On the integrability of the sine-Gordon system

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Preface

This thesis is derived from research done by the author between November 1993 and January 1997 at the University of Durham, financially supported by an Engineering and Physical Sciences Research Council studentship. No part of it has been submitted previously for any degree at any university.

In chapter 1 the sine-Gordon system of nonlinear partial differential equations is introduced. For this system and with $N \in \mathbb{Z}$, $P, Q \in \mathbb{R}$ the sets of initial-boundary value problems $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ are defined. No claim of originality is made for any of the material concerning the set $\mathbf{A}_N$. However, it is believed that all the results regarding the set $\mathbf{B}_{P,Q}$ are original to this thesis. The starting point for this whole investigation was the author's paper [1].

Acknowledgements

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Abstract

This thesis investigates the integrability of the sine-Gordon system of nonlinear partial differential equations when the dependent variables are subject to some very particular boundary conditions. In chapter 1 the sine-Gordon system is introduced and, with $N \in \mathbb{Z}$, $P, Q \in \mathbb{R}$, the sets of initial-boundary value problems $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ are defined. In the set $\mathbf{A}_N$ the spatial variable x is unbounded and the boundary conditions are fixed by initially choosing the topological charge N. This set of problems is the one usually associated with the sine-Gordon system. In the set $\mathbf{B}_{P,Q}$ the spatial coordinate is constrained to the semi-line $(-\infty, 0]$ and there exists two boundary parameters $P, Q \in \mathbb{R}$ to be chosen a priori. It is the study of this second set of initial-boundary value problems for arbitrary P, Q which forms *all* the original work of this dissertation.

The study presented here is primarily concerned with the development of *three* separate inverse scattering methods for solving these sets of initial-boundary value problems. The first of these is developed in chapter 3 and is applicable to a subset of the problems in $\mathbf{A}_N$. The method is the one usually associated with the sine-Gordon system and studies the asymptotics of the initial data as $x \to \pm\infty$. It is included in this thesis for completeness and as background for the original material which follows. Next, in chapters 4 and 5, the inverse scattering methods appropriate to initial-boundary value problems in subsets of $\mathbf{B}_{P,0}$ and $\mathbf{B}_{P,Q\neq 0}$ are constructed. In these cases it is important to realise that it is only possible to study the asymptotics of the initial data as $x \to -\infty$.

Once these three methods have been formulated they are used to find soliton solutions and infinite sets of integrals of motion for these boundary value problems. When a boundary is present at $x = 0$ the interaction of the solitons with this boundary is

studied. These topics are addressed in chapter 6. Finally in chapter 7 the question of the integrability of both sets of problems is addressed. By interpreting the various inverse scattering methods in terms of canonical coordinate transformations of phase space it is seen that the existence of such methods can be viewed as a constructive proof of the integrability of these boundary value problems.

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Chapter 1

Introduction

1.1 Subject overview

This thesis concerns the theory of integrable nonlinear partial differential evolution equations for variables dependent on two spacetime coordinates $(x, t) \in \mathrm{D} \subset \mathbb{R}^2$. Such systems are often called $1+1$ dimensional integrable field theories in the theoretical physics literature.

One of the main techniques for solving initial-boundary value problems for these equations is the inverse scattering method. This involves a nonlinear change of variables (phase space coordinates) which is invertible and which makes the equation(s) linear and explicitly solvable. It is the existence of such a transformation that prompts the use of the term 'integrable'. Many of the systems solved in this way possess a Hamiltonian structure. That is, the equations defining them are infinite-dimensional analogues of Hamilton's equations in classical mechanics. In such cases the inverse scattering method can be interpreted as a canonical transformation with respect to this structure so that the variables which linearise the system have the meaning of action-angle variables.

Since its inception in the pioneering work [2], the inverse scattering method has been used to solve many different integrable equations of this type. Examples of these are the well known Korteweg-de Vries, nonlinear Schrödinger and sine-Gordon systems. Originally the method was developed to solve problems for which phase space is related to the functional class of 'rapidly decreasing' $\mathbb{C}$ valued functions. In this case one is able to calculate the multi-soliton solutions known to applied mathematicians for many years. The original solutions to initial-boundary value problems of this type for the Korteweg-de Vries, nonlinear Schrödinger and sine-Gordon systems are to be found in [2, 3, 4] respectively.

Then, approximately 20 years ago, the method was successfully modified to accommodate phase spaces related to the functional class of (quasi)periodic $\mathbb{C}$ valued functions. The resulting theory was termed finite gap integration [5] and using this the analogue of soliton solutions were calculated. As opposed to the solitons, however, these solutions are expressed in terms of theta functions common in algebraic geometry, and so were given the name finite gap or algebro-geometric solutions.

In addition to the many explicit solutions calculated for these equations, the inverse scattering method provided the key to a construction of a complete set of action-angle variables for such systems. In many cases, however, this construction required a truncation of the original phase space to a space defined in terms of the scattering data of the problem. This will be explained in more detail in subsequent chapters.

The integrability of any of these equations when subject to specific 'local' boundary conditions is a much more open question and has received considerable attention in the mathematical literature over the past decade. One would not expect any set of boundary conditions to preserve integrability and the question of how to categorise those which do is an interesting question which has only been partially answered. The study of this problem was initiated in [6] which contained important steps in fitting such 'integrable boundary conditions' into the **r**-matrix (Hamiltonian) picture of the inverse scattering method. However, this work provided no clue as to how to modify the theory so as to explicitly solve the resulting problems. Such a modification was first developed for systems defined on the semi-infinite interval, $x \in (-\infty, 0]$ with the dependent variable(s) decreasing 'rapidly' as $x \to -\infty$ and satisfying an integrable boundary condition at $x = 0$. These are related to the case when $x \in \mathbb{R}$ with the dependent variables rapidly decreasing as $|x| \to \infty$. Therefore it is not surprising that there exists multi-soliton solutions to these problems. The literature developing this semi-infinite inverse scattering method is extensive but a few key references are [7].

The situation when space is finite eg $x \in [-1, 0]$ with the dependent variable(s) satisfying an integrable boundary condition at each end is far less clear. Once again the original results [6] are useful but the calculation of explicit solutions to the problems requires far more work. Only in the last five years has progress been made for a small number of systems, specifically the nonlinear Schrödinger equation and the sine-Gordon system. These were solved by appropriately adapting the finite gap integration theory for the (quasi)periodic problem to the particular system at hand. To do this similar ideas to those for semi-infinite boundary value problems were used [8].

Despite such progress, the analysis of integrable partial differential equations on a semi-infinite or finite spatial interval is at an early stage and much work has still to be done. Besides a thorough development of inverse scattering techniques to these problems it remains to investigate how certain conjectured properties of integrable systems (eg the Painlevé tests) must be modified when boundaries are present.

Following this general discussion of integrability and the inverse scattering method, the rest of the thesis will concentrate on the study of two particular types of boundary value problems for the sine-Gordon system. The reasons for this particular specialisation are two-fold. In the first instance the problems of Type $\mathbf{A}_N$ have a long history of applications to many branches of physics. Secondly, over the last few years many applications of the problems of Type $\mathbf{B}_{P,Q}$ to condensed matter systems with boundaries and impurities have been recognised and theoretical physicists are currently studying the quantum field theory associated with this set of problems.

A subset of the problems of Type $\mathbf{A}_N$ are known to be integrable and a thorough analysis of these was carried out more than 20 years ago [4]. However the situation is far less clear for problems of Type $\mathbf{B}_{P,Q}$ and only partial results have been obtained for these systems. In the following sections the two sets of problems are stated in full and some of the known results regarding them are given. From this information an open problem regarding the integrability of the set $\mathbf{B}_{P,Q}$ is identified for further investigation.

1.2 Initial-boundary value problems of Type $\mathbf{A}_N$

This set of problems is the one usually associated with the sine-Gordon system. There exists an extensive literature discussing the underlying mathematics of these and their applications to physics. A precise statement of the problems is given by the definitions below. In addition some comments regarding them are made which should be kept in mind during subsequent chapters.

Notation 1.1 *For $a, b \in \mathbb{R}$ let $\mathcal{S}(\mathbb{R}, a, b)$ be the set of smooth functions $f : \mathbb{R} \to \mathbb{R}$ which have the asymptotics*

$$\lim_{x \to +\infty} f(x) = b \quad , \quad \lim_{x \to -\infty} f(x) = a \qquad \textit{rapidly in } x. \tag{1.2.1}$$

The term 'rapidly in x' is to indicate that all the spatial derivatives of f decay faster than any power of x as $x \to \pm\infty$. In addition let $\mathcal{S}'(\mathbb{R}, a, a) \subset \mathcal{S}(\mathbb{R}, a, a)$ denote the subset of functions which satisfy the involution $f(x) = f(-x)$ $\forall x \in \mathbb{R}$. Finally let $\mathcal{M}_N \stackrel{\text{def}}{=} \mathcal{S}(\mathbb{R}, 0, 2\pi N) \times \mathcal{S}(\mathbb{R}, 0, 0)$ and $\mathcal{M}'_0 \stackrel{\text{def}}{=} \mathcal{S}'(\mathbb{R}, 0, 0) \times \mathcal{S}'(\mathbb{R}, 0, 0)$.

With this notation it is possible to define a set of initial-boundary value problems for the sine-Gordon system which are parameterised by elements of the *phase space* $\mathcal{M}_N$. The study of these problems forms a major part of this thesis.

Definition 1.2 *With $N \in \mathbb{Z}$, an 'initial-boundary value problem of Type $\mathbf{A}_N$' is the problem of determining the functions φ, $\varpi : (x, t) \mapsto \mathbb{R}$ with $(x, t) \in \mathbb{R}^2$ which satisfy:*

- *the sine-Gordon system*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varpi, \\ \frac{\partial \varpi}{\partial t} &= \frac{\partial^2 \varphi}{\partial x^2} - \sin \varphi \qquad \forall\, x, t \in \mathbb{R}. \end{aligned} \tag{1.2.2}$$

- *the boundary conditions*

$$(\varphi(\cdot, t), \varpi(\cdot, t)) \in \mathcal{M}_N \qquad \forall t \in \mathbb{R}. \tag{1.2.3}$$

- *the 'initial' conditions*

$$(\varphi(\cdot, 0), \varpi(\cdot, 0) = (\varphi_N, \varpi_0) \in \mathcal{M}_N. \tag{1.2.4}$$

From this definition it is clear that specifying the initial data $(\varphi_N, \varpi_0) \in \mathcal{M}_N$ for some $N \in \mathbb{Z}$ is equivalent to specifying a particular problem of Type $\mathbf{A}_N$.

There are a couple of technical remarks which need to be made regarding these initial-boundary value problems.

- The question of existence and uniqueness of the solution to arbitrary problems of Type $\mathbf{A}_N$ remains unanswered throughout this thesis. That is, for arbitrary $(\varphi_N, \varpi_0) \in \mathcal{M}_N$, does there exist a unique solution to the problem of Type $\mathbf{A}_N$ which satisfies (1.2.4).

- It may be the case that the boundary conditions in definition 1.2 are redundant, i.e once an element of $\mathcal{M}_N$ is chosen as initial data then the dynamics of the problem may leave this space invariant and the imposition of boundary conditions is not needed.

Such technicalities are not really in the spirit of this thesis. However, regarding the first of these comments, since one of the aims of this thesis is to explicitly solve problems of this Type it will be assumed that all such problems possess a unique solution. For a subset of these problems, to be introduced in chapter 3, it will be seen that the assumption is indeed valid.

1.2.1 Physical applications and quantum field theory

A detailed discussion of the many applications the set $\mathbf{A}_N$ have to physics would take many pages and is beyond the scope of this thesis. However, a few key applications are to the modelling of Josephson junctions in condensed matter physics [9], optical pulse propagation through a dielectric medium [10] and elementary particles [11]. Excellent background reading regarding these applications is [12].

The interaction of elementary particles is described by quantum field theory. The quantum field theory associated with the set $\bigcup_{N\in\mathbb{Z}} \mathbf{A}_N$ has many very interesting properties and remains an active area of research to this day. Probably the most intriguing result regarding this quantum field theory is its strong-weak coupling 'duality' with the massive Thirring model of Majorana fermions [11]. In addition the model is 'quantum integrable'. By definition this means that there exists an infinite number of operators in the theory which mutually commute with one another and with the quantum Hamiltonian. This infinite family constrains the system sufficiently so that the exact solution (in the sense of quantum field theory) can be conjectured using factorised S-matrix/bootstrap/form factor techniques [13].

1.2.2 Inverse scattering theory for problems of Type $\mathbf{A}_N$

The inverse scattering theory for a subset of the problems of Type $\mathbf{A}_N$ was formulated in [4] the subset being defined by a reduced phase space $\check{\mathcal{M}}_N \subset \mathcal{M}_N$. When restricted to this phase space, the inverse scattering method becomes an *invertible* transformation of 'coordinates' and, in terms of the transformed set, the time evolution is easily determined.

From this solution emerged an interpretation of the excitation spectrum in terms of relativistic field theory and it was the semiclassical quantisation of this analysis that led to the concept of an integrable quantum field theory.

The inverse scattering method for solving this subset of problems is explained in detail in chapter 3. This explanation follows closely that given in [14].

1.3 Initial-boundary value problems of Type $\mathbf{B}_{P,Q}$

In this section a set of initial-boundary value problems for the sine-Gordon system with $x \in (-\infty, 0]$ is defined. All the subsequent analysis concerning these problems is original to this thesis.

Notation 1.3 *For $P, Q \in \mathbb{R}$ and denote by (f, g) a pair of smooth functions f, g : $(-\infty, 0] \to \mathbb{R}$ which satisfy the boundary conditions*

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} + P \sin\frac{f(0)}{2} - Q\cos\frac{f(0)}{2} = 0,$$

$$\left.\frac{\partial g}{\partial x}\right|_{x=0} + \frac{P}{2} g(0)\cos\frac{f(0)}{2} + \frac{Q}{2} g(0)\sin\frac{f(0)}{2} = 0,$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = 0 \qquad \textit{rapidly in } x.$$

Let $\mathcal{N}_{P,Q}$ denote the space of such pairs of functions.

It is now possible to define the set of initial-boundary value problems for the sine-Gordon system which are parameterised by elements of the phase space $\mathcal{N}_{P,Q}$.

Definition 1.4 *With $P, Q \in \mathbb{R}$ an 'initial-boundary value problem of Type $\mathbf{B}_{P,Q}$' is the problem of determining the functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ with $(x,t) \in (-\infty, 0] \times \mathbb{R}$ which satisfy:*

- *the sine-Gordon system*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varpi, \\ \frac{\partial \varpi}{\partial t} &= \frac{\partial^2 \varphi}{\partial x^2} - \sin \varphi \qquad \forall\, (x,t) \in (-\infty, 0] \times \mathbb{R}. \end{aligned} \tag{1.3.1}$$

- *the boundary conditions*

$$(\varphi(\cdot, t), \varpi(\cdot, t)) \in \mathcal{N}_{P,Q} \qquad \forall t \in \mathbb{R}. \tag{1.3.2}$$

- *the 'initial' conditions*

$$(\varphi(\cdot, 0), \varpi(\cdot, 0)) = (\varphi_{P,Q}, \varpi_{P,Q}) \in \mathcal{N}_{P,Q}. \tag{1.3.3}$$

Once again it is clear that specifying the initial data $(\varphi_{P,Q}, \varpi_{P,Q})$ for some $P, Q \in \mathbb{R}$ is equivalent to specifying a particular problem of Type $\mathbf{B}_{P,Q}$.

Regarding the existence and uniqueness of solutions to problems of this Type, the same comments as those made for problems of Type $\mathbf{A}_N$ can also be made here. The same assumptions as made in section 1.2 will also be made for the problems $\mathbf{B}_{P,Q}$. However, it will be seen in chapters 4 and 5 that, for a subset of the problems of this Type, the assumption that a unique solution exists is indeed a valid one.

1.3.1 On the integrability of the set $\mathbf{B}_{P,Q}$

It was in the theoretical physics literature that the question of the integrability of the set $\mathbf{B}_{P,Q}$ was first considered. In [15] the first nontrivial integral of motion (i.e. not the Hamiltonian) was constructed. As a consequence of this, the system was *conjectured* to possess an infinity of such integrals an infinite subset of which then survive quantisation. In short, the quantum theory associated to the set $\mathbf{B}_{P,Q}$ was assumed to be quantum integrable.

In addition to this result and conjecture Ghoshal and Zamolodchikov [15] (following the original idea of Cherednik [16]) went on to develop S-matrix / bootstrap ideas for systems with a single spatial boundary by exploiting the factorisation of scattering when space is semi-infinite. (It is unclear whether or not analogous ideas can be developed when space is finite although this is commonly *assumed* when dealing with periodic systems). Having developed these techniques Ghoshal and Zamolodchikov went on to apply them to the conjectured integrable quantum field theory associated to $\mathbf{B}_{P,Q}$ and calculated so-called reflection matrices for the interaction of the quantum solitons with the $x = 0$ boundary. With these results the concept of an *integrable boundary quantum field theory* was introduced as a new object of study for integrable field theorists and as a result the investigation of such systems has recently become quite fashionable.

The conjecture of the classical integrability of the problems was further supported in [17] where a particular element of $\mathbf{B}_{P,Q}$ was solved. This 'kink' solution was found using Hirota's method and allowed the calculation of a 'time delay' experienced by a kink when it interacts with the boundary. A semiclassical quantisation of these results was found to agree with the full quantum results of [15].

This evidence certainly seems compelling and to many researchers in the field the existence of a nontrivial 'higher' conservation law or a soliton solution to a particular problem would count as a proof of the integrability of the whole set. This is not the point of view taken in this thesis and it is only through the development of an inverse scattering method for (a subset of) such problems that their integrability can rigorously be proved.

1.3.2 Inverse scattering theory for problems in $\mathbf{B}_{P,0}$, $\mathbf{B}_{0,Q}$

Subsets of the problems with P and/or Q equal to zero have already been studied in the mathematical literature. The set $\mathbf{B}_{P,0}$ was first considered in [6] where the **r**-

matrix approach to the inverse scattering method was used to construct an infinite number of involutive integrals of motion. Following this the inverse scattering method was modified to solve a subset of the problems of Types $\mathbf{B}_{P,0}$ and $\mathbf{B}_{0,Q}$. The required modifications were deduced by using 'gauge transformations' to constrain solutions to problems of Type $\mathbf{A}_N$ [7]. These ideas will be explained in detail in the subsequent chapters.

1.4 An open question requiring further study

It is clear from the section 1.3 that much work has been done and many results obtained for problems of Type $\mathbf{B}_{P,Q}$. Much of this is due to the original paper [15] which, as was stressed earlier, made the *assumption* of the integrability of this set of problems. Meanwhile, as explained in subsection 1.3.2, there exist well developed techniques for the incorporation of $\mathbf{B}_{P,0}$, $\mathbf{B}_{0,Q}$ into the inverse scattering method. Putting these results together it is natural to ask the question:

For general P, $Q \in \mathbb{R}$ can an inverse scattering method be developed for solving problems of Type $\mathbf{B}_{P,Q}$ thus proving that that such a set form an integrable system ?

It is the study of this problem that forms all the original work contains in this thesis.

1.5 Motivation and objectives

This section is of a more subjective nature. It is here that I want to explain my motivation for studying the problem detailed in the previous section for a PhD thesis and the objectives I wished to achieve as a result.

In the first instance, I was determined to do research in the broad field known as mathematical physics and to achieve a reasonable degree of mathematical rigour as regards my results. Also, I wished to work in an area which draws upon results from many different disciplines of mathematics. Thus the field of integrable systems in general and the inverse scattering method in particular was a very natural place to look for some open problems. The underlying theory of this method calls upon results from analysis, algebra, geometry and physics and it is the way these ideas come together to solve concrete equations in applied mathematics that attracted me to the study of inverse scattering theory.

Then, just prior to me starting my research, the paper [15] appeared and, as a result, the subject of integrable boundary field theories was becoming fashionable amongst the theoretical physics community. In [15] the relationship between boundary conditions and integrability was raised as an interesting and open problem. In addition the integrability of the set $\mathbf{B}_{P,Q}$ was conjectured.

At the same time I read an interview article with Professor Stephen Smale where he advised PhD students to study a narrow subject in depth rather than a broad one without understanding. This comment and the conjectured integrability of $\mathbf{B}_{P,Q}$ became fixed in my mind and when I discovered that there existed some background material on questions of this type to get me started I had found the subject of my thesis.

Why concentrate on these particular problems ? The question of their integrability and the inclusion of such systems into the scheme of inverse scattering is a very neat and concise one. Also, this set of problems have been shown to have applications to condensed matter physics and as a result are currently a fashionable area of research.

The main objective of my PhD research into the mathematical content of such problems was to develop my own understanding of the concept of integrability. When considering mechanical models (no dependence on x) there is a specific definition of an integrable Hamiltonian system and there exist theorems about the properties of these. However, no such definitions and theorems exist when considering problems depending on x (partial differential equations) and to some extent an integrable system of this type means different things to different people. This is a less than ideal situation for a mathematical term to find itself in. As a result I was determined to fix my own definition of integrability for these systems and to investigate to what extent some particular problems were able to meet this criterion.

With these ideas and objectives in place it was natural to turn my attention to the inverse scattering method which, at the present time, provides the best setting in which to address such problems.

1.6 Thesis outline

The study of the sets $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ occupy chapters 2-7 of this thesis. Chapter 8 poses some interesting problems which require further investigation. All the results concerning the set $\mathbf{B}_{P,Q}$ are original to this thesis.

The contents of each of the eight chapters is as follows:

1. This chapter has been an introduction to some of the literature and ideas with which this thesis is concerned. Initial-boundary value problems of Types $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ have been defined and an open question regarding the latter set has been identified. This question provides the main direction of research for this thesis.

2. In this chapter the problems of Types $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ are 'linearised' and the resulting linear systems are solved by the Fourier transform method. The main ideas underlying this solution are important for later chapters when developing the inverse scattering method for solving the full nonlinear problems. Also, the relationship between the two sets of linear problems should be kept in mind to help with the subsequent analysis.

3. The inverse scattering method is developed for solving a subset of the problems of Type $\mathbf{A}_N$ - this subset being defined by a restricted phase space $\check{\mathcal{M}}_N \subset \mathcal{M}_N$. The whole chapter mirrors very closely the formulation presented in [14].

4. The inverse scattering method developed in chapter 3 is modified to solve a subset of the problems of Type $\mathbf{B}_{P,0}$. Once again this subset is defined by a restricted phase space $\check{\mathcal{N}}_{P,0} \subset \mathcal{N}_{P,0}$. The analysis leading to this modification is presented in some detail. It is not particularly elegant but along with the similar results of chapter 5, it forms the backbone of the original work contained in this thesis.

5. This chapter mirrors chapter 4 but instead considers the set of problems $\mathbf{B}_{P,Q\neq 0}$ Only the main results are presented as the working follows that presented in the previous chapter.

6. The inverse scattering method formulated in chapter 3 and its subsequent modifications in chapters 4 and 5 are used to solve some particular problems of Type $\mathbf{A}_0$, $\mathbf{A}_{\pm 1}$, $\mathbf{B}_{P,0}$, $\mathbf{B}_{P,Q\neq 0}$. Some of the properties of these 'soliton' solutions are studied and compared with the results of [17].

7. This chapter introduces the reader to the theory of finite dimensional integrable systems. A rigorous definition of 'integrability' is developed and this is formally extended to the infinite dimensional Hamiltonian systems $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$. The direct and inverse scattering transforms, (two of the three stages in the inverse scattering method), can then be interpreted as coordinate transformations for the phase spaces $\check{\mathcal{M}}_N$, $\check{\mathcal{N}}_{P,Q}$. In terms of the new coordinates (scattering data) the initial-boundary value problems can be solved explicitly so that the sets of problems defined by these phase spaces form integrable systems. Following this, an infinite set of integrals of motion for these systems are constructed and a brief discussion of the 'action-angle' coordinates for the space $\check{\mathcal{M}}_N$ is given.

8. Some interesting open problems regarding the sine-Gordon system are discussed.

To conclude, it is the intention of this thesis to give a detailed description of the current state of inverse scattering theory for the sine-Gordon system. Therefore it should be mentioned that much of the analysis in chapters 2-7 is of a highly technical nature. *This thesis is not an easy read.*

Chapter 2

The method of Fourier transforms for solving the linearised problems

2.1 Introduction

The inverse scattering method can be viewed as a nonlinear extension to the method of Fourier transforms for solving initial-boundary value problems for linear partial differential equations. Indeed, when a nonlinear equation solvable by inverse scattering method has a 'coupling' parameter in front of its nonlinear part, it reduces to the method of Fourier transforms as this parameter approaches zero.

In order to develop the ideas of chapters 3,4 and 5 it will be helpful to consider the Fourier transform method applied to the linearised forms of the initial-boundary value problems of Types $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$. By doing this many of the ideas of the full inverse scattering method for solving the nonlinear problems can be seen much more clearly and the difficulties in certain aspects of its formulation can be identified. Also, the way the inverse scattering method must be modified in order to solve the nonlinear initial-boundary value problems of Type $\mathbf{B}_{P,Q}$ is made more transparent by first considering the relationship between the linearised forms of these problems and the linearisations of those of Type $\mathbf{A}_N$.

2.2 The classical Fourier transform

Definition 2.1 *A function $f : \mathbb{R} \to \mathbb{C}$ is called absolutely integrable if $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.*

Definition 2.2 *For any absolutely integrable function f, define its Fourier transform $\tilde{f}$ by*

$$\tilde{f}(k) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx. \tag{2.2.1}$$

Definition 2.3 *A sequence of functions (f_n) is said to converge uniformly to F on an interval I if for each $\epsilon > 0$ there exists a number N depending on ϵ but not on x, such that $|f_n(x) - F(x)| < \epsilon$ for all $n > N$ and all $x \in I$.*

Lemma 2.4 *For an absolutely integrable function f let*

$$\tilde{f}_M(k) \stackrel{\text{def}}{=} \int_{-M}^{M} f(x)e^{-ikx}\,dx,$$

then $\tilde{f}_M(k)$ converges uniformly to $\tilde{f}(k)$.

Definition 2.5 *A function $f : \mathbb{R} \to \mathbb{C}$ is called piecewise smooth if all its derivatives exist and are continuous except (possibly) at a set of points $x_1, x_2, \ldots$ such that any finite interval contains only a finite number of the x_i, and if the function and all its derivatives have, at worst, finite jump discontinuities there.*

Remark 2.6 *All elements of $\mathcal{S}(\mathbb{R}; 0, 0)$ are absolutely integrable and piecewise smooth.*

Theorem 2.7 (Inversion Theorem) *If f is absolutely integrable, continuous, and piecewise smooth, then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dx. \tag{2.2.2}$$

Notation 2.8 *For $a, b \in \mathbb{C}$ let $\mathcal{T}(\mathbb{R}; a, b)$ be the set of smooth complex valued functions f of a single real variable which have the asymptotics*

$$\lim_{x\to+\infty} f(x) = b \quad , \quad \lim_{x\to-\infty} f(x) = a \qquad \textit{rapidly in } x. \tag{2.2.3}$$

In addition let $\tilde{\mathcal{T}}(\mathbb{R}; a, \bar{a}) \subset \mathcal{T}(\mathbb{R}; a, \bar{a})$ be the subset of functions which satisfy the involution $\bar{f}(x) = f(-x)\ \forall x \in \mathbb{R}$.

Proposition 2.9 *According to definition 2.2 and remark 2.6, if $f \in \mathcal{S}(\mathbb{R}; 0, 0)$ then $\tilde{f} \in \tilde{\mathcal{T}}(\mathbb{R}; 0, 0)$.*

The proofs of many of these results are to be found in [18] and the references therein.

2.3 Solving the linearised form of the set A_0 using Fourier transforms

The sine-Gordon system as it appears in definition 1.2 does not possess a coupling constant in front of its nonlinear part. This is also the case for the Korteweg-de Vries equation when written in its standard form [19]. However, as with this equation, it is possible to introduce an arbitrary non-zero coupling into problems for the sine-Gordon system by scaling the dependent variables. The introduction of this parameter is essentially a trivial one since all solutions for an arbitrary coupling can immediately be related to those when this parameter is set equal to one. Despite this, the introduction of a parameter in this way makes it possible to linearise these problems by considering the limit as the coupling tends to zero.

To include the parameter $\beta \in \mathbb{R}$ into this set of problems define new dependent variables $(\beta\Phi, \beta\Pi) \stackrel{\text{def}}{=} (\varphi, \varpi)$ and substitute these into the bulk sine-Gordon systems of definitions 1.2, 1.4. Therefore $(\Phi(\cdot,t), \Pi(\cdot,t)) \in \mathcal{M}_{N/\beta}\ \forall t \in \mathbb{R}$. With this parameter in place it is possible to consider the linearised forms of the problems of Type $\mathbf{A}_N$. Considering $\beta \to 0$ in this set (obviously rewritten in the new variables (Φ, Π)) forces the specialisation to $N = 0$ and a Taylor expansion shows that to order β they reduce to solving an initial-boundary value problem for the linear Klein-Gordon equation with phase space $\mathcal{M}_0$. That is the problem of determining the functions Φ, $\Pi : \mathbb{R}^2 \to \mathbb{R}$ which satisfy:

- the Klein-Gordon system

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \Pi, \\ \frac{\partial \Pi}{\partial t} &= \frac{\partial^2 \Phi}{\partial x^2} - \Phi \qquad \forall x, t \in \mathbb{R}. \end{aligned} \tag{2.3.1}$$

- the boundary conditions

$$(\Phi(\cdot,t), \Pi(\cdot,t)) \in \mathcal{M}_0 \qquad \forall t \in \mathbb{R}. \tag{2.3.2}$$

- the 'initial' conditions

$$(\Phi(\cdot,0), \Pi(\cdot,0)) = (\varphi_0, \varpi_0) \in \mathcal{M}_0. \tag{2.3.3}$$

Notice that $\mathcal{M}_0$ is an infinite dimensional real vector space as required by the linearity of the problem.

For an arbitrary initial condition $(\varphi_0, \varpi_0) \in \mathcal{M}_0$ this linearised problem is solved by using the results of section 2.2 in three well defined stages outlined in the following three subsections.

2.3.1 Stage 1: the direct Fourier transform

Definition 2.10 *With $(f,g) \in \mathcal{M}_0$ denote by $\tilde{f}, \tilde{g}$ the Fourier transforms of f, g respectively. Let* **dFt** *denote the 'direct Fourier transform' map*

$$\mathbf{dFt} : \mathcal{M}_0 \to \tilde{\mathcal{T}}(\mathbb{R}; 0, 0) \times \tilde{\mathcal{T}}(\mathbb{R}; 0, 0),$$

defined by

$$\mathbf{dFt}(f,g) = (\tilde{f}, \tilde{g}).$$

From theorem 2.7 and remark 2.6 it is obvious that the inverse of this map exists. For consistency with later chapters it will be convenient to call this the 'inverse Fourier transform' map $\mathbf{iFt} \stackrel{\text{def}}{=} (\mathbf{dFt})^{-1}$.

Apply the map **dFt** to the initial configuration (φ_0, ϖ_0) by defining

$$\tilde{\Phi}(k,0) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \varphi_0(x) e^{-ikx} dx, \qquad \tilde{\Pi}(k,0) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \varpi_0(x) e^{-ikx} dx. \tag{2.3.4}$$

The pair $(\tilde{\Phi}(\cdot,0), \tilde{\Pi}(\cdot,0)) = \mathbf{dFt}(\varphi_0, \varpi_0)$ are said to constitute the 'initial Fourier data' for the problem.

2.3.2 Stage 2: the time evolution of the Fourier data

Suppose the solution (Φ, Π) to the problem (2.3.1)-(2.3.3) is already known. Applying **dFt** to the pair $(\Phi(\cdot,t), \Pi(\cdot,t))$ at a general time $t \in \mathbb{R}$ gives

$$\int_{-\infty}^{\infty} (\Phi_{tt} - \Phi_{xx} + \Phi) e^{-ikx} dx = 0, \qquad \int_{-\infty}^{\infty} (\Pi_{tt} - \Pi_{xx} + \Pi) e^{-ikx} dx = 0 \tag{2.3.5}$$

$\forall\, t, k \in \mathbb{R}$ with subscripts denoting differentiation. Integrating by parts, using (2.3.2) and the uniform convergence of the integrals to take the differentiations under the integral signs, shows the time evolution of the Fourier data for this solution to be governed by the equations

$$\begin{aligned} \frac{\partial^2 \tilde{\Phi}(k,t)}{\partial t^2} &= -(k^2+1)\tilde{\Phi}(k,t) \\ \frac{\partial^2 \tilde{\Pi}(k,t)}{\partial t^2} &= -(k^2+1)\tilde{\Pi}(k,t). \end{aligned} \tag{2.3.6}$$

Note that these equations are *linear* and once the initial conditions (2.3.4) have been imposed they have the solution

$$\tilde{\Phi}(k,t) = (k^2+1)^{-1/2}\tilde{\Pi}(k,0)\sin(k^2+1)^{1/2}t + \tilde{\Phi}(k,0)\cos(k^2+1)^{1/2}t$$
$$\tilde{\Pi}(k,t) = \tilde{\Pi}(k,0)\cos(k^2+1)^{1/2}t - (k^2+1)^{1/2}\tilde{\Phi}(k,0)\sin(k^2+1)^{1/2}t, \quad (2.3.7)$$

so that by construction $(\Phi(\cdot,t)\Pi(\cdot,t)) \in \tilde{\mathcal{T}}(\mathbb{R};0,0) \times \tilde{\mathcal{T}}(\mathbb{R};0,0)\ \ \forall t \in \mathbb{R}$.

Definition 2.11 *Let ς_t denote the bijective map*

$$\varsigma_t : \tilde{\mathcal{T}}(\mathbb{R};0,0) \times \tilde{\mathcal{T}}(\mathbb{R};0,0) \to \tilde{\mathcal{T}}(\mathbb{R};0,0) \times \tilde{\mathcal{T}}(\mathbb{R};0,0), \quad (2.3.8)$$

such that

$$\varsigma_t : (\tilde{\Phi}(\cdot,0), \tilde{\Pi}(\cdot,0)) \mapsto (\tilde{\Phi}(\cdot,t), \tilde{\Pi}(\cdot,t)),$$

with $\tilde{\Phi}(\cdot,t)$, $\tilde{\Pi}(\cdot,t)$ given by (2.3.7).

2.3.3 Stage 3: the inverse Fourier transform

Having deduced how the initial Fourier data must evolve in time, act with the inverse Fourier transform **iFt** on this time evolved data in order to find the solution (Φ, Π). In particular

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\Phi}(k,t)e^{ikx}dk, \quad (2.3.9)$$

so that

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\{\tilde{\Phi}(k,0)\cos((k^2+1)^{1/2}t)$$
$$+(k^2+1)^{-1/2}\tilde{\Pi}(k,0)\sin((k^2+1)^{1/2}t)\Big\}e^{ikx}dk. \quad (2.3.10)$$

Repeating this for $\Pi(x,t)$ shows $\Pi(x,t) = \Phi_t(x,t)$ which must be the case by construction.

Thus the stages 1-3 have enabled the calculation of the pair $(\Phi(\cdot,t), \Pi(\cdot,t))$ which evolve from the initial configuration (φ_0, ϖ_0) according to the linearised form of the problems

$$
\begin{tikzcd}
(\varphi_0, \varpi_0) \arrow[rr, "\mathbf{dFt}"] \arrow[d, "\mathrm{T}_t"'] & & (\tilde{\Phi}(\cdot,0), \tilde{\Pi}(\cdot,0)) \arrow[d, "\varsigma_t"] \\
(\Phi(\cdot,t), \Pi(\cdot,t)) & & (\tilde{\Phi}(\cdot,t), \tilde{\Pi}(\cdot,t)) \arrow[ll, "\mathbf{iFt} = (\mathbf{dFt})^{-1}"]
\end{tikzcd}
$$

$\mathrm{T}_t \stackrel{\mathrm{def}}{=}$ time evolution governed by the *linear* problem (2.3.1)-(2.3.3)

$\varsigma_t =$ time evolution governed by the *linear* o.d.e's (2.3.6)

Figure 2.1: **The Fourier transform method for solving the linearised form of problems in the set $\mathbf{A}_0$**

of Type $\mathbf{A}_0$. The method of solution can be expressed as the commutative diagram in figure 2.1.

It should be mentioned that the transformations **dFt** (stage 1) and **iFt** (stage 3) can be applied to φ_0, (resp. $\tilde{\Phi}(\cdot,t)$) and ϖ_0, (resp. $\tilde{\Pi}(\cdot,t)$) separately. The direct (resp. inverse) Fourier transforms does not mix this data together which is a reflection of the linearity of the problem. When solving the full nonlinear problem with $\beta \neq 0$ using the inverse scattering method it will be seen that there exists an analogous diagrammatic representation for this solution also. However, it is no longer the case that the 'coordinates' and 'momentum' can be transformed independently in the nonlinear analogue of the direct Fourier transform or that the set that replaces the Fourier data at time t can be separated and the inverse transformation applied to these individual pieces.

In conclusion, solving linear problems of this Type by the Fourier transform method can be thought of as determining the time evolution map T_t of figure 2.1. From the above analysis this transform can be written as the composite map

$$\mathrm{T}_t = \mathbf{iFt} \circ \varsigma_t \circ \mathbf{dFt}. \tag{2.3.11}$$

2.4 The linearised form of the problems $\mathbf{B}_{P,0}$

This section shows how the method of Fourier transforms can be used to solve the linearised form of initial-boundary value problems in the set $\mathbf{B}_{P,Q}$ by considering the problems with $P \in \mathbb{R}, Q = 0$ in general and by concentrating on those with $P, Q = 0$ in particular. As with the earlier material in this chapter, the analysis presented here is designed to make the results and ideas of chapters 4 and 5 more transparent.

Notation 2.12 *For $P \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function which satisfies the boundary conditions*

$$\frac{\partial f}{\partial x}(x = 0) + \frac{Pf(0)}{2} = 0,$$
$$\lim_{x \to -\infty} f(x) = 0 \qquad \textit{rapidly in } x.$$

Let $\mathcal{L}_P$ denote the infinite dimensional real vector space of such functions and $\mathcal{N}_P^{lin} \stackrel{\text{def}}{=} \mathcal{L}_P \times \mathcal{L}_P$.

The vector space $\mathcal{N}_P^{lin}$ results from $\mathcal{N}_{P/\beta,Q/\beta}$ by carrying out the linearisation procedure ($\beta \to 0$) described for the set of problems $\mathbf{A}_0$ in the previous section. Note that, just as N was forced to be zero in section 2.3, this process forces the specialisation to $Q = 0$, and reduces the set $\mathbf{B}_{P,0}$ to a set of initial-boundary value problems for the linear Klein-Gordon equation with phase space $\mathcal{N}_P^{lin}$. An element of this set is the problem of determining the functions Φ, $\Pi : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ which satisfy:

- the Klein-Gordon system

$$\frac{\partial \Phi}{\partial t} = \Pi,$$
$$\frac{\partial \Pi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} - \Phi \qquad \forall (x,t) \in (-\infty, 0] \times \mathbb{R}. \tag{2.4.1}$$

- the boundary conditions

$$(\Phi(\cdot, t), \Pi(\cdot, t)) \in \mathcal{N}_P^{lin} \qquad \forall t \in \mathbb{R}. \tag{2.4.2}$$

- the 'initial' conditions

$$(\Phi(\cdot, 0), \Pi(\cdot, 0)) = ({}^P\varphi, {}^P\varpi) \in \mathcal{N}_P^{lin}. \tag{2.4.3}$$

2.4.1 Solving the linearised form of the problems in $\mathbf{B}_{0,0}$

This subsection starts with some fairly obvious but important results.

Lemma 2.13 *The vector spaces $\mathcal{S}'(\mathbb{R}, 0, 0)$ and $\mathcal{L}_0$ are isomorphic. An arbitrary function $f \in \mathcal{S}'(\mathbb{R}, 0, 0)$ satisfies the 'symmetry relation' $f(x) = f(-x)\ \forall x \in \mathbb{R}$. Using this relation it can be seen that a unique element of $\mathcal{L}_0$ can be constructed by restricting the domain of f. Similarly from an arbitrary function $g \in \mathcal{L}_0$ a unique element of $\mathcal{S}'(\mathbb{R}, 0, 0)$ results by defining $g(-x) \stackrel{\text{def}}{=} g(x)\ \forall x \in \mathbb{R}^- \cup \{0\}$.*

Therefore (Φ, Π), the solution to (2.3.1)-(2.3.3) for some $(\varphi_0, \varpi_0) \in \mathcal{M}_0$, which satisfies

$$(\Phi(\cdot, t), \Pi(\cdot, t)) \in \mathcal{M}_0' \qquad \forall t \in \mathbb{R}, \tag{2.4.4}$$

so that in particular

$$(\varphi_0, \varpi_0) \in \mathcal{M}_0',$$

is such that the restriction

$$(\Phi(\cdot, t), \Pi(\cdot, t))|_{(-\infty,0]} \in \mathcal{N}_0^{lin} \qquad \forall t \in \mathbb{R}.$$

It follows that the restriction of (Φ, Π) to the domain $x \in (-\infty, 0]$ solves (2.4.1)-(2.4.3) with the initial conditions

$$({}^0\varphi, {}^0\varpi) = (\varphi_0, \varpi_0)|_{(-\infty,0]}\,.$$

In terms of the Fourier data $(\tilde{\Phi}(\cdot, t), \tilde{\Pi}(\cdot, t))$ constructed from $(\Phi(\cdot, t), \Pi(\cdot, t))$ at a general time $t \in \mathbb{R}$ using **dFt**, (2.4.4) translates into

$$(\tilde{\Phi}(\cdot, t), \tilde{\Pi}(\cdot, t)) \in \mathcal{M}_0' \quad \forall t \in \mathbb{R}. \tag{2.4.5}$$

When Φ, Π evolve according to the Klein-Gordon system (2.3.1), the time evolution of this Fourier data is given by the map ς_t (see definition 2.11). This leaves $\mathcal{M}_0'$ invariant so that if some initial Fourier data in this space is given, it will generate time evolved Fourier data which is also in this space. This is due to the $x \to -x$ invariance of the problem (2.3.1)-(2.3.3).

Notation 2.14 *Let* $\mathbf{iFt}|_{(-\infty,0]}$ *denote the map* $\mathbf{iFt}$ *with the parameter* x *restricted to the semi-infinite interval* $(-\infty, 0]$.

Given an element of $\mathcal{M}'_0$ as initial Fourier data, then the composite map $\mathbf{iFt}|_{(-\infty,0]} \circ \varsigma_t$ applied to this produces a solution to the linearised form of a problem of Type $\mathbf{B}_{0,0}$.

This completes the development of stages 2 and 3 in the Fourier transform method for solving problems of the form (2.4.1)- (2.4.3) when $P = 0$. It remains to develop the first stage of such a method and it is this formulation that is addressed below.

The inverse Fourier transform map $\mathbf{iFt}|_{(-\infty,0]}$ takes the form

$$\mathbf{iFt}|_{(-\infty,0]} : \mathcal{M}'_0 \to \mathcal{N}_0^{lin},$$

and stage 1 of the Fourier transform method is the construction of

$$\mathbf{dFt}|_{(-\infty,0]} \stackrel{\text{def}}{=} \left(\mathbf{iFt}|_{(-\infty,0]}\right)^{-1}.$$

To do this, fix $({}^0\varphi, {}^0\varpi) \in \mathcal{N}_0^{lin}$ and use lemma 2.13 to deduce a unique element $(\varphi_0, \varpi_0) \in \mathcal{M}'_0$. Then apply $\mathbf{dFt}$ to this to find initial scattering data in this space also. That is, in obvious notation,

$$(\tilde{\Phi}(\cdot, 0), \tilde{\Pi}(\cdot, 0)) = \mathbf{dFt}(\varphi_0, \varpi_0) = 2\int_{-\infty}^{0} ({}^0\varphi(x), {}^0\varpi(x)) \cos(\cdot)x \, dx.$$

This procedure gives a map : $\mathcal{N}_0^{lin} \to \mathcal{M}'_0$ and by theorem 2.7 it is the inverse of $\mathbf{dFt}|_{(-\infty,0]}$. Therefore

$$\mathbf{dFt}|_{(-\infty,0]} : \mathcal{N}_0^{lin} \to \mathcal{M}'_0,$$

is given by

$$\mathbf{dFt}|_{(-\infty,0]}({}^0\varphi, {}^0\varpi) = 2\int_{-\infty}^{0} ({}^0\varphi(x), {}^0\varpi(x)) \cos(\cdot)x \, dx,$$

which is the well known cosine transform.

The result of these considerations is that the solution to the linearised form of initial-boundary value problems of type $\mathbf{B}_{0,0}$ is

$$\begin{aligned}\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} &\left\{\tilde{\Phi}(k,0)\cos((k^2+1)^{1/2}t)\right. \\ &\left. +(k^2+1)^{-1/2}\tilde{\Pi}(k,0)\sin((k^2+1)^{1/2}t)\right\} e^{ikx} dk, \qquad (2.4.6)\end{aligned}$$

$\Pi = \Phi_t$, with the initial Fourier data given by

$$\tilde{\Phi}(k,0) = 2\int_{-\infty}^{0} {}^{0}\varphi(x)\cos kx\,dx, \qquad \tilde{\Pi}(k,0) = 2\int_{-\infty}^{0} {}^{0}\varpi(x)\cos kx\,dx, \tag{2.4.7}$$

and $({}^{0}\varphi, {}^{0}\varpi) \in \mathcal{N}_0^{lin}$.

2.4.2 Constraining the Fourier data so as to solve the linearised problems in $\mathbf{B}_{P\neq 0,0}$

Of crucial importance to all the considerations of the previous subsection was lemma 2.13, which related $\mathcal{N}_0^{lin}$ to $\mathcal{M}_0'$. When considering the linearised form of the problems of type $\mathbf{B}_{P\neq 0,0}$ it is not clear that there exists an analogue of this lemma relating the whole of $\mathcal{N}_{P\neq 0}^{lin}$ to some subspace of $\mathcal{M}_0$. So, it is unclear just how to proceed in order to isolate a subset of Fourier data, which is invariant under ς_t, and which is appropriate to a solution of these problems.

However, it is possible to make some progress by virtue of (2.3.10), the closed form of the general solution to the linearised problems of type $\mathbf{A}_0$. Using this it is possible to deduce the generalisation of the constraint (2.4.5) when $P \neq 0$. Demanding that Φ, as given by (2.3.10), be an element of $\mathcal{L}_P$ forces

$$\int_{-\infty}^{\infty} (ik + P)\,\tilde{\Phi}(k,t)dk = 0 \qquad \forall t \in \mathbb{R}. \tag{2.4.8}$$

An identical equation for $\tilde{\Pi}(k,t)$ obviously follows and a *subset* of solutions are those pairs $(\tilde{\Phi}, \tilde{\Pi})$ such that

$$(\tilde{\Phi}(-k,t), \tilde{\Pi}(-k,t)) = \frac{ik+P}{ik-P}((\tilde{\Phi}(k,t), \tilde{\Pi}(k,t)) \qquad \forall k,t \in \mathbb{R}. \tag{2.4.9}$$

It should be noted, however, that there may exist $({}^{P}\varphi, {}^{P}\varpi) \in \mathcal{N}_P^{lin}$ such that (Φ, Π), the solution to (2.4.1)-(2.4.3) may have Fourier data satisfying (2.4.8) but not (2.4.9).

According to (2.3.7) if the relation (2.4.9) holds at $t = 0$ it will continue to do so at all other times, so that $\mathbf{iFt}|_{(-\infty,0]} \circ \varsigma_t$ applied to such initial data will solve (2.4.1)-(2.4.3) for some $({}^{P}\varphi, {}^{P}\varpi) \in \mathcal{N}_P^{lin}$.

This is as much as will be said about the linear problem of this Type. Without a result such as lemma 2.13 to relate $\mathcal{M}_0$ to $\mathcal{N}_{P\neq 0}^{lin}$ it is not easy to formulate the analogue of the cosine transform which produces initial data satisfying (2.4.9) at $t = 0$ thus completing the development of a Fourier transform method for solving a *subset* of these problems.

2.5 Concluding remarks

The Fourier transform method for solving a system of linear partial differential equations involves 3 separate stages. Indeed it can be expressed as a commutative diagram such as figure 2.1.

When modifying the results of section 2.3 so as to be applicable to initial-boundary value problems with $x \in (-\infty, 0]$ it is necessary to make the following changes:

- restrict to $x \in (-\infty, 0]$ by introducing $\mathbf{iFt}|_{(-\infty,0]}$.
- translate the boundary conditions at $x = 0$ into symmetry relations such as (2.4.9) for the Fourier data. The time evolution map ς_t must leave these relations invariant.
- using a result such as lemma 2.13 construct a map $\mathbf{dFt}|_{(-\infty,0]}$ such that for Fourier data which respects the symmetry relations, $\mathbf{dFt}|_{(-\infty,0]} = \left(\mathbf{iFt}|_{(-\infty,0]}\right)^{-1}$.

All the points should be absorbed before proceeding to chapters 4 and 5. In these chapters identical ideas are re-encountered as modifications of the full inverse scattering method developed in chapter 3.

Chapter 3

The inverse scattering method for solving problems of Type $\mathbf{A}_N$

3.1 Introduction

An initial-boundary value problem of Type $\mathbf{A}_N$ is defined by a pair of functions $(\varphi_N, \varpi_0) \in \mathcal{M}_N$. For such a pair, the associated problem is to determine the functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ with $(x,t) \in \mathbb{R}^2$ which satisfy:

- the sine-Gordon system

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varpi, \\ \frac{\partial \varpi}{\partial t} &= \frac{\partial^2 \varphi}{\partial x^2} - \sin\varphi \qquad \forall\, x,t \in \mathbb{R}. \end{aligned} \tag{3.1.1}$$

- the boundary conditions

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \in \mathcal{M}_N \qquad \forall t \in \mathbb{R}. \tag{3.1.2}$$

- the 'initial' conditions

$$(\varphi(\cdot,0), \varpi(\cdot,0) = (\varphi_N, \varpi_0) \in \mathcal{M}_N. \tag{3.1.3}$$

This chapter contains a detailed analysis of the inverse scattering method for solving a *subset* of these problems - the subset being defined by the restricted phase space $\check{\mathcal{M}}_N \subset \mathcal{M}_N$. As with the Fourier transform method, for solving the linearised form of the set $\mathbf{A}_0$ and studied in chapter 2, the inverse scattering method will turn out to be composed of three stages. Section 3.2 details the first these which will turn out to be the nonlinear analogue of the direct Fourier transform **dFt**. At a fixed value of the time parameter $t \in \mathbb{R}$ the 'direct scattering transform' is constructed as the map

$$\mathbf{dst} : \bigcup_{p \in \mathbb{Z}} \mathcal{M}_{2p+N} \to \mathcal{D}_{N\mathrm{mod}2},$$

the set $\mathcal{D}_{N\mathrm{mod}2}$ being called the space of 'scattering data'. This scattering data is analogous to the Fourier data introduced in chapter 2.

In section 3.3 the 'inverse scattering transform' is constructed as the map

$$\mathbf{ist} : \bigcup_{r \in \mathbb{N}} \hat{\mathcal{D}}^{q,r}_{q\mathrm{mod}2} \to \bigcup_{p \in \mathbb{Z}} \mathcal{M}_{2p+q},$$

with

$$\bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{q\mathrm{mod}2} \subset \mathcal{D}_{q\mathrm{mod}2},$$

and in section 3.4 it is shown that once the domain of **dst** is suitably restricted then actually $\mathbf{ist} = (\mathbf{dst})^{-1}$ which justifies its name.

Section 3.6 translates the time evolution of $(\varphi(\cdot,t),\varpi(\cdot,t))$ as dictated by the sine-Gordon system into a time evolution for the scattering data $\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t))$. In terms of this data the evolution is governed by a set of *linear* ordinary differential equations which are easily solved in terms of the initial scattering data $\mathbf{dst}(\varphi_N,\varpi_0)$ to define the time evolution map τ_t.

Section 3.7 pieces together the maps **dst**, **ist**, τ_t to form the inverse scattering method for solving a subset of $\mathbf{A}_N$. It should be clear from this construction why the inverse scattering method is a nonlinear analogue of the Fourier transform method. Some features of this method are also discussed.

3.2 The direct scattering transform for problems of Type $\mathbf{A}_N$

In section 3.7 it will be seen how the direct scattering transform can be thought of as a nonlinear analogue of the direct Fourier transform and as such will form the first stage in a solution to a *subset* of the initial-boundary value problems in the set $\mathbf{A}_N$.

For the moment, however, the transform will be developed as the map

$$\mathbf{dst} : \bigcup_{p\in\mathbb{Z}} \mathcal{M}_{2p+N} \to \mathcal{D}_{N\mathrm{mod}2} \tag{3.2.1}$$

for arbitrary $N \in \mathbb{Z}$, and with $\mathcal{D}_{N\mathrm{mod}2}$ the space of 'scattering data'.

Throughout this section suppose that the time parameter $t \in \mathbb{R}$ is fixed and that $(\varphi(\cdot,t),\varpi(\cdot,t))$ is an arbitrary element of $\mathcal{M}_N$. From these functions construct the $\mathbb{C}$

valued matrix

$$U(x,t,\lambda) \stackrel{\text{def}}{=} \frac{1}{4i}\left(\varpi(x,t)\sigma_3 + \left(\lambda + \frac{1}{\lambda}\right)\sin\frac{\varphi(x,t)}{2}\sigma_1 + \left(\lambda - \frac{1}{\lambda}\right)\cos\frac{\varphi(x,t)}{2}\sigma_2\right), \tag{3.2.2}$$

where $\lambda \in \mathbb{C}\setminus\{0\}$ is the so-called spectral parameter and

$$\sigma_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2.3}$$

Next, define the transition matrix $T(x,y,t,\lambda)$ to be the unique 2×2 matrix solution of the initial value problem

$$\frac{\partial T}{\partial x}(x,y,t,\lambda) = U(x,t,\lambda)T(x,y,t,\lambda), \qquad T(y,y,t,\lambda) = \mathbb{I}, \tag{3.2.4}$$

where $\mathbb{I}$ is the 2×2 unit matrix. So $T(x,y,t,\cdot)$ is analytic in $\mathbb{C}\setminus\{0\}$ but has essential singularities at $\lambda = 0,\infty$ and satisfies

$$T(x,z,t,\lambda)T(z,y,t,\lambda) = T(x,y,t,\lambda) \Rightarrow T(x,y,t,\lambda) = T^{-1}(y,x,t,\lambda). \tag{3.2.5}$$

In addition

$$\frac{\partial}{\partial x}\det T(x,y,t,\lambda) = \operatorname{tr} U(x,y,t,\lambda)\det T(x,y,t,\lambda) = 0, \tag{3.2.6}$$

and therefore $\det T(x,y,t,\lambda) = \det T(y,y,t,\lambda) = 1$ so that the transition matrix is unimodular. The reality of the pair $(\varphi(x,t),\varpi(x,t))$ and the form of the matrix $U(x,t,\lambda)$ imply the involutions

$$\begin{aligned} \bar{T}(x,y,t,\bar{\lambda}) &= \sigma_2 T(x,y,t,\lambda)\sigma_2, \\ T(x,y,t,-\lambda) &= \sigma_3 T(x,y,t,\lambda)\sigma_3. \end{aligned} \tag{3.2.7}$$

The Jost functions $T_\pm(x,t,\lambda)$ are defined for $\lambda \in \mathbb{R}\setminus\{0\}$ and are built from the transition matrix according to

$$T_\pm(x,t,\lambda) \stackrel{\text{def}}{=} \lim_{y\to\pm\infty} T(x,y,t,\lambda)E_\pm(y,\lambda), \tag{3.2.8}$$

with

$$E_\pm(x,\lambda) = \frac{1}{\sqrt{2}}(i\sigma_3)^{N_\pm}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\exp\left(\frac{1}{i}k_1(\lambda)x\sigma_3\right), \tag{3.2.9}$$

and $k_1(\lambda) = \frac{1}{4}\left(\lambda - \frac{1}{\lambda}\right)$, $N_+ = N$, $N_- = 0$. These unimodular matrices satisfy

$$\frac{\partial T_\pm(x,t,\lambda)}{\partial x} = U(x,t,\lambda)T_\pm(x,t,\lambda), \tag{3.2.10}$$

and have the asymptotic behaviour

$$T_\pm(x,t,\lambda) \to E_\pm(x,\lambda), \qquad \text{as } x \to \pm\infty. \tag{3.2.11}$$

Next postulate the integral representations

$$\begin{aligned} T_+(x,t,\lambda) &= \Omega(x,t)E_-(x,\lambda) + \int_x^\infty \Omega(x,t)\left(\Gamma_+^{(1)}(x,y,t) + \frac{1}{\lambda}\Gamma_+^{(2)}(x,y,t)\right)E_-(y,\lambda)\,dy \\ T_-(x,t,\lambda) &= \Omega(x,t)E_-(x,\lambda) + \int_{-\infty}^x \Omega(x,t)\left(\Gamma_-^{(1)}(x,y,t) + \frac{1}{\lambda}\Gamma_-^{(2)}(x,y,t)\right)E_-(y,\lambda)\,dy, \end{aligned} \tag{3.2.12}$$

with

$$\Omega(x,t) \stackrel{\text{def}}{=} \exp\left(\frac{i\varphi(x,t)}{4}\sigma_3\right). \tag{3.2.13}$$

Substituting these into (3.2.10), (3.2.11) yields a system of linear partial differential equations for the kernels $\Gamma_\pm^{(1),(2)}(x,y,t)$. This system is presented in [14] and can be related to a system of linear Volterra integral equations which determine $\Gamma_\pm^{(1),(2)}(x,y,t)$ uniquely. These solutions can be found explicitly by exploiting the absolute convergence of iterations for such equations when $(\varphi(\cdot,t), \varpi(\cdot,t) \in \mathcal{M}_N$.

This argument can then be reversed so that the existence of this unique set $\Gamma_\pm^{(1),(2)}(x,y,t)$ through the Volterra integral equations proves the existence of the integral representations (3.2.12). More details of this idea plus explicit formulae for the (technically simpler) nonlinear Schrödinger equation are to be found in [14].

From (3.2.12) it is possible to deduce the following properties of the columns $T_\pm^{(l)}(x,t,\cdot)$, $l = 1,2$ of $T_\pm(x,t,\cdot)$. The columns $T_-^{(1)}(x,t,\cdot), T_+^{(2)}(x,t,\cdot)$ can be analytically continued into the upper half of the λ plane, whereas $T_+^{(1)}(x,t,\cdot), T_-^{(2)}(x,t,\cdot)$ can be continued into the lower half. They also possess the asymptotic behaviour

$$\begin{aligned} e^{-\frac{\lambda x}{4i}}T_-^{(1)}(x,t,\lambda) &= \frac{1}{\sqrt{2}}\Omega(x,t)\begin{pmatrix} 1 \\ i \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right) \\ e^{\frac{\lambda x}{4i}}T_+^{(2)}(x,t,\lambda) &= \frac{1}{\sqrt{2}}\Omega(x,t)\begin{pmatrix} i \\ 1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right) \qquad \mathrm{Im}\lambda \geq 0, \end{aligned}$$

$$e^{-\frac{\lambda x}{4i}} T_+^{(1)}(x,t,\lambda) = \frac{1}{\sqrt{2}}\Omega(x,t)\begin{pmatrix} 1 \\ i \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right)$$

$$e^{\frac{\lambda x}{4i}} T_-^{(2)}(x,t,\lambda) = \frac{1}{\sqrt{2}}\Omega(x,t)\begin{pmatrix} i \\ 1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right) \qquad \mathrm{Im}\lambda \leq 0, \qquad (3.2.14)$$

as $|\lambda| \to \infty$. Similarly for $|\lambda| \to 0$ the columns satisfy

$$e^{\frac{x}{4\lambda i}} T_-^{(1)}(x,t,\lambda) = \frac{1}{\sqrt{2}}\Omega^{-1}(x,t)\begin{pmatrix} 1 \\ i \end{pmatrix} + O(|\lambda|)$$

$$e^{-\frac{x}{4\lambda i}} T_+^{(2)}(x,t,\lambda) = \frac{(-1)^N}{\sqrt{2}}\Omega^{-1}(x,t)\begin{pmatrix} i \\ 1 \end{pmatrix} + O(|\lambda|) \qquad \mathrm{Im}\lambda \geq 0,$$

$$e^{\frac{x}{4\lambda i}} T_+^{(1)}(x,t,\lambda) = \frac{(-1)^N}{\sqrt{2}}\Omega^{-1}(x,t)\begin{pmatrix} 1 \\ i \end{pmatrix} + O(|\lambda|)$$

$$e^{-\frac{x}{4\lambda i}} T_-^{(2)}(x,t,\lambda) = \frac{1}{\sqrt{2}}\Omega^{-1}(x,t)\begin{pmatrix} i \\ 1 \end{pmatrix} + O(|\lambda|) \qquad \mathrm{Im}\lambda \leq 0. \qquad (3.2.15)$$

The involutions (3.2.7) give the relations

$$\bar{T}_\pm^{(1)}(x,t,\lambda) = i\sigma_2 T_\pm^{(2)}(x,t,\bar{\lambda})$$

$$\bar{T}_\pm^{(2)}(x,t,\lambda) = i\sigma_2 T_\pm^{(1)}(x,t,\bar{\lambda})$$

$$\bar{T}_\pm^{(1,2)}(x,t,-\bar{\lambda}) = -i\sigma_1 T_\pm^{(1,2)}(x,t,\lambda), \qquad (3.2.16)$$

where λ is restricted to lie in the appropriate domain of analyticity.

Next, use the Jost functions to construct the reduced transition matrix $T(\lambda,t)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ according to

$$T(\lambda,t) \stackrel{\mathrm{def}}{=} T_+^{-1}(x,t,\lambda)T_-(x,t,\lambda) = \lim_{x,-y\to\infty} E_+^{-1}(x,\lambda)T(x,y,t,\lambda)E_-(y,\lambda). \qquad (3.2.17)$$

This matrix is unimodular, independent of x and satisfies the involution

$$\bar{T}(\lambda,t) = \sigma_2 T(\lambda,t)\sigma_2 \qquad (3.2.18)$$

which ensures that it has the form

$$T(\lambda,t) = \begin{pmatrix} a(\lambda,t) & -\bar{b}(\lambda,t) \\ b(\lambda,t) & \bar{a}(\lambda,t) \end{pmatrix}, \qquad (3.2.19)$$

subject to the constraint

$$|a(\lambda,t)|^2 + |b(\lambda,t)|^2 = 1 \quad \forall\lambda \in \mathbb{R}. \qquad (3.2.20)$$

From (3.2.17) the 'transition coefficient' $a(\cdot,t)$ can be defined by

$$a(\cdot,t) \stackrel{\text{def}}{=} \det\left(T_-^{(1)}(x,t,\cdot), T_+^{(2)}(x,t,\cdot)\right), \tag{3.2.21}$$

which implies that it has an analytic continuation into the upper half plane such that

$$a(-\bar{\lambda},t) = \bar{a}(\lambda,t). \tag{3.2.22}$$

In addition it has the asymptotics

$$\begin{aligned} a(\lambda,t) &= 1 + O\left(\frac{1}{|\lambda|}\right), \qquad |\lambda| \to \infty \\ a(\lambda,t) &= (-1)^N + O(|\lambda|), \qquad |\lambda| \to 0. \end{aligned} \tag{3.2.23}$$

Similarly the transition coefficient

$$b(\cdot,t) \stackrel{\text{def}}{=} \det\left(T_+^{(1)}(x,t,\cdot), T_-^{(1)}(x,t,\cdot)\right), \tag{3.2.24}$$

has no analytic continuation away from $\mathbb{R} \setminus \{0\}$ but satisfies the involution

$$b(-\lambda,t) = \bar{b}(\lambda,t) \quad \forall \lambda \in \mathbb{R}. \tag{3.2.25}$$

Also the relation (3.2.24) and the integral representations (3.2.12) imply that

$$\lim_{|\lambda|\to 0,\infty} b(\lambda,t) = 0 \qquad \text{rapidly in } \lambda. \tag{3.2.26}$$

Now make the following definition regarding the structure of the transition coefficient $a(\cdot,t)$.

Notation 3.1 *At time $t \in \mathbb{R}$ let the function $a(\cdot,t)$:*

- *have $n_1(t) \in \mathbb{N}$[1] zeroes $\lambda_1(t), \ldots, \lambda_{n_1(t)}(t)$ satisfying*

$$\mathrm{Re}(\lambda_j(t)) = 0, \; j = 1, \ldots, n_1(t).$$

Therefore $\lambda_j(t) = i\kappa_j(t)$ with $\kappa_j(t) > 0$.

[1] Throughout this thesis it is assumed that $0 \in \mathbb{N}$.

- *have* $n_2(t) \in \mathbb{N}$ *zeroes* $\lambda_{n_1(t)+1}(t), \ldots, \lambda_{n_1(t)+n_2(t)}(t)$ *satisfying*

$$\mathrm{Re}(\lambda_j(t)) > 0,\ \mathrm{Im}(\lambda_j(t)) > 0,\ j = n_1(t)+1, \ldots, n_1(t)+n_2(t).$$

The involution (3.2.22) implies that

$$(\lambda_{n_1(t)+n_2(t)+1}(t) \stackrel{\text{def}}{=} -\bar{\lambda}_{n_1(t)+1}(t)), \ldots, (\lambda_{n_1(t)+2n_2(t)}(t) \stackrel{\text{def}}{=} -\bar{\lambda}_{n_1(t)+n_2(t)}(t)),$$

are the zeroes such that

$$\mathrm{Re}(\lambda_j(t)) < 0,\ \mathrm{Im}(\lambda_j(t)) > 0,\ j = n_1(t)+n_2(t), \ldots, n_1(t)+2n_2(t).$$

- *have* $n_3(t) \in \mathbb{N}$ *zeroes* $\lambda_{n_1(t)+2n_2(t)+1}(t), \ldots, \lambda_{n_1(t)+2n_2(t)+n_3(t)}(t)$ *which are real and positive. The involution (3.2.22) implies that*

$$(\lambda_{n_1(t)+2n_2(t)+n_3(t)+1}(t) \stackrel{\text{def}}{=} -\lambda_{n_1(t)+2n_2(t)+1}(t)), \ldots,$$

$$\ldots, (\lambda_{n_1(t)+2n_2(t)+2n_3(t)}(t) \stackrel{\text{def}}{=} -\lambda_{n_1(t)+2n_2(t)+n_3(t)}(t)),$$

are the zeroes which are real and negative.

The total number of zeroes of the transition coefficient $a(\cdot, t)$ *is therefore* $n(t) \stackrel{\text{def}}{=} n_1(t) + 2n_2(t) + 2n_3(t)$.

At the points $\lambda_j(t)$, $j = 1, \ldots n(t)$ it can be seen from (3.2.21) that

$$\det\left(T_-^{(1)}(x,t,\lambda_j(t)), T_+^{(2)}(x,t,\lambda_j(t))\right) = 0, \tag{3.2.27}$$

which implies

$$T_-^{(1)}(x,t,\lambda_j(t)) = \gamma_j(t) T_+^{(2)}(x,t,\lambda_j(t)), \quad j = 1, \ldots, n(t), \tag{3.2.28}$$

thus defining the normalisation coefficients

$$\begin{aligned}
&\bar{\gamma}_j(t) = \gamma_j(t), && j = 1, \ldots, n_1(t), \\
&\bar{\gamma}_k(t) = \gamma_{k+n_2(t)}(t), && k = n_1(t)+1, \ldots, n_1(t)+n_2(t), \\
&\bar{\gamma}_l(t) = \gamma_{l+n_3(t)}(t), && l = n_1(t)+2n_2(t)+1, \ldots, n_1(t)+2n_2(t)+n_3(t).
\end{aligned} \tag{3.2.29}$$

Definition 3.2 *At time $t \in \mathbb{R}$ and for a given $N \in \mathbb{Z}$ let $(a(\cdot,t), b(\cdot,t))$ denote a pair of transition coefficients so that they possess the properties (3.2.20)-(3.2.26) deduced above. In addition suppose the function $a(\cdot,t)$ has $n_1(t)+2n_2(t)+2n_3(t)$ zeroes as specified by notation 3.1 and let $\gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)+2n_3(t)}(t)$ be the associated normalisation coefficients satisfying (3.2.28), (3.2.29). Define the sets $\mathcal{D}^{n_1(t),n_2(t),n_3(t)}_{N\text{mod}2}$, $\mathcal{D}_{N\text{mod}2}$ by*

$$\mathcal{D}^{n_1(t),n_2(t),n_3(t)}_{N\text{mod}2} \stackrel{\text{def}}{=} \left\{(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)+2n_3(t)}(t))\right\}, \tag{3.2.30}$$

$$\mathcal{D}_{N\text{mod}2} \stackrel{\text{def}}{=} \bigcup_{q,r,s\in\mathbb{N}} \mathcal{D}^{q,r,s}_{N\text{mod}2}. \tag{3.2.31}$$

This allows the definition of a map

$$\mathbf{dst} : \mathcal{M}_N \to \mathcal{D}_{N\text{mod}2}, \tag{3.2.32}$$

given explicitly by

$$\mathbf{dst}(\varphi(\cdot,t), \varpi(\cdot,t)) = (a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)+2n_3(t)}(t)). \tag{3.2.33}$$

However, the above considerations hold identically if N is replaced by $2p+N$ for arbitrary $p \in \mathbb{Z}$. Therefore the domain of **dst** can be extended,

$$\mathbf{dst} : \bigcup_{p\in\mathbb{Z}} \mathcal{M}_{2p+N} \to \mathcal{D}_{N\text{mod}2}, \tag{3.2.34}$$

as promised in the introductory remarks to this section.

3.2.1 Some subsets of $\mathcal{D}_{N\text{mod}2}$

To enable a clear formulation of the inverse scattering transform as stage 3 of the inverse scattering method it is convenient to constrain the function $a(\cdot,t)$ and the normalisation coefficients $\gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)+2n_3(t)}(t)$ to have some particular properties. These pick out the subset $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\text{mod}2} \subset \mathcal{D}^{n_1(t),n_2(t),0}_{N\text{mod}2}$ which in turn is used to specify a subset of configurations

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \in \hat{\mathcal{M}}_N \subset \mathcal{M}_N.$$

Definition 3.3 *Let the space* $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\bmod 2} \subset \mathcal{D}^{n_1(t),n_2(t),0}_{N\bmod 2}$ *be comprised of scattering data*

$$(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \mathcal{D}^{n_1(t),n_2(t),0}_{N\bmod 2} \tag{3.2.35}$$

such that:

- *the number* $n(t) = n_1(t) + 2n_2(t)$ *is finite.*
- *all the* $n(t)$ *zeroes of* $a(\cdot,t)$ *are simple.*
- *none of the normalisation coefficients* $\gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)$ *are zero.*

For the set $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\bmod 2}$ there exists a particularly nice representation due to the following corollary to definition 3.3.

Corollary 3.4 *For the scattering data (3.2.35) which satisfies the constraints of definition 3.3 the transition coefficient* $a(\cdot,t)$ *can be uniquely determined in terms of the coefficient* $b(\cdot,t)$ *and the zeroes* $\lambda_1(t), \ldots \lambda_{n_1(t)+2n_2(t)}(t)$ *to be*

$$a(\lambda,t) = \prod_{j=1}^{n_1(t)} \frac{\lambda - i\kappa_j(t)}{\lambda + i\kappa_j(t)} \prod_{k=n_1(t)+1}^{n_1(t)+n_2(t)} \frac{\lambda - \lambda_k(t)}{\lambda - \bar{\lambda}_k(t)} \cdot \frac{\lambda + \bar{\lambda}_k(t)}{\lambda + \lambda_k(t)} \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |b(\mu,t)|^2)}{\mu - \lambda} d\mu \right\}, \tag{3.2.36}$$

when $\operatorname{Im}\lambda > 0$*. This formula can then be analytically continued down to the real line using the Sochoki-Plemelj formula for generalised functions,*

$$\frac{1}{\mu - \lambda} \to \frac{1}{\mu - \lambda - i0} = \text{p.v.} \frac{1}{\mu - \lambda} + i\pi\delta(\mu - \lambda). \tag{3.2.37}$$

The proof of these results is to be found in [14].

Therefore, it follows from the asymptotic expression (3.2.23) and the involution (3.2.25) that $n_1(t)$, the number of purely imaginary zeroes of $a(\cdot,t)$ is such that

$$n_1(t) \equiv N \pmod 2. \tag{3.2.38}$$

As a result of corollary 3.4, for $n_1(t) \in 2\mathbb{N} + N \bmod 2$, $n_2(t) \in \mathbb{N}$, the set $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\bmod 2}$ can be represented as the set

$$\{(b(\cdot,t) : \kappa_j(t), \gamma_j(t); j = 1\ldots, n_1(t) : \lambda_k(t), \gamma_k(t); k = n_1(t) + 1\ldots, n_1(t) + n_2(t))\}. \tag{3.2.39}$$

Due to this simple representation it will be useful to define the subspace $\hat{\mathcal{M}}_N \subset \mathcal{M}_N$ as follows.

Definition 3.5 *For arbitrary $N \in \mathbb{Z}$ let the space*

$$\bigcup_{p\in\mathbb{Z}} \hat{\mathcal{M}}_{2p+N} \subset \bigcup_{q\in\mathbb{Z}} \mathcal{M}_{2q+N} \tag{3.2.40}$$

be the inverse image of the map **dst**,

$$\mathbf{dst}^{-1}\left(\bigcup_{q\in 2\mathbb{N}+N\,\mathrm{mod}2, r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{N\mathrm{mod}2}\right) = \bigcup_{p\in\mathbb{Z}} \hat{\mathcal{M}}_{2p+N}. \tag{3.2.41}$$

3.2.2 Some results regarding the restriction of dst to $\hat{\mathcal{M}}_N$

The reason for this subsection is two-fold. Firstly, the analysis developed here is used in subsection 3.2.3 to prove that, when restricted to a suitable domain, the map **dst** is injective. In addition, subsection 3.3.1 uses these facts to formulate a map **ist** which, for suitable restrictions on the domain/range, is deduced to be the inverse of **dst** in section 3.4.

For a configuration $(\varphi(\cdot,t), \varpi(\cdot,t)) \in \hat{\mathcal{M}}_N$ so that $\mathbf{dst}(\varphi(\cdot,t), \varpi(\cdot,t)) \in \hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\mathrm{mod}2}$ for some $n_1(t) \in 2\mathbb{N} + N \bmod 2$, $n_2(t) \in \mathbb{N}$, the relation (3.2.17) can be rewritten as

$$G_+(x,t,\lambda)G_-(x,t,\lambda) = G(x,t,\lambda) \qquad \forall\,\lambda \in \mathbb{R} \tag{3.2.42}$$

with, in obvious notation

$$\begin{aligned}
G_+(x,t,\lambda) &= a(\lambda,t)e^{-ik_1(\lambda)x\sigma_3}\left(T_-^{(1)}(x,t,\lambda), T_+^{(2)}(x,t,\lambda)\right)^{-1}, \\
G_-(x,t,\lambda) &= \left(T_+^{(1)}(x,t,\lambda), T_-^{(2)}(x,t,\lambda)\right)e^{ik_1(\lambda)x\sigma_3}, \\
G(x,t,\lambda) &= \begin{pmatrix} 1 & -e^{-2ik_1(\lambda)x}\bar{b}(\lambda,t) \\ -e^{2ik_1(\lambda)x}\bar{b}(\lambda,t) & 1 \end{pmatrix}.
\end{aligned} \tag{3.2.43}$$

The matrices $G_\pm(x,t,\lambda), G(x,t,\lambda)$ have the following properties:

1) The matrix $G(x,t,\lambda)$ is Hermitian and due to (3.2.26) satisfies,

$$\bar{G}(x,t,-\lambda) = G(x,t,\lambda), \qquad \lim_{|\lambda|\to 0,\infty} G(x,t,\lambda) = \mathbb{I} \qquad \text{rapidly in } \lambda. \tag{3.2.44}$$

2) The matrices $G_+(x,t,\lambda)$ and $G_-(x,t,\lambda)$ admit an analytic continuation into the upper and lower half planes respectively, and in these domains they satisfy the involutions

$$
\begin{aligned}
&G_+^\dagger(x,t,\lambda) = G_-(x,t,\bar\lambda),\\
&\bar G_+(x,t,-\bar\lambda) = iG_+(x,t,\lambda)\sigma_1\\
&\bar G_-(x,t,-\bar\lambda) = -i\sigma_1 G_-(x,t,\lambda).
\end{aligned}
\tag{3.2.45}
$$

3) In their domains of analyticity $G_\pm(x,t,\lambda)$ have the asymptotic behaviour

$$
\begin{aligned}
&G_+(x,t,\lambda) = \mathcal{E}^{-1}\Omega^{-1}(x,t)\left(\mathbb{I} + O(|\lambda|^{-1})\right),\\
&G_-(x,t,\lambda) = \Omega(x,t)\mathcal{E}\left(\mathbb{I} + O(|\lambda|^{-1})\right)
\end{aligned}
\tag{3.2.46}
$$

as $|\lambda| \to \infty$ and

$$
\begin{aligned}
&G_+(x,t,\lambda) = (-\sigma_3)^N\mathcal{E}^{-1}\Omega(x,t)\left(\mathbb{I} + O(|\lambda|)\right),\\
&G_-(x,t,\lambda) = \Omega^{-1}(x,t)\mathcal{E}(-\sigma_3)^N\left(\mathbb{I} + O(|\lambda|)\right)
\end{aligned}
\tag{3.2.47}
$$

as $|\lambda| \to 0$ with $\mathcal{E} = \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_1)$.

4) The matrices $G_+(x,t,\lambda)$ and $G_-(x,t,\lambda)$ are nondegenerate in their domains of analyticity except for the points $\lambda = \lambda_j(t)$ and $\lambda = \bar\lambda_j(t)$, respectively, where

$$
\operatorname{Im} G_+(x,t,\lambda_j(t)) = N_j^+(x,t), \qquad \operatorname{Ker} G_-(x,t,\bar\lambda_j(t)) = N_j^-(x,t), \qquad j = 1,\dots,n(t).
\tag{3.2.48}
$$

The $N_j^+(x,t)$ and $N_j^-(x,t)$ are one-dimensional subspaces in $\mathbb{C}^2$ spanned respectively by the vectors

$$
\begin{pmatrix} 1 \\ -\gamma_j(x,t) \end{pmatrix} \text{and} \begin{pmatrix} \bar\gamma_j(x,t) \\ 1 \end{pmatrix},
\tag{3.2.49}
$$

where $\gamma_j(x,t) \stackrel{\text{def}}{=} e^{\frac{1}{2}(\lambda_j(t) - \frac{1}{\lambda_j(t)})x}\gamma_j(t)$ and $n(t) = n_1(t) + 2n_2(t)$.

Notice that from (3.2.2), (3.2.8), (3.2.43) it follows that the original configuration $(\varphi(\cdot,t), \varpi(\cdot,t))$ can be recovered from the matrix $G_+(x,t,\lambda)$ according to

$$
\begin{aligned}
&\varphi(x,t) = 2\arcsin \operatorname{tr}\left(i\sigma_1\frac{dG_+^{-1}}{dx}(x,t,1)G_+(x,t,1)\right)\\
&\varpi(x,t) = 2i\operatorname{tr}\left(\sigma_3\frac{dG_+^{-1}}{dx}(x,t,1)G_+(x,t,1)\right).
\end{aligned}
\tag{3.2.50}
$$

This results of this subsection will be relied upon heavily in subsections 3.2.3, 3.3.1.

3.2.3 dst is injective (when its domain is suitably restricted)

In this subsection it will be proved that the map

$$\mathbf{dst} : \bigcup_{p\in\mathbb{Z}} \hat{\mathcal{M}}_{2p+N} \to \bigcup_{q\in 2\mathbb{N}+N\mathrm{mod}2, r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{N\mathrm{mod}2} \tag{3.2.51}$$

is injective (i.e 1-1). Consider two configurations

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \in \hat{\mathcal{M}}_N, \qquad (\varphi'(\cdot,t), \varpi'(\cdot,t)) \in \hat{\mathcal{M}}_{N'}$$

such that $N \equiv N' \pmod 2$ and suppose that under the map **dst** both these lead to the same element of $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\mathrm{mod}2}$ denoted by

$$(b(\cdot,t) : \kappa_j(t), \gamma_j(t); j = 1\ldots n_1(t) : \lambda_k(t), \gamma_k(t); k = n_1(t)+1\ldots n_1(t)+n_2(t))\,.$$

The relation (3.2.42) implies

$$G_+(x,t,\lambda)G_-(x,\lambda) = G'_+(x,t,\lambda)G'_-(x,\lambda) \qquad \forall\lambda\in\mathbb{R}, \tag{3.2.52}$$

where $G'_\pm(x,t,\lambda)$ are the matrices which result from the replacement $(\varphi(\cdot,t), \varpi(\cdot,t)) \to (\varphi'(\cdot,t), \varpi'(\cdot,t))$ in the construction of the matrices $G_\pm(x,t,\lambda)$ defined by (3.2.43). From this it follows that

$$\Psi(x,t,\lambda) = G_+^{-1}(x,t,\lambda)G'_+(x,t,\lambda) = G_-(x,t,\lambda)G'^{-1}_-(x,t,\lambda) \quad \forall\lambda\in\mathbb{R}, \tag{3.2.53}$$

with the left hand side analytic in the upper half plane except at $\lambda = \lambda_j(t)$, and the right hand side analytic in the lower half plane except at $\lambda = \bar{\lambda}_j(t), j = 1,\ldots,n(t)$.

In the neighbourhood of $\lambda = \lambda_j(t)$ the matrices $G_+(x,t,\lambda)$, $G_+^{-1}(x,t,\lambda)$ have the expansions

$$\begin{aligned} G_+(x,t,\lambda) &= A(x,t) + O(|\lambda-\lambda_j(t)|) \\ G_+^{-1}(x,t,\lambda) &= \frac{B(x,t)}{\lambda-\lambda_j(t)} + O(1), \end{aligned} \tag{3.2.54}$$

with

$$A(x,t)B(x,t) = B(x,t)A(x,t) = 0. \tag{3.2.55}$$

Since $\mathrm{Im}A(x,t) = \mathrm{Im}G_+(x,t,\lambda_j(t)) = N_j^+(x,t)$ it follows that the subspace $N_j^+(x,t)$ is contained in $\mathrm{Ker}B(x,t)$ and both spaces being one dimensional they must coincide with each other:

$$N_j^+(x,t) = \mathrm{Ker}B(x,t). \tag{3.2.56}$$

There is a similar expansion for $G'_+(x,t,\lambda)$, $G'^{-1}_+(x,t,\lambda)$ so that

$$\mathrm{Im}A'(x,t) = N_j^+(x,t) = \mathrm{Ker}B(x,t). \tag{3.2.57}$$

It is clear that the residue of $G_+^{-1}(x,t,\lambda)G'_+(x,t,\lambda)$ equals $B(x,t)A'(x,t)$ and therefore vanishes. As a result $\Psi(x,t,\lambda)$ has no singularities at $\lambda = \lambda_j(t), \bar{\lambda}_j(t)$ and hence is an entire function. From (3.2.47),(3.2.46) it follows that

$$\Psi(x,t,0) = \Omega^{-1}(x,t)\Omega'(x,t)$$
$$\Psi(x,t,\infty) = \Omega(x,t)\Omega'^{-1}(x,t). \tag{3.2.58}$$

Hence by the famous Liouville theorem [20]

$$\Omega^2(x,t) = \Omega'^2(x,t) \quad \forall x \in \mathbb{R}, \tag{3.2.59}$$

so that

$$\varphi(\cdot,t) \equiv \varphi'(\cdot,t) \tag{3.2.60}$$

an obvious consequence of which is $N = N'$. The identity (3.2.60) implies $\Psi(x,t,\lambda) = \mathbb{I}$ so that $G_\pm(x,t,\lambda) = G'_\pm(x,t,\lambda)\,\forall x,\lambda \in \mathbb{R}$. Therefore using (3.2.50) it follows that $(\varphi(\cdot,t),\varpi(\cdot,t)) \equiv (\varphi'(\cdot,t),\varpi'(\cdot,t))$ and it is proved that $\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t)) \neq \mathbf{dst}(\varphi'(\cdot,t),\varpi'(\cdot,t))$ for any two distinct $(\varphi(\cdot,t),\varpi(\cdot,t)),(\varphi'(\cdot,t),\varpi'(\cdot,t)) \in \bigcup_{p\in\mathbb{Z}}\hat{\mathcal{M}}_p$ from which it follows that the map given by (3.2.51) is injective.

3.3 The inverse scattering transform for a subset of the problems in $\mathbf{A}_N$

In this section a map **ist** is constructed such that for arbitrary $t \in \mathbb{R}$ and $n_1(t) \in \mathbb{N}$,

$$\mathbf{ist} : \bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{n_1(t),r}_{n_1(t)\mathrm{mod}2} \to \bigcup_{q\in\mathbb{Z}} \mathcal{M}_{2q+n_1(t)}. \tag{3.3.1}$$

In section 3.7 it will be seen how this map can be thought of as a nonlinear analogue of the inverse Fourier transform and as such forms the third and final stage in a solution to *some* of the initial-boundary value problems in the set $\mathbf{A}_N$.

The transformation **ist** takes the form of a two parameter (x, t) family of matrix Riemann-Hilbert problems often studied in analytic function theory. In this section there will be no restriction on the parameter $x \in \mathbb{R}$. However, for future work it is important to notice that the transformation is 'pointwise' so that the domain of x can be restricted at will. This observation was also made in chapter 2 regarding the inverse Fourier transform. There the linearised form of the problems in the set $\mathbf{B}_{P,0}$ were solved by restricting the domain of the parameter x appearing in this map to $(-\infty, 0]$. This idea will be adopted once more in chapters 4 and 5 but this time for the inverse scattering transform **ist** in order to to solve the full nonlinear problems in the set $\mathbf{B}_{P,Q}$.

3.3.1 The formulation of ist

Let

$$(a(\cdot, t), b(\cdot, t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)), \tag{3.3.2}$$

be some element of $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}$ with $n_1(t), n_2(t) \in \mathbb{N}$ and $n(t) = n_1(t) + 2n_2(t)$. At this point it will not be specified whether there exists a $(\varphi(\cdot, t), \varpi(\cdot, t)) \in \mathcal{M}_N$, $N \in 2\mathbb{Z} + n_1(t) \bmod 2$ such that $\mathbf{dst}(\varphi(\cdot, t), \varpi(\cdot, t))$ equals (3.3.2). However in section 3.4 it will be seen that such a configuration does exist and that, in some suitable domains/ranges, **ist** is in fact the inverse of **dst**.

From the scattering data (3.3.2) and the results of subsection 3.2.2 construct the matrix $G(x, t, \lambda)$ and the subspaces $N_j^{\pm}(x, t)$. These now constitute the input data for a family of Riemann-Hilbert problems parameterised by the spacetime coordinates $(x, t) \in \mathbb{R}^2$. These problems are to find the matrices $g_{\pm}(x, t, \lambda)$ satisfying

$$G(x, t, \lambda) = g_+(x, t, \lambda) g_-(x, t, \lambda) \quad \forall \lambda \in \mathbb{R}, \tag{3.3.3}$$

such that $g_{\pm}(x, t, \lambda)$ extend analytically into the domains $\pm\mathrm{Im}\,\lambda \geq 0$ and are nondegenerate in these domains except for points $\lambda = \lambda_j(t)$ and $\lambda = \bar{\lambda}_j(t)$, respectively, where

$$\mathrm{Im}\, g_+(x,t,\lambda_j(t)) = N_j^+(x,t), \qquad \mathrm{Ker}\, g_-(x,t,\bar{\lambda}_j(t)) = N_j^-(x,t), \qquad j = 1,\dots,n(t). \tag{3.3.4}$$

Also, they are required to be normalised according to

$$\begin{aligned} g_+(x,t,\lambda) &= \mathcal{E}^{-1}\tilde{\Omega}^{-1}(x,t)\left(\mathbb{I} + O(|\lambda|^{-1})\right), \\ g_-(x,t,\lambda) &= \tilde{\Omega}(x,t)\mathcal{E}\left(\mathbb{I} + O(|\lambda|^{-1})\right) \end{aligned} \tag{3.3.5}$$

as $|\lambda| \to \infty$ and

$$\begin{aligned} g_+(x,t,\lambda) &= (-\sigma_3)^{n_1(t)}\mathcal{E}^{-1}\tilde{\Omega}(x,t)\left(\mathbb{I} + O(|\lambda|)\right), \\ g_-(x,t,\lambda) &= \tilde{\Omega}^{-1}(x,t)\mathcal{E}(-\sigma_3)^{n_1(t)}\left(\mathbb{I} + O(|\lambda|)\right) \end{aligned} \tag{3.3.6}$$

as $|\lambda| \to 0$, $\mathcal{E} = \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_1)$ with $\tilde{\Omega}(x,t)$ constrained to be diagonal, continuous and to satisfy

$$\lim_{x\to-\infty} \tilde{\Omega}(x,t) = \mathbb{I}. \tag{3.3.7}$$

From this information it is possible to make the following assertions:

1″) For the family of Riemann problems stated above there exists a solution $g_\pm(x,t,\lambda)$ such that

$$\begin{aligned} g_+^\dagger(x,t,\lambda) &= g_-(x,t,\bar{\lambda}), \\ \bar{g}_+(x,t,-\bar{\lambda}) &= i g_+(x,t,\lambda)\sigma_1 \\ \bar{g}_-(x,t,\bar{\lambda}) &= -i\sigma_1 g_-(x,t,-\lambda), \end{aligned} \tag{3.3.8}$$

$\forall\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda \geq 0$.

2″) This solution is unique.

3″) The matrices $F_\pm(x,t,\lambda)$ constructed from this solution according to

$$F_+(x,t,\lambda) = g_+^{-1}(x,t,\lambda)e^{-ik_1(\lambda)x\sigma_3}, \qquad F_-(x,\lambda) = g_-(x,t,\lambda)e^{-ik_1(\lambda)x\sigma_3} \tag{3.3.9}$$

satisfy the auxiliary linear equation

$$\frac{dF_\pm}{dx}(x,t,\lambda) = \frac{1}{4i}\left(\varpi(x,t)\sigma_3 + \left(\lambda + \frac{1}{\lambda}\right)\sin\frac{\varphi(x,t)}{2}\sigma_1 + \left(\lambda - \frac{1}{\lambda}\right)\cos\frac{\varphi(x,t)}{2}\sigma_2\right)F_\pm(x,t,\lambda), \tag{3.3.10}$$

with $\varpi(x,t), \varphi(x,t) \in \mathbb{R}$ and the matrix function $\hat{\Omega}$ taking the form

$$\hat{\Omega}(x,t) = \exp\left(\frac{i\varphi(x,t)}{4}\sigma_3\right). \tag{3.3.11}$$

4″) The pair of functions $(\varphi(\cdot,t), \varpi(\cdot,t))$ are an element of $\mathcal{M}_N$ for some $N \in \mathbb{Z}$ such that $N \equiv n_1(t) \pmod 2$.

Proofs of assertions $1'' - 4''$ are to be found in [14] and so will not be reproduced here but it should be mentioned that the proof of assertion 2″ mirrors the analysis (3.2.52)-(3.2.60). It is clear from assertion 3″ that the pair $(\varphi(\cdot,t), \varpi(\cdot,t))$ can be extracted from the solution according to

$$\varphi(x,t) = 2\arcsin\ \mathrm{tr}\left(i\sigma_1\frac{dg_+^{-1}}{dx}(x,t,1)g_+(x,t,1)\right) \tag{3.3.12}$$

$$\varpi(x,t) = 2i\,\mathrm{tr}\left(\sigma_3\frac{dg_+^{-1}}{dx}(x,t,1)g_+(x,t,1)\right). \tag{3.3.13}$$

A more convenient representation for the function $\varphi(\cdot,t)$ can be found by investigating the assertion 1″. In [14] this assertion is proved by mapping the Riemann problems onto a family of simpler problems with a single normalisation at $\lambda = \infty$. This family is closely related to the one which arises when developing the inverse scattering transform for the nonlinear Schrödinger equation and existence/uniqueness theorems for its solutions can be proved.

Let $\check{g}_\pm(x,t,\lambda)$ be a solution of the family of Riemann problems

$$G(x,t,\lambda) = \check{g}_+(x,t,\lambda)\check{g}_-(x,t,\lambda), \tag{3.3.14}$$

where

a) the matrices $\check{g}_\pm(x,t,\lambda)$ extend analytically into the half planes $\pm\ \mathrm{Im}\lambda \geq 0$ and are normalised to $\mathbb{I}$ at $\lambda = \infty$,

$$\check{g}_\pm(x,t,\lambda) = \mathbb{I} + O\left(\frac{1}{|\lambda|}\right). \tag{3.3.15}$$

b) As for $g_\pm(x,t,\lambda)$ the matrices $\check{g}_\pm(x,t,\lambda)$ are required to be nondegenerate everywhere except $\lambda = \lambda_j(t)$ and $\lambda = \bar{\lambda}_j(t)$, respectively, and

$$\mathrm{Im}\,\check{g}_+(x,t,\lambda_j(t)) = N_j^+(x,t), \qquad \mathrm{Ker}\,\check{g}_-(x,t,\bar{\lambda}_j(t)) = N_j^-(x,t), \qquad j = 1,\ldots,n(t). \tag{3.3.16}$$

In [14] it is established that for each set of time dependent scattering data there exists a unique solution $\check{g}_\pm(x,t,\lambda)$ satisfying $\det \check{g}_+(x,t,\lambda) = a(\lambda,t)$. In addition to this, and as a consequence of the structure of the input data the solution satisfies $\det \check{g}_+(x,t,0) = a(0,t) = (-1)^{n_1(t)}$, the involutions

$$\begin{aligned} \check{g}_+^\dagger(x,t,\bar{\lambda}) &= \check{g}_-(x,t,\lambda) \\ \bar{\check{g}}_\pm(x,t,-\bar{\lambda}) &= \check{g}_\pm(x,t,\lambda) \end{aligned} \tag{3.3.17}$$

and the relation

$$\check{g}_+(x,t,0)\check{g}_-(x,t,0) = \mathbb{I}. \tag{3.3.18}$$

Using these results and the asymptotic form of $\check{g}_+(x,t,\lambda)$ as $x \longrightarrow -\infty$ the matrix

$$\tilde{\Omega}^2(x,t) \stackrel{\text{def}}{=} \mathcal{E}(-\sigma_3)^{n_1(t)}\check{g}_+(x,t,0)\mathcal{E}^{-1} \tag{3.3.19}$$

can be deduced to have the form

$$\tilde{\Omega}^2(x,t) = \exp\left(\frac{i\varphi(x,t)}{2}\sigma_3\right), \tag{3.3.20}$$

so that $\tilde{\Omega}(x,t)$ can be uniquely determined to have the form (3.3.11) due to the requirement (3.3.7).

It is now clear that the matrices

$$\begin{aligned} g_+(x,t,\lambda) &= \check{g}_+(x,t,\lambda)\mathcal{E}^{-1}\tilde{\Omega}^{-1}(x,t), \\ g_-(x,t,\lambda) &= \tilde{\Omega}(x,t)\mathcal{E}\check{g}_-(x,t,\lambda) \end{aligned} \tag{3.3.21}$$

provide a solution to the Riemann problem in question in terms of the Riemann problem with the standard normalisation. This solution is unique by virtue of the uniqueness of the $\check{g}_\pm(x,t,\lambda)$ for a given set of scattering data as the input of the problem.

As $g_\pm(x,t,\lambda)$ can be expressed in terms of the solutions $\check{g}_\pm(x,t,\lambda)$ it is much less laborious to calculate $\varphi(x,t)$ directly from these simple solutions using

$$\exp\left(\frac{i\varphi(x,t)}{2}\sigma_3\right) = \mathcal{E}(-\sigma_3)^{n_1(t)}\check{g}_+(x,t,0)\mathcal{E}^{-1}, \tag{3.3.22}$$

with the condition

$$\lim_{x\longrightarrow-\infty} \varphi(x,t) = 0,$$

rather than to follow steps (3.3.21), (3.3.12). This observation will be exploited in the later chapters where some explicit expressions for $\check{g}_{\pm}(x,t,\lambda)$ are given. Using (3.3.22) it will be shown that these lead to the 'multi-soliton' solutions to specific problems.

This completes the construction of the transform **ist** given by (3.3.1). From the details of this formulation and assertions $1''-4''$, all of which are proved in [14], it is clear that this map is injective. As a final point, and for the benefit of the next section, it will be useful to make the following definition.

Definition 3.6 *For arbitrary $t \in \mathbb{R}$ and $n_1(t) \in \mathbb{N}$ define the subset $\check{\mathcal{M}}_N \subset \mathcal{M}_N$ implicitly in terms of the image of the map* **ist**. *That is,*

$$\bigcup_{q\in\mathbb{Z}} \check{\mathcal{M}}_{2q+n_1(t)} \stackrel{\text{def}}{=} \mathbf{ist}\left(\bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{n_1(t),r}_{n_1(t)\mathrm{mod}2}\right). \tag{3.3.23}$$

It should be stressed that this definition only gives $\check{\mathcal{M}}_N$ implicitly in terms of the specific form of a subset of scattering data.

3.4 $\mathbf{ist} = (\mathbf{dst})^{-1}$

In section 3.2 the map

$$\mathbf{dst} : \bigcup_{p\in\mathbb{Z}} \mathcal{M}_{2p+N} \to \mathcal{D}_{N\mathrm{mod}2},$$

was constructed for $N \in \mathbb{Z}$ with

$$\bigcup_{p\in\mathbb{Z}} \hat{\mathcal{M}}_{2p+N} \stackrel{\text{def}}{=} \mathbf{dst}^{-1}\left(\bigcup_{q\in 2\mathbb{N}+N\mathrm{mod}2,r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{N\mathrm{mod}2}\right),$$

and such that the restriction

$$\mathbf{dst} : \bigcup_{p\in\mathbb{Z}} \hat{\mathcal{M}}_{2p+N} \xrightarrow{1-1} \bigcup_{q\in 2\mathbb{N}+N\mathrm{mod}2,r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{N\mathrm{mod}2}.$$

In section 3.3 the map

$$\mathbf{ist} : \bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{n_1(t),r}_{n_1(t)\mathrm{mod}2} \xrightarrow{1-1} \bigcup_{q\in\mathbb{Z}} \mathcal{M}_{2q+n_1(t)},$$

was formulated for $t \in \mathbb{R}$, $n_1(t) \in \mathbb{N}$ and

$$\bigcup_{q\in\mathbb{Z}} \check{\mathcal{M}}_{2q+n_1(t)} \stackrel{\mathrm{def}}{=} \mathbf{ist}\left(\bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{n_1(t),r}_{n_1(t)\mathrm{mod}2}\right).$$

The purpose of this section is to make assertion $1'''$ below.

$1'''$) For an arbitrary element of $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}$ denoted

$$(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)),$$

let $g_\pm(x,t,\lambda)$ be the solution to the Riemann problem (3.3.3)-(3.3.7). Then:

- the matrices $F_\pm(x,t,\lambda)$ constructed from $g_\pm(x,t,\lambda)$ according to (3.3.9) satisfy the auxiliary linear problem (3.3.10) (assertion $3''$).

- these matrices are composed of the time dependent Jost solutions $T_\pm(x,t,\lambda)$, (which are defined by the asymptotics (3.2.11)), of the auxiliary linear problem (3.3.10) as

$$\begin{aligned} F_+(x,t,\lambda) &= \frac{1}{a(\lambda,t)}\left(T^{(1)}_-(x,t,\lambda),T^{(2)}_+(x,t,\lambda)\right) \\ F_-(x,t,\lambda) &= \left(T^{(1)}_+(x,t,\lambda),T^{(2)}_-(x,t,\lambda)\right), \end{aligned} \tag{3.4.1}$$

  where $a(\lambda,t)$ is given in terms of the input scattering data by (3.2.36). Of course, these solutions will obey the involutions already deduced in section 3.2 eg. $T_\pm(x,t,-\lambda)=-\sigma_3T_\pm(x,t,\lambda)\sigma_2,\ldots$.

- the time dependent reduced transition matrix constructed from the Jost solutions according to (3.2.17) has transition coefficients $a(\cdot,t),b(\cdot,t)$ and normalisation coefficients $\gamma_1(t),\ldots\gamma_{n_1(t)+2n_2(t)}(t)$ at the zeroes $\lambda_1(t),\ldots\lambda_{n_1(t)+2n_2(t)}(t)$ of the transition coefficient $a(\cdot,t)$.

This assertion is proved in [14]. By applying **ist** then **dst** in turn at an arbitrary time $t\in\mathbb{R}$ it shows that $\check{\mathcal{M}}_N\subset\hat{\mathcal{M}}_N$ and that for configurations $(\varphi(\cdot,t),\varpi(\cdot,t))\in\check{\mathcal{M}}_N$ the maps **dst** and **ist** are in fact the inverse of one another.

3.5 A brief recap of the results so far

Before moving on to the time dependence of the scattering data it is a good idea to recap and clarify the results obtained for the maps **dst** and **ist**.

For $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \check{\mathcal{M}}_N$ a set of Jost solutions $T_\pm(x,t,\lambda)$ are fixed uniquely by

$$\begin{aligned} \frac{\partial T_\pm(x,t,\lambda)}{\partial x} &= U(x,t,\lambda)T_\pm(x,t,\lambda), \\ T_\pm(x,t,\lambda) \to E_\pm(x,\lambda), &\qquad \text{as } x \to \pm\infty, \end{aligned} \tag{3.5.1}$$

where $U(x,t,\lambda), E_\pm(x,\lambda)$ are given by (3.2.2), (3.2.9) respectively. From these Jost solutions follows a set of scattering data which is unique for different pairs $T_\pm(x,t,\lambda)$. The determination of this data constitutes the direct scattering transform **dst**.

Now consider the inverse scattering transform **ist**. Given the set of scattering data just constructed, the map **ist** allows $T_\pm(x,t,\lambda)$ to be reconstructed and therefore $(\varphi(\cdot,t),\varpi(\cdot,t))$ can be determined from the differential equation (3.5.1).

The relevance of these direct and inverse scattering transforms to the initial-boundary value problems of Type $\mathbf{A}_N$ will become evident when the time evolution of $(\varphi(\cdot,t),\varpi(\cdot,t))$ as governed by the sine-Gordon system (3.1.1) is translated into a time evolution for the scattering data which results from an arbitrary set of Jost solutions. This is the subject of the next section.

3.6 The time evolution of the scattering data

In this section the time evolution of the scattering data $\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t))$ will be studied when (φ,ϖ) are forced to evolve according to the initial-boundary value problem of Type $\mathbf{A}_N$ which is defined by the initial conditions

$$(\varphi(\cdot,0),\varpi(\cdot,0)) = (\varphi_N,\varpi_0).$$

The result will be the bijective time evolution map

$$\tau_t : \mathcal{D}^{q,r,s}_{N \bmod 2} \xrightarrow{1-1} \mathcal{D}^{q,r,s}_{N \bmod 2}, \tag{3.6.1}$$

from a set of 'initial' scattering data $\mathbf{dst}(\varphi_N, \varpi_0)$ to a set of 'time evolved' scattering data $\mathbf{dst}(\varphi(\cdot,t), \varpi(\cdot,t))$.

It is τ_t which replaces the time evolution map ς_t introduced in chapter 2 for the linearised problems and as such forms the second stage in the inverse scattering method for solving problems in the set $\mathbf{A}_N$. There is, however, one crucially important similarity between τ_t and ς_t. Namely, both of them result from solving Cauchy problems for a set of *linear* ordinary differential equations ((2.3.7) in the case of ς_t).

3.6.1 The zero curvature representation for the sine-Gordon equation

Let (φ, ϖ) be the solution to the problem of Type $\mathbf{A}_N$ defined by the initial data $(\varphi_N, \varpi_0) \in \mathcal{M}_N$. From this construct the $\mathbb{C}$ valued matrix $V(x,t,\lambda)$

$$V(x,t,\lambda) \stackrel{\text{def}}{=} \frac{1}{4i}\left(\frac{\partial\varphi}{\partial x}(x,t)\sigma_3 + \left(\lambda - \frac{1}{\lambda}\right)\sin\frac{\varphi(x,t)}{2}\sigma_1 + \left(\lambda + \frac{1}{\lambda}\right)\cos\frac{\varphi(x,t)}{2}\sigma_2\right). \tag{3.6.2}$$

This and the matrix $U(x,t,\lambda)$ defined in (3.2.2) must satisfy a remarkable relation given by the following proposition.

Proposition 3.7 *As a result of*

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= \varpi\\ \frac{\partial\varpi}{\partial t} &= \frac{\partial^2\varphi}{\partial x^2} - \sin\varphi, \qquad \forall x,t \in \mathbb{R},\end{aligned} \tag{3.6.3}$$

it follows that

$$\left(\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V]\right) = 0, \qquad \forall\lambda \in \mathbb{C}\setminus\{0\}. \tag{3.6.4}$$

Therefore the overdetermined system of linear equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= U(x,t,\lambda)f\\ \frac{\partial f}{\partial t} &= V(x,t,\lambda)f,\end{aligned} \tag{3.6.5}$$

for a $\mathbb{C}^2$ valued column vector f are compatible.

It should be noted that variants of (3.6.4) also hold for all other equations solvable by the inverse scattering method.

The linear system (3.6.5) have a natural geometric interpretation. The matrix functions U, V may be considered as local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ where the space-time $\mathbb{R}^2$ is the base and the matrix f takes values in the fibre $\mathbb{C}^2$ and λ is a subsidiary complex parameter. Equations (3.6.5) show that f is a covariantly constant vector while (3.6.4) amounts to saying that the (U, V) connection has zero curvature. For this reason a representation of a nonlinear equation in the form (3.6.4) is called a 'zero curvature' representation.

Let ψ denote *any* 2×2 complex valued matrix of rank 2 which solves

$$\frac{\partial \psi}{\partial x} = U(x, t, \lambda)\psi \tag{3.6.6}$$

$$\frac{\partial \psi}{\partial t} = V(x, t, \lambda)\psi, \tag{3.6.7}$$

$\forall (x, t) \in \mathbb{R}^2$, $\lambda \in \mathbb{C} \setminus \{0\}$. All such solutions are related by a change in normalisation $\psi(x, t, \lambda) \to \psi(x, t, \lambda)N(\lambda)$ with $N(\lambda)$ an arbitrary matrix. Also, (3.6.6), (3.6.7) imply

$$\begin{aligned} \frac{\partial}{\partial x} \det \psi &= \mathrm{tr} U(x, t, \lambda) \det \psi = 0 \\ \frac{\partial}{\partial t} \det \psi &= \mathrm{tr} V(x, t, \lambda) \det \psi = 0. \end{aligned} \tag{3.6.8}$$

3.6.2 Determining the evolution of the scattering data

The transition matrix introduced in section 3.2 can be uniquely constructed from any solution ψ as

$$T(x, y, t, \lambda) = \psi(x, t, \lambda)\psi^{-1}(y, t, \lambda), \tag{3.6.9}$$

so that with (φ, ϖ) a solution to the sine-Gordon equation $T(x, y, t, \lambda)$ must satisfy

$$\frac{\partial T}{\partial t}(x, y, t, \lambda) = V(x, t, \lambda)T(x, y, t, \lambda) - T(x, y, t, \lambda)V(y, t, \lambda). \tag{3.6.10}$$

It then follows that the time evolution of the Jost solutions is given by

$$\frac{\partial T_\pm}{\partial t}(x, t, \lambda) = V(x, t, \lambda)T_\pm(x, t, \lambda) - \frac{1}{4i}(\lambda + \lambda^{-1})T_\pm(x, t, \lambda)\sigma_3 \quad \forall \lambda \in \mathbb{R} \setminus \{0\}, \tag{3.6.11}$$

so that

$$\frac{\partial T_-^{(1)}}{\partial t}(x,t,\lambda) = V(x,t,\lambda)T_-^{(1)}(x,t,\lambda) - \frac{1}{4i}(\lambda+\lambda^{-1})T_-^{(1)}(x,t,\lambda)$$
$$\frac{\partial T_+^{(2)}}{\partial t}(x,t,\lambda) = V(x,t,\lambda)T_+^{(2)}(x,t,\lambda) + \frac{1}{4i}(\lambda+\lambda^{-1})T_+^{(2)}(x,t,\lambda), \qquad (3.6.12)$$

will hold $\forall \lambda \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Im}\lambda \geq 0$.

The reduced transition matrix $T(\lambda,t)$ satisfies

$$\frac{\partial T}{\partial t}(\lambda,t) = \frac{1}{4i}\left(\lambda + \frac{1}{\lambda}\right)[\sigma_3, T(\lambda,t)], \quad \forall \lambda \in \mathbb{R} \setminus \{0\}, \qquad (3.6.13)$$

so that the time dependence of the transition coefficients is governed by the *linear* ordinary differential equations

$$\frac{\partial a}{\partial t}(\lambda,t) = 0 \qquad \operatorname{Im}\lambda \geq 0 \qquad (3.6.14)$$

$$\frac{\partial b}{\partial t}(\lambda,t) = \frac{i}{2}(\lambda+\lambda^{-1})b(\lambda,t) \qquad \lambda \in \mathbb{R}. \qquad (3.6.15)$$

Obviously, since $(\varphi(\cdot,0),\varpi(\cdot,0)) \equiv (\varphi_N,\varpi_0)$, the initial data appropriate to these equations are the transition coefficients $a(\cdot,0), b(\cdot,0)$ appearing in

$$(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)+2n_3(0)}(0)) \stackrel{\text{def}}{=} \mathbf{dst}(\varphi_N,\varpi_0), \qquad (3.6.16)$$

so that

$$a(\lambda,t) = a(\lambda,0) \qquad \operatorname{Im}\lambda \geq 0, \qquad (3.6.17)$$

$$b(\lambda,t) = e^{\frac{i}{2}(\lambda+\frac{1}{\lambda})t}b(\lambda,0) \qquad \lambda \in \mathbb{R}. \qquad (3.6.18)$$

From the first of these it follows that the number and position of the zeroes of $a(\cdot,t)$ remains fixed as (φ,ϖ) evolve according to the problem defined by initial data (φ_N,ϖ_0),

$$n_i(t) = n_i(0) \qquad \forall t \in \mathbb{R} \qquad i = 1,2,3$$

$$\lambda_j(t) = \lambda_j(0), \qquad \forall t \in \mathbb{R} \qquad j = 1,\ldots,n(0).$$

So, (3.2.28) becomes

$$T_-^{(1)}(x,t,\lambda_j(0)) = \gamma_j(t)T_+^{(2)}(x,t,\lambda_j(0)), \quad j = 1,\ldots,n(0), \qquad (3.6.19)$$

and consistency of this with (3.6.12) implies the *linear* ordinary differential equations for the time evolution of the normalisation coefficients

$$\frac{\partial\gamma_j}{\partial t}(t) = \frac{i}{2}(\lambda_j(0) + \lambda_j^{-1}(0))\gamma_j(t), \qquad j = 1, \ldots, n(0) \tag{3.6.20}$$

and once again the normalisation coefficients appearing in (3.6.16) must be taken as initial data so that

$$\gamma_j(t) = e^{\frac{i}{2}\left(\lambda_j(0)+\frac{1}{\lambda_j(0)}\right)t}\gamma_j(0), \qquad j = 1, \ldots, n(0). \tag{3.6.21}$$

3.6.3 The time evolution map τ_t

Definition 3.8 *For $t \in \mathbb{R}$ let $\tau_t, t \in \mathbb{R}$ denote the bijection*

$$\tau_t : \mathcal{D}^{n_1(0),n_2(0),n_3(0)}_{N\text{mod}2} \xrightarrow{1-1} \mathcal{D}^{n_1(0),n_2(0),n_3(0)}_{N\text{mod}2}, \tag{3.6.22}$$

from the initial scattering data defined in (3.6.16) to the time evolved data given by (3.6.17), (3.6.18) and (3.6.21). That is,

$$\begin{aligned}\tau_t : (a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)+2n_3(0)}(0)) \mapsto \\ (a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(0)+2n_2(0)+2n_3(0)}(t)).\end{aligned} \tag{3.6.23}$$

3.7 Piecing together the inverse scattering method

The results of the three preceeding sections can now be pieced together to form the inverse scattering method for solving a *subset* of the problems in the set $\mathbf{A}_N$.

Definition 3.9 *For $N \in \mathbb{Z}$ let $\check{\mathbf{A}}_N$ denote the subset of problems of Type $\mathbf{A}_N$ which have an initial condition in the subspace $\check{\mathcal{M}}_N \subset \mathcal{M}_N$.*

In sections 3.2-3.6 it has been proved that when the initial data (φ_N, ϖ_0) is an element of $\check{\mathcal{M}}_N$ then the image of the composite map $\mathbf{ist} \circ \tau_t \circ \mathbf{dst}$ is such that

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = (\mathbf{ist} \circ \tau_t \circ \mathbf{dst})(\varphi_N, \varpi_0),$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\breve{\mathcal{M}}_N \arrow[r, "\mathbf{dst}"] \arrow[d, "\mathrm{T}_t"'] & \mathcal{D}_{N\mathrm{mod}2} \arrow[d, "\tau_t"] \\
\breve{\mathcal{M}}_N & \mathcal{D}_{N\mathrm{mod}2} \arrow[l, "\mathbf{ist} = (\mathbf{dst})^{-1}"]
\end{tikzcd}$$

$\mathrm{T}_t \stackrel{\mathrm{def}}{=}$ time evolution map defined by a *nonlinear* problem of Type $\mathbf{A}_N$

$\tau_t =$ bijective time evolution map governed by a set of *linear* o.d.e's

Figure 3.1: **The inverse scattering method for solving problems in $\breve{\mathbf{A}}_N$**

is also an element of $\breve{\mathcal{M}}_N$ and the resulting functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ solve the initial-boundary value problem of Type $\mathbf{A}_N$ with initial data $(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_N, \varpi_0)$. Notice that, although **dst** was formulated for arbitrary initial data in $\mathcal{M}_N$, the inverse map (if one exists) was not and, as a result, attention must focus on the subset $\breve{\mathcal{M}}_N$.

As with the method of Fourier transforms studied in chapter 2 the time evolution map

$$\mathrm{T}_t : \breve{\mathcal{M}}_N \to \breve{\mathcal{M}}_N,$$

giving the solution to the set of problems $\breve{\mathbf{A}}_N$ as

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = \mathrm{T}_t(\varphi_N, \varpi_0),$$

can be expressed by the commutative diagram in figure 3.1.

However, this is not quite the whole story. It remains to recall that in definition 3.6 $\breve{\mathcal{M}}_N$ is only defined implicitly in terms of the map **ist** applied to a subset of $\mathcal{D}_{N\mathrm{mod}2}$. Currently, the only way to deduce an element of $\breve{\mathcal{M}}_N$, (which can then be used as initial data), is to choose suitable scattering data in $\hat{\mathcal{D}}^{q,r}_{q\mathrm{mod}2}$ for some $q, r \in \mathbb{N}$ and then to apply the inverse scattering transform **ist** to this. In other words, the inverse scattering method developed in this chapter *cannot* be used to solve the problem in the set $\mathbf{A}_N$ defined by initial data (φ_N, ϖ_0) for *any* $(\varphi_N, \varpi_0) \in \mathcal{M}_N$ given *a priori* ! But, if it is only required that the initial configuration be deduced *a posteriori* then problems in the set $\breve{\mathbf{A}}_N$ can be solved by applying the composite map $\mathbf{ist} \circ \tau_t$ to appropriate elements of the space of initial scattering data. This will be seen explicitly in chapter 6 where sets of *soliton* scattering data will be chosen at $t = 0$ and the resulting solutions found by applying $\mathbf{ist} \circ \tau_t$ to these.

This completes a detailed discussion of the inverse scattering method for 'solving' *some* elements of the set $\mathbf{A}_N$.

Chapter 4

The inverse scattering method for solving problems of Type $\mathbf{B}_{P,0}$

4.1 Introduction

In chapter 3 the inverse scattering method was developed for solving a subset of the initial-boundary value problems of Type $\mathbf{A}_N$. In this chapter the method is developed so that it can be used to solve a subset of the problems of Type $\mathbf{B}_{P,0}$. Chapter 5 continues this development so that some of the problems of Type $\mathbf{B}_{P,Q\neq 0}$ can also be solved in this way.

Recall from chapter 1 that an initial-boundary value problem of Type $\mathbf{B}_{P,0}$ is defined by a pair of functions $(\varphi_{P,0}, \varpi_{P,0}) \in \mathcal{N}_{P,0}$. The problem associated with this pair is to determine the functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ with $(x,t) \in (-\infty, 0] \times \mathbb{R}$ which satisfy:

- the sine-Gordon system

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varpi, \\ \frac{\partial \varpi}{\partial t} &= \frac{\partial^2 \varphi}{\partial x^2} - \sin\varphi \qquad \forall\,(x,t) \in (-\infty,0] \times \mathbb{R}. \end{aligned} \tag{4.1.1}$$

- the boundary conditions

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \in \mathcal{N}_{P,0} \qquad \forall t \in \mathbb{R}. \tag{4.1.2}$$

- the 'initial' conditions

$$(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_{P,0}, \varpi_{P,0}) \in \mathcal{N}_{P,0}. \tag{4.1.3}$$

This chapter formulates the inverse scattering method for solving a *subset* of these problems - the subset being defined by a restricted phase space $\check{\mathcal{N}}_{P,0} \subset \mathcal{N}_{P,0}$.

4.2 Certain solutions to problems of Type $\mathbf{A}_N$ also solve problems in $\mathbf{B}_{P,0}$

The purpose of this section is to show how the solutions to certain problems in the set $\mathbf{A}_N$ also solve problems of Type $\mathbf{B}_{P,0}$.

For some $N \in \mathbb{Z}$ suppose (φ, ϖ) is the solution to the initial-boundary value problem of Type $\mathbf{A}_N$ defined by the initial conditions

$$(\varphi(\cdot, 0), \varpi(\cdot, 0)) = (\varphi_N, \varpi_0) \in \hat{\mathcal{M}}_N.$$

According to the results of chapter 3 it follows that at any time $t \in \mathbb{R}$ the pair $(\varphi(\cdot, t), \varpi(\cdot, t))$ is an element of $\hat{\mathcal{M}}_N$ and so

$$\mathbf{dst}(\varphi(\cdot, t), \varpi(\cdot, t)) = (a(\cdot, t), b(\cdot, t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}^{n_1(t), n_2(t)}_{N \bmod 2} \tag{4.2.1}$$

for some $n_1(t) \in 2\mathbb{N} + N \bmod 2$, $n_2(t) \in \mathbb{N}$, $n(t) = n_1(t) + 2n_2(t)$ and $a(\cdot, t)$ determined by $b(\cdot, t)$ and a set of simple zeroes

$$\left\{\lambda_1(t), \ldots, \lambda_{n_1(t)+2n_2(t)}(t)\right\},$$

by (3.2.36).

In addition let $\tilde{\varphi}$, $\tilde{\varpi} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$(\tilde{\varphi}(x, t), \tilde{\varpi}(x, t)) = (\varphi(-x, t), \varpi(-x, t)), \qquad \forall t \in \mathbb{R},$$

so that $(\tilde{\varphi}, \tilde{\varpi})$ also satisfy the sine-Gordon system (1.2.2).

4.2.1 The zero curvature representation and gauge transformations

Suppose that $U(x, t, \lambda), V(x, t, \lambda)$, (resp. $\tilde{U}(x, t, \lambda), \tilde{V}(x, t, \lambda)$) are the matrices introduced in (3.2.2), (3.6.2) and constructed from $(\varphi(x, t), \varpi(x, t))$, (resp. $(\tilde{\varphi}(x, t), \tilde{\varpi}(x, t))$). Since both (φ, ϖ) and $(\tilde{\varphi}, \tilde{\varpi})$ solve the sine-Gordon equation (1.2.2) it follows from proposition 3.7 that the systems of linear equations

$$\begin{aligned}
\frac{\partial f}{\partial x} &= U(x, t, \lambda) f, \\
\frac{\partial f}{\partial t} &= V(x, t, \lambda) f, \\
\frac{\partial \tilde{f}}{\partial x} &= \tilde{U}(x, t, \lambda) \tilde{f}, \\
\frac{\partial \tilde{f}}{\partial t} &= \tilde{V}(x, t, \lambda) \tilde{f},
\end{aligned} \tag{4.2.2}$$

are compatible $\forall x, t \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

As explained in section 3.6, the matrices $(U(x,t,\lambda), V(x,t,\lambda))$, $(\tilde{U}(x,t,\lambda), \tilde{V}(x,t,\lambda))$ can be regarded as local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$. A local change of linear frame in the fibre induced by matrix $\mathcal{G}(x,t,\lambda)$

$$f \to \hat{f} = \mathcal{G} f, \tag{4.2.3}$$

is accompanied by the transformation

$$\begin{aligned} U \to \hat{U} &= \frac{\partial \mathcal{G}}{\partial x} + \mathcal{G} U \mathcal{G}^{-1} \\ V \to \hat{V} &= \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G} V \mathcal{G}^{-1}, \end{aligned} \tag{4.2.4}$$

The zero curvature condition is invariant under such *gauge transformations.*

Let $\psi, \tilde{\psi}$ denote any 2×2 matrices of rank 2 which solve

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= U(x,t,\lambda)\psi, \\ \frac{\partial \psi}{\partial t} &= V(x,t,\lambda)\psi, \\ \frac{\partial \tilde{\psi}}{\partial t} &= \tilde{U}(x,t,\lambda)\tilde{\psi}, \\ \frac{\partial \tilde{\psi}}{\partial t} &= \tilde{V}(x,t,\lambda)\tilde{\psi}, \end{aligned} \tag{4.2.5}$$

$\forall x, t \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0\}$. All the solutions to these equations are related by a change in normalization

$$\psi(x,t,\lambda) \to \psi(x,t,\lambda) N(\lambda),$$

$$\tilde{\psi}(x,t,\lambda) \to \tilde{\psi}(x,t,\lambda) \tilde{N}(\lambda),$$

with $N(\lambda)$, $\tilde{N}(\lambda)$ arbitrary 2×2 matrices. Therefore let $\psi(x,t,\lambda)N(\lambda)$, $\tilde{\psi}(x,t,\lambda)\tilde{N}(\lambda)$ denote the general solution to (4.2.5).

4.2.2 Constraining solutions by demanding a gauge relation

Throughout the rest of this section suppose that $N(\lambda)$, $\tilde{N}(\lambda)$ can be chosen so that

$$\tilde{\psi}(x,t,\lambda)\tilde{N}(\lambda)N^{-1}(\lambda)\psi^{-1}(x,t,\lambda) = L(x,t,\lambda) \qquad \forall x, t \in \mathbb{R},\ \lambda \in \mathbb{C} \setminus \{0\}, \tag{4.2.6}$$

where

$$
\begin{aligned}
L(x,t,\lambda) \stackrel{\text{def}}{=} & \Big((\lambda^2-\lambda^{-2})\cos a_1(x,t)+a_2(x,t)\Big)\,\mathbb{I} \\
& +i\Big((\lambda^2+\lambda^{-2})\sin a_1(x,t)+a_3(x,t)\Big)\,\sigma_3 \\
& +i\Big((\lambda-\lambda^{-1})d_1(x,t)+(\lambda+\lambda^{-1})f_1(x,t)\Big)\,\sigma_1 \\
& +i\Big((\lambda+\lambda^{-1})d_2(x,t)+(\lambda-\lambda^{-1})f_2(x,t)\Big)\,\sigma_2, \qquad (4.2.7)
\end{aligned}
$$

for some functions $a_i, d_j, f_j,\ i=1,\ldots,3,\ j=1,2$ such that $\forall\, t\in\mathbb{R}$

$$
d_1(0,t)=\lim_{x\to\infty} d_1(x,t)=\lim_{x\to\infty} f_2(x,t)=0. \qquad (4.2.8)
$$

This constraint is the same as demanding that $\psi(x,t,\lambda)N(\lambda)$ and $\tilde{\psi}(x,t,\lambda)\tilde{N}(\lambda)$ can be related by a gauge transformation of the form (4.2.7), (4.2.8). The matrices $\psi(x,t,\lambda)$ and $\tilde{\psi}(x,t,\lambda)$ are calculated according to subsection 4.2.1 and consistency of (4.2.6) and (4.2.7) forces all the coefficients to be real valued with d_1, d_2 'even' in the variable x and a_1, a_2, a_3, f_1, f_2 'odd'.

Demanding that relations (4.2.6), (4.2.7) hold amounts to imposing a constraint on the solution (φ, ϖ). The consequences of this will be studied in the next subsection.

4.2.3 Solutions which meet the constraint also solve a problem in $\mathbf{B}_{P,0}$

The solutions to problems of Type $\mathbf{A}_N$ which satisfy the constraint (4.2.6), (4.2.7) for some choice of $N(\lambda)$, $\tilde{N}(\lambda)$ also solve an initial boundary value problem of Type $\mathbf{B}_{P,0}$ for some $P\in\mathbb{R}$. To see this substitute (4.2.6) into (4.2.5) so that for all $x,t\in\mathbb{R}, \lambda\in\mathbb{C}\setminus\{0\}$, the functions $a_i, d_j, f_j,\ i=1,\ldots,3,\ j=1,2$ and any admissible solution must satisfy

$$
\begin{aligned}
\frac{\partial L}{\partial x}(x,t,\lambda) &= \tilde{U}(x,t,\lambda)L(x,t,\lambda)-L(x,t,\lambda)U(x,t,\lambda), \\
\frac{\partial L}{\partial t}(x,t,\lambda) &= \tilde{V}(x,t,\lambda)L(x,t,\lambda)-L(x,t,\lambda)V(x,t,\lambda), \qquad (4.2.9)
\end{aligned}
$$

and other relations which result from additional differentiations. Some of these constraints are

$$
\begin{aligned}
\frac{\partial\varphi}{\partial x}(x,t) &= d_1(x,t)\cos\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right) - d_2(x,t)\sin\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right),\\
\frac{\partial\varpi}{\partial x}(x,t) &= \frac{\partial d_1}{\partial t}(x,t)\cos\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right) - \frac{\partial d_2}{\partial t}(x,t)\sin\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right)\\
&\qquad -\frac{d_1}{4}(x,t)(\varpi(-x,t)+\varpi(x,t))\sin\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right)\\
&\qquad -\frac{d_2}{4}(x,t)(\varpi(-x,t)+\varpi(x,t))\cos\left(\frac{\varphi(-x,t)+\varphi(x,t)}{4}\right),\\
\frac{\partial d_2}{\partial t}(x,t) &= \frac{a_2}{4}(x,t)\left(\sin\frac{\varphi}{2}(x,t)-\sin\frac{\varphi}{2}(-x,t)\right),\\
\frac{\partial d_1}{\partial t}(x,t) &= \frac{a_2}{4}(x,t)\left(\cos\frac{\varphi}{2}(x,t)-\cos\frac{\varphi}{2}(-x,t)\right),
\end{aligned}
\tag{4.2.10}
$$

so that on defining $P \in \mathbb{R}$ by

$$
P \stackrel{\text{def}}{=} d_2(0,t) \tag{4.2.11}
$$

the functions φ, ϖ must satisfy

$$
\begin{aligned}
\left.\frac{\partial\varphi}{\partial x}(x,t)\right|_{x=0} + P\sin\frac{\varphi}{2}(0,t) &= 0,\\
\left.\frac{\partial\varpi}{\partial x}(x,t)\right|_{x=0} + \frac{P}{2}\varpi(0,t)\cos\frac{\varphi}{2}(0,t) &= 0 \qquad \forall t \in \mathbb{R}.
\end{aligned}
\tag{4.2.12}
$$

Fixing $t \in \mathbb{R}$ it follows that upon a restriction of the domain to $x \in (-\infty, 0]$,

$$
(\varphi(\cdot,t),\varpi(\cdot,t))|_{(-\infty,0]} \in \mathcal{N}_{P,0}, \tag{4.2.13}
$$

with $P = d_2(0,t)$.

So, if a solution to an initial-boundary value problem of Type $\mathbf{A}_N$ is such that constraint (4.2.6), (4.2.7), (4.2.8) can be made to hold by an appropriate choice of $N(\lambda), \check{N}(\lambda)$ then it will also satisfy the differential equations (4.2.9) and so solve a problem of Type $\mathbf{B}_{P,0}$ for some $P \in \mathbb{R}$.

4.2.4 The constraint picks out subspaces of scattering data

In this subsection $t \in \mathbb{R}$ is fixed and at this time the constraint (4.2.6), (4.2.7), (4.2.8) is reformulated as a constraint on the scattering data (4.2.1).

The transition matrix built from $(\varphi(\cdot,t),\varpi(\cdot,t))$ is uniquely constructed from *any* ψ according to

$$T(x,y,t,\lambda)=\psi(x,t,\lambda)\psi^{-1}(y,t,\lambda).$$

As a consequence of $\tilde{U}(x,t,\lambda)=U(-x,t,\lambda)$ and the involution $-\sigma_2 U(x,t,\lambda^{-1})\sigma_2=U(x,t,\lambda)$ it follows that the constraint (4.2.6), (4.2.7), (4.2.8) can be rewritten in terms of this transition matrix as

$$T(x,y,t,\lambda)=j(-x,t,\lambda^{-1})T(-x,-y,t,\lambda^{-1})j^{-1}(-y,t,\lambda^{-1})\quad \forall x,y,\in\mathbb{R},\lambda\in\mathbb{C}\setminus\{0\}, \tag{4.2.14}$$

with

$$j(x,t,\lambda)\stackrel{\text{def}}{=}\sigma_2 L(x,t,\lambda). \tag{4.2.15}$$

In turn this relation coupled with the involution $T_+(x,t,\lambda)=-\sigma_3 T_+(x,t,-\lambda)\sigma_2$ for the Jost solution implies that for $\lambda\in\mathbb{R}\setminus\{0\}$,

$$T_+(x,t,-\lambda^{-1})=-\sigma_3 j^{-1}(x,t,\lambda^{-1})T_-(-x,t,\lambda)\tilde{j}^{-1}(t,\lambda^{-1}), \tag{4.2.16}$$

where

$$\tilde{j}(t,\lambda^{-1})\stackrel{\text{def}}{=}\frac{1}{2}\lim_{y\to-\infty}\sigma_2\exp\left(\frac{iy}{4}(\lambda-\lambda^{-1})\sigma_3\right)(1-i\sigma_1)(-i\sigma_3)^N$$
$$j^{-1}(-y,t,\lambda^{-1})(1+i\sigma_1)\exp\left(-\frac{iy}{4}(\lambda-\lambda^{-1})\sigma_3\right). \tag{4.2.17}$$

As a result of (3.6.11) for the time evolution of $T_\pm(x,t,\lambda)$, the relation (4.2.16) continues to imply (4.2.9) and therefore (4.2.13).

To proceed it is necessary to consider the cases of N odd/even separately.

N odd

With $N\in 2\mathbb{Z}+1$,

$$\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t))=(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t))\in\hat{\mathcal{D}}_1^{n_1(t),n_2(t)}, \tag{4.2.18}$$

for some $n_1(t)\in 2\mathbb{N}+1$, $n_2(t)\in\mathbb{N}$.

From a detailed analysis of (4.2.9) it is possible to deduce that a_1, a_2, a_3, d_2, f_1 must have the asymptotics

$$a_1(x,t) \to \mp\frac{\pi N}{2},\ a_2(x,t) \to 0,\ a_3(x,t) \to \pm i^{N+1}\beta_3,$$

$$f_1(x,t) \to \pm i^{N+1}(\Lambda^{-1} - \Lambda),\ d_2(x,t) \to 0, \qquad x \to \pm\infty \quad \forall t \in \mathbb{R}, \tag{4.2.19}$$

with $\beta_3 \in \mathbb{R}, \Lambda \in \mathbb{R}^+$.

Using the equality $\det L(0,t,\lambda) = \det L(+\infty,t,\lambda)$ implied by (4.2.6) it follows that $\beta_3 = 2$,

$$L(0,t,\lambda) = (\lambda + \lambda^{-1})((\lambda - \lambda^{-1})\mathbb{I} + iP\sigma_2), \tag{4.2.20}$$

$$L(+\infty,t,\lambda) = i^{N+2}(\lambda + \lambda^{-1})((\lambda + \lambda^{-1})\sigma_3 + (\Lambda^{-1} - \Lambda)\sigma_1), \tag{4.2.21}$$

and $P^2 = (\Lambda + \Lambda^{-1})^2$ so that $|P|$ is constrained to be ≥ 2. The matrix $\jmath(t,\lambda^{-1})$ as defined by (4.2.17) is found to be

$$\jmath(t,\lambda^{-1}) = \frac{i((\lambda + \lambda^{-1})\sigma_1 + (\Lambda^{-1} - \Lambda)\sigma_2)}{(\lambda + \lambda^{-1})((\lambda + \lambda^{-1})^2 + (\Lambda^{-1} - \Lambda)^2)}, \tag{4.2.22}$$

so that both $L(0,t,\lambda)$ and $\jmath(t,\lambda^{-1})$ are independent of t.

Now consider the Jost solutions evaluated at $x = 0$. Denoting these by

$$T_\pm(0,t,\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} a_\pm(\lambda,t) & b_\pm(\lambda,t) \\ c_\pm(\lambda,t) & d_\pm(\lambda,t) \end{pmatrix}, \tag{4.2.23}$$

the constraint (4.2.16) translates into the relations

$$\begin{aligned}
a_+(-\lambda^{-1},t) &= \frac{(\lambda - \lambda^{-1})d_-(\lambda,t) - Pb_-(\lambda,t)}{\lambda + \lambda^{-1} + i\Lambda - i\Lambda^{-1}} \\
b_+(-\lambda^{-1},t) &= \frac{Pa_-(\lambda,t) - (\lambda - \lambda^{-1})c_-(\lambda,t)}{\lambda + \lambda^{-1} - i\Lambda + i\Lambda^{-1}} \\
c_+(-\lambda^{-1},t) &= \frac{(\lambda - \lambda^{-1})b_-(\lambda,t) + Pd_-(\lambda,t)}{\lambda + \lambda^{-1} + i\Lambda - i\Lambda^{-1}} \\
d_+(-\lambda^{-1},t) &= -\frac{(\lambda - \lambda^{-1})a_-(\lambda,t) + Pc_-(\lambda,t)}{\lambda + \lambda^{-1} - i\Lambda + i\Lambda^{-1}},
\end{aligned} \tag{4.2.24}$$

to be satisfied $\forall \lambda \in \mathbb{R} \setminus \{0\}$.

In sections 3.2, 3.6 it was deduced that $a_-(\cdot,t), c_-(\cdot,t), b_+(\cdot,t), d_+(\cdot,t)$ extend analytically into the upper half of the complex plane whereas $a_+(\cdot,t), c_+(\cdot,t), b_-(\cdot,t), d_-(\cdot,t)$

extend analytically into the lower half. The relations (4.2.24) are consistent with this analytic continuation so that they can be understood to hold for all λ in the appropriate domains of analyticity. As a result

$$\begin{aligned}&\operatorname{sign}(P)b_+(i\Lambda,t)-id_+(i\Lambda,t)=0\\&\operatorname{sign}(P)a_-(i\Lambda,t)-ic_-(i\Lambda,t)=0.\end{aligned}\tag{4.2.25}$$

The transition coefficient $a(\cdot,t)$ is defined in the domain $\mathrm{Im}\lambda\geq 0$ as

$$a(\lambda,t)=a_-(\lambda,t)d_+(\lambda,t)-c_-(\lambda,t)b_+(\lambda,t),\tag{4.2.26}$$

so that (4.2.24) imply that this coefficient satisfies the symmetry condition

$$a(-\lambda^{-1},t)=-h(\lambda)a(\lambda,t),\tag{4.2.27}$$

with

$$h(\lambda)\stackrel{\text{def}}{=}\frac{(\lambda+i\Lambda)(\lambda-i\Lambda^{-1})}{(\lambda-i\Lambda)(\lambda+i\Lambda^{-1})}.\tag{4.2.28}$$

For $\lambda\in\mathbb{R}$ the transition coefficient $b(\cdot,t)$ is given in terms of the coefficients appearing in (4.2.23) by

$$\begin{aligned}b(\lambda,t)&=c_-(\lambda,t)a_+(\lambda,t)-a_-(\lambda,t)c_+(\lambda,t)\\&=b_+(-\lambda,t)d_-(-\lambda,t)-d_+(-\lambda,t)b_-(-\lambda,t),\end{aligned}\tag{4.2.29}$$

so that (4.2.24) imply

$$b(\lambda,t)=-b(\lambda^{-1},t).\tag{4.2.30}$$

Now recall that since $(\varphi(\cdot,t),\varpi(\cdot,t))\in\hat{\mathcal{M}}_N$ it follows that for $\mathrm{Im}\lambda\geq 0$ the coefficient $a(\lambda,t)$ can be expressed in terms of $b(\lambda,t)$ and its zeroes

$$\left\{\lambda_1(t),\ldots,\lambda_{n_1(t)+2n_2(t)}(t)\right\},$$

through the dispersion relation (3.2.36). Therefore by (4.2.25), (4.2.27), (4.2.29) it is required that

$$a(i\Lambda,t)=0,\qquad a(i\Lambda^{-1},t)\neq 0,$$

and the sequence of purely imaginary zeroes of $a(\cdot,t)$ must take the form

$$\lambda_1(t),\ldots,\lambda_{n_1(t)}(t)=i\Lambda, i\kappa_2(t),\ldots,i\kappa_{m(t)}(t),\frac{i}{\kappa_2(t)},\ldots,\frac{i}{\kappa_{m(t)}(t)}, \tag{4.2.31}$$

with $m(t)=\frac{n_1(t)-1}{2}$. As for the set of zeroes

$$\left\{\lambda_{n_1(t)+1}(t),\ldots,\lambda_{n_1(t)+n_2(t)}(t)\right\},$$

which have real and imaginary parts greater than zero; letting

$$\left\{\lambda_{n_1(t)+1}(t),\ldots,\lambda_{n_1(t)+n_a(t)}(t)\right\},$$

denote the subset which do not lie on the unit circle $|\lambda|^2=1$ it follows from (4.2.27), (4.2.30) that the integer $n_a(t)$ must be even and that this set must take the form

$$\left\{\lambda_{n_1(t)+1}(t),\ldots,\lambda_{n_1(t)+n_a(t)/2}(t),\frac{1}{\bar{\lambda}_{n_1(t)+1}(t)},\ldots,\frac{1}{\bar{\lambda}_{n_1(t)+n_a(t)/2}(t)}\right\}.$$

However, denoting by

$$\left\{\lambda_{n_1(t)+n_a(t)+1}(t),\ldots,\lambda_{n_1(t)+n_2(t)}(t)\right\},$$

the subset which do lie on the circle $|\lambda|^2=1$, symmetry conditions (4.2.27), (4.2.30) do not imply any constraints on the position of these zeroes and $n_2(t)$ can be odd or even.

The set of zeroes

$$\left\{-\bar{\lambda}_{n_1(t)+1}(t),\ldots,-\bar{\lambda}_{n_1(t)+n_a(t)}(t)\right\},$$

which have positive imaginary part, negative real part and which do not lie on the circle $|\lambda|^2=1$ take the form

$$\left\{-\bar{\lambda}_{n_1(t)+1}(t),\ldots,-\bar{\lambda}_{n_1(t)+n_a(t)/2}(t),-\frac{1}{\lambda_{n_1(t)+1}(t)},\ldots,-\frac{1}{\lambda_{n_1(t)+n_a(t)/2}(t)}\right\},$$

where as before the $n_2(t)-n_a(t)$ zeroes in this domain which do lie on the circle $|\lambda|^2=1$ are not constrained.

The above reasoning shows that for each zero $\lambda_j(t)$, $j=2,\ldots,n_1(t)+2n_2(t)$ present in the scattering data (4.2.18) there must exist another one given by $-\frac{1}{\lambda_j(t)}$.

By (3.6.19), (4.2.24)-(4.2.25) the normalization coefficients associated with all the zeroes must satisfy (in obvious notation),

$$\gamma_{i\Lambda}(t) \in \mathbb{R} \setminus \{0\}, \tag{4.2.32}$$

$$\gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = -1 \qquad \lambda_j(t) \neq i\Lambda. \tag{4.2.33}$$

Now using notation 3.1 and (3.2.29) for the normalization coefficients, (4.2.33) implies that $n_a(t) = n_2(t) \in 2\mathbb{N}$.

So when N is odd the constraint (4.2.6), (4.2.7), equivalently expressed as (4.2.16), implies that the scattering data (4.2.18) must be an element of a subspace of $\hat{\mathcal{D}}_1^{n_1(t),n_2(t)}$. This subspace is defined below.

Definition 4.1 *For arbitrary* $t \in \mathbb{R}$ *fix* $n_1(t) \in 2\mathbb{N}+1$, $n_2(t) \in 2\mathbb{N}$, $\Lambda \in \mathbb{R}^+$ *and let the subspace* $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)} \subset \hat{\mathcal{D}}_1^{n_1(t),n_2(t)}$ *denote sets of scattering data*

$$(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_1^{n_1(t),n_2(t)},$$

such that

- *in their domains of analyticity the transition coefficients* $a(\cdot,t), b(\cdot,t)$ *satisfy*

$$\begin{aligned} b(\lambda,t) &= -b(\lambda^{-1},t) \\ a(\lambda,t) &= -\frac{(\lambda - i\Lambda)(\lambda + i\Lambda^{-1})}{(\lambda + i\Lambda)(\lambda - i\Lambda^{-1})} a(-\lambda^{-1},t), \end{aligned}$$

  so $a(i\Lambda,t) = 0$ *and unless* $\Lambda = 1$, *for all the* $\lambda_j(t) \neq i\Lambda$ *such that* $a(\lambda_j(t),t) = 0$ *it follows that* $a(-\lambda_j^{-1}(t),t) = 0$ *also.*

- *the normalization coefficients at these zeroes are such that*

$$\gamma_{i\Lambda}(t) \in \mathbb{R} \setminus \{0\}, \qquad \gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = -1 \quad \lambda_j(t) \neq i\Lambda.$$

In the next subsection it will be seen how the inverse scattering transform **ist** can be used to prove that *all* scattering data in this subset leads to Jost solutions satisfying (4.2.16) and therefore (4.2.9), (4.2.13). However, to complete this subsection it remains to turn to the case when N is even.

N **even**

With $N \in 2\mathbb{Z}$,

$$\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t)) = (a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_0^{n_1(t),n_2(t)}, \quad (4.2.34)$$

for some $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$.

From (4.2.9) follows the asymptotics,

$$a_1(x,t) \to \mp\frac{\pi N}{2},\ a_2(x,t) \to 0,\ a_3(x,t) \to 0,$$

$$d_2(x,t) \to i^N(\xi+\xi^{-1}),\ f_1(x,t) \to 0, \qquad x \to \pm\infty \quad \forall t \in \mathbb{R}, \quad (4.2.35)$$

with

$$\xi \in \mathbb{C} : |\xi| = 1 \ \text{ or } \ \xi \in \mathbb{R}\setminus\{0\}.$$

In addition there exists the constraint

$$P^2 = (\xi+\xi^{-1})^2. \quad (4.2.36)$$

In this case the relations (4.2.24) are replaced by

$$\begin{aligned}
a_+(-\lambda^{-1},t) &= \frac{(\lambda-\lambda^{-1})d_-(\lambda,t) - Pb_-(\lambda,t)}{\lambda-\lambda^{-1}-i\xi-i\xi^{-1}}\\
b_+(-\lambda^{-1},t) &= \frac{Pa_-(\lambda,t) - (\lambda-\lambda^{-1})c_-(\lambda,t)}{\lambda^{-1}-\lambda-i\xi-i\xi^{-1}}\\
c_+(-\lambda^{-1},t) &= \frac{(\lambda-\lambda^{-1})b_-(\lambda,t) + Pd_-(\lambda,t)}{\lambda-\lambda^{-1}-i\xi-i\xi^{-1}}\\
d_+(-\lambda^{-1},t) &= -\frac{(\lambda-\lambda^{-1})a_-(\lambda,t) + Pc_-(\lambda,t)}{\lambda^{-1}-\lambda-i\xi-i\xi^{-1}},
\end{aligned} \quad (4.2.37)$$

to be satisfied $\forall\lambda \in \mathbb{R}\setminus\{0\}$. Using these in place of (4.2.24) the subsequent reasoning mirrors that detailed for the case of N odd and the constraint (4.2.6), (4.2.7) implies that the scattering data (4.2.34) must be an element of one of two subspaces of $\hat{\mathcal{D}}_0^{n_1(t),n_2(t)}$ depending on whether $\mathrm{Im}(\xi) = 0$ or not. These spaces are defined below.

Definition 4.2 *For arbitrary $t \in \mathbb{R}$ fix $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in 2\mathbb{N}$ and parameter $\xi \in \mathbb{R}\setminus\{0\}$. Let the subspace $\mathcal{F}_{0,\xi}^{n_1(t),n_2(t)} \subset \hat{\mathcal{D}}_0^{n_1(t),n_2(t)}$ denote sets of scattering data*

$$(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_0^{n_1(t),n_2(t)},$$

such that

- *in their domains of analyticity the transition coefficients* $a(\cdot,t), b(\cdot,t)$ *satisfy*

$$a(\lambda,t) = a(-\lambda^{-1},t)$$
$$b(\lambda,t) = \left[\frac{\lambda-\lambda^{-1}+i(\xi+\xi^{-1})}{\lambda-\lambda^{-1}-i(\xi+\xi^{-1})}\right] b(\lambda^{-1},t),$$

  so that for all the $\lambda_j(t)$ *such that* $a(\lambda_j(t),t) = 0$ *it necessarily follows that* $a(-\lambda_j^{-1}(t),t) = 0$ *also. However since* $n_1(t) \in 2\mathbb{N}$ *it follows that* $a(i,t) \neq 0$.

- *the normalisation coefficients at these zeroes are such that*

$$\gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = \left[\frac{\lambda_j(t)-\lambda_j^{-1}(t)+i(\xi+\xi^{-1})}{\lambda_j(t)-\lambda_j^{-1}(t)-i(\xi+\xi^{-1})}\right].$$

Alternatively if $\xi \in \mathbb{C} : |\xi| = 1$, $\mathrm{Im}(\xi) \neq 0$ the scattering data must be an element of the subspace $\hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$ given by the following definition.

Definition 4.3 *For arbitrary* $t \in \mathbb{R}$ *fix* $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$ *and parameter*

$$\xi \in \mathbb{C} : |\xi| = 1, \ \mathrm{Im}(\xi) \neq 0.$$

Let the subspace $\hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)} \subset \hat{\mathcal{D}}_0^{n_1(t),n_2(t)}$ *denote sets of scattering data*

$$(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_0^{n_1(t),n_2(t)},$$

such that

- *in their domains of analyticity the transition coefficients* $a(\cdot,t), b(\cdot,t)$ *satisfy*

$$a(\lambda,t) = a(-\lambda^{-1},t)$$
$$b(\lambda,t) = \left[\frac{\lambda-\lambda^{-1}+i(\xi+\xi^{-1})}{\lambda-\lambda^{-1}-i(\xi+\xi^{-1})}\right] b(\lambda^{-1},t),$$

  so that for all the $\lambda_j(t)$ *such that* $a(\lambda_j(t),t) = 0$ *it necessarily follows that* $a(-\lambda_j^{-1}(t),t) = 0$ *also. However since* $n_1(t) \in 2\mathbb{N}$ *it follows that* $a(i,t) \neq 0$.

- *the normalisation coefficients at these zeroes are such that*

$$\gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = \left[\frac{\lambda_j(t)-\lambda_j^{-1}(t)+i(\xi+\xi^{-1})}{\lambda_j(t)-\lambda_j^{-1}(t)-i(\xi+\xi^{-1})}\right].$$

This subsection is now complete. It has been shown that some of the problems of Type $\mathbf{A}_N$ have solutions which also solve a problem in the set $\mathbf{B}_{P,0}$ and that the scattering data of these solutions at a fixed time t must take a particular form.

In the next subsection the converse question will be addressed. It will be established that given scattering data in one of the subspaces $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)}, \mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}, \hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$ then there does exist a relation such as (4.2.16), (4.2.7), (4.2.8) for the Jost solutions constructed form this data and so relations (4.2.9), (4.2.13) hold.

4.2.5 Subspaces of scattering data imply the constraint

In the previous subsection it was established that if a constraint such as (4.2.16), (4.2.17) is satisfied by a pair of Jost solutions then this implies (4.2.13) *and* the scattering data constructed from these solutions must be an element of one of the subsets $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)}, \mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}, \hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$. The construction of subsection 4.2.4 also ensures that given

$$(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)}(0)) \in \mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}, \mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}, \hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}$$

then

$$\begin{aligned}(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(0)+2n_2(0)}(t)) &\stackrel{\text{def}}{=} \\ &\tau_t(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)}(0)) \qquad (4.2.38)\end{aligned}$$

is an element of $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}, \mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}, \hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}$ respectively.

In this subsection the results of sections 3.3, 3.4 are used to prove that for arbitrary

$$(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)}(0)) \in \mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)},$$

the Jost solutions to the auxiliary problem (3.3.10) which are constructed from the time dependent data following from (4.2.38) satisfy the relation

$$T_+(x,t,-\lambda^{-1}) = -\sigma_3 L^{-1}(x,t,\lambda^{-1})\sigma_2 T_-(-x,t,\lambda)\tilde{\jmath}^{-1}(t,\lambda^{-1}), \qquad \lambda \in \mathbb{R}\setminus\{0\} \qquad (4.2.39)$$

with

$$\tilde{\jmath}(t,\lambda^{-1}) = \frac{i((\lambda+\lambda^{-1})\sigma_1 + (\Lambda^{-1}-\Lambda)\sigma_2)}{(\lambda+\lambda^{-1})((\lambda+\lambda^{-1})^2 + (\Lambda^{-1}-\Lambda)^2)} \tag{4.2.40}$$

and $L(x,t,\lambda)$ taking the form (4.2.7), (4.2.8) for some functions $a_{1,2,3}, d_{1,2}, f_{1,2}$. As a result of (4.2.38) these Jost solutions will evolve in time according to (3.6.11) and so the matrix $L(x,t,\lambda)$ will satisfy (4.2.9). So, fixing $t \in \mathbb{R}$ a relation such as (4.2.39), (4.2.40) will imply (4.2.13). Similar analysis can also be carried out for the subspaces $\mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}$, $\hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$. The conclusion remains the same.

At a particular time $t \in \mathbb{R}$ construct the matrix $G(x,t,\lambda)$ from the scattering data $(a(\cdot,t), b(\cdot,t)\,:\,\gamma_1(t),\ldots,\gamma_{n_1(0)+2n_2(0)}(t))$ according to (3.2.43) and for $\lambda \in \mathbb{R}\setminus\{0\}$ define

$$A(\lambda) \stackrel{\text{def}}{=} \alpha(\lambda)\sigma_2 = -\left((\lambda^2+1)(\lambda^{-1}+i\Lambda)(\lambda^{-1}-i\Lambda^{-1})\right)^{-1}\sigma_2.$$

Since the scattering data $(a(\cdot,t), b(\cdot,t)\,:\,\gamma_1(t),\ldots,\gamma_{n_1(0)+2n_2(0)}(t))$ is an element of $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}$ it follows that

$$G(x,t,\lambda) = A(\lambda)G(-x,t,-\lambda^{-1})A^{-1}(\lambda), \qquad \lambda \in \mathbb{R}. \tag{4.2.41}$$

Therefore, according to subsection 3.3.1, there must exist a matrix function $L(x,t,\cdot)$ which is regular throughout $\mathbb{C}\setminus\{0\}$ with a double pole at $\lambda=0$ such that the solution to the family of Riemann-Hilbert problems (3.3.3) satisfies

$$\begin{aligned} g_+(x,t,\lambda) &= -A(\lambda)g_+(-x,t,-\lambda^{-1})\sigma_2 L(x,t,-\lambda)\sigma_3 \qquad \operatorname{Im}\lambda \geq 0, \\ g_-(x,t,\lambda) &= -\sigma_3 L^{-1}(x,t,-\lambda)\sigma_2 g_-(-x,t,-\lambda^{-1})A^{-1}(\lambda) \qquad \operatorname{Im}\lambda \leq 0. \end{aligned} \tag{4.2.42}$$

Using $\det g_+(x,t,\lambda) = a(\lambda,t)$ it follows that

$$L(x,t,\lambda) = -\sigma_3 L^t(x,t,\lambda)\sigma_3. \tag{4.2.43}$$

Analyticity and the asymptotics (3.3.5), (3.3.6), (3.3.11) constrain $L(x,t,\lambda)$ to have the form

$$L(x,t,\lambda) = \sum_{p=-2}^{2} L_p(x,t)\lambda^p, \tag{4.2.44}$$

with

$$\begin{aligned} L_2(x,t) &= \exp(ia_1(x,t)\sigma_3), \\ L_{-2}(x,t) &= -\exp(-ia_1(x,t)\sigma_3), \end{aligned} \tag{4.2.45}$$

and $a_1(x,t)$ expressed in terms of

$$(\varphi(\cdot,t),\varpi(\cdot,t)) \stackrel{\text{def}}{=} \mathbf{ist}(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(0)+2n_2(0)}(t)),$$

as $a_1(x,t)=\frac{1}{4}(\varphi(-x,t)-\varphi(x,t))$.

Assertion 1‴) in section 3.4 ensures that the Jost solutions $T_\pm(x,t,\lambda)$ to the auxiliary linear problem (3.3.10) follow from the columns of $g_\pm(x,t,\lambda)$ according to (3.3.9), (3.4.1). Therefore rewriting (4.2.42) in terms of these solutions (and remembering that the scattering data is an element of $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}$) follows (4.2.39), (4.2.40). It remains to deduce that $L(x,t,\lambda)$ does indeed have the form (4.2.7), (4.2.8). At order $\lambda^{\pm 2}$ this is already evident from (4.2.45). However, to deduce the structure of the remaining coefficients it is necessary to use the involutions satisfied by $T_\pm(x,t,\lambda)$, the asymptotics of these solutions (3.2.11) and their unimodularity. For $\lambda\in\mathbb{R}\setminus\{0\}$ the involutions

$$\begin{aligned}\bar{T}_\pm(x,t,\lambda) &= \sigma_2 T_\pm(x,t,\lambda)\sigma_2,\\ \bar{T}_\pm(x,t,-\lambda) &= -i\sigma_1 T_\pm(x,t,\lambda)\end{aligned} \tag{4.2.46}$$

imply

$$L(x,t,-\lambda)=\sigma_3 L(x,t,\lambda)\sigma_3, \tag{4.2.47}$$

$$\bar{L}(x,t,\lambda)=\sigma_1 L(x,t,-\lambda)\sigma_1. \tag{4.2.48}$$

These relations in conjunction with (4.2.43), (4.2.44), (4.2.45) imply that $L(x,t,\lambda)$ has the form (4.2.7). Finally

$$\det T_\pm(x,t,\lambda)=1$$

$$T_\pm(x,t,\lambda)\to E_\pm(x,\lambda),\qquad \text{as } x\to\pm\infty,$$

enforce the conditions (4.2.8).

Similar arguments hold for the subspaces $\mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}$, $\hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}$ so that *all* scattering data in $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)}$, $\mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}$, $\hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$ leads (via the map **ist**) to a pair of functions $(\varphi(\cdot,t),\varpi(\cdot,t))$ satisfying (4.2.13).

With this conclusion it is time to use these results to develop the inverse scattering method for solving a subset of the problems of Type $\mathbf{B}_{P,0}$.

4.3 The inverse scattering method for solving problems of Type $\mathbf{B}_{P,0}$

In this section the results of chapter 3 and section 4.2 will be used to develop the inverse scattering method for solving a *subset* of the problems of Type $\mathbf{B}_{P,0}$.

In chapter 3 it was seen how the inverse scattering method could only be applied to the problems of Type $\mathbf{A}_N$ which are defined by initial data in the subspace $\check{\mathcal{M}}_N \subset \mathcal{M}_N$. However, at the present time, $\check{\mathcal{M}}_N$ can only be defined implicitly in terms of scattering data (see definition 3.6). Until this subspace is realised in terms of pairs of functions (φ, ϖ) the direct scattering transform **dst** is, to some extent, redundant. In section 3.7 it was mentioned how the composite transform $\mathbf{ist} \circ \tau_t$ could be used to solve problems of Type $\mathbf{A}_N$ by an appropriate choice of scattering data at $t = 0$. The initial conditions leading to this solution can then be deduced *a posteriori.*

This reasoning repeats itself when considering the initial-boundary value problems of Type $\mathbf{B}_{P,0}$. Therefore it makes sense to formulate the inverse scattering transform $(\mathbf{ist}|_{(-\infty,0]})$ first in subsection 4.3.1 so that the space $\check{\mathcal{N}}_{P,0} \subset \mathcal{N}_{P,0}$ which is the analogue of $\check{\mathcal{M}}_N$ is introduced immediately. The direct scattering transform $(\mathbf{dst}|_{(-\infty,0]})$ which follows in subsection 4.3.2 will then be formulated entirely in terms of this subspace. From the results of chapter 3 and section 4.2 it will be clear that $(\mathbf{ist}|_{(-\infty,0]})$ and $(\mathbf{dst}|_{(-\infty,0]})$ are the inverse of one another. As has already been mentioned the time evolution map τ_t introduced in section 3.6 leaves the subspaces $\mathcal{F}^{n_1(t),n_2(t)}_{1,\Lambda}$, $\mathcal{F}^{n_1(t),n_2(t)}_{0,\xi}$ and $\hat{\mathcal{F}}^{n_1(t),n_2(t)}_{0,\xi}$ invariant. This will be developed more explicitly in subsection 4.3.3 with the result that the inverse scattering method for solving problems of Type $\mathbf{B}_{P,0}$ can be pieced together as in subsection 4.3.4.

4.3.1 The inverse scattering transform for a subset of problems in $\mathbf{B}_{P,0}$

In chapter 3 the inverse scattering transform was formulated as the injective map

$$\mathbf{ist}\left(\bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{q\bmod 2}\right) = \bigcup_{p\in\mathbb{Z}} \check{\mathcal{M}}_{2p+q},$$

for arbitrary $q \in \mathbb{N}$. Using the results of section 4.2 it is now possible to develop some restrictions on **ist** so that it can be used as the third stage in a solution to some of the problems of Type $\mathbf{B}_{P,0}$.

Recall that at any time $t \in \mathbb{R}$ the inverse scattering transform **ist** comprises a one parameter family (x) of Riemann-Hilbert factorization problems. When considering initial-boundary value problems of Type $\mathbf{A}_N$ it is necessary to consider $x \in \mathbb{R}$. However, due to the 'pointwise' nature of the transform, the appropriate domain for x when considering the problems of Type $\mathbf{B}_{P,0}$ is the semi-line $(-\infty, 0]$.

Definition 4.4 *Let* $\mathbf{ist}|_{(-\infty,0]}$ *denote the map* **ist** *with the parameter* x *restricted to the semi-line* $(-\infty, 0]$.

This definition and the results of section 4.2 make it possible to state the following lemma.

Lemma 4.5 *Fix* $t \in \mathbb{R}$, $n_1(t) \in 2\mathbb{N}+1$, $n_2(t) \in 2\mathbb{N}$, $\Lambda \in \mathbb{R}^+$ *and let*

$$\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right) \in \mathcal{F}^{n_1(t),n_2(t)}_{1,\Lambda},$$

then

$$(\varphi(\cdot,t), \varpi(\cdot,t))|_{(-\infty,0]} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right),$$

is an element of $\mathcal{N}_{P,0}$ *with* $P = i^{n(t)+1}(\Lambda + \Lambda^{-1})$ *and* $n(t) = n_1(t) + 2n_2(t)$.

It follows from section 4.2 that

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \stackrel{\text{def}}{=} \mathbf{ist}\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right),$$

will satisfy the constraint (4.2.6), (4.2.7) for some $P = d_2(0,t) \in \mathbb{R}$ and the Jost solutions following from this will obey (4.2.16), (4.2.17). As a result the reduced transition matrix constructed form $(\varphi(\cdot,t), \varpi(\cdot,t))$ can be written as

$$T(\lambda,t) = -\tilde{\jmath}(t,-\lambda)T_-^{-1}(0,t,-\lambda^{-1})\jmath(0,t,-\lambda)\sigma_3 T_-(0,t,\lambda), \tag{4.3.1}$$

so, recalling the form of the transition matrix in terms of the coefficients $a(\cdot,t)$, $b(\cdot,t)$, the involutions which these coefficients satisfy and the form they must take so that the scattering data is an element of $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)}$, it follows that

$$\begin{aligned} T(1,t) &= \begin{pmatrix} a(1,t) & 0 \\ 0 & -h(1)a(1,t) \end{pmatrix}, \\ a(1,t) &= (i)^{n(t)-1}(1-i\Lambda)(1+i\Lambda)^{-1}. \end{aligned} \tag{4.3.2}$$

Evaluating (4.3.1) at $\lambda = 1$ yields $P = (\Lambda - \Lambda^{-1} - 2i)a(1,t)$ so that $P = (i)^{n(t)+1}(\Lambda + \Lambda^{-1})$ which is consistent with the constraint $P^2 = (\Lambda + \Lambda^{-1})^2$ deduced earlier. Lemma 4.5 is proved.

Identical reasoning with $\mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}$ replacing $\mathcal{F}_{1,\Lambda}^{n_1(t),n_2(t)}$ leads to:

Lemma 4.6 *Fix $t \in \mathbb{R}$, $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in 2\mathbb{N}$ and a parameter $\xi \in \mathbb{R} \setminus \{0\}$. Let*

$$\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right) \in \mathcal{F}_{0,\xi}^{n_1(t),n_2(t)},$$

then

$$(\varphi(\cdot,t), \varpi(\cdot,t))|_{(-\infty,0]} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right),$$

is an element of $\mathcal{N}_{P,0}$ with $P = i^{n(t)}(\xi + \xi^{-1})$ and $n(t) = n_1(t) + 2n_2(t)$.

Finally with $\hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)}$ replacing $\mathcal{F}_{0,\xi}^{n_1(t),n_2(t)}$ it is now possible to prove:

Lemma 4.7 *Fix $t \in \mathbb{R}$, $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$ and a parameter*

$$\xi \in \mathbb{C} : |\xi| = 1, \ \operatorname{Im}(\xi) \neq 0.$$

Let

$$\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right) \in \hat{\mathcal{F}}_{0,\xi}^{n_1(t),n_2(t)},$$

then

$$(\varphi(\cdot,t),\varpi(\cdot,t))|_{(-\infty,0]} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)\right),$$

is an element of $\mathcal{N}_{P,0}$ *with* $P = i^{n(t)}(\xi+\xi^{-1})$ *and* $n(t) = n_1(t)+2n_2(t)$.

The results of lemmas 4.5 - 4.7 can be collected together in proposition 4.9. First it is necessary to make the following definition.

Definition 4.8 *Given* $\Lambda \in \mathbb{R}^+$ *or* $\xi \in \mathbb{C} : |\xi| = 1, \mathrm{Im}(\xi) \neq 0$ *define the sets of scattering data* $\mathcal{F}_\Lambda$, $\mathcal{F}'_\xi$ *as*

$$\mathcal{F}_\Lambda \stackrel{\text{def}}{=} \bigcup_{p,q\in 2\mathbb{N}} \left(\mathcal{F}^{p+1,q}_{1,\Lambda} \cup \mathcal{F}^{p,q}_{0,\Lambda} \cup \mathcal{F}^{p,q}_{0,-\Lambda}\right),$$

$$\mathcal{F}'_\xi \stackrel{\text{def}}{=} \bigcup_{p,q\in \mathbb{N}} \left(\hat{\mathcal{F}}^{2p,q}_{0,\xi} \cup \hat{\mathcal{F}}^{2p,q}_{0,-\xi}\right).$$

Proposition 4.9 *Given* $\Lambda \in \mathbb{R}^+$ *or* $\xi \in \mathbb{C} : |\xi| = 1, \mathrm{Im}(\xi) \neq 0$ *then*

$$\mathbf{ist}_{(-\infty,0]} : \mathcal{F}_\Lambda \xrightarrow{1-1} \mathcal{N}_{(\Lambda+\Lambda^{-1}),0} \cup \mathcal{N}_{-(\Lambda+\Lambda^{-1}),0},$$

$$\mathbf{ist}_{(-\infty,0]} : \mathcal{F}'_\xi \xrightarrow{1-1} \mathcal{N}_{(\xi+\xi^{-1}),0} \cup \mathcal{N}_{-(\xi+\xi^{-1}),0}.$$

The proof of this proposition can be pieced together using lemmas 4.5 - 4.7 and the results of chapter 3. Just as with definition 3.6 where the subspace $\breve{\mathcal{M}}_N$ was introduced, the following definition is of crucial importance regarding the problems of Type $\mathbf{B}_{P,0}$. In effect the subspaces introduced in this definition define the subset of problems in $\mathbf{B}_{P,0}$ which can be solved by the inverse scattering method.

Definition 4.10 *Define the respective subspaces*

$$\breve{\mathcal{N}}_{(\Lambda+\Lambda^{-1}),0},\ \ \breve{\mathcal{N}}_{-(\Lambda+\Lambda^{-1}),0},\ \ \breve{\mathcal{N}}_{(\xi+\xi^{-1}),0} \subset \mathcal{N}_{(\Lambda+\Lambda^{-1}),0},\ \ \mathcal{N}_{-(\Lambda+\Lambda^{-1}),0},\ \ \mathcal{N}_{(\xi+\xi^{-1}),0},$$

by the images of the map $\mathbf{ist}_{(-\infty,0]}$. *Namely,*

$$\breve{\mathcal{N}}_{(\Lambda+\Lambda^{-1}),0} \cup \breve{\mathcal{N}}_{-(\Lambda+\Lambda^{-1}),0} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(\mathcal{F}_\Lambda\right),$$

$$\breve{\mathcal{N}}_{(\xi+\xi^{-1}),0} \cup \breve{\mathcal{N}}_{-(\xi+\xi^{-1}),0} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(\mathcal{F}'_\xi\right). \tag{4.3.3}$$

This completes the formulation of two separate restrictions for the map $\mathbf{ist}|_{(-\infty,0]}$. In subsection 4.3.4 it will be seen how these are to become the third stage in a solution to a subset of the problems of Type $\mathbf{B}_{P,0}$ - this subset being defined by the subspace $\breve{\mathcal{N}}_{P,0}$.

4.3.2 The direct scattering transform for a subset of problems in $\mathbf{B}_{P,0}$

In definition 4.8 the subsets

$$\mathcal{F}_\Lambda,\ \mathcal{F}'_\xi \subset \bigcup_{p,q\in\mathbb{N}} \hat{\mathcal{D}}^{p,q}_{p\mathrm{mod}2},$$

were introduced According to proposition 4.9 and definition 4.10 the restriction of the inverse scattering transform $\mathbf{ist}|_{(-\infty,0]}$ to these subsets yields an element of $\breve{\mathcal{N}}_{P,0}$ with P given in terms of Λ or ξ.

This subsection is not concerned with a general formulation

$$\mathbf{dst}|_{(-\infty,0]} : \mathcal{N}_{P,0} \to \mathcal{D}_0 \cup \mathcal{D}_1,$$

but only with the restricted transforms

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \to \mathcal{F}_\Lambda,\ \ \mathcal{F}'_\xi, \tag{4.3.4}$$

with the parameter in the image fixed in terms of the given $P \in \mathbb{R}$.

At time $t \in \mathbb{R}$ let $(\varphi(\cdot,t), \varpi(\cdot,t) \in \breve{\mathcal{N}}_{P,0}$ for some $P \in \mathbb{R}$. For $x \leq 0,\ \lambda \in \mathbb{R}\backslash\{0\}$ use this data to construct the Jost solution $T_-(x,t,\lambda)$ according to the prescription outlined in chapter 3 and define the matrix

$$j_1(\lambda) \stackrel{\mathrm{def}}{=} -i(\lambda+\lambda^{-1})((\lambda-\lambda^{-1})\sigma_1 + P\sigma_3). \tag{4.3.5}$$

To proceed it is necessary to consider the cases $|P| < 2,\ |P| > 2,\ |P| = 2$ separately.

$\underline{|P| < 2}$

For $\xi \in \mathbb{C} : |\xi| = 1,\ \mathrm{Im}(\xi) \neq 0$ let

$$j_2(\lambda) \stackrel{\text{def}}{=} \frac{i((\lambda - \lambda^{-1})\sigma_1 - (\xi + \xi^{-1})\sigma_2)}{(\lambda + \lambda^{-1})((\lambda - \lambda^{-1})^2 + (\xi + \xi^{-1})^2)}, \tag{4.3.6}$$

and, with $\lambda \in \mathbb{R} \setminus \{0\}$, *define* the Jost solution $T_+(0,t,\cdot)$ by

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1}) T_-(0,t,\lambda) j_2^{-1}(\lambda^{-1}). \tag{4.3.7}$$

With this definition the free parameter ξ must be chosen so that $T_+(0,t,\cdot)$ meets certain requirements. These are to ensure that it is compatible with the analysis of chapter 3 and so with the construction of a set of scattering data in $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}$ for some $n_1(t),\ n_2(t) \in \mathbb{N}$. Namely, the parameter ξ must be chosen so that:

1. $\det T_+(0,t,\lambda) = 1 \quad \forall \lambda \in \mathbb{R} \setminus \{0\}$.

2. the columns of $T_+(0,t,\cdot)$ have the appropriate analytic properties.

3. the resulting $T_\pm(0,t,\cdot)$ lead to scattering data in $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}$ with $n_1(t),\ n_2(t) \in \mathbb{N}$.

It is straightforward to deduce that the first requirement forces $(\xi + \xi^{-1})^2 = P^2$. However, this does not determine ξ uniquely and constraints 2 and 3 must also be imposed. Defining

$$T_\pm(0,t,\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} a_\pm(\lambda,t) & b_\pm(\lambda,t) \\ c_\pm(\lambda,t) & d_\pm(\lambda,t) \end{pmatrix}, \tag{4.3.8}$$

it follows from (4.3.7) that

$$\begin{aligned} a_+(-\lambda^{-1},t) &= \frac{(\lambda - \lambda^{-1})d_-(\lambda,t) - Pb_-(\lambda,t)}{\lambda - \lambda^{-1} - i\xi - i\xi^{-1}} \\ b_+(-\lambda^{-1},t) &= \frac{Pa_-(\lambda,t) - (\lambda - \lambda^{-1})c_-(\lambda,t)}{\lambda^{-1} - \lambda - i\xi - i\xi^{-1}} \\ c_+(-\lambda^{-1},t) &= \frac{(\lambda - \lambda^{-1})b_-(\lambda,t) + Pd_-(\lambda,t)}{\lambda - \lambda^{-1} - i\xi - i\xi^{-1}} \\ d_+(-\lambda^{-1},t) &= -\frac{(\lambda - \lambda^{-1})a_-(\lambda,t) + Pc_-(\lambda,t)}{\lambda^{-1} - \lambda - i\xi - i\xi^{-1}}, \end{aligned} \tag{4.3.9}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. Now demand that $b_+(\cdot,t)$, $d_+(\cdot,t)$ have analytic continuations into the half plane $\mathrm{Im}(\lambda) \geq 0$ so that with

$$\vartheta \stackrel{\text{def}}{=} \frac{1}{2}\left(|P| + i\sqrt{4 - P^2}\right), \tag{4.3.10}$$

it follows that

$$b_+(i\vartheta, t) = 0 \quad \Leftrightarrow \quad d_+(i\vartheta, t) = 0,$$

$$b_+(i\vartheta^{-1}, t) = 0 \quad \Leftrightarrow \quad d_+(i\vartheta^{-1}, t) = 0.$$

Therefore, if either of $b_+(i\vartheta^{\pm 1}, t) = 0$ then (3.2.21) implies that the transition coefficient $a(\cdot, t)$ is such that $a(i\vartheta^{\pm 1}, t) = 0$. However, in addition to this, (3.2.28) implies that $\gamma_{i\vartheta^{\pm 1}}(t)$ must become unbounded so that requirement 3 would be violated. Therefore it is required that $b_+(i\vartheta^{\pm 1}, t) \neq 0$.

With

$$\tilde{\Gamma}(\lambda, t) \stackrel{\text{def}}{=} Pa_-(\lambda, t) - (\lambda - \lambda^{-1})c_-(\lambda, t), \tag{4.3.11}$$

it is clear that

$$\tilde{\Gamma}(i\vartheta, t) = 0 \quad \Leftrightarrow \quad \tilde{\Gamma}(i\vartheta^{-1}, t) = 0,$$

so it is only necessary to consider the two possibilities $\tilde{\Gamma}(i\vartheta, t) = 0$ or $\tilde{\Gamma}(i\vartheta, t) \neq 0$. Since

$$b_+(-\lambda^{-1}, t) \stackrel{\text{def}}{=} \frac{\tilde{\Gamma}(\lambda, t)}{\lambda^{-1} - \lambda - i\xi - i\xi^{-1}}, \tag{4.3.12}$$

it follows that if $(\varphi(\cdot, t), \varpi(\cdot, t)$ is such that $\tilde{\Gamma}(i\vartheta, t) = 0$ then ξ must be chosen to ensure $\vartheta + \vartheta^{-1} + \xi + \xi^{-1} = 0$ also. It is sufficient to make the definition $\xi \stackrel{\text{def}}{=} -\vartheta$. Alternatively, if $(\varphi(\cdot, t), \varpi(\cdot, t)$ is such that $\tilde{\Gamma}(i\vartheta, t) \neq 0$ then ξ must be chosen so that $\vartheta + \vartheta^{-1} + \xi + \xi^{-1} \neq 0$ also and $\xi \stackrel{\text{def}}{=} \vartheta$ is required.

All this reasoning ensures that the Jost solutions $T_\pm(0, t, \lambda)$ meet the requirements 1-3 stated above. The scattering data associated with $(\varphi(\cdot, t), \varpi(\cdot, t))$ is then found using these Jost solutions in the standard manner. By construction this data has all the required properties.

The above formulation constitutes a map $\mathbf{dst}|_{(-\infty,0]}$ such that

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \to \bigcup_{q,r \in \mathbb{N}} \hat{\mathcal{F}}^{2q,r}_{0,-\vartheta} \subset \mathcal{F}'_{-\vartheta},$$

if $(\varphi(\cdot, t), \varpi(\cdot, t))$ is such that $\tilde{\Gamma}(i\vartheta, t) = 0$. Alternatively,

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \to \bigcup_{q,r \in \mathbb{N}} \hat{\mathcal{F}}^{2q,r}_{0,\vartheta} \subset \mathcal{F}'_{\vartheta},$$

if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i\vartheta,t)\neq 0$.

$\underline{|P|>2}$

There now exists an added complication. In addition to determining the form that a parameter must take it is also necessary to distinguish between two possible definitions of the Jost solution $T_+(0,t,\cdot)$.

For $\xi\in\mathbb{R}\setminus\{0,1\}$ let

$$j_2(\lambda)\stackrel{\text{def}}{=}\frac{i((\lambda-\lambda^{-1})\sigma_1-(\xi+\xi^{-1})\sigma_2)}{(\lambda+\lambda^{-1})((\lambda-\lambda^{-1})^2+(\xi+\xi^{-1})^2)},\tag{4.3.13}$$

and for $\Lambda\in\mathbb{R}^+\setminus\{1\}$ let

$$j_3(\lambda)\stackrel{\text{def}}{=}\frac{i((\lambda+\lambda^{-1})\sigma_1+(\Lambda^{-1}-\Lambda)\sigma_2)}{(\lambda+\lambda^{-1})((\lambda+\lambda^{-1})^2+(\Lambda-\Lambda^{-1})^2)}.\tag{4.3.14}$$

The Jost solution $T_+(0,t,\cdot)$ is *defined* by either

$$T_+(0,t,-\lambda^{-1})\stackrel{\text{def}}{=}j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_2^{-1}(\lambda^{-1}),\tag{4.3.15}$$

or

$$T_+(0,t,-\lambda^{-1})\stackrel{\text{def}}{=}j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_3^{-1}(\lambda^{-1}),\tag{4.3.16}$$

depending on the particular choice of $(\varphi(\cdot,t),\varpi(\cdot,t))$. This is explained as follows. With two possible definitions of $T_+(0,t,\cdot)$ it must be deduced which one is appropriate (and, in addition, the form of the parameter ξ or Λ), such that the previously specified criteria 1-3 are met. Requirement 1 simply amounts to the parameter ξ (resp. Λ) being such that $(\xi+\xi^{-1})^2=P^2$ (resp. $(\Lambda+\Lambda^{-1})^2=P^2$).

Suppose ϑ solves $\vartheta+\vartheta^{-1}=|P|$ and that one of the three possibilities

$$\tilde{\Gamma}(i\vartheta,t)=0,\qquad \tilde{\Gamma}(i\vartheta^{-1},t)=0\tag{4.3.17}$$

$$\tilde{\Gamma}(i\vartheta,t)=0,\qquad \tilde{\Gamma}(i\vartheta^{-1},t)\neq 0\tag{4.3.18}$$

$$\tilde{\Gamma}(i\vartheta,t)=0,\qquad \tilde{\Gamma}(i\vartheta^{-1},t)\neq 0,\tag{4.3.19}$$

is satisfied. Since $|P|\neq 2$ these relations distinguish ϑ from ϑ^{-1}. Once again defining

$$T_\pm(0,t,\lambda)\stackrel{\text{def}}{=}\begin{pmatrix} a_\pm(\lambda,t) & b_\pm(\lambda,t)\\ c_\pm(\lambda,t) & d_\pm(\lambda,t)\end{pmatrix},\tag{4.3.20}$$

it follows that the definition (4.3.15) leads to

$$b_+(-\lambda^{-1},t) \stackrel{\text{def}}{=} \frac{\tilde{\Gamma}(\lambda,t)}{\lambda^{-1}-\lambda-i\xi-i\xi^{-1}}, \tag{4.3.21}$$

whereas the alternative definition (4.3.16) yields

$$b_+(-\lambda^{-1},t) \stackrel{\text{def}}{=} \frac{\tilde{\Gamma}(\lambda,t)}{\lambda+\lambda^{-1}+i\Lambda^{-1}-i\Lambda}. \tag{4.3.22}$$

Using exactly the same reasoning as with the case of $|P| < 2$ it follows that for the second and third of the criteria to hold, (4.3.17) implies that (4.3.15) must be chosen and that $\xi \stackrel{\text{def}}{=} -\vartheta$. However if (4.3.19) is satisfied then (4.3.15) is the correct definition once more but this time with $\xi \stackrel{\text{def}}{=} \vartheta$. Finally, if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that (4.3.18) holds then it is necessary to choose (4.3.16) as the definition of the Jost solution $T_+(0,t,\cdot)$ with $\Lambda \stackrel{\text{def}}{=} \vartheta$. Thus the appropriate Jost solution can be deduced and the pair $T_\pm(0,t,\cdot)$ are such that requirements 1-3 are satisfied. The scattering data associated with $(\varphi(\cdot,t),\varpi(\cdot,t))$ follows in the standard manner and, by construction, it is of the required form.

The above formulation constitutes a map $\mathbf{dst}|_{(-\infty,0]}$ such that

$$\mathbf{dst}|_{(-\infty,0]} : \check{\mathcal{N}}_{P,0} \to \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q,r}_{0,-\vartheta} \subset \mathcal{F}_\vartheta,$$

if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i\vartheta,t)$, $\tilde{\Gamma}(i\vartheta^{-1},t) = 0$. Alternatively, if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i\vartheta,t)$, $\tilde{\Gamma}(i\vartheta^{-1},t) \neq 0$ then

$$\mathbf{dst}|_{(-\infty,0]} : \check{\mathcal{N}}_{P,0} \to \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q,r}_{0,\vartheta} \subset \mathcal{F}_\vartheta.$$

Finally, if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i\vartheta,t) = 0$ whereas $\tilde{\Gamma}(i\vartheta,t) \neq 0$ then

$$\mathbf{dst}|_{(-\infty,0]} : \check{\mathcal{N}}_{P,0} \to \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q+1,r}_{1,\vartheta} \subset \mathcal{F}_\vartheta.$$

It remains to examine the situation when $|P| = 2$. This is done as follows.

$\underline{|P| = 2}$

When considering $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \check{\mathcal{N}}_{P,0}$ such that $|P| = 2$ there is a further level of complexity in the determination of the form of the map $\mathbf{dst}|_{(-\infty,0]}$. Defining the matrices

$$j_4(\lambda) \stackrel{\text{def}}{=} \frac{i((\lambda-\lambda^{-1})\sigma_1 - 2\sigma_2)}{(\lambda+\lambda^{-1})^3}, \tag{4.3.23}$$

$$j_5(\lambda) \stackrel{\text{def}}{=} \frac{i((\lambda-\lambda^{-1})\sigma_1 + 2\sigma_2)}{(\lambda+\lambda^{-1})^3}, \tag{4.3.24}$$

$$j_6(\lambda) \stackrel{\text{def}}{=} \frac{i\sigma_1}{(\lambda-\lambda^{-1})^2}, \tag{4.3.25}$$

it must be deduced which of the three definitions

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_4^{-1}(\lambda^{-1}), \tag{4.3.26}$$

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_5^{-1}(\lambda^{-1}), \tag{4.3.27}$$

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_6^{-1}(\lambda^{-1}),, \tag{4.3.28}$$

is appropriate for a construction of the necessary scattering data. That is which of these three possible definitions is such that requirements 2 and 3 are satisfied. Defining

$$\begin{aligned} \tilde{\Gamma}(\lambda,t) &\stackrel{\text{def}}{=} Pa_-(\lambda,t) - (\lambda-\lambda^{-1})c_-(\lambda,t), \\ \hat{\tilde{\Gamma}}(\lambda,t) &\stackrel{\text{def}}{=} P\frac{da_-}{d\lambda}(\lambda,t) - (\lambda-\lambda^{-1})\frac{dc_-}{d\lambda}(\lambda,t), \end{aligned} \tag{4.3.29}$$

the result depends on which of

$$\tilde{\Gamma}(i,t) \neq 0, \tag{4.3.30}$$

$$\tilde{\Gamma}(i,t) = 0, \qquad \hat{\tilde{\Gamma}}(i,t) = 0, \tag{4.3.31}$$

$$\tilde{\Gamma}(i,t) = 0, \qquad \hat{\tilde{\Gamma}}(i,t) \neq 0,, \tag{4.3.32}$$

is satisfied for a given $(\varphi(\cdot,t),\varpi(\cdot,t))$.

If $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that (4.3.30) occurs then (4.3.26) must be chosen as the definition of the Jost solution $T_+(0,t,\cdot)$ so that requirements 1-3 are satisfied. Alternatively, if (4.3.31) occurs then (4.3.27) must be chosen as the appropriate definition. Finally, if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that (4.3.32) holds then (4.3.28) must be selected as the definition of the Jost solution at $x = 0$.

Following on from these ideas the scattering data for $(\varphi(\cdot,t),\varpi(\cdot,t))$ can be calculated as detailed in chapter 3. By construction it has all the required properties so that the map $\mathbf{dst}|_{(-\infty,0]}$ takes the form

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \rightarrow \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q,r}_{0,-1} \subset \mathcal{F}_1,$$

if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i,t) \neq 0$. Alternatively, if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i,t)$, $\hat{\hat{\Gamma}}(i,t) = 0$ then

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \rightarrow \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q,r}_{0,1} \subset \mathcal{F}_1.$$

Finally, $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that $\tilde{\Gamma}(i,t) = 0$, $\hat{\hat{\Gamma}}(i,t) \neq 0$ then

$$\mathbf{dst}|_{(-\infty,0]} : \breve{\mathcal{N}}_{P,0} \rightarrow \bigcup_{q,r\in 2\mathbb{N}} \mathcal{F}^{q+1,r}_{1,1} \subset \mathcal{F}_1.$$

This completes the formulation of the map $\mathbf{dst}|_{(-\infty,0]}$ for all possible $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \breve{\mathcal{N}}_{P,0}$ with $P \in \mathbb{R}$. From the analysis presented in chapter 3 and section 4.2 it is easily deduced that the maps $\mathbf{dst}|_{(-\infty,0]}$ and $\mathbf{ist}|_{(-\infty,0]}$ are such that

$$\mathbf{ist}|_{(-\infty,0]} = \left(\mathbf{dst}|_{(-\infty,0]}\right)^{-1}.$$

4.3.3 Time evolving the scattering data

With $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \breve{\mathcal{N}}_{P,0}$ this subsection will study the time evolution of the scattering data

$$(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)) \stackrel{\text{def}}{=} \mathbf{dst}|_{(-\infty,0]}(\varphi(\cdot,t),\varpi(\cdot,t)) \in \mathcal{F}'_{\pm\vartheta(P)},\ \mathcal{F}_{\vartheta(P)}, \tag{4.3.33}$$

when (φ,ϖ) is forced to evolve according to an initial-boundary value problem of Type $\mathbf{B}_{P,0}$. For the sake of brevity only the situation with $|P| < 2$ ($\Rightarrow \mathcal{F}'_{\pm\vartheta(P)}$) will be considered, the results for $|P| \geq 2$ ($\mathcal{F}_{\vartheta(P)}$) following trivially.

For $P \in (2,-2)$ let (φ,ϖ) be the solution to a problem of Type $\mathbf{B}_{P,0}$. The equation governing the time evolution of the Jost solution $T_-(0,t,\lambda)$ constructed from (φ,ϖ), follows from (3.6.2)-(3.6.11). That is

$$\frac{\partial T_-}{\partial t}(0,t,\lambda) = V(0,t,\lambda)T_-(0,t,\lambda) - \frac{1}{4i}(\lambda+\lambda^{-1})T_-(0,t,\lambda)\sigma_3 \quad \forall\lambda\in\mathbb{R}\setminus\{0\}. \quad (4.3.34)$$

The Jost solution $T_+(0,t,\lambda)$ is *defined* by (4.3.7) with $\xi = \pm\vartheta(P)$ already determined. Using this definition and (4.3.34) it is straightforward to deduce that $T_+(0,t,\lambda)$ evolves in time according to

$$\frac{\partial T_+}{\partial t}(0,t,\lambda) = V(0,t,\lambda)T_+(0,t,\lambda) - \frac{1}{4i}(\lambda+\lambda^{-1})T_+(0,t,\lambda)\sigma_3 \quad \forall\lambda\in\mathbb{R}\setminus\{0\}. \quad (4.3.35)$$

Note that use has been made of $(\varphi(\cdot,t),\varpi(\cdot,t))\in\mathcal{N}_{P,0}$ in deriving this equation and that by construction the Jost solutions $T_\pm(0,t,\lambda)$ evolve exactly as they did before when considering problems of Type $\mathbf{A}_N$.

The differential equations for the time evolution of $T_\pm(0,t,\lambda)$ are consistent with the appropriate analytic continuations and imply that the transition coefficients $a(\cdot,t), b(\cdot,t)$ appearing in (4.3.33) evolve in time according to

$$\frac{\partial a}{\partial t}(\lambda,t) = 0 \qquad \operatorname{Im}\lambda\geq 0$$

$$\frac{\partial b}{\partial t}(\lambda,t) = \frac{i}{2}(\lambda+\lambda^{-1})b(\lambda,t) \qquad \lambda\in\mathbb{R},$$

so that the transition coefficients at any time $T\in\mathbb{R}$ can be deduced from those at time t according to,

$$a(\lambda,T) = a(\lambda,t) \qquad \operatorname{Im}\lambda\geq 0,$$

$$b(\lambda,T) = e^{-\frac{i}{2}(\lambda+\frac{1}{\lambda})(t-T)}b(\lambda,t) \qquad \lambda\in\mathbb{R}.$$

From these equations it follows that $n_1(T)$, $(n_2(T))$, the number of purely imaginary zeroes of the coefficient $a(\cdot,T)$, (the number of zeroes of $a(\cdot,T)$ with positive real and imaginary part), is such that

$$n_{1,2}(T) = n_{1,2}(t),$$

and the position of these zeroes is given by

$$\lambda_j(T) \stackrel{\text{def}}{=} \lambda_j(t), \qquad j=1,\ldots,n(T),$$

with $n(T) \stackrel{\text{def}}{=} n_1(T)+2n_2(T)$.

The normalisation coefficients at the zeroes $\lambda_j(t)$ are defined by

$$T_-^{(1)}(0,t,\lambda_j(t)) = \gamma_j(t)T_+^{(2)}(0,t,\lambda_j(t)), \qquad j=1,\ldots,n(t),$$

and (4.3.34), (4.3.35) imply that these coefficients must evolve in time according to

$$\frac{\partial \gamma_j}{\partial t}(t) = \frac{i}{2}(\lambda_j(t)+\lambda_j^{-1}(t))\gamma_j(t), \qquad j=1,\ldots,n(t).$$

Therefore, at any time $T \in \mathbb{R}$, the normalization coefficients $\gamma_1(T),\ldots,\gamma_{n(T)}(T)$ are given by

$$\gamma_j(T) = e^{-\frac{i}{2}\left(\lambda_j(t)+\frac{1}{\lambda_j(t)}\right)(t-T)}\gamma_j(t), \qquad j=1,\ldots,n(T).$$

To summarise, so far in this subsection it has been deduced that if the pair (φ,ϖ) solves a problem of Type $\mathbf{B}_{P,0}$ *and* is such that $(\varphi(\cdot,T),\varpi(\cdot,T)) \in \breve{\mathcal{N}}_{P,0}$ at any time T then $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \breve{\mathcal{N}}_{P,0}$ for all times $t \in \mathbb{R}$. In addition it can be seen that the ordinary differential equations governing the time evolution of the scattering data (4.3.33) are identical to those appearing in subsection 3.6.2 when considering problems of Type $\mathbf{A}_N$ and which were used in subsection 3.6.3 to define the time evolution map τ_t.

Therefore, using these observations, it is clear that with

$$(\varphi(\cdot,0),\varpi(\cdot,0)) = (\varphi_{P,0},\varpi_{P,0}) \in \breve{\mathcal{N}}_{P,0},$$

so that

$$(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)) = \mathbf{dst}|_{(-\infty,0]}(\varphi_{P,0},\varpi_{P,0}), \tag{4.3.36}$$

then the bijective time evolution map τ_t gives the evolution of initial scattering data appropriate for a solution to some of the problems of Type $\mathbf{B}_{P,0}$. Necessarily τ_t is such that

$$\tau_t : \mathcal{F}'_{\pm\vartheta(P)} \xrightarrow{1-1} \mathcal{F}'_{\pm\vartheta(P)}. \tag{4.3.37}$$

This completes the development of a time evolution map for the initial scattering data (4.3.36) when $P \in (2,-2)$. Identical reasoning for problems with $|P| \geq 2$ leads to τ_t once more but in place of (4.3.37) there is the bijection

$$\tau_t : \mathcal{F}_{\vartheta(P)} \xrightarrow{1-1} \mathcal{F}_{\vartheta(P)}. \tag{4.3.38}$$

4.3.4 Piecing together the inverse scattering method

Piecing together subsections 4.3.1-4.3.3 leads to the inverse scattering method for solving a *subset* of the problems in the set $\mathbf{B}_{P,0}$.

Definition 4.11 *For $P \in \mathbb{R}$ let $\breve{\mathbf{B}}_{P,0}$ denote the subset of problems of Type $\mathbf{B}_{P,0}$ which have an initial condition in the subspace $\breve{\mathcal{N}}_{P,0} \subset \mathcal{N}_{P,0}$.*

It has been proved that when the initial data $(\varphi_{P,0}, \varpi_{P,0})$ is an element of $\breve{\mathcal{N}}_{P,0}$ then the image of the composite map $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t \circ \mathbf{dst}|_{(-\infty,0]}$ is such that

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = (\mathbf{ist}|_{(-\infty,0]} \circ \tau_t \circ \mathbf{dst}|_{(-\infty,0]})(\varphi_{P,0}, \varpi_{P,0}),$$

is also an element of $\breve{\mathcal{N}}_{P,0}$ and the resulting functions φ, ϖ : $(x,t) \mapsto \mathbb{R}$ solve the initial-boundary value problem of Type $\mathbf{B}_{P,0}$ with initial data $(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_{P,0}, \varpi_{P,0})$. The time evolution map

$$\mathrm{T}'_t : \breve{\mathcal{N}}_{P,0} \to \breve{\mathcal{N}}_{P,0},$$

giving the solution to the set of problems $\breve{\mathbf{B}}_{P,0}$ as

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = \mathrm{T}'_t(\varphi_{P,0}, \varpi_{P,0}),$$

can be expressed by the commutative diagram in figure 4.1.

Recall from definition 4.10 that $\breve{\mathcal{N}}_{P,0}$ is only defined implicitly in terms of the map $\mathbf{ist}|_{(-\infty,0]}$ applied to $\mathcal{F}_{\vartheta(P)}$ or $\mathcal{F}'_{\vartheta(P)}$. Therefore, the inverse scattering method developed in this chapter *cannot* be used to solve the problem in the set $\mathbf{B}_{P,0}$ defined by initial data $(\varphi_{P,0}, \varpi_{P,0})$ for *any* $(\varphi_{P,0}, \varpi_{P,0}) \in \mathcal{N}_{P,0}$ given beforehand. This is identical to the situation for problems of Type $\mathbf{A}_N$ discussed in chapter 3. But, if it is only required that the initial configuration be deduced once a solution is known then solutions to problems in the set $\breve{\mathbf{B}}_{P,0}$ can be solved by applying the composite map $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ to appropriate elements of the space of initial scattering data. This idea will be adopted in chapter 6 where sets of *soliton* scattering data will be chosen at $t = 0$ and the resulting solutions found by applying $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ to these.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\breve{\mathcal{N}}_{P,0} \arrow[r, "\mathbf{dst}|_{(-\infty,0]}"] \arrow[d, "\mathrm{T}'_t"'] & \mathcal{F}'_{\pm\vartheta(P)}, \mathcal{F}_{\vartheta(P)} \arrow[d, "\tau_t"] \\
\breve{\mathcal{N}}_{P,0} & \mathcal{F}'_{\pm\vartheta(P)}, \mathcal{F}_{\vartheta(P)} \arrow[l, "\mathbf{ist}|_{(-\infty,0]} = \left(\mathbf{dst}|_{(-\infty,0]}\right)^{-1}"]
\end{tikzcd}
$$

$\mathrm{T}'_t \stackrel{\mathrm{def}}{=}$ time evolution map defined by a *nonlinear* problem of Type $\mathbf{B}_{P,0}$

τ_t = bijective time evolution map governed by a set of *linear* o.d.e's

Figure 4.1: **The inverse scattering method for solving problems in $\breve{\mathbf{B}}_{P,0}$**

This completes the development of the inverse scattering method for solving the initial-boundary value problems in the set $\breve{\mathbf{B}}_{P,0}$.

Chapter 5

The inverse scattering method for solving problems of Type $\mathbf{B}_{P,Q\neq 0}$

5.1 Introduction

Throughout this chapter suppose that $Q \in \mathbb{R} \setminus \{0\}$ and recall from chapter 1 that an initial-boundary value problem of Type $\mathbf{B}_{P,Q}$ is the problem of determining the functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ with $(x,t) \in (-\infty, 0] \times \mathbb{R}$ which satisfy:

- the sine-Gordon system

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varpi, \\ \frac{\partial \varpi}{\partial t} &= \frac{\partial^2 \varphi}{\partial x^2} - \sin\varphi \qquad \forall\, (x,t) \in (-\infty, 0] \times \mathbb{R}. \end{aligned} \tag{5.1.1}$$

- the boundary conditions

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \in \mathcal{N}_{P,Q} \qquad \forall t \in \mathbb{R}. \tag{5.1.2}$$

- the 'initial' conditions

$$(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_{P,Q}, \varpi_{P,Q}) \in \mathcal{N}_{P,Q}. \tag{5.1.3}$$

Following on from chapter 4 this chapter develops the inverse scattering method for solving a subset of the problems of this Type - the subset being defined by a reduced phase space $\check{\mathcal{N}}_{P,Q} \subset \mathcal{N}_{P,Q}$. The analysis follows very closely that formulated for solving problems in the set $\mathbf{B}_{P,0}$ but is more complicated. As a result, only the most important changes to the results of chapter 4 will be detailed here. It is hoped that with these pointers the reader can fill in the gaps appropriately.

5.2 Certain solutions to problems of Type $\mathbf{A}_N$ also solve problems in $\mathbf{B}_{P,Q}$

For some $N \in \mathbb{Z}$ suppose (φ, ϖ) is the solution to the initial-boundary value problem of Type $\mathbf{A}_N$ defined by the initial conditions

$$(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_N, \varpi_0) \in \hat{\mathcal{M}}_N.$$

According to the results of chapter 3 it follows that at any time $t \in \mathbb{R}$ the pair $(\varphi(\cdot,t), \varpi(\cdot,t))$ is an element of $\hat{\mathcal{M}}_N$ and so

$$\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t)) = (a(\cdot,t), b(\cdot,t) : \gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}^{n_1(t),n_2(t)}_{N\mathbf{mod}2} \quad (5.2.1)$$

for some $n_1(t) \in 2\mathbb{N} + N \bmod 2$, $n_2(t) \in \mathbb{N}$, $n(t) = n_1(t) + 2n_2(t)$ and $a(\cdot,t)$ determined by $b(\cdot,t)$ and a set of simple zeroes

$$\left\{\lambda_1(t),\ldots,\lambda_{n_1(t)+2n_2(t)}(t)\right\},$$

by (3.2.36).

In addition let $\tilde{\varphi}$, $\tilde{\varpi} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$(\tilde{\varphi}(x,t), \tilde{\varpi}(x,t)) = (\varphi(-x,t), \varpi(-x,t)), \qquad \forall t \in \mathbb{R},$$

so that $(\tilde{\varphi}, \tilde{\varpi})$ also satisfy the sine Gordon system (1.2.2).

5.2.1 Constraining solutions by demanding a gauge relation

In place of the gauge constraint (4.2.6), (4.2.7), (4.2.8), constrain (φ, ϖ) by demanding that there exist $N(\lambda)$, $\tilde{N}(\lambda)$ such that

$$\tilde{\psi}(x,t,\lambda)\tilde{N}(\lambda)N^{-1}(\lambda)\psi^{-1}(x,t,\lambda) = L(x,t,\lambda) \qquad \forall x,t \in \mathbb{R},\ \lambda \in \mathbb{C}\setminus\{0\}, \qquad (5.2.2)$$

with

$$\begin{aligned} L(x,t,\lambda) \stackrel{\text{def}}{=} & \Big((\lambda^2 - \lambda^{-2})\cos a_1(x,t) + a_2(x,t)\Big)\,\mathbb{I} \\ & + i\Big((\lambda^2 + \lambda^{-2})\sin a_1(x,t) + a_3(x,t)\Big)\,\sigma_3 \\ & + i\Big((\lambda - \lambda^{-1})d_1(x,t) + (\lambda + \lambda^{-1})f_1(x,t)\Big)\,\sigma_1 \\ & + i\Big((\lambda + \lambda^{-1})d_2(x,t) + (\lambda - \lambda^{-1})f_2(x,t)\Big)\,\sigma_2, \end{aligned} \qquad (5.2.3)$$

for some matrix coefficients a_i, d_j, f_j, $i = 1,\ldots,3$, $j = 1,2$ such that $\forall\, t \in \mathbb{R}$

$$d_1(0,t) \neq 0. \qquad (5.2.4)$$

With

$$P \stackrel{\text{def}}{=} d_2(0,t), \qquad Q \stackrel{\text{def}}{=} d_1(0,t) \neq 0, \qquad (5.2.5)$$

equations (4.2.10) imply the constraint

$$\frac{\partial \varphi}{\partial x}(x,t)\bigg|_{x=0} + P\sin\frac{\varphi}{2}(0,t) - Q\cos\frac{\varphi}{2}(0,t) = 0,$$

$$\frac{\partial \varpi}{\partial x}(x,t)\bigg|_{x=0} + \frac{P}{2}\varpi(0,t)\cos\frac{\varphi}{2}(0,t) + \frac{Q}{2}\varpi(0,t)\sin\frac{\varphi}{2}(0,t) = 0 \qquad \forall t \in \mathbb{R}, \quad (5.2.6)$$

so that, fixing $t \in \mathbb{R}$ and restricting to the domain to $x \in (-\infty, 0]$,

$$(\varphi(\cdot,t), \varpi(\cdot,t))|_{(-\infty,0]} \in \mathcal{N}_{P,Q}, \tag{5.2.7}$$

with $P = d_2(0,t)$, $Q = d_1(0,t) \neq 0$.

Therefore, if a solution to an initial-boundary value problem of Type $\mathbf{A}_N$ is such that constraint (5.2.2), (5.2.3), (5.2.4) can be made to hold by an appropriate choice of $N(\lambda), \tilde{N}(\lambda)$ it will also solve a problem of Type $\mathbf{B}_{P,Q}$ for some P, $Q \in \mathbb{R}$.

5.2.2 The constraint picks out subspaces of scattering data

For the rest of this section fix $t \in \mathbb{R}$. According to (4.2.16), (4.2.17) the gauge constraint translates into the relation

$$T_+(x,t,-\lambda^{-1}) = -\sigma_3 j^{-1}(x,t,\lambda^{-1})T_-(-x,t,\lambda)\tilde{\jmath}^{-1}(t,\lambda^{-1}), \tag{5.2.8}$$

with $\lambda \in \mathbb{R} \setminus \{0\}$, $j(x,t,\lambda) \stackrel{\text{def}}{=} \sigma_2 L(x,t,\lambda)$ and

$$\tilde{\jmath}(t,\lambda^{-1}) \stackrel{\text{def}}{=} \frac{1}{2}\lim_{y\to-\infty}\left\{\sigma_2 \exp\left(\frac{iy}{4}(\lambda-\lambda^{-1})\sigma_3\right)(1-i\sigma_1)(-i\sigma_3)^N \right.$$
$$\left. \cdot j^{-1}(-y,t,\lambda^{-1})(1+i\sigma_1)\exp\left(-\frac{iy}{4}(\lambda-\lambda^{-1})\sigma_3\right)\right\}. \tag{5.2.9}$$

Once again it is necessary to consider the cases of N even/odd separately.

N **odd**

With $N \in 2\mathbb{Z}+1$,

$$\mathbf{dst}(\varphi(\cdot,t),\varpi(\cdot,t)) = (a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_1^{n_1(t),n_2(t)} \tag{5.2.10}$$

for some $n_1(t) \in 2\mathbb{N}+1$, $n_2(t) \in \mathbb{N}$.

From a detailed analysis of (4.2.9) it is possible to deduce that the coefficients of the matrix $L(x,t,\lambda)$ must have the asymptotics

$$\begin{gathered}
a_1(x,t) \to \mp\frac{\pi N}{2}, \quad a_3(x,t) \to \pm i^{N+1}((\omega+\omega^{-1})(\chi+\chi^{-1})-2) \\
d_1(x,t) \to i^{N+1}\epsilon((\omega+\omega^{-1})-(\chi+\chi^{-1})), \\
a_2(x,t),\ d_2(x,t),\ f_{1,2}(x,t) \to 0, \quad x \to \pm\infty \quad \forall t \in \mathbb{R},
\end{gathered} \tag{5.2.11}$$

with

$$\epsilon = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1,\ \mathrm{Im}\,\chi \neq 0.$$

From these asymptotics, $\det L(0,t,\lambda) = \det L(\infty,t,\lambda)$ implies that

$$\begin{aligned}
Q^2 &= (\mathrm{Im}\,(\chi))^2(\omega-\omega^{-1})^2 \\
P^2 &= (\mathrm{Re}\,(\chi))^2(\omega+\omega^{-1})^2,
\end{aligned} \tag{5.2.12}$$

the matrices $j(0,t,\lambda^{-1})$, $\tilde{j}(t,\lambda^{-1})$ taking the form

$$\begin{gathered}
j(0,t,\lambda^{-1}) = iP(\lambda+\lambda^{-1})\mathbb{I} - (\lambda^2-\lambda^{-2})\sigma_2 - Q(\lambda-\lambda^{-1})\sigma_3, \\
\tilde{j}(t,\lambda^{-1}) = \begin{pmatrix} 0 & -p^{-1}(-\lambda^{-1}) \\ -p^{-1}(\lambda^{-1}) & 0 \end{pmatrix},
\end{gathered}$$

with

$$p(\lambda) \stackrel{\mathrm{def}}{=} i((\lambda-\lambda^{-1}) - i\epsilon(\omega+\omega^{-1}))((\lambda-\lambda^{-1}) + 2i\epsilon\mathrm{Re}\,(\chi)),$$

and clearly both $j(0,t,\lambda^{-1})$ and $\tilde{j}(t,\lambda^{-1})$ are independent of t.

In place of (4.2.24) there exist more complicated relations such as

$$\begin{aligned}
b_+(-\lambda^{-1},t) &= \frac{(P(\lambda+\lambda^{-1}) - iQ(\lambda-\lambda^{-1}))a_-(\lambda,t) - (\lambda^2-\lambda^{-2})c_-(\lambda,t)}{(\lambda-\lambda^{-1}-i\epsilon(\omega+\omega^{-1}))(\lambda-\lambda^{-1}+2i\epsilon\mathrm{Re}\,(\chi))}, \\
d_+(-\lambda^{-1},t) &= -\frac{(\lambda^2-\lambda^{-2})a_-(\lambda,t) + (P(\lambda+\lambda^{-1}) + iQ(\lambda-\lambda^{-1}))c_-(\lambda,t)}{(\lambda-\lambda^{-1}-i\epsilon(\omega+\omega^{-1}))(\lambda-\lambda^{-1}+2i\epsilon\mathrm{Re}\,(\chi))},
\end{aligned} \tag{5.2.13}$$

to be satisfied $\forall\lambda \in \mathbb{R}\setminus\{0\}$. Once again these are compatible with the analytic continuation of $a_-(\cdot,t), c_-(\cdot,t), b_+(\cdot,t), d_+(\cdot,t)$ into the upper half of the complex plane and so can be understood to hold for all λ in this domain.

Proceeding as in chapter 4 it is found that the constraint (5.2.2), (5.2.3), (5.2.4) forces the scattering data (5.2.10) to be an element of a subspace of $\hat{\mathcal{D}}_1^{n_1(t),n_2(t)}$. This subspace is the analogue of that introduced in definition 4.1 and is given by:

Definition 5.1 *For arbitrary $t \in \mathbb{R}$ fix $n_1(t) \in 2\mathbb{N}+1$, $n_2(t) \in \mathbb{N}$ and parameters*

$$\epsilon, \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \operatorname{Im} \chi \neq 0.$$

Let the subspace $\mathcal{F}_{1,\epsilon,\omega,\chi}^{n_1(t),n_2(t),\rho} \subset \hat{\mathcal{D}}_1^{n_1(t),n_2(t)}$ denote sets of scattering data

$$(a(\cdot,t),\, b(\cdot,t) : \gamma_1(t), \ldots \gamma_{n_1(t)+n_2(t)}(t)) \in \hat{\mathcal{D}}_1^{n_1(t),n_2(t)},$$

such that

- *in their domains of analyticity the transition coefficients $a(\cdot,t)$, $b(\cdot,t)$ satisfy*

$$a(\lambda,t) = -a(-\lambda^{-1},t)$$

$$b(\lambda,t) = -\left[\frac{((\lambda-\lambda^{-1}) - i(\omega+\omega^{-1}))((\lambda-\lambda^{-1}) + i(\chi+\chi^{-1}))}{((\lambda-\lambda^{-1}) + i(\omega+\omega^{-1}))((\lambda-\lambda^{-1}) - i(\chi+\chi^{-1}))}\right]^{\epsilon} b(\lambda^{-1},t),$$

  so $a(i,t) = 0$ and for all the $\lambda_j(t)$ such that $a(\lambda_j(t),t) = 0$ it follows that $a(-\lambda_j^{-1}(t),t) = 0$ also.

- *the normalisation coefficients at these zeroes are such that*

$$\gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = -\left[\frac{((\lambda_j(t)-\lambda_j^{-1}(t)) - i(\omega+\omega^{-1}))((\lambda_j(t)-\lambda_j^{-1}(t)) + i(\chi+\chi^{-1}))}{((\lambda_j(t)-\lambda_j^{-1}(t)) + i(\omega+\omega^{-1}))((\lambda_j(t)-\lambda_j^{-1}(t)) - i(\chi+\chi^{-1}))}\right]^{\epsilon},$$

  and

$$\operatorname{sign}(\gamma_i(t)) = \rho.$$

With this result it is time to move on to the case of N even.

N **even**

With $N \in 2\mathbb{Z}$,

$$\mathbf{dst}(\varphi(\cdot,t), \varpi(\cdot,t)) = (a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}_0^{n_1(t),n_2(t)} \qquad (5.2.14)$$

for some $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$.

From (4.2.9) and the equality

$$\det L(\infty,t,\lambda) = \det L(-\infty,t,\lambda) = \det L(0,t,\lambda),$$

there follows the asymptotics

$$a_1(x,t) \to \mp\frac{\pi N}{2}, \qquad a_2(x,t) \to \pm i^N \varepsilon(\eta - \eta^{-1})(\xi + \xi^{-1})$$

$$d_2(x,t) \to i^N(\xi + \xi^{-1}), \qquad f_2(x,t) \to \pm i^N \varepsilon(\eta^{-1} - \eta)$$

$$a_3(x,t),\ d_1(x,t),\ f_1(x,t) \to 0 \tag{5.2.15}$$

as $x \to \pm\infty \quad \forall t \in \mathbb{R}$ with

$$\varepsilon = \pm 1, \quad \eta \in (0,1) \quad \xi \in \mathbb{C} : |\xi| = 1, \ \mathrm{Im}\,\xi \neq 0,$$

and the constraints

$$\begin{aligned} P^2 &= (\mathrm{Re}(\xi))^2(\eta + \eta^{-1})^2 \\ Q^2 &= (\mathrm{Im}(\xi))^2(\eta - \eta^{-1})^2. \end{aligned} \tag{5.2.16}$$

This time the matrices appearing in (5.2.8) take the form

$$j(0,t,\lambda^{-1}) = iP(\lambda + \lambda^{-1})\mathbb{I} - (\lambda^2 - \lambda^{-2})\sigma_2 - Q(\lambda - \lambda^{-1})\sigma_3,$$

$$\tilde{j}(t,\lambda^{-1}) = \begin{pmatrix} 0 & -q^{-1}(-\lambda^{-1}) \\ -q^{-1}(\lambda^{-1}) & 0 \end{pmatrix},$$

with

$$q(\lambda) \stackrel{\mathrm{def}}{=} i((\lambda + \lambda^{-1}) - i\varepsilon(\eta - \eta^{-1}))((\lambda - \lambda^{-1}) + 2i\mathrm{Re}\,(\xi)),$$

and once again both $j(0,t,\lambda^{-1})$ and $\tilde{j}(t,\lambda^{-1})$ are independent of t.

In place of (4.2.24) there exist a set of relations similar to those for N odd. In particular

$$\begin{aligned} b_+(-\lambda^{-1},t) &= -\frac{(P(\lambda + \lambda^{-1}) - iQ(\lambda - \lambda^{-1}))a_-(\lambda,t) - (\lambda^2 - \lambda^{-2})c_-(\lambda,t)}{(\lambda + \lambda^{-1} - i\varepsilon(\eta^{-1} - \eta))(\lambda - \lambda^{-1} + 2i\mathrm{Re}\,(\xi))}. \\ d_+(-\lambda^{-1},t) &= \frac{(\lambda^2 - \lambda^{-2})a_-(\lambda,t) + (P(\lambda + \lambda^{-1}) + iQ(\lambda - \lambda^{-1}))c_-(\lambda,t)}{(\lambda + \lambda^{-1} - i\varepsilon(\eta^{-1} - \eta))(\lambda - \lambda^{-1} + 2i\mathrm{Re}\,(\xi))}. \end{aligned} \tag{5.2.17}$$

to be satisfied $\forall\lambda \in \mathbb{R} \setminus \{0\}$. These are compatible with the analytic continuation of $a_-(\cdot,t), c_-(\cdot,t), b_+(\cdot,t), d_+(\cdot,t)$ into the upper half of the complex plane and so can be understood to hold for all λ in this domain.

Once again, proceeding as in chapter 4, it is found that the constraint (5.2.2), (5.2.3), (5.2.4) forces the scattering data (5.2.14) to be an element of a subspace of $\hat{\mathcal{D}}_0^{n_1(t),n_2(t)}$. This is the analogue of those subspaces introduced in definitions 4.2 and 4.3. It is given by:

Definition 5.2 *For arbitrary $t \in \mathbb{R}$ fix $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$ and parameters*

$$\varepsilon, \varrho = \pm 1, \quad \eta \in (0,1), \quad \xi \in \mathbb{C} : |\xi| = 1, \ \mathrm{Im}\,\xi \neq 0.$$

Let the subspace $\mathcal{F}_{0,\varepsilon,\eta,\xi}^{n_1(t),n_2(t),\varrho} \subset \hat{\mathcal{D}}_0^{n_1(t),n_2(t)}$ denote sets of scattering data

$$(a(\cdot,t),\, b(\cdot,t) : \gamma_1(t), \ldots \gamma_{n_1(t)+n_2(t)}(t)) \in \hat{\mathcal{D}}_0^{n_1(t),n_2(t)},$$

such that

- *in their domains of analyticity the transition coefficients $a(\cdot,t)$, $b(\cdot,t)$ satisfy*

$$a(\lambda,t) = \left[\frac{\lambda + \lambda^{-1} + i(\eta - \eta^{-1})}{\lambda + \lambda^{-1} - i(\eta - \eta^{-1})}\right]^{\varepsilon} a(-\lambda^{-1},t)$$

$$b(\lambda,t) = \left[\frac{\lambda - \lambda^{-1} + i(\xi + \xi^{-1})}{\lambda - \lambda^{-1} - i(\xi + \xi^{-1})}\right] b(\lambda^{-1},t),$$

  so $a(i\eta^{-\varepsilon},t) = a(i,t) = 0$ and for all the $\lambda_j(t) \neq i\eta^{-\varepsilon}$ such that $a(\lambda_j(t),t)) = 0$ it follows that $a(-\lambda_j^{-1}(t),t) = 0$ also.

- *the normalisation coefficients at these zeroes are such that:*

$$\gamma_{\lambda_j(t)}(t)\gamma_{-\lambda_j^{-1}(t)}(t) = \left[\frac{\lambda_j(t) - \lambda_j^{-1}(t) + i(\xi + \xi^{-1})}{\lambda_j(t) - \lambda_j^{-1}(t) - i(\xi + \xi^{-1})}\right],$$

  for $\lambda_j(t) \neq i\eta^{-\varepsilon}$ and

$$\gamma_{i\eta^{-\varepsilon}}(t) \in \mathbb{R} \setminus \{0\}, \qquad \mathrm{sign}(\gamma_i(t)) = \varrho.$$

This subsection is now complete. It has been shown that some of the problems of Type $\mathbf{A}_N$ have solutions which also solve a problem in the set $\mathbf{B}_{P,Q\neq 0}$ and that the scattering data of these solutions at a fixed time t must take a particular form.

5.2.3 Subspaces of scattering data imply the constraint

Just as in subsection 4.2.5 it can be shown (using assertion $1'''$ of section 3.4) that given scattering data in one of the two subspaces $\mathcal{F}^{n_1(t),n_2(t),\rho}_{1,\epsilon,\omega,\chi}$, $\mathcal{F}^{n_1(t),n_2(t),\varrho}_{0,\varepsilon,\eta,\xi}$ then there does exist a relation such as (5.2.8), (5.2.3), (5.2.4) for the Jost solutions constructed from this data and so relation (5.2.7) holds.

5.3 The inverse scattering method for solving problems of Type $\mathbf{B}_{P,Q}$

In chapter 3 it was seen how the inverse scattering method could only be applied to the problems of Type $\mathbf{A}_N$ which are defined by initial data in the subspace $\check{\mathcal{M}}_N \subset \mathcal{M}_N$. When developing this method in order to solve problems in the set $\mathbf{B}_{P,0}$, this drawback prompted the formulation of the inverse scattering transform $\mathbf{ist}|_{(-\infty,0]}$ and the subspace $\check{\mathcal{N}}_{P,0} \subset \mathcal{N}_{P,0}$ before introducing the direct scattering transform $\mathbf{dst}|_{(-\infty,0]}$ (see chapter 4). This reasoning repeats itself here when developing the method to solve problems of Type $\mathbf{B}_{P,Q\neq 0}$.

5.3.1 The inverse scattering transform for a subset of problems in $\mathbf{B}_{P,Q}$

In chapter 3 the inverse scattering transform was formulated as the injective map

$$\mathbf{ist}\left(\bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{q,r}_{q\mathrm{mod}2}\right) = \bigcup_{p\in\mathbb{Z}} \check{\mathcal{M}}_{2p+q},$$

for arbitrary $q \in \mathbb{N}$. In chapter 4 the notation $\mathbf{ist}|_{(-\infty,0]}$ was introduced to denote this map with the parameter x restricted to the semi-line $(-\infty,0]$. Following on from these results it is now possible to develop some different restrictions on $\mathbf{ist}|_{(-\infty,0]}$ so that it can be used as the third stage in a solution to some of the problems of Type $\mathbf{B}_{P,Q}$.

Lemma 4.5 is replaced by:

Lemma 5.3 *Fix $t \in \mathbb{R}$, $n_1(t) \in 2\mathbb{N}+1$, $n_2(t) \in \mathbb{N}$ and parameters*

$$\epsilon, \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \mathrm{Im}\,\chi \neq 0.$$

Let

$$\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right) \in \mathcal{F}^{n_1(t),n_2(t),\rho}_{1,\epsilon,\omega,\chi},$$

then

$$(\varphi(\cdot,t), \varpi(\cdot,t))|_{(-\infty,0]} \stackrel{\mathrm{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right),$$

is an element of $\mathcal{N}_{P,Q}$ with

$$P = i^{n(t)+1}\mathrm{Re}\,(\chi)(\omega + \omega^{-1})$$
$$Q = \epsilon\rho|(\mathrm{Im}\,(\chi))(\omega - \omega^{-1})|,$$

and $n(t) = n_1(t) + 2n_2(t)$.

The proof of this lemma follows by exactly the same methods as those used in order to prove lemma 4.5. Identical reasoning with $\mathcal{F}^{n_1(t),n_2(t),\varrho}_{0,\varepsilon,\eta,\xi}$ replacing $\mathcal{F}^{n_1(t),n_2(t),\rho}_{1,\epsilon,\omega,\chi}$ leads to:

Lemma 5.4 *Fix $t \in \mathbb{R}$, $n_1(t) \in 2\mathbb{N}$, $n_2(t) \in \mathbb{N}$ and parameters*

$$\varepsilon, \varrho = \pm 1, \quad \eta \in (0,1), \quad \xi \in \mathbb{C} : |\xi| = 1, \ \mathrm{Im}\,\xi \neq 0.$$

Let

$$\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right) \in \mathcal{F}^{n_1(t),n_2(t),\varrho}_{0,\varepsilon,\eta,\xi},$$

then

$$(\varphi(\cdot,t), \varpi(\cdot,t))|_{(-\infty,0]} \stackrel{\mathrm{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)\right),$$

is an element of $\mathcal{N}_{P,Q}$ with

$$P = i^{n(t)}\mathrm{Re}(\xi)(\eta + \eta^{-1})$$
$$Q = -\varepsilon\varrho|\mathrm{Im}\,(\xi)(\eta - \eta^{-1})|,$$

and $n(t) = n_1(t) + 2n_2(t)$.

The results of lemmas 5.3 and 5.4 can be collected together in proposition 5.6. First it is necessary to make the following (rather horrible) definition.

Definition 5.5 *With*

$$\epsilon, \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \mathrm{Im}\,\chi \neq 0,$$

define the subset

$$\mathcal{G}^{\epsilon,\rho}_{\omega,\chi} \subset \bigcup_{p,q \in \mathbb{N}} \hat{\mathcal{D}}^{p,q}_{p\mathrm{mod}2},$$

by

$$\begin{aligned} \mathcal{G}^{\epsilon,\rho}_{\omega,\chi} \stackrel{\text{def}}{=} \bigcup_{p,q\in\mathbb{N}} \Big(& \mathcal{F}^{2p+1,q,\rho}_{1,\epsilon,\omega,\chi} \cup \mathcal{F}^{2p+1,q,\rho}_{1,\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p+1,q,-\rho}_{1,-\epsilon,\omega,\chi} \cup \mathcal{F}^{2p+1,q,-\rho}_{1,-\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p+1,q,-\rho}_{1,\epsilon,\omega,\chi} \cup \mathcal{F}^{2p+1,q,-\rho}_{1,\epsilon,\omega,-\chi} \\ & \cup \mathcal{F}^{2p+1,q,\rho}_{1,-\epsilon,\omega,\chi} \cup \mathcal{F}^{2p+1,q,\rho}_{1,-\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p,q,\rho}_{0,\epsilon,\omega,\chi} \cup \mathcal{F}^{2p,q,\rho}_{0,\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p,q,-\rho}_{0,-\epsilon,\omega,\chi} \\ & \cup \mathcal{F}^{2p,q,-\rho}_{0,-\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p,q,-\rho}_{0,\epsilon,\omega,\chi} \cup \mathcal{F}^{2p,q,-\rho}_{0,\epsilon,\omega,-\chi} \cup \mathcal{F}^{2p,q,\rho}_{0,-\epsilon,\omega,\chi} \cup \mathcal{F}^{2p,q,\rho}_{0,-\epsilon,\omega,-\chi} \Big) \end{aligned} \tag{5.3.1}$$

In chapter 3 it was seen that for arbitrary $n_1(t) \in \mathbb{N}$,

$$\mathbf{ist} : \bigcup_{r\in\mathbb{N}} \hat{\mathcal{D}}^{n_1(t),r}_{n_1(t)\mathrm{mod}2} \xrightarrow{1-1} \bigcup_{q\in\mathbb{Z}} \mathcal{M}_{2q+n_1(t)}.$$

Therefore:

Proposition 5.6 *Choosing parameters*

$$\epsilon, \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \mathrm{Im}\,\chi \neq 0,$$

then

$$\mathbf{ist}|_{(-\infty,0]} : \mathcal{G}^{\epsilon,\rho}_{\omega,\chi} \xrightarrow{1-1} \mathcal{N}_{A,B} \cup \mathcal{N}_{-A,B} \cup \mathcal{N}_{A,-B} \cup \mathcal{N}_{-A,-B},$$

where

$$\begin{aligned} A &= \Big|\mathrm{Re}(\chi)(\omega + \omega^{-1})\Big| \\ B &= \Big|\mathrm{Im}(\chi)(\omega - \omega^{-1})\Big|. \end{aligned} \tag{5.3.2}$$

The proof of this proposition is easily pieced together using the results of chapter 3 and lemmas 5.3 and 5.4. Next introduce the subspace $\breve{\mathcal{N}}_{P,Q\neq 0}$.

Definition 5.7 *With*

$$\epsilon, \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \operatorname{Im}\chi \neq 0,$$

let

$$A = \left|\operatorname{Re}(\chi)(\omega + \omega^{-1})\right|, \qquad B = \left|\operatorname{Im}(\chi)(\omega - \omega^{-1})\right|,$$

and define the four subspaces $\check{\mathcal{N}}_{\pm A, \pm B}$, $\check{\mathcal{N}}_{\pm A, \mp B}$ *via the image of the map* $\mathbf{ist}|_{(-\infty,0]}$. *Namely,*

$$\check{\mathcal{N}}_{A,B} \cup \check{\mathcal{N}}_{-A,B} \cup \check{\mathcal{N}}_{A,-B} \cup \check{\mathcal{N}}_{-A,-B} \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}(\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}).$$

This completes the development of a restriction of the transform $\mathbf{ist}|_{(-\infty,0]}$. From the discussions of earlier chapters it is evident how this will form part of the inverse scattering method for solving a subset of the problems in $\mathbf{B}_{P,Q\neq 0}$; the subset being defined by the reduced phase space $\check{\mathcal{N}}_{P,Q}$. Notice that the analysis appearing in this chapter mirrors that developed (in more detail) in chapter 4 for the problems with $Q = 0$. This will continue to be the case throughout the rest of this chapter.

In the next subsection attention is turned to a formulation of the direct scattering transform.

5.3.2 The direct scattering transform for a subset of problems in $\mathbf{B}_{P,Q}$

In definition 5.5 the subset

$$\mathcal{G}^{\epsilon,\rho}_{\omega,\chi} \subset \bigcup_{p,q \in \mathbb{N}} \hat{\mathcal{D}}^{p,q}_{p \bmod 2},$$

was introduced. According to proposition 5.6 and definition 5.7 the restriction of the inverse scattering transform $\mathbf{ist}|_{(-\infty,0]}$ to this subset yields an element of $\check{\mathcal{N}}_{P,Q}$ with P and Q given in terms of ω, χ, ϵ, ρ.

In this subsection the direct scattering transform $\mathbf{dst}|_{(-\infty,0]}$ is developed so that

$$\mathbf{dst}|_{(-\infty,0]} : \check{\mathcal{N}}_{P,Q} \longrightarrow \mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$$

with the parameters in the image fixed in terms of the given P, $Q \in \mathbb{R}$. When considering the set of problems of Type $\mathbf{B}_{P,0}$ it was found that $\mathbf{dst}|_{(-\infty,0]}$ must be formulated separately for the cases $|P| < 2$, $|P| > 2$ and $|P| = 2$. However, when $Q \neq 0$ no such complications arise and it is possible to develop $\mathbf{dst}|_{(-\infty,0]}$ in a single stage.

At time $t \in \mathbb{R}$ let $(\varphi(\cdot,t), \varpi(\cdot,t) \in \breve{\mathcal{N}}_{P,Q}$ for some $P \in \mathbb{R}$, $Q \in \mathbb{R} \setminus \{0\}$. For $x \leq 0$, $\lambda \in \mathbb{R} \setminus \{0\}$ use this data to construct the Jost solution $T_-(x,t,\lambda)$ as outlined in chapter 3 and define the matrix $j_1(\lambda)$ by

$$j_1(\lambda) \stackrel{\text{def}}{=} -i((\lambda^2 - \lambda^{-2})\sigma_1 + P(\lambda + \lambda^{-1})\sigma_3 - iQ(\lambda - \lambda^{-1})\mathbb{I}). \tag{5.3.3}$$

For

$$\epsilon = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1, \ \operatorname{Im}\chi \neq 0,$$

let

$$j_{2,3}(t,\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -p_{2,3}^{-1}(-\lambda) \\ -p_{2,3}^{-1}(\lambda) & 0 \end{pmatrix}, \tag{5.3.4}$$

with

$$p_2(\lambda) \stackrel{\text{def}}{=} i((\lambda - \lambda^{-1}) - i\epsilon(\omega + \omega^{-1}))((\lambda - \lambda^{-1}) + 2i\epsilon \operatorname{Re}(\chi)),$$
$$p_3(\lambda) \stackrel{\text{def}}{=} i((\lambda + \lambda^{-1}) - i\epsilon(\omega - \omega^{-1}))((\lambda - \lambda^{-1}) + 2i\operatorname{Re}(\chi)), \tag{5.3.5}$$

and, depending on the choice of $(\varphi(\cdot,t), \varpi(\cdot,t)$, *define* the Jost solution $T_+(0,t,\cdot)$ by either

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_2^{-1}(\lambda^{-1}), \tag{5.3.6}$$

or

$$T_+(0,t,-\lambda^{-1}) \stackrel{\text{def}}{=} j_1^{-1}(\lambda^{-1})T_-(0,t,\lambda)j_3^{-1}(\lambda^{-1}). \tag{5.3.7}$$

To determine which definition is appropriate, and the form the parameters ϵ, ω, χ must take, it is necessary to demand:

1. $\det T_+(0,t,\lambda) = 1 \quad \forall \lambda \in \mathbb{R} \setminus \{0\}$.
2. the columns of $T_+(0,t,\cdot)$ have the appropriate analytic properties.
3. the resulting $T_\pm(0,t,\cdot)$ lead to scattering data in $\hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}$ with $n_1(t)$, $n_2(t) \in \mathbb{N}$.

Defining

$$\Gamma(\lambda,t) \overset{\text{def}}{=} (P(\lambda+\lambda^{-1}) - iQ(\lambda-\lambda^{-1}))a_-(\lambda,t) - (\lambda^2-\lambda^{-2})c_-(\lambda,t),$$
$$\tilde{\Gamma}(\lambda,t) \overset{\text{def}}{=} (\lambda^2-\lambda^{-2})a_-(\lambda,t) + (P(\lambda+\lambda^{-1}) + iQ(\lambda-\lambda^{-1}))c_-(\lambda,t),$$

and

$$\omega \overset{\text{def}}{=} \frac{1}{2}(D - [D^2-4]^{1/2}),$$

with

$$D = \frac{1}{2}(P^2+Q^2+[(Q^2-P^2+4)^2+4P^2Q^2]^{1/2}),$$

it follows that

$$\Gamma(i\omega,t)=0 \qquad \Leftrightarrow \qquad \tilde{\Gamma}(i\omega,t)=0.$$

To deduce the appropriate definition for $T_+(0,t,\cdot)$ notice that (5.3.6) implies that for $\lambda \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned} b_+(-\lambda^{-1},t) &\overset{\text{def}}{=} i\frac{\Gamma(\lambda,t)}{p_2(-\lambda^{-1})} \\ d_+(-\lambda^{-1},t) &\overset{\text{def}}{=} -i\frac{\tilde{\Gamma}(\lambda,t)}{p_2(-\lambda^{-1})}, \end{aligned} \tag{5.3.8}$$

whereas (5.3.7) yields

$$\begin{aligned} b_+(-\lambda^{-1},t) &\overset{\text{def}}{=} i\frac{\Gamma(\lambda,t)}{p_3(-\lambda^{-1})} \\ d_+(-\lambda^{-1},t) &\overset{\text{def}}{=} -i\frac{\tilde{\Gamma}(\lambda,t)}{p_3(-\lambda^{-1})}. \end{aligned} \tag{5.3.9}$$

Therefore, with

$$\vartheta \overset{\text{def}}{=} \left|\frac{P}{\omega+\omega}\right| + i\left|\frac{Q}{\omega-\omega}\right|,$$

consider the eight possibilities,

(i)	$\Gamma(i\omega,t)=0,$	$\Gamma(i\omega^{-1},t)=0,$	$\Gamma(i\vartheta,t)=0,$
(ii)	$\Gamma(i\omega,t)=0,$	$\Gamma(i\omega^{-1},t)=0,$	$\Gamma(i\vartheta,t)\neq 0,$
(iii)	$\Gamma(i\omega,t)=0,$	$\Gamma(i\omega^{-1},t)\neq 0,$	$\Gamma(i\vartheta,t)=0,$
(iv)	$\Gamma(i\omega,t)=0,$	$\Gamma(i\omega^{-1},t)\neq 0,$	$\Gamma(i\vartheta,t)\neq 0,$
(v)	$\Gamma(i\omega,t)\neq 0,$	$\Gamma(i\omega^{-1},t)=0,$	$\Gamma(i\vartheta,t)=0,$

$$
\begin{array}{llll}
(vi) & \Gamma(i\omega,t)\neq 0, & \Gamma(i\omega^{-1},t)=0, & \Gamma(i\vartheta,t)\neq 0,\\
(vii) & \Gamma(i\omega,t)\neq 0, & \Gamma(i\omega^{-1},t)\neq 0, & \Gamma(i\vartheta,t)=0,\\
(vii) & \Gamma(i\omega,t)\neq 0, & \Gamma(i\omega^{-1},t)\neq 0, & \Gamma(i\vartheta,t)\neq 0,
\end{array}
$$

depending on the particular configuration $(\varphi(\cdot,t),\varpi(\cdot,t))$. By demanding the constraints 1-3 it is easy to deduce that if $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that (i) holds then it is necessary to choose definition (5.3.6) with $\epsilon=1$ and $\chi=-\vartheta$. The scattering data can then be found in the standard manner and, by construction, this data has all the appropriate properties. Imposing criteria 1-3 when $(\varphi(\cdot,t),\varpi(\cdot,t))$ is such that (ii) holds shows (5.3.6) to be the correct definition once more, but this time with $\epsilon=1$, $\chi=\vartheta$. The form of $T_+(0,t,\cdot)$ for all eight scenarios can be summarised as:

$$
\begin{array}{rcl}
(i) & \Rightarrow & (5.3.6)\text{ with } \epsilon=1,\ \chi=-\vartheta,\\
(ii) & \Rightarrow & (5.3.6)\text{ with } \epsilon=1,\ \chi=\vartheta,\\
(iii) & \Rightarrow & (5.3.7)\text{ with } \epsilon=-1,\ \chi=-\vartheta,\\
(iv) & \Rightarrow & (5.3.7)\text{ with } \epsilon=-1,\ \chi=\vartheta,\\
(v) & \Rightarrow & (5.3.7)\text{ with } \epsilon=1,\ \chi=-\vartheta,\\
(vi) & \Rightarrow & (5.3.7)\text{ with } \epsilon=1,\ \chi=\vartheta,\\
(vii) & \Rightarrow & (5.3.6)\text{ with } \epsilon=-1,\ \chi=-\vartheta,\\
(viii) & \Rightarrow & (5.3.6)\text{ with } \epsilon=-1,\ \chi=\vartheta.
\end{array}
$$

With this reasoning

$$
\begin{array}{rcl}
(i) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,-\vartheta}^{1,\,\mathbf{sign}(Q)},\\
(ii) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,\vartheta}^{1,\,\mathbf{sign}(Q)},\\
(iii) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,-\vartheta}^{-1,\,\mathbf{sign}(Q)},\\
(iv) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,\vartheta}^{-1,\,\mathbf{sign}(Q)},\\
(v) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,-\vartheta}^{1,\,-\mathbf{sign}(Q)},\\
(vi) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,\vartheta}^{1,\,-\mathbf{sign}(Q)},\\
(vii) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,-\vartheta}^{-1,\,-\mathbf{sign}(Q)},\\
(viii) & \Rightarrow & \mathbf{dst}|_{(-\infty,0]}:\check{\mathcal{N}}_{P,Q}\longrightarrow\mathcal{G}_{\omega,\vartheta}^{-1,\,-\mathbf{sign}(Q)}.
\end{array}
$$

These eight possibilities constitute a transformation from $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \check{\mathcal{N}}_{P,Q}$ to some element of a set of scattering data. From the analysis presented in chapter 3 and the results of section 5.2 it is easily deduced that the restricted maps $\mathbf{dst}|_{(-\infty,0]}$ and $\mathbf{ist}|_{(-\infty,0]}$ developed in subsections 5.3.2, 5.3.1 respectively are such that

$$\mathbf{ist}|_{(-\infty,0]} = \left(\mathbf{dst}|_{(-\infty,0]}\right)^{-1}.$$

5.3.3 Time evolving the scattering data

With $(\varphi(\cdot,t),\varpi(\cdot,t)) \in \check{\mathcal{N}}_{P,Q\neq 0}$ this subsection outlines the determination of the time evolution of

$$(a(\cdot,t),b(\cdot,t):\gamma_1(t),\ldots,\gamma_{n_1(t)+2n_2(t)}(t)) \stackrel{\text{def}}{=} \mathbf{dst}|_{(-\infty,0]}(\varphi(\cdot,t),\varpi(\cdot,t)) \in \mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)}, \tag{5.3.10}$$

when (φ,ϖ) evolves according to an initial-boundary value problem of Type $\mathbf{B}_{P,Q\neq 0}$.

For $P \in \mathbb{R}$, $Q \in \mathbb{R}\setminus\{0\}$ let (φ,ϖ) be the solution to a problem of Type $\mathbf{B}_{P,Q}$. The equation governing the time evolution of the Jost solution $T_-(0,t,\lambda)$ constructed from (φ,ϖ) is

$$\frac{\partial T_-}{\partial t}(0,t,\lambda) = V(0,t,\lambda)T_-(0,t,\lambda) - \frac{1}{4i}(\lambda+\lambda^{-1})T_-(0,t,\lambda)\sigma_3 \quad \forall\lambda\in\mathbb{R}\setminus\{0\}. \tag{5.3.11}$$

With the Jost solution $T_+(0,t,\lambda)$ *defined* by either (5.3.6) or (5.3.7), ($\epsilon,\rho,\omega,\chi$ are already determined in terms of P and Q), it must evolve in time according to

$$\frac{\partial T_+}{\partial t}(0,t,\lambda) = V(0,t,\lambda)T_+(0,t,\lambda) - \frac{1}{4i}(\lambda+\lambda^{-1})T_+(0,t,\lambda)\sigma_3 \quad \forall\lambda\in\mathbb{R}\setminus\{0\}. \tag{5.3.12}$$

The proceeding analysis is identical to that appearing in subsection 4.3.3 for the problems with $P \in (-2,2)$, $Q = 0$. Once again it is deduced that the map τ_t gives the evolution of the initial scattering data when (φ,ϖ) solves a problem of Type $\mathbf{B}_{P,Q}$, and that this map is a bijection

$$\tau_t : \mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)} \xrightarrow{1-1} \mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)}. \tag{5.3.13}$$

This completes the development of the third and final stage required for an inverse scattering solution to some of the initial-boundary value problems in the set $\mathbf{B}_{P,Q\neq 0}$. In the next subsection this method will be pieced together.

5.3.4 Piecing together the inverse scattering method

Subsections 5.3.1, 5.3.2, 5.3.3 lead to the inverse scattering method for solving a *subset* of the problems in the set $\mathbf{B}_{P,Q\neq 0}$.

Definition 5.8 *For $P \in \mathbb{R}$, $Q \in \mathbb{R}\setminus\{0\}$ let $\breve{\mathbf{B}}_{P,Q}$ denote the subset of problems of Type $\mathbf{B}_{P,Q}$ which have an initial condition in the subspace $\breve{\mathcal{N}}_{P,Q} \subset \mathcal{N}_{P,Q}$.*

When the initial data $(\varphi_{P,Q}, \varpi_{P,Q})$ is an element of $\breve{\mathcal{N}}_{P,Q}$ then the image of the composite map $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t \circ \mathbf{dst}|_{(-\infty,0]}$ is such that

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = (\mathbf{ist}|_{(-\infty,0]} \circ \tau_t \circ \mathbf{dst}|_{(-\infty,0]})(\varphi_{P,Q}, \varpi_{P,Q}),$$

is also an element of $\breve{\mathcal{N}}_{P,Q}$ and the resulting functions φ, $\varpi : (x,t) \mapsto \mathbb{R}$ solve the initial-boundary value problem of Type $\mathbf{B}_{P,Q}$ with initial data $(\varphi(\cdot,0), \varpi(\cdot,0)) = (\varphi_{P,Q}, \varpi_{P,Q})$.

The time evolution map

$$\mathrm{T}'_t : \breve{\mathcal{N}}_{P,Q} \to \breve{\mathcal{N}}_{P,Q},$$

giving the solution to the set of problems $\breve{\mathbf{B}}_{P,Q}$ as

$$(\varphi(\cdot,t), \varpi(\cdot,t)) = \mathrm{T}'_t(\varphi_{P,Q}, \varpi_{P,Q}),$$

can be expressed by the commutative diagram in figure 5.1.

Once again, recall that $\breve{\mathcal{N}}_{P,Q}$ is only defined implicitly in terms of the map $\mathbf{ist}|_{(-\infty,0]}$ applied to $\mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)}$. So, as before, the inverse scattering method developed here *cannot* be used to solve the problem in the set $\mathbf{B}_{P,Q}$ defined by initial data $(\varphi_{P,Q}, \varpi_{P,Q})$ for *any* $(\varphi_{P,Q}, \varpi_{P,Q}) \in \mathcal{N}_{P,Q}$ given beforehand. This is identical to the situation for problems in $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,0}$ discussed in chapters 3 and 4. However, arbitrary solutions to problems in the set $\breve{\mathbf{B}}_{P,Q}$ can be found by applying the composite map $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ to appropriate elements of the space of initial scattering data. The particular problem for which this is the solution can then be deduced by setting $t = 0$. This idea is developed in the next chapter.

This completes the development of the inverse scattering method for solving the initial-boundary value problems in the set $\breve{\mathbf{B}}_{P,Q\neq 0}$. In the next chapter the analysis presented

$$
\begin{tikzcd}
\breve{\mathcal{N}}_{P,Q} \arrow[rrr, "\mathbf{dst}|_{(-\infty,0]}"] \arrow[d, "\mathrm{T}'_t"'] & & & \mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)} \arrow[d, "\tau_t"] \\
\breve{\mathcal{N}}_{P,Q} & & & \mathcal{G}^{\epsilon(P,Q),\rho(P,Q)}_{\omega(P,Q),\chi(P,Q)} \arrow[lll, "\mathbf{ist}|_{(-\infty,0]} = \left(\mathbf{dst}|_{(-\infty,0]}\right)^{-1}"]
\end{tikzcd}
$$

$\mathrm{T}'_t \overset{\text{def}}{=}$ time evolution map defined by a *nonlinear* problem of Type $\mathbf{B}_{P,Q\neq 0}$

τ_t = bijective time evolution map governed by a set of *linear* o.d.e's

Figure 5.1: **The inverse scattering method for solving problems in $\breve{\mathbf{B}}_{P,Q\neq 0}$**

in chapters 3-5 will be used to find explicit solutions for some of the problems in $\breve{\mathbf{A}}_N$ and $\breve{\mathbf{B}}_{P,Q}$.

Chapter 6

Soliton solutions

6.1 Introduction

In this chapter the solutions to particular initial-boundary value problems in the sets $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,0}$ and $\breve{\mathbf{B}}_{P,Q\neq 0}$ are found using the inverse scattering method developed in chapters 3, 4 and 5 respectively. As has already been seen, this method involves three distinct stages in order to solve the problem defined by the initial configuration condition $(\varphi_N, \varpi_0) \in \breve{\mathcal{M}}_N$, (resp.$(\varphi_{P,Q}, \varpi_{P,Q}) \in \breve{\mathcal{N}}_{P,Q}$). However, as has already been explained, a description of $\breve{\mathcal{M}}_N$, $(\breve{\mathcal{N}}_{P,Q})$ in terms of such pairs of functions is unknown at the present time and as a result the first stage of the method (the maps **dst**, $\mathbf{dst}|_{(-\infty,0]}$) cannot be used.

These problems can be avoided, however, if it is only required that the initial configuration defining a solution is deduced *a posteriori* and not specified *a priori*. This idea is possible since the space of scattering data which single out $\breve{\mathcal{M}}_N$, $(\breve{\mathcal{N}}_{P,Q})$ can be defined precisely. Therefore, in order to find solutions to problems in the sets $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,0}$ and $\breve{\mathbf{B}}_{P,Q\neq 0}$ it suffices to choose an element of the appropriate set of scattering data at time $t = 0$ and to apply the composite map $\mathbf{ist} \circ \tau_t$, $(\mathbf{ist}|_{(-\infty,0]} \circ \tau_t)$ to this.

6.2 The inverse scattering transform applied to soliton scattering data

The inverse scattering transform **ist** was formulated at arbitrary $x, t \in \mathbb{R}$ in section 3.3 and was represented as the map (3.3.1). In section 3.7 it was explained how **ist** constitutes the third stage in the inverse scattering method for solving initial-boundary value problems of Type $\breve{\mathbf{A}}_N$. Chapters 4 and 5 proceeded to develop this method so that it could be used to solve problems of Type $\breve{\mathbf{B}}_{P,Q}$. The modification of the third stage, i.e the map **ist**, simply required a restriction of the parameter x to the semi-line $(-\infty, 0]$ and this restriction was denoted $\mathbf{ist}|_{(-\infty,0]}$.

This section gives a detailed application of the map **ist** to the scattering data

$$(a(\cdot,t), b(\cdot,t) \equiv 0 : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \in \hat{\mathcal{D}}^{n_1(t),n_2(t)}_{n_1(t)\mathrm{mod}2}, \tag{6.2.1}$$

for some $n_1(t)$, $n_2(t) \in \mathbb{N}$ and $n(t) = n_1(t) + 2n_2(t)$. It is only in the cases when $b(\cdot, t) \equiv 0$, ('soliton' scattering data), that the inverse scattering transform can be executed analytically.

When $b(\cdot, t) \equiv 0$, (3.2.43) yields $G(x, t, \lambda) = \mathbb{I}$ so that the solution to the standard Riemann problem (3.3.14) can be written as

$$\check{g}_+(x, t, \lambda) = \Xi(x, t, \lambda), \qquad \check{g}_-(x, t, \lambda) = \Xi^{-1}(x, t, \lambda). \tag{6.2.2}$$

The matrix $\Xi^\dagger(x, t, \lambda) = \Xi^{-1}(x, t, \bar{\lambda})$ is constrained to be analytic in the upper half λ plane, to have the asymptotic (3.3.15) and to satisfy

$$\Xi^{-1}(x, t, \bar{\lambda}_j(t))N_j^-(x, t) = \Xi^\dagger(x, t, \lambda_j(t))N_j^-(x, t) = 0, \qquad j = 1, \dots, n(t), \tag{6.2.3}$$

where the subspace $N_j^-(x, t)$ is defined by (3.2.49). From these constraints it follows that the matrix $\Xi^{-1}(x, t, \lambda)$ can be resolved into partial fractions as

$$\Xi^{-1}(x, t, \lambda) = \mathbb{I} + \sum_{j=1}^{n(t)} \frac{A_j(x, t)}{\lambda - \lambda_j(t)}, \tag{6.2.4}$$

with some matrix coefficients $A_j(x, t)$. Asymptotically expanding this representation as $\lambda \to \lambda_j(t)$ yields

$$\begin{aligned} \Xi^{-1}(x, t, \lambda) &= \frac{A_j(x, t)}{\lambda - \lambda_j(t)} + O(1) \\ \Xi(x, t, \lambda) &= B_j(x, t) + O(|\lambda - \lambda_j(t)|), \end{aligned} \tag{6.2.5}$$

so that $A_j(x, t)B_j(x, t) = B_j(x, t)A_j(x, t) = 0$. This together with $\mathrm{Im}B_j(x, t) = N_j^+(x, t)$ implies that the matrices $A_j(x, t)$ are rank one and can be represented as

$$A_j(x, t) = z_j(x, t)\left(N_j^-(x, t)\right)^\dagger, \tag{6.2.6}$$

with

$$z_j(x, t) \stackrel{\text{def}}{=} \begin{pmatrix} p_j(x, t) \\ q_j(x, t) \end{pmatrix}, \qquad \left(N_j^-(x, t)\right)^\dagger = (\gamma_j(x, t), 1), \tag{6.2.7}$$

Substituting (6.2.6) into (6.2.4) and then into (6.2.3) gives a system of linear algebraic equations for the unknowns $p_j(x, t), q_j(x, t)$. These can be represented as

$$\begin{aligned} \mathbf{M}(x, t)\vec{p}(x, t) &= -\vec{\gamma}(x, t) \\ \mathbf{M}(x, t)\vec{q}(x, t) &= -\vec{1}, \end{aligned} \tag{6.2.8}$$

where $\mathbf{M}(x,t)$ is the $n(t) \times n(t)$ matrix with entries

$$(\mathbf{M}(x,t))_{j,k} = \frac{1 + \bar{\gamma}_j(x,t)\gamma_k(x,t)}{\bar{\lambda}_j(t) - \lambda_k(t)}, \qquad j,\, k = 1, \ldots, n(t), \tag{6.2.9}$$

the $n(t)$ component column vectors $\vec{p}(x,t)$, $\vec{q}(x,t)$, $\vec{\gamma}(x,t)$ take the form

$$\vec{p}(x,t) = \begin{pmatrix} p_1(x,t) \\ \vdots \\ p_{n(t)}(x,t) \end{pmatrix}, \quad \text{etc.} \tag{6.2.10}$$

and $\vec{1}$ denotes the $n(t)$ component column vector with 1 in each entry.

From (6.2.6), (6.2.7) and (6.2.8) it is straightforward to construct the matrices $A_j(x,t)$ and so the solution $\Xi(x,t,\lambda)$ to the standard Riemann problem (3.3.14) according to (6.2.4). From this solution an element of $\breve{\mathcal{M}}_N$ can be deduced using (3.3.22), (3.3.13) and the requirement that

$$\lim_{x \to -\infty} \varphi(x,t) = 0. \tag{6.2.11}$$

In this section it has been seen how the inverse scattering transform map **ist** can be applied to the soliton scattering data (6.2.1). These results will be used in the subsequent sections when finding 'soliton solutions' to the particular problems in the sets $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,Q}$ which are defined by initial scattering data of this form.

6.3 Single soliton solutions to problems of Type $\mathbf{A}_0$, $\mathbf{A}_{\pm 1}$ and $\mathbf{B}_{P,Q}$

This section considers the solutions to problems of Type $\breve{\mathbf{A}}_0$, $\breve{\mathbf{A}}_{\pm 1}$ and $\breve{\mathbf{B}}_{P,Q}$ for particular initial conditions $(\varphi_0, \varpi_0) \in \breve{\mathcal{M}}_0$, $(\varphi_{\pm 1}, \varpi_0) \in \breve{\mathcal{M}}_{\pm 1}$ and $(\varphi_{P,Q}, \varpi_{P,Q}) \in \breve{\mathcal{N}}_{P,Q}$. These conditions are the ones for which the scattering data $\mathbf{dst}(\varphi_0, \varpi_0)$, $\mathbf{dst}(\varphi_0, \varpi_{\pm 1})$ or $\mathbf{dst}|_{(-\infty,0]}(\varphi_{P,Q}, \varpi_{P,Q})$ is such that $b(\cdot, 0) \equiv 0$ and $a(\cdot, 0)$ has a single zero $\lambda_1(0)$ such that $\operatorname{Im}\lambda_1(0) > 0$, $\operatorname{Re}\lambda_1(0) \geq 0$. Because of the structure of this initial scattering data, the solutions which result from applying $\mathbf{ist} \circ \tau_t$, $(\mathbf{ist}|_{(-\infty,0]} \circ \tau_t)$ to it are called *single soliton* solutions. The characteristics of these solutions are very different depending on whether $\operatorname{Re}\lambda_1(0) = 0$ or not.

6.3.1 Single soliton solutions of 'kink' type which solve problems in $\mathbf{A}_{\pm 1}$

Fixing $\epsilon_1 = \pm 1$ and $\kappa_1(0)$, $|\gamma_1(0)| \in \mathbb{R}^+$ suppose $(\varphi_{\epsilon_1}, \varpi_0) \in \check{\mathcal{M}}_{\epsilon_1}$ is such that

$$\mathbf{dst}(\varphi_{\epsilon_1}, \varpi_0) = \left(a(\lambda, 0) = \frac{\lambda - i\kappa_1(0)}{\lambda + i\kappa_1(0)},\ b(\cdot, 0) \equiv 0 : \gamma_1(0) = -\epsilon_1 |\gamma_1(0)| \right).$$

Applying $\mathbf{ist} \circ \tau_t$ to this data (see (6.2.2)-(6.2.11)) gives the solution

$$\begin{aligned} \varphi(x,t) &= 4\epsilon_1 \arctan \frac{1}{|\gamma_1(x,t)|} \\ \varpi(x,t) &= -\frac{2((\kappa_1(0))^2 - 1)\gamma_1(x,t)}{\kappa_1(0)(1 + (\gamma_1(x,t))^2)}, \end{aligned} \tag{6.3.1}$$

$$\gamma_1(x,t) = \exp\left\{ -\frac{1}{2} \left[\left(\kappa_1(0) + \kappa_1(0)^{-1} \right) x + \left(\kappa_1(0) - \kappa_1(0)^{-1} \right) t \right] \right\} \gamma_1(0),$$

and the principal branch of $\arctan x$ is taken. Therefore $\varpi(x,t) = \varphi_t(x,t)$ as it must be by construction and

$$\begin{aligned} \varphi(x,t) &= 4\epsilon_1 \arctan \exp \left(\frac{x - vt - x_1}{\sqrt{1 - v^2}} \right) \\ v &= \frac{1 - (\kappa_1(0))^2}{1 + (\kappa_1(0))^2}, \quad |v| < 1 \\ x_1 &= \sqrt{1 - v^2} \log |\gamma_1(0)|. \end{aligned} \tag{6.3.2}$$

There exists a natural interpretation of this solution as a relativistic particle moving to the right with velocity v and whose centre of inertia coordinate at $t = 0$ is x_1. It is called a soliton solution of 'kink' type and carries the topological charge $N = \epsilon_1$. Particular solution with $N = 1$ are often referred to as 'kinks' whilst those with $N = -1$ are 'antikinks'. This interpretation will be explored more closely in subsection 7.5.4 where the mass, energy and momentum of the particle will be calculated using a set of 'trace identities' for the problems of Type $\check{\mathbf{A}}_N$.

6.3.2 Single soliton solutions of 'breather' type which solve problems in the set $\mathbf{A}_0$

Fix $\lambda_1(0) \in \mathbb{C}$ such that $\mathrm{Im}\lambda_1(0)$, $\mathrm{Re}\lambda_1(0) > 0$ and $\gamma_1(0) \in \mathbb{C}$ and consider the initial configuration $(\varphi_0, \varpi_0) \in \breve{\mathcal{M}}_0$ such that

$$\mathbf{dst}(\varphi_0, \varpi_0) = \left(a(\lambda, 0) = \frac{\lambda - \lambda_1(0)}{\lambda - \bar{\lambda}_1(0)} \cdot \frac{\lambda + \bar{\lambda}_1(0)}{\lambda + \lambda_1(0)},\ b(\cdot, 0) \equiv 0 : \gamma_1(0) \right). \tag{6.3.3}$$

Proceeding exactly as before gives $\varpi(x,t) = \varphi_t(x,t)$ and the solution

$$\varphi(x,t) = 4 \arctan \frac{\nu}{\zeta} \frac{\sin\left(\frac{\omega_1(t-vx)}{\sqrt{1-v^2}} + \phi_1\right)}{\cosh\left(\frac{\omega_2(x-vt-x_1)}{\sqrt{1-v^2}}\right)} \tag{6.3.4}$$

with

$$\zeta = \mathrm{Re}\,\lambda_1(0), \quad \nu = \mathrm{Im}\,\lambda_1(0), \quad v = \frac{1 - |\lambda_1(0)|^2}{1 + |\lambda_1(0)|^2}, \quad \phi_1 = \arg \gamma_1(0), \tag{6.3.5}$$

$$\omega_1 = \frac{\zeta}{|\lambda_1(0)|}, \quad \omega_2 = \frac{\nu}{|\lambda_1(0)|}, \quad x_1 = \frac{\sqrt{1-v^2}}{\omega_2} \log |\gamma_1(0)|. \tag{6.3.6}$$

This soliton solution is specified by four real parameters and describes a particle-like solution with internal degrees of freedom. It is called a soliton of 'breather' type. Along with the translational motion of a relativistic particle with velocity v and initial centre of inertia coordinate x_1, the breather oscillates in both space and time with frequencies $\frac{v\omega_1}{\sqrt{1-v^2}}, \frac{v\omega_2}{\sqrt{1-v^2}}$ respectively. The parameter ϕ_1 plays the role of an initial phase. The mass, energy and momentum of this type of particle are also calculated using trace identities in subsection 7.5.4.

6.3.3 Single kink type solutions to problems in $\mathbf{B}_{P,0}$

Fixing $\epsilon_1 = \pm 1$ and Λ, $|\gamma_1(0)| \in \mathbb{R}^+$ then according to definition 4.1 the scattering data

$$\left(a(\lambda, 0) = \frac{\lambda - i\Lambda}{\lambda + i\Lambda},\ b(\cdot, 0) \equiv 0 : \gamma_1(0) = -\epsilon_1 |\gamma_1(0)| \right),$$

is an element of $\mathcal{F}^{1,0}_{1,\Lambda}$. Applying the composite transform $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ to this data gives the solution $\varpi = \varphi_t$ with

$$\varphi(x,t) = 4\epsilon_1 \arctan \exp\left(\frac{x - vt - x_1}{\sqrt{1-v^2}}\right)$$

$$v = \frac{1-\Lambda^2}{1+\Lambda^2}, \quad |v| < 1$$
$$x_1 = \sqrt{1-v^2}\log|\gamma_1|, \tag{6.3.7}$$

$\forall x \in (-\infty, 0]$, $t \in \mathbb{R}$. This is a kink or antikink (depending on the choice of ϵ_1) moving with a fixed velocity. According to lemma 4.5 the function φ satisfies the boundary condition

$$\frac{\partial\varphi}{\partial x}(x,t)\bigg|_{x=0} - (\Lambda + \Lambda^{-1})\sin\frac{\varphi}{2}(0,t) = 0 \qquad \forall t \in \mathbb{R}, \tag{6.3.8}$$

and it is straightforward to verify that this is indeed the case. The boundary condition satisfied by ϖ follows directly.

Depending on sign(v) this solution can be interpreted as a soliton (particle) of kink type either being emitted from or absorbed by the boundary at $x = 0$. The energy of this particle is found in subsection 7.5.4 using a different set of trace identities applicable to problems of Type $\breve{\mathbf{B}}_{P,Q}$.

6.3.4 Single breather type solutions to problems in $\mathbf{B}_{P,0}$

Fixing parameters $A \in (-2,2)$, $\alpha \in (0, \frac{\pi}{2})$ such that $\sin\alpha > \frac{A}{2}$ and $\gamma_1(0) \in \mathbb{C}$ such that

$$|\gamma_1(0)|^2 = \frac{2\sin\alpha + A}{2\sin\alpha - A},$$

it follows from definition 4.3 that the scattering data

$$\left(a(\lambda,0) = \frac{\lambda - ie^{-i\alpha}}{\lambda + ie^{i\alpha}} \cdot \frac{\lambda - ie^{i\alpha}}{\lambda + ie^{-i\alpha}},\ b(\cdot,0) \equiv 0 : b(\cdot) \equiv 0 : \gamma_1(0)\right)$$

is an element of $\hat{\mathcal{F}}^{0,1}_{0,\xi}$ with

$$\xi = \frac{1}{2}(A + i\sqrt{4 - A^2}).$$

Therefore

$$(\varphi(\cdot,t), \varpi(\cdot,t)) \overset{\text{def}}{=} (\mathbf{ist}|_{(-\infty,0]} \circ \tau_t)\left(a(\lambda,0) = \frac{\lambda - ie^{-i\alpha}}{\lambda + ie^{i\alpha}} \cdot \frac{\lambda - ie^{i\alpha}}{\lambda + ie^{-i\alpha}},\ b(\cdot,0) \equiv 0 : \gamma_1(0)\right),$$

is such that $\varpi = \varphi_t$ and

$$\varphi(x,t) = 4\arctan\left(\tan\alpha \frac{\sin(t\cos\alpha + \phi_1)}{\cosh(\sin\alpha(x - x_1))}\right), \tag{6.3.9}$$

with

$$x_1 \stackrel{\text{def}}{=} \frac{1}{2\sin\alpha}\log\left(\frac{2\sin\alpha + A}{2\sin\alpha - A}\right), \qquad \phi_1 \in [0, 2\pi).$$

According to lemma 4.7 this function satisfies the boundary condition

$$\frac{\partial\varphi}{\partial x}(x,t)\bigg|_{x=0} - A\sin\frac{\varphi}{2}(0,t) = 0 \qquad \forall t \in \mathbb{R}, \tag{6.3.10}$$

and the boundary condition satisfied by ϖ follows directly.

This solution will be termed a 'boundary-breather' soliton solution to the sine-Gordon system on the semi-line $x \in (-\infty, 0]$. It is clearly a stationary soliton of breather type when regarded as a solution on the whole line $x \in \mathbb{R}$, but when space is restricted to the semi-line $(-\infty, 0]$ it is a time dependent solution which remains localized in the vicinity of the $x = 0$ boundary.

Remark 6.1 *It follows from (6.3.8) that there only exists a $(\varphi_{P,0}, \varpi_{P,0}) \in \breve{\mathcal{N}}_{P,0}$ which evolves into a single soliton solution of kink type (6.3.7) when $P \leq -2$. When $|P| < 2$ there are no single soliton solutions of this type but there are ones of boundary breather type (6.3.9). Finally, when $P \geq 2$ there does not exist a problem in $\breve{\mathbf{B}}_{P,0}$ which has a single soliton as its solution.*

6.3.5 Single soliton solutions to problems in $\mathbf{B}_{P,Q\neq 0}$

It follows from definitions 5.1, 5.2, 5.5 that the only single soliton scattering data in $\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$ has the form

$$\left(a(\lambda, 0) = \frac{\lambda - i}{\lambda + i},\ b(\cdot, 0) \equiv 0 : \gamma_1(0) = \rho|\gamma_1(0)|\right),$$

with fixed parameters

$$\epsilon,\ \rho = \pm 1, \qquad \omega \in (0,1), \qquad \chi \in \mathbb{C} : |\chi| = 1,\ \operatorname{Im}\chi \neq 0,$$

and

$$|\gamma_1(0)| = (f(\omega,\chi))^{\frac{\epsilon}{2}}),$$

$$f(\omega,\chi) \stackrel{\text{def}}{=} \frac{(\omega + \omega^{-1} - 2)(2 + \chi + \chi^{-1})}{(\omega + \omega^{-1} + 2)(2 - \chi - \chi^{-1})} \in \mathbb{R}^+.$$

Applying the map $(\mathbf{ist}|_{(-\infty,0]} \circ \tau_t)$ to this data gives $\varpi \equiv 0$ and

$$\varphi(x,t) = \varphi(x) = 4\rho \arctan \exp(x - x_1),$$
$$x_1 = \frac{\epsilon}{2} \log f(\omega, \chi), \tag{6.3.11}$$

$\forall x \in (-\infty, 0]$. According to lemma 5.3 this function satisfies the boundary condition

$$\left.\frac{\partial \varphi}{\partial x}(x)\right|_{x=0} - \mathrm{Re}(\chi)(\omega + \omega^{-1}) \sin \frac{\varphi}{2}(0) - \epsilon\rho|\mathrm{Im}(\chi)(\omega - \omega^{-1})| \cos \frac{\varphi}{2}(0) = 0. \tag{6.3.12}$$

Remark 6.2 *Notice that, in contrast to the $Q = 0$ case there is sufficient freedom in the choice of the parameters $\epsilon, \rho, \omega, \chi$ so as to realise a single soliton solution which satisfies the $x = 0$ boundary condition for all possible choices of boundary parameters $P \in \mathbb{R}$, $Q \in \mathbb{R} \setminus \{0\}$.*

6.4 Multi-soliton solutions to problems of Type $\mathbf{B}_{P,Q}$ and boundary scattering

This section considers certain properties of the solutions to problems of Type $\breve{\mathbf{B}}_{P,Q}$ defined by particular initial conditions $(\varphi_{P,Q}, \varpi_{P,Q}) \in \breve{\mathcal{N}}_{P,Q}$. These initial configurations are such that the transition coefficient $a(\cdot, 0)$ appearing in $\mathbf{dst}|_{(-\infty,0]}(\varphi_{P,Q}, \varpi_{P,Q})$ has multiple simple zeroes $\{\lambda_1(0), \ldots, \lambda_{n(0)}(0)\}$ satisfying $\mathrm{Im}\,\lambda_j(0) > 0$, $\mathrm{Re}\,\lambda_j(0) \geq 0$ $\forall j = 1, \ldots, n(0)$ and the transition coefficient $b(\cdot, 0) \equiv 0$. These solutions are termed the $n(0)$ *soliton (multi-soliton)* solutions to the problems on the semi-line.

6.4.1 Kink scattering from the boundary when $Q = 0$

Consider the scattering data

$$\left(a(\lambda, 0) = \frac{\lambda - i\kappa(0)}{\lambda + i\kappa(0)} \cdot \frac{\lambda - i(\kappa(0))^{-1}}{\lambda + i(\kappa(0))^{-1}},\ b(\cdot, 0) \equiv 0 :\ \gamma_1(0) = e^{a_1},\ \gamma_2(0) = \epsilon_2 e^{a_2} \right), \tag{6.4.1}$$

with parameters $\kappa(0) \in (0,1)$, $\epsilon_2 = \pm 1$, $a_{1,2} \in \mathbb{R}$. Fixing a parameter $A \in \mathbb{R}$ and defining

$$\xi = \frac{1}{2}(A + \sqrt{A^2 - 4}),$$

it follows from definition 4.2 that for this data to be an element of $\mathcal{F}^{2,0}_{0,\xi}$ the parameters ϵ_2, a_1, a_2 must satisfy

$$\begin{aligned}\epsilon_2 &= \operatorname{sign}\left(\frac{\kappa(0)+(\kappa(0))^{-1}+A}{\kappa(0)+(\kappa(0))^{-1}-A}\right),\\ a_1+a_2 &= \log\left|\frac{\kappa(0)+(\kappa(0))^{-1}+A}{\kappa(0)+(\kappa(0))^{-1}-A}\right|.\end{aligned}\tag{6.4.2}$$

Constraining these three parameters in this way and applying the composite transform $(\mathbf{ist}|_{(-\infty,0]}\circ\tau_t)$ to the data (6.4.1) gives $\varpi=\varphi_t$ and

$$\varphi(x,t)=4\arctan\left(\frac{1}{v}\left[\frac{\epsilon_2\exp\left(-\frac{(x+vt)}{\sqrt{1-v^2}}+a_2\right)-\exp\left(-\frac{(x-vt)}{\sqrt{1-v^2}}+a_1\right)}{1+\epsilon_2\exp\left(-\frac{2x}{\sqrt{1-v^2}}+a_1+a_2\right)}\right]\right),\tag{6.4.3}$$

$\forall\,(x,t)\in(-\infty,0]\times\mathbb{R}$, with

$$v=\frac{1-(\kappa(0))^2}{1+(\kappa(0))^2},$$

so that

$$\begin{aligned}\epsilon_2 &= \operatorname{sign}\left(\frac{4-A^2(1-v^2)}{(2-A\sqrt{1-v^2})^2}\right)\\ a_1+a_2 &= \log\left(\frac{|4-A^2(1-v^2)|}{(2-A\sqrt{1-v^2})^2}\right).\end{aligned}\tag{6.4.4}$$

By lemma 4.6 the function φ satisfies the boundary condition

$$\left.\frac{\partial\varphi}{\partial x}(x,t)\right|_{x=0}-A\sin\frac{\varphi}{2}(0,t)=0\qquad\forall t\in\mathbb{R}.$$

Now consider the asymptotic $x,t\to-\infty$ in the solution (6.4.3). Since $v>0$ it follows that

$$\varphi(x,t)\sim 4\arctan\exp\left(\frac{(x-vt)}{\sqrt{1-v^2}}-\tilde{a}_1\right),\tag{6.4.5}$$

as $x,t\to-\infty$, with $\tilde{a}_1\stackrel{\text{def}}{=}a_1+\log v$. Therefore, by comparison with (6.3.2), it can be seen that asymptotically as $x,t\to-\infty$ the solution (6.4.3) represents a kink moving to the right with speed v. It is useful to rewrite (6.4.5) as

$$\varphi(x,t)\sim 4\arctan\exp\left(\frac{x-v(t+\Delta_1)}{\sqrt{1-v^2}}\right)\qquad x,t\to-\infty,\tag{6.4.6}$$

where

$$\Delta_1\stackrel{\text{def}}{=}\frac{\sqrt{1-v^2}}{v}\tilde{a}_1.$$

The asymptotic $x \to -\infty, t \to \infty$ yields

$$\varphi(x,t) \sim -4\epsilon_2 \arctan\exp\left(\frac{(x+vt)}{\sqrt{1-v^2}} - \tilde{a}_2\right), \tag{6.4.7}$$

so that in this limit (6.4.3) coincides with the form of a single soliton of kink type moving to the left with speed v and possessing topological charge $(-\epsilon_2)$. Rewriting this asymptotic as

$$\varphi(x,t) \sim -4\epsilon_2 \arctan\exp\left(\frac{x+v(t-\Delta_2)}{\sqrt{1-v^2}}\right) \qquad x, -t \to -\infty, \tag{6.4.8}$$

where

$$\Delta_2 \stackrel{\text{def}}{=} \frac{\sqrt{1-v^2}}{v}\tilde{a}_2,$$

and comparing with (6.4.6) it is straightforward to identify the 'time delay' experienced by the incoming kink at $t = -\infty$ as a result of its interaction with the boundary at $x = 0$ as

$$\Delta = \Delta_1 + \Delta_2 = \frac{\sqrt{1-v^2}}{v}(\tilde{a}_1 + \tilde{a}_2).$$

From (6.4.4) it follows that this can be written in terms of the speed $v > 0$ and the boundary parameter A as

$$\Delta = \frac{\sqrt{1-v^2}}{v}\log\left(\frac{v^2|4 - A^2(1-v^2)|}{(2 - A\sqrt{1-v^2})^2)^2}\right). \tag{6.4.9}$$

The topological charge of the reflected (anti)kink depends on these parameters according to

$$N_{\text{outgoing}} = \text{sign}\left(A^2(1-v^2) - 4)\right).$$

There are a couple of points to note regarding the preceeding analysis:

- since $0 < v < 1$ it follows that choosing $A \in (-2, 2)$ for the boundary parameter will always cause the incoming kink to reflect as an antikink and it is only when $|A| \geq 2$ that a diagonal component to the 'boundary scattering matrix' appears.
- when A is fixed such that $|A| \geq 2$, the time delay Δ develops a logarithmic singularity at velocities satisfying $v^2 = \frac{A^2-4}{A^2}$. This reflects the existence of single fixed velocity kink solutions such as (6.3.7) at this value of boundary parameter, which can either be emitted from or absorbed by the boundary at $x = 0$ without violating energy conservation.

6.4.2 Kink scattering from the boundary when $Q \neq 0$

Fixing parameters

$$\theta \in \mathbb{R}^+, \qquad \rho = \pm 1, \qquad a_1, a_2 \in \mathbb{R}, \qquad \mu \in (0, 2\pi) \setminus \{\pi\}, \qquad \nu \in \mathbb{R}^+,$$

and consider the scattering data

$$\left(a(\lambda, 0) = \frac{\lambda - i}{\lambda + i} \cdot \frac{\lambda^2 - 2i\lambda \cosh\theta - 1}{\lambda^2 + 2i\lambda\cosh\theta - 1},\, b(\cdot, 0) \equiv 0 :\ \gamma_1 = \rho e^{a_1},\, \gamma_2 = -e^{a_2},\, \gamma_3 = \epsilon_3 e^{a_3} \right), \tag{6.4.10}$$

where, for the moment, $\epsilon_3 = \pm 1$, $a_3 \in \mathbb{R}$ remain arbitrary. Defining

$$\chi \stackrel{\text{def}}{=} -e^{i\mu}, \qquad \omega \stackrel{\text{def}}{=} e^{-\nu},$$

it follows from definition 5.1 that this data will be an element of $\mathcal{F}^{3,0,\rho}_{1,\epsilon,\omega,-\chi}$ with $\epsilon = \pm 1$ if the parameters ϵ_3, a_3 are chosen to satisfy

$$\begin{aligned} \epsilon_3 &= \operatorname{sign}\left(\frac{\tanh\frac{1}{2}(\theta+\nu)\tanh\frac{1}{2}(\theta-\nu)}{\tanh\frac{1}{2}(\theta+i\mu)\tanh\frac{1}{2}(\theta-i\mu)} \right), \\ a_3 + a_2 &= \epsilon \log\left| \frac{\tanh\frac{1}{2}(\theta+\nu)\tanh\frac{1}{2}(\theta-\nu)}{\tanh\frac{1}{2}(\theta+i\mu)\tanh\frac{1}{2}(\theta-i\mu)} \right|. \end{aligned} \tag{6.4.11}$$

Constraining ϵ_3, a_3 in this way and applying the composite transform $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ gives a rather complicated expression for $\varphi(x,t)$ and $\varpi(x,t)$. However, it follows from lemma 5.3 that the function φ satisfies the boundary condition

$$\left.\frac{\partial\varphi}{\partial x}(x,t)\right|_{x=0} + 2\cos\mu\cosh\nu\sin\frac{\varphi}{2}(0,t) - 2\epsilon\rho|\sin\mu\sinh\nu|\cos\frac{\varphi}{2}(0,t) = 0 \qquad \forall t \in \mathbb{R}.$$

Just as in the previous subsection when considering a solution with $Q = 0$, the asymptotics of $x, \pm t \to -\infty$ of this solution can be studied and the time delay Δ deduced. The results of this analysis are:

- as $x, t \to -\infty$ the solution $\varphi(x,t)$ has the form of a (right-moving) kink with velocity $v = \tanh\theta$.

- as $x, -t \to -\infty$ the solution takes the form of a soliton of kink type moving with velocity $-v$. The topological charge of this reflected (anti)kink is

$$N_{\text{outgoing}} = -\operatorname{sign}\left(\frac{\tanh\frac{1}{2}(\theta+\nu)\tanh\frac{1}{2}(\theta-\nu)}{\tanh\frac{1}{2}(\theta+i\mu)\tanh\frac{1}{2}(\theta-i\mu)} \right).$$

- the time delay Δ experienced by the soliton due to its interaction with the boundary at $x = 0$ is given by

$$\Delta = \frac{1}{\sinh\theta}\log\left(\tanh^2\theta\tanh^2\frac{\theta}{2}\left|\frac{\tanh\frac{1}{2}(\theta+\nu)\tanh\frac{1}{2}(\theta-\nu)}{\tanh\frac{1}{2}(\theta+i\mu)\tanh\frac{1}{2}(\theta-i\mu)}\right|^{\epsilon}\right).$$

It must be mentioned that this solution was first found in [17] using τ function methods.

6.4.3 Regarding the existence of boundary-breathers

In subsection 6.3.4 a two parameter family (α, ϕ_1) of single 'boundary-breather' solutions were found. These solve a two parameter family of initial-boundary value problems of Type $\breve{\mathbf{B}}_{P,0}$ with $P \in (-2, 2)$.

Multi-boundary-breather solutions will be defined as being generated by scattering data in the subsets $\mathcal{F}_\Lambda$, $\mathcal{F}'_\xi$, $\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$ (see definitions 4.8, 5.5) which have zeroes of the transition coefficient $a(\cdot, t)$, (i.e $a(\lambda_j(t), t) = 0$)) such that $\mathrm{Re}(\lambda_j(t)) \neq 0$ *and* $|\lambda_j(t)|^2 = 1$. However it can be seen that:

- the set $\mathcal{F}_\Lambda$ does not contain any elements which have zeroes of this type.
- as was seen in subsection 6.3.4 some elements of $\mathcal{F}'_\xi$ do possess such zeroes but they must all satisfy the constraint $\mathrm{Im}(\lambda_j(t)) > \mathrm{Re}(\xi)$.
- certain elements of $\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$ also have zeroes of this form but this time they are constrained by the inequality $\mathrm{Im}(\lambda_j(t)) > \mathrm{Re}(\chi)$.

These comments show that if $Q = 0$ and $|P| \geq 2$ there do not exist any soliton solutions of boundary-breather type which solve an initial-boundary value problem in $\breve{\mathbf{B}}_{P,0}$. However, as soon as Q is perturbed away from zero then such solutions can exist for all values of the parameter P. The $Q \to 0$ limit of the solutions to the problems with $Q \neq 0$ will be studied in the next section but this does not affect the results found here.

6.5 Investigating the $Q \to 0$ limit of the solutions to problems of Type $\mathbf{B}_{P,Q}$

Up to now the problems of Type $\mathbf{B}_{P,Q}$ with $Q = 0$ have been considered separately from those with $Q \neq 0$. This section explains why this is necessary and reasons as to why the $Q \to 0$ limit of the $Q \neq 0$ analysis actually reproduces the same solutions as those found using the $Q = 0$ results when this limit is taken sufficiently carefully.

According to proposition 5.6 the $Q \to 0$ limit of the subset of scattering data $\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$ corresponds to either $\omega \to 1$ or $\chi \to \pm 1$. However, in these limits the normalization coefficient $\gamma_i(t)$ becomes either unbounded or equal to zero and the formulation of the inverse scattering transform **ist** and therefore its restriction $\mathbf{ist}|_{(-\infty,0]}$ forbids such situations. Therefore, it is comforting that the analysis of chapters 4 and 5 naturally differentiates between the cases $Q = 0$, $Q \neq 0$.

Once solutions to problems of Type $\breve{\mathbf{B}}_{P,Q\neq 0}$ have been found by applying $\mathbf{ist}|_{(-\infty,0]} \circ \tau_t$ to elements of $\mathcal{G}^{\epsilon,\rho}_{\omega,\chi}$ it is then possible to examine the $Q \to 0$ limit explicitly in these formulae. For certain solutions (i.e choices of ϵ, $\mathrm{Re}(\chi) = \pm 1$) this limit is 'sick' and the resulting expression no longer satisfy the asymptotic boundary condition as $x \to -\infty$. However, in other cases this limit is well defined and the solutions found agree with those resulting directly from the $Q = 0$ analysis (without the transition coefficient $a(\cdot, t)$ having a zero at $\lambda(t) = i$).

It is satisfying to see that, although the inverse scattering theory for solving problems in the set $\breve{\mathbf{B}}_{P,Q}$ with $Q = 0$ is distinct from that developed for the problems with $Q \neq 0$, the 'non-sick' limits of the solutions to the latter reproduce the solutions to the former - the sick limits corresponding to a violation of the boundary condition $\varphi(x,t) \to 0$ as $x \to -\infty$.

The points outlined above will now be illustrated by considering the simplest example of the static (anti)kink solution (6.3.11) which satisfies the boundary condition (6.3.12). When $\omega \to 1$ or $\chi \to -1$ so that $Q \to 0$ in this boundary condition it follows that $f(\omega, \chi) \to 0$. Alternatively, when $\chi \to 1$, $f(\omega, \chi) \to \infty$. The centre of inertia coordinate for this kink solution is $x = x_1 = \frac{\epsilon}{2} \log f(\omega, \chi)$ and allowing this to tend to $-\infty$ clearly violates the imposed boundary condition $\varphi(x) \to 0$ as $x \to -\infty$. So, when considering the limits $\omega \to 1$ or $\chi \to -1$ it is necessary to choose the solution with $\epsilon = -1$. The limit $\chi \to 1$ requires $\epsilon = 1$ and the solution to the $Q = 0$ problem resulting from these limiting procedures is the 'zero soliton' solution $\varphi \equiv 0$ - the static soliton disappears infinitely far 'behind the wall'.

Such reasoning can be repeated to study how the scattering solution of subsection 6.4.2 reduces to that of subsection 6.4.1 when certain parameters in the former are correctly chosen so as to preserve the asymptotic boundary condition as $x \to -\infty$ when the $Q \to 0$ limit is taken. For these solutions, it is found that when $\epsilon = 1$ in subsection 6.4.2 the $Q \to 0$ limit forces $P \geq 2$ whereas when $\epsilon = -1$ this limit forces $P < 2$, and in these limits the solution reduces to that of subsection 6.4.1 with $P = -A$.

6.6 Relaxing the constraints on $a(\cdot, t)$

Recall from chapters 3-5 that the subspaces $\check{\mathcal{M}}_N$ and $\check{\mathcal{N}}_{P,Q}$ are defined in terms of scattering data $(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t))$ such that:

- the number $n(t) = n_1(t) + 2n_2(t)$ is finite.
- all the $n(t)$ zeroes of $a(\cdot, t)$ are simple.
- none of the normalisation coefficients $\gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)$ are zero.

Recall also that all the solutions found so far in this chapter have scattering data with these properties but that the third restriction was formally relaxed in some of the analysis of section 6.5.

Consider the scattering data (6.3.3) leading to the solution (6.3.4) and suppose $\phi_1 = 0$ so that the solution reads

$$\varphi(x,t) = 4 \arctan \frac{\nu}{\zeta} \frac{\sin\left(\frac{\omega_1(t-vx)}{\sqrt{1-v^2}}\right)}{\cosh\left(\frac{\omega_2(x-vt-x_1)}{\sqrt{1-v^2}}\right)}.$$

Now take the limit $\lambda_1(0) \to i$ in this formula to find

$$\varphi(x,t) \to 4\arctan\left(\frac{t}{\cosh(x-x_1)}\right) \tag{6.6.1}$$

and the scattering data (6.3.3) becomes

$$\mathbf{dst}(\varphi_0, \varpi_0) \to \left(a(\lambda,0) = \left(\frac{\lambda - i}{\lambda + i}\right)^2, b(\cdot,0) \equiv 0 : \gamma_1(0) \equiv \bar{\gamma}_1(0)\right).$$

Therefore, at least on a formal level, a solution to a problem of Type $\mathbf{A}_0$ has been constructed for which the transition coefficient $a(\cdot,t)$ does not just have simple zeroes. This is perhaps not too surprising as the observant reader will have noticed that this requirement was only imposed in chapters 3-5 for notational simplicity. Finally notice that (6.6.1) also solves a problem of Type $\mathbf{B}_{-2\tanh x_1, 0}$ upon a restriction of x to the semiline $(-\infty, 0]$.

Chapter 7

Integrable systems theory and the sets $\mathbf{A}_N$, $\mathbf{B}_{P,Q}$

7.1 Introduction

In this chapter the concept of an integrable mechanical system is developed, first for ordinary differential equations (finite dimensional phase spaces) and then for partial differential equations (infinite dimensional phase spaces). Having formulated this notion in a precise way it is possible to make the assertion at the end of subsection 7.4.1. Namely, that the restricted phase spaces $\breve{\mathcal{M}}_N$, $\breve{\mathcal{N}}_{P,Q}$ define an *integrable sine-Gordon system.*

Section 7.5 is a construction of an infinite set of 'local integrals of motion' for the sine-Gordon systems $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,Q}$, (this is often taken as a definition of integrability). A set of 'trace identities' is then constructed for these integrals. Section 7.6 briefly introduces the idea of action-angle coordinates for the phase space of an integrable mechanical system.

7.2 Analytical mechanics and finite dimensional integrable systems

7.2.1 Mechanical systems

One of the basic problems in classical mechanics is to solve Newton's equations of motion for a mechanical system with n degrees of freedom. Namely,

$$\frac{d^2 y_k}{dt^2}(t) = -\frac{\partial}{\partial y_k(t)} V(y_1(t), \ldots, y_n(t)), \qquad k = 1, \ldots, n \qquad (7.2.1)$$

for real valued functions $\{y_k\}$ of a time variable $t \in \mathbb{R}$. At any time t the set $\{y_k(t)\}$ are to be thought of as coordinates on an n dimensional manifold $\mathcal{W}^n(t)$ called the configuration space. As a consequence of the time translation invariance of the system (7.2.1), the form of $\mathcal{W}^n(t)$ is independent of t so that $\mathcal{W}^n(t) \equiv \mathcal{W}^n$ for all $t \in \mathbb{R}$.

The system (7.2.1) can be derived from the Lagrangian function $L : T\mathcal{W}^n \to \mathbb{R}$ which, when written in terms of the coordinates $\{y_k(t)\}$ takes the form

$$L = \frac{1}{2}\sum_{k=1}^{n} \dot{y}_k^2(t) - V(y_1(t), \ldots, y_n(t)), \tag{7.2.2}$$

by applying the Lagrange equations

$$\frac{d}{dt}\frac{dL}{d\dot{y}_k(t)} = \frac{dL}{dy_k(t)}, \qquad k = 1, \ldots, n. \tag{7.2.3}$$

Since $(\dot{y}_1(t), \ldots, \dot{y}_n(t))$ is a tangent vector to $\mathcal{W}^n$, it follows that the 'generalized momentum'

$$(y_{n+1}(t), \ldots, y_{2n}(t)) \stackrel{\text{def}}{=} \left(\frac{dL}{d\dot{y}_1(t)}, \ldots, \frac{dL}{d\dot{y}_n(t)}\right), \tag{7.2.4}$$

is a cotangent vector. Therefore the system (7.2.1) can be written as

$$\begin{aligned} \frac{dy_k}{dt}(t) &= y_{n+k}(t) \\ \frac{dy_{n+k}}{dt}(t) &= -\frac{\partial V}{\partial y_k(t)}, \qquad k = 1, \ldots, n, \end{aligned} \tag{7.2.5}$$

and the 'phase space' with coordinates $(y_1(t), \ldots, y_{2n}(t))$ is the $2n$ dimensional cotangent bundle $T^*\mathcal{W}^n(t) \equiv T^*\mathcal{W}^n \ \ \forall t \in \mathbb{R}$.

7.2.2 Integrable systems and symplectic geometry

The theory of finite dimensional integrable systems can be expressed in the language of symplectic geometry and this allows a formulation of some very general results regarding their properties. Those integrable systems which are mechanical in nature then appear as special cases of this general formalism.

Definition 7.1 *Let M^{2n} be a $2n$, $n \in \mathbb{N}$ dimensional real manifold. A symplectic structure on M^{2n} is a closed nondegenerate differential 2-form ω^2:*

$$d\omega^2 = 0 \ \ \text{and} \ \ \forall \xi \in TM_x^{2n} \neq 0 \ \exists \eta \in TM_x^{2n} : \omega^2(\xi, \eta) \neq 0, \tag{7.2.6}$$

with TM_x^{2n} the tangent space to M^{2n} at the point x. The pair (M^{2n}, ω^2) is called a symplectic manifold.

One of the major reasons for introducing such manifolds is made clear by following theorem.

Theorem 7.2 *The cotangent bundle $T^*\mathcal{W}^n (\equiv T^*\mathcal{W}^n(t)\ \forall t \in \mathbb{R})$ has a natural symplectic structure. In the local coordinates $\{y_p(t) : p = 1, \ldots, 2n\}$ defined above, this symplectic structure is given by*

$$\omega^2(t) = \sum_{k=1}^{n} dy_k(t) \wedge dy_{n+k}(t). \tag{7.2.7}$$

The proof of this theorem and many more details regarding the rest of this section are to be found in the classic textbook [21].

There exists a natural isomorphism I between 1 forms and vector fields on a symplectic manifold (M^{2n}, ω^2). This is defined by the relation

$$\omega^2(\eta, I\omega^1) = \omega^1(\eta) \tag{7.2.8}$$

for all $\eta \in TM^{2n}$.

Definition 7.3 *Let (M^{2n}, ω^2) be a symplectic manifold and let F_1, F_2 be two functions $: M^{2n} \to \mathbb{R}$. Denote the algebra of such functions by $\mathcal{A}$. The symplectic structure gives a natural map $: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$:*

$$(F_1, F_2)_{Pb} = \omega^2(I\,dF_2,\ I\,dF_1). \tag{7.2.9}$$

This is called the Poisson bracket of the functions F_1, F_2.

The properties of the exterior derivative and the skew symmetry of the wedge product imply the following theorem.

Theorem 7.4 *The Poisson bracket is skew symmetric,*

$$(F_1, F_2)_{Pb} + (F_2, F_1)_{Pb} = 0,$$

and satisfies the Jacobi identity

$$((F_1, F_2)_{Pb}, F_3)_{Pb} + ((F_2, F_3)_{Pb}, F_1)_{Pb} + ((F_3, F_1)_{Pb}, F_2)_{Pb} = 0.$$

In addition it has the properties

$$(F_1, F_2F_3)_{Pb} - (F_1, F_2)_{Pb}\, F_3 - F_2\,(F_1, F_3)_{Pb} = 0,$$

$$(J(F_1), K(F_2))_{Pb} = J'(F_1)\,(F_1, F_2)_{Pb}\, K'(F_2).$$

It follows from theorem 7.4 that the Poisson bracket of two arbitrary functions F_1, F_2 : $M^{2n} \longrightarrow \mathbb{R}$ can be calculated once the 'fundamental' Poisson brackets between a set of coordinate functions $y_i : M^{2n} \longrightarrow \mathbb{R}$, $i = 1, \dots, 2n$ are known. For the mechanical systems discussed above (7.2.7), (7.2.9) lead to the 'canonical' form for the coordinates $\{y_k(t) : k = 1, \dots, 2n\}$,

$$(y_i(t), y_j(t))_{Pb} = (y_{n+i}(t), y_{n+j}(t))_{Pb} = 0 \quad (y_{i+n}(t), y_j(t))_{Pb} = \delta_{ij}, \qquad i, j = 1, \dots, n. \tag{7.2.10}$$

With this definition of the Poisson bracket, the concept of a Hamiltonian system can be introduced.

Definition 7.5 *At an arbitrary time $t \in \mathbb{R}$ let $\{y_p(t) : p = 1 \dots 2n\}$ be coordinates on $M^{2n}(\equiv M^{2n}(t)\ \forall t \in \mathbb{R})$ where $(M^{2n}, \omega^2(t))$ is a symplectic manifold. A set of first order ordinary differential equations is said to form a Hamiltonian system if there exists a function $H : M^{2n} \longrightarrow \mathbb{R}$ such that the system possesses a Hamiltonian structure, i.e it can be represented as*

$$\frac{dy_p}{dt}(t) = (H, y_p(t))_{Pb} \qquad p = 1, \dots, 2n \tag{7.2.11}$$

$\forall t \in \mathbb{R}$. *These are Hamilton's equations of motion for the system and H is called the Hamiltonian function.*

Using the symplectic structure of $T^*\mathcal{W}^n$ defined in theorem 7.2 it is possible to rewrite the mechanical system (7.2.5) in the Hamiltonian form

$$\frac{dy_p}{dt}(t) = (H, y_p(t))_{Pb} \qquad p = 1, \dots, 2n, \tag{7.2.12}$$

$\forall t \in \mathbb{R}$ with the Hamiltonian function expressed in coordinates as

$$H(y_1(t), \dots, y_{2n}(t)) = \frac{1}{2}\sum_{k=1}^{n} y_{k+n}^2(t) + V(y_1(t), \dots, y_n(t)). \tag{7.2.13}$$

Following these preliminaries it is possible to define an integrable Hamiltonian system of ordinary differential equations.

Definition 7.6 *A (finite dimensional) Hamiltonian system is said to be integrable by quadratures ('integrable') if there exists a set of coordinates* $\{Y_q(t) : q = 1, \ldots, 2n\}$ *for the manifold* $M^{2n}(t)$ *such that on making the transformation* $y_p(t) = y_p(\{Y_q(t)\})$ *the system (7.2.11) decouples and so can be solved, up to the evaluation of integrals and the inversion of functions, by a separation of variables.*

This is a precise definition of the integrability of a Hamiltonian system. With a couple more definitions it is possible to formulate one of the most important results regarding systems of this type i.e Liouville's theorem. This is one of the major theorems of analytical mechanics and can be used to establish the integrability of Hamiltonian systems.

Definition 7.7 *A function* $F : M^{2n} \to \mathbb{R}$ *is a first integral of a Hamiltonian system with Hamiltonian function* H *if the Poisson bracket* $(H, F)_{Pb} \equiv 0 \ \forall t \in \mathbb{R}$. *Two functions* F_1, F_2 *are said to be 'in involution' if* $(F_1, F_2)_{Pb} \equiv 0 \ \forall t \in \mathbb{R}$. *All first integrals are therefore in involution with the Hamiltonian function.*

Definition 7.8 *The* n *functions* $F_1, \ldots, F_n \in \mathcal{A}$ *are independent at* $y(t) \in M^{2n}$ *if the* n *1 forms* $dF_1, \ldots, dF_n \in TM^{2n}_{y(t)}$ *are linearly independent. Equivalently this amounts to the requirement*

$$rank \left\{ \frac{\partial F_k}{\partial y_p(t)} \right\} \bigg|_{y(t)} = n, \tag{7.2.14}$$

where $k = 1, \ldots, n$, $p = 1, \ldots, 2n$ *so that the left hand side of this equation represents the rank of an* $n \times 2n$ *matrix.*

Theorem 7.9 (Liouville) *If in a Hamiltonian system with* n *degrees of freedom and configuration space* $M^{2n}(\equiv M^{2n}(t) \ \forall t \in \mathbb{R})$, n *first integrals in involution are known which are independent on a dense subspace of* M^{2n}, *then the system is integrable by quadratures.*

A detailed proof of this theorem and much more information regarding finite dimensional mechanical systems is to be found in [21].

7.3 Mechanical systems and the sets $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$

In this section it will be seen how boundary value problems of Types $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$ can formally be viewed as mechanical systems of ordinary differential equations each with a continuous infinity of degrees of freedom parameterised by $x \in \mathbb{R}$ ($x \in \mathbb{R}^- \cup \{0\}$) as opposed to $n \in \mathbb{N}$. The phase spaces of these systems are clearly of infinite dimension. It will also be seen that these problems have a Hamiltonian structure.

7.3.1 The sets $\mathbf{A}_N$, $\mathbf{B}_{P,Q}$ can be viewed as infinite dimensional mechanical systems

With $t \in \mathbb{R}$ fixed and $(\varphi(\cdot,t), \varpi(\cdot,t)) \in \mathcal{M}_N$, the set $\{(\varphi(x,t), \varpi(x,t)) : x \in \mathbb{R}\}$ can, at least on a formal level, be regarded as coordinates on the infinite dimensional 'cotangent bundle' $T^*\mathcal{S}(\mathbb{R}; 0, 2\pi N) \stackrel{\text{def}}{=} \mathcal{M}_N$.

Comparing the sine-Gordon system

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(x,t) &= \varpi(x,t) \\ \frac{\partial \varpi}{\partial t}(x,t) &= \frac{\partial^2 \varphi}{\partial x^2}(x,t) - \sin\varphi(x,t), \end{aligned} \tag{7.3.1}$$

with (7.2.5) it is clear that the set $\mathbf{A}_N$ takes the form of a mechanical system with phase space $\mathcal{M}_N$.

Identical reasoning with the problems of Type $\mathbf{B}_{P,Q}$ leads to a *definition* of the cotangent bundle $\mathcal{N}_{P,Q}$ with coordinates $\left\{(\varphi(x,t), \varpi(x,t)) : x \in \mathbb{R}^- \cup \{0\}\right\}$. Once again the sine-Gordon system can be thought of as a mechanical system but this time with the infinite dimensional phase space $\mathcal{N}_{P,Q}$.

This formal identification with the $n \to \infty$ limit of a mechanical system can be continued with the introduction of functional differential forms and vector fields on these 'cotangent bundles'. However, such concepts are not needed for this thesis and so will not be mentioned further. The interested reader is referred to [22, 23].

7.3.2 The Hamiltonian structure of $\mathbf{A}_N$, $\mathbf{B}_{P,Q}$

The notion of a Poisson bracket and Hamiltonian function can be formally extended to infinite dimensional mechanical systems such as $\mathbf{A}_N$ and $\mathbf{B}_{P,Q}$. For arbitrary functionals

$$F_1, F_2 : \mathcal{M}_N \to \mathbb{C},$$

$$F_3, F_4 : \mathcal{N}_{P,Q} \to \mathbb{C},$$

define the Poisson brackets $(F_1, F_2)_{Pb}$, $(F_3, F_4)_{Pb}$ by

$$\begin{aligned}
(F_1, F_2)_{Pb} &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \left(\frac{\delta F_1}{\delta \varpi(x,t)} \frac{\delta F_2}{\delta \varphi(x,t)} - \frac{\delta F_1}{\delta \varphi(x,t)} \frac{\delta F_2}{\delta \varpi(x,t)} \right) dx, \\
(F_3, F_4)_{Pb} &\stackrel{\text{def}}{=} \int_{-\infty}^{0} \left(\frac{\delta F_3}{\delta \varpi(x,t)} \frac{\delta F_4}{\delta \varphi(x,t)} - \frac{\delta F_3}{\delta \varphi(x,t)} \frac{\delta F_4}{\delta \varpi(x,t)} \right) dx.
\end{aligned} \tag{7.3.2}$$

This definition is a formal generalisation of the Poisson bracket resulting from (7.2), (7.3). The Poisson brackets between coordinates can be formally written as

$$(\varpi(x,t), \varpi(y,t))_{Pb} = 0, \quad (\varphi(x,t), \varphi(y,t))_{Pb} = 0, \quad (\varpi(x,t), \varphi(y,t))_{Pb} = \delta(x-y), \tag{7.3.3}$$

for x, y in the appropriate domain. The form of these fundamental Poisson brackets is due to the functional derivative being formally defined as a derivative 'in the direction of' the Dirac δ 'function' which is neither a function nor a regular distribution. Therefore, a rigorous formulation of these ideas along the lines of section 7.2, and including the introduction of a Jacobi identity, must be carried out in terms of such generalised functions [18]. These important points will re-emerge in section 7.6 when discussing the construction of a set of action-angle coordinates for the phase space $\check{\mathcal{M}}_N$.

For the sine-Gordon mechanical systems defined by phase spaces $\mathcal{M}_N$, (resp. $\mathcal{N}_{P,Q}$) Hamilton's equations of motion take the form

$$\frac{\partial \varphi}{\partial t}(x,t) = (H, \varphi(x,t))_{Pb}, \qquad \frac{\partial \varpi}{\partial t}(x,t) = (H, \varpi(x,t))_{Pb}, \tag{7.3.4}$$

with $H : \mathcal{M}_N \ (N_{P,Q}) \to \mathbb{R}$ the Hamiltonian functional

$$H[\varphi(\cdot,t), \varpi(\cdot,t)] = \int_{-\infty}^{\infty} \left[\frac{1}{2} \varpi^2(x,t) + \frac{1}{2} \left(\frac{\partial \varphi}{\partial x}(x,t) \right)^2 + 1 - \cos \varphi(x,t) \right] dx, \tag{7.3.5}$$

for the system defined by $\mathcal{M}_N$ and

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0}\left[\frac{1}{2}\varpi^2(x,t)+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2+1-\cos\varphi(x,t)dx\right]$$
$$-2Q\sin\frac{\varphi}{2}(0,t)-2P\cos\frac{\varphi}{2}(0,t), \quad (7.3.6)$$

for the system defined by $\mathcal{N}_{P,Q}$.

7.4 The sets $\breve{\mathbf{A}}_N$ and $\breve{\mathbf{B}}_{P,Q}$ are integrable systems

In this section the definition 7.6 is extended to include infinite dimensional mechanical systems which have the same form as those introduced in section 7.3. It is then straightforward to deduce that the existence of the inverse scattering methods developed in chapters 3, 4 and 5 is a proof that the sets of sine-Gordon problems defined by the restricted phase spaces $\breve{\mathcal{M}}_N$, $\breve{\mathcal{N}}_{P,Q}$ are infinite dimensional integrable systems.

Definition 7.10 *Let $M^\infty(\equiv M^\infty(t)\ \forall t\in\mathbb{R})$ be some infinite dimensional function space with coordinates $\{(\varphi(x,t),\varpi(x,t)) : x\in D\subset\mathbb{R}\}$ at time t. Suppose that M^∞ is the phase space of some mechanical system expressible as*

$$\frac{\partial\varphi}{\partial t}(x,t)=(H,\varphi(x,t))_{Pb}, \qquad \frac{\partial\varpi}{\partial t}(x,t)=(H,\varpi(x,t))_{Pb} \qquad \forall t\in\mathbb{R},$$

for some Hamiltonian functional H. This system is said to be integrable by quadratures ('integrable') on M^∞ if there exists a set of coordinates for this space such that, in terms of these, Hamilton's equations of motion decouple and so can be solved, up to the evaluation of integrals and the inversion of functions, by a separation of variables.

7.4.1 $\breve{\mathcal{M}}_N$ and $\breve{\mathcal{N}}_{P,Q}$ define integrable sine-Gordon systems

The direct/inverse scattering transform **dst/ist** developed in chapter 3 and the restrictions made in chapters 4 and 5 can be viewed as (invertible) coordinate transformations for the phase spaces $\breve{\mathcal{M}}_N$, $\breve{\mathcal{N}}_{P,Q}$. In terms of the new coordinates (essentially the real and imaginary parts of the scattering data), the time evolution is governed by a set of decoupled linear ordinary differential equations which can easily be solved to define the time evolution map τ_t. Therefore, *the existence of the transformations* $\mathbf{dst}/\mathbf{ist}$ *and* $\mathbf{dst}|_{(-\infty,0]}/\mathbf{ist}|_{(-\infty,0]}$ *and the form of the time evolution map* τ_t *prove that the sine-Gordon systems defined by the restricted phase space* $\breve{\mathcal{M}}_N$ *and* $\breve{\mathcal{N}}_{P,Q}$ *are integrable.*

7.5 An infinite set of first integrals for $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,Q}$ and their trace identities

As has already been seen in section 7.2 there is a deep relationship between the number of first integrals of a Hamiltonian system and its integrability. Unfortunately, for infinite dimensional Hamiltonian systems such as those defined by the sets $\breve{\mathbf{A}}_N$, $\breve{\mathbf{B}}_{P,Q}$ there is no analogue of Liouville's theorem 7.9 in order to prove integrability. In spite of this, the integrability of the sine-Gordon system with restricted phase spaces $\breve{\mathcal{M}}_N$ and $\breve{\mathcal{N}}_{P,Q}$ has been established directly in section 7.4.

In this section an infinite number of first integrals of motion are constructed for the integrable Hamiltonian systems $\breve{\mathbf{A}}_N$ and $\breve{\mathbf{B}}_{P,Q}$. Subsection 7.5.1 concerns the set $\breve{\mathbf{A}}_N$ and the first integrals are found to be 'local' i.e they have the form

$$I_p[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{\infty} g_p(\varphi(x,t),\varpi(x,t))\,dx \qquad p\in\mathbb{Z}, \tag{7.5.1}$$

where g_p is a real valued function of $\varphi(x,t)$, $\varpi(x,t)$, $\varphi_x(x,t)$, $\varpi_x(x,t)$, $\varphi_{xx}(x,t),\ldots$. The Hamiltonian functional (7.3.5) is clearly of this form.

Subsections 7.5.2 and 7.5.3 consider the sets $\breve{\mathbf{B}}_{P,0}$ and $\breve{\mathbf{B}}_{P,Q\neq 0}$ respectively. Once more an infinite set of local integrals are constructed for these systems. In both cases these integrals have the form

$$\tilde{I}_p[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0} \tilde{g}_p(\varphi(x,t),\varpi(x,t))\,dx + \tilde{h}_p(\varphi(0,t),\varpi(0,t)) \qquad p\in\mathbb{Z}, \tag{7.5.2}$$

with $\tilde{g}_p$ a complex valued function of $\varphi(x,t)$, $\varpi(x,t)$, $\varphi_x(x,t)$, $\varpi_x(x,t)$, $\varphi_{xx}(x,t),\ldots$ and $\tilde{h}_p$ a complex function of $\varphi(0,t)$, $\varpi(0,t)$, $\varphi_x(0,t)$, $\varpi_x(0,t)$, $\varphi_{xx}(0,t),\ldots$. The real and imaginary parts of these integrals can then viewed as distinct real valued first integrals for the systems $\breve{\mathbf{B}}_{P,0}$, $\breve{\mathbf{B}}_{P,Q\neq 0}$.

7.5.1 First integrals for the set $\breve{\mathbf{A}}_N$

Let $(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)}(0))$ be a general element of $\hat{\mathcal{D}}^{n_1(0),n_2(0)}_{n_1(0)\mathrm{mod}2}$ for some $n_1(0)$, $n_2(0) \in \mathbb{N}$. Define $n(0) \stackrel{\text{def}}{=} n_1(0) + 2n_2(0)$ and

$$(\varphi_N, \varpi_0) \stackrel{\text{def}}{=} \mathbf{ist}(a(\cdot,0), b(\cdot,0) : \gamma_1(0), \ldots, \gamma_{n_1(0)+2n_2(0)}(0)),$$

where $N \in \mathbb{Z}$ is such that $N \equiv n(0) \bmod 2$. By definition the pair (φ_N, ϖ_0) is an element of $\breve{\mathcal{M}}_N$.

For the integrable sine-Gordon system $\breve{\mathbf{A}}_N$ there exist two infinite families of first integrals. These can be constructed by considering an asymptotic expansion of the reduced transition matrix $T(\lambda, 0)$ as $\lambda \to 0, \infty$, which is constrained to have the form (3.2.19). When (φ_N, ϖ_0) evolve in time according to the sine-Gordon system the evolution of $a(\cdot,0)$, $b(\cdot,0)$ is given by the time evolution map τ_t. That is

$$a(\lambda,t) = a(\lambda,0) \qquad \operatorname{Im}\lambda \geq 0,$$

$$b(\lambda,t) = e^{\frac{i}{2}\left(\lambda+\frac{1}{\lambda}\right)t} b(\lambda,0) \qquad \lambda \in \mathbb{R},$$

and it is the time independence of the diagonal elements of $T(\lambda, 0)$ that will be crucial in constructing the time independent first integrals. It will only be necessary to consider the family of integrals resulting from the $|\lambda| \to \infty$ asymptotic as the family resulting from the $|\lambda| \to 0$ expansion can immediately be deduced.

From (φ_N, ϖ_0) construct the transition matrix $T(x, y, 0, \lambda)$ as detailed in chapter 3 and suppose that as $|\lambda| \to \infty$ the asymptotic form of this matrix can be written as

$$T(x,y,0,\lambda) = \Omega(x)\left(\mathbb{I} + W(x,\lambda)\right) \exp Z(x,y,\lambda) \left(\mathbb{I} + W(y,\lambda)\right)^{-1} \Omega^{-1}(y) \qquad (7.5.3)$$

$\bmod O(|\lambda|^{-\infty})$, where $W(x,\lambda)$ is an off diagonal matrix and $Z(x,y,\lambda)$ is a diagonal matrix satisfying $Z(x,x,\lambda) = \mathbb{I}$ and

$$\Omega(x) \stackrel{\text{def}}{=} \exp\left(\frac{i\varphi_N(x)}{4}\sigma_3\right).$$

From (3.2.4) it follows that $Z(x,y,\lambda)$ and $W(x,\lambda)$ must be related through

$$Z(x,y,\lambda) = \frac{1}{4i}\int_y^x \left(\theta(x')\sigma_3 + \left(\lambda - \frac{1}{\lambda}e^{-i\varphi_N(x)\sigma_3}\right)\sigma_2 W(x',\lambda)\right)dx', \tag{7.5.4}$$

where

$$\theta(x) \stackrel{\text{def}}{=} \varpi_0(x) + \frac{d\varphi_N}{dx}(x). \tag{7.5.5}$$

This relation is consistent with the requirement that $Z(x,y,\lambda), W(x,\lambda)$ be diagonal and off diagonal respectively. In addition (3.2.4) implies that $W(x,\lambda)$ satisfies the differential equation

$$\frac{dW}{dx} = \frac{1}{2i}\theta\sigma_3 W + \frac{\lambda}{4i}(\sigma_2 - W\sigma_2 W) - \frac{1}{4i\lambda}(\sigma_2 e^{i\varphi_N\sigma_3} - W\sigma_2 e^{i\varphi_N\sigma_3}W). \tag{7.5.6}$$

Now suppose that the matrix $W(x,\lambda)$ has the asymptotic expansion

$$W(x,\lambda) = \sum_{m=0}^{\infty}\frac{W_m(x)}{\lambda^m} \tag{7.5.7}$$

as $|\lambda| \to \infty$ where, by virtue of (3.2.7),

$$W_m(x) = \begin{pmatrix} 0 & -\bar{w}_m(x) \\ w_m(x) & 0 \end{pmatrix}. \tag{7.5.8}$$

Substituting this into (7.5.6) yields

$$w_0(x) = i, \tag{7.5.9}$$

$$\begin{aligned} w_{m+1}(x) = -2i\frac{dw_m}{dx}(x) - \theta(x)w_m(x) &= \frac{i}{2}\sum_{k=1}^{m} w_k(x)w_{m+1-k}(x) \\ &- \frac{i}{2}e^{-i\varphi_N(x)}\sum_{k=0}^{m-1} w_k(x)w_{m-k-1}(x) - \frac{i}{2}e^{i\varphi_N(x)}\delta_{m,1}, \end{aligned} \tag{7.5.10}$$

so that, by (7.5.4), $Z(x,y,\lambda)$ has the asymptotic expansion

$$Z(x,y,\lambda) = \frac{\lambda(x-y)}{4i}\sigma_3 + i\sum_{m=1}^{\infty}\frac{Z_m(x,y)}{\lambda^m}, \tag{7.5.11}$$

as $|\lambda| \to \infty$ with

$$\begin{aligned} Z_m(x,y) &= \begin{pmatrix} z_m(x,y) & 0 \\ 0 & -\bar{z}_m(x,y) \end{pmatrix} \\ z_m(x,y) &= \frac{i}{4}\int_y^x (w_{m+1}(x') - e^{-i\varphi_N(x')}w_{m-1}(x'))dx'. \end{aligned} \tag{7.5.12}$$

To derive the asymptotic expansion of the reduced transition matrix $T(\lambda,0)$ the limits $x,-y\to\infty$ must be taken as prescribed by (3.2.17). The matrices $W_m(x)$, $m\geq 1$ vanish as $|x|\to\infty$ so that as $|\lambda|\to\infty$

$$T(\lambda,0)=e^{P(\lambda)}+O(|\lambda|^{-\infty}), \tag{7.5.13}$$

where

$$\begin{aligned} P(\lambda)&=i\begin{pmatrix} p(\lambda) & 0\\ 0 & -\bar{p}(\lambda)\end{pmatrix}\\ p(\lambda)&=\lim_{x,-y\to\infty}\left(\sum_{m=1}^{\infty}\frac{\hat{z}_m(x,y)}{\lambda^m}\right), \end{aligned} \tag{7.5.14}$$

and

$$\begin{aligned} \hat{z}_1(x,y)&\stackrel{\text{def}}{=} z_1(x,y)-\frac{1}{4}(x-y),\\ \hat{z}_m(x,y)&\stackrel{\text{def}}{=} z_m(x,y), \qquad\qquad m>1. \end{aligned} \tag{7.5.15}$$

From (7.5.9)-(7.5.12) it follows that

$$p(\lambda)=\sum_{m=1}^{\infty}\frac{I_m[\varphi_N,\varpi_0]}{\lambda^m}, \tag{7.5.16}$$

with

$$I_1[\varphi_N,\varpi_0]=-\frac{1}{4}\int_{-\infty}^{\infty}\left(\frac{(\theta(x))^2}{2}+1-\cos\varphi_N(x)\right)dx, \tag{7.5.17}$$

and for arbitrary $m>1$

$$I_m[\varphi_N,\varpi_0]=\frac{i}{4}\int_{-\infty}^{\infty}\left(w_{m+1}(x)-e^{-i\varphi_N(x)}w_{m-1}(x)\right)dx. \tag{7.5.18}$$

The unimodularity of $T(\lambda,0)$ implies that $\operatorname{tr}P(\lambda)=0$ so that the quantities $I_m[\varphi_N,\varpi_0]$ are real.

Notice that the diagonal property of $P(\lambda)$ is in agreement with (3.2.26). Comparing (7.5.13)-(7.5.16) with (3.2.19) leads to

$$\log a(\lambda,0)=i\sum_{m=1}^{\infty}\frac{I_m[\varphi_N,\varpi_0]}{\lambda^m}, \tag{7.5.19}$$

as $|\lambda|\to\infty$.

To derive the asymptotic expansion of $T(\lambda, 0)$ as $|\lambda| \to 0$ it suffices to use the invariance of $U(x, 0, \lambda)$ as defined by (3.2.2), under the transformations $\varpi_0 \to \varpi_0, \varphi_N \to -\varphi_N, \lambda \to -\frac{1}{\lambda}$. Denoting by $\acute{T}(x, y, 0, \lambda)$ the transition matrix constructed from the initial configuration $(\acute{\varphi}_{-N}, \acute{\varpi}_0) \stackrel{\text{def}}{=} (-\varphi_N, \varpi_0)$ it follows that

$$\acute{T}\left(x, y, 0, -\frac{1}{\lambda}\right) = T(x, y, 0, \lambda). \tag{7.5.20}$$

Therefore, the $|\lambda| \to \infty$ expansion of $\acute{T}(\lambda, 0)$ yields

$$\log a(\lambda, 0) = i \sum_{m=0}^{\infty} I_{-m}[\varphi_N, \varpi_0] \lambda^m \tag{7.5.21}$$

as $|\lambda| \to 0$ with

$$\begin{aligned} I_0[\varphi_N, \varpi_0] &= \pi N \pmod{2\pi} \\ I_{-m}\left[\varphi_N, \varpi_0\right] &= (-1)^m I_m\left[-\varphi_N, \varpi_0\right], \quad m = 1, 2, \ldots. \end{aligned} \tag{7.5.22}$$

Using the involution $a(-\lambda, 0) = \bar{a}(\lambda, 0)$, $\lambda \in \mathbb{R}$ it follows that all the $\{I_{2m}\left[\varphi_N, \varpi_0\right]\}$ with $m \in \mathbb{Z} \setminus \{0\}$ are identically zero.

Now suppose that (φ, ϖ) is the solution to the problem of Type $\breve{\mathbf{A}}_N$ defined by the initial condition $(\varphi_N, \varpi_0) \in \breve{\mathcal{M}}_N$. According to the analysis of section 3.6, when (φ_N, ϖ_0) evolves into (φ, ϖ) the transition coefficient $a(\cdot, 0)$ evolves as $a(\cdot, t) = a(\cdot, 0)\ \forall t \in \mathbb{R}$. Therefore (7.5.19), (7.5.21) imply that

$$I_m[\varphi(\cdot, t), \varpi(\cdot, t)] = I_m[\varphi_N, \varpi_0], \qquad \forall m \in \mathbb{Z},\ t \in \mathbb{R}, \tag{7.5.23}$$

so the infinite set of functionals $\{I_{2m+1} : m \in \mathbb{Z}\}$ are the nontrivial, time independent first integrals for the integrable system $\breve{\mathbf{A}}_N$. These integrals clearly have the form (7.5.1) and the Hamiltonian functional (7.3.5) can be constructed from the set $\{I_{2m+1}\}$ according to

$$H[\varphi(\cdot, t), \varpi(\cdot, t)] = 2\left(I_{-1}[\varphi(\cdot, t), \varpi(\cdot, t)] - I_1[\varphi(\cdot, t), \varpi(\cdot, t)]\right). \tag{7.5.24}$$

The conserved 'momentum' functional $P[\varphi(\cdot, t), \varpi(\cdot, t)]$ takes the form

$$P[\varphi(\cdot, t), \varpi(\cdot, t)] \stackrel{\text{def}}{=} -\int_{-\infty}^{\infty} \varpi(x, t) \frac{\partial \varphi}{\partial x}(x, t) dx = 2\left(I_{-1}[\varphi(\cdot, t), \varpi(\cdot, t)] + I_1[\varphi(\cdot, t), \varpi(\cdot, t)]\right). \tag{7.5.25}$$

Finally, recall that the coefficient $a(\cdot,t)$ is given in terms of the coefficient $b(\cdot,t)$ and a set of simple zeroes by the dispersion relation (3.2.36). Substituting this into (7.5.19) and (7.5.21) and expanding as $\lambda \to \infty$, 0 respectively gives the so-called 'trace identities',

$$\begin{aligned}\operatorname{sign}(2m+1) I_{2m+1}\left[\varphi(\cdot,t), \varpi(\cdot,t)\right] = &\frac{1}{\pi}\int_0^\infty \log(1-|b(\lambda,0)|^2)\lambda^{2m} d\lambda \\ &-\frac{2(-1)^m}{2m+1}\sum_{j=1}^{n_1(0)} (\kappa_j(0))^{2m+1} \\ &-\frac{2i}{2m+1}\sum_{k=n_1(0)+1}^{n_1(0)+n_2(0)} (\bar{\lambda}_k(0))^{2m+1} - (\lambda_k(0))^{2m+1},\end{aligned} \tag{7.5.26}$$

for all $t \in \mathbb{R}$, $m \in \mathbb{Z}$. These will be used in subsection 7.5.4 to find energies, masses and momenta for the particle-like soliton solutions previously found in chapter 6. Attention is now turned to a construction of an infinite set of first integrals for the integrable sine-Gordon system defined by the phase space $\breve{\mathcal{N}}_{P,Q}$.

7.5.2 First integrals for the set $\breve{\mathbf{B}}_{P,0}$

To construct an infinite set of first integrals for the integrable sine-Gordon system defined by the phase space $\breve{\mathcal{N}}_{P,0}$ it is necessary to consider the three cases,

$$(a(\cdot,0), b(\cdot,0) : \gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)) \in \mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)},\ \mathcal{F}_{0,\xi}^{n_1(0),n_2(0)},\ \hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}$$

separately from one another.

$\underline{\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}}$

For some $\Lambda \in \mathbb{R}^+$, $n_1(0) \in 2\mathbb{N}+1$, $n_2(0) \in 2\mathbb{N}$ let the scattering data $(a(\cdot,0), b(\cdot,0) :$ $\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0))$ be a general element of $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}$. According to lemma 4.5,

$$(\varphi(\cdot,0), \varpi(\cdot,0)) \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,0), b(\cdot,0) : \gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)\right),$$

is an element of $\breve{\mathcal{N}}_{P,0}$ with $P = i^{n(0)+1}(\Lambda+\Lambda^{-1})$ and $n(0) = n_1(0)+2n_2(0)$.

From $(\varphi(\cdot,0), \varpi(\cdot,0))$ construct the transition matrix $T(0,y,0,\lambda)$ as outlined in chapter 3 with $y \le 0$. By (7.5.3) this matrix has the asymptotic expansion

$$T(0,y,0,\lambda) = \Omega(0)\left(\mathbb{I}+W(0,\lambda)\right)\exp Z(0,y,\lambda)\left(\mathbb{I}+W(y,\lambda)\right)^{-1}\Omega^{-1}(y), \tag{7.5.27}$$

mod $O(|\lambda|^{-\infty})$ as $|\lambda| \to \infty$ with

$$\Omega(x) \stackrel{\text{def}}{=} \exp\left(\frac{i\varphi(x,0)}{4}\sigma_3\right).$$

Calculating the Jost solution $T_-(0,0,\cdot)$ from $T(0,y,0,\lambda)$ in the usual way it follows from the results of chapter 4 that the reduced transition matrix can be written in terms of this solution as

$$T(\lambda,0) = j_3(-\lambda)T_-^{-1}(0,0,-\lambda^{-1})j_1(-\lambda)T_-(0,0,\lambda), \tag{7.5.28}$$

with

$$\begin{aligned} j_1(\lambda) &\stackrel{\text{def}}{=} -i(\lambda+\lambda^{-1})((\lambda-\lambda^{-1})\sigma_1 + P\sigma_3), \\ j_3(\lambda) &\stackrel{\text{def}}{=} \frac{i((\lambda+\lambda^{-1})\sigma_1 + (\Lambda^{-1}-\Lambda)\sigma_2)}{(\lambda+\lambda^{-1})((\lambda+\lambda^{-1})^2 + (\Lambda-\Lambda^{-1})^2)}. \end{aligned} \tag{7.5.29}$$

Alternatively,

$$\sigma_2 T_-^t(0,0,\lambda^{-1})\mathrm{L}(\lambda)T_-(0,0,\lambda) = \mathrm{M}(\lambda)T(\lambda,0), \tag{7.5.30}$$

with

$$\mathrm{L}(\lambda) \stackrel{\text{def}}{=} -\sigma_2 j_1(\lambda)\sigma_3,$$

$$\mathrm{M}(\lambda) \stackrel{\text{def}}{=} (\lambda+\lambda^{-1})((\lambda+\lambda^{-1})\sigma_3 - i(\Lambda^{-1}-\Lambda)\mathbb{I}).$$

Now recall that $\lim_{\lambda\to 0,\infty} b(\lambda,0) = 0$ rapidly in λ so that an asymptotic expansion of the left hand side of (7.5.30) must be diagonal modulo $O(|\lambda|^{-\infty})$ terms as $|\lambda| \to \infty$ and modulo $O(|\lambda|^{\infty})$ terms as $|\lambda| \to 0$. This is indeed the case since $(\varphi(\cdot,0),\varpi(\cdot,0)) \in \breve{\mathcal{N}}_{P,0}$. As opposed to the set of problems studied in the previous subsection, the relation (7.5.30) only leads to a single infinite set of first integrals. The asymptotic expansions of this equation as $|\lambda| \to 0,\infty$ coincide and so they do not give different sets of integrals of motion as was the case for $\breve{\mathbf{A}}_N$. To find the asymptotic expansion of the left hand side of (7.5.30) as $|\lambda| \to \infty$ rewrite this equation as

$$\sigma_2 \acute{T}_-^t(0,0,-\lambda)\mathrm{L}(\lambda)T_-(0,0,\lambda) = \mathrm{M}(\lambda)T(\lambda,0),$$

where $\acute{T}_-(0,0,\cdot)$ is the Jost solution constructed from $(-\varphi(\cdot,0),\varpi(\cdot,0))$. The asymptotic expansions of $T_-(0,0,\cdot)$ and $\acute{T}_-(0,0,\cdot)$ can be deduced from (7.5.27) and

$$\acute{T}(0,y,0,\lambda) = \Omega^{-1}(0)\left(\mathbb{I} + \acute{W}(0,\lambda)\right)\exp \acute{Z}(0,y,\lambda)\left(\mathbb{I} + \acute{W}(y,\lambda)\right)^{-1}\Omega(y),$$

$\mathrm{mod}\, O(|\lambda|^{-\infty})$ as $|\lambda| \to \infty$, the functions $\acute{W}, \acute{Z}$ resulting from W, Z by making the replacement $(\varphi(\cdot,0), \varpi(\cdot,0)) \to (-\varphi(\cdot,0), \varpi(\cdot,0))$.

In place of (7.5.13) there is the relation

$$\mathrm{M}(\lambda)T(\lambda,0) = \lambda^2 \sigma_3 e^{Q(\lambda)} + O(|\lambda|^{-\infty}), \tag{7.5.31}$$

as $|\lambda| \to \infty$ with

$$Q(\lambda) = i \begin{pmatrix} q(\lambda) & 0 \\ 0 & -\bar{q}(\lambda) \end{pmatrix},$$

$$q(\lambda) = \sum_{m=1}^{\infty} \frac{\tilde{I}_m[\varphi(\cdot,0), \varpi(\cdot,0)]}{\lambda^m},$$

$$\tilde{I}_m[\varphi(\cdot,0), \varpi(\cdot,0)] = \left(\lim_{y \to -\infty} (\hat{z}_m(0,y) - (-1)^m \hat{\acute{z}}_m(0,y)\right) + \tilde{h}_m(\varphi(0,0), \varpi(0,0)),$$

and $\tilde{h}_m$ a function of $\varphi(0,0)$, $\varpi(0,0)$, $\varphi_{xx}(0,0)$, $\varpi_{xx}(0,0)$, $\varphi_{xxxx}(0,0)\ldots$. Using

$$\mathrm{tr}\, Q(\lambda) = \log\left(-\lambda^{-4} \det \mathrm{M}(\lambda)\right),$$

it can be seen that $\{\tilde{I}_{2m-1}\}$ are real valued functionals whereas $\{\tilde{I}_{2m}\}$ have an imaginary component.

With $f(\lambda) \stackrel{\mathrm{def}}{=} \lambda^{-2}(\lambda + \lambda^{-1})((\lambda + \lambda^{-1}) - i(\Lambda^{-1} - \Lambda))$ the asymptotic expression (7.5.19) is replaced by

$$\log a(\lambda,0) + \log f(\lambda) = i \sum_{m=1}^{\infty} \frac{\tilde{I}_m[\varphi(\cdot,0), \varpi(\cdot,0)]}{\lambda^m}, \tag{7.5.32}$$

as $|\lambda| \to \infty$.

Now suppose that (φ, ϖ) is the solution to the problem of Type $\breve{\mathbf{B}}_{P,0}$ defined by the initial condition $(\varphi(\cdot,0), \varpi(\cdot,0)) \in \breve{\mathcal{N}}_{P,0}$. According to the results of chapter 4, when $(\varphi(\cdot,0), \varpi(\cdot,0))$ evolves into $(\varphi(\cdot,t), \varpi(\cdot,t))$ the transition coefficient $a(\cdot,0)$ evolves as $a(\cdot,t) = a(\cdot,0)$ $\forall t \in \mathbb{R}$. Therefore (7.5.32) implies

$$\tilde{I}_m[\varphi(\cdot,t), \varpi(\cdot,t)] = \tilde{I}_m[\varphi(\cdot,0), \varpi(\cdot,0)], \qquad \forall m \in \mathbb{N}+1,\ t \in \mathbb{R}, \tag{7.5.33}$$

and so the real and imaginary parts of the infinite set of functionals $\left\{\tilde{I}_m : m \in \mathbb{N}+1\right\}$ are the time independent first integrals for the integrable system defined by the space $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}$. These integrals clearly have the form (7.5.2) and $-2\tilde{I}_1[\varphi(\cdot,0), \varpi(\cdot,0)]$ is found to be the Hamiltonian functional

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0}\left[\frac{1}{2}\varpi^2(x,t)+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2+1-\cos\varphi(x,t)dx\right]$$
$$-2i^{n(0)+1}(\Lambda+\Lambda^{-1})\cos\frac{\varphi}{2}(0,t). \quad (7.5.34)$$

Substituting $f(\lambda)$ and the dispersion relation (3.2.36) into (7.5.32) and expanding as $|\lambda|\to\infty$, this Hamiltonian can be written in terms of $b(\cdot,0)$, Λ and the zeroes of $a(\cdot,0)$ as

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = -\frac{2}{\pi}\int_0^{\infty}\log(1-|b(\lambda,0)|^2)\,d\lambda$$
$$+4\sum_{j=1}^{n_1(0)}(\kappa_j(0))+8\sum_{k=n_1(0)+1}^{n_1(0)+n_2(0)}\operatorname{Im}(\lambda_k(0))-2(\Lambda-\Lambda^{-1}).$$
$$(7.5.35)$$

The higher 'trace identities' can be found in a similar manner but a simple closed form for these such as (7.5.26) does not exist.

$\mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}$

For some $\xi\in\mathbb{R}\setminus\{0\}$, $n_1(0)$, $n_2(0)\in 2\mathbb{N}$ suppose the scattering data $(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0))$ is an element of $\mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}$. According to lemma 4.6,

$$(\varphi(\cdot,0),\varpi(\cdot,0)) \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)\right),$$

is an element of $\breve{\mathcal{N}}_{P,0}$ with $P=i^{n(0)}(\xi+\xi^{-1})$ and $n(0)=n_1(0)+2n_2(0)$.

Once again the results of chapter 4 imply the relation (7.5.30) but this time with

$$\mathrm{M}(\lambda)\stackrel{\text{def}}{=}(\lambda+\lambda^{-1})((\lambda-\lambda^{-1})\sigma_3+i(\xi+\xi^{-1})\mathbb{I}).$$

An infinite set of first integrals follow from (7.5.30) in exactly the same way as before. These are identical in form to those deduced for the subspace $\mathcal{F}_{1,\Lambda}^{n_1(0),n_2(0)}$ but instead of making the replacement $P=i^{n(0)+1}(\Lambda+\Lambda^{-1})$ this boundary parameter is given by $P=i^{n(0)}(\xi+\xi^{-1})$. The Hamiltonian functional

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0}\left[\frac{1}{2}\varpi^2(x,t)+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2+1-\cos\varphi(x,t)dx\right]$$
$$-2i^{n(0)}(\xi+\xi^{-1})\cos\frac{\varphi}{2}(0,t), \quad (7.5.36)$$

can be written in terms of $b(\cdot,0)$, ξ and the zeroes of $a(\cdot,0)$ as the trace identity

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = -\frac{2}{\pi}\int_0^\infty \log(1-|b(\lambda,0)|^2)\,d\lambda + 4\sum_{j=1}^{n_1(0)}(\kappa_j(0)) + 8\sum_{k=n_1(0)+1}^{n_1(0)+n_2(0)} \mathrm{Im}(\lambda_k(0)) - 2(\xi+\xi^{-1}). \tag{7.5.37}$$

$\underline{\hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}}$

Identical reasoning to above but with $n_1(0)\in 2\mathbb{N}$, $n_2(0)\in\mathbb{N}$, $\xi\in\mathbb{C}:|\xi|=1$, $\mathrm{Im}(\xi)\neq 0$ and the scattering data $(a(\cdot,0),b(\cdot,0)\ :\ \gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0))$ an element of $\hat{\mathcal{F}}_{0,\xi}^{n_1(0),n_2(0)}$, leads to first integrals and trace identities which have exactly the same form as those found for $\mathcal{F}_{0,\xi}^{n_1(0),n_2(0)}$.

7.5.3 First integrals for the set $\breve{\mathbf{B}}_{P,Q\neq 0}$

To construct an infinite set of first integrals for the integrable sine-Gordon system defined by the phase space $\breve{\mathcal{N}}_{P,Q\neq 0}$ the two situations,

$$(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0))\in\mathcal{F}_{1,\epsilon,\omega,\chi}^{n_1(0),n_2(0),\rho},\quad \mathcal{F}_{0,\epsilon,\omega,\chi}^{n_1(0),n_2(0),\rho}$$

must be considered separately.

$\underline{\mathcal{F}_{1,\epsilon,\omega,\chi}^{n_1(0),n_2(0),\rho}}$

For arbitrary $n_1(0)\in 2\mathbb{N}+1$, $n_2(0)\in\mathbb{N}$ and parameters

$$\epsilon,\rho=\pm 1,\qquad \omega\in(0,1),\qquad \chi\in\mathbb{C}\ :\ |\chi|=1,\ \mathrm{Im}\,\chi\neq 0,$$

suppose that

$$(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0))\in\mathcal{F}_{1,\epsilon,\omega,\chi}^{n_1(0),n_2(0),\rho}.$$

According to lemma 5.3,

$$(\varphi(\cdot,0),\varpi(\cdot,0)) \stackrel{\text{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)\right),$$

is an element of $\breve{\mathcal{N}}_{P,Q}$ with

$$P = i^{n(0)+1}\mathrm{Re}\,(\chi)(\omega+\omega^{-1})$$
$$Q = \epsilon\rho|(\mathrm{Im}\,(\chi))(\omega-\omega^{-1})|,$$

and $n(0) = n_1(0) + 2n_2(0)$.

Using this scattering data as a starting point, the construction of subsection 7.5.2 can be repeated to find an infinite set of first integrals for these problems. The Hamiltonian functional has the form

$$\begin{aligned} H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0} \left[\frac{1}{2}\varpi^2(x,t) + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2 + 1 - \cos\varphi(x,t)\right]\, dx \\ -2i^{n(0)+1}\mathrm{Re}(\chi)(\omega+\omega^{-1})\cos\frac{\varphi}{2}(0,t) - 2\epsilon\rho|\mathrm{Im}(\chi)(\omega-\omega^{-1})|\sin\frac{\varphi}{2}(0,t), \end{aligned} \tag{7.5.38}$$

and the trace identity for this functional is

$$\begin{aligned} H[\varphi(\cdot,t),\varpi(\cdot,t)] = -\frac{2}{\pi}\int_0^\infty \log(1-|b(\lambda,0)|^2)\,d\lambda + 4\sum_{j=1}^{n_1(0)}(\kappa_j(0)) \\ + 8\sum_{k=n_1(0)+1}^{n_1(0)+n_2(0)} \mathrm{Im}(\lambda_k(0)) + 2\epsilon(\omega+\omega^{-1}-2\mathrm{Re}(\chi)). \end{aligned} \tag{7.5.39}$$

$\underline{\mathcal{F}^{n_1(0),n_2(0),\rho}_{0,\epsilon,\omega,\chi}}$

Finally, consider $n_1(0)\in 2\mathbb{N}$, $n_2(0)\in\mathbb{N}$ and parameters

$$\epsilon,\rho=\pm 1, \qquad \omega\in(0,1), \qquad \chi\in\mathbb{C} : |\chi| = 1,\ \mathrm{Im}\,\chi\neq 0,$$

and suppose that

$$(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)) \in \mathcal{F}^{n_1(0),n_2(0),\rho}_{0,\epsilon,\omega,\chi}.$$

Lemma 5.4 shows that

$$(\varphi(\cdot,0),\varpi(\cdot,0)) \stackrel{\mathrm{def}}{=} \mathbf{ist}|_{(-\infty,0]}\left(a(\cdot,0),b(\cdot,0):\gamma_1(0),\ldots,\gamma_{n_1(0)+2n_2(0)}(0)\right),$$

is an element of $\breve{\mathcal{N}}_{P,Q}$ with

$$P = i^{n(0)}\mathrm{Re}\,(\chi)(\omega + \omega^{-1})$$

$$Q = -\epsilon\rho|(\mathrm{Im}\,(\chi))(\omega - \omega^{-1})|,$$

and $n(0) = n_1(0) + 2n_2(0)$.

The Hamiltonian functional

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{0}\left[\frac{1}{2}\varpi^2(x,t) + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2 + 1 - \cos\varphi(x,t)\right]\,dx$$
$$-2i^{n(0)}\mathrm{Re}(\chi)(\omega+\omega^{-1})\cos\frac{\varphi}{2}(0,t) + 2\epsilon\rho|\mathrm{Im}(\chi)(\omega-\omega^{-1})|\sin\frac{\varphi}{2}(0,t), \tag{7.5.40}$$

satisfies the trace identity

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = -\frac{2}{\pi}\int_0^\infty \log(1-|b(\lambda,0)|^2)\,d\lambda + 4\sum_{j=1}^{n_1(0)}(\kappa_j(0))$$
$$+\,8\sum_{k=n_1(0)+1}^{n_1(0)+n_2(0)}\mathrm{Im}(\lambda_k(0)) - 2(\epsilon(\omega^{-1}-\omega) + 2\mathrm{Re}(\chi)). \tag{7.5.41}$$

7.5.4 Soliton solutions revisited

As was seen in subsection 7.5.1, when considering problems of Type $\breve{\mathbf{A}}_N$ the Hamiltonian and momentum functionals

$$H[\varphi(\cdot,t),\varpi(\cdot,t)] = \int_{-\infty}^{\infty}\left[\frac{1}{2}\varpi^2(x,t) + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}(x,t)\right)^2 + 1 - \cos\varphi(x,t)\right]\,dx,$$
$$P[\varphi(\cdot,t),\varpi(\cdot,t)] = -\int_{-\infty}^{\infty}\varpi(x,t)\frac{\partial\varphi}{\partial x}(x,t)\,dx, \tag{7.5.42}$$

are independent of the time t at which they are evaluated. For the relativistic particles of kink type (6.3.2),

$$H_k = \frac{8}{\sqrt{1-v^2}}, \qquad P_k = \frac{8v}{\sqrt{1-v^2}}, \tag{7.5.43}$$

so that a static soliton of kink type has an energy, to be identified with its mass M_k, which is equal to 8 units. Notice that relativistic energy-momentum relation $H_k^2 = P_k^2 + M_k^2$ holds by virtue of the Poincare invariance of the problems $\mathbf{A}_N$.

Evaluating the Hamiltonian and momentum functionals for the breather solution (6.3.4) yields

$$H_b = \frac{16}{\sqrt{1-v^2}} \sin \arg \lambda_1, \qquad P_b = \frac{16v}{\sqrt{1-v^2}} \sin \arg \lambda_1, \tag{7.5.44}$$

so that the mass of the breather M_b is equal to $16 \sin \arg \lambda_1$ units and $H_b^2 = P_b^2 + M_b^2$. Notice that the mass of a particle of breather type is less than twice the mass of a particle of kink type. Often the breather is interpreted as a bound state of a kink/antikink pair.

Turning to the problems of Type $\breve{\mathbf{B}}_{P,Q}$, the solution (6.3.7) is to be interpreted as a soliton particle of kink type either being emitted from or absorbed by the boundary at $x = 0$. The speed of this soliton is such that the Hamiltonian functional for the particle + boundary system,

$$H[\varphi(\cdot,t), \varpi(\cdot,t)] = \int_{-\infty}^{0} \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial t}(x,t) \right)^2 + \frac{1}{2} \left(\frac{\partial \varphi}{\partial x}(x,t) \right)^2 + 1 - \cos \varphi(x,t) \right] dx$$
$$+2(\Lambda + \Lambda^{-1}) \cos \frac{\varphi}{2}(0,t), \tag{7.5.45}$$

remains conserved throughout the emission (absorption) process. Note that these processes can be made to occur at an arbitrary time by choosing $\gamma_1(0)$ appropriately. Using the trace identities (7.5.34)-(7.5.37) it is found that the solution (6.3.7) has energy $2(\Lambda + \Lambda^{-1})$, the same as the 'vacuum' solution $(\varphi, \varpi) \equiv (0,0)$. Indeed, this degeneracy of the Hamiltonian functional is common when considering the set of problems $\breve{\mathbf{B}}_{P,Q}$ for general P, $Q \in \mathbb{R}$.

For the solution (6.3.12) the Hamiltonian

$$H[\varphi(\cdot,t), \varpi(\cdot,t)] = \int_{-\infty}^{0} \left\{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial t}(x,t) \right)^2 + \frac{1}{2} \left(\frac{\partial \varphi}{\partial x}(x,t) \right)^2 + 1 - \cos \varphi(x,t) \right\} dx$$
$$+2\mathrm{Re}(\chi)(\omega + \omega^{-1}) \cos \frac{\varphi}{2}(0,t) - 2\epsilon\rho |\mathrm{Im}(\chi)(\omega - \omega^{-1})| \sin \frac{\varphi}{2}(0,t), \tag{7.5.46}$$

is equal to $2[2 + \epsilon((\omega + \omega^{-1}) - (\chi + \chi^{-1}))]$.

7.6 Action-angle coordinates for integrable mechanical systems

This section introduces the concept of *action-angle* coordinates for the phase space of an integrable mechanical system. The next subsection develops these ideas for finite dimensional phase spaces by introducing canonical coordinate transformations. Theorem 7.12 asserts the existence of a set of action-angle coordinates for the phase space of a finite dimensional integrable system. Subsection 7.6.2 briefly discusses the construction of action angle coordinates for the phase space $\check{\mathcal{M}}_N$.

7.6.1 Finite dimensional integrable mechanical systems

The ideas of section 7.2 and in particular the results of theorem 7.9 can be extended by exploiting canonical transformations of the manifold (M^{2n}, ω^2).

Definition 7.11 *A canonical transformation of a symplectic manifold (M^{2n}, ω^2) is a differentiable mapping $g : M^{2n} \longrightarrow M^{2n}$ which preserves the 2 form ω^2, namely $g^*(\omega^2) = \omega^2$ where g^* is the pullback map induced by g. If in terms of coordinates this transformation is expressed as $g : y_p \mapsto Y_p,\ \ p = 1, \ldots, 2n$ then $\omega^2|_{\{Y_p\}} = \omega^2|_{\{y_p\}}$.*

Using these transformations theorem 7.9 can be extended to give another of the major theorems regarding integrable systems.

Theorem 7.12 (Arnold) *Under the hypothesis of Liouville's theorem, at any time $t \in \mathbb{R}$ there exists a canonical transformation to coordinates $(I_k(t), \theta_k(t)),\ \ k = 1, \ldots, n$ such that the first integrals depend only on the set $\{I_k(t)\}$.*

This theorem has far reaching consequences for the Hamiltonian systems which satisfy the hypothesis of Liouville's theorem and so are integrable. This can be seen by considering mechanical systems so that $M^{2n} \equiv T^*\mathcal{W}^n$ and the symplectic form at time $t \in \mathbb{R}$ is given by (7.2.7). As was deduced in section 7.2 these lead to the fundamental 'canonical' Poisson brackets for the coordinate functions $\{y_p(t) : p = 1, \ldots, 2n\}$,

$$(y_i(t), y_j(t))_{Pb} = (y_{n+i}(t), y_{n+j}(t))_{Pb} = 0 \quad (y_{i+n}(t), y_j(t))_{Pb} = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{7.6.1}$$

Let F_i, $i = 1, \ldots, n$ represent the n first integrals which appear in the hypothesis of Liouville's theorem. Clearly the Hamiltonian function must be a linear combination of these. Theorem 7.12 says that there exists a canonical transformation to new momentum coordinates $\{I_i(t)\}$,

$$F_i(\{y_p(t)\}) = I_i(t), \qquad i = 1, \ldots, n \tag{7.6.2}$$

and that there exists the 'canonically conjugate' variables

$$\theta_i(t) = \theta_i(\{y_p(t)\}), \qquad i = 1, \ldots, n \tag{7.6.3}$$

such that

$$\omega^2(t) = \sum_{i=1}^{n} d\theta_i(t) \wedge dI_i(t). \tag{7.6.4}$$

In terms of these new coordinates the fundamental Poisson brackets become

$$(\theta_i(t), \theta_j(t))_{Pb} = (I_i, (t) I_j(t))_{Pb} = 0 \quad (I_i(t), \theta_j(t))_{Pb} = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{7.6.5}$$

Since, in these new coordinates, the Hamiltonian function has the form $H = H(\{I_i(t)\})$ Hamilton's equations of motion for the system are

$$\begin{aligned} \frac{dI_i}{dt}(t) &= (H(\{I_j(t)\}), I_i(t))_{Pb} = 0, \\ \frac{d\theta_i}{dt}(t) &= (H(\{I_j(t)\}), \theta_i(t))_{Pb} = f_i(\{I_j(t)\}), \qquad i, j = 1, \ldots, n \end{aligned} \tag{7.6.6}$$

for some functions f_i. The first set of equations is a statement that the $\{I_i(t)\}$ are constant functions of the time variable. This observation allows the second set to be integrated so that in terms of the new coordinates $\{I_k(t), \theta_k(t)\}$ the time evolution takes the simple form,

$$\begin{aligned} I_i(t) &= \alpha_i \\ \theta_i(t) &= f_i(\{\alpha_j\})t + \beta_i, \qquad i = 1, \ldots, n \end{aligned} \tag{7.6.7}$$

and the $\{\alpha_i, \beta_i\}$ are fixed by the initial conditions.

Thus the time evolution can be completely determined, in principle, by canonically transforming to a set of coordinates $\{I_k(t), \theta_k(t)\}$ which linearise Hamilton's equations of motion and the sets which have this property are called action-angle coordinates. It is necessary to include the words 'in principle' in the above statement because in practice it may be impossible to the determine the canonical transformation to these variables and subsequently to invert the solution to obtain the evolution as

$$y_p(t) = y_p(\{\alpha_j\}, \{\beta_j\}, t), \qquad p = 1, \ldots, 2n. \tag{7.6.8}$$

Theorems 7.9, 7.12 only assert the existence of solutions. The actual construction then involves various ingenious procedures.

7.6.2 Action-angle coordinates for $\check{\mathcal{M}}_N$

For $t \in \mathbb{R}$ the pair $(\varphi(\cdot, t), \varpi(\cdot, t)) \in \check{\mathcal{M}}_N$ has 'coordinates' $\{(\varphi(x, t), \varpi(x, t)) : x \in \mathbb{R}\}$. These coordinate functionals satisfy the canonical Poisson bracket relations

$$(\varpi(x,t), \varpi(y,t))_{Pb} = 0, \quad (\varphi(x,t), \varphi(y,t))_{Pb} = 0, \quad (\varpi(x,t), \varphi(y,t))_{Pb} = \delta(x-y), \tag{7.6.9}$$

with $x, y \in \mathbb{R}$.

Let A and B be two matrix functionals, i.e 2×2 matrices whose elements are $\mathbb{C}$ valued functionals. Defining

$$(A \overset{\otimes}{,} B)_{Pb} \overset{\text{def}}{=} \int_{-\infty}^{\infty} \left(\frac{\delta A}{\delta \varpi(z,t)} \otimes \frac{\delta B}{\delta \varphi(z,t)} - \frac{\delta A}{\delta \varphi(z,t)} \otimes \frac{\delta B}{\delta \varpi(z,t)} \right) dz, \tag{7.6.10}$$

then the transition matrix $T(x, y, t, \lambda)$ constructed from $(\varphi(\cdot, t), \varpi(\cdot, t))$ with $y < x$ satisfies

$$(T(x,y,t,\lambda) \overset{\otimes}{,} T(x,y,t,\mu))_{Pb} = [\mathbf{r}(\lambda,\mu), T(x,y,t,\lambda) \otimes T(x,y,t,\mu)], \tag{7.6.11}$$

with the *classical* $\mathbf{r}$-*matrix*

$$\mathbf{r}(\lambda,\mu) \overset{\text{def}}{=} -\frac{1}{16(\lambda^2-\mu^2)} \left[(\lambda^2+\mu^2)\sigma_3 \otimes \sigma_3 + 2\lambda\mu(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \right].$$

Next introduce the scattering data

$$(a(\cdot,t), b(\cdot,t) : \gamma_1(t), \ldots, \gamma_{n_1(t)+2n_2(t)}(t)) \stackrel{\text{def}}{=} \mathbf{dst}(\varphi(\cdot,t), \varpi(\cdot,t)).$$

It follows from (7.6.11) with $\lambda, \mu > 0$ and the generalized function identity

$$\lim_{y\to\pm\infty} \text{p.v.}\frac{e^{\frac{i}{2}\left(\lambda-\frac{1}{\lambda}-\mu+\frac{1}{\mu}\right)y}}{\lambda-\mu} = \pm i\pi\delta(\lambda-\mu), \tag{7.6.12}$$

which holds for such λ, μ, that the variables

$$\rho(\lambda,t) \stackrel{\text{def}}{=} -\frac{4}{\pi\lambda}\log(1-|b(\lambda,t)|^2), \qquad \phi(\lambda,t) \stackrel{\text{def}}{=} -\arg b(\lambda,t), \qquad \lambda \in \mathbb{R}^+ \cup \{0\},$$

$$p_j(t) \stackrel{\text{def}}{=} -8\log(-i\lambda_j(t)), \qquad q_j(t) \stackrel{\text{def}}{=} \log|\gamma_j(t)|, \qquad j = 1, \ldots, n_1(t),$$

$$\xi_k(t) \stackrel{\text{def}}{=} -16\log|\lambda_k(t)|, \qquad \eta_k(t) = \log|\gamma_k(t)|, \qquad k = n_1(t)+1, \ldots, n_1(t)+n_2(t),$$

and

$$\varrho_k(t) = 16\arg\lambda_k(t), \qquad \phi_k(t) = \arg\gamma_k(t), \qquad k = n_1(t)+1, \ldots, n_1(t)+n_2(t),$$

form a canonical family, i.e. they satisfy

$$(\rho(\lambda,t), \phi(\mu,t))_{Pb} = \delta(\lambda-\mu), \qquad \lambda, \mu \geq 0,$$

$$(p_i(t), q_j(t))_{Pb} = \delta_{ij}, \qquad (\xi_k(t), \eta_l(t))_{Pb} = (\varrho_k(t), \phi_l(t))_{Pb} = \delta_{kl},$$

where $i, j = 1, \ldots, n_1(t)$ and $k, l = n_1(t)+1, \ldots, n_1(t)+n_2(t)$. Therefore the direct scattering transform **dst** previously developed in chapter 3 can be regarded as a canonical coordinate transformation of the phase space $\breve{\mathcal{M}}_N$. This construction is of a rather formal nature requiring the use of generalized function identities.

In addition the infinite set of local integrals $\{I_{2m+1}\}$ constructed in subsection 7.5.1 depend only on the 'action' coordinates $\rho(\lambda,t)$, $p_j(t)$, $\xi_k(t)$, $\varrho_k(t)$, according to

$$\begin{aligned}\operatorname{sign}(2l+1)I_{2l+1}\left[\varphi(\cdot,t), \varpi(\cdot,t)\right] = &-\frac{1}{4}\int_0^\infty \rho(\lambda,t)\lambda^{2l+1}d\lambda \\ &+\frac{2(-1)^{l-1}}{2l+1}\sum_{j=1}^{n_1(t)} e^{-\frac{2l+1}{8}p_j(t)} \\ &-\frac{4}{2l+1}\sum_{k=n_1(t)+1}^{n_1(t)+n_2(t)} e^{-\frac{2l+1}{16}\xi_k(t)}\sin\frac{2l+1}{16}\varrho_k(t),\end{aligned} \tag{7.6.13}$$

with $m \in \mathbb{Z}$. It is clear that the map **dst** can be viewed as a transformation to an infinite set of action-angle coordinates for $\breve{\mathcal{M}}_N$. In terms of these coordinates the dynamics of the sine-Gordon system is linear and therefore easily solvable. The map **ist** is interpreted as the inverse coordinate transformation.

Chapter 8

Some further lines of study

8.1 Open problems for the sine-Gordon system

This chapter briefly discusses some open questions connected with the work of this thesis. It is hoped that the interested reader will be encouraged to study such matters.

8.1.1 The phase spaces $\breve{\mathcal{M}}_N$ and $\breve{\mathcal{N}}_{P,Q}$

By now it should be clear that the major problem with the inverse scattering methods developed in chapters 3-5 is the realization of the phase spaces $\breve{\mathcal{M}}_N \subset \mathcal{M}_N$ and $\breve{\mathcal{N}}_{P,Q} \subset \mathcal{N}_{P,Q}$ in terms of pairs of initial data (φ_N, ϖ_0), (resp. $(\varphi_{P,Q}, \varpi_{P,Q})$). Some ideas as to how such a construction might be attempted are developed in [14] for one of the phase spaces appropriate to the nonlinear Schrödinger equation. However, this question has yet to receive any attention when considering the spaces $\breve{\mathcal{M}}_N$, $\breve{\mathcal{N}}_{P,Q}$.

8.1.2 Action-angle coordinates for $\breve{\mathcal{N}}_{P,Q}$

An interesting (and much easier) problem is the construction of a set of action-angle coordinates for $\breve{\mathcal{N}}_{P,Q}$ by proceeding in a similar fashion to subsection 7.6.2. Some results in this direction have already been obtained by the author. Let C and D be two matrix functionals, i.e. 2×2 matrices whose elements are maps : $\breve{\mathcal{N}}_{P,Q} \to \mathbb{C}$ and define

$$(C \overset{\otimes}{,} D)_{Pb} \stackrel{\text{def}}{=} \int_{-\infty}^{0} \left(\frac{\delta C}{\delta \varpi(z,t)} \otimes \frac{\delta D}{\delta \varphi(z,t)} - \frac{\delta C}{\delta \varphi(z,t)} \otimes \frac{\delta D}{\delta \varpi(z,t)} \right) dz. \tag{8.1.1}$$

With this definition the transition matrix $T(0,y,t,\lambda)$ constructed from $(\varphi(\cdot,t), \varpi(\cdot,t)) \in \breve{\mathcal{N}}_{P,Q}$ satisfies

$$(T(0,y,t,\lambda) \overset{\otimes}{,} T(0,y,t,\mu))_{Pb} = [\mathbf{r}(\lambda,\mu), T(0,y,t,\lambda) \otimes T(0,y,t,\mu)], \tag{8.1.2}$$

with (obviously) $y < 0$. Recalling that

$$L(0,t,\lambda) = (\lambda^2 - \lambda^{-2})\mathbb{I} + iQ(\lambda - \lambda^{-1})\sigma_1 + iP(\lambda + \lambda^{-1})\sigma_2,$$

it follows that the matrix

$$\mathcal{T}(y,t,\lambda) \stackrel{\text{def}}{=} T^t(0,y,t,\lambda^{-1}) L(0,t,\lambda) T(0,y,t,\lambda),$$

satisfies

$$(\mathcal{T}(y,t,\lambda) \overset{\otimes}{,} \mathcal{T}(y,t,\mu))_{Pb} = [\mathbf{r}(\lambda,\mu),\, \mathcal{T}(y,t,\lambda) \otimes \mathcal{T}(y,t,\mu)]$$
$$+(\mathbb{I} \otimes \mathcal{T}(y,t,\mu))\mathbf{r}^{t_2}(\lambda,\mu^{-1})(\mathcal{T}(y,t,\lambda) \otimes \mathbb{I})$$
$$-(\mathcal{T}(y,t,\lambda) \otimes \mathbb{I})\mathbf{r}^{t_2}(\lambda,\mu^{-1})(\mathbb{I} \otimes \mathcal{T}(y,t,\mu)), \quad (8.1.3)$$

with $(C \otimes D)^{t_2} \overset{\text{def}}{=} C \otimes D^t$, so that

$$(\operatorname{tr} \mathcal{T}(y,t,\lambda), \operatorname{tr} \mathcal{T}(y,t,\mu))_{Pb} = 0.$$

The relation (8.1.3) along with the generalized function identity (7.6.12) are the appropriate starting points for a formal construction of the action-angle coordinates for $\check{\mathcal{N}}_{P,Q}$. It is anticipated that this calculation can be done without too much difficulty.

8.1.3 The sine-Gordon system on a finite spatial interval

In [6, 8] the sine-Gordon system with $x \in [-1, 0]$ is studied with the phase space variables $(\varphi(\cdot,t), \varpi(\cdot,t))$ subject to the nonlinear boundary conditions

$$\frac{\partial\varphi}{\partial x}(x,t)\bigg|_{x=0} = P_1 \sin\frac{\varphi}{2}(0,t), \qquad \frac{\partial\varphi}{\partial x}(x,t)\bigg|_{x=-1} = P_2 \sin\frac{\varphi}{2}(-1,t),$$
$$\frac{\partial\varpi}{\partial x}(x,t)\bigg|_{x=0} = \frac{P_1}{2}\varpi(0,t)\cos\frac{\varphi}{2}(0,t), \qquad \frac{\partial\varpi}{\partial x}(x,t)\bigg|_{x=-1} = \frac{P_2}{2}\varpi(-1,t)\cos\frac{\varphi}{2}(-1,t),$$
$$(8.1.4)$$

with $P_{1,2} \in \mathbb{R}$. In [6] an infinite set of local integrals of motion are found for this system using a relation similar to (8.1.3). Following on from this, the inverse scattering method for solving initial-boundary value problems of this type was developed in [8]. The basis of this development is the finite gap integration technique already applicable to quasiperiodic problems. The boundary conditions (8.1.4) were incorporated into this technique by exploiting the idea of 'triangularity of $V(0,t,\lambda_0)$' at some points $\lambda_0 \in \mathbb{C}$. This condition can then be shown to manifest itself as a certain (anti)involution requirement for a Riemann surface.

A generalization of these results by introducing more general boundary conditions, with terms such as $Q_1 \cos\frac{\varphi}{2}(0,t)$ and $Q_2 \cos\frac{\varphi}{2}(-1,t)$ added onto the respective conditions for φ, has yet to be fully achieved. However, in [1] it is seen how an infinite set of integrals of motion can be constructed for this system in exactly the same way as [6]. Also it can be shown that the 'triangularity of $V(0,t,\lambda_0)$' method continues to work as a starting point in the development of an inverse scattering method for solving these initial-boundary value problems, and that these more general boundary conditions lead to another (anti)involution requirement for a Riemann surface. Much work is still needed to finish this construction.

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