Extension maps and the moduli spaces of rank 2 vector bundles over an algebraic curve

Gronow, Michael Justin

Extension Maps and the Moduli Spaces of Rank 2 Vector Bundles over an Algebraic Curve

by

Michael Justin Gronow

A thesis presented for

the degree of Doctor of Philosophy

June 1997

Department of Mathematical Sciences,

University of Durham,

South Road, Durham,

DH1 3LE, England

Abstract.

Extension Maps and the Moduli Spaces of

Rank 2 Vector Bundles over an Algebraic Curve

Michael Justin Gronow

Let $\mathcal{SU}_C(2,\Lambda)$ be the moduli space of rank 2 vector bundles with determinant Λ on an algebraic curve C. This thesis investigates the properties of a rational map $\mathbb{P}U_{d,\Lambda} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$ where $\mathbb{P}U_{d,\Lambda}$ is a projective bundle of extensions over the Jacobian $J^d(C)$. In doing so the degree of the moduli space $\mathcal{SU}_C(2,\mathcal{O}_C)$ is calculated for non-hyperelliptic curves of genus four (3.4.2). Information about trisecants to the Kummer variety $\mathcal{K} \subset \mathcal{SU}_C(2,\mathcal{O}_C)$ is obtained in sections 4.3 and 4.4. These sections describe the varieties swept out by these trisecants in the fibres of $\mathbb{P}U_{1,\mathcal{O}_C} \to J^1(C)$ for curves of genus 3, 4 and 5. The fibres of ϵ_d over $E \in \mathcal{SU}_C(2,\Lambda)$ are then studied. For certain values of d these correspond to the family of maximal line subbundles of E. These are either zero or one dimensional and a complete description of when these families are smooth is given (5.4.9), (5.4.10). In the one dimensional case its genus is also calculated (if connected) (5.5.5). Finally a correspondence on the curve fibres is shown to exist (5.6.2) and its degree is calculated (5.6.5). This in turn gives some information about multisecants to projective curves (5.7.4), (5.7.7).

Preface.

Extension Maps and the Moduli Spaces of

Rank 2 Vector Bundles over an Algebraic Curve

Michael Justin Gronow

This work has been sponsored by the U.K. Engineering and Physical Sciences Research Council.

The thesis is based on research carried out between October 1993 and June 1997 under the supervision of Dr W.M.Oxbury. It has not been submitted for any other degree either at Durham or at any other University.

No claim of originality is made for the material presented in chapter 1 or chapter 2 apart from (2.2.8).

The material in chapters 3, 4 and 5 is original apart from the introductory remarks at the beginning of each chapter, sections 3.1, 3.3 and 4.1 which review the main results of [NR2], [BNR] and [OPP] that will be needed in this thesis and sections 4.2 and 5.1 which give a short account of elementary results on ruled surfaces and determinantal varieties respectively. It has also been necessary to include other authors results throughout these chapters and it is clearly stated when this occurs.

Acknowledgements.

First and foremost I would like to thank my supervisor Bill Oxbury for many helpful suggestions and conversations throughout my time at Durham.

I would also like to express my gratitude to my friends and colleagues in the Mathematics Department for making my time in Durham all the more enjoyable. Special thanks must go to Helen Fawley and Karen McGaul for entertaining lunchtimes and getting me addicted to cryptic crosswords, Michael Young and Mansour Aghasi for their technical advice and Alistair MacIntyre for his enviable quality of being able to cheer me up through the hard times.

Last but by no means least I would like to thank my parents for their support and encouragement. This thesis is dedicated to them.

Contents

CHAPTER 1

Introduction

This thesis studies the moduli spaces of rank 2 bundles with fixed determinant bundle Λ from the viewpoint of extensions or short exact sequences. In particular there exists a projective bundle of extensions $\mathbb{P}U_{d,\Lambda}$ over the Jacobian of degree d line bundles on C and a rational map to the moduli space:

$$\mathbb{P}U_{d,\Lambda} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda).$$

It is this map that will be the main object of study throughout and the thesis is structured in the following way:

The rest of this chapter gives an introduction to moduli spaces of vector bundles and reviews some well known results. The projective bundle of extensions and the map ϵ_d are then formally defined and in the last section the rational cohomology of $\mathcal{SU}_C(2,\Lambda)$ and the Verlinde formulae for rank 2 bundles is discussed.

Chapter 2 looks at ϵ_d restricted to a single fibre of $\mathbb{P}U_{d,\Lambda} \to J^d(C)$ and reviews work of Bertram [Be] and Thaddeus [Th] done in this area. The final part of the chapter shows how Thaddeus' "flips" construction is equivalent, in the genus 2 case, to a classical construction showing the rationality of the quadratic line complex (2.2.6), (2.2.7) and (2.2.8).

Chapters 3 and 4 study ϵ_d over the whole of $\mathbb{P}U_{d,\Lambda}$ but restricts to the special case

$d = 1, \Lambda = \mathcal{O}_C$. ϵ_1 then has the particularly nice property that it is linear and injective on the fibres of $\mathbb{P}U_{1,\mathcal{O}_C} \to J^1(C)$ with respect to a line bundle $\mathcal{L}$ on $\mathcal{SU}_C(2, \mathcal{O}_C)$ (2.0.4). This then allows one to calculate the degree of the image of ϵ_1 (3.2.9) (with respect to $\mathcal{L}$) and, in the case when $g = 4$, the degree of the moduli space itself (3.4.2). It has also been shown [OPP] that the trisecants to the Kummer variety $\mathcal{K} \subset \mathcal{SU}_C(2, \mathcal{O}_C)$ are also trisecants to the curve C in some fibre of $\mathbb{P}U_{1,\mathcal{O}_C} \to J^1(C)$. Chapter 4 describes the variety of trisecants in each of these fibres for curves of genus 3, 4 and 5.

The final chapter is concerned with the fibres of ϵ_d over a bundle $E \in \mathcal{SU}_C(2, \Lambda)$. For certain values of d these correspond to the maximal line subbundles of E and are either zero or one dimensional. (5.4.9) and (5.4.10) describes exactly when these fibres are smooth. In the smooth one dimensional case the degree of its canonical line bundle is calculated (5.5.3) (and hence its genus if the fibre is connected.) A correspondence is then shown to exist (5.6.2) on the curve fibres and its degree is calculated (5.6.5). It will then be shown how this can lead to information about certain multisecants to projective curves (5.7.4), (5.7.7). Finally a list of sufficient conditions (5.8.1) for the one dimensional fibres to be connected is given.

Unless otherwise stated the following notation will be used throughout:

C – a non-hyperelliptic compact Riemann surface of genus $g \geq 2$.

S^iC – the i-th symmetric power of C.

E, F – rank two holomorphic vector bundles on C.

Λ – determinant line bundle. In particular $\Lambda = \mathcal{O}_C$ or $\mathcal{O}_C(p)$ (for some $p \in C$) depending on the degree of the rank 2 bundle.

$J^d(C)$ or $\mathrm{Pic}^d C$ – abelian variety of line bundles on C with first Chern class d.

Θ – the theta divisor on $J^{g-1}(C)$.

K – the canonical line bundle on C.

$\mathcal{P}$ – a Poincaré line bundle of degree d on $C \times J^d(C)$.

$\mathcal{SU}_C(2,\Lambda)$ – the moduli space of rank two bundles with determinant Λ.

$\mathcal{K}$ – the Kummer variety of $\mathcal{SU}_C(2,\Lambda)$.

$\mathcal{L}$ – the ample generator of the Picard group of $\mathcal{SU}_C(2,\Lambda)$.

1.1. Moduli spaces of vector bundles

A holomorphic vector bundle E on C is said to be stable (resp. semistable) if for all proper subbundles $L \subset E$:

$$\frac{\deg L}{\mathrm{rank} L} < \frac{\deg E}{\mathrm{rank} E} \qquad \left(\text{resp. } \frac{\deg L}{\mathrm{rank} L} \leq \frac{\deg E}{\mathrm{rank} E}.\right).$$

Any semistable vector bundle E admits a filtration:

$$E = E_0 \supset E_1 \supset \cdots \supset E_r = 0$$

such that $\frac{E_{i-1}}{E_i}$ is a stable vector bundle and:

$$\frac{\deg E_{i-1}}{\mathrm{rank} E_{i-1}} = \frac{\deg E}{\mathrm{rank} E}$$

for all $i = 1, \ldots, r$. The graded bundle:

$$GrE \stackrel{\mathrm{def}}{=} \bigoplus_{i=1}^{r} \frac{E_{i-1}}{E_i}$$

is well-defined up to isomorphism and two semistable bundles are said to be S-equivalent if their graded bundles are isomorphic.

Clearly if E and E' are both *stable* then E is S-equivalent to E' if and only if they are isomorphic.

Restricting to this special class of vector bundles eliminates any *unstable* bundles having "bad" properties. In particular a Hausdorff moduli space of such bundles can be constructed.

Theorem 1.1.1. *There exists a normal, irreducible, projective variety $\mathcal{U}_C(r,n)$ parametrising the set of S-equivalence classes of semistable bundles of rank r and degree n on C. The dimension of $\mathcal{U}_C(r,n)$ is $r^2(g-1)+1$.*

For a proof of this theorem see [Mul].

Remark 1.1.2. $\mathcal{U}_C(r,n)$ is a *coarse* moduli space i.e. for all families $\mathcal{X} \to C \times S$ of vector bundles of rank r and degree n on C, where S is some algebraic variety, the map:

$$S \xrightarrow{f} \mathcal{U}_C(r,n)$$

$$s \longmapsto \mathcal{X}\Big|_{C\times\{s\}}$$

is a morphism of algebraic varieties.

Moreover if r and n are coprime then $\mathcal{U}_C(r,n)$ is a *fine* moduli space i.e. there is a family of vector bundles $\mathcal{U} \to C \times \mathcal{U}_C(r,n)$ such that for all families of vector bundles as above:

$$(1_C \times f)^*\mathcal{U} \cong \mathcal{X} \otimes q^*L$$

where $q : C \times S \to S$ is projection and L is a line bundle on S. In particular if S a point this says that:

$$\mathcal{U}\Big|_{C\times\{E\}} \cong E$$

for all $E \in \mathcal{U}_C(r,n)$.

Tensoring $E \in \mathcal{U}_C(r,n)$ by any line bundle of degree n' preserves semistability. This induces an isomorphism:

$$\mathcal{U}_C(r,n) \cong \mathcal{U}_C(r,n+rn').$$

Thus there exist r distinct moduli spaces $\mathcal{U}_C(r,n)$, $0 \leq n \leq r-1$.

The moduli spaces $\mathcal{U}_C(r,n)$ are locally factorial and admit Cartier divisors $\Theta_{r,n}$ [DN] such that the Picard group of $\mathcal{U}_C(r,n)$ is given by:

$$\mathrm{Pic}\,\mathcal{U}_C(r,n) \cong \mathrm{Pic}\,J^n(C) \oplus \mathbb{Z}\{\Theta_{r,n}\}.$$

More precisely the Cartier divisor constructed in [DN] is contained in the moduli space $\mathcal{U}_C(r,r(g-1))$ where it is shown to be supported on the closure of the set:

$$\{V \in \mathcal{U}_C^s(r,r(g-1)) \mid h^0(V) = h^1(V) \neq 0\}.$$

If $r \mid n$ then the divisor $\Theta_{r,n}$ is obtained by the pullback of $\Theta_{r,r(g-1)}$ via the map:

$$\mathcal{U}_C(r,n) \xrightarrow{\otimes M} \mathcal{U}_C(r,r(g-1))$$

where M is some line bundle on C of suitable degree. In fact [DN] generalises this to show the existence of a Cartier divisor $\Theta_{r,n}$ for the case $r \nmid n$. Of course $\Theta_{r,n}$ depends on the line bundle M but Drezet and Narasimham showed that its restriction to the fibre of the determinant map:

$$\mathcal{U}_C(r,n) \xrightarrow{\det} J^n(C)$$

is independent of M.
The remainder of this thesis will concentrate on the subvarieties $\mathcal{SU}_C(2,\Lambda) \subset \mathcal{U}_C(2,\Lambda)$ of rank 2 vector bundles having fixed determinant Λ i.e. the fibres of the determinant map.

The two moduli spaces in the rank two case will be denoted by:

$$\mathcal{SU}_C(2) \stackrel{\mathrm{def}}{=} \mathcal{SU}_C(2,\mathcal{O}_C)$$
$$\mathcal{SU}_C(2,1) \stackrel{\mathrm{def}}{=} \mathcal{SU}_C(2,\mathcal{O}_C(p))$$

for some $p \in C$. These moduli spaces have dimension $3g-3$.

Remark 1.1.3. The notation $\mathcal{SU}_C(r)$ is standard following a result of Narasimhan and Seshadri [NS] showing that the points of $\mathcal{SU}_C(r)$ are in one to one correspondence with isomorphism classes of representations of the fundamental group of C by the group $SU(r)$.

The line bundles associated to the restriction of the divisors $\Theta_{2,0}$ and $\Theta_{2,1}$ to $\mathcal{SU}_C(2)$ and $\mathcal{SU}_C(2,1)$ respectively will both be denoted by $\mathcal{L}$. The restricted divisors $\Theta_{\mathcal{L}}$ are called the generalised theta divisors of $\mathcal{SU}_C(2,\Lambda)$.

The moduli space $\mathcal{SU}_C(2,1)$ is smooth for all genus whereas for $g \geq 3$ the singular locus of $\mathcal{SU}_C(2)$ is precisely the semistable boundary or the Kummer variety $\mathcal{K} \subset \mathcal{SU}_C(2)$ [NR1, Thm 1] :

$$\begin{aligned} J^0 &\longrightarrow \mathcal{SU}_C(2) \\ M &\longmapsto M \oplus M^{-1}. \end{aligned}$$

For $g = 2$ $\mathcal{SU}_C(2)$ is smooth and isomorphic to $\mathbb{P}^3$ [NR1, Thm 2] . Let $\phi_{\mathcal{L}}$ be the natural map:

$$\mathcal{SU}_C(2) \longrightarrow \mathbb{P}H^0(\mathcal{SU}_C(2),\mathcal{L})^*.$$

The linear system $|\mathcal{L}|$ has no base points [R] so $\phi_{\mathcal{L}}$ is defined everywhere. If one defines a map $\mathcal{SU}_C(2) \xrightarrow{\phi} |2\Theta|$ by:

$$E \mapsto \{L \in J^{g-1}(C) \mid h^0(C, L\otimes E) \geq 1\}$$

then there is an isomorphism [B] :

$$H^0(\mathcal{SU}_C(2),\mathcal{L})^* \cong H^0(J^{g-1}(C), 2\Theta)$$

and the following diagram is commutative:

$$\begin{tikzcd} & \mathcal{SU}_C(2) \arrow[dl, "\phi"'] \arrow[dr, "\phi_{\mathcal{L}}"] & \\ {|2\Theta|} \arrow[rr, "\sim"] & & {|\mathcal{L}|^*} \end{tikzcd}$$

Hence the dimension of the linear system $|\mathcal{L}|$ is $2^g - 1$. The map ϕ restricts to the embedding:

$$J^0/\pm \overset{\phi}{\hookrightarrow} |2\Theta|$$

$$M \longmapsto \Theta_M + \Theta_{M^{-1}}$$

on the Kummer variety, where Θ_M is the translate of $\Theta \subset J^{g-1}(C)$ by M.

Note that Wirtinger duality gives the isomorphism:

$$H^0(J^{g-1}(C), 2\Theta) \cong H^0(J^0(C), 2\Theta_C)^*$$

where $2\Theta_C$ is in the linear equivalence class of $\Theta_{L^{-1}} + \Theta_{K^{-1}L}$ for some $L \in J^{g-1}(C)$ (the linear equivalence class being independent of L.) The corresponding isomorphism:

$$H^0(\mathcal{SU}_C(2), \mathcal{L}) \cong H^0(J^0(C), 2\Theta_C)$$

is also given by restricting the sections of $\mathcal{L}$, or non-abelian theta functions on $\mathcal{SU}_C(2)$, to the Kummer variety $J^0/\pm \hookrightarrow \mathcal{SU}_C(2)$ [B].

Remark 1.1.4. Tensoring by a fixed theta characteristic κ on C gives an isomorphism of $\mathcal{SU}_C(2)$ with $\mathcal{SU}_C(2, K)$. $\mathcal{SU}_C(2, K)$ maps into the dual space $|2\Theta|^* \cong |2\Theta_C|$ via:

$$E \longmapsto \{M \in J^0(C) \mid h^0(C, M \otimes E) \geq 1\}$$

and again restricts to the corresponding embedding on the Kummer variety J^{g-1}/i, where i corresponds to the involution sending L to KL^{-1}.

1.2. Stratification of the moduli space

Associated to any vector bundle E of rank 2 is its Segre invariant:

$$s(E) \overset{\text{def}}{=} c_1(E) - 2\max\, c_1(L)$$

where the maximum is taken over all line subbundles $L \subset E$.

By the definition of semistability of E it is clear that:

$$E \text{ is stable (resp. semistable)} \iff s(E) > 0 \quad (\text{resp. } s(E) \geq 0).$$

As a function on the moduli space of vector bundles of fixed degree the Segre invariant is lower semicontinuous and hence gives a stratification of $\mathcal{U}_C(2,n)$ into locally closed subsets. Define

$$\mathcal{U}_C^m(2,n) = \{E \in \mathcal{U}_C(2,n) \mid s(E) \leq m\}$$

then [LN, Prop 3.1.] says:

Proposition 1.2.1. *For $1 \leq m \leq g$ and $m \equiv n \pmod 2$, $\mathcal{U}_C^m(2,n)$ is an irreducible algebraic variety of dimension $3g+m-2$ (resp. $4g-3$) if $m \leq g-2$ (resp. $m = g-1$ or $m = g$).*

Since the Segre invariant is lower semicontinuous and $\dim \mathcal{U}_C(2,n) = 4g-3$ it follows from (1.2.1) that $s(E) \leq g$ for all rank 2 vector bundles on C.

1.3. Extension maps

A Poincaré line bundle of degree d for C is a line bundle on $C \times J^d(C)$ whose restriction to $C \times \{L\}$ is isomorphic to L for all $L \in J^d(C)$. For a fixed $p_0 \in C$ there exists a unique degree d Poincaré line bundle whose restriction to $\{p_0\} \times J^d(C)$ is trivial. Denote this line bundle by $\mathcal{P}$ and let $\sigma : C \times J^d(C) \to C$, $\pi : C \times J^d(C) \to J^d(C)$ be projections onto the first and second factors respectively. Consider the higher direct image:

$$U_{d,\Lambda} = R^1\pi_*(\mathcal{P}^{-2} \otimes \sigma^*\Lambda^{-1})$$

on $J^d(C)$. For $d \geq 1$ this is a vector bundle of rank $2d+n+g-1$ and the fibre of $\mathbb{P}U_{d,\Lambda}$ can be identified with $\mathbb{P}H^1(C, L^{-2}\Lambda^{-1})$ i.e. isomorphism classes of non-trivial extensions of the form:

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L\Lambda \longrightarrow 0 \tag{e}$$

A rational map:

$$\mathbb{P}U_{d,\Lambda} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$$

can then be defined by mapping this extension to the rank two bundle E. To be more precise, if $\nu : \mathbb{P}U_{d,\Lambda} \to \mathrm{Pic}^d C$ is projection then there exists [L] a universal extension on $C \times \mathbb{P}U_{d,\Lambda}$:

$$0 \longrightarrow (1_C \times \nu)^*\mathcal{P}^{-1} \otimes \mathcal{O}_{\mathbb{P}U_{d,\Lambda}}(1) \longrightarrow \mathcal{E} \longrightarrow (1_C \times \nu)^*(\mathcal{P}^{-1} \otimes \sigma^*\Lambda) \longrightarrow 0 \tag{1}$$

with the property that (1) restricted to $C \times \{(e)\} = (e)$ for all extensions $(e) \in \mathbb{P}U_{d,\Lambda}$. The universal property of the coarse moduli space $\mathcal{SU}_C(2,\Lambda)$ now implies that there exists a morphism:

$$\mathbb{P}U^o_{d,\Lambda} \longrightarrow \mathcal{SU}_C(2,\Lambda)$$

where $\mathbb{P}U^o_{d,\Lambda} \subset \mathbb{P}U_{d,\Lambda}$ is some open, dense subset.

1.4. Cohomology and the Verlinde formulae

There are two questions that naturally arise from studying the moduli spaces $\mathcal{SU}_C(2,\Lambda)$ and their embeddings:

1. What are the dimensions of the spaces of k-th order non-abelian theta functions $H^0(\mathcal{SU}_C(2,\Lambda),\mathcal{L}^k)$?

2. Can one describe the rational cohomology ring $H^*(\mathcal{SU}_C(2,\Lambda),\mathbb{Q})$?

If one restricts to the fine, odd degree moduli space $\mathcal{SU}_C(2,1)$ then the second question has been answered. In particular if one decomposes the second Chern class of End $\mathcal{U}$

(where $\mathcal{U}$ is the Poincaré bundle on $\mathcal{SU}_C(2,1) \times C$) into its Künneth components:

$$2\alpha \in H^2(\mathcal{SU}_C(2,1), \mathbb{Z})$$

$$4\psi_i \in H^3(\mathcal{SU}_C(2,1), \mathbb{Z})$$

$$-\beta \in H^4(\mathcal{SU}_C(2,1), \mathbb{Z})$$

then it has been shown [Ne2] that $H^*(\mathcal{SU}_C(2,1), \mathbb{Q})$ is generated over $\mathbb{Q}$ by α, β and $\{\psi_i\}$. Furthermore it is known that $c_1(\mathcal{L}) = \alpha$ and also what relations the generators of $H^*(\mathcal{SU}_C(2,1), \mathbb{Z})$ satisfy.

The dimension of $H^0(\mathcal{SU}_C(2,1), \mathcal{L}^k)$ may then be calculated using the Hirzebruch-Riemann-Roch formula:

$$\sum_i (-1)^i h^i(\mathcal{L}^k) = ch(\mathcal{L}^k).td(\mathcal{SU}_C(2,\Lambda))[\mathcal{SU}_C(2,\Lambda)] \tag{2}$$

The right hand side is now relatively easy to calculate and the higher cohomology on the left hand side vanishes by the Kodaira vanishing theorem (since $\mathcal{L}$ is ample and the canonical bundle of $\mathcal{SU}_C(2,1)$ is negative.) Thus (2) gives a formula for $h^0(\mathcal{SU}_C(2,1), \mathcal{L}^k)$ hence answering question **(1)**:

$$\dim H^0(\mathcal{SU}_C(2,1), \mathcal{L}^k) = (k+1)^{g-1} \sum_{i=1}^{2k+1} \frac{(-1)^{i+1}}{(\sin \frac{i\pi}{2k+2})^{2g-2}}.$$

A general formula (the Verlinde formula) for the dimension of $H^0(\mathcal{SU}_C(r,\Lambda), \mathcal{L}^k)$ was conjectured by mathematical physicists working on conformal field theory. It has subsequently been proved by numerous mathematicians including Thaddeus [Th] . In the rank 2 even degree case it simplifies to:

$$\dim H^0(\mathcal{SU}_C(2), \mathcal{L}^k) = \left(\frac{k}{2}+1\right)^{g-1} \sum_{i=1}^{k+1} \frac{1}{(\sin \frac{i\pi}{k+2})^{2g-2}}.$$

Putting $k=1$ in this formula gives the number 2^g as expected.

Calculating the rational cohomology and relations of any generators for the even degree moduli space seems somewhat harder. In particular it would be nice to know the degree of the moduli space $\mathcal{SU}_C(2)$ i.e. $c_1(\mathcal{L})^{3g-3}$. On the bright side there is a conjecture due to Witten and Szenes [Sz, Conj 4.2] giving a formula for the intersection numbers of moduli spaces of principal G-bundles over a curve C. Taking $G = SL_2$ this formula can be simplified to give:

Conjecture 1.4.1. *The degree of $\mathcal{SU}_C(2) \subset |2\Theta|$ is given by:*

$$deg\mathcal{SU}_C(2) = (3g-3)!\, 2^{g-1} \left| Res_{z=0} \left(\frac{\cot z}{(2z)^{2g-2}} dz \right) \right| .$$

Calculating this residue gives the following degrees:

Genus	2	3	4	5	6
Degree	1	4	96	6336	873600

agreeing with the degree of $\mathcal{SU}_C(2)$ for $g = 2$ and 3 [NR1], [NR2]. In Chapter 3 a new geometric proof of the degree of $\mathcal{SU}_C(2)$ in the genus 4 case will be given, again agreeing with the calculation above.

CHAPTER 2

The fibrewise extension map

Let the fibre of the extension bundle $\mathbb{P}U_{d,\Lambda} \to J^d(C)$ (identified with $\mathbb{P}H^1(C, L^{-2}\Lambda^{-1})$) be denoted by $\mathbb{P}_L$ for $L \in J^d(C)$. C maps naturally into $\mathbb{P}_L$ via the linear system $|KL^2\Lambda|$:

$$C \to \mathbb{P}H^0(KL^2\Lambda)^* \cong \mathbb{P}_L.$$

This chapter will look at the extension map ϵ_d restricted to the fibres $\mathbb{P}_L$. This approach was taken by Newstead [Ne3] , Bertram [Be] and Thaddeus [Th] to prove various results concerning $\mathcal{SU}_C(r, \Lambda)$. The following gives a review of their work and the final part of the chapter shows how Thaddeus' flips construction reduces in genus 2 case to a classical construction showing the rationality of the quadratic line complex.

If $n = \deg \Lambda$ then by Riemann-Roch:

$$\mathbb{P}_L \cong \mathbb{P}^{g+2d+n-2}.$$

Thus if $\deg L^2\Lambda = 2g-1$ then $\mathbb{P}_L$ and $\mathcal{SU}_C(2,\Lambda)$ have the same dimension. Newstead [Ne3] , [Ne4] showed that if $\deg \Lambda$ is odd then the map $\mathbb{P}_L \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$ is a birational morphism. The fact that $\mathcal{SU}_C(2,\Lambda)$ was rational for $\deg \Lambda$ odd was known earlier than [Ne3] and follows from a result of Tjurin [Tj]. When $\deg \Lambda$ is *even* it is not generally known whether $\mathcal{SU}_C(2,\Lambda)$ is rational or not, although for $g = 2$ Narasimhan and

Ramanan [NR1] have shown that:

$$\mathcal{SU}_C(2) \xrightarrow{\phi} \mathbb{P}H^0(J^1(C), 2\Theta)$$

is an isomorphism.

Remark 2.0.2. The rational map ϵ_d fails to be defined when an extension of $\mathbb{P}U_{d,\Lambda}$ corresponds to an unstable bundle. To see when this occurs suppose $\Lambda = \mathcal{O}_C$, $\overline{D} \subset \mathbb{P}H^1(L^{-2}) \cong \mathbb{P}H^0(KL^2)^*$ is the span of some divisor D and $(e) \in \mathbb{P}H^1(C, L^{-2})$ is the extension:

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Then $(e) \in \overline{D}$ if and only if $(e) \in \mathbb{P}(\ker \pi_{\overline{D}})$ where $\pi_{\overline{D}}$ is given by:

$$\pi_{\overline{D}} : H^0(KL^2)^* \longrightarrow H^0(KL^2(-D))^*.$$

Thus the image extension:

$$0 \longrightarrow L^{-1} \longrightarrow F \longrightarrow L(-D) \longrightarrow 0$$

splits. One checks that F is given by the fibred product $E \times_L L(-D)$ so that there exists a non-zero homomorphism $L(-D) \to E$. Thus E has a line subbundle M with $\deg M \geq \deg L - \deg D$ and the Segre invariant of E satisfies:

$$\begin{aligned} s(E) &\leq c_1(E) - 2\deg M \\ &\leq c_1(E) - 2(\deg L - \deg D). \end{aligned}$$

Since $\deg E = 0$ then $\deg D \leq \deg L$ implies that $s(E) \leq 0$ i.e. E is not stable. It can be concluded that if E corresponds to an extension $(e) \in \mathrm{Sec}_d C \subset \mathbb{P}_L$ then E is not stable.

Remark 2.0.3. The argument given in (2.0.2) can be generalised to give a stratification of $\mathbb{P}_L$ in terms of the Segre invariant. More precisely:

$$s(E) \geq s \iff (e) \notin \mathrm{Sec}_{\frac{1}{2}(2d+n+s-2)} C$$

See [LN, Prop 1.1] for details.

A more precise description of when ϵ_d fails to be defined is given by Bertram [Be, Thm 1] in which he shows that ϵ_d may be resolved by a sequence of blow-ups of secant varieties of $C \to \mathbb{P}_L$. Moreover he also shows that there exists an isomorphism:

$$H^0(\mathcal{SU}_C(2), \mathcal{L}) \cong H^0(\mathbb{P}_L, \mathcal{I}_C^{d-1}(d))$$

where $\mathcal{I}_C$ is the ideal sheaf of $C \to \mathbb{P}_L$. This says that via the composition:

$$\mathbb{P}U_{d,\mathcal{O}_C} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2) \xrightarrow{\phi_{\mathcal{L}}} |2\Theta|$$

the pullback of any hyperplane is a hypersurface of degree d vanishing to order $d-1$ along the curve $C \to \mathbb{P}_L$. To see why this should be the case let H_M denote the hyperplane in $|2\Theta|$ given by a generic line bundle $M \in J^{g-1}(C)$ (cf (1.1.4)) and suppose $d = \deg L \geq 2$. By generic we shall mean a line bundle $M \in J^{g-1}(C)$ such that:

$$h^0(L^{-1}M(p)) = 0 \quad \text{and} \quad h^0(KL^{-1}M^{-1}(p)) = h^1(LM(-p)) = 0$$

for all $p \in C$. In particular $h^0(L^{-1}M) = h^1(LM) = 0$.
The map $\mathcal{SU}_C(2) \xrightarrow{\phi} |2\Theta|$ is defined by:

$$E \longmapsto D_E = \{M \in J^{g-1}(C) \mid h^0(M \otimes E) \geq 1\},$$

so the pullback of a hyperplane is given by:

$$(\phi_{\mathcal{L}}.\epsilon_d)^{-1}(H_M) = \{(e) \in \mathbb{P}_L \mid h^0(M \otimes E) \geq 1\}$$

where $(e) \in \mathbb{P}_L$ corresponds to the non-split extension $0 \to L^{-1} \to E \to L \to 0$. Tensoring by M gives:

$$0 \longrightarrow L^{-1}M \longrightarrow M \otimes E \longrightarrow LM \longrightarrow 0 \tag{1}$$

and taking the long exact cohomology sequence gives:

$$0 \to H^0(L^{-1}M) \longrightarrow H^0(M \otimes E) \longrightarrow H^0(LM) \xrightarrow{\delta(e)} H^1(L^{-1}M) \to \dots \tag{2}$$

where $\delta(e)$ is the non-zero coboundary map given by $(e) \in H^1(L^{-2})$. As M was chosen generically such that $h^0(L^{-1}M) = 0 = h^1(LM)$ then by Riemann-Roch both $H^0(LM)$ and $H^1(L^{-1}M)$ are d dimensional vector spaces and there exists a linear map:

$$\begin{aligned} H^1(L^{-2}) &\longrightarrow \mathrm{Hom}(H^0(LM), H^1(L^{-1}M)) \\ (e) &\longmapsto \delta(e) \end{aligned}$$

Thus $\delta(e)$ has non-zero kernel when the $d \times d$ matrix representing $\delta(e)$ has zero determinant i.e. $(\phi_{\mathcal{L}}.\epsilon_d)^{-1}(H_M)$ is a hypersurface of degree d. If (e) corresponds to a point $p \in C \to \mathbb{P}_L$ then E sits in the extension $0 \to L^{-1}(p) \to E \to L(-p) \to 0$. Tensoring by M and taking the cohomology sequence we can deduce that:

$$\begin{aligned} h^0(M \otimes E) &\leq h^0(ML^{-1}(p)) + h^0(ML(-p)) \\ &= h^0(ML(-p)) \\ &= d - 1 \end{aligned}$$

(if M was chosen generically as above.) Bertram shows that equality must hold above i.e. $\operatorname{rank} \delta(e) = 1$ on the curve $C \to \mathbb{P}_L$ and $(\phi_{\mathcal{L}}.\epsilon_d)^{-1}(H_M)$ vanishes to order $d-1$ along the curve.

Remark 2.0.4. If $\Lambda = \mathcal{O}_C$ and $d = 1$ then ϵ_1 is *linear* on $\mathbb{P}_L$ with respect to $\mathcal{L}$. It is injective on each fibre too: by (2.0.2) *every* extension of $\mathbb{P}_L$ ($L \in J^1(C)$) corresponds to a semistable bundle $E \in \mathcal{SU}_C(2)$ with points on the curve $C \to \mathbb{P}_L$ mapping injectively to the semistable boundary $\mathcal{K}$ via:

$$p \mapsto L^{-1}(p) \oplus L(-p).$$

Points not on $C \to \mathbb{P}_L$ correspond to bundles with Segre invariant 2 and the fact that these map injectively into $\mathcal{SU}_C(2)$ follows from (5.3.2).

2.1. Moduli spaces of pairs

This section will give a review of some work done by Thaddeus [Th] on pairs (E, ϕ) where E is a rank 2 vector bundle over a curve C with fixed determinant Γ and $\phi \in H^0(C, E)$ is a *non-zero* section of the bundle. These pairs admit a semistability condition depending on some parameter $\sigma \in \mathbb{Q}$ and Thaddeus has constructed moduli spaces $\mathcal{M}(\sigma, \Gamma)$ of these pairs. Furthermore as σ varies these moduli spaces undergo a sequence of "flips". The next section will show that in the specific case when C is a curve of genus 2 and Γ has odd degree then this sequence of "flips" will correspond exactly to the construction given after (2.2.6).

Definition 2.1.1. *The pair (E, ϕ) is σ-semistable if for all line subbundles $L \subset E$:*

$$\begin{aligned} &(1). \quad \deg L \leq \frac{1}{2}\deg E - \sigma \quad \textit{if } \phi \in H^0(C, L) \\ &(2). \quad \deg L \leq \frac{1}{2}\deg E + \sigma \quad \textit{if } \phi \notin H^0(C, L). \end{aligned}$$

If the inequalities are both strict then (E, ϕ) is said to be σ-stable.

Theorem 2.1.2. *There exists a projective moduli space $\mathcal{M}(\sigma, \Gamma)$ of σ-semistable pairs (E, ϕ) which is non-empty if and only if $\sigma \leq \frac{d}{2}$, where $d = \deg \Gamma$.*

It is easy to see that σ-semistability of a pair (E, ϕ) will remain unchanged over certain intervals of the real line. In fact a pair remains *σ-stable* for any $\sigma \in (\max(0, \frac{d}{2} - i - 1), \frac{d}{2} - i)$, with $i \in \mathbb{Z}$ and $0 \leq i \leq \frac{d-1}{2}$ i.e. the moduli space $\mathcal{M}(\sigma, \Gamma)$ remains unchanged as σ varies in the interval above. Thus write $\mathcal{M}_i$ for the moduli space $\mathcal{M}(\sigma, \Gamma)$ with $\sigma \in (\max(0, \frac{d}{2} - i - 1), \frac{d}{2} - i)$. Since i is bounded as above there exists $\omega + 1$ such moduli spaces where $\omega = \left[\frac{d-1}{2}\right]$.

Proposition 2.1.3. *There exists an isomorphism:*

$$\mathcal{M}_0 \cong \mathbb{P}H^1(C, \Gamma^{-1}).$$

PROOF. Suppose $(E, \phi) \in \mathcal{M}_0$. If $i = 0$ then $\sigma \in (\max(0, \frac{d}{2} - 1), \frac{d}{2})$. Thus the first semistability condition of (2.1.1) implies that for all line subbundles $L \subset E$, $\deg L \leq 0$ if $\phi \in H^0(L)$. Hence equality must hold, $L = \mathcal{O}_C$ and E occurs in the extension:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \Gamma \longrightarrow 0 \tag{3}$$

with $\phi \in H^0(\mathcal{O}_C)$ the constant section. The result now follows if (3) is non-split. This follows from the second semistability condition which says that E has no line subbundles of degree $\geq d$. $\square$

Definition 2.1.4. *Let $\rho_1 : C \times S^iC \to C$ and $\rho_2 : C \times S^iC \to S^iC$ be projections and $\Delta \subset C \times S^iC$ the universal divisor. Define:*

$$W_i^- = (R^0\rho_2)_*\mathcal{O}_\Delta(\rho_1^*\Gamma \otimes \mathcal{O}(-\Delta))$$
$$W_i^+ = (R^1\rho_2)_*\mathcal{O}_{C\times S^iC}(\rho_1^*\Gamma^{-1} \otimes \mathcal{O}(2\Delta)).$$

W_i^- and W_i^+ are vector bundles on S^iC of rank i and $d + g - 1 - 2i$ respectively.

Proposition 2.1.5. *For $i \leq \frac{d-1}{2}$ there is a family over $\mathbb{P}W_i^-$ (resp. $\mathbb{P}W_i^+$) parametrising exactly those pairs which are represented in $\mathcal{M}_{i-1}$ but not $\mathcal{M}_i$ (resp. $\mathcal{M}_i$ but not $\mathcal{M}_{i-1}$).*

Theorem 2.1.6. *The moduli space $\mathcal{M}_i$ is obtained from $\mathcal{M}_{i-1}$ by a blow-up followed by a blow-down in another direction. More precisely, if $\widetilde{\mathcal{M}}_i^+$ is the blow-up of $\mathcal{M}_i$ along $\mathbb{P}W_i^+$ and $\widetilde{\mathcal{M}}_i^-$ is the blow-up of $\mathcal{M}_{i-1}$ along $\mathbb{P}W_i^-$ then there exists an isomorphism:*

$$\widetilde{\mathcal{M}}_i^+ \cong \widetilde{\mathcal{M}}_i^-.$$

Proposition 2.1.7. *If $\mathcal{M}_\omega$ is the last moduli space of pairs, where $\omega = \left[\frac{d-1}{2}\right]$, then there is a natural map $\mathcal{M}_\omega \to \mathcal{SU}(2,\Gamma)$ with fibre $\mathbb{P}H^0(C,E)$ over a stable bundle $E \in \mathcal{SU}_C(2,\Gamma)$.*

PROOF. If $i = \omega$ then $\sigma \in (0, \frac{1}{2})$ or $\sigma \in (0,1)$ (if d is odd or even respectively). With these constraints then it is clear that σ-semistability of a pair (E, ϕ) implies ordinary semistability of the bundle E. So there is a well defined map $\mathcal{M}_\omega \to \mathcal{SU}(2,\Gamma)$. Conversely if E is a stable bundle i.e. $\deg L < \frac{d}{2}$ for all line subbundles $L \subset E$ then (E, ϕ) is *σ-stable* for any $\phi \in H^0(C, E)$ and since $(E, \phi) \sim (E, c\phi)$ for any $c \in \mathbb{C}^*$ the fibre of the map $\mathcal{M}_\omega \to \mathcal{SU}_C(2,\Gamma)$ over a stable bundle E is $\mathbb{P}H^0(C, E)$. $\square$

(2.1.6) and (2.1.7) are summarised in the following diagram:

$$\begin{array}{ccccccccc}
 & \widetilde{\mathcal{M}}_2 & & \widetilde{\mathcal{M}}_3 & & & \widetilde{\mathcal{M}}_\omega & & \\
\swarrow & & \searrow \quad \swarrow & & \searrow & \swarrow & & \searrow & \\
\mathcal{M}_1 & & \mathcal{M}_2 & & \mathcal{M}_3 \cdots \mathcal{M}_{\omega-1} & & & & \mathcal{M}_\omega \\
\downarrow & & & & & & & & \downarrow \\
\mathcal{M}_0 & & & & & & & & \mathcal{SU}_C(2,\Gamma)
\end{array}$$

where $\widetilde{\mathcal{M}}_i$ is the blow-up of $\mathcal{M}_{i-1}$ along $\mathbb{P}W_i^-$. Note that W_1^- is a line bundle i.e. $\mathbb{P}W_1^- \cong C$ so after blowing up $\mathbb{P}W_1^-$ it cannot be blown down in another direction.

Bertram's blow-ups of the secant varieties $\mathrm{Sec}_i C$ in the extension space $\mathbb{P}_L$ are closely related to Thaddeus' flips construction (the first moduli space of pairs $\mathcal{M}_0$ is isomorphic to the extension space $\mathbb{P}H^1(C, L^{-2}\Lambda^{-1})$ for some line bundle L.) Bertram's blow-ups are all equivalent to Thaddeus'. However Thaddeus also constructs a sequence of blow-downs and ends with the morphism $\mathcal{M}_\omega \to \mathcal{SU}_C(2,\Gamma)$. Bertram continues to blow-up his secant varieties until he obtains a morphism $\widetilde{\mathbb{P}}_L \to \mathcal{SU}_C(2,\Lambda)$ agreeing with the extension map ϵ_d away from the proper transforms of the blow-ups.

2.2. The genus 2, odd degree moduli space

The following will give a brief overview of some of the geometry of $\mathcal{SU}_C(2,1)$ for C a curve of genus two.

By definition there exists a unique double cover:

$$C \xrightarrow{2:1} \mathbb{P}^1$$

branched at 6 points $\lambda_1, \ldots, \lambda_6$. The corresponding pencil of quadrics in $\mathbb{P}^5$, modulo $\mathbb{P}GL_6(\mathbb{C})$, is given by:

$$Q_\lambda : \quad \sum_{i=1}^{6} (\lambda - \lambda_i) x_i^2 = 0 \qquad \lambda \in \mathbb{P}^1.$$

Moreover this construction is invertible i.e. every smooth quadric Q_λ contains two irreducible 3-dimensional families of 2-planes. Thus there is a natural 2:1 map branched at the 6 *singular* quadrics Q_{λ_i}. The main result of this section due to Newstead [Ne1] is:

Theorem 2.2.1. *$\mathcal{SU}_C(2,1)$ is isomorphic to the intersection of the pencil of quadric hypersurfaces $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$.*

Given a point f in the intersection of the pencil of quadric hypersurfaces, a rank 2 vector bundle is constructed by choosing a quadric Q_λ and an irreducible 3-dimensional family of 2-planes contained in it. One shows that for each $x \in C$ there exists a $\mathbb{P}^1$ of these two-planes passing through f (denoted by $\mathbb{P}^1_x$.) Then

$$\bigcup_{x \in C} \mathbb{P}^1_x$$

is shown to be a $\mathbb{P}^1$-bundle over C and hence isomorphic to $\mathbb{P}(F)$ for some rank two bundle F. F then has the required properties that it is stable and of odd degree.

If $l \subset \mathcal{SU}_C(2,1)$ is a line through f then for each $x \in C$ there is a unique two-plane of $\mathbb{P}^1_x$ containing l. This defines a section of $\mathbb{P}(F)$ which corresponds to a line subbundle

$M \subset F$ of degree zero (necessarily of maximal degree by the stability of F.)

Proposition 2.2.2. *The variety of lines on $\mathcal{SU}_C(2,1)$ is parametrised by the Jacobian $J^0(C)$. Moreover suppose l_M is a line on $\mathcal{SU}_C(2,1)$ given by $M \in J^0(C)$ then:*

$$f \in l_M \iff M \subset F \text{ is a line subbundle of maximal degree.}$$

For details of the proof of this proposition see [Ne1, Prop 4] and [O].

Remark 2.2.3. Note that the tangent plane at a point $f \in \mathcal{SU}_C(2,1) \subset \mathbb{P}^5$ will cut out a pencil of quadric cones in $\mathbb{P}^3$ with vertex f. Thus their intersection will generically be four lines and so (2.2.2) says that a generic $F \in \mathcal{SU}_C(2,1)$ has four maximal line subbundles.

Remark 2.2.4. By taking one quadric of the pencil $\{Q_\lambda\}$ to be the Plücker embedding of the Grassmanian $G(2,4)$ the intersection given in (2.2.1) corresponds to a family of lines in $\mathbb{P}^3$, namely the quadratic line complex (see [GH] for a discussion of this line complex.)

It can be shown that the normal bundle N_l of a line $l \subset \mathcal{SU}_C(2,1)$ has trivial first Chern class. Thus by the classification theorem of vector bundles on $\mathbb{P}^1$:

$$N_l = \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$$

for some $n \in \mathbb{Z}$. Generically $n = 0$ and N_l will be trivial but for some lines it can occur that $n = 1$. A line $l \subset \mathcal{SU}_C(2,1)$ is said to be *special* if its normal bundle N_l jumps.

Definition 2.2.5. *Let X_l be the closure of the set $\{l' \subset \mathcal{SU}_C(2,1) \mid l \neq l', l \cap l' \neq \emptyset\}$ i.e. the set of lines in $\mathcal{SU}_C(2,1)$ meeting a given fixed line l.*

If l is special then it can be shown that $l \in X_l$.

Proposition 2.2.6. [GH, p796] *There exists a birational map $f_l : \mathcal{SU}_C(2,1) - l \to \mathbb{P}^3$ and the image of $X_l \subset \mathcal{SU}_C(2,1)$ is a quintic curve.*

PROOF. Regarding $\mathcal{SU}_C(2,1)$ as the intersection of a pencil of quadrics in $\mathbb{P}^5$ the map $f_l : \mathcal{SU}_C(2,1) - l \to \mathbb{P}^3$ is just projection away from the line l onto a complementary 3-plane. The map f_l is easily seen to be one to one away from the locus of lines meeting l (and hence a birational map): if any 2-plane through l contains two distinct points p, q of $\mathcal{SU}_C(2,1)$ then the line $\overline{pq}$ also meets l. So $\overline{pq}$ meets $\mathcal{SU}_C(2,1)$ in three points and so lies on each quadric Q_λ. Hence $\overline{pq}$ lies on $\mathcal{SU}_C(2,1)$.
Let X be the image of X_l. To see that X has degree 5 note that the points of intersection of X with a generic 2-plane V_2 correspond to lines meeting l *and* lying in the hyperplane spanned by l and V_2. But for generic V_2 the intersection $\mathcal{SU}_C(2,1) \cap \overline{lV_2}$ will be a smooth intersection of a pencil of quadrics in $\mathbb{P}^4$. This variety contains 16 lines [GH, p550] all of which meet exactly five other lines. Hence X is a quintic. □

If $\pi_l : \mathcal{M}_l \to \mathcal{SU}_C(2,1)$ is the blow-up of $\mathcal{SU}_C(2,1)$ along l, then f_l can be extended to a holomorphic map:

$$\tilde{f}_l : \mathcal{M}_l \longrightarrow \mathbb{P}^3.$$

If l is not special then $\tilde{f}_l$ is one to one away from the proper transforms of the lines meeting l. $\tilde{f}_l$ maps each of these lines onto the corresponding point of the quintic X. Thus there is a diagram:

$$\begin{tikzcd} & \mathcal{M}_l \arrow[dl, "\tilde{f}_l"'] \arrow[dr, "\pi_l"] & \\ \mathbb{P}^3 & & \mathcal{SU}_C(2,1) \end{tikzcd}$$

where $\tilde{f}_l$ is the blow-up of the quintic curve $X \subset \mathbb{P}^3$.

We now return to Thaddeus' construction in the last section and restrict attention to genus 2 curves with Γ having degree $d = 3$. Then $\omega = \left[\frac{d-1}{2}\right] = 1$ so there exist two moduli spaces of pairs $\mathcal{M}_0 \cong \mathbb{P}^3$ (by (2.1.3) and Riemann-Roch) and $\mathcal{M}_1$. Thus the flips diagram in the last section reduces to:

$$\begin{tikzcd} & \mathcal{M}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{M}_0 & & \mathcal{SU}_C(2,\Gamma) \end{tikzcd}$$

The claim is that this corresponds exactly with the construction given after the proof of (2.2.6). (2.1.6) says that the map $\mathcal{M}_1 \to \mathcal{M}_0$ is given by the blow-up of $\mathbb{P}W_1^- \cong C \subset \mathcal{M}_0$. This can explicitly be seen in the following:

Proposition 2.2.7. *$\mathcal{M}_1 \to \mathcal{M}_0$ is the blow-up of $C \hookrightarrow \mathbb{P}^3$ embedded as a quintic via the linear system $|K\Gamma|$.*

PROOF. By (2.1.3) all pairs $(F, \phi) \in \mathcal{M}_0$ are such that F lies in the extension:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F \longrightarrow \Gamma \longrightarrow 0 \tag{4}$$

and $\phi \in H^0(C, \mathcal{O}_C)$. To find pairs that are in $\mathcal{M}_0$ but not in $\mathcal{M}_1$ apply the second σ-semistability condition of (2.1.1). For $i = 0$, $\sigma \in (\frac{1}{2}, \frac{3}{2})$ so take $\sigma = 1$, for $i = 1$, $\sigma \in (0, \frac{1}{2})$ so take $\sigma = \frac{1}{4}$. Then for any line subbundle M (2.1.1) says:

$$\begin{aligned} \mathcal{M}_0 : \quad & \deg M \leq \frac{5}{2} \quad \text{if } \phi \notin H^0(C, M) \\ \mathcal{M}_1 : \quad & \deg M \leq \frac{7}{4} \quad \text{if } \phi \notin H^0(C, M) \end{aligned}$$

Thus $(F, \phi) \in \mathcal{M}_0 \setminus \mathcal{M}_1$ if F occurs in the extension:

$$0 \longrightarrow M \longrightarrow F \longrightarrow M^{-1}\Gamma \longrightarrow 0 \tag{5}$$

with $\deg M = 2$ and $\phi \notin H^0(C, M)$. From the induced long exact cohomology sequence of (5) there exists a non-zero section $\phi' \in H^0(C, M^{-1}\Gamma)$ namely the image of $\phi \in H^0(C, F)$. Thus $M^{-1}\Gamma = \mathcal{O}_C(x)$ for some $x \in C$ and (5) becomes:

$$0 \longrightarrow \Gamma(-x) \longrightarrow F \longrightarrow \mathcal{O}_C(x) \longrightarrow 0$$

Thus pairs $(F, \phi) \in \mathcal{M}_0 \setminus \mathcal{M}_1$ correspond to those extensions $s \in \mathbb{P}H^1(C, \Gamma^{-1})$ such that the map $\Gamma(-x) \to \Gamma$ lifts to F.

Now twisting (4) by $\Gamma^{-1}(x)$ and taking the long exact cohomology sequence yields:

$$0 \longrightarrow H^0(F \otimes \Gamma^{-1}(x)) \longrightarrow H^0(\mathcal{O}_C(x)) \xrightarrow{\otimes s} H^1(\Gamma^{-1}(x)) \longrightarrow \dots$$

Thus given $\gamma_x : \Gamma(-x) \to \Gamma$ i.e. $\gamma_x \in H^0(\mathcal{O}_C(x))$, then it lifts to a map $\Gamma(-x) \to F$ if and only if $\gamma_x \otimes s = 0$. Alternatively $s \in H^1(C, \Gamma^{-1})$ is in the kernel of the map:

$$\gamma_x : H^1(\Gamma^{-1}) \longrightarrow H^1(\Gamma^{-1}(x))$$

or, by Serre duality, the map:

$$\gamma_x : H^0(K\Gamma)^* \longrightarrow H^0(K\Gamma(-x))^*. \tag{6}$$

This is dual to the injection $H^0(K\Gamma(-x)) \hookrightarrow H^0(K\Gamma)$ and so γ_x is surjective. By Riemann-Roch $\dim\ker\gamma_x = h^0(K\Gamma(-x)) - h^0(K\Gamma) = 1$ i.e. given any $x \in C$ there exists a *unique* $s \in \mathbb{P}H^1(\Gamma^{-1})$ such that $\gamma_x \otimes s = 0$. Now $s \in \mathbb{P}H^1(\Gamma^{-1})$ corresponds to $\phi_{|K\Gamma|}(x)$ where $\phi_{|K\Gamma|}$ is the embedding of C via $|K\Gamma|$: recall that s is in the kernel of (6) where s is thought of as a linear functional on $H^0(C, K\Gamma)$. But evaluation at x is clearly a linear functional in the kernel of γ_x and so by uniqueness must be equal to s. □

Proposition 2.2.8. *If $\Gamma \neq K(-x)$ for any $x \in C$ then the map $\mathcal{M}_1 \to \mathcal{SU}(2, \Gamma)$ is given by the blow-up of the line $l_{\Gamma K^{-1}} \subset \mathcal{SU}(2, \Gamma)$.*

PROOF. By (2.2.2) $F \in \mathcal{SU}(2, \Gamma)$ lies on a line l_M if and only if $M \subset F$ is a line subbundle of maximal degree. By the semistability of F a maximal line subbundle must have degree 1. Thus the natural choice is to take $M = \Gamma K^{-1}$. By (2.1.7) the fibre of the map $\mathcal{M}_1 \to \mathcal{SU}(2,\Gamma)$ is $\mathbb{P}H^0(C,F)$. Thus the problem reduces to calculating $h^0(C,F)$ on and off the line $l_{\Gamma K^{-1}} \subset \mathcal{SU}(2,\Gamma)$. Now:

$$\begin{aligned} F \text{ lies on the line } l_{\Gamma K^{-1}} &\iff \text{there exists an extension } 0 \to \Gamma K^{-1} \to F \to K \to 0 \\ &\iff h^0(C, K \otimes F^*) \neq 0 \\ &\iff h^1(C,F) \neq 0 \\ &\iff h^0(C,F) \geq 2 \qquad \text{(by Riemann-Roch)} \end{aligned}$$

Thus if F does not lie on the line $l_{\Gamma K^{-1}}$ then $h^0(C,F) \leq 1$, but by Riemann-Roch $h^0(C,F) \geq 1$ and so equality holds.

If F lies on $l_{\Gamma K^{-1}}$ then there exists the exact sequence:

$$0 \to H^0(\Gamma K^{-1}) \to H^0(F) \to H^0(K) \to H^1(\Gamma K^{-1}) \to \ldots$$

with $\deg \Gamma K^{-1} = 1$. Now, by hypothesis, Γ was chosen so that $h^0(\Gamma K^{-1}) = 0 = h^1(\Gamma K^{-1})$ so $H^0(C,F) \cong H^0(C,K)$ i.e. $h^0(F) = 2$. $\square$

CHAPTER 3

The g-plane ruling

The last chapter concentrated on the extension map:

$$\mathbb{P}U_{d,\Lambda} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$$

when restricted to a single fibre of $\mathbb{P}U_{d,\Lambda} \xrightarrow{\nu} J^d(C)$. Now consider the extension map over the *whole* of the projective bundle but restrict to the case when $d = 1$ and $\Lambda = \mathcal{O}_C$. Then $\mathbb{P}U_1 \stackrel{\text{def}}{=} \mathbb{P}U_{1,\mathcal{O}_C}$ is a bundle of g-planes over $J^1(C)$ and each fibre $\mathbb{P}_L$ ($L \in J^1(C)$) maps linearly and injectively into $\mathcal{SU}_C(2) \subset \mathbb{P}H^0(\mathcal{SU}_C(2),\mathcal{L})^*$ by (2.0.4). Furthermore the curve $C \subset \mathbb{P}_L$ (mapped via the linear system $|KL^2|$) is mapped to the Kummer variety $\mathcal{K} = J^0/\pm$ by:

$$p \longmapsto L^{-1}(p) \oplus L(-p).$$

The image of:

$$\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$$

clearly parametrises those bundles with Segre invariant ≤ 2 and by (1.2.1) this is an irreducible algebraic subvariety $\mathcal{V} \subset \mathcal{SU}_C(2)$ of codimension $g-3$.

Looking at this extension map and its image was the approach taken by Narasimhan and Ramanan [NR2] to prove that for C a non-hyperelliptic curve of genus 3 the moduli space $\mathcal{SU}_C(2) \subset \mathbb{P}^7$ is isomorphic to a quartic hypersurface. This chapter gives a brief account of the methods of Narasimhan and Ramanan. These ideas are then extended to calculate the degree of $\mathcal{SU}_C(2) \subset \mathbb{P}^{15}$ for C a non-hyperelliptic curve of genus 4 (cf (1.4.1)). In particular $\mathcal{V}$, the image of $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$, is a Cartier divisor. The degree of $\mathcal{V}$ is computed and using a spectral curve construction its fundamental class is calculated. The degree of $\mathcal{SU}_C(2) \subset \mathbb{P}^{15}$ is then easily deduced.

3.1. The genus 3, even degree moduli space

The main result of [NR2] says:

Theorem 3.1.1. *If C is non-hyperelliptic of genus 3 then $\mathcal{SU}_C(2)$ is isomorphic to a quartic hypersurface in $\mathbb{P}^7$.*

For $g = 3$ not only can the fibre of $\mathbb{P}U_1$ be identified with $\mathbb{P}H^1(C, L^{-2})$ so defining a surjective map $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$ but it can also be identified with the subspace of $H^0(J^2(C), 2\Theta)$ consisting of those sections which vanish on $W_1(C) + L \subset J^2(C)$. This defines a map $\mathbb{P}U_1 \xrightarrow{\delta_1} \mathbb{P}H^0(J^2(C), 2\Theta)$. In fact the bundle V_1 on $J^1(C)$ with fibre given by the subspace above sits in the short exact sequence:

$$0 \longrightarrow V_1 \longrightarrow H^0(J^2(C), 2\Theta) \otimes \mathcal{O}_{J^1} \longrightarrow Q_1 \longrightarrow 0$$

and U_1 is isomorphic to V_1 up to a twist by some line bundle $\mathcal{N}$ on $J^1(C)$ (cf (3.2.8).) Narasimhan and Ramanan then showed that there exists a commutative diagram:

$$\begin{array}{ccc} & \mathcal{SU}_C(2) & \\ \epsilon_1 \nearrow & & \searrow \phi_{\mathcal{L}} \\ \mathbb{P}U_1 & \xrightarrow{\delta_1} & \mathbb{P}H^0(J^2(C), 2\Theta) \end{array}$$

from which they deduced the injectivity of $\phi_{\mathcal{L}}$ and moreover that $\phi_{\mathcal{L}}$ was an embedding. The degree of $\mathcal{SU}_C(2) \subset \mathbb{P}^7$ is calculated by showing that:

$$[c_1(T)]^6 = 32$$

where $T \longrightarrow \mathbb{P}U_1$ is the tautological line bundle. The result then follows from the fact that ϵ_1 is generically 8:1 (cf (5.5.6).)

3.2. Degree of g-plane ruling

The following will calculate the degree of the g-plane ruling (or the subvariety $\mathcal{V} \subset \mathcal{SU}_C(2)$ of bundles with Segre invariant ≤ 2):

$$\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$$

as a subvariety of $\mathbb{P}H^0(\mathcal{SU}_C(2), \mathcal{L})^* \cong \mathbb{P}^{2^g-1}$.

While ϵ_1 was generically 8:1 in the genus 3 case, a result due to Lange and Narasimhan [LN, Prop 3.3.] shows that:

Proposition 3.2.1. *If $g \geq 4$ then ϵ_1 is injective on an open dense subset of $\mathbb{P}U_1$.*

Proposition 3.2.2. *The Chern character of $U_d \longrightarrow J^d(C)$ is given by:*

$$ch(U_d) = (2d + g - 1) + 4\theta$$

where $\theta \in H^2(J^d, \mathbb{Z})$ is the class of a translate of the theta divisor $\Theta \subset J^{g-1}(C)$.

PROOF. Recall from section (1.3) that if $\mathcal{P}$ is a Poincaré line bundle of degree d on $C \times J^d(C)$ then U_d is defined as the first direct image bundle:

$$U_d = R^1\pi_*\mathcal{P}^{-2}$$

where $\pi : C \times J^d(C) \longrightarrow J^d(C)$ is projection onto the second factor. Grothendieck-Riemann-Roch gives:

$$ch(\pi_!\mathcal{P}^{-2}) \cdot td(J^d(C)) = \pi_*(ch(\mathcal{P}^{-2}) \cdot td(C \times J^d(C))) \qquad (1)$$

Let ξ be the pullback of the class of a point on C, ς the class of $c^{1,1} \in H^1(C, \mathbb{Z}) \otimes H^1(J^d(C), \mathbb{Z})$ and θ the pullback to $C \times J^d(C)$ of the class $\theta \in H^2(J^d(C), \mathbb{Z})$. Then, following the argument in Arbarello et al [ACGH, pages 334–336] :

$$td(J^d(C)) = 1 \quad , \quad R^0\pi_*\mathcal{P}^{-2} = 0$$

$$td(C \times J^d(C)) = 1 + (1-g)\xi \quad , \quad c_1(\mathcal{P}^{-2}) = -2d\xi - 2\varsigma$$

and the following relations hold:

$$\varsigma^3 = \xi.\varsigma = 0$$

$$\varsigma^2 = -2\xi\theta.$$

Thus:

$$\begin{aligned} ch(\mathcal{P}^{-2}) = \exp(c_1(\mathcal{P}^{-2})) &= \sum_{k=0}^{\infty} \frac{(-2d\xi - 2\varsigma)^k}{k!} \\ &= 1 - 2d\xi - 2\varsigma + \frac{4\varsigma^2}{2} \\ &= 1 - 2d\xi - 2\varsigma - 4\xi\theta. \end{aligned}$$

Substituting into (1):

$$\begin{aligned} ch(-U_d) &= \pi_*[(1 - 2d\xi - 2\varsigma - 4\xi\theta)(1 + (1-g)\xi)] \\ &= \pi_*[1 - 2d\xi - 2\varsigma + (1-g)\xi - 4\xi\theta] \\ &= (1 - g - 2d) - 4\theta. \qquad \square \end{aligned}$$

Corollary 3.2.3. *The Chern classes of U_d are given by:*

$$c_k(U_d) = \frac{(4\theta)^k}{k!}.$$

PROOF. Following the argument in [ACGH, p336] let the Chern polynomial of U_d be given by:

$$c_t(U_d) = \prod_{i=1}^{2d+g-1} (1 + \lambda_i t)$$

then the Chern character is defined as:

$$ch(U_d) = \sum_i e^{\lambda_i}.$$

Thus by (3.2.2) the k-th homogenous term is:

$$ch_k(U_d) = \frac{1}{k!}\sum_i \lambda_i^k = \begin{cases} 4\theta, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases} \tag{2}$$

Now

$$\begin{aligned} c_t(U_d) &= \exp(\log(\prod(1+\lambda_i t))) \\ &= \exp(\sum \log(1+\lambda_i t)) \\ &= \exp(\sum \lambda_i t) \qquad \text{(expanding logs and using (2))} \\ &= \exp(4\theta t). \end{aligned}$$

The result now follows from the expansion of this polynomial. □

The Chern classes of U_d satisfy the following simple relation:

Lemma 3.2.4. *Write c_k for $c_k(U_d)$. Then for $k \geq 2$:*

$$\sum_{i=1}^{k-1}(-1)^{i-1}c_i c_{k-i} + (-1)^{k-1}c_k = c_k.$$

PROOF. If k is odd then there is nothing to prove since all terms of the sum on the left hand side cancel except $(-1)^{k-1}c_k$. Now suppose that k is even. From (3.2.3):

$$\begin{aligned} \sum_{i=1}^{k-1}(-1)^{i-1}c_i c_{k-i} + (-1)^{k-1}c_k &= \frac{(4\theta)}{1!}\frac{(4\theta)^{k-1}}{(k-1)!} - \frac{(4\theta)^2}{2!}\frac{(4\theta)^{k-2}}{(k-2)!} + \cdots + (-1)^{k-1}\frac{(4\theta)^k}{k!} \\ &= (4\theta)^k\left(\frac{1}{1!(k-1)!} - \frac{1}{2!(k-2)!} + \cdots - \frac{1}{k!}\right) \\ &= \frac{(4\theta)^k}{k!}\left(k - \frac{k(k-1)}{2!} + \cdots + k - 1\right). \end{aligned}$$

The expression in the brackets is obtained by expanding out $1-(1-x)^k$ and putting $x = 1$. Thus:

$$\sum_{i=1}^{k-1}(-1)^{i-1}c_i c_{k-i} + (-1)^{k-1}c_k = \frac{(4\theta)^k}{k!} = c_k$$

as asserted. □

Let $T \to \mathbb{P}U_1$ be the tautological line bundle. In particular $T\big|_{\mathbb{P}_L} = \mathcal{O}_{\mathbb{P}_L}(-1)$.

Lemma 3.2.5. *Let $\eta = c_1(T)$. Then:*

$$\eta^{g+k}\theta^{g-k} = c_k\eta^g\theta^{g-k}.$$

Proof. By [GH, p606] η satisfies the following relation:

$$\eta^{g+1} = c_1\eta^g - c_2\eta^{g-1} + \cdots - (-1)^{g+1}c_{g+1} \tag{3}$$

Use this and induction to prove that the coefficient of η^g in η^{g+k} is c_k. First note that by (3) this statement is trivially true for $k = 1$. Now assume true for all $n \leq k-1$. Then:

$$\begin{aligned}
\eta^{g+k} &= \eta^{k-1}\eta^{g+1} \\
&= c_1\eta^{g+k-1} - c_2\eta^{g+k-2} + \cdots + (-1)^{k-1}c_k\eta^g + \cdots - (-1)^{g+1}c_{g+1}\eta^{k-1}.
\end{aligned}$$

Now using the induction hypothesis it is clear that the coefficient of η^g in η^{g+k} is given by:

$$c_1c_{k-1} - c_2c_{k-2} + \cdots + (-1)^{k-2}c_{k-1}c_1 + (-1)^{k-1}c_k$$

which by (3.2.4) is equal to c_k. The proof of the lemma follows easily from this, (3.2.4) again and the fact that $c_{k+j}\theta^{g-k} = 0$ for all $j \geq 1$:

$$\begin{aligned}
\eta^{g+k}\theta^{g-k} &= (c_1\eta^{g+k-1} - c_2\eta^{g+k-2} + \cdots + (-1)^{k-1}c_k\eta^g)\theta^{g-k} \\
&= (c_1c_{k-1} - c_2c_{k-2} + \cdots + (-1)^{k-1}c_k)\eta^g\theta^{g-k} \\
&= c_k\eta^g\theta^{g-k}
\end{aligned}$$

as asserted. □

Remark 3.2.6. By changing variables (3.2.5) is equivalent to:

$$\eta^{2g-k}\theta^k = c_{g-k}\eta^g\theta^k.$$

Lemma 3.2.7. *Let f_1 denote the composition:*

$$\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2) \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}H^0(\mathcal{SU}_C(2), \mathcal{L})^*.$$

Then:

$$f_1^* \mathcal{O}_{\mathbb{P}^{2^g-1}}(1) = T^{-1} \otimes \nu^* \mathcal{O}(2\Theta_{p_0}),$$

where $\nu : \mathbb{P}U_1 \to J^1(C)$ is projection and $\mathcal{O}(2\Theta_{p_0})$ is the pullback of $\mathcal{O}(2\Theta_C)$ by the translation $J^1 \xrightarrow{t-p_0} J^0$.

Proof. The lemma is equivalent to proving that:

$$\epsilon_1^* \mathcal{L} = T^{-1} \otimes \nu^* \mathcal{O}(2\Theta_{p_0}).$$

The Picard group of $\mathbb{P}U_1$ is given by:

$$\operatorname{Pic} \mathbb{P}U_1 = \nu^* \operatorname{Pic} J^1(C) \otimes \mathbb{Z}\{T^{-1}\}$$

and by (2.0.4) $\epsilon_1^* \mathcal{L}\big|_{\mathbb{P}_L} = \mathcal{O}_{\mathbb{P}_L}(1)$. Thus:

$$\epsilon_1^* \mathcal{L} = T^{-1} \otimes \nu^* \mathcal{N}$$

for some line bundle $\mathcal{N} \to J^1(C)$. It just remains to show that $\mathcal{N} = \mathcal{O}(2\Theta_{p_0})$. To see this consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}U_1 & \xrightarrow{\epsilon_1} & \mathcal{SU}_C(2) \\ s\uparrow\downarrow\nu & & \uparrow \mathrm{Kum} \\ J^1 & \xrightarrow{t-p_0} & J^0 \end{array}$$

where s is a section of $\mathbb{P}U_1 \to J^1(C)$ given by associating to each $L \in J^1(C)$ the point $p_0 \in C \to \mathbb{P}_L$ (where C is mapped to the fibre $\mathbb{P}_L$ via the linear system $|KL^2|$.) By commutativity:

$$\begin{aligned} s^*\epsilon_1^*\mathcal{L} &= t_{-p_0}^*\mathrm{Kum}^*\mathcal{L} \\ &= t_{-p_0}^*\mathcal{O}_{J^0}(2\Theta_C) \\ &= \mathcal{O}_{J^1}(2\Theta_{p_0}). \end{aligned}$$

But:

$$\begin{aligned} s^*\epsilon_1^*\mathcal{L} &= s^*(T^{-1}\otimes\nu^*\mathcal{N}) \\ &= s^*T^{-1}\otimes\mathcal{N}. \end{aligned}$$

Thus the proof is complete if it can be shown that s^*T^{-1} is the trivial line bundle on $J^1(C)$. The section s above corresponds to some line subbundle $\mathcal{M}\subset U_1$ and $s^*T^{-1}=\mathcal{M}$. Now s was defined by fixing p_0 in each fibre $\mathbb{P}_L$ and allowing $L\in J^1(C)$ to vary i.e. p_0 corresponds to the one-dimensional kernel of:

$$H^1(L^{-2})\longrightarrow H^1(L^{-2}(p_0)).$$

This comes from the exact sequence:

$$0\longrightarrow L^{-2}\longrightarrow L^{-2}(p_0)\longrightarrow L^{-2}(p_0)\Big|_{p_0}\longrightarrow 0 \tag{4}$$

i.e. the one-dimensional kernel is given by $H^0(L^{-2}(p_0)\Big|_{p_0})$. Globalising (4) to an exact sequence on $C\times J^1(C)$ gives:

$$0\longrightarrow\mathcal{P}^{-2}\longrightarrow\mathcal{P}^{-2}(\Gamma)\longrightarrow\mathcal{P}^{-2}(\Gamma)\Big|_{\Gamma}\longrightarrow 0$$

where Γ is the product divisor $\{p_0\}\times J^1(C)$ and $\mathcal{P}$ is a Poincaré line bundle on $C\times J^1(C)$ restricting to the trivial line bundle on Γ. Then by construction $\mathcal{M}=\pi_*\mathcal{P}^{-2}(\Gamma)\Big|_{\Gamma}$ where $\pi: C\times J^1(C)\to J^1(C)$ is projection. Now $\Gamma\Big|_{\Gamma}=N_{\Gamma/C\times J^1}=\mathcal{O}_{C\times J^1}$ and by the definition of $\mathcal{P}$:

$$\mathcal{M}=\pi_*\mathcal{P}^{-2}\Big|_{\Gamma}=\pi_*\mathcal{O}_{\Gamma}\cong\mathcal{O}_{J^1}$$

as required. $\square$

Remark 3.2.8. Analogous to the discussion in section 3.1 U_1 is in fact isomorphic to $V_{g-2} \otimes \mathcal{O}_{J^1}(2\Theta_p)$ where V_{g-2} is a bundle on $J^1(C)$ sitting in the exact sequence:

$$0 \longrightarrow V_{g-2} \longrightarrow H^0(J^{g-1}(C), 2\Theta) \otimes \mathcal{O}_{J^1} \longrightarrow Q_{g-2} \longrightarrow 0$$

the fibre of V_{g-2} over $L \in J^1(C)$ being given by the sections of $H^0(J^{g-1}(C), 2\Theta)$ vanishing on $W_{g-2}(C) + L \subset J^{g-1}(C)$. See [OP] for details.

Proposition 3.2.9. *Let $g \geq 4$. Then the subvariety $\mathcal{V} \subset \mathcal{SU}_C(2)$ of bundles with Segre invariant ≤ 2 has degree:*

$$\deg \mathcal{V} = (-4)^g \sum_{k=0}^{g} (-1)^k \frac{k!}{2^k} \binom{g}{k} \binom{2g}{k}$$

Proof. The image $\phi_{\mathcal{L}}(\mathcal{V})$ represents a homology class in $H_{4g}(\mathbb{P}^{2^g-1}, \mathbb{Z})$ so the degree of $\phi_{\mathcal{L}}(\mathcal{V})$ is given by:

$$\begin{aligned}\deg \phi_{\mathcal{L}}(\mathcal{V}) &= h^{2g} \cdot \eta_{\mathcal{V}} \\ &= (-h)^{2g} \cdot \eta_{\mathcal{V}}\end{aligned}$$

where h is the class of a hyperplane in $\mathbb{P}^{2^g-1}$ and $\eta_{\mathcal{V}}$ is the fundamental class of $\phi_{\mathcal{L}}(\mathcal{V})$. By (3.2.1) and the pullback map:

$$H^*(\mathbb{P}^{2^g-1}, \mathbb{Z}) \longrightarrow H^*(\mathbb{P}U_1, \mathbb{Z})$$

this is equivalent to calculating $f_1^*(-h)^{2g}$. By (3.2.7) and the fact that θ^j vanishes for all $j \geq g+1$ we have:

$$\begin{aligned}f_1^*(-h)^{2g} &= (\eta - 2\theta)^{2g} \\ &= \sum_{k=0}^{g} (-2)^k \binom{2g}{k} \eta^{2g-k}\theta^k \in H^{4g}(\mathbb{P}U_1, \mathbb{Z}) \cong \mathbb{Z}.\end{aligned} \tag{5}$$

By (3.2.3), (3.2.6) and using the fact that $\theta^g = g!$ in $H^{2g}(J^1, \mathbb{Z}) \cong \mathbb{Z}$ (see [ACGH, p21] for example) and $\eta^g = (-1)^g$ in $H^{2g}(\mathbb{P}U_1, \mathbb{Z}) \cong \mathbb{Z}$ we have:

$$\begin{aligned}
\eta^{2g-k}\theta^k &= c_{g-k}(U_1)\eta^g\theta^k \\
&= \frac{(4\theta)^{g-k}}{(g-k)!}(-1)^g\theta^k \\
&= \frac{(-1)^g 4^{g-k} g!}{(g-k)!} \\
&= (-1)^g 4^{g-k} k! \binom{g}{k}.
\end{aligned}$$

Substituting this into (5) gives the required result. □

By (1.2.1) and the locally factorial property of $\mathcal{SU}_C(2)$ [DN] the variety $\mathcal{V} \subset \mathcal{SU}_C(2)$ is a *Cartier* divisor for $g = 4$. Since the Picard group of $\mathcal{SU}_C(2)$ is infinite cyclic and generated by $\mathcal{L}$, $\mathcal{V}$ will be in the linear equivalence class $|m\Theta_{\mathcal{L}}|$ for some $m \in \mathbb{Z}$ and so given by the complete intersection of $\mathcal{SU}_C(2)$ with some hypersurface of degree m in $\mathbb{P}^{2^g-1}$. The degree of $\mathcal{SU}_C(2)$ is then given by $\frac{\deg \mathcal{V}}{m}$. Section (3.4) calculates the integer m.

3.3. Spectral Curves

This section will review some background material needed for the computations of the class of $\mathcal{V}$ given in the next section. This is the work done by Beauville, Narasimhan, and Ramanan [BNR] providing a way of transferring calculations from the non-abelian moduli space $\mathcal{SU}_C(r)$ to the abelian Jacobian of some curve B. More precisely this so-called spectral curve B comes equipped with a natural r-sheeted covering of C. A generic vector bundle on C of rank r is then the direct image of some line bundle on B.

While their results have been proved for arbitrary rank r, and degree n bundles, the following will restrict to the case $r = 2$, $n = 0$. The main result of [BNR] is:

Theorem 3.3.1. *For C a smooth, irreducible, projective curve of genus $g \geq 2$ there exists a $2:1$ covering:*

$$q : B \longrightarrow C$$

branched in $4g-4$ points, with B smooth and irreducible, such that the direct image map:

$$q_* : \widehat{J}_B^{2g-2} \longrightarrow \mathcal{U}_C(2,0)$$

is dominant. Here $\widehat{J}_B^{2g-2}$ denotes the set of line bundles $N \in J_B^{2g-2}$ for which q_*N is semistable.

Remark 3.3.2. To construct B explicitly consider the projective bundle:

$$\mathbb{P} \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O}_C \oplus K) \xrightarrow{q} C$$

on C and let $s_1 \in H^0(K)$, $s_2 \in H^0(K^2)$ be generic sections. The bundles $q^*K \otimes \mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{\mathbb{P}}(1)$ have canonical sections x and y respectively. More explicitly y is given by the pullback of the canonical section of:

$$q_*\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_C \oplus K^{-1}$$

i.e. the constant section of $\mathcal{O}_C$. Similarly x is given by the pullback of the canonical section of:

$$q_*(q^*K \otimes \mathcal{O}_{\mathbb{P}}(1)) \cong K \oplus \mathcal{O}_C.$$

Now define $B \subset \mathbb{P}(\mathcal{O}_C \oplus K)$ to be the zero-scheme of the section:

$$x^2 + q^*s_1 x.y + q^*s_2 y^2 \in H^0(\mathbb{P}, q^*K^2 \otimes \mathcal{O}_{\mathbb{P}}(2)).$$

Clearly q restricted to B is 2:1. Furthermore $\mathcal{O}_{\mathbb{P}}(1)\big|_B = \mathcal{O}_B$, thus:

$$q_*\mathcal{O}_B = \mathcal{O}_C \oplus K^{-1}.$$

Note that by Riemann-Roch the genus of B is given by:

$$\begin{aligned} 1 - g(B) &= \chi(B, \mathcal{O}_B) \\ &= \chi(C, q_*(\mathcal{O}_B)) \\ &= \chi(\mathcal{O}_C) + \chi(K^{-1}) \\ &= (-g+1) + (-2g+2-g+1) \end{aligned}$$

i.e. $g(B) = 4g - 3$ which is the dimension of $\mathcal{U}_C(2,0)$.

In order to calculate the class of $\mathcal{V} \subset \mathcal{SU}_C(2)$ one needs to restrict to bundles q_*N with trivial determinant. If $\mathrm{Nm} : J_B^{2g-2} \to J_C^{2g-2}$ is the norm map associated to q then:

$$\begin{aligned} \det q_*N &= \mathrm{Nm}N \otimes \det q_*\mathcal{O}_B \\ &= \mathrm{Nm}N \otimes K^{-1} \end{aligned}$$

and so the subvariety one needs to restrict to is:

$$P \stackrel{\text{def}}{=} \mathrm{Nm}^{-1}K$$

i.e. the Prym variety associated to the double cover $B \xrightarrow{2:1} C$.

Proposition 3.3.3. *The direct image map of (*3.3.1*) induces a dominant rational map $P \to \mathcal{SU}_C(2)$.*

Now the induced map $q^* : J_C \to J_B$ is injective [BNR, Remark 3.10] so the following result [BNR, Prop 2.6] can be applied to the double cover $B \to C$ above:

Proposition 3.3.4. *Let $q : B \to C$ be a finite morphism of projective, nonsingular curves such that the induced map $q^* : J_C \to J_B$ is injective. If Δ is the ramification divisor and $P = Nm^{-1}(\det q_*\mathcal{O}(\Delta))$, then there exists a line bundle $\tau \to P$ such that:*

$$H^0(J_C^{g_C-1}, \mathcal{O}(2\Theta_C)) \cong H^0(P, \tau)^*.$$

Remark 3.3.5. The proposition above is proved by considering the addition map:

$$\alpha_{g-1} : P \times q^* J_C^{g-1} \longrightarrow J_B^{4g-4}$$

and showing that that pullback line bundle $\alpha_{g-1}^* \mathcal{O}(\Theta_B)$ is isomorphic to $p_1^* \tau \otimes p_2^*(2\Theta_C)$ where p_1, p_2 are projections of $P \times q^* J_C^{g-1}$ onto the first and second factors respectively and τ is some line bundle on P. The pullback of the natural section of $\mathcal{O}(\Theta_B)$ then gives rise to a non-degenerate element of $H^0(P, \tau) \otimes H^0(J_C^{g-1}, \mathcal{O}(2\Theta_C))$ and the isomorphism of (3.3.4) follows. Moreover τ is the pullback of the line bundle $\mathcal{O}(\Theta_{\mathcal{L}})$ on $\mathcal{SU}_C(2)$ via the direct image map $P \to \mathcal{SU}_C(2)$ (see [BNR, 5.5].)

3.4. The class of the g-plane ruling

Assume that the genus of C is even. By (1.2.1) the subvariety $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{SU}_C^{g-2} \subset \mathcal{SU}_C(2)$ of bundles with Segre invariant $\leq g-2$ has dimension equal to $3g-4$ i.e. $\mathcal{D}$ is a Cartier divisor on $\mathcal{SU}_C(2)$. If $g = 4$ then clearly the divisors $\mathcal{D}$ and the image of the g-plane ruling $\mathcal{V}$ coincide. The following calculates the class of $\mathcal{D}$:

Proposition 3.4.1. *The divisor $\mathcal{D} = \mathcal{SU}_C^{g-2}(2) \subset \mathcal{SU}_C(2)$ of bundles with Segre invariant $\leq g-2$ is a member of the linear system $|2^g \Theta_{\mathcal{L}}|$.*

PROOF. Let $q : B \xrightarrow{2:1} C$ be the double cover of the last section. By (3.3.5) calculating the class of $\mathcal{D} \subset \mathcal{SU}_C(2)$ with respect to $\Theta_{\mathcal{L}}$ is equivalent to calculating the class of the pre-image in P with respect to the line bundle $\tau \to P$.

Suppose $E \in \mathcal{SU}_C(2)$ is in the image of the dominant rational map $P \xrightarrow{q_*} \mathcal{SU}_C(2)$. Then $E = q_*\zeta$ for some $\zeta \in P$ and the condition for $E \in \mathcal{D}$ is given by:

$$H^0(C, L \otimes q_*\zeta) \neq 0 \qquad \text{for some } L \in J^{\frac{g-2}{2}}(C)$$

which, by the projection formula, is equivalent to:

$$H^0(B, q^*L \otimes \zeta) \neq 0 \qquad \text{for some } q^*L \in q^*J^{\frac{g-2}{2}}(C).$$

Consider the addition map:

$$q^*J^{\frac{g-2}{2}}(C) \times P \xrightarrow{\alpha} J^{3g-4}(B)$$
$$(q^*L, \zeta) \longmapsto q^*L \otimes \zeta$$

then $H^0(B, q^*L \otimes \zeta) \neq 0$ if and only if $q^*L \otimes \zeta \in W_{3g-4}(B)$. Hence the class of $(q_*)^{-1}(\mathfrak{D})$ is given by:

$$(p_2)_*\alpha^*[W_{3g-4}(B)]$$

where $p_2 : q^*J^{\frac{g-2}{2}}(C) \times P \longrightarrow P$ is projection onto the second factor. By the standard formula [ACGH, p212]:

$$[W_{3g-4}(B)] = \frac{\theta_B^{g+1}}{(g+1)!}.$$

By (3.3.5):

$$\begin{aligned}
\alpha^*[W_{3g-4}(B)] &= \frac{(p_1^*2\theta_C + p_2^*c_1(\tau))^{g+1}}{(g+1)!} \\
&= \frac{1}{(g+1)!}(p_1^*(2\theta_C)^{g+1} + (g+1)p_1^*(2\theta_C)^g p_2^*c_1(\tau) + \cdots + p_2^*c_1^{g+1}(\tau)) \\
&\in H^{2g+2}(q^*J^{\frac{g-2}{2}}(C) \times P, \mathbb{Z}).
\end{aligned}$$

The second term is the only one which will contribute anything under the Gysin homomorphism:

$$(p_2)_* : H^*(q^*J^{\frac{g-2}{2}}(C) \times P, \mathbb{Z}) \longrightarrow H^{*-2g}(P, \mathbb{Z})$$

and since $(p_2)_*p_1^*(2\theta_C)^g = 2^g \cdot g!$ we have:

$$\begin{aligned}
(p_2)_*\alpha^*[W_{3g-4}(B)] &= \frac{(g+1)\cdot 2^g \cdot g!}{(g+1)!}c_1(\tau) \\
&= 2^g c_1(\tau).
\end{aligned}$$

By (3.3.5) this implies that $\mathfrak{D} \in |2^g\Theta_{\mathcal{L}}|$. $\square$

As predicted by (1.4.1):

Proposition 3.4.2. *If C is a non-hyperelliptic curve of genus 4, then the moduli space $\mathcal{SU}_C(2) \subset \mathbb{P}^{15}$ has degree 96.*

PROOF. By (3.4.1) $\mathfrak{D} \in |16\Theta_{\mathcal{L}}|$ and so $\deg \mathfrak{D} = 16 \deg \mathcal{SU}_C(2)$, but by (3.2.9) $\deg \mathfrak{D} = 1536$. Thus:

$$\deg \mathcal{SU}_C(2) = \frac{1536}{16} = 96. \quad \square$$

CHAPTER 4

Trisecants to the Kummer variety

This chapter considers the trisecants to the Kummer variety $\mathcal{K} \subset \mathcal{SU}_C(2)$. Oxbury, Pauly and Previato [OPP] have recently studied the Brill-Noether loci of $\mathcal{SU}_C(2, K)$ and in doing so have shown that these so-called Fay trisecants are all contained in the g-plane ruling $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$ as trisecants to the curve $C \to \mathbb{P}_L$ for some $L \in J^1$. This chapter is concerned with what varieties these trisecants sweep out in each fibre $\mathbb{P}_L$ for curves of genus 3, 4 and 5.

4.1. Fay trisecants

The Kummer variety $\mathcal{K} = J^0/\pm \overset{\phi}{\hookrightarrow} |2\Theta|$ admits a four dimensional family of trisecants and the existence of such a family characterises Jacobians of algebraic curves among all principally polarised abelian varieties. More precisely this family of trisecants is given by the fibred product:

$$\begin{array}{ccc} \mathfrak{F} & \longrightarrow & S^4C \\ \big\downarrow & & \big\downarrow{\scriptstyle Abel-Jacobi} \\ J^2(C) & \xrightarrow{sq} & J^4(C) \end{array}$$

i.e. pairs $(M, D) \in J^2(C) \times S^4C$ such that $M^2 = \mathcal{O}_C(D)$. If $D = p+q+r+s$ then it can be shown [Mu2] that:

$$\phi(M(-p-q)) = \phi(M(-r-s))$$

$$\phi(M(-p-r)) = \phi(M(-q-s))$$

$$\phi(M(-q-r)) = \phi(M(-p-s))$$

are 3 collinear points of $\mathcal{K} \subset \mathbb{P}^{2^g-1}$.

The following gives a brief account of why each of these trisecants is also a trisecant of $C \to \mathbb{P}_L$ for some $L \in J^1(C)$. More importantly (4.1.2) gives a condition for a divisor of degree 3 to correspond to such a trisecant. See [OPP] for details.

First recall the Hecke correspondence between $\mathcal{SU}_C(2)$ and $\mathcal{SU}_C(2, \mathcal{O}_C(p))$: any bundle $F \in \mathcal{SU}_C(2, \mathcal{O}_C(p))$ occurs in an extension of the form:

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_p \longrightarrow 0$$

where $\mathbb{C}_p$ is the skyscraper sheaf supported at p. There exists a $\mathbb{P}^1$ of these extensions and it can be shown that the kernel E is semistable. Hence this defines a Hecke line $l_F \subset \mathcal{SU}_C(2)$ (cf (2.2.4).)

Proposition 4.1.1. [OPP, Thm 1.4] *Let $l_F \subset \mathcal{SU}_C(2)$ be a Hecke line. Then:*

1. There exists a bijection between points of intersection of $l_F \cap \mathcal{K}$, counted with multiplicity, and line bundles $N \subset F$ with $\deg N = 0$.

2. $l_F \cap \mathcal{K} \neq \emptyset$ if and only if l_F is contained in a g-plane $\mathbb{P}_L$ for some $L \in J^1(C)$. Moreover if C is non-hyperelliptic and $l_F \cap \mathcal{K}$ has cardinality k then the number of such g-planes is $1 + \binom{k}{2}$.

Lemma 4.1.2. [OPP, Thm 2.1] *The Fay trisecants are precisely the Hecke lines trisecant to the Kummer variety. In particular if $p, q, r \in C$ then $\overline{pqr}$ is a trisecant of $C \to \mathbb{P}_L$ if and only if:*

$$L^2(s) = \mathcal{O}_C(p+q+r)$$

for some $s \in C$.

This result is used extensively throughout the chapter and we give the proof for the readers convenience:

PROOF. Let $l_F \subset \mathcal{SU}_C(2)$ be any Hecke line, with $\det F = \mathcal{O}_C(p)$. We look for the condition for l_F to be a trisecant of $\mathcal{K}$. If l_F intersects the Kummer variety then there is an exact sequence:

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_p \longrightarrow 0$$

where E is in the S-equivalence class of $N \oplus N^{-1}$ for some line bundle N of degree zero. So either N or N^{-1} is a line subbundle of F (necessarily maximal by the stability of F.) Suppose N is a line subbundle of F then by (4.1.1) l_F is trisecant to $\mathcal{K}$ if and only if F has two further maximal line subbundles. Now a result of Lange and Narasimhan [LN] shows that there exists a bijection between maximal line subbundles of a rank 2 bundle and certain secant line bundles (cf (5.2.1).) In the present case the maximal line subbundles of F (other than N) correspond to points of C mapped to the extension class of:

$$0 \longrightarrow N \longrightarrow F \longrightarrow N^{-1}(p) \longrightarrow 0$$

under $C \to \mathbb{P}H^0(C, KN^{-2}(p))^*$. Thus l_F is a trisecant to $\mathcal{K}$ if and only if the image of C has a node. This occurs if and only if:

$$\begin{aligned} h^0(KN^{-2}(p-q-r)) &\geq h^0(KN^{-2}(p)) - 1 \\ &= g-1 \end{aligned}$$

for some $p, q, r \in C$, i.e. $h^0(N^2(-p+q+r)) \geq 1$ or $N^2(-p+q+r) = \mathcal{O}_C(s)$ for some $s \in C$. By taking $L = N^{-1}(p)$ one sees that l_F is a trisecant $\overline{pqr}$ of $C \to \mathbb{P}_L$ if and only if $L^2(s) = \mathcal{O}_C(p+q+r)$. Now taking $M = L(s)$ it is easy to see that the points of intersection of l_F with the Kummer variety are given by $M(-p-q)$, $M(-p-r)$ and $M(-q-r)$. $\square$

Remark 4.1.3. Suppose C is non-hyperelliptic and $L^2(s) = \mathcal{O}_C(p+q+s)$ so that $C \xrightarrow{\phi_L} \mathbb{P}_L$ has a double point at $\phi_L(p) = \phi_L(q)$. Then taking $L = N^{-1}(s)$ gives:

$$N^2(-s+p+q) = \mathcal{O}_C(s) \tag{1}$$

i.e. $\mathbb{P}H^0(KN^{-2}(s))^*$ has a node. By the bijection of Lange and Narasimhan (5.2.1) this node corresponds to two maximal line subbundles of some rank 2 bundle F. Explicitly these line subbundles are given as:

$$N_1 = N^{-1}(s-p) \quad \text{and} \quad N_2 = N^{-1}(s-q).$$

But by (1) $N_1 = N_2^{-1}$ and so by (4.1.1) the trisecant corresponding to $L^2(s)$ is tangential to the Kummer variety at $N_1 \oplus N_1^{-1} = N_2 \oplus N_2^{-1}$.

Corollary 4.1.4. [OPP, Cor 2.2] ***1.*** *If C is non-hyperelliptic then no Fay trisecant can have more than 3 intersection points with $\mathcal{K}$.*

2. *If C is hyperelliptic then every Fay trisecant is in fact a quadrisecant.*

Remark 4.1.5. If C is hyperelliptic then, with the same notation as in (4.1.2), the quadrisecants to C are given by the divisors $p+q+r+\iota(s)$ where ι is the hyperelliptic involution.

The following result [OPP, Prop 1.2] will also be useful in determining the degree of the variety of Fay trisecants in the moduli space $\mathcal{SU}_C(2)$ itself:

Proposition 4.1.6. *For L, $M \in J^1(C)$ the intersection $\mathbb{P}_L \cap \mathbb{P}_M$ (in the moduli space $\mathcal{SU}_C(2)$) is given by:*

1. *the secant line $\overline{pq}$ of the curve (in either $\mathbb{P}_L$ or $\mathbb{P}_M$) if $LM = \mathcal{O}_C(p+q)$*

2. *the point $L(-p) \oplus L^{-1}(p) \in \mathcal{K}$ if $h^0(C, LM) = 0$ and $LM^{-1} = \mathcal{O}_C(p-q)$*

3. *empty otherwise.*

4.2. Ruled Surfaces

Most of the new results in this chapter concern ruled surfaces over an algebraic curve. This section will fix the notation used and also review some of the results needed. All these results on ruled surfaces can be found in either [H] or [GH].

Any ruled surface on C can be written as the projectivization of some rank two bundle $V \xrightarrow{p} C$ i.e. the points in the fibre $(\mathbb{P}V)_x$ $(x \in C)$ correspond to the one-dimensional linear subspaces of the two-dimensional vector space V_x. Moreover the Picard group and the group of numerical equivalence classes of divisors are given by:

$$Pic\,\mathbb{P}V = \mathbb{Z}\{C_0\} \oplus p^* Pic\,C$$

$$Num\,\mathbb{P}V = \mathbb{Z} \oplus \mathbb{Z}$$

where $C_0 \subset \mathbb{P}V$ is some section. Thus if f is a fibre of p and $\mathfrak{b}$ is a divisor of C write $aC_0 + \mathfrak{b}f$ for a divisor on $\mathbb{P}V$ and $aC_0 + bf$ for a numerical divisor.

The bundle V is said to be normalised if $H^0(C, V) \neq 0$ and $H^0(C, L \otimes V) = 0$ for all line bundles L on C with $\deg L < 0$. If V is normalised then there exists a section $C_0 \subset \mathbb{P}V$ such that $\mathcal{O}_{\mathbb{P}V}(C_0) = \mathcal{O}_{\mathbb{P}V}(1)$. Moreover numerically $C_0.C_0 = -\deg V$ and there exists the following isomorphisms:

$$p_*(\mathcal{O}_{\mathbb{P}V}(1)) \cong V^*$$

$$H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1) \otimes p^*M) \cong H^0(C, V^* \otimes M)$$

where M is any line bundle on C.

Now restrict to *rational* ruled surfaces. A classical result due to Grothendieck says that any vector bundle on $\mathbb{P}^1$ is decomposable. Thus write:

$$S_n \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}) \quad \text{for } n \geq 0.$$

Denote by $E_0 \subset S_n$ the zero section of S_n i.e. the section corresponding to the section $(0,1)$ of $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$. Note that $E_0.E_0 = n$. Let $E_\infty \subset S_n$ be the closure of the curve given by the section $(\sigma, 0)$ of $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ (i.e. $(\sigma,0)$ gives a curve away from the zeros of σ.) It can be shown that $E_\infty.E_\infty = -n$.

Let $\phi_{k,n}$ be the map $S_n \to \mathbb{P}H^0(S_n, E_0 + kf)^*$ given by the linear system $|E_0 + kf|$ ($k \in \mathbb{Z}$), and denote the image by $S_{k,n}$. Then these are the rational ruled scrolls, which are the surfaces of minimal degree:

$$\begin{aligned}(E_0 + kf).(E_0 + kf) &= E_0.E_0 + 2E_0.f \\ &= n + 2k\end{aligned}$$

in $\mathbb{P}^{n+2k+1}$.

Let D_0 and D_∞ be the images $\phi_{k,n}(E_0)$ and $\phi_{k,n}(E_\infty)$ respectively. D_∞ is referred to as the directrix of the rational ruled scroll and is unique if $n > 0$.

Remark 4.2.1. The scrolls $S_{k,n}$ can be described more geometrically as the union of straight lines joining points on the rational normal curve $D_0 \subset V_{n+k}$ with corresponding points on the directrix $D_\infty \subset V_k$, where V_{n+k} and V_k are disjoint linear subspaces of $\mathbb{P}^{n+2k+1}$.

4.3. Quadrisecants to hyperelliptic Kummers

The results of the next two sections can be summarised in the two tables at the end of this chapter.

As stated in (4.1.4) hyperelliptic curves are special in the sense that trisecants to the Kummer are in fact *quadrisecants.* This section describes the variety of quadrisecants in each $\mathbb{P}_L$ for hyperelliptic curves of any genus. Throughout this section let $\iota : C \to C$

be the hyperelliptic involution and H a hyperelliptic divisor on C.

Proposition 4.3.1. *Suppose C is hyperelliptic of genus 3 and $h^0(L^2) = 0$. Then the variety of quadrisecants to $C \to \mathbb{P}H^0(C, KL^2)^*$ is a smooth quadric surface.*

PROOF. This follows from a result in [ACGH, p221, Ex C-3], namely that if a curve $C \to \mathbb{P}^3$ is hyperelliptic and linearly normal then the linear series cut out on C by quadrics has vector space dimension ≤ 9 i.e. the restriction map:

$$Sym^2 H^0(KL^2) \longrightarrow H^0(K^2L^4) \cong \mathbb{C}^{10}$$

has a kernel and hence C lies on a quadric. Clearly every quadrisecant of C must lie on this quadric too. To see that the quadric is smooth note that no two quadrisecants can meet. Otherwise the two lines would span a two-plane containing seven or eight points of the curve (depending on whether the quadrisecants meet at the curve or not.) This is a contradiction since the curve has degree six. □

For a more direct proof of this result see (4.4.5).

Proposition 4.3.2. *Let C be any curve of genus 4 and $h^0(L^2) = 0$. Then the variety of trisecants to $C \xrightarrow{\phi_L} \mathbb{P}H^0(C, KL^2)^* \cong \mathbb{P}^4$ is a finite number of lines. In particular if C is hyperelliptic then it has a unique quadrisecant.*

PROOF. By (4.1.2) trisecants to $C \to \mathbb{P}_L$ are given by effective divisors of the form $L^2(s)$ for some $s \in C$. Now $h^0(L^2) = 0$ so $L^2(s) \notin W_3^1(C)$ for any $s \in C$. Also note that $L^2(s)$ is not contained in $W_3(C)$ *for all* $s \in C$ otherwise the set equality [ACGH, p266] :

$$W_2(C) = \bigcap_{s \in C} (W_3(C) - u(s))$$

would imply that $h^0(L^2) = 1$. Now varying s gives a copy of $C \subset J^3(C)$, namely $L^2 + W_1(C)$, which will intersect the theta divisor a finite number of times away from the singular locus. The actual number in general can be calculated using their respective cohomology classes i.e.

$$\frac{\theta^3}{3!}.\theta = 4.$$

These four trisecants correspond to one quadrisecant if C is hyperelliptic: note that by (4.1.5) quadrisecants are given by divisors of the form $L^2(s + \iota(s)) = \mathcal{O}_C(D)$. Suppose $L^2(t + \iota(t)) = \mathcal{O}_C(D')$ corresponds to another quadrisecant ($s \neq t$). Then they either:

1. Span a hyperplane of $\mathbb{P}H^0(KL^2)^* \cong \mathbb{P}^4$ i.e. are skew;

2. Meet at a point of the curve $C \subset \mathbb{P}^4$; or

3. Meet away from the curve.

(1) cannot occur otherwise $L^2(s + \iota(s)) + L^2(t + \iota(t)) \sim KL^2$ i.e. $L^2 = H$.

(2) cannot occur otherwise $h^0(KL^2(-D'')) = 2$ where D'' is a divisor of degree seven.

(3) cannot occur otherwise $h^0(KL^2(-D - D')) = h^0(HL^{-2}) = 2$.

Hence $C \longrightarrow \mathbb{P}_L$ has a unique quadrisecant. □

See (4.4.10) and (4.4.11) for a discussion of these trisecants for non-hyperelliptic curves of genus 4.

Proposition 4.3.3. *Let C be any curve of genus $g \geq 5$ and suppose that $h^0(L^2) = 0$. Then $C \longrightarrow \mathbb{P}H^0(C, KL^2)^*$ admits a finite number of trisecants if and only if $L^2 \in W_3(C) - W_1(C)$. Furthermore if $W_4^1(C)$ is empty then it has a unique trisecant.*

Proof. By (4.1.2) $L^2(s)$ corresponds to a trisecant if and only if $L^2(s) \in W_3(C)$ for some $s \in C$ i.e. $L^2 \in W_3(C) - W_1(C)$. If this is the case then by the argument in the proof of (4.3.2) $(L^2 + W_1(C)) \cap W_3(C)$ is a proper intersection missing the singular locus of $W_3(C)$. Thus $C \longrightarrow \mathbb{P}_L$ has a finite number of trisecants.

If $L^2(s) = \mathcal{O}_C(D) \in W_3(C)$ and $L^2(t) = \mathcal{O}_C(D_1) \in W_3(C)$ for some $t \neq s$ then:

$$D + t \sim D_1 + s$$

i.e. $L^2(s+t) \in W_4^1(C)$. $\square$

Remark 4.3.4. If $g \geq 7$ then the Brill-Noether number $\rho(d,r) = g-(r+1)(g-d+r)$ of W_4^1 is negative. Thus a generic curve of genus $g \geq 7$ will have a unique trisecant in each $\mathbb{P}_L$ when $h^0(L^2) = 0$ and $L^2 \in W_3(C) - W_1(C)$.

Example 4.3.5. If C is hyperelliptic of genus $g \geq 5$ and $h^0(L^2) = 0$ with $L^2(s) = \mathcal{O}_C(p+q+r) \in W_3(C)$ for some $s \in C$ then $\overline{pqr\,\iota(s)}$ is the *unique* quadrisecant to $C \to \mathbb{P}_L$: by the proof of (4.3.3) it has more than one trisecant if:

$$p + q + r + t \sim D_1 + s$$

for some $t \neq s$. Since C is hyperelliptic a g_4^1 is obtained by adding 2 base points to the g_2^1. Thus $p+q+r+t \in g_4^1$ if and only if $t = \iota(p), \iota(q)$, or $\iota(r)$ (Note $p+q \notin g_2^1$ for example, otherwise $L^2(s) = H + r \in W_3^1(C)$ i.e. $h^0(L^2) = 1$.) So, for example:

$$L^2(s+\iota(p)) \sim p+q+r+\iota(p) \sim s+q+r+\iota(s)$$

and $L^2(\iota(p)) \sim q+r+\iota(s)$ corresponds to a trisecant of C. But $\overline{qr\,\iota(s)} = \overline{pqr\,\iota(s)}$ and so the original *quadrisecant* is recovered.

See (4.4.13) for a discussion of the number of trisecants to non-hyperelliptic curves of genus 5.

Proposition 4.3.6. *Let C be a hyperelliptic curve of genus $g \geq 3$ and suppose that $h^0(L^2) \geq 1$. Then the variety of quadrisecants to $C \xrightarrow{\phi_L} \mathbb{P}H^0(C, KL^2)^*$ is:*

1. A cone over a rational normal curve in $\mathbb{P}^{g-1}$ with vertex at the double point of $C \subset \mathbb{P}^g$ if $h^0(L^2) = 1$.

2. $Sec_2 C$ if $h^0(L^2) = 2$.

PROOF. If $h^0(L^2) = 1$ then $L^2 = \mathcal{O}_C(x+y)$ for some $x, y \in C$. C clearly has a double point at $\phi_L(x) = \phi_L(y)$ and by (4.1.2) the quadrisecants correspond to lines through this double point intersecting the curve again in the points s and $\iota(s)$. Projection of the curve away from the double point is clearly the hyperelliptic cover and the quadrisecants sweep out a cone over the rational normal curve $C \to \mathbb{P}H^0(K)^*$.
If $h^0(L^2) = 2$ then $L^2 = H$ and the map ϕ_L is hyperelliptic mapping C two to one onto a rational normal curve of degree g. By (4.1.2) all quadrisecants come from the linear system $|2H|$. These correspond to *bisecants* of the curve $\phi_L(C)$. □

Remark 4.3.7. Note that for genus 3 Sec_2C actually sweeps out the whole of $\mathbb{P}_L$, otherwise projecting from a point not on Sec_2C would give a planar curve of genus three and degree six with *no* singularities, contradicting the degree-genus formula.

4.4. Trisecants to non-hyperelliptic Kummers

This section fills in the gaps in the tables at the end of this chapter that weren't covered in the last section, namely the description of the variety of trisecants in $\mathbb{P}_L$ for non-hyperelliptic curves of genus 3, 4, and 5. The first result requires two preliminary lemmas:

Lemma 4.4.1. *Let ρ_1 and ρ_2 be the two projections of $C \times C$ onto the first and second factors respectively and Δ the diagonal of $C \times C$. Then:*

$$(R^1\rho_1)_*\mathcal{O}_{C\times C}(\Delta) \cong N_{C/J^1}$$

where N_{C/J^1} is the normal bundle of $C \hookrightarrow J^1(C)$.

PROOF. Taking the exact higher direct image of the sequence:

$$0 \longrightarrow \mathcal{O}_{C\times C} \longrightarrow \mathcal{O}_{C\times C}(\Delta) \longrightarrow \mathcal{O}_\Delta(\Delta) \longrightarrow 0$$

yields:

$$0 \to (\rho_1)_* \mathcal{O}_{C\times C} \to (\rho_1)_* \mathcal{O}_{C\times C}(\Delta) \to (\rho_1)_* \mathcal{O}_\Delta(\Delta) \xrightarrow{\alpha} (R^1\rho_1)_* \mathcal{O}_{C\times C}$$
$$\to (R^1\rho_1)_* \mathcal{O}_{C\times C}(\Delta) \to 0$$

By [ACGH, p171] there are isomorphisms:

$$(R^1\rho_1)_* \mathcal{O}_{C\times C} \cong T_{J^1}\Big|_C \quad ; \quad (\rho_1)_* \mathcal{O}_\Delta(\Delta) \cong T_C.$$

Thus in the present case α is injective and

$$(R^1\rho_1)_* \mathcal{O}_{C\times C}(\Delta) \cong \frac{T_{J^1}\Big|_C}{T_C} \cong N_{C/J^1}. \quad \square$$

Lemma 4.4.2. *Let C be a curve of genus 3 and L a line bundle of degree one on C with $h^0(L^2(s)) = 1$ for all $s \in C$. Then with the same notation as the last lemma the line bundle:*

$$N \overset{def}{=} ((\rho_1)_*(\rho_2^* L^2 \otimes \mathcal{O}_{C\times C}(\Delta)))^*$$

has degree two.

Proof. Let ξ be the pullback of the class of a point on C (via ρ_2), x the pullback of the class of a point on C (via ρ_1) and γ the $(1,1)$ part of the class of the diagonal Δ. We apply the Grothendieck-Riemann-Roch formula to the line bundle $M \overset{\text{def}}{=} \rho_2^* L^2 \otimes \mathcal{O}_{C\times C}(\Delta)$ i.e.

$$ch((\rho_1)_! M).td(C) = (\rho_1)_*[ch(M).td(C \times C)]. \tag{2}$$

In the present case $td(C) = 1 - 2\xi$ and $ch(M) = 1 + c_1(M) = 1 + 3\xi + \gamma + x$ (cf [ACGH, p338].) Since $h^1(L^2(s)) = 0$ for all $s \in C$ (2) reduces to:

$$\begin{aligned} chN^* &= (\rho_1)_*[(1 + 3\xi + \gamma + x)(1 - 2\xi)] \\ &= 1 - 2x. \end{aligned}$$

Thus N is a line bundle of degree two. $\square$

4.4.1. Genus 3.

Proposition 4.4.3. *Let C be non-hyperelliptic of genus 3 and suppose that $h^0(L^2) = 0$. Then the variety of trisecants to $C \to \mathbb{P}H^0(C, KL^2)^*$ is a ruled surface of degree 8, triple along the curve.*

PROOF. By (4.1.2) $\overline{pqr}$ is a trisecant of $C \hookrightarrow \mathbb{P}_L$ if and only if:

$$L^2(s) = \mathcal{O}_C(p+q+r)$$

for some $s \in C$ i.e. $\overline{pqr}$ is given by the projectivization of the kernel of the surjective map:

$$H^0(C, KL^2)^* \xrightarrow{\sigma} H^0(C, K(-s))^* \cong H^1(C, \mathcal{O}_C(s))$$

where σ is given by a non-zero element of $H^0(L^2(s))^* \cong \mathbb{C}$. So varying $s \in C$ gives a ruled surface of trisecants $\mathbb{P}V$ where V is a rank two vector bundle given by the short exact sequence:

$$0 \longrightarrow V \longrightarrow H^0(KL^2)^* \otimes \mathcal{O}_C \longrightarrow (R^1\rho_1)_*\mathcal{O}(\Delta) \otimes N \longrightarrow 0, \qquad (3)$$

and $N = ((\rho_1)_*(\rho_2^*L^2 \otimes \mathcal{O}_{C\times C}(\Delta)))^*$. By Lemmas (4.4.1) and (4.4.2) we see that:

$$\begin{aligned} \deg V &= -\deg(N_{C/J^1} \otimes N) \\ &= -8. \end{aligned}$$

So we have defined a map:

$$\mathbb{P}V \xrightarrow{\beta} \mathbb{P}H^0(KL^2)^*$$

given by some sublinear system of $|\mathcal{O}_{\mathbb{P}V}(1)|$. Thus the degree of the image of β satisfies:

$$\begin{aligned} (\deg\beta).(\deg \operatorname{im}\beta) &= \mathcal{O}_{\mathbb{P}V}(1).\mathcal{O}_{\mathbb{P}V}(1) \\ &= -\deg V \\ &= 8. \end{aligned}$$

Now $\deg \beta = 1$ since the trisecants of $C \hookrightarrow \mathbb{P}_L$ are parametrised by the curve $L^2 + W_1(C) \cong C$ and $L^2(s) = L^2(t)$ if and only if $s = t$. Thus $\deg \operatorname{im} \beta = 8$.

The surface is triple along $C \hookrightarrow \mathbb{P}H^0(C, KL^2)^*$ since there exist 3 trisecants through every point $p \in C$ i.e. project the curve away from $p \in C$ to obtain a planar curve of degree 5 and genus 3. The trisecants through p then correspond to the double points of this curve which is given by the standard degree-genus formula. Also note that this is the *only* singular locus of the ruled surface since its intersection with a generic two-plane is a planar curve of degree 8 and genus 3 having 6 triple points (corresponding to the points of intersection of $C \hookrightarrow \mathbb{P}^3$ with the two plane) which, by the degree-genus formula, are its only singular points. $\square$

Remark 4.4.4. Let $\mathcal{T} \subset \mathcal{SU}_C(2)$ be the reduced variety of Fay trisecants to $\mathcal{K} \subset \mathcal{SU}_C(2) \subset \mathbb{P}^7$. Then (4.4.3) can be used to calculate the degree of $\mathcal{T}$ i.e. (4.4.3) says that $\mathcal{T}$ cuts out a ruled surface of degree 8 in the generic fibre $\mathbb{P}_L$ of $\mathbb{P}U_1 \to J^1(C)$ (recall that $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$ is a linear map on the fibres of $\mathbb{P}U_1 \to J^1(C)$) with some possible extra components coming from the Fay trisecants *not* contained in $\mathbb{P}_L$. But each Fay trisecant is contained in *some* $\mathbb{P}_M$ $M \in J^1(C)$, so any extra components are given by (4.1.6). Suppose these extra components have degree n in $\mathbb{P}_L$ then provided each $\mathbb{P}_L \subset \mathcal{SU}_C(2)$ is not tangent to $\mathcal{T}$ it may be deduced that $\deg \mathcal{T} = 8 + n$.

Remark 4.4.5. Note that (4.3.1) may also be proved in a similar vein to that of (4.4.3). The fibres of the bundle $\mathbb{P}V$ correspond to trisecants of the curve but for hyperelliptic curves we have seen that the fibres in fact map to *quadrisecants.* Thus each quadrisecant corresponds to 4 trisecants and the map of the projective bundle into $\mathbb{P}^3$ is 4:1. Thus the image is a quadric surface which is smooth by the same argument as that in the proof of (4.3.1).

Remark 4.4.6. The quadric above can be constructed explicitly: by (4.1.5) the quadrisecants are given by divisors from the linear system $|HL^2|$ on the curve $C \subset$

$\mathbb{P}H^0(KL^2)^*$. More explicitly each quadrisecant is given by $\mathbb{P}(\ker\delta) \cong \mathbb{P}^1$ where δ is given by:

$$H^0(KL^2)^* \xrightarrow{\delta} H^0(H)^* \longrightarrow 0$$

δ is induced by a divisor from $|HL^2| \cong \mathbb{P}^1$. The linear system $|HL^2|$ has no base points and so varying the divisors in this linear system sweeps out the smooth quadric:

$$\mathbb{P}(\ker\delta) \times \left|HL^2\right| \cong \mathbb{P}^1 \times \mathbb{P}^1$$

in $\mathbb{P}H^0(KL^2)^*$. If $h^0(L^2) = 1$ then $|HL^2|$ has two base points corresponding to the double point of $C \to \mathbb{P}H^0(KL^2)^*$. The quadric is then singular with vertex at this double point (cf (4.3.6).)

Proposition 4.4.7. *Suppose C is non-hyperelliptic of genus 3 and $h^0(L^2) = 1$. Then the variety of trisecants to $C \xrightarrow{\phi_L} \mathbb{P}H^0(C, KL^2)^*$ is either:*

1. *The union of a cone over the canonical curve and a quadric cone if $L^2 = K(-2u)$ for some $u \in C$.*

2. *The union of a cone over the canonical curve and a smooth quadric if $L^2 \neq K(-2u)$ for any $u \in C$.*

PROOF. First note that since $h^0(L^2) = 1$ then $L^2 = \mathcal{O}_C(x + y)$ for some $x, y \in C$ and $C \to \mathbb{P}^3$ will have a double point at $\phi_L(x) = \phi_L(y)$. Thus $\overline{xys}$ corresponds to a trisecant for all $s \in C$. Projection away from $\phi_L(x) = \phi_L(y)$ maps C onto the canonical curve.

1. (see [LN, Prop 5.4]) If $L^2 = K(-2u)$ for some $u \in C$ then by (4.1.2) trisecants of $C \subset \mathbb{P}_L$ correspond to divisors in the complete linear system $|L^2(u)| \cong |K(-u)| \cong \mathbb{P}^1$. Consider the map given by this complete linear system:

$$f : C \xrightarrow{3:1} \mathbb{P}^1.$$

Pulling back $\mathcal{O}_{\mathbb{P}^1}(2)$ gives an injection:

$$H^0(\mathbb{P}^1, \mathcal{O}(2)) \hookrightarrow H^0(C, K^2(-2u)) = H^0(C, KL^2)$$

and $H^0(\mathbb{P}^1, \mathcal{O}(2))$ is of codimension one in $H^0(C, KL^2)$ so defines a point $(e) \in \mathbb{P}_L$. Projection from this point makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{|KL^2|} & \mathbb{P}_L \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle proj} \\ \mathbb{P}^1 & \xrightarrow{|\mathcal{O}_{\mathbb{P}^1}(2)|} & \mathbb{P}^2 \end{array}$$

and it is clear that the trisecants to $C \subset \mathbb{P}_L$ given by $|K(-u)|$ sweep out a cone over $\mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(2)} \mathbb{P}^2$ with vertex (e).

2. Again a trisecant to $C \to \mathbb{P}^3$ is given by the condition $L^2(s) = \mathcal{O}_C(p+q+r)$. If $L^2 = K(-u-v)$, $u \neq v$, then there exists two families of such trisecants, namely those corresponding to divisors in the linear systems $|K(-u)| \cong \mathbb{P}^1$ and $|K(-v)| \cong \mathbb{P}^1$.
Let $p_i + q_i + r_i \in |K(-u)|$ and l_i the corresponding trisecant then:

(a) *l_i meets no other trisecant from $|K(-u)|$*: firstly l_i and l_j don't meet at any point of C since C is non-hyperelliptic. Now suppose $l_i \cap l_j \neq \emptyset$ then $\overline{l_i l_j} \cong \mathbb{P}^2$ and so:

$$\begin{aligned} p_i + q_i + r_i + p_j + q_j + r_j \sim KL^2 &\Rightarrow K^2(-2u) \sim KL^2 \\ &\Rightarrow L^2 \sim K(-2u) \qquad \text{(Contradiction)} \end{aligned}$$

(b) *l_i meets every trisecant coming from $|K(-v)|$*: if $x+y+z \in |K(-v)|$ corresponds to some trisecant l' then $p_i + q_i + r_i + x + y + z \sim K^2(-u-v) \sim KL^2$. Hence l_i and l' span a hyperplane and so meet.
Thus the surface swept out by trisecants from $|K(-u)|$ and $|K(-v)|$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ i.e. a smooth quadric surface. □

Remark 4.4.8. Conversely if $C \to \mathbb{P}_L$ lies on a quadric cone then projecting from the vertex gives a 3:1 map $f : C \to \mathbb{P}^1$. Since C has genus 3 then every pencil of degree 3 is of the form $K(-u)$ for some $u \in C$ i.e. $f^*\mathcal{O}_{\mathbb{P}^1}(1) = K(-u)$. Then $f^*\mathcal{O}_{\mathbb{P}^1}(2) = KL^2 = K^2(-2u)$ i.e. $L^2 = K(-2u)$.

4.4.2. Genus 4.

Proposition 4.4.9. *Let C be a non-hyperelliptic curve of genus 4 and suppose that $h^0(L^2) = 0$. Then $C \subset \mathbb{P}_L$ admits 4 trisecants (in general).*

PROOF. Exactly the same as (4.3.2). □

Remark 4.4.10. As one might expect none of the four trisecants from (4.4.9) meet in a generic g-plane $\mathbb{P}_L$: if $L^2(s) = \mathcal{O}_C(D)$ and $L^2(t) = \mathcal{O}_C(D')$ correspond to two distinct trisecants meeting at a point away from $C \subset \mathbb{P}H^0(KL^2)^*$ then they span a two-plane and so:

$$\begin{aligned} h^0(KL^2(-D-D')) &= h^0(KL^{-2}(-s-t)) \\ &= 2. \end{aligned}$$

Since $\deg KL^{-2}(-s-t) = 2$ this implies that C is hyperelliptic. Two trisecants cannot meet at a point of C in a generic g-plane either: suppose two such trisecants exist (and meet at a point $r \in C$) i.e.

$$L^2(s) = \mathcal{O}_C(p+q+r) \quad \text{and} \quad L^2(t) = \mathcal{O}_C(r+u+v).$$

Then the condition on L for this to occur is given by:

$$\begin{aligned} p+q+r+t \sim u+v+r+s &\iff p+q+t \sim u+v+s \\ &\iff L^2(s+t-r) \in W_3^1(C) \\ &\iff L^2(s+t) \in W_3^1(C) + W_1(C) \\ &\iff L^2 \in ((W_3^1 + W_1) - W_1) - W_1). \end{aligned}$$

Now $W^1_3(C)$ is zero-dimensional for C non-hyperelliptic so two trisecants meet at a point of $C \subset \mathbb{P}_L$ if and only if L^2 lies in the 3-dimensional subset $((W^1_3 + W_1) - W_1) - W_1) \subset J^2(C)$.

Remark 4.4.11. There exists ≤ 3 trisecants of $C \subset \mathbb{P}_L$ if either the tangent space to $L^2 + W_1(C) \subset J^3(C)$ at $L^2(s)$ is contained in the tangent space to $W_3(C) \subset J^3(C)$ at $L^2(s)$, or $L^2 + W_1(C)$ passes through a singular point of $W_3(C)$. The latter case is ruled out since $h^0(L^2) = 0$. To see when the former case occurs note that if $L^2(s) = \mathcal{O}_C(p+q+r)$ is a point of $W_3(C)$ (necessarily smooth) then by the Riemann-Kempf singularity theorem:

$$\mathbb{P}T_{L^2(s)}W_3(C) = \overline{\phi_K(pqr)} \subset \mathbb{P}H^0(K)^* \cong \mathbb{P}^3$$

and:

$$\mathbb{P}T_sW_1(C) = \phi_K(s) \in C \subset \mathbb{P}H^0(K)^*.$$

Thus $T_{L^2(s)}(L^2 + W_1(C)) \subset T_{L^2(s)}W_3(C)$ if and only if $\overline{pqrs}$ spans a 2-plane on the canonical curve i.e. $h^0(K(-p-q-r-s)) = 1$. By Riemann-Roch this is the case if and only if $L^2(2s) \in W^1_4(C)$. Thus $C \subset \mathbb{P}_L$ has ≤ 3 trisecants if and only if:

$$\begin{aligned} L^2 &\in \bigcup_{s\in C}(W^1_4(C) - u(2s)) \setminus W_2(C) \\ &= \bigcup_{s\in C}(K(-2s) - W_2(C)) \setminus W_2(C). \end{aligned}$$

If x and y are the two residual points cut out by $\overline{pqrs}$ on the canonical curve then:

$$L^2(x) = \mathcal{O}_C(p+q+r-s+x) = K(-2s-y)$$

$$L^2(y) = \mathcal{O}_C(p+q+r-s+y) = K(-2s-x)$$

correspond to the two other trisecants of $C \subset \mathbb{P}_L$.

Proposition 4.4.12. *Let C be non-hyperelliptic of genus 4. Then the variety of trisecants to $\phi_L : C \to \mathbb{P}H^0(C, KL^2)^* \cong \mathbb{P}^4$ is given by:*

1. A cone over the canonical curve if $h^0(L^2) = 1$ and $L^2(s) \notin W^1_3(C)$ for any $s \in C$.

2. *The cone from case (1) and the rational normal scroll* $S_{1,1}$ *if* $L^2(s) \in W_3^1(C)$ *for some* $s \in C$.

PROOF. As before trisecants $\overline{pqr}$ of $C \subset \mathbb{P}H^0(C, KL^2)^*$ are given by the condition that $L^2(s) = \mathcal{O}_C(p+q+r)$ for some $s \in C$.

1. Exactly the same proof as the beginning of (4.4.7).

2. If $L^2(s) \in W_3^1(C)$ for some $s \in C$ then $h^0(L^2) = 1$ $(L^2 = \mathcal{O}_C(x+y))$ and we obtain the cone over the canonical curve as in case (**1**). As well as this sextic cone the divisors coming from the pencil $|L^2(s)| \cong \mathbb{P}^1$ all correspond to trisecants too. Note that if $L^2(s) \in W_3^1(C)$ then there exists no other point of C, $t \neq s$, with the same property. If this were the case then $\overline{xys}$ and $\overline{xyt}$ would correspond to the same ruling on the quadric containing the canonical curve $C \subset \mathbb{P}^3$, which implies that $h^0(K(-x-y-s-t)) = 2$ i.e. C is hyperelliptic. Denote the surface swept out by the pencil of trisecants coming from the complete linear system $|L^2(s)|$ by S. To see what this pencil of trisecants sweeps out first note that if $L^2(s) = \mathcal{O}_C(p+q+r)$ then $\overline{pqr}$ is also a trisecant of $C \to \mathbb{P}H^0(C, KL^2(s))^* \cong \mathbb{P}^5$ since:

$$h^0(KL^2(s)(-p-q-r)) = h^0(K) = 4.$$

So the pencil $|L^2(s)|$ also sweeps out some rational ruled surface $S' \subset \mathbb{P}^5$. S is then obtained by the projection of S' away from $s \in C \to \mathbb{P}H^0(KL^2(s))^*$. Secondly note that S is mapped, via projection away from the double point $\phi_L(x) = \phi_L(y)$, to the unique quadric Q containing the canonical curve $C \subset \mathbb{P}H^0(C,K)^* \cong \mathbb{P}^3$ (the pencil of trisecants being mapped to a ruling of this quadric.) In short there exists a commutative diagram:

$$\begin{tikzcd}
& S' \subset \mathbb{P}H^0(KL^2(s))^* \arrow[dl, "\pi_s"'] \arrow[dr, "\pi_{xys}"] & \\
S \subset \mathbb{P}H^0(KL^2)^* \arrow[rr, "\pi_{xy}"] & & \mathbb{P}H^0(K)^* \supset Q
\end{tikzcd}$$

where π_{xys} is projection away from the trisecant $\overline{xys} \subset \mathbb{P}^5$. If C has no vanishing theta null then the quadric Q is smooth and is given (in the usual notation of section 4.2) by:

$$\mathbb{P}V = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0+f|} \mathbb{P}H^0(\mathbb{P}V, E_0 + f)^* \cong \mathbb{P}H^0(K)^*.$$

Since π_{xys} is projection away from a fibre f of S' the commutative diagram above may be rewritten as:

$$\begin{tikzcd} & S' \subset \mathbb{P}H^0(\mathbb{P}V, E_0 + 2f)^* \arrow[dl, "\pi_s"'] \arrow[dr, "\pi_{xys}"] & \\ S \subset \mathbb{P}W^* \arrow[rr, "\pi_{xy}"] & & \mathbb{P}H^0(\mathbb{P}V, E_0 + f)^* \supset Q \end{tikzcd}$$

where $W \subset H^0(\mathbb{P}V, E_0 + 2f)$ is some sublinear system. Hence it can be concluded that S is the projection away from $s \in C \subset \mathbb{P}^5$ of the rational normal scroll:

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0+2f|} \mathbb{P}^5.$$

If C has a vanishing theta null then Q is singular and so given by:

$$\mathbb{P}V' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0'|} \mathbb{P}H^0(\mathbb{P}V', E_0')^* \cong \mathbb{P}H^0(K)^*.$$

Thus by the same argument S is given by projection away from $s \in C \subset \mathbb{P}^5$ of the rational normal scroll:

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0'+f|} \mathbb{P}^5.$$

The rational normal scrolls in these two cases are isomorphic to $S_{2,0}$ and $S_{1,2}$ respectively. In the first case the directrix of the scroll is not unique and so, by a standard computation, projection from a point on $S_{2,0}$ gives the rational normal scroll $S_{1,1} \subset \mathbb{P}^4$ (see [GH, p520] for details.) In the second case if $s \in C$ does not lie on the directrix of the scroll then by a similar calculation projection from s will give the scroll $S_{1,1} \subset \mathbb{P}^4$ again. If $s \in C$ lies on the directrix then projection from s gives the rational ruled surface $S_{0,3}$ i.e. the image of the map:

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0''|} \mathbb{P}^4.$$

This is a cone and every trisecant passes through the vertex. Thus any two trisecants span a two-plane and so:

$$h^0(KL^2 \otimes L^{-2}(-s) \otimes L^{-2}(-s)) = 2.$$

This implies that C is hyperelliptic. A contradiction. $\square$

4.3.3. Genus 5.

As was seen in (4.3.3) if $h^0(L^2) = 0$ then a curve $C \to \mathbb{P}_L$ of genus $g \geq 5$ has a trisecant if and only if $L^2 \in W_3(C) - W_1(C)$. It is unique if $W_4^1(C)$ is empty. If $W_4^1(C)$ is non-empty and $L^2(s) \in W_4^1(C) - W_1(C)$ then trisecants other than $L^2(s) = \mathcal{O}_C(D)$ are given by divisors of the type $L^2(t)$ where:

$$D + t \sim D' + s$$

for some divisor D' of degree 3.

Example 4.4.13. If C is non-hyperelliptic of genus 5 then $W_4^1(C)$ is non-empty and a g_4^1 is cut out by a ruling of two planes on a quadric of rank 3 or 4 containing the canonical curve $C \subset \mathbb{P}^4$ (see [ACGH, p208] .) In fact if the quadric has rank 4 then there exists another ruling cutting out a residual g_4^1 (if the quadric is of rank 3 the g_4^1 is autoresidual.)

Suppose $L^2(s) = \mathcal{O}_C(D)$ and $D + t \in g_4^1$ for some $t \in C$. Then $\overline{D+t}$ spans a two-plane on the canonical curve and this two-plane lies on a quadric containing the curve. There exists a unique two plane of this ruling passing through s and so $D+t \sim s+D'$ for some divisor D' of degree 3. Hence there exist 2 trisecants to $C \to \mathbb{P}_L$ namely those given by $L^2(s)$ and $L^2(t)$.

There are more than 2 trisecants if $\overline{D+t} = \overline{D+t+t'}$ for some $t' \neq t$. This is the case if and only if $h^0(K(-D-t-t'))=2$ i.e. C contains a g^1_3. We look at the case when C is a non-hyperelliptic trigonal curve separately. In this case $W^1_4(C)$ is isomorphic to two copies of C. One copy is cut out by lines through a fixed point on C given as a plane quintic with one double point (denote this curve by Γ_C) (cf [ACGH, p208]) and the other copy is given by linear systems of the form g^1_3+c, $c \in C$ i.e. cut out by a base point c and the pencil of lines through the double point of Γ_C. Then there are two ways for $D+t=p+q+r+t$ to be in a g^1_4 for some $t \in C$. Either $\overline{pqr}$ is collinear on Γ_C, or one of $\overline{pq}$, $\overline{pr}$, $\overline{qr}$ passes through the double point of Γ_C. Suppose $L^2(s)=\mathcal{O}_C(p+q+r)$ is collinear on Γ_C, cutting out two residual points t and t' (neither equal to s.) Then:

$$p+q+r+t \sim D_1+s$$

$$p+q+r+t' \sim D_2+s$$

for some divisors D_1, D_2 of degree 3. Thus:

$$L^2(s)=\mathcal{O}_C(D) \quad , \quad L^2(t)=\mathcal{O}_C(D_1) \quad , \quad L^2(t')=\mathcal{O}_C(D_2)$$

correspond to 3 distinct trisecants of $C \to \mathbb{P}H^0(KL^2)^*$.

Now suppose $L^2(s)=\mathcal{O}_C(p+q+r)$ and $\overline{pq}$ passes through the double point of Γ_C. Then $p+q+t \in g^1_3$ for some $t \in C$ $(t \neq s.)$ Thus:

$$\begin{aligned} L^2(s+t) &\sim p+q+r+t \\ &\sim a+b+s+r \end{aligned}$$

for some $a, b \in C$ where $\overline{abs}$ passes through the double point of Γ_C; i.e. $L^2(s)$ and $L^2(t)=\mathcal{O}_C(a+b+r)$ correspond to 2 distinct trisecants of $C \to \mathbb{P}H^0(KL^2)^*$. If $t=s$ then it is clear that $L^2(s)$ is the unique trisecant of the curve.

Proposition 4.4.14. *Suppose C is a non-trigonal curve of genus 5 and $h^0(L^2)=1$. Then the variety of trisecants to $C \xrightarrow{\phi_L} \mathbb{P}H^0(C,KL^2)^* \cong \mathbb{P}^5$ is a cone over the canonical* *curve.*

PROOF. If $h^0(L^2) = 1$ then the image of $C \xrightarrow{\phi_L} \mathbb{P}^0(KL^2)^*$ is a curve of degree 10 with a double point $\phi_L(x) = \phi_L(y)$ (where $L^2 = \mathcal{O}_C(x + y)$.) As before the only trisecants to this curve are those lines through this double point hitting the curve again i.e. a cone over the canonical curve. □

Proposition 4.4.15. *Suppose C is a non-hyperelliptic, trigonal curve of genus 5 and $L^2 = \mathcal{O}_C(x + y)$ for some $x, y \in C$. Then the variety of trisecants to $C \xrightarrow{\phi_L} \mathbb{P}H^0(C, KL^2)^* \cong \mathbb{P}^5$ is:*

1. A cone over the canonical curve if $L^2(s) \notin W_3^1(C)$ for any $s \in C$.

2. A cone over the canonical curve and the rational ruled surface $S_{2,0}$ if $L^2(s) \in W_3^1(C)$ for some $s \in C$.

PROOF. The proof of case (**1**) is exactly the same as in (4.4.14). Now consider case (**2**): if C is trigonal then it is well known that it has a *unique* g_3^1 and that the Serre dual of this g_3^1 maps C onto a plane quintic with one double point i.e.

$$C \xrightarrow{\phi'_L} \mathbb{P}H^0(C, KL^{-2}(-s))^* \cong \mathbb{P}^2.$$

If $\phi'_L(a) = \phi'_L(b)$ corresponds to the double point then the g_3^1 is cut out by the pencil of lines through this double point. In terms of rational ruled surfaces a pencil of lines such as this is given by the map:

$$\mathbb{P}V \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{|E_0|} \mathbb{P}H^0(\mathbb{P}V, E_0)^* \cong \mathbb{P}^2.$$

Now if $L^2(s) = g_3^1$ then, as in the proof of (4.4.12), each divisor of $|L^2(s)|$ also corresponds to a trisecant of $\mathbb{P}H^0(C, KL^2(s))^* \cong \mathbb{P}^6$. Thus there exists the following commutative diagram:

$$\begin{tikzcd} & S' \subset \mathbb{P}H^0(KL^2(s))^* \arrow[dl, "\pi_s"'] \arrow[dr, "\pi_1"] & \\ S \subset \mathbb{P}H^0(KL^2)^* \arrow[rr, "\pi_2"] & & \mathbb{P}H^0(KL^{-2}(-s))^* \supset \mathbb{P}V \end{tikzcd}$$

where π_1 is given by projection away from the span of two trisecants (one of which is $\overline{xys}$.) This maps each trisecant onto a line through the double point of $C \to \mathbb{P}H^0(KL^{-2}(-s))^*$. π_2 is given by projection away from the span of a trisecant and the double point $\phi_L(x) = \phi_L(y)$, and π_s is projection away from $s \in C$. Identifying $\mathbb{P}H^0(KL^{-2}(-s))^*$ with $\mathbb{P}H^0(E_0)^*$ the map π_1 can be rewritten as:

$$S' \subset \mathbb{P}H^0(\mathbb{P}V, E_0 + 2f)^* \xrightarrow{\pi_1} \mathbb{P}H^0(\mathbb{P}V, E_0)^*.$$

The rational ruled surface:

$$S' = \mathbb{P}V \xrightarrow{|E_0+2f|} \mathbb{P}^6$$

in terms of the notation of section 4.2 is $S_{2,1}$. Now S is obtained by projection of this surface away from s. If s does not lie on the directrix then the surface obtained is $S_{2,0}$, if s lies on the directrix then $S_{1,2}$ is obtained. To see which surface actually describes the surface of trisecants we use (4.2.1): this says that $S_{2,0}$ is the union of straight lines joining points on $D_0 \subset V_2$ with corresponding points on the directrix $D_\infty \subset V_2'$, whilst $S_{1,2}$ is the union of straight lines joining points on $D_0 \subset V_3$ with corresponding points on the directrix $D_\infty = V_1$. Now consider the two-plane spanned by the line $\overline{ab}$ and the double point $\phi_L(x) = \phi_L(y)$ of $C \to \mathbb{P}_L$. Every trisecant $\overline{pqr} \subset \mathbb{P}_L$ coming from the linear system $|L^2(s)|$ intersects this two-plane since:

$$\begin{aligned} h^0(KL^2(-a-b-x-y-p-q-r)) &= h^0(KL^{-2}(-a-b-s)) \\ &= 2 \end{aligned}$$

(recall $\phi_L'(a) = \phi_L'(b)$ is the double point of $C \to \mathbb{P}^2$.) This pencil of trisecants will either sweep out a conic or a line in this two-plane. Now the line $\overline{ab}$ meets the two trisecants of $|L^2(s)|$ passing through a and b (corresponding to the two divisors cut out by the two tangent lines to the node of $C \to \mathbb{P}^2$) but not every trisecant meets the line $\overline{ab}$ otherwise we would have $h^0(KL^2(-p-q-r-a-b)) = 3$ i.e. $h^0(K(-a-b-s)) = 3$. By Riemann-Roch this is the case if and only if $h^0(\mathcal{O}_C(a+b+s)) = 2$ i.e. $\mathcal{O}_C(a+b+s) = L^2(s)$ (since the g^1_3 is unique), but this implies that $x+y \sim a+b$ i.e. C is hyperelliptic. Hence it can be concluded that the pencil of trisecants sweeps out a conic in the two-plane $\overline{abxy}$ and the rational ruled surface $S_{2,0}$ in $\mathbb{P}_L$. □

Remark 4.4.16. Suppose C is a generic curve of genus 6 i.e. it has 5 tetragonal pencils. If $h^0(L^2) = 0$ then by (4.3.3) $C \to \mathbb{P}_L$ admits a trisecant if and only if $L^2(s) \in W_3(C)$ for some $s \in C$. Moreover if $L^2(s+t) \in W^1_4(C)$ for some $t \neq s$ then it has 2 distinct trisecants. To see this note that C may be embedded as a planar curve of degree 6 with 4 nodes (no 3 of which are collinear.) The 5 tetragonal pencils are then cut out by the pencils of lines through each double point and the pencil of conics through all 4 double points. If $L^2(s) = \mathcal{O}_C(p+q+r)$ and p, q, r lie on such a line (or conic) then it cuts out one residual point t on C and there exists a unique line (or conic) from this pencil passing through s. By the proof of (4.3.3) this implies that $C \to \mathbb{P}_L$ has two trisecants given by $L^2(s)$ and $L^2(t)$.

Genus	Type of curve	$h^0(L^2)$	Description of trisecants to $C \to \mathbb{P}_L$	Corresponding Results
3	Non-hyper	0	Degree 8 surface triple along the curve	(4.4.3)
		1	Union of a cone over the canonical curve and a singular quadric if $L^2 = K(-2u)$	(4.4.7)
			Union of a cone over the canonical curve and a smooth quadric if $L^2 \neq K(-2u)$	
	Hyper	0	Smooth quadric surface	(4.3.1)
		1	Quadric cone	(4.3.6)
		2	All of $\mathbb{P}_L$	
4	Non-hyper	0	≤ 4 lines	(4.4.9)
		1	Cone over canonical curve if $L^2(s) \notin W_3^1(C)$	(4.4.12)
			Union of a cone over the canonical curve and the rational ruled surface $S_{1,1}$ if $L^2(s) \in W_3^1(C)$	
	Hyper	0	A unique line	(4.3.2)
		1	Cone over a rational normal curve in $\mathbb{P}^3$	(4.3.6)
		2	$\mathrm{Sec}_2 C$	

Genus	Type of curve	$h^0(L^2)$	Description of trisecants to $C \to \mathbb{P}_L$	Corresponding Results
5	Non-trig	0	A finite number of lines if $L^2 \in W_3(C) - W_1(C)$, empty otherwise	(4.3.3), (4.4.13)
		1	Cone over canonical curve	(4.4.14)
	Trig	0	A finite number of lines if $L^2 \in W_3(C) - W_1(C)$, empty otherwise	(4.3.3), (4.4.13)
		1	Cone over canonical curve if $L^2(s) \notin W_3^1(C)$	(4.4.15)
			Union of cone over canonical curve and the rational ruled surface $S_{2,0}$ if $L^2(s) \in W_3^1(C)$	
	Hyper	0	A unique line if $L^2 \in W_3(C) - W_1(C)$, empty otherwise	(4.3.3)
		1	Cone over a rational normal curve in $\mathbb{P}^4$	(4.3.6)
		2	$\mathrm{Sec}_2 C$	
6	Generic	0	A finite number of lines if $L^2 \in W_3(C) - W_1(C)$, empty otherwise	(4.3.3), (4.4.16)
≥ 7	Generic	0	A unique line if $L^2 \in W_3 - W_1$, empty otherwise	(4.3.4)

CHAPTER 5

Maximal line subbundles

The last two chapters considered the images of the extension maps:

$$\mathbb{P}U_{d,\Lambda} \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$$

in order to obtain information about subvarieties of $\mathcal{SU}_C(2,\Lambda)$ whether it was the g-plane ruling, the variety of trisecants to the Kummer or the moduli space itself. This chapter however will concentrate on the *fibres* of ϵ_d. For certain values of d the fibre $\epsilon_d^{-1}(E)$ over $E \in \mathcal{SU}_C(2,\Lambda)$ will correspond to the set of maximal line subbundles of E, denoted by W_E. These turn out to be either zero or one dimensional and have a determinantal structure. By using the notion of a *very stable* vector bundle a complete description of whether these fibres are smooth or not is given (5.4.9), (5.4.10). The existence of a correspondence on W_E will then be shown and a calculation of its degree will be given (5.6.2), (5.6.5). This in turn leads to information about multisecants to certain projective curves (5.7.4), (5.7.7). The final part of this chapter discusses the connectedness of the curve W_E.

Before describing the sets W_E scheme theoretically this chapter will start with a brief review of the theory of determinantal varieties and some results already known about maximal subbundles.

5.1. Determinantal Varieties

Let

$$h : V \longrightarrow W$$

be a homomorphism of holomorphic vector bundles of rank n and m respectively over an analytic space X.

The k-th determinantal variety associated to h is supported on the set:

$$\begin{aligned} X_k(h) &= \{p \in X \mid \text{ rank } h_p \leq k\} \\ &= \{p \in X \mid \wedge^{k+1} h_p = 0\}. \end{aligned}$$

By choosing local trivializations for V and W one can formally define $X_k(h)$ as the preimage of the set of $m \times n$ matrices with rank at most k under the map:

$$f : U \longrightarrow M(m, n)$$

for some open set $U \subset X$, in which case it can be seen that:

$$\text{codim} X_k(h) \leq (m - k)(n - k).$$

To calculate the class of the determinantal variety define:

$$\Delta_{p,q}(a_t) \stackrel{\text{def}}{=} \det \begin{pmatrix} a_p & \cdots & a_{p+q-1} \\ \vdots & & \vdots \\ a_{p-q+1} & \cdots & a_p \end{pmatrix}$$

where

$$a_t = \sum_{k=-\infty}^{\infty} a_k t^k$$

is any formal series. Then:

Proposition 5.1.1 (Porteous' formula). *If $X_k(h)$ is empty or has expected dimension the class of $X_k(h)$ is given by:*

$$x_k = \Delta_{m-k,n-k}(c_t(W - V)).$$

5.2. Preliminary results on maximal subbundles

One of the most useful results concerning maximal line subbundles and one which will be used frequently throughout this chapter is a result by Lange and Narasimhan [LN, Prop 2.4] giving a bijection between between maximal line subbundles of a rank 2 bundle and certain secant line bundles to the curve C. More precisely, $N \stackrel{\text{def}}{=} \mathcal{O}_C(D)$ is a d-secant line bundle with respect to the map $\phi_L : C \to \mathbb{P}H^0(C, KL^2\Lambda)^*$ if the span of the effective, degree d divisor D has dimension $d-1$. If $D = p+q$ is a node or a cusp of $\phi_L(C)$ then the dimension of $\overline{D}$ is considered to be zero dimensional. Then:

Proposition 5.2.1. *Given a point* $(e) \in \mathbb{P}H^0(C, KL^2\Lambda)^*$ *i.e. an extension* $0 \to L^{-1} \to E \to L\Lambda \to 0$ *with* $s(E) = c_1(L^2\Lambda) = d \geq 1$*, there is a canonical bijection between:*

1. Maximal subbundles M^{-1} *of* E *different from* L^{-1}.

2. d*-secant line bundles* $N = \mathcal{O}_C(D)$ *of* $C \to \mathbb{P}H^0(C, KL^2\Lambda)^*$ *such that* $(e) \in \overline{D} \subset \mathbb{P}H^0(KL^2\Lambda)^*$.

The bijection can be seen explicitly as follows: if $M^{-1} \neq L^{-1}$ is a maximal line subbundle of E then the composition $M^{-1} \to E \to L\Lambda$ is non-zero i.e. $ML\Lambda = \mathcal{O}_C(D)$ for some $D \in S^dC$. It is easily checked that this is the same secant line bundle as in (5.2.1) and hence that $M^{-1} = L\Lambda(-D)$, see [LN, Lem 2.3.]

(5.7.1) will give an improvement of this result for bundles E with maximal Segre invariant.

The following results describe how the dimension of the set of maximal line subbundles $W_E \subset J^d(C)$ varies as the Segre invariant $s(E)$ is varied. First a result due to Maruyama [M] says that:

Proposition 5.2.2. *1. If $E \in \mathcal{SU}_C(2,\Lambda)$ has Segre invariant g then* $\dim W_E = 1$

2. If $E \in \mathcal{SU}_C(2,\Lambda)$ has Segre invariant $g-1$ then there is an open, dense subset of $\mathcal{SU}_C(2,\Lambda)$ such that every vector bundle in this subset has only a finite number of maximal line subbundles.

If $1 \le s(E) \le g-2$ then [LN, Prop 3.3] gives:

Proposition 5.2.3. *Suppose $1 \le m \le g-2$ and* $\deg \Lambda \equiv m \pmod 2$. *Then there is an open, dense subset of $\mathcal{SU}_C^m(2,\Lambda)$ such that every vector bundle in this subset has exactly one maximal line subbundle.*

It should be noted that there are examples of bundles E for which W_E does *not* have the expected dimension:

Example 5.2.4. Let C be non-hyperelliptic of genus 3. Suppose $E \in \mathcal{SU}_C(2)$ has a maximal line subbundle $L^{-1} \subset E$ of degree -1 i.e. $s(E) = 2$ (thus we are in the situation of (5.2.2).) The following finds a $J_2(C)$-orbit of 64 such bundles of $\mathcal{SU}_C(2)$ which have a one dimensional family of maximal line subbundles parametrised by C. By (5.2.1) maximal line subbundles of E (other than L^{-1}) correspond to 2-secant line bundles of $C \subset \mathbb{P}_L \cong \mathbb{P}^3$ passing through (e). If E is generic then projecting away from (e) maps C birationally onto a plane sextic. The 2-secant line bundles (which are distinct since C is non-hyperelliptic) then correspond to nodes of this curve, of which there are 7. Thus E has a finite number of maximal line subbundles as expected (8 to be precise.) However, it can occur that C does not map birationally as above but that for some E it maps either:

1. 2:1 onto a plane cubic; or

2. 3:1 onto a conic.

In which case E has a one dimensional family of maximal line subbundles.

Suppose now that C is non-bielliptic so that the first case cannot occur. Now $C \subset \mathbb{P}_L$ will map 3:1 onto a conic if and only if it lies on a quadric cone, thus $L^2 = K(-2u)$ for some $u \in C$ by (4.4.8). By the argument above the bundles $E \in \mathcal{SU}_C(2)$ with a one dimensional family of maximal line subbundles clearly correspond to the vertices of these cones. Now consider the exact sequence:

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L \longrightarrow 0. \qquad (1)$$

This extension corresponds to some point in $\mathbb{P}_L$. The following gives the condition that (1) is the vertex of the quadric cone above. Twist (1) by a theta characteristic κ to obtain:

$$0 \longrightarrow \mathcal{O}_C(u) \longrightarrow \kappa \otimes E \longrightarrow K(-u) \longrightarrow 0 \qquad (2)$$

By Serre duality the coboundary map $H^0(K(-u)) \xrightarrow{\delta(E)} H^1(\mathcal{O}_C(u))$ is an element of $S^2H^1(\mathcal{O}_C(u))$. The claim is that if (1) is the vertex of a cone then rank $\delta(E) = 0$. To see this consider the map $C \to \mathbb{P}^1$ given by the complete linear system $|K(-u)|$. Pulling back sections of $\mathcal{O}_{\mathbb{P}^1}(2)$ gives:

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \hookrightarrow H^0(C, K^2(-2u)) \cong H^0(C, KL^2).$$

Thus taking the dual gives a surjection:

$$H^0(C, KL^2)^* \xrightarrow{\delta} S^2H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^* \cong S^2H^1(C, \mathcal{O}_C(u)).$$

Thus $\delta(E)$ can be identified (up to scalar) with the image of (1) under the map δ. If (1) is the vertex of the cone then by definition δ vanishes on (1) and hence rank $\delta(E) = 0$. From (2):

$$\begin{aligned} h^0(\kappa \otimes E) &= h^0(u) + h^0(K(-u)) \\ &= 3. \end{aligned}$$

Now there exists a *unique* stable bundle W with canonical determinant and $h^0(W) = 3$ [La] i.e. the normal bundle of the curve $C \hookrightarrow J^1(C)$. Thus there exists 64 stable bundles $E \in \mathcal{SU}_C(2)$ such that W_E has dimension one, namely the bundles:

$$\{E = \kappa^{-1} \otimes W\}_{\kappa \in \vartheta(C)}$$

5.3. Fibres of the extension map

In light of (5.2.3) where an almost complete description of maximal line subbundles of a bundle E with Segre invariant satisfying $1 \leq s(E) \leq g-2$ is given, the remainder of this chapter will concentrate on the Zariski open set $\mathcal{SU}_C^{\max}(2,\Lambda) \subset \mathcal{SU}_C(2,\Lambda)$ consisting of those bundles with *maximal* Segre invariant. By (1.2.1) this subvariety is dense in $\mathcal{SU}_C(2,\Lambda)$.

There are four cases to consider:

$$s(E) = \begin{cases} g, & \text{if } c_1(E) + g \equiv 0 \pmod 2; \\ g-1, & \text{if } c_1(E) + g \equiv 1 \pmod 2. \end{cases}$$

The degree of the maximal line subbundles are given by $-d$ where:

$$d = \frac{s(E) - c_1(E)}{2}$$

In the four cases being considered this reduces to:

$$d = \begin{cases} \frac{g}{2}, & \text{if } E \in \mathcal{SU}_C(2) \text{ and } g \text{ is even;} \\ \frac{g-1}{2}, & \text{if } E \in \mathcal{SU}_C(2) \text{ and } g \text{ is odd;} \\ \frac{g-2}{2}, & \text{if } E \in \mathcal{SU}_C(2,1) \text{ and } g \text{ is even;} \\ \frac{g-1}{2}, & \text{if } E \in \mathcal{SU}_C(2,1) \text{ and } g \text{ is odd.} \end{cases} \qquad (3)$$

Assume for the remainder of this chapter (unless otherwise stated) that d has been chosen (as above) such that the fibres of ϵ_d correspond to *maximal* line subbundles.

The extension map ϵ_d is clearly surjective onto $\mathcal{SU}_C^{\max}(2,\Lambda)$ and by Riemann-Roch:

$$\dim \mathbb{P}U_{d,\Lambda} = \begin{cases} 3g-2, & \text{if } c_1(E)+g \equiv 0 \pmod 2; \\ 3g-3, & \text{if } c_1(E)+g \equiv 1 \pmod 2. \end{cases}$$

Since $\dim \mathcal{SU}_C(2) = \dim \mathcal{SU}_C(2,1) = 3g-3$ the dimension of the *generic* fibre is given by:

$$\dim W_E = \begin{cases} 1, & \text{if } c_1(E)+g \equiv 0 \pmod 2; \\ 0, & \text{if } c_1(E)+g \equiv 1 \pmod 2. \end{cases} \tag{4}$$

Remark 5.3.1. If $L^{-1} \subset E$ is a maximal line subbundle then $h^0(L \otimes E) = 1$. This follows from the following result of Lange and Narasimhan [LN, Lem 2.1]:

Lemma 5.3.2. *Let $E \in \mathcal{SU}_C(2,\Lambda)$ be stable. Then the projection of the fibre $\epsilon_d^{-1}(E)$ into the Jacobian $J^d(C)$ is injective.*

Thus the fibre $\epsilon_d^{-1}(E)$ maps bijectively, via projection, onto its image $W_E \subset J^d(C)$. The following describes the sets W_E scheme-theoretically i.e. as the zero locus of some section of a vector bundle on $J^d(C)$:

Proposition 5.3.3. *Let $k \equiv c_1(E)+g \pmod 2$. Then W_E is the $(3g-3-k)$-th determinantal variety associated to the bundle map $\mathcal{E} \xrightarrow{\gamma} \mathcal{F}$ where:*

$$\mathcal{E} = \pi_*(\mathcal{P} \otimes \sigma^*(K \otimes E)) \quad ; \quad \mathcal{F} = \pi_*((\mathcal{P} \otimes \sigma^*(K \otimes E))\big|_\Gamma$$

$\pi : C \times J^d(C) \to J^d(C)$, $\sigma : C \times J^d(C) \to C$ are projections and Γ is the product divisor $D \times J^d(C)$ for some smooth divisor $D \in |K|$.

PROOF. Set theoretically W_E is given by:

$$W_E = \{L \in J^d(C) \mid h^0(C, L \otimes E) \neq 0\} \tag{5}$$

Consider the following exact sequence on C:

$$0 \longrightarrow E \longrightarrow K \otimes E \longrightarrow K \otimes E\big|_D \longrightarrow 0.$$

Twisting this exact sequence by a line bundle $L \in J^d(C)$ and globalising over $J^d(C)$ gives:

$$0 \longrightarrow \mathcal{P} \otimes \sigma^* E \longrightarrow \mathcal{P} \otimes \sigma^*(K \otimes E) \longrightarrow (\mathcal{P} \otimes \sigma^*(K \otimes E))\big|_\Gamma \longrightarrow 0 \qquad (6)$$

Taking the exact higher direct image of (6) with respect to π yields:

$$0 \to \pi_*(\mathcal{P} \otimes \sigma^* E) \to \mathcal{E} \xrightarrow{\gamma} \mathcal{F} \to R^1\pi_*(\mathcal{P} \otimes \sigma^* E) \to 0.$$

This exact sequence is functorial with respect to base change. For example let S be a point parametrising a line bundle $L \in J^d(C)$, $v : S \to J^d(C)$ the inclusion map, $C \times S \xrightarrow{q_1} S$ and $C \times S \xrightarrow{q_2} C$ projections. Then:

$$(1_C \times v)^* \mathcal{P} \cong L$$

and the kernel and cokernel of $v^*\mathcal{E} \xrightarrow{v^*\gamma} v^*\mathcal{F}$ are isomorphic to $(q_1)_*(L \otimes q_2^* E)$ and $(R^1 q_1)_*(L \otimes q_2^* E)$ respectively (cf [ACGH, p178] .) In particular $v^* R^1\pi_*(\mathcal{P} \otimes \sigma^* E)$ i.e. the restriction of $R^1\pi_*(\mathcal{P} \otimes \sigma^* E)$ to $L \in J^d(C)$ can be identified with $H^1(C, L \otimes E)$. Now rank $\mathcal{F} = h^0(KL \otimes E\big|_D) = 4g - 4$ and by Riemann-Roch and (3):

$$\begin{aligned} \text{rank } \mathcal{E} = h^0(KL \otimes E) &= 2(2g - 2 + d) + c_1(E) - 2(g-1) \\ &= \begin{cases} 3g - 2, & \text{if } c_1(E) + g \equiv 0 \pmod 2; \\ 3g - 3, & \text{if } c_1(E) + g \equiv 1 \pmod 2. \end{cases} \end{aligned}$$

Thus if $k \equiv c_1(E) + g \pmod 2$ then:

$$\begin{aligned} W_E &= \{L \in J^d(C) \mid h^0(L \otimes E) = 1\} \\ &= \{L \in J^d(C) \mid h^1(L \otimes E) = g - 1 + k\} \\ &= \{L \in J^d(C) \mid \text{rank } \gamma_L \leq 4g - 4 - (g - 1 + k) = 3g - 3 - k\} \end{aligned}$$

i.e. W_E is the $(3g - 3 - k)$-th determinantal variety associated to γ. □

Remark 5.3.4. Note that the lower bound on the dimension of this determinantal variety is given by:

$$\dim W_E \geq g - (\mathrm{rank}\mathcal{E} - (3g-3-k))(\mathrm{rank}\mathcal{F} - (3g-3-k))$$
$$= \begin{cases} 1, & \text{if } k = c_1(E) + g \equiv 0 \pmod 2; \\ 0, & \text{if } k = c_1(E) + g \equiv 1 \pmod 2. \end{cases}$$

5.4. Smoothness of the fibres

The main result of this section (5.4.9) shows that if E is very stable then the space of its maximal line subbundles W_E is smooth. A sketch proof of this is given by Laumon [Lau1] . The following fills in the details of that proof and in the last part of this section a new result (5.4.10) concerning the smoothness of fibres over bundles that are not very stable is given.

Definition 5.4.1. *A stable vector bundle $E \in \mathcal{SU}_C(2,\Lambda)$ is said to be very stable if the space of sections $H^0(C, K \otimes EndE)$ has no nilpotent elements i.e. if $\alpha \in H^0(C, K \otimes EndE)$ then:*

$$\alpha^n \equiv 0 \iff \alpha \equiv 0 \quad \textit{for all } n \in \mathbb{Z}.$$

Remark 5.4.2. The existence of such bundles was shown by Laumon [Lau2] in which he also shows that they form an open, dense subset of $\mathcal{SU}_C(2,\Lambda)$.

Proposition 5.4.3. *$E \in \mathcal{SU}_C(2,\Lambda)$ is not very stable if and only if there exists a line subbundle $L^{-1} \subset E$ with $h^0(C, KL^{-2}\Lambda^{-1}) \neq 0$.*

PROOF. ($\Leftarrow$) Suppose there exists a line bundle L as above i.e. E lies in the extension $0 \to L^{-1} \to E \to L\Lambda \to 0$ and $\beta \in H^0(KL^{-2}\Lambda^{-1}) - 0$ is given by a non-zero homomorphism $L\Lambda \xrightarrow{\beta} KL^{-1}$. Let α be the composition $E \twoheadrightarrow L\Lambda \xrightarrow{\beta} KL^{-1} \hookrightarrow K \otimes E$

i.e. $\alpha \in H^0(K \otimes EndE)$. Then it is clear from the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1} & \longrightarrow & E & \longrightarrow & L\Lambda & \longrightarrow & 0 \\
 & & & & \downarrow \alpha & & \downarrow \beta & & \\
0 & \leftarrow & KL\Lambda & \leftarrow & K \otimes E & \leftarrow & KL^{-1} & \leftarrow & 0 \\
 & & \downarrow \beta & & \downarrow \alpha & & & & \\
0 & \rightarrow & K^2L^{-1} & \rightarrow & K^2 \otimes E & \rightarrow & K^2L\Lambda & \rightarrow & 0
\end{array}$$

that $\alpha^2 \equiv 0$ i.e. the composition $KL^{-1} \to K \otimes E \to KL\Lambda$ is zero by definition. Hence E is not very stable.

($\Rightarrow$) Conversely suppose that $\alpha \in H^0(C, K \otimes EndE) - 0$ with $\alpha^2 \equiv 0$ i.e. the composition $E \xrightarrow{\alpha} K \otimes E \xrightarrow{\alpha'} K^2 \otimes E$ is zero (where α' is induced by α.) Then there exists a line subbundle $L^{-1} \subset E$ such that the image of E under α is contained in KL^{-1} and α' is zero on KL^{-1}. From the exact sequence:

$$0 \longrightarrow KL^{-1} \longrightarrow K \otimes E \longrightarrow KL\Lambda \longrightarrow 0$$

this implies that α' factors through $KL\Lambda \to K^2L^{-1}$ i.e. $h^0(KL^{-2}\Lambda^{-1}) \neq 0$. $\square$

Definition 5.4.4. *A first-order deformation of a line bundle L is a family of line bundles $\mathfrak{T} \to C \times Spec\,\mathbb{C}[t]/(t^2)$ such that $i_1^*\mathfrak{T} = L$, where $i_1 \colon C \to C \times Spec\,\mathbb{C}[t]/(t^2)$ is the inclusion map sending a point $p \in C$ to $p \times (t)$.*

The set of equivalence classes of first-order deformations of L is in one to one correspondence with the tangent space $T_L(J^d(C))$ after the identification:

$$T_L(J^d(C)) = Hom(S, (J^d(C), L))$$

writing $Spec\,\mathbb{C}[t]/(t^2)$ as S.

The following describes the tangent space $T_L(W_E)$ to $W_E \subset J^d(C)$ at the point L:

Lemma 5.4.5. *Let $s \in H^0(C, L \otimes E)$ be a section of $L \otimes E$. The set of tangent vectors $\varphi \in T_L(J^d(C)) \cong H^1(C, \mathcal{O}_C)$ such that s can be extended to a section $\tilde{s} \in H^0(C \times S, \mathcal{T} \otimes E)$, (where $\mathcal{T}$ is the first-order deformation of L corresponding to φ) is given by:*

$$T_L(W_E) = \{\varphi \in H^1(C, \mathcal{O}_C) \mid \varphi.s = 0 \text{ in } H^1(C, L \otimes E)\}.$$

PROOF. Let $\{U_\alpha\}$ be an open cover for C, $\{g_{\alpha\beta}\}$ transition functions for L, $\{h_{\alpha\beta}\}$ transition functions for E, $\{s_\alpha\}$ holomorphic functions on $\{U_\alpha\}$ representing s, and $\{\varphi_{\alpha\beta}\}$ holomorphic functions representing the cocycle φ. Then transition functions for $\mathcal{T}$ are given by:

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta}(1 + \varepsilon\varphi_{\alpha\beta})$$

defined on $(U_\alpha \cap U_\beta) \times S$. An extension of s to $\tilde{s}$ is given locally by:

$$\tilde{s}_\alpha = s_\alpha + \varepsilon s'_\alpha$$

satisfying $\tilde{s}_\alpha = \tilde{g}_{\alpha\beta} h_{\alpha\beta} \tilde{s}_\beta$ i.e substituting in for $\tilde{s}_\alpha$, $\tilde{g}_{\alpha\beta}$, and comparing coefficients we have:

$$s_\alpha = g_{\alpha\beta} h_{\alpha\beta} s_\beta$$

$$s'_\alpha = g_{\alpha\beta} h_{\alpha\beta} \varphi_{\alpha\beta} s_\beta + g_{\alpha\beta} h_{\alpha\beta} s'_\beta$$

i.e.

$$\varphi_{\alpha\beta} s_\alpha = s'_\alpha - g_{\alpha\beta} h_{\alpha\beta} s'_\beta$$

where $\varphi_{\alpha\beta} s_\alpha$ is a cocycle representing the cup-product $\varphi \cdot s \in H^1(L \otimes E)$, and $s'_\alpha - g_{\alpha\beta} h_{\alpha\beta} s'_\beta$ is the coboundary $\delta s'$. $\square$

Lemma 5.4.6 (Hopf). *Let A, B, C be complex vector spaces and let:*

$$\nu : A \otimes B \to C$$

be a linear map, injective on each factor separately. Then:

$$\dim \nu(A \otimes B) \geq \dim A + \dim B - 1.$$

Proposition 5.4.7. *Suppose W_E has codimension $g-1+k$. Then W_E is smooth at $L \in W_E$ if and only if the Petri map:*

$$\mu_L : H^0(C, L \otimes E) \otimes H^0(C, KL^{-1} \otimes E^*) \longrightarrow H^0(C, K)$$

is injective on each factor.

PROOF. ($\Leftarrow$) From the proof of (5.3.3) W_E was described as the $(3g-3-k)$-th determinantal variety of a homomorphism of vector bundles $\mathcal{E} \xrightarrow{\gamma} \mathcal{F}$ where $k \equiv c_1(E) + g \pmod 2$ and $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of rank $3g-2-k$ and $4g-4$ respectively over the Jacobian of C. For any $L \in W_E$ we have:

$$\text{codim} T_L W_E \leq \text{codim} W_E = g-1+k.$$

By (5.4.5) $T_L W_E$ consists of those tangent vectors $\varphi \in H^1(C, \mathcal{O}_C)$ such that $\varphi.s = 0$ for *every* $s \in H^0(C, L \otimes E)$. Denote the Serre-duality pairing by $\langle\ ,\ \rangle$ then:

$$\begin{aligned}
\varphi.s = 0 &\iff \langle \varphi.s, t\rangle = 0 \qquad \text{for all } t \in H^0(KL^{-1} \otimes E^*)\\
&\iff \langle \varphi, s.t\rangle = 0 \qquad \text{for all } t \in H^0(KL^{-1} \otimes E^*)\\
&\iff \langle \varphi, \mu_L(s \otimes t)\rangle = 0 \qquad \text{for all } t \in H^0(KL^{-1} \otimes E^*).
\end{aligned}$$

Thus $T_L W_E = (\text{Image } \mu_L)^{\perp}$. Since μ_L is injective then by (5.4.6):

$$\begin{aligned}
\text{codim} T_L W_E = \text{rank } \mu_L &\geq h^0(L \otimes E) + h^0(KL^{-1} \otimes E^*) - 1\\
&= 1 + (g-1+k) - 1\\
&= g-1+k.
\end{aligned}$$

Thus:

$$g-1+k \leq \text{codim} T_L W_E \leq \text{codim} W_E = g-1+k.$$

Hence equality holds throughout and W_E is smooth at L.

($\Rightarrow$) Now suppose W_E is smooth at $L \in W_E$ then:

$$\mathrm{codim} T_L W_E = \mathrm{codim} W_E = g - 1 + k$$

i.e. the rank of the Petri map:

$$\mu_L : H^0(C, L \otimes E) \otimes H^0(C, KL^{-1} \otimes E^*) \longrightarrow H^0(C, K)$$

is equal to $g-1+k$. Since $L \in W_E$ then $h^0(L \otimes E) = 1$ (cf (5.3.1)) and $h^0(KL^{-1} \otimes E^*) = g - 1 + k$ so μ_L must be injective on each factor. □

Lemma 5.4.8. *Let E be very stable and $L \in W_E$. Then the Petri map:*

$$\mu_L : H^0(C, L \otimes E) \otimes H^0(C, KL^{-1} \otimes E^*) \longrightarrow H^0(C, K)$$

is injective on each factor.

Proof. Regard $s \in H^0(C, L \otimes E)$ and $t \in H^0(C, KL^{-1} \otimes E^*)$ as homomorphisms:

$$L^{-1} \xrightarrow{s} E \quad ; \quad E \xrightarrow{t} KL^{-1}$$

and consider the composition:

$$\alpha : E \xrightarrow{t} KL^{-1} \xrightarrow{id \otimes s} K \otimes E.$$

Suppose $\mu_L(s \otimes t) = 0$ i.e. $L^{-1} \to E \to KL^{-1}$ is the zero homomorphism, then:

$$\alpha^2 : E \longrightarrow KL^{-1} \xrightarrow{id \otimes s} K \otimes E \xrightarrow{t} K^2 L^{-1} \longrightarrow K^2 \otimes E$$

is zero. Since E is very stable this implies that $\alpha \equiv 0$.

Now suppose that t is non-zero. Then the image of E is the line bundle $KL^{-1}(-D)$, where D is the divisor given by the zero locus of t. Since $\alpha \equiv 0$ this means $s = 0$. Now suppose $s \in H^0(C, L \otimes E)$ is non-zero. Since $KL^{-1} \hookrightarrow K \otimes E$ is an injection, it must be the case that $t = 0$ if the composition α is to be zero. □

Corollary 5.4.9. *If $E \in \mathcal{SU}_C^{max}(2,\Lambda)$ is very stable and W_E has codimension $g-1+k$ then W_E is smooth.*

PROOF. Follows immediately from (5.4.7) and (5.4.8). $\square$

The rest of this section will be concerned with whether the converse of (5.4.9) is true or not.

Proposition 5.4.10. *Suppose $E \in \mathcal{SU}_C^{max}(2,\Lambda)$ is not very stable and W_E has codimension $g-1+k$. Then W_E is not smooth if and only if there exists $L \in W_E$ with:*

$$h^0(C, KL^{-2}\Lambda^{-1}) \neq 0.$$

PROOF. ($\Leftarrow$) Since $L \in W_E$ there exists an extension:

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L\Lambda \longrightarrow 0.$$

Twisting this by $KL^{-1}\Lambda^{-1}$ and taking the long exact cohomology sequence gives:

$$0 \longrightarrow H^0(KL^{-2}\Lambda^{-1}) \longrightarrow H^0(KL^{-1} \otimes E^*) \xrightarrow{s} H^0(K) \longrightarrow \dots \tag{7}$$

where the last map is given by some non-zero section $s \in H^0(C, L \otimes E)$. Since $h^0(KL^{-2}\Lambda^{-1}) \neq 0$ then it follows from (7) that:

$$\{s\} \otimes H^0(KL^{-1} \otimes E^*) \longrightarrow H^0(K)$$

is not injective i.e. the Petri map is not injective on each factor. Thus by (5.4.7) W_E is singular at L.

($\Rightarrow$) Now suppose that *for all* $L \in W_E$, $h^0(KL^{-2}\Lambda^{-1}) = 0$ then from (7):

$$\{s\} \otimes H^0(KL^{-1} \otimes E^*) \longrightarrow H^0(K)$$

is injective i.e. $\langle s,t\rangle \neq 0$ for all $t \in H^0(KL^{-1}\otimes E^*) - 0$. Furthermore:

$$\begin{aligned}\langle s,t\rangle \neq 0 \quad \forall \text{ non-zero } t &\iff \langle cs,t\rangle \neq 0 \quad \forall \text{ non-zero } t \text{ and } \forall\, c \in \mathbb{C}^* \\ &\iff \langle s',t\rangle \neq 0 \quad \forall \text{ non-zero } t \text{ and } \forall\, s' \in H^0(L\otimes E) - 0\end{aligned}$$

since $H^0(L\otimes E) \cong \mathbb{C}$ by (5.3.1) i.e. the Petri map μ_L is injective on *each* factor for all $L \in W_E$ and hence W_E is smooth by (5.4.7). $\square$

Remark 5.4.11. Note that by (5.4.3) E is not very stable if and only if there exists *some* line subbundle $M^{-1} \subset E$ with $h^0(KM^{-2}\Lambda^{-1}) \neq 0$. (5.4.10) says that W_E is not smooth if and only if the line bundle M^{-1} is *maximal*.

Example 5.4.12. Suppose C has genus 2 and consider the 4:1 map $\mathbb{P}U_0 \xrightarrow{\epsilon_0} \mathcal{SU}_C(2,1)$ (cf (2.2.3).) By (5.4.3) $F \in \mathcal{SU}_C(2,1)$ is not very stable if and only if $h^0(KL^{-2}\Lambda^{-1}) \neq 0$ for some line subbundle $L^{-1} \subset F$ i.e. $\deg L = 0$ and $L^2\Lambda = \mathcal{O}_C(x)$ for some $x \in C$. The corresponding fibre $\mathbb{P}_L$ of $\mathbb{P}U_0 \to J^0(C)$ maps to a line of not very stable bundles of $\mathcal{SU}_C(2,1)$. Varying $x \in C$ gives the locus of not very stable bundles as a ruled surface and this ruled surface is precisely the ruled surface of special lines of $\mathcal{SU}_C(2,1)$ or the locus of bundles which have ≤ 3 maximal line subbundles. See [O] for more details.

Now consider the extension map $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$ for a curve C of genus 2. The cohomological condition of (5.4.3) gives that E is not very stable if and only if κ^{-1} is a line subbundle of E for some theta characteristic κ. Again the branch locus is precisely the locus of not very stable bundles and the 16 fibres $\mathbb{P}_\kappa$ map via ϵ_1 to the classical $(16)_6$ configuration of the Kummer surface $\mathcal{K} \subset \mathcal{SU}_C(2) \cong \mathbb{P}^3$. See [O] for more details.

Example 5.4.13. Consider the case of the 8:1 map $\mathbb{P}U_1 \xrightarrow{\epsilon_1} \mathcal{SU}_C(2)$ for $g = 3$ (cf (5.2.4)). Suppose that $E \in \mathcal{SU}_C(2)$ is not very stable and that there exists $L \in J^1(C)$

satisfying the condition of (5.4.10). The following shows explicitly that ϵ_1 is branched at such an E. Now $L \in W_E$ and:

$$h^0(KL^{-2}) \neq 0 \iff h^0(L^2) \neq 0.$$

Thus the copy of the curve in the fibre $\mathbb{P}_L$, (given by the complete linear system $|KL^2|$), has a double point. By (5.2.1) maximal subbundles of E other than L^{-1} correspond to 2-secant line bundles of this curve passing through (e). By projecting $C \subset \mathbb{P}_L$ from (e) onto a planar curve and counting double points it is seen that there are seven such line bundles (cf (5.2.4)), but this includes the original double point of the curve (which does not correspond to a maximal line subbundle.) Thus in total (after counting L^{-1}) E has seven maximal line subbundles.

If $c_1(E) + g \equiv 0 \pmod 2$ and $L \in W_E$ then by Riemann-Roch the cohomological condition of (5.4.10) holds if and only if $L^2\Lambda \in W_g^1(C)$.
Similarly if $c_1(E) + g \equiv 1 \pmod 2$ and $L \in W_E$ then it holds if and only if $L^2\Lambda \in W_{g-1}(C) \cong \Theta$.
So if $k \equiv c_1(E) + g \pmod 2$ then the branch locus of $\mathbb{P}U_d \xrightarrow{\epsilon_d} \mathcal{SU}_C^{\max}(2, \Lambda)$ is an open, dense subset of the image of the subvariety:

$$\mathcal{W} \stackrel{\text{def}}{=} \bigcup_L \mathbb{P}_L \subset \mathbb{P}U_d \quad , \quad L^2\Lambda \in W_{g-k}^{1-k}(C).$$

Example 5.4.14. In the case $g = 3$, $\Lambda = \mathcal{O}_C$ the degree of the branch locus $\mathbb{P}U_1 \to \mathcal{SU}_C(2)$ can be calculated easily. Since ϵ_1 is 8:1 the degree is given by:

$$\deg \epsilon_1(\mathcal{W}) = \frac{1}{8} c_1(\epsilon_1^*\mathcal{L})^5.\nu^*\mathrm{sq}^*(\theta)$$

where $J^1(C) \xrightarrow{\mathrm{sq}} J^2(C)$ is the square map and ν is the projection $\mathbb{P}U_1 \to J^1(C)$. By (3.2.7):

$$\deg \epsilon_1(\mathcal{W}) = \frac{1}{8}(2\theta - \eta)^5.4\theta$$

Using (3.2.3), $\eta^3 = -1$ and the identity:

$$\eta^4 = c_1(U_1)\eta^3 - c_2(U_1)\eta^2 + c_3(U_1)\eta - c_4(U_1)$$

this reduces to:

$$\begin{aligned}
\deg \epsilon_1(\mathcal{W}) &= \frac{1}{8}(2\theta - \eta)^5.4\theta \\
&= \frac{1}{8}(-\eta^5 + 5\eta^4(2\theta) - 10\eta^3(2\theta)^2).4\theta \\
&= \frac{1}{8}[(c_1^2 - c_2) - 10\theta c_1 + 40\theta^2].4\theta \\
&= 4\theta^3 \\
&= 24.
\end{aligned}$$

Remark 5.4.15. Unlike the genus 2 case (5.4.12), where the branch locus is precisely the locus of not very stable bundles, the calculation above shows that this is not the case for genus 3. While the branch locus above corresponds to not very stable bundles, $E \in \mathcal{SU}_C(2)$ is also not very stable if it occurs as an extension:

$$0 \longrightarrow \kappa^{-1} \longrightarrow E \longrightarrow \kappa \longrightarrow 0$$

for any theta characteristic κ. The locus of such bundles is the 64 translates (by the 64 theta characteristics) of the generalised theta divisor $\Theta_{2,4} \subset \mathcal{SU}_C(2, K)$, i.e. hyperplane sections of $\mathcal{SU}_C(2) \subset \mathbb{P}^7$, all of which have degree 4.

5.5. Degree of canonical line bundle on curve fibres

This section calculates the degree of the canonical line bundle K_{W_E} for the smooth curve of maximal line subbundles of $E \in \mathcal{SU}_C(2, \Lambda)$ and so its genus if it is connected. First we shall need to calculate the Chern character of the bundle $\mathcal{E}$ defined in (5.3.3):

Lemma 5.5.1. *Let E be a vector bundle of rank r on C, π and σ the projections $C \times J^d(C) \to J^d(C)$, $C \times J^d(C) \to C$. Then the Chern classes of the bundle $\mathcal{E} =$*

$\pi_*(\mathcal{P} \otimes \sigma^*(K \otimes E))$ *are given by:*

$$c_k(\mathcal{E}) = (-1)^k \frac{(r\theta)^k}{k!}$$

Proof. The following is a sketch proof as it is very similar to the calculation done in (3.2.2). Let $\mathcal{E} = \pi_* F$ and suppose that $\deg(K \otimes E) = m$. Then by Grothendieck-Riemann-Roch and using the same notation as in (3.2.2):

$$\begin{aligned} ch(\mathcal{E}) = ch(\pi_! F) &= \pi_*[ch(F).td(C \times J^d(C))] \\ &= \pi_*[ch(\mathcal{P}).ch(\sigma^*(K \otimes E)).td(C \times J^d(C))] \\ &= \pi_*[(1 + d\xi + \varsigma - \xi\theta).(r + m\xi).(1 + (1-g)\xi)] \\ &= \pi_*[(r + m - rg + dr)\xi + r + \varsigma r - r\xi\theta] \\ &= (rd + m - r(g-1)) - r\theta \end{aligned}$$

Note that $rd + m - r(g-1) = h^0(KL \otimes E)$ is the rank of $\mathcal{E}$ where $L \in J^d(C)$. The result then follows from a calculation similar to that in (3.2.3). □

Lemma 5.5.2. *Let $W \to X$ be a vector bundle on a complex manifold X with $s \in H^0(X, W)$ a non-zero section. Suppose $(s) = Y \subset X$ is the smooth zero locus of s with $codim_X Y =$ rank W. Then:*

$$N_{Y/X} = W\Big|_Y.$$

Proof. Let $\{g_{\alpha\beta}\}$ be transition functions for W and $\{s_\alpha\}$ holomorphic sections representing $s \in H^0(X, W)$ such that:

$$s_\alpha = g_{\alpha\beta} s_\beta.$$

Then:

$$ds_\alpha = dg_{\alpha\beta} s_\beta + g_{\alpha\beta} ds_\beta$$

and this is everywhere non-zero since Y is smooth. Thus along Y we have $ds_\alpha = g_{\alpha\beta} ds_\beta$ and so $\{ds_\alpha\}$ corresponds to a non-zero section $ds \in H^0(Y, (T_X^* \otimes W)\big|_Y)$ i.e. a non-zero homomorphism:

$$ds : T_X\big|_Y \longrightarrow W\big|_Y.$$

Now $\ker ds = T_Y$ so we have the exact sequence:

$$0 \longrightarrow T_Y \longrightarrow T_X\big|_Y \longrightarrow W\big|_Y$$

and since $T_X\big|_Y / T_Y = N_{Y/X}$ we have the injection:

$$N_{Y/X} \hookrightarrow W\big|_Y.$$

The result then follows from the fact that $\operatorname{rank} N_{Y/X} = \operatorname{codim}_X Y$ and $\operatorname{codim}_X Y = \operatorname{rank} W$ by hypothesis. $\square$

Proposition 5.5.3. *Suppose W_E is smooth. Then the degree of the canonical line bundle of W_E is given by:*

$$\deg K_{W_E} = 2^{g+1}(g-1).$$

Proof. Recall from the proof of (5.3.3) that (for $c_1(E) + g \equiv 0 \pmod 2$) W_E is the $(3g-3)$-rd determinantal variety associated to the homomorphism of vector bundles:

$$\mathcal{E} \xrightarrow{\gamma} \mathcal{F}$$

i.e. the zero locus of a section of the bundle $\bigwedge^{3g-2} \mathcal{E}^* \otimes \bigwedge^{3g-2} \mathcal{F}$. We wish to use (5.5.2) but it cannot be used directly since:

$$\operatorname{codim}_{W_E} J^d(C) = g - 1 \neq \binom{4g-4}{3g-2} = \operatorname{rank} \textstyle\bigwedge^{3g-2} \mathcal{E}^* \otimes \bigwedge^{3g-2} \mathcal{F}.$$

Instead, let $T \to \mathbb{P}(\mathcal{E})$ be the tautological line bundle on $\mathbb{P}(\mathcal{E})$ and consider the following diagram:

$$\begin{array}{ccc} T^{-1} \otimes \rho^*\mathcal{F} & & \mathcal{E}^* \otimes \mathcal{F} \\ \downarrow & & \downarrow \\ \widetilde{W}_E \subset \mathbb{P}(\mathcal{E}) & \xrightarrow{\rho} & J^d(C) \supset W_E \end{array}$$

where:

$$\widetilde{W}_E = \{(L, l) \in \mathbb{P}(\mathcal{E}) \mid L \in W_E, l \subset \ker \gamma_L\}$$
$$= \{(L, l) \in \mathbb{P}(\mathcal{E}) \mid \tilde{\gamma}((L, l)) = 0\}$$

and $\tilde{\gamma} \in H^0(\mathbb{P}(\mathcal{E}), T^{-1} \otimes \rho^*\mathcal{F})$. Note that $\widetilde{W}_E \cong W_E$ since W_E is smooth and $h^0(C, L \otimes E) = 1$ for all $L \in W_E$ (5.3.1.)

The codimension of $\widetilde{W}_E$ and the rank of $T^{-1} \otimes \rho^*\mathcal{F}$ now coincide. By (5.5.2) the normal bundle of $\widetilde{W}_E \subset \mathbb{P}(\mathcal{E})$ is given by:

$$N_{\widetilde{W}_E/\mathbb{P}(\mathcal{E})} = \operatorname{Hom}(T, \rho^*\mathcal{F})\Big|_{\widetilde{W}_E} \stackrel{\text{def}}{=} \widehat{E}\Big|_{\widetilde{W}_E}.$$

The degree of K_{W_E} is obtained from the exact sequence:

$$0 \longrightarrow T_{\widetilde{W}_E} \longrightarrow T_{\mathbb{P}(\mathcal{E})}\Big|_{\widetilde{W}_E} \longrightarrow N_{\widetilde{W}_E/\mathbb{P}(\mathcal{E})} \longrightarrow 0$$

i.e.

$$c_1(T_{\widetilde{W}_E}) = (c_1(T_{\mathbb{P}(\mathcal{E})}) - c_1(\widehat{E})) \cdot \eta_{\widetilde{W}_E}$$

where $\eta_{\widetilde{W}_E}$ is the cohomology class of $\widetilde{W}_E \subset \mathbb{P}(\mathcal{E})$. Since

$$c_1(T_{\mathbb{P}\mathcal{E}}) = c_1(Hom(T, \rho^*\mathcal{E}/T))$$

then:

$$c_1(T_{\mathbb{P}(\mathcal{E})}) = c_1(T^{-1}) \cdot \operatorname{rank}(\rho^*\mathcal{E}/T) + c_1(\rho^*\mathcal{E}/T) \cdot \operatorname{rank} T^{-1}$$
$$= (3g-3)\lambda + \rho^*c_1(\mathcal{E}) - c_1(T)$$
$$= (3g-2)\lambda + \rho^*c_1(\mathcal{E})$$

where $\lambda = c_1(T^{-1})$. Since $\mathcal{F}$ is a trivial bundle of rank $4g-4$ over $J^d(C)$ then:

$$c_1(\widehat{E}) = c_1(T^{-1} \otimes \rho^*\mathcal{F}) = (4g-4)\lambda$$

and:

$$c_1(T_{\widetilde{W}_E}) = ((2-g)\lambda + \rho^* c_1(\mathcal{E})) \cdot \eta_{\widetilde{W}_E} \tag{8}$$

The class of $\widetilde{W}_E$ is calculated using Porteous' formula (5.1.1):

$$\begin{aligned} \eta_{\widetilde{W}_E} &= \Delta_{m-k,n-k}(c_t(\rho^*\mathcal{F} - T)) \\ &= \Delta_{4g-4,1}(c_t(\rho^*\mathcal{F} - T)) \\ &= \Delta_{1,4g-4}(c_t(T - \rho^*\mathcal{F})) \\ &= \Delta_{1,4g-4}(-\lambda t) \\ &= \lambda^{4g-4} \end{aligned} \tag{9}$$

where $m = \operatorname{rank} \rho^*\mathcal{F}$, $n = \operatorname{rank} T$, and $k = 0$. Via the Gysin homomorphism:

$$\rho_* : H^*(\mathbb{P}(\mathcal{E}), \mathbb{Z}) \longrightarrow H^{*-2r+2}(J^d(C), \mathbb{Z}) \qquad r = \operatorname{rank} \mathcal{E}$$

$\rho_*\lambda^{3g-3+i} = c_i(-\mathcal{E})$ (cf [ACGH, p318]) and so applying ρ_* to (8) we obtain:

$$\begin{aligned} c_1(T_{W_E}) = \rho_*(c_1(T_{\widetilde{W}_E})) &= (2-g)\rho_*\lambda^{4g-3} + c_1(\mathcal{E})\rho_*\lambda^{4g-4} \\ &= (2-g)c_g(-\mathcal{E}) + c_{g-1}(-\mathcal{E}) \cdot c_1(\mathcal{E}) \\ &= (2-g)\frac{(2\theta)^g}{g!} + \frac{(2\theta)^{g-1}}{(g-1)!}(-2\theta) \qquad \text{(by (5.5.1))} \\ &= (2-g)2^g - 2^g g \\ &= 2^{g+1}(1-g). \end{aligned}$$

Thus $\deg K_{W_E} = 2^{g+1}(g-1)$ as asserted. □

Remark 5.5.4. By (9) the class of W_E is given by:

$$\rho_*\lambda^{4g-4} = \frac{(2\theta)^{g-1}}{(g-1)!}.$$

(This is also proved in [Lau1].)

Remark 5.5.5. If W_E is *connected* then (5.5.3) immediately gives the genus of W_E as:

$$g(W_E) = 1 + 2^g(g-1).$$

When W_E is finite the above calculation can be used to give its degree. This is well known and has been calculated in [G] for example.

Proposition 5.5.6. *If E is very stable and $c_1(E) + g \equiv 1 \pmod 2$ then E has 2^g distinct maximal subbundles.*

PROOF. Since E is very stable then by (5.4.9) W_E is smooth. Following the method of the proof of (5.5.3) the class of W_E just needs to be calculated. By Porteous' formula:

$$\begin{aligned} \eta_{W_E} &= \Delta_{m-k,n-k}(c_t(\rho^*\mathfrak{F} - T)) \\ &= \lambda^{4g-4} \end{aligned}$$

where $m = \text{rank}\rho^*\mathfrak{F} = 4g-4$, $n = \text{rank}T = 1$, and $k = 0$. This time the Gysin homomorphism gives:

$$\rho_*\lambda^{3g-4+i} = c_i(-\mathcal{E})$$

Thus by (5.5.1), $\rho_*\lambda^{4g-4} = c_g(-\mathcal{E}) = \frac{(2\theta)^g}{g!} = 2^g$. □

Example 5.5.7. It has already been seen (2.2.3) that for $g = 2$ and general $E \in \mathcal{SU}_C(2,1)$ the fibre of $\mathbb{P}U_0 \to \mathcal{SU}_C(2,1)$ consists of $2^2 = 4$ points. For $g = 3$ and $E \in \mathcal{SU}_C(2)$ a general point (i.e. a very stable vector bundle) the fibre of $\mathbb{P}U_1 \to \mathcal{SU}_C(2)$ consists of $2^3 = 8$ points (cf (5.2.4)).

Now consider the curve fibre of $\mathbb{P}U_1 \to \mathcal{SU}_C(2)$ for genus two curves. In this case $W_E = \{L \in J^1(C) \mid h^0(L \otimes E) \neq 0\}$ is a 2Θ divisor. If it is smooth (which it is if E is very stable) then the degree of its canonical bundle is given by the adjunction formula:

$$K_{W_E} = (K_J \otimes [2\Theta])\Big|_{2\Theta}$$

i.e. $\deg K_{W_E} = 4\theta^2 = 8$ as expected from (5.5.3).

5.6. Correspondence on curve fibres

The motivation for this section comes from the theory of Prym-Tyurin varieties; namely that if X is a smooth, projective curve and $\mathcal{C} : X \to X$ a symmetric correspondence on X (inducing a symmetric endomorphism i on $J(X)$) satisfying:

$$\mathcal{C}^2 + (m-2)\mathcal{C} - (m-1) = 0$$

for some $m \in \mathbb{Z}$ then im $(i-1)$ is a Prym-Tyurin variety of $J(X)$ (see [LB] for more details.)

The following shows that there is such a correspondence on the curve of maximal subbundles of $E \in \mathcal{SU}_C(2)$ when the genus of C is 2. The correspondence is then shown to exist for higher genus and its degree is calculated. Unfortunately it has not been possible to find a polynomial identity as for the genus 2 case. However the correspondence does have a nice geometrical interpretation in terms of multisecants to projective curves.

First consider the smooth projective curve of maximal line subbundles of a very stable vector bundle E over a curve of genus 2. This curve is supported on the set $W_E = \{L \in J^1(C) \mid h^0(L \otimes E) \neq 0\}$ so by Riemann-Roch:

$$L \in W_E \iff KL^{-1} \in W_E.$$

This defines a symmetric correspondence of degree one on W_E explicitly given by:

$$\mathcal{C} : W_E \longrightarrow W_E$$
$$L \longmapsto KL^{-1}.$$

$\mathcal{C}$ satisfies the polynomial identity $\mathcal{C}^2 - 1 = 0$ and if $W_E \subset J^1(C)$ lies away from the 16 theta characteristics of C then the involution $\mathcal{C}$ defines a 2:1 unbranched map $W_E \to X$ onto a curve of genus 3 (corresponding to the 2:1 map onto a smooth planar quartic in the Kummer surface.) If i is the induced involution on the Jacobian $J(W_E)$ then $P \stackrel{\text{def}}{=} \text{im}\,(i-1)$ is the Prym variety of the double cover $W_E \stackrel{2:1}{\to} X$.

The question arises whether the correspondence $\mathcal{C}$ can be generalised for curves C of higher genus. Let E be very stable, $c_1(E)+g \equiv 0 \pmod 2$ and W_E the smooth curve of maximal line subbundles of E. Consider the following subset of $W_E \times W_E$:

$$\begin{aligned}\mathfrak{D} &= \{(L,M) \in W_E \times W_E \mid h^1(C, LM\Lambda) \geq 1\} \quad , \quad LM\Lambda \in J^g(C) \\ &= \{(L,M) \in W_E \times W_E \mid h^0(C, KL^{-1}M^{-1}\Lambda^{-1}) \geq 1\}\end{aligned}$$

i.e. $\mathfrak{D}$ is given by the intersection of $W_{g-2}(C)$ with the image of $W_E \times W_E$ under the composition:

$$\begin{aligned} W_E \times W_E &\longrightarrow J^g(C) \longrightarrow J^{g-2}(C) \\ (L,M) &\longmapsto LM\Lambda \longmapsto KL^{-1}M^{-1}\Lambda^{-1}\end{aligned} \tag{10}$$

This construction is easily seen to reduce to the correspondence described earlier for $g = 2$.

Remark 5.6.1. We assume for the remainder of this and the next section that $E \in \mathcal{SU}_C(2,\Lambda)$ is generic in the sense that the image of $W_E \times W_E$ under the composition (10) does not intersect $W^1_{g-2}(C)$. To see that a generic bundle has this property we count dimensions: first suppose that $\Lambda = \mathcal{O}_C$ (a similar calculation holds if $\Lambda = \mathcal{O}_C(p)$.) Fixing $L \in J^{g/2}(C)$ there exists (for generic C) a $(g-6)$-dimensional variety $\widetilde{W} \subset J^{g/2}(C)$ such that for every $M \in \widetilde{W}$ we have $KL^{-1}M^{-1} \in W^1_{g-2}(C)$. The bundles (or extensions) in the fibre of the projective bundle $\mathbb{P}U_{g/2} \to J^{g/2}(C)$ over L (denoted by $\mathbb{P}_L \cong \mathbb{P}^{2g-2}$) which do not satisfy the assumption are those that occur in the extension space $\mathbb{P}_M$. By (5.2.1) each of these extensions lies in the linear span

$\overline{D} \subset \mathbb{P}_M$ for some $D \in |LM|$. Since $\overline{D} \cong \mathbb{P}^{g-1}$ and $h^0(LM) = 3$ the locus of such extensions is $(g+1)$-dimensional. Now varying $L \in J^{g/2}(C)$ we see that the locus of bundles of $\mathcal{SU}_C(2)$ satisfying the assumption has dimension $(g+1)+(g-6)+g = 3g-5 < \dim \mathcal{SU}_C(2)$.

Now:

$$\dim W_E \times W_E = \operatorname{codim}_{J(C)} W_{g-2}.$$

Hence $\mathfrak{D}$ has *expected* dimension zero. In fact:

Proposition 5.6.2. *Suppose E is very stable. Then the subset $\mathfrak{D} \subset W_E \times W_E$ has codimension one.*

Lemma 5.6.3. *If E is very stable and $L \in W_E$ then $KL^{-2}\Lambda^{-1} \neq \mathcal{O}_C(D)$ for any $D \in S^{g-2}C$.*

PROOF. This is just (5.4.3) restated. $\square$

PROOF OF (5.6.2). If the image of $(L, M) \in W_E \times W_E$ under the composition (10) lies on $W_{g-2}(C)$ then $KL^{-1}M^{-1}\Lambda^{-1} = \mathcal{O}_C(D)$ for some divisor of degree $g-2$ on C. Thus by fixing $L \in W_E$ the problem may be re-formulated by asking:

Do there exist divisors $D \in S^{g-2}C$ for which $M \stackrel{\text{def}}{=} KL^{-1}\Lambda^{-1}(-D)$ is an element of W_E?

i.e. divisors $D \in S^{g-2}C$ for which $h^0(C, KL^{-1}\Lambda^{-1}(-D) \otimes E) \neq 0$?

Let $\rho_1 : C \times S^{g-2}C \to C$ and $\rho_2 : C \times S^{g-2}C \to S^{g-2}C$ be projections onto the first and second factors and consider the exact sequence:

$$\begin{aligned} 0 \longrightarrow \rho_1^*(KL^{-1}\Lambda^{-1} \otimes E) \otimes \Delta^{-1} \longrightarrow & \rho_1^*(KL^{-1}\Lambda^{-1} \otimes E) \\ & \longrightarrow \rho_1^*(KL^{-1}\Lambda^{-1} \otimes E)\big|_{\Delta} \longrightarrow 0 \end{aligned} \tag{11}$$

where $\Delta \subset C \times S^{g-2}C$ is a universal divisor. Now take the exact higher direct image sequence of (11) with respect to ρ_2 to obtain:

$$0 \to (\rho_2)_*(\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E) \otimes \Delta^{-1}) \longrightarrow \mathcal{A} \xrightarrow{\gamma} \mathcal{B}$$
$$\xrightarrow{\delta} (R^1\rho_2)_*(\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E) \otimes \Delta^{-1}) \to (R^1\rho_2)_*\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E) \to 0$$

Note that:

$$h^0(KL^{-1}\Lambda^{-1} \otimes E) - h^1(KL^{-1}\Lambda^{-1} \otimes E) = g-2$$

but $h^1(KL^{-1}\Lambda^{-1} \otimes E) = h^0(L\Lambda \otimes E^*) = h^0(L \otimes E) = 1$ since $L \in W_E$. Thus

$$\mathcal{A} = (\rho_2)_*(\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E))$$

is a vector bundle on $S^{g-2}C$ of rank $g-1$.

$$\mathcal{B} = (\rho_2)_*(\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E)\big|_{\Delta})$$

is a vector bundle of rank $2g-4$ on $S^{g-2}C$, its fibre over $D \in S^{g-2}C$ being identified with the vector space $H^0(F/F(-D))$ where $F = KL^{-1}\Lambda^{-1} \otimes E$. As in (5.3.3) this construction is functorial with respect to base change and:

$$\begin{aligned}
Y_L \stackrel{\text{def}}{=} &\{D \in S^{g-2}C \mid h^0(KL^{-1}\Lambda^{-1}(-D) \otimes E) = 1\} \\
= &\{D \in S^{g-2}C \mid h^1(KL^{-1}\Lambda^{-1}(-D) \otimes E) = g-1\} \\
= &\{D \in S^{g-2}C \mid \mathrm{rank}\delta_D = g-2\} \\
= &\{D \in S^{g-2}C \mid \mathrm{rank}\gamma_D = (2g-4)-(g-2) = g-2\}
\end{aligned}$$

i.e. Y_L is the $(g-2)$-nd determinantal variety associated to $\mathcal{A} \xrightarrow{\gamma} \mathcal{B}$. If Y_L is non-empty then the codimension of Y_L satisfies the inequality:

$$\mathrm{codim}_{S^{g-2}C} Y_L \le [(2g-4)-(g-2)][(g-1)-(g-2)] = g-2$$

and $\mathfrak{D}\big|_{\{L\}\times W_E}$ is given by the image of Y_L under the Abel-Jacobi map so $\dim \mathfrak{D}\big|_{\{L\}\times W_E} \ge 0$. Equality holds if $\mathfrak{D}\big|_{\{L\}\times W_E} \ne \{L\} \times W_E$. Since Y_L is closed this is the case if there exists a single point of $\{L\} \times W_E$ whose image via (10) does not lie on $W_{g-2}(C)$. By (5.6.3) this point is given by (L, L). It just remains to show that Y_L is non-empty. This follows from the following result (5.6.5) calculating the class y on which Y_L is supported. □

Remark 5.6.4. It follows from (5.6.1) and the fact that $\dim \mathfrak{D}\big|_{\{L\}\times W_E} = 0$ that $\dim Y_L = 0$ when Y_L is non-empty.

Again let the correspondence given above be denoted by $\mathcal{C}$ where:

$$\mathcal{C} : W_E \longrightarrow W_E$$

$$L \longmapsto \{M \in W_E \mid h^1(C, LM\Lambda) \geq 1\}$$

Proposition 5.6.5. *The degree of the correspondence $\mathcal{C}$ is given by:*

$$\deg \mathcal{C} = 1 + 2^{g-1}(g-2).$$

PROOF. Fix $L \in W_E$. The cohomology class of $Y_L \subset S^{g-2}C$ defined in the proof of (5.6.2) gives the number of divisors $D \in S^{g-2}C$ such that:

$$M \overset{\text{def}}{=} KL^{-1}\Lambda^{-1}(-D) \in W_E \tag{12}$$

whereas the degree of $\mathcal{C}$ is given by the number of *line bundles* M. By the assumption in (5.6.1) these numbers are equal.

Now Y_L is empty or has expected dimension (5.6.4) and so its class is given by Porteous' formula (5.1.1) i.e.

$$y = \Delta_{g-2,1}(c_t(\mathcal{B} - \mathcal{A})).$$

Now $\mathcal{A}$ is trivial of rank $g-1$ and to calculate the Chern polynomial of:

$$\mathcal{B} = (\rho_2)_*(\rho_1^*(KL^{-1}\Lambda^{-1} \otimes E)\big|_{\Delta}) \overset{\text{def}}{=} (\rho_2)_*\mathcal{B}'\big|_{\Delta}$$

use the Grothendieck-Riemann-Roch formula:

$$ch((\rho_2)_!\mathcal{B}'\big|_{\Delta}) \cdot td(S^{g-2}C) = (\rho_2)_*(ch(\mathcal{B}'\big|_{\Delta}) \cdot td(C \times S^{g-2}C)) \tag{13}$$

The calculation of this Chern character is similar to [ACGH, p340, Lemma 2.5.] From the exact sequence:

$$0 \to \mathcal{O}_{C\times S^{g-2}C}(\mathcal{B}' \otimes (-\Delta)) \to \mathcal{O}_{C\times S^{g-2}C}(\mathcal{B}') \to \mathcal{O}_{\Delta}(\mathcal{B}') \to 0$$

and the fact that $\deg KL^{-1}\Lambda^{-1} \otimes E = 3g-4$:

$$\begin{aligned} ch(\mathcal{B}'\big|_{\Delta}) &= ch(\mathcal{B}') - ch(\mathcal{B}') \cdot ch(-\Delta) \\ &= (2+(3g-4)\xi)(1-e^{-\delta}) \end{aligned}$$

where ξ is the class of the pullback of a point, and δ is the class of the universal divisor $\Delta \subset C \times S^{g-2}C$. By [ACGH, p 338] :

$$\delta = (g-2)\xi + \varsigma + x$$

where ς is the class of the diagonal in $C \times C$, and x is the class of $C_q = q + C_{g-3} \subset S^{g-2}C$, $q \in C$. The following relations hold:

$$\varsigma^2 = -2\xi\theta \quad , \quad \xi^2 = \xi\varsigma = \varsigma^3 = 0$$

where θ is the pullback of the class $\theta \in H^2(J(C), \mathbb{Z})$ to $S^{g-2}C$. Thus:

$$\begin{aligned} ch(\mathcal{B}'\big|_{\Delta}) &= [2+(3g-4)\xi][1-e^{-(g-2)\xi-\varsigma-x}] \\ &= [(2+(3g-4)\xi][1-(1-(g-2)\xi)(1-\varsigma-\xi\theta)e^{-x}] \\ &= 2+(3g-4)\xi-(2+g\xi)(1-\varsigma-\xi\theta)e^{-x}. \end{aligned}$$

Substituting into (13) (and cancelling $td(S^{g-2}C)$ terms) gives:

$$\begin{aligned} ch(\mathcal{B}) &= (\rho_2)_*[(2+(3g-4)\xi-(2+g\xi)(1-\varsigma-\xi\theta)e^{-x}) \cdot (1+(1-g)\xi)] \\ &= (\rho_2)_*[2+(g-2)\xi+(-2+(g-2)\xi+2\varsigma+2\xi\theta)e^{-x}] \\ &= (g-2)+[(g-2)+2\theta]e^{-x}. \end{aligned}$$

Now $(g-2)+2\theta$ is the Chern character of a vector bundle G of rank $g-2$ having Chern polynomial $e^{2t\theta}$. So $[(g-2)+2\theta]e^{-x}$ is the Chern character of the tensor product of such a bundle with a line bundle having first Chern class $-x$. If

$$c_t(G) = e^{2t\theta} = \prod_{i=1}^{g-2}(1+\beta_i t)$$

then:

$$\begin{aligned} c_t(\mathcal{B}) &= \prod_{i=1}^{g-2}(1-(x-\beta_i)t) \\ &= (1-xt)^{g-2}\prod_{i=1}^{g-2}\left(1+\frac{\beta_i t}{1-xt}\right) \\ &= (1-xt)^{g-2}\exp\left(\frac{2t\theta}{1-xt}\right). \end{aligned}$$

Now $y = \Delta_{g-2,1}(c_t(\mathcal{B}-\mathcal{A})) =$ coefficient of t^{g-2} in $c_t(\mathcal{B})$.

$$c_t(\mathcal{B}) = (1-xt)^{g-2} + (2t\theta)(1-xt)^{g-3} + \cdots + \frac{(2t\theta)^{g-3}}{(g-3)!}(1-xt) + \frac{(2t\theta)^{g-2}}{(g-2)!} + \ldots$$

Thus:

$$\begin{aligned} \text{coefficient of } t^{g-2} = (-1)^{g-2}x^{g-2} &+ (-1)^{g-3}(2\theta)x^{g-3} + \ldots \\ &\cdots + \frac{(2\theta)^{g-3}}{(g-3)!}(-x) + \frac{(2\theta)^{g-2}}{(g-2)!} \end{aligned} \tag{14}$$

If $u : S^{g-2}C \to J^{g-2}(C)$ is the Abel-Jacobi map then $u_* x = [W_{g-3}] = \frac{\theta^3}{3!}$ and more generally:

$$u_* x^i = [W_{g-2-i}] = \frac{\theta^{2+i}}{(2+i)!} \quad \text{for } i \geq 0.$$

Thus applying u_* to (14) gives:

$$\begin{aligned} \deg \mathcal{C} &= (-1)^{g-2} + \frac{(-1)^{g-3}2g!}{(g-1)!} + \cdots - \frac{2^{g-3}g!}{(g-3)!3!} + \frac{2^{g-2}g!}{(g-2)!2!} \\ &= (-1)^{g-2}\sum_{k=0}^{g-2}(-2)^k\binom{g}{k} \\ &= (-1)^{g-2}[(1-2)^g - \frac{(-2)^{g-1}g!}{(g-1)!} - (-2)^g] \\ &= 1 + 2^{g-1}g - 2^g \\ &= 1 + 2^{g-1}(g-2) \end{aligned}$$

as asserted. □

5.7. Multisecants to projective curves

We start this section with an improvement of the bijection of Lange and Narasimhan (5.2.1) for rank 2 bundles with maximal Segre invariant:

Proposition 5.7.1. *Given a point $(e) \in \mathbb{P}H^0(KL^2\Lambda)^*$ corresponding to a generic bundle E of maximal Segre invariant, i.e. $s(E) = c_1(L^2\Lambda) = d = g$ or $g-1$, there is a canonical bijection between:*

1. Maximal line subbundles $M^{-1} \subset E$ (different from L^{-1}.)

2. Divisors $D \in S^dC$ such that $(e) \in \overline{D} \subset \mathbb{P}H^0(KL^2\Lambda)^*$.

Proof. Since E has maximal Segre invariant then generically it will have more than one maximal line subbundle (cf (5.2.2), (5.5.6).) Suppose $M^{-1} \subset E$ is such a line subbundle then by (5.2.1) this corresponds to the line bundle $LM\Lambda$ and $(e) \in \overline{D} \subset \mathbb{P}H^0(KL^2\Lambda)^*$ for some $D \in |LM\Lambda|$. We show that D is the *unique* divisor from this linear system with this property. First tensor the extension (e) by M:

$$0 \longrightarrow ML^{-1} \longrightarrow M \otimes E \longrightarrow LM\Lambda \longrightarrow 0.$$

Since $M \neq L$ taking the long exact cohomology sequence gives:

$$0 \longrightarrow H^0(M \otimes E) \longrightarrow H^0(LM\Lambda) \xrightarrow{\delta(e)} H^1(ML^{-1}) \longrightarrow \dots$$

where the coboundary map is induced by the element $(e) \in H^1(L^{-2}\Lambda^{-1})$. By (5.3.1) $h^0(M \otimes E) = 1$ so the coboundary map $\delta(e)$ has one-dimensional kernel. This means that (e) is in the kernel of the map:

$$\mathbb{P}H^0(KL^2\Lambda)^* \longrightarrow \mathbb{P}H^0(KL^2\Lambda(-D))^* \cong \mathbb{P}H^1(ML^{-1})$$

for a unique $D \in |LM\Lambda|$ i.e. $(e) \in \overline{D} \subset \mathbb{P}H^0(KL^2\Lambda)^*$. □

Definition 5.7.2. *By a d-secant to $C \to \mathbb{P}^n$ we shall mean an effective divisor of degree d on C spanning a $(d-2)$-plane.*

Lemma 5.7.3. *Let $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^{2g-2-k}$ be a curve of genus $g \geq 3$ and degree $3g-2-k$ ($k = 0, 1$) and suppose that $N \notin W^{1-k}_{g-k}(C)$. Then the number of $(g-k)$-secants to C is $1-k$.*

PROOF. $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^{2g-2-k}$ has a $(g-k)$-secant given by a divisor D if and only if:

$$\begin{aligned} h^0(KN(-D)) &= (2g-k-1) - (g-k-1) \\ &= g \end{aligned}$$

i.e. $\mathcal{O}_C(D) = N$. By hypothesis $N \notin W^{1-k}_{g-k}(C)$ so if $k = 1$ then $C \to \mathbb{P}H^0(KN)^*$ has no $(g-1)$-secants and if $k = 0$ then it has a *unique* g-secant. $\square$

Proposition 5.7.4. *Let $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^{2g-3}$ be a curve of genus $g \geq 3$ and degree $3g-3$ with $N \in J^{g-1}(C)$ a generic point (in particular $N \notin W_{g-1}(C)$.) Then the projection of C from a generic point $(e) \in \mathbb{P}^{2g-3}$ gives a curve $C \to \mathbb{P}^{2g-4}$ with exactly $2^g - 1$ $(g-1)$-secants.*

PROOF. By (5.7.3) the curve $C \to \mathbb{P}H^0(KN)^*$ has no $(g-1)$-secants. So all $(g-1)$-secants to $C \to \mathbb{P}^{2g-4}$ come from divisors D of degree $g-1$ spanning a $(g-2)$-plane in $\mathbb{P}H^0(KN)^*$ and passing through (e). We now identify $\mathbb{P}H^0(KN)^*$ with the space of extensions $\mathbb{P}H^1(L^{-2}\Lambda^{-1})$ by choosing L and Λ appropriately i.e. $\Lambda = \mathcal{O}_C(p)$ for any $p \in C$ and $L^2 = N(-p)$ if g is even; and $\Lambda = \mathcal{O}_C$ and $L^2 = N$ if g is odd. By a generic $N \in J^{g-1}(C)$ we shall mean an N such that the locus of not very stable bundles in the extension space $\mathbb{P}H^1(N^{-1}) \cong \mathbb{P}^{2g-3}$ has codimension one or more. If this were not the case for generic N then via the finite to one extension map:

$$\mathbb{P}U_d \xrightarrow{\epsilon_d} \mathcal{SU}_C(2, \Lambda)$$

($d = \frac{g-1-i}{2}$ where $i = \deg \Lambda$) one sees that the locus of not very stable bundles in

$\mathcal{SU}_C(2,\Lambda)$ has dimension $(2g-3)+g = \dim \mathcal{SU}_C(2,\Lambda)$, contradicting (5.4.2). In particular $N \notin W_{g-1}(C)$ otherwise *every* extension of $\mathbb{P}H^1(N^{-1})$ would correspond to a not very stable bundle (cf (5.4.3).) Now $(e) \in \mathbb{P}H^1(L^{-2}\Lambda^{-1})$ is chosen generically such that the corresponding bundle is very stable, has maximal Segre invariant $g-1$ i.e. $(e) \notin Sec_{g-2}C$ (cf (2.0.3)) and has a finite number of maximal line subbundles (cf (5.2.2).) Then by (5.7.1) divisors D of the type above are in one to one correspondence with maximal line subbundles of E (other than L^{-1}). By (5.5.6) there are $2^g - 1$ of these. □

Example 5.7.5. If C has genus 3 then using (5.7.4) we recover the fact that the projection of $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^3$ away from a generic point of $\mathbb{P}^3$ gives a planar curve of degree 6 with 7 nodes.

If C has genus 4 then (5.7.4) says that the projection of $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^5$ away from a generic point of $\mathbb{P}^5$ gives a curve $C \to \mathbb{P}^4$ of degree 9 with exactly 15 trisecants. This is backed up by a classical result of Berzolari [LeB] which says that a curve in $\mathbb{P}^4$ of genus g and degree d has:

$$\binom{d-2}{3} - g(d-4)$$

trisecants.

The correspondence $\mathcal{C}$ on the curve of maximal line subbundles given in section 5.6 has a nice geometric interpretation in terms of g-secants to projective curves. By (5.7.1) every maximal line subbundle of E (other than L^{-1}) is given by a divisor D such that $\deg D = g$ and $\overline{D} \cong \mathbb{P}^{g-1}$ passes through $(e) \in \mathbb{P}_L$. If $M, N \in W_E$ then the corresponding divisors satisfy:

$$LM\Lambda = \mathcal{O}_C(D) \quad \text{and} \quad LN\Lambda = \mathcal{O}_C(D'). \tag{15}$$

Proposition 5.7.6. *$M, N \in W_E$ are in correspondence with each other if and only if $\overline{D+D'} \subset \mathbb{P}_L$ spans a hyperplane.*

PROOF. By (10) and (5.6.1) $N \in \mathcal{C}(M)$ if and only if $h^0(KM^{-1}N^{-1}\Lambda^{-1}) = 1$. Substituting in (15) this is equivalent to:

$$h^0(KL^2\Lambda(-D-D')) = 1$$

i.e. $\overline{D+D'}$ spans a hyperplane in $\mathbb{P}H^0(KL^2\Lambda)^* = \mathbb{P}_L$. □

Proposition 5.7.7. *Let $C \to \mathbb{P}H^0(KN)^* \cong \mathbb{P}^{2g-2}$ be a curve of genus $g \geq 3$ and degree $3g-2$ with $N \in J^g(C)$ a generic point (in particular $N \notin W^1_g(C)$.) Then projection of C from a generic point $(e) \in \mathbb{P}^{2g-2}$ gives a curve $C \to \mathbb{P}^{2g-3}$ with a one dimensional family of g-secants. If this curve of g-secants is connected then it has genus $g' = 1 + 2^g(g-1)$. Moreover if C is generic then a generic g-secant meets exactly $1 + 2^{g-1}(g-2)$ other g-secants away from the curve.*

PROOF. Again identify $\mathbb{P}H^0(KN)^*$ with the space of extensions $\mathbb{P}H^1(L^{-2}\Lambda^{-1})$ taking $\Lambda = \mathcal{O}_C(p)$ for any $p \in C$ and $L^2 = N(-p)$ if g is odd; and $\Lambda = \mathcal{O}_C$ and $L^2 = N$ if g is even. By a generic $N \in J^g(C)$ we shall mean an N such that the locus of not very stable bundles in the extension space $\mathbb{P}H^1(N^{-1}) \cong \mathbb{P}^{2g-2}$ has codimension one or more. If this were not the case for generic N then via the extension map (which generically has one-dimensional fibre):

$$\mathbb{P}U_d \xrightarrow{\epsilon_d} \mathcal{SU}_C(2,\Lambda)$$

($d = \frac{g-i}{2}$ where $i = \deg\Lambda$) one see that the locus of not very stable bundles in $\mathcal{SU}_C(2,\Lambda)$ has dimension $(2g-2)+g-1 = \dim\mathcal{SU}_C(2,\Lambda)$ contradicting (5.4.2). In particular $N \notin W^1_g(C)$ otherwise *every* extension of $\mathbb{P}H^1(N^{-1})$ would correspond to a not very stable bundle (cf (5.4.3).) By a generic point $(e) \in \mathbb{P}H^1(L^{-2}\Lambda^{-1})$ we mean an extension corresponding to a very stable bundle E with Segre invariant g. By (5.7.1) and (5.7.3) there is a one to one correspondence between g-secants to $C \to \mathbb{P}^{2g-3}$ and maximal line subbundles of E. By (5.5.5) this is a curve of genus $1 + 2^g(g-1)$ (if connected.)

We now wish to apply the correspondence $\mathcal{C}$ to this curve of g-secants so suppose also that C and E satisfy the conditions of (5.6.1). By (5.7.6) two maximal line subbundles M^{-1}, N^{-1} of E are in correspondence with each other if and only if $\overline{D+D'}$ spans a hyperplane in $\mathbb{P}_L$. Projecting from (e) shows the corresponding g-secants of $\mathbb{P}^{2g-3}$ meet. Either they meet away from C which will be the case generically or D and D' share a common point p i.e. the hyperplane $\overline{D+D'}$ cuts the curve $C \to \mathbb{P}H^1(L^{-2}\Lambda^{-1})$ with multiplicity 2 at p. Thus by (5.6.5) the number of g-secants meeting a generic g-secant away from the curve is given by the degree of the correspondence $\mathcal{C}$ i.e. $1 + 2^{g-1}(g-2)$. $\square$

Example 5.7.8. If $g = 3$ and $E \in \mathcal{SU}_C(2,1)$ then the one dimensional family of maximal line subbundles of E gives a one dimensional family of trisecants to a curve of degree 7 in $\mathbb{P}^3$. (5.5.5) says that if connected this curve of trisecants has genus 17. This is backed up by a result of Gruson and Peskine [GP, Thm 3.6] which says that for a smooth space curve of genus g and degree d the curve of trisecants to it has geometric genus:

$$g' = \frac{1}{6}(d-4)(d-5)(2d-3) + \frac{g}{2}(d^2 - 9d + 24 - 2g).$$

Moreover a generic trisecant of $C \to \mathbb{P}^3$ meets 5 other trisecants away from the curve. By projecting away from a point on the curve and counting double points it is seen that there are 7 trisecants through every point of C. Thus in total there are $6 + 6 + 6 + 5 = 23$ trisecants meeting a generic trisecant.

5.8. Connectedness of fibres

Section 5.4 described exactly when the fibres of the extension map $\epsilon_d : \mathbb{P}U_d \to \mathcal{SU}_C(2,\Lambda)$ were smooth and section 5.5 calculated the degree of its canonical line bundle in the curve fibre case, which in turn gives the genus of W_E if it is connected. This section gives a list of conditions that are sufficient for these curves to be connected. First note that in the genus 2 case W_E is clearly connected since it is a 2Θ divisor. For higher genus we describe W_E in a slightly different way to that in (5.3.3) but keep the same notation. If $L^{-1} \subset E$ is a maximal line subbundle then by Riemann-Roch:

$$h^0(L \otimes E) = 1 \iff h^0(KL^{-1}\Lambda^{-1} \otimes E) = g-1.$$

Taking the exact higher direct image of the sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{P}^{-1} \otimes \sigma^*(K\Lambda^{-1} \otimes E) \longrightarrow \mathcal{P}^{-1} \otimes \sigma^*(K^2\Lambda^{-1} \otimes E) \\ \xrightarrow{\gamma_1} (\mathcal{P}^{-1} \otimes \sigma^*(K^2\Lambda^{-1} \otimes E))\big|_{\Gamma} \longrightarrow 0 \end{aligned}$$

as in (6) gives a homomorphism $\mathcal{E}_1 \xrightarrow{\gamma_1} \mathcal{F}_1$, where rank $\mathcal{E}_1 = 5g-6$, rank $\mathcal{F}_1 = 4g-4$ and W_E is given as the $(4g-5)$-th determinantal variety associated to γ_1. The reason for this construction is so that $e = \mathrm{rank}\mathcal{E}_1 \geq \mathrm{rank}\mathcal{F}_1 = f$ and we may write down the complex (16) below. Now:

Proposition 5.8.1. *W_E is connected if either of the following hold:*

1. *$\mathcal{E}_1^*$ is ample,*
2. *$H^i(J^d(C), \bigwedge^{4g-5+i}\mathcal{E}_1) = 0$ for all i with $1 \leq i \leq g-1$.*

Proposition 5.8.2. *If C is a curve of genus 3 then W_E is connected if any of the following hold:*

1. *$\mathcal{E}_1^*$ is ample,*

2. $H^2(J^1(C), \mathcal{E}_1 \otimes \det^{-1}\mathcal{E}_1) = 0$,

3. $\mathcal{O}_{\mathbb{P}\mathcal{E}_1}(10)$ *is nef and big.*

The condition that $\mathcal{E}_1^*$ is ample follows immediately from the connectedness theorem of Fulton and Lazarsfeld [ACGH, p311] (recall that $\mathcal{F}_1$ is a trivial vector bundle.) The second condition of (5.8.1) comes from the Eagon-Northcott complex [EN] :

$$0 \to K^{e-f} \to \cdots \to K^1 \to K^0 \to \mathcal{I}_{W_E} \to 0 \tag{16}$$

where $\mathcal{I}_{W_E}$ is the ideal sheaf of $W_E \subset J^d(C)$ and

$$K^i \stackrel{\text{def}}{=} S^i\mathcal{F}_1^* \otimes \det \mathcal{F}_1^* \otimes \wedge^{f+i}\mathcal{E}_1.$$

Now W_E is connected if $H^1(J^d(C), \mathcal{I}_{W_E}) = 0$. From (16) this is equivalent to showing:

$$H^i(K^{i-1}) = 0 \qquad \text{for all } i.$$

Or more explicitly, since $\mathcal{F}_1$ is trivial, that:

$$H^i(S^{i-1}\mathcal{F}_1^* \otimes \wedge^{4g-5+i}\mathcal{E}_1) = 0 \qquad 1 \le i \le g-1$$

but this is the case if and only if:

$$H^i(\wedge^{4g-5+i}\mathcal{E}_1) = 0 \qquad 1 \le i \le g-1.$$

In the genus 3 case this last condition reduces to:

$$H^2(\det \mathcal{E}_1) = H^1(\det^{-1}\mathcal{E}_1) = 0 \quad \text{and} \quad H^2(\mathcal{E}_1 \otimes \det^{-1}\mathcal{E}_1) = 0.$$

By a Grothendieck-Riemann-Roch calculation the class of $\det^{-1}\mathcal{E}_1$ is given as 2θ. Thus $H^i(\det^{-1}\mathcal{E}_1) = 0$ for all $i \ge 1$.

The third condition of (5.8.2) comes from the isomorphism:

$$\begin{aligned} H^2(J^d(C), \mathcal{E}_1 \otimes \det^{-1}\mathcal{E}_1) &\cong H^2(\mathbb{P}\mathcal{E}_1, \mathcal{O}_{\mathbb{P}\mathcal{E}_1}(1) \otimes \pi^*\det^{-1}\mathcal{E}_1) \\ &\cong H^2(\mathbb{P}\mathcal{E}_1, K_{\mathbb{P}\mathcal{E}_1} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}_1}(10)). \end{aligned}$$

A theorem of Kawamata-Viehweg [SS] says that this vanishes if $\mathcal{O}_{\mathbb{P}\mathcal{E}_1}(10)$ is nef and big.

Remark 5.8.3. By the discussion in the last section the maximal line subbundles of $E \in \mathcal{SU}_C(2,1)$ for a curve C of genus 3 correspond to trisecants to a space curve of degree 7. It is therefore interesting to see if the scheme of trisecants to such a curve is connected. The answer to this question is yes and follows from a theorem of Ballico [Ba] which says that the scheme of trisecants to a smooth, connected space curve with non-special hyperplane sections is connected. It should be noted that this result does not imply the connectedness of W_E in the genus 3 case since the scheme of trisecants above is obtained by *projecting* the 3-secants to $C \to \mathbb{P}_L$ (which correspond to the maximal line subbundles of E) from a point $(e) \in \mathbb{P}_L$. However irreducibility of the scheme of trisecants *would* imply the connectedness of W_E.

Bibliography

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris; *Geometry of Algebraic Curves*; Springer, 1985;

[AS] E. Arrondo, I. Sols; *Bounding sections of bundles on curves*; Complex Projective Geometry, L.M.S Lecture Note Series **179** ;

[Ba] E. Ballico; *On the connectedness of the scheme of multisecants to a projective curve*; Geom. Ded. **53** (1994), 327–332;

[B] A. Beauville; *Fibrés de rang 2 sur les courbes fibré déterminant et fonctions thêta*; Bull. Soc. math. France **116** (1988), 431–448;

[Be] A. Bertram; *Moduli of rank 2 vector bundles, theta divisors, and the geometry of curves in projective space*; J. Differential Geometry **35** (1992), 429–469;

[BNR] A. Beauville, M. S. Narasimhan, S. Ramanan; *Spectral curves and the generalised theta divisor*, J. reine angew. Math. **398** (1989), 169–179;

[DN] J. -M. Drezet, M. S. Narasimhan; *Groupe de Picard des variétiés modules de fibrés semi-stables sur les courbes algébriques*, Invent. math. **97** (1989), 53–94;

[EN] J. Eagon, D. G. Northcott; *Ideals defined by matrices and a certain complex associated to them*, Proc. Royal. Soc. **A 269** (1962), 188–204;

[G] F. Ghione; *Quelques résultats de Corrado Segre sur les surfaces réglées*, Math. Ann. **255** (1981), 77–95;

[GH] P. A. Griffiths, J. Harris; *Principles of Algebraic Geometry*; Wiley, 1978;

[GP] L. Gruson, C. Peskine; *Courbes de l'espace projectif: varietes de secantes*, Enumerative Geometry and Classical Algebraic Geometry; Birkhäuser, 1982;

[H] R. Hartshorne; *Algebraic Geometry*; Springer, 1977;

[L] H. Lange; *Universal families of extensions*, J. Algebra **83** (1983), 101–112;

[LB] H. Lange, C. Birkenhake; *Complex Abelian Varieties*; Grun. der. math. Wiss. **302** Springer-Verlag;

[LN] H. Lange, M. S. Narasimhan; *Maximal subbundles of rank two vector bundles on curves*, Math. Ann **266** (1983), 55–72;

[La] Y. Laszlo; *Un théorème de Riemann pour les diviseurs theta sur les espaces de modules de fibrés stables*, Duke Math. J. **64** (1991), 333–347;

[Lau1] G. Laumon; *Letters to M. S. Narasimhan* (1987);

[Lau2] G. Laumon; *Un analogue global du cône nilpotent*, Duke Math. J. **52** (1988), 667–671;

[LeB] P. Le Barz; *Formules Multisecantes pour les courbes gauches quelconques*, Enumerative Geometry and Classical Algebraic Geometry; Birkhäuser, 1982;

[M] M. Maruyama; *On classification of ruled surfaces*, Lectures in Mathematics, Kyoto Univ. No 3. (1970);

[Mu1] D. Mumford; *Projective invariants of projective structures and applications*, Proc. Int. Cong. of Mathematicians, Stockholm (1962), 526–530;

[Mu2] D. Mumford; *Curves and their Jacobians*, Ann Arbor, (1975);

[NR1] M. S. Narasimhan, S. Ramanan; *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. **89** (1969), 19–51;

[NR2] M. S. Narasimhan, S. Ramanan; 2Θ*-linear systems on abelian varieties*; in *Vector bundles on algebraic varieties*, Tata Institute, Bombay 1984, Oxford (1988);

[NS] M. S. Narasimhan, C. S. Seshadri; *Stable and unitary vector bundles on a compact Riemann surface*; Ann. Math. **82** (1965), 540–567;

[Ne1] P. E. Newstead; *Stable bundles of rank 2 and odd degree over a curve of genus 2*; Topology 7 (1968), 205–215;

[Ne2] P. E. Newstead; *Characteristic classes of stable bundles of rank 2 over an algebraic curve*, Trans. Amer. Math. Soc. **169** (1972), 337–345;

[Ne3] P. E. Newstead; *Rationality of moduli spaces of stable bundles*, Math Ann. **215** (1975), 251–268;

[Ne4] P. E. Newstead; *Correction to "Rationality of moduli spaces of stable bundles"*, Math Ann. **249** (1980), 281–282;

[O] W. M. Oxbury; *Stable bundles and branched coverings over Riemann surfaces*, Oxford D.Phil (1987);

[OP] W. M. Oxbury, C. Pauly; *Heisenberg invariant quartics and $\mathcal{SU}_C(2)$ for a curve of genus 4*, preprint alg-geom/9703026;

[OPP] W. M. Oxbury, C. Pauly, E. Previato; *Subvarieties of $\mathcal{SU}_C(2)$ and 2Θ divisors in the Jacobian*, preprint alg-geom/9701001 to appear in Trans. AMS;

[R] M. Raynaud; *Sections des fibrés vectoriels sur une courbe*, Bull. Soc. math. France **110** (1982), 103–125;

[SS] B. Shiffman, A. Sommese; *Vanishing Theorems on Complex Manifolds*; Birkhäuser, 1992;

[Sz] A. Szenes; *The Combinatorics of the Verlinde Formulas*; in *Vector Bundles in Algebraic Geometry*, LMS Lecture Note Series **208**;

[Th] M. Thaddeus; *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), 317–353;

[Tj] A. N. Tjurin; *On the classification of two-dimensional vector bundles over an algebraic curve of arbitrary genus*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 21–52.