Totally real submanifolds of the nearly kaehler 6-sphere

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of the Nearly Kaehler 6-sphere

by

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Abstract

Totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere are the main topic of this thesis. Having introduced preliminaries on the theory of complex and almost complex manifolds, the nearly Kaehler structure of S^6 and the non existence of almost complex, 4-dimensional submanifolds of the 6-sphere [G3], the results of N. Ejiri [Ej1] on orientability, minimality and characterization by means of constant sectional curvature are given.

Results concerning the pinching of the sectional curvature in the compact case are coming next (see: [D.O.V.V1], [D.V.V2]). These results are obtained by using the integral formulae of A. Ros [R], formulae which play a crucial role in global Riemannian geometry. After a discussion on a new Riemannian invariant δ, introduced by B.Y.Chen in [Ch2], for submanifolds of real space forms and the inequality (which is the best possible) satisfied by δ, we focus on the case where the inequality becomes an equality. In this case the shape operator of the immersion attains a special form and this helps with the classification. In particular, if M is a 3-dimensional totally real submanifold of S^6 then the Chen's equality becomes $\delta_M = 2$, and if M is assumed to be of constant scalar curvature, we classify M by two explicitely described immersions of S^3 in S^6, [C.D.V.V1]. By assuming that the complementary distribution of a certain distribution of M is integrable, M is characterized in terms of a warped product of a minimal, totally real, non-totally geodesic surface immersion in S^6, which lies linearly full in a totally geodesic S^5 [C.D.V.V2]. Furthermore, with respect to totally real 3- dimensional submanifolds satisfying Chen's equality ([D.V]): if M is contained in a totally geodesic S^5, then M can be classified in terms of complex curves in $\mathbb{C}P^2(4)$ lifted via the Hopf fibration. These submanifolds satisfy always Chen's equality. In case M lies linearly full in S^6 and satisfies Chen's equality classification has been in terms of tubes of radius $\frac{\pi}{2}$, in the direction of the second normal space, over an almost complex curve. Finally, local converses of the last two results are proved.

Contents

PREFACE

This thesis is work done in the Department of Mathematical Sciences of the University of Durham under the supervision of Dr. J. Bolton and it is a discussion of the results obtained in the direction of the totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere. The thesis is in its main part based on the following papers:

- [Ej1] on minimality and characterization in terms of constant sectional curvature.
- [D.O.V.V1] and [D.V.V2] on the problem of the characterization by means of pinched sectional curvature in the compact case.
- [Ch2] on the invariant δ and Chen's inequality-equality for immersions in real space forms.
- [C.D.V.V1] on the classification in the case of constant scalar curvature with Chen's equality satisfied, by means of two equivariant totally real immersions of S^3,
- [C.D.V.V2] on the classification, in case Chen's equality occurs together with some extra assumptions of integrability on a certain distribution, in terms of warped product of totally real, minimal, non-totally geodesic, surface immersions lying linearly full in a totally geodesic S^5 and
- [D.V] on the classification of such submanifolds satisfying Chen's equality in terms of complex curves in $\mathbb{C}P^2(4)$ lifted via the Hopf fibration and of tubes of radius $\pi/2$ in the direction of the second normal space over almost complex curves in S^6.

No part of this thesis has been previously submitted for a degree in the University of Durham or any other University.

Preliminaries

It is well known that a 6-dimensional sphere S^6 does not admit any Kaehlerian structure, and whether S^6 does or not admit any complex structure, as far as we know, is still an open question. However, by considering S^6 as an hypersphere of $\mathbb{R}^7$ and identifying $\mathbb{R}^7$ with the imaginary Caley numbers $Im\mathcal{O}$, an almost complex structure J can be defined on S^6 in a natural way (see for instance [Ca2]) and is, in a sense made precise in [C.G], the "best" almost complex structure on S^6.

Together with the standard metric g on S^6, J determines a **nearly Kaehler structure** in the sense of A.Gray [G1], i.e: $(\tilde{\nabla}_X J)X = 0, \quad \forall X \in \mathfrak{X}(S^6)$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on S^6. The construction of the almost complex structure on S^6 is realized in terms of a vector cross product defined in $\mathbb{R}^7$, a product which is defined using the Cayley multiplication, and is related also with the usual metric g on S^6. This vector cross product is preserved by a subgroup of $SO(7)$, namely by elements of the group G_2 of automorphisms of $Im\mathcal{O}$, and G_2 is a compact Lie group which acts transitively on S^6 and preserves both J and g. With respect to J, two particular types of submanifolds M of S^6 can be investigated: the **almost complex**, for which the tangent space remains invariant under the action of J, and the **totally real**, for which the tangent space of M is mapped by J into the normal space of M .

In [G3] A.Gray proved that, with respect to the canonical nearly Kaehler structure, S^6 has no 4-dimensional almost complex submanifolds and thus there only exist 2-dimensional almost complex submanifolds in S^6 and these are always minimal. Almost complex surfaces will play a crucial role in the classification of totally real 3-dimensional submanifolds of S^6 satisfying the **Chen's equality** and curvature properties of such surfaces were first obtained by K. Sekigawa in [S] where it was proved that: if $M \xrightarrow{\psi} S^6$ is an almost complex and non totally geodesic immersion of a surface M then the degree of the mapping ψ is 3, if M has constant sectional curvature K then $K \in \{\frac{1}{6}, 1\}$, and in case M is compact for a pinching of the sectional curvature $K > \frac{1}{6}$ on M then $K \equiv 1$ on M, and when $\frac{1}{6} \leq K < 1$ on M then $K \equiv \frac{1}{6}$ on M. Moreover, by considering the subset $M' = \{p \in M : \quad \sigma(X, Y) \neq 0\}$ of M and defining the function $G = \frac{1}{2}(\|\nabla^{\perp}\sigma\|^2)$, where $\nabla^{\perp}$ denotes the connection in the normal bundle of M and σ the second fundamental form of the immersion ψ, in case M is compact, K.Sekigawa proved that:

if $-\frac{1}{4} \leq G \leq 0$ on M, then $G = 0$ or $-\frac{1}{4}$ and furthermore, M is homeomorphic to a 2-dimensional torus (resp. a 2-dimensional sphere) if $G = 0$ on M (resp. $G = -\frac{1}{4}$ on M). K. Sekigawa provided examples of almost complex submanifolds of S^6 corresponding to the cases $K = 1, \frac{1}{6}$ and 0 (in case M is of constant sectional curvature). In the same direction, and with respect to the pinching of the sectional curvature in the compact case, F.Dillen, B.Opozda, L.Verstraelen and L.Vrancken proved, in [D.O.V.V2], that, for an almost complex surface in S^6: if $\frac{1}{6} \leq K \leq 1$, then either $K = 1$ or $K = \frac{1}{6}$ and in [D.V.V1] F.Dillen, L.Verstraelen and L.Vrancken proved that if $0 \leq K \leq \frac{1}{6}$, then either $K = 0$ or $K = \frac{1}{6}$. The method used in [D.O.V.V1] and [D.V.V1] was based on the integral formulae of A.Ros [R] which provide a powerful tool in global Riemannian geometry and there is a certain analogy with Stokes' theorem and Hopf's lemma.

More recently, by using the method of harmonic sequences, an idea which goes back to Laplace (see: L.Darboux, *Lecons sur la theorie generale des surfaces*, Gauthier-Villar, Paris, 1915), J. Bolton and L. M. Woodward studied, in [B.W], the general situation of harmonic maps of a Riemann surface into $\mathbb{C}P^n$ and deduced congruence theorems for these maps, as well as for harmonic maps from a Riemann surface into S^n. The same authors together with L. Vrancken in [B.V.W1] studied almost complex curves in S^6. These are non-constant smooth maps from a Riemann surface into S^6, whose differential is complex linear and it is well-known that any such map is a weakly conformal harmonic map or, equivalently, a weakly conformal minimal immersion. They classified almost complex curves, by means of $O(7)$-congruence, in the following 4 different types:

(I) linearly full in S^6 and superminimal, (II) linearly full in S^6 but not superminimal, (III) linearly full in some totally geodesic S^5 in S^6 (thus, by [Ca2], necessarily not superminimal) and (IV) totally geodesic, where the case (IV) is trivial and consists of curves with image of the form $S^6 \cap V$ where $V \subset Im\mathcal{O}$ is an associative 3-plane. Of the remaining 3 types the best understood is that of type (I). R.L.Bryant dealt with this case in [B1], he gave a "Weierstrass representation" theorem for such curves and proved that there are compact almost complex curves of type (I) of every genus. Almost complex curves of genus 0, which are necessarily of type (I) or (IV), have been studied by N. Ejiri in [Ej2], who described all S^1 - symmetric examples. In the case of constant Gaussian curvature K, as we have seen already, the values 0, $\frac{1}{6}$, 1 obtained by K.Sekigawa in [S], correspond to almost complex curves of type (III), (I) and (IV) respectively.

The almost complex curves of type (II) and (III) are more difficult to deal with, since they are not superminimal (**i.e**: the associated harmonic sequence does not terminate). In the same paper, criteria for recognising when a weakly conformal map, from a Riemann surface into S^6, is $O(7)$-congruent to an almost complex curve of the above 4 types, are given and by considering almost complex curves of type (III) the relation with totally real harmonic maps into $\mathbb{C}P^2$ has been investigated.

Although almost complex surfaces and totally real 3-dimensional submanifolds of S^6 are minimal, totally real surfaces are not. In the direction of the totally real non-minimal surfaces very few things can be said. In the direction of totally real minimal surfaces M results were obtained by F.Dillen, B. Opozda, L. Verstraelen and L. Vrancken in [D.O.V.V3]. They showed that if M is homeomorphic to a sphere then M is totally geodesic and consequently, if M is compact and has non-negative Gaussian curvature K, then either $K = 0$ or $K = 1$, deriving from these that if M is of constant Gaussian curvature K then, either $K = 0$ or $K = 1$. In Chapter 4 and in order to obtain theorem (25) which classifies totally real 3-dimensional submanifolds of S^6 (satisfying the so-called "Chen's equality" and some extra assumptions of integrability) in terms of warped product, of totally real, minimal, non-totally geodesic surface immersions in S^6 whose ellipse of curvature is a circle (and consequently lies linearly full in a totally geodesic S^5), we shall use a part of the method followed by the authors in [D.O.V.V3].

Recently in [B.V.W2], J. Bolton, L. Vrancken and L. M. Woodward studied totally real surfaces of S^6 with non-circular ellipse of curvature. In particular, they showed that such surfaces cannot be linearly full in S^6, each of them is an open subset of a complete totally real minimally immersed $\mathbb{R}^2$ and each complete one is "equivariant" in the sense that it is invariant under a 1-parameter subgroup of G_2.

In the present thesis, results concerning totally real 3-dimensional submanifolds of S^6 are discussed starting from some preliminaries on the theory of complex, almost complex manifolds, the non integrable nearly Kaehler structure and the non existence of 4-dimensional almost complex submanifolds of S^6. We discuss in details the results of N. Ejiri on the orientability, minimality and classification in terms of constant sectional curvature and also the results of F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken with respect to the problem of characterization by means of pinched curvature in the compact case. Furthermore, the very important and relatively new Riemannian invariant δ, introduced

by B. Y. Chen in [Ch2], with respect to submanifolds of real space forms, is discussed independently. This invariant satisfies always an inequality involving the main Riemannian invariants of M and when this turns out to be an equality the shape operator of the immersion attains a very "symmetric" form which provides information on the second fundamental form of the immersion. Using the hypothesis that the so-called **Chen's equality** is satisfied on a 3-dimensional totally real submanifold M of S^6 and extra assumptions of constancy of the scalar curvature and integrability of the complement of a certain distribution $\mathcal{D}$, classification has been obtained in [C.D.V.V1], [C.D.V.V2] in terms of two equivariant totally real immersions of S^3 in S^6 and tubes around totally real minimal immersions in S^6 (in the direction of the second normal space), respectively. In the last part we discuss the results of F. Dillen, L. Vrancken [D.V] who classsified totally real submanifolds of of S^6, which are contained in a totally gedesic S^5, in terms of almost complex curves in $\mathbb{C}P^2(4)$ lifted via the Hopf fibration $S^5 \longrightarrow \mathbb{C}P^2(4)$ and showed that such submanifolds always satisfy Chen's equality. In case the 3-dimensional totally real submanifold lies linearly full in S^6 the classification obtained in [D.V] is in terms of tubes of radius $\pi/2$ in the direction of the second normal space.

Chapter 1

General theory

1.1 Preliminaries, the Frobenius theorem

In Chapter 1 and specifically within the first two sections, after introducing notation and some necessary concepts, we state the Frobenius theorem and we go on with the fundamental equations, standard rigidity theorems and basic facts from the theory of minimal submanifolds of Riemannian manifolds together with a certain number of examples. In the third section we extensively refer on the theory of almost complex manifolds and in the last section we focus on the nearly Kaehler S^6 and present the proof of the non existence of almost complex 4 - dimensional suline-number-mode bmanifolds of S^6.

Let M be a differentiable manifold. We denote by $\mathcal{F}(M)$ the set of all , locally defined on M, differentiable functions $f \in \mathcal{C}^\infty(M, \mathbb{R})$, by $\mathfrak{X}(M)$ the $\mathcal{F}(M)$ - module of differentiable vector fields on M and by $\mathcal{C}^\infty(M, N)$ the set of all differentiable mappings from M into another differentiable manifold N.
If $f \in \mathcal{C}^\infty(M, N)$ the **differential** of f is defined by

$$(f_\star(X))_p(g) = X_p(g \circ f), \quad (X, g) \in T_pM \times \mathbb{C}^\infty(N, \mathbb{R}) \tag{1.1.1}$$

and the **transpose** of $(f_\star)_p$ defined by the

$$\left(f^*\omega'\right)(X_1, \ldots, X_q) = \omega'(f_\star X_1, \ldots, f_\star X_q) \tag{1.1.2}$$

for any $X_1, \ldots, X_q \in T_pM$ and $\omega' \in \mathcal{D}^q(N)$, where $\mathcal{D}^q(N)$ denotes the set of all q-forms on N.

Let f be a differentiable mapping from the m-dimensional differentiable manifold M into the n-dimensional N.

Definition 1 *The mapping f is said to be an* **immersion** *if and only if the differential $(f_\star)_p$ is injective for every point p of M and in this case M will be called an* **immersed** *submanifold of N. When an immersion f is injective it will be called an* **imbedding** *and M will be said to be an* **imbedded** *submanifold of N.*
An open subset of a manifold can always be considered as a submanifold in a natural manner and in this case will be called an **open piece** *of the ambient manifold.*

In order to state the Frobenius theorem we recall the following concepts

Definition 2 *A q- dimensional* **distribution** *on a differentiable manifold M is a mapping $\underline{D}$ defined on M which assigns to each point p in M a q-dimensional linear subspace $\underline{D}_p$ of T_pM.*
The q-dimensional distribution $\underline{D}$ will be called **differentiable** *if there exist differentiable vector fields, defined on a neighborhood of the point p, which, for each point η in this neighborhood, form a basis of $\underline{D}$. The set of these q vector fields with this property is said to be a* **local basis** *of the distribution $\underline{D}$.*
A vector field $X \in \mathfrak{X}(M)$ is said to be an element of $\underline{D}$ if $X_p \in \underline{D}_p$, $\forall p \in M$.
The distribution $\underline{D}$ will be called **involutive** *if and only if $[X,Y] \in \underline{D}$. $\forall (X,Y) \in \underline{D} \times \underline{D}$.*
An imbedded submanifold $f : M' \longrightarrow M$ of the manifold M will be called an **integral submanifold** *of the distribution $\underline{D}$ if and only if: $f_\star(T_pM')) = \underline{D}_{f(p)}$. $\forall p \in M'$ where f is the imbedding of M' in M.*
If there exists no integral submanifold of $\underline{D}$ which contains properly M', then M' will be called a **maximal integral submanifold** *of $\underline{D}$.*
A distribution $\underline{D}$ is said to be **integrable** *if and only if, for every point $p \in M$, there exists an integral submanifold of $\underline{D}$ containing p.*

We can now state the classical Frobenius theorem in the following form(see [Ch1], pg:29-30)

Theorem 1 *An involutive distribution $\underline{D}$ on a differentiable manifold M is integrable. Furthermore, through every point $p \in M$ there passes a unique maximal integral submanifold*

of $\underline{D}$ *and every other integral submanifold containing* p *is an open submanifold of this maximal one.*

An alternative way to state the above theorem is the following. Let us at first define

$$\Omega = \{\omega \in \mathcal{D}^1(M) : \omega(X_p) = 0, \quad \forall (X, p) \in \underline{D} \times M\} \tag{1.1.3}$$

and let $I(\Omega)$ denote the ideal generated by Ω in the ring of the exterior differentials on M. Then theorem (1) can be stated as follows

Theorem 2 *The distribution* $\underline{D}$ *is involutive if and only if* $d\Omega$ *is contained in the ideal* $I(\Omega)$ *where* Ω *is given by (1.1.3).*

1.2 Fundamental equations, minimal submanifolds

Let $\widetilde{M}$ be a differentiable manifold. A **Riemannian metric** on $\widetilde{M}$ is a (0.2)-type tensor field $\tilde{g}$ satisfying the following conditions
(i) $\tilde{g}$ is symmetric : $\tilde{g}(X, Y) = \tilde{g}(Y, X), \quad \forall X, Y \in \mathfrak{X}(\widetilde{M})$,
(ii) $\tilde{g}$ is positive - definite : $\tilde{g}(X, X) \geq 0, \quad \forall X \in \mathfrak{X}(\widetilde{M})$ and $\tilde{g}(X, X) = 0$ if and only if $X = 0$.

Let M be an n - dimensional manifold immersed in an m - dimensional Riemannian manifold $\widetilde{M}$ with $m > n$. In a local discussion we can assume, without loss of generality, that M is imbedded in $\widetilde{M}$.
If $\{V, u^A\}$, $\{\mathcal{U}, x^h\}$, where $A = 1, \ldots, m$ and $h = 1, \ldots, n$, are coordinate systems on $\widetilde{M}$ and on M respectively, then

$$u^A = u^A(x^h). \tag{1.2.1}$$

Suppose that $f : M \hookrightarrow \widetilde{M}$ is the immersion. Let $X \in \mathfrak{X}(M)$ and identify X with its image $f_*(X)$. If $X = \sum_{h=1}^{n} X^h(\frac{\partial}{\partial x^h})$ then under this identification, we get:

$$X = \sum_{A=1}^{m} \sum_{h=1}^{n} X^h \cdot \frac{\partial u^A}{\partial x^h} \cdot (\frac{\partial}{\partial u^A}). \tag{1.2.2}$$

If $\tilde{g}$ is the metric tensor on the ambient space $\widetilde{M}$ we define the **induced** metric g on M by setting

$$g(X, Y) = \tilde{g}(X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{1.2.3}$$

Henceforth we use the same notation for the induced and for the metric tensor of the ambient space and we also identify $x \in M$ with $f(x) \in \widetilde{M}$.

Definition 3 *If for the vector* $\xi \in T_x(\widetilde{M})$, $x \in M$ *the equality* $g(X,\xi) = 0$ *holds* $\forall X \in T_x M$, *then* ξ *is said to be a* **normal vector** *of* M *in* $\widetilde{M}$ *at the point* x.

Let $\perp M$ be the vector bundle of all normal vectors of M in $\widetilde{M}$. Then the decomposition $T\widetilde{M}|_M = TM \oplus \perp M$ holds. We denote by $\widetilde{\nabla}$ the Levi - Civita connection on $\widetilde{M}$. We need the following

Lemma 1 *([Ch1], pg:37-38) If* $X, Y \in \mathfrak{X}(M)$ *and* $\widetilde{X}, \widetilde{Y}$ *are extensions of* X *and* Y *respectively then* $[\widetilde{X}, \widetilde{Y}]|_M$ *and* $(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y})|_M$ *do not depend on the extension.*

Under the aspect of lemma (1) we can proceed introducing the following concepts

Definition 4 *If* $X, Y \in \mathfrak{X}(M)$, *with* M *is an immersed submanifold of the Riemannian manifold* $\widetilde{M}$, *let:*

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.2.4}$$

where $\nabla_X Y$ *denotes the tangential and* $h(X,Y)$ *the normal component of* $\widetilde{\nabla}_X Y$. *We shall call* ∇ *the* **induced connection** *and* h *the* **second fundamental form** *of the submanifold* M.
If ξ *is a normal vector field we set*

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.2.5}$$

where $-A_\xi X$ *denotes the tangential part and* $\nabla^\perp_X \xi$ *the normal part of* $\widetilde{\nabla}_X \xi$ *with respect to* M. *Then* A *will be called the* **shape operator** *and* $\nabla^\perp$ *the* **normal connection** *([Ch1], pg:41).*

The shape operator is self-adjoint, the second fundamental form is a symmetric one and they satisfy the following relation

$$g(h(X,Y),\xi) = g(A_\xi X, Y), \quad \forall X, Y \in \mathfrak{X}(M), \quad \forall \xi \in \mathfrak{X}^\perp(M). \tag{1.2.6}$$

Definition 5 *A normal vector field ξ on M will be called* **parallel** *in the normal bundle if and only if :* $\nabla^{\perp}_X \xi = 0, \quad \forall X \in \mathfrak{X}(M)$.

An immersion $f : M \longrightarrow \widetilde{M}$ is said to be **totally geodesic** *if and only if $h \equiv 0$ on M.*

If $\xi \in \mathfrak{X}^{\perp}(M)$ and $A_\xi = a \cdot I$ for some function a, then ξ is called an **umbilical section** *on M. If M is umbilical with respect to every local normal section of M, then M is called* **totally umbilical**.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis in T_xM. The **mean curvature vector** *H of M is defined by:* $H = \frac{1}{n} \cdot \sum_{i=1}^{n} h(e_i, e_i)$.

If $H \equiv 0$ on M, then M is said to be **minimal** .

We define the covariant derivative of the second fundamental form by setting

$$(\nabla_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.2.7}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

If the covariant derivative of h vanishes identically on M then h is said to be **parallel**.

Denote by $\widetilde{R}, R$ the curvature tensor fields of $\widetilde{M}$ and M respectively, then :

$$\begin{aligned}
\widetilde{R}(X, Y)Z &= \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_{[X,Y]} Z \\
&= \widetilde{\nabla}_Y (\nabla_X Z + h(X, Z)) - \widetilde{\nabla}_X (\nabla_Y Z + h(X, Z)) - (\nabla_{[X,Y]} Z + h([X, Y], Z)) \\
&= \nabla_Y \nabla_X Z + h(Y, \nabla_X Z) - A_{h(X,Z)} Y + \nabla^{\perp}_Y h(X, Z) \\
&\quad - (\nabla_X \nabla_Y Z + h(X, \nabla_Y Z) - A_{h(Y,Z)} X + \nabla^{\perp}_X h(Y, Z)) \\
&\quad - \nabla_{[Y,X]} Z - h([Y, X], Z)
\end{aligned}$$

and using the equation (1.2.7) we get

$$\widetilde{R}(X, Y) Z = R(X, Y) Z - A_{h(X,Z)} Y + A_{h(Y,Z)} X + (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z) \tag{1.2.8}$$

where $X, Y, Z \in \mathfrak{X}(M)$. Taking $W \in \mathfrak{X}(M)$ and using (1.2.8) we deduce the **Gauss equation**

$$\begin{aligned}
&g\left(\widetilde{R}(X, Y) Z, W\right) - g(R(X, Y) Z, W) = \\
&g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)).
\end{aligned} \tag{1.2.9}$$

and by considering the normal component in equation (1.2.8) and the definition of the covariant derivative of h we deduce the **Codazzi equation**

$$\left(\widetilde{R}(X, Y) Z\right)^{\perp} = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z). \tag{1.2.10}$$

Definition 6 *Let the* **curvature tensor of the normal bundle** *of M be:*

$$R^{\perp}(X,Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi \tag{1.2.11}$$

for all $X, Y \in \mathfrak{X}(M), \quad \forall \xi \in \mathfrak{X}^{\perp}(M)$.

Let us take $X, Y \in \mathfrak{X}(M), \quad \xi \in \mathfrak{X}^{\perp}(M)$ and compute

$$\begin{aligned}
\widetilde{R}(X,Y)\xi &= \widetilde{\nabla}^{\perp}_X \widetilde{\nabla}^{\perp}_Y \xi - \widetilde{\nabla}^{\perp}_Y \widetilde{\nabla}^{\perp}_X \xi - \widetilde{\nabla}_{[X,Y]}\xi \\
&= \widetilde{\nabla}_X\left(-A_\xi Y + \nabla^{\perp}_Y \xi\right) - \widetilde{\nabla}_Y\left(-A_\xi X + \nabla^{\perp}_X \xi\right) - \left(-A_\xi [X,Y] + \nabla^{\perp}_{[X,Y]}\xi\right) \\
&= -\nabla_X A_\xi Y - h(X, A_\xi Y) - A_{\nabla^{\perp}_X \xi} X + \nabla^{\perp}_X \nabla^{\perp}_Y \xi \\
&+ \nabla_Y (A_\xi X) + h(Y, A_\xi X) + A_{\nabla^{\perp}_X \xi} Y - \nabla^{\perp}_Y \nabla^{\perp}_X \xi + A_\xi [X,Y] - \nabla^{\perp}_{[X,Y]}\xi \Longrightarrow \\
\widetilde{R}(X,Y)\xi &= R^{\perp}(X,Y)\xi - h(X, A_\xi Y) + h(X, A_\xi Y) \\
&- (\nabla_X A)_\xi Y + (\nabla_Y A)_\xi X
\end{aligned}$$

Taking $\eta \in \mathfrak{X}^{\perp}(M)$ we deduce the **Ricci equation**

$$g\left(\widetilde{R}(X,Y)\xi, \eta\right) = g\left(R^{\perp}(X,Y)\xi, \eta\right) + g\left([A_\eta, A_\xi]X, Y\right) \tag{1.2.12}$$

Remark 1 *If $R^{\perp} \equiv 0$ then the normal connection of M is said to be* **flat** .
In particular, if the ambient space $\widetilde{M}$ is of constant sectional curvature c then

$$\widetilde{R}(X,Y)Z = c \cdot [g(X,Z)\, g(Y,W) - g(X,W)\, g(Y,Z)] \tag{1.2.13}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. Hence, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, the $\widetilde{R}(X,Y)Z$ is tangent to M and thus the fundamental equations reduce to the equations

$$g(R(X,Y)Z, W) = c \cdot [g(X,Z) \cdot g(Y,W) - g(Y,Z) \cdot g(X,W)] +$$
$$g(h(X,Z), h(Y,W)) - g(h(Y,Z), h(X,W)), \tag{1.2.14}$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{1.2.15}$$
$$g\left(\widetilde{R}^{\perp}(X,Y)\xi, \eta\right) = g([A_\eta, A_\xi]X, Y). \tag{1.2.16}$$

of Gauss, Codazzi and Ricci respectively.

Definition 7 *Let M be an n-dimensional submanifold of the m-dimensional manifold $\widetilde{M}$. At each point $x \in M$ we define the* **first normal** *space $\mathcal{N}_1(x)$ to be the image Imh of the tangent space at x under the second fundamental form.*

Note: A q-plane bundle over a manifold M is called a **Riemannian q-plane bundle** if it is equipped with a Riemannian metric and a compatible connection.
Let us now state the **fundamental theorems** for submanifolds (see [Ch1] and for the proof [B.C]).

Theorem 3 *(**Existence**) Let M be a simply connected n-dimensional Riemannian manifold with a Riemannian q-plane bundle E over M equipped with a second fundamental form h and associated shape operator A. If the equations of Gauss (1.2.14), Codazzi (1.2.15) and Ricci (1.2.16)are satisfied then M can be isometrically immersed in an $(n+q)$-dimensional space form $\mathbb{R}^{n+q}(c)$ of curvature c with normal bundle E.*

Theorem 4 *(**Rigidity**) Let f_1. $f_2 : M \hookrightarrow \mathbb{R}^m(c)$ be two isometric immersions of an n-dimensional Riemannian manifold in an m-dimensional space form $\mathbb{R}^m(c)$ with normal bundles E_1 and E_2 respectively. equipped with their canonical bundle metrics. connections and second fundamental forms. Suppose that there exists an isometry $f : M \hookrightarrow \widetilde{M}$ such that f can be covered by a bundle map $\widetilde{f} : E_1 \longrightarrow E_2$ which preserves the bundle metrics. the connections and the second fundamental forms. Then. there exists a rigid motion F of the space form such that: $F \circ f_1 = f_2 \circ f$.*

In the last part of the present section some basic results and a number of examples concerning minimal submanifolds are included since the totally real 3-dimensional submanifolds of the Nearly Kaehler 6 - sphere are always minimal.

- A minimal submanifold M in a Riemannian manifold N is an extremal for the integral of volume ([Ch1]. pg:75).

- If M is a minimal submanifold of an Euclidean space E^m the Ricci tensor is negative semidefinite and M is totally geodesic if and only if its scalar curvature is zero [T].

- Let $X^1, \ldots, X^{n+1}$ be the standard coordinates of the Euclidean space E^{n+1}. A hypersurface M which can be globally represented under the form
  $X^{n+1} = X^{n+1}(X^1, \ldots, X^n)$
  will be called a **non parametric hypersurface** . It has been proved that:
  If $n \leq 7$ then a non parametric hypersurface in E^{n+1} is necessarily linear. If $n > 7$

then examples of non linear non parametric hypersurfaces have been found ([Ch1], pg:81).

- A submanifold of an Euclidean space E^m is minimal if and only if the position vector field is harmonic [T].

- If M is a submanifold of a small hypersphere S^{m-1} of E^m centered at the point C then M is minimal in S^{m-1} if and only if $\Delta\tilde{X} = c \cdot \tilde{X}$ for some constant c, where $\tilde{X}$ denotes the position vector field of M in E^m with respect to the point C and Δ the Laplacian on M (see:[T]).

- Suppose that M is an n-dimensional submanifold of an Euclidean space E^m. If the position vector field $\tilde{X}$ of M in E^m with respect to a point $C \in E^m$ is parallel to H, then the submanifold M is either a minimal submanifold of E^m or a minimal submanifold of a small hypersphere of E^m centered at the point C ([Ch1], pg:81).

- Let M be an n-dimensional complete minimal submanifold of $S^m(a)$ with non-negative sectional curvature and suppose that the normal connection of M is flat. If the scalar curvature of M is constant, then M is either a great sphere of $S^m(a)$ or a pythagorean product of the form
$S^{p_1}(r_1) \times \ldots \times S^{p_N}(r_N), \quad \sum p_i = n, \quad 1 < N \leq m - n + 1$
with essential codimension $N - 1$, where $r_i = a(\frac{p_i}{n})^{\frac{1}{2}}, \quad i \in \{1, \ldots, N\}$ [Y.I].

- Let M be an n-dimensional compact minimal submanifold of $S^m(a)$. If M has non-negative sectional curvature and the normal connection of M is flat then, we have that M is either a great sphere of $S^m(a)$ or a pythagorean product of the form given in the previous case [Y.I].

Examples of minimal submanifolds

Example 1 *Every totally geodesic submanifold of a Riemannian manifold is a minimal submanifold. In particular, every great hypersphere of a space form $\mathbb{R}^m(k)$ is a minimal submanifold ([Ch1], pg:86).*

Example 2 *Let a be a non - zero constant and M the subset of E^3 given by* $M = \{(r\cos\phi, \quad r\sin\phi, \quad a\cosh^{-1}(\frac{r}{a})) : \quad \phi, r \in \mathbb{R}\}$. *$M$ is a minimal surface in E^3 and it is called a* **catenoid** .

Example 3 *Let a be a non zero constant and define the subset M_1 of E^3 by $M_1 = \{(r\cos\theta,\ r\sin\theta,\ a\theta) :\ r,\theta \in \mathbb{R} - \{0\}\}$. Then M_1 is a minimal surface in E^3 called the* **right helicoid**.

Example 4 *Let $S^q(r)$ be the q - dimensional sphere in E^{q+1} of radius r and n, p be positive integers such that $p < n$. The product manifold $M_{p,n-p}$ is given by the $M_{p,n-p} = S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$. We imbed $M_{p,n-p}$ in S^{n+1} in the following way. Take $(X_1, X_2) \in M_{p,n-p} \times M_{p,n-p}$ where X_1 is a vector of E^{p+1} of lenghth $\sqrt{\frac{p}{n}}$ and X_2 is vector in E_{n-p+1} of length $\sqrt{\frac{n-p}{n}}$. Consider (X_1, X_2) as a unit vector in $E^{n+2} = E^{p+1} \times E^{n-p+1}$. Then $M_{p,n-p}$ is a minimal submanifold of the $(n+1)$ -dimensional unit sphere and will be called a* **Clifford minimal torus**.
In the specific case where $n = 2$ and $p = 1$ then $M_{1,1}$ is a flat minimal surface in S^3 called the **Clifford torus**.

Example 5 *If (x, y, z) denote the standard coordinates in $\mathbb{R}^3$ and $(u^1, \ldots, u^5)$ the standard coordinates in E^5 we consider the mapping defined by*

$$u^1 = \frac{1}{\sqrt{3}}yz, \quad u^2 = \frac{1}{\sqrt{3}}zx, \quad u^3 = \frac{1}{\sqrt{3}}xy$$
$$u^4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u^5 = \frac{1}{6}(x^2 + y^2 - 2z^2).$$

This map restricts to give an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$ and two antipodal points are mapped into the same point. Thus this map defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane will be called the **Veronese surface** *and is a minimal surface of $S^4(1)$.*

Example 6 *Every complex submanifold M of a Kaehlerian manifold $\widetilde{M}$ is minimal (for the proof see: §1.4 lemma (6)).*

1.3 Complex and almost complex manifolds

In this section we deal with the theory of complex and almost complex manifolds. Standard results will be quoted together with a certain number of examples. Main references for the material presented in (§1.3) are [Y.K1] and [Mat].

Definition 8 *Let M be a Hausdorff topological space. $\{\mathcal{U}_\alpha\}$ with $\alpha \in A$ an open covering of M and suppose that for each $\mathcal{U}_\alpha$ there is an homeomorphism φ_α from $\mathcal{U}_\alpha$ onto an open subset of $\mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is a holomorphic mapping for any pair $(\alpha, \beta) \in A \times A$. Then M is said to be a* **complex manifold of dimension n** *and the family $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called a* **holomorphic coordinate neighborhood** *system.*

In this stage we quote some linear algebraic results on real and complex vector spaces which will be applied to tangent spaces of manifolds.

Let V be a vector space over $\mathbb{R}$. The **complexification** of V is defined to be the set $V^C = \{X + iY : X, Y \in V\}$ and V^C can be given the structure of a complex vector space by defining the operations $(X + iY) + (\overline{X} + i\overline{Y}) = (X + \overline{X}) + i(Y + \overline{Y})$ and $(a + ib)(X + iY) = (aX - bY) + i(bX + aY)$ for all $a + ib \in \mathbb{C}$, $X + iY$, $\overline{X} + i\overline{Y} \in V^C$

Identifying $V^C \ni X + i0 \equiv X \in V$ we may consider V as a subset of V^C.

For any $Z = X + iY \in V^C$ we define its **conjugate** by: $\overline{Z} = X - iY \in V^C$ and the operation of **complex conjugation** from V^C onto itself satisfies:

$\overline{Z + W} = \overline{Z} + \overline{W}$, $\overline{\lambda Z} = \overline{\lambda}\overline{Z}$, $\forall Z, W \in V^C$, $\forall \lambda \in \mathbb{C}$. Let V be n-dimensional vector space, $\{e_1, \ldots, e_n\}$ a basis of V over the reals. If $X = \sum_j a^j e_j$ and $Y = \sum_j b^j e_j$ are elements of V then we can write: $X + iY = \sum_j (a^j + ib^j)e_j = \sum_j \lambda^j e_j$ and by considering the e_j as elements of the complexification we can see that they form a basis of V^C over the complex numbers.

A linear endomorphism J of a real vector space V satisfying $J^2 = -I$ is said to be a **complex structure** on V, where I denotes the identity transformation of V.

Suppose that V is a real vector space with almost complex structure J. If for any $\lambda = a + ib \in \mathbb{C}$, $X \in V$ we set $\lambda X = aX + bJX$, then V can be considered as a complex vector space (clearly V has to be even dimensional).

Conversely, given an n-dimensional complex vector space V we can always consider V as an $2n$-dimensional real vector space and by defining the linear endomorphism $JX = iX$, $\forall X \in V$ we construct a complex structure on V. Let us extend this discussion on the complexification V^C.

If V is a real vector space with complex structure J we can extend J to a **complex linear** endomorphism of V^C by setting: $J(X + iY) = JX + iJY$ and clearly we shall have $J^2 = -I$.

Suppose $2n$ is the real dimension of V and $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ a basis of V. If we put $Z_k = \frac{1}{2}(X_k - iJX_k)$, $\overline{Z_k} = \frac{1}{2}(X_k + iJX_k)$, $\forall k = 1, \ldots, n$ then:

$\{Z_1, \ldots, Z_n, \overline{Z_1}, \ldots \overline{Z_n}\}$ is a basis of V^C and applying J we get

$JZ_k = iZ_k$, $J\overline{Z_k} = -i\overline{Z_k}, \forall k \in \{1, \ldots, \}$.

If we set $V^{1,0} = \{Z \in V^C : \quad JZ = iZ\}$, $V^{0,1} = \{Z \in V^C : \quad JZ = -iZ\}$ we get the decomposition into direct sum: $V^C = V^{1,0} \oplus V^{0,1}$ and in particular, by writing

$Z = \frac{1}{2}(Z - iJZ) + \frac{1}{2}(Z + iJZ)$, $\forall Z \in V^C$, then the components of the analysis of V^C in direct sum are of the form

$V^{1,0} = \{X - iJX : X \in V\}$. $V^{0,1} = \{X + iJX : X \in V\}$.

Let V^* be the dual of the vector space V, that is the real vector space of all linear mappings from V into $\mathbb{R}$. Each element $f \in V^*$ can be extended to a linear functional $\widetilde{f}$ of $V^{\mathbb{C}}$ by setting: $\widetilde{f}(X + iY) = f(X) + if(Y)$, $\forall X + iY \in V^{\mathbb{C}}$.

Using this extension we see that if $\{f^1, \ldots, f^n\}$ is a basis of V^* then $\{\widetilde{f^1}, \ldots, \widetilde{f^n}\}$ is a basis of $(V^C)^*$.

On the other hand, for $h = f + ig \in (V^*)^C$ we can define an element $\widetilde{h}$ of $(V^*)^C$ by setting $\widetilde{h}(Z) = \widetilde{f}(Z) + i\widetilde{g}(Z)$, $\forall Z \in V^{\mathbb{C}}$, where $\widetilde{f}, \widetilde{g}$ are the extensions on $(V^*)^C$ of f and g respectively.

The linear mapping $(V^*)^C \ni h \longrightarrow \widetilde{h} \in (V^C)^*$ is an isomorphism and in conclusion the spaces $(V^*)^C$ and $(V^C)^*$ can be identified.

If J is a complex structure on V then J induces a complex structure on V^* by setting

$Jf(X) = f(JX)$, $\forall (X, f) \in V \times V^*$.

Similarly with the case of V^C, the decomposition in direct sum $(V^*)^C = V_{1,0} \oplus V_{0,1}$ holds and the components of the sum can be described by

$V_{1,0} = \{f \in (V^*)^C : f(X) = 0, \quad \forall X \in V^{0,1}\}$,

$V_{0,1} = \{f \in (V^*)^C : f(X) = 0, \quad \forall X \in V^{1,0}\}$.

We can now return to the generic case of differentiable manifolds.

Suppose M is an n - dimensional complex manifold, $\{z^1, \ldots, z^n\}$ complex local coordinates around a point p of M and x^j, y^j the real and imaginary part respectively of the coordinate z^j. In this case $\{(\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial y^1})_p, \ldots, (\frac{\partial}{\partial x^n})_p, (\frac{\partial}{\partial y^n})_p\}$ is a basis of T_pM and $\{(dx^1)_p, (dy^1)_p, \ldots, (dx^n)_p, (dy^n)_p\}$ a basis of the dual T_p^*M. We set

$(\frac{\partial}{\partial z^j})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^j})_p - i(\frac{\partial}{\partial y^j})_p\}$, $(\frac{\partial}{\partial \bar{z}^j})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^j})_p + i(\frac{\partial}{\partial y^j})_p\}$ and

$(dz^j)_p = (dx^j)_p + i(dy^j)_p$, $(d\bar{z}^j)_p = (dx^j)_p - i(dy^j)_p$, $\forall j = 1, \ldots, n$.

Then $\{(\frac{\partial}{\partial z^j})_p, (\frac{\partial}{\partial \tilde{z}^j})_p\}$ and $\{(dz^j)_p, (d\tilde{z}^j)_p\}$ result to be bases of $T_p^C M$ and $(T_p^* M)^C$ respectively. Furthermore, we have

$$(dz^k)_p \left(\left(\frac{\partial}{\partial z^j} \right)_p \right) = (d\tilde{z}^k)_p \left(\left(\frac{\partial}{\partial d\tilde{z}^j} \right)_p \right) = \delta_j^k. \tag{1.3.1}$$

$$(dz^k)_p \left(\left(\frac{\partial}{\partial \tilde{z}^j} \right)_p \right) = (d\tilde{z}^k)_p \left(\left(\frac{\partial}{\partial z^j} \right)_p \right) = 0. \tag{1.3.2}$$

Definition 9 *Let M be a real differentiable manifold. A tensor field J on M will be called an* **almost complex structure** *on M if and only if J is an endomorphism of the tangent space $T_x M$ at each point $x \in M$ and $J^2 = -I$.*
A manifold M with a fixed almost complex structure will be called an **almost complex manifold**.

Let M be a complex manifold, $\{z^1, \ldots, z^n\}$ complex local coordinates around a point p, and $z^j = x^j + iy^j, j = 1, \ldots, n$. Define $J_p \in End(T_p M)$ by setting:

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) = -\left(\frac{\partial}{\partial x^j}\right), \quad \forall j = 1, \ldots, n \tag{1.3.3}$$

and we can extend J on $T_p^C M$ by

$$J\left(\frac{\partial}{\partial z^j}\right) = i\left(\frac{\partial}{\partial z^j}\right), \quad J\left(\frac{\partial}{\partial \tilde{z}^j}\right) = -i\left(\frac{\partial}{\partial \tilde{z}^j}\right), \quad \forall j = 1, \ldots, n. \tag{1.3.4}$$

It is not hard to verify that an element Z of $T_p^C M$ belongs to $\mathcal{L}\{(\frac{\partial}{\partial z^j})\}$ (where $\mathcal{L}$ denotes the set of linear combinations) if and only if $JZ = iZ$ and belongs to $\mathcal{L}\{(\frac{\partial}{\partial \tilde{z}^j}\}$ if and only if $JZ = -iZ$. Suppose that $\{w^j\}$ is another local complex coordinates system around the same point p and let $w^j = u^j + iv^j$. Define the endomorphism J_1 of $T_p M$ by

$$J_1\left(\frac{\partial}{\partial u^j}\right) = \frac{\partial}{\partial v^j}, \quad J_1\left(\frac{\partial}{\partial v^j}\right) = -\left(\frac{\partial}{\partial u^j}\right) \tag{1.3.5}$$

and extend on the complexification $T_p^C M$ by setting

$$J_1\left(\frac{\partial}{\partial w^j}\right) = i\left(\frac{\partial}{\partial w^j}\right), \quad J_1\left(\frac{\partial}{\partial \tilde{w}^j}\right) = -i\left(\frac{\partial}{\partial \tilde{w}^j}\right). \tag{1.3.6}$$

If $z^k = z^k(w^1, \ldots, w^n), \quad \forall k = 1, \ldots, n$ we have:

$$\left(\frac{\partial}{\partial w^k}\right)_p = \sum_j \left(\frac{\partial z^j}{\partial w^k}\right)(p) \left(\frac{\partial}{\partial z^j}\right)_p. \tag{1.3.7}$$

$$\left(\frac{\partial}{\partial \tilde{w}^k}\right)_p = \sum_j \left(\frac{\partial \tilde{z}^j}{\partial w^k}\right)(p) \left(\frac{\partial}{\partial \tilde{z}^j}\right)_p \tag{1.3.8}$$

for all $k = 1, \ldots, n$.

Since $\frac{\partial}{\partial w^k}$ and $\frac{\partial}{\partial \widetilde{w}^k}$ result to be linear combinations of $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \widetilde{z}^j}$ respectively, we can verify, by applying J, that J and J_1 coincide at each point $p \in M$. Hence, J is independent of coordinates and will be called the **almost complex structure** attached to M.

Definition 10 *Let $M, \widetilde{M}$ be almost complex manifolds with almost complex structures J and $\widetilde{J}$ respectively. A mapping $f : M \longrightarrow \widetilde{M}$ will be called* **almost complex** *if and only if $\widetilde{J} \circ f_\star = f_\star \circ J$.*

Suppose that $M, \widetilde{M}$ are complex manifolds and f a mapping from M into $\widetilde{M}$. Let $\{z^k\}_{k=1}^n, \{w^j\}_{j=1}^m$ be local coordinate systems around $p \in M$, $f(p) \in \widetilde{M}$. We ask for conditions on the mapping f in order to be an almost complex map.

Since f is a mapping then for the complex local coordinates on $M, \widetilde{M}$ we write:

$u^j = u^j(x^1, y^1, \ldots, x^n, y^n)$, $v^j = y^j(x^1, y^1, \ldots, x^n, y^n)$ for any $j = 1, \ldots, m$ where $f^\star u^j$ has been identified with x^j and $f^\star v^j$ with v^j. Consequently we can write the following equations:

$$f_\star\left(\frac{\partial}{\partial x^k}\right) = \sum_j \left(\frac{\partial u^j}{\partial x^k}\right)(p)\left(\frac{\partial}{\partial u^j}\right) + \sum_j \left(\frac{\partial v^j}{\partial x^k}\right)(p)\left(\frac{\partial}{\partial v^j}\right), \tag{1.3.9}$$

$$f_\star\left(\frac{\partial}{\partial y^k}\right) = \sum_j \left(\frac{\partial u^j}{\partial y^k}\right)(p)\left(\frac{\partial}{\partial u^j}\right) + \sum_j \left(\frac{\partial v^j}{\partial y^k}\right)(p)\left(\frac{\partial}{\partial v^j}\right). \tag{1.3.10}$$

Compairing $f_\star(J(\frac{\partial}{\partial x^k}))$ with $\widetilde{J}(f_\star(\frac{\partial}{\partial x^k}))$ and $f_\star(J(\frac{\partial}{\partial y^k}))$ with $\widetilde{J}(f_\star(\frac{\partial}{\partial y^k}))$ we see that f is an almost complex map if and only if the $(\frac{\partial u^j}{\partial x^k})(p) = (\frac{\partial v^j}{\partial y^k})(p)$ and $(\frac{\partial u^j}{\partial y^k})(p) = -(\frac{\partial v^j}{\partial x^k})(p)$ are satisfied for-all j, k, i.e: existence of the Cauchy-Riemann holomorphicity conditions for f. Hence, for almost complex mappings between complex manifolds we have the following:

Proposition 1 *Let $M, \widetilde{M}$ be complex manifolds, the mapping $f : M \longrightarrow \widetilde{M}$ results to be almost complex, with respect to the complex structures on M and $\widetilde{M}$, if and only if f is holomorphic.*

The space $T_p^C M$ will be called the **complex tangent space** of M at p.

Let (M, J) be an almost complex manifold, J its almost complex structure. Then the complex tangent space at p can be decomposed into the direct sum

$$T_p^C(M) = T_p^{1,0}(M) \dotplus T_p^{0,1}(M). \tag{1.3.11}$$

where $T_p^{1,0}(M)$, $T_p^{0,1}(M)$ are the eigenspaces of J corresponding to the eigenvalues $+i$ $-i$ respectively and it can be verified that a complex tangent vector $Z \in T_p^{1,0}(M)$ (resp. $\in T_p^{0,1}(M)$) if and only if $Z = X - iJX$ (resp. $X + iJX$), for some $X \in T_p(M)$.

Remark 2 *([Mat], pg:112) Let us at first recall that a complex manifold M is said to be a* **complex submanifold** *of the complex manifold $\widetilde{M}$ if and only if M is a submanifold of $\widetilde{M}$ when considered as real differentiable manifolds and the immersion of M into $\widetilde{M}$ is holomorphic.*
In II §10 of [Mat] it has been shown that a compact manifold is diffeomorphic to a closed submanifold of $\mathbb{R}^N$. Furthermore, this assertion holds for an arbitrary paracompact manifold (Whitney's theorem). However, it is not true in general that an arbitrary paracompact complex manifold is holomorphically isomorphic to a closed complex submanifold of $\mathbb{C}^N$. In particular, we have the following result ([Mat], pg:112):

Proposition 2 *A compact connected complex manifold in $\mathbb{C}^N$ consists of one single point.*

In order to obtain the above result we use the **maximum principle** *for holomorphic functions. Specifically, any holomorphic function defined on a connected complex manifold M and having its modulus attain a local maximum at a point p_0 of M is a constant function. By considering a standard coordinate system on $\mathbb{C}^N$ the restriction of each coordinate function on M is holomorphic and since M is assumed to be compact follows that each coordinate function has to be constant.*

Let us focus for the rest of the section on almost complex manifolds and quote some basic results in order to rend more natural the pass towards the unit 6-sphere, which carries an almost complex structure but it is not a complex manifold (the almost complex structure is not integrable).
As we have already seen an almost complex structure can be attached to any complex manifold. Conversely, we have the following

Theorem 5 *Let M be a $2n$ - dimensional differentiable manifold with an almost complex structure J and suppose that there is an open cover of M satisfying the following:*
There is a local coordinate system $\{x^1, y^1, \ldots, x^n, y^n\}$ on each open set U of the cover

such that, for each point $q \in M$ *the:* $J_q(\frac{\partial}{\partial x^k})_q = (\frac{\partial}{\partial y^k})_q$ *and* $J_q(\frac{\partial}{\partial y^k})_q = -(\frac{\partial}{\partial x^k})$, $\forall \quad k = 1, \ldots, n$ *are satisfied. Then* M *is a complex submanifold and* J *is the almost complex structure attached to the complex structure.*

Proof: Let $\{x^1, y^1, \ldots, x^n, y^n\}$, $\{u^1, v^1, \ldots, u^n, v^n\}$ be local coordinate systems on U, V respectively. On $U \cap V$ we set: $u^j = u^j(x^1, y^1, \ldots, x^n, y^n)$, $v^j = v^j(x^1, y^1, \ldots, x^n, y^n\}$, $\forall j = 1, \ldots, n$ and consequently the following relations hold

$$\frac{\partial}{\partial x^j} = \sum_k \left(\frac{\partial u^k}{\partial x^j}\right) \frac{\partial}{\partial u^k} + \sum_k \left(\frac{\partial v^k}{\partial x^j}\right) \frac{\partial}{\partial v^k}, \tag{1.3.12}$$

$$\frac{\partial}{\partial y^j} = \sum_k \left(\frac{\partial u^k}{\partial y^j}\right) \frac{\partial}{\partial u^k} + \sum_k \left(\frac{\partial v^k}{\partial y^j}\right) \frac{\partial}{\partial v^k} \tag{1.3.13}$$

for any $j = 1, \ldots, n$. Applying the almost complex structure on the above equations and compairing them we deduce that the

$$\frac{\partial u^k}{\partial x^j} = \frac{\partial v^k}{\partial y^j}, \quad \frac{\partial u^k}{\partial y^j} = -\frac{\partial v^k}{\partial x^j} \tag{1.3.14}$$

must be satisfied. If we put $z^k = x^k + iy^k$, $w^k = u^k + iv^k$ then on $U \cap V$ exist the $w^k = u^k(z^1, \ldots, z^n) + iv^k(z^1, \ldots, z^n)$ and w^k must be holomorphic in the $z^1, \ldots, z^n$ in virtue of the equations (1.3.14). Hence M is a complex manifold.

Let M be an almost complex manifold with J its almost complex structure. For any vector field X on M define a vector field JX by setting $(JX)_p = J_p X_p$, $\forall p \in M$. The map $X \longrightarrow JX$ results to be a linear transformation of the vector space $\mathfrak{X}(M)$ satisfying $J^2(X) = -X$, $\forall X \in \mathfrak{X}(M)$.

Consider $\mathfrak{X}(M)$ as a real vector space and denote by $\mathfrak{X}^C(M)$ the complexification of $\mathfrak{X}(M)$. An element $Z = X + iY \in \mathfrak{X}^C(M)$ will be called a **complex vector field** and at each point p of M is defined by $Z_p = X_p + iY_p \in T_p^C M$. The linear transformation J can be extended to a linear transformation of $\mathfrak{X}^C(M)$ by $J(X + iY) = JX + iJY$ and the bracket is defined by $[Z, \bar{Z}] = ([X, \bar{X}] - [Y, \bar{Y}]) + i([X, \bar{Y}] + [Y, \bar{X}]$, $\forall Z = X + iY$, $\bar{Z} = \bar{X} + i\bar{Y} \in \mathfrak{X}^C(M)$. In this way the complexification $\mathfrak{X}^C(M)$ becomes a Lie algebra over $\mathbb{C}$.

Definition 11 *If* $Z \in \mathfrak{X}^C(M)$ *is a complex vector field and* $Z_p \in T_p^{1,0}(M)$, $\forall p \in M$ *(resp.* $\in T_p^{0,1}(M)$*) then* Z *is said to be of* **holomorphic** *(resp.* **antiholomorphic***) type. We denote the set of vector fields of holomorphic and antiholomorphic type by* $\mathfrak{X}^{1,0}(M)$

and $\mathfrak{X}^{0,1}(M)$ *respectively.x* $\mathfrak{X}^{C}(M)$ *is the direct sum of these subspaces.*
The almost complex structure J *will be called* **integrable** *if and only if:*
$[Z_1, Z_2] \in \mathfrak{X}^{1,0}(M), \quad \forall (Z_1, Z_2) \in \mathfrak{X}^{1,0}(M) \times \mathfrak{X}^{1,0}(M).$

Note: Since the conjugate of the Lie bracket is the bracket of the conjugates and $\tilde{\mathfrak{X}}^{1,0}(M) = \mathfrak{X}^{0,1}(M)$ we see that if J is integrable then $[\overline{Z_1}, \overline{Z_2}] \in \mathfrak{X}^{0,1}(M), \quad \forall (\bar{Z}_1, \bar{Z}_2) \in \mathfrak{X}^{0,1}(M) \times \mathfrak{X}^{0,1}(M)$.

Definition 12 $N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \quad \forall X, Y \in \mathfrak{X}(M)$ *is said to be the* **torsion** *tensor field with respect to the almost complex structure* J *on an almost complex manifold* Mx.

The following theorem links the concepts of integrable structure and torsion tensor field.

Theorem 6 *An almost complex structure* J *is integrable if and only if:*
$N(X, Y) = 0, \quad \forall X, Y \in \mathfrak{X}(M)$.

Proof : Let X and Y be real vector fields and set $Z = [X - iJX, Y - iJY]$. Then, J is integrable if and only if $Z \in \mathfrak{X}^{1,0}(M), \forall X, Y \in \mathfrak{X}(M)$. On the other hand, it is easy to check that: $Z + iJZ = N(X, Y) + iJN(X, Y), \quad \forall X, Y \in \mathfrak{X}(M)$.
Observing that $Z + iJZ = 0$ is equivalent to $N(X, Y) = 0$ and simultaneously to the condition of Z being of holomorphic type (since $Z \in \mathfrak{X}^{1,0}(M)$ is equivalent to $JZ = iZ$) we get the required assertion.

The next theorem is an "integration" of theorem (6) connecting the integrability of an almost complex structure with the condition that an almost complex manifold is a complex manifold (for the proof see: [Mat])

Theorem 7 *Let* M *an almost complex manifold with almost complex structure* J. *Then* J *is a complex structure if and only if the torsion tensor field with respect to* J *vanishes identically on* M.

In the last part of the section we provide some further examples of complex manifolds

Example 7 *Let* $\mathbb{C}^n$ *be the* n *- dimensional complex vector space, and set* $z^k = x^k + iy^k, \quad k = 1, \ldots, n$ *.Identify* $\mathbb{C}^n$ *with* $\mathbb{R}^{2n}$ *by*

$(z^1, \ldots, z^n) \longrightarrow (x^1, \ldots, x^n, y^1, \ldots, y^n)$ *and define a complex structure* J_0 *on* $\mathbb{R}^{2n}$ *setting:*
$(x^1, \ldots, x^n, y^1, \ldots, y^n) \longrightarrow (y^1, \ldots, y^n, -x^1, \ldots, -x^n)$,
the so-called **canonical complex structure** *of* $\mathbb{R}^{2n}$. *In terms of the natural basis,* J *is given by the matrix*

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = J_0.$$

Example 8 *A Lie algebra* $\mathcal{F}$ *over* $\mathbb{C}$ *is called a* **complex Lie algebra**. *Consider* $\mathcal{F}$ *as a real vector space and define a complex structure* J *on* $\mathcal{F}$ *by setting :* $JX = iX, \quad \forall X \in \mathcal{F}$. *Then* J *satisfies the* $[JX, Y] = [X, JY] = J[X, Y], \quad \forall X, Y \in \mathcal{F}$. *Conversely, if* $\mathcal{F}$ *is a real Lie algebra with a map* J *satisfying* $[JX, Y] = [X, JY] = J[X, Y], \quad \forall X, Y \in \mathcal{F}$ *and by defining* $(a + ib)X = aX + bJX$ *we get:*
$[(a + ib)X, Y] = (a + ib)[X, Y], \quad \forall X, Y \in \mathcal{F}$. *Hence,* $\mathcal{F}$ *is a complex Lie algebra.*

Example 9 *A* **complex Lie group** *G is a group which is at the same time a complex manifold such that the mappings*
$G \times G \ni (a, b) \longrightarrow a \cdot b \in G$ *and* $G \ni a \longrightarrow a^{-1} \in G$ *are both holomorphic.*

- $GL(n, \mathbb{C})$ *is a complex Lie group.*
- $\mathbb{C}^n$ *is an additive complex Lie group.*
- *The direct product of two complex Lie groups is a complex Lie group.*
- *Every even-dimensional commutative Lie group is a complex Lie group (see:[Y.K1]).*

Example 10 *(see:[Y.K1]) Let* M *be a complex manifold and let* $\widetilde{M}$ *be a covering space over* M *with projection* $p : \widetilde{M} \longrightarrow M$. *Let* $\{U_j\}$ *be an open cover of* $\widetilde{M}$ *such that* U_j *is mapped by* p *holomorphically onto* $p(U_j)$ *for all* j.
Denote by J *the complex structure of* M *and by* p_j *the restriction of* p *in* U_j.
Define $J_j = (p_j^{-1})_* \circ J \circ (p_j)_*$. *On* $U_k \cap U_j$ *the* J_k *and* J_j *coincide since* $p_j^{-1} \circ p_k$ *is the identity on* $U_k \cap U_j$. *Therefore, the operator on* $\widetilde{M}$, *having as its restrictions the* J_j, *defines a complex structure* $\widetilde{J}$ *on* $\widetilde{M}$. *By construction* p *is holomorphic with respect to* $\widetilde{J}$.

Example 11 *Consider* $\mathbb{C}^n$ *as a real* $2n$*-dimensional vector space and let* $\{a_1, \ldots, a_{2n}\}$ *be a basis. Define* $\Gamma = \{\sum_{j=1}^{2n} m_j a_j : m_j \in Z\}$ *and thinking about* $\mathbb{C}^n$ *as an additive abelian*

group Γ *is a subgroup of* $(\mathbb{C}^n, +)$.

Denote $T = \mathbb{C}^n/\Gamma$ *and let* $\pi : \mathbb{C}^n \longrightarrow T$ *be the natural projection.* $\mathcal{U} \subset T$. *Defining* $\mathcal{U}$ *to be an open in* T *if and only if* $\pi^{-1}(\mathcal{U})$ *is open in* $\mathbb{C}^n$, *we confer* T *with a topological structure. Each point* $\pi(a) \in T$, $a \in \mathbb{C}^n$, *has a neighborhood homeomorphic to a neighborhood of the point* $a \in \mathbb{C}^n$. *In this way* T *can be regarded as a complex manifold of complex dimension* n *and will be called the* **complex torus**.

Example 12 *In* $\mathbb{C}^{n+1} - \{0\}$ *is defined the equivalence relation*
$(z^k) \sim (w^k)$ *if and only if* $\exists \quad \lambda \in \mathbb{C}\ \{0\} : z^k = \lambda \cdot w^k, \quad \forall k \in \{0, \ldots, n\}$.
The quotient $\mathbb{C}P^n = \mathbb{C}^{n+1} - \{0\}/\sim$ *will be called the* **complex projective space** *and the topology on* $\mathbb{C}P^n$ *is defined in terms of the quotient topology.*
Set $\mathcal{U}_j^\star = \{(z^h) \in \mathbb{C}^{n+1} - \{0\} : z^j \neq 0\}$. $\pi(\mathcal{U}_j^\star) = \mathcal{U}_j$ *where* $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n$ *denotes the natural projection. Define the maps:*
$\phi_j : \mathcal{U}_j \longrightarrow \mathbb{C}^n$ *such that* $o_\star([z]) = (\frac{z_0}{z_j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}), \quad \forall [z] \in \mathcal{U}_j$.
Then $\{(\mathcal{U}_j, \phi_j)_{j=1}^n\}$ *is a complex coordinate neighborhood of* $\mathbb{C}P^n$.
The local coordinate system $\{\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}\}$ *is called the* **inhomogeneous** *and* $\{z^0, \ldots, z^n\}$ *the* **homogeneous** *coordinate system of* $\mathbb{C}P^n$.

Example 13 *Let* S^{2p+1} *and* S^{2q+1} *be two unit spheres , then the product manifold* $S^{2p+1} \times S^{2q+1}$ *admits a complex structure (see[C.E]).*

In the next section we discuss in detail the special case of the 6-dimensional unit sphere, which admits a non integrable almost complex structure. For the time being we limit ourselves in simply quoting the example.

1.4 Hermitian manifolds and the nearly Kaehler 6-sphere

Before considering the case of the 6-sphere we need to recall some results from the general theory on the almost Hermitian and Hermitian manifolds. Furthermore, we quote results on the existence of r-fold vector cross products on manifolds, results which elucidate the

special character of S^6 and justify the emphasis given on the study of this particular space form.

Definition 13 *Let M be an almost complex manifold with almost complex structure J. A* **Hermitian** *metric on M is a Riemannian metric g such that*
$g(JX, JY) = g(X, Y), \quad \forall (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$.
An almost complex manifold endowed with a Hermitian metric is said to be an **almost Hermitian** *manifold. A complex manifold with a Hermitian metric will be called* **Hermitian** .

Remark 3 *Every almost complex manifold M admits a Hermitian metric. Indeed, if h is a Riemannian metric on M then, by setting:*
$g(X, Y) = h(X, Y) + h(JX, JY), \quad \forall (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$, *$g$ is a Hermitian metric on M.*

Definition 14 *Let M be an almost Hermitian manifold, J its almost complex structure and g the metric. Define the* **Kaehler 2 - form** *by*
$\Phi(X, Y) = g(JX, Y), \quad \forall (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$.

Remark 4 *Since g is positive - definite and the endomorphism J is not singular at each point p of M, follows that: $\Phi^k = \Phi \wedge \ldots \wedge \Phi$, ($k$ - times). $1 \leq k \leq n = dim M$ is non zero at each point of M. We conclude that an almost Hermitian manifold is always orientable.*

In order to both state and prove a theorem connecting conditions of parallelism of the almost complex structure, the Kaehler 2-form with the integrability of the structure (equivalently with the vanishing of the torsion tensor field N), we need the following:

Lemma 2 *Let M be an almost Hermitian manifold with almost complex structure J and Hermitian metric g. Then for all vector fields in $\mathfrak{X}(M)$ the following relation holds*

$$2 \cdot g\left((\nabla_X J) Y, Z\right) - g\left(JX, N(Y, Z)\right) = 3 \cdot d\Phi(X, JY, JZ) - 3 \cdot d\Phi(X, Y, Z). \tag{1.4.1}$$

Proof: To verify the above equation we need to compute the differential of the Kaehler 2-form Φ recalling the definition (12) of the torsion tensor field and that the covariant

derivative of J is given by: $(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M)$.

For the calculation of $d\Phi$ let us at first recall ([Y.K1], pg:17) the following formulae for the exterior differential of an 1-form ω_1 and of a 2-form ω_2:

$$2d\omega_1(X,Y) = X(\omega_1(Y)) - Y(\omega_1(X)) - \omega_1([X,Y]), \tag{1.4.2}$$

$$3d\omega_2(X,Y,Z) = X(\omega_2(Y,Z)) + Y(\omega_2(Z,X)) + Z(\omega_2(X,Y))$$
$$-\omega_2([X,Y],Z) - \omega_2([Y,Z],X) - \omega_2([Z,X],Y). \tag{1.4.3}$$

Using the relation (1.4.3) and observing that

$(\nabla_X J)JY = -J(\nabla_X J)Y, (\nabla_X \Phi)(Y,Z) = g(Y,(\nabla_X J)Z)$ and

$N(Y,Z) = (\nabla_{JY}J)Z - (\nabla_{JZ}J)Y + J(\nabla_Z J)Y - J(\nabla_Y J)Z$

for any $X, Y, Z \in \mathfrak{X}(M)$ we can verify the required assertion.

Theorem 8 *Let M be an almost Hermitian manifold with almost complex structure J and Hermitian metric g. Then the following conditions are equivalent*

(1) $\nabla J = 0$

(2) $\nabla \Phi = 0$

(3) $N = 0$ *and* $d\Phi = 0$.

Proof: We have $(\nabla_X \Phi)(Y,Z) = g(Y,(\nabla_X J)Z), \forall X, Y, Z \in \mathfrak{X}(M)$. Thus $\nabla J = 0$ if and only if $\nabla \Phi = 0$. If $\nabla \Phi = 0$ then $d\Phi = 0$ and using lemma (2) we get that the torsion tensor field must vanish.

Conversely, if $N = 0$ and $d\Phi = 0$ hold simultaneously then using again lemma (2) we obtain that the covariant derivatives of the almost complex structure and of the Kaehler 2-form vanish.

Definition 15 *A Hermitian metric g on an almost complex manifold M is said to be a* **Kaehlerian metric** *if and only if the Kaehler 2-form Φ is closed and in this case M is called an* **almost Kaehlerian manifold**.

A complex manifold with a Kaehlerian metric is called a **Kaehlerian manifold** *and in view of theorem (8) M is Kaehlerian if and only if $\nabla J = 0$.*

An almost Hermitian manifold M with almost complex structure J is said to be a **nearly Kaehler manifold** *if and only if: $(\nabla_X J)X = 0, \quad \forall X \in \mathfrak{X}(M)$ (or equivalently $(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M)$.)*

From the general theory on nearly Kaehler manifolds we quote the following result concerning the torsion and the curvature tensor ([Y.K1], pg:145).

Lemma 3 *Let M be a nearly Kaehler manifold with J almost complex structure, g metric and R curvature tensor field. Then:*

(1) $g(R(X,Y)Z,W) = g(R(X,Y)JZ,JW) - g((\nabla_X J)Y,(\nabla_Z J)W)$

(2) $g(R(X,Y)Y,X) = g(R(X,Y)JY,JX) + g((\nabla_X J)Y,(\nabla_X J)Y)$

(3) $g(R(X,Y)Z,W) = g(R(JX,JY)JZ,JW)$

for all vector fields $X,Y,Z,W \in \mathfrak{X}(M)$

Let us now focus on the 6-dimensional unit sphere and its (nearly Kaehler) almost complex structure. We recall some results on the existence of r-fold vector cross products on manifolds given that an almost complex structure is an 1-fold vector cross product.
An r - fold **vector cross product** X [B.G] on an n-dimensional vector space V is a continuous map $X : V^r \longrightarrow V$. $1 \leq r \leq n-1$ satisfying:
$< X(a_1,\ldots,a_r), a_i >= 0. \quad \forall i \in \{1,\ldots,r\}$
$< X(a_1,\ldots,a_r), X(a_1,\ldots,a_r) >= det((a_i,a_j))$ where $<,>$ denotes an ordinary positive definite inner product on V.
In [Wh] it has been proved that an r-fold vector cross product exists in precisely the following cases
n is even and r=1,
n is arbitrary and r = n-1,
n = 4 or 8 and r = 3 (the discussion of the problem in [Wh] is in terms of algebraic topology, for a purely algebraic approach see [J] or [Ek]). The concept of r-fold vector cross product can be extended on the tangent space of a manifold (see:[G2]) and the following result holds

Theorem 9 *([G2]) Let S^n be the n - dimensional unit sphere in $\mathbb{R}^{n+1}$ and $<,>$ the metric tensor on S^n, induced from the usual positive definite of $\mathbb{R}^{n+1}$. If S^n has a globally defined r-fold vector cross product then, in the vector space sense, there is an (r+1)-fold vector cross product on $\mathbb{R}^{n+1}$.*

Proof: Let X_m denote the r - fold vector cross product on S^n at the point $m \in S^n$. Define the map $\mathcal{P} : (\mathbb{R}^{n+1})^{r+1} \longrightarrow \mathbb{R}^{n+1}$ as follows:
let $a_1, \ldots, a_r \in \mathbb{R}^{n+1}, a_{r+1} = b + c$ where b is the component of a_{r+1} normal to $a_1, \ldots, a_r$.
If $b = 0$ we set $\mathcal{P}(a_1, \ldots, a_{r+1}) = 0$ and if $b \neq 0$ set
$\mathcal{P}(a_1, \ldots, a_{r+1}) = \|b\| \cdot X_d(a_1, \ldots, a_r)$ where $d = \|b\|^{-1} \cdot b$.
It is not hard to check that $\mathcal{P}$ defines actually an (r+1)-fold product on $\mathbb{R}^{n+1}$. with respect to the induced metric tensor, $\mathcal{P}$ is linear in $a_1, \ldots, a_r$ but in general is continuous only in a_{r+1}.

Combining theorem (9) with the result of ([Wh]) on the existence of r-fold vector cross products and recalling that an almost complex structure is an 1-fold vector cross product, we get:

Corollary 1 *The only spheres with an almost complex structure are S^2 and S^6. Furthermore, there does not exist a 3 - fold vector cross product on S^8.*

Under the view of the corollary (1) we now focus our attention on the case of the 6-dimensional unit sphere. An almost complex structure can be constructed on S^6 using the **Cayley** numbers.

Remark 5 *([Wi], pg:163) A* **quaternion** *is a number of the form $q = w + xi + yj + zk$ where $w, x, y, z \in \mathbb{R}$. Addition is defined in the standard way and multiplication is given by:*
$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j, \quad ij + ji = jk + kj = ki + ik = 0$.
Define the **conjugate** $\bar{q}$ *by: $\bar{q} = w - xi - yj - zk$ and the algebra of quaternions results to be associative but not commutative.*
A **Cayley number** *$x = (q_1, q_2)$ is an ordered pair of quaternions and their set $\mathcal{O}$ is an 8 - dimensional non - associative algebra over the real numbers. Addition and multiplication in $\mathcal{O}$ are defined as follows*
$(q_1, q_2) \pm (q_1', q_2') = (q_1 \pm q_1', q_2 \pm q_2')$.
$(q_1, q_2)(q_1', q_2') = (q_1 q_1' - \bar{q}_2' q_2, q_2' q_1 + q_2 \bar{q}_1')$
and the conjugate of x is $\bar{x} = (\bar{q}_1, -q_2)$. Then $x\bar{x} = (q_1\bar{q}_1 + \bar{q}_2 q_2, 0)$ and by setting $|x|^2 = q_1\bar{q}_1 + \bar{q}_2 q_2$ we obtain $|x| > 0$ unless $x = 0$. Although associativity for the multipli-

cation of Cayley numbers does not hold. we have the:

$x(xx') = (xx)x', (x'x)x = x'(xx)$, *for all* $x, x' \in \mathcal{O}$.

Moreover the $|xx'| = |x||x'|$ *holds. so the* $xx' = 0$ *implies that either* $x = 0$ *or* $x' = 0$.

A Caley number $x = (q_1, q_2)$ *is defined to be* **real** *if* q_1 *is real and* $q_2 = 0$ *and is said to be* **purely imaginary** *if* q_1 *is a purely imaginary quaternion.*

The multiplication on the Cayley algebra $\mathcal{O}$ may be used to define a vector cross product and to express the usual inner product in $\mathbb{R}^7 \equiv Im(\mathcal{O})$ by setting:

$$u \times v = \frac{1}{2}(uv - vu), \quad < u, v >= -\frac{1}{2}(uv + vu) \tag{1.4.4}$$

for all $u, v \in Im(\mathcal{O})$.

Conversely, the Cayley multiplication on $Im\mathcal{O}$, in terms of the inner and vector cross product in $\mathbb{R}^7$, is given by $(r + u)(s + v) = rs - < u, v > + rv + su + (u \times v)$ where $(u, v) \in Im(\mathcal{O}) \times Im(\mathcal{O}), (r, s) \in Re(\mathcal{O}) \times Re(\mathcal{O})$. The following lemma holds

Lemma 4 *Identifying the imaginary part of the Cayley numbers with* $\mathbb{R}^7$ *then the inner product, the vector cross product and the Cayley multiplication (regarding in the same time* $\times$ *as an* $\mathfrak{X}(\mathcal{R}^7)$ *linear mapping* $\times : \mathfrak{X}(\mathbb{R}^7) \times \mathfrak{X}(\mathbb{R}^7) \longrightarrow \mathfrak{X}(\mathbb{R}^7)$ *) satisfy*

(1) $u \times v = -v \times u$,

(2) $< u \times v, w >=< u, v \times w >$,

(3) $(u \times v) \times w + u \times (v \times w) = 2 < u, w > v - < v, w > u - < u, v > w$,

(4) $D_u(v \times w) = (D_u v) \times w + v \times (D_u w)$.

for any $u, v, w \in \mathfrak{X}(\mathbb{R}^7)$. *where* D *denotes the Riemannian connection on* $\mathfrak{X}(\mathbb{R}^7)$.

We can approach the Cayley numbers in a slightly different way (see: [D.O.V.V3]).

Let $\{e_0, e_1, \ldots, e_7\}$ be the standard basis of $\mathbb{R}^8$ and write each point α of $\mathbb{R}^8$ in the form $\alpha = Ae_0 + x$ where $A \in \mathbb{R}$ and x is a linear combination of $e_1, \ldots, e_7$. We can consider α as a Cayley number and will be called purely imaginary if $A = 0$. For any pair $(u, v) \in Im(\mathcal{O}) \times Im(\mathcal{O})$ we define the Cayley multiplication $\cdot$ by

$$u \cdot v = - < u, v > e_0 + u \times v. \tag{1.4.5}$$

where $<,>$ is the usual inner product in $\mathbb{R}^7$ and $u \times v$ can be determined by the matrix

$$\begin{pmatrix} 0 & e_3 & -e_2 & e_5 & -e_4 & e_7 & -e_6 \\ -e_3 & 0 & e_1 & e_6 & -e_7 & -e_4 & e_5 \\ e_2 & -e_1 & 0 & -e_7 & -e_6 & e_5 & e_4 \\ -e_5 & -e_6 & e_7 & 0 & e_1 & e_2 & -e_3 \\ e_4 & e_7 & e_6 & -e_1 & 0 & -e_3 & -e_2 \\ -e_7 & e_4 & -e_5 & -e_2 & e_3 & 0 & e_1 \\ e_6 & -e_5 & -e_4 & e_3 & e_2 & -e_1 & 0 \end{pmatrix} = [(e_i \times e_j)_{i,j=1}^7].$$

An almost complex structure on S^6

Consider $S^6 = \{p \in \mathbb{R}^7 : < p, p >= 1\}$ as a hypersurface of $\mathbb{R}^7$, where $\mathbb{R}^7$ is identified with $Im(\mathcal{O})$. Each point $p \in S^6$ may be regarded as the unit normal vector to S^6 and the tangent space can be identified with the linear subspace of $\mathbb{R}^7$ orthogonal to p. Define

$J : \mathfrak{X}(S^6) \longrightarrow \mathfrak{X}(S^6) : J_p X = p \times X, \quad \forall (p, X) \in S^6 \times T_p S^6.$

Then J_p is an endomorphism of $T_p S^6$ at each p and moreover, using the standard properties of the vector cross product (lemma 4), for all $X, Y \in T_p S^6$ we obtain :

$J_p^2 X = p \times (p \times X) = -X$ and

$< J_p X, J_p Y >=< p \times X, p \times Y >= - < X, p \times (p \times Y) >=< X, Y >.$

Recalling the definition we conclude that $(S^6, <,>, J)$ is an almost Hermitian manifold with respect to the almost complex structure J.

If $p \in S^6$ and $X \in T_p \mathbb{R}^7$ we let $\mathcal{P}(X)$ be the orthogonal projection of X onto $T_p S^6$. Then, if $X \in \mathfrak{X}(S^6)$:

$$\begin{aligned} \left(\tilde{\nabla}_X J\right) X &= \tilde{\nabla}_X JX - J\left(\tilde{\nabla}_X X\right) \\ &= \tilde{\nabla}_X (p \times X) - p \times \tilde{\nabla}_X X \\ &= \mathcal{P}\{D_X (p \times X)\} - p \times \tilde{\nabla}_X X \\ &= \mathcal{P}\{(D_X p) \times X + p \times D_X X\} - p \times \tilde{\nabla}_X X \\ &= \mathcal{P}\{X \times X + p \times D_X X\} - p \times \tilde{\nabla}_X X \\ &= \mathcal{P}\{p \times D_X X\} - p \times \tilde{\nabla} \\ &= p \times D_X X - p \times \tilde{\nabla}_X X \\ &= p \times \left(\tilde{\nabla}_X X + < D_X X, p > p\right) - p \times \tilde{\nabla}_X X = 0 \end{aligned}$$

and it is proved that S^6 is a **nearly Kaehler manifold** but not Kaehlerian since the second Betti number of S^6 is zero (see:[G1]). In order to see that S^6 is not Kaehlerian we could use a result of A.Gray ([G1], pg:280) according to which any orientable hypersurface

of $\mathbb{R}^7$, with the almost complex structure induced by the Cayley numbers, is Kaehlerian if and only if it is totally geodesic.

In the last part of the section and of the chapter we are going to state and prove a result of A.Gray on the non existence of 4 - dimensional almost complex submanifolds of S^6. In order to proceed we need some preliminaries and we introduce them preserving the notation and terminology used by A.Gray in [G2].

Definition 16 *Let $(\widetilde{M}, <,>, J)$ be an almost Hermitian manifold and M a submanifold of $\widetilde{M}$. Then, M is called:*
an almost complex submanifold *if and only if $J(T_pM) \subset T_pM, \quad \forall \quad p \in M$*
a totally real submanifold *if and only if $J(T_pM) \subset \perp_p M, \quad \forall \quad p \in M$.*

Let $M, \widetilde{M}$ be Riemannian manifolds with M isometrically imbedded in $\widetilde{M}$ and denote by $\mathfrak{X}(M), \widetilde{\mathfrak{X}}(M)$ the Lie algebras of vector fields of M and restrictions of vector fields of $\widetilde{M}$ on M respectively. Then the decomposition $\widetilde{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}^{\perp}(M)$ holds.

Definition 17 *Let $M, \widetilde{M}$ be Riemannian manifolds as above and $\nabla, \widetilde{\nabla}$ their Riemannian connections respectively. Define the* **configuration** *tensor*
$T : \mathfrak{X}(M) \times \widetilde{\mathfrak{X}}(M) \longrightarrow \widetilde{\mathfrak{X}}(M)$ *by setting*
$T(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y, \quad \forall (X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$
$T(X,\xi) = \mathcal{P}\widetilde{\nabla}_X \xi, \quad \forall (X,\xi) \in \mathfrak{X}(M) \times \mathfrak{X}\perp(M),$
where $\mathcal{P} : \widetilde{\mathfrak{X}}(M) \longrightarrow \mathfrak{X}(M)$ is the orthogonal projection.

The next lemma can be verified by direct computation.

Lemma 5 *The configuration tensor satisfies the following properties*
$T_X Y = T_Y X, \quad < T_X Y, \xi >= - < T_X \xi, Y >,$
$T_X(\mathfrak{X}(M)) \subset \mathfrak{X}^{\perp}(M). \quad T_X(\mathfrak{X}^{\perp}(M)) \subset \mathfrak{X}(M), \quad \forall (X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M). \quad \forall \xi \in \mathfrak{X}^{\perp}(M).$

In the following two lemmata we impose further assumptins on $M, \widetilde{M}$.

Lemma 6 *Suppose that $(\widetilde{M}, <,>, J)$ is a nearly Kaehler manifold and M is an almost complex submanifold of $\widetilde{M}$ and let $\widetilde{\nabla}, \nabla$ be the Riemannian connections on $\widetilde{M}, M$ respectively. Then:*

(a): M is a nearly Kaehler manifold ($M \in \mathcal{NK}$)

(b): $T_X JX = JT_X X, \quad \forall \quad X \in \mathfrak{X}(M)$

(c): M is minimal in $\widetilde{M}$.

Proof: $M \in \mathcal{NK} \Longrightarrow (\tilde{\nabla}_X J)X = 0, \quad \forall \quad X \in \mathfrak{X}(M) \Longrightarrow$

$\tilde{\nabla}_X JX - J\tilde{\nabla}_X X = T_X JX + \nabla_X JX - JT_X X - J\nabla_X X = 0 \Longrightarrow$

$(\nabla_X J)X + T_X JX - JT_X X = 0, \quad \forall X \in \mathfrak{X}(M).$

Since $(\nabla_X J)X \in \mathfrak{X}(M), \quad T_X JX \in \mathfrak{X}^{\perp}(M)$ and $JT_X X \in \mathfrak{X}^{\perp}(M)$, by taking tangential and normal parts with respect to M, we get:

$(\nabla_X J)X = 0$ and $T_X JX = JT_X X. \quad \forall X \in \mathfrak{X}(M)$ and the (a), (b) are proved.

In order to prove (c) we observe that since $T_X JX = JT_X X, \quad \forall X \in \mathfrak{X}(M)$ and T is symmetric, we have:

$T_X X = -JT_X JX = -JT_{JX} X. \quad \forall X \in \mathfrak{X}(M) \Longrightarrow T_X X + T_{JX} JX = 0, \quad \forall X \in \mathfrak{X}(M).$

Choosing an orthonormal frame defined on an open subset of M to be of the form $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ we deduce the required assertion.

Note: In case the ambient manifold is Kahlerian and M is an almost complex submanifold then M is minimal since any Kaehlerian manifold is nearly Kaehler.

Lemma 7 *Let M be a 4-dimensional nearly Kaehler manifold, then M is Kaehlerian.*

Proof: Choose $\{X, JX, Y, JY\}$ to be an orthonormal frame field defined on an open subset of M. But M being Kaehler is equivalent, by definition, to: $(\nabla_X J)Y + (\nabla_Y J)X = 0 \quad \forall X, Y \in \mathfrak{X}(M)$ and the following hold:

$$< JY, JY >= 1 \Longrightarrow < \nabla_X JY, JY >= 0 \Longrightarrow \nabla_X JY \perp JY, \tag{1.4.6}$$

$$< Y, Y >= 1 \Longrightarrow < \nabla_X Y, Y >= < J\nabla_X Y, JY >= 0. \tag{1.4.7}$$

Combining equations (1.4.6), (1.4.7) we obtain that $(\nabla_X J)Y$ is normal to JY for all vector fields $X, Y \in \mathfrak{X}(M)$.

From the definition of nearly Kaehler manifold and repeating similar steps we have: $(\nabla_X J)Y = -(\nabla_Y J)X \perp JX$ thus $(\nabla_X J)Y$ is normal to JX.

On the other hand, it is easy to verify that $(\nabla_X J)Y = J(\nabla_X J)JY. \quad \forall X, Y \in \mathfrak{X}(M).$

Since $(\nabla_X J)Y$ is normal to the $span\{X, Y\}$ we can write

$$(\nabla_X J)\, JY \perp JY \Longrightarrow J\,(\nabla_X J)\, JY = (\nabla_X J)\, Y \perp Y \tag{1.4.8}$$

$$(\nabla_Y J)\, JX \perp JX \Longrightarrow -\,(\nabla_X J)\, JY \perp JX \Longrightarrow$$

$$-J\,(\nabla_X J)\, JY = -\,(\nabla_X J)\, Y \perp X \tag{1.4.9}$$

hence, $(\nabla_X J)Y$ is normal to the $span\{X, JX, Y, JY\}$. But M is 4-dimensional, therefore $(\nabla_X J)Y$ has to vanish for any vector fields $X, Y \in \mathfrak{X}(M)$ and the assertion is proved.

Theorem 10 *With respect to the usual almost complex structure on S^6 there are no 4-dimensional almost complex submanifolds.*

Proof: Let M be an almost complex 4-dimensional submanifold of S^6. Take: $p \in M$, $X \in T_pM$ such that $< X, X >= 1$ with $T_X X \neq 0$. Since S^6 is nearly Kaehler follows from (b) in lemma(6) that: $T_X JX = JT_X X \neq 0, \quad \forall X \in T_pM$.
Define the function $f(X) = \|T_X X\|^2$ on the unit tangent space $\mathcal{U}_pM$ at the point p and suppose that f attains its maximum at $\widetilde{X} \in T_pM$. We claim that :

$$< T_{\widetilde{X}} Y, T_{\widetilde{X}} \widetilde{X} >= 0, \quad \forall Y \in \mathcal{U}_pM. \quad \text{and} \quad Y \perp span\{\widetilde{X}, J\widetilde{X}\}. \tag{1.4.10}$$

To verify the equation (1.4.10) we proceed in the following way: take any $Y \in \mathcal{U}_pM$. such that $Y \perp span\{\widetilde{X}, J\widetilde{X}\}$ and define the function

$$\alpha(t) = f\left(cos\,(t)\,\widetilde{X} + sin\,(t)\,Y\right). \tag{1.4.11}$$

Being $\widetilde{X}$ a point of maximum for the function f implies:
$\frac{d}{dt}\alpha|_{t=0} = 0 \Longrightarrow \frac{d}{dt}\|T_{cos(t)\widetilde{X}+sin(t)Y} cos(t)\widetilde{X} + sin(t)Y\|^2|_{t=0} = 0 \Longrightarrow$
$\frac{d}{dt}\|cos^2(t)T_{\widetilde{X}}\widetilde{X} + sin(2t)T_{\widetilde{X}}Y + sin^2(2t)T_Y Y\|^2|_{t=0} = 0,$
and the rest consists in computing first order derivatives with respect to t and substituting the value $t = 0$. The only term which survives is the term involving $< T_{\widetilde{X}}\widetilde{X}, T_{\widetilde{X}}Y >$ which has to vanish, and equation (1.4.10) has been proved.
Considering the unit vector $U = \frac{\widetilde{X}+J\widetilde{X}}{\sqrt{2}}$, with $Y \perp \{U, JU\}$, it is easy to check that $JT_{\widetilde{X}}\widetilde{X} = T_U U$ which implies that U is another point in the unit tangent space $\mathcal{U}_pM$ where the function f attains a maximum and under the aspect of the preceding discussion we conclude that $T_{\widetilde{X}}Y$ has to be normal to $JT_{\widetilde{X}}\widetilde{X}$. Working similarly for $T_Y JY$ we

deduce

$$< T_{\tilde{X}}Y, JT_{\tilde{X}}\tilde{X} >=< T_{\tilde{X}}JY, T_{\tilde{X}}\tilde{X} >=< T_{\tilde{X}}JY, JT_{\tilde{X}}\tilde{X} >= 0. \tag{1.4.12}$$

Combining the equations (1.4.11), (1.4.12) with the fact that the normal space $\perp_p M$ has to be spanned by $\{T_{\tilde{X}}\tilde{X}, JT_{\tilde{X}}\tilde{X}\}$, since $T_{\tilde{X}}\tilde{X} \neq 0$ by assumption, we get that $T_{\tilde{X}}Y$ and $T_{\tilde{X}}JY$ must vanish.

In this point let us recall lemma (3) and denote by $\tilde{R}, \tilde{K} = 1$ the curvature operator and the constant sectional curvature on S^6 respectively and by $\tilde{\nabla}$ the Riemaniann connection on the 6-sphere.

For $\|Y\| = 1$ and taking $Z = \tilde{X} - J\tilde{X}, W = Y - JY$ we have

$$\tilde{K} =< \tilde{R}\left(\tilde{X}, Y\right) Z, W > - < \tilde{R}\left(\tilde{X}, Y\right) J\tilde{X}, JY >= \| \left(\tilde{\nabla}_{\tilde{X}}J\right) Y\|^2 \Longrightarrow$$
$$\tilde{K} = \| \left(\nabla_{\tilde{X}}J\right) Y + T_{\tilde{X}}JY - JT_{\tilde{X}}Y\|^2$$

where $(\nabla_{\tilde{X}}J)Y = 0$ because of lemma (7) and of equations (1.4.11), (1.4.12). This is a contradiction and since the initial assumption was the existence of a unit vector in $\mathcal{U}_p M$ for an arbitrary p where $T_X X$ does not vanish, we can deduce that $T_X X = 0, \quad \forall X \in \mathfrak{X}(M)$. Consequently, M is totally geodesic in S^6 therefore M has to be an open submanifold of a 4-dimensional sphere of constant curvature $\tilde{K}$. But, in the same time, M has to be a flat manifold, given that : M is Kaehlerian (see:lemma (7)) and any Kaehlerian manifold of constant curvature is a flat manifold, provided $dim M > 2$ (see:[Y.K1], pg:131). The assertion has been proved.

In view of the above theorem we deduce that the almost complex submanifolds of S^6 are almost complex surfaces and these surfaces are always minimal in S^6.

Henceforth we focus on totally real 3 -dimensional submanifolds of S^6 but almost complex and also totally real minimal surfaces will be taken under consideration within Chapter 4 in order to construct and classify totally real 3-dimensional submanifolds which satisfy Chen's equality and some further properties.

Chapter 2

Totally real 3 - dimensional submanifolds in S^6

2.1 Introduction

In this chapter we focus on the first results concerning 3-dimensional totally real submanifolds of S^6. We shall prove that any such submanifold M of S^6 is orientable, minimal and we are going to present the first classification results by means of constant and suitably pinched sectional curvature. In particular, Chapter 2 is structured as follows:

In §2.2 we include preliminary notation concerning the second fundamental form, the shape operator and the basic equations of Gauss, Godazzi and Ricci adapted in the case of submanifolds of a space form. We also define a (1,2)-type tensor field $\mathcal{G}$ on S^6 relating covariant differentiation and almost complex structure and give a very useful geometrical interpretation for $\mathcal{G}$. Furthermore, we relate this tensor field with the normal connection, the second fundamental form and the shape operator of the submanifold M. We finally prove an important symmetry property concerning $< h(X,Y), JZ >$ for X, Y, Z tangent vector fields of M.

In §2.3 we give a proof of the orientability and minimality of M and we present the first classification result in terms of constant sectional curvature. The main reference for both the first two paragraphs is [Ej1].

Some very important integral relations, due to A.Ros, are mentioned in the beginning of the fourth section and then are followed by a description of the way a number of integral

relations on compact manifolds are obtained. This enables us to give the proof of a classification result, for compact totally real 3-dimensional submanifolds, by means of pinched sectional curvature. The main reference is [D.O.V.V1]. An improvement of the result of (§2.3) is presented in the last section. The main reference is [D.V.V2].

2.2 Preliminaries, the tensor field $\mathcal{G}$

Let $(S^6, <,>, J)$ be the nearly Kaehler 6-sphere and M a totally real 3-dimensional submanifold. We denote by $\mathbf{D}$, $\tilde{\nabla}$ and ∇ the Riemannian connections on the 7-dimensional Euclidean space, on the 6-sphere and on the submanifold M respectively. Since S^6 is a space form of constant sectional curvature 1 then the fundamental equations of Gauss, Godazzi and Ricci attain respectively the reduced form given by the system of equations (1.2.15), (1.2.16) and (1.3.1) of the first chapter.

Definition 18 *Let* $\mathcal{G}(X,Y) = (\nabla_X J)(Y), \quad \forall X, Y \in \mathfrak{X}(S^6)$

It is easy to check that the above defined $\mathcal{G}$ is an $(1,2)$ - type tensor field on S^6 and it is actually the covariant derivative of the almost complex structure J of the 6-sphere.
In the following lemma we give a geometrical interpretation for $\mathcal{G}$ and we deduce some useful properties linking $\mathcal{G}$ with the almost complex structure, the metric tensor and the connection on S^6. Moreover, we prove that if X and Y are tangential to M then $\mathcal{G}(X,Y)$ is normal to M and we find its relation with the normal connection and the shape operator of M.

Lemma 8 *Let $X, Y, Z \in T_pS^6$. Then:*

$$\mathcal{G}(X,Y) = X \times Y - < X \times Y, p > p \tag{2.2.1}$$

$$\mathcal{G}(X,JY) = -J\mathcal{G}(X,Y) \tag{2.2.2}$$

$$\left(\tilde{\nabla}_X \mathcal{G}\right)(Y,Z) = < Y, JZ > X + < X, Z > JY - < X, Y > JZ \tag{2.2.3}$$

$$< \mathcal{G}(X,Y), Z > + < \mathcal{G}(X,Z), Y >= 0 \tag{2.2.4}$$

$$< \mathcal{G}(X,Y), \mathcal{G}(Z,W) >= < X, Z >< Y, W > - < X, W >< Y, Z > +$$

$$< JX, Z >< Y, JW > - < JX, W >< Y, JZ >, \tag{2.2.5}$$

Further, if M is a totally real 3-dimensional submanifold of S^6, then:

$$\mathcal{G}(X,Y) = X \times Y \in \mathfrak{X}^{\perp}(M), \quad \forall X, Y \in \mathfrak{X}(M), \tag{2.2.6}$$

where $\times$ denotes the vector cross product in $\mathbb{R}^7$.

Proof: Since computations concerning the proof of the above relations are not carried out explicitely in [Ej1] and [D.O.V.V1] we shall give some details.
For $\mathcal{P} : \mathfrak{X}(\mathbb{R}^7) \longrightarrow \mathfrak{X}(S^6)$ (orthogonal projection), $p \in S^6$, $(X,Y) \in \mathfrak{X}(S^6) \times \mathfrak{X}(S^6)$, and D the Levi-Civita connection in $\mathbb{R}^7$, we can write:

$$\mathcal{G}(X,Y) = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y = \tilde{\nabla}_X(p \times Y) - p \times \tilde{\nabla}_X Y =$$
$$= \mathcal{P}\left[(D_X p) \times Y + p \times D_X Y\right] - p \times \tilde{\nabla}_X Y = \mathcal{P}\left[X \times Y\right] \Longrightarrow$$
$$\mathcal{G}(X,Y) = X \times Y - < X \times Y, p > p \tag{2.2.7}$$

and the equation (2.2.1) is proved. The second equality can be obtained by using standard properties of the vector cross product. To prove the equation (2.2.3) we use the definition of the covariant derivative for an (1.2) type tensor field and repeat similar steps as in proof of equation (2.2.1) observing that:

$$\tilde{\nabla}_X \left(Y \times Z - < Y \times Z, p > p\right) = \tilde{\nabla}_X \left(Y \times Z\right) - \mathcal{P} D_X \left(< Y \times Z, p > p\right),$$
$$\mathcal{P}[D_X \left(< Y \times Z, p > p\right)] = D_X \left(< Y \times Z, p > p\right) - < D_X \left(< Y \times Z, p > p\right), p > p$$

Since $D_X p = X$ and X is normal to p it is clear that equation (2.2.4) follows directly from equation (2.2.1).
In order to obtain the equation (2.2.5) we use (2.2.1), standard properties of the vector

cross product and write:

$$< \mathcal{G}(X,Y), \mathcal{G}(Z,W) > = < X \times Y, Z \times W > - < Z \times W, p > \cdot < X \times Y, p >$$
$$- < X \times Y, p > - < Z \times W, p >$$
$$+ < X \times Y, p > \cdot < Z \times W, p >$$

where $< X \times Y, Z \times W > = < X, Z > \cdot < Y, W > - < X, W > \cdot < Y, Z >$.

To prove that $\mathcal{G}(X,Y) \in \mathfrak{X}^{\perp}(M)$ for any $X, Y \in \mathfrak{X}(M)$, we follow a different method, with respect to the process in [Ej1], by using properties of the vector cross product and its relation with the Caley multiplication and the standard inner product in $\mathbb{R}^7$. Let $X, Y \in \mathfrak{X}(M)$ and use equation (2.2.1) and the fact that M is totally real, in order to get:

$$< X, JY > = 0 \Longrightarrow < X, p \times Y > = 0 \Longrightarrow$$
$$< X \times Y, p > = 0 \Longrightarrow X \times Y \in \mathfrak{X}(S^6) \Longrightarrow$$
$$\mathcal{G}(X,Y) = X \times Y$$

and the first part of the (2.2.6) is proved.

Using once more the properties of the vector cross product we can easily check that $\mathcal{G}(X,Y)$ is normal to the $span\{X, Y, JX, JY\}$ and it remains only to prove that $\mathcal{G}(X,Y)$ is also normal to JZ.

Assume, without loss of generality, that $\{X, Y, Z\}$ is an orthonormal frame field defined on an open subset of M and recall the relations between vector cross product, inner product and Cayley multiplication given by the (1.4.4) in Chapter 1. We compute:

$$< X \times Y, Z > = \frac{1}{2}\{(X \times Y) \cdot Z - Z \cdot (X \times Y)\}$$
$$= \frac{1}{2}\{< X \cdot Y, Z > - < Y \cdot X, Z >\}$$
$$= \frac{1}{4}\{(X \cdot Y) \cdot Z - Z \cdot (X \cdot Y)\} - \frac{1}{4}\{(Y \cdot X) \cdot Z - Z \cdot (Y \cdot X)\}.$$

Observing that $< X, Y > = 0$ if and only if $X \cdot Y = Y \cdot X$ we have $< X \times Y, Z > = 0$ and the assertion is proved.

Lemma 9 *Let M be as in the previous lemma, then the following equations hold:*

$$\nabla^{\perp}_X JY = \mathcal{G}(X,Y) + J(\nabla_X Y), \tag{2.2.8}$$

$$A_{JX}Y = -Jh(X,Y), \tag{2.2.9}$$

$$< h(X,Y), JZ > = < h(X,Z), JY >. \tag{2.2.10}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Proof : Let $X, Y \in \mathfrak{X}(M)$, then:

$$\mathcal{G}(X,Y) = (\tilde{\nabla}_X J)Y = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y =$$
$$-A_{JY}X + \nabla^{\perp}_X JY - \nabla_X Y - h(X,Y) \Longrightarrow$$
$$\mathcal{G}(X,Y) = -A_{JY}X + \nabla^{\perp}_X Y - J(\nabla_X Y) - Jh(X,Y)$$

where $\mathcal{G}(X,Y)$ is normal to M. Taking the normal and the tangential component we deduce equations (2.2.8) and (2.2.9). A proof of the third symmetry relation appears in lemma 4.1 of [Ej1], pg:760. We shall obtain it in a slightly different and easier way by observing that for all $X, Y, Z \in \mathfrak{X}(M)$, we have:

$$\begin{aligned} 0 &= < \left(\tilde{\nabla}_X J\right) Y, Z > \\ &= < -A_{JY}X + \nabla^{\perp}_X Y - Jh(X,Y) - J(\nabla_X Y), Z > \\ &= < -A_{JY}X, Z > - < Jh(X,Y), Z > \end{aligned}$$

and on the other hand, $< h(X,Y), \xi > = < A_\xi X, Y >, \quad \forall X, Y \in \mathfrak{X}(M). \quad \forall \xi \in \mathfrak{X}^{\perp}(M)$. Since the inner product in S^6 is Hermitian, we get the assertion.

2.3 Orientability, minimality and classification in terms of constant sectional curvature

Theorem 11 *([Ej1], pg:760-762) Any 3-dimensional, totally real submanifold of* S^6 *is orientable and minimal.*

Proof: To prove that M is orientable we recall standard properties of the vector cross product and observe that:

$< \mathcal{G}(X,Y), X > = < X \times Y, X > = - < Y, X \times X > = 0,$

$< \mathcal{G}(X,Y), Y > = < X \times Y, Y > = - < X, Y \times Y > = 0.$

$\mathcal{G}(X,Y) = X \times Y \neq 0. \quad X, Y \in \mathfrak{X}(M) - \{O\}$ with X and Y mutually orthogonal.

To prove the minimality we use the definition of the covariant derivative of $\mathcal{G}$ and the

expression for the shape operator given by the equation (2.2.9). Then, the

$$\left(\tilde{\nabla}_X \mathcal{G}\right)(Y,Z) = \tilde{\nabla}_X \mathcal{G}(Y,Z) - \mathcal{G}\left(\tilde{\nabla}_X Y, Z\right) - \mathcal{G}\left(Y, \tilde{\nabla}_X Z\right),$$
$$\tilde{\nabla}_X \mathcal{G}(Y,Z) = -A_{\mathcal{G}(Y,Z)}X + \nabla^{\perp}_X \mathcal{G}(Y,Z)$$

hold, and combining them we deduce:

$$\left(\tilde{\nabla}_X \mathcal{G}\right)(Y,Z) = -A_{\mathcal{G}(Y,Z)}X + \nabla^{\perp}_X \mathcal{G}(Y,Z) - \mathcal{G}\left(\tilde{\nabla}_X Y, Z\right) - \mathcal{G}\left(Y, \tilde{\nabla}_X Z\right) \tag{2.3.1}$$

Using the (2.2.8), (2.2.9) of lemma (9) we can also get:

$$-A_{\mathcal{G}(Y,Z)}X = Jh\left(J\mathcal{G}(Y,Z), X\right). \tag{2.3.2}$$

$$\nabla^{\perp}_X \mathcal{G}(Y,Z) = -\left(\mathcal{G}\left(J\mathcal{G}(Y,Z), X\right) + J\left(\nabla_X \mathcal{G}(Y,Z)\right)\right). \tag{2.3.3}$$

Combining the equations (2.3.1), (2.3.2), (2.3.3) and (2.2.2) of Lemma (8), and for any $X, Y, Z \in \mathfrak{X}(M)$, we obtain:

$$\left(\tilde{\nabla}_X \mathcal{G}\right)(X,Y) = Jh\left(Jh(Y,Z), X\right) + J\mathcal{G}\left(X, \mathcal{G}(Y,Z)\right) -$$
$$J\left(\nabla_X J\mathcal{G}\right)(Y,Z) - \mathcal{G}\left(h(X,Y), Z\right) - \mathcal{G}\left(Y, h(X,Z)\right). \tag{2.3.4}$$

In this point we modify from the process followed in [Ej1] by recalling the equation (2.2.3) of lemma (8) and considering, without loss of generality, the vector fields $X, Y, Z \in \mathfrak{X}(M)$ to be mutually orthogonal. Then the right hand side in equation (2.3.4) vanishes and thus, by taking its tangential component, we get:

$$h\left(X, J\mathcal{G}(Y,Z)\right) + J\mathcal{G}\left(h(X,Y), Z\right) + J\mathcal{G}\left(Y, h(X,Z)\right) = 0. \tag{2.3.5}$$

Under the aspect of equation (2.3.5) let us choose $\{e_1, e_2, e_3\}$ to be local orthonormal frame fields on M such that the conditions : $J\mathcal{G}(e_1, e_2) = e_3$, $J\mathcal{G}(e_2, e_3) = e_1$ and $J\mathcal{G}(e_3, e_1) = e_2$ are satisfied. Applying the equation (2.3.5), for this particular choice of frame, we deduce the required minimality and the proof is completed.

We are going to present the first classification result in the case of a totally real submanifold in S^6 with constant sectional curvature.

A description of the process followed in [Ej1] will be given but our intention is to focus on certain points where suggestions are omitted and the method used seems to be obscure.

Theorem 12 *Let M be a totally real, 3-dimensional submanifold of S^6. If M has constant sectional curvature then the values attained are: 1 if M is totally geodesic, or $\frac{1}{16}$.*

Proof: Being the submanifold M of constant sectional curvature c and the ambient space the unit 6-sphere, the equation of Gauss reduces to:

$$(1-c)\left[< X,Z >< Y,W > - < X,W >< Y,Z >\right]$$
$$+ < h(X,Z), h(Y,W) > - < h(X,W), h(Y,Z) >= 0. \tag{2.3.6}$$

If $c = 1$ then $h \equiv 0$ on M and M is totally geodesic in S^6. Assume from now on that $c \neq 1$.

Let $x \in M$ and define the function

$\mu(X) =< h(X,X), JX >, \quad \forall X \in \mathcal{U}_x M = \{X \in T_x M : \ < X,X >= 1\}$.

The subset of the unit tangent bundle $\mathcal{U}_x M$ is compact, let us assume that μ attains its maximum at the point $X \in \mathcal{U}_x M$. We claim that:

$$< h(X,X), JY >= 0 \tag{2.3.7}$$

for any $Y \in \mathcal{U}_x M$ with Y normal to X. In order to prove the relation (2.3.7) we use the method of the **maximalization**, as we already did in order to prove theorem (9) of Chapter 1. Let us define the function

$$\tilde{\mu}(\theta) = \mu(cos(\theta)X + sin(\theta)Y) \tag{2.3.8}$$

where $(X,Y) \in \mathcal{U}_x M \times \mathcal{U}_x M$. $\quad \theta \in \mathbb{R}$, $\quad X \perp Y$ and in particular, X is choosen to be the point of the unit tangent bundle at which the function μ attains its maximum. Being X a point of maximum of μ implies that the first derivative of the function $\tilde{\mu}$ evaluated at $\theta = 0$ has to vanish. Therefore,

$$\frac{d}{d\theta} < h(cos\theta X + sin\theta Y, cos\theta X + sin\theta Y), cos\theta JX + sin\theta JY > |_{\theta=0} = 0$$
$$\Longrightarrow \frac{d}{\theta}\{cos^2\theta sin\theta < h(X,X), JY >\}|_{\theta=0} = 0$$
$$\Longrightarrow \{\left(-sin2\theta sin\theta + cos^3\theta\right) < h(X,X), JY >\}|_{\theta=0} = 0$$
$$\Longrightarrow < h(X,Y), JY >= 0 \tag{2.3.9}$$

and the assertion (2.3.7) is proved. It is also clear that $h(X,X)$ has to be parallel to JX. Furthermore, the function μ cannot be a constant. Indeed, M is minimal and non totally

geodesic and $h(X,X)$ is parallel to JX. Let us assume that μ is a constant. Then, for any element $w = \frac{u}{\|u\|}$ of the unit tangent bundle, the equalities

$$< h(w,w), Jw >=< h(\frac{u}{\|u\|}, \frac{u}{\|u\|}), J\frac{u}{\|u\|} >= lim_{\|u\|\longrightarrow\infty}(\frac{1}{\|u\|^3}) \cdot < h(u,u), Ju >$$

must be satisfied, since μ is supposed to be constant. We conclude that $< h(u,u), Ju >= 0$ for all $u \in \mathcal{U}_x M$. This is a contradiction consisting in that $h(X,X)$ is, from one side parallel to JX and on the other, if μ is a constant, $h(X,X)$ has to be perpendicular to JX.

At this stage we will give some explanations regarding the method used in [Ej1] in order to elucidate details for which there are no suggestions in the relevant paper.

According to [Ej1], at each point $x \in M$ we choose an orthonormal basis of the tangent space by taking e_1 to be the maximum point of the function μ in $T_x M$ and, modifying in this stage the process, let us consider any pair X, Y of orthonormal vectors such that $\{e_1, X, Y\}$ results to be an orthonormal basis of $T_x M$. Using the equations (2.3.7) and (2.2.10) we can write:

$$\begin{aligned} h(e_1,e_1) &= \alpha_1 Je_1, \\ h(e_1,X) &= \alpha_2 JX + \beta JY, \\ h(e_1,Y) &= \beta JX + \alpha_3 JY. \end{aligned} \tag{2.3.10}$$

Applying Gauss equation (2.3.6) to e_1, e_1, X, X and e_1, e_1, Y, Y we get the system

$$\begin{aligned} 1 - c - \beta^2 + \alpha_1\alpha_2 - \alpha_2^2 &= 0. \\ 1 - c - \beta^2 + \alpha_1\alpha_3 - \alpha_3^2 &= 0. \end{aligned} \tag{2.3.11}$$

From the above equations we deduce that:

$\alpha_1\alpha_2 - \alpha_2^2 = \alpha_1\alpha_3 - \alpha_3^2 \Longrightarrow \alpha_1(\alpha_2 - \alpha_3) = (\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3)$.

Suppose that $\alpha_2 \neq \alpha_3$, then: $\alpha_1 = \alpha_2 + \alpha_3$ and in the same time the $\alpha_1 = -(\alpha_2 + \alpha_3)$ must be satisfied, because of the minimality of M. Consequently α_1 must be zero.

But, if $\alpha_1 = 0$ then $h(e_1, e_1) = 0$ and this implies that h has to vanish, given that e_1 is an eigenvector of the shape operator in the direction Je_1 and moreover e_1 has a maximum property between all the eigenvectors.

Hence, by assuming $\alpha_1 \neq 0$ we obtain $\alpha_2 = \alpha_3$ i.e: $< h(X,X), Je_1 >=< h(Y,Y), Je_1 >$

which implies that every tangent vector, normal to e_1, is an eigenvector of A_{Je_1} (and this is actually the crucial point in order to proceed further with the computation of the second fundamental form).

Choose $e_2 \in T_xM$ to be the maximum point of the restriction of the function μ in the orthogonal complement of e_1 with respect to T_xM and let e_3 be another unit tangent vector such that the $\{e_1, e_2, e_3\}$ forms an orthonormal basis of T_pM. In view of the previous discussion both e_2 and e_3 are eigenvectors of the shape operator in the direction Je_1. From the special choice of the orthonormal frame and using equation (2.2.10) we get:

$$< h(e_2, e_2), Je_3 >=0. \tag{2.3.12}$$

$$< h(e_2, e_3), Je_2 >=0. \tag{2.3.13}$$

Preserving the notation let us define $\alpha_i =< h(e_i, e_i), Je_1 >, \quad \forall i = 1, 2, 3$ and since M is minimal in S^6 we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{2.3.14}$$

In order to compute the quantities α_i we apply Gauss equation (2.3.7) for $X = e_1 = Z$ and $Y = e_2 = W$, obtaining

$$1 - c + < h(e_1, e_1), h(e_2, e_2) > - < h(e_1, e_2), h(e_1, e_2) >= 0. \tag{2.3.15}$$

By virtue of equation (2.3.12) and since e_1, e_2, e_3 are eigenvectors of A_{Je_1} we can write :

$$h(e_1, e_1) = \alpha_1 Je_1, \quad h(e_1, e_2) = \alpha_2 Je_2, \quad h(e_1, e_3) = \alpha_3 Je_1. \tag{2.3.16}$$

Using equations (2.3.12), (2.3.13), (2.3.14) and the symmetry property given by the equation (2.2.10) we get

$$< h(e_1, e_1), h(e_2, e_2) >= \alpha_1 \cdot \alpha_2, \quad < h(e_1, e_2), h(e_1, e_2) >= \alpha_2^2. \tag{2.3.17}$$

Working similarly and applying the Gauss equation for $X = e_1 = Z, \quad Y = e_3 = W$ we deduce the following system

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ 1 - c + \alpha_1 \cdot \alpha_2 - \alpha_2^2 &= 0, \\ 1 - c + \alpha_1 \cdot \alpha_3 - \alpha_3^2 &= 0. \end{aligned} \tag{2.3.18}$$

This system provides the following values for the unknownls:

$$\alpha_1 = 2\sqrt{\frac{1-c}{3}}, \quad \alpha_2 = -\sqrt{\frac{1-c}{3}}, \quad \alpha_3 = -\sqrt{1-c} \tag{2.3.19}$$

and by direct substitution we find:

$$h(e_1, e_1) = 2 \cdot \sqrt{\frac{1-c}{3}} \cdot Je_1. \tag{2.3.20}$$

Combining the (2.3.12), (2.3.16), (2.3.19) and by using the minimality of M we may set

$$h(e_2, e_2) = -\sqrt{\frac{1-c}{3}} Je_1 + \lambda Je_2, \tag{2.3.21}$$

$$h(e_3, e_3) = -\sqrt{\frac{1-c}{3}} Je_1 - \lambda Je_2, \tag{2.3.22}$$

$$h(e_2, e_3) = -\lambda Je_3. \tag{2.3.23}$$

Applying once more the Gauss equation for $X = e_2 = W$ and $Y = e_3 = Z$ we compute λ and the second fundamental form of M is given by:

$$\begin{aligned} &h(e_1, e_1) = 2\sqrt{\frac{1-c}{3}} \cdot Je_1, \quad h(e_2, e_2) = -\sqrt{\frac{1-c}{3}} \cdot Je_1 + \sqrt{\frac{2(1-c)}{3}} \cdot Je_2 \\ &h(e_3, e_3) = -\sqrt{\frac{1-c}{3}} \cdot Je_1 - \sqrt{\frac{2(1-c)}{3}} \cdot J_2, \quad h(e_1, e_2) = -\sqrt{\frac{1-c}{3}} \cdot Je_2 \\ &h(e_1, e_3) = -\sqrt{\frac{1-c}{3}} \cdot Je_3, \quad h(e_2, e_3) = -\sqrt{\frac{2(1-c)}{3}} \cdot Je_3. \end{aligned} \tag{2.3.24}$$

The last step consists in computing the connection on M by using the equation (1.2.15) of Godazzi, from lemma (9) the equation (2.2.8) and the fact that the basis $\{e_1, e_2, e_3\}$ is choosen to be orthonormal. After computing the connection a direct calculation gives $R(e_1, e_2)e_1 = \frac{1}{16} \cdot e_2$ and this proves the theorem.

Remark 6 *It is clear that the crucial point in the whole discussion of the proof of the above theorem consists in computing the values of the coefficients α_i of the second fundamental form in the direction of the normal vector Je_1 and this is obtained by choosing in a proper manner the vectors e_1, e_2 in the unit tangent bundle. We could obtain the same values for the coefficients α_i by choosing the vector e_1 to be the maximum point of the function μ and taking e_2 and e_3 to be simply eigenvectors of A_{Je_1}. Using equation*

(2.2.10) h can be written in the form (see [D.D.V.V]):

$$h(e_1,e_1) = \alpha_1 Je_1, \quad h(e_1,e_2) = \alpha_2 Je_1, \quad h(e_1,e_3) = \alpha_3 Je_1$$
$$h(e_2,e_2) = \alpha_2 Je_1 + aJe_2 + bJe_3, \quad h(e_2,e_3) = bJe_2 - aJe_3$$
$$h(e_3,e_3) = \alpha_3 Je_1 - aJe_2 - bJe_3, \tag{2.3.25}$$

where: $\alpha_1 + \alpha_2 + \alpha_3 = 0$. $\alpha_1 > 0$. $\alpha_1 - 2\alpha_2 \geq 0$. $\alpha_1 - 2\alpha_3 \geq 0$.
Imposing that M *is of constant curvature and applying the reduced Gauss equation for* $X = e_1 = W$, $Y = e_2 = Z$ *and for* $X = e_1 = W$, $Y = e_3 = Z$ *we get the first and the second equation of the system (2.3.18), the rest of the steps are the same.*

2.4 A. Ros' formulae and pinching in compact case

In this section results concerning the classification of totally real submanifolds of S^6 by means of pinched curvature are presented. In order to proceed we need some results on the integration on compact manifolds.

Let M be a compact Riemannian manifold, UM its unit tangent bundle, UM_p the fibre of M over the point $p \in M$ and we denote by dp, du, du_p the canonical measures on M, UM, UM_p respectively. For any continuous function $f : UM \longrightarrow \mathbb{R}$ and any k-covariant tensor field T on M, we have (see:[R])

$$\int_{UM} f du = \int_M \left[\int_{UM_p} f du_p \right] dp \tag{2.4.1}$$

$$\int_{UM} (\nabla T)(u,u,.....u)\, du = 0 \tag{2.4.2}$$

$$\int_{UM} \left[\sum_{i=1}^{n} (\nabla T)(e_i,e_i,u,....,u) \right] dux = 0 \tag{2.4.3}$$

where $\{e_1,.....,e_n\}$ is an orthonormal basis of the the tangent space and ∇ denotes the Riemannian connection on M.

Remark 7 *The above formulae play a crucial role in the rest of the chapter and provide a powerful tool in obtaining results in global Riemannian geometry.*

They are used very often for the deduction of a number of integral relations by defining certain covariant tensor fields and functions on M, calculating their "covariant"derivatives and Laplacians respectively and finally applying the quoted formulae of A. Ros.
Then by a combination of the deduced relations it is proved, in case M is compact, a very useful integral relation involving the covariant derivative of the second fundamental form, the shape operator and the sectional curvature of the manifold.
We do not intend to give here explicitely the proof of all the prerequisite relations which are used in order to get the basic integral relation, but, since in [D.O.V.V1] details regarding the calculations are omitted, and the suggestions given, in particular as far as it concerns the computation of the Laplacian, seem to be obscure, we think that a description in generic lines of the way the Laplacian of a function can be computed is necessary. Furthermore, we intend to give few details concerning the computation of the covariant derivative of tensor fields defined on the unit tangent bundle of M since this is necessary in order to apply the above integral formulae.

We are going to present a method of computing the Laplacian of functions defined on the unit tangent bundle of a Riemannian 3-dimensional manifold.
Let us take u to be an element of the fibre of UM through the point p in the unit tangent bundle and take orthonormal unit vectors $\{e_2, e_3\}$ to be normal to the vector u. Regard $\{e_2, e_3\}$ as an orthonormal basis of $T_u(M_p)$ and the vectors $\{u, e_2, e_3\}$ as an orthonormal basis of the tangent space at the point p. We choose vectors $\{u, e_2, e_3\}$ such that $\mathcal{G}(e_2, e_3) = Ju, \quad \mathcal{G}(e_3, u) = Je_2, \quad \mathcal{G}(u, e_2) = Je_3$ and denote by Δ the Laplacian on the fibre through the point. Then, for any differentiable function f defined on the fibre, we can write:

$$\Delta f = e_2 e_2 f + e_3 e_3 f. \tag{2.4.4}$$

In order to describe a way of computing the Laplacian of a particular function defined in the [D.O.V.V1] (lemma2, pg:744), since the suggestions given seem to be obscure, we need, at first, to discuss briefly some generalities about the differential operators of the **divergence** of a vector field and of the **gradient** of a function ([Car], pg:83-87).
Let M be a Riemannian manifold of dimension n and $p \in M$. It can be shown that there exists a neighborhood $\mathcal{U}$ of the point p and n vector fields $\{E_1, ..., E_n\}$, defined on an open neighborhood and orthonormal at each point of U, such that, the $\nabla_{E_i} E_j(p) = 0$ is

satisfied. Such a family of vector fields will be called a (local) **geodesic frame** at p.

Definition 19 *Let $X \in \mathfrak{X}(M)$ be a vector field and $f \in \mathcal{D}(M)$ a differentiable function on M. We call* **divergence** *of the vector field X a function $divX : M \longrightarrow \mathbb{R}$ given by*
$divX(p) = Trace\{Y(p) \longrightarrow \nabla_Y X(p)\}, \quad \forall p \in M.$
The **gradient** *of f is a vector field* **gradf** *defined on M by:*
$< gradf(p), u >= df_p u \in \mathbb{R}, \quad \forall u \in T_p(M), \quad p \in M.$

Let us consider $\{E_1, \ldots, E_n\}$ to be a geodesic frame defined in a neighborhood of the point $p \in M$. If $X = \sum_{i=1}^{n} f_i E_i$ and f any differentiable function on M, it can be proved that:

$$\begin{aligned} divX\,(p) &= \sum_{i=1}^{n} \left[E_i\,(f_i)\,(p)\right], \\ gradf\,(p) &= \sum_{i=1}^{n} \left[(E_i\,(f))\,E_i\,(p)\right]. \end{aligned} \tag{2.4.5}$$

Let now M denote a Riemannian manifold as above and define on M the following operator:

Definition 20 $\Delta f : \mathcal{D}(M) \longrightarrow \mathcal{D}(M)$ *denotes the* **Laplacian** *of M, given by:*
$\Delta f = divgradf, \quad \forall f \in \mathcal{D}(M).$

Consider a geodesic frame $\{E_1, \ldots, E_n\}$ around the point p and $f \in \mathcal{D}(M)$. It follows from the (2.4.5) that:

$$\Delta f\,(p) = \sum_{i=1}^{n} E_i\,(E_i\,(f))\,(p). \tag{2.4.6}$$

Let us consider the following function ([D.O.V.V1], Lemma 2, pg:744)

$$f\,(u, u) = < h\,(u, u)\,, Ju >^2. \tag{2.4.7}$$

The domain of the function is the fibre in the unit tangent bundle over the point p of a 3-dimensional Riemannian manifold. In order to compute the Laplacian of this function we recall the equation (2.4.4) and under this aspect it will suffice to calculate separetely each term of the equality. We shall describe a method of computing by referring to the first term, for the the second the calculation can be carried out similarly.

By assumption the vectors u, e_2, e_3 form an orthonormal basis of the tangent space of M and moreover satisfy the relations:

$$\mathcal{G}(e_2, e_3) = Ju, \quad \mathcal{G}(u, e_2) = Je_3, \quad \mathcal{G}(e_3, u) = Je_2. \tag{2.4.8}$$

In the unit tangent bundle we consider the following vectors:

$$u(\theta) = cos(\theta)\, u + sin(\theta)\, e_2, \quad e_2(\theta) = -sin(\theta)\, u + cos(\theta)\, e_2. \tag{2.4.9}$$

Observe that the first term in the expression (2.4.4) of the Laplacian can be equivalently written:

$$e_2 e_2 f(u) = \frac{d^2}{d\theta^2} \left[< h(u_\theta, u_\theta), Ju_\theta > \right] |_{\theta=0} \tag{2.4.10}$$

and we can verify by straightforward calculation that

$$\begin{aligned} &\frac{d}{d\theta} \left[< h(u_\theta, u_\theta), Ju_\theta > \right] |_{\theta=0} = 3 \cdot < h(u, u), Je_2 > \\ &\frac{d^2}{d\theta^2} \left[< h(u_\theta, u_\theta), Ju_\theta > \right] |_{\theta=0} = -3 \cdot < h(u, u), Ju > +6 \cdot < h(u, e_2), Je_2 > . \end{aligned} \tag{2.4.11}$$

On the other hand we can easily compute

$$\begin{aligned} &\frac{d}{d\theta} \left[< h(u_\theta, u_\theta), Ju_\theta >^2 \right] |_{\theta=0} = 2 \cdot \left[\frac{d}{d\theta} \left(< h(u_\theta, u_\theta), Ju_\theta > \right) \right]^2 + \\ &2 \cdot < h(u_\theta, u_\theta), Ju_\theta > \cdot \frac{d^2}{d\theta^2} < h(u_\theta, u_\theta), Ju_\theta > |_{\theta=0}. \end{aligned} \tag{2.4.12}$$

Combining the equations (2.4.11), (2.4.12) we deduce the following expression

$$\begin{aligned} &\frac{d^2}{d\theta^2} \left[< h(u_\theta, u_\theta), Ju_\theta > \right] |_{\theta=0} = 18 \cdot < h(u, u), Je_2 >^2 + \\ &12 \cdot < h(u, e_2), Je_2 > \cdot < h(u, u), Ju > -6 \cdot < h(u, u), Ju >^2 \end{aligned} \tag{2.4.13}$$

which actually is the first term in the equation (2.4.4) expressing the Laplacian. Similarly working we can compute the remaining term $e_3 e_3 f(u)$ and finally we get:

$$\Delta f(u) = 18 \cdot < h(u, u), h(u, u) >^2 - 42 \cdot f(u, u). \tag{2.4.14}$$

Integrating the Laplacian over the compact submanifold UM_p, and recalling Stokes' theorem, we deduce the following integral relation:

$$3 \cdot \int_{UM_p} < h(u, u), h(u, u) >^2 = 7 \cdot \int_{UM_p} < h(u, u), Ju >^2 \tag{2.4.15}$$

According to ([D.O.V.V1], lemma3, pg:744) we define the (0,7)- type tensor field $\mathcal{T}_1$ by setting:

Definition 21 *Let* $T_1(X_1,.....,X_7) =< \mathcal{G}(X_1,X_2), h(X_3,X_4) > \cdot < h(X_5,X_6), JX_7 >$.

Let u be a vector in the unit tangent bundle and, in order to apply the integral formulae of A.Ros, we need to compute $\sum_{i=1}^{3}[(\nabla T_1)(e_i,e_i,e_u,e_u,e_u,e_u,e_u,e_u)]$. We shall carry out in few details only the calculation involving the first term of the summation, since the other two terms behave similarly.

From the definitions of the tensor field T_1 and of the covariant derivative of an (0,q)-type tensor field we have:

$$\nabla_{e_1}\left[T_1\left(e_1,u,\ldots,u\right)\right] = \nabla_{e_1}\left[< \mathcal{G}\left(e_1,u\right), h\left(u,u\right) >< h\left(u,u\right), Ju >\right] =$$

$$\left[\nabla_{e_1} < \mathcal{G}\left(e_1,u\right), h\left(u,u\right) >\right] < h\left(u,u\right), Ju > +$$

$$< \mathcal{G}\left(u,u\right), h\left(u,u\right) > \left[\nabla_{e_1} < h\left(u,u\right), Ju >\right] =$$

$$< \nabla^{\perp}_{e_1}\mathcal{G}\left(e_1,u\right), h\left(u,u\right) >< h\left(u,u\right), Ju > +$$

$$< \mathcal{G}\left(e_1,u\right), \nabla^{\perp}_{e_1}h\left(u,u\right) >< h\left(u,u\right), Ju > +$$

$$< \mathcal{G}\left(e_1,u\right), h\left(u,u\right) >< \nabla^{\perp}_{e_1}h\left(u,u\right), Ju > +$$

$$< \mathcal{G}\left(e_1,u\right), h\left(u,u\right) >< h\left(u,u\right), \nabla^{\perp}_{e_1}Ju > . \qquad (2.4.16)$$

$$T_1\left(\nabla_{e_1}e_1,u,\ldots,u\right) =< \mathcal{G}\left(\nabla_{e_1}e_1,u\right), h\left(u,u\right) >< h\left(u,u\right), Ju > . \qquad (2.4.17)$$

$$T_1\left(e_1,\nabla_{e_1}u,u,\ldots,u\right) =< \mathcal{G}\left(e_1,\nabla_{e_1}u\right), h\left(u,u\right) >< h\left(u,u\right), Ju > . \qquad (2.4.18)$$

$$2T_1\left(e_1,u,\nabla_{e_1}u,u,\ldots,u\right) =< \mathcal{G}\left(e_1,u\right), h\left(\nabla_{e_1}u,u\right), Ju > . \qquad (2.4.19)$$

$$2T_1\left(e_1,u,u,u,\nabla_{e_1}u,,u,u\right) =< \mathcal{G}\left(e_1,u\right), h\left(\nabla_{e_1}u,u\right), Ju > . \qquad (2.4.20)$$

$$T_1\left(e_1,u,\ldots,u,\nabla_{e_1}u\right) =< \mathcal{G}\left(e_1,u\right), h\left(u,u\right) >< h\left(u,u\right), J\nabla_{e_1}u > . \qquad (2.4.21)$$

From the above relations, by simply considering common factors, we deduce:

$$
\begin{aligned}
&(\nabla T_1)(e_1, e_1, u, \ldots, u) = < \nabla^{\perp}_{e_1} \mathcal{G}(e_1, u) - \mathcal{G}(\nabla_{e_1} e_1, u) - \mathcal{G}(e_1, \nabla_{e_1} u), h(u,u) > \\
&< h(u,u), Ju > + < \mathcal{G}(e_1, u), \nabla^{\perp}_{e_1} h(u,u) - 2h(\nabla_{e_1} u, u) >< h(u,u), Ju > + \\
&< \mathcal{G}(e_1, u), h(u,u) >< \nabla^{\perp}_{e_1} h(u,u) - 2h(\nabla_{e_1} u, u), Ju > + \\
&< \mathcal{G}(e_1, u), h(u,u) >< h(u,u), \nabla^{\perp}_{e_1} Ju - J\nabla_{e_1} u >= \\
&< (\nabla_{e_1} \mathcal{G})(e_1, u), h(u,u) >< h(u,u), Ju > + \\
&< \mathcal{G}(e_1, u), (\nabla_{e_1} h)(u,u) >< h(u,u), Ju > + \\
&< \mathcal{G}(e_1, u), h(u,u) >< h(u,u), \mathcal{G}(e_1, u) > . \qquad (2.4.22)
\end{aligned}
$$

Recalling the equation (2.2.3) and repeating the same steps for the covariant derivative of T_1 with respect to e_2 and e_3 we finally get the following expression:

$$
\begin{aligned}
&\sum_{i=1}^{3} [(\nabla T_1)(e_i, e_i, u, u, u, u, u, u)] = \\
&< h(u,u), h(u,u) > -2 \cdot < h(u,u), Ju >^2 + \\
&+ \sum_{i=1}^{3} [< (\nabla)(e_i, u, u), Ju > \cdot < \mathcal{G}(e_i, u), h(u,u) >] . \qquad (2.4.23)
\end{aligned}
$$

Note: We could alternatively, in order to simplify the calculations, start by proving that the above sum does not depend on the particular choice of the basis and then assume that u is an element of a basis $\{e_1, e_2, e_3\}$ satisfying
$\mathcal{G}(e_1, e_2) = Je_3, \quad \mathcal{G}(e_2, e_3) = Je_1, \quad \mathcal{G}(e_3, e_1) = e_2, \quad h(e_1, e_1) = \alpha_1 Je_1.$
If we impose $u = e_1$, where e_1 is choosen to be an eigenvector of the shape operator in the direction $J\bar{e_1}$, then, follows from the method of proof of theorem (12), that $h(e_1, e_1) =< h(e_1, e_1), Je_1 > Je_1$ and this simplifies considerably the calculations.

Integrating (2.4.23) over the unit tangent bundle recalling the 2.4.1 integral formula of A. Ros and using the (2.4.15) we obtain:

$$
\begin{aligned}
&\int_{UM} \sum_{i=1}^{3} (< (\nabla h)(e_i, u, u), Ju > \cdot < \mathcal{G}(u, e_i), (u,u) >) = \\
&\frac{1}{3} \cdot \int_{UM} \left[< h(u,u), Ju >^2 \right] . \qquad (2.4.24)
\end{aligned}
$$

The basic integral relation. On UM we define the following function:

$$
g(u) =< h(u,u), Ju > \cdot < (\nabla_{e_2} h)(u,u,u,u), Ju > . \qquad (2.4.25)
$$

Moving on the same line, as in the previous case, we compute the Laplacian of the function g recalling the definition of the second order covariant derivative of the second fundamental form

$$\left(\nabla^2 h\right)(u,u,u,u) = \nabla^{\perp}_u (\nabla h)(u,u,u) - (\nabla h)(\nabla_u u,u,u) -$$
$$(\nabla h)(u,\nabla_u u,u) - (\nabla h) \quad (u,u,\nabla_u u,u). \tag{2.4.26}$$

Next we describe a way to obtain integral relations involving the terms of the equation (2.4.25). Define the (0,6)-type tensor field:

$$T_2(X_1,....,X_6) = < h(X_1,X_2), JX_3 > \cdot < h(X_4,X_5), JX_6 > . \tag{2.4.27}$$

It is not hard to see, following the method previously described, that the relation below holds:

$$\left(\nabla^2 T_2\right)(u,u,u,u,u,u,u,u) =$$
$$2\cdot < \left(\nabla^2 h\right)(u,u,u,u), Ju > \cdot < h(u,u) > +2\cdot < (\nabla h)(u,u,u), Ju >^2 . \tag{2.4.28}$$

Integrating over the unit tangent bundle we get, in virtue of equation (2.4.3), the following relation:

$$\int_{UM} < \left(\nabla^2 h\right)(u,u,u,u), Ju > \cdot < h(u,u), Ju > +$$
$$\int_{UM} < (\nabla h)(u,u,u), Ju >^2 = 0 \tag{2.4.29}$$

Define the (0,4)-type tensor field T_3 by setting :

$$T_3(X_1,.....,X_4) = < h(X_1,X_2), h(X_3,X_4) > . \tag{2.4.30}$$

It can be verified that the following holds:

$$\left(\nabla^2 T_3\right)(u,u,u,u,u,u) =$$
$$2\cdot < \left(\nabla^2 h\right)(u,u,u,u), h(u,u) > +2\cdot < (\nabla h)(u,u,u), (\nabla h)(u,u,u) > . \tag{2.4.31}$$

Integrating over UM and using the integral formulae (2.4.1) we get:

$$\int_{UM} < \left(\nabla^2 h\right)(u,u,u,u), h(u,u) > +$$
$$\int_{UM} < (\nabla h)(u,u,u), (\nabla h)(u,u,u) >= 0. \tag{2.4.32}$$

Recall the definition of the tensor field T_2 and compute :

$$\sum_{i=1}^{3}\left(\left(\nabla^2 T_2\right)(e_i,e_i,u,u,u,u,u,u)\right)=2\cdot<\left(\nabla^2 h\right)(e_i,e_i,u,u),Ju>\cdot<h(u,u),Ju>+$$
$$2\cdot<(\nabla h)(u,u,u),(\nabla h)(u,u,u)>. \tag{2.4.33}$$

If we integrate over the $\mathcal{UM}$, reasoning as in the previous cases, we deduce:

$$\int_{\mathcal{UM}}<(\nabla h)(u,u,u),(\nabla h)(u,u,u)>+$$
$$+\int_{\mathcal{UM}}\sum_{i=1}^{3}\left[<\left(\nabla^2 h\right)(e_i,e_i,u,u),Ju>\cdot<h(u,u),Ju>\right]=0. \tag{2.4.34}$$

Recall the definition of the function g. As we have already mentioned we can compute the Laplacian of the function g using the methodology proposed in the beginning of our discussion. After some very long and tedious calculations we get the following result for the Laplacian:

$$(\Delta g)(u)=-72\cdot g(u)+30\cdot<h(u,u),\left(\nabla^2 h\right)(u,u,u,u)>-24\cdot<h(u,u),Ju>^2+$$
$$30\cdot<h(u,u),h(u,u)>-48\cdot\sum_{i=1}^{3}\left[<(\nabla h)(u,u,e_i),Ju>\cdot<h(u,u),\mathcal{G}(u,e_i)>\right]-$$
$$18\cdot R(u,A_{Ju}u,A_{Ju}u,u)+8\cdot\sum_{i=1}^{3}<h(u,u),Ju>\cdot<\left(\nabla^2 h\right)(e_i,e_i,u,u)>. \tag{2.4.35}$$

Integrating Δg over the compact $\mathcal{UM}$, using Stokes' theorem and observing that:

$$\int_{\mathcal{UM}}g(u)=-\int_{\mathcal{UM}}<(\nabla h)(u,u,u),Ju>^2, \tag{2.4.36}$$
$$\int_{\mathcal{UM}}^{-}<h(u,u),\left(\nabla^2 h\right)(u,u,u,u)>=$$
$$-\int_{\mathcal{UM}}<(\nabla h)(u,u,u).(\nabla h)(u,u,u)>, \tag{2.4.37}$$
$$\int_{\mathcal{UM}}<h(u,u),h(u.u)>=\frac{7}{3}\cdot\int_{\mathcal{UM}}<h(u,u),Ju>^2, \tag{2.4.38}$$
$$\int_{\mathcal{UM}}\sum_{i=1}^{3}<(\nabla h)(u,u,e_i),Ju>\cdot<h(u,u),\mathcal{G}(u,e_i)>=$$
$$\frac{1}{3}\cdot\int_{\mathcal{UM}}<h(u,u),Ju>^2, \tag{2.4.39}$$
$$\int_{\mathcal{UM}}\sum_{i=1}^{3}<h(u,u).Ju>\cdot<\left(\nabla^2 h\right)(u,u,e_i,e_i),Ju>=$$
$$-\int_{\mathcal{UM}}<(\nabla h)(u.u.u).(\nabla h)(u.u.u)>, \tag{2.4.40}$$

we get the following equality:

$$72 \cdot \int_{UM} < (\nabla h)(u,u,u), Ju >^2 + 30 \cdot \int_{UM} < h(u,u), Ju >^2 -$$
$$38 \cdot \int_{UM} < (\nabla h)(u,u,u), (\nabla h)(u,u,u) > -18 \cdot \int_{UM} R(u, A_{Ju}u, A_{Ju}u, u) = 0. \quad (2.4.41)$$

Moreover, it is not hard to verify that

$$< (\nabla h)(X,Y,Z), JW >=< (\nabla h)(X,Y,W), JZ > -$$
$$< h(Y,Z), \mathcal{G}(X,W) > - < h(Y,W), \mathcal{G}(X,Z) >, \quad (2.4.42)$$

and writing $\|(\nabla h)(u,u,u)\|^2 =< (\nabla h)(u,u,u), Ju >^2 + < (\nabla h)(u,u,u), Je_2 >^2 +$ $< (\nabla h)(u,u,u), Je_3 >^2$, by using the equation (2.4.42), we obtain:

$$\| (\nabla h)(u,u,u) \|^2 = \sum_{i=1}^{3} < (\nabla h)(e_i,u,u), Ju >^2 -$$
$$2 \cdot \sum_{i=1}^{3} < (\nabla h)(u,u,e_i), Ju > \cdot < h(u,u), \mathcal{G}(u,e_i) > +$$
$$\|h(u,u)\|^2 - < h(u,u), Ju >^2 . \quad (2.4.43)$$

Integrating the expression (2.4.43) over UM . using the equations (2.4.23). (2.4.33) and also the integral formula of A.Ros (2.4.3) we deduce :

$$\int_{UM} \| (\nabla h)(u,u,u) \|^2 = \int_{UM} \sum_{i=1}^{3} < (\nabla h)(e_i,u,u), Ju >^2 . \quad (2.4.44)$$

As the final step let us define the function $\mathcal{K}(u) =< (\nabla h)(u,u,u), Ju >^2$ on the unit tangent bundle and compute its Laplacian obtaining

$$(\Delta \mathcal{K})(u) = -72 \cdot \mathcal{K}(u) + 2 \cdot \| (\nabla h)(u,u,u) \|^2 + 30 \cdot \sum_{i=1}^{3} < (\nabla h)(e_i,u,u), Ju >^2 -$$
$$12 \cdot \sum_{i=1}^{3} [< (\nabla h)(e_i,u,u), Ju > \cdot < \mathcal{G}(u,e_i), h(u,u) >]. \quad (2.4.45)$$

An integration over the compact M and using the equations (2.4.39) and (2.4.44) gives

$$\int_{UM} \sum_{i=1}^{3} < (\nabla h)(e_i,u,u), Ju >^2 =$$
$$\frac{9}{4} \cdot \int_{UM} < (\nabla h)(u,u,u), Ju >^2 + \frac{1}{12} \cdot \int_{UM} < h(u,u), Ju >^2 . \quad (2.4.46)$$

Applying the relations (2.4.44), (2.4.46) and compairing their terms we deduce the following equation

$$\int_{UM} \| (\nabla h)(u,u,u) \|^2 =$$
$$\frac{9}{4} \cdot \int_{UM} < (\nabla h)(u,u,u), Ju >^2 + \frac{3}{4} \cdot \int_{UM} < h(u,u), Ju >^2 . \qquad (2.4.47)$$

Applying once more the equation (2.4.47) in equation (2.4.41) we get :

$$\frac{3}{4} \cdot \int_{UM} < (\nabla h)(u,u,u), Ju >^2 - \frac{1}{12} \cdot \int_{UM} < h(u,u), Ju >^2 +$$
$$\int_{UM} R(u, A_{Ju}u, A_{Ju}u, u) = 0. \qquad (2.4.48)$$

Lemma 10 *The following relation holds:*

$$\frac{3}{4} \int_{UM} < (\nabla h)(u,u,u), Ju >^2 +$$
$$\int_{UM} \left[R(u, A_{Ju}u, A_{Ju}, u) - \frac{1}{16} \left(\|A_{Ju}u\|^2 - < A_{Ju}u, u >^2 \right) \right] = 0.$$

Proof : In order to obtain this equation it is enough to apply equations (2.4.15) and (2.4.33) in equation (2.4.35).

We can now state and prove a proposition from which, almost straighforwardly, we shall obtain the main result of this section.

Proposition 3 *Let M be a totally real 3-dimensional, compact submanifold of the nearly Kaehler 6-sphere. Suppose that all its sectional curvatures satisfy the inequality* $\mathbf{K \geq 1/16}$. *Then* for all $u \in UM$,

(1) $< (\nabla h)(u,u,u), Ju >= 0.$

(2) $R(u, A_{Ju}u, A_{Ju}u, u) = \frac{1}{16} \cdot [\|A_{Ju}u\|^2 - < A_{Ju}u, u >^2]$.

Proof :([D.O.V.V1], pg:748) By assumption $K \geq \frac{1}{16}$ therefore, from the definition of the sectional curvature we get: $R(u, A_{Ju}u, A_{Ju}u, u)/(\|A_{Ju}\|^2 - < A_{Ju}u, u >^2) \geq \frac{1}{16}$ and the assertion follows directly from lemma (10).

Let us now state and prove the main result of this section.

Theorem 13 *Let M be a totally real 3-dimensional submanifold of S^6 and suppose that all the sectional curvatures of M satisfy the inequality $\mathbf{K} > \frac{1}{16}$. Then $K \equiv 1$ on M.*

Proof:([D.O.V.V1], pg:748) By assumption $K > \frac{1}{16}$ and using **(2)** of the proposition (3), if $A_{Ju}u$ is not parallel to u, then $span\{u, A_{Ju}u\}$ is a 2 - dimensional plane of sectional curvature $\frac{1}{16}$ and this is a contradiction. By the contradiction we get that all the $A_{Ju}u$ have to be parallel to the vectors u. But, as we have seen (2.2.9), in the beginning of the chapter, the equality $A_{Ju}u = -Jh(u,u)$ holds. This equality together with the fact that $A_{Ju}u$ is parallel u imply that:
$\|h(u,u)\|^2 =< h(u,u), Ju >^2, \quad \forall u \in UM_p, \quad \forall p \in M$
and recalling the equation (2.4.15) we see that the second fundamental form must vanish on M. The assertion is proved.

In the last part of this section section we give a first example of a 3-dimensional, totally real, totally geodesic submanifold of S^6.

Example 14 *Let $M = \{\mathbf{x} \in S^6 : \mathbf{x} = x_1 \cdot e_1 + x_3 \cdot e_3 + x_5 \cdot e_5 + x_7 \cdot e_7\}$.*
The immersion $j : M \hookrightarrow S^6$, where $\mathbf{j}$ denotes the natural injection, it is a totally real, totally geodesic immersion of the 3-dimensional submanifold M. The submanifold M is actually the intersection of S^6 with the coassociative 4-plane spanned by $\{e_1, e_3, e_5, e_7\}$ and this intuitive approach to the construction explains the why the submanifold under consideration is totally real and totally geodesic. In order to prove that M is actually a totally real submanifold of the 6-sphere it will be enough to consider $p \in M$, $p = \sum_{i=1}^{7} p_i e_i$, $v \in T_pM : v = v_1e_1 + v_3e_3 + v_5e_5 + v_7e_7$ such that $< p, v >= 0$, and compute J_pv.

2.5 Classification in compact case for sectional curvature satisfying $K \geq \frac{1}{16}$

The result which is going to be presented in this section can be found in [D.V.V2]. A lot of the details included in [D.V.V2] will be omitted since, in certain cases, the way information about the second fundamental form is obtained is based in sorting out the solutions of a large system of equations which, on its turn, is formed by applying the Gauss equation and the conditions of the proposition (3).

Furthermore, another crucial point of the whole discussion consists in choosing a special orthonormal frame, in order to compute the second fundamental form, and consequently the connection, and this choice is based on applying the method of **maximalization**. Since we have already analyzed this method, details will not be given.

Let M be a totally real, compact, 3-dimensional submanifold of S^6 and $p \in M$. On the fibre of the unit tangent bundle through the point p we define the function:

$$f_1(u) = < h(u,u), Ju > . \tag{2.5.1}$$

The set $\mathcal{U}M_p$ is compact, therefore we can suppose that f_1 attains its maximum at a point $u \in \mathcal{U}M_p$. We know that for any point $w \in \mathcal{U}M_p$ such that $< u, w >= 0$ it will be: $< h(u,u), Jw >= 0$ and at each point $p \in M$ we choose e_1 to be the point of maximum for the function f_1. Denote by f_2 the restriction of f_1 to the orthogonal complement of e_1 in the unit tangent bundle at each point p of M. If the restriction is identically zero we take e_2 to be an eigenvector of the shape operator A_{Je_1}, otherwise we choose e_2 to be the point where f_2 attains its absolute maximum. Finally we consider e_3 to be a unit vector satisfying the condition $\mathcal{G}(e_1, e_2) = e_3$.

Using the minimality of M and the symmetry of $< h(X,Y), JZ >$, it is easy to see that the second fundamental form can be expressed in the following way:

$$\begin{aligned}
&h(e_1,e_1) = a \cdot Je_1, \quad h(e_2,e_2) = b \cdot Je_1 + d \cdot Je_3, \\
&h(e_3,e_3) = -(a+b) \cdot Je_1 - d \cdot Je_2, \quad h(e_1,e_2) = b \cdot Je_2 + c \cdot Je_3 \\
&h(e_1,e_3) = c \cdot Je_2 - (a+d) \cdot Je_3, \quad h(e_2,e_3) = c \cdot Je_1 - d \cdot Je_3,
\end{aligned} \tag{2.5.2}$$

where $a \geq b \geq 0$ (since maximum conditions have been imposed on e_1, e_2) and $c, d \in \mathbb{R}$.

At this point we shall simply state the following lemma which gives the possible values of the coefficients of the second fundamental form. The method used in ([D.V.V2], pg:572-574) is based in recalling proposition (3), imposing on that all the sectional curvatures of M satisfy the inequality $K \geq \frac{1}{16}$, applying this together with the Gauss equation and finally decomposing any vector of the tangent space at the point p, with respect to the choosen special orthonormal basis.
Having resolved a large system of equations, the following is obtained :

Lemma 11 *If M is a totally real, compact, 3-dimensional submanifold of S^6 and all its sectional curvatures satisfy the $K \geq \frac{1}{16}$, then at each point $p \in M$ there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM such that either: $h \equiv 0$ on M (i.e: M is totally geodesic in S^6), or*

$$h(e_1, e_1) = \frac{\sqrt{5}}{2} \cdot Je_1, \quad h(e_2, e_2) = -\frac{\sqrt{5}}{2} \cdot Je_1 + \frac{\sqrt{10}}{4} \cdot Je_2$$
$$h(e_3, e_3) = -\frac{\sqrt{5}}{4} \cdot Je_1 - \frac{\sqrt{10}}{4} \cdot Je_2, \quad h(e_1, e_2) = -\frac{\sqrt{5}}{4} \cdot Je_2$$
$$h(e_1, e_3) = -\frac{\sqrt{5}}{4} \cdot Je_3, \quad h(e_2, e_3) = -\frac{\sqrt{10}}{4} \cdot Je_3, \tag{2.5.3}$$

or

$$h(e_1, e_1) = \frac{\sqrt{5}}{2} \cdot Je_1, \quad h(e_2, e_2) = -\frac{\sqrt{5}}{4} \cdot Je_1$$
$$h(e_3, e_3) = -\frac{\sqrt{5}}{4} \cdot Je_1, \quad h(e_1, e_2) = -\frac{\sqrt{5}}{4} \cdot Je_2$$
$$h(e_1, e_3) = -\frac{\sqrt{5}}{4} \cdot Je_3, \quad h(e_2, e_3) = 0. \tag{2.5.4}$$

Proposition 4 *Let M be a totally real, compact, 3-dimensional submanifold of S^6 such that all its sectional curvatures satisfy $K \geq \frac{1}{16}$. Then the following cases can occur:*

(1) $K(p) = 1$ *if $h \equiv 0$ on M.*

(2) $K(p) = \frac{1}{16}$, *if the (2.5.3) hold.*

(3) $\frac{1}{16} \leq K(p) \leq \frac{21}{16}$, *if the (2.5.4) hold,*

where the values $\frac{1}{16}$ and $\frac{21}{16}$ are actually obtained.

Proof: In the first case the point p is a totally geodesic point and an easy application of the Gauss equation gives $K(p) = 1$.

In the second case we observe that h_p has the same form as in theorem(11), relative to the classification by means of constant sectional curvature $K = \frac{1}{16}$ (it is enough to substitute the value of $c = \frac{1}{16}$ in the system (2.3.24)). Thus, $K(p) = \frac{1}{16}$ in virtue of the Gauss equation.

In the third case, by employing once more the Gauss equation, the (2.5.4) and straight-

forward computation, we can verify that:

$$R(e_1,e_2)\,e_2 = R(e_1,e_3)\,e_3 = \frac{1}{16}\cdot e_1, \quad R(e_2,e_3)\,e_3 = \frac{21}{16}\cdot e_2$$
$$R(e_1,e_2)\,e_3 = R(e_2,e_3)\,e_1 = R(e_3,e_1)\,e_2 = 0. \tag{2.5.5}$$

Let us consider any plane section π of the tangent space T_pM with an orthonormal basis $\{X,Y\}$, such that:

$$X = cos\theta e_2 + sin\theta e_3, \quad Y = sin\phi e_1 - cos\phi sin\theta e_2 + cos\phi\cdot cos\theta e_3.$$

By direct calculation we see that:
$R(X,Y,X,Y) = \frac{1}{16} + \frac{20}{16}\cdot cos^2(\phi) \Longrightarrow K(\pi) = \frac{1}{16} + \frac{20}{16}\cdot cos^2(\phi) \Longrightarrow \frac{1}{16} \leq K \leq \frac{21}{16}$.
where the value $\frac{1}{16}$ is attained when $cos\phi = 0$ (ie: when π passes through e_1) and the value $\frac{21}{16}$ is attained for $\|cos\phi\| = 1$ (ie: when $\pi = span\{e_2, e_3\}$). The proof is completed. Observing that any plane section of M may either pass through e_1 or coincide with $span\{e_1, e_2\}$ we deduce, using proposition (4), the following

Corollary 2 *If M is a totally real, compact, 3-dimensional submanifold of S^6 and all its sectional curvatures satisfy either $\frac{1}{16} \leq K \leq 1$ or $\frac{1}{16} \leq K < \frac{21}{16}$ then:*
either $K \equiv 1$ and M is totally geodesic, or $K = \frac{1}{16}$ on M.

Let us now consider the third case of the proposition (4).

Proposition 5 *Let M be a totally real, compact, 3-dimensional submanifold of S^6 , of non constant sectional curvature satisfying the inequality $K \geq \frac{1}{16}$. Then, there exists a global tangent vector field E_1 and local tangent vector fields E_2, E_3 such that:*

(1) *$\{E_1, E_2, E_3\}$ is a local orthonormal frame such that $\mathcal{G}(E_2, E_3) = JE_1$.*

(2) *$\forall p \in M : f_1(u) =< h(u,u), Ju >$ attains its maximum at $E_1(p)$.*

(3) $h(E_1,E_1) = \frac{\sqrt{5}}{2}\cdot JE_1, \quad h(E_2,E_2) = h(E_3,E_3) = -\frac{\sqrt{5}}{4}\cdot JE_1,$
$h(E_1,E_2) = -\frac{\sqrt{5}}{4}\cdot JE_2, \quad h(E_2,E_2) = 0, \quad h(E_1,E_3) = -\frac{\sqrt{5}}{4}\cdot JE_3.$

(4) $\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3 = 0.$
$\nabla_{E_1}E_2 = -\frac{11}{4}\cdot E_3, \quad \nabla_{E_2}E_1 = \frac{1}{4}\cdot E_3, \quad \nabla_{E_1}E_3 = \frac{11}{4}\cdot E_2,$
$\nabla_{E_3}E_1 = -\frac{1}{4}\cdot E_2, \quad \nabla_{E_2}E_3 = -\nabla_{E_3}E_2 = -\frac{1}{4}\cdot E_1.$

Proof: (see:[D.V.V2], pg:576) Since $K \geq \frac{1}{16}$ and K is not constant it follows from the proposition (5) and from lemma (11) that the maximum point of f_1 defines globally on M a vector field.

Let $\{E_2, E_3\}$ be locally defined vector fields, orthogonal to E_1. By a change of sign of E_3, if necessary, and by using lemma(11) once more we see that the first three conditions of the proposition are satisfied. It remains to determine the connection.

Using the orthonormality of the vector fields and the compatibility of the Riemannian metric, it is easy to check that the connection can be written in the form:

$$\begin{aligned}
\nabla_{E_1} E_1 &= a_{12} E_2 + a_{13} E_3, \quad \nabla_{E_2} E_2 = a_{21} E_1 + a_{23} E_3 \\
\nabla_{E_3} E_3 &= a_{31} E_1 + a_{32} E_2, \quad \nabla_{E_1} E_2 = -a_{12} E_1 + a_{11} E_3 \\
\nabla_{E_2} E_1 &= -a_{21} E_2 + a_{22} E_2, \quad \nabla_{E_3} E_1 = -a_{31} E_3 + a_{33} E_2
\end{aligned} \tag{2.5.6}$$

where the $\{a_{ij}\}$ are locally defined functions. Recalling (2.2.8) of lemma (9), by using the Codazzi equations

$$(\nabla h)(E_1, E_2, E_1) = (\nabla h)(E_2, E_1, E_1), \quad (\nabla h)(E_1, E_3, E_1) = (\nabla h)(E_3, E_1, E_1), \tag{2.5.7}$$

we get : $a_{12} = a_{21} = a_{13} = a_{31} = 0$ and $a_{22} = \frac{1}{4}$, $a_{33} = -\frac{1}{4}$.

By a direct substitution, of the found values for the a'_{ij} s, the system 2.5.6 becomes

$$\begin{aligned}
&\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_2 = a_{23} E_3, \quad \nabla_{E_3} E_3 = a_{32} E_2 \\
&\nabla_{E_1} E_2 = a_{11} E_3, \quad \nabla_{E_2} E_1 = \frac{1}{4} \cdot E_2, \quad \nabla_{E_3} E_1 = -\frac{1}{4} \cdot E_2,
\end{aligned} \tag{2.5.8}$$

and on the other hand, direct computation gives:

$R(e_1, e_2, e_2, e_1) = R(e_1, e_3, e_3, e_2) = 0$. $\quad R(e_2, e_3, e_3, e_2) = \frac{21}{16}$.

Combining the last relations for the curvature tensor, the values obtained for the coefficients a_{ij} and applying Gauss equation, we obtain:

$$\begin{aligned}
E_1(a_{23}) - E_2(a_{11}) + a_{32} \cdot \left(a_{11} - \frac{1}{4}\right) &= 0, \\
E_1(a_{32}) - E_3(a_{11}) + \left(\frac{1}{4} - a_{11}\right) \cdot a_{23} &= 0, \\
E_2(a_{32}) + E_3(a_{23}) - \frac{1}{2} \cdot a_{11} - a_{23}^2 - a_{32}^2 &= \frac{22}{16}.
\end{aligned} \tag{2.5.9}$$

If θ is an arbitrary function defined locally on M we set:

$$U_1 = E_1, \quad U_2 = cos\theta E_2 + sin\theta E_3, \quad U_3 = -sin\theta E_2 + cos\theta E_3. \tag{2.5.10}$$

It is easy to see that $\{U_1, U_2, U_3\}$ satisfy the first three required conditions. Hence it remains to search for a basis satisfying the last two conditions concerning the connection. We retain the expression given for the $\{U_i\}$ in terms of the local function θ and of the $E_i's$. Let us impose that the $U_i's$ satisfy the required conditions on the connection.
Recalling standard properties of the connection, the definition (2.5.10) of the $U_i's$, the conditions (2.5.8) on the connection and observing that, for instance, we can write:

$$\begin{aligned} &\nabla_{E_2} cos\,(\theta) \cdot E_2 = cos\,(\theta) \cdot \nabla_{E_2} E_2 + E_2\,(cos\,(\theta)) \cdot E_2 = \\ &E_2\,(cos\,(\theta)) \cdot E_2 = -sin\,(\theta) \cdot E_2\,(\theta) \cdot E_2, \end{aligned} \tag{2.5.11}$$

where $E_2(\theta) = d\theta(E_2)$, we see that if the U_i satisfy the last two conditions of the proposition under discussion, then the following must hold:

$$d\theta\,(E_1) + a_{11} + \frac{11}{4} = 0, \quad d\theta\,(E_2) + a_{23} = 0, \quad d\theta\,(E_3) - a_{32} = 0. \tag{2.5.12}$$

Conversely, if the function θ satisfies the system (2.5.12) then the $\{U_i's\}$ satisfy the conditions on the connection. The system (2.5.12) has locally a solution if and only if the differential form:

$$\omega = (a_{11} + \frac{11}{4}) \cdot \theta_1 + a_{23} \cdot \theta_2 + a_{32} \cdot \theta_3$$

is a closed form, where $\{\theta_1, \theta_2, \theta_3\}$ is the dual of $\{E_1, E_2, E_3\}$.
In order to make clear the last conclusion, given that further explanations are omitted in [D.V.V2], it will be enough to observe that the local existence of a function θ, which is a solution of the (2.5.12), is equivalent to the following system :

$$\begin{aligned} &E_1\,(\theta) = -\left(a_{11} + \frac{11}{4}\right) = -\left(a_{11} + \frac{11}{4}\right) \cdot \theta_1\,(E_1)\,, \\ &E_2\,(\theta) = a_{23} = a_{23} \cdot \theta_2\,(E_2)\,, \quad E_3\,(\theta) = a_{32} = a_{32} \cdot \theta_3\,(E_3)\,. \end{aligned} \tag{2.5.13}$$

On the other hand, the assumption of the differential form ω being closed (i.e. $d\omega = 0$), is equivalent to the existence of the relations (2.5.9). Indeed:

$d\omega = d\{(a_{11}+\frac{11}{4})\cdot\theta_1+a_{23}\cdot\theta_2-a_{32}\cdot\theta_3\}$ and as an indication of the way the process works we shall give details ommited in [D.V.V2]. Recalling the (1.4.2) from Chapter 1 and imposing that ω is, in our case, a closed form, is equivalent to: $d\omega(E_i, E_j) = 0. \quad \forall i, j = 1, \ldots, n$ with $i \neq j$. As an application we compute:

$$
\begin{aligned}
&d\omega(E_1, E_2) = E_1(\omega(E_2)) - E_2(\omega(E_1)) - \omega([E_1, E_2]) = \\
&E_1(a_{23}) - E_2\left(a_{11} + \frac{11}{4}\right) - \omega(\nabla_{E_1}E_2 - \nabla_{E_2}E_1) = \\
&E_1(a_{23}) - E_2(a_{11}) - \omega\left(a_{11}\cdot E_3 - \frac{1}{4}\cdot E_2\right) = \\
&E_1(a_{23}) - E_2(a_{11}) - \frac{1}{4}\cdot a_{23} - a_{32}\cdot a_{11} = 0. \qquad (2.5.14)
\end{aligned}
$$

Working similarly we can complete the proof of the proposition.

Examples of totally real, 3 - dimensional, compact submanifolds of S^6, satisfying $K \geq \frac{1}{16}$.

Example 15 *Let* $S^3 = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$ *and consider the vector fields*

$X_1(y_1, y_2, y_3, y_4) = (y_2, -y_1, y_4, -y_3)$,

$X_2(y_1, y_2, y_3, y_4) = (y_3, -y_4, -y_1, y_2)$,

$X_3(y_1, y_2, y_3, y_4) = (y_4, y_3, -y_2, -y_1)$, *which form a basis of* $\mathfrak{X}(S^3)$. *It is easy to check that:*

$[X_1, X_2] = 2\cdot X_3, \quad [X_2, X_3] = 2\cdot X_1, \quad [X_3, X_1] = 2\cdot X_2$.

Define a metric $<,>$ *on* S^3 *by setting:*

$< X_i, X_j >= 0$ *if* $i \neq j$, $< X_1, X_1 >= \frac{4}{9}$, $< X_2, X_2 >= \frac{8}{3}$ $< X_3, X_3 >= \frac{8}{3}$.

If we put $E_1 = \frac{3}{2}\cdot X_1, \quad E_2 = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\cdot X_2, \quad E_3 = -\frac{1}{2}\cdot\sqrt{\frac{3}{2}}\cdot X_3$, *then* $\{E_1, E_2, E_3\}$ *is an orthonormal basis on* S^3. *We denote by* ∇ *the Levi-Civitax connection with respect to the metric* $<,>$. *Using the Koszul's formula:*

$$
\begin{aligned}
< \nabla_X Y, Z > = \frac{1}{2}\cdot\{&X\cdot < Y, Z > + Y\cdot < Z, X > - Z\cdot < X, Y > \\
&- < [Y, Z], X > - < [X, Z], Y > - < [Y, X], Z >\}
\end{aligned}
$$

and the definition of the curvature tensor, by a straightforward computation, we get:

Lemma 12

(1) $\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3 = 0$.

(2) $\nabla_{E_1}E_2 = -\frac{11}{4}\cdot E_3,\quad \nabla_{E_2}E_1 = \frac{1}{4}\cdot E_3,\quad \nabla_{E_1}E_3 = \frac{11}{4}\cdot E_2,\quad \nabla_{E_3}E_1 = -\frac{1}{4}\cdot E_2.$

(3) $\nabla_{E_2}E_3 = -\nabla_{E_3}E_2 = -\frac{1}{4}\cdot E_1,$

(4) $R(E_1,E_2)E_3 = R(E_2,E_3)E_1 = R(E_3,E_1)E_2 = 0.$

(5) $R(E_1,E_2)E_2 = R(E_1,E_3)E_3 = \frac{1}{16}\cdot E_1,\quad R(E_2,E_3)E_3 = \frac{21}{16}\cdot E_2.$

Observing now that

$$R(E_1,E_2)E_2 = \frac{1}{16}\cdot E_1,\quad R(E_1,E_2)E_1 = -\frac{1}{16}\cdot E_2,\quad R(E_1,E_2)E_3 = 0$$
$$R(E_1,E_3)E_1 = -\frac{1}{16}\cdot E_3,\quad R(E_1,E_3)E_3 = \frac{1}{16}\cdot E_1,\quad R(E_1,E_3)E_2 = 0$$
$$R(E_2,E_3)E_2 = -\frac{21}{16}\cdot E_3,\quad R(E_2,E_3)E_3 = \frac{1}{16}\cdot E_2,\quad R(E_2,E_3)E_1 = 0$$

and since the vector fields $\{E_1,E_2,E_3\}$ form a basis on $\mathfrak{X}(S^3)$, by expressing any tangent vector fields X , Y , Z , W with respect to this basis, it is not hard to verify the following

Lemma 13 *If X,Y,Z,W are vector fields on S^3 and $V^\perp$ denotes the orthogonal complement of a vector V with respect to $<,>$, then:*

$$< R(X,Y)W,Z >= \frac{1}{16}\cdot(< X,Z > \cdot < Y,W > - < X,W > \cdot < Y,Z >) +$$
$$\frac{20}{16}\cdot\left(< X^\perp,Z^\perp > \cdot < Y^\perp,W^\perp > - < X^\perp,W^\perp > \cdot < Y^\perp,Z^\perp >\right).$$

In order to prove the above relation we could proceed, according to the suggestion given in ([D.V.V2], pg:578), using the fact that both the members of the equality in lemma (13) are curvature-like and thus, it should be enough to prove that their sectional curvatures are equal. Denote the right hand side by **Q** *, take a plane section π in S^3 and let $\{X,Y\}$ be an orthonormal basis of the plane section, such that:*
$X = cos\theta E_2 + sin\theta E_3,\ Y = sin\phi E_1 - cos\phi sin\phi E_2 + cos\phi cos\theta E_3$, *where* $\phi,\ \theta \in \mathbb{R}$.
It is a matter of direct calculation to verify that:
$R(X,Y,Y,X) = Q(X,Y,Y,X) = \frac{1}{16} + \frac{20}{16}\cdot cos^2(\phi)$.
From the last equation we see that the sectional curvature corresponding to the plane section π is given by: $K(\pi) = \frac{1}{16} + \frac{20}{16}\cdot cos^2(\phi)$ and as an obvious consequence we get: $\frac{1}{16} \le K(\pi) \le \frac{21}{16}$, where the value $\frac{1}{16}$ is attained for any plane section containing E_1 and the value $\frac{21}{16}$ if $\pi = span\{E_2,E_3\}$.

Let $f : S^3 \longrightarrow S^6, \quad (y_1, y_2, y_3, y_4) \longrightarrow (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

be the mapping having as an explicit parametrization:

$$x_1 = \frac{1}{9}\left(5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1\right), \quad x_2 = -\frac{2}{3}y_2,$$
$$x_3 = 2\frac{\sqrt{5}}{9}\left(y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1\right), \quad x_4 = \frac{\sqrt{39}}{\sqrt{2}}\left(-10y_3y_1 - 2y_3 - 10y_2y_4\right),$$
$$x_5 = \frac{\sqrt{15}}{9}\sqrt{2}\left(2y_1y_4 - 2y_4 - 2y_3y_2\right), \quad x_6 = \frac{\sqrt{15}}{9}\sqrt{2}\left(2y_1y_3 - 2y_3 + 2y_2y_4\right),$$
$$x_7 = -\frac{\sqrt{3}}{9}\sqrt{2}\left(10y_1y_4 + 2y_4 - 10y_2y_3\right).$$

The following theorem can be obtained directly from the above parametrization and recalling the way the frame $\{E_1, E_2, E_3\}$ has been constructed.

Theorem 14 *The mapping f is an isometric, totally real embedding satisfying the*

$$h(E_1, E_2) = \frac{\sqrt{5}}{2} \cdot Jf_\star(E_1), \quad \mathcal{G}(f_\star E_1, f_\star E_2) = Jf_\star(E_3)$$

and the frame field $\{E_1, E_2, E_3\}$ satisfies the conditions of the proposition (4).

Remark 8 *In [Mas] K.Mashimo classified the 3-dimensional, compact, totally real submanifolds of S^6, which are obtained as orbits of closed subgroups of G_2 and has proved that one of them has constant sectional curvature $1/16$. The following example is the explicit expression for this case ([D.V.V2], pg:580) and the image of the immersed $S^3(1/16)$ which is given, is nothing but such an orbit.*

Example 16 *Let $S^3(1/16) = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1^2 + y_2^2 + y_3^2 + y_4^2 = 16\}$ and we define*

the mapping: $x : S^3(\frac{1}{16}) \longrightarrow \mathbb{R}^7$ *by setting :*

$$x_1 = \frac{\sqrt{15}}{2^{10}} (y_1y_3 + y_2y_4)(y_1y_4 - y_2y_3)\left(y_1^2 + y_2^2 - y_3^2 - y_4^2\right).$$

$$x_2 = 2^{-12}\left[-\sum_j y_j^6 + 5\sum_{i<j}\left(y_i^2 + y_j^2\right) y_i^2 y_j^2 - 30\sum_{i<j<k} y_i^2 y_j^2 y_k^2\right],$$

$$x_3 = 2^{-10}\left[y_3y_4\left(y_3^2 - y_4^2\right)\left(y_3^2 + y_4^2 - 5y_1^2 - 5y_2^2\right) + y_1y_2\left(y_1^2 - y_2^2\right)\left(y_1^2 + y_2^2 - 5y_3^2 - 5y_4^2\right)\right],$$

$$x_4 = 2^{-12}\left[y_2y_4\left(y_2^4 + 3y_3^4 - y_4^4 - 3y_1^4\right) + y_1y_3\left(y_3^4 + 3y_2^4 - y_1^4 - 3y_4^4\right)\right]$$

$$+\ 2^{-12}\left[(y_1y_3 - y_2y_4)\left(y_1^2\left(y_2^2 + 4y_4^2\right) - y_3^2\left(y_4^2 + 4y_2^2\right)\right)\right],$$

$$x_5 = x_4(y_2, -y_1, y_3, y_4),$$

$$x_6 = 2^{-12}\sqrt{6}\left[y_1y_3\left(y_1^4 + 5y_2x^4 - y_3^4 - 5y_4^4\right) - y_2y_4\left(y_2^4 + 5y_1^4 - y_4^4 - 5y_3^4\right)\right]$$

$$+10.2^{-12}\sqrt{6}\left[(y_1y_3 - y_2y_4)\left(y_3^2y_4^2 - y_1^2y_2^2\right)\right],$$

$$x_7 = x_6(y_2, -y_1, y_3, y_4).$$

By direct computation we can prove the following :

Theorem 15 *The mapping* $x : S^3(\frac{1}{16}) \longrightarrow S^6$ *is a totally real isometric immersion.*

Let $p = (4, 0, 0, 0) \in S^3(\frac{1}{16})$, then $x(p) = (0, -1, 0, \ldots, 0)$ and we shall show that there are exactly 23 other points in $S^3(\frac{1}{16})$ which are mapped onto the same point.
Since $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 16$ it is not hard to see that
$x_2(y_1, y_2, y_3, y_4) = -1 + 2^{-9} \cdot N(y_1, y_2, y_3, y_4)$, where N is given from the relation
$N(y_1, y_2, y_3, y_4) = \sum_{i<j} y_i^2 \cdot y_j^2(y_i^2 + y_j^2) - 3 \cdot \sum_{i<j<k} y_i^2 \cdot y_j^2 \cdot y_k^2$.
In order to prove that the mapping x is actually 24 - fold, it should be enough to show that there exist exactly 24 solutions of the form (y_1, y_2, y_3, y_4), such that: $N(y_1, y_2, y_3, y_4) = 0$ and $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 16$.
By performing the substitution $\lambda_i = y_i^2$ and supposing that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, the condition $N \equiv 0$ is equivalent to

$$(\lambda_1 - \lambda_2)(\lambda_2(\lambda_1 - \lambda_2) + \lambda_3(\lambda_1 - \lambda_4) + \lambda_4(\lambda_1 - \lambda_2)) +$$

$$(\lambda_2 - \lambda_3)(\lambda_2(\lambda_1 - \lambda_4) + \lambda_3(\lambda_2 - \lambda_4) + \lambda_1\lambda_2 - \lambda_3\lambda_4) +$$

$$(\lambda_3 - \lambda_4)(\lambda_1(\lambda_3 - \lambda_1) + \lambda_2(\lambda_3 - \lambda_4) + \lambda_3(\lambda_2 - \lambda_4)) = 0.$$

This equation has the non zero solutions: either $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, \lambda, \lambda, \lambda)$, where $\lambda > 0$ or solutions of the form

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, 0, 0, 0)$. with $\lambda > 0$.

Since N invariant under permutation of $\{y_1, y_2, y_3, y_4\}$ and changing the sign of one or more of the $y_k's$, if necessary, we obtain that the set of the solutions of the equation under discussion is :

$S = \{(4,0,0,0), (-4,0,0,0), (0,4,0,0), \ldots, (0,0,0,-4), (2,2,2,2), (-2,2,2,2), \ldots$

$\ldots, (-2,-2,-2,-2)\}$ and obviously $|S| = 24$.

We need the following lemma (see: [D.V.V2]), which actually is the local version of the Cartan, Ambrose, Hicks theorem ([Wo], pg:30).

Lemma 14 *Let $M^n, \widetilde{M}^n$ be Riemannian manifolds with Levi-Civita connections $\nabla, \widetilde{\nabla}$ respectively. Suppose that there are c_{ij}^k, for $i, j, k \in \{1, 2, \ldots\ldots, n\}$ such that $\forall p \in M, \forall \widetilde{p} \in \widetilde{M}$ there exist orthonormal frame fields $\{E_i\}$ around the point p and $\{\widetilde{E}_i\}$ around $\widetilde{p}$, such that :*

$$\nabla_{E_i} E_j = \sum_k (c_{ij}^k E_k) \quad \text{and} \quad \nabla_{\widetilde{E}_i} \widetilde{E}_j = \sum_k (c_{ij}^k \widetilde{E}_k).$$

Then, for every $p \in M, \widetilde{p} \in \widetilde{M}$ there exists a local isometry $\mathbf{f}$ which maps a neighborhood of p onto a neighbourhood of $\widetilde{p}$ and E_i on $\widetilde{E}_i$.

Let : $x_1 : M_1 \longrightarrow S^6$, $x_2 : M_2 \longrightarrow S^6$, $x_3 : M_3 \longrightarrow S^6$ be the immersions corresponding to the examples (15), (16) and (14) respectively.

Theorem 16 *(see:[D.V.V2], pg:582) Let $x : M^3 \longrightarrow S^6$ be a totally real, isometric immersion of a 3-dimensional complete Riemannian manifold into S^6. If the sectional curvatures K of M satisfy $K \geq \frac{1}{16}$, then either M is simply connected and x is congruent, either to:*

(1) $x_1 : M_1 \longrightarrow S^6$. **ie:** $\frac{1}{16} \leq K \leq \frac{21}{16}$.

(2) $x_3 : M_3 \longrightarrow S^6$. **ie:** $K \equiv 1$.

or $\widetilde{x}$ (the composition of the universal covering map of M with x), is congruent to:

$x_2 : M_2 \longrightarrow S^6$, **ie:** $K = \frac{1}{16}$.

Proof: Let $\widetilde{x} = x \circ \pi$, where π denotes the universal covering map $\pi : \widetilde{M} \longrightarrow M$.

From the Bonnet-Myers theorem we know that, since M is compact, the same is true for

$\widetilde{M}$. From proposition (4) we obtain that either $\widetilde{M}$ is totally geodesic, and in this case $\tilde{x}$ is congruent to x_3, or $\widetilde{M}$ has constant sectional curvature $\frac{1}{16}$,and $\tilde{x}$ is congruent to x_2, or that the sectional curvatures K, of $\widetilde{M}$, vary between $\frac{1}{16}$ and $\frac{21}{16}$.

In the last case, from (4), (5) of the proposition (5), using the (1), (2), (3) of lemma (11) together with lemma (12), we obtain that $\widetilde{M}$ is homogeneous and locally isometric to M_1. But since M_1 is analytic, there will be an isometry between M_1 and $\widetilde{M}$. Therefore there exist, an orthonormal basis $\{E_1, E_2, E_3\}$ of M_1, an orthonormal basis $\{F_1, F_2, F_3\}$ of $\widetilde{M}$, both defined globally and satisfying proposition (5), and an isometry $o : M_1 \longrightarrow \widetilde{M}$, such that:

$$\phi_* E_i = F_i, \quad \forall i = 1, 2, 3.$$

If we denote by ϕ the map between the normal bundles of M_1 and $\widetilde{M}$, defined by $\phi(JE_i) = JF_i$, then we see that o preserves the bundle metric, the second fundamental form and the normal connection. By the rigidity theorem (4) of Chapter 1, $\tilde{x}$ and x_1 are congruent and since x_1 is an embedding, follows that $\tilde{x}$ is an embedding, in the corresponding case, and consequently x_1 is an isometry. The proof is completed.

Chapter 3

Chen's inequality and a Riemannian invariant for submanifolds in space forms

3.1 Introduction

In this chapter we discuss an inequality due to B.Y. Chen ([Ch2]) which involves the main invariants of a submanifold of a space form. This inequality is an improvement of an inequality proved by Chen in [Ch1] and it is actually the best possible, since examples of submanifolds of Euclidean space forms are given for which the inequality becomes an equality. Specifically, Chen proved that:

$$\inf K \geq \frac{1}{2}\left[\tau - \frac{n^2(n-2)}{(n-1)} \cdot \|H\|^2 - (n+1)\cdot(n-2)\cdot c\right], \tag{3.1.1}$$

where τ denotes the scalar curvature of an n-dimensional Riemannian manifold ($n \geq 2$) immersed in an m-dimensional space form $\mathbb{R}^m(c)$ and at each $p \in M$, for π running over the set of all 2 - plane sections in T_pM, the function **infK** is defined by:

$$\inf K(p) = \inf\{K(\pi) : \forall \pi \subset T_pM\}. \tag{3.1.2}$$

Chen defined a new Riemannian invariant of a submanifold M of a space form, namely : $\delta_M : M \longrightarrow \mathbb{R}$, which is given by :

$$\delta_M(p) = \frac{\tau(p)}{2} - \inf K(p). \tag{3.1.3}$$

and in this case Chen's inequality attains the form

$$\delta_M \leq \frac{(n-2)}{2} \cdot \{\frac{n^2}{n-1} \cdot \|H\|^2 + (n+1) \cdot c\}. \tag{3.1.4}$$

We also investigate the situation of a submanifold for which the inequality becomes an equality. In particular, Chapter 3 is structured in the following way:

In (§3.2) we introduce preliminary notation and basic definitions, and give a proof of a relation existing between, scalar curvature, length of the mean curvature and of the second fundamental form, the sectional curvature of the space form and the dimension of the submanifold. An algebraic lemma, which plays a crucial role in obtaining Chen's inequality, will be stated as well.

In (§3.3), after proving the inequality, we focus on the case where the inequality becomes an equality. It will be shown that, by an appropriate choice of the orthonormal frame, the shape operator attains a very "nice" form.

In (§3.4) we further insist in the case of the equality by defining and studying a distribution on the submanifold M. Main reference for the sections (§3.2), (§3,3) and (§3.4) is [Ch2].

In section (§3.5) we consider the case of totally real 3-dimensional submanifolds of S^6 and give the very elementary results about the form the shape operator and the second fundamental form attains, and finally we state and prove an existence and uniqueness result which will be useful for the next chapter.

The last section (§3.6) is devoted to the presentation of some examples of totally real submanifolds of S^6 satisfying the Chen's equality. The further study of the implications of the examples and of their influence in the classification will be emphalsized in the next chapter and in the present section we only deduced some first conclusions. The main reference for the sections (§3.5) and (§3.6) are [C.D.V.V1] and [C.D.V.V2].

3.2 Preliminaries

Let $\mathbb{R}^m(c)$ denote an m-dimensional space form of constant sectional curvature c and M an n-dimensional Riemannian manifold immersed in $\mathbb{R}^m(c)$.

Consider an orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ defined on an open subset of the ambient space such that span $\{e_1, \ldots, e_n\} = T_pM$ at each point $p \in M$. Then, the orthonormal frame will be called an **adapted frame**. If $\{h_{ij}^{\alpha}\}$ are the coefficients of the

second fundamental form, with respect to an adapted orthonormal frame, we can write :

$$h(e_i, e_j) = \sum_{\alpha=n+1}^{m} h_{ij}^{\alpha} e_{\alpha}, \quad H = \frac{1}{n} \cdot \sum_{i=1}^{n} h(e_i, e_i),$$
$$\|h\|^2 = \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{m} \left(h_{ij}^{\alpha}\right)^2, \quad \|H\|^2 = \frac{1}{n^2} \cdot \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{m} h_{ii}^{\alpha} \cdot h_{jj}^{\alpha}. \tag{3.2.1}$$

We are going to prove a relation between the scalar curvature τ of M, the length of the second fundamental form, the length of the mean curvature vector, and the sectional curvature of the ambient space form.

Lemma 15 *Let M be a Riemannian manifold immersed in the m-dimensional space form $\mathbb{R}^m(c)$. Then the following relation holds:*

$$\tau = n^2 \cdot \|H\|^2 - \|h\|^2 + n \cdot (n-1) \cdot c. \tag{3.2.2}$$

Proof : Let $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ be an adapted frame on $\mathbb{R}^m(c)$. We denote by $R, \widetilde{R}$ the curvature operators on M and $\mathbb{R}^m(c)$ respectively and use the notation involving the coefficients of the second fundamental form given by the system (3.2.1).
By hypothesis $\mathbb{R}^m(c)$ is a c-space form and thus $\widetilde{R}(e_i, e_j, e_i, e_j) = c, \quad \forall i \neq j$. On the other hand $\tau = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j)$. Hence, by using the Gauss equation and in the same time keeping in mind that from the n^2 pairs (e_i, e_j) only $n \cdot (n-1)$ of them contribute to the sum giving τ, we obtain (3.2.2).
Recall in this point the following ([Ch1]):

Definition 22 *Let M an n-dimensional submanifold of a Riemannian manifold. An involutive distribution on M will be called a* **foliation** *on M. Let ν be a subbundle of the normal bundle of M. Then ν is said to be a* **parallel normal subbundle** *if and only if: $\nabla_X^{\perp} \xi \in \nu, \quad \forall X \in T_pM, p \in M$ and for any section ξ of the subbundle ν, where $\nabla^{\perp}$ denotes the Levi-Civita connection in the normal bundle of M.*

In order to proceed with the discussion of the main result of this chapter we shall state the following algebraic lemma ([Ch2])

Lemma 16 *Let $a_1, a_2, ..., a_n, c$ be real numbers, $n \geq 2$ and suppose that the equation $(\sum_{i=1}^{n} a_i)^2 = (n-1) \cdot (\sum_{i=1}^{n} a_i^2 + c)$ holds. Then $2a_1a_2 \geq c$ and the equality occurs if and only if: $a_1 + a_2 = a_k, \quad \forall k = 3, 4, ..., n$.*

3.3 Chen's equality and the shape operator

Let M and $\mathbb{R}^m(c)$ be as in the previous section, $\{e_1,\dots,e_m\}$ an adapted orthonormal frame and $\pi = span\{e_1,e_2\}$. Furthermore, let us assume that $H/\|H\| = e_{n+1} \neq 0$, where H denotes the mean curvature vector, and set:

$$\delta = \tau - \frac{n^2\cdot(n-2)}{n-1}\cdot\|H\|^2 - (n+1)\cdot(n-2)\cdot c. \tag{3.3.1}$$

were the δ of the above expression is not the same with the one used in the introduction of the current chapter, but its double. Using the relations, (3.2.2) of lemma (15), (3.2.1) and (3.3.1) we get:

$$n^2\cdot\|H\|^2 = (n-1)\cdot\|h\|^2 + (n-1)\cdot(\delta - 2c)\,. \tag{3.3.2}$$

From the expressions (3.2.1) for the length of the second fundamental form and for the mean curvature vector we obtain:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\cdot\{\sum_{i=1}^{n}\left(h_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{\alpha=n+2}^{m}\left(h_{ij}^{\alpha}\right)^2 + \delta - 2\cdot c\}. \tag{3.3.3}$$

Recalling lemma (16) and applying it we deduce:

$$2\cdot h_{11}^{n+1}\cdot h_{22}^{n+1} \geq \sum_{i\neq j}\left[\left(h_{ij}^{n+1}\right)^2\right] + \sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{m}\left[\left(h_{ij}^{\alpha}\right)^2\right] + \delta - 2\cdot c. \tag{3.3.4}$$

On the other hand it is easy to verify, using the relations (3.2.1), the Gauss equation and the definition of the sectional curvature, that:

$$K(\pi) = c + \sum_{\alpha=n+1}^{m} h_{11}^{n+1}h_{22}^{n+1} - \sum_{\alpha=n+1}^{m}\left(h_{12}^{\alpha}\right)^2. \tag{3.3.5}$$

Combining the relations (3.3.5) and (3.3.4) we get

$$K(\pi) \geq \sum_{\alpha=n+1}^{m}\sum_{j>2}\{\left(h_{1j}^{\alpha}\right)^2 + \left(h_{2j}^{\alpha}\right)^2\} + \frac{1}{2}\cdot\sum_{i\neq j>2}\left(h_{ij}^{n+1}\right)^2 +$$
$$\frac{1}{2}\cdot\sum_{\alpha=n+2}^{m}\sum_{j>2}\left(h_{ij}^{\alpha}\right)^2 + \frac{1}{2}\cdot\sum_{\alpha=n+2}^{m}\left[\left(h_{11}^{\alpha}+h_{22}^{\alpha}\right)^2\right] + \frac{\delta}{2}, \tag{3.3.6}$$

and it is obvious that the inequality $inf K \geq \frac{1}{2}\cdot\delta$ holds.

Now, let us consider the case where the equality occurs in (3.3.6) and choose $\{e_1,e_2\}$ such that the sectional curvature K attains its minimum for the plane section π spanned

by e_1 and e_2. Moreover, we can choose a basis $\{e_1, \ldots, e_n\}$ which diagonalizes the shape operator in the direction e_{n+1}. Under these circumstances, $infK = \frac{\delta}{2}$ and it can be applied the second part of lemma (16).

In order to apply it we first observe that, in case the equality occurs, the inequalities (3.3.4) and (3.3.6) become equalities and the following sequence of equalities must be satisfied:

$$\frac{\delta}{2} = c + h_{11}^{n+1} \cdot h_{22}^{n+1} - \sum_{\alpha=n+2}^{m} (h_{12}^{\alpha})^2,$$

$$\frac{\delta}{2} = \sum_{\alpha=n+1}^{m} \sum_{j>2} \left((h_{1j}^{\alpha})^2 + (h_{2j}^{\alpha})^2 \right) + \frac{1}{2} \cdot \sum_{\alpha=n+2}^{m} \sum_{j>2} (h_{ij}^{n+1})^2 +$$

$$\frac{1}{2} \cdot \sum_{\alpha=n+2}^{m} (h_{11}^{\alpha} + h_{22}^{\alpha})^2 + \frac{\delta}{2}. \tag{3.3.7}$$

The second of the above equations, after the simplification of the quantity $\frac{\delta}{2}$, becomes zero and it is nothing but a sum of squares. Therefore, each of its terms must vanish and thus we deduce the following information about the coefficients of the second fundamental form:

$$h_{i3}^{n+1} = \ldots = h_{in}^{n+1} = 0, \quad h_{i \neq j > 2}^{n+1} = 0$$

$$h_{1j}^{r} = h_{2j}^{r} = h_{ij}^{r} = 0, \quad r = n+2, \ldots, m, \quad i, j = 3, \ldots, n$$

$$h_{11}^{n+2} + h_{22}^{n+2} = \ldots = h_{11}^{m} + h_{22}^{m} = 0, \quad h_{12}^{n+1} = 0. \tag{3.3.8}$$

The first two equations of the system (3.3.8) are obtained using the vanishing of the terms $(h_{ij}^{n+1})^2$ for any $j > 2$, $\alpha = n+2, \ldots, m$ and the symmetry of h. The fourth equation is a consequence of the equalities $(h_{11}^{\alpha} + h_{22}^{\alpha})^2 = 0$, where $\alpha = n+2, \ldots, m$. The third has been deduced by imposing : $(h_{1j}^{r})^2 + (h_{2j}^{r})^2 = 0$, for $r = n+2, \ldots, m$ and $i, j = 3, \ldots, n$. Finally, the fifth is implied by the assumption on the diagonalization of the shape operator by the basis $\{e_1, \ldots, e_n\}$ with respect to the direction $e_{n+1} = H/\|H\|$.

Setting : $a_i = h_{ii}^{n+1}, i = 1, 2, \ldots, n$ we can apply the second part of lemma (16) and obtain:

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}. \tag{3.3.9}$$

Under the aspect of the above discussion and combining (3.3.7), (3.3.8) and (3.3.9) we can state the following ([Ch2], pg:570-571)

Theorem 17 *Let M be an n-dimensional Riemannian manifold immersed into the c-space form $\mathbb{R}^m(c)$. Then the inequality (3.1.1) holds.*
If equality occurs in (3.1.1) there is an adapted orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ such that:

$$\begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix} = A_{n+1}, \quad \text{where} \quad a + b = \mu,$$

and

$$\begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \ldots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix} = A_r, \quad \forall r = n+2, \ldots, m.$$

where A_{n+1} denotes the shape operator in the direction of the normal vector e_{n+1} and A_r in the direction determined by the rest of the vectors $e_{n+2}, \ldots, e_m$ of the normal space.

It is clear that in the case of a 3-dimensional totally real submanifold of S^6, since M has to be minimal, the invariant δ satisfies the $\delta \leq 2$. In particular, the form attained by the shape operators, when the Chen's equality $\delta_M = 2$ occurs, plays a crucial role in the investigation of such submanifolds and this is going to be illustrated in due chapter 4.

3.4 An integrable distribution

In this section we study further the generic case of submanifolds M of a real space form and specifically we focus on the properties of a distribution defined on M when the Chen's equality occurs on M.

Let M be a Riemannian manifold immersed in the real space form $\mathbb{R}^m(c)$ and suppose that at each point $p \in M$ the equality in (3.1.1), i.e: the

$$inf K = \frac{1}{2} \cdot \{\tau - (\frac{n^2 \cdot (n-2)}{(n-1)}) \cdot \|H\|^2 - (n+1) \cdot (n-2) \cdot c\}$$

holds, (where the notation concerns the "initial" δ of the introduction).
If this is the case then, the shape operators attain the form described in theorem (17).
At each point p of M we consider the following subset of the tangent space

Definition 23 $\mathcal{D}(p) = \{X \in T_pM : (n-1) \cdot h(X,Y) = n \cdot H < X,Y >, \quad \forall Y \in T_pM\}$.

Let us assume, from now on, that the dimension of $\mathcal{D}(p)$ does not depend on the point p. Then $\mathcal{D}(p)$ defines a distribution on M.

Remark 9 *The distribution $\mathcal{D}$ defined on M provides information on the existence of totally umbilical submanifolds of M which foliate M. In this case, if k denotes the dimension of the distribution, M will be called a* **k-ruled** *submanifold of the space form and this means that is foliated (ie: an involutive distribution is given on M) by a family of totally geodesic k-dimensional submanifolds. If this is the case then (see:[Ch2]), M will be generated by the motion of a totally geodesic k-dimensional submanifold along an (n-k)-dimensional manifold.*
An analogous situation, probably a motivation for Chen to define the distribution $\mathcal{D}$, is the well known case of the ruled surfaces in the 3-dimensional Euclidean space and in particular we could mention the examples of the cylinder, cone, hyperboloid, hyperbolic paraboloid (saddle), of any ruled surface and more specifically of any developable surface.

Using the information provided by theorem (17), concerning the form attained by the shape operator when Chen's equality occurs on M, we shall study further the distribution $\mathcal{D}$.

- It is evident that:

$$h(e_j, e_j) = \mu \cdot e_{n+1}, \ \forall j = 3, \ldots, n$$
$$h(e_j, e_k) = 0, \ \forall j = 3, \ldots, n, \ \forall k = 1, \ldots, n, \ k \neq j.$$

  Therefore, at each point $p \in M$, the vectors $\{e_3, \ldots, e_n\}$ are elements of $\mathcal{D}(p)$ and thus, $dim\mathcal{D}(p) \geq n-2$.
  If $\mathcal{D}(p) = span\{e_3, \ldots, e_n\}$ then $dim\mathcal{D}(p) = n-2$.

- Suppose that $dim\mathcal{D}(p) = n-1$. If $e_2 \in \mathcal{D}(p)$ then:

$$(n-1)h(e_2,e_2) = n < e_2, e_1 > \cdot H = 0 \Longrightarrow$$
$$h(e_2,e_1) = 0 \Longrightarrow h^r_{12} = h^r_{21} = 0,$$
$$\forall r \in \{n+2, \ldots, m\}$$

But, since $a+b = \mu$ and $e_1 \notin \mathcal{D}(p)$, and moreover $H = (n-1)\cdot\mu\cdot e_{n+1}$, we get that $a = 0, \quad b = \mu$ and further, from the form attained by the shape operators, in this particular case, we obtain $Imh = span\{e_{n+1}\}$.

Let $X, Y, Z \in T_pM$ and use the Codazzi equation (1.2.15, Chapter1) to get:

$$\nabla^\perp_X h(Y,Z) - \nabla^\perp_Y h(X,Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$$
$$- h(\nabla_Y X, Z) - h(X, \nabla_Y Z).$$

If we choose $X, Z \in \mathcal{D}(p)$ to be mutually orthogonal then, by using the definition (23) of $\mathcal{D}(p)$, we see that $h(X,Z) = 0$ and moreover:

$$\nabla^\perp_X h(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$$
$$- h(\nabla_Y X, Z) - h(X, \nabla_Y Z) \in span\{e_{n+1}\},$$

i.e: Imh is a **parallel** normal subbundle of the normal bundle.

- Suppose that $dim\mathcal{D}(p) = n$ ie: $\mathcal{D}(p) = span\{e_1, e_2, \ldots, e_n\}$. If this is the case, then for the coefficients of the second fundamental form, by taking $X, Y \in T_pM$ to be mutually orthogonal and applying the defining property of $\mathcal{D}$, we find that:

$h^r_{ij} = 0, \quad \forall i,j = 1, \ldots, n, \quad \forall r = n+1, \ldots, m.$

In conclusion, M will be a totally geodesic submanifold of the space form $\mathbb{R}^m(c)$ and $\mathcal{D}$ trivially integrable since, $dimImh = 0$ and each integral submanifold of the distribution is totally geodesic.

- Consider again the cases:

$dim\mathcal{D}(p) = n-1, \quad Imh = span\{e_{n+1}\}, \quad \mathcal{D}^\perp(p) = span\{e_1\}$ and

$dim\mathcal{D}(p) = n-2, \quad \mathcal{D}^\perp(p) = span\{e_1, e_2\}.$

Choose $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Then $h(X,Z) = h(Y,Z) = 0$, thus

$$(\nabla_X h)(Y,Z) = -h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$
$$(\nabla_Y h)(X,Z) = -h(\nabla_Y X, Z) - h(X, \nabla_Y Z)$$

and since the torsion is zero we deduce:

$$h([X,Y],Z) = h(\nabla_X Y, Z) - h(\nabla_Y X, Z), \quad \text{with} X, Y \in \mathcal{D} \Longrightarrow$$
$$h([X,Y],Z) = \mu \cdot \{< X, \nabla_Y Z > - < Y, \nabla_X Z >\} \cdot e_{n+1}.$$

Using the Codazzi equation we finally obtain:

$$h([X,Y],Z) = \mu \cdot < [X,Y], Z > \cdot e_{n+1} \Longrightarrow [X,Y] \in \mathcal{D}(p)$$

and it is proved (recalling the trivial case of being M totally geodesic in $\mathbb{R}^m(c)$) that the distribution $\mathcal{D}$ is **integrable**.

- Let us consider a connected component (or equivalently: a maximal integral submanifold) of $\mathcal{D}(p)$. If $dim\mathcal{D}(p) = n$ then each connected component is totally geodesic and consequently will be totally umbilical.
  If $dim\mathcal{D}(p) = n-2$ then $\mathcal{D}(p) = \{e_3, \ldots, e_n\}$, using theorem (17), and for $(k,r) \in \{1, \ldots, n\} \times \{n+2, \ldots, m\}$, we have:

$$A_{e_{n+1}} e_k = \mu e_k, \quad \forall k \quad \text{and} \quad A_{e_r} e_k = 0, \quad \forall k \geq 3, \quad \forall r.$$

  It is obvious that the connected component under consideration is **totally umbilical**.
  If $dim\mathcal{D}(p) = n-1$ then, by interchanging e_1 with e_2 if necessary, we get:

$$\mathcal{D}(p) = span\{e_2, \ldots, e_n\}, \quad h^r_{12} = h^r_{21} = 0, \quad a = 0, \quad b = \mu$$

  and it is easy to check that $\mathcal{D}(p)$ is totally umbilical.

Note: In the discussion of the totally umbilicity of the distribution $\mathcal{D}$ we have modified the process with respect to [Ch2], making a direct use of the form attained by the shape operator. We can state the following:

Lemma 17 *Let M be an n-dimensional submanifold of a real space form $\mathbb{R}^m(c)$. Suppose that the Chen's equality holds and moreover we assume that the dimension of $\mathcal{D}$ does not depend on point $p \in M$. Then, exactly one of the following cases occurs:*

(1) *$\mathcal{D}$ is an (n-2)-dimensional distribution*

(2) $\mathcal{D} = TM$ *and* M *is totally geodesic in* $\mathbb{R}^m(c)$

(3) *The first normal subbundle* Imh *is 1-dimensional and if we choose* $e_{n+1} \in Imh$, *then* A_{n+1} *has exactly two distinct eigenvalues* $0, \mu$ *with multiplicities 1 and n-1 respectively. Thus,* $\mathcal{D}$ *gives rise to an (n-1)-dimensional distribution.*

Furthermore, the distribution $\mathcal{D}$ *is integrable and each connected component of* $\mathcal{D}$ *is a totally umbilical submanifold of the ambient space form.*

Let $M \xrightarrow{i} \widetilde{M}$ be an immersion of the the n-dimensional manifold M. Let $\mathcal{D}$ be an integrable distribution on M and $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$ in the tangent bundle of M. Then:

Lemma 18 $h(\mathcal{D}, \mathcal{D}^\perp) = 0$ *if and only if* $\perp M|_L$ *is a parallel normal subbundle of* L *in* $\widetilde{M}$ *for any* L, *where* L *is a connected component of the distribution* $\mathcal{D}$.

Proof: If $\widehat{A}, \widehat{\nabla}^\perp$ denote the shape operator and the normal connection of the connected component (of $\mathcal{D}$) L in $\widetilde{M}$, then for any $X \in L$, and $Z \in \perp L$ (in M) we have

$$-\widehat{A}_Z X + \widehat{\nabla}^\perp_X Z = \widetilde{\nabla}_X Z = \nabla_X Z + h(X, Z).$$

Therefore, $h(X, Z) = 0$ if and only if $\widehat{\nabla}^\perp_X Z \in \mathcal{D}^\perp$.
It is clear now that, $h(\mathcal{D}, \mathcal{D}^\perp) = 0$ if and only if $\perp L$ (in M) is a parallel normal subbundle of $\perp L$ in $\widetilde{M}$, for any $L \in \mathcal{D}$. But, $\perp M|_L = (\mathcal{D}^\perp|_L)^\perp$ and the assertion is proved.

3.5 Chen's equality, totally real 3-dimensional submanifolds of S^6 and an existence and uniqueness theorem.

In this section we are going to focus on totally real 3-dimensional submanifolds of S^6 satisfying Chen's equality. A certain number of examples and basic facts, which are prerequisites for the discussion of the results of the next chapter, will be given. Infact,

this section can be characterized as an introduction for Chapter 4. The main references are [C.D.V.V1] and [C.D.V.V2].

Let $M \hookrightarrow \widetilde{M}$ be an immersion of the n-dimensional Riemannian manifold M into the m-dimensional manifold space form $\widetilde{M}$ and we assume that the immersion is minimal. In this case $H \equiv 0$, therefore Chen's inequality becomes:

$$\delta_M \leq \frac{1}{2} \cdot (n+1) \cdot (n-2) \tag{3.5.1}$$

and the shape operators attain the following form (see:[C.D.V.V1])

$$\begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \dots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \dots & 0 \\ 0 & 0 & \vdots & \dots & \vdots \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} = A_r, \quad \forall r = n+1, \dots, m.$$

Indeed, since we have choosen $\|H\|H = e_{n+1}$ if $H \neq 0$ and if $H = 0$ then e_{n+1} is a direction which diagonalizes the shape operator, given that M is by hypothesis minimal, follows from theorem (17) that the condition $a+b=0$ must be satisfied and the assertion is an obvious consequence. From now on we shall focus on totally real immersions of 3-dimensional submanifolds of S^6. In this case the immersion is minimal and **Chen's inequality** becomes: $\delta_M \leq 2$.

The next theorem gathers the first results concerning the Chen's inequality, the form of the shape operator when the equality $\delta_M = 2$ occurs, and it is a direct application of the above observations.

Theorem 18 *(see:[C.D.V.V1]) Let $M \xrightarrow{\iota} S^6$ be a totally real immersion of the 3-dimensional manifold M. Then $\delta_M \leq 2$ and equality holds, at a point p of M, if and only if there is a tangent basis $\{e_1, e_2, e_3\}$ and a normal basis $\{e_4, e_5, e_6\}$ such that the shape operators attain the form:*

$$\begin{pmatrix} h^r_{11} & h^r_{12} & 0 \\ h^r_{12} & -h^r_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_r, \quad r = 4, 5, 6.$$

The following theorem provides further information on the second fundamental form, at a point $p \in M$, when the Chen's equality $\delta_M(p) = 2$ occurs.

Theorem 19 *(see:[C.D.V.V1]) Let M be a totally real 3-dimensional submanifold of the 6-sphere. Then $\delta_M \leq 2$ and the equality holds if and only if there is a basis $\{e_1, e_2, e_3\}$ of T_pM such that:*

$$h(e_1,e_1) = \lambda Je_1, \quad h(e_1,e_2) = -\lambda Je_2, \quad h(e_2,e_2) = -\lambda Je_1,$$
$$h(e_1,e_3) = h(e_2,e_3) = h(e_3,e_3) = 0. \tag{3.5.2}$$

where $\lambda \in \mathbb{R}^+ - \{0\}$ satisfies the $2\lambda^2 = 3 - \tau(p)$.

Proof: Let us assume that the equality $\delta_M(p) = 2$ holds at $p \in M$. If p is a totally geodesic point there is nothing to prove. We assume p to be non totally geodesic.
On $UM_p = \{u \in T_pM : \ < u,u >= 1\}$ is defined the function $f_p(u) =< h(u,u), Ju >$ and since UM_p is compact, we can assume, using the method of **maximalization**, that f_p attains its maximum at the point $u \in UM_p$. It follows, repeating the same steps as we did in similar previous cases, that:

$$f_p(u) > 0, \quad < h(u,u), Jw >= 0, \quad \forall w \in UM_p \quad \text{such that} \quad < u,w >= 0.$$

But, M being totally real implies that $A_{JY}X = -Jh(X,Y)$ [see:(2.2.9) in lemma (9) of Chapter 2]. Combining these facts we obtain that u is an eigenvector of A_{Ju}.
Set $u = e_1$ and choose e_2, e_3 such that, the set $\{e_1, e_2, e_3\}$ is an orthonormal basis of T_pM and e_i is an eigenvector of A_{Je_i} of corresponding eigenvalue λ_i. From the theorem (18) follows that the image of the operator A_{Je_1} is a subspace of the tangent space, at most 2-dimensional, and moreover, since e_1 belongs to this basis (rearranging the terms, if necessary), we can assume :

$$A_{Je_1}e_1 = \lambda e_1, \quad A_{Je_1}e_2 = -\lambda Je_2, \quad A_{Je_3}e_3 = 0.$$

Using once more theorem (18), recalling the symmetry of $< h(u,v), Jw >$ and the fact that $\{Je_1, Je_2, Je_3\}$ is an orthonormal basis of $\perp_pM$, we obtain

$$< A_{Je_i}e_j, e_3 >= 0, \quad \forall i,j = 1,2,3$$

and combining with the fact of being M minimal in S^6, we get:

$$< A_{Je_3}e_3, e_2 >=< A_{Je_1}e_1, e_2 >=< A_{Je_2}e_2, e_2 >= 0.$$

Therefore, the second fundamental form has been completely determined as a function of the eigenvalue λ.

Applying the Gauss equation (1.2.14) for all the 2-plane sections $\{e_i, e_j\}$ we obtain:

$$2\tau = 6 - 2 < h, h > \quad \text{and} \quad < h, h >= \lambda^2 \Longrightarrow 2\lambda_2 = 3 - \tau.$$

and the proof is completed.

The theorem on the uniqueness and existence of totally real immersions in S^6 will be splited in two parts (see:[C.D.V.V1])

Theorem 20 (Uniqueness) *Let $x^1, x^2 : M \hookrightarrow S^6$, be two totally real, isometric immersions of a 3-dimensional Riemannian manifold $(M, <,>)$, with second fundamental forms h^1, h^2 respectively and the same orientation (induced by $J\mathcal{G}$, where $\mathcal{G}$ is the tensor field $\mathcal{G}(X,Y) = (\widetilde{\nabla}_X J)Y$ defined on the 6-sphere).*
Suppose $< h^1(X,Y), Jx^1_\star Z >=< h^2(X,Y), Jx^2_\star Z >, \quad \forall X, Y, Z \in \mathfrak{X}M$. Then, there exists an isometry $S^6 \xrightarrow{A} S^6$ such that $x^1 = x^2 \circ A$.

Proof: We have: $\nabla^{\perp}_X JY = J\nabla_X Y + \mathcal{G}(X,Y), \quad A_{JY}X = -Jh(X,Y), \quad \forall X, Y \in \mathfrak{X}(M)$, (see:(2.2.8), (2.2.9), and we can check that the normal connections and the shape operators of the immersions x^1, x^2 coincide. Applying standard uniqueness results for minimal immersions in real space forms we have the required assertion.

Theorem 21 (Existence) *Let $(M, <,>)$ be a simply connected, oriented , 3-dimensional Riemannian manifold and $\wedge : TM \times TM \longrightarrow TM$ a skew symmetric operator assigning, to each pair $\{X, Y\}$ of linearly independent vectors, the unique vector $X \wedge Y$ of length $\sqrt{< X, X > \cdot < Y, Y > - < X, Y >^2}$ which is orthogonal to both X, Y and such that the set $\{X, Y, X \wedge Y\}$ is a positively oriented basis.*
Let α be a symmetric bilinear TM - valued form defined on M and such that:

(1) $Tr\alpha = 0$.

(2) $< \alpha(X,Y), Z >$ *is totally symmetric.*

(3) $(\nabla\alpha)(X,Y,Z) + X \wedge \alpha(Y,Z)$ *is totally symmetric.*

(4) $R(X,Y)Z =< Y, Z > \cdot X - < X, Z > \cdot Y + \alpha(\alpha(Y,Z), X) - \alpha(\alpha(X,Z), Y)$.

Then, there exists a totally real immersion $x : M \hookrightarrow S^6$ such that the second fundamental form h satisfies: $h(X,Y) = J\alpha(X,Y)$ and moreover $\mathcal{G}(X,Y) = J(X \wedge Y)$.

Proof: For all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and for any $\{Y_1, Y_2, Y_3\}$ orthonormal vector fields on M we have:

$$\begin{aligned} &< X \wedge Y, Z > + < X \wedge Z, Y >= 0, \\ &< Y_3 \wedge Y_1, Y_3 \wedge Y_2 >= 0. \end{aligned} \tag{3.5.3}$$

Take X, Y to be orthonormal, then, since $< X \wedge Y, X >=< X \wedge Y, Y >= 0$, we get:

$$\begin{aligned} &< \nabla_Z (X \wedge Y), X >= - < X \wedge Y, \nabla_Z X >=< (\nabla_Z X) \wedge Y, X >, \\ &< X \wedge \nabla_Z Y, X >= 0 \Longrightarrow < \nabla_Z (X \wedge Y), X >= \\ &< (\nabla_Z X) \wedge Y, X > + < X \wedge (\nabla_Z Y), X > . \end{aligned} \tag{3.5.4}$$

Similarly we obtain:

$$\begin{aligned} &< \nabla_Z (X \wedge Y), Y >= - < X \wedge Y, \nabla_Z Y >, \\ &< (\nabla_Z X) \wedge Y, Y >= 0 \Longrightarrow < \nabla_Z (X \wedge Y), Y >= \\ &< (\nabla_Z X) \wedge Y + X \wedge (\nabla_Z Y), X \wedge Y > . \end{aligned} \tag{3.5.5}$$

We also observe that

$$\begin{aligned} &< X \tilde{\wedge} Y, X \wedge Y >= 1 \Longrightarrow < \nabla_Z (X \wedge Y), X \wedge Y >= 0 \Longrightarrow \\ &< \nabla_Z (X \wedge Y), X \wedge Y >=< (\nabla_Z X) \wedge Y + X \wedge (\nabla_Z Y), X \wedge Y > . \end{aligned} \tag{3.5.6}$$

Combining the relations (3.5.4), (3.5.5) and (3.5.6) we obtain

$$\nabla_Z (X \wedge Y) = (\nabla_Z X) \wedge Y + X \wedge (\nabla_Z Y) \tag{3.5.7}$$

and it is easy to check that (3.5.7) is valid for arbitrary vector fields.
Identify NM with TM via J_0 and define a connection $\nabla^\perp$ on NM by setting

$$\nabla^\perp_X J_0 Y = J_0 \nabla_X Y + J_0 (X \wedge Y) \tag{3.5.8}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$, where J_0 is the identification between TM and NM. Define a second fundamental form $\mathbf{h}$ and a shape operator A_{J_0X} for any $X, Y \in \mathfrak{X}(M)$ by setting :

$$h(X,Y) = J_0\alpha(X,Y) \tag{3.5.9}$$

$$A_{J_0X}Y = \alpha(X,Y). \tag{3.5.10}$$

The equations of Gauss, Codazzi and Ricci are satisfied hence, by the existence and uniqueness theorems for immersions in real space forms, there exists an isometric immersion $x : M \hookrightarrow S^6$, with second fundamental form h, normal connection $\nabla^\perp$ and shape operator A_{J_0X}. It remains only to be shown that the immersion x is totally real.
Consider S^6 be immersed in $\mathbb{R}^7$ by the inclusion map and define a vector cross product on $\mathbb{R}^7$ by setting

$$\begin{aligned} &p \times X = J_0X, \quad p \times J_0X = -X, \quad X \times Y = J_0(X \wedge Y), \\ &(J_0X) \times Y = X \wedge Y - \langle X, Y \rangle \cdot p, \quad J_0X \times J_0Y = -X \wedge Y, \end{aligned} \tag{3.5.11}$$

for any vector fields $X, Y \in \mathfrak{X}\mathbb{R}^7$. Denote by D the Levi-Civita connection on $\mathbb{R}^7$ and using (3.5.8), (3.5.9), (3.5.10) and (3.5.11) we get:

$$\begin{aligned} D_Y(p \times X) - (D_Y p) \times X - p \times D_Y X &= D_Y(J_0X) - Y \times X - J_0 D_Y X \\ &= \nabla^\perp_Y(J_0X) - A_{J_0X}Y - J_0(X \wedge Y) \\ &\quad - J_0\nabla_Y X - J_0 h(X,Y) \\ &= J_0\nabla_Y X + J_0(X \wedge Y) - \alpha(X,Y) \\ &\quad - J_0(X \wedge Y) - J_0\nabla_Y X - \alpha(X,Y) \\ &= 0. \end{aligned}$$

Similarly working we can compute the rest of the equations in order to prove that the defined vector cross product, is actually parallel.
Recalling the uniqueness of the vector cross product on the 6-sphere (see:[Ca1]), and by applying an arbitrary element of $SO(7)$, we deduce that M is totally real and the tensor field J_0 coincides with the usual almost complex structure of S^6. This completes the proof of the theorem.
As the final step in this section, by observing that the dimension of $\mathcal{D}(p)$ is either 3 or 1 and from the form attained by the shape operators, we can state the following:

Lemma 19 *(see: [D.V]) Let M be a totally real 3-dimensional submanifold of S^6 with second fundamental form h. Then $\delta_M(p) = 2$ if and only if there exists a tangent vector $v \in T_pM$, such that : $h(v, w) = 0, \quad \forall w \in T_pM$.*

3.6 Chen's equality and examples

This section is actually the continuation of the previous one and provides some examples of totally real submanifolds satisfying Chen's equality. These examples will be used within chapter 4 for the classification of totally real submanifolds of S^6 which satisfy Chen's equality and in particular they will play a basic role in the classification of submanifolds having constant scalar curvature as well as in the case where further conditions of integrability on the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are imposed.

Example 17 *Let $S^3 = \{(y_1, y_2, y_3, y_4) \in E^4 : y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$ be the unit sphere in E^4, where S^3 is considered as an immersed submanifold of $\mathbb{R}^4$ by the inclusion map. Let us define the vector fields $X_1, X_2, X_3 \in \mathfrak{X}(S^3)$ by:*

$$\begin{aligned} X_1(y_1, y_2, y_3, y_4) &= (y_2, -y_1, y_4, -y_3), \\ X_2(y_1, y_2, y_3, y_4) &= (y_3, -y_4, -y_1, y_2), \\ X_3(y_1, y_2, y_3, y_4) &= (y_4, y_3, -y_2, -y_1). \end{aligned} \tag{3.6.1}$$

It is easy to verify that:

$$[X_1, X_2] = 2 \cdot X_3, \quad [X_2, X_3] = 2 \cdot X_1, \quad [X_3, X_1] = 2 \cdot X_1. \tag{3.6.2}$$

A metric $<,>$ and an orthonormal basis $\{E_1, E_2, E_3\}$, with respect to this metric, can be defined on S^3 by setting:

$$\begin{aligned} &< X_i, X_j >= 0, \quad \forall i \neq j, \\ &< X_1, X_1 >=< X_2, X_2 >= 6, \quad < X_3, X_3 >= 36, \\ &E_1 = \frac{1}{\sqrt{6}} \cdot X_1, \quad E_2 = \frac{1}{\sqrt{6}} \cdot X_2, \quad E_3 = \frac{1}{6} \cdot X_3. \end{aligned} \tag{3.6.3}$$

Using the formula of Koszul we compute the connection on S^3 and get

$$\begin{aligned}
&\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3,\\
&\nabla_{E_1}E_2 = -\nabla_{E_2}E_1 = E_3, \quad \nabla_{E_1}E_3 = -E_2,\\
&\nabla_{E_3}E_1 = -\frac{2}{3}\cdot E_1, \quad \nabla_{E_2}E_3 = E_1, \quad \nabla_{E_3}E_2 = \frac{2}{3}\cdot E_1,
\end{aligned} \tag{3.6.4}$$

and a straightforward computation yields

$$\begin{aligned}
&R(E_1,E_2)E_3 = R(E_2,E_3)E_1 = R(E_3,E_1)E_2 = 0,\\
&R(E_1,E_2)E_2 = -\frac{7}{3}\cdot E_1, \quad R(E_1,E_3)E_3 = E_1, \quad R(E_2,E_3)E_3 = E_2.
\end{aligned} \tag{3.6.5}$$

Using theorem (21) we can define a symmetric bilinear form α on the tangent bundle by:

$$\begin{aligned}
&\alpha(E_1,E_1) = \sqrt{\frac{5}{3}}\cdot E_1, \quad \alpha(E_1,E_2) = -\sqrt{\frac{5}{3}}\cdot E_2, \quad \alpha(E_2,E_2) = -\sqrt{\frac{5}{3}}\cdot E_1,\\
&\alpha(E_3,E_1) = \alpha(E_3,E_2) = \alpha(E_3,E_3) = 0.
\end{aligned} \tag{3.6.6}$$

Using (3.6.4), (3.6.5) and direct calculation we can show the symmetric bilinear operator, defined by (3.6.6), satisfies the conditions of the existence theorem (21) and hence there exists a totally real isometric immersion:

$$(S^3, <,>) \xrightarrow{\varphi_1} S^6 \quad \textit{such that} \quad h(X,Y) = J\alpha(X,Y), \quad \forall X,Y \in \mathfrak{X}(S^3).$$

In particular, by direct calculation we get $\tau = -\frac{1}{3}$. $infK = -\frac{7}{3}$ and thus $\delta = 2$.

Note: *The immersion φ_1 is a totally real immersion of constant scalar curvature $\tau = -\frac{1}{3}$ satisfying Chen's equality.*

We proceed with the construction of a second totally real immersion of the 3-dimensional unit sphere in S^6 which satisfies Chen's equality and moreover, is of constant sectional curvature.

Example 18 *Consider again the S^3 as a submanifold of $\mathbb{R}^4$, the immersion given by the inclusion map. Take the vector fields X_1, X_2, X_3 as in example (17) and define a new metric $<,>$ on S^3 by setting:*

$$\begin{aligned}
&< X_i, X_j >= 0, \quad \forall i,j \in \{1,\ldots,n\}, \quad i \neq j \quad \textit{and}\\
&< X_1, X_1 >=< X_2, X_2 >= 2, \quad < X_3, X_3 >= 4.
\end{aligned} \tag{3.6.7}$$

Define the orthonormal basis $\{E_1, E_2, E_3\}$ *as in example (17) and by repeating analogous steps we compute the connection and the curvature operator:*

$$\begin{aligned}
&\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3, \quad \nabla_{E_1}E_2 = -\nabla_{E_2}E_1 = -E_3,\\
&\nabla_{E_1}E_3 = -E_2, \quad \nabla_{E_3}E_1 = 0, \quad \nabla_{E_2}E_3 = -E_1, \quad \nabla_{E_3}E_2 = 0,\\
&R(E_1,E_2)E_2 = -E_1, \quad R(E_1,E_3)E_3 = E_1, \quad R(E_2,E_3)E_3 = E_2,\\
&R(E_1,E_2)E_3 = R(E_2,E_3)E_1 = R(E_3,E_1)E_2 = 0.
\end{aligned} \tag{3.6.8}$$

Define a bilinear symmetric form on TS^3 *by setting*

$$\begin{aligned}
&\alpha(E_1,E_1) = E_1, \quad \alpha(E_1,E_2) = -E_2, \quad \alpha(E_2,E_2) = -E_1,\\
&\alpha(E_1,E_3) = \alpha(E_2,E_3) = \alpha(E_3,E_3) = 0.
\end{aligned} \tag{3.6.9}$$

It is easy to verify that the conditions of the existence theorem (21) are satisfied. Therefore, there exists a totally real immersion:

$$(S^3, <,>) \xrightarrow{\psi_2} S^6, \quad \textit{such that} \quad h(X,Y) = J\alpha(X,Y), \quad \forall X, Y \in \mathfrak{X}(S^3).$$

By direct calculation we find: $\tau = 1, \quad inf K = -1, \quad \delta = 2.$

Note: *ψ_2 is a totally real immersion satisfying Chen's equality and of constant scalar curvature.*

Example 19 *Let N be any unit vector in $\mathbb{R}^7$ and S^5 be the unit sphere in the linear subspace orthogonal to N. We consider an arbitrary surface immersion $f : M^2 \hookrightarrow S^5$ and define*

$$x : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{sint} M^2 \hookrightarrow S^5 : x(t,p) = sin(t)N + cos(t)f(p).$$

Then x is an example of a minimal, totally real immersion and it is easy to verify, using theorem (18), that satisfies Chen's equality and furthermore, **it is not of constant scalar curvature.**

Note: *The mapping x is a warped product immersion of the surface immersion $f : M^2 \hookrightarrow S^5$ with the unit normal N to this S^5. More details on the warped product immersions there are in the next chapter, in the last part of the section §4.3.*

Chapter 4

Chen's equality and classification of totally real 3-dimensional submanifolds of S^6

4.1 Introduction

In this chapter classification results are discussed, concerning totally real, 3-dimensional submanifolds of S^6, imposing at first, that they all satisfy Chen's equality. In particular, chapter 4 is structured in the following way:

In (§4.2) the condition of being a totally real immersion of constant scalar curvature is studied and it is proved that in this case the immersion is either totally geodesic or locally congruent to one of the immersions υ_1, υ_2, (therefore satisfying Chen's equality) given in the examples (17), (18) respectively, of the section (§3.6). The main reference for this section is [C.D.V.V1].

In (§4.3), totally real, 3-dimensional submanifolds of S^6 satisfying Chen's equality are investigated, under the extra assumptions that the subspaces $\mathcal{D}(p)$, given by definition (23) has constant dimension (and thus $\mathcal{D}$ results to be a distribution) and moreover that the complementary distribution $\mathcal{D}^\perp$ is also integrable. In this case the **warped** product immersion given in the example (19) of (§3.6) characterizes such immersions, satisfying the imposed conditions. The main reference for this section is [C.D.V.V2].

In (§4.4) it is proved that, starting from a holomorphic curve $N_1 \xrightarrow{\phi} \mathbb{C}P^2(4)$, and using the

Hopf's fibration $S^5 \xrightarrow{\pi} \mathbb{C}P^2(4)$, we can find a totally geodesic embedding $S^5 \xrightarrow{i} S^6$ such that the immersion $PN_1 \xrightarrow{i\circ\psi} S^6$, where ψ denotes a lifting of the holomorphic curve to a unit circle bundle over N_1, is a 3-dimensional totally real immersion in S^6 which satisfies Chen's equality.

In (§4.5) almost complex curves without totally geodesic points are taken under consideration and by defining a mapping from their unit tangent bundle into S^6, totally real immersions (possibly branched) satisfying Chen's equality are produced. Furthermore, by defining tubes of radius $\frac{\pi}{2}$ over almost complex immersions in S^6, new examples of totally real 3-dimensional submanifolds are obtained.

In (§4.6) local converses of the above theorems are proved and more specifically:

for any totally real immersion of a 3-dimensional manifold in S^6, which is not linearly full in S^6 and satisfies Chen's equality, there exists a totally geodesic S^5 and a holomorphic curve $S \xrightarrow{\phi} \mathbb{C}P^2(4)$ such that the totally real immersion is congruent to the map ψ which is obtained from ϕ in the way described in the fourth section of this chapter.

for any totally real immersion of a 3-dimensional manifold, which is linearly full in S^6, satisfying Chen's equality for a non totally geodesic point of the immersed submanifold, there exists an almost complex curve in S^6 such that the initial immersion is congruent, in a neighbourhood of the point p, to an immersion $\tilde{\psi}$ obtained in the way described in the second part of (§4.5). The main reference for the three last sections is [D.V].

4.2 Chen's equality and constant scalar curvature

As we have already mentioned in the introduction the basic assumptions on the totally real, 3-dimensional immersion are that of satisfying Chen's equality $\delta_M = 2$ and that on the constancy of the scalar curvature. The classification will be obtained in terms of immersions congruent to the immersions given in the examples (17) and (18) of §3.6.

Let $M^3 \xrightarrow{\psi} S^6$ be a totally real immersion of constant scalar curvature τ such that $\delta_M(p) = 2$ at each point $p \in M$, and moreover assume M^3 to be non totally geodesic.

If we recall theorem (19) then, by using the assumption that Chen's equality is satisfied on M, we can assert the existence of an orthonormal basis of the tangent space of M at

each point p of M such that:

$$h(e_1, e_1) = \lambda Je_1, \quad h(e_1, e_2) = -\lambda Je_2, \quad h(e_2, e_2) = -\lambda Je_1$$
$$h(e_1, e_3) = h(e_2, e_3) = h(e_3, e_3) = 0. \tag{4.2.1}$$

where $2 \cdot \lambda^2 = 3 - \tau(p)$.

Before we state and give the proof of the main theorem of this section we need to prepare lemmas.

Let us consider the function $f_q(u) =< h(u,u), Ju >$ defined on the unit tangent bundle UM_q where $q \in M$.

Lemma 20 *Let $q \in M$ and s_q be a critical value of the function f_q at the point q. Then $s_q \in \{-\lambda, 0, \lambda\}$.*

Proof: Since s_q is a critical value of f_q, there will be $u \in UM_q$ such that:
$f_q(u) = s_q$ and $< h(u,u), Jw >= 0. \quad \forall w \in UM_p \quad$ with $\quad w \perp u$.
Take an orthonormal basis $\{e_1, e_2, e_3\}$ satisfying the relations (4.2.1) and put

$$u = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad \text{where} \quad \sum_{i=1}^{3} \alpha_i^2 = 1. \tag{4.2.2}$$

The following steps are only a matter of easy calculations. First we note that

$$h(u,u) = \lambda\left(\alpha_1^2 - \alpha_2^2\right) Je_1 - 2\lambda\alpha_1\alpha_2 Je_2, \tag{4.2.3}$$

so that $s_q = < h(u,u), Ju >= \lambda\alpha_1(\alpha_1^2 - 3\alpha_2^2)$ is a critical value of f_q if and only if the following system is satisfied:

$$< h(u,u), \alpha_2 Je_1 - \alpha_1 Je_2 >= 3\left(\alpha_1^2 - \alpha_2^2\right)\lambda\alpha_2 = 0, \tag{4.2.4}$$
$$< h(u,u), \alpha_3 Je_1 - \alpha_1 Je_3 >= \lambda\left(\alpha_1^2 - \alpha_2^2\right)\alpha_3 = 0, \tag{4.2.5}$$

where $\lambda \neq 0$ since M is assumed to be non totally geodesic.

Let us consider the distinct cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$.

If $\alpha_2 = 0$ then the equation (4.2.5) becomes $\alpha_1\alpha_3 = 0$ and since $\sum_{i=1}^{3} \alpha_i^2 = 1$ we get:

$$u \in \{\pm e_1, \pm e_3\} \quad \text{and thus} < h(u,u), Ju > \in \{\pm\lambda, 0\}.$$

If $\alpha_2 \neq 0$ then the equations (4.2.4), (4.2.5) imply $\alpha_1\alpha_3 = 0$ and $\alpha_2^2 = 3\alpha_1^2$, therefore:

$$\alpha_3 = 0, \quad \alpha_1^2 = \frac{1}{4}, \quad \alpha_3^2 = \frac{3}{4}.$$

Similarly working in the remaining cases we complete the proof.

In the next lemma it is shown that $\{e_1, e_2, e_3\}$ can be extended to orthonormal vector fields $\{E_1, E_2, E_3\}$ satisfying similar conditions.

Lemma 21 *If $p \in M$ there exist orthonormal vector fields $\{E_1, E_2, E_3\}$ around p such that:*

$$h(E_1, E_1) = \lambda JE_1, \quad h(E_1, E_2) = -\lambda JE_2, \quad h(E_2, E_2) = -\lambda JE_1,$$
$$h(E_1, E_3) = h(E_2, E_3) = h(E_3, E_3) = 0. \tag{4.2.6}$$

Proof: Suppose that at the point $p \in M$ the function f_p attains an absolute maximum at u_0. Let $\{U_1, U_2, U_3\}$ be a locally defined in a neighborhood $\mathcal{U}$ of p, differentiable orthonormal basis such that, $U_i(p) = e_i$, with $e_1 = u$, satisfies the conditions (3.5.2) of theorem (19) in chapter 3. Define the function γ by setting:

$$\gamma : \mathbb{R}^3 \times \mathcal{U} \longrightarrow \mathbb{R}^3 : \gamma(a_1, a_2, a_3, q) = (b_1, b_2, b_3) \tag{4.2.7}$$

where $b_k = \sum_{i,j=1}^{3} a_i a_j < h(U_i, U_j), JU_k > -\lambda a_k, \quad \forall k = 1, 2, 3$. Using the conditions (3.5.2) in Chapter 3, satisfied by the elements of the orthonormal basis, we get:

$$\frac{\partial b_k}{\partial a_m}(1, 0, \ldots, 0, p) = 2 < h(U_1(p), U_k(p)), JU_m(p) > -\lambda\delta_{km}(p) = \begin{array}{ll} 0, & k \neq m \\ \lambda, & k = m = 1 \\ -3\lambda, & k = m = 2 \\ -\lambda, & k = m = 3 \end{array}$$

By the implicit function theorem we can assert the existence of differentiable functions a_1, a_2, a_3 defined on a neighborhood of the point p and such that:
the local vector field $V = a_1 \cdot V_1 + a_2 \cdot V_2 + a_3 \cdot V_3$ satisfies $V(p) = u$ and $h(u, u) = \lambda \cdot JW$. Thus, the vector field $W = \frac{V}{\sqrt{<V,V>}}$ satisfies the relation $h(W, W) = \frac{\lambda}{\sqrt{<V,V>}} \cdot JW$ and consequently, in a neighborhood of p, the function f_p attains a relative extremum in $W(p)$. From lemma (20) we know that the set of the critical values of f_q is finite and we also observe that the quantity $\frac{\lambda}{\sqrt{<V,V>}}$ changes continuously, since the second fundamental

form and the almost complex structure change continuously. Thus the vector V has unit length at each point.

If we take $E_1 = V$ we can extend u differentiably to a vector field V, defined on a neighborhood $\mathcal{U}$, such that $< V, V >= 1$ and moreover, at each point $q \in \mathcal{U}$ the function f_q attains its absolut maximum at $V(q)$.

The differentiable extension of the vector u to a unit vector field $V = E_1$, for which the function f_q attains its maximum in $V(q)$, completes one part of the proof. Let us focus on the part concerning the second fundamental form.

In the way E_1 has been chosen we know that $A_{JE_1}E_1 = \lambda \cdot JE_1$, ie: λ is an eigenvalue of the shape operator and as it is well known the shape operator has two more eigenspaces. The first is 1-dimensional, corresponding to the eigenvalue $-\lambda$ and the second is again 1-dimensional, corresponding to the eigenvalue 0. Since these three eigenvalues are different and have constant multiplicities we can assert, using standard results, the local existence, around the point p, of vector fields $\{E_2, E_3\}$ such that:

$$A_{JE_1}E_2 = -\lambda \cdot E_2, \quad A_{JE_1}E_3 = 0. \tag{4.2.8}$$

Combining with the well known relation $A_{JY}X = -Jh(X, Y)$, (2.2.9, chapter 2), we easily complete the proof.

Let us now assume that for the orthonormal frame field $\{E_1, E_2, E_3\}$ we have:

$$\mathcal{G}(E_1, E_2) = JE_3, \quad \mathcal{G}(E_2, E_3) = JE_1, \quad \mathcal{G}(E_3, E_1) = JE_2. \tag{4.2.9}$$

In order to determine the connection on M we need the following two lemmata. In the first of them we shall clarify some points which are not explicitely carried out in [C.D.V.V1].

Lemma 22 *Let $\{E_1, E_2, E_3\}$ be as in theorem (19) of Chapter 3, then:*

$$\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3, \tag{4.2.10}$$

$$< \nabla_{E_1}E_2 + \nabla_{E_2}E_1, E_3 >= 0, \tag{4.2.11}$$

$$\nabla_{E_3}E_1 = -\frac{1}{3} \cdot (1 + < \nabla_{E_1}E_2, E_3 >) \cdot E_2. \tag{4.2.12}$$

Proof: Using (4.2.6) of lemma (21) the Codazzi equation yields:

$$\begin{aligned}(\nabla h)(E_1, E_3, E_3) &= (\nabla h)(E_3, E_1, E_3)\\ &\Longrightarrow \nabla^{\perp}_{E_1} h(E_3, E_1) - h(\nabla_{E_1} E_3, E_3) - h(E_3, \nabla_{E_1} E_3)\\ &= \nabla^{\perp}_{E_3} h(E_1, E_3) - h(\nabla_{E_3} E_1, E_3) - h(E_1, \nabla_{E_3} E_3)\\ &\Longrightarrow h(E_1, \nabla_{E_3} E_3) = 0\end{aligned}$$

and thus

$$\Longrightarrow \nabla_{E_3} E_3 \quad \text{is parallel to} \quad E_3 \quad \text{and} \quad E_3 \perp \nabla_{E_3} E_3 \Longrightarrow \nabla_{E_3} E_3 = 0. \tag{4.2.13}$$

Moreover

$$\begin{aligned}(\nabla h)(E_3, E_1, E_1) &= (\nabla h)(E_1, E_3, E_1)\\ &\Longrightarrow \nabla^{\perp}_{E_3} h(E_1, E_1) - 2 \cdot h(\nabla_{E_3} E_1, E_1)\\ &= \nabla^{\perp}_{E_1} h(E_3, E_1) - h(\nabla_{E_1} E_3, E_1) - h(E_3, \nabla_{E_1} E_1),\end{aligned}$$

and we deduce that

$$\lambda \cdot \nabla^{\perp}_{E_3} J E_1 - 2 \cdot J A_{JE_1} \nabla_{E_3} E_1 = J A_{E_1} \nabla_{E_1} E_1 + J A_{JE_3} \nabla_{E_1} E_1, \tag{4.2.14}$$

where

$$\nabla^{\perp}_{E_3} J E_1 = J \nabla_{E_3} E_1 + \mathcal{G}(E_3, E_1) \Longrightarrow \nabla^{\perp}_{E_3} J E_1 = J \nabla_{E_3} E_1 - J E_2$$

and

$$-2h(\nabla_{E_3} E_1, E_1) = 2 \cdot < \nabla_{E_3} E_1, E_1 > \lambda J E_2,$$

since: $\nabla_{E_3} E_1 \perp E_1$, $h(E_3, \nabla_{E_3} E_1) = 0$, and $A_{JE_1} E_2 = -\lambda \cdot E_2$.
Working similarly for the right hand side and recalling that because of the (4.2.13) we have $\nabla_{E_3} E_3 = 0$, we get:

$$\begin{aligned}\lambda \cdot J \nabla_{E_3} E_1 + \lambda \cdot J E_2 + 2 \cdot < \nabla_{E_3} E_1, E_2 > \cdot \lambda J E_2 = &- < \nabla_{E_1} E_3, E_1 > \cdot \lambda J E_1\\ &+ < \nabla_{E_1} E_3, E_2 > \cdot \lambda J E_2\end{aligned}$$

therefore:

$$\begin{aligned}&< \nabla_{E_1} E_1, E_3 >= 0 \quad \text{and}\\ &3 \cdot < \nabla_{E_3} E_1, E_2 >= - < \nabla_{E_1} E_2, E_3 > - 1.\end{aligned} \tag{4.2.15}$$

Using the equation of Codazzi we have:

$$(\nabla h)(E_3, E_2, E_2) = (\nabla h)(E_2, E_3, E_2) \Longrightarrow < \nabla_{E_2} E_2, E_3 >= 0 \tag{4.2.16}$$

$$(\nabla h)(E_2, E_1, E_1) = (\nabla h)(E_1, E_2, E_1) \Longrightarrow$$

$$< \nabla_{E_2} E_1 + \nabla_{E_1} E_2, E_3 >= 0, \quad < \nabla_{E_1} E_1, E_2 >= 0, \quad < \nabla_{E_2} E_2, E_1 >= 0. \tag{4.2.17}$$

Combining (4.2.15), (4.2.16) and (4.2.17) we complete the proof of the lemma.

Lemma 23 *The basis $\{E_1, E_2, E_3\}$ given in lemma (21) satisfies either:*

$$\begin{aligned} &\nabla_{E_1} E_2 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0 \\ &\nabla_{E_1} E_3 = -E_2, \quad \nabla_{E_2} E_3 = E_1, \quad \nabla_{E_1} E_2 = E_3 \\ &\nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_3} E_1 = -\frac{2}{3} E_2, \quad \nabla_{E_3} E_2 = \frac{2}{3} E_1 \end{aligned} \tag{4.2.18}$$

with $\tau = -\frac{1}{3}$, ie: $\lambda = \sqrt{\frac{5}{3}}$, or :

$$\begin{aligned} &\nabla E_1 E_1 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0, \\ &\nabla_{E_1} E_3 = -E_2, \quad \nabla_{E_2} E_3 = E_1, \quad \nabla_{E_1} E_2 = E_3 \\ &\nabla_{E_2} E_1 = E_3, \quad \nabla_{E_3} E_1 = -\frac{2}{3} E_2, \quad \nabla_{E_3} E_2 = \frac{2}{3} E_1, \end{aligned} \tag{4.2.19}$$

with $\tau = 1$, ie: $\lambda = 1$.

Proof: In virtue of lemma (22) we have

$$\begin{aligned} &< \nabla_{E_1} E_2, E_3 > = - < \nabla_{E_2} E_1, E_3 >, \\ &< \nabla_{E_1} E_2, E_2 > = 0, \\ &< \nabla_{E_2} E_1, E_1 > = - < \nabla_{E_2} E_2, E_1 >= 0 \end{aligned}$$

and thus:

$$\nabla_{E_1} E_2 = b \cdot E_3, \quad \nabla_{E_2} E_1 = -b \cdot E_3, \tag{4.2.20}$$

where b is a locally defined function. Applying Gauss equation for $X = E_1$ and $Y = Z = E_3$ we obtain

$$< R(E_1, E_3) E_3, E_3 >= 1. \tag{4.2.21}$$

From the relations

$$< \nabla_{E_1} E_3, E_1 >= - < E_3, \nabla_{E_3} E_1 >= 0,$$
$$< \nabla_{E_1} E_3, E_2 >= - < E_3, \nabla_{E_1} E_2 >,$$
$$[E_3, E_1] = \nabla_{E_3} E_1 - \nabla_{E_1} E_3, \tag{4.2.22}$$

by using the (4.2.12) of lemma(22) we deduce:

$$1 = - < \nabla_{E_3} E_1, E_2 > (< \nabla_{E_3} E_2, E_1 > + < \nabla_{E_2} E_3, E_1 >)$$
$$+ < \nabla_{E_3} E_1, E_2 >< \nabla_{E_2} E_3, E_1 >$$

and consequently

$$1 = b\left(\frac{1}{3}(b+1) + b\right) - \frac{1}{3}b(b+1) = b^2, \tag{4.2.23}$$

and combining with equation (4.2.20) we conclude that the local function b is a constant and moreover satisfies $b^2 = 1$.

In order to determine λ we apply Gauss equation for $X = W = E_1$ and $Y = Z = E_2$:

$$1 - 2\lambda^2 =< R(E_1, E_2) E_1, E_2 >$$
$$=< \nabla_{E_1} \nabla_{E_2} E_2 - \nabla_{E_2} \nabla_{E_1} E_2 - \nabla_{\nabla_{E_1} E_2 - \nabla_{E_2} E_1} E_2, E_1 >$$
$$= -b < \nabla_{E_2} E_3, E_1 > -2b < \nabla_{E_3} E_2, E_1 >$$
$$= -b^2 - \frac{2}{3}b \cdot (b+1)$$

and so we get:

$$1 - 2 \cdot \lambda^2 = -\frac{5}{3} - \frac{2}{3}b. \tag{4.2.24}$$

If $b = 1$ then $\lambda^2 = \frac{5}{3}$ and if $b = -1$ then $\lambda^2 = 1$ hence, M is locally isometric either to $(S^3, <>_1)$, or to $(S^3, <>_2)$ where $<>_1$ and $<>_2$ are the metrics constructed in the examples (14) and (15) of chapter 3 repsectively. We can now state the following

Theorem 22 *Let $M \xrightarrow{x} S^6$ be a totally real immersion of the 3-dimensional manifold M. Then $\delta_M \leq 2$. If M has constant scalar curvature τ and the equality $\delta_M = 2$ holds identically then, either x is totally geodesic or locally congruent to one of the immersions ψ_1, ψ_2.*

Proof: It sufficies to recall the existence theorem (21) of chapter 3 and combine it with the lemma (21).

4.3 Chen's equality and integrability of $\mathcal{D}^{\perp}(p)$

In this section, totally real 3-dimensional submanifolds of S^6 satisfying

- $\delta_M(p) = 2$
- $dim\mathcal{D}(p)$ is constant and $\mathcal{D}^{\perp}$ is integrable

are classified using a minimal, non totally geodesic, totally real immersion of a surface M^2 into S^6, whose ellipse of curvature is a circle. This immersion is linearly full in a totally geodesic S^5, the associated warped product provides the required 3-dimensional, totally real immersion in S^6 satisfying the initial assumptions and furthermore, this process is invertible.

Let $f : M^2 \hookrightarrow S^5$ be a minimal immersion of a surface, S^5 totally geodesic in S^6. The associated warped product ([N]) immersion is given by:

$$x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{cos(t)} M^2 \hookrightarrow S^6 : (t, p) \longrightarrow sin\,(t)\, N + cos\,(t)\, f\,(p) \tag{4.3.1}$$

Let $X \in \mathfrak{X}(M^2)$, then the following equalities hold:

$$\begin{aligned} &x_{\star}\left(\frac{\partial}{\partial t}\right) = cos\,(t)\, N - sin\,(t)\, f\,(p)\,, \\ &x_{\star}\,(X) = cos\,(t)\, f_{\star}\,(X)\,, \quad Jx_{\star}\left(\frac{\partial}{\partial t}\right) = N \times f\,(p)\,, \\ &Jx_{\star}\,(X) = cos\,(t)\, sin\,(t)\, N \times f_{\star}\,(X) + sin^2\,(t)\, Jf_{\star}\,(X)\,. \end{aligned} \tag{4.3.2}$$

Using the (4.3.2) it is easy to verify that the existence of the :

$$< N \times f\,(p)\,, f_{\star}\,(X) >= 0. \tag{4.3.3}$$

$$< N \times f_{\star}\,(p)\,, f_{\star}\,(Y) >= 0. \tag{4.3.4}$$

$$< Jf_{\star}\,(X)\,, f_{\star}\,(Y) >= 0. \tag{4.3.5}$$

for all vector fields $X, Y \in \mathfrak{X}(M^2)$, is equivalent to the fact of being the immersion f totally real.

On the other hand, recalling that $\mathcal{G}(X, Y)$ is the tangential part, with respect to S^6, of

the vector cross product $X \times Y$ in $\mathbb{R}^7$, by using standard properties of the vector cross product we get

$$< \mathcal{G}(f_*(X), f_*(Y)), N >= 0, \tag{4.3.6}$$

$$< Jf_*(X), N >= 0 \tag{4.3.7}$$

and the condition that the warped immersion x be totally real has been reduced to conditions depending only on the surface immersion f.

Let us identify M^2 with its image $f(M^2)$ and differentiate the equation (4.3.3) with respect to Y keeping in mind that, the vector N is constant and $N \times p$ is perpendicular to X for all $X \in \mathfrak{X}(M^2)$, given that the surface is assumed to be totally real. We obtain

$$< N \times p, X >= 0 \Longrightarrow < N \times \tilde{\nabla}_Y p, X > + < N \times p, \tilde{\nabla}_Y X >= 0,$$

therefore

$$< N \times Y, X > + < N \times p, \nabla_Y X + h(X, Y) >= 0, \tag{4.3.8}$$

since $N \times p \perp \nabla_Y X \in \mathfrak{X}(M^2)$. Hence,

$$< N \times Y, X > + < N \times p, h(X, Y) >= 0, \tag{4.3.9}$$

where $span\{N, JX, JY\} = \perp_p M$ in S^5 since M^2 is contained in a totally geodesic S^5.

But $< N \times Y, X > + < N \times p, h(X, Y) >= 0 =< N \times X, Y > + < N \times p, h(X, Y) >$, therefore $< N \times Y, X >=< N \times p, h(X, Y) >= 0$. **i.e:** equation (4.3.3) implies (4.3.4).

Take $\{e_1, e_2\}$ to be an orthonormal basis of $T_p M^2$. It is clear that $\mathcal{G}(e_1, e_2)$ is perpendicular to the span of the vectors $\{e_1, e_2, Je_1, Je_2, p\}$ and from equation (4.3.4) we have that is also normal to the constant vector N. Furthermore, from the equation (4.3.3) we see that $N \times p$ is perpendicular to the span of the vectors e_1, e_2, Je_1, Je_2 and in conclusion

$$\mathcal{G}(e_1, e_2) = \pm p \times N. \tag{4.3.10}$$

After changing the sign (if necessary) we make sure that

$$e_1 \times e_2 = JN, \quad e_1 \times N = -Je_2, \quad e_2 \times N = Je_1. \tag{4.3.11}$$

Note: The normal space of M^2 in S^5 is spanned by the vectors $\{Je_1, Je_2, JN\}$. On the other hand $< N \times X, Y >= 0$ and thus

$$< N \times h(Y, Z), X > + < N \times Y, h(X, Z) >= 0, \quad \forall X, Y, Z \in \mathfrak{X}(M). \tag{4.3.12}$$

By setting $X = e_1, Y = e_2$ we obtain:

$$< Je_2, h(e_2, Z) > + < Je_1, h(e_1, Z) >= 0.$$
$$< h(e_2, e_2), JZ > + < h(e_1, e_1), JZ >= 0. \tag{4.3.13}$$

where the equations (4.3.12) and (4.3.11) imply that the mean curvature vector of M^2 in S^5 is orthogonal to both Je_1 and Je_2. If we recall equation (4.3.9) we see that N is normal to H and clearly the mean curvature vector has to vanish.
Again from equation (4.3.9) and by recalling that $p \times N = \pm \mathcal{G}(e_1, e_2)$ we see that h attains the form:

$$h(e_1, e_1) = \alpha Je_2 + \beta Je_2,$$
$$h(e_2, e_2) = -\alpha Je_1 - \beta Je_2,$$
$$h(e_1, e_2) = \beta Je_1 - \alpha Je_2, \tag{4.3.14}$$

and this actually means that the ellipse of curvature of M^2 is a circle (possibly a point). Conversely, let us assume that M^2 is a minimal, totally real surface in S^6, whose ellipse of curvature is a circle. In order to proceed we need the following (see:[D.O.V.V3]).

Lemma 24 *Let M^2 be a totally real surface in S^6, denote with ∇ the Riemannian connection on M^2. Then:*

$$A_{JY}X = -J(h(X, Y))^t, \tag{4.3.15}$$
$$\nabla^{\perp}_X JY = \mathcal{G}(X, Y) + J(\nabla_X Y) + J(h(X, Y))^n, \tag{4.3.16}$$
$$< h(X, Y), JZ >=< h(X, Z), JY >. \tag{4.3.17}$$

for all $(X, Y) \in \mathfrak{X}(M^2) \times \mathfrak{X}(M^2)$, where t denotes the tangential and n the normal part with respect to M^2 in S^6.

Proof: It is $\tilde{\nabla}_X JY = -A_{JY}X + \nabla^{\perp}_X JY$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on S^6, and from the definition of the tensor field $\mathcal{G}$:

$$\begin{aligned}\mathcal{G}(X, Y) &= \left(\tilde{\nabla}_X J\right) Y \\ &= \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y \\ &= -A_{JY}X + \nabla^{\perp}_X JY - J\nabla_X Y - Jh(X, Y)\end{aligned}$$

$$\mathcal{G}(X,Y) = -A_{JY}X + \nabla^{\perp}_{X}JY - J\nabla_{X}Y - Jh(X,Y) \tag{4.3.18}$$

for all vector fields $X, Y \in \mathfrak{X}(M^2)$. On the other hand it is easy to check that $\mathcal{G}(X,Y)$ is normal to M^2. Hence, by taking normal and tangential components of (4.3.18) we deduce the equations (4.3.15) and (4.3.16).

In order to deduce the equation (4.3.17) we use the (4.3.15) and get:

$$< h(X,Y), JZ >= - < (Jh(X,Y))^{t}, Z >=< A_{JX}Y, Z >$$

where

$$\tilde{\nabla}_{JX} < Y, Z >= 0 \Longrightarrow < A_{JX}Y, Z >=< Y, A_{JX}Z >$$

and thus

$$< h(X,Y), JZ >=< Y, A_{JX}Z >=< h(X,Z), JY > .$$

The proof is completed.

We now return to the case of the totally real minimal surface in S^6 with ellipse of curvature a circle. Take $\{e_1, e_2\}$ to be an orthonormal basis of the tangent space at the point $p \in M^2$. It is easy to check that $\perp_p M^2$ is spanned by the set of vectors $\{Je_1, Je_2, e_1 \times e_2, J(e_1 \times e_2)\}$. Using (4.3.17) in lemma (24) and the minimality of M^2 we get that the second fundamental form can be written as

$$\begin{aligned} h(e_1,e_1) &= a_1 Je_1 + a_2 Je_2 + a_3(e_1 \times e_2) + a_4 J(e_1 \times e_2), \\ h(e_1,e_2) &= a_2 Je_1 - a_1 Je_2 + c(e_1 \times e_2) + dJ(e_1 \times e_2). \end{aligned} \tag{4.3.19}$$

To get further information on the coefficients of the second fundamental form we observe (using minimality and the assumption that the ellipse of curvature is a circle) that, for any i = 1,2 and for any $\phi \in \mathbb{R}$, the following relations must be satisfied:

$$\|h\left(\frac{e_1+e_2}{\sqrt{2}}, \frac{e_1+e_2}{\sqrt{2}}\right)\|^2 = \|h(e_i,e_i)\|^2 \Longrightarrow \|h(e_1,e_2)\|^2 = \|h(e_i,e_i)\|^2.$$

$$\begin{aligned} \|h(cos\phi e_1 + sin\phi e_2, cos\phi e_1 + sin\phi e_2)\|^2 &= \|\left(cos^2\phi - sin^2\phi\right) h(e_1,e_1) + sin2\phi h(e_1,e_2)\|^2 \\ &= \|cos2\phi h(e_1,e_1) + sin2\phi h(e_1,e_2)\|^2 \\ &= \|h(e_i,e_i)\|^2. \end{aligned}$$

and as consequence we get

$$\|h(e_1, e_2)\|^2 = \|h(e_i, e_i)\|^2. \tag{4.3.20}$$

$$< h(e_1, e_2), h(e_i, e_i) >= 0. \tag{4.3.21}$$

From (4.3.19), (4.3.20) and (4.3.21) we get

$$a_3^2 + a_4^2 = c^2 + d^2, \quad a_3 c + a_4 d = 0. \tag{4.3.22}$$

The next step consists in the use of the Codazzi equation. In order to apply this equation let us at first write the connection ∇ on M^2 in the form:

$$\nabla_{e_1} e_1 = Ae_2, \quad \nabla_{e_2} e_2 = Be_1, \quad \nabla_{e_1} e_2 = -Ae_2, \quad \nabla_{e_2} e_1 = -Be_1, \tag{4.3.23}$$

and consider the system

$$(\nabla_{e_1} h)(e_2, e_1) = (\nabla_{e_2} h)(e_1, e_1). \tag{4.3.24}$$

$$(\nabla_{e_2} h)(e_1, e_2) = (\nabla_{e_1} h)(e_2, e_2). \tag{4.3.25}$$

Compairing the coefficients of the terms, on both sides of the above system, corresponding to the same vectors of the normal space, we get

$$e_1(a_2) - e_2(a_1) + 2da_3 - 2ca_4 - a_3 = 0, \tag{4.3.26}$$

$$e_2(a_1) + 2a_3 d + a_3 - 2ca_4 - e_1(a_2) = 0. \tag{4.3.27}$$

Adding the equations (4.3.26) and (4.3.27) we get

$$a_3 d - ca_4 = 0, \tag{4.3.28}$$

and from the system formed by the equations (4.3.22), (4.3.28) we easily obtain

$$a_3 = a_4 = c = d = 0. \tag{4.3.29}$$

The (4.3.29) prove that $h(X, Y) \in JTM^2, \quad \forall X, Y \in \mathfrak{X}(M^2)$ and more specifically, the second fundamental form becomes

$$h(e_1, e_1) = a_1 Je_1 + a_2 Je_2, \quad h(e_1, e_2) = a_2 Je_1 - a_1 Je_2. \tag{4.3.30}$$

Under the aspect of the relations (4.3.15), (4.3.16) of lemma (24) it is clear that the terms which merit a special attention, in resolving the system formed by the Codazzi equations, are those involving the calculation of the terms $\nabla^{\perp}_{e_k}(e_1 \times e_2)$ and $\nabla^{\perp}_{e_k} J(e_1 \times e_2)$, $k = 1, 2$, where $\nabla^{\perp}$ denotes the normal connection of M^2. As an example let us compute:

$$\nabla^{\perp}_{e_1}(e_1 \times e_2) = \tilde{\nabla}_{e_1}(e_1 \times e_2) - < \tilde{\nabla}_{e_1}(e_1 \times e_2), e_1 > \cdot e_1 - < \tilde{\nabla}_{e_1}(e_1 \times e_2), e_2 > \cdot e_2$$

where

$$\begin{aligned}\tilde{\nabla}_{e_1}(e_1 \times e_2) &= D_{e_1}(e_1 \times e_2) - < D_{e_1}(e_1 \times e_2), p > \cdot p \\ &= D_{e_1}(e_1 \times e_2),\end{aligned}$$

since the assumption of being M^2 totally real implies

$$\begin{aligned}&< e_1 \times e_2, p >= 0 \Longrightarrow < D_{e_1}(e_1 \times e_2), p >= - < e_1 \times e_2, D_{e_1} p >= \\ &- < e_1 \times e_2, e_1 > \Longrightarrow < D_{e_1}(e_1 \times e_2), p >= 0.\end{aligned} \tag{4.3.31}$$

Therefore,

$$\begin{aligned}\tilde{\nabla}_{e_1}(e_1 \times e_2) &= D_{e_1}(e_1 \times e_2) \\ &= (D_{e_1} e_1) \times e_2 + e_1 \times D_{e_1} e_2 \\ &= \{\nabla_{e_1} e_1 + h(e_1, e_1)\} \times e_2 + e_1 \times \{\nabla_{e_1} e_2 + h(e_1, e_2)\} \\ &= \{A e_2 + a_1 J e_1 + a_2 J e_2 + a_3 (e_1 \times e_2) + a_4 J(e_1 \times e_2)\} \times e_2 \\ &+ e_1 \times \{-A e_2 + a_2 J e_1 - a_1 J e_2 + c(e_1 \times e_2) + d J(e_1 \times e_2)\} \\ &= -a_1 J(e_1 \times e_2) - a_2 p - a_3 e_1 + a_4 J e_1 - A(e_1 \times e_2) \\ &+ a_2 p + a_1 J(e_1 \times e_2) - c e_2 + d J e_2,\end{aligned}$$

so:

$$D_{e_1}(e_1 \times e_2) = -a_3 e_1 - c e_2 + a_4 J e_1 + d J e_2 - A(e_1 \times e_2)$$

and finally we get:

$$\nabla^{\perp}_{e_1}(e_1 \times e_2) = a_4 J e_1 + d J e_2 - A(e_1 \times e_2).$$

Let $\{E_1, E_2\}$ be an orthonormal basis of the tangent bundle of M^2. Define the subbundle $\mathbf{B}$ of the normal bundle by setting:

$$B(p) = J(T_p M) \oplus span\{\mathcal{G}(E_1, E_2)\}, \tag{4.3.32}$$

Let us recall at this stage the definition (22) in chapter 3. of a q-dimensional **parallel normal subbundle** of the normal bundle. Using the relations (4.3.15), (4.3.16) of lemma(24) and the equation (4.3.30) we can show that the 3-dimensional normal subbundle **B** is parallel in the normal bundle. Since computations are not carried out explicitely in [C.D.V.V2] we calculate indicatively the following:

$$\nabla^{\perp}_{E_1} JE_1 = J\nabla_{E_1} E_1 + \mathcal{G}(E_1, E_1) + (Jh(E_1, E_1))^n \Longrightarrow$$
$$\nabla^{\perp}_{E_1} JE_1 = AJE_2 - (-a_1 E_1 - a_2 E_2)^n = AJE_2 \in B$$

and

$$\nabla^{\perp}_{E_1}(E_1 \times E_2) = \tilde{\nabla}_{E_1}(E_1 \times E_2) - < \tilde{\nabla}_{E_1}(E_1 \times E_2), E_1 > E_1$$
$$- < \tilde{\nabla}_{E_1}(E_1 \times E_2), E_2 > E_2$$

where for the terms involving the covariant derivative of the vector cross product we have

$$\begin{aligned}
\tilde{\nabla}_{E_1}(E_1 \times E_2) &= D_{E_1}(E_1 \times E_2) \\
&= (D_{E_1}) \times E_2 + E_1 \times D_{E_1} E_2 \\
&= \{\nabla_{E_1} E_1 + h(E_1, E_1)\} \times E_2 + E_1 \times \{\nabla_{E_1} E_2 + h(E_1, E_2)\} \times E_2 \\
&= \{AE_2 + a_1 JE_1 + a_2 JE_2\} \times E_2 + E_1 \times \{-AE_2 + a_2 JE_1 - a_1 JE_2\} \\
&= -A(E_1 \times E_2) \in B.
\end{aligned}$$

Analogoously working for the remaining cases we verify that **B** it is actually parallel in the normal bundle.

Since $\mathbf{B}^{-}$is a 3-dimensional subbundle of the normal bundle of M^2 we may use the following theorem, due to J.Erbacher [Er]:

Theorem 23 *Let $M^n \hookrightarrow \widetilde{M}^{n+p}(c)$be an isometric immersion, M^n be connected, and suppose that the first normal space $N_1(x)$ is contained in a subbundle N of the normal bundle. If N is invariant under parallel translation with respect to the connection in the normal bundle and the dimension of N is a constant l then, there exists a totally geodesic submanifold N^{n+l} of $\widetilde{M}^{n+p}(c)$ such that: $\iota(M^n) \subset N^{n+l}$.*

Using the above theorem we conclude that M^2 lies in a 5-dimensional totally geodesic hypersphere of S^6 .

Let N be a unit vector orthogonal to this S^5. By construction JX is tangent to S^5 and hence orthogonal to N for all X tangent to M^2. Therefore the (4.3.3) is satisfied, the (4.3.4) follows from (4.3.3) and (4.3.5) is true because M^2 is, by hypothesis, totally real in S^6.

Even if M^2 is totally geodesic we can determine uniquely this S^5 in the following way: for any point $p \in M$ this S^5 is the unique great hypersphere of S^6 passing through p and tangent to $T_pM \oplus B(p)$.

If M^2 is not totally geodesic then it can not be contained in a totally geodesic 4-sphere.

The above discussion can be crystalized in the next theorem ([C.D.V.V2]):

Theorem 24

(1) *Let $f : M^2 \hookrightarrow S^6$ be a minimal, non totally geodesic, totally real immersion in S^6 , whose ellipse of curvature is a circle. Then M^2 is contained in a unique totally geodesic S^5 and the warped product immersion (4.3.1) is totally real.*

(2) *Let f, x be as in (4.3.1), then x is totally real if and only if f is totally real and $J(f_\star X)$ is tangent to S^5 for all vector fields $X \in \mathfrak{X}(M)$.*

(3) *Let f, x be as in (4.3.1). If x is totally real, then f is totally real, minimal and has as ellipse of curvature a circle.*

We need the following

Example 20 *Let $F : M^2 \hookrightarrow S^6$ be a linearly full, superminimal (as described in §4.5), almost complex immersion. Let U, V be local orthonormal vector fields defined on a neighborhood W, which span the second normal bundle. For any $\gamma \in (0, \pi)$ we define the tube of radius γ, in the direction of the second normal bundle, by setting:*

$$F_\gamma : W \times S^1 \hookrightarrow S^6 : (x, \theta) \longrightarrow cos(\gamma) f(x) + sin(\gamma)(cos(\theta) U + sin(\theta) V). \tag{4.3.33}$$

The mapping F_γ defines a totally real immersion if and only if: either $cos(\gamma) = 0$ or $tan^2(\gamma) = \frac{4}{5}$.

Let us now assume that $M \hookrightarrow S^6$ is a totally real immersion of the 3-dimensional manifold M. From lemma (19) of Chapter 3 we know that Chen's equality $\delta_M(p) = 2$ is satisfied

at the point p of M if and only if there is an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM such that:

$h(e_1, e_1) = \lambda Je_1$, $h(e_1, e_2) = -\lambda Je_2$, $h(e_2, e_2) = -\lambda Je_1$, and $h(e_k, e_3) = 0$, $\forall k = 1, 2, 3$ where $2\lambda^2 = 3 - \tau(p)$.

Defining $\mathcal{D}(p) = \{X \in T_pM : h(X, Y) = 0, \quad \forall Y \in T_pM\}$ we get that $\mathcal{D}(p)$ is either 3-dimensional (and p is a totally geodesic point) or 1-dimensional.

From now on we assume that the dimension of $\mathcal{D}$ is constant on the manifold M. According to lemma (21) we can assert the existence of local orthonormal vector fields $\{E_i\}_{i=1}^3$, defined on a neighborhood of the point p, and satisfying the conditions

$h(E_1, E_1) = \lambda JE_1$, $h(E_1, E_2) = -\lambda JE_2$, $h(E_2, E_2) = -\lambda JE_1$, and $h(E_k, E_3) = 0$, $\forall k = 1, 2, 3$, where λ is a local function satisfying $2\lambda^2 = 3 - \tau(p)$.

Suppose $\mathcal{G}(E_1, E_2) = JE_3$, $\mathcal{G}(E_2, E_3) = JE_1$, $\mathcal{G}(E_3, E_1) = JE_2$ and moreover we assume that $dim\mathcal{D}(p) = 1$, $\forall p \in M$ and $\mathcal{D}^\perp$ to be integrable.

Let $p \in M$ and define the local functions

$$\gamma_{ij}^k = < \nabla_{E_i} E_j, E_k >, \quad i, j, k = 1, 2, 3. \tag{4.3.34}$$

We have the following

Lemma 25 *The above defined functions γ_{ij}^k satisfy the relations*

$$\gamma_{ij}^k + \gamma_{ik}^j = 0, \quad \gamma_{33}^1 = \gamma_{33}^2 = 0, \quad \gamma_{11}^3 = \gamma_{22}^3, \quad \gamma_{12}^3 = -\gamma_{21}^3, \quad \gamma_{31}^2 = -\frac{1}{3}\left(\gamma_{12}^3 + 1\right) \tag{4.3.35}$$

$$E_1(\lambda) = -3 \cdot \lambda\gamma_{21}^2, \quad E_2(\lambda) = -3 \cdot \lambda\gamma_{11}^2, \quad E_3(\lambda) = -\lambda\gamma_{13}^1. \tag{4.3.36}$$

Proof : From the Codazzi equation we have:

$$(\nabla h)(E_1, E_3, E_3) = (\nabla h)(E_3, E_1, E_3) \Longrightarrow$$

$$\nabla_{E_1}^\perp h(E_3, E_3) - 2h(\nabla_{E_1} E_3, E_3) = \nabla_{E_3}^\perp h(E_1, E_3) - h(\nabla_{E_3} E_1, E_3) - h(E_1, \nabla_{E_3} E_3)$$

and thus

$$h(E_1, \nabla_{E_3} E_3) = 0. \tag{4.3.37}$$

If we put $\nabla_{E_3} E_3 = \kappa E_1 + \mu E_2$ and use (4.3.37) we get

$$h(E_1, \nabla_{E_3} E_3) = 0 \Longrightarrow \lambda\{\kappa JE_1 - \mu JE_2\} = 0 \Longrightarrow$$

$$\kappa = \mu = 0 \Longrightarrow \nabla_{E_3} E_3 = 0 \tag{4.3.38}$$

and the equation (4.3.38) implies that : $\gamma^1_{33} = \gamma^2_{33} = 0$.

$$(\nabla h)(E_3, E_1, E_1) = (\nabla h)(E_1, E_3, E_1) \Longrightarrow$$
$$\nabla^{\perp}_{E_3} \lambda J E_1 - 2h(\nabla_{E_3} E_1, E_1) = -h(\nabla_{E_1} E_3, E_1) - h(E_3, \nabla_{E_1} E_1), \tag{4.3.39}$$

where $\nabla^{\perp}_{E_3} \lambda J E_1 = E_3(\lambda) J E_1 + \lambda J \nabla_{E_3} E_1 + \lambda J E_2$. Using the (4.2.12) of lemma (22) we can write

$$E_3(\lambda) J E_1 + \lambda J \nabla_{E_3} E_1 + \lambda J E_2 + 2 < \nabla_{E_3} E_1, E_2 > \lambda J E_2 =$$
$$- < \nabla_{E_1} E_3, E_1 > \lambda J E_1 + < \nabla_{E_1} E_3, E_2 > \lambda J E_2, \tag{4.3.40}$$

and since $< \nabla_{E_3} E_3, E_1 >= 0 = - < E_3, \nabla_{E_3} E_1 >$, by comparing components on both sides of the (4.3.40), we obtain

$$E_3(\lambda) = \lambda \cdot < \nabla_{E_1} E_1, E_3 >.$$
$$3 < \nabla_{E_3} E_1, E_2 >= - < \nabla_{E_1} E_2, E_3 > -1, \Longrightarrow$$
$$E_3(\lambda) = \lambda \gamma^3_{11}, \quad 3\gamma^2_{31} = -\gamma^3_{12} - 1. \tag{4.3.41}$$

Applying once more the Codazzi equation to the ordered triple E_2, E_1, E_1 we have

$$(\nabla h)(E_2, E_1, E_1) = (\nabla h)(E_1, E_2, E_1) \Longrightarrow < \nabla_{E_2} E_1 + \nabla_{E_1} E_2, E_3 >= 0,$$
$$E_2(\lambda) = 3\lambda < \nabla_{E_1} E_1, E_2 >, \quad E_1(\lambda) = -3\lambda < \nabla_{E_2} E_2, E_1 > . \tag{4.3.42}$$

Combining (4.3.42) with (4.3.39), (4.3.40) and (4.3.41) we complete the proof.
In order to simplify the notation we introduce the local functions :
$a = \gamma^3_{11}, \quad b = \gamma^3_{12}, \quad c = \gamma^2_{11}, \quad d = \gamma^2_{21}$. Then, by lemma (25), we get:

$$\nabla_{E_1} E_1 = cE_2 + aE_3, \quad \nabla_{E_1} E_2 = -cE_1 + bE_3, \quad \nabla_{E_1} E_3 = -aE_1 - bE_2,$$
$$\nabla_{E_2} E_1 = dE_2 - bE_3, \quad \nabla_{E_2} E_2 = -dE_1 + aE_3, \quad \nabla_{E_2} E_3 = bE_1 - aE_2,$$
$$\nabla_{E_3} E_1 = -\frac{1}{3}(b+1) \cdot E_2, \quad \nabla_{E_3} E_2 = \frac{1}{3}(b+1) E_1, \quad \nabla_{E_3} E_3 = 0,$$
$$E_1(\lambda) = -3\lambda d, \quad E_2(\lambda) = 3\lambda c, \quad E_3(\lambda) = \lambda a. \tag{4.3.43}$$

By assumption $\mathcal{D}^{\perp}$ is integrable and $\mathcal{D}^{\perp} = span\{E_1, E_2\}$. Since $\mathcal{D}^{\perp}$ is integrable we have:

$$[E_1, E_2] = \nabla_{E_1} E_2 - \nabla_{E_2} E_1 =$$
$$-cE_1 + bE_3 - dE_2 + bE_3 \in span\{E_1, E_2\} \Longrightarrow b = 0. \tag{4.3.44}$$

Lemma 26 *The local function a, under the above assumptions, satisfies the conditions:* $E_1(a) = 0, \quad E_2(a) = 0, \quad E_3(a) = 1 + a^2$.

Proof: From the Gauss equation and using the (4.3.43) for $b = 0$, we obtain:

$$0 =< R(E_1, E_2)\, E_1, E_3 >= -c \cdot a - E_2(a) + c \cdot a \Longrightarrow E_2(a) = 0.$$

$$0 =< R(E_2, E_2)\, E_2, E_3 >\Longrightarrow E_1(a) = 0.$$

$$1 =< R(E_1, E_3)\, E_3, E_1 >\Longrightarrow E_3(a) - a^2 = 1,$$

and the proof is completed.

In order to state and prove the next lemma we remind well known facts about warped product immersions, especially S. Hiepko's condition [Hi], as it is stated in [N], about the decomposition of a Riemannian manifold into a warped product.

Let M be a Riemannian manifold, with Levi-Civita connection ∇, isometrically immersed into the Riemannian manifold N. A subbundle E, of the tangent bundle TM, will be called:

(a) **parallel** if $\nabla_X Y \in E, \ \forall (X, Y) \in TM \times E$,

(b) **autoparallel** if $\nabla_X Y \in E, \ \forall (X, Y) \in E \times E$,

(c) **totally umbilical** if there exists $H \in E^{\perp}$ such that:

$< \nabla_X Y, Z >=< X, Y > \cdot < H, Z >, \ \forall (X, Y, Z) \in E \times E \times E^{\perp}$ and in this case H will be called the **mean curvature normal** of E,

(d) **spherical** if it is totally umbilical and its mean curvature normal H satisfies the condition $< \nabla_X H, Z >= 0, \ \forall (X, Z) \in E \times E^{\perp}$.

If E is autoparallel, totally umbilical or spherical, then it is involutive and all the leaves of the foliation of M (induced by E) are: totally geodesic, totally umbilical or spherical respectively.

Let $M_0, M_1, \ldots, M_k$ be Riemannian manifolds and $M = M_0 \times \ldots \times M_k$ their product. Let $\pi_i : M \longrightarrow M_i, \quad \forall i = 0, 1, \ldots, k$ be the canonical projection, L_i the foliation of M canonically induced by M_i and $TM \longrightarrow TL_i$ the vector bundle projection. If $\rho_1, \ldots, \rho_k : M_0 \longrightarrow \mathbb{R}_+$ functions, then:

$$< X, Y >=< (\pi_0)_* X, (\pi_0)_* Y > + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 ((\pi_i)_* X, (\pi_i)_* Y >$$

defines a Riemannian metric on M and $(M, <,>)$ will be called the **warped product** $M_0 \times_{\rho_1} M_1 \times \ldots \times_{\rho_k} M_k$ of $M_0, \ldots, M_k$ and $\rho_1, \ldots, \rho_k$ the **warping functions**.

TM splits orthogonally, with respect to the metric $<,>$, i.e: $TM = \oplus_0^k TL_i$.

In [Hi] S. Hiepko proved the following "condition":

Let M be a Riemannian manifold with $TM = \oplus_0^k E_i$ an orthogonal decomposition into non trivial vector subbundles such that E_i is spherical and $E_i^{\perp}$ is autoparallel for $i = 1, \ldots, k$.

Then :

(a) For every point $\widetilde{p} \in M$ there is an isometry ψ of a warped product $M_0 \times_{\rho_1} \times \ldots \times_{\rho_k} M_k$ onto a neighbourhood of $\widetilde{p}$ in M such that:

$\rho_1(\widetilde{p}_0) = \ldots = \rho_k(\widetilde{p}_0) = 1 \quad (A)$

where $\widetilde{p}_0$ is the component of $\psi^{-1}(\widetilde{p})$ in M_0, and such that:

$\psi(\{p_0\} \times \ldots \times \{p_{i-1}\} \times M_i \times \{p_{i+1}\} \times \ldots \times \{p_k\})$ is an integral submanifold of E_i for $i = 0, \ldots, k$ and for all $p_0 \in M_0, \ldots, p_k \in M_k \quad (B)$

(b) If M is simply connected and complete, then for every point $\widetilde{p} \in M$ there is an isometry ψ of a warped product $M_0 \times_{\rho_1} M_1 \times \ldots \times_{\rho_k} M_k$ onto all of M with the properties (A) and (B). (for more details on the warped product immersions and representations of the standard spaces of constant curvature in terms of warped products, see [N]).

Lemma 27 *Let M be as above and $p \in M$. Then, in a neighborhood of p, M is a warped product of an interval $(-\epsilon, +\epsilon)$ and of the connected component N^2 of $\mathcal{D}^{\perp}$ through p.*

Proof: Under the aspect of the above quoted result, and in order to prove that M is actually a warped product of the asserteds form, we need to show that: for the components of the orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$, of the tangent bundle into the non trivial subbundle $\mathcal{D}$ and its orthogonal complement $\mathcal{D}^{\perp}$, the distribution $\mathcal{D}$ is spherical and $\mathcal{D}^{\perp}$ is autoparallel.

From the relations (4.3.43) we obtain $\nabla_{E_3} E_3 = 0$. Therefore $\mathcal{D}^{\perp}$ is totally geodesic and, by setting $b = 0$ in (4.3.43), we obtain

$$< \nabla_{E_i} E_j, E_3 >= \delta_{ij} a \cdot E_3, \quad \forall i, j \in = 1, 2.$$

Hence, $\mathcal{D}^{\perp}$ is totally umbilical in M with mean curvature vector $\eta = a \cdot E_3$. Moreover, since $E_1(a) = E_2(a) = 0$ we deduce that the mean curvature vector is parallel and consequently $\mathcal{D}^{\perp}$ will be spherical. Using Hiepko's condition (see:[Hi], theorem 16) we have the required

assertion.

We can now state and prove the main theorem of the section (see: [C.D.V.V2])

Theorem 25 *Let $f : M^2 \hookrightarrow S^6$ be a minimal, non totally geodesic totally real immersion in S^6 whose ellipse of curvature is a circle. Then M^2 is linearly full in a totally geodesic S^5. Let N be a unit vector perpendicular to this S^5. The map*

$$x : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{cost} M^2 \hookrightarrow S^6 : (t, p) \longrightarrow sin(t)N + cos(t)p$$

is a totally real, non totally geodesic immersion satisfying Chen's equality. Conversely, every totally real, non totally geodesic immersion of a 3-dimensional manifold L into S^6 such that: (i) $\delta_L = 2$, (ii) $dim\mathcal{D} = 1$ and (iii) $\mathcal{D}^\perp$ is integrable,
can be locally obtained in this way.

Proof: Using the theorem (24) we obtain the direct part of the assertion. Conversely, from the lemma (26) we get that L is locally (for $M^2 = N^2$) a warped product and the distributions on L, determined by the the product structure, locally coincide with $\mathcal{D}$ and $\mathcal{D}^\perp$. Moreover, since $h(\mathcal{D}, \mathcal{D}^\perp) = 0$, by using lemma (18) of Chapter 3, we obtain that L is locally immersed as a warped product, with its first factor totally geodesic. Therefore, we can assume that the first factor of the corresponding decomposition is 1-dimensional and this decomposition is unique up to isometries. Hence, L is immersed in the way described by the (4.3.1).

4.4 Sasakian structure on S^5, Hopf lifting and classification.

In this section we are going to present the way a totally real 3-dimensional immersion in S^6 can be constructed, starting from a holomorphic curve ϕ in $\mathbb{C}P^2(4)$, lifting ϕ on the circle bundle, over the domain of the curve induced by the Hopf fibration, to an invariant immersion in S^5 and proving the existence of a suitable imbedding of the 5-sphere into S^6.

Let us at first quote some basic facts on the theory of Sasakian manifolds and in partic-

ular on the Sasakian structure of the S^5 induced from the complex structure of $\mathbb{C}^3$. As references for the theory on the Sasakian manifolds we quote [Y.K2] and [Y.K1].

Definition 24 *Let M be an odd-dimensional differentiable manifold, $\{\phi, \xi, \eta\}$ an (1.1)-type tensor field, a vector field and an 1-form respectively, such that,*

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 0,$$
$$\forall (X, \xi) \in \mathfrak{X}(M) \times \mathfrak{X}^{\perp}(M). \tag{4.4.1}$$

Then M is said to be **an almost contact manifold** *with* **almost contact structure** *$\{\phi, \xi, \eta\}$. Define the* **Nijenhuis** *torsion tensor N_ϕ, corresponding to the tensor field ϕ by setting*

$$N_\phi(X, Y) = [\phi X, \phi Y] - [X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y], \quad \forall X, Y \in \mathfrak{X}(M).$$

The almost contact structure on M is said to be **normal** *if and only if $N_\phi + d\eta \otimes \xi = 0$.*

Suppose that a Riemannian metric tensor field g is given on an almost contact manifold M, then:

Definition 25 *If the almost contact structure satisfies*

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad \forall X, Y \in \mathfrak{X}(M), \quad \forall \xi \in \mathfrak{X}^{\perp}(M)$$

then $\{\phi, \xi, \bar{\eta}, g\}$ is called an **almost contact metric structure** *on M and M an almost contact metric manifold.*
An almost contact metric structure is called **a contact metric structure** *if:*

$$d\eta(X, Y) = g(\phi X, Y), \quad \forall (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M).$$

Moreover, if a contact metric structure on M is in the same time normal, M will be called a **Sasakian manifold**.

Let M be an (2n+1)-dimensional contact metric manifold with associated contact metric structure $\{\phi, \xi, \eta, g\}$. We have the following

Theorem 26 *([Y.K1], pg:272) An almost contact metric structure (ϕ, ξ, η, g) on M is a Sasakian structure if and only if :*

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X, \forall X, Y \in \mathfrak{X}(M). \tag{4.4.2}$$

Let us put $Y = \xi$ in the equation (4.4.2). Using the definition of the covariant derivative of an (1,1)-type tensor field we get:

$$(\nabla_X \phi)\xi = g(X, \xi)\xi - \eta(Y)X \Longrightarrow \nabla_X \phi(\xi) - \phi(\nabla_X \xi) = g(X, \xi)\xi - \eta(Y)X$$

and from the conditions of the definition (25) we have

$$g(X, \xi) = g(\phi X, \phi\xi) - \eta(X)\eta(\xi) = -\eta(X),$$

since $\eta(\xi) = 1$ and $\phi\xi = 0$. Combining the above relations we obtain:

$$\nabla_X \xi = -\phi X, \quad \forall (X, \xi) \in \mathfrak{X}(M) \times \mathfrak{X}(M). \tag{4.4.3}$$

Let us consider the following examples of Sasakian manifolds.

Example 21 *Let $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \|z\| = 1\}$ be the (2n+1)-dimensional unit sphere and for any $z \in S^{2n+1}$ we put $\xi = Jz$, where J denotes the multiplication by i in $\mathbb{C}^{n+1}$. Let $\pi : T_z(\mathbb{C}^{n+1}) \longrightarrow T_z(S^{2n+1})$ be the orthogonal projection.*
Setting $\phi = \pi \circ J$ we obtain a Sasakian structure $\{\phi, \xi, \eta, g\}$ on S^{2n+1} where, η is an 1-form dual to ξ and g is the standard metric tensor field on S^{2n+1}.

In the next example we discuss the Sasakian structure induced on S^5 by the complex structure of $\mathbb{C}^3$ and also sketch its relation with the vector cross product on $\mathbb{R}^7$([D.V]).

Example 22 *Let $i : S^5 \hookrightarrow S^6 \subset \mathbb{R}^7$ be the inclusion map and consider S^5 as an hypersphere of S^6 given by $x_4 = 0$. Let j be the map defined by*
$S^5 \xrightarrow{j} \mathbb{C}^3 : (x_1, x_2, x_3, 0, x_5, x_6, x_7) \longrightarrow (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7)$.
At each point $p = (x_1, \ldots, x_7) \in S^5$ we define the **structural** *vector field ξ by setting*

$$\xi(p) = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3) = e_4 \times p,$$

and for any tangent vector field $v = (v_1, v_2, v_3, 0, v_5, v_6, v_7) \in T_p(S^5)$, *with* v *orthogonal to* ξ, *we have*

$$\phi(v) = (-v_5, -v_6, -v_7, 0, v_1, v_2, v_3) = v \times e_4.$$

It is not hard to check that $(S^5, \phi, \xi, \eta, g)$, *where* η *is dual to* ξ, *is a Sasakian manifold and furthermore, the following hold*

$$\phi\xi = 0 \Longrightarrow \phi\xi(p) = 0, \quad \forall p \in T_pS^5 \Longrightarrow (e_4 \times p) \times e_4 - < (e_4 \times p) \times e_4, p > p = 0.$$

and thus

$$\phi w = w \times e_4 - < w \times e_4, p > \cdot p, \quad \forall w \in T_pS^5. \tag{4.4.4}$$

In order to state and present the main result of this section it is necessary to quote some generalities on the theory of invariant submanifolds of Sasakian manifolds.

Definition 26 *Let* $\widetilde{M}$ *be a (2m+1)-dimensional Sasakian manifold with Sasakian structure* (ϕ, ξ, η, g). *An (2n+1)-dimensional submanifold* M *of* $\widetilde{M}^{2m+1}$ *will be called an* **invariant** *submanifold if and only if*

(1) $\xi \in \mathfrak{X}(M)$, *everywhere on* M *and*

(2) $\phi X \in \mathfrak{X}(M)$, $\forall (p, X) \in M \times \mathfrak{X}(M)$, *that is:* $\phi T_p(M) \subset T_p(M)$, $\forall p \in M$.

If M is an invarint submanifold of the Sasakian manifold $\widetilde{M}$, with induced structure tensor fields denoted also by (ϕ, ξ, η, g) it can be verified that M is a Sasakian manifold. Let $\widetilde{\nabla}, \nabla$ denote the covariant differentiation corresponding to the Levi-Civita connection on $\widetilde{M}$ and M respectively and assume that M is an invariant submanifold of $\widetilde{M}$. The M being invariant means that the structural vector field ξ is tangent to M, recalling the (4.4.3), for any tangent vector field X to M we can write:

$$-\phi X = \widetilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi) = \nabla_X \xi,$$

and by compairing tangential and normal components on both sides we get:

$$h(X, \xi) = 0, \quad \forall (X, \xi) \in \mathfrak{X}(M) \times \widetilde{\mathfrak{X}}(M). \tag{4.4.5}$$

Let us return to the specific case of the 5-dimensional unit sphere. Suppose that M^3 is a 3-dimensional invariant submanifold of S^5. The structural tensor field $\xi = e_4 \times p$ of S^5 will be tangent to M^3.

If $p_r : \mathbb{C}^3 - \{0\} \longrightarrow \mathbb{C}P^2(4)$ denotes the canonical projection from $\mathbb{C}^3 - \{0\}$ onto the quotient complex projective 2-dimensional space , then the so called **Hopf fibration** is the composed mapping $\pi : S^5 \longrightarrow \mathbb{C}P^2(4)$ such that $\pi =: p_r \circ j$ and the inverse image, of each point of $\mathbb{C}P^2(4)$ under π, is a circle. The Hopf fibration annihilates ξ, **i.e:** $d\pi(\xi) = 0$. Then, if M^3 is an invariant submanifold of S^5, the image $\pi(M^3)$ is a holomorphic curve. Conversely, if $N_1 \xrightarrow{\phi} \mathbb{C}P^2(4)$ is a holomorphic curve and PN_1 is the circle bundle over N_1, induced by the Hopf fibration π, then, if the diagram

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5 \\ \downarrow & \pi & \downarrow \quad \pi \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array}$$

commutes, ψ is an invariant immersion in the Sasakian space form S^5 with structure vector field ξ tangent along ψ.

Before we state and prove the main result of the section we recall lemma (19) of Chapter 3.

Theorem 27 *Let $N_1 \xrightarrow{\phi} \mathbb{C}P^2(4)$ be a holomorphic curve in $\mathbb{C}P^2(4)$, PN_1 be the circle bundle over N_1, induced by the Hopf fibration $S^5 \xrightarrow{\pi} \mathbb{C}P^2(4)$, and let $PN_1 \xrightarrow{\psi} S^5$ be an isometric immersion such that the diagram*

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5 \\ \downarrow & \pi & \downarrow \quad \pi \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array}$$

commutes. Then there exists a totally geodesic imbedding $S^5 \xrightarrow{\imath} S^6$ such that the mapping $PN_1 \xrightarrow{\imath \circ \psi} S^6$ is a (3-dimensional) totally real immersion in S^6 satisfying Chen's equality.

Proof: Let $N_1 \xrightarrow{\phi} \mathbb{C}P^2(4)$ be a holomorphic curve, lift ϕ to an invariant immersion $PN_1 \xrightarrow{\psi} S^5$ and identify PN_1 with its image $\psi(PN_1) \subset S^5$.

If h denotes the second fundamental form of the immersion ψ then from (4.4.5) we have $h(X, \xi) = 0$ for any tangent vector field X. It follows, from the above quoted lemma (19), that PN_1 is a minimal submanifold of S^5 satisfying Chen's equality.

It remains to prove the existence of an embedding $S^5 \xrightarrow{i} S^6$ such that the composed mapping $PN_1 \xrightarrow{i\circ\psi} S^6$ is a totally real immersion.

Consider S^5 as the hypersphere in $S^6 \subset \mathbb{R}^7$ given by the equation $x_4 = 0$. Let $p \in PN_1$ and let X be a unit tangent vector of PN_1 such that X is orthogonal to the structure vector field ξ, where $\xi = e_4 \times p$.

In this case $\{X, \phi X, \xi\}$ is an orthonormal basis for the tangent space of PN_1.

But $\xi = e_4 \times p$ and $J\xi = p \times (e_4 \times \xi) = e_4 \perp S^5$ therefore $J\xi \perp PN_1$.

On the other hand $\phi X = X \times e_4 - < X \times e_4, p > \cdot p$ and the following hold:

$$< p \times X, X >= 0, \quad < p \times X, \xi >=< p \times X, e_4 \times p >= - < X, e_4 >= 0$$
$$< p \times X, \phi X >=< p \times X, X \times e_4 >=< p, X \times (X \times e_4) >= - < p, e_4 >= 0 \quad (4.4.6)$$

and from the relations (4.4.6) we get that

$$JX \perp \psi_\star(T_pPN_1), \quad \forall X \perp \xi \quad \text{and} \quad J\xi \perp \psi_\star(T_pPN_1) \quad \text{at each} \quad p \in PN_1$$

and the assertion is proved.

4.5 Almost complex curves in S^6 and classification

Let $\tilde{\phi} : N^2 \longrightarrow S^6$ be an almost complex curve, $\tilde{\phi}$ denotes its position vector field in $\mathbb{R}^7$. For notation's convention we indicate with α the second fundamental form of $\tilde{\phi}$, by J the pullback of the almost complex structure to N^2 and recall that, for any vector fields X, Y tangent to the curve, the following formulae hold (see:[D.V])

$$\alpha(X, JY) = J\alpha(X, Y), \quad A_{J\eta} = JA_\eta = -A_\eta J, \quad \nabla^\perp_X J\eta = \mathcal{G}(X, Y) + J\nabla^\perp_X \eta,$$
$$(\nabla\alpha)(X, Y, JZ) = J(\nabla\alpha)(X, Y, Z) + \mathcal{G}\left(\tilde{\phi}_\star X, \alpha(Y, Z)\right). \quad (4.5.1)$$

Let $p \in N^2$ be a non totally geodesic point, V an arbitrary unit tangent vector field defined on a neighborhod W of p.

Define the (non-zero) function $\mu = \|\alpha(V, V)\|$. Since N^2 is almost complex in S^6 it follows that μ does not depend on the particular choice of V. Let $U = JV$ and define the

following vectors:

$$F_1 = \tilde{\phi}, \quad F_2 = \tilde{\phi}_* V, \quad F_3 = J\tilde{\phi}_* V, \quad F_4 = \frac{\alpha(V,V)}{\mu},$$
$$F_5 = \frac{\alpha(V,JV)}{\mu} = \frac{J\alpha(V,V)}{\mu} = F_1 \times F_4,$$
$$F_6 = F_2 \times \frac{\alpha(V,V)}{\mu}, \quad F_7 = F_3 \times \frac{\alpha(V,V)}{\mu}. \tag{4.5.2}$$

Then $\{F_1, F_2, \ldots, F_7\}$ is a $\mathcal{G}_2$ frame, given that $\mathcal{G}_2$ preserves the vector cross product. Since N^2 is an almost complex curve the vectors $\{F_2, F_3\}$ span its tangent space and consequently $\{F_4, F_5, F_6, F_7\}$ form a basis of the normal space along N^2. Thus, there exist functions $\{\alpha_1, \ldots, \alpha_4\}$ such that:

$$(\nabla\alpha)(V,V,V) = \mu(\alpha_1 F_4 + \alpha_2 F_5 + \alpha_3 F_6 + \alpha_4 F_7). \tag{4.5.3}$$

Using the equations (4.5.1), and in particular the $(\nabla\alpha)(X,Y,JZ) = J(\nabla\alpha)(X,Y,Z) + \mathcal{G}(\tilde{\phi}_* X, \alpha(Y,Z))$, we obtain (given that by hypothesis $U = JV$) that:

$$(\nabla\alpha)(V,V,U) = \mu(-\alpha_2 F_4 + \alpha_1 F_5 + (1+\alpha_4) F_6 - \alpha_3 F_7). \tag{4.5.4}$$

On the other hand, we note that N^2 is:

- Superminimal if and only if $(\nabla\alpha)(V,V,V)$ is perpendicular to $(\nabla\alpha)(V,V,U)$ and both have the same length.
- Linearly full in S^6 if and only if the components of $(\nabla\alpha)(V,V,V)$ and $(\nabla\alpha)(V,V,U)$, orthogonal to F_4 and F_5, are linearly independent

of the preliminaries that there are

We recall from Page 5 four types of almost complex curves (see [B.V.W1]). We now see, from the equations (4.5.3) and (4.5.4), that N^2 is an almost complex surface of type (I), **i.e** superminimal and linearly full, if and only if $\alpha_3 = 0$ and $\alpha_4 = -\frac{1}{2}$.
Similarly, N^2 is an almost complex surface of type (III), **i.e**: linearly full in a totally geodesic S^5, if and only if $\alpha_4 + \alpha_3^2 + \alpha_4^2 = 0$.
Let μ_1, μ_2 be functions, locally defined on N^2 by:

$$\nabla_V V = \mu_1 U, \quad \nabla_V U = \mu_2 V, \quad \nabla_V U = -\mu_1 V, \quad \nabla_U V = -\mu_2 U. \tag{4.5.5}$$

Recalling, from (4.5.2), that $\alpha(V,V) = \mu \cdot F_4$, the definition of the covariant derivative of the second fundamental form α and equation (4.5.3), we obtain:

$$\begin{aligned}\alpha_1\mu^2 &= <(\nabla\alpha)(V,V,V), \alpha(V,V)> \\ &= \frac{1}{2}V<\alpha(V,V), \alpha(V,V)> -2\cdot\mu_1<\alpha(V,U), \alpha(V,V)>,\end{aligned}$$

therefore:

$$\alpha_1\mu^2 = \frac{1}{2}V\left(\mu^2\right) = \mu\cdot V(\mu). \tag{4.5.6}$$

Working similarly with the equation (4.5.4), using the equality $\alpha(V,JV) = \mu\cdot F_5$ we finally obtain

$$\alpha_1 = \frac{V(\mu)}{\mu}, \quad \alpha_2 = \frac{-U(\mu)}{\mu}. \tag{4.5.7}$$

In order to proceed further we need the following technical lemma

Lemma 28 *If D denotes the standard connection on $\mathbb{R}^7$ then*

(1) $D_V(\mu F_4) = \mu\cdot(-\mu F_2 + \alpha_1 F_4 + (\alpha_2 + 2\mu_1)F_5 + \alpha_3 F_6 + \alpha_4 F_7)$

(2) $D_U(\mu F_4) = \mu\cdot(-\mu F_3 - \alpha_2 F_4 + (\alpha_1 - 2\mu_2)F_5 + (1+\alpha_4)F_6 - \alpha_3 F_7$

(3) $D_V(\mu F_5) = \mu\cdot(-\mu F_3 - (\alpha_2 + 2\mu_1)F_4 + \alpha_1 F_5 + (1+\alpha_4)F_6 - \alpha_3 F_7)$

(4) $D_U(\mu F_5) = \mu\cdot(-\mu F_2 - (\alpha_1 - 2\mu_2)F_4 - \alpha_2 F_5 - \alpha_3 F_6 - \alpha_4 F_7)$

(5) $D_V(\mu F_6) = \mu\cdot(-\alpha_3 F_4 - (\alpha_4+1)F_5 + \alpha_1 F_6 + (\alpha_2 + 3\mu_1)F_7)$

(6) $D_U(\mu F_6) = \mu\cdot(-(\alpha_4+1)F_4 + \alpha_3 F_5 - \alpha_2 F_6 + (\alpha_1 - 3\mu_2)F_7)$

(7) $D_V(\mu F_7) = \mu\cdot(-\alpha_4 F_4 + \alpha_3 F_5 - (\alpha_2 + 3\mu_1)F_6 + \alpha_1 F_7)$

(8) $D_U(\mu F_7) = \mu\cdot(\alpha_3 F_4 + \alpha_4 F_5 + (3\mu_2 - \alpha_1)F_6 - \alpha_2 F_7)$

Proof : Although in [D.V] there are some computations, with respect to the way the above equations can be deduced, we shall give some more details concerning the process. First, let us observe that since $\widetilde{\phi}$ is the position vector field of $\widetilde{\phi}(N^2) \subset S^6$, it follows that:

$$<\alpha(V,V), \widetilde{\phi}> = 0 \Longrightarrow <D_V\alpha(V,V), \widetilde{\phi}> = -<\alpha(V,V), V>,$$

and decomposing the connection D of $\mathbb{R}^7$, with respect to the connection on $o(N^2)$, the normal connection and the second fundamental form α of the immersion $\tilde{\phi}$, we have

$$D_V\alpha(V,V) = -\tilde{\phi}_*\left(A_{\alpha(V,V)}V\right) + \nabla^{\perp}_V\alpha(V,V) - < V, \alpha(V,V) > \cdot\tilde{\phi} \tag{4.5.8}$$

$$\nabla^{\perp}_V\alpha(V,V) = (\nabla\alpha)(V,V,V) - 2\alpha(\nabla_V V, V) \Longrightarrow$$

$$\nabla^{\perp}_V\alpha(V,V) = \mu\cdot(\alpha_1 F_1 + \alpha_2 F_5 + \alpha_3 F_6 + \alpha_4 F_7) - 2\alpha(\mu_1 U, V), \tag{4.5.9}$$

and from the (4.5.3), (4.5.5) we get

$$\begin{aligned} A_{\alpha(V,V)}V &= < A_{\alpha(V,V)}, V > \cdot V + < A_{\alpha(V,V)}, U > \cdot U \\ &= < \alpha(V,V), \alpha(V,V) > \cdot V + < \alpha(U,V), \alpha(V,V) > \cdot U \\ &= < \mu F_4, \mu F_4 > \cdot V + < \mu F_5, \mu F_5 > \cdot U. \end{aligned}$$

Hence,

$$A_{\alpha(V,V)}V = \mu^2\cdot V \tag{4.5.10}$$

It is also clear that $< V, \alpha(V,V) >= 0$ and by replacing (4.5.9), (4.5.10) in (4.5.8) and combining with (4.5.3) we obtain:

$$D_V(\alpha(V,V)) = -\mu^2\cdot\tilde{\phi}_*V + (\nabla\alpha)(V,V,V) - 2\mu_1\alpha(U,V) \Longrightarrow$$

so

$$D_V(\alpha(V,V)) = -\mu^2 F_2 + \mu\cdot(\alpha_1 F_4 + \alpha_2 F_5 + \alpha_3 F_6 + \alpha_4 F_7 + 2\mu_1 F_5), \tag{4.5.11}$$

of Lemma 28

and (4.5.11) is actually the equation (1). Similarly we can compute the equations (2), (3) and (4) of lemma (28). In order to deduce the fifth equation let us observe that:

$$\begin{aligned} D_V(\mu F_6) &= D_V(F_2\times\alpha(V,V)) \\ &= (D_V F_2)\times\alpha(V,V) + F_2\times(D_V\alpha(V,V)) \end{aligned}$$

and thus

$$\begin{aligned} D_V(\mu F_6) &= \{\nabla_V V + \alpha(V,V) - < V, V > \cdot\tilde{\phi}\}\times\alpha(V,V) \\ &+ F_2\times\{\mu(-\mu F_2 + \alpha_1 F_4 + (\alpha_2 + 2\mu_1)F_5 + \alpha_3 F_6 + \alpha_4 F_7)\}, \end{aligned} \tag{4.5.12}$$

where $\nabla_V V = \mu_1 U$, and $\tilde{\phi} = F_1$. Using (4.5.3) we get the fifth equation and analogously we work for the rest of the cases.

Theorem 28 *Let $\tilde{\varphi} : N^2 \hookrightarrow S^6$ be an almost complex curve, α denotes the second fundamental form of the immersion $\tilde{\varphi}$, without totally geodesic points. Let UN_2 be the unit tangent bundle of the curve and define the mapping*

$$\tilde{\psi} : UN^2 \longrightarrow S^6 \quad \text{such that} \quad UN^2 \ni v \xrightarrow{\tilde{\psi}} \tilde{\varphi}_*(v) \times \frac{\alpha(v,v)}{\|\alpha(v,v)\|}. \tag{4.5.13}$$

Then, $\tilde{\psi}$ is a (possibly branched) totally real immersion into S^6 satisfying Chen's equality and moreover is linearly full in S^6.

Proof: Consider vector fields U, V as in lemma (28). Each element v of UN^2 can be expressed, with respect to the basis $\{U, V\}$, in the form $v = cos(\frac{t}{3}) \cdot V + sin(\frac{t}{3}) \cdot U$ and it is easy to see that:

$$\tilde{\varphi}_*(v) = cos\frac{t}{3}\tilde{\varphi}_* V + sin\frac{t}{3}\tilde{\varphi}_* U. \tag{4.5.14}$$

$$\alpha(v,v) = cos^2\frac{t}{3}\alpha(V,V) + sin\frac{2t}{3}\alpha(V,U) + sin^2\frac{t}{3}\alpha(U,U). \tag{4.5.15}$$

Using (4.5.2), the definition (4.5.13) of the mapping $\tilde{\psi}$ and standard properties of the vector cross product it is not hard to deduce the following local parametrization for $\tilde{\psi}$, defined in a neighborhood W of a point q :

$$\tilde{\psi}(q,t) = costF_6(q) + sintF_7(q). \tag{4.5.16}$$

for all $(q,t) \in W \times \mathbb{R}$.

In order to find conditions such that $\tilde{\psi}$ is an immersion of the unit tangent bundle of the almost complex curve $\tilde{\varphi}$ into S^6, we need to compute the image under $\tilde{\psi}_*$ of the basis $\{V, U, \frac{\partial}{\partial t}\}$ of UN^2.

From equation (4.5.16), by differentiation with respect to t, we get

$$\tilde{\psi}_*\left(\frac{\partial}{\partial t}\right) = -sintF_6 + costF_7. \tag{4.5.17}$$

Let us compute the image of the vectors V, U. At first we observe that:

$\tilde{\varphi}_*(V) = costD_V F_6 + sintD_V F_7$ and $\tilde{\psi}_*(U) = costD_U F_6 + sintD_U F_7$.

On the other hand $D_V(\mu F_i) = V(\mu)F_i + \mu D_V F_i, \quad \forall i \in \{1, \ldots, 7\}$.

Recalling (4)-(8) of lemma(28), combining with the above observations, and repeating

analogously the process followed in the case of the vector U, we find :

$$\tilde{\psi}_\star(V) = (-\alpha_3 cost - \alpha_4 sint)\, F_4 + (\alpha_3 sint - (\alpha_4 + 1)\, cost)\, F_5 + \left(3\mu_1 - \frac{U(\mu)}{\mu}\right) \tilde{\psi}_\star\left(\frac{\partial}{\partial t}\right). \tag{4.5.18}$$

$$\tilde{\psi}_\star(U) = (\alpha_3 sint - (1 + \alpha_4)\, cost)\, F_4 + (\alpha_3 cost + \alpha_4 sint)\, F_5 + \left(-3\mu_2 + \frac{V(\mu)}{\mu}\right) \cdot \tilde{\psi}_\star\left(\frac{\partial}{\partial t}\right). \tag{4.5.19}$$

From the equations (4.5.19), (4.5.18) and (4.5.17) we see that $\tilde{\psi}$ is an immersion at the point $(q, t) \in W \times \mathbb{R}$ if and only if the following condition is satisfied:

$$(\alpha_3(q)\, cost + \alpha_4(q)\, sint)^2 + (\alpha_3(q)\, sint - (1 + \alpha_4(q))\, cost)^2 \neq 0. \tag{4.5.20}$$

Remark 10 *If N^2 is linearly full and superminimal (i.e: of type (I)) then the equation (4.5.20) is always satisfied. Indeed, let us choose a vector V such that, at the point q of the neighborhood W, the $\|(\nabla\alpha)(u, u, u)\|^2$ attains an absolute maximum at $V(q)$.*
In this case we can always ensure that the relations $\alpha_3(q) \neq 0$, $\alpha_4(q) \neq 0$ will hold and consequently the (4.5.20) will be satisfied, unless the $\alpha_4 = -1$ and $sin(t) = 0$ occur. This means that the branching points of the immersion $\tilde{\psi}_\star$ are two antipodal points on the circle corresponding to the points on N^2 where the second normal space has no maximal rank.

We restrict now to the open dense subset on which $\tilde{\psi}$ is an immersion. From the (4.5.17), (4.5.18) and (4.5.19) we get that $\{-sintF_6 + costF_7,\quad F_4,\quad F_5\}$ is a basis of $\tilde{\psi}_\star(\mathcal{U}N^2)$ along $\mathcal{U}N^2$. Since $\tilde{\psi}(q, t) = costF_6(q) + sintF_7(q)$, by a direct application of the definition of the almost complex structure, we deduce:

$$J(-sintF_6 + costF_7) = (costF_6 + sintF_7) \times (-sintF_6 + costF_7) = -F_1.$$

$$J(F_4) = (costF_6 + sintF_7) \times F_4 = -costF_2 - sintF_7.$$

$$J(F_5) = (costF_6 + sintF_7) \times F_5 = -sintF_2 + costF_3.$$

Hence, $\tilde{\psi}$ is a **totally real** immersion in S^6.
Using the (4.5.17), (4.5.18), (4.5.19) we obtain:

$$D_{\frac{\partial}{\partial t}}\left(\tilde{\psi}_\star\left(\frac{\partial}{\partial t}\right)\right) = -costF_6 - sintF_7 = -U. \tag{4.5.21}$$

$$D_{\frac{\partial}{\partial t}} F_4 = D_{\frac{\partial}{\partial t}} F_5 = 0. \tag{4.5.22}$$

where $\tilde{\psi}$ is actually the position vector field of $\mathcal{U}N^2$ in $\mathbb{R}^7$ and D denotes the Levi-Civita connection in $\mathbb{R}^7$. If h denotes the second fundamental form of the immersion $\tilde{\psi}$ then:

$$h(\tilde{\psi}_*(\frac{\partial}{\partial t}), \tilde{\psi}_* w) = 0, \quad \forall w \in \mathcal{U}N^2.$$

Recalling lemma (19) of chapter 3 we conclude that $\mathcal{U}N^2$ (with its induced metric) satisfies Chen's equality.

It remains only to check that $\mathcal{U}N^2$ lies linearly full in S^6. In order to prove the full linearity in S^6 we need to compute the first normal space of the immersion $\tilde{\psi}$. Given that details for the computation of the first normal space are ommited in [D.V], we intend to carry out explicitely some of them. Since

$$< \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right), \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right) >= 1 \Longrightarrow < D_V \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right), \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right) >= 0,$$
$$< D_V \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right), F_4 >= - < \tilde{\psi}_* \left(\frac{\partial}{\partial t}\right), D_V F_4 >$$

(in analogy we can deduce information for the covariant derivative with respect to the vector U and also for the component in the direction of the vector F_5) hold, we see that the first normal space of the immersion ψ can be computed by taking the normal component (in S^6) of the $D_V F_4$, $D_V F_5$, $D_U F_4$ and $D_U F_5$ where D denotes the Riemannian connection on $\mathbb{R}^7$. Let us consider for example the term $D_V F_4$. Recalling lemma (28) we compute

$$D_V(\mu F_4) = V(\mu) F_4 + \mu D_V F_4 = \frac{1}{\mu}\{D_V(\mu F_4) - V(\mu) F_4\} =$$
$$\frac{1}{\mu}\{\mu(-\mu F_2 + \alpha_1 F_4 + (\alpha_2 + 2\mu_1) F_5 + \alpha_3 F_6 + \alpha_4 F_7) - V(\mu) F_4\}$$

and by using (4.5.7) we deduce that:

$$D_V F_4 = -\mu F_2 + (\alpha_2 + 2\mu_1) F_5 + \alpha_3 F_6 + \alpha_4 F_7.$$

Therefore, the component of $D_V F_4$ which contributes to the first normal space of $\tilde{\psi}$ is the vector $X = -\mu F_2$. Similarly we see that the first normal space is actually spanned by the vectors $X = -\mu F_2$ and $Y = \mu F_3$.

Repeating the same procedure we find

$$D_V F_2 = \mu_1 F_3 + \mu F_4 - F_1.$$

and since F_1 is an element of the second normal space we obtain that $\widetilde{\nu}(\mathcal{U}N^2)$ is linearly full in S^6 and the proof is completed.

Suppose that, a non totally geodesic almost complex curve is branched or has totally geodesic points. In this case we can extend the immersion described by (4.5.13) in the following way:
Let $N^2 \xrightarrow{\phi} S^6$ be the almost complex curve and consider two, mutually orthogonal, unit vectors u_1 and u_2, in the direction of the second normal space of the curve. An element of the (unit) second normal bundle will be of the form $w = costu_1 + sintu_2$ and the tube SN^2 of radius $\frac{\pi}{2}$ around the (almost complex) surface immersion $\widetilde{\phi}$ (in the direction determined by the vectors u_1, u_2) can be parametrized by the mapping:

$$SN^2 \ni cost\, u_1(p) + sint\, u_2(p) \longrightarrow cos\left(\frac{\pi}{2}\right)\widetilde{\phi}(p) + sin\left(\frac{\pi}{2}\right)(cost\, u_1(p) + sint\, u_2(p)) \in S^6.$$

Adapting the method used in the proof of theorem (28) in case where the unit tangent bundle is replaced by the second normal bundle we obtain

Theorem 29 *Let $\widetilde{\phi} : N^2 \longrightarrow S^6$ be a (branched) almost complex immersion. Then the above described tube SN^2 is a 3 - dimensional (possibly branched) totally real submanifold of S^6 satisfying Chen's equality.*

Remark 11 *The parametrization of $\widetilde{\nu}$ given by the equation (4.5.16) is nothing but the tube of radius $\frac{\pi}{2}$ around the almost complex surface immersion $\widetilde{\phi}$, in the direction determined by the vectors F_6 and F_7 of the unit tangent bundle.*

4.6 Local converses

We are going to present local converses of the theorems (27), (28) and (29).

Theorem 30 *Let $F : M^3 \longrightarrow S^6$ be a totally real immersion which is not linearly full in S^6. Then M^3 automatically satisfies Chen's equality and there exists a totally geodesic S^5 and a holomorphic immersion $\phi : N_1 \xrightarrow{\phi} \mathbb{C}P^2$ such that F is congruent to ν, which is obtained from ϕ in the way described in theorem (27).*

Proof: $M^3 \xrightarrow{\psi} S^6$ (totally real) is by hypothesis non linearly full in S^6, thus there exists a vector $u \in \mathbb{R}^7$ which is normal to the image of M^3 under ψ. Without loss of generality we assume that $e_4 = u$.

Since M^3 is totally real and $e_4 \perp M^3$ we get that $\xi = e_4 \times p$ is a tangential vector field. Therefore we can consider the diagram

$$\begin{array}{ccc} M^3 \xrightarrow{\psi} S^5 & \subset & S^6 \\ \downarrow \pi & & \downarrow \pi \\ N_1 \xrightarrow{\phi} & & \mathbb{C}P^2 \end{array}$$

where $\pi : M^3 \longrightarrow N_1$ denotes the Hopf fibration from S^5 onto $\mathbb{C}P^2$ and $N_1 = \pi(M^3)$.

Since ξ is tangential then N_1 is well defined and in order to show that $\phi : N_1 \longrightarrow \mathbb{C}P^2$ is holomorphic it will be suffient to verify that ϕ is an almost complex mapping which by definition means $(\phi \circ \pi)_*(T_p M^3) \subset T_p M^3, \forall p \in M^3$, i.e: $\phi(M^3)$ is an invariant submanifold of the Sasakian space form S^5.

Take a unit vector field $X \in \mathfrak{X}(M^3)$ such that X is normal to $\xi = -Je_4$. Since $\mathcal{G}(U, V)$ is normal to M^3 for all $U, V \in \mathfrak{X}(M^3)$ we get

$\mathfrak{X}^{\perp}(M^3) = span\{e_4, JX, \mathcal{G}(X, \xi)\}$ where $\mathcal{G}(X, \xi) = Je_4 \times \xi$. Recalling (see:example 22) that the (1,1)-type tensor field ϕ satisfies $\phi(w) = w \times e_4 - < w \times e_4, p > \cdot p, \forall w \in T_p M^3$ we obtain

$$\phi(X) = X \times e_4 - < X \times e_4, p > \cdot p = X \times e_4 = X \times e_4$$

since $\mathcal{G}(X, \xi) \in \mathfrak{X}(M^3)$. On the other hand

$$< \phi(X), JX > = < X \times e_4, p \times X > = - < p, e_4 > = 0$$

$$< \phi(X), e_4 > = < X \times e_4, e_4 > = 0$$

$$< \phi(X), (p \times e_4) \times X > = < X \times e_4, (p \times e_4) \times X > = 0$$

and consequently $\phi(X) \in \mathfrak{X}(M^3)$. $\forall X \in \mathfrak{X}(M^3)$, i.e: M^3 is actually an invariant submanifold of S^5 and the proof of the theorem is completed.

Let us now take under consideration the case where the totally real 3 - dimensional submanifold is contained in a totally geodesic S^5.

Theorem 31 *Let $F : M^3 \longrightarrow S^6$ be a totally real, linearly full, 3 - dimensional immersion sastisfying Chen's equality and $p \in M^3$ be a non totally geodesic point. Then, there exists an almost complex curve $N^2 \xrightarrow{\widetilde{\phi}} S^6$ such that, F is locally congruent to $\widetilde{\psi}$ (around p) which is obtained from $\widetilde{\phi}$ in the way described in theorem (27).*

Proof: Let $p \in M^3$ be a non totally geodesic point and choose (see:lemma 21) an orthonormal frame field $\{E_1, E_2, E_3\}$, defined on an open neighborhood of p, such that:

$$h(E_1, E_2) = \lambda JF_\star E_1, \quad h(E_1, E_2) = -\lambda JF_\star E_2, \quad h(E_2, E_2) = -\lambda JF_\star E_1,$$
$$h(E_1, E_3) = h(E_2, E_3) = h(E_3, E_3) = 0,$$
$$\mathcal{G}(F_\star E_1, F_\star E_2) = JF_\star E_3, \quad \mathcal{G}(F_\star E_2, F_\star E_3) = JF_\star E_1, \quad \mathcal{G}(F_\star E_3, F_\star E_1) = JF_\star E_2$$

where $2 \cdot \lambda = 3 - \tau^2$ and

$$\nabla_{E_1} E_1 = cE_2 + aE_3, \quad \nabla_{E_1} E_2 = -cE_1 + bE_3, \quad \nabla_{E_1} E_3 = -aE_1 - bE_2,$$
$$\nabla_{E_2} E_1 = dE_2 - bE_3, \quad \nabla_{E_2} E_2 = -dE_1 + aE_3, \quad \nabla_{E_2} E_3 = bE_1 - aE_2,$$
$$\nabla_{E_3} E_1 = -\frac{1}{3}(b+1) E_2, \quad \nabla_{E_3} E_2 = \frac{1}{3}(b+1) E_1, \quad \nabla_{E_3} E_3 = 0,$$

where $E_1(\lambda) = -3\lambda d$, $E_2(\lambda) = 3\lambda c$ and $E_3(\lambda) = \lambda a$.
Identify a neighborhood of the point p with a neighborhood $I \times W_1$ of the origin in $\mathbb{R}^3$ with coordinates (t, u, v), such that $p = (0, 0, 0)$, and $E_3 = \frac{\partial}{\partial t}$.
There exist functions α_1 and α_2 defined on W_1 such that the vectors $E_1 + \alpha_1 E_3$ and $E_2 + \alpha_2 E_3$ form a basis of the tangent space to $W_1 \subset M^3$ at the point $q = (0, u, v)$. Let us consider at first the case where $a = 0$ and $b = -1$. Then, if $\widetilde{\nabla}$ denotes the Riemannian connection on S^6, we get:

$$\widetilde{\nabla}_{E_1} JF_\star E_3 = J\widetilde{\nabla}_{E_1} F_\star E_3 + \mathcal{G}(F_\star E_1, F_\star E_3) = J(\nabla_{E_1} F_\star E_3 + h(E_1, E_3)) - JF_\star E_2 =$$
$$-aJF_\star E_1 - (b+1) JF_\star E_2 \Longrightarrow \widetilde{\nabla}_{E_1} JF_\star E_3 = 0 \quad \text{and}$$
$$\widetilde{\nabla}_{E_2} JF_\star E_3 = J\widetilde{\nabla}_{E_2} F_\star E_3 + \mathcal{G}(F_\star E_2, F_\star E_3) = J(\nabla_{E_2} F_\star E_3 + h(E_2, E_3)) + JF_\star E_1$$
$$\Longrightarrow \widetilde{\nabla}_{E_2} JF_\star E_3 = (b+1) JF_\star E_1 - aJF_\star E_2 \Longrightarrow \widetilde{\nabla}_{E_2} JF_\star E_3 = 0.$$

Hence, $JF_\star E_3$ is a constant vector field along M^3.
It is also clear that the first normal space is spanned by $JF_\star E_1, JF_\star E_2$ and in the same

time is a parallel subbundle of the normal bundle since

$< \tilde{\nabla}_{E_i} JF_\star E_1, JF_\star E_3 >=< \tilde{\nabla}_{E_2} JF_\star E_i, JF_\star E_2 >= 0$ for $i = 1, 2$.

Using the Erbacher's theorem (23) on the reduction of the codimension we deduce that M^3 has to be contained in a totally geodesic S^5 and this actually contradicts our assumption. Similarly working under the hypothesis of being $a = 0$ and $b = -1$ we get another contradiction.

Under the aspect of the preceding discussion we can suppose that the set W of non totally geodesic points , such that $a^2 + (b+1)^2 \neq 0$, is an open dense subset of M^3 and $W \cap W_1$ is open and dense in W_1.

Since $\nabla_{E_3} E_3 = 0$ and $h(E_3, E_3) = 0$ we can view $E_3 = \frac{\partial}{\partial t}$ as a constant vector field, restricted on I and evaluated at the point $t = 0$, and thus we deduce the following parametrization for $F(M^3)$:

$$F(t, u, v) = costF(0, u, v) + sintF_\star(E_3(0, u, v)). \tag{4.6.1}$$

Let us now define the mapping $\tilde{\phi}$ by :

$$W_1 \ni (u, v) \xrightarrow{\tilde{\phi}} JF_\star (E_3 (0, u, v)) \in S^6. \tag{4.6.2}$$

Recalling that $\{E_1 + \alpha_1 E_3, E_2 + \alpha_2 E_3\}$ is a basis for the tangent space of $W_1 \subset M^3$ at the points $q = (0, u, v)$ we get

$$\tilde{\phi}(0, u, v) = JF_\star (q) E_3 (q). \tag{4.6.3}$$

$$\begin{aligned} \tilde{\phi}_\star (E_1 + \alpha_1 E_3) &= D_{E_1 + \alpha_1 E_3} JF_\star E_3 \\ &= -a (q) JF_\star E_1 (q) - (b+1) (q) JF_\star E_3 \end{aligned}$$

hence

$$\tilde{\phi}_\star (E_1 + \alpha_1 E_3) = -a (q) F (q) \times F_\star E_1 (q) - (b+1) (q) F (q) \times F_\star E_2 (q). \tag{4.6.4}$$

and finally from the equalities

$$\begin{aligned} \tilde{\phi}_\star (E_2 + \alpha_2 E_3) &= D_{E_2 + \alpha_2 E_3} JF_\star E_3 \\ &= ((b+1) JF_\star E_3 - a JF_\star E_2) (q) \end{aligned}$$

we obtain

$$\widetilde{\phi}_{\star}\left(E_2+\alpha_2 E_3\right)=(b+1)(q)\,F(q)\times F_{\star}E_1(q)-a(q)\,F(q)\times F_{\star}E_2(q). \tag{4.6.5}$$

Hence $\widetilde{\phi}$ is an immersion at points where $a(q)\neq 0$ and $b(q)\neq -1$ and in this case the $\widetilde{\phi}_{\star}(T_qW_1)$ is spanned by the vectors $JF_{\star}E_1(q)$ and $JF_{\star}E_2(q)$.

Using standard properties of the vector cross product and (4.6.3)-(4.6.5) we can write

$$\begin{aligned}\widetilde{\phi}\times\widetilde{\phi}_{\star}\left(E_1+\alpha_1 E_3\right)&=(F\times F_{\star}E_3)\times(-aF\times F_{\star}E_1-(b+1)\,F\times F_{\star}E_2)\\&=aF\times F_{\star}E_2-(b+1)\,F\times F_{\star}E_1,\end{aligned}$$

and from the above equalities we get:

$$\widetilde{\phi}\times\phi_{\star}\left(E_1+\alpha_1 E_3\right)=-\widetilde{\phi}_{\star}\left(E_2+\alpha_2 E_3\right) \tag{4.6.6}$$

and thus $\widetilde{\phi}$ is a (possibly branched) almost complex immersion in S^6.

Recalling that for an almost complex curve $\phi: N_2\longrightarrow S^6$ with second fundamental form α we have $\alpha(X,JY)=J\alpha(X,Y)$ we see that, in order to compute the first normal space, it will be sufficient to compute the components of $D_{E_1+\alpha_1E_3}JF_{\star}E_1$ and $D_{E_2+\alpha_2E_3}JF_{\star}E_1$ normal to $\widetilde{\phi}(W_1)$ and tangential to S^6. Using lemma (22) we deduce

$$\begin{aligned}D_{E_1+\alpha_1E_3}JF_{\star}E_1&=\left(c+\alpha_1-\frac{1}{3}(b+1)\,\alpha_1\right)JF_{\star}E_2-\lambda F_{\star}E_1+aJF_{\star}E_3,\\D_{E_2+\alpha_2E_3}JF_{\star}E_1&=\left(d+\alpha_2-\frac{1}{3}(b+1)\,\alpha_2\right)JF_{\star}E_2+\lambda F_{\star}E_2-(b+1)\,JF_{\star}E_3,\end{aligned} \tag{4.6.7}$$

and it is clear that the first normal space to $\phi(W_1)$ at the point q is spanned by $F_{\star}E_1(0,u,v)$ and $F_{\star}E_2(0,u,v)$ where the $F(0,u,v)$ and $F_{\star}E_1(0,u,v)$ are mutually orthogonal vector fields which are both orthogonal to the tangent space and the first normal space of the immersion ϕ. But, for X and Y orthonormal vector fields along an almost complex immersion φ which are orthogonal to both the tangent and first normal space, by taking a different parameter t, if necessary, instead of the parametrization (4.5.17), the map φ can be locally expressed under the form:

$$\widetilde{\varphi}(t,q)=cost\,X(q)+sint\,Y(q).$$

Since the complex line bundles determined by $\widetilde{\phi}(N_2)$ and the first normal space can be extended in the points where ϕ is not an immersion, the totally real immersion $\widetilde{\psi}$ (corresponding to ϕ by theorem (28) or (29)) can be written as

$$\widetilde{\psi}(t,u,v) = costF(0,u,v) + sintF_{\star}E_3(0,u,v)$$

where $\widetilde{\psi} = F$ and the proof is completed.

EPILOGUE

Recently, quasi-Einstein, i.e: the Ricci tensor has an eigenvalue with multiplicity at least 2, totally real submanifolds of S^6 have been investigated in [D.D.V.V], where examples of 3-dimensional quasi- Einstein totally real submanifolds have been constructed by considering tubes with different radii. Specifically, starting from an almost complex curve $N^2 \xrightarrow{\phi} S^6$ without totally geodesic points, the mapping

$$\psi_\gamma : \mathcal{U}N^2 \longrightarrow S^6, \quad v \longrightarrow cos\gamma\phi + sin\gamma v \times \frac{\alpha(v,v)}{\|\alpha(v,v)\|},$$

defined on the unit tangent bundle $\mathcal{U}N^2$, where α denotes the second fundamental form of the surface N^2, is an immersion on an open dense subset of $\mathcal{U}N^2$ and moreover, ψ is totally real if an only if either: (a) $\gamma = \frac{\pi}{2}$ or (b) $cos^2\gamma = \frac{5}{9}$ and N^2 is superminimal. In both cases the immersion ψ defines a quasi-Einstein metric on $\mathcal{U}N^2$ and if (a) holds then, with respect to this metric, $\delta_{\mathcal{U}N^2} = 2$ and if (b) holds then $\delta_{\mathcal{U}N^2} < 2$.

Furthermore, a converse of the above result is given.

In [V] L. Vrancken studied Langrangian immersions $M^3 \xrightarrow{F} S^6$ which admit a unit Killing vector field E_3, i.e:

$$E_3(< Y, Z >) =< [E_3, Y], Z > + < Y, [E_3, Z] >, \quad \forall Y, Z \in \mathfrak{X}(M^3),$$

(see: [Mat], pg:88), whose integral curves are great circles. Then, there exists an open and dense subset U of M^3 such that each $p \in U$ has a neighborhood V such that $V \xrightarrow{F} S^6$ is obtained either as:

(1) a Hopf lift of a holomorphic curve in $\mathbb{C}P^2(4)$ in the way described in theorem (27),

(2) a tube of radius $\frac{\pi}{2}$ in the direction of the second normal bundle on an almost complex superminimal surface in the way described in theorem (28),

(3) a tube of radius $\frac{\pi}{2}$ in the direction of the first normal bundle on an almost complex superminimal surface $N^2 \xrightarrow{\phi} S^6$, described by

$$\mathcal{U}N^2 \xrightarrow{\phi} S^6, \quad v \longrightarrow \frac{\alpha(v,v)}{\|\alpha(v,v)\|}.$$

The complete classification of the 3-dimensional totally real submanifolds of the nearly Kaehler S^6, a problem which has been resolved (see:[B.V.W1]) with respect to the almost

complex surfaces, is still open, only partial results. as the results quoted in the present thesis, are known and the problem seems to be a hard one.

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