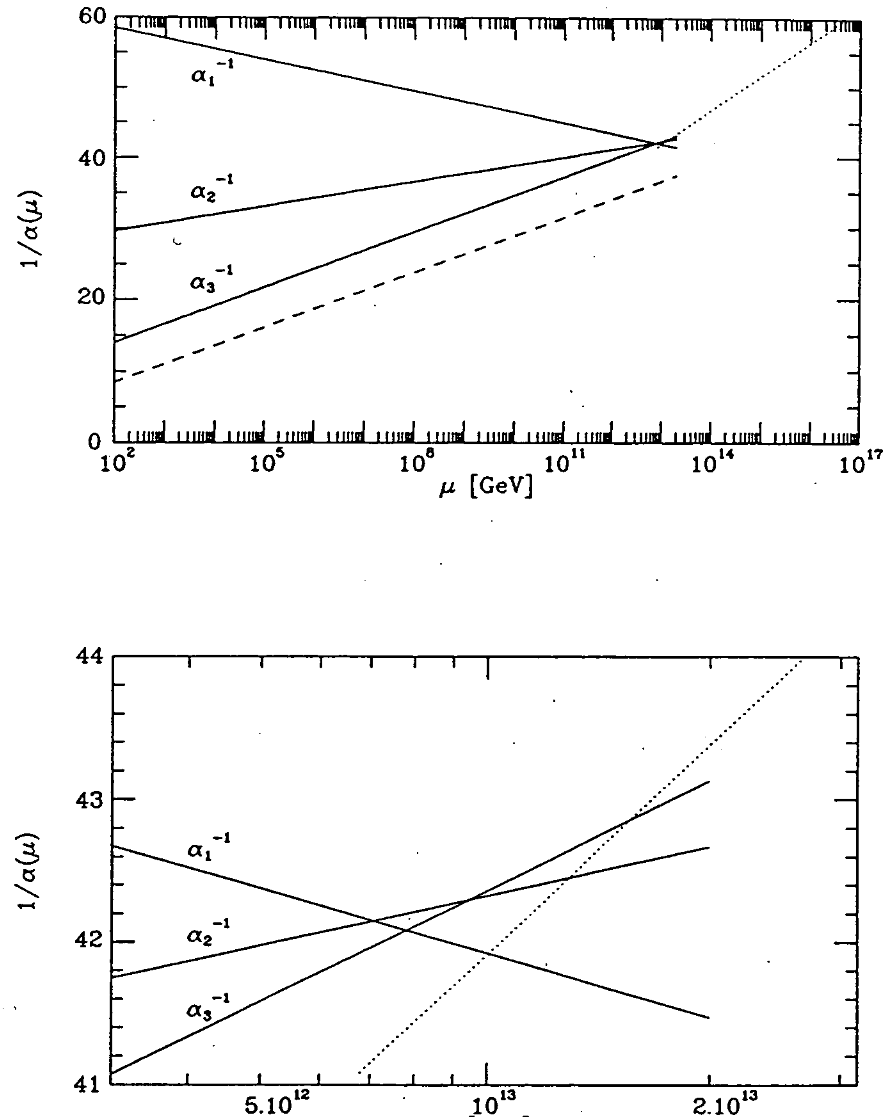


Figure 6.1: (a) Evolution of the couplings for the case $M_H/M_X = 10^3$, $M_S/M_X = 10^{-3}$. The solid lines show the evolution of the couplings that is required for unification, for the given values of $\alpha_1(M_Z)$ and $\alpha_2(M_Z)$. the dotted line is the evolution of the corresponding grand unified coupling above M_X. The dashed line represents the evolution of the experimental value of $\alpha_3(M_Z)$. (b) Enlargement of (a) showing the behaviour of the couplings in the threshold region.

6.3 SO(10) model

It is also possible to embed the Standard Model in a minimal left-right symmetric model [35, 36, 37]. The left-right symmetry breaking is characterised by a scale M_R, and accordingly the running of the couplings is modified above M_R. This model can in turn be embedded in SO(10) grand unification, if the couplings evolve to a single unification point [38].
The symmetry group of the left-right model would be

$$SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow SU(3) \times SU(2)_L \times U(1)_Y \quad (6.29)$$

and so there is a symmetry breaking

$$SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y \quad (6.30)$$

which is analogous to

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_Q. \quad (6.31)$$

This then implies the introduction of a right-handed neutrino for each generation. These three neutrino states would have a mass $\approx M_R$, the scale of left-right symmetry breaking. In the left-right symmetric model the fermion content of one generation is simply

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} u \\ d \end{pmatrix}_R \begin{pmatrix} e \\ \nu \end{pmatrix}_L \begin{pmatrix} e \\ \nu \end{pmatrix}_R . \quad (6.32)$$

The large mass differences between left- and right-handed neutrinos can be accommodated in a natural way through the see-saw mechanism.
There will also be three new gauge bosons associated with the $SU(2)_R$ - an extra W^+,W^- and Z that couple to the right handed fermions only. These too will have a mass $\approx M_R$.
The restoration of parity invariance at some higher energy is an appealing idea in itself, especially as it can be achieved within a relatively simple model with only a few additional particle states. However, it is tempting to suppose that this is only a staging post on the way to a grand unified theory.
The possible chain of symmetry breaking is

$$\begin{aligned} SO(10) &\rightarrow SU(4) \times SU(2)_L \times SU(2)_R \\ &\rightarrow SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \\ &\rightarrow SU(3) \times SU(2)_L \times U(1)_Y \\ &\rightarrow SU(3) \times U(1)_Q. \end{aligned} \quad (6.33)$$

The only other possible chain involves the already mentioned SU(5) model.

$$SO(10) \rightarrow SU(5) \times U(1). \tag{6.34}$$

For simplicity, we will consider the SO(10) model with $M_{X'} = M_X$ *i.e.* with SO(10) breaking directly to the minimal left-right symmetric extension of the standard model. Then there is only one scale M_R between M_Z and M_X. Grand unification requires a single gauge coupling above M_X. If threshold effects around M_X are neglected, this implies that the couplings must be coincident at M_X. The couplings at M_Z are now known to a much greater degree of accuracy from LEP measurements. Hence it is possible to evolve the couplings from M_Z, and see if there is an M_R for which the couplings *do* meet at a single unification point M_X. Furthermore, as the lifetime of the proton depends on M_X, the value of M_X must be high enough to be consistent with the lower bound on the proton lifetime. For the decay channel $p \rightarrow e^+ + \pi^0$ the limit on the partial lifetime is $\tau_p/B \geq 5.5 \times 10^{32}$ years (90% c.l.) [33].
Three scalar fields are needed, corresponding to the three scales of symmetry breaking. A **210** dimensional scalar Φ_{210} for the first stage at M_X, where SO(10) breaks directly to the minimal L-R model. (If this happens in two stages as in eqn.(5), then a **45** and a **54** can be used respectively.) Then a five-index antisymmetric Φ_{126} is needed, which contains two triplets Δ_L and Δ_R of $SU(2)_L$ and $SU(2)_R$. Δ_R is responsible for the left-right symmetry breaking at M_R. A Φ_{10} contains two doublets, one of which is the usual Higgs doublet of the electroweak model.
For the Standard Model, *i.e.* for $M_Z \leq \mu \leq M_R$, the β-function coefficients b_i,b_{ij} were given in section 6.2.
For $M_R \leq \mu \leq M_X$ the couplings $\alpha_1 = \alpha_{BL}$ and $\alpha_2 = \alpha_R = \alpha_L$. The generator of $U(1)_{BL}$ must be correctly normalised within the grand unified model

$$\frac{1}{\alpha_Y(M_R)} = \frac{3}{5}\frac{1}{\alpha_R(M_R)} + \frac{2}{5}\frac{1}{\alpha_{BL}(M_R)}. \tag{6.35}$$

The minimal left-right symmetric model has 2 extra scalar triplets, an extra doublet, 3 extra (right-handed) neutrinos and 3 extra gauge bosons. The β-function coefficients are, taking $g_R = g_L$ [36]

$$b_i = \begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix} + N_{fam} \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} \tag{6.36}$$

$$+N_H \begin{pmatrix} 0 \\ 1/3 \\ 0 \end{pmatrix} + N_\Delta \begin{pmatrix} 3 \\ 2/3 \\ 0 \end{pmatrix} \tag{6.37}$$

$$b_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -136/3 & 0 \\ 0 & 0 & -102 \end{pmatrix} + N_{fam} \begin{pmatrix} 7/6 & 3 & 4/3 \\ 1/2 & 49/3 & 4 \\ 1/6 & 3 & 76/3 \end{pmatrix} \tag{6.38}$$
$$+N_H \begin{pmatrix} 0 & 0 & 0 \\ 0 & 22/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + N_\Delta \begin{pmatrix} 54 & 72 & 0 \\ 12 & 56/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where N_H is the number of pairs of scalar doublets and N_Δ is the number of pairs of triplets. It is assumed that $N_H = 1$ and $N_\Delta = 1$.
Evolving the couplings from M_Z it turns out that a single unification point is obtained for $M_R = 10^{10.44\pm0.33}$ GeV. The unification scale is $M_X = 10^{15.20\pm0.25}$ GeV. (see figs (6.2) - (6.5)). It might be supposed that the fact that such a solution exists is rather unremarkable. Given that the evolved couplings of the minimal Standard Model do not meet at a single point, it might be thought that the introduction of a further parameter, in this case M_R, is bound to lead to a single unification point. However, this is not neccessarily the case. It could happen, for example, that at M_R the β-functions change in such a way that the couplings never meet. Even if unification can be achieved, there are additional constraints on the possible values of both the unification scale M_X and the intermediate scale M_R. M_X is constrained on the one hand by the lower limit on the proton lifetime $\tau_p \geq 5.5 \times 10^{31}$ years, corresponding to $M_X \geq 10^{15}$ GeV, and on the other hand by requiring M_X to be less than the Planck mass M_P. So 10^{15} GeV $\leq M_X \leq 10^{19}$ GeV. The scale M_R must obviously be in the range $M_Z \leq M_R \leq M_X$.
The fact that a physically acceptable solution *does* exist is noteworthy in itself, when compared to minimal SU(5) [27] for which this is not the case. For the SO(10) model considered here, the proton lifetime is given by [34]

$$\tau_p = 2.757 \times \left(\frac{M_X}{10^{15}}\right) \times 10^{31} \text{ years.} \tag{6.39}$$

For $M_X = 10^{15.20\pm0.25}$ GeV this implies that

$$\tau_p = 10^{32.25\pm1.0} \text{ years.} \tag{6.40}$$

This can be compared with the experimental lower bound of $\tau_p \geq 1.7 \times 10^{32}$ years [33], assuming a branching ratio for the decay channel $p \to e^+ + \pi^0$ in SO(10) of 30% [34] *i.e.*

$$\tau_p \geq 10^{32.23} \text{ years.} \tag{6.41}$$

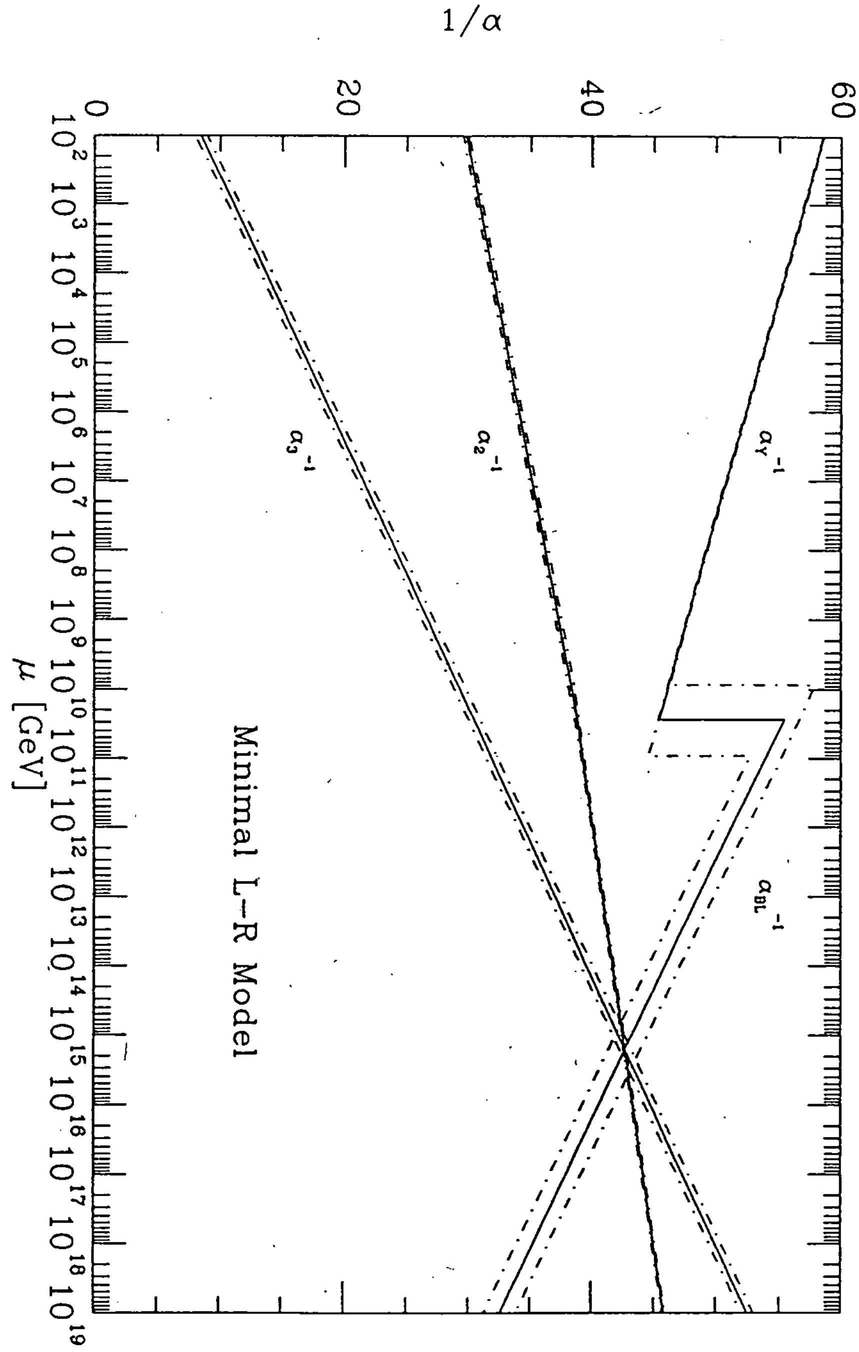


Figure 6.2: Coupling unification in the minimal left-right symmetric model described in the text. The dot-dash bands correspond to the experimental errors on the couplings at M_Z, as given in eqn.(6.17).

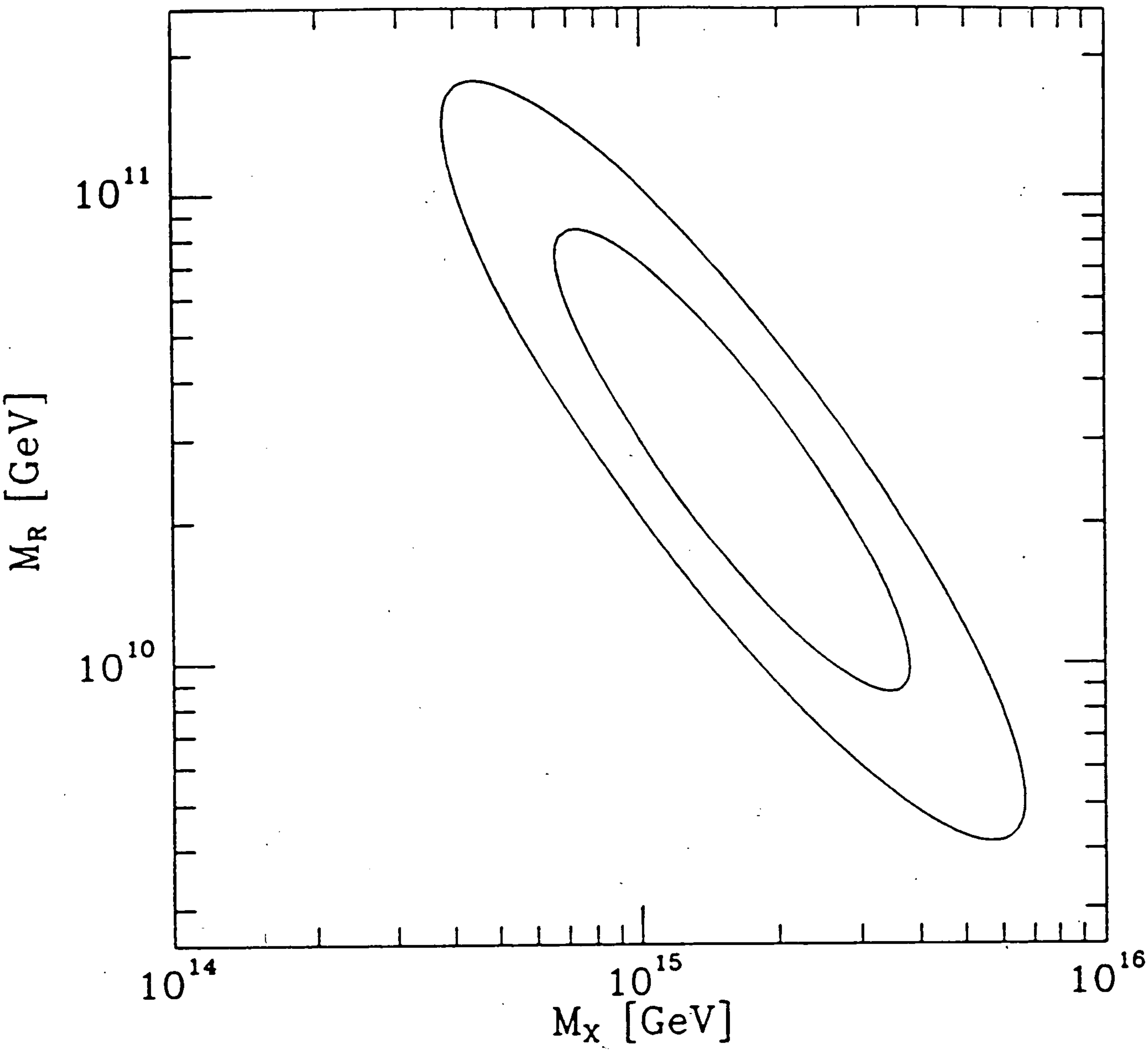


Figure 6.3: Contours corresponding to 68% and 95% confidence levels in the M_R - M_X plane.

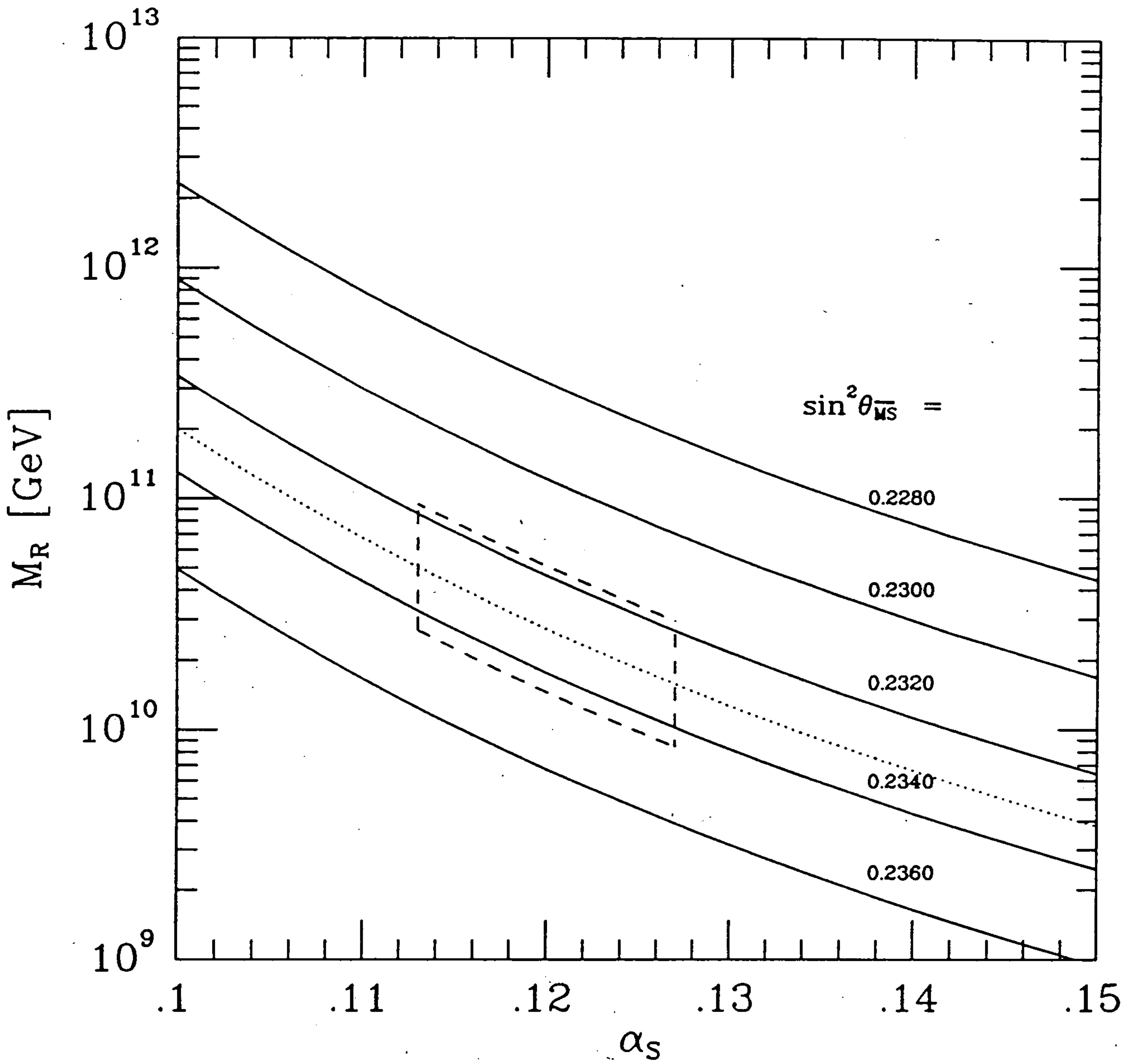


Figure 6.4: Dependence of M_R on $\alpha_s(M_Z)$, for different $\sin^2\theta_{\overline{MS}}(M_Z)$. The dotted line corresponds to the measured value of 0.2331 given in eqn.(6.12), and the dashed box to the experimental errors on these parameters given in eqns.(6.12,6.17).

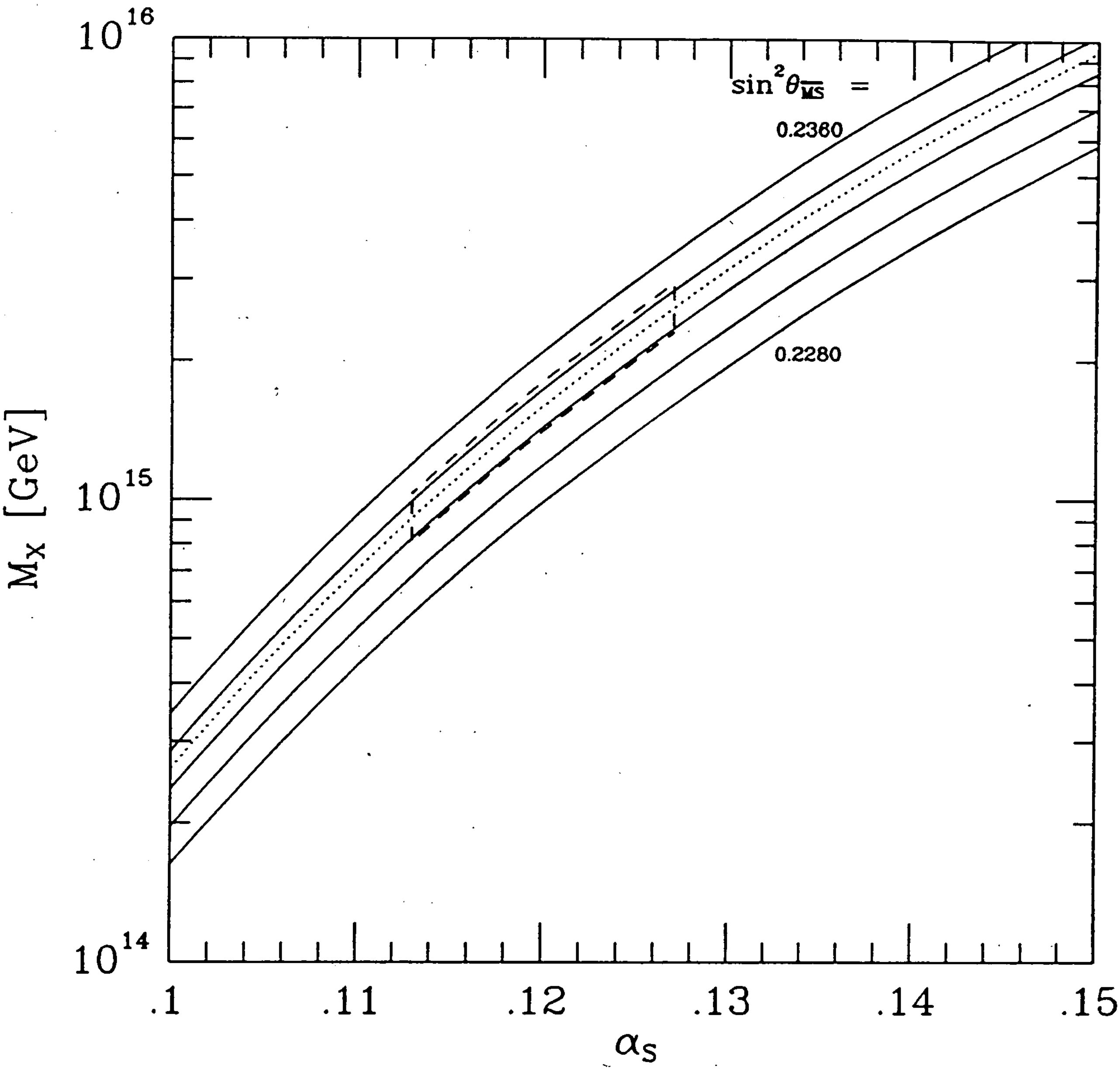


Figure 6.5: Dependence of M_X (SO(10) model) on $\alpha_s(M_Z)$, for different $\sin^2\theta_{\overline{\rm MS}}(M_Z)$. The dotted line corresponds to the measured value of 0.2331 given in eqn.(6.12), and the dashed box to the experimental errors on these parameters given in eqns.(6.12,6.17).

6.4 Supersymmetry

If supersymmetry is restored at an intermediate stage rather than left-right symmetry, *i.e.* in mimimal SUSY SU(5), the same starting values as used above imply that $M_Z \leq M_{SUSY} \leq 10^{3.9}$ GeV and $10^{15.7}$ GeV $\leq M_X \leq 10^{16.4}$ GeV (see figs.(6.6),(6.7)) and consequently that

$$\tau_p = 10^{34\pm1.5} \text{ years,} \tag{6.42}$$

a longer lifetime than in the minimal L-R/SO(10) case and safely clear of the lower bound. The errors in the two lifetimes are comparable. In contrast to the large value of M_R, M_{SUSY} is quite small. This has caused much excitement, since it is potentially within the sights of planned experiments (at LHC), and also because it is exactly the range required for the SUSY breaking scale to allow naturally soft SUSY breaking, *i.e.* to remove most of the fine tuning problems associated with GUTS.

However, the error on M_{SUSY} is somewhat larger than that for M_R. Moreover, the assumption is made that all the new (light) supersymmetric particle states have a common mass M_{SUSY}. If a more realistic mass spectrum is adopted then, in general, predictive power is lost. One exception to this is if new assumptions are made which are associated with Supergravity/Superstring derived GUTs. This has recently been studied by Ross and Roberts [29], who find that this results in an increase by a factor of 3 - 10 in the effective SUSY scale, as compared to the value $M_{SUSY} = 10^{3.0\pm1.0}$ GeV reported by Amaldi *et al.* [27].

In contrast to the SUSY SU(5) model, the problems associated with the Minimal L-R/SO(10) model are not so much to do with the range of intermediate masses, but the fact that this pattern of symmetry breaking is not unique within SO(10). This is ameliorated somewhat by the work of Buccella *et al.* [34] who carried out a first order analysis of SO(10) models with different intermediate symmetries (assuming only one intermediate scale). They found that the other models imply lower values of M_X than the minimal L-R model.

The value of M_X determined (to 2^{nd} order) in the minimal L-R model leads to a proton lifetime of $\tau_p = 10^{32.25\pm1.0}$ years, which is already rather close to the lower bound. This raises the possibility that as the lower bound rises, it may be possible to rule out a large class of non-supersymmetric SO(10) models.

However, there is a problem affecting SO(10) models that will confound any such attempts, namely threshold effects. Because of the high dimensionality of the scalar representations of SO(10) which are needed to carry out the required symmetry breaking, threshold effects at the GUT scale can be quite large, causing an additional uncertainty in the proton lifetime of up to several orders of magnitude (see for example Dixit and Sher [39]). Therefore SO(10) models are not likely to be within reach of experiment in the near future. In addition to this, the work of earlier chapters shows the importance of constructing models that are free from quadratic divergences, and further implies that this is only possible for a restricted class of models i.e. supersymmetric gauge theories.

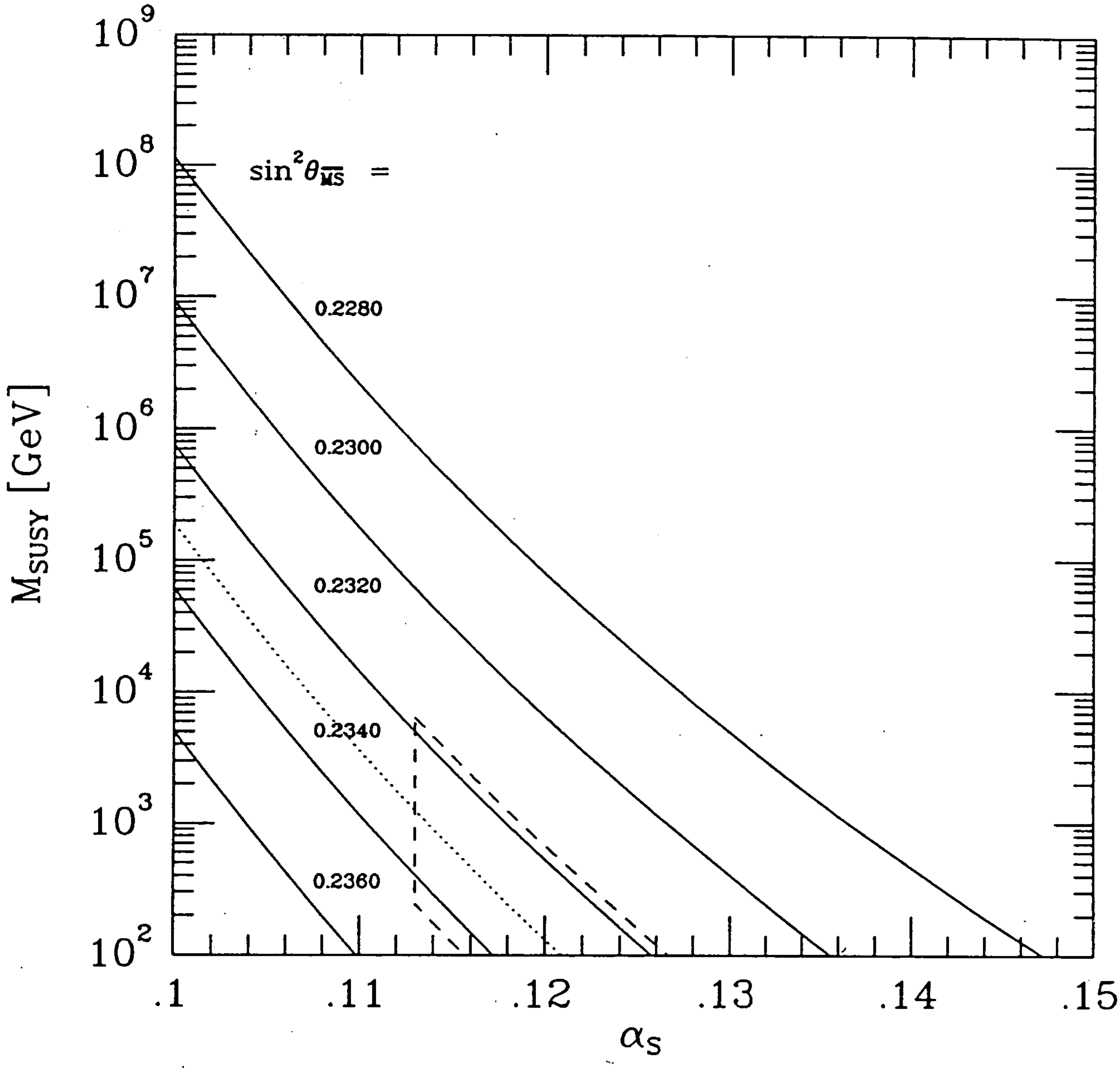


Figure 6.6: Dependence of M_{SUSY} on $\alpha_s(M_Z)$, for different $\sin^2\theta_{\overline{\rm MS}}(M_Z)$. The dotted line corresponds to the measured value of 0.2331 given in eqn.(6.12), and the dashed box to the experimental errors on these parameters given in eqns.(6.12,6.17).

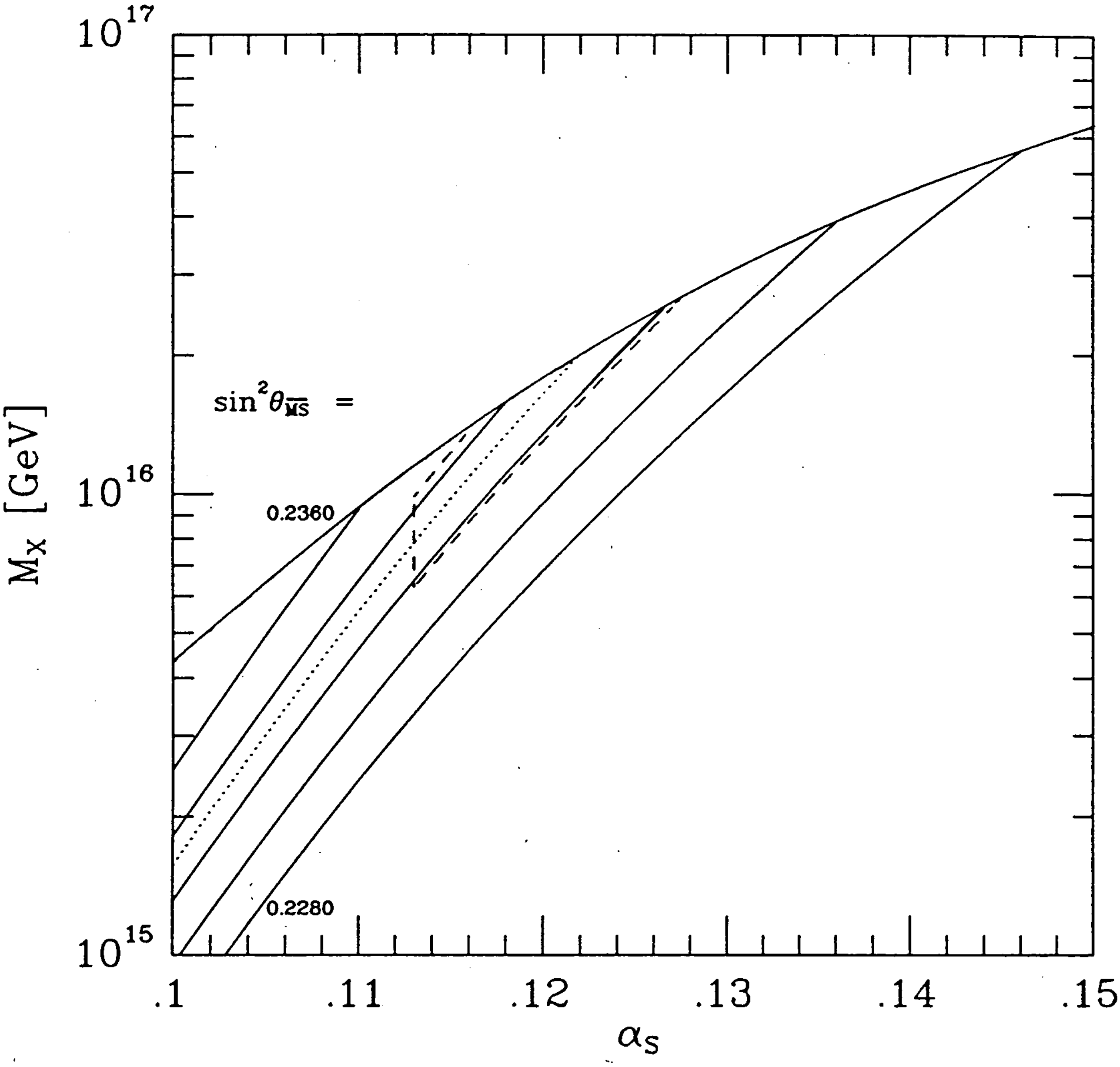


Figure 6.7: Dependence of M_X (SUSY SU(5)) on $\alpha_s(M_Z)$, for different $\sin^2\theta_{\overline{\rm MS}}(M_Z)$. The dotted line corresponds to the measured value of 0.2331 given in eqn.(6.12), and the dashed box to the experimental errors on these parameters given in eqns.(6.12,6.17).

Appendix A

The Gamma Function

The function $\Gamma(z)$ can be defined by one of the following three expressions [1, 2];

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt = \int_0^1 (\log 1/t)^{z-1}dt \tag{A.1}$$

Re $z > 0$

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)} \\ &= \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+z/2)\dots(1+z/n)} \\ &= z^{-1}\prod_{n=1}^{\infty}[(1+1/n)^z(1+z/n)^{-1}] \end{aligned} \tag{A.2}$$

$$1/\Gamma(z) = ze^{\gamma z}\prod_{n=1}^{\infty}[(1+z/n)e^{-z/n}] \tag{A.3}$$

where

$$\gamma = \lim_{m\to\infty}(\sum_{n=1}^{m} 1/n - \log m) = 0.5772156649\dots \tag{A.4}$$

denotes Euler's or Mascheroni's constant. The definition (A.1) was used by Euler, (A.2) (in a different notation) by Gauss, and (A.3) by Weierstrass. From (A.3) and (A.4) it can be seen that the gamma function is an analytic function of z, whose only finite singularities are at z = 0, -1, -2, -3,...
From (A.1) it follows that

$$\Gamma(z) = \int_0^1 e^{-t}t^{z-1}dt + \int_1^\infty e^{-t}t^{z-1}dt = P(z) + Q(z) \tag{A.5}$$

Q(z) is an integral function. Expanding e^{-t} in a power series and integrating term by term gives

$$P(z) = \sum_{n=0}^{\infty}(-1)^n[n!(z+n)]^{-1} \qquad \text{(A.6)}$$

Hence it follows that $(-1)^n/n!$ is the residue of $\Gamma(z)$ at the simple pole $z = -n$ $(n = 0, 1, 2, \ldots)$.
It can be shown that the expressions (A.1), (A.2) and (A.3) represent the same function.
For a positive integer n and Re$z > 0$ repeated integration by parts yields

$$\int_0^n (1-\frac{t}{n})t^{z-1}dt = \frac{n!n^z}{z(z+1)(z+2)\ldots(z+n)} \qquad \text{(A.7)}$$

so that (by Tannery's theorem)

$$\lim_{n\to\infty}\int_0^n (1-\frac{t}{n})t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt \qquad \text{(A.8)}$$

and therefore (A.1) is equivalent to (A.2).

$$\int_0^\infty e^{-t}t^{z-1}dt = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\ldots(z+n)} \qquad \text{(A.9)}$$

(A.3) can be deduced from (A.2) as follows. By (A.2)

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} z(1+z)(1+z/2)\ldots(1+z/n)e^{-z\log n} \qquad \text{(A.10)}$$

or

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \lim_{n\to\infty}[z(1+z)e^{-z}(1+z/2)e^{-z/2}\ldots(1+z/n)e^{-z/n}] \\ &\times e^{z(1+1/2+1/3\ldots+1/n-\log n)}\end{aligned} \qquad \text{(A.11)}$$

which implies that

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}[(1+z/n)e^{-z/n}] \qquad \text{(A.12)}$$

If the real part of z is negative, and $n+1 > \text{Re}\, z > n$ $(n = 0, 1, 2, \ldots)$, $\Gamma(z)$ can be represented by an integral due to Cauchy and Saalschutz [2, 3]

$$\Gamma(z) = \int_0^\infty \left[e^{-t} - \sum_{m=0}^{n}\frac{(-t)^m}{m!}\right]t^{z-1}dt \qquad \text{(A.13)}$$

$$-(n+1) < \Re\, z < -n$$

This is to be compared to (A.1)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{A.1}$$

$\Re z > 0$

It should be emphasized here that although $\Gamma(z)$ is an analytic function except at the points $z = 0, -1, -2, \ldots$ the same is not true of the integral functions on the RHS of (A.1) and (A.13). The expression

$$\int_0^\infty e^{-t} t^{z-1} dt \tag{A.14}$$

is an analytic function only in the domain $\Re z > 0$. Likewise

$$\int_0^\infty \left[e^{-t} - \sum_{m=0}^{n} \frac{(-t)^m}{m!} \right] t^{z-1} dt \tag{A.15}$$

is an analytic function in the domain $-(n+1) < \Re z < -n$.

Appendix B

The Beta Function

B.1 Standard Form

The beta function can be defined by the integral

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{B.1}$$

$\Re x > 0, \Re y > 0$

Substituting $t = v/(1+v)$, the relation

$$\beta(x,y) = \int_0^\infty v^{x-1}(1+v)^{-x-y}dv \tag{B.2}$$

$\Re x > 0, \Re y > 0$

is obtained, and from this

$$\beta(x,y) = \int_0^\infty (v^{x-1} + v^{y-1})(1+v)^{-x-y}dv \tag{B.3}$$

$\Re x > 0, \Re y > 0$

can be deduced. It follows that $\beta(x,y) = \beta(y,x)$.
$\beta(x,y)$ can be expressed in terms of gamma functions. From

$$\Gamma(z) = s^z \int_0^\infty e^{-st}t^{z-1}dt \tag{B.4}$$

$\Re z > 0$, s real and +ve

it follows that

$$\int_0^\infty e^{-(1+v)t}t^{x+y-1}dt = \frac{\Gamma(x+y)}{(1+v)^{x+y}} \tag{B.5}$$

$\Re (x+y) > 0$

Multiplying (B.5) by v^{x-1} and integrating with respect to v between 0 and ∞ gives

$$\int_0^\infty dv \int_0^\infty dt\, e^{-(1+v)t} t^{x+y-1} v^{x-1} = \int_0^\infty dv\, \Gamma(x+y) v^{x-1} (1+v)^{-x-y} \quad \text{(B.6)}$$

Interchanging the order of integration, the LHS becomes

$$\int_0^\infty dt\, t^{x+y-1} e^{-t} \int_0^\infty dv\, e^{-vt} v^{x-1} \quad \text{(B.7)}$$

which is equal to

$$\int_0^\infty dt\, t^{x+y-1} e^{-t} t^{-x} \Gamma(x) \quad \text{(B.8)}$$

$$\Re x > 0$$

and therefore

$$\int_0^\infty dt\, t^{y-1} e^{-t} \Gamma(x) = \Gamma(x)\Gamma(y) \quad \text{(B.9)}$$

$$\Re x > 0, \Re y > 0$$

so that

$$\int_0^\infty v^{x-1}(1+v)^{-x-y} dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{(B.10)}$$

$$\Re x > 0, \Re y > 0$$

or

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{(B.11)}$$

$$\Re x > 0, \Re y > 0$$

In principle the beta function is still only defined for $\Re x > 0, \Re y > 0$. However, as $\Gamma(z)$ is analytic except at $z = 0, -1, -2, \ldots$ (B.11) can be used to define the analytic continuation of $\beta(x,y)$ in the domain $\Re x < 0$ and in $\Re y < 0$. Alternatively (B.11) can be taken as the definiton of the beta function, and then its integral representations worked out accordingly. Either approach gives

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{(B.12)}$$

for all x,y

It is important to note that this does not mean that the integral representation of the beta function, the LHS of (B.10) can also be analytically continued. It is still the case that

$$\beta(x,y) = \int_0^\infty v^{x-1}(1+v)^{-x-y} dv \quad \text{(B.13)}$$

$$\Re x > 0, \Re y > 0$$

B.2 Non-Standard Representation

This can be demonstrated explicitly by considering the case $\Re(x+y) > 0, \Re y > 0, -(n+1) < \Re x < -n$ and returning to (B.5)

$$\int_0^\infty e^{-(1+v)t} t^{x+y-1} dt = \frac{\Gamma(x+y)}{(1+v)^{x+y}}$$

This time multiply both sides by $v^{x-1}F(v, x+y)$ where $F(v, x+y)$ is a function to be determined. Integrating with respect to v between 0 and ∞ gives

$$\int_0^\infty dv\, v^{x-1} F(v, x+y) \int_0^\infty dt\, e^{-(1+v)t} t^{x+y-1}$$
$$= \int_0^\infty dv\, v^{x-1} F(v, x+y)(1+v)^{-x-y}\Gamma(x+y) \qquad \text{(B.14)}$$

If the function $F(v, x+y)$ is taken such that it satisfies the equation

$$F(v, x+y) \int_0^\infty dt\, e^{-(1+v)t} t^{x+y-1} =$$
$$\int_0^\infty dt\, e^{-(1+v)t} t^{x+y-1} (1 - e^{vt} \sum_{m=0}^{n} (-vt)^m / m!) \qquad \text{(B.15)}$$

then the LHS of (B.14) becomes

$$\text{LHS} = \int_0^\infty dv\, v^{x-1} \int_0^\infty dt\, e^{-(1+v)t} t^{x+y-1} (1 - e^{vt} \sum_{m=0}^{n} (-vt)^m / m!) \qquad \text{(B.16)}$$

Interchanging the order of integration gives

$$\int_0^\infty dt\, e^{-t} t^y \int_0^\infty dv\, e^{-vt} (vt)^{x-1} (1 - e^{vt} \sum_{m=0}^{n} (-vt)^m / m!) \qquad \text{(B.17)}$$

$$= \int_0^\infty dt\, e^{-t} t^{y-1} \int_0^\infty dv\, e^{-v} (v)^{x-1} (1 - e^{v} \sum_{m=0}^{n} (-v)^m / m!) \qquad \text{(B.18)}$$

$$= \int_0^\infty dt\, e^{-t} t^{y-1} \Gamma(x) \qquad \text{(B.19)}$$

$$= \Gamma(y)\Gamma(x) \qquad \text{(B.20)}$$

$$\Re y > 0, -(n+1) < \Re x < -n$$

From (B.20) and (B.14)

$$\Gamma(x)\Gamma(y) = \Gamma(x+y) \int_0^\infty dv\, v^{x-1} (1+v)^{-x-y} F(v, x+y) \qquad \text{(B.21)}$$

$$\Re y > 0, -(n+1) < \Re x < -n$$

So that from (B.12)

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty dv\, v^{x-1}(1+v)^{-x-y}F(v,x+y) \tag{B.22}$$

$$\Re y > 0, -(n+1) < \Re x < -n$$

The function $F(v,x+y)$ is defined by (B.15)

$$F(v,x+y) = \frac{\int_0^\infty dt\, e^{-(1+v)t}t^{x+y-1}(1 - e^{vt}\sum_{m=0}^{n}(-vt)^m/m!)}{\int_0^\infty dt\, e^{-(1+v)t}t^{x+y-1}} \tag{B.23}$$

$$\Rightarrow F(v,x+y) = 1 - \frac{1}{\Gamma(x+y)}(1+v)^{x+y}\int_0^\infty dt\, t^{x+y-1}e^{-t}\sum_{m=0}^{n}(-vt)^m/m! \tag{B.24}$$

$$\Rightarrow F(v,x+y) = 1 - \frac{1}{\Gamma(x+y)}(1+v)^{x+y}\sum_{m=0}^{n}\Gamma(m+x+y)(-v)^m/m! \tag{B.25}$$

If in (B.22) the substitution $v = 1/t$ is made, it follows that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty dt\, t^{y-1}(1+t)^{-x-y}F(1/t,x+y) \tag{B.26}$$

$$\Re y > 0, -(n+1) < \Re x < -n$$

Appendix C

Integrals in D Dimensions

C.1 Basic Forms

From (1.18) and (1.22)

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+m^2)^n} = \frac{(m^2)^{\omega-n}}{(4\pi)^\omega} \frac{\Gamma(n-\omega)}{\Gamma(n)} \tag{C.1}$$

$$\text{for } \Re(n-\omega) > 0,\ \Re\omega > 0$$
$$\equiv 0 < \Re\omega < \Re n$$

From this other integrals follow.
In the region $0 < \Re\omega < \Re n$, the left hand side of (C.1) is well behaved, and translational invariance (1.16) can be applied. The translation

$$k \to k + p \tag{C.2}$$

implies that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+2p.k+m^2)^n} = \frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega} \frac{\Gamma(n-\omega)}{\Gamma(n)} \tag{C.3}$$

$$0 < \Re\omega < \Re n$$

Differentiating (C.3) with respect to p_μ, then gives

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k^2+2p.k+m^2)^n} = \frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega} \frac{\Gamma(n-\omega)}{\Gamma(n)}(-p_\mu) \tag{C.4}$$

$$0 < \Re\omega < \Re(n-1)$$

The differentiation has had the effect of reducing the size of the domain of the functional equality (C.4) as compared to that of (C.3) in the ω plane.

However, from the analytic properties of the RHS of (C.4) it is expected that (C.4) will be correct in the larger domain. This will be examined shortly. Differentiating with respect to p_μ again gives

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2 + 2p.k + m^2)^n}$$
$$= \frac{(m^2 - p^2)^{\omega - n}}{(4\pi)^\omega} \frac{\Gamma(n-1-\omega)}{\Gamma(n)}$$
$$\times \left(p_\mu p_\nu (n - \omega - 1) + \frac{1}{2}\delta_{\mu\nu}(m^2 - p^2)\right) \tag{C.5}$$

$$0 < \Re\omega < \Re(n-2)$$

For $p_\mu = 0$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2 + m^2)^n} = \frac{(m^2)^{\omega - n}}{(4\pi)^\omega} \frac{\Gamma(n-1-\omega)}{\Gamma(n)} \frac{1}{2}\delta_{\mu\nu} \tag{C.6}$$

$$0 < \Re\omega < \Re(n-2)$$

Again, differentiation has caused the domain to shrink in (C.5) and (C.6). From the form of the RHS of (C.6) it looks like the domain should be $0 < \Re\omega < \Re(n-1)$ rather than $0 < \Re\omega < \Re(n-2)$. If $k_\mu k_\nu \approx k^2$, then comparison with (C.1) supports this view. More exactly, since the LHS of (C.6) is of the form

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu k_\nu f(k^2) \tag{C.7}$$

The lorentz structure implies that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu k_\nu f(k^2) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\delta_{\mu\nu}}{2\omega} k^2 f(k^2) \tag{C.8}$$

and hence

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2 + m^2)^n} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\frac{\delta_{\mu\nu}}{2\omega} k^2}{(k^2 + m^2)^n} \tag{C.9}$$

Using the identity $k^2 = k^2 + m^2 - m^2$ in the numerator, this becomes

$$= \frac{\delta_{\mu\nu}}{2\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left(\frac{1}{(k^2 + m^2)^{n-1}} - \frac{m^2}{(k^2 + m^2)^n}\right) \tag{C.10}$$

then from (C.1)

$$= \frac{\delta_{\mu\nu}}{2\omega} \frac{(m^2)^{\omega - n + 1}}{(4\pi)^\omega} \left(\frac{\Gamma(n-1-\omega)}{\Gamma(n-1)} - \frac{\Gamma(n-\omega)}{\Gamma(n)}\right) \tag{C.11}$$

$$0 < \Re\omega < \Re(n-1)$$

$$= \frac{1}{2}\delta_{\mu\nu}\frac{(m^2)^{\omega-n+1}}{(4\pi)^\omega}\frac{\Gamma(n-1-\omega)}{\Gamma(n)} \tag{C.12}$$

$$0 < \Re\,\omega < \Re\,(n-1)$$

A proof can be given for integer n using exponential parametrisation. Starting with the LHS of (C.3)

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(k^2+2k.p+m^2)^n}$$

$$= \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp[-(k^2+2k.p+m^2)(x_1+\ldots x_n)]$$

$$= \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \frac{1}{(4\pi)^\omega}\frac{1}{(x_1+\ldots x_n)^\omega} \exp[-(p^2-m^2)(x_1+\ldots x_n)]$$

$$= \frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)} \tag{C.13}$$

$$0 < \Re\,\omega < n$$

Now multilpying by $(-p_\mu)$

$$(-p_\mu)\frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)}$$

$$= \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \frac{1}{(4\pi)^\omega}\frac{1}{(x_1+\ldots x_n)^\omega}(-p_\mu) \exp[-(p^2-m^2)(x_1+\ldots x_n)] \tag{C.14}$$

but (2.53) is

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu \exp(-xk^2 - 2k.px) = \frac{1}{(4\pi x)^\omega}(-p_\mu)\exp(xp^2) \tag{C.15}$$

and therefore

$$(-p_\mu)\frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)}$$

$$= \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu exp[-(k^2+2p.k+m^2)(x_1+\ldots x_n)] \tag{C.16}$$

$$\text{for } 0 < \Re\,\omega < n$$

Integrating over the x_i gives

$$(-p_\mu)\frac{(m^2-p^2)^{\omega-n}}{(4\pi)^\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{k_\mu}{(k^2+2p.k+m^2)^n} \tag{C.17}$$

$$\text{for } 0 < \Re\,\omega < n \text{ (n integer)}$$

C.2 Extended Forms

The technique of Feynman parametrisation is based on

$$\frac{1}{a^{\alpha}b^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} \tag{C.18}$$

$\Re\,\alpha > 0; \Re\,\beta > 0$

This expression is derived from the integral representation of the beta function (B.2)

$$\beta(\alpha,\beta) = \int_0^{\infty} v^{\alpha-1}(1+v)^{-\alpha-\beta}dv \tag{C.19}$$

$\Re\,\alpha > 0,\ \Re\,\beta > 0$

or, after $v \to 1/v$

$$\beta(\alpha,\beta) = \int_0^{\infty} v^{\beta-1}(1+v)^{-\alpha-\beta}dv \tag{C.20}$$

$\Re\,\alpha > 0,\ \Re\,\beta > 0$

letting $v = (b/a)u$ implies

$$\beta(\alpha,\beta) = a^{\alpha}b^{\beta} \int_0^{\infty} du u^{\beta-1}(a+bu)^{-\alpha-\beta} \tag{C.21}$$

$\Re\,\alpha > 0,\ \Re\,\beta > 0$

then $u = (1-x)/x$ gives

$$\beta(\alpha,\beta) = a^{\alpha}b^{\beta} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} \tag{C.22}$$

$\Re\,\alpha > 0,\ \Re\,\beta > 0$

Now integrals of the type

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[(k+p)^2]^m[k^2]^n} \tag{C.23}$$

can be evaluated. Using (C.18), (C.23) becomes

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{x^{m-1}(1-x)^{n-1}}{[(k+p)^2x+k^2(1-x)]^{m+n}} \tag{C.24}$$

for $\Re\,m > 0,\ \Re\,n > 0$

Interchanging the order of integration, and rearranging the denominator gives

$$\frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\int_0^1 dx x^{m-1}(1-x)^{n-1}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{[(k+xp)^2+p^2x(1-x)]^{m+n}} \tag{C.25}$$

for $\Re m>0$, $\Re n>0$

Provided $0<\Re\omega<\Re(m+n)$, the integral over k is well defined and after the translation $k\rightarrow k-xp$, is of the form (C.1)

$$\frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\int_0^1 dx x^{m-1}(1-x)^{n-1}\frac{(p^2x(1-x))^{\omega-(m+n)}}{(4\pi)^\omega}\frac{\Gamma(m+n-\omega)}{\Gamma(m+n)} \tag{C.26}$$

for $\Re m>0$, $\Re n>0$, $0<\Re\omega<\Re(m+n)$

$$=\frac{\Gamma(n+m-\omega)}{\Gamma(n)\Gamma(m)}\frac{(p^2)^{\omega-(m+n)}}{(4\pi)^\omega}\int_0^1 dx x^{\omega-n-1}(1-x)^{\omega-m-1} \tag{C.27}$$

for $\Re m>0$, $\Re n>0$, $0<\Re\omega<\Re(m+n)$

The integral over x is of the form (C.1), therefore provided $\Re(\omega-m)>0$, $\Re(\omega-n)>0$

$$\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{[(k+p)^2]^m[k^2]^n}=\frac{\Gamma(n+m-\omega)}{\Gamma(n)\Gamma(m)}\frac{(p^2)^{\omega-(m+n)}}{(4\pi)^\omega}\beta(\omega-n,\omega-m) \tag{C.28}$$

for $0<\Re m<\Re\omega$, $0<\Re n<\Re\omega$, $0<\Re\omega<\Re(m+n)$

For $\omega=2-\epsilon/2$ ($D=4-\epsilon$) the only integer values of m and n which (C.28) satisfies are $m=1, n=1$.

Using the same prescription as in (C.23) to (C.28), the following formulae can be obtained;

$$\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{k_\mu}{[(k+p)^2]^m[k^2]^n}$$

$$=\frac{\Gamma(n+m-\omega)}{\Gamma(n)\Gamma(m)}\frac{(p^2)^{\omega-(m+n)}}{(4\pi)^\omega}(-p_\mu)\beta(\omega-m,\omega-n+1) \tag{C.29}$$

for $0<\Re m<\Re\omega$, $0<\Re n<\Re(\omega+1)$, $0<\Re\omega<\Re(m+n)$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{[(k+p)^2]^m [k^2]^n}$$

$$= \frac{(p^2)^{\omega-(m+n)}}{(4\pi)^\omega \Gamma(n)\Gamma(m)}$$

$$\times(\frac{1}{2}\delta_{\mu\nu}p^2\Gamma(m+n-1-\omega)\beta(\omega-m+1,\omega-n+1)$$

$$+p_\mu p_\nu \Gamma(m+n-\omega)\beta(\omega-m,\omega-n+2) \qquad \text{(C.30)}$$

for $0 < \Re m < \Re\omega$, $0 < \Re n < \Re(\omega+1)$, $0 < \Re\omega < \Re(m+n-1)$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu k_\lambda}{[(k+p)^2]^m [k^2]^n}$$

$$= \frac{(p^2)^{\omega-(m+n)}}{(4\pi)^\omega \Gamma(n)\Gamma(m)}$$

$$\times(-\frac{1}{2}p^2(\delta_{\mu\nu}p_\lambda + \delta_{\mu\lambda}p_\nu + \delta_{\nu\lambda}p_\mu)\Gamma(m+n-1-\omega)\beta(\omega-m+1,\omega-n+2)$$

$$-p_\mu p_\nu p_\lambda \Gamma(m+n-\omega)\beta(\omega-m,\omega-n+3)) \qquad \text{(C.31)}$$

for $0 < \Re m < \Re\omega$, $0 < \Re n < \Re(\omega+2)$, $0 < \Re\omega < \Re(m+n-1)$

Appendix D

Translations

D.1 Logarithmic Divergence

Consider the logarithmically divergent integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^4} \tag{D.1}$$

Using the identity $(k+p-p)^2/k^2 = 1$ this equals

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{(k+p-p)^2}{(k+p)^4 k^2} \tag{D.2}$$

$$= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left(\frac{1}{(k+p)^2 k^2} - \frac{(p^2+2k.p)}{(k+p)^4 k^2} \right) \tag{D.3}$$

$$= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left(\frac{1}{k^4} - \frac{(p^2+2k.p)}{(k+p)^2 k^4} - \frac{(p^2+2k.p)}{(k+p)^4 k^2} \right) \tag{D.4}$$

$$(I_1 + I_2) \qquad\qquad (I_3 + I_4)$$

Using Feynman parametrisation

$$I_1 = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^2 k^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} 2 \int_0^1 dx \frac{(1-x)}{[k^2+p^2x(1-x)]^3} \tag{D.5}$$

$$I_3 = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^4 k^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} 2 \int_0^1 dx \frac{x}{[k^2+p^2x(1-x)]^3} \tag{D.6}$$

which implies that

$$I_1 + I_3 = \int_0^1 dx \frac{1}{(4\pi)^\omega}(p^2 x(1-x))^{\omega-3}\Gamma(3-\omega) \tag{D.7}$$

$$0 < \Re\omega < 3$$

after interchanging integrations and using (C.1).
Similarly

$$I_2 = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k+p)^2 k^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} 2\int_0^1 dx \frac{(1-x)(k_\mu - xp_\mu)}{[k^2 + p^2 x(1-x)]^3} \tag{D.8}$$

$$I_4 = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k+p)^4 k^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} 2\int_0^1 dx \frac{x(k_\mu - xp_\mu)}{[k^2 + p^2 x(1-x)]^3} \tag{D.9}$$

imply that

$$I_2 + I_4 = \int_0^1 dx \frac{1}{(4\pi)^\omega}(p^2 x(1-x))^{\omega-3}\Gamma(3-\omega)(-xp_\mu) \tag{D.10}$$

$$0 < \Re\omega < 3$$

Now

$$2p_\mu(I_2+I_4)+p^2(I_1+I_3) = \frac{1}{(4\pi)^\omega}(p^2)^{\omega-2}\Gamma(3-\omega)\int_0^1 dx(1-2x)x^{\omega-3}(1-x)^{\omega-3} \tag{D.11}$$

$$0 < \Re\omega < 3$$

The integral over x is antisymmetric and therefore vanishes, so

$$2p_\mu(I_2 + I_4) + p^2(I_1 + I_3) = 0 \tag{D.12}$$

and

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k)^4} \tag{D.13}$$

D.2 Linear Divergence

Consider the linearly divergent integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu + p_\mu}{(k+p)^4} \tag{D.14}$$

Using the same identity as before this is equal to

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{(k_\mu + p_\mu)(k+p-p)^2}{(k+p)^4 k^2} \tag{D.15}$$

$$= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{k_\mu}{(k)^4} - \underbrace{\frac{k_\mu(p^2+2k.p)}{(k+p)^2 k^4}}_{I_1} - \underbrace{\frac{k_\mu(p^2+2k.p)}{(k+p)^4 k^2}}_{I_2} \right.$$

$$\left. + \underbrace{\frac{p_\mu}{(k+p)^2 k^2}}_{I_3} - \underbrace{\frac{p_\mu(p^2+2k.p)}{(k+p)^4 k^2}}_{I_4} \right\} \tag{D.16}$$

Feynman parametrisation and translation give

$$I_1 + I_2 = -2 \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{2p.kk_\mu + p^2 p_\mu x(2x-1)}{[k^2 + p^2 x(1-x)]^3} \tag{D.17}$$

where (C.4) has been used to eliminate the linear terms, and it has been assumed that the integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{[k^2+m^2]^3} \tag{D.18}$$

is translationally invariant for $1 < \Re\omega < 2$. The integral I_4 is

$$I_4 = -2 \int_0^1 dx p^2 p_\mu x(1-2x) \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + p^2 x(1-x)]^3} \tag{D.19}$$

and so

$$I_1 + I_2 + I_4 = -2 \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{2p.kk_\mu}{[k^2 + p^2 x(1-x)]^3} \tag{D.20}$$

Using (C.6) with the domain taken to be $0 < \Re\omega < \Re(n-1)$ rather than $0 < \Re\omega < \Re(n-2)$ (i.e. (C.12)) this becomes

$$I_1 + I_2 + I_4 = -\frac{1}{(4\pi)^\omega}(p^2)^{\omega-2}(p_\mu)\Gamma(2-\omega) \int_0^1 dx (x(1-x))^{\omega-2} \tag{D.21}$$

The integral I_3 is

$$I_3 = p_\mu \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + p^2 x(1-x)]^2} \tag{D.22}$$

$$= \frac{1}{(4\pi)^{\omega}}(p^2)^{\omega-2}(p_\mu)\Gamma(2-\omega)\int_0^1 dx(x(1-x))^{\omega-2} \tag{D.23}$$

Therefore

$$I_3 + (I_1 + I_2 + I_4) = 0 \tag{D.24}$$

and

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu + p_\mu}{(k+p)^4} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k)^4} \tag{D.25}$$

D.3 Quadratic Divergence

For the quadratically divergent integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^2} \tag{D.26}$$

rearrangement gives

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{(k+p-p)^2}{(k+p)^2k^2} \tag{D.27}$$

$$= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{1}{(k)^2} - \frac{2k.p}{(k+p)^2k^2} - \frac{p^2}{(k+p)^2k^2} \right\} \tag{D.28}$$

From (C.28) and (C.29) paying attention to the limits

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{p^2}{(k+p)^2k^2} + \frac{2k.p}{(k+p)^2k^2} \right\}$$

$$= \frac{1}{(4\pi)^{\omega}}(p^2)^{\omega-1}\Gamma(2-\omega)\beta(\omega-1,\omega-1)$$

$$- \frac{1}{(4\pi)^{\omega}}2(p^2)^{\omega-1}\Gamma(2-\omega)\beta(\omega-1,\omega) \tag{D.29}$$

$$= 0$$

since $\beta(\omega-1,\omega) = \frac{1}{2}\beta(\omega-1,\omega-1)$. Therefore it follows that

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k+p)^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k)^2} \tag{D.30}$$

D.4 Cubic Divergence

The integrand of the cubically divergent integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu + p_\mu}{(k+p)^2} \tag{D.31}$$

can be rearranged to give

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left\{ \frac{k_\mu}{(k)^2} + \frac{p_\mu}{(k)^2} - \frac{2p.kk_\mu}{(k)^4} \right.$$

$$\left. + \underset{(I_1)}{\frac{4(p.k)^2 k_\mu}{(k+p)^2 k^4}} + \underset{(I_2)}{\frac{2p.kp^2 k_\mu}{(k+p)^2 k^4}} - \underset{(I_3)}{\frac{p^2 k_\mu}{(k+p)^2 k^2}} \right\} \tag{D.32}$$

where (D.30) has been used to obtain the second term. Assuming

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu k_\lambda}{[k^2+m^2]^3} = 0 \tag{D.33}$$

$$1 < \Re\omega < 2$$

it follows that

$$I_1 = 4 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^1 dx 2x(1-x) \frac{(-2p^2 p.kk_\mu - (p.k)^2 p_\mu - x^2 p^4 p_\mu)}{[k^2 + p^2 x(1-x)]^3} \tag{D.34}$$

$$I_2 = 2 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^1 dx 2(1-x) \frac{(p^2 p.kk_\mu + x^2 p^4 p_\mu)}{[k^2 + p^2 x(1-x)]^3} \tag{D.35}$$

$$I_3 = - \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^1 dx \frac{p^2(-xp_\mu)}{[k^2 + p^2 x(1-x)]^2} \tag{D.36}$$

Carrying out the 2ω dimensional integration

$$I_1 = - \int_0^1 dx \frac{1}{(4\pi)^\omega} x(1-x)$$

$$\times (\Gamma(2-\omega)(p^2 x(1-x))^{\omega-2} 6p^2 p_\mu + \Gamma(3-\omega)(p^2 x(1-x))^{\omega-3} 4x^2 p^4 p_\mu) \tag{D.37}$$

$$I_2 = - \int_0^1 dx \frac{1}{(4\pi)^\omega} (1-x)$$

$$\times (\Gamma(2-\omega)(p^2 x(1-x))^{\omega-2} p^2 p_\mu + \Gamma(3-\omega)(p^2 x(1-x))^{\omega-3} 2x^2 p^4 p_\mu) \tag{D.38}$$

$$I_3 = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(p^2)^{\omega-1}p_\mu\beta(\omega,\omega-1) \quad \text{(D.39)}$$

In (D.39) the x integration has also been done. Integrating over x in I_1 and I_2 gives

$$I_1 + I_2 = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(p^2)^{\omega-1}p_\mu$$

$$\times\{\beta(\omega+1,\omega-1)(6-4(2-\omega))$$

$$+\beta(\omega,\omega-1)(-6-1+2(2-\omega)) + \beta(\omega-1,\omega-1)\} \quad \text{(D.40)}$$

$$= \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(p^2)^{\omega-1}p_\mu(-\beta(\omega,\omega-1)) \quad \text{(D.41)}$$

which implies that

$$I_1 + I_2 + I_3 = 0 \quad \text{(D.42)}$$

and so

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{k_\mu + p_\mu}{(k+p)^2} =$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left\{\frac{k_\mu}{(k)^2} + \frac{p_\mu}{(k)^2} - \frac{2p.kk_\mu}{(k)^4}\right. \quad \text{(D.43)}$$

Bibliography

[1] Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc. (1964), ch. 6, p255.

[2] A. Erdelyi et al., Higher Transcendental Functions, Vol.1, McGraw-Hill Book Co., Inc. (1953), ch. 1, p1.

[3] Whittaker - Watson, (1927), p243.

[4] Kenneth G. Wilson, Phys. Rev. D7 no. 10 (1973) 2911.

[5] D. Bailin and A. Love, Introduction to Gauge Field Theory, Adam Hilger (1986).

[6] Lewis H. Ryder, Quantum Field Theory, Cambridge University Press (1985).

[7] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.

[8] George Leibbrandt, Rev. Mod. Phys. 47 no.4 (1975) 849.

[9] G. Leibbrandt and D. M. Capper, J. Math. Phys. 15 (1974) 82.

[10] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill (1964) p154.

[11] W. Pauli and F. Villars, Rev. Mod. Phys. 21 no. 3 (1949) 434.

[12] Warren Siegel, Phys. Lett. 84B no.2 (1979) 193.

[13] D. M. Capper, D. R. T. Jones and P. van Nieuwenhuizen, Nucl. Phys. B167 (1980) 479.

[14] Warren Siegel, Phys. Lett. 94B no.1 (1980) 37.

[15] L. V. Avdeev and A. A. Vladimirov, Nucl. Phys. B219 (1983) 262.

[16] M. Veltman, Acta Phys. Polon. B12 (1981) 437.

[17] I. Jack and D. R. T. Jones, Phys. Lett. B234 no. 3 (1990) 321.

[18] I. Jack and D. R. T. Jones, Nucl. Phys. B342 (1990) 127.

[19] M. S. Al-Sarhi, I. Jack and D. R. T. Jones, Nucl. Phys. B345 (1990) 431.

[20] A. Sirlin, Phys. Rev. D22 (1980) 971; W. J. Marciano and A. Sirlin, *ibid.* 22 (1980) 2695.

[21] G. Degrassi, S. Fanchiotti and A. Sirlin, Nucl. Phys. B351 (1991) 49.

[22] H. Plothow-Besch, CERN-PPE/91-206 (1991).

[23] J. Carter, plenary talk given at the joint Lepton-Photon Symposium and EPS Conference, Geneva (1991).

[24] T. Hebbeker, plenary talk given at the joint Lepton-Photon Symposium and EPS Conference, Geneva (1991).

[25] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32 (1974) 438.

[26] A. J. Buras, J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135 (1978) 66.

[27] U. Amaldi, W. de Boer, H. Furstenau, Phys. Lett. 260B (1991) 447.

[28] J. Ellis, S. Kelley and D. V. Nanopoulos, Phys. Lett. 249B (1990) 441, Phys. Lett. 260B (1991) 131, and CERN preprint CERN-TH.6140/91 (1991).
P. Langacker, talk given at the PASCOS '90 conference, Boston (1990), Penn. Univ. preprint UPR-0435T (1990).
P. Langacker and M. Luo, Phys. Rev. D44 (1991) 817.
F. Anselmo, L. Cifarelli, A. Peterman, A. Zichichi, preprint CERN-PPE-91-123 (1991).
A. Giveon, L. J. Hall and U. Sarid, Phys. Lett. 271B (1991) 138.
R. Barbieri and L. J. Hall, Phys. Rev. Lett. 68 (1992) 752.
P. Ramond, Florida Univ. preprint UFIFT-92-4 (1992).

[29] G. G. Ross and R. G. Roberts, RAL preprint RAL-92-005 (1992).

[30] M. B. Einhorn and D. R. T. Jones, Nucl. Phys. B196 (1982) 475.

[31] L. Hall, Nucl. Phys. B178 (1981) 75.

[32] S. Weinberg, Phys. Lett. 91B (1980) 51.

[33] IMB-3 Collaboration, R. Becker-Szendy *et al.*, Phys. Rev. D42 (1990) 2974, see also: Review of Particle Properties, Phys. Lett. 239B (1990).

[34] F. Buccella, G. Miele, L. Rosa, P. Santorellim and T. Tuzi, Phys. Lett. 233B (1989) 178.

[35] T. W. B. Kibble, G. Lazarides, and Q. Shafi, Phys. Rev. D26 (1982) 435.

[36] W. E. Caswell, J. Milutinovic and G. Senjanovic, Phys. Rev. D26 (1982) 161.

[37] T. G. Rizzo and G. Senjanovic, Phys. Rev. D24 (1981) 704; *ibid.* 25 (1982) 235.

[38] H. Murayama and T. Yanagida, Tohoku Univ. preprint TU-370 (1991).
T. G. Rizzo, Argonne Lab. preprint ANL-HEP-PR-91-92 (1991).
U. Amaldi, W. de Boer, P. H. Frampton, H. Furstenau and J. T. Liu, UNC preprint IFP-415-UNC (1991).

[39] V. V. Dixit and M. Sher, Phys. Rev. D40 (1989) 3765.