Chiral gauge theories and their applications

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by

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Preface

This thesis summarises work done by the author between October 1994 and April 1998 at the Department of Mathematical Sciences of the University of Durham and at CERN theory division under the supervision of David Fairlie. No part of this work has been previously submitted for any degree at this or any other university.

Chapter one serves as an introduction and no claim is made for originality. Chapters two, three and four are believed to contain mostly original work by the author with the obvious exception of the introductory sections. (Work that is not that of the author shall be properly acknowledged.) The material of chapter two is published in Physics Letters B403 (1997) 250; chapter three is based on a paper published in Physics Letters B409 (1997) 153 and the work presented in chapter four is due to appear in Nuclear Physics B.

Abstract

This thesis is concerned with so called chiral gauge theories, also known as self dual gauge theories. In particular, the aim of this thesis to investigate the role that chiral gauge theories play in duality symmetries in lower dimensions through dimensional reduction.

Chapter one serves as an introduction to the notions of duality in field and string theory. The problems of formulating well defined actions for self-dual gauge theories are introduced as well as a brief presentation of the different approaches used to over come these problems.

Chapter two introduces dimensional reduction and demonstrates how duality symmetries arise from the dimensional reduction of self-dual theories in a variety of dimensions and on different compact spaces. Examples are presented where the couplings of the resulting theories are calculated explicitly in terms of the geometrical data of the compact space. The duality generators acting on these couplings are also calculated explicitly and related to the geometry/topology of the compact space.

Chapter three deals with the idea of duality manifest actions and their relation to the self-dual theories in higher dimensions. Non-linear Born-Infeld type actions are introduced and again dimensional reduction is shown to play a role in the duality of the Born-Infeld action. This leads to a duality manifest version of the Born-Infeld action.

Chapter four describes perhaps the main application of this thesis. The effective action of the M-theory five brane wrapped on a torus is identified with the effective action of the IIB D-3 brane dimensionally reduced on a circle (after some appropriate world volume dualizations). The IIB S-duality then arises as a result of the modular symmetry of the torus.

The final chapter contains a brief summary and a hint of further directions for research that were outside the scope of this thesis.

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I would like to dedicate this thesis to my parents.

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Chapter 1

Introduction

The study of symmetries in nature is the heart of contempory theoretical physics. Geometry provides a natural and intuitive way of expressing these symmetries. New unexpected symmetries, known as dualities are now playing a central role. It seems natural that a geometric understanding of these symmetries is available. This thesis will examine the role of chiral/self-dual gauge theories in providing a geometric description of the origin of these hidden duality symmetries. The main application of this will be in relating the M-theory five brane compactified on a torus, which contains on its world volume a chiral two form, with the IIB D-3 brane. The S-duality of the IIB theory will then arise as a result of the geometry of the compact space and the nature of the chiral gauge theory. We begain by introducing the notions of duality in field theory, sigma models and string theory. We then move on to introduce chiral gauge theories themselves.

1.1 Duality in field theory

We begin by considering an Abelian free p-form theory in D-dimensions in the absence of sources. The action is given by:

$$S = \frac{1}{2}\int_{M^D} F(A) \wedge^* F(A) \tag{1.1}$$

Where $F = dA$ is the field strength associated with the potential, $A \in \Lambda^p(M^D)$. Where $\Lambda^p(M^D)$ denotes the space of p-forms on M^D and d is the exterior derivative. See appendix A for form conventions. The equations of motion and Bianchi identities are given respectively as follows:

$$d^*F(A) = 0 \tag{1.2}$$

$$dF(A) = 0 \tag{1.3}$$

One notes immediately that the Bianchi identity and equations of motion are interchanged for a potential whose field strength is the Hodge dual of F. That is for $H = {}^*F$ where $H = dB$, equations 1.2 and 1.3 become respectively:

$$dH = 0 \tag{1.4}$$

$$d^*H = 0 \tag{1.5}$$

One can see that equation 1.4 which is the equation of motion for F is the Bianchi identity for H and vice versa for equation 1.5. We note that $H \in \Lambda^{D-p-1}(M^D)$ from the action of the Hodge dual and so $B \in \Lambda^{D-p-2}(M^D)$. Hence a p-form potential is dual to a D-p-2 form potential. Recall that a p-form in D-dimensions has n degrees of freedom where

$$n = \frac{D!}{(D-p)!p!} \tag{1.6}$$

However, these are not all physical degrees of freedom. The number of physical degrees of freedom of a massless field is associated with removing the longitudinal polarisations and so to calculate the physical degrees of freedom n_{phys} we consider the number of degrees of freedom of a p-form in $D-2$, hence,

$$n_{phys} = \frac{(D-2)!}{(D-2-p)!p!} \tag{1.7}$$

From this we see, as one would expect, that the number of physical degrees of freedom of a p-form and its dual, a D-2-p form are equal. In fact, both formulations must be physically equivalent as the equations 1.2 and 1.3 are equivalent to 1.4 and 1.5. We can move between these different, so called dual, descriptions using an action formalism as follows. First, in the action 1.1 we replace the field strength $F(A)$ which is a closed p+1 form with an arbitrary not necessarily closed p+1 form, F. We then impose its closure through the use of a Lagrange multiplier, $B \in \Lambda^{D-p-2}(M^D)$ so that we add to the action the term:

$$S_c = (-1)^p \int dB \wedge F \tag{1.8}$$

The sign in front of the constraint is simply a convenient convention. This then gives the so called parent action

$$S_P = \int_{M^D} \frac{1}{2} F \wedge {}^*F + (-1)^p dB \wedge F \tag{1.9}$$

First we will show that S_P is equivalent to the action 1.1. Rewrite the S_c as

$$S_c = (-1)^p \int d(B \wedge F) + (-1)^{D-p-1} B \wedge dF \tag{1.10}$$

The first term is a total derivative and can be ignored in topologically trivial situations. The equations of motion for B imply $dF = 0$ which we can solve locally, to give $F = dA$. Upon substituting this into the action S_P we recover the action 1.1. This is a classical equivalence. In order to examine the equivalence at the quantum level it is necessary to consider the partition function and integration over the fields to be eliminated. This is discussed later.

Now we shall derive the dual description of S by varying the action S_P with respect to F. One sees the equation of motion for F implies that

$$ {}^*F = (-1)^{(p+1)(D-p)} dB \tag{1.11} $$

We then put this into the action 1.9 to recover the dual action:

$$ S_D = \frac{1}{2} \int_{M^D} H(B) \wedge {}^*H(B) \tag{1.12} $$

where $H = dB$ the field strength for B.

We shall now describe an alternative way to demonstrate equivalence between the action and its dual. The following argument is found in [1]. This method may be more readily applied to other situations; in particular it allows us to generate dual target spaces in sigma models. This is known as T-duality and will be discussed later. The first step is to observe that the action 1.1, apart from the usual gauge symmetry $A \to A + d\chi$, where $\chi \in \Lambda^{p-1}(M^D)$ exhibits a global symmetry. This trivial symmetry is given by: $A \to A + C$, where C is a closed p form. Now we wish to extend the action 1.1 in order to make this symmetry local. This is achieved as follows, taking now $C \in \Lambda^p(M^D)$ not to be closed. We see under:

$$ A \to A + C \tag{1.13} $$

$$ F \to F + dC \tag{1.14} $$

We now introduce a new field strength

$$ G \equiv F - D \tag{1.15} $$

where $D \in \Lambda^{p+1}(M^D)$ transforms under 1.13 as follows:

$$ D \to D + dC \tag{1.16} $$

Should we now replace F in action 1.1 with G given by equation 1.15 we see that the action will be invariant under transformations 1.13, 1.14. This is because G is invariant. So we see that D acts like a gauge potential for the symmetry 1.13, 1.14 in that we have introduced it and its transformation to keep the action invariant under 1.13,1.14. Now it is clear that the new theory possessing the local symmetry is certainly not equivalent to the original theory. To recover an equivalent description of the original theory one may constrain D to be pure gauge. When a field is pure gauge, a gauge choice may be made to gauge it away and so the original theory will be recovered with only the usual local symmetry for A.

And so we introduce a new field $B \in \Lambda^{d-p-2}$ with field strength, $H = dB$. Then, to constrain D to be pure gauge ie. closed we add the following term to the action:

$$S_C = (-1)^p \int_{M^D} H(B) \wedge D \tag{1.17}$$

After integration by parts we obtain

$$S_C = (-1)^p \int_{M^D} d(B \wedge D) + (-1)^{D-p-1} B \wedge dD \tag{1.18}$$

Now the equations of motion of B imply that $dD = 0$. Again the total derivative term is topological and so as before we will ignore it for the moment. The fact that D is closed implies it is pure gauge and so we may set $D = 0$ without changing the physics. With D set to zero the action now only possess the original local symmetry and the original action 1.1 is recovered. To generate the dual action we can instead integrate out D. The equations of motion for D imply:

$$D = F + (-1)^{p+1*} H \tag{1.19}$$

Substituting this into the action produces the dual action as before:

$$S_D = \frac{1}{2} \int_{M^D} H(B) \wedge {}^* H(B) \tag{1.20}$$

Or one may simply fix a gauge for A using 1.13 so that $A = 0$. Then integrating out D is the same as integrating out F from 1.9. To summarise, we take a global symmetry and make it local by introducing an additional gauge field. We then constrain this gauge field to be pure gauge through use of a Lagrange multiplier. After integrating out the new gauge field we express the action in terms of the Lagrange multiplier which we now call the dual field.

1.1.1 Global Questions

One can think of the field A as being a connection on a U(1) fibre bundle whose base space is M^D. A non-trivial fibration will allow us to have monopoles. This view of monopoles and gauge theories follows [2] as opposed to the perhaps more physical approach of Dirac [3]. We briefly recall how this arises. Consider the magnetic charge given by the integral of the flux through a closed cycle as follows:

$$g = \int_{\Sigma^{p+1}} F \tag{1.21}$$

Where Σ^{p+1} is a closed p+1 cycle. We have $\Sigma^{p+1} = \Sigma^+ \cup \Sigma^-$ and allow $A^\pm$ to be the connection on $\Sigma^\pm$ so

$$g = \int_{\Sigma^+} dA^+ + \int_{\Sigma^-} dA^- = \int_{\partial\Sigma^+} A^+ - A^- \tag{1.22}$$

Where we have used Stokes theorem: $\int_M d\omega = \int_{\partial M} \omega$ and that $\partial\Sigma^+ = -\partial\Sigma^-$, due to the orientation properties of the boundary operator. The difference between connections on overlapping patches will be given by the transition function that is $A^+ - A^- = d\chi$. This implies

$$g = \int_{\partial\Sigma^+} d\chi \tag{1.23}$$

The group element, $G = e^{i\chi}$ must be single valued. This constrains the periods of χ to be integer multiples of 2π. Therefore, $\int_\Sigma d\chi = 2\pi n \qquad n \in \mathbf{Z}$ and so:

$$g = 2\pi n \qquad n \in \mathbf{Z} \tag{1.24}$$

This is the Dirac quantization condition. It states that magnetic charge g, is an integral multiple of 2π, we have implicitly set the electric charge to 1 as no coupling appears in 1.1. As an aside, the magnetic charge 1.21 is related to the first Chern class, $c_1(L)$ of the bundle, L, by $g = 2\pi c_1(L)$.

It is clear that the constraint $dD = 0$, which follows from integrating out B from the action 1.18, does not imply one can use transformation 1.16 to gauge it away globally. Essentially, for $D \in H^{p+1}(M^D)$, that is for D a representative of the p+1 cohomology of M^D, we will not be able to gauge away D even though $dD = 0$. First we note that the first term in the action 1.18, (the total derivative) actually constrains the periods of D to be integer multiples of 2π. We then note that the transformation 1.14 allows a shift in the periods of F by integer multiples of 2π. Hence, any non-trivial $D \in H^{p+1}(M^D, \mathbf{Z})$ may be removed from the action by a shift in F using the transformation 1.14.

We now move to the specific case of electromagnetism in 4-dimensions. This is the action 1.1 with $A \in \Lambda^1(M^4)$. We are free to add total derivatives to the action without affecting the equations of motion. Consider the term $\frac{\theta}{4\pi^2}\int_{M^4} F \wedge F$. This term despite being a total derivative is important in the presence of monopoles. (See [4] and references therein for a discussion.) It has been shown that the effect of this term is to charge the monopoles electrically so that they in fact become dyonic. In fact a fundamental monopole will carry electric charge $\frac{\theta}{2\pi}$. Extending the arguments for Dirac quantization to include objects that carry both electric and magnetic charges (dyons) one finds the equivalent quantisation condition, due to Zwanziger [5] is

$$e_1 g_2 - e_2 g_1 = 2\pi n \hbar \qquad n \in \mathbf{Z} \tag{1.25}$$

Where e_i is the electric charge carried by object i and g_i is the magnetic charge carried by object i. From now on we will set $\hbar$ to be one.

Reinstating the coupling and introducing the theta term in the action we have:

$$S = \int_{M^4} \frac{1}{2e^2} F \wedge {}^*F - \frac{\theta}{16\pi^2} F \wedge F \tag{1.26}$$

In order to repeat the duality procedure for this action described above we rewrite it in the following way.

$$S = \frac{1}{16\pi} \int_{M^4} \Im\{\tau F^{(+)} \wedge {}^*F^{(+)}\} \qquad (1.27)$$

Where we have introduced the complex coupling

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \qquad (1.28)$$

and the (anti-)self-dual field strength:

$$F^{(\pm)} = F \pm i^*F \qquad (1.29)$$

We can write the action 1.27 as:

$$S = \frac{-i}{32\pi} \int_{M^4} \tau F^{(+)} \wedge {}^*F^{(+)} - \bar{\tau} F^{(-)} \wedge {}^*F^{(-)} \qquad (1.30)$$

With the action in this form it is then trivial to repeat the duality procedure given above by introducing a (anti-) self-dual field strength $H^{(\pm)}$ for the Lagrange multiplier field B. This gives for the parent action:

$$S_P = \frac{-i}{32\pi} \int_{M^4} \tau F^{(+)} \wedge {}^*F^{(+)} - \bar{\tau} F^{(-)} \wedge {}^*F^{(-)} - H^{(+)} \wedge F^{(+)} + H^{(-)} \wedge F^{(-)} \qquad (1.31)$$

Therefore, after finding the equations of motion of $F^{(\pm)}$ and substituting this into the action to express the action in term of the dual field strength we find:

$$S_D = \frac{1}{16\pi} \int_{M^4} \Im\{\left(\frac{-1}{\tau}\right) H^{(+)} \wedge {}^*H^{(+)}\} \qquad (1.32)$$

The result of this is that the dual theory has the same form as the original theory but with the complex coupling inverted. In the absence of the theta term the dual theory has coupling $\tilde{e} = 2\frac{2\pi}{e}$ which we remark is the same as the monopole charge. [1]

Hence one can think of the dual theory as being a theory in which monopoles are the fundamental objects and electrons are the topologically non-trivial objects. This is what one would imagine from exchanging the roles of Bianchi identities and equations of motion when moving between dual pictures. Hence, we have shown that

$$\tau \to \frac{-1}{\tau} \tag{1.33}$$

is a duality symmetry. We also know that there ought to be a trivial symmetry created by shifts in $\Re\{\tau\}$ as $F \wedge F$ is a total derivative. In fact one can show that a shift

$$\tau \to \tau + 1 \tag{1.34}$$

leaves the partition function invariant. First note that the quantisation condition 1.24 implies $F \wedge F$ will be quantised in multiples of $4(2\pi)^2$, [1]. Therefore the θ term in the action 1.26 will give a contribution of $\theta m, \quad m \in \mathbf{Z}$. Therefore, $\theta \to \theta + 2\pi$ leaves the partition function invariant. Translating this invariance in terms of τ we see that this corresponds to the transformation 1.34.

Combinations of these two generators gives an element of the group SL(2,Z). Where the group acts as a fractional linear transformation on τ as follows:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.35}$$

where $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$.

In fact this is the modular group of the torus with modular parameter τ.

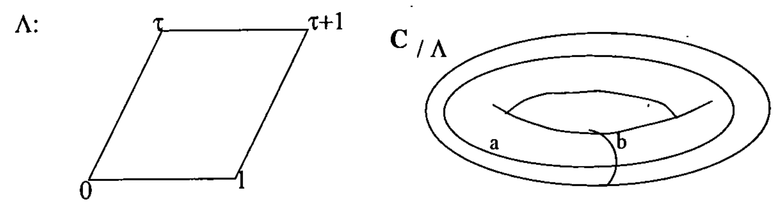

Figure 1.1: torus

Recall that a torus may be described as the complex plane modulo a lattice, Λ. The conformal part of the lattice is defined by the modular parameter τ as shown in the figure 1.1. Any transformation of τ that leaves the lattice invariant is a symmetry of the torus. The two independent transformations are given by 1.33 and 1.34. Geometrically one sees that the first generator essentially switches the definition of a and b cycles while the second is just a shift by a lattice vector.

We will now move on to consider aspects of the quantum theory and the dualisation procedure. This follows arguments given in [1, 6]. Suppressing the overall normalization factor arising form the regularisation procedure and the factor of the volume of the gauge group, recall that the partition function for action 1.26 may be written as follows:

$$Z(\tau) = (\tau_2)^{\frac{1}{2}(b_1 - b_0)} \int \mathcal{D}A e^{-S_\tau(A)} \tag{1.36}$$

Where $b_i = \dim(\text{ Harm}^i(M^4))$ is the i-th Betti number.

The term in front of the functional requires some explanation. This is added to remove the integration over the zero modes of the fields in the partition function. There will be b_1 zero modes of A and b_0 zero modes of the the gauge transformation parameter. The prefactor cancels the contribution to Z from integrating over these modes.

So introducing the parent action 1.31 and after gauging away A we are left

[1]The factor of 2 occurs because of the normalisation of the charge. The reason for this choice is that often the action 1.26 is envisaged as an effective theory of a spontaneously broken SU(2) theory. The monopoles are then of a 't Hooft-Polyakov type [7] which have magnetic charge, $g_{SU(2)} = 2g_{dirac}$

with:

$$Z(\tau) = (\tau_2)^{\frac{1}{2}(b_1 - b_0)} \int \mathcal{D}D^+ \mathcal{D}D^- \mathcal{D}B^+ \mathcal{D}B^- e^{-S_\tau(D^\pm, B^\pm)} \tag{1.37}$$

(Note, the action in 1.36 and 1.37 is in Euclidean space.) We now integrate out $D^\pm$. As the action is Gaussian, we see that integrating out D is the same as using the equations of motion for D to eliminate it from the action. Therefore, the integration produces the dual action 1.32.

We now have to be concerned with the Jacobian factor. For each mode of D^- we will have $\tau^{-\frac{1}{2}}$ and for each mode of D^+ we will have $\bar{\tau}^{-\frac{1}{2}}$. Therefore, after the integration the partition function will be, and removing zero modes as before:

$$Z(\tau) = (\tau_2)^{\frac{1}{2}(b_1 - b_0)} \tau^{-\frac{1}{2}b^-} \bar{\tau}^{-\frac{1}{2}b^+} \int \mathcal{D}B^+ \mathcal{D}B^- e^{-S_{-\frac{1}{\tau}}(B^\pm)} \tag{1.38}$$

Where $b^\pm = \dim(\ \mathrm{Harm}^{2\pm}(M^4))$ is the dimension of the space of (anti-) self dual Harmonic two forms on M^4.

Recall from above that if we write the partition function 1.36 in dual variables we have:

$$Z\Big(\frac{1}{\tau}\Big) = \Big(\frac{\tau_2}{|\tau|^2}\Big)^{\frac{1}{2}(b_1 - b_0)} \int \mathcal{D}B e^{-S_{\frac{1}{\tau}}(B)} \tag{1.39}$$

We rewrite the prefactor as follows:

$$\Big(\frac{\tau_2}{|\tau|^2}\Big)^{\frac{1}{2}(b_1 - b_0)} = (\tau_2)^{\frac{1}{2}(b_1 - b_0)} (\bar{\tau}\tau)^{\frac{1}{2}(b_0 - b_1)} \tag{1.40}$$

The comparison with the prefactor for $Z(\tau)$ implies:

$$Z(\tau) = \bar{\tau}^{\frac{1}{2}(b_0 - b_1 + b^+)} \tau^{\frac{1}{2}(b_0 - b_1 + b^-)} Z\Big(\frac{1}{\tau}\Big) \tag{1.41}$$

Standard identities for the Euler characteristic:$\chi = \sum_{i=0}^{D} b_i(-1)^i$ and the Hirzebruch signature $\sigma = b^+ - b^-$ imply:

$$Z(\tau) = \bar{\tau}^{\frac{1}{4}(\chi+\sigma)} \tau^{\frac{1}{4}(\chi-\sigma)} Z\Big(\frac{1}{\tau}\Big) \tag{1.42}$$

Hence we see that when quantum effects associated with non-trivial topology of the base manifold are taken into account the partition function is not duality invariant but instead acts like a modular form whose weights are given by the topological data of the M^4.

1.2 Duality in σ-models

We begin with a discussion of bosonic σ models. The fields are maps from a d-dimensional space M into a D-dimensional target space T.

$$X^I : M \to T \tag{1.43}$$

We return to component notation so as not to confuse operations in the base and target spaces. The action is given by:

$$S = \int_{M^d} d^d\sigma \sqrt{-\eta}\eta^{\mu\nu} \partial_\mu X^I \partial_\nu X^J G_{IJ}(X) \tag{1.44}$$

Where, $\{\sigma^\mu\}$ $\mu = 0..d-1$ are coordinates on M. $\eta_{\mu\nu}$ is the metric on M. We can interpret $\{X^I\}$ $I = 0..D-1$ as coordinates on T with metric on T given by G_{IJ}. The equations of motion of $X^I(\sigma)$ are given by:

$$D^\mu \partial_\mu X^I = 0 \tag{1.45}$$

Where D_μ is the covariant derivative on T pulled back to M. The pullback given by $D_\mu = \partial_\mu X^I D_I$ The equation of motion implies that $X^I(\sigma)$ must be a harmonic function. Thus, X^I is a harmonic map from T to M.

Assume that T has an isometry. This implies the Lie derivative of the metric $G_{\mu\nu}$ in the direction of some vector field $v(\sigma)$ vanishes. That is

$$\delta_v G_{\mu\nu} = 0, \qquad \delta_v X^I = V^I \tag{1.46}$$

One can show that the action 1.44 is invariant under the transformation 1.46 provided V is a Killing vector.

$$\delta_v S = \int_{M^d} d^d\sigma\sqrt{-\eta}\eta^{\mu\nu}\partial_\mu X^I \partial_\nu X^J V^K \partial_K(G_{IJ}) + 2\partial_\mu V^I \partial_\nu X^J G_{IJ} \qquad (1.47)$$

Which becomes

$$\delta_v S = \int_{M^d} d^d\sigma\sqrt{-\eta}\eta^{\mu\nu}\partial_\mu X^I \partial_\nu X^J (V^K \partial_K G_{IJ} + 2G_{KJ}\partial_I V^K) \qquad (1.48)$$

After some manipulations we find

$$\delta_v S = \int_{M^d} d^d\sigma\sqrt{-\eta}\eta^{\mu\nu}\partial_\mu X^I \partial_\nu X^J D_{(I}V_{J)} \qquad (1.49)$$

This vanishes if v obeys the Killing equation $D_{(I}V_{J)} = 0$. Hence, an isometry of the target space, T is also a symmetry of the action 1.44.

We now wish to find the dual action. We proceed as before by introducing a parent action constructed in the same way as in the field theory case. This approach to duality in σ-models follows the discussion in [8]. First, we choose adaptive coordinates such that $V = \frac{\partial}{\partial X^0}$. We rewrite the action in terms of $W_\mu = \partial_\mu X^0$ a sort of field strength associated with the field X^0. To generate the parent action, W is taken to a generic field and a Lagrange multiplier term is introduced to constrain $dW = 0$. We will again ignore global issues. Hence we find the following parent action, up to total derivatives:

$$S_P = \int_{M^d} d^d\sigma\sqrt{-\eta}G_{00}W^\mu W_\mu + 2G_{0i}W^\mu\partial_\mu X^i + G_{ij}\partial^\mu X^i \partial_\mu X^j$$

$$+\epsilon^{\mu_1\mu_2..\mu_{d-2}\alpha\beta}B_{\mu_1..\mu_{d-2}}\partial_\alpha W_\beta \qquad (1.50)$$

$B \in \Lambda^{d-2}(M^d)$

If we integrate out B we recover the original action 1.44. If we integrate out W we obtain the following dual action:

$$S_D = \int_{M^d} d^d\sigma\sqrt{-\eta}\frac{1}{G_{00}}H^{\mu_1\mu_2..\mu_{d-1}}H_{\mu_1\mu_2..\mu_{d-1}} + \frac{G_{0i}}{G_{00}}{}^*H^\mu\partial_\mu X^i$$

$$+\Big[G_{ij} - \frac{G_{i0}G_{0j}}{G_{00}}\Big]\partial_\mu X^i \partial^\mu X^j \qquad (1.51)$$

Where $H = dB$ is the field strength of B and $*$ is the Hodge dual on M^d. Note, the field B does not have any immediate interpretation, as X did, as a coordinate on T. Thus the duality transformation associated with an isometry on T results in replacing a coordinate on T with a field on M. The transformation also results in a change in the couplings that were interpreted as the metric on T. Note that in the case of d=2 then B will be a scalar field and the duality transformation maps a scalar to a scalar. The form of the action is then left invariant only the couplings have changed. Interpreting the couplings as a metric on the target space one sees that the dual theory is again a σ-model with dual target space $\tilde{T}$ whose metric may be read off from 1.51.

1.2.1 T-duality in string theory

We now wish to concentrate on the case that is relevant to string theory. No attempt will be made here to review the whole of string theory. One simply notes that a string action, in so called Polyakov form, has the form of an *extended* σ-model whose base space M is the 2-dimensional world sheet, with coordinates $\sigma^\mu \doteq (\tau, \sigma)$, and the target space T has an interpretation as space time. The σ-model is extended in the sense that the fields X also couple to a two form background field B through a pull-back to the world sheet. We will assume we are working with a superstring theory whose critical dimension is 10 though in what follows we only deal with the bosonic part. We also assume that the strings are closed such that $\sigma = \sigma + 2\pi$. The string action is

$$S_{string} = \frac{1}{4\pi\alpha}\int_\Sigma d^2\sigma(\sqrt{-\eta}\eta^{\mu\nu}G_{IJ}(X) + \epsilon^{\mu\nu}B_{IJ}(X))\partial_\mu X^I \partial_\nu X^J \tag{1.52}$$

$(\frac{1}{\alpha})$ has an interpretation of the string tension and so has dimensions of $\frac{1}{(length)^2}$. String theory is a conformal field theory. However, in order for the the above action to be conformally invariant, that is have vanishing beta function, it is necessary to impose some constraints on the couplings G and

B [9]. Vanishing of the beta function to one loop implies that the coupling G obeys Einstien's equation and that B is harmonic on T. This is precisely what one requires to interpret G as the space-time metric and B a background field. There is also an additional allowed term that couples the world sheet to a scalar field, known as the dilaton.

$$S_{dil} = \frac{1}{8\pi}\int_\Sigma \Phi R^{(\Sigma)} \tag{1.53}$$

We now assume as before that T possess an isometry. For an isometry to be present, in addition to the Lie derivative of the metric vanishing, the Lie derivative of B and Φ must vanish. It is trivial to demonstrate that in the presence of such an isometry the action 1.52 is invariant under $\delta_V X^I = \epsilon V^I$. Where V is the Killing vector that generates the isometry. This is a global symmetry of the action. Global in this instance implies independent of the world sheet (of course V may depend on space-time). As was seen in the field theory section, given a global symmetry we may construct a dual action using the following method. First, make the symmetry local by introducing an additional gauge style field with appropriate transformation properties. Therefore,

$$\partial_\mu X^I \rightarrow D_\mu X^I \equiv \partial_\mu X^I + A_\mu V^I \tag{1.54}$$

where under

$$\delta_V X^I = \epsilon(\sigma) V^I \qquad \delta_V A_\mu = -\partial_\mu \epsilon \tag{1.55}$$

We then constrain the curvature of A to vanish by introducing the Lagrange multiplier term:

$$S_C = \frac{1}{4\pi\alpha}\int_\Sigma \partial_\mu Y A_\nu \epsilon^{\mu\nu} \tag{1.56}$$

This term includes a total derivative that constrains the periods of A to be integer multiples of 2π and a term that constrains A to be closed. Hence, integrating out Y allows us to recover the original action as A may be globally gauged away. (See discussion in field theory section about global issues.) To generate the dual action first go to adaptive coordinates such that $V^I = V\delta^{0I}$.

Then use transformation 1.55 to set $X^0 = 0$. Finally, integrate out A. To make contact with other conventions we go to complex coordinates $(\tau, \sigma) \rightarrow z, \bar{z}$, where $z = \sigma + i\tau$, and introduce the background matrix that combines the metric and two form potential as follows: $E_{IJ} = G_{IJ}(X) + B_{IJ}(X)$. It is also natural to absorb the tension into the target space coordinates to give dimensionless fields $X' = \frac{X}{\sqrt{\alpha}}$. The action written in these variables after the gauging away of X^0 is

$$S = \frac{1}{4\pi} \int_{\Sigma} d^2 z E_{ij} \partial X^i \bar{\partial} X^j + E_{0j} A \bar{\partial} X^j + E_{i0} \partial X^i \bar{A}$$

$$+ A \bar{\partial} Y - \bar{A} \partial Y \qquad (1.57)$$

And so we set $X^0 = 0$ and integrating out $A, \bar{A}$ to give the dual action:

$$S_D = \frac{1}{4\pi} \int_{\Sigma} d^2 z (\partial Y - \partial X^i E_{i0}) E_{00}^{-1} (\bar{\partial} Y + E_{0i} \bar{X}^i) + E_{ij} \partial X^i \bar{\partial} X^j \qquad (1.58)$$

We define coordinates on a dual target space $\tilde{T}$ by taking $Y = \tilde{X}^0$ and $X^i = \tilde{X}^i$. The dual target space then has a metric and background field given by:

$$\tilde{G}_{00} = \frac{1}{G_{00}} \qquad \tilde{G}_{0i} = \frac{G_{0i}}{G_{00}} \qquad \tilde{G}_{ij} = G_{ij} - (G_{0i} G_{0j} - B_{0i} B_{0j}) \frac{1}{G_{00}} \qquad (1.59)$$

$$\tilde{B}_{0i} = \frac{G_{0i}}{G_{00}} \qquad \tilde{B}_{ij} = B_{ij} + (G_{0i} B_{0j} - B_{0i} G_{0j}) \frac{1}{G_{00}} \qquad (1.60)$$

These are known as the Buscher rules for T-dual string backgrounds, see [10]. As yet we have not considered any quantum considerations. Equivalence of the partition functions will determine whether the dual theories are equivalent at the quantum level. For ease of exposition, consider the case $G_{0i} = 0$

and $B_{IJ} = 0$. The partition function, after suppressing the overall normalization from the regularisation, is given by

$$Z_G = (G_{00})^{1/2b_0} \int \mathcal{D}X e^{S(G)} \tag{1.61}$$

The term in front of the functional is to remove the integration over the zero modes. (Here we are assuming that G_{00} is a constant). To obtain to dual action introduce the parent action as before. Integrating out W from the parent action implies we must have a factor of $(G_{00})^{-1/2b_1}$ to eliminate its zero mode. The action obviously transforms as expected by the classical calculation given that the integral is Gaussian. A comparison is then made with the partition function written in dual variables, Z_D to give:

$$Z = (G_{00})^{b_0 - 1/2b_1} Z_D \tag{1.62}$$

Recall for a two dimensional surface, the Euler characteristic is given by $\chi = 2b_0 - b_1$ and so 1.62 becomes:

$$Z = (G_{00})^{1/2\chi} Z_D \tag{1.63}$$

This is in the absence of the dilaton term. Including this term, one sees that it is independent of the classical transformations and so the classical discussion goes through without change. Recall via the Gauss Bonnet theorem that $\frac{1}{4\pi}\int_\Sigma R = \chi$. Hence the dilaton couples to the Euler number of the world sheet. (This is the origin, from the string theory point of view, of the dilaton as the loop counting parameter in string scattering calculations.) This term then allows us to exactly identify the partition function with the duality transformed partition function by inducing a duality transformation for the dilaton. Hence under duality one requires [2] :

$$\tilde{\Phi} = \Phi - ln(G_{00}) \tag{1.64}$$

The duality is then at the level of the partition functions and so the two quantum theories are therefore equivalent. If the background field B is reinstated then there is no change and the dilaton transforms in the same way.

[2]Sometimes the dilaton coupling is written with a factor of $\frac{1}{4\pi}$ and so the dilaton transformation will differ by a factor of two in this case.

The equivalence of the partition functions under duality was not true in the field theory case for a general manifold. The main reason it can work here is due to the ability of the dilaton to transform in such a way as to cancel the factor appearing in the partition function duals.

As discussed above, the field equations for the background fields must correspond to the vanishing of the beta-function. The shift of the dilaton ensures that the beta-function of the dual couplings also vanishes; without this shift the dual theory would not be consistent. (The Jacobian factor that induces the dilaton shift under duality is a one loop effect. This is what one would expect given that the shifted dilaton solves the background field equations that were derived from the one loop beta function.)

Essentially we take the isometry to be related to a compact coordinate on T. That is $X^0 = X^0 + 2\pi$. The duality transformation then takes

$$R \to \tilde{R} = \frac{1}{R} \tag{1.65}$$

Thus, a string in a space-time propagating in a spacetime, $M^9 \times S^1$ is dual and so physically equivalent to a string propagating in a space $M^9 \times \tilde{S}^1$ where $\tilde{S}$ has radius, $\tilde{R}$ and S^1 has radius R.

Now the duality group is actually bigger than the Z_2. Integral shifts in $B_{\mu\nu}$ leave the partition function invariant. ($B_{\mu\nu}$ is analogous to the theta coupling in field theory). Hence the actually duality group is $SL(2, \mathbf{Z})$. To generalize the above discussion to a larger number of isometries abelian corresponding to a space time $M^d \times T^{10-d}$ follows very naturally. Now the duality group is $O(d, d, \mathbf{Z})$. See [12] for a review.

The above discussion brings out the connection between T-duality in string theory and S-duality field theory. We will now approach T-duality from an entirely different perspective, mostly following [12]. For simplicity we ignore B in what follows. Call X the coordinate of the circle. Consider the equations of motion of X from the sigma model point of view. X must satisfy the free wave equation. In a flat background, the solutions will naturally split up into right and left movers.

$$X(\sigma,\tau) = X_R(\sigma-\tau) + X_L(\sigma+\tau) \tag{1.66}$$

Carry out a mode expansion

$$X_R(\sigma-\tau) = x_R - p_R(\sigma-\tau) + i\sum_{l\neq 0}\frac{1}{l}\alpha_l e^{il(\sigma-\tau)} \tag{1.67}$$

$$X_L(\sigma+\tau) = x_L - p_L(\sigma+\tau) + i\sum_{l\neq 0}\frac{1}{l}\tilde{\alpha}_l e^{il(\sigma+\tau)} \tag{1.68}$$

where $x = x_L + x_R$ and

$$p_R = \frac{1}{\sqrt{2}}(\frac{n}{R} - mR) \tag{1.69}$$

$$p_L = \frac{1}{\sqrt{2}}(\frac{n}{R} + mR) \tag{1.70}$$

The total momentum is $P = p_L + p_R$. Where $m, n \in \mathbf{Z}$. Recall that for objects propagating on a circle of radius R, momentum must be quantized in units of $\frac{1}{R}$. This the origin of the $(\frac{n}{R})$ terms in 1.70; n is the quantum number associated with momentum around S^1. The other term mR, requires some further explanation. It corresponds to the contribution from a string wrapping around the S^1 m times. Recall, for $X = X + 2\pi R$ and $\sigma = \sigma + 2\pi$:

$$\int_{S^1} dX = mR \tag{1.71}$$

where m is the winding number of the string around the S^1. The contribution to the Hamiltonian, H may be written as

$$H = L_{0L} + L_{0R} \tag{1.72}$$

where

$$L_{0R} = \frac{1}{2}p_R^2 + \sum_{l=1}\alpha_{-l}\alpha_l \tag{1.73}$$

$$L_{0L} = \frac{1}{2}p_L^2 + \sum_{l=1} \tilde{\alpha}_{-l}\tilde{\alpha}_l \tag{1.74}$$

These are conserved independently because of conformal invariance. Now we observe that under

$$R \to \frac{1}{R} \qquad m \leftrightarrow n \tag{1.75}$$

L_{0R} and L_{0L} are invariant as

$$p_R \to -p_R \qquad p_L \to p_L \tag{1.76}$$

under the transformation 1.75.

From this we see that the T-duality transformation $R \to \frac{1}{R}$ must be accompanied by exchanging momentum and winding numbers for the string. This result indicates how an extended object such as a string behaves very differently in the presence of compact dimensions. Essentially one can exchange a momentum mode in a compact direction for a wrapping mode with a compensating change the geometry to leave the physics invariant. This result will be generalized in some small way when considering objects whose extension is in more then one dimension.

As discussed above, the equations that imply the vanishing of the beta function become the field equations for the background spacetime. Via this method one may construct a low energy effective action for the string theory. Varying this action with respect to the background fields implies the necessary equations of motion such that the beta-function for the σ-model vanishes. (Low energy implies stringy effects are suppressed.) It is well known that this gives a supergravity action. For closed strings with N=2 supersymmetries there two possibilities depending on whether the fermions are have the same or opposite chirality. IIA denotes that fermions have different chiralities and IIB denotes they have the same chirality. This will be discussed in more detail later. If we dimensionally reduce the IIA theory on a circle the resulting theory is simply type II in 9 dimensions. (Spinors are non-chiral in 9-dimensions.) Similarly, IIB theory dimensionally reduced on a circle must also be type II in 9 dimensions. So both type two theories when dimensionally reduced must give the same theory in 9 dimensions. By equating the IIA on S^1 with IIB on $\tilde{S}^1$ it is possible to identify how to move between different, physically equivalent descriptions of the single 9-dimensional theory. These different descriptions correspond to T-dual descriptions. One important observation is the dilaton shift that emerged from quantum correction to the duality transformation in the partition function now is a result of identifying the two classical theories after dimensional reduction. Hence the low energy effective actions seem to contain quantum information of the string at the classical level.

1.3 Self-dual/Chiral Gauge theories

A self-dual gauge theory is a standard p-form theory whose field strength is self-dual. That is

$$H = {}^*H \tag{1.77}$$

where $H = dB$ and $B \in \Lambda^p(M^D)$. By considering the properties of the Hodge dual we obtain the following restrictions (see appendix A for some properties of the Hodge dual):

$$H \in \Lambda^{D/2}(M^D) \Rightarrow B \in \Lambda^{(D/2-1)}(M^D) \tag{1.78}$$

For a D-dimensional spacetime with Lorentzian signature, the Hodge dual acting on a $D/2$ form is unipotent only for

$$D = 2 \quad mod \quad 4. \tag{1.79}$$

Case D=2: a scalar theory with $\partial_t\phi = \partial_x\phi$ which implies there are only left moving modes. This is the origin of the name chiral. The self-duality condition in 2-dimensions is equivalent to projecting out right moving modes. Such (anti-)chiral scalars are the foundation of the Heterotic string which treats left and right moving modes differently.

Case D=6: a two-form theory in 6-dimensions. This theory will be discussed in detail later on. In particular as it arises as the world volume theory on the M-theory 5-brane.

Case D=10: a 4-form theory in 10 dimensions. This theory arises in the Ramond Ramond sector of the IIB string theory. This is the potential that couples minimally to the D-3 brane.

No other cases will be considered as the next instance would be with $D = 14$ which does not have any applications. However, it is perhaps worth mentioning that one may alter these conditions by changing the signature and so work in Euclidean space or in spacetimes with two or more time like directions. (This observation is related to theories such as F-theory [15] where there is an SO(10,2) structure so allows for a self dual 5-form or the speculative S-theory with a SO(11,3) structure [16, 17])

There arise immediate problems when trying to naively write down an action for such a self-dual field theory. First observe that the field strength for potentials that meet the conditions given above 1.78, 1.79 are necessarily odd forms. The usual kinetic term for an action given by 1.1 would then identically vanish for H obeying the self duality equation 1.77, as

$$H \wedge *H = H \wedge H = 0 \tag{1.80}$$

One might consider writing an action for general a generic H with the self-duality appearing as a constraint on H through use of a Lagrange multiplier. This however proves fruitless as the Lagrange multiplier becomes propagating [18, 19, 20, 21]. This is obviously unphysical. There are several approaches to these problems. First, it is not clear why one would require an action: the equations of motion present no difficulty being given by 1.77 and the Bianchi identity for B. One requirement for an action would be to quantize the system, most obviously, via a path integral formalism. This turns out not to be possible [22, 23]. Hence it is not clear whether an action makes sense. Another possibility is to have an action for a non-chiral field and then impose the self-duality projection afterwards. If an action is required then there are two possibilities. One possibility is not to have an action that is *manifestly* Lorentz invariant. Another possibility is to allow additional auxiliary fields in the action. (This is similar to the Lagrange multiplier approach but the action is set up to avoid the problem of extraneous propagating degrees of freedom.)

The approach due to Verlinde [6] and based on a formulation of Henneaux [20] is as follows.

The action for a generic p-form field in its first order form is as follows:

$$\int_{M^D} H \wedge dB + \frac{1}{2}\int_{M^D} H \wedge *H \tag{1.81}$$

This action is akin to the parent actions that we used to generate dual theories in the previous sections. $H \in \Lambda^{D/2}(M^D)$ and $B \in \Lambda^{D/2-1}(M^D)$ are treated as arbitrary independent forms. The relationship between H and B only arises from taking the equations of motion of either field.

Impose the self-duality constraint by the following projection equation:

$$i_V(H - dB) = 0 \tag{1.82}$$

V is an auxiliary vector that projects out half of the degrees of freedom carried by B. These equations lead to a similar action approach to the Henneaux formulation if $V = \frac{\partial}{\partial t}$. One may specify V and then insert the results of the projection 1.82 into the action 1.81. This results is a loss of manifest Lorentz invariance in the action 1.81. One may show that Lorentz invariance is still present but it is not apparent at the level of the action. Defining forms as being parallel or perpendicular to V as follows: $i_V X^{\perp} = 0$ and $i_V X^{\parallel} \neq 0$ one may decompose $\Lambda^p(M^D) = \Lambda^{p\parallel}(M^D) \oplus \Lambda^{p\perp}(M^D)$ Hence, we write for $H = H^{\perp} + H^{\parallel}$ and similarly $dB = dB^{\perp} + dB^{\parallel}$. Substitute this into 1.82 we obtain that

$$H^{\perp} = dB^{\perp} \tag{1.83}$$

We may then insert this into the action 1.81. The result of this is that only $H^{\parallel}$ is then free. $dB^{\parallel}$ only appears in the action as a total derivative and can be ignored. The result of this manipulation is that we have halved the number of degrees of freedom. The equations of motion of $H^{\|}$ then imply

$$H^{\|} = {}^{*}dB^{\perp} \tag{1.84}$$

Combining 1.83 and 1.84 we see that

$$H^{\|} = {}^{*}H^{\perp} \tag{1.85}$$

This is equivalent to equation 1.77 with a particular direction singled out. For example, consider a two form theory with $V = \frac{\partial}{\partial t}$ in flat space. Note, in component form: $H^{\|}$ is H_{0ij} and $H^{\perp}$ is H_{ijk}, where we have split up $\{x^{\mu}\}$ into $\{x^0, x^i\}$. This inspires a natural generalization of the definition of electric and magnetic fields for p-form theories. The electric field being defined as $E^{ij} \equiv H^{0ij}$ and the magnetic fields as $B^{ij} \equiv \frac{1}{3!}\epsilon^{ijklm}B_{klm}$. Therfore one sees that equation 1.85 becomes

$$E^{ij} = B^{ij} \tag{1.86}$$

This is the non-covariant form of $H = {}^{*}H$. Indeed one can now make contact with the Henneaux, Teitelboim action [20]. Their action is given by:

$$S = \int_{M^D} E \cdot B - B^2 \tag{1.87}$$

Varying this action then implies the equation 1.86. Returning to equations 1.81 and 1.82, it is important to note that nothing physical depends on V. It simply provides a projection direction for the equation 1.85. The Henneaux formulation which is not manifestly covariant corresponds to a particular choice of V. Other choices of V produce the same equation 1.86 Lorentz transformed.

It is possible to add sources and still maintain the chirality condition provided that the electric and magnetic charges are equal [24]. For definiteness we will consider the two form case in six dimensions. We recall that a two form couples to a string world sheet. The current associated with a string is:

$$J^{\mu\nu} = e\int_{\Sigma} \delta^6(X - x_0)dX^{\mu} \wedge dX^{\nu} \tag{1.88}$$

Where Σ is a string world sheet $X^\mu(\tau,\sigma)$ is the embedding of the string into the in 6-dimensions. The magnetic string which would be akin to a Dirac monopole must now be constrained to have charge $g = e$. Then via the Wu-Yang or Dirac arguments (see section on field theory) the usual quantization condition gives $e^2 = nh$. The current may be coupled minimally in the action as follows

$$S_{source} = \int_{M^6} B \wedge {}^*J \tag{1.89}$$

provided one also redefines the electric and magnetic fields as follows:

$$E^{ij} = -H^{0ij} + \frac{1}{3!}\epsilon^{ijklm}G_{klm} \tag{1.90}$$

$$B^{ij} = \frac{1}{3!}\epsilon^{ijklm}H_{klm} - G^{0ij} \tag{1.91}$$

The 3-form G is completely determined by the current as follows:

$$d^*G = -J \tag{1.92}$$

1.3.1 PST approach

This follows the work of [25] generalized for a p-form. Consider the Lorentz invariant action for the p-form B with field strength $H = dB$ given by:

$$S_{PST} = \frac{1}{2}\int_{M^D} H \wedge {}^*H + \frac{H^{+,u} \wedge {}^*H^{+,u}}{u^2} + u \wedge dW \tag{1.93}$$

where we have introduced an auxiliary fields u and W and defined $H^{+,u} = i_u H^+$ where H^+ satisfies the self duality condition 1.77. The action is invariant under the following local transformations:

$$\delta B = d\chi, \qquad \delta W = d\phi \tag{1.94}$$

these are the usual gauge transformations associated with a massless p-form field. There are also the following non-standard symmetries:

$$\delta B = u \wedge \psi \qquad \delta W = -\frac{1}{u^2}\psi \wedge u \wedge H^{+,u} \quad \delta u = 0 \tag{1.95}$$

and

$$\delta u = d\psi, \quad \delta B = -\frac{\psi}{u^2}H^{+,u}, \quad \delta W = \frac{\psi}{(u^2)^2}H^{+,u} \wedge H^{+,u} \tag{1.96}$$

The equations of motion of W imply u is closed. This implies locally that u is exact. If one ignores global issues then one may write $u = da$ where a is now an auxiliary scalar. (This is the usual way the action is written [25]). If one now chooses a time-like u such as by choosing $a = t$ we recover the Henneaux action. If one chooses a space like u then one recovers a Henneaux type action but with space-like projection. Here we note that the u of the PST approach which enters as an auxiliary one form plays the same role as the v in the Verlindé approach. It provides a projection direction. Therefore the action 1.93 is an action for a self-dual two form with manifest Lorentz invariance. The price for this is the addition of an auxiliary field. Finally with u constrained to be closed we write the action with W integrated out as:

$$S = \frac{1}{2}\int_{M^D} H \wedge {}^*H + \frac{H^{+,u} \wedge {}^*H^{+,u}}{2u^2} \tag{1.97}$$

The action possesses the same symmetries as above but now with W absent and u closed. The first of the non-standard so called PST symmetries then becomes simply

$$\delta B = u \wedge \psi \tag{1.98}$$

Therefore one may use this to gauge away half the degrees of freedom carried by B. To connect this with the previous formulation we write $B = B^{\perp} + B^{\|}$ Where $\perp$ and $\|$ are with respect to u. Hence, the gauge symmetry, 1.98 allows a complete gauging away of $B^{\|}$. This is the same result as for the Verlinde approach where only $B^{\perp}$ becomes physical.

To summarise, the PST formulation brings in a new gauge symmetry that implies $B^{||}$ is pure gauge. The Verlinde approach uses a projection equation 1.82 that means $B^{||}$ is not present in the action except as in a total derivative term and so is not physical. Similarly, the Henneaux approach has $B^{||}$ being non physical through a non-covariant action.

It is possible to include the presence of sources in the PST approach much in the same way as with Henneaux et al. [26] First introduce the modified field strength:

$$F = H + {}^*G \tag{1.99}$$

where H=dB as before. G is defined to satisfy the following equation:

$$d^*G - j = 0 \tag{1.100}$$

where j is the current density. One now repeats the PST construction with F in place of H and add the term:

$$S_{coupling} = -B \wedge {}^*j \tag{1.101}$$

A simple check demonstrates that with the usual gauge choice of $u = dt$ we recover the Henneaux action in the presence of sources.

Chapter 2

Dimensional reduction of self-dual theories

2.1 Introduction

One of the first attempts at unification of the known forces was the the programme instigated by Kaluza and Klein [27] [28]. Their principal idea was that the local symmetries that are central to gauge theories might appear as the result of some hidden dimensions of space-time. The first case studied was that of 5-dimensional gravity where the vacuum is taken to be some 4 dimensional manifold times a circle whose radius is assumed to be small. By small we mean that we can ignore variation of fields on such a length scale and so the derivatives of fields in the compact direction are taken to vanish. (One may take into account variations of the fields by carrying out a Fourier analysis of the fields on the circle. When this is done, one obtains an infinite tower of massive fields.) The zeroth mode ie. the constant mode corresponds to the massless field with higher modes corresponding to fields with mass.

Taking just the zeroth mode, the 5-dimensional Einstein Hilbert action reduces to the 4-dimensional Einstein-Hilbert action together with a Maxwell action for electromagnetism with a scalar field. The $U(1)$ gauge group now arises out of the geometrical symmetry of the compact space.

Due to a number of problems, this program was abandoned until string theory appeared and required a critical dimension of 26 in the case of Bosonic strings and 10 in the case of Superstrings. In order to make contact with the empirical world it was necessary to go from the critical dimension down to d=3+1. As such the notion of hidden compact dimensions of space time became a natural consideration for the string theory program. The 10-dimensional theories however are now more complex; they include various Abelian p-form field theories as well as the gravitational sector. An approximation of the low energy theory of strings is the so called infinite tension limit where, $\alpha \to 0$. In this limit the whole of the gravitational sector of the theory decouples and we are left with only the field theory sector. This leads us to consider Kaluza-Klein reductions of p-form gauge theories in the absence of dynamical gravity.

This idea finds other applications when considering effective brane actions. Recall that the world volume actions of branes do not contain dynamical gravity. They do however contain field theories. D-branes have Abelian 1 form potentials on the the world volume of the brane and the M-5 brane has a self-dual 2-form theory on its world volume. Therefore, simple field theories act as toy models for brane world volume theories. Wrapping branes on compact dimensions leads to considering dimensional reductions of p-form theories in the absence of dynamical gravity. (Wrapping a brane is also known as double dimensional reduction; the space time and world volume of the brane are both reduced, a la Kaluza and Klein leading to the interpretation of a wrapped object around the compact space.) In what follows we shall examine the dimensional reduction of free self-dual p-from theories. The applications to string theory will be highlighted, though the application to brane physics will not be explored until later.

2.2 Reducing self-dual p-forms

The theory we wish to consider is a self-dual p-from theory on a manifold M^{D+d}. To meet the self duality conditions given in the previous section we require $D + d = 2(p + 1) = 2 \bmod 4$. In what follows we use the Verlinde approach to self-dual gauge theories described in the previous chapter. The action is repeated here for the readers convenience.

$$S = \frac{1}{2\pi}\int_{M^{D+d}} dB \wedge H + \frac{1}{2}H \wedge {}^*H \tag{2.1}$$

The prefactor of $\frac{1}{2\pi}$ is a normalisation that was not present in the action given in the previous section. Its presence will not effect the equations of motion but will be relevant for *theta* type terms that emerge after the reduction. The self duality constraint is given by

$$i_v(H - dB) = 0 \tag{2.2}$$

where

$$H \in \Lambda^{p+1}(M^{D+d}) \qquad B \in \Lambda^p(M^{D+d}) \tag{2.3}$$

Here we also note that the field B is compact by this one means it is a true U(1) theory and so B may have non-trivial periods. The manifold M^{D+d} is taken to be a Cartesian product of a compact D-dimensional space which we label, K^D and a d-dimensional space-time which we label N^d. Hence,

$$M^{D+d} = N^d \times K^D \tag{2.4}$$

The metric on M^{D+d} is taken to be the direct sum of the metric on N^d and the metric on K^D. This is a consistent truncation. It truncates any fields charged with respect to the Kaluza-Klein vector fields. (In the absence of dynamical gravity these fields will be non-dynamical anyway.) We carry out the following decomposition on $\frac{D+d}{2}$ forms as follows (recall from above $p + 1 \equiv \frac{D+d}{2}$):

$$\Lambda^{p+1}(M^{D+d}) = \Lambda^{d/2}(N^d) \bigwedge \Lambda^{D/2}(K^D)$$
$$\bigoplus_i [\Lambda^i(N^d) \bigwedge \Lambda^{p+1-i}(K^D) \oplus \Lambda^{p+1-i}(N^d) \bigwedge \Lambda^i(K^D)] \tag{2.5}$$

Where i runs from 0 to $i \leq d$ and $i \leq D$ but omitting the case of $i = d/2$. The space has been split up according to the action of the Hodge star. The first term in the decomposition is Hodge self-dual. That is, elements in this space are mapped to other elements in this space under the Hodge star. The other spaces are paired so that an element in one space is mapped onto its pair under the hodge star and vice versa. Hence, the pair of spaces are Hodge self-dual. A self-dual theory must have a field strength that decomposes naturally in this way.

We now carry out the Kaluza-Klein reduction for H and B keeping only the zero modes on the compact space K^D. This implies that the forms on the compact space must be harmonic. We denote $\{\gamma_I(q)\}$ to be a canonical basis of harmonic q-forms on K^D with $I = 1..b^q(K^D)$. This leads to a Kaluza-Klein ansatz for the system 2.1, 2.2 as follows:

$$H = \sum_{q=0}^{p+1} \sum_{I=1}^{b^q} F_q^I \wedge \gamma_I(q) \tag{2.6}$$

and

$$B = \sum_{q=0}^{p} \sum_{I=1}^{b^q} C_q^I \wedge \gamma_I(q) \tag{2.7}$$

where $F_q^I \in \Lambda^{p+1-q}(N^d)$ and $C_q^I \in \Lambda^{p-q}(N^d)$. Recall, that B is compact, this implies C_q^I is also compact. This is completely general. One may truncate the ansatz in anyway consistent with the decomposition given in 2.5. We now substitute this ansatz for H and B into the equations 2.1 and 2.2. The vector field v is so far unspecified. We recall that no physical quantities can depend on our choice of v. We have two distinct possibilities. One may choose v to lie in the compact space, K^D or the space-time, N^d. We have shown in the previous section that a choice of v in a space time direction will break the manifest Lorentz invariance of the theory. Hence, we will pick v to be a global vector field in the compact space. Such a choice is not always possible. If the space has zero first Betti number then v will not exist globally. For the moment we will assume that such a choice exists. To get the reduced theory one makes a specific choice of v, and so obtain the equation $H^{\perp} = dB^{\perp}$ which we substitute into the action 2.1. $H^{\perp}$ is now dynamic but $H^{\|}$ appears in the action only algebraically and so is an auxiliary field that may be integrated out. Therefore, $H^{\|}$ is integrated out to give the final reduced theory. We will carry out this procedure for a variety of specific cases.

2.3 A two form theory in 6 dimensions

2.3.1 Compact space: a two torus

First we introduce the canonical basis of one forms on T^2 which we denote by a and b. These are Poincare dual to the non-trivial homology a,b cycles of the torus. The intersection matrix is given by:

$$L = \int_{T^2} \begin{pmatrix} a \wedge a & a \wedge b \\ b \wedge a & b \wedge b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.8}$$

and the period matrix is:

$$M = \int_{T^2} \begin{pmatrix} a \wedge^* a & a \wedge^* b \\ b \wedge^* a & b \wedge^* b \end{pmatrix} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{2.9}$$

Where the torus has modular parameter τ and volume normalised to unity. Following 2.6, 2.7 we have

$$H = F_D \wedge a + F \wedge b + G + J \wedge a \wedge b \tag{2.10}$$

and

$$B = A_D \wedge a + A \wedge b + C + \phi a \wedge b \tag{2.11}$$

where $F, F_D, C \in \Lambda^2(M^4)$, $J, A, A_D \in \Lambda^1(M^4)$, $\phi \in \Lambda^0(M^4)$ and $G \in \Lambda^3(M^4)$.

Now one needs to specify a choice for v. v is chosen such that $i_v a = 0$. Inserting 2.11 and 2.10 into the projection equation 2.2 we obtain.

$$F = dA \quad G = dC \tag{2.12}$$

Where d is now an exterior derivative in 4 dimensions. The four dimensional action will now split up into two sectors as indicated by the decomposition 2.5. The first will be a theory of the one form field, A and the other will be of the two form field C.

We deal first with the one form action. This was considered by Verlinde in [6] Substitute the equation 2.12 for F into the action 2.1 and factor out the integrals over the torus and space time:

$$S = \int_{T^2} a \wedge b \int_{M^4} -dA \wedge F_D + dA \wedge dA_D$$
$$+\frac{1}{2}\Big(\int_{T^2} a \wedge {}^*a \int_{M^4} F_D \wedge {}^*F_D + \int_{T^2} a \wedge {}^*b \int_{M^4} F_D \wedge {}^*dA$$
$$+ \int_{T^2} b \wedge {}^*a \int_{M^4} dA \wedge {}^*F_D + \int_{T^2} b \wedge {}^*b \int_{M^4} dA \wedge {}^*dA\Big) \tag{2.13}$$

The integrals over the compact space are then evaluated using 2.8 and 2.9. The term involving A_D is a total derivative and so is thrown away. (The equation of motion from varying A_D in the action 2.1 is, dF=0 which is solved by the equation 2.12). The resulting action is:

$$S_4 = \int_{M^4} -dA \wedge F_D + \frac{1}{2}(F_D, dA) M \wedge {}^* \begin{pmatrix} F_D \\ dA \end{pmatrix} \tag{2.14}$$

F_D is now auxiliary and so must be eliminated either in the path integral (the integral is Gaussian) or by substituting the equations of motion for F_D into the action. Only the classical level will be considered so Jacobian factors in the partition function will be ignored. The equations of motion of F_D give:

$$F_D = \tau_2{}^* dA - \tau_1 dA \tag{2.15}$$

This identifies F_D with the dual field strength. Inserting equation 2.15 into the action 2.14 we obtain the final form of the action:

$$\int_{M^4} -\tau_2 F \wedge^* F + \tau_1 F \wedge F \tag{2.16}$$

This is Maxwell theory in 4-dimensions with a theta term. Compare this action with that given in the field theory section 1.26. (Some overall factor of $\frac{-1}{8\pi}$ is required to make the identification exact.) Should this procedure be repeated with $i_v b = 0$ and then with F integrated out then one recovers the dual action. That is action 2.16 with $\tau \to -\frac{1}{\tau}$. Hence, one obtains the Maxwell action and its dual from the two form self-dual theory reduced on a torus by making different choices of v.

Moving to the two form sector we repeat the procedure for the field G. Substitute the equation for G from the projection equation 2.2 into the action 2.1. After factoring out and integrating over the torus one is left with:

$$S_C = \int_{M^4} J \wedge dC + \frac{1}{2} dC \wedge * dC + J \wedge {}^* J \tag{2.17}$$

J is auxiliary and can be eliminated leaving the final action for C:

$$S_C = \frac{1}{2} \int_{M^4} dC \wedge {}^* dC \tag{2.18}$$

This is independent of the choice of v. Recall, a one form theory couples to point like objects and a two form theory couples to strings. A string wrapping the a-cycle is electrically charged with respect to A and a string wrapping the b-cycle is electrically charged with respect to A_D. We show this as follows. Let the spatial coordinate of the string is given by σ. The string is closed and σ is chosen to have unit period, that is $\sigma \equiv \sigma + 1$. Recall the coupling term in component form is given by

$$S_{coupling} = \int_{M^6} J^{\hat{\mu}\hat{\nu}} B_{\hat{\mu}\hat{\nu}} \qquad (2.19)$$

where $J^{\hat{\mu}\hat{\nu}} = \int_{M^6} dX^{\hat{\mu}} \wedge dX^{\hat{\nu}} \delta(x - x_0)$. $\{X^{\hat{\mu}}\}$ are coordinates on M^6. Reducing this on T^2 gives the following action:

$$S_{coupling} = \int_{M^4} j^{\mu\nu} C_{\mu\nu} + \int_{M^4} j^{\mu 1} A_{\mu 1} + j^{\mu 2} A_{\mu 2} \qquad (2.20)$$

where X^1 and X^2 are coordinates on the torus in the directions of the a and b cycles respectively. (They are taken to have unit periods). Note we have truncated the scalar which couples to j^{12} because it is removed by the self-duality projection. Thus we can identify A_1 with A and A_2 with A_D. The current coupling to (A, A_D) is given by the usual current coupling to a one form $q \int dX^{\mu} \delta(x - x^0)$ where q is

$$q = (\int dX^1, \int dX^2) = (n, m) \qquad (2.21)$$

where (n,m) are the winding numbers of the string around the a and b cycles of the torus [24]. Hence, the wound string is charged with respect to the four dimensional one form theory. The unwrapped string is neutral (as one would require for different decoupling sectors). However, we see from the equations of motion 2.15 that A_D is the dual potential. This implies that electric coupling to A_D is equivalent to magnetic coupling to A (and vice versa). Therefore states charged with respect to A_D will be magnetically charged with respect to A. Hence, the self-dual string wrapped on cycles of T^2 provides a full dyonic spectrum of states by wrapping the (a,b) cycles (n,m) times to give dyonic states with charge (n, m). Under the SL(2,Z) transformation $\tau \to \frac{a\tau + b}{c\tau + d}$, these states will transfrom as follows:

$$\begin{pmatrix} n \\ m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \qquad (2.22)$$

One can see this geometrically considering the generators of $SL(2,\mathbf{Z})$. Recall S-duality , $\tau \rightarrow \frac{-1}{\tau}$ is generated in the formalism described above by swapping the a and b cycles of the Kaluza-Klein torus. The above shows that the string wrapped around the a and b cycles corresponds to the electric and magnetically charged particles, hence it is no surprise that swapping a and b cycles interchanges electric and magnetic charges from the four dimensional point of view.

One notes that for a string with tension, T the mass of the charged particles in the four dimensional theory will have a spectrum given $M^2 = T^2|Z|^2$ where $Z \equiv (n+\tau m)$. This formula is essentially $tension \times length$ (Oscillator modes are neglected because the Kaluza Klein ansatz keeps only zero modes). It is interesting that this is the same formula that arises for BPS states in Super Yang-Mills theory. This method of producing dyonic states by wrapping the self dual string has been applied in string theory by [31] and [30]; where dyonic p-branes were produced from wrapping self-dual p+1 branes around appropriate cycles and truncating the neutral non-wrapped branes.

2.3.2 Compact Space: a four manifold

One can obtain a σ-model by reducing the self-dual 6-dimensional theory down to d=2 on an internal four manifold denoted M^4, [6]. Assume that M^4 has an even intersection form, and $b^+ = b^-$. As usual the reduction on the compact space is imposed by taking only the zeroth Fourier mode such that fields forms on M^4 are taken to be harmonic. We introduce a basis for harmonic forms on M^4 as follows. The set $\{\alpha_i\}$ is a basis of self dual two forms on M^4 and the set $\{\beta^i\}$ is a basis of anti self dual two forms on M^4, where $i = 1..b^{\pm}(M^4)$. The canonical basis will have intersection form:

$$L = \int_{M^4} \alpha_i \wedge \beta^j = \delta_i^j \tag{2.23}$$

The period matrix is given by:

$$M = \int_{M^4} \begin{pmatrix} \alpha_i \wedge^* \alpha_j & \alpha_i \wedge^* \beta^j \\ \beta^i \wedge^* \alpha_j & \beta^i \wedge^* \beta^j \end{pmatrix} = \begin{pmatrix} G_{ij} - B_{ik}G^{kl}B_{lj} & B_{il}G^{lj} \\ G^{il}B_{lj} & G^{ij} \end{pmatrix} \quad (2.24)$$

The world sheet will be denoted by Σ. We introduce $\Pi^i, \Pi_i^D \in \Lambda^1(\Sigma)$ and $X^i, X_i^D \in \Lambda^0(\Sigma)$ Follow a similar Kaluza-Klein ansatz for H and B as before with:

$$H = \alpha_i \wedge \Pi^i + \beta^i \wedge \Pi_i^D, \quad B = \alpha_i \wedge X^i + \beta^i \wedge X_i^D \quad (2.25)$$

V is chosen such that

$$i_v \beta^i = 0, \quad \forall i \quad (2.26)$$

The existence of such a V is an assumption, not all manifolds will admit a global choice for V. This then implies that $\Pi^i = dX^i$. Inserting this into the action 2.1, and carrying out the integrals over the four dimensional surface using the data from the intersection and period matrices one obtains the following action:

$$S_2 = \int_\Sigma \Pi_i^D \wedge dX^i + \frac{1}{2}(dX^i \Pi_i^D) M \wedge^* \begin{pmatrix} dX^j \\ \Pi_i^D \end{pmatrix} \quad (2.27)$$

Π_i^D remains auxiliary in the action 2.27 and must be eliminated. Its equation of motion becomes:

$$\Pi_i^D = G_{ij}{}^* dX^i - B_{ij} dX^i \quad (2.28)$$

After substituting this equation into the action 2.27 to eliminate Π^D one obtains the following action:

$$S_2 = \int_\Sigma dX^i \wedge {}^* dX^j G_{ij} + dX^i \wedge dX^j B_{ij} \quad (2.29)$$

One can interpret this action as a σ-model with a $b_2/2$ dimensional target space. This target space will have metric G_{ij} and background field B_{ij}.

It is important to recall that this metric from the point of view of the 6-dimensional theory is the metric on the space of Harmonic two forms. It is not the metric of the compact space. A simple observation concerning the dimensions of the relative spaces shows the surprising feature of this system. From the point of view of the original 6-dimensional theory the compact space is 4-dimensional however from the point of view of the σ-model the target space is $b_2/2$ dimensional. How should one interpret this? In fact this sort of ambiguity will play a central role in understanding string and M-theory dualities when looking at branes on compact spaces. We will simply outline how this works. Recall from the review of T-duality in string theory that there was an ambiguity in the geometry of the compact space that arose from exchanging momentum modes with winding modes of the string. Here we have a similar situation. The data from the period matrix of two cycles corresponds to the wrapping of non-trivial two periods of H on the two cycles of the compact space. It is this period matrix that becomes the metric in the *dual* σ-model. Hence this is really another example where momentum is exchanged for a winding of the field. More formally, recall that the winding of the string was associated with representatives of the first cohomology on the target space, $H^1(T, \mathbf{Z})$ which we exchanged for momenta on the dual target space, $\tilde{T}$. In the 6-dimensional case we have representatives of $H^2(M^4, \mathbf{Z})$ corresponding to non-trivial fluxes through 2-cycles of M^4, that we exchange for momenta in the σ-model. Thus, we are identifying periods of the two form field C around non-trivial two cycles, Σ_i of M^4, with X^i.

2.4 A four form theory in ten dimensions

We now move to the next case of a self dual four form theory in ten dimensions [32]. To reduce down to 4-dimensions, the compact space is taken to be 6 dimensional. The simplest case, though of little phenomenological interest, will be a six torus. We note that under the decomposition given in equations 2.5 we can have fields in the paired space that transform under the duality transformation as well as fields in the first self-dual space. We shall begin our analysis with fields in the first space. It is necessary to have a basis of $H^3(T^6, \mathbf{Z})$ Let $\{\gamma_I\}$ be such a basis. $I = 1$ to $b_3(T^6)$ where $b_3(T^6)$ is the 3rd Betti number of T^6. A canonical basis is chosen such that the intersection matrix

$$Q_{IJ} = \int_{T^6} \gamma_I \wedge \gamma_J \tag{2.30}$$

is antidiagonal. The period matrix is:

$$G_{IJ} = \int_{T^6} \gamma_I \wedge^* \gamma_J \tag{2.31}$$

Where, the $\{\gamma_I\}$ are the basis of 3-forms in T^6. For a torus, the three form basis may be written in terms of a product of the one form basis. The action in 10 dimensions is given as before by:

$$S_{10} = \int_{M^{10}} dC \wedge H + \frac{1}{2} \int_{M^{10}} H \wedge^* H \tag{2.32}$$

with also the self duality equation:

$$i_v(H - dC) = 0 \tag{2.33}$$

Taking zero modes of the fields on T^6 implies the usual Kaluza-Klein ansatz for the fields.

$$H = \sum_{I=1}^{20} F^I \gamma_I, \qquad C = \sum_{I=1}^{20} A^I \gamma_I \tag{2.34}$$

where $F^I \in \Lambda^2(M^4)$ and $A^I \in \Lambda^1(M^4)$. We then proceed as before. Picking out a particular v and then decomposing the 3-form basis, $\{\gamma_I\}$ into parallel and perpendicular parts via the equations: $i_v\gamma_a \neq 0$ for γ_a in the space *parallel* to v and $i_v\gamma_i = 0$ for γ_i in the space *perpendicular* to v. From now on the indices a,b will indicate γ parallel and i,j will indicate γ perpendicular. This projection decomposes the 3 form basis into 10 parallel and 10 perpendicular basis 3-forms. Hence, when we substitute the ansatz 2.34 into the self duality equation 2.33 we arrive at 10 equations:

$$F^a = dA^a. \tag{2.35}$$

First compactify M^{10} on $M^4 \times T^6$ which involves substituting in the ansatz 2.34 into the action 2.32. The necessary integrals over T^6 are given by the period and intersection matrices 2.31 2.30. We then arrive at a 4-dimensional action. We then substitute in the equations 2.35 derived from the self duality equation 2.33 for a particular choice of v. This gives the action:

$$S_4 = \int_{M^4} dA^a \wedge F^i Q_{ai} + \frac{1}{2}\int_{M^4} dA^a \wedge^* dA^b G_{ab} + 2F^i \wedge^* dA^a G_{ia} + F^i \wedge^* F^j G_{ij} \tag{2.36}$$

As before we must now integrate out the auxiliary fields, F^i. Again, the integrals will be Gaussian. Doing the integration, or equivalently eliminating from the action 2.36 using the equations of motion gives the following action for an Abelian gauge theory:

$$S_4 = \int dA^a \wedge dA^b \sigma_{ab} + dA^a \wedge^* dA^b \tau_{ab} \tag{2.37}$$

Where the coupling matrices τ and σ are given by:

$$\tau_{ab} = G_{ab} + Q_{ai} G^{ij} Q_{jb} - G_{ai} G^{ij} G_{jb} \tag{2.38}$$

$$\sigma_{ab} = Q_{ai} G^{ij} G_{ib} + G_{ai} G^{ij} Q_{jb} \tag{2.39}$$

The raised indices indicate the inverse matrix, such that $G^{ab} = G^{-1}_{ab}$ and not the *parallel* components of G^{-1}_{IJ}. Note that apart from the usual curvature squared term there is also a topological term that is a generalization of the theta term for a U(1) gauge theory. To see how the couplings τ and σ transform under duality one considers a different choice of V and examine how the action transforms. This will be discussed later when we calculate the more general case discussed below.

Before, we noted that it may be possible to construct theories that are self dual that contain fields of different form rank (We mean the rank of the fields after compactification; before compactification it is clear that the fields must have the same rank). These fields will live in the sum of paired spaces discussed earlier. As such, the ansatz considered previously may be viewed as a degenerate case; in which the pair of the space is the space itself. We now move on to consider the case where we have a sum of fields that live in such a pair of spaces. Replace the ansatz 2.34 with the following:

$$H = \sum_I A^I \mu_I + B^I \nu_I \tag{2.40}$$

$$C = \sum_I a^I \mu_I + b^I \nu_I \tag{2.41}$$

where

$$A^I \in \Lambda^1(M^4) \quad B^I \in \Lambda^3(M^4) \quad a \in \Lambda^0(M^4) \quad b \in \Lambda^2(M^4) \tag{2.42}$$

and $\{\mu_I\}$ is the canonical basis of $H^4(T^6, \mathbf{Z})$ and $\{\nu_I\}$ is the canonical basis of $H^2(T^6, \mathbf{Z})$. Note $b^2(T^6) = b^4(T^6) = 15$, hence, I=1..15. As before we construct the period matrices associated with both bases. Let G be the period matrix of the 4-form basis, $\{\mu_I\}$ and F be the period matrix of the 2-form basis $\{\nu_I\}$. There will also be an intersection matrix Q defined by: $Q_{IJ} = \int_{T^6} \mu_I \wedge \nu_J$ which in the canonical basis will be antidiagonal. We now define *parallel* and *perpendicular* bases for both the 2 and 4 forms as before. The indices a,b indicate parallel 2-form and i,j indicate perpendicular 2-form. $\tilde{a}, \tilde{b}$ denotes parallel 4-form and $\tilde{i}, \tilde{j}$ denotes perpendicular 4-forms. It can be seen for any given one form there are 5 parallel and 10 perpendicular 2-forms and 10 parallel, 5 perpendicular 4-forms. Hence substituting in the ansätze 2.40, 2.41 into the self duality equation 2.33 we have for a particular choice of v a set of 15 equations:

$$A^{\tilde{a}} - da^{\tilde{a}} = 0 \tag{2.43}$$

$$B^a - db^a = 0 \tag{2.44}$$

We compactify $S_{M^{10}}$ as before, performing the necessary T^6 integrals which introduce the period and intersection matrices defined above. Then substitute in the self duality equations 2.43, 2.44 into the compactified action. After throwing away irrelevant total derivatives, we integrate out all auxiliary fields, B^i and $A^{\bar{i}}$. This leaves, the following 4-dimensional action:

$$S_4 = \int_{M^4} \frac{1}{2}[da^{\tilde{a}} \wedge^* da^{\tilde{b}}\tilde{\tau}_{\tilde{a}\tilde{b}} + db^a \wedge^* db^b \tau_{ab}] - da^{\tilde{a}} \wedge db^b \sigma_{\tilde{a}b}, \tag{2.45}$$

where we have the following couplings:

$$\tilde{\tau}_{\tilde{a}\tilde{b}} = G_{\tilde{a}\tilde{b}} + Q_{\tilde{a}i}F^{ij}Q_{j\tilde{b}} - G_{\tilde{a}\bar{i}}G^{\bar{i}\bar{j}}G_{\bar{j}\tilde{b}} \tag{2.46}$$

$$\tau_{ab} = F_{ab} + Q_{a\bar{i}}G^{\bar{i}\bar{j}}Q_{\bar{j}b} - F_{ai}F^{ij}F_{jb} \tag{2.47}$$

$$\sigma_{\tilde{a}b} = Q_{\tilde{a}i}F^{ij}F_{jb} + G_{\tilde{a}\bar{i}}G^{\bar{i}\bar{j}}Q_{\bar{j}b}. \tag{2.48}$$

The action contains the usual kinetic terms for scalar fields and two form fields. There is also the topological term which is a generalization of the theta term that couples the scalar and two form fields.

Note that if $G = F$ then we have the same equation for the coupling as before, 2.46, 2.47. This confirms that the previous case is a degenerate version of the more general situation in which we have a paired space. Also, one can easily check that this formula for the coupling of the Abelian gauge theory in terms of the period matrix of the compactified space reproduces the simple T^2 result. In this instance the period matrix is two dimensional and so the perpendicular and parallel parts are one dimensional and hence no matrix inverses are involved.

Now we wish to construct the generators for the duality transformation for the theory described above. We follow the previous S-duality example, by choosing different directions for v and then determining how the theory changes. It is obvious from equations 2.45 and 2.46, 2.47, 2.48 that only the coupling matrices change when a different direction for v is chosen. Hence, the duality related theories will have the same form of the action with only the couplings being different. This implies of course that the equations of motion of the duality related theories will only differ by the value of the coupling matrices given in the action. The equations of motion for the action 2.45 are simply the free field equations for scalar, one form and two form fields. Also each field strength has the usual Bianchi identity. The topological coupling between the scalar and two form fields will be transparent to the classical equations of motion but will play a role in the partition function (by analogy with the usual theta term).

We will now consider a concrete example where the T^6 is taken to be a Cartesian product of three orthogonal T^2 each one with area one. (Orthogonal tori implies that the metric of the T^6 would be of block diagonal form, each block being 2×2, corresponding to the metric on each torus which would be related to τ its modular parameter). Obviously, $b^1(T^6) = 6$, so there are six possible choices for v. Each choice will give a duality related theory.

The coupling constant matrices were calculated explicitly for each choice of v. These gave:

For $v = a1$ (corresponding to the a cycle of the 1st torus)

$$\tau = \frac{1}{({}^1\tau_{22})} diag({}^1\tau_{22}, {}^2\tau_{11}, {}^2\tau_{22}, {}^3\tau_{11}, {}^3\tau_{22}) \tag{2.49}$$

$$\tilde{\tau} = 1 \oplus \begin{pmatrix} {}^2\tau_{11}{}^3\tau_{11} & {}^2\tau_{12}{}^3\tau_{12} \\ {}^2\tau_{12}{}^3\tau_{12} & {}^2\tau_{22}{}^3\tau_{22} \end{pmatrix} \oplus \begin{pmatrix} {}^2\tau_{11}{}^3\tau_{22} & {}^2\tau_{12}{}^3\tau_{12} \\ {}^2\tau_{12}{}^3\tau_{12} & {}^2\tau_{22}{}^3\tau_{11} \end{pmatrix}$$

$$\oplus 1 \oplus \frac{1}{({}^1\tau_{22})} diag({}^2\tau_{11}, {}^2\tau_{22}, {}^3\tau_{11}, {}^3\tau_{22}) \tag{2.50}$$

$$\sigma = \frac{{}^1\tau_{12}}{{}^1\tau_{22}} M \tag{2.51}$$

The direct sum refers to blockdiagonal decomposition of the matrices. M is a 5×10 matrix which has a 4×4 identity matrix in the last block and zeros elsewhere. ${}^a\tau_{ij}$ refers to the ijth element of the period matrix of the a th 2-torus. This shows how the moduli of the tori combine to give the coupling constants for the 4 dimensional theory. The coupling matrices calculated for different choices of v can be related to the above coupling matrices (calculated for v lying in the $a1$ direction), as follows:

For $v = b1$ (corresponding to the b cycle of the first torus)

$$ {}^1\tau_{22} \leftrightarrow {}^1\tau_{11} \tag{2.52} $$

For $v = a2$ (corresponding to the a cycle of the second torus)

$$ {}^2\tau_{ij} \leftrightarrow {}^1\tau_{ij} \tag{2.53} $$

For $v = b2$

$$ {}^1\tau_{11} \leftrightarrow {}^2\tau_{22} \tag{2.54} $$

$$ {}^1\tau_{12} \leftrightarrow {}^2\tau_{12} \tag{2.55} $$

For $v = a3$

$$ {}^1\tau_{ij} \leftrightarrow {}^3\tau_{ij} \tag{2.56} $$

For $v = b3$

$$ {}^1\tau_{11} \leftrightarrow {}^3\tau_{22} \tag{2.57} $$

$$ {}^1\tau_{12} \leftrightarrow {}^3\tau_{12} \tag{2.58} $$

These form a set of duality generators that act on the couplings of the theory. Note, the transformation $\tau_{11} \leftrightarrow \tau_{22}$ is equivalent to the imaginary part of

$\tau \leftrightarrow \frac{-1}{\tau}$. So the duality generators we have above correspond to the obvious generalization of the coupling inversion duality generator of SL(2,Z).

The difference between the S-duality (coupling constant inversion) and the transformations we describe above is that part of the coupling constant matrix is left invariant by the duality transformation. In the terms of our scheme for calculating these generators, this is a result of having cycles that contain both projection directions. These are left invariant by the duality transformation; cycles that contain neither are of course projected out and so do not appear. The cycles that contain one of the directions are those that are transformed under duality. Simple counting of the number of 2 and 4 cycles with these properties confirms this picture. To determine the other generators it is necessary to look at the theta type coupling, σ and determine the generators that corresponds to the SL(2,Z), $\tau \to \tau + 1$ Following the same arguments that lead the theta term being invariant in the partition function under $\tau \to \tau + m$ (where m is integer), we conclude that $\sigma \to \sigma + m$ leaves the partition function invariant. In terms of τ this is equivalent to

$$^1\tau_{12} \to ^1 \tau_{12} + m^1\tau_{22} \tag{2.59}$$

(for v chosen to be the a direction of the 1st torus) and

$$^1\tau_{12} \to ^1 \tau_{12} + m^1\tau_{11} \tag{2.60}$$

(for v chosen to be in the b direction of the 1st torus).

Considering combinations of these generators allows one to construct the duality group. In fact the generators given above are over complete. For example, transformation 2.55 can be formed by composing 2.52 and 2.53. The minimal set of the above will be given by 2.52, 2.53, 2.56 and 2.59. These generate as one would expect $SL(2,\mathbf{Z}) \times SL(2,\mathbf{Z}) \times SL(2,\mathbf{Z})$. Note, using the techniques in chapter 1, one could dualize the two form fields into scalars. This dual action would then have the appearance of a σ-model. The base space would be four dimensional and the target space would be 15 dimensional. The metric would be given by the period matrices of 2-forms and 4 forms on T^6.

2.4.1 Summary

One concludes from this analysis that a p-form theory on M^D, dimensionally reduced on a compact space denoted by K^d has a $D-d$ dimensional reduced space-time with $b^q(K^d)$ p-q form Abelian gauge theories. If the original p-form theory is self dual then one has half the number. The couplings for these theories are given by the relevant period and intersection matrices for the compact space. Self-dual theories have a natural set of duality generators formed by considering different self-duality projections in the compact space.

Chapter 3

Duality manifest actions and non-linear theories

3.1 Introduction

Duality in field and string theory as presented in the introductory chapter is a *hidden* symmetry. Hidden implies that the symmetry is not *manifest* in the action. It emerges from either manipulations of the partition function or from considering symmetries of the equations of motion and Bianchi identities. Often it is stated that S-duality in Maxwell theory is not present as a symmetry of the action. It is not quite correct that the Maxwell action is not duality invariant, because one cannot vary the field strength, F as an independent variable. One must vary the gauge potentials and so know how the gauge potential transforms under duality. This is done in a paper by Henneaux and Deser [24], [33]. This is in contrast with the Lorentz symmetry under which the action is *manifestly* invariant. In constructing self-dual theories this problem has in some way been already addressed. Consider the case of free Maxwell theory in 4-dimensions. This theory is known to be invariant under $F \rightarrow {}^*F$, that is in terms of electric and magnetic fields, $E \rightarrow B$, $B \rightarrow -E$. In attempting to introduce this symmetry at the level of the action we are faced with the same choice as when trying to construct self-dual theories: either break manifest Lorentz invariance or introduce auxiliary fields. In this chapter we shall explore the first approach, sacrificing Lorentz invariance for duality invariance. This was first explored by Schwarz and Sen in [34] and then extended for Born-Infeld theory by [35] and others in [38], [36, 37]. It should be stressed that it is only the loss of manifest Lorentz symmetry in the action. The theory is still Lorentz invariant it is just that this symmetry is now *hidden.* This hidden Lorentz symmetry is equivalent to the status if the duality symmetry in the usual formulation.

It has been discussed how to move between the Verlinde approach and the Henneaux approach for self dual theories by making a specific choice for the projection direction V. V in the time direction implies the usual Henneaux form. In the previous section where there were hidden dimensions V was taken to lie in the compact space so that the reduced theory would be Lorentz invariant. Should V be taken to lie in the reduced space time, for example in the time direction then we would recover the dimensionally reduced Henneaux formalism. Lorentz invariance would be broken as before but the duality symmetry would be manifest.

3.2 Duality manifest Maxwell theory in four dimensions

Instead of the approach followed in [34], the action with manifest duality will be derived from dimensional reduction of the Henneaux action for a self dual two form in six dimensions. Thus we consider the theory on $M^6 = M^4 \times T^2$; with a 6-dimensional metric that is a direct sum of the M^4 and T^2 metrics. The usual Kaluza Klein reduction is used, taking only zero modes of the fields on the torus. Recall the Henneaux action in component form is:

$$\int_{M^6} E^{\mu\nu} B_{\mu\nu} - B^{\mu\nu} B_{\mu\nu} \tag{3.1}$$

Where $\mu, \nu = 1..5$, are the spatial components of M^6. The field strength,

$H = dC$ allows one to define electric and magnetic fields as follows:

$$E_{\mu\nu} = H_{0\mu\nu}, \quad B^{\mu\nu} = \frac{1}{3!}\epsilon^{\mu\nu\alpha\beta\gamma}H_{\alpha\beta\gamma} \tag{3.2}$$

Harmonic one forms on T^2 are introduced as before, $\{a, b\}$ with period and intersection matrix given by 2.8 and 2.9. The two form C is then decomposed:

$$C = A_1 \wedge a + A_2 \wedge b \tag{3.3}$$

This truncates the possible two form field. Inserting this into 3.2 and 3.1, one finds the following action. (Where the integrals over the torus have been done using the period, M and intersection matrix, L.)

$$S = \int_{M^4} \vec{E}^i \cdot \vec{B}^j L_{ij} - \vec{B}^i \cdot \vec{B}^j M^{-1}_{ij} \tag{3.4}$$

Where $\vec{E}^i$ is the usual electric field in 4-dimensions for gauge potential $\vec{A}_i$ and $\vec{B}^i$ is the usual magnetic field in 4-dimensions for gauge field $\vec{A}_i$ and Dot denotes a three dimensional dotproduct of a spatial three vector. Lorentz invariance is certainly not manifest; the action being expressed in terms of magnetic and electric fields in a non Lorentz invariant way. However, it is duality manifest in the following sense. A_1 and A_2, which will shown to be duals appear in the action on the same footing. We will examine the duality invariance of this action 3.4 after first demonstrating its equivalence to the Maxwell action. Note that the duality invariant form requires the introduction of more gauge fields. It is possible however to eliminate one of these fields in favour of the other, in doing so one returns to the usual from of the action. Observe, the action 3.4 has two local symmetries :

$$\delta A^i_0 = \chi^i, \quad \delta \vec{A}^i = \vec{\nabla}\psi^i \tag{3.5}$$

The second is the usual gauge invariance but the first is an extra local symmetry not usually present. Using this one may choose χ^i so that:

$$A^i_0 = 0 \tag{3.6}$$

Note that A_0^i only appears in the action in a total derivative term so the equations of motion are obviously unchanged. The equations of motion for the $\vec{A}^2$ field now give:

$$\vec{\nabla}(\vec{B}^i M_{i2}^{-1} - \vec{E}^1) = 0 \tag{3.7}$$

This is independent of time derivatives of $\vec{A}^2$ which means that $\vec{A}^2$ may be treated as auxiliary. Hence we can use its equation of motion to eliminate it from the action. Solving the above equation implies:

$$\vec{B}^i M_{i2}^{-1} - \vec{E}^1 = \vec{\nabla}\phi \tag{3.8}$$

Using the remaining gauge invariance in ψ^1, one can remove the $\vec{\nabla}\phi$ term leaving the equation:

$$\vec{B}^i M_{i2}^{-1} = \vec{E}^1 \tag{3.9}$$

If one sets M to be the unit matrix then we may observe that the above equation sets the magnetic field of A^2 to be equal to the electric field of A^1. This is what one would expect from dual potentials. Substituting equation 3.9 into the action 3.4 we get:

$$S = \int_{M^4} \frac{1}{M_{22}^{-1}}(\vec{E}^1 \cdot \vec{E}^1 - \vec{B}^1 \cdot \vec{B}^1) + \frac{M_{12}^{-1}}{M_{22}^{-1}}\vec{E}^1 \cdot \vec{B}^1 \tag{3.10}$$

Note that Gauss' Law is now a consequence of the Bianchi identity for the dual potential. Inserting the values of M_{ij}^{-1} in terms of the modular parameter of the torus, τ into the above and rewriting in a Lorenz invariant form using the field strength F associated with the gauge potential A^1, this becomes:

$$S = \int_{M^4} -\tau_2 F \wedge {}^*F + \tau_1 F \wedge F \tag{3.11}$$

Which is of course the usual Maxwell action with theta term.

This is the action produced by eliminating A^2. If instead one eliminates A^1, it is no surprise that this produces the S-dual action:

$$S = \int_{M^4} -\frac{\tau_2}{|\tau|^2} F \wedge {}^*F - \frac{\tau_1}{|\tau|^2} F \wedge F \tag{3.12}$$

where now F is the field strength associated with the potential A^2.

This is very similar to the Verlinde approach where eliminating potentials that are associated with the a,b cycles of the torus produce the Maxwell action or its dual. Returning to the action 3.4, one can check that the action is manifestly symmetric under the following $SL(2, \mathbf{R})$ transformation:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \to \omega \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad \omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.13}$$

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1 \tag{3.14}$$

Under transformation 3.14 the period matrix M transforms as follows:

$$M \to \omega M \omega^T \tag{3.15}$$

Note that ω is $SL(2, \mathbf{R})$ so that

$$\omega^T L \omega = L \tag{3.16}$$

Now note that

$$M^{-1} = L^T M L \tag{3.17}$$

So the action 3.4 may be written (suppressing i,j indices):

$$S = \int_{M^4} \vec{E}^i \cdot L_{ij} \vec{B}^j - \vec{B}^i (L^T M L)_{ij} \cdot \vec{B}^j \tag{3.18}$$

The action is now manifestly invariant given the identities: 3.15, 3.16 and transformation of the fields given by 3.13. The $SL(2, \mathbf{R})$ duality is a classical result, by considering the invariance of the partition function one would expect that the duality only persists for the $SL(2, \mathbf{Z})$ sub group.

The action 3.4 is also manifestly symmetric under rotations. However, due to the special role of time, the action does not appear to be invariant under Lorentz boosts. Taking M to be the identity, one sees that in the $A_0^i = 0$ gauge, the action is invariant under [34] :

$$\delta A_\nu^i = x^o v^\mu \partial_\mu A_\nu^i + v \cdot x L_{ij} \epsilon_\nu{}^{\mu\rho} \partial_\mu A_\rho^j \tag{3.19}$$

where v is an arbitrary 3-vector. Using the equations of motion 3.9, the above transformation becomes:

$$\delta A_\nu^1 = x^o v^\mu \partial_\mu A_\nu^1 + v \cdot x \partial_0 A_\rho^1 \tag{3.20}$$

which is the usual transformation law of A_μ^1 in the $A_0^1 = 0$ gauge. Thus the Lorentz symmetry is restored at the level of the equations of motion. Hence, as stated in the introduction the Lorentz symmetry is now the *hidden* symmetry.

This construction generalises in a number of obvious ways. By carrying out the dimensional reduction process described in the previous chapter on a variety of manifolds for the different self dual theories one may recover a host of duality symmetric actions. We shall not repeat the calculation for the examples given in the previous chapter. Instead one notes that formalism allows one to dimensionally reduce on manifolds that previously were forbidden because of topological considerations. Recall that in the previous section one required the existence of a global vector field v, on the compact space so that one could carry out the self duality projection in a direction in the compact space and recover a manifestly Lorentz invariant theory. If we are not worried about the reduced theory being manifestly invariant then there is no reason why the projection direction should be taken in the direction of the compact space. For compact manifolds that do not admit globally defined vector fields we are forced into this route. Hence, in these instances one uses the Henneaux formalism to carry out dimensional reduction instead of the Verlinde formalism we used previously. One interesting application is the reduction of the self-dual two form in six dimensions on a K3 manifold.

3.3 Reduction of self-dual two form on K3

The starting point will be the Henneaux action in six dimensions, see action 3.1. First it is necessary to recall some facts about K3 surfaces [39].

$$\chi = 24, \quad \sigma = b^+ - b^- = -16,$$
$$b^0(K^3) = b^4(K^3) = 1, \quad b^1(K^3) = b^3(K^3) = 0, \quad b^2(K^3) = 22. \tag{3.21}$$

Where χ is the Euler characteristic, σ the Hirzebruch signature and $b^i(K^3)$ are the Betti numbers. It is Ricci flat and so makes a suitable compactification manifold. (Despite being Ricci flat, it has half the number of Killing spinors of flat space.) As usual one introduces a basis for $H^2(K3, \mathbf{Z})$ which we denote by $\{\gamma_I\}$, where I=1..22. As $b^1(K3) = 0$ there are no harmonic 1 forms or 3 forms on K3. This fact is related to the lack of a global vector field on K3. The periods of these forms provides the data through which one may specify the K3 surface.

A K3 surface has intersection form that is even and self dual. Recall some general properties of the intersection form and period matrix, [39]:

$$L^{-1} = L^{IJ} = \#(\Sigma_I, \Sigma_J), \quad L^I{}_J \equiv G^{IK} L_{KJ}, \quad L^I{}_K L^K{}_J = \delta^I_J \tag{3.22}$$

The eigenvalues of $L^I{}_J$ are ± 1 corresponding to elements of $H^{\pm}(K3, \mathbf{Z})$. Where $H^{\pm}(K3, \mathbf{Z})$ denotes the space of (anti) self dual harmonic 2 forms on K3. The dimension of this space is denoted $b^{\pm}$. Note that $b^2 = b^+ + b^-$, and so using the properties listed in 3.21 above one may deduce $b^+ = 3$ and $b^- = 19$. This implies that the signature of L is (3,19). Thus L is a self dual even lattice with signature (3,19). It is known that such a lattice, unique upto isomorphisms is given by [39]

$$L = -E_8 \oplus -E_8 \oplus U \oplus U \oplus U \tag{3.23}$$

Where E_8 denotes the 8×8 Cartan matrix of the Lie algebra of E_8 and

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.24}$$

So with $M^6 = M^2 \times K3$, we reduce the action 3.1 as follows. To take the zero modes in the Kaluza Klein reduction one takes the fields on K3 to be Harmonic and so:

$$C = X^I \wedge \gamma_I \tag{3.25}$$

Where X^I is a compact scalar on M^2. Therefore the electric and magnetic fields will be given by:

$$E = \partial_\tau X^I \wedge \gamma_I, \qquad B = \partial_\sigma X^I \wedge ({}^*\gamma_I) \tag{3.26}$$

where (τ, σ) are coordinates on M^2.

Inserting the 3.26 into the action 3.1 and then evaluating the K3 integrals using the period and intersection matrices produces the reduced action:

$$S_2 = \int_{M^2} \partial_\tau X^I \partial_\sigma X^J L_{IJ} - \partial_\sigma X^I \partial_\sigma X^J M_{IJ} \tag{3.27}$$

How should one interpret this action? It is obviously a sort of string action as it has the from of a 2-dimensional sigma model. First one takes the equations of motion of X^I and observe they imply (up to gauge transformations), [14]:

$$M\partial_\sigma X - L\partial_\tau X = 0 \tag{3.28}$$

Multiplying the above equation by M^{-1} produces:

$$\partial_\sigma X - M^{-1} L \partial_\tau X = 0 \tag{3.29}$$

One then uses the above properties of the basis of (anti) self dual that $M^{-1}LX^I = \pm X^I$. This implies the above equation decomposes into:

$$\partial_\sigma X^i - \partial_\tau X^i = 0, \quad M^{-1}LX^i = +X^i, \quad i = 1..3 \tag{3.30}$$

$$\partial_\sigma X^a + \partial_\tau X^a = 0, \quad M^{-1}LX^a = -X^a, \quad a = 1..19 \tag{3.31}$$

One may write these more succinctly by introducing the projection operator:

$$P_{\pm I}^{J} = \frac{1}{2}(\delta_I^J \pm L^{JL} M_{LI}) \tag{3.32}$$

$P_\pm$ projects on to the $b^\pm$ dimensional subspace of X^I. As before, define $\partial_\pm = \frac{1}{2}(\partial_\sigma \pm \partial_\tau)$. Therefore the equations 3.30 and 3.31 may be written [40]:

$$P_+\partial_- X = 0 \qquad P_-\partial_+ X = 0 \tag{3.33}$$

These are the equations of motion for (anti) chiral scalars. Recall $\partial_- X = 0$ is the equation for a left mover and $\partial_+ X = 0$ is the equation for a right mover. Therefore, P_+ projects out right movers and the P_- projects out left movers. Thus the above equations describe 3 right movers and 19 left movers. Recall that the heterotic [41, 42] string reduced on T^n has n right movers and 16+n left movers, from the Narain lattice [43]. Thus we have recovered the (anti) chiral bosons of the Narain lattice of the heterotic string on T^3. This is in fact not at all the complete picture because there are no Virasoro constraint conditions, nor dilaton, nor the remaining 7 ordinary bosons. To rectify this and obtain the correct action for the heterotic string it will be necessary to begin with a different action than just the simple free self-dual two form theory in six dimensions [40]. This will have a natural M-theory interpretation that will be introduced later. The duality group will be associated with the modular transformations of the K3 surface. Note that the Moduli space for the Heterotic string on T^3, associated with the Narain lattice is identical to the moduli space of K3 surfaces [39].

3.4 Non-linear theories and duality

3.4.1 Duality in Born-Infeld theory

So far all the theories examined have been free theories with simple actions that give linear equations of motion. Now we wish to move on to describe duality in a non-linear theory. The theory under consideration will be Born-Infeld theory [44]. This has an action given by the following:

$$S = \int_{M^4} d^4x - \sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})} \tag{3.34}$$

Where $\eta_{\mu\nu} = diag(-1,1,1,1)$ the Lorentz metric in 4 dimensions and $F_{\mu\nu}$ is the antisymmetric field strength given as usual by the exterior derivative of a one form A_μ. The negative sign in the square root occurs because the determinant is negative. The original motivation behind the theory was to reformulate Maxwell's theory so as to avoid having infinite fields at the origin of an electric point like source, such as an electron [44]. There are a number of interesting solutions to the equations of motion, see [45] for a review. The reformulation proved fruitless however and the theory did not receive much attention until the advent of string theory. As shown by a number of authors [46], the effective action of open string theory is of Born-Infeld type. This is related to the action for a D-brane [47], [48], [49], [50], [51] which is of so called Dirac-Born-Infeld type through a sequence of T-dualities.

To dualize this action we follow the methods described in chapter 1. The field strength F will be taken to be a generic two form that is constrained to be closed by the introduction of the Lagrange multiplier term:

$$S_c = \frac{1}{2} F \wedge F^D \tag{3.35}$$

where $F^D = dA^D$ as usual. Integrating out A_D implies F is closed and hence one recovers (locally) 3.34. To obtain the dual action, one integrates out F. For the free theories discussed so far the equation of motion of F yielded a linear relation between F and F^D. This will not be the case here. One uses the following identity (true for any antisymmetric tensor F):

$$det(\eta_{\mu\nu} + F_{\mu\nu}) = det(\eta)(1 - \frac{1}{2} trF^2 + \frac{1}{8}(trF^2)^2 - \frac{1}{4} trF^4) \tag{3.36}$$

to obtain the following equation of motion for F:

$$\epsilon^{\mu\nu\alpha\beta} F^D_{\alpha\beta} = \frac{F^{\mu\nu} - \frac{1}{2}(F_{\rho\sigma}F^{\rho\sigma})F^{\mu\nu} + F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}}{\sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})}} \tag{3.37}$$

One remarks that the above equation becomes at first order in F, the usual equation relating a field strength with its dual, $F = {}^*F^D$. It is surprising given the complexity of the relation 3.37 that when it is inverted and used to eliminate F one recovers the same form of action for F^D. So that the dual action becomes:

$$S_D = \int_{M^4} d^4x - \sqrt{-det(\eta_{\mu\nu} + F^D_{\mu\nu})} \tag{3.38}$$

This has been reported in [52], [53]. One can also add a theta type term to the action and introduce a coupling for F so that the action to quadratic order in F corresponds to the action 1.26 given in Chapter 1. As before one introduces a complex coupling which is denoted here as $\lambda = \lambda_1 + i\lambda_2$. It appears in the action as follows:

$$S = \int_{M^4} d^4x - \sqrt{-det(\eta_{\mu\nu} + \sqrt{\lambda_2}F_{\mu\nu})} + \frac{1}{2}\lambda_1 F \wedge F \tag{3.39}$$

Under the duality transformation described above the dual action is of the same form but now the complex coupling is inverted:

$$\lambda \to \frac{-1}{\lambda} \tag{3.40}$$

The shifting of λ_1 by a constant is also a classical symmetry. Combining these generators produces the usual $SL(2,\mathbf{R})$. Born-Infeld with complex coupling as given above was shown to posses $SL(2,\mathbf{R})$ in [36], [37]. (By considering the partition function one would only expect that the $SL(2,\mathbf{R})$ would be broken to $SL(2,\mathbf{Z})$ as usual, however the integral is not Gaussian so this cannot be proven explicitly. Replacing η by a general metric leaves the duality transformation untouched [52], [45].

We have shown in the previous chapter how duality can result from dimensional reduction of some self-dual theory on a compact space. The duality symmetry emerges from the geometry of the compact space. One would like to extend this idea to include include such non-linear theories. It is clear that it is impossible to get a non-linear theory from a linear one through dimensional reduction. So to derive the Born-Infeld theory one would require a parent theory that is also of a similar non-linear form. (The parent theory is the theory before dimensional reduction out of which the four dimensional Born-Infeld theory and its dual must be produced.)

As we require the parent theory be self-dual, it must be a D = 2 mod 4 dimensional theory. The duality group exhibited here is the modular group of the torus, and so it is natural to imagine that the origin of the Born-Infeld theory is some non-linear self-dual theory in six dimensions dimensionally reduced on a torus. This is in fact the case as first reported in [35]. Actually it is worth while considering what is meant by duality in a non-linear theory. The conditions derived for the existence of a self dual theory given in Chapter 1 are really only valid for a linear dual theory. That is one where the dual is given by the Hodge star. However for the theory we wish to consider this is not the case; the duality relation is of the form given by equation 3.37. However, one assumes that to quadratic order in the fields the action will approximate to the usual Yang-Mills action and the duality relation at this order will be the usual Hodge star. With these assumption one may extrapolate the conclusions for the existence of the linear theory to the non-linear theory.

3.4.2 Non-linear self dual 2-form theory in six dimensions

The starting point will be the action determined recently, [77], [56] in the context of searching for the M-theory five brane action. The full 5-brane action will be introduced in the next chapter. Consider the following six dimensional Born-Infeld type action.

$$S = -\int_{M^6} d^6x \sqrt{-det(G_{\mu\nu} + i\frac{\tilde{H}_{\mu\nu}}{\sqrt{|v|^2}})} - \frac{\tilde{H}^{\mu\nu} H_{\mu\nu\rho} v^\rho}{4v^2} \quad (3.41)$$

$G_{\mu\nu}$ is the metric in six dimensions and $\tilde{H} \in \Lambda^2(M^6)$, $v \in \Lambda^1(M^6)$. We define $\tilde{H}$ by the following:

$$\tilde{H} = {}^*(H \wedge v) \quad (3.42)$$

Where * is the Hodge dual acting in 6 dimensions. $H \in \Lambda^3(M^6)$ is the field strength of the Abelian potential $B \in \Lambda^2(M^6)$ defined by the usual relation $H = dB$.

The field v is constrained to be closed ie. $dv = 0$. It is a completely auxiliary field introduced to preserve the manifest Lorentz invariance in the action. Usually the above action is written with $v = da$ so that the closure of v is trivial, however, in what follows we will take v to be on T^2 which obviously has non-trivial first cohomology. Hence, the constraint condition on v is left unsolved.

The properties of this action are discussed in detail in [77]. There are two symmetries that will prove relevant to the following analysis. One is the usual gauge symmetry for an abelian potential, $\delta B = d\chi$. The other is the non trivial gauge symmetry introduced by the new auxiliary field :

$$\delta B = \psi \wedge v \quad (3.43)$$

where $\psi \in \Lambda^1(M^6)$ is the gauge parameter.

Note that despite the presence of i in the argument of the determinant 3.42 the action is real. This is because $\tilde{H}$ is antisymmetric and so only occurs in the expansion of the determinant in even powers. Also $\tilde{H}$ is a degenerate matrix of rank four so the polynomial under the square root will be order four in $\tilde{H}$. (To see this observe that $\tilde{H}$ has the zero eigenvector, v, by definition 3.42 and so its determinant will vanish.)

3.4.3 Dimensional Reduction on T^2

Now, we will double dimensionally reduce this action on a torus, keeping only the zero modes. Thus: $M^6 \to M^4 \times T^2$. The metric G, is taken to be given by the following direct sum:

$$G = \eta \oplus \pi \tag{3.44}$$

where π is the metric on the torus and η the metric in four dimensions. (This is taken to be flat.) This is in fact a truncation (consistent) where we do not consider the possible Kaluza-Klein fields corresponding to the compact dimensions (of which there should be two). Our ansatz for the gauge field B is again truncated. We have only included a part that couples to the conformal part of the torus. The reason for this truncation is that we are trying to recover the Born-Infeld action which has only one-form potentials. Hence,

$$\sum_I B = A^I \wedge \gamma_I \quad \Rightarrow \quad H = \sum_I F^I \wedge \gamma_I \tag{3.45}$$

where $A^I \in \Lambda^1(M^4)$, $F^I = dA^I$ and γ_I are the canonical one forms associated with the non trivial homology one cycles one the torus. Hence, they form a basis for $H^1(T^2, Z)$. There are two such one cycles, hence $I = 1, 2$. This is just a renaming of the basis used previously, with $\gamma_1 \equiv a$ and $\gamma_2 \equiv b$.

Now we have two natural possibilities for the auxiliary field v. It can be chosen such that $v \in \Lambda^1(T^2)$ or $v \in \Lambda^1(M^4)$. We will look at the consequences of both choices. Though of course, both possibilities must be physically equivalent. In the first instance, we find the following for $\tilde{H}$:

$$\tilde{H} = {}^*F^{I*}(\gamma_I \wedge v) \tag{3.46}$$

where the Hodge star in front of F acts in M^4 and the Hodge star in front of the parentheses acts in T^2. This gives $\tilde{H} \in \Lambda^2(M^4)$. We can now factorize the determinant, using

$$det(A \oplus B) = det(A)det(B). \tag{3.47}$$

So that

$$S = \int_{M^4} d^4x \int_{T^2} d^2\sigma - \sqrt{\pi}\sqrt{-det(\eta_{\alpha\beta} + i\frac{{}^*F^I{}_{\alpha\beta}{}^*(\gamma_I \wedge v)}{\sqrt{|v|^2}})}$$

$$-\frac{{}^*F^{I\mu\nu}F^J{}_{\mu\nu}{}^*(\gamma_I \wedge v)^*(\gamma_J \wedge^* v)}{4v^2} \qquad (3.48)$$

To investigate this action we will now make a gauge choice for v. A natural choice is to take $v \in H^1(T^2, Z)$. So suppose we choose v to be γ_L. The local symmetry 3.41 then allows us to gauge away one of the fields, A^L. It only remains to evaluate the terms in the action such as $\gamma_L \wedge \gamma_I$ and $\gamma_L \wedge^* \gamma_I$. We can evaluate these using an explicit basis for $H^1(T^2, Z)$. These terms are proportional to the volume form Ω as follows:

$$\gamma_I \wedge^* \gamma_J = \frac{M_{IJ}\Omega}{\mathcal{V}}, \qquad \gamma_I \wedge \gamma_J = \frac{L_{IJ}\Omega}{\mathcal{V}}. \qquad (3.49)$$

where $\mathcal{V} = \int_{T^2}\Omega$ and M_{IJ} and L_{IJ} are the period and intersection matrices defined as follows:

$$M = \int_{T^2}\begin{pmatrix} \gamma_1 \wedge^* \gamma_1 & \gamma_1 \wedge^* \gamma_2 \\ \gamma_2 \wedge^* \gamma_1 & \gamma_2 \wedge^* \gamma_2 \end{pmatrix} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \qquad (3.50)$$

$$L_{IJ} = \int_{T^2} \gamma_I \wedge \gamma_J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (3.51)$$

We substitute in 3.49,3.50,3.51 into 3.48 and integrate over the torus. The area of the torus is then shifted into the argument of the determinant. We now carry out a global space-time scaling to absorb this factor as follows:

$$X' = \mathcal{V}^{-1/4}X, \quad \eta'_{\alpha\beta} = \sqrt{\mathcal{V}}\eta_{\alpha\beta} \qquad (3.52)$$

This then gives the following action:

$$S = \int_{M^4} d^4x - \sqrt{-det(\eta'_{\alpha\beta} + i^*F_{\alpha\beta}\omega)} - \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\rho \tag{3.53}$$

Where ω and ρ depend on the specific choice of v. Note that after the scaling ω and ρ become independent of the torus. The two independent choices for v give the following:

$$v = \gamma_1 \quad \Rightarrow \quad \omega = \sqrt{\frac{\tau_2}{|\tau|^2}}, \quad \rho = \frac{-\tau_1}{|\tau|^2} \tag{3.54}$$

$$v = \gamma_2 \quad \Rightarrow \quad \omega = \sqrt{\tau_2}, \quad \rho = \tau_1 \tag{3.55}$$

Redefining, $F' = i^*F$ allows us to identify the action 3.53 with the Born-Infeld action given in 3.34. With this identification we then compare the action 3.53 for different choices of v with the action 3.39 and its dual implied by 3.40. For choice $v = \gamma_1$ we identify 3.53 with 3.39 and for $v = \gamma_2$ we identify 3.53 with the dual theory. These identifications imply simply that we must identify the complex coupling, λ with the modular parameter of the torus, τ. That is,

$$\lambda = \tau \tag{3.56}$$

The duality transformation that inverts λ is then given by making a different choice for v. Hence, we see how duality becomes a gauge symmetry of this theory. This result is identical that of the linear theory discussed in Chapter 2. The Verlinde auxiliary field v was used to provide a projection direction in the compact space. In this formalism the PST field v provides us with a gauge symmetry that allows us to gauge away the degrees of freedom removed by the Verlinde projection. Thus it is no surprise that the different choices of v give the S-dual related theories.

The other possibility mentioned above is that $v \in \Lambda^1(M^4)$. Note, that once such a choice is made, manifest Lorentz invariance is broken as v picks out a direction in space time. We will go immediately to the obvious choice

$v = dt$. Other choices for v, will not be related by duality as in the previous case but by Lorentz transformations. We use the same ansätze as before for the metric and the two form gauge field B, however now we find that the matrix, $G + i\tilde{H}$ does not decompose into block diagonal form and so the determinant will not immediately factorise. Hence, we explicitly expand out the determinant using the following identity (where $H_{\mu\nu}$ is an antisymmetric tensor in 6 dimensions of rank 4):

$$det(G_{\mu\nu} + iH_{\mu\nu}) = detG(1 + \frac{1}{2}trH^2 + \frac{1}{8}(trH^2)^2 - \frac{1}{4}trH^4) \tag{3.57}$$

We now define the magnetic and electric field strengths in the usual way:

$$B_i = \frac{1}{2}\epsilon_i{}^{jk}F_{jk}, \quad E_i = F_{i0} \tag{3.58}$$

where $i, j = 1, 2, 3$. We now substitute in the E and B fields into the action expanded out using the above identity. We also have used the period and intersection matrices of the torus as before and integrated over the torus. We also make the same scaling of the metric so to absorb the factor from the area of the torus.

$$S = \int_{M^4} d^4x\sqrt{-\eta'}\sqrt{-P(B,M)} + E_i^1 B^{2i} - E_i^2 B^{1i} \tag{3.59}$$

$$\begin{aligned} P(B,M) \equiv 1 + B_i^I B^{Ji} M_{IJ}^{-1} &+ \frac{1}{2} B_i^I B^{Ji} B_j^L B^{Kj} M_{IJ}^{-1} M_{LK}^{-1} \\ &- \frac{1}{2} B_i^I B^{Li} B_j^J B^{Kj} M_{IJ}^{-1} M_{LK}^{-1} \end{aligned} \tag{3.60}$$

(Where we have used the identity: $M^{-1} = L^T M L$.) This may be written in a more succinct way if one notes the following [57]. From the definition of the determinant (detM = +1)

$$L_{IJ} detM = M^{-1}_{IK} L_{KM} M^{-1}_{MJ}, \tag{3.61}$$

we can obtain the following result

$$L_{IJ} L_{LK} = M^{-1}_{IL} M^{-1}_{JK} - M^{-1}_{IK} M^{-1}_{JL}. \tag{3.62}$$

The quartic terms in the action may be written as follows:

$$\frac{1}{2} B^I_i B^{Li} B^J_j B^{Kj} (M^{-1}_{IL} M^{-1}_{JK} - M^{-1}_{IK} M^{-1}_{JL}) \tag{3.63}$$

Using the identity 3.62 this can be written:

$$\frac{1}{2} (B^I_i L_{IJ} B^J_j)(B^{Li} L_{LK} B^{Kj}), \tag{3.64}$$

which is equal to the following:

$$\frac{1}{4} (\epsilon^{ijk} B^I_j L_{IJ} B^J_k)^2. \tag{3.65}$$

This gives the final form for the manifestly dual action:

$$S = \int_{M^4} d^4x \sqrt{-\eta'} \sqrt{-(1 + B^I_i (L^T M L)_{IJ} B^{Ji} + \frac{1}{4} (\epsilon^{ijk} B^I_j L_{IJ} B^J_k)^2)} \tag{3.66}$$

It is this action that we claim is the Born-Infeld equivalent of the Schwarz-Sen action. It has also been derived in [38] from a careful Hamiltonian analysis of the Born-Infeld action. Note, the action 3.66 is manifestly dual under the $SL(2, \mathbf{R})$ transformations given in 3.13, 3.14, 3.15 just as for Maxwell theory. The two magnetic fields, B^1 and B^2 appear in the action symmetrically. (Obviously, a choice of space-like v would give a pair of electric fields.) These fields are related to each other by S-duality, as we will show when we demonstrate the equivalence of the above action to Born-Infeld. Several authors, using very different approaches to those described here, have produced duality manifest actions for Born-Infeld theory [58]. This prompts the question whether a six dimensional self dual theory could be lifted from their result.

We will now go to the case where $\tau = i$ as this will ease our calculation greatly. We will reinstate the couplings later. We will now follow the method of described in the previous section [34], to show that this action gives the Born-Infeld in 4-dimensions [35]. First use gauge invariance to set $A_0 = 0$. Then, as discussed in one of the A_i^L field becomes auxiliary and may be eliminated in favour of the other. Let us work in the concrete case where we will eliminate A^2 from the action . We find the equation of motion for A^2 by varying the action (setting M equal to the identity):

$$\vec{\nabla} \wedge (\vec{M}(B^1, B^2) - \vec{E}^1) = 0 \tag{3.67}$$

Where

$$\vec{M}(B^1, B^2) = \frac{\vec{B}^2 - (\vec{B}^2 \cdot \vec{B}^1)\vec{B}^1 + (\vec{B}^1 \cdot \vec{B}^1)\vec{B}^2}{\sqrt{1 + (\vec{B}^1)^2(\vec{B}^2)^2 - (\vec{B}^1 \cdot \vec{B}^2)^2 + (\vec{B}^1)^2 + (\vec{B}^2)^2}} \tag{3.68}$$

with $\vec{B}$ being a vector in 3 dimensions. We can solve this by writing

$$\vec{M}(B^1, B^2) - \vec{E}^1 = \vec{\nabla}\psi \tag{3.69}$$

We still have some gauge symmetry left $\delta\vec{A}^1 = \vec{\nabla}\chi$ to eliminate $\nabla\psi$. Leaving the equation:

$$\vec{M}(B^1, B^2) - \vec{E}^1 = 0 \tag{3.70}$$

The equivalent equation in Schwarz Sen approach to Maxwell theory is simply $\vec{B}^2 = \vec{E}^1$, which greatly facilitates the calculation and explicitly shows that the pair of Electric and Magnetic fields are related by duality.

The next step is to solve this equation for $\vec{B}^2$. After some manipulations we find

$$\vec{B}^2 = \frac{\vec{E}^1 + (\vec{E}^1 \cdot \vec{B}^1)\vec{B}^1}{\sqrt{1 + (\vec{B}^1)^2(\vec{B}^2)^2 - (\vec{B}^1 \cdot \vec{B}^2)^2 + (\vec{B}^1)^2 + (\vec{B}^2)^2}} \tag{3.71}$$

As a simple check we can see that this equation for B^2 reduces to to Maxwell case to first order in fields.

We now substitute this into the action (9) and find:

$$S = \int_{M^4} d^4x \sqrt{-\eta'} \sqrt{(1 + (\vec{B}^1)^2 - (\vec{E}^1)^2 - (\vec{E}^1 \cdot \vec{B}^1)^2)} \tag{3.72}$$

This becomes after rewriting in terms of a four dimensional determinant:

$$S = \int_{M^4} d^4x \sqrt{-det(\eta'_{\mu\nu} + F_{\mu\nu})} \tag{3.73}$$

This is of course the Born-Infeld with trivial background fields. If we reinstate the coupling and repeat the above procedure we see that we get the expected coupling dependence. That is, we recover the action 3.40.

We generate the dual theory by repeating the process but instead we integrate out A^1 instead of A^2. This gives the same action but (as expected) with the coupling inverted. So in this description of the theory, duality is a symmetry of the action. The two duality related theories are given by eliminating different fields from the action. It is a nice check that the two routes, one with v in the compact space and one with v in space time give (as they obviously should) the same results.

3.4.4 Conclusions

We have shown how duality manifest actions of a Schwarz Sen type arise naturally from the dimensional reduction of self-dual theories. In particular, we have shown that this idea extends to non-linear theories of a Born-Infeld type. Again the duality symmetry of the theory arises from the modular symmetry of the torus.

One might speculate how the duality manifest Born-Infeld theory presented here might be related to other formulations. In [60] a duality symmetric action was presented that involved an infinite number of auxiliary fields. This is based on the approach of Mclain, Wu and Yu [59] for chiral scalars and generalised in [61] for p-forms and developed by [63]. It has been shown [62] that one may obtain the a PST type action from this approach, that is an action with only one auxiliary field, by a consistent truncation (or gauge fixing) of the Mclain Wu action. Therefore it seems natural that the above action presented here before v is fixed might be related to that presented in [60] via the same process [64]. We have not presented the Mclain, Wu approach for chiral fields and so will not pursue this connection.

Chapter 4

The M-theory five brane and the IIB D-three brane

"...Is that the end?"
"No, let me think. We need a closing with a *pointe.*"
"A what?"
"Yes, an act of the intellect that expresses the inconceivable correspondence between two objects, beyond all belief..."
Umberto Eco, The Island of the Day before.

4.1 Introduction

There is a great deal of literature on M-theory and string dualities, citing all the relevant work in this enormous field is outside the scope of this thesis. For a review one may consult [65, 66, 67, 68]. Type II strings are closed strings with N=2 supersymmetry in ten dimensions. There are two possible N=2 theories in ten dimensions. There is a non-chiral theory, IIA in which the the spinors have both left and right chiralitites and a chiral theory, IIB in which there are two spinors of the same chirality. (The chirality operator is given by Γ^{11}, the product of the ten gamma matrices.) The low energy effective theory of type II strings is N=2A,B supergravity. The bosonic sector partitions naturally into two parts, the so called Neveu-Schwarz, Neveu-Schwarz sector and the Ramond, Ramond sector. The Neveu-Schwarz, Neveu Schwarz sector (denoted from now on as NS) contains a two form potential and its electromagnetic dual a six form potential. (This is true of both A and B type supergravities.) The Ramond, Ramond sector (denoted RR) however contains odd form potentials for IIA and even forms for IIB. The RR sector in the supergravity action does not couple to the dilaton as does the NS sector (in the string frame). This is symptomatic of the RR states being non-perturbative from the string theory point of view. A fundamental string couples electrically to the NS two form, through the usual minimal coupling. As such the string is a fundamental NS object.

The object that couples to the 6-form potential must be a 5-brane (assuming the usual minimal coupling). This is known as the solitonic 5-brane. The name solitonic implies that one should be able to find it as a solution of the supergravity equations of motion (with fields finite at the core of the solution) and indeed that is the case. One also notes that its tension is proportional to $1/g_{string}^2$. This is typical of solitons, for example the 't Hooft-Polyakov monopole has a mass proportional to $1/g_{YM}^2$.

The RR potentials couple to so called D-branes [47, 48, 49, 50, 51]. As a p-form potential couples to a p-1 brane one infers that there are even dimensional D-branes in IIA and odd dimensional D-branes in IIB. These branes are also solutions of the supergravity equations of motion that break half the supersymmetries (such states are known as BPS). Their existence can usually be deduced from the existence of central charges in the supersymmetry algebra. As these solutions break half the supersymmetries one expects that an effective action for these solutions, to be given by a zero mode analysis, will have 8 on-shell supercharges. Recall that a Weyl/Majorana spinor in ten dimensions has 16 real components, putting it on shell halves the number of components leaving 8 real components. Thus we require for the effective action of the D-brane a field content with 8 bosonic and 8 fermionic (on-shell) degrees of freedom. One can check that this conspires nicely to produce a vector multiplet on the brane. The scalar fields of the multiplet have the obvious interpretation of being Goldstone bosons corresponding to the broken translation modes. Hence for a p-dimensional D-brane there are 10-p scalar fields. A vector field in p dimensions carries p-2 on shell degrees of freedom. Hence one observes a that single vector field plus the (10-p) scalars corresponding to the (10-p) transverse dimensions gives the requisite 8 bosonic degrees of freedom on the D-brane. (This independent of p, the dimension of the brane). The zero mode origin of the vector field is not so clear from this perspective.

The origin of the vector field on the D-brane is slightly more apparent if one examines D-branes from a slightly different approach. If one considers open strings, the usual boundary conditions are Neumann boundary conditions- that is, $\partial_\sigma X^I|_{\sigma=0,\pi} = 0$. As was discussed in chapter one, string theory possess a so called T-duality. The result of T-duality on a open string with Neumann boundary conditions is to exchange it for Dirichlet boundary conditions- that is $\partial_\tau X^I|_{\sigma=0,\pi} = 0$. This implies that the ends of the string are fixed on some hyper plane. This plane one interprets as the D-brane. The field on the brane that couples to the end of the strings will be the vector field. Note that it is defined only on the brane; it is not a pull back from some field in the ambient space-time on to the brane.

In analogy with the Green Schwarz string, [70] we wish to describe the effective action of a D-brane in a manifestly space-time supersymmetric covariant way. This is achieved by considering the brane embedded in a super-space using supersymmetric invariant one forms. Thus, as shown in [71, 72] the D-brane action (in a flat superspace background), in the string metric may be given by:

$$S = -e^{-\phi} \int_{M^D} \sqrt{-det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} + e^{-\phi} I_{WZ} \tag{4.1}$$

Where $G_{\mu\nu}$ is the induced metric on the brane given by:

$$G_{\mu\nu} = \Pi^I_\mu \Pi^J_\nu \eta_{IJ} \tag{4.2}$$

$$\Pi^I_\mu = \partial_\mu X^I - \bar{\theta}\Gamma^I \partial_\mu \theta \tag{4.3}$$

X^I are space-time coordinates, and θ are space time spinors. They are 16 component Majorana and Weyl spinors, spinorial indices are always suppressed. There are two of them for N=2 supersymmetry as discussed above with chirality dependent on whether the theory is IIA or IIB. Thus θ^α, $\alpha = 1, 2$, the α index is often suppressed with an implicit sum over this index. In the IIB case, the indices will be contracted with τ_3 and τ_1.

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - b_{\mu\nu} \tag{4.4}$$

where $F = dA$ is the field strength of the vector field A on the brane.

$$b = -\bar{\theta}\Gamma_{11}\Gamma_M d\theta(dX^M + \frac{1}{2}\bar{\theta}\Gamma^M d\theta) \tag{4.5}$$

(For the IIB theory the Γ_{11} is replaced by τ_3. This is true in the expressions that follow as well as above.) b is such that its exterior derivative is a superinvariant. That is one may check:

$$db = d\bar{\theta} \wedge \Gamma_{11}\Pi^M \Gamma_M \wedge d\theta \tag{4.6}$$

From this one sees that the variation b under supersymmetry is exact. Hence its variation may be cancelled by a suitable SUSY transformation of A. Thus the combination $\mathcal{F} = F - b$ is also a SUSY invariant. One notes that b is the usual term added to the Green Schwarz string to add κ invariance to the action. The second term in the action 4.1, I_{WZ} is added to ensure the κ symmetry of the D-brane action. It is most naturally given, as was the case for the string, by a SUSY invariant defined in one dimension higher than the brane dimension. Decomposing into positive and negative chirality spinors. One may write the action as for the IIA theory:

$$I_{WZ} = \int_{M^{D+1}} d\bar{\theta} e^{\mathcal{F}} \begin{pmatrix} cosh\gamma & sinh\gamma \\ -sin\gamma & -cos\gamma \end{pmatrix} d\theta \tag{4.7}$$

$\partial M^{D+1} \equiv M^D$, where M^D is the D-brane world volume. $\gamma \equiv \Pi^I \Gamma_I$, is the induced gamma matrix on the world volume.

(A similar expression exists for the IIB theory where the matrix acts on the two sorts of spinors rather than on the different chiral spinors.)

One may write I_{WZ} as the integral of an exact form and then use Stokes theorem to obtain an action on M^D. This will be a much more complicated expression though is always derivable from the simpler I_{WZ} form up to exact pieces. The action on the brane is denoted by the formal expression:

$$\int_{M^D} e^{\mathcal{F}} \wedge C \tag{4.8}$$

Where C is is a formal sum of forms given by $\sum_i C_{(i)}$ where $C_{(i)}$ is a form of rank i, where $C_{(i)}$ will have an expression in terms of X^I and θ that may be read off from 4.7.

Note that if one sets all fermions to zero and makes a coordinate choice such that brane coordinates and spacetime coordinates coincide then the D-brane action becomes the so called Dirac Born-Infeld action:

$$S = \int_{M^D} -\sqrt{det(\eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j G_{ij} + F_{\mu\nu})} \tag{4.9}$$

Where i=1,..,10-p. This is the same as the Born-Infeld action considered before only with the addition of the scalars corresponding to fluctuations in the brane in transverse directions.

Recall that IIA and IIB string theories are T-dual. That is there is an equivalence of IIA string theory on M^9 times a circle of radius R and IIB string theory on M^9 times a circle of radius $\frac{1}{R}$. To make this picture consistent one must see how the D-branes of the two theories match up. First consider wrapping a D-brane on a circle of radius R. To wrap the brane one identifies the brane and space-time coordinate associated with the circle and then carries out a simultaneous Kaluza-Klein reduction. The fields are taken to be independent of the circle coordinate; that is we take only zero modes. This means that the brane is not charged with respect to the space-time Kaluza-Klein vector field. The result will be a brane of one less world volume dimension in 9 dimensions. We now consider a so called direct reduction of a D-brane in the other type II theory. By direct reduction one implies that the brane is not wrapped around the compact direction in space-time. As such one simply separates out through a coordinate choice the tenth, compact dimension of radius $\tilde{R}$. The result of this will be to pull out a compact scalar on the world volume of the D-brane corresponding to fluctuations in the tenth, compact direction. Note that from the nine dimensional point of view the information regarding the chirality of the spinors is lost because there are no Weyl spinors in nine dimensions. One may now identify the wrapped brane and the directly reduced brane. On the wrapped brane:

$$\begin{aligned} A &\rightarrow (A, \phi) \\ b &\rightarrow (b_{(2)}, b_{(1)}) \\ \Pi^I &\rightarrow (\Pi^i, C') \\ C_{(i)} &\rightarrow (C_i, C_{(i-1)}) \end{aligned} \tag{4.10}$$

And similarly for the non-wrapped brane:

$$\begin{aligned} A &\rightarrow A \\ b &\rightarrow (b_{(2)}, b_{(1)}) \\ \Pi^I &\rightarrow (\Pi^i, d\phi + C') \\ C_{(i)} &\rightarrow (C_i, C_{(i-1)}) \end{aligned} \tag{4.11}$$

After some manipulations of the reduced actions one can show that the following identifications of the fields are required to identify the the reduced brane with the wrapped brane. Writing the wrapped brane fields on the left and the unwrapped fields on the right:

$$A \equiv A,$$

$$\begin{aligned}
b_{(2)} &\equiv b_{(2)} \\
b_{(1)} &\equiv C' \\
C' &\equiv b_{(1)} \\
\Pi^i &\equiv \Pi^i \\
C_{(i-1)} &\equiv C_{(j)}, \quad j = i-1 \\
C_{(j)} &\equiv C_{(i-1)}, \quad j = i-1 \\
R &\equiv \frac{1}{\bar{\bar{R}}} \qquad (4.12)
\end{aligned}$$

Thus given a nine dimensional theory, there is in ambiguity in the lifting to ten dimensions. One can lift to either IIA or IIB. It is ambiguity that is the T-duality between IIA and IIB theories on a circle. This notion generalises to other situations. For example one may identify the type two theory reduced on a $K3$ with the heterotic string reduced on T^4.

It is known that IIA supergravity may be obtained from N=1 supergravity in eleven dimensions. Recall eleven dimensional spinors are 32 component Majorana spinors. Dimensionally reducing eleven dimensional supergravity on a circle, one decomposes the spinors into chiral and anti-chiral parts according to the action of Γ^{11} to give IIA supergravity in D=10. The string coupling emerges from the compactification radius of the eleventh dimension. It is for this reason that the the eleven dimensions are not apparent from string perturbation theory. Small string coupling implies small radius. The eleven dimensions only become apparent at strong coupling where perturbation theory breaks down. Thus it is said that eleven dimensional supergravity is the low energy effective action of some theory that is equivalent to the strong coupling limit of IIA string theory. This theory is often called M-theory. In what follows we will often mix up M-theory, which is still not properly formulated and eleven dimension supergravity. Just as one labels IIA,B supergravities in ten dimensions from their string theory origins. Recall that the bosonic sector of eleven dimensional supergravity contains, along with the graviton, a three form potential and its magnetic dual a six form potential. Solutions of the supergravity effective equations allow a membrane which is charged electrically with respect to the 3-form and a five brane which is charged with respect to its six form electromagnetic dual. These solutions are BPS and so break half the number of supersymmetries. We therefore require 8 on-shell fermionic and bosonic degrees of freedom. There are eight transverse directions to the membrane and so there is no requirement to add any world volume fields other than the scalars that arise naturally from the pull back of the space-time metric on to the world volume. Thus the effective world volume theory of the membrane will be given by:

$$S = \int_{M^3} -\sqrt{det(G_{\mu\nu})} - b_{(3)} \tag{4.13}$$

Where $b_{(3)}$ is the pullback of the three form potential on to the brane as is usual for minimal coupling. $G_{\mu\nu}$ is the pull back of the space-time metric on to the membrane. By choosing coordinates one sees that $G_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j g_{ij}$, where i,j=1..8.

The M-theory five brane [22, 55, 74, 75, 76, 84] however will require additional world volume fields, just as did the D-branes. Note that there are 5 transverse directions to the five brane hence there are 5 scalars on the world volume. This leaves a deficit of 3 bosonic degrees of freedom. This can be made up by adding a self dual two form potential living on the world volume of the brane. (Note, this field is not the pull back onto the brane of any field in the eleven dimensional supergravity.) Recall that a two form in 6 dimensions has six physical degrees of freedom. The self-duality constraint halves this number to give the requisite three. In fact, it can be shown that the field content forms a tensor multiplet on the brane. As noted above the origin of the vector field on the D-brane may be thought of as a result of a string ending on the D-brane; the string ends couple to the vector field. There is a similar situation in M-theory. The five brane is analogous to a D-brane in that a membrane may end on a five brane. The boundary of a membrane will be a string which must couple to a two form potential in the five world volume.

It is evident that the IIA D-branes and the fundamental string must come from a reduction of the M-theory membrane and five brane on a circle. This may be demonstrated for the D-2 brane as follows. Consider doing a world volume duality on the vector field on the D-2 brane. Its dual is a compact scalar. This scalar may then be identified with the scalar field on the M-theory membrane that is associated with fluctuations in the compact eleventh dimension. Thus the D-2 brane is the direct reduction of the membrane with a scalar field on the brane dualized. A similar origin has been proven for the D-4 brane. The M-5 is wrapped on a circle and its two form potential is dualized on the world volume to give a vector field. This can then be identified with the vector field on the D-4 brane. In each case after reduction (either direct or wrapped) there is a world volume dualization of one of the fields on the world volume that allows the M-theory/IIA duality to be identified.

It is not possible to obtain IIB supergravity from any compactification of D=11 supergravity because of the chiral nature of the IIB supergravity. However, if one reduced the eleven dimensional supergravity down to nine dimensions by dimensionally reducing on a torus, T^2, then one would obtain the same nine dimensional theory that allowed us to identify IIA/IIB T-duality. Therefore, one may construct a M-theory/IIB *T-duality* by such a reduction process. Again one would anticipate it being necessary to world volume dualize to identify the appropriate fields in the different pictures.

The M-theory 5-brane has also been related to the heterotic string. A double dimensional reduction of the 5-brane on K3 has been identified with the heterotic string compactified on T^3 [40]. This also produced a reformulation of the heterotic string in which the Narain duality was manifest in the action. This is related to the calculation presented in chapter 3 where the dimensional reduction of a chiral two from on K3 produces, 3 chiral and 19 anti-chiral scalars.

This chapter will involve the relationship between M-theory and IIB string theory. As the discussed above, reducing M-theory on a torus ought to be identified with the IIB theory reduced on a circle. For example the 11-

dimensional membrane wrapped around one cycle of the torus will be identified with the IIB fundamental string and the membrane wrapped around the other cycle will be identified with the D-string. IIB string theory is conjectured to posses an SL(2,Z) S-duality that exchanges RR states for NS while inverting the string coupling. As such, the IIB SL(2,Z) duality which mixes Ramond Ramond and Neveu-Schwarz sectors may be seen as a geometrical consequence of the torus in the M-theory picture. More concretely, under the SL(2,Z) transformations, the R-R and NS-NS two forms transform as an SL(2,Z) doublet while the axion-dilaton undergoes an SL(2,Z) fractional linear transformation and the R-R 4-form is left invariant.

The IIB string theory also possesses other branes apart from the fundamental string and D-string. The theory also contains a self-dual D-3 brane, a D-5 brane and a solitonic 5-brane. The self-dual three brane, so called because it couples to the self-dual, SL(2,Z) inert, Ramond Ramond 4-form, will be the main topic of this chapter. In particular, we will investigate its relationship to the M-theory 5-brane. For completeness we state that the D-5 and solitonic 5-brane couple magnetically to the R-R, NS-NS two forms respectively and so should transform into each other under SL(2,Z). This is investigated in [80]. It would be interesting to see how these five branes are related to the M-theory 5-brane and how their duality properties appear. (However, we will not do so here).

Given the relationship between M-theory and IIB, we expect the M-theory 5-brane wrapped on the torus to be identified the direct reduction of the IIB self-dual three brane after an appropriate world volume dualization [71]. (By direct reduction we imply that the brane's world volume is not reduced). The duality properties of the 3-brane should then arise as a consequence of the modular symmetry of the torus in the M-theory picture.

In [35] this identification was carried out for the Born-Infeld action ie. in the absence of R-R fields and without reference to the background space-time. This is related to the calculation presented in the previous chapter only there the scalar field was truncated. Here we will include the R-R fields as well as the embedding in a superspace background and make the identification in 9-dimensions. This identification of M theory and IIB string theory has been discussed in detail for the low energy effective theories in [81] and with a view to extended objects in [82, 83].

First we will introduce our notation and describe the M5-brane action. No efforts will be made to compare our results with the interesting and indeed powerful 5-brane approach [84, 85, 86] based solely on the equations of motion. We will then carry out the double dimensional reduction on T^2. Following this we will describe the the direct reduction of the IIB three brane on S^1. To compare the two actions it will be necessary to make world volume duality transformations of some of the fields on the brane.

This duality procedure, for the case given above is far from trivial. We will make a variety of truncations that will enable us to construct the dual actions for the truncated cases. These duality transformations are of a similar type as those described in some detail in [52, 53, 78].

The point of the transformations is that we will be able to identify the dualized reduced 5-brane with the reduced D-3 brane. In doing so we will be able to explicitly identify the fields and construct the SL(2,Z) duality properties of the IIB theory from the M-theory picture. In particular, the SL(2,Z) transformation of the three brane will arise out of a gauge choice made on the 5-brane world volume. This is in precise analogy with the discussion of chapter two where the S-duality arose from different choices of v for the self duality projection.

The M-theory/type II relationship is summarised in figure 4.1. D-denotes world volume dualization, W denotes wrapping, R denotes direct reduction and S denotes S-duality. S^1 implies dimensional reduction on a circle. T^2 implies dimensional reduction on a torus.

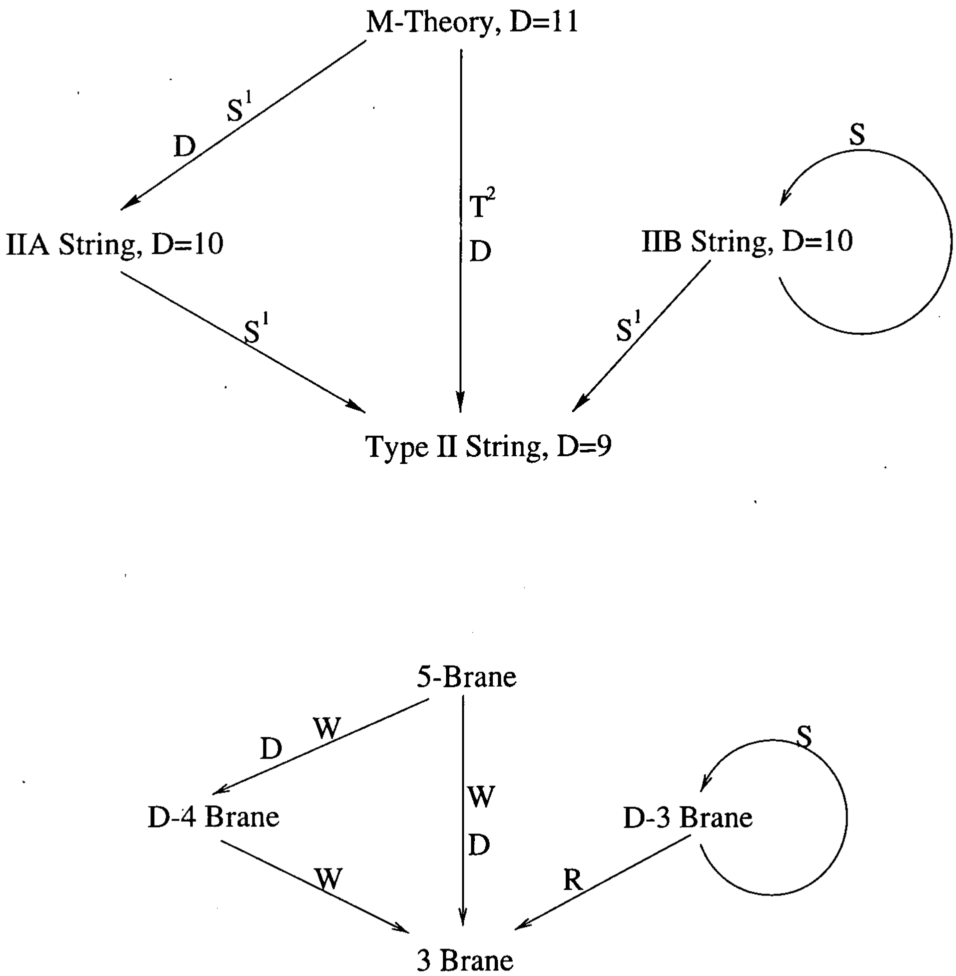

Figure 4.1: M-theory/typeII dualities

4.2 The M-theory 5-brane

The kappa symmetric action for the 5-brane [55, 76] is as follows. We work with a flat Minkowski background, using a metric, $\eta = diag(-1, +1, +1, ..)$. The θ coordinates are 32 component Majorana spinors and X^M are 11-dimensional space-time coordinates $(M, N = 0..9, 11)$. We will follow [76] and use the convention where the Clifford algebra for the Γ matrices is $\{\Gamma^M \Gamma^N\} = 2\eta^{MN}$. The global supersymmetry transformations may written as:

$$\delta\theta = \epsilon, \qquad \delta X^M = \bar{\epsilon}\Gamma^M\theta. \tag{4.14}$$

The action is written in terms of the following supersymmetric invariant one forms

$$d\theta, \quad \Pi^M = dX^M + \bar{\theta}\Gamma^M d\theta \tag{4.15}$$

where $d = d\sigma^{\hat{\mu}}\partial_{\hat{\mu}}$; the exterior derivative pulled back to the brane. $\sigma^{\hat{\mu}}$ are the coordinates of the brane, $\hat{\mu} = 0..5$. (We use the convention that $d\sigma^\mu$ is odd with respect to the grassmann variables so that $d\theta = d\sigma^\mu\partial_\mu\theta = -\partial_\mu\theta d\sigma^\mu$). The action will also contain a world volume self dual two form gauge field, B whose field strength is as usual given by $H = dB$. In order to ensure supersymmetry this is extended as follows;

$$\mathcal{H} = H - b_3 \tag{4.16}$$

where b_3 is the 11 dimensional 3-form potential pulled back to the brane defined as follows:

$$b_3 = \frac{1}{2}\bar{\theta}\Gamma_{MN}d\theta(dX^M dX^N + dX^M\bar{\theta}\Gamma^N d\theta + \frac{1}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta) \tag{4.17}$$

We are implicitly assuming wedge products for forms unless stated otherwise. The action for the 5-brane will be written as follows:

$$S = -\int_{M^6} d^6x \sqrt{-det\Big(G_{\hat{\mu}\hat{\nu}} + i\frac{\tilde{\mathcal{H}}_{\hat{\mu}\hat{\nu}}}{\sqrt{v^{\hat{\mu}}v_{\hat{\mu}}}}\Big)} - \frac{\sqrt{-G}\tilde{\mathcal{H}}^{\hat{\mu}\hat{\nu}}H_{\hat{\mu}\hat{\nu}\hat{\rho}}v^{\hat{\rho}}}{4v^2} + S_{WZ} \tag{4.18}$$

where:

$$\tilde{\mathcal{H}}_{\hat{\mu}\hat{\nu}} = \frac{1}{6}G_{\hat{\mu}\hat{\alpha}}G_{\hat{\nu}\hat{\beta}}\frac{\epsilon^{\hat{\alpha}\hat{\beta}\hat{\delta}\hat{\gamma}\hat{\rho}\hat{\sigma}}}{\sqrt{-G}}H_{\hat{\delta}\hat{\gamma}\hat{\rho}}v_{\hat{\sigma}} \tag{4.19}$$

and

$$G_{\hat{\mu}\hat{\nu}} = \Pi^M_{\hat{\mu}}\Pi^N_{\hat{\nu}}\eta_{MN} \tag{4.20}$$

$G = detG_{\hat{\mu}\hat{\nu}}$; v is a completely auxiliary closed one form field introduced to allow the self-duality condition to be imposed in the action while maintaining Lorentz invariance [1]. See the references [55] for a discussion on this Lorentz invariant formulation.

The S_{WZ} is the so called Wess Zumino part of the action that is introduced to ensure the kappa symmetry of the action and in analogy with the usual Wess-Zumino type action may be written more conveniently as an exact form over a manifold whose boundary corresponds to the five brane world volume. That is:

$$S_{WZ} = \int_{M^7} I_7 \tag{4.21}$$

where $dI_7 = 0$ and $\partial M^7 = M^6$ which implies locally we may write $I_7 = d\Omega_6$. Thus we can write S_{WZ} as an integral over the world volume, $S_{WZ} = \int_{M^6} \Omega_6$.

$$I_7 = -\frac{1}{4}\mathcal{H} d\bar{\theta}\not{\psi}\not{\psi} d\theta - \frac{1}{120} d\bar{\theta}\not{\psi}^5 d\theta \tag{4.22}$$

where $\not{\psi} = \Gamma_M \Pi^M$ the induced Gamma matrix. Integrating we find:

$$\Omega_6 = C_6 + \mathcal{H} \wedge b_3 \tag{4.23}$$

where b_3 is the same form that appears in combination with H above. (We will not need an explicit form for C_6). This action has been shown to have all the properties required of the 5-brane [76]. Apart from the usual gauge symmetries associated with the gauge potential B and the background field C, this action has additional local, so called *PST* symmetries one of which we will use later to eliminate half the degrees of freedom of the two form gauge field.

$$\delta B = \chi \wedge v \tag{4.24}$$

[1]Usually the action 4.18 is written with $v = da$; however this is only locally correct as v is constrained to be closed but not necessarily exact.

This will be the action that we will double dimensionally reduce on T^2. And so we send, $M^6 \to M^4 \times T^2$ and $M^{11} \to M^9 \times T^2$. We will identify

$$(X^{11}, X^9) = (\sigma^4, \sigma^5) = (y^1, y^2) \tag{4.25}$$

Where (y^1, y^2) are the coordinates on the space-time torus. In these coordinates we will identify $y^1 = y^1 + 1$ and $y^2 = y^2 + 1$. Despite reducing to 9 dimensions we will not decompose the spinors as it will be convenient in what follows to leave them. We will drop all functional dependence of the fields on the compact coordinates, that is taking only the zero modes. $m, n = 0..8$ will be the non compact space-time indices, $i, j = 1, 2$ will be torus coordinate indices and $\mu, \nu = 0..3$ will be the coordinates of the non-wrapped 5-brane world volume. The space-time metric will be written as

$$\eta_{MN} \to \eta_{mn} \oplus \eta_{ij} \tag{4.26}$$

This truncates the space-time Kaluza Klein fields associated with the torus. This is because we are only interested in the M-5 brane/D-3 relationship. Such Kaluza-Klein fields in the M-theory picture are associated with the wrapped D and fundamental string in IIB. We will take η_{mn} to be flat Minkowski metric and take the metric on the torus to be given by

$$\eta_{ij} dy^i \otimes dy^j = \frac{V}{\tau_2}(dy^1 \otimes dy^1 + \tau_1 dy^2 \otimes dy^1 + \tau_1 dy^1 \otimes dy^2 + |\tau|^2 dy^2 \otimes dy^2) \tag{4.27}$$

$\tau = \tau_1 + i\tau_2$ is the complex structure of the torus and V is the area of the torus. The reduction of the brane metric G from 4.20 follows.

$$\begin{aligned} G_{\hat{\mu}\hat{\nu}} d\sigma^{\hat{\mu}} \otimes d\sigma^{\hat{\nu}} &= (\Pi^m_\mu \Pi^n_\nu \eta_{mn} + C^i_\mu C^j_\nu \eta_{ij}) d\sigma^\mu \otimes d\sigma^\nu \\ &+ C_{i\mu} d\sigma^\mu \otimes dy^i + C_{j\nu} dy^j \otimes d\sigma^\nu + \eta_{ij} dy^i \otimes dy^j \end{aligned} \tag{4.28}$$

Where

$$C^i_\mu = -\bar{\theta}\Gamma^i_T \partial_\mu \theta \tag{4.29}$$

Γ_T are the Gamma matrices on the torus. As we have identified the space-time coordinates (X^{11}, X^9) with (y^1, y^2) the torus coordinates we have $(\Gamma_T^1, \Gamma_T^2) = (\Gamma^{11}, \Gamma^9)$. Now the background three form potential will reduce as follows:

$$b_3 = b_{(3)} + b_{(2)i}dy^i + b_{(1)}dy^1 \wedge dy^2 \quad (4.30)$$

Where

$$b_{(3)} = \frac{1}{2}\bar{\theta}\Gamma_{mn}d\theta(dX^m dX^n + dX^m\bar{\theta}\Gamma^n d\theta + \frac{1}{3}\bar{\theta}\Gamma^m d\theta\bar{\theta}\Gamma^n d\theta)$$

$$+\frac{1}{2}\bar{\theta}\Gamma_m\Gamma_{Ti}d\theta(dX^m\bar{\theta}\Gamma_T^i d\theta + \frac{2}{3}\bar{\theta}\Gamma^m d\theta\bar{\theta}\Gamma_T^i d\theta) + \frac{1}{6}\bar{\theta}\Gamma_{Tij}d\theta(\bar{\theta}\Gamma_T^i d\theta\bar{\theta}\Gamma_T^j d\theta) \quad (4.31)$$

$$b_{(2)i} = \frac{1}{2}\bar{\theta}\Gamma_{Ti}\Gamma_n d\theta(2dX^n + \bar{\theta}\Gamma^n d\theta) + \frac{1}{2}\bar{\theta}\Gamma_{Tij}d\theta\bar{\theta}\Gamma^j d\theta \quad (4.32)$$

$$b_{(1)} = \bar{\theta}\Gamma_{T12}d\theta \quad (4.33)$$

As usual Γ_{pq} implies $\Gamma_{[p}\Gamma_{q]}$, where square brackets on the indices mean antisymmetrisation (without any weighting factor). Similarly, we reduce the world volume gauge field as follows:

$$B = B_{(0)}dy^1 \wedge dy^2 + B_{(1)i} \wedge dy^i + B_{(2)} \quad (4.34)$$

so that the we may write for $\mathcal{H} = H - b$

$$\mathcal{H} = \mathcal{J} + \mathcal{F}_i \wedge dy^i + \mathcal{L}dy^1 \wedge dy^2 \quad (4.35)$$

Where we have defined:

$$\mathcal{J} = dB_{(2)} - b_{(3)} \qquad \mathcal{F}_i = dB_{(1)i} - b_{(2)i} \qquad \mathcal{L} = dB_{(0)} - b_{(1)} \quad (4.36)$$

We now need to determine whether the auxiliary one form will be in T^2 only or in M^4 only. The two choices are physically equivalent. The restriction simply corresponds to a partial gauge fixing. In what follows we will take v to be a member of the first cohomology on T^2. We will consider the specific choices $v = dy^1$ and $v = dy^2$. These two independent gauge choices are what will eventually generate the S-duality on the 3-brane. Should we put v in M^4, for example $v = dt$ then the SL(2,Z) symmetry of the 3-brane will become manifest in the action but we will lose manifest Lorentz invariance. This will give an action of type given in [34], [35] and discussed in the previous chapter. The relationship between the formulation of the reduced action and the different gauge choices for the PST one form was discussed in the previous chapter. For now we will take the torus to be have $\tau = 1$ and $V = 1$; we will reinstate the dependence on V and τ when required. So with the specific gauge choice

$$v = dy^2 \tag{4.37}$$

this implies:

$$\tilde{\mathcal{H}}^{\hat{\mu}\hat{\nu}} = ({}^*\mathcal{F}^{\mu\nu}, {}^*\mathcal{J}^{\mu 1}) \tag{4.38}$$

Therefore,

$$\tilde{\mathcal{H}}_{\mu\nu} = {}^*\mathcal{F}_{\mu\nu} + C_\mu{\cdot}C_\rho{}^*\mathcal{F}^\rho{}_\nu + {}^*\mathcal{F}_\mu{}^\rho C_\rho{\cdot}C_\nu + C_\mu{\cdot}C_\sigma C_\nu{\cdot}C_\rho{}^*\mathcal{F}^{\sigma\rho} - C_{1[\mu}{}^*\mathcal{J}_{\nu]} \tag{4.39}$$

$$\tilde{\mathcal{H}}_{\mu i} = \eta_{i1}({}^*\mathcal{J}^\mu + C_\mu{\cdot}C_\rho{}^*\mathcal{J}^\rho) - C_{i\rho}{}^*\mathcal{F}^\rho_\mu - C_{i\rho}{}^*\mathcal{F}^{\rho\sigma}C_\sigma{\cdot}C_\nu - C_{i\rho}{}^*\mathcal{J}^\rho C_{1\mu} \tag{4.40}$$

$$\tilde{\mathcal{H}}_{ij} = C_{i\mu}C_{j\nu}{}^*\mathcal{F}^{\mu\nu} - C_{2\rho}{}^*\mathcal{J}^\rho \tag{4.41}$$

$$v^2 = 1 + (C_2)^2 \tag{4.42}$$

Where we use the notation $C_\mu{\cdot}C_\nu = C^i_\mu \eta_{ij} C^j_\nu$ and * is the Hodge dual in 4 dimensions. Combining the above equations with the reduced metric 4.28 we have for, M, the matrix inside the determinant of action 4.18:

$$M = (G_{\mu\nu} + C_{\mu}{\cdot}C_{\nu} + \frac{i\tilde{\mathcal{H}}_{\mu\nu}}{\sqrt{1+(C_2)^2}})d\sigma^{\mu}\otimes d\sigma^{\nu}$$

$$+(C_{i\nu} - \frac{i\tilde{\mathcal{H}}_{\nu i}}{\sqrt{1+(C_2)^2}})d\sigma^{i}\otimes d\sigma^{\nu} + (C_{\mu j} + \frac{i\tilde{\mathcal{H}}_{\mu i})}{\sqrt{1+(C_2)^2}})d\sigma^{\mu}\otimes d\sigma^{i}$$

$$+(\eta_{ij} + \frac{i\tilde{\mathcal{H}}_{ij}}{\sqrt{1+(C_2)^2}})d\sigma^{i}\otimes d\sigma^{j} \qquad (4.43)$$

Importantly, we remark that M occurs in the action only in the determinant and so we are allowed to manipulate M in anyway that leaves the determinant invariant. Our goal will be to compare with the D-3 brane, hence it is natural to express the above as a four dimensional determinant. Using the well known identities:

$$det\begin{pmatrix} L & P \\ Q & J \end{pmatrix} = det\begin{pmatrix} L - Q^T J^{-1} P & 0 \\ 0 & J \end{pmatrix} \qquad (4.44)$$

and

$$det(A \oplus B) = det(A)det(B) \qquad (4.45)$$

We have

$$detM = det(M_{ij})det(M_{\mu\nu} - M^T_{\mu i}(M^{-1})^{ij}M_{j\nu}) \qquad (4.46)$$

which gives after numerous cancellations:

$$det(M_{\hat{\mu}\hat{\nu}}) = det(M_{ij})det\Big(G_{\mu\nu} + \frac{i^*\mathcal{F}_{\mu\nu}}{\sqrt{1+(C_2)^2}} + \frac{P_{[\mu}C_{2\rho}{}^*\mathcal{F}^{\rho}{}_{\nu]}C_{2\alpha}P^{\alpha}(1+(C_2)^2)}{1+(C_2)^2-(C_{2\beta}P^{\beta})^2}$$

$$- \frac{(P_{\mu}P_{\nu} + C_{2\rho}{}^*\mathcal{F}^{\rho}{}_{\mu}C_{2\sigma}{}^*\mathcal{F}^{\sigma}{}_{\nu})}{1+(C_2)^2-(C_{2\beta}P^{\beta})^2}\Big) \qquad (4.47)$$

Where

$$P_\mu = {}^*\mathcal{J}_\mu - C_{1\rho}{}^*\mathcal{F}^\rho{}_\mu \tag{4.48}$$

and explicitly,

$$detM_{ij} = \frac{1+(C_2)^2 - (C_{2\rho}P^\rho)^2}{1+(C_2)^2} \tag{4.49}$$

We will now turn to reducing the Wess-Zumino term. First, we note that

$$\psi \to (\psi, \Gamma_{Ti}C^i, \Gamma_{Ti}dy^i) \tag{4.50}$$

Using this and the reduction for $\tilde{\mathcal{H}}$ we calculate the reduced WZ terms by substituting these into I_7. Doing the reduction for I_7 is equivalent to doing the reduction for Ω_6 provided that the compact space has no boundary, which is of course the case for a torus. We produce for I_5 where $S_{WZ^5} = \int_{M^5} I_5$ and $\partial M^5 = M^4$. Taking care with factors this produces:

$$I_5 = -\frac{1}{3!}d\bar{\theta}\psi^3\Gamma_{T12}d\theta - \frac{1}{2}\mathcal{F}_{[i}(d\bar{\theta}\psi\Gamma_{Tj]}d\theta + d\bar{\theta}\Gamma_{Tl}C^l\Gamma_{Tj]}d\theta)$$

$$-\frac{1}{4}\mathcal{J}d\bar{\theta}\Gamma_{T12}d\theta + \frac{1}{4}b_{(1)}d\bar{\theta}(\psi^2 + \psi\Gamma_{Tl}C^l + \Gamma_{Tk}C^k\Gamma_{Tm}C^m)d\bar{\theta} \tag{4.51}$$

Next, we will examine the PST term, the second term in action 4.18. Upon dimensional reduction this term naturally splits into a sum of two parts. The first part $I^{(1)}_{PST}$, consists of terms that look like terms in the Wess-Zumino term and a total derivative (corresponding to the theta term). The second part, $I^{(2)}_{PST}$ is distinct and will be associated with a term arising from dualizing the $\mathcal{J}$ field.

$$I^{(1)}_{PST} = \int_{M^4} \frac{1}{2}(\mathcal{F}_i \wedge \mathcal{F}_j + \mathcal{J} \wedge \mathcal{L}_{ij})\gamma^{ij}(v) \tag{4.52}$$

$$I^{(2)}_{PST} = -P^\mu \mathcal{F}_{\mu\nu(i)}C^{\nu(j)}\frac{v_j\epsilon^{il}v_l}{v^2} \tag{4.53}$$

where

$$\gamma^{il}(v) = \frac{1}{v^2}\epsilon^{ij}v_j G^{lm}v_m. \tag{4.54}$$

$F \wedge F$ is a theta type term that may contribute. In fact it is this term that we will later identify with the axion coupling in the 3-brane. For a specific choice of $v = dy^i$, we may gauge away L and F_i but this will not gauge away the fields $b_{(1)}$ and $b_{i(2)}$ that must be kept. And so we integrate the Wess-Zumino terms and combine them with the relevant PST terms using,

$$d(\Omega - I^{(1)}_{PST}) = I_5. \tag{4.55}$$

And so in terms of fields given in 4.31, 4.32, 4.33 this gives the interaction term for the reduced action: For choice dy^1:

$$\Omega = b_{(4)} + b_{(2)1} \wedge \mathcal{F} - {}^*P \wedge b_{(1)} - \frac{1}{2}\frac{\tau_1}{|\tau|^2}\mathcal{F} \wedge \mathcal{F} \tag{4.56}$$

For choice $v = dy^2$:

$$\Omega = b_{(4)} - b_{(2)2} \wedge \mathcal{F} - {}^*P \wedge b_{(1)} + \frac{1}{2}\tau_1 \mathcal{F} \wedge \mathcal{F} \tag{4.57}$$

Here we remark that the index i is associated with the torus coordinates $\{y^i\}$, see equation 4.32. Now, so that we may compare with the D3-brane we will rewrite the above expression in terms of orthonormal coordinates $\bar{y}^{\bar{i}}$ on the torus. Using the equation,

$$b_{(2)i} = e_i{}^{\bar{i}}\bar{b}_{(2)\bar{i}} \tag{4.58}$$

where

$$e_i{}^{\bar{i}} = \sqrt{\frac{V}{\tau_2}}\begin{pmatrix} 1 & 0 \\ \tau_1 & \tau_2 \end{pmatrix} \tag{4.59}$$

is the zweibein of the torus whose metric is given by 4.27. We then carry out a space time, Weyl scaling

$$X' = X\eta^{1/8} \qquad \theta' = \theta\eta^{1/16} \tag{4.60}$$

We will discuss the relevance of this scaling later. And so when we substitute this into the above, we find: For $v = dy^1$:

$$\Omega = \bar{b}_{(4)} - \bar{b}_{(2)\bar{2}} \wedge \bar{b}_{(2)\bar{1}} - \frac{1}{2}\frac{\tau_1}{\tau_2}\bar{b}_{(2)\bar{1}} \wedge \bar{b}_{(2)\bar{1}} + \frac{1}{\sqrt{\tau_2}}\bar{b}_{(2)\bar{1}} \wedge \mathcal{F} - \eta^{\frac{3}{8}} {}^*P \wedge \bar{b}_{(1)} - \frac{1}{2}\frac{\tau_1}{|\tau|^2}\mathcal{F} \wedge \mathcal{F} \tag{4.61}$$

and

$$\mathcal{F} = F + \frac{\tau_1}{\sqrt{\tau_2}}\bar{b}_{(2)1} + \sqrt{\tau_2}\bar{b}_{(2)2} \tag{4.62}$$

For $v = dy^2$

$$\Omega = \bar{b}_{(4)} - \sqrt{\tau_2}\bar{b}_{(2)\bar{2}} \wedge \mathcal{F} - \eta^{\frac{3}{8}} {}^*P \wedge \bar{b}_{(1)} + \frac{1}{2}\tau_1 F \wedge F \tag{4.63}$$

and

$$\mathcal{F} = F - \frac{1}{\sqrt{\tau_2}}\bar{b}_{(2)\bar{1}} \tag{4.64}$$

We remark that all the terms in Ω depend on either τ or η so they form essentially independent couplings. This will be true when we consider the first part of the action, see below. We also have the extra term, $I^{(2)}_{PST}$ which becomes for the choice $v = dy^i$:

$$I^{(2)}_{PST} = \frac{-1}{1 + (C^i)^2} P^\mu \mathcal{F}_{\mu\nu} C^{\nu(i)} \tag{4.65}$$

Consider the truncation where one sets $\theta = 0$. (This is a consistent truncation). We will also explicitly reinstate the general metric η_{ij} of the torus and leave the auxiliary field v unspecified. (Apart from the fact that it is a closed one form on the torus.) This gives for the first part of the action:

$$S_{5-2} = -\int_{T^2}\int_{M^4}\sqrt{\eta}\sqrt{-det(G_{\mu\nu} + i\alpha^i(v)^*F_{(i)\mu\nu} - \beta(v)^*J_\mu{}^*J_\nu)} + \frac{1}{2}F_i \wedge F_j\gamma^{ij}(v) \tag{4.66}$$

where $\alpha^i(v)$ and $\beta(v)$ and $\gamma(v)^{ij}$ are constants that remain to be evaluated and will be dependent on our choice of v.

However, before evaluating them we will put the $\sqrt{\eta}$ inside the determinant. This becomes $\eta^{\frac{1}{4}}$ inside the determinant. We will then carry out a Weyl scaling as before, see equation 4.60 so that we absorb this factor into the rescaled metric. That is

$$G'_{\mu\nu} = G_{\mu\nu}\eta^{\frac{1}{4}} \tag{4.67}$$

We then rewrite the action in this rescaled metric taking care with factors of η. The T^2 integral is trivial.

We will use the symmetry given by equation (5) to eliminate half the degrees of freedom contained in the gauge fields. For the choice $v = dy^L$ we gauge away $F_{(L)}$ and L_{12}. This leaves only one vector gauge field in the action, with field strength F, and one two form gauge field, with field strength J. The PST part of the action will then contribute a total derivative that we shall be able to identify it with an axion coupling. We will now write the action in its final form as follows:

$$S_{5-2} = -\int_{M^4}\sqrt{-det(G'_{\mu\nu} + i\alpha(v)^*F_{\mu\nu} - \beta^*J_\mu{}^*J_\nu)} + \frac{1}{2}F_i \wedge F_j\gamma^{ij}(v) \tag{4.68}$$

We now consider the two natural independent gauge choices for v and evaluate the coefficients, α, β and γ.

For $v = dy^1$:

$$\alpha = \sqrt{\frac{\tau_2}{|\tau|^2}} \qquad \beta = \eta^{3/4} \qquad \gamma = -\frac{\tau_1}{|\tau|^2} \tag{4.69}$$

for $v = dy^2$:

$$\alpha = \sqrt{\tau_2} \qquad \beta = \eta^{3/4} \qquad \gamma = \tau_1 \tag{4.70}$$

Note that the vector fields couple only to the complex structure of the torus. That is the couplings are completely determined by the shape of the torus and are independent of its size. Different choices of v give different couplings. The opposite is true for the two form fields. The coupling for the two form field is independent of the choice of v and is dependent only on the area of the torus. Note that this is frame dependent statement that is reliant on the Weyl rescaling. Combining $\tau = \tau_1 + i\tau_2$ we see the different choices of v generate the transformation $\tau \rightarrow \frac{-1}{\tau}$ in the vector field couplings. This corresponds to one of the generators of SL(2,Z) the modular group of the torus. The other generator will arise from an integral shift in τ_1 which will cause a trivial shift in the total derivative term. Later when we compare with the 3 brane on S^1, we will identify the complex structure of the torus with the axion-dilaton and the area of the torus will be related to the radius of the compact dimension as given in [82].

4.3 The D-3 brane

Starting with the 10 dimensional IIB three brane action in 10 dimensions [53],[71] we will directly reduce the action on a circle. We have two space-time spinors, θ^α, $\alpha = 1, 2$. These are Majorana, Weyl spinors in 10 dimensions with the same chirality. The natural group acting this index is SL(2,R). In the actions below, following the conventions in [71, 72], we will combine these spinors using the Pauli matrices τ_3 and τ_1. The indices labelling the different spinors will be suppressed (as will the actual spinor indices). We will also take $2\pi\alpha' = 1$. The action (in the Einstein frame) is written:

$$S_3 = -\int d^4\sigma \sqrt{-det(G_{\mu\nu} + e^{-\frac{\phi}{2}}\mathcal{F}_{\mu\nu})} + \int_{M^5} I_5 \tag{4.71}$$

where

$$\mathcal{F} = F - e^{\frac{\phi}{2}} b \tag{4.72}$$

where

$$b = -\bar{\theta}\tau_3\Gamma_m d\theta(dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta) \quad (4.73)$$

and F is the field strength of an abelian vector field A. As before,

$$G_{\mu\nu} = \Pi^m_\mu \Pi^n_\nu g_{mn} \quad (4.74)$$

The Wess-Zumino term is given by:

$$I_5 = \frac{1}{6}d\bar{\theta}\tau_3\tau_1\psi\!\!\!/^3 d\theta + d\theta\tau_1\mathcal{F}\psi\!\!\!/ d\theta = d(C_4 + e^{\frac{-\phi}{2}}C_2 \wedge \mathcal{F}) \quad (4.75)$$

and we may add a term coupling to the axion as follows:

$$I_{td} = \frac{1}{2}C_0 F \wedge F \quad (4.76)$$

We will reduce this action directly implying we will not identify any of the brane coordinates with the compact dimension. Hence, we will write $X^9 = X^9 + 1 = \phi$ and so decompose the background metric $g_{mn} \to g_{mn} \oplus R^2$ where R is the circumference of the compact dimension. That is as before we truncate out the space time Kaluza Klein field. (On the M-theory side this corresponds to truncating the wrapped membrane). Therefore,

$$\Pi^m_\mu = (\Pi^m_\mu, \Pi^9_\mu) \quad (4.77)$$

where $\Pi^9_\mu = \partial_\mu\phi + C'_\mu$ and $C'_\mu = -\bar{\theta}\Gamma^9\partial_\mu\theta$. This gives for the induced world volume metric:

$$G_{\mu\nu} \to G_{\mu\nu} + R^2(\partial_\mu\phi + C'_\mu)(\partial_\nu\phi + C'_\nu) \quad (4.78)$$

The world volume gauge field is left invariant. The NS 2 form

$$b \to b - \bar{\theta}\tau_3\Gamma_9 d\theta(d\phi + \frac{1}{2}\bar{\theta}\Gamma^9 d\theta) \quad (4.79)$$

which we will write as

$$b \to b + b^R \wedge d\phi \quad (4.80)$$

where b^R corresponds to the NS two form reduced to a one form in 9 dimensions. It is this field that a wrapped fundamental string would couple to. The Wess-Zumino part becomes:

$$I_5 = \frac{1}{6}d\bar{\theta}\tau_3\tau_1\psi^3 d\theta + d\theta\tau_1\mathcal{F}\psi d\theta + \frac{1}{2}d\bar{\theta}\tau_3\tau_1\psi^2\chi d\theta + d\bar{\theta}\tau_1\mathcal{F}\chi d\theta \qquad (4.81)$$

where $\chi = (d\phi + C')\Gamma_9$ So the final reduced action for the three brane becomes:

$$\begin{aligned} S_{3,(S^1)} &= -\int d^4\sigma\sqrt{-det(G_{\mu\nu} + e^{\frac{-\phi}{2}}\mathcal{F}_{\mu\nu} - b^R_{[\mu}\partial_{\nu]}\phi + R^2(\partial_\mu\phi + C'_\mu)(\partial_\nu\phi + C'_\nu))} \\ &+ \int_{M^4}\frac{1}{2}C_0 F\wedge F + C_4 + e^{\frac{-\phi}{2}}C_2\wedge\mathcal{F} + R^2(C_3 + C_R\wedge\mathcal{F})\wedge d\phi \qquad (4.82) \end{aligned}$$

We wish to compare the wrapped 5-brane with different choices of v with the 3-brane and its S-dual. The S-dual 3-brane is determined by dualizing the vector field on the brane using the same method as described below for dualizing the scalar field. This has been carried out in [53], hence we simply quote the result:

$$\begin{aligned} S &= -\int d^4\sigma\sqrt{-det(G_{\mu\nu} + \frac{e^{-\frac{\phi}{2}}}{(C_0^2 + e^{-2\phi})}\mathcal{F}_{\mu\nu})} + \int_{M^4} C_{(4)} - C_{(2)}\wedge b \\ &- \frac{1}{2}C_0 e^{\frac{-\phi}{2}} b\wedge b + e^{\frac{\phi}{2}} b\wedge\mathcal{F} - \frac{C_0}{2(C_0^2 + e^{-2\phi})}\mathcal{F}\wedge\mathcal{F} \qquad (4.83) \end{aligned}$$

and $\mathcal{F} = (F + e^{-\frac{\phi}{2}}C_{(2)} + e^{\frac{\phi}{2}}C_0 b)$

The direct reduction would follow as before. The items to note are the, as expected, inversion of the the coupling $\lambda \to \frac{-1}{\lambda}$ where $\lambda = C_0 + ie^{-\phi}$ and the slightly altered form of $\mathcal{F}$ and the Wess Zumino terms.

In order to exactly identify the reduced 3-brane action with the 5-brane wrapped action we will first need to do a world volume duality transformation on the field ϕ. This is in the spirit of [78] whereby world volume dual actions are associated with the M-theory picture of the brane. To do this we follow the techniques of [52, 53, 78] and discussed in chapter one.

We will first deal with the bosonic truncation before moving on to consider the more general case. This gives the standard Dirac Born-Infeld action.

$$S = -\int d^4\sigma\sqrt{-det(G_{\mu\nu} + F_{\mu\nu} + R^2\partial_\mu\phi\partial_\nu\phi)} \tag{4.84}$$

We will dualize the scalar field ϕ by replacing its field strength $d\phi$ with l and then adding an additional constraint term to the action $S_c = H \wedge (d\phi - l)$. H is a lagrange multiplier ensuring that $l = d\phi$. To find the dual we first find the equations of motion for ϕ and solve. This implies $dH = 0$ which means we may locally write $H = dB$. Then we must find the equations of motion for l and solve in terms of H. We simplify the problem by working in the frame in which F is in Jordan form with eigenvalues f_1 and f_2. l_i are the components of l and h_i are the components of the dual of H. The equations of motion for l are:

$$h_1 = \frac{-(1+f_2^2)}{\sqrt{-detM}}l_1R^2 \qquad h_2 = \frac{(1+f_2^2)}{\sqrt{-detM}}l_2R^2 \tag{4.85}$$

$$h_3 = \frac{(1-f_1^2)}{\sqrt{-detM}}l_3R^2 \qquad h_4 = \frac{(1-f_1^2)}{\sqrt{-detM}}l_4R^2 \tag{4.86}$$

where

$$M_{\mu\nu} = G_{\mu\nu} + F_{\mu\nu} + R^2 l_\mu l_\nu \tag{4.87}$$

We then invert these equations to solve for l_i. The solutions are:

$$l_1 = \frac{(f_1^2-1)}{\sqrt{-det\tilde{M}}}\frac{h_1}{R^2} \qquad l_2 = \frac{-(f_1^2-1)}{\sqrt{-det\tilde{M}}}\frac{h_2}{R^2} \tag{4.88}$$

$$l_3 = \frac{(1+f_2^2)}{\sqrt{-det\tilde{M}}}\frac{h_3}{R^2} \qquad l_4 = \frac{(1+f_2^2)}{\sqrt{-det\tilde{M}}}\frac{h_4}{R^2} \tag{4.89}$$

Where

$$\tilde{M}_{\mu\nu} = G_{\mu\nu} + i^*F_{\mu\nu} - \frac{1}{R^2}(^*H)_\mu(^*H)_\nu \tag{4.90}$$

When we substitute these equations into the action we find, reinstating dilaton dependence and the axion term:

$$S_D = -\int d^4\sigma\sqrt{-det\Big(G_{\mu\nu} + ie^{\frac{-\phi}{2}}{}^*F_{\mu\nu} - \frac{1}{R^2}(^*H)_\mu(^*H)_\nu\Big)} + \frac{1}{2}C_0 F\wedge F \tag{4.91}$$

The axion term goes through untouched. Note how the radius which acts as a coupling for the scalar field is inverted in the dual action. We are now in a position to compare the dualized, directly reduced on S^1, IIB D-3 brane action with the double dimensionally reduced on T^2, M5 brane action.

In fact, we shall compare the reduced three brane with the with the vector fields dualized and non dualized with the wrapped 5-brane with the two different gauge choices described above. And so we compare equations 4.91,4.83 with 4.68, 4.69, 4.70 given above.

In doing so must identify the fields and the moduli of the two theories appropriately. When we compare with the usual M-theory predictions given in [82] concerning the relationship between the moduli of the IIB theory in 9 dimensions with the geometrical properties of the torus used in the M-theory compactification we have agreement. The scaling of the metric given in equation 4.67 implies

$$G^B_{\mu\nu} = Area(T^2)^{\frac{1}{2}} G^M_{\mu\nu} \tag{4.92}$$

From both the coefficient in front of F in the determinant and the coefficient in front of the $F\wedge F$ term, we identify the axion-dilaton of the IIB theory (in the 10 dimensional Einstein frame) with the complex structure of the torus.

$$\lambda = C_0 + ie^{-\phi} = \tau \tag{4.93}$$

From comparing the coefficient in front of $^*H$, the radius of the the 10th dimension in IIB becomes:

$$R_B = Area(T^2)^{-\frac{3}{4}} \tag{4.94}$$

Where have identified the gauge field on the reduced 5-brane with the gauge field on the reduced D-3. The dualized scalar on the D-3 brane becomes identified with the three form on the reduced M-5 brane.

We will reinstate the truncated fields and attempt to identify these fields between the dual pictures. The duality transformation now becomes a great deal more complicated; it is essentially the terms involving b^R that prevents us from dualizing the 3-brane action as above. We could however take advantage of the fact that the dualized action ought to be our reduced 5-brane action by carrying out the following consistency check. We can obtain an algebraic expression for H from the equations of motion of L_μ from the reduced three brane. Instead of inverting these equations to obtain an expression for L we may simply insert our expression for H into the reduced 5-brane action and check that this action is the same as the original three brane action. This is essentially the method used in [53] to check the relationship between the 5-brane and 4-brane. This is algebraically extremely involved in this case and does not provide much insight. However, for the case in which the $b^R = 0$ can be dealt with directly. Recall, the integrated Wess-Zumino term:

$$\int_{M^4} C_{(4)} + C_{(2)} \wedge \mathcal{F} + (C_{(3)} + C^R \wedge \mathcal{F}) \wedge d\phi \tag{4.95}$$

With the b^R term vanishing from the determinant in $S_{3,(S^1)}$ we can see that the first term in the action is of the same form as that for the case $\theta = 0$ already considered. As already described, we replace $d\phi$ in the action with a generic one form L and add the constraint $H \wedge (d\phi - L)$. Then integrating out ϕ implies H is closed and we are left with the term $-H \wedge L$. Before we simply integrated out L leaving an action in terms of H. Now we will combine the terms outside the square root that are linear in L as follows:

$$S = -(H - C_{(3)} - C^R \wedge \mathcal{F}) \wedge (L + C') - (H - C_{(3)} - C^R \wedge \mathcal{F}) \wedge C' \tag{4.96}$$

We can now integrate out the combination $L+C'$ which appears in the action in favour of $\mathcal{H} \equiv (H - C_{(3)} - C^R \wedge \mathcal{F})$ using equations 4.89. This gives the following dual action, (reinstating R dependence):

$$S = -\sqrt{-det(G_{\mu\nu} + i^*\mathcal{F}_{\mu\nu} - \frac{1}{R^2}{}^*\mathcal{H}_\mu{}^*\mathcal{H}_\nu)} + C_{(4)} + C_{(2)} \wedge \mathcal{F} - \frac{1}{R}\mathcal{H} \wedge C' \quad (4.97)$$

By comparing 4.97 with 4.64, corresponding to the case $v = dy^2$, we make the following identifications to equate this action with the reduced 5-brane action. Writing IIB fields on the left and M-fields after scaling and converting to orthonormal frame, see 4.58, 4.60, on the right:

$$b_{(4)} = C_{(4)} \qquad b_{(3)} = C_{(3)} \qquad b_{(2)1} = b \qquad b_{(2)2} = C_{(2)} \quad (4.98)$$

$$C^1 = C^R \qquad b_{(1)} = C' \qquad J = H \qquad F = F \quad (4.99)$$

To make these identifications which are very natural we have set $C^2 = 0$ on the 5-brane side, this significantly simplifies the 5-brane action.

For the case $v = dy^1$ we compare with the S dual action 4.83 after reduction and set $C^1 = 0$ on the 5-brane side to make the corresponding simplification required in order to dualize the scalar field. See equations 4.61, 4.69, 4.70 and 4.83. The identifications required to equate this action are the same as above with $C^2 = b^R$. This is a requirement of consistency.

We now wish to consider cases where the duality transformation of the scalar field differs from above because of the interaction term with the b^R field (or C^R in the S-dual case) inside the determinant. Using the technique described above, once we know how the the Dirac Born Infeld part in the brane action transforms under duality we can recover how the full brane action including the Wess-Zumino terms transforms. Hence in what follows we drop the Wess-Zumino terms as the duality transformation to include them follows immediately. (This is essentially because adding terms that are linear in dualizing field does not change the form of the dual action.)

First, we consider the approximation whereby the Born-Infeld term is replaced with a Yang-Mills term. This gives, keeping only the scalar corresponding to compact direction:

$$S = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\mu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mathcal{F}^{\mu\nu}b^R_{[\nu}\partial_{\mu]}\phi - \frac{1}{4}b^R_{[\mu}\partial_{\nu]}\phi b^{R[\nu}\partial^{\mu]}\phi \qquad (4.100)$$

We now dualize ϕ following the same procedure as before to obtain the following dual action:

$$S_D = \frac{1}{(1+(b^R)^2)}\Big(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\mu} - \frac{1}{2}H_\mu H^\mu - \frac{1}{2}b^{R*}_\mu\mathcal{F}^{\mu\nu}b^{R*}_\rho\mathcal{F}^\rho{}_\nu - \frac{1}{2}(H^\mu b^R_\mu)^2 - H_\mu\mathcal{F}^{\mu\nu}b^R_\nu\Big)$$

$$(4.101)$$

Should we make the same approximation to the 5-brane action, ie. replacing the first term by a field strength squared term, we find that we recover directly the above action. Note the peculiar factor $\frac{1}{(1+(b^R)^2)}$ in front of the action which comes in the 5-brane case from the $\frac{1}{v^2}$ factor is a result of dualizing the scalar field in the D3 brane. The final term in the action is identified with $I^{(2)}_{PST}$.

Constructing the dual action directly for the full DBI action 4.82 is difficult as discussed above. However, with the rather specific case of vanishing $\mathcal{F}$ we can construct the dual theory exactly.

And so for the reduced D3 brane, writing out the determinant exactly we have:

$$S_1 = -\int\sqrt{\Big(1+\partial_\mu\phi\partial^\mu\phi - (b^R_\mu\partial^\mu\phi)^2 + (b^R_\mu\partial^\mu\phi)^2\Big)} \qquad (4.102)$$

Adding the usual constraint term and and integrating out ϕ we have the following equations of motion for l_μ:

$$^*H_\mu = \frac{l_\mu(1 - b^R_\mu(b\cdot l) + (b^R)^2)}{\sqrt{1+l^2-(b^R\cdot l)^2+(b^R)^2 l^2}} \qquad (4.103)$$

which we can invert to give an expression for l_μ:

$$l_\mu = \frac{({}^*H_\mu + ({}^*H \cdot b^R) b^R_\mu)}{\sqrt{(1 + (b^R)^2)(1 + (b^R)^2 - H^2 - (b^R \cdot {}^*H)^2)}} \tag{4.104}$$

Inserting this in the action 4.102 provides the dual:

$$S_D = -\int \frac{\sqrt{(1 + (b^R)^2 - {}^*H^2 - ({}^*H \cdot b^R)^2)}}{\sqrt{1 + (b^R)^2}} \tag{4.105}$$

which we may write as follows:

$$S = -\int Q \sqrt{det\Big(G_{\mu\nu} - \frac{{}^*H_\mu {}^*H_\nu}{1 + (b^R)^2 - (b^R \cdot {}^*H)^2}\Big)} \tag{4.106}$$

where

$$Q = \sqrt{\frac{1 + (b^R)^2 - (b^R \cdot {}^*H)^2}{1 + (b^R)^2}} \tag{4.107}$$

This is identical to the reduced 5-brane action with $\mathcal{F}$ set to zero, see equation 3.70, once we make the following identifications:

$$P_\mu = {}^*H_\mu \qquad C_{2\mu} = b^R_\mu \tag{4.108}$$

This again is consistent with 4.99.

4.4 Conclusions

We have shown that the action 4.18 for the M theory five brane, under double dimensional reduction on a torus produces the self-dual three brane of IIB directly reduced on a circle. The S-duality of IIB becomes transparent as the modular symmetry of the torus. The different gauge choices for $v \in H^1(T^2)$ correspond to different S-dual formulations of the 3-brane. The identification of the moduli and the fields of the two theories has been shown to be in agreement with work considering the ambient supergravity [81] and the identification of the string with the partially wrapped membrane [82]. In order to make this identification it was necessary to dualize the scalar corresponding to fluctuations in the compact direction. This duality transformation acts non-trivially on the action. In fact, in the most general case the dual action is extremely difficult to construct explicitly; even proving the equivalence with the reduced 5-brane which ought to be an algebraic exercise proves to be difficult due to the complexity of the duality transformation. However, by making approximations to the Born-Infeld part or by truncating fields we explicitly construct dual actions to the reduced three brane in these cases. It should be noted that the results are essentially classical and with a very specific choice of world volume topology for the 5-brane, hence we do not encounter the problems reported in [22, 23].

Recently, there has been an attempt to rewrite the 5-brane action with an auxiliary metric as one does for the string so as to make the action linear [89]. This essentially shifts the complexity of the action into the equations of motion for the auxiliary metric. Again the duality transformation becomes difficult to implement exactly.

One of the aspects not explored explicitly in this chapter is the role which the five brane may have in a reformulation of the three brane in which the S-duality of IIB is manifest, as reported in the recent work [90, 91]. In [35], by taking v to be a one form in M^4 instead of T^2 an action was produced that has the S-duality manifest [34],[37],[38].(Recently this has been explored in detail for the five brane, [87]. The disadvantage with this approach is that the Lorentz invariance is then not manifest. It is not clear if a connection can be made between these two approaches. It would be interesting if one could give some physical interpretation to the auxiliary field v which plays a crucial role in encoding the self-duality condition in the action. We remark that other relevant work regarding the five brane in an action formulation and its relationship to duality is given in [92],[93], [94].

Chapter 5

Conclusions and further speculations

This thesis has attempted to demonstrate the role of self dual, also known as chiral, gauge theories in duality symmetries. The dimensional reduction of these theories provides a geometric interpretation of S-duality that allows a Kaluza-Klein type understanding of the duality symmetry. From this perspective, the duality arises from the geometric properties of the compact space. This has been explored for both two form and four form self dual theories on a variety of spaces. This has also been applied to non-linear Born-Infeld type theories and in doing so has inspired a reformulation of the Born-Infeld theory to allow the duality symmetry to appear *manifestly* in the action.

Perhaps the most natural application is to the M-theory, IIB string theory relationship. The IIB self-dual D-3 brane is shown to emerge after double dimensional reduction (wrapping) of the M-theory five brane and a world volume dualization of the two form field. The S-duality of the IIB theory is now a consequence of the modular symmetry of the torus. The relationship between the PST formulation of the five brane and the different S-dual descriptions of the three brane is discussed in detail along with the necessary world volume dualization of the full non-linear, supersymmetric theory.

It would be interesting to consider a more complicated surface other than a simple torus on which to wrap the five brane. In particualar, it has been shown that the effective theory of N=2 super Yang-Mills arises from the five brane-truncated to quadratic order in field strengths- and wrapped on a Riemann surface [95, 96]. This Riemann surface may then be related to the Seiberg-Witten curve of the corresponding gauge theory. So far this has not been derived in an action formalism of the five brane. One of the reasons for this is that to a generic Riemann surface does not meet the requiste topological restrictions. One way round this might be to choose v to lie in a space-time direction and then derive a duality manifest version of super Yang-Mills from which one would have to perform the sort of elimination carried out in chapter 3, section 3.2 .

In this discussion there has been no attempt made to obtain results valid at the quantum level. As such the duality symmetry obtained is only true classically and may not be preserved by the partition function. In fact as has been shown in [22, 23] and briefly mentioned in the introductory chapter, the duality symmetry for Yang-Mills theory is not a symmetry of the partition function once global properties are taken into account. In the discussion presented here it was shown that often global properties are necessary in formulating self-dual theories. One example of this would be the requirement of a non-vanishing first Betti number. In fact it is well known that partition functions for self-dual gauge theories are particularly problematic [22, 23]. Recently, a five-brane partition function has been presented with the required modular properties [97], though the approach there does not use any action for the five brane. Elsewhere [98] there have been attempts to study the diffeomorphism anomaly that arises in eleven dimensions as a result of the presence of the five brane, the anomaly could only be resolved with certain topological restrictions, (such as the existence of a global vector field on the 5-brane submanifold). This is suggestive of the connection between the problems of the diffeomorphism anomaly induced by the five brane and the difficulty in forming a partition function from an action describing a self-dual gauge field. Related ideas are discussed in [99].

It has demonstrated how classical S-duality of the D-3 brane arises from the dimensional reduction of the M-5 brane. As the D-3 brane action after suitable quadratic approximation is meant to be N=4 super Yang-Mills one expects that the D-3 brane partition function (though it is not entirely clear what such a partition function means for a D-brane) will transform under S-duality as a modular form whose weights are dependent on the global properties of the three brane. This leads to the natural speculation that the lack of three brane partition function modular invariance is related to the diffeomorphism anomaly of the five brane. One might imagine that there is some term that has been missed from the three brane action that would cancel the S-duality anomaly and restore the S-duality to a full quantum symmetry for the 3 brane. Such a term might be related to terms required to cancel the 5-brane anomaly. This is highly speculative and is intended to be the subject of future work.

Appendix A

Differential forms

We present our conventions for forms as follows.

A p-form F on a d-dimensioal manifold, M^d. Is given by:

$$F = \frac{1}{p!} F_{\mu_1 \mu_2 .. \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdot \cdot \wedge dx^{\mu_p} \tag{A.1}$$

Where $\{dx^\mu\}$ are a set of basis one forms on the manifold M^d and $\{\vec{x}_\mu\}$ are a set of dual basis vectors. Note, $\vec{x}_\mu dx^\nu = \delta^\nu_\mu$.

We denote the space of p-forms on M^d by $\Lambda^p(M^d)$

The wedge product between basis one forms is defined as follows:

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \tag{A.2}$$

and in general for $A \in \Lambda^p(M^d)$ and $B \in \Lambda^q(M^d)$:

$$A \wedge B = \frac{1}{p!q!} A_{[\mu_1 .. \mu_p} B_{\nu_1 .. \nu_q]} dx^{\mu_1} \wedge .. \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge .. \wedge dx^{\nu_q} \tag{A.3}$$

where square brackets denote complete antisymmetrisation.

That is

$$A_{[\mu} B_{\nu]} = \frac{1}{2}(A_\mu B_\nu - B_\nu A_\mu) \tag{A.4}$$

$$A_{[\mu}B_\nu C_{\rho]} = \frac{1}{3!}(A_\mu B_\nu C_\rho - A_\mu B_\rho C_\nu - A_\nu B_\mu C_\rho + A_\nu B_\rho C_\nu + A_\rho B_\mu C_\nu - A_\rho B_\nu C_\mu) \tag{A.5}$$

etc.

The exterior derivative, acts as follows:

$$dF = \frac{1}{p!}\partial_{[\mu_1} F_{\mu_2\mu_3..\mu_{p+1}]} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdot\cdot \wedge dx^{\mu_{p+1}} \tag{A.6}$$

The inner product on a p-form F is a contraction with a vector $\vec{v}$ is denoted as follows:

$$i_v F = \frac{1}{(p-1)!} v^\mu F_{\mu\nu_1..\nu_{p-1}} dx^{\nu_1} \wedge dx^{\nu_2} \wedge .. \wedge dx^{\nu_{p-1}} \tag{A.7}$$

where $\vec{v} = v^\mu \vec{x}_\mu$

We take the totally antisymmetric tensor to be defined with indices up.

This means the Hodge dual is defined as follows:

$$({}^*F)_{\mu_1\mu_2..\mu_p} = \frac{1}{(d-p)!} g_{\mu_1\nu_1} g_{\mu_2\nu_2} .. g_{\mu_p\nu_p} \epsilon^{\nu_1\nu_2..\nu_p\nu_{p+1}..\nu_d} F_{\nu_{p+1}\nu_{p+2}..\nu_d} \frac{1}{|g|^{\frac{1}{2}}} \tag{A.8}$$

where $g_{\mu\nu}$ is the metric on M^d and $g = det g_{\mu\nu}$

Remark that

$$^{**}F = (-1)^{p(d-1)+s} F \tag{A.9}$$

where M^d has signature s.

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