Gauge theory constraints on the fermion-boson vertex

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Gauge Theory Constraints on the Fermion-Boson Vertex

by

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Abstract

In this thesis we investigate the role played by fundamental properties of QED in determining the non-perturbative fermion-boson vertex. These key features are gauge invariance and multiplicative renormalisability. We use the Schwinger-Dyson equations as the non-perturbative tool to study the general structure of the fermion-boson vertex in QED. These equations, being an infinite set, have to be truncated if they are to be solved. Such a truncation is made possible by choosing a suitable non-perturbative ansatz for the fermion-boson vertex. This choice must satisfy these key properties of gauge invariance and multiplicative renormalisability.

In this thesis we develop the constraints, in the case of massless unquenched QED, that have to be fulfilled to ensure that both the fermion and photon propagators are multiplicatively renormalisable–at least as far as leading and subleading logarithms are concerned. To this end, the Schwinger-Dyson equations are solved perturbatively for the fermion and photon wave-function renormalisations. We then deduce the conditions imposed by multiplicative renormalisability for these renormalisation functions. As a last step we compare the two results coming from the solution of the Schwinger-Dyson equations and multiplicative renormalisability in order to derive the necessary constraints on the vertex function. These constitute the main results of this part of the thesis.

In the weak coupling limit the solution of the Schwinger-Dyson equations must agree with perturbation theory. Consequently, we can find additional constraints on the 3-point vertex by perturbative calculation. Hence, the one loop vertex in QED is then calculated in arbitrary covariant gauges as an analytic function of its momenta. The vertex is decomposed into a longitudinal part, that is fully responsible for ensuring the Ward and Ward-Takahashi identities are satisfied, and a transverse part. The transverse part is decomposed into 8 independent components each being separately free of kinematic singularities in **any** covariant gauge in a basis that modifies that proposed by Ball and Chiu. Analytic expressions for all 11 components of the $O(\alpha)$ vertex are given explicitly in terms of elementary functions and one Spence function. These results greatly simplify in particular kinematic regimes. These are the new results of the second part of this thesis.

Dedicated to my mum, dad and my brother
to whom I owe everything
and are my light in the darkness.

We receive three educations,
one from our parents,
one from our schoolmasters,
and one from the world.
The third contradicts all that
the first two teach us.

-Montesquieu-

Declaration

This thesis is based on research by the author, carried out between 1992 and 1995. The material presented has not been submitted previously for any degree in either this or any other University. No claim of originality is made for the work contained in either Chapter One or Chapter Two. Chapter Three, Chapter Four and Chapter Five are summarised in the paper which is in preparation in collaboration with my supervisor M.R. Pennington :

Studying multiplicatively renormalisable Schwinger-Dyson equations to construct the full vertex in arbitrary gauge in unquenched QED

Chapter Six is based on the paper of the author in collaboration with M.R. Pennington and M. Reenders of the University of Groningen. This work has, in fact, been performed independently in Durham and Groningen and the authors joined forces only to compare and check their answer and to write the paper :

One loop QED vertex in any covariant gauge : its complete analytic form,
A.Kızılersü, M.Reenders and M.R. Pennington, Phys. Rev. D52 1242 (1995).

Statement of Copyright

Acknowledgment

This is the only place in this thesis where I can express my feelings, without using any equations, to the people whose contribution made this thesis complete and the process bearable.

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Chapter 1

Introduction

The sciences do not try to explain,
they hardly even try to interpret,
they mainly make models.
By a model is meant a mathematical construct which,
with the addition of certain verbal interpretations,
describes observed phenomena.
The justification of such a mathematical construct is
solely and precisely that it is expected to work.

-J.V. Neumann-

1.1 "The Jewel of Physics"

The physicist's endless quest is to understand nature. A major success is Quantum Electrodynamics, the quantum theory of the interaction of light and matter. According to R. Feynman it is the " jewel of physics" and the "physicist's proudest possession". Quantum Electrodynamics (QED) is the best tested of all theories of the fundamental interactions. It is a gauge theory which predicts experimental results to great precision.

1.2 Gauge Field Theories [1, 2]

The main interest of high energy physics is the particles and their interactions, which are governed by symmetry groups. These determine conservation laws and the invariances of the physics. For instance, when we transform quantities by translations or rotations, if they remain unchanged we can talk about symmetries and conservation laws. In mathematical language, this lack of change means physical laws are invariant under certain groups of transformations, the symmetry groups. Basically, these symmetry groups [3] are:

1- **Space-time symmetries** which include the Lorentz and Poincaré groups. In quantum field theory (QFT), particles are described by fields which obey appropriate equations. Invariance of these equations under the space-time symmetry groups leads to the conservation of energy, momentum and angular momentum. The momentum four-vector is the generator of space-time translation and angular momentum is that of space-time rotations. One of the most important examples of this kind of invariance is the principle of relativity, which requires that the equations of motion should be invariant under the Poincaré group. The fields can only be scalars, spinors, vectors and tensors under this group. Such a Poincaré transformation is :

$$x^\mu \rightarrow x'^\mu = a^\mu + l^\mu_\nu x^\nu \;, \quad \text{(space-time translations plus rotations)} \,. \tag{1.2.1}$$

For instance, vector and Dirac fields transform under this group as :

$$\begin{aligned} \text{Vector field} \;&:\; \phi'^\mu(x') = l^\mu_\nu \phi^\nu(x) \quad , \\ \text{Dirac field} \;&:\; \Psi'(x') = D(\epsilon)\Psi(x), \; D(\epsilon) = e^{-\frac{i}{4}\sigma^{\mu\nu}\epsilon_{\mu\nu}} \,. \end{aligned} \tag{1.2.2}$$

2-**Internal symmetries :** Experiments in particle physics have shown that as well as being representations of the Poincaré group, the fields also have internal degrees of freedom. This is due to the invariance of the equations of motion under certain groups of transformations such as the $U(1)$ and $SU(2)$ phase transformation groups related to internal quantum numbers (isospin, flavour, color, etc.).

These symmetries can be either *global* or *local.* Global transformations are independent of space-time coordinates. Local transformations will be different from one point to another (i.e. they depend on space-time coordinates). An example of such a symmetry is the $U(1)$ gauge group of electromagnetic theory. This symmetry ensures that the observables do not depend on the phase of the field; the physics constructed with $\Psi(x)$ is the same as the physics constructed with the fields $\Psi'(x)$ where :

$$\Psi'(x) = e^{ie\Lambda}\,\Psi(x) \quad , \tag{1.2.3}$$

If Λ has no dependence on x then we have a *global* symmetry, but if $\Lambda = \Lambda(x)$ then we have a *local* symmetry. As we shall see all the forces of nature (the strong and electroweak forces) are well described by local gauge theories.

1.2.1 Quantum Electrodynamics : QED as a Gauge Theory

As mentioned before, in a QFT, particles are identified by the fields which obey relativistic wave equations. These fields interact with each other under the influence of four fundamental forces by the exchange of gauge bosons. One of these forces is the **Electromagnetic Force** which has been developed over a long period with contributions from Faraday, Maxwell, Einstein, Feynman etal. Today the nature of the electromagnetic interaction is better known than any other. The unification of Maxwell's electromagnetism with the quantum field theory is called **Quantum Electrodynamics** or **QED** [4, 5].

In QED, the relativistic wave equations for spin-1/2 and spin-1 particles are given below :

I. Vector (Maxwell) Field :

Photons, the spin-1 or vector particles, are described by Maxwell's equations. In covariant form these are :

$$\partial_\mu F^{\mu\nu} = 0 \ , \tag{1.2.4}$$

where the field strength tensor $F^{\mu\nu}$ can be expressed in terms of the vector potential A^μ by :

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ , \tag{1.2.5}$$

where A_μ denotes the photon field. The appropriate Lagrangian is :

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ . \tag{1.2.6}$$

II. Dirac Spinor Field :

Fermions (i.e. spin-1/2 particles obeying Fermi-Dirac statistics) are described by the Dirac equation, which in covariant form is :

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = 0 \ , \tag{1.2.7}$$

where Ψ denotes spinor (fermion) fields and γ_μ are Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Eqn. (1.2.7) is the equation of motion of the Dirac Lagrangian, which is :

$$\mathcal{L}_{Dirac} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi \ . \tag{1.2.8}$$

If one looks for invariances of spin-1/2 fields, the Dirac Lagrangian for a free theory is found to be invariant under :

$$\Psi \rightarrow e^{ie\Lambda}\,\Psi \quad , \quad \overline{\Psi} \rightarrow \overline{\Psi}\,e^{-ie\Lambda} \quad , \tag{1.2.9}$$

where Λ here is a <u>constant</u>. Thus the transformation in Eqn. (1.2.9) is global. In order to generalize the global symmetry to a local one, the Lagrangian must be invariant under the local transformation :

$$\Psi \longrightarrow \Psi' = e^{ie\Lambda(x)}\,\Psi \quad . \tag{1.2.10}$$

When this transformation is applied to the Dirac Lagrangian, Eqn. (1.2.8), for the free theory, it yields :

$$\mathcal{L}'_{Dirac} = \mathcal{L}_{Dirac} - e\overline{\Psi}\,\gamma^\mu\,\Psi\,\partial_\mu\Lambda \ . \tag{1.2.11}$$

The Lagrangian is clearly not invariant under local gauge transformations. However, invariance is achieved by replacing the derivative operator ∂^μ by D^μ, the covariant derivative, defined by :

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu \quad , \tag{1.2.12}$$

where A_μ, the vector field, is required to transform as :

$$A'_\mu = A_\mu + \partial_\mu \Lambda \quad , \tag{1.2.13}$$

and then Dirac Lagrangian can be written as :

$$\mathcal{L} = \overline{\Psi} \left(i\gamma^\mu D_\mu - m \right) \Psi \quad . \tag{1.2.14}$$

Now Eqn. (1.2.14) is locally gauge invariant. When the Maxwell field term, $-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$, is added to this Lagrangian, it gives the Lagrangian of QED :

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + e\,\overline{\Psi}\gamma^\mu A_\mu \Psi + \overline{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi \,. \tag{1.2.15}$$

The covariant derivative automatically generates the interaction term. It is a feature of all local gauge symmetries that they generate the basic interactions.

In summary, QED is a local gauge theory describing the electromagnetic interactions of fermions and photons, and is determined by the given Lagrangian. Thus, QED is invariant under the local (gauge) transformations :

$$\begin{aligned} A'_\mu &= A_\mu + \partial_\mu \Lambda(x) \quad , \\ \Psi' &= e^{ie\Lambda(x)}\, \Psi \quad , \\ \overline{\Psi}' &= \overline{\Psi}\, e^{-ie\Lambda(x)} \quad . \end{aligned} \tag{1.2.16}$$

1.2.2 Gauge Fixing

The property of gauge invariance means that A^μ and A'^μ are equally good as photon fields. Therefore, when we perform a functional integration over A_μ , this lack of uniqueness of the vector potential causes overcounting. To overcome this, we fix the gauge and thereby eliminate this infinity of choices. The Lagrangian is then no longer gauge invariant. To do this, we introduce the Lagrange multiplier (or gauge fixing term)

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \quad , \tag{1.2.17}$$

into the Lagrangian. ξ is the covariant gauge parameter in this term. As a result, there will be many different, but physically equivalent gauge conditions. Each has its own advantages. Then, the QED Lagrangian is modified to :

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e\overline{\Psi}\gamma^\mu A_\mu \Psi + \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \quad . \tag{1.2.18}$$

The different choices of gauge fixing term does not alter the physics.

1.3 Path Integral Formalism

In this section, we shall present the derivation of the Schwinger-Dyson equations (SD) to make this thesis self-contained. In order to do this, the path integral technique [4, 6, 7] will be used. Let us briefly review this formalism:

This technique strongly depends on fundamental quantum mechanics. The Path integral (or functional integral) can be thought of as the sum of contributions of all the possible paths that a particle can travel between initial point a with position x_i and final point b with position x_f. Instead of calculating the certain motion of a particle as in classical mechanics, only probabilities are calculable in quantum mechanics. The probability, that a particle is created by a source, then moves in space-time, interacts and is then destroyed by observing it is given by summing over all possible paths. If we divide up each path into N intermediate points, and then sum over all paths, by integrating over the positions of these intermediate points, we can write :

$$\sum_{paths} = \lim_{N\to\infty}\prod_{n=1}^{N}\delta\phi_n \to \int \mathcal{D}\phi \quad . \tag{1.3.1}$$

The integral $\int \mathcal{D}$ is an infinite product of integrals, taken over all possible paths. All intermediate points between a and b are connected by what are called the Green's functions

G of the theory. In other words, path integrals are a compact form of Green's functions. If S is the action for a path, then the amplitude is :

$$\exp\left[iS(field)\right] \quad , \tag{1.3.2}$$

and the integral over all paths is given by :

$$\int \exp\left[i\, S\,(field)\right] \prod \mathcal{D}(field) \quad . \tag{1.3.3}$$

In general, the path integral of a field theory is given by the addition of the term $\int dx source(x) field(x)$ as :

$$Z\,[source] = \int \mathcal{D}\,[field]\exp\left[i\left(S + \int dx(source)\,(field)\right)\right] \quad , \tag{1.3.4}$$

where the source represents particle creation or annihilation. The functional Z has a Taylor series, $Z(source) = \sum_0^\infty \delta^n Z(0)/\delta[source(x_1)].....\delta[source(x_n)]$, and is known as the generating functional because it generates all Green's functions of the theory by taking functional derivatives with respect to the source term :

$$G^{(n)} = \frac{1}{i^n}\frac{\delta^n Z_0[J]}{\delta J(x_1).........\delta J(x_n)}\Big|_{J=0} \quad , \tag{1.3.5}$$

where J is the source of some scalar field. For instance, the 2-point Green's function gives the boson propagator in a free scalar field theory :

$$G(x,y) = -\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)}\Big|_{J=0} = i\Delta_F(x-y) \ , \quad \text{(2-point function)} \quad , \tag{1.3.6}$$

and the fermion propagator in the case of free fields is :

$$G(x,y) \quad = \quad -\frac{\delta^2 Z_0[\eta,\overline{\eta}]}{\delta\eta(x)\,\delta\overline{\eta}(y)}\Big|_{\eta=\overline{\eta}=0} = iS_F(x-y) \quad , \tag{1.3.7}$$

where η and $\overline{\eta}$ are the sources of fermion and anti-fermion fields, and are Grassmann variables. When the free action is replaced by the complete one by the addition of the interaction term, Eqn. (1.3.4) is true for the interacting fields as well. The generating functional Z generates both disconnected and connected graphs. When this formalism is applied to the amplitude for a physical process, one wants to distinguish these two types of graphs and find another functional that generates only connected graphs. This is denoted by W and is related to Z by :

$$Z\,[source] \;=\; e^{i\,W[source]} \quad , \tag{1.3.8}$$

or

$$W[source] \;=\; -i\ln Z\,[source] \quad . \tag{1.3.9}$$

The second derivative of W with respect to sources is

$$\frac{\delta^2 W}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} = -i\frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} = iG_c(x_1,x_2) \quad , \tag{1.3.10}$$

where G_c is now the connected Green's function. The next step is to find a new generating functional for the proper (1-particle irreducible) vertex : Γ, with no external propagator factors. It can be derived from W by the functional Legendre transformation:

$$W[source] = \Gamma\,[field] + \int dx(source)\,(field) \quad , \tag{1.3.11}$$

where the sources and fields satisfy :

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x) \quad , \qquad \frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x) \quad . \tag{1.3.12}$$

Making use of the above expressions, we obtain two useful relations :

$$iG_c(x_1,x_2) = \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = \frac{\phi(x)}{J(y)} \quad , \tag{1.3.13}$$

$$\Gamma(x,y) = \frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\,\delta\phi(y)} = -\frac{J(x)}{\phi(y)} \quad . \tag{1.3.14}$$

If we generalize Eqn. (1.3.13), we can write $(n+2m)$-point Green's function in QED as :

$$(-i)^{n+2m-1} \qquad \frac{\delta^{n+2m}W[J,\eta,\overline{\eta}]}{\delta J_{\mu_1}(x_1).....J_{\mu_n}(x_n)\,\eta(y_1)......\eta(y_m)\,\overline{\eta}(z_1).......\overline{\eta}(z_m)}\bigg|_{J=\,\eta=\,\overline{\eta}=\,0}$$

$$= G_c^{(n+2m)}(x_1,.....,x_n,y_1,.....y_m,z_1,......z_m) \quad . \tag{1.3.15}$$

1.3.1 Derivation of the Schwinger-Dyson Equations in QED

The exact equations for the Green's functions of the theory can be found by using the same path integral method. These relations are called the **Schwinger – Dyson equations.** They provide the starting point for the study of non-perturbative physics. Their derivation is based on a simple fact that the functional integral of a complete derivative is zero :

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi} \equiv 0 \quad . \tag{1.3.16}$$

If we apply this to the example of a scalar theory :

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi} \exp \left\{ i \left(S(\phi) + \int dx \, J\phi \right) \right\} = 0 \quad , \tag{1.3.17}$$

$$\int \mathcal{D}\phi \, i \, \left[S'(\phi) + J \right] \exp i \left(S(\phi) + \int dx \, J\phi \right) = 0 \quad . \tag{1.3.18}$$

This can be rewritten as a differential equation :

$$\left[S' \left(-i \frac{\delta}{\delta J} \right) \right] Z[J] = 0 \quad . \tag{1.3.19}$$

Now, as an example let us derive the Schwinger-Dyson equation for the gauge boson which relates 2-point Green's functions to the 3-point one. First of all, referring to Eqn. (1.3.4), the generating functional can be written for QED as :

$$\begin{aligned} Z[J,\eta,\overline{\eta}] &= \exp \left(i \, W[J,\eta,\overline{\eta}] \right) \quad , \\ Z[J,\eta,\overline{\eta}] &= \int \mathcal{D}[A] \, \mathcal{D}[\Psi] \, \mathcal{D}[\overline{\Psi}] \\ & \qquad \times \exp \left\{ i \left[S(A_\mu, \Psi, \overline{\Psi}) + \int dx \, \left(J^\mu A_\mu + \eta \overline{\Psi} + \overline{\eta} \Psi \right) \right] \right\}, \end{aligned} \tag{1.3.20}$$

where the action S, including gauge fixing term, can be written as :

$$S = \int dx \left\{ - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\Psi} \left(i \, \partial\!\!\!/ - m \right) \psi - \frac{1}{2\xi} \left(\partial_\mu A^\mu \right)^2 - e \overline{\Psi} \gamma_\mu A^\mu \Psi \right\} \quad . \tag{1.3.21}$$

Making use of Eqn. (1.3.1) yields the following differential equation :

$$\left[\frac{\delta S}{\delta A_\mu(x)}\left(-i\frac{\delta}{\delta J_\nu},-i\frac{\delta}{\delta\overline{\eta}},+i\frac{\delta}{\delta\eta}\right)+J_\mu(x)\right]Z\left[J,\eta,\overline{\eta}\right]=0 \quad . \tag{1.3.22}$$

Taking the derivative of the QED action, Eqn. (1.3.21), with respect to the boson field, A_μ, we have :

$$\frac{\delta S}{\delta A_\mu(x)}=\left[\Box g_{\mu\nu}-(1-\xi^{-1})\,\partial_\mu\partial_\nu\right]A^\nu-e\,\overline{\Psi}\gamma_\mu\Psi \quad . \tag{1.3.23}$$

Substituting this equation into Eqn. (1.3.22) gives

$$\left[J_\mu+\left(\Box g_{\mu\nu}-(1-\xi^{-1})\,\partial_\mu\partial_\nu\right)\left(\frac{-i\delta}{\delta J_\nu}\right)-e\left(\frac{i\delta}{\delta\eta}\right)\gamma_\mu\left(\frac{-i\delta}{\delta\overline{\eta}}\right)\right]e^{i\,W[J,\eta,\overline{\eta}]}=0\,, \tag{1.3.24}$$

and after differentiating, we obtain the following equation in terms of connected Green's functions :

$$J_\mu+\left[\Box g_{\mu\nu}-\left(1-\xi^{-1}\right)\partial_\mu\partial_\nu\right]\frac{\delta W}{\delta J_\nu}-ie\frac{\delta}{\delta\eta}\left(\gamma_\mu\frac{\delta W}{\delta\overline{\eta}}\right)+e\frac{\delta W}{\delta\eta}\gamma_\mu\frac{\delta W}{\delta\overline{\eta}}=0\,. \tag{1.3.25}$$

Now, by using Eqn. (1.3.11) to replace W with the proper vertex function, Γ, leads to :

$$W\left[J_\mu,\eta,\overline{\eta}\right]=\Gamma\left[A_\mu,\Psi,\overline{\Psi}\right]+\int dx\left(J_\mu A^\mu+\eta\overline{\Psi}+\overline{\eta}\Psi\right)\,, \tag{1.3.26}$$

which satisfies;

$$\begin{aligned}\frac{\delta W[J]}{\delta J_\mu}&=A^\mu\,, & \frac{\delta W[\eta]}{\delta\overline{\eta}}&=\Psi\,, & \frac{\delta W[\overline{\eta}]}{\delta\eta}&=-\overline{\Psi}\,,\\ \frac{\delta\Gamma[A]}{\delta A_\mu}&=-J^\mu\,, & \frac{\delta\Gamma[\overline{\Psi}]}{\delta\overline{\Psi}}&=-\eta\,, & \frac{\delta\Gamma[\Psi]}{\delta\overline{\Psi}}&=\overline{\eta}\,,\end{aligned} \tag{1.3.27}$$

where

$$\frac{\delta^2 W}{\delta\eta\,\delta\overline{\eta}}=\frac{\delta\Psi}{\delta\eta}\qquad,\qquad\frac{\delta^2\Gamma}{\delta\Psi\,\delta\overline{\Psi}}=-\frac{\delta\eta}{\delta\Psi}\quad . \tag{1.3.28}$$

After making use of the above relations for Eqn. (1.3.25), we find

$$\left.\frac{\delta\Gamma}{\delta A_\mu(x)}\right|_{\Psi=\overline{\Psi}=0}=\left[\Box g_{\mu\nu}-\left(1-\xi^{-1}\right)\partial_\mu\partial_\nu\right]A^\nu(x)-ie\gamma_\mu\left(\left.\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\,\delta\Psi(x)}\right|_{\Psi=\overline{\Psi}=0}\right)^{-1}. \tag{1.3.29}$$

Now taking the derivative of this equation with respect to $A_\mu(y)$, and setting the fields equal to zero, we have

$$\left.\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta A_\mu(x)}\right|_{A=\Psi=\overline{\Psi}=0}$$
$$= \left[\Box g_{\mu\nu} - \left(1-\xi^{-1}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y) - ie\gamma_\mu \frac{\partial}{\partial A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\,\delta\Psi(x)}\right|_{\Psi=\overline{\Psi}=0}\right)^{-1} . \tag{1.3.30}$$

Then the derivative of the inverse matrix with respect to A_ν is :

$$\frac{\delta M^{-1}}{\delta A_\mu} = -M^{-1}\frac{\delta}{\delta A_\mu}MM^{-1} \quad ,$$

where

$$M = \frac{\delta^2\Gamma}{\delta\Psi(z_1)\,\delta\overline{\Psi}(z_2)} \quad .$$

Inserting this relation into Eqn. (1.3.1), we find :

$$\left.\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta A_\mu(x)}\right|_{A=\Psi=\overline{\Psi}=0} = \left[\Box g_{\mu\nu} - \left(1-\xi^{-1}\right)\partial_\mu\,\partial_\nu\right]\delta^4(x-y) + \Pi_{\mu\nu}(z_1,z_2)\,, \tag{1.3.31}$$

where

$$\Pi_{\mu\nu}(z_1,z_2) = ie\int dz_1 dz_2\gamma_\mu\left(\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\delta\Psi(z_1)}\right)^{-1}\frac{\delta}{\delta A_\nu(y)}\frac{\delta^2\Gamma}{\delta\overline{\Psi}(z_1)\,\delta\Psi(z_2)}\left(\frac{\delta^2\Gamma}{\delta\Psi(z_2)\,\delta\overline{\Psi}(x)}\right)^{-1} . \tag{1.3.32}$$

Making use of Eqn. (1.3.15) to write derivatives of the proper vertex in terms of connected Green's functions :

$$-G_c^{(1,1)}(y,z_1,z_2) = \left.\frac{\delta^3\Gamma}{\delta A_\nu(y)\,\delta\overline{\Psi}(z_1)\,\delta\Psi(z_2)}\right|_{A=\Psi=\overline{\Psi}=0} = e\Lambda_\nu(y,z_1,z_2) \quad , \tag{1.3.33}$$

$$-iG_c^{(0,1)}(x,y) = \left.\left(\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\,\delta\Psi(y)}\right)^{-1}\right|_{\Psi=\overline{\Psi}=0} = S_F(x,y) \quad , \tag{1.3.34}$$

$$-iG_c^{(2,0)}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta A_\mu(x)}\right|_{A=0} = \Delta^{-1}_{\mu\nu}(x,y) \quad , \tag{1.3.35}$$

Eqn. (1.3.31) finally becomes :

$$i\Delta^{-1}_{\mu\nu}(x,y) = \left[i\left(\Delta_0\right)_{\mu\nu}(x,y)\right]^{-1} - e^2 \int dz_1 dz_2 \gamma_\mu S_F(x,z_1)\Lambda_\nu(y;z_1,z_2)S_F(z_2,x)\,. \tag{1.3.36}$$

This equation can be represented diagramatically as :

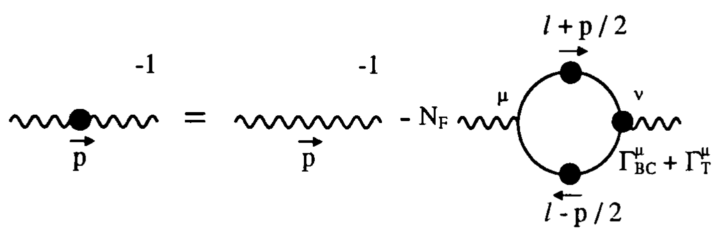


Figure 1.1: Schwinger-Dyson equation for the gauge boson

1.3.2 Ward-Takahashi Identities

We are now going to derive identities which are an important consequence of the gauge invariance of the generating functional. The best known, the Ward-Takahashi identity [7, 4], relates the fermion propagator to the proper fermion-boson vertex. Such identities are essential in proving the equality of renormalisation constants $Z_1 = Z_2$. The generating functional for QED given earlier in Eqn. (1.3.20) is invariant under gauge transformations, Eqn. (1.2.16). This is because changing variables in an integral has no effect on its value. The only term, which is not gauge invariant, is the gauge fixing term and the coupling to the sources. But since the generating functional Z is gauge invariant and only a function of $\eta, \overline{\eta}$ and J , these terms must vanish. Therefore, variation of the generating functional gives :

$$\begin{aligned} Z + \delta Z \;\; = \;\; & \int \mathcal{D}[A_\mu]\,\mathcal{D}[\Psi]\,\mathcal{D}[\overline{\Psi}] \Big\{ \exp i \int dx \Big[\mathcal{L}(A_\mu,\Psi,\overline{\Psi}) + A_\mu J^\mu + \overline{\eta}\Psi + \eta\overline{\Psi}\Big] \\ & + \left(-\frac{1}{\xi}\left(\partial_\mu A^\mu\right)\Box + J^\mu\left(\partial_\mu \Lambda\right) + ie\Lambda\left(\eta\overline{\Psi} - \overline{\eta}\Psi\right)\right)\Big\}\,. \end{aligned} \tag{1.3.37}$$

Expanding the exponential to the first order in Λ and integrating by parts gives :

$$\int \mathcal{D}[A_\mu]\,\mathcal{D}[\Psi]\,\mathcal{D}[\overline{\Psi}]\left\{-\frac{\Box}{\xi}\left(\partial_\lambda A^\lambda(x)\right) + ie\left(\eta(x)\overline{\Psi}(x) - \overline{\eta}(x)\Psi(x)\right) - \partial_\mu J^\mu\right\}$$
$$\times \exp i\left[S + \int dx\left(A_\mu J^\mu + \overline{\eta}\Psi + \eta\overline{\Psi}\right)\right] = 0\,. \tag{1.3.38}$$

Rewriting this expression as a functional differential equation for Z, using Eqn. (1.3.1), we get :

$$i\Lambda\left[-\partial_\mu J^\mu + e\left(\eta\frac{\delta}{\delta\eta(x)} - \overline{\eta}\frac{\delta}{\delta\overline{\eta}(x)}\right) - \frac{\Box}{\xi}\partial_\mu\frac{1}{i}\frac{\delta}{\delta J_\mu}\right] Z[J,\eta,\overline{\eta}] = 0 \quad . \tag{1.3.39}$$

After replacing Z by the connected Green's functions yields

$$-\frac{\Box}{\xi}\partial_\mu\frac{\delta W}{\delta J_\mu(x)} - \partial_\mu J^\mu(x) + ie\left(\eta\frac{\delta W}{\delta\eta(x)} - \overline{\eta}\frac{\delta W}{\delta\overline{\eta}(x)}\right) = 0 \quad . \tag{1.3.40}$$

We then write this in terms of the proper vertex function using Eqn. (1.3.26) :

$$-\frac{\Box}{\xi}\partial_\mu A^\mu(x) + \partial_\mu\frac{\delta\Gamma}{\delta A_\mu}(x) + ie\left(\overline{\Psi}\frac{\delta\Gamma}{\delta\overline{\Psi}(x)} - \Psi\frac{\delta\gamma}{\delta\Psi(x)}\right) = 0 \quad . \tag{1.3.41}$$

Taking derivatives with respect to Ψ and $\overline{\Psi}$ and setting $A = \Psi = \overline{\Psi} = 0$ leads to :

$$-\partial_{x\mu}\frac{\delta^3\Gamma}{\delta\overline{\Psi}(z)\,\delta\Psi(y)\,\delta A_\mu(x)}\bigg|_{A=\Psi=\overline{\Psi}=0}$$
$$= ie\frac{\delta^2}{\delta\overline{\Psi}(z)\delta\Psi(y)}\left(\overline{\Psi}\frac{\delta\Gamma}{\delta\overline{\Psi}(x)} - \Psi\frac{\delta\Gamma}{\delta\Psi}(x)\right)\bigg|_{\Psi=\overline{\Psi}=A=0} \quad ,$$
$$= ie\left[\delta^4(x-y)\frac{\delta^2\Gamma}{\delta\overline{\Psi}(z)\,\delta\Psi(x)} - \delta^4(x-z)\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\,\delta\Psi(y)}\right]_{A=\Psi=\overline{\Psi}=0} \quad . \tag{1.3.42}$$

To transform this equation into momentum space, we make a Fourier transformation where

$$\int dx\int dy e^{(ipx-iky)}\frac{\delta^2\Gamma}{\delta\overline{\Psi}(x)\,\delta\Psi(y)}\bigg|_{A=\Psi=\overline{\Psi}=0} = i(2\pi)^4\delta(p-k)S_F^{-1}(k) \quad , \tag{1.3.43}$$

and

$$\int dx \int dy \int dz e^{i(px-ky-qz)} \frac{\delta^3 \Gamma}{\delta \overline{\Psi}(x)\, \delta \Psi(y)\, \delta A^{\mu}(z)} \Bigg|_{A=\Psi=\overline{\Psi}=0}$$

$$= ie\,(2\pi)^4\, \delta(p-k-q)\, \Gamma(k,p,q), \qquad (1.3.44)$$

Eqn. (1.3.42) then gives the relation between the inverse fermion propagator and the 3-point vertex function in momentum space :

$$q^{\mu}\Gamma^{\mu}(p,q,p+q) = S'^{-1}_{F}(p+q) - S'^{-1}_{F}(p) \quad . \qquad (1.3.45)$$

This is known as the Ward-Takahashi identity. If we differentiate Eqn. (1.3.42) with respect to $A_{\mu}(y)$ and set $A = 0$ it gives :

$$-\frac{\Box}{\xi}\, \partial_{x\mu}\delta(x-y) = \partial_{x\mu} \frac{\delta^2 \Gamma}{\delta A_{\nu}(y)\, \delta A_{\mu}(x)} \Bigg|_{A=0} \quad . \qquad (1.3.46)$$

The Fourier transform of this equation gives :

$$q^{\mu}\Delta^{-1}_{\mu\nu}(q) = q_{\nu}\frac{q^2}{\xi} \quad , \qquad (1.3.47)$$

which is a Ward-Takahashi identity for the photon.

This short description of how to derive the SD-equations and the simplest Ward-Takahashi identities provides us with the main tools for this thesis, which we shall use in the coming chapters.

The structure of this thesis is as follows :

- In Chapter 2, we are going to give a brief introduction to using the SD-equations. The commonest way to deal with these equations is to make an ansatz for the fermion-boson vertex. We discuss some simple choices for this ansatz in the rest of this chapter.
- The SD-equations are solved perturbatively for the fermion and photon wave-function renormalisations in Chapter 3.
- In chapter 4 we compute the most general form for the fermion and photon wave-function renormalisations for them to be multiplicatively renormalisable.

- In Chapter 5, we make a comparison, order by order, between the results of Chapter 3 and Chapter 4 to determine the constraints on the fermion-boson vertex imposed by multiplicative renormalisability.

- In Chapter 6, the one-loop vertex in QED is calculated in arbitrary covariant gauges as an analytic function of its momenta.

- Finally, we give our conclusions in Chapter 7.

Chapter 2

Constructing Non-Perturbative Vertices

Although this may seem a paradox,
all exact science is dominated
by the idea of approximation.

-B. Russell-

2.1 Introduction

In this section, we shall discuss non-perturbative methods to address some of the problems in Quantum Field Theory (QFT) that cannot be addressed by perturbation theory [8].

The calculation of a scattering process by using path integral methods involves a term which contains the exponential of the action of the relevant theory. As we have seen, in Eqn. (1.2.15) the action of the theory usually contains two parts : a free and an interaction part. The free piece contains the non-interacting term obtained by taking all couplings to zero. It gives a Gaussian integral which can be computed exactly. The exponential of the interaction term may be expanded in powers of the coupling constant. Then, the calculation of $e^{-}e^{-}$ scattering, for instance, is approximated at low orders by a few Feynman diagrams. This perturbative approach to QED works very well. In this regard, QED is known to be the best understood QFT. However, perturbation theory is not the whole story of a QFT. There are problems which cannot be solved by the procedure just described. For example, Quantum Chromodynamics (QCD), the theory of strong interaction physics, possesses a property known as confinement [9, 10, 11]. In the long distance or infrared region, the potential energy between two quarks increases linearly as $V(r) \simeq kr$. As a result of this property, if we try to separate a quark and an antiquark in a meson, then, instead of having a free quark and antiquark, we have two separate mesons. In this way, we believe quarks are confined and free quarks never observed. Only in the short range or high energy limit, when the potential energy of two quarks is $V(r) \simeq a/r$ do quarks behave as though they are free. It is, of course, impossible to give a proof of confinement, a large-distance or low-energy phenomenon, in the context of perturbative QCD. Non-perturbative methods have to be sought to understand this phenomenon.

In perturbation theory, if one starts with zero mass in the Lagrangian, fermion fields will always remain massless. However, there is a strong evidence of a new phase of QED when the coupling is strong ($\alpha \simeq 1$). Then, fermion masses may be generated dynamically by chiral symmetry breaking [12, 13, 14, 15, 16, 17, 18] : "start from nothing to get something" [19]. Therefore, QED has to be treated non-perturbatively in the strong coupling region where the chiral symmetry breaking occurs.

In this thesis we study the non-perturbative framework for investigating QED in the continuum [20, 21]. The natural vehicle for this is the system of Schwinger-Dyson equations [7].

2.2 Schwinger-Dyson Equations

The Schwinger-Dyson (SD) equations are [9] coupled integral equations which relate the Green's functions of a field theory to each other. Solving these field equations provides a solution of the theory. Once all the n-point Green's functions of a field theory are known then everything possible is known about that field theory. Below we show the first few of the infinite tower of SD equations :

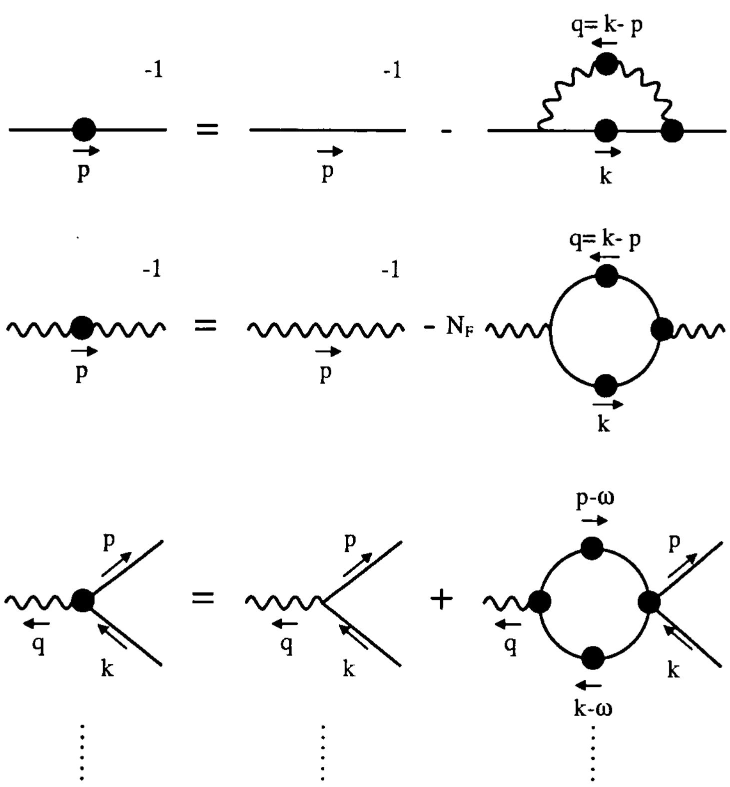


Figure 2.1: Schwinger-Dyson equations

The solid dots indicate full quantities, which include all possible fermion-gauge boson emission and recapture and lines without dots indicate bare quantities. Straight lines represent the fermion propagator, $S_F(p)$, wavy lines the photon propagator, $\Delta_{\mu\nu}(p)$, and N_F is the number of flavours of fermion.

Obviously, as seen from these graphs, the full set of SD-equations for any particular field theory contains an infinite number of equations which relate 2-point functions to 3-point vertices, 3-point vertices to the 4-point functions, and in general n-point Green functions to the $(n+1)$-point Green functions. It is therefore not possible to solve this infinite set of equations simultaneously. However, we can make progress by truncating or approximating this system of equations [22, 23, 9], to arrive at a solvable problem from which we can hope to extract all the necessary information. The best known truncation is perturbation theory. In the limit of sufficiently small coupling constant, when $\alpha << 1$, the SD-equations are the usual perturbative expansion of the S-matrix. As previously discussed, in this small coupling regime we need only calculate a few diagrams for a scattering amplitude and the rest will be relatively smaller. However for $\alpha \simeq 1$, this picture breaks down, as the next terms are of the same order and so cannot be neglected. In this region, these equations should be solved non-perturbatively. Thus if one wants to study the non-perturbative behaviour of any Green function, which may demonstrate dynamical mass generation or quark confinement, this requires some (non-perturbative) approximation or truncation rather than a perturbative one. Moreover by making approximations we may lose uncontrolled key pieces of the physics. Since gauge invariance and multiplicative renormalisability are two basic requirements of a gauge theory any successful model must maintain these two important properties [22, 15]. How to achieve this is what this thesis is about.

The most common way to deal with the SD-equations is to replace the fermion-gauge boson vertex by a suitable ansatz. In other words, instead of solving the SD-equation for the vertex (the third graph in Fig. (2.1)) we approximate its solution. Then the problem is reduced to solve the coupled equations for the fermion and gauge boson propagators. The idea behind this truncation is to choose the vertex ansatz in a clever way so that we maintain all the relevent information lost by decoupling the equations for the propagators from the rest of the equations. As a result of the truncation, this vertex ansatz has to satisfy certain criteria that the solution of the vertex equation must itself satisfy. These criteria [24, 25, 9] are listed below :

Any ansatz for the vertex function :

- I)- must satisfy the Ward-Takahashi identity;

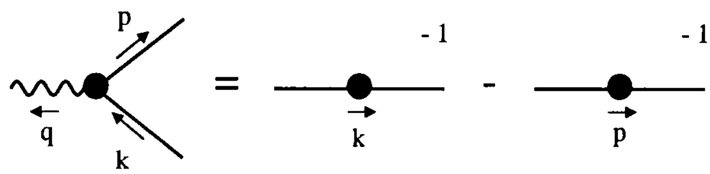


Figure 2.2: Ward-Takahashi identity

$$q_\mu \Gamma^\mu(k^2,p^2,q^2) = S_F^{-1}(k) - S_F^{-1}(p) \quad . \tag{2.2.1}$$

In gauge theories Ward-Takahashi identities [7, 4] are consequences of gauge invariance. They are not only satisfied at every order of perturbation theory, but are also true non-perturbatively, as we have described in Sect. 1.3.2.

- II)- must be free of kinematic singularities. As a result of the Ward identity,

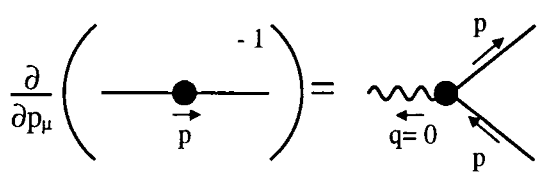


Figure 2.3: Ward identity

$$\Gamma^\mu(p,p) = \lim_{k\to p} \Gamma^\mu(k,p) = \frac{\partial S_F(p)}{\partial p_\mu} \ , \tag{2.2.2}$$

the vertex should have a unique form in the limit $k \to p$ [26].

- III)- must reduce to the bare vertex in the free field limit; in other words, when full propagators are replaced by bare ones, it should reduce to the bare vertex.

- IV)- must have the same transformation properties as the bare vertex γ_μ, under charge conjugation C [25, 27];

$$C\,\Gamma_\mu(k,p)\,C^{-1} = -\Gamma_\mu^T(-p,-k) \quad . \tag{2.2.3}$$

- V)- should ensure the multiplicative renormalisability of the SD-equations [28, 22, 29].

- VI)- should ensure local gauge covariance of the propagator and vertex [30, 9].

2.3 Importance of the Vertex Ansatz

The aim is then to try to find a suitable vertex ansatz which satisfies the above criteria. With this in mind, we shall, in this section, build up a picture of where we presently are. We shall see how this idea works, where we have reached today and what more can be done. Before starting to develop these ideas, some terminology and conventions have to be introduced.

- **Ladder Approximation** means the full vertex is replaced by the bare one.

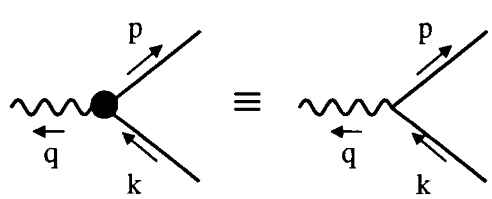


Figure 2.4: Ladder approximation

- **Quenched Approximation** is when the corrections to the photon propagator are not taken into account (i.e. the second graph of Fig. (2.1) does not contribute). It is equivalent to taking the number of flavours to zero ($N_F = 0$). In this approximation, the full photon propagator is replaced by the bare one.

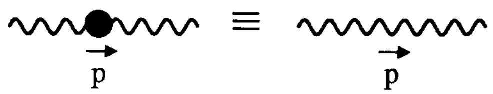


Figure 2.5: Quenched approximation

- $S_F(p)$ is the complete (full or dressed) fermion propagator carrying momentum p. It involves two functions of p^2. This fact follows from the spin structure of the fermion propagator. These two functions can be chosen to be $F(p^2)$, the wave-function renormalisation, and $M(p^2)$, the mass function, so that

  ⋆ **Full Fermion Propagator :**

$$iS_F(p) = i\frac{F(p^2)}{\not p - M(p^2)} \quad . \tag{2.3.1}$$

  [This can be (and often is) written in a variety of other ways, e.g. $S_F(p)^{-1} = \alpha(p^2)\ \not p + \beta(p^2)$, etc, always involving two independent scalar functions.] Since $S_F(p)$ is a gauge-variant quantity, these functions $F(p^2)$, $M(p^2)$ will in general depend on the gauge. They can be calculated, in principle, at each order in perturbation theory. At lowest order $F(p^2) = 1$, $M(p^2) = m$, the bare mass. Therefore,

  ⋆ **Bare Fermion Propagator :**

$$i\, S_F^0(p) = \frac{i}{\not p - m} \quad . \tag{2.3.2}$$

- The full photon propagator involves a photon wave-function renormalisation, $G(p^2)$, analogous to $F(p^2)$, then,

  ⋆ **Full Photon Propagator :**

$$i\,\Delta_{\mu\nu}(p) \ = \ -\frac{i}{p^2}\left[G(p^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^4}\right] \quad . \tag{2.3.3}$$

  At the lowest order $G(p^2) = 1$. So,

⋆**Bare Photon Propagator :**

$$i\,\Delta_{\mu\nu}(p) \;=\; -\frac{i}{p^2}\left[g_{\mu\nu}+(\xi-1)\,\frac{p_\mu p_\nu}{p^2}\right]\;, \tag{2.3.4}$$

where ξ is the arbitrary covariant gauge parameter introduced in Sect. 1.2.2. We shall see in the next section that the same functions $F(p^2), M(p^2)$ also occur in the complete fermion-gauge boson vertex, since the Ward-Takahashi identity relates the 3-point Green's function to the fermion propagator in a well-known way.

- ⋆**Full Vertex :**

$$-ie\,\Gamma^\mu_F \quad , \tag{2.3.5}$$

⋆**Bare Vertex :**

$$-ie\,\gamma^\mu \quad . \tag{2.3.6}$$

After introducing our notation, we can now start to solve the SD-equations in the simplest approximation, called the rainbow approximation.

2.3.1 Rainbow approximation

The Rainbow approximation [29, 14, 31, 32, 33] is the name given to the combination of the Ladder, Fig. 2.4, and Quenched approximations, Fig. 2.5, where the vertex and photon propagator are bare and only the fermion propagator is full. This approximation can be represented as

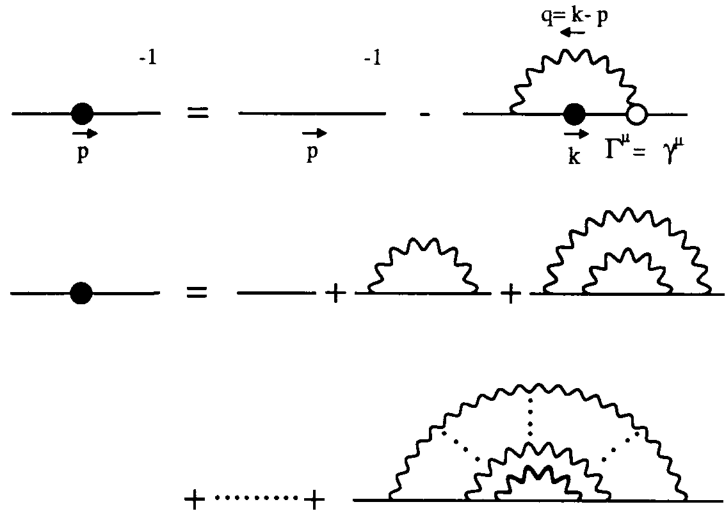


Figure 2.6: Rainbow approximation

This SD-equation, Fig. 2.1, can be solved for the wave-function renormalisation and the mass function. Making use of the Feynman rules for Fig. 2.6,

$$-iS_F^{-1}(p) = -iS_F^{0^{-1}}(p) - \int_M \frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)\; i\,S_F(k)\;(-ie\gamma^\nu)\; i\Delta^0_{\mu\nu}(q)\,, \qquad (2.3.7)$$

where M denotes that this integral is to be performed in Minkowski space, e is the bare coupling and $q = k - p$. Using the expressions for the full and bare fermion propagators, and for the bare photon propagator, the above equation can be written as,

$$i\frac{(\not p - M(p^2))}{F(p^2)} = i\,(\not p - m) - \frac{e^2}{(2\pi)^4}\int_M \frac{d^4k}{q^2}\,\frac{F(p^2)}{(k^2 - M^2(k))} \times \left[\gamma^\mu\left(\not k + M(p^2)\right)\gamma^\nu\left(g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right)\right]. \qquad (2.3.8)$$

Multiplying by $\not p/4$ and taking the trace (and noting that the trace of an odd number of γ-matrices is zero), Eqn. (2.3.8) gives the inverse fermion wave-function renormalisation :

$$\frac{1}{F(p^2)} = 1 + \frac{i\alpha}{16\,\pi^3\,p^2} \int_M \frac{d^4k}{q^2}\, \frac{F(k^2)}{(k^2 - M^2(k^2))}\, \mathrm{Tr}\left[\not p \gamma_\mu \not k \gamma^\mu \left(g_{\mu\nu} + (\xi - 1)\, \frac{q_\mu q_\nu}{q^2} \right) \right] . \tag{2.3.9}$$

Related traces are calculated by using :

$$\begin{aligned} \mathrm{Tr}\,(\not p \gamma^\mu \not k \gamma^\nu) &= -8\, k \cdot p\,, \\ \mathrm{Tr}\,(\not p \gamma^\mu \not k \gamma^\nu q_\mu q_\nu) &= 4\,[(k^2 + p^2)\; k \cdot p - 2k^2 p^2]\,. \end{aligned} \tag{2.3.10}$$

Substituting Eqn. (2.3.10) into Eqn. (2.3.9) and neglecting M with respect to k and p, we obtain,

$$\frac{1}{F(p^2)} = 1 + \frac{i\alpha}{4\pi^3 p^2} \int_M \frac{d^4k}{k^2 q^2} F(k^2) \left\{ -2\, k \cdot p + \frac{(\xi - 1)}{q^2} \left((k^2 + p^2)\, k \cdot p - 2k^2 p^2 \right) \right\} . \tag{2.3.11}$$

The ratio of the mass function to wave-function renormalisation is found by taking the trace of Eqn. (2.3.8) :

$$\frac{M(p^2)}{F(p^2)} = m - \frac{i\,\alpha}{16\,\pi^3} \int_M \frac{d^4k}{q^2}\, \frac{F(k^2)\, M(k^2)}{(k^2 - M^2(k^2))}\, \mathrm{Tr} \left[\gamma^\mu \gamma^\nu \left(g_{\mu\nu} + (\xi - 1)\, \frac{q_\mu q_\nu}{q^2} \right) \right] , \tag{2.3.12}$$

with

$$\begin{aligned} \mathrm{Tr}\,(\gamma^\mu \gamma^\nu g_{\mu\nu}) &= 16 \quad , \\ \mathrm{Tr}\,(\gamma^\mu \gamma^\nu q_\mu q_\nu) &= 4\, q^2 \quad . \end{aligned} \tag{2.3.13}$$

Letting the bare mass go to zero, Eqn. (2.3.12) can be written as :

$$\frac{M(p^2)}{F(p^2)} = - \frac{i\,\alpha}{4\,\pi^3} \int_M \frac{d^4k}{q^2}\, \frac{F(k^2)\, M(k^2)}{(k^2 - M^2(k^2))}\, (3 + \xi) \quad . \tag{2.3.14}$$

Usually, it is easier to perform the integration in Euclidean space rather than Minkowski space. In order to do this, we rotate the k_0'-plane by $\pi/2$ radians, a procedure proposed by Wick. We can interpret this in mathematical language as :

$$\begin{array}{ccc} \text{Minkowski Space} & \longrightarrow & \text{Euclidean Space} \\ k_0 & \longrightarrow & i k_0 \\ k_i \ (i=1,2,3) & \longrightarrow & k_i \\ d^4k' & \longrightarrow & i d^4 k' \\ k_M^2 = k_0^2 - k_i & \longrightarrow & -k_E^2 = -\left(k_0^2 + k_i^2\right) \quad . \end{array} \tag{2.3.15}$$

So, we shall now evaluate the k-integration in Euclidean space by employing a Wick rotation, assuming the necessary analytic properties hold.

It is then appropriate to describe our Euclidean coordinate system. We choose the external momentum as

$$p^\mu \;=\; (p,\,0,\,0,\,0)\ , \tag{2.3.16}$$

and use spherical polar coordinates in 4-dimensions. The internal momentum k is then given by the 4-vector

$$k^\mu \;=\; (k\cos\psi,\ k\sin\psi\sin\theta\cos\phi, k\sin\psi\sin\theta\sin\phi, k\sin\psi\cos\theta)\ , \tag{2.3.17}$$

where

$$k\,:\,[0,\infty]\quad ,\quad \psi\,:\,[0,\,\pi]\quad ,\quad \theta\,:\,[0,\,\pi]\quad \text{and}\quad \phi\,:\,[0,\,2\pi]\quad .$$

The θ and ϕ integrals are trivial, since our integrals have only ψ dependence, because they have terms like $k\cdot p = kp\cos\psi$. The radial and angular parts of the k-integration can be performed by writing the Jacobian as :

$$d^4k = 2\,\pi\,k^2\,dk^2\,\sin^2\psi\,d\psi \quad .$$

This leads to

$$\begin{aligned} \frac{1}{F(p^2)} \;&=\; 1 - \frac{\alpha}{2p^2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi}\frac{dk^2}{q^2}\sin^2\psi\,d\psi F(k^2) \\ &\qquad \times\left\{-2k\cdot p + \frac{(\xi-1)}{q^2}\left((k^2+p^2)\,k\cdot p - 2k^2p^2\right)\right\}. \end{aligned} \tag{2.3.18}$$

Referring to Appendix B for these angular integrations, one can see that Eqn. (2.3.18) takes the form,

$$\frac{1}{F(p^2)} = 1 - \frac{\alpha}{2p^2\pi^2}\int_0^{\Lambda^2} dk^2\,F(k^2)\,(X_1 + \xi\,X_2)\ , \tag{2.3.19}$$

where

$$X_1 \equiv -2\, I_{1,1} - \left(k^2 + p^2\right)\, I_{1,2} + 2p^2 k^2\, I_{0,2}\,,$$
$$X_2 \equiv \left(k^2 + p^2\right)\, I_{1,2} - 2p^2 k^2 I_{0,2}\quad . \tag{2.3.20}$$

Splitting the above integral into two regions, we have :

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha}{4p^2\pi}\left[\int_0^{p^2} dk^2 F(k)\,\xi\,\frac{k^2}{p^2} + \int_{p^2}^{\Lambda^2} dk^2\, F(k)\,\xi\,\frac{p^2}{k^2}\right]\,, \tag{2.3.21}$$

Similarly, Eqn. (2.3.12) becomes

$$\frac{M(p^2)}{F(p^2)} = \frac{\alpha}{4\pi}\Bigg[\int_0^{p^2} dk^2\,\frac{1}{p^2}\,F(k^2)\,M(k^2)\,(3+\xi) + \int_{p^2}^{\Lambda^2} dk^2\,\frac{1}{k^2}\,F(k^2)\,M(k^2)\,(3+\xi)\Bigg]\,. \tag{2.3.22}$$

We now use an iterative procedure appropriate to perturbation theory. Starting from $F(k^2) = 1$ and collecting only leading order terms for the sake of simplicity, Eqn. (2.3.21) yields :

$$F(p^2) = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2} + \frac{3}{2}\left(\frac{\alpha\xi}{4\pi}\right)^2 \ln^2\frac{p^2}{\Lambda^2} + \frac{5}{2}\left(\frac{\alpha\xi}{4\pi}\right)^3 \ln^3\frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha^4)\,. \tag{2.3.23}$$

One of the checks of whether the rainbow approximation for the vertex is good or not is to see whether the above solution for $F(p^2)$ ensures its multiplicative renormalisability. As a necessary requirement of multiplicative renormalisability, the wave-function renormalisation should be of the form :

$$F(p^2) = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2} + \frac{1}{2!}\left(\frac{\alpha\xi}{4\pi}\right)^2 \ln^2\frac{p^2}{\Lambda^2} + \frac{1}{3!}\left(\frac{\alpha\xi}{4\pi}\right)^3 \ln^3\frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha^4)\,. \tag{2.3.24}$$

As we can see these two expressions Eqns. (2.3.23, 2.3.24) are not equal to each other except in the Landau gauge, $\xi = 0$. Therefore this solution of the SD-equation is **not** multiplicatively renormalisable for an arbitrary covariant gauge. Moreover, as we shall shortly discuss, this approximation violates the Ward-Takahashi identity [23, 29], and hence one of the requirements of gauge covariance, in all but the Landau gauge $\xi = 0$.

In any acceptable truncation of the field theory, the physical observables should be gauge independent, such as the mass of the particle, the critical coupling at which the mass is generated etc. In Fig. 2.7, the Euclidean mass, defined by $m = M(m)$ from Eqn. (2.3.22), is plotted versus the coupling constant for different gauges in the rainbow approximation [29]. The bare mass is zero. The perturbative and non-perturbative solutions agree in the region of $\alpha < \alpha_c$, but once we reach the critical coupling, these two solutions separate from each other. The mass remains zero in perturbation theory, but if $\alpha > \alpha_c$ then a nonzero value can be dynamically generated [34]. Obviously in this plot the Euclidean mass has different values for different gauges, which means that the mass has a gauge dependence in this approximation. This is clearly not acceptable.

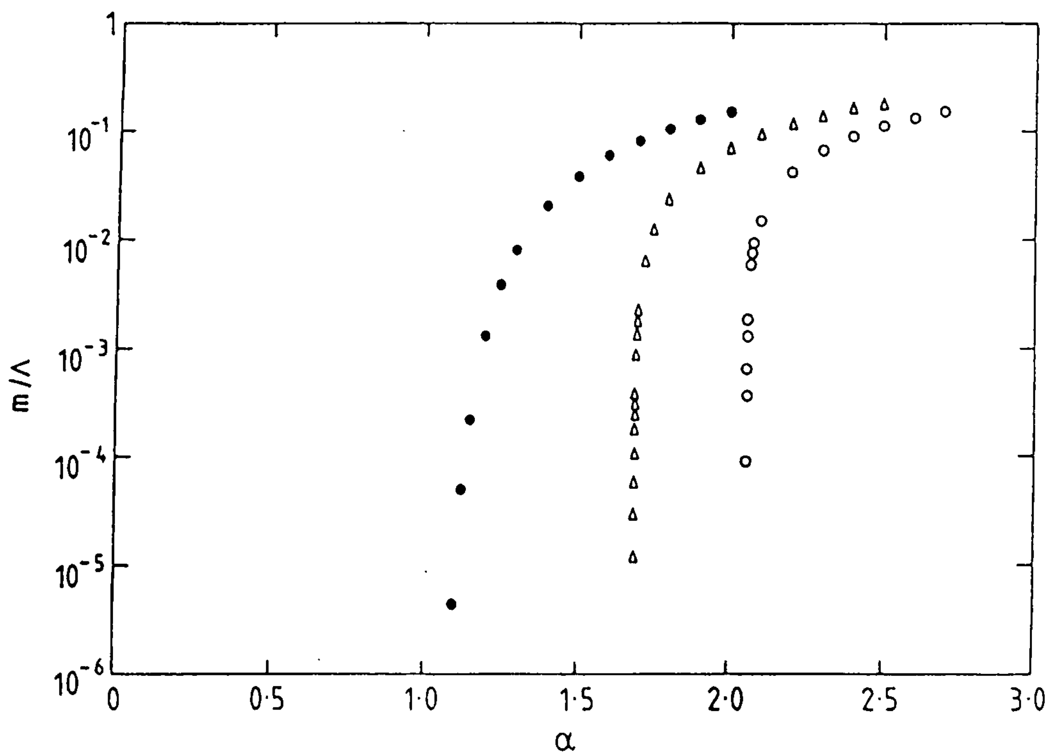


Figure 2.7: Dynamical mass generation in the rainbow approximation with Landau (ξ=0) •, Feynman (ξ=1) o, Yennie (ξ=3) ▵

The next step is to improve on the rainbow approximation by trying to re-establish the gauge invariance of the theory which is lost by approximating the vertex by its bare form. Since the Ward-Takahashi identity is an exact relation between the inverse full fermion propagator and the 3-point vertex function, one can use it to try to extract some information about the vertex function.

$$q_\mu \Gamma^\mu(k^2, p^2, q^2) = S_F^{-1}(k) - S_F^{-1}(p) \quad . \tag{2.2.1}$$

2.3.2 Ball-Chiu Vertex Ansatz

The standard approach to representing the vertex is to divide it into two pieces which are called **Transverse** and **Longitudinal** parts [26, 22],

$$\Gamma_F^\mu(k,p) = \Gamma_T^\mu(k,p) + \Gamma_L^\mu(k,p) \quad , \tag{2.3.25}$$

with

$$q_\mu \Gamma_T^\mu(k,p) = 0 \quad , \tag{2.3.26}$$

where q_μ is the photon momenta. Because of Eqn.(2.3.26), the transverse part of the vertex does not know anything about the Ward-Takahashi identity.

In order to satisfy Eqn. (2.2.1), an obvious first try for a possible vertex function could be :

$$\Gamma^\mu(k,p,q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) V_\nu + \frac{q^\mu}{q^2}\left(S_F^{-1}(k^2) - S_F^{-1}(p^2)\right) \quad . \tag{2.3.27}$$

The above choice fulfills the Ward-Takahashi identity and Eqn. (2.3.26) for every value of V_ν, but if we consider the limit $k \longrightarrow p$ then Eqn. (2.3.27) should have a unique limit given by the Ward identity, Eqn. (2.2.2). When this limit of Eqn. (2.3.27) is taken, we find

$$\Gamma^\mu(p,p) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) V_\nu(p,p) + \frac{q^\mu q^\nu}{q^2}\left(\frac{\partial S_F^{-1}(p)}{\partial p^\nu}\right) \quad . \tag{2.3.28}$$

This result can be equal to Eqn. (2.2.2) if and only if

$$V_\nu(p,p) = \frac{\partial S_F^{-1}(p)}{\partial p^\nu} \quad , \tag{2.3.29}$$

i.e. the function V_ν is specified to be the vertex itself. This happens because the full vertex is free of kinematic singularities; so the singularities appearing in Eqn. (2.3.28) must cancel between the longitudinal and transverse parts.

As a second try one can apply the Ward identity to the inverse fermion propagator, which can be written in general as :

$$S_F^{-1} = a(p^2)\ \not p + b(p^2)\ , \tag{2.3.30}$$

and when the Ward identity is applied, we find : .

$$\begin{aligned} \Gamma^\mu(p,p) &= \frac{\partial S_F^{-1}}{\partial p_\mu} \\ &= a(p^2)\gamma_\mu + 2\,a'(p^2)p^\mu \not p + 2\,b'(p^2)p^\mu\ . \end{aligned} \tag{2.3.31}$$

We write this expression in terms of k and p in a careful way so as not to introduce any kinematic singularity, which was not originally in Eqn. (2.2.1). Thus

$$\begin{aligned} &\Gamma^\mu(p,p) \\ &= \lim_{k\to p}\left[\frac{\gamma_\mu}{2}\left(a(k^2)+a(p^2)\right)+\frac{1}{2}(k+p)^\mu(\not k+\not p)\frac{a(k^2)-a(p^2)}{k^2-p^2}+(k+p)^\mu\frac{b(k^2)-b(p^2)}{k^2-p^2}\right] \\ &= \Gamma_T^\mu(p,p)+\Gamma_L^\mu(p,p)\quad . \end{aligned} \tag{2.3.32}$$

On multiplying Eqn. (2.3.32) by q_μ, we can easily see that it satisfies the Ward-Takahashi identity automatically. Hence, Eqn. (2.3.32) can be expressed as :

$$\begin{aligned} \Gamma^\mu(k,p) &= \frac{\gamma_\mu}{2}\left(a(k^2)+a(p^2)\right)+\frac{1}{2}(k+p)^\mu(\not k+\not p)\frac{a(k^2)-a(p^2)}{k^2-p^2} \\ &+ (k+p)^\mu\frac{b(k^2)-b(p^2)}{k^2-p^2}+\Gamma_T^\mu\quad . \end{aligned} \tag{2.3.33}$$

Consequently, by choosing $a(p^2)$ and $b(p^2)$ in Eqn. (2.3.33) as

$$a(p^2)=\frac{1}{F(p^2)}\,,\qquad\qquad b(p^2)=\frac{M(p^2)}{F(p^2)}\,,$$

Ball and Chiu [26] express the non-perturbative structure of the part of the vertex (a part conventionally called the longitudinal component) that fulfills the Ward-Takahashi identity in terms of two non-perturbative functions describing the fermion propagator. The vertex proposed by them can be written as

$$\begin{aligned}\Gamma^\mu_{BC}(k,p) &= \Gamma^\mu_L(k,p)\\ &= \frac{\gamma_\mu}{2}\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\\ &+\frac{1}{2}\frac{(k^\mu+p^\mu)(\not k+\not p)}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)\\ &+\frac{(k^\mu+p^\mu)}{k^2-p^2}\left(\frac{M(k^2)}{F(k^2)}-\frac{M(k^2)}{F(p^2)}\right) \quad . \end{aligned} \tag{2.3.34}$$

In addition, the Ward-Takahashi and Ward identities put some natural constraints on the transverse vertex : one is that the photon momentum is orthogonal to the transverse vertex,

$$q_\mu \Gamma^\mu_T(k,p) = 0 \ , \tag{2.3.35}$$

and the second is that in the limit of $q_\mu \to 0$, the transverse piece vanishes,

$$\Gamma^\mu_T(p,p) = 0 \quad . \tag{2.3.36}$$

It is this condition that is not satisfied by our first example in Eqn. (2.3.27). Since the full vertex and its longitudinal component are free of kinematic singularities, this requires that the same property must hold for the transverse vertex as well.

One of the deficiencies of the rainbow approximation is, therefore, overcome by the BC vertex, namely, respecting the Ward-Takahashi identity [24]. We now see if it is any good for the multiplicative renormalisability of the fermion propagator [23, 24, 31, 35, 36]. Substituting the BC vertex in the SD equation for the fermion propagator,

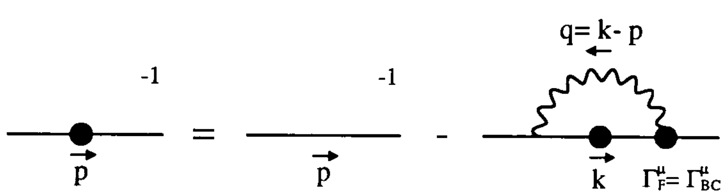


Figure 2.8: Ball-Chiu approximation

$$-iS_F^{-1}(p) = -iS_F^{0\,-1}(p) - \int_M \frac{d^4k}{(2\pi)^4} (-ie\Gamma^\mu_{BC})\ i\,S_F(k)\ (-ie\gamma^\nu)\ i\Delta^0_{\mu\nu}(q^2)\,. \tag{2.3.37}$$

Making use of Eqns. (2.3.4, 2.3.1) and Eqn. (2.3.34), we get

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 + \frac{i\alpha}{16\,p^2\,\pi^2}\int_M \frac{d^4k}{k^2q^2}\,F(k^2)\\ &\times\Bigg\{A\,\mathrm{Tr}\left[\not p\,\gamma^\mu\,\not k\gamma^\nu\left(g_{\mu\nu} + (\xi-1)\,\frac{q_\mu q_\nu}{q^2}\right)\right]\\ &+B\,\mathrm{Tr}\left[\not p\,(\not k + \not p)\,(k+p)^\mu\,\not k\,\gamma^\nu\left(g_{\mu\nu} + (\xi-1)\,\frac{q_\mu q_\nu}{q^2}\right)\right]\Bigg\}\,,\end{aligned} \tag{2.3.38}$$

where

$$\begin{aligned} A &= \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right)\\ B &= \frac{1}{2\,(k^2-p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\quad . \end{aligned} \tag{2.3.39}$$

Using the traces

$$\begin{aligned} \mathrm{Tr}\,[\not p\,(\not k + \not p)\,(k+p)^\mu\,\not k\,\gamma^\nu g_{\mu\nu}] &= 4\left[\left(k^2+p^2\right)k\cdot p + 2k^2p^2\right]\\ \mathrm{Tr}\,[\not p\,(\not k + \not p)\,(k+p)^\mu\,\not k\,\gamma^\nu q_\mu q_\nu] &= 4\left(k^2-p^2\right)^2 k\cdot p\,, \end{aligned} \tag{2.3.40}$$

and Eqn. (2.3.40) in Eqn. (2.3.38) and performing a Wick rotation, we have

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 - \frac{\alpha}{4\,p^2\,\pi^3}\int_E \frac{d^4k}{k^2q^2}\,F(k^2)\\ &\times\Bigg\{A\Bigg[\left(-2k\cdot p - \frac{1}{q^2}\left[\left(k^2+p^2\right)k\cdot p - 2k^2p^2\right]\right)\\ &\quad+\frac{\xi}{q^2}\left(\left(k^2+p^2\right)k\cdot p - 2k^2p^2\right)\Bigg]\\ &+B\Bigg[\left(\left(k^2+p^2\right)k\cdot p + 2k^2p^2 - \frac{1}{q^2} - \frac{1}{q^2}\left(k^2-p^2\right)^2 k\cdot p\right)\\ &\quad+\frac{\xi}{q^2}\left(k^2-p^2\right)^2 k\cdot p\Bigg]\Bigg\}\,. \end{aligned} \tag{2.3.41}$$

This expression can be written as

$$\frac{1}{F(p^2)} = 1 - \frac{\alpha}{2\,p^2\,\pi^2}\int_0^{\Lambda^2} dk^2\, F(k^2)\,[A\,(X_1 + \xi X_2) + B\,(Y_1 + \xi Y_2)]\ , \tag{2.3.42}$$

where

$$\begin{aligned} Y_1 &\equiv \left(k^2+p^2\right)\, I_{1,1} + 2k^2p^2\, I_{0,1} - \left(k^2-p^2\right)^2\, I_{1,2} \\ Y_2 &\equiv \left(k^2-p^2\right)^2\, I_{1,2}\quad , \end{aligned} \tag{2.3.43}$$

and

$$\begin{aligned} X_1 &\equiv -2\, I_{1,1} - \left(k^2+p^2\right)\, I_{1,2} + 2k^2p^2\, I_{0,2} \\ X_2 &\equiv \left(k^2+p^2\right)\, I_{1,2} - 2k^2p^2\, I_{0,2}\quad . \end{aligned} \tag{2.3.20}$$

We refer to Appendix B for the evalution of these angular integrals. Splitting Eqn. (2.3.42) into two regions to compute the k-integral, we find the expression for $1/F(p^2)$:

$$\begin{aligned} \frac{1}{F(p^2)} &= 1 - \frac{\alpha}{2\,p^2\,\pi} \\ &\times \left\{ \int_0^{p^2} dk^2\, F(k^2)\left[\xi\, A\left(-\frac{k^2}{2\,p^2}\right) + B\,\frac{3}{4}\,\frac{k^2}{p^2}\left(p^2+k^2\right) - \xi\, B\,\frac{k^2}{p^2}\left(k^2-p^2\right)\right]\right. \\ &\left. + \int_{p^2}^{\Lambda^2} dk^2\, F(k^2)\left[\xi\, A\left(-\frac{p^2}{2\,k^2}\right) + B\,\frac{3}{4}\,\frac{p^2}{k^2}\left(p^2+k^2\right) + \xi\, B\frac{p^2}{k^2}\left(k^2-p^2\right)\right]\right\}. \end{aligned} \tag{2.3.44}$$

If we substitute Eqn. (2.3.39) into the above expression and tidy up, we find

$$\begin{aligned} \frac{1}{F(p^2)} = 1 - \frac{\alpha}{4\,p^2\,\pi}\Bigg\{ &\int_0^{p^2} dk^2\Bigg[\ \xi\left(3 - \frac{F(k^2)}{F(p^2)}\right)\left(\frac{k^2}{2\,p^2}\right) \\ &\qquad + \left(1 - \frac{F(k^2)}{F(p^2)}\right)\frac{3}{4}\,\frac{k^2}{p^2}\frac{(p^2+k^2)}{(k^2-p^2)}\Bigg] \\ &+ \int_{p^2}^{\Lambda^2} dk^2\Bigg[\ \xi\,\frac{F(k^2)}{F(p^2)}\left(-\frac{p^2}{k^2}\right) \\ &\qquad + \left(1 - \frac{F(k^2)}{F(p^2)}\right)\frac{3}{4}\,\frac{p^2}{k^2}\frac{(p^2+k^2)}{(k^2-p^2)}\Bigg]\Bigg\}\ . \end{aligned} \tag{2.3.45}$$

Again, using on iterative procedure, we find

$$F(p^2) = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2}\left(\frac{\xi^2}{2}-\frac{3\xi}{8}\right)+\left(\frac{\alpha\xi}{4\pi}\right)^2\ln^2\frac{p^2}{\Lambda^2}$$
$$+ \left(\frac{\alpha\xi}{4\pi}\right)^3\left(\frac{\xi^3}{6}-\frac{3\xi^2}{8}+\frac{3\xi}{16}\right)\ln^3\frac{p^2}{\Lambda^2}+\mathcal{O}(\alpha^4)\,. \tag{2.3.46}$$

Again looking at Eqn. (2.3.24) we see that result is not multiplicative renormalisable. There are some extra terms in the above expression which can only be cancelled by the transverse component of the vertex, since the longitudinal part is fixed by the Ball-Chiu construction. Thus the natural next step will be to try to find a suitable transverse vertex.

2.3.3 Curtis-Pennington Vertex Ansatz

We have seen in the previous section that multiplicative renormalisability puts a condition on the transverse component of the vertex [22, 37]. It is only the transverse vertex that can restore multiplicative renormalisability of the fermion propagator and hence it should know about $F(p^2), M(p^2)$. In order to construct a suitable non-perturbative transverse structure of the vertex, perturbation theory can be used as a guide. Let us anticipate a perturbative result we shall derive in Chapter 6, namely :

$$\Gamma_T = -\frac{\alpha\xi}{8\pi}\gamma_\mu\ln\frac{k^2}{p^2}+\frac{\alpha\xi}{8\pi k^2}\left(k^\mu\,\not{k}+p^\mu\,\not{k}-k^\mu\,\not{p}\right)\ln\frac{k^2}{p^2}\,,\qquad \text{when}\quad k^2>>p^2\,. \tag{6.2.48}$$

One can try to establish a non-perturbative ansatz that gives the above result in the weak coupling limit when $k^2 >> p^2$. In perturbation theory, if only leading log terms are considered then the wave-function renormalisation can be written as :

$$F(p^2) = 1+\frac{\alpha\,\xi}{4\,\pi}\ln\frac{p^2}{\Lambda^2}+\cdots\quad . \tag{2.3.47}$$

Note that the following form gives Eqn. (6.2.48) apart from the γ-matrix structure :

$$\frac{1}{2}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]\cong-\frac{\alpha\,\xi}{8\,\pi}\ln\frac{k^2}{p^2}\quad . \tag{2.3.48}$$

As we shall see, the correct γ-matrix structure, in the limit $k^2 >> p^2$, is offered by the basis vector :

$$T_6^\mu(k^2 >> p^2) = -k^2\,\gamma^\mu + k^\mu\,\not{k} - k^\mu\,\not{p} + p^\mu\,\not{k}\,. \tag{2.3.49}$$

Consequently, the transverse vertex can be written as

$$\Gamma_T^\mu \overset{k^2\to\infty}{\simeq} -\frac{\alpha\,\xi}{8\,\pi}\,T_6^\mu\,\frac{\ln\left(k^2/p^2\right)}{k^2} \quad . \tag{2.3.50}$$

One can then write a possible non-perturbative transverse vertex as :

$$\Gamma_T^\mu = \frac{1}{2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right]\frac{T_6^\mu}{k^2} \quad . \tag{2.3.51}$$

The full vertex satisfies charge conjugation relation, Eqn. (2.2.3), as a result of this the transverse is symmetric in k, p. Thus one can only think of having a k^2 term in the denominator in the limit $k^2 >> p^2$. However if we account for all symmetry properties and the correct dimensions (since the vertex by itself is dimensionless) the above expression can be rewritten as the so called Curtis-Pennington (CP) vertex [22], in which the massless case is :

$$\Gamma_{CP}^\mu = \frac{T_6^\mu}{2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right]\frac{(k^2+p^2)}{(k^2-p^2)^2}\,. \tag{2.3.52}$$

Obviously, by defining the transverse vertex in such a form a kinematic singularity has been introduced when $k^2 \to p^2$. In the massive case the CP vertex takes the form

$$\Gamma_{CP}^\mu = \frac{1}{2}\,\frac{T_6^\mu}{d(k,p)}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right]\,, \tag{2.3.53}$$

where

$$d = \frac{(k^2-p^2)^2 + \left(M^2(k^2) + M^2(p^2)\right)^2}{k^2+p^2} \quad . \tag{2.3.54}$$

By using this vertex to solve the SD-equation,

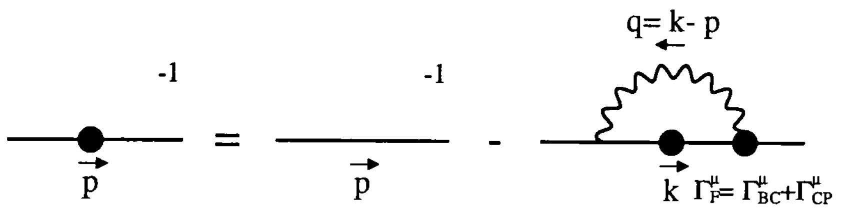


Figure 2.9: Curtis-Pennington ansatz

If we add the CP-vertex to the BC-vertex in Eqn. (2.3.37), we find :

$$-iS_F^{-1}(p) = -iS_F^{0\,-1}(p) - \int_M \frac{d^4k}{(2\pi)^4}\left(-ie\Gamma^\mu_{BC+CP}\right)\, i\,S_F(k)\,\left(-ie\gamma^\nu\right)\, i\Delta^0_{\mu\nu}(q)\,. \tag{2.3.55}$$

Substituting the fermion and photon propagators into this expression, we get,

$$\begin{aligned}\frac{1}{F(p^2)} &= 1+\mathcal{F}_L+\frac{i\,\alpha}{16\,p^2\,\pi^3}\int_M \frac{d^4k}{k^2\,q^2}\,F(k^2)\frac{(k^2+p^2)}{(k^2-p^2)}\,B\\ &\times\mathrm{Tr}\Big[\,\not p\left(\gamma^\mu(p^2-k^2)+(k+p)^\mu\,\not q\right)\,\not k\gamma^\nu\Big]\left(g_{\mu\nu}+(\xi-1)\,\frac{q_\mu q_\nu}{q^2}\right)\,,\end{aligned} \tag{2.3.56}$$

where recall B is defined in Eqn. (2.3.39) and $\mathcal{F}_L$ is the contribution coming from the longitudinal part of the vertex given in Eqn. (2.3.34). After taking related traces,

$$\begin{aligned}\mathrm{Tr}\left[\not p\left(\gamma^\mu\left(p^2-k^2\right)+(k+p)^\mu\,\not q\right)\,\not k\gamma^\nu g_{\mu\nu}\right] &= 12\left(k^2-p^2\right)\,k\cdot p\,,\\ \mathrm{Tr}\left[\not p\left(\gamma^\mu\left(p^2-k^2\right)+(k+p)^\mu\,\not q\right)\,\not k\gamma^\nu q_\mu q_\nu\right] &= 0\quad,\end{aligned} \tag{2.3.57}$$

in Eqn. (2.3.56) and performing the Wick rotation, we obtain :

$$\frac{1}{F(p^2)} = 1 + \mathcal{F}_L + \frac{3\,\alpha}{4\,p^2\,\pi^3}\int_E \frac{d^4k}{k^2\,q^2}\,F(k^2)\,B\left(k^2+p^2\right). \tag{2.3.58}$$

Evaluating the angular integration we have :

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 + \mathcal{F}_L \\ &+ \frac{3\,\alpha}{8\,p^2\,\pi}\left[\int_0^{p^2} dk^2\,F(k)\,B\left(k^2+p^2\right)\frac{k^2}{p^2} + \int_{p^2}^{\Lambda^2} dk^2\,F(p)\,B\left(k^2+p^2\right)\frac{p^2}{k^2}\right],\end{aligned} \tag{2.3.59}$$

and after tidying up, this expression can be rewritten as :

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 + \mathcal{F}_L \\ &+ \frac{\alpha}{4\,p^2\,\pi^2}\left\{\int_0^{p^2} dk^2\left(1-\frac{F(k^2)}{F(p^2)}\right)\frac{3}{4}\,\frac{k^2}{p^2}\,\frac{(k^2+p^2)}{(k^2-p^2)}\right. \\ &+ \left.\int_{p^2}^{\Lambda^2} dk^2\left(1-\frac{F(k^2)}{F(p^2)}\right)\frac{3}{4}\,\frac{p^2}{k^2}\,\frac{(k^2+p^2)}{(k^2-p^2)}\right\}.\end{aligned} \tag{2.3.60}$$

Referring back to Eqn. (2.3.45) for the form for $\mathcal{F}_L$, we see, the necessary cancellation between the longitudinal and transverse parts has taken place. We then find :

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 - \frac{\alpha\,\xi}{4\,p^2\,\pi^2}\left\{\int_0^{p^2} dk^2\left(3-\frac{F(k^2)}{F(p^2)}\right)\left(\frac{k^2}{p^2}\right)\right. \\ &\qquad\left. + \int_{p^2}^{\Lambda^2} dk^2\left(1-\frac{F(k^2)}{F(p^2)}\right)\left(-\frac{p^2}{k^2}\right)\right\}.\end{aligned} \tag{2.3.61}$$

Eventually, this solution of the wave-function renormalisation comes out to be multiplicatively renormalisable :

$$F(p^2) = 1 + \frac{\alpha\,\xi}{4\,\pi}\ln\frac{p^2}{\Lambda^2} + \frac{1}{2!}\left(\frac{\alpha\,\xi}{4\,\pi}\right)^2\ln^2\frac{p^2}{\Lambda^2} + \frac{1}{3!}\left(\frac{\alpha\,\xi}{4\,\pi}\right)^3\ln^3\frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha^4). \tag{2.3.62}$$

and the mass function for this vertex is,

$$M(p^2) = m_0 \left[1 - \frac{3\,\alpha}{4\,\pi} \ln \frac{p^2}{\Lambda^2} + \cdots \right] . \tag{2.3.63}$$

which is gauge independent at least for leading logs.

We have already seen in Fig. 2.7 that the mass function was dependent on the gauge parameter in the bare vertex approximation. We also mentioned that any candidate for a vertex ansatz has to produce physical observables independent of the gauge. Numerical studies of non-perturbative quenched QED [29, 24, 34, 31, 35] using the CP vertex in SD-equation, Eqn. (2.3.55), have shown that the Euclidean mass is no longer strongly dependent on the gauge parameter. As a result of this, in analogy with Fig. 2.7, the mass function is plotted against the coupling constant [29], using CP vertex, in the figure below :

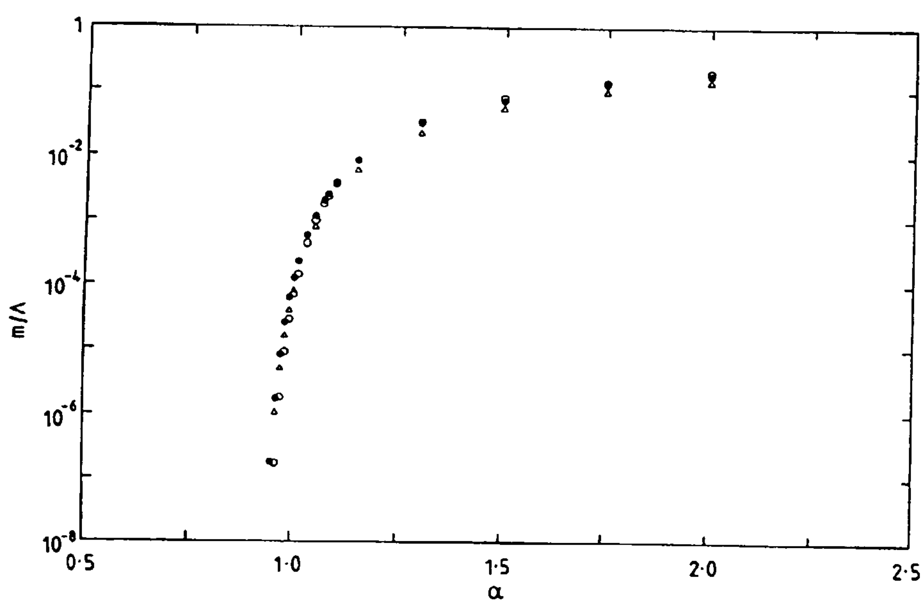


Figure 2.10: Mass generation with Curtis-Pennington ansatz

In these studies, the critical coupling constant is also plotted versus gauge parameter [34], then it looks like :

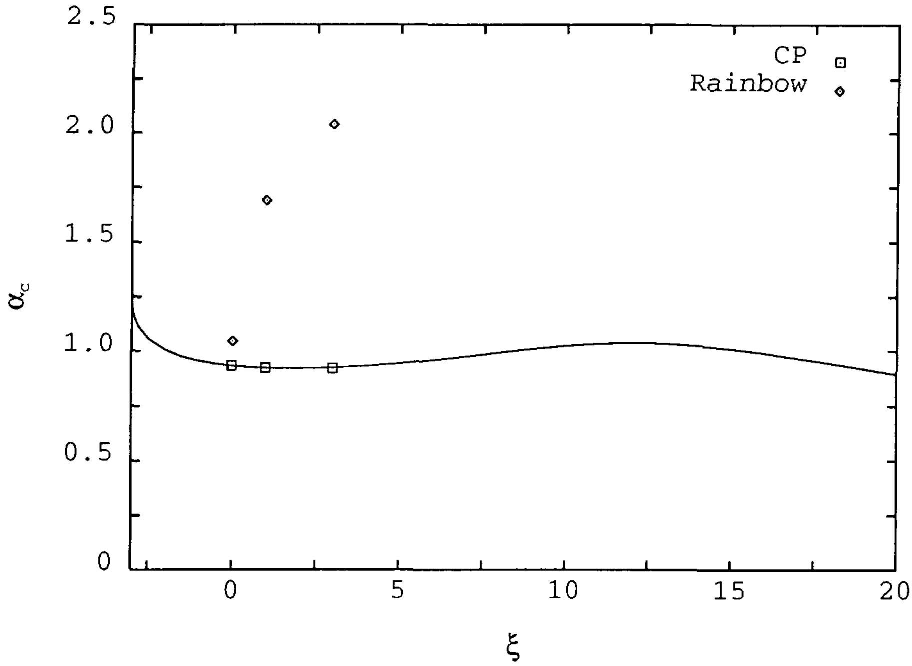


Figure 2.11: Gauge dependence of the coupling constant with CP ansatz

This vertex (CP) makes the critical coupling weakly dependent on the gauge parameter. There is appreciable improvement compared with the Rainbow approximation where the critical coupling is strongly gauge dependent. Since the aim is to find a vertex ansatz that makes the critical coupling and other observables completely gauge independent, there is still room for improvement. This may come from the contribution of the other basis vectors in the transverse vertex since only one of them is used in the CP-vertex. In this regard, Bashir and Pennington have recently suggested a different ansatz for the transverse component. All this work is in the context of quenched QED.

2.3.4 Bashir-Pennington Vertex Ansatz

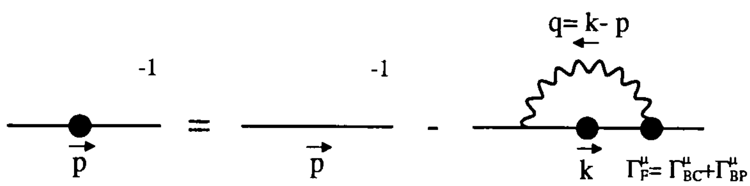


Figure 2.12: Bashir-Pennington ansatz

Bashir and Pennington have taken into account four of the basis tensors in the transverse piece which involve an odd number of γ-matrices to study the SD-equation. Referring to Ball-Chiu's work [26],

$$\Gamma_T^\mu(k,p) = \sum_{i=2,3,6,8} \tau_i \left(k^2, p^2, q^2\right) T_i^\mu(k,p) , \tag{2.3.64}$$

where

$$\begin{aligned} T_2^\mu &= \left(p^\mu (k \cdot q) - k^\mu (p \cdot q) \right) (\not{k} + \not{p}) , \\ T_3^\mu &= q^2 \gamma^\mu - q^\mu \not{q} , \\ T_6^\mu &= \gamma^\mu (p^2 - k^2) + (p+k)^\mu \not{q} , \\ T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not{p} - p^\mu \not{k} \\ \text{with} \qquad \sigma_{\mu\nu} &= \frac{1}{2} \left[\gamma_\mu, \gamma_\nu \right] . \end{aligned} \tag{2.3.65}$$

They have tried to solve the SD-equation for the wave-function renormalisation $F(p^2)$ and mass function $M(p^2)$ in quenched QED by demanding a gauge independent critical coupling constant using bifurcation analysis. They managed to write the transverse vertex in terms of two unknown functions W_1 and W_2 which have some constraints to obey. Multiplicative renormalisability of the fermion propagator imposes the constraint

$$\int_0^1 dx W_1(x) = 0 \quad , \tag{2.3.66}$$

whereas, the gauge-independence of the critical coupling places the following constraint on W_2,

$$\int_0^1 \frac{dx}{\sqrt{x}} W_2(x) = 0 \quad . \tag{2.3.67}$$

Then, a non-perturbative multiplicatively renormalisable solution of the wave-function renormalisation and the mass function are

$$\begin{aligned} F(p^2) &= A(p^2)^{\nu} \quad , \\ M(p^2) &= B(p^2)^{-s} \quad , \end{aligned} \qquad (2.3.68)$$

the latter only holds in the neighbourhood of the bifurcation point $\nu = \alpha\xi/4\pi$. Coefficients of the basis tensors are then given in terms of the functions W_1 and W_2 [38] :

With

$$\overline{\tau}(k^2,p^2) = \frac{1}{4}\frac{1}{k^2-p^2}\frac{1}{s_1(k^2,p^2)}\left[W_1\left(\frac{k^2}{p^2}\right) - W_1\left(\frac{p^2}{k^2}\right)\right] ,$$

then

$$\begin{aligned} \tau_6(k^2,p^2) = & -\frac{1}{2}\frac{k^2+p^2}{(k^2-p^2)^2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) + \frac{1}{3}\frac{k^2+p^2}{k^2-p^2}\overline{\tau}(k^2,p^2) \\ & +\frac{1}{6}\frac{1}{k^2-p^2}\frac{1}{s_1(k^2,p^2)}\left[W_1\left(\frac{k^2}{p^2}\right) + W_1\left(\frac{p^2}{k^2}\right)\right] , \end{aligned}$$

$$\begin{aligned} \tau_2(k^2,p^2) = & \frac{2\xi}{(k^2-p^2)^2}\frac{q_2(k^2,p^2)}{s_2(k^2,p^2)} - 6\frac{\tau_6(k^2,p^2)}{(k^2-p^2)} \\ & -\frac{1}{(k^2-p^2)^2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right] \\ & -\frac{k^2+p^2}{(k^2-p^2)^3}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] , \end{aligned}$$

$$\begin{aligned} \tau_3(k^2,p^2) = & -\frac{k^2+p^2}{k^2-p^2}\tau_6(k^2,p^2) \\ & +\frac{1}{k^2-p^2}\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}\tau_2\left(\frac{k^2}{p^2}\right) - \frac{\xi}{3}q_3(k^2,p^2)\right] \\ & -\frac{1}{6}\frac{k^2+p^2}{(k^2-p^2)^2}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) + W_2\left(\frac{p^2}{k^2}\right)\right] \\ & +\frac{1}{6}\frac{k^4+p^4-6k^2p^2}{(k^2-p^2)^3}\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right) - W_2\left(\frac{p^2}{k^2}\right)\right] , \end{aligned}$$

and

$$\begin{aligned}
\tau_8(k^2,p^2) &= -2\,\frac{k^2+p^2}{k^2-p^2}\,\tau_6(k^2,p^2)+\overline{\tau}(k^2,p^2) \\
&\quad -\frac{1}{k^2-p^2}\,\frac{1}{s_2(k^2,p^2)}\left[\frac{1}{2}\,r_2\left(\frac{k^2}{p^2}\right)-\frac{\xi}{3}\,q_8(k^2,p^2)\right] \\
&\quad -\frac{1}{3}\,\frac{k^2+p^2}{(k^2-p^2)^2}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)+W_2\left(\frac{p^2}{k^2}\right)\right] \\
&\quad -\frac{2}{3}\,\frac{k^4+p^4}{(k^2-p^2)^3}\,\frac{1}{s_2(k^2,p^2)}\left[W_2\left(\frac{k^2}{p^2}\right)-W_2\left(\frac{p^2}{k^2}\right)\right] \quad ,
\end{aligned}$$

where

$$\begin{aligned}
r_1\left(\frac{k^2}{p^2}\right) &= \left(\frac{k^2}{p^2}\right)\left[1-\left(\frac{k^2}{p^2}\right)^{\nu}\right]-\left(\frac{p^2}{k^2}\right)\left[1-\left(\frac{p^2}{k^2}\right)^{\nu}\right] \quad , \\
r_2\left(\frac{k^2}{p^2}\right) &= \left(\frac{k^2}{p^2}\right)^{\frac{1}{2}-s_c}\left[1-\left(\frac{k^2}{p^2}\right)^{\nu}\right]-\left(\frac{p^2}{k^2}\right)^{\frac{1}{2}-s_c}\left[1-\left(\frac{p^2}{k^2}\right)^{\nu}\right] \quad , \\
s_1(k^2,p^2) &= \frac{k^2}{p^2}\,F(k^2)+\frac{p^2}{k^2}\,F(p^2) \quad , \\
s_2(k^2,p^2) &= \frac{k}{p}\,\frac{\mathcal{M}(k^2)}{\mathcal{M}(p^2)}\,F(k^2)+\frac{p}{k}\,\frac{\mathcal{M}(p^2)}{\mathcal{M}(k^2)}\,F(p^2) \quad ; \\
q_2(k^2,p^2) &= \frac{1}{k^2-p^2}\left[\frac{k^3}{p}\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}-\frac{p^3}{k}\,\frac{\mathcal{M}(p^2)F(p^2)}{\mathcal{M}(k^2)F(k^2)}\right] \quad , \\
q_3(k^2,p^2) &= \frac{kp}{(k^2-p^2)^2}\left[(p^2-3k^2)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}-(k^2-3p^2)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}\right] \quad , \\
q_8(k^2,p^2) &= \frac{1}{(k^2-p^2)^2}\left[\frac{k}{p}(3k^4+p^4)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}-\frac{p}{k}(k^4+3p^4)\,\frac{\mathcal{M}(k^2)F(k^2)}{\mathcal{M}(p^2)F(p^2)}\right] \quad .
\end{aligned}$$

Bashir and Pennington also give a simple example for W_1 and W_2. However, these functions are not unique. Still more information is required in order to pin them down. Hopefully, this information may be extracted from the perturbation theory calculation in arbitrary covariant gauges, which we introduce later in this thesis.

2.4 Our Aim

Our aim in the first part of this thesis is to find non-perturbative constraints on the fermion-gauge boson vertex by using the fermion $F(p)$ and photon $G(p)$ wave-function renormalisations for **unquenched** QED. The next step will be to make a simple suggestion for constructing a non-perturbative vertex ansatz which satisfies these constraints. To represent the main steps of this procedure, we display in the following flow diagram the calculations of Chapters 3-6.

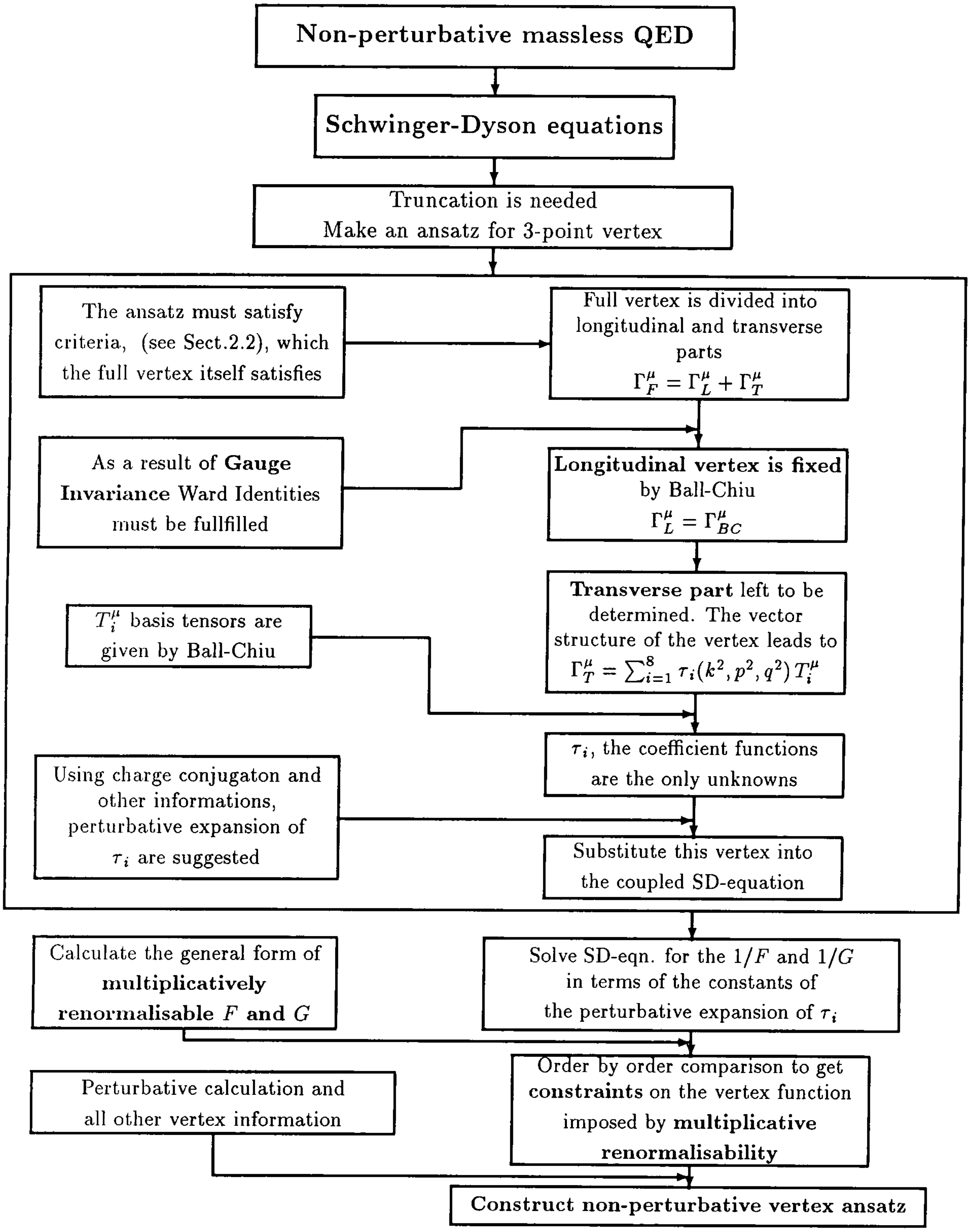


Figure 2.13: Flow diagram of the Schwinger-Dyson calculation

Chapter 3

Solving Unquenched Schwinger-Dyson Equations in Massless QED

On the mountains of truth
you can never climb in vain :
either you will reach a point higher up today,
or you will be training your powers so that
you will be able to climb higher tomorrow.

-Nietzsche-

3.1 Introduction

Our main intention is to study the fermion-gauge boson interaction in massless QED by using the **unquenched** SD-equations. One can approximate the fermion-gauge boson vertex by an ansatz satisfying the criteria which were mentioned previously in Eqns. (2.2.1–2.2.3). The longitudinal part of this vertex is fixed completely by Ball and Chiu [26]. However, the transverse part remains undetermined. It was seen in the last Chapter, how multiplicative renormalisability plays an important role in fixing this [22, 29]. It is crucial to know the constraints imposed by multiplicative renormalisability [28, 39] on the vertex function. This would help us find a better approximation for the fermion-gauge boson vertex function. In this context, we shall devote this chapter to two main topics in the framework of massless QED :

(1) studying the unquenched fermion SD-equation for the fermion wave-function renormalisation in an arbitrary covariant gauge,

(2) studying the unquenched gauge-boson SD-equation for the photon wave-function renormalisation in an arbitrary covariant gauge.

These SD-equations are displayed below :

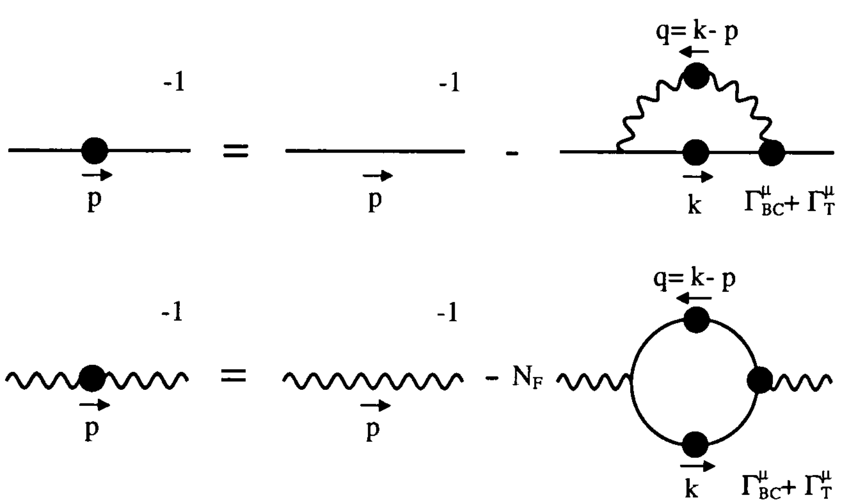


Figure 3.1: Unquenched Schwinger-Dyson equations for fermion and gauge boson

We shall write out the most general perturbative expansion of the coefficients τ_i appearing in the representation of the transverse vertex. This will yield the perturbative expansion for the wave-function renormalisation and photon functions upto next-to-leading log terms of the expansion parameters of τ_i. Then, we shall see how multiplicative renormalisability constraints these parameters. This will help us construct the non-perturbative transverse vertex.

Let us start with the unquenched SD-equation for the fermion.

3.2 Unquenched Schwinger-Dyson Equation for the Fermion in massless QED

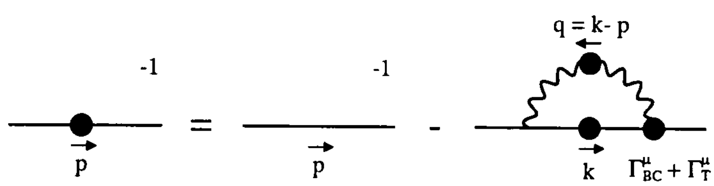


Figure 3.2: Unquenched Schwinger-Dyson equations for fermion

Making use of the Feynman rules, Fig. 3.2 above can be written as :

$$-iS_F^{-1}(p) = -iS_F^{0^{-1}}(p) - \int_M \frac{d^4k}{(2\pi)^4}(-ie\Gamma_F^\mu)\, iS_F(k)\,(-ie\gamma^\nu)\, i\Delta_{\mu\nu}(q)\,. \qquad (3.2.1)$$

Recalling the fermion and photon propagators from Chapter 2 in the massless case,

$$iS_F(p) \;=\; i\,\frac{F(p^2)}{\not p}\;, \qquad (2.3.1)$$

$$\begin{aligned} i\Delta_{\mu\nu}(q) &= -i\,\frac{G(q^2)}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\,\xi\,\frac{q_\mu q_\nu}{q^4} \\ &= -i\,\Delta^T_{\mu\nu} - i\,\xi\,\frac{q_\mu q_\nu}{q^4}\,, \end{aligned} \qquad (2.3.3)$$

and employing the Ward-Takahashi identity for the longitudinal part of the photon propagator,

$$q^\mu\Gamma_\mu(k,p,q) = S_F^{-1}(k) - S_F^{-1}(p)\,, \qquad (2.2.1)$$

we can rewrite Eqn. (3.2.1) as :

$$\begin{aligned} iS_F^{-1}(p) &= iS_F^{-1}(p) - e^2 \int_M \frac{d^4k}{(2\pi)^4} \Big\{ \Gamma_F^\mu \, S_F(k)\, \gamma^\nu \, \Delta^T_{\mu\nu}(q) \\ & \qquad + \xi \, \left(S_F^{-1}(k) - S_F^{-1}(p)\right) \, S_F(k) \, \frac{\not q}{q^4} \Big\} , \\ &= iS_F^{-1}(p) - e^2 \int_M \frac{d^4k}{(2\pi)^4} \Big\{ \Gamma_F^\mu \, S_F(k)\, \gamma^\nu \, \Delta^T_{\mu\nu}(q) \\ & \qquad + \xi \left(\frac{\not q}{q^4} - S_F^{-1}(p)\, S_F(k) \, \frac{\not q}{q^4} \right) \Big\} . \end{aligned} \tag{3.2.2}$$

Being an odd integral, the second term in the integral is zero :

$$\int \frac{d^4k}{(2\,\pi)^4} \, \frac{\not q}{q^4} = 0 \, . \tag{3.2.3}$$

After substituting the fermion and photon propagators, Eqn. (2.3.1, 2.3.3), in Eqn. (3.2.2),we get

$$\begin{aligned} \frac{\not p}{F(p^2)} &= \not p + \frac{ie^2}{(2\pi)^4} \int_M d^4k \Big\{ \Gamma_F^\mu \, \frac{F(k^2)}{\not k} \gamma^\nu \, \frac{G(q^2)}{q^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \\ & \qquad - \xi \, \frac{\not p}{F(p^2)} \, \frac{F(k^2)}{\not k} \, \frac{\not q}{q^4} \Big\} . \end{aligned} \tag{3.2.4}$$

Multiplying by $\not p/4$ and taking the trace, we find :

$$\begin{aligned} \frac{1}{F(p^2)} &= 1 + \frac{i\,\alpha}{16\,\pi^3\,p^2} \int_M \frac{d^4k}{k^2 q^2} \mathrm{Tr} \Big\{ \not p \Gamma_F^\mu \, \not k \, \gamma^\nu \, F(k^2)\, G(q^2) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \\ & \qquad - \xi \, \frac{p^2}{q^2} \, \not k \, \not q \, \frac{F(k^2)}{F(p^2)} \Big\} . \end{aligned} \tag{3.2.5}$$

As always, we divide the full vertex into two pieces,

$$\Gamma_F^\mu = \Gamma_L^\mu + \Gamma_T^\mu \quad . \tag{3.2.6}$$

Recall the definition of the longitudinal piece given in Chapter 2 :

$$\begin{aligned} \Gamma_L^\mu(k,p) &= \Gamma_{BC}^\mu(k,p) \\ &= A\,\gamma^\mu + B\,\left(k^\mu + p^\mu\right)\left(\not k + \not p\right) , \end{aligned}$$

where

$$A = \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) ,$$

$$B = \frac{1}{2(k^2-p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) . \tag{2.3.38}$$

Also recall the transverse piece is :

$$\Gamma_T^\mu(k,p,q) = \sum_{i=2,3,6,8} \tau_i(k^2,p^2,q^2)\, T_i^\mu , \tag{2.3.64}$$

where the τ_i are coefficient functions depending on momenta k^2, p^2 and q^2, which are as yet undetermined, and the T's are the basis tensors defined by Ball and Chiu [26],

$$\begin{aligned}
T_2^\mu &= (\, p^\mu(k\cdot q) - k^\mu(p\cdot q)\,)\,(\not{k} + \not{p}) ,\\
T_3^\mu &= q^2\gamma^\mu - q^\mu \not{q} ,\\
T_6^\mu &= \gamma^\mu(p^2-k^2) + (p+k)^\mu \not{q} ,\\
T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not{p} - p^\mu \not{k} ,\\
\text{with} \qquad \sigma_{\mu\nu} &= \frac{1}{2}\,[\gamma_\mu, \gamma_\nu] .
\end{aligned} \tag{2.3.65}$$

Making use of this full vertex function in Eqn. (3.2.5), we have :

$$\begin{aligned}
\frac{1}{F(p^2)} = {} & 1 + \frac{i\,\alpha}{16\pi^3 p^2}\int_M \frac{d^4k}{k^2q^2}\, F(k^2) \\
& \times \Bigg\{ -\xi\,\frac{p^2}{q^2}\,\frac{1}{F(p^2)}\,\mathrm{Tr}(\not{k}\,\not{q}) \\
& + A\,G(q^2)\,\mathrm{Tr}\Big[\,\not{p}\gamma^\mu\,\not{k}\gamma^\nu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Big] \\
& + B\,G(q^2)\,\mathrm{Tr}\Big[\,\not{p}(k+p)^\mu\,(\not{k}+\not{p})\,\not{k}\gamma^\nu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Big] \\
& + G(q^2)\quad \mathrm{Tr}\Big[\,\not{p}\Gamma_T^\mu\,\not{k}\gamma^\nu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Big]\Bigg\} .
\end{aligned} \tag{3.2.7}$$

Let us compute the necessary traces below :

$$
\begin{aligned}
\mathrm{Tr}\ [\not p\, A\, \gamma_\mu\ \not k\, \gamma^\nu g_{\mu\nu}] &= -\ 8\, A\, k\cdot p \quad , \\
\mathrm{Tr}\ [\not p\, A\, \gamma_\mu\ \not k \gamma^\nu q_\mu q_\nu] &= 4\, A\ ((k^2+p^2)\, k\cdot p - 2\, k^2p^2) \quad , \\
\mathrm{Tr}\ [\not p\, B\, (k+p)^\mu (\not k + \not p)\ \not k\, \gamma^\nu g_{\mu\nu}] &= 4\, B\ ((k^2+p^2)\, k\cdot p + 2\, k^2p^2) \quad , \\
\mathrm{Tr}\ [\not p\, B\, (k+p)^\mu (\not k + \not p)\ \not k\, \gamma^\nu q_\mu q_\nu] &= 4\, B\, (k^2-p^2)^2\, k\cdot p \quad , \\
\mathrm{Tr}\ [\not p\, T_2^\mu\ \not k\, \gamma^\nu\, g_{\mu\nu}] &= 4(k^2+p^2)\,(k^2p^2 - (k\cdot p)^2) \quad , \\
\mathrm{Tr}\ [\not p\, T_2^\mu\ \not k\, \gamma^\nu\, q_\mu q_\nu] &= 0 \quad , \\
\mathrm{Tr}\ [\not p\, T_3^\mu\ \not k\, \gamma^\nu\, g_{\mu\nu}] &= 16\,(k\cdot p)^2 + 8\, k^2p^2 - 12\,(k^2+p^2)\, k\cdot p \quad , \\
\mathrm{Tr}\ [\not p\, T_3^\mu\ \not k\, \gamma^\nu\, q_\mu q_\nu] &= 0 \quad , \\
\mathrm{Tr}\ [\not p\, T_6^\mu\ \not k\, \gamma^\nu\, g_{\mu\nu}] &= 12\,(k^2-p^2)\, k\cdot p \quad , \\
\mathrm{Tr}\ [\not p\, T_6^\mu\ \not k\, \gamma^\nu\, q_\mu q_\nu] &= 0 \quad , \\
\mathrm{Tr}\ [\not p\, T_8^\mu\ \not k\, \gamma^\nu\, g_{\mu\nu}] &= 8\ (k^2p^2 - (k\cdot p)^2) \quad , \\
\mathrm{Tr}\ [\not p\, T_8^\mu\ \not k\, \gamma^\nu\, q_\mu q_\nu] &= 0 \quad , \\
\mathrm{Tr}\ [\not k\ \not q] &= 4\,(k^2 - k\cdot p) \quad .
\end{aligned}
$$

Using these traces, Eqn. (3.2.7) acquires the form :

$$
\begin{aligned}
\frac{1}{F(p^2)} &= 1 + \frac{i\,\alpha}{4\,\pi^3\, p^2} \int_M \frac{d^4k}{k^2\, q^2}\, F(k^2) \\
&\times \Bigg\{ -\ \xi\, \frac{p^2}{q^2}\, \frac{1}{F(p^2)} \left(k^2 - k\cdot p\right) \\
&\quad + G(q^2) \left[A\left(-2\, k\cdot p - \frac{1}{q^2}\ \left[(k^2+p^2)\, k\cdot p - 2k^2p^2 \right] \right) \right] \\
&\quad + G(q^2) \left[B\left((k^2+p^2)\, k\cdot p + 2\, k^2p^2 - \frac{1}{q^2}(k^2-p^2)^2\, k\cdot p \right) \right] \\
&\quad + G(q^2) \Bigg[\quad \tau_2\, (k^2+p^2)\ \left(k^2p^2 - (k\cdot p)^2\right) \\
&\qquad + \tau_3 \left(4\,(k\cdot p)^2 + 2k^2p^2 - 3(k^2+p^2)\, k\cdot p \right) \\
&\qquad + \tau_6 \left(3\,(k^2-p^2)\, k\cdot p \right) \\
&\qquad + \tau_8 \left(2\, k^2p^2 - 2\,(k\cdot p)^2 \right) \Bigg] \Bigg\} \quad .
\end{aligned}
\tag{3.2.8}
$$

To perform these integrals, we move to Euclidean space using the Wick rotation. Tidying up leads to :

$$
\begin{aligned}
\frac{1}{F(p^2)} &= 1-\frac{\alpha}{4\,\pi^3\,p^2}\int_E \frac{d^4k}{k^2\,q^2} \\
&\times \Bigg\{ -\xi\,\frac{F(k^2)}{F(p^2)}\,\frac{p^2}{q^2}\,(k^2-k\cdot p) \\
&\quad + F(k^2)\,G(q^2)\Bigg[\,\frac{A}{q^2}\left(4(k\cdot p)^2+2k^2p^2-3\,k\cdot p\,(k^2+p^2)\right) \\
&\qquad + \frac{B}{q^2}\left(2\,(k^2+p^2)\left(k^2p^2-(k\cdot p)^2\right)\right)\Bigg] \\
&\quad + F(k^2)\,G(p^2)\Bigg[\quad \tau_2\,(k^2+p^2)\left(k^2p^2-(k\cdot p)^2\right) \\
&\qquad - \tau_3\left(4\,(k\cdot p)^2+2\,k^2p^2-3(k^2+p^2)\,k\cdot p\right) \\
&\qquad - \tau_6\left(3\,(k^2-p^2)\,k\cdot p\right) \\
&\qquad - \tau_8\left(2\,k^2p^2-2\,(k\cdot p)^2\right)\quad\Bigg]\Bigg\}.
\end{aligned}
\tag{3.2.9}
$$

To proceed further, we introduce the following general form of the perturbative leading logarithmic expansion of the fermion and photon wave-function renormalisations :

$$
\begin{aligned}
F(p^2) = 1 \;&+\; \alpha\,\left(A_{11}\ln\frac{p^2}{\Lambda^2}+A_{10}\right)+\alpha^2\left(A_{22}\ln^2\frac{p^2}{\Lambda^2}+A_{21}\ln\frac{p^2}{\Lambda^2}\right) \\
&+\; \alpha^3\left(A_{33}\ln^3\frac{p^2}{\Lambda^2}+A_{32}\ln^2\frac{p^2}{\Lambda^2}\right)+\mathcal{O}(\alpha^4)\quad ,
\end{aligned}
\tag{3.2.10}
$$

and

$$
\begin{aligned}
G(q^2) = 1 \;&+\; \alpha\,\left(B_{11}\ln\frac{q^2}{\Lambda^2}+B_{10}\right)+\alpha^2\left(B_{22}\ln^2\frac{q^2}{\Lambda^2}+B_{21}\ln\frac{q^2}{\Lambda^2}\right) \\
&+\; \alpha^3\left(B_{33}\ln^3\frac{q^2}{\Lambda^2}+B_{32}\ln^2\frac{q^2}{\Lambda^2}\right)+\mathcal{O}(\alpha^4)\quad ,
\end{aligned}
\tag{3.2.11}
$$

where we keep only leading and next-to-leading logarithms. Eqn. (3.2.9) can then be re-expressed as follows after plugging in these expansions and splitting up the angular and radial part of the integral :

$$
\begin{aligned}
\frac{1}{F(p^2)} &= 1 - \frac{\alpha}{2\pi^2 p^2} \int_0^{\Lambda^2} \int_0^{\pi} dk^2\, d\psi\, \sin^2\psi \\
&\times \Bigg\{ \qquad\qquad - \xi \frac{F(k^2)}{F(p^2)} \frac{p^2}{q^4} (k^2 - k\cdot p) \\
&+ F(k^2) \left[(1+\alpha B_{10}) + (\alpha B_{11} + \alpha^2 B_{21}) \ln\frac{q^2}{\Lambda^2} + \alpha^2 B_{22} \ln^2\frac{q^2}{\Lambda^2} + \cdots \right] \\
&\qquad \times \Bigg[\frac{A}{q^4} \left\{ 4(k\cdot p)^2 + 2k^2p^2 - 3\, k\cdot p\, (k^2+p^2) \right\} \\
&\qquad + \frac{B}{q^4} \left\{ 2\,(k^2+p^2)\,(k^2p^2 - (k\cdot p)^2) \right\} \Bigg] \\
&+ F(k^2) \left[(1+\alpha B_{10}) + (\alpha B_{11} + \alpha^2 B_{21}) \ln\frac{q^2}{\Lambda^2} + \alpha^2 B_{22} \ln^2\frac{q^2}{\Lambda^2} + \cdots \right] \\
&\qquad \times \Bigg[\frac{\tau_2}{q^2} (k^2+p^2) \left\{ k^2p^2 - (k\cdot p)^2 \right\} \\
&\qquad - \frac{\tau_3}{q^2} \left\{ 4\,(k\cdot p)^2 + 2k^2p^2 - 3(k^2+p^2)\, k\cdot p \right\} \\
&\qquad - \frac{\tau_6}{q^2} \left\{ 3\,(k^2-p^2)\, k\cdot p \right\} \\
&\qquad - \frac{\tau_8}{q^2} \left\{ 2\, k^2p^2 - 2\,(k\cdot p)^2 \right\} \Bigg] \qquad \Bigg\} . \qquad (3.2.12)
\end{aligned}
$$

Before we divide this equation into four pieces and for convenience deal with these separately, we introduce the following simplified notation for the angular integrals. We define

$$
I_{n,m} = \int_0^{\pi} d\psi\, \sin^2\psi\, \frac{(k\cdot p)^n}{(q^2)^m} \quad . \qquad (3.2.13)
$$

We then separate Eqn. (3.2.12) into terms with an explicit gauge dependence L_ξ, the contribution of the "A" and "B" parts of the longitudinal vertex, L_A, L_B, and the contribution of the transverse vertex, T, such that

$$
\frac{1}{F(p^2)} = 1 - \left(L_\xi^f + L_A^f + L_B^f + T^f \right) . \qquad (3.2.14)
$$

We collect the terms containing ξ in L_ξ^f,

$$
L_\xi^f = -\frac{\alpha\,\xi}{2\pi^2} \int_0^{\Lambda^2} dk^2\, \frac{F(k^2)}{F(p^2)} \left(k^2 I_{0,2} - I_{1,2} \right) . \qquad (3.2.15)
$$

The other terms are :

$$
\begin{aligned}
L_A^f &= \frac{1}{2\,\pi^2\,p^2}\int_0^{\Lambda^2} dk^2\,F(k^2)\,A \\
&\times \Bigg\{ (\alpha+\alpha^2\,B_{10})\left\{4\,I_{2,2}+2\,k^2p^2\,I_{0,2}-3\,(k^2+p^2)\,I_{1,2}\right\} \\
&\quad + \int_0^{\pi} d\psi\,\sin^2\psi\,\left((\alpha^2\,B_{11}+\alpha^3\,B_{21})\,\ln\frac{q^2}{\Lambda^2}+\alpha^3\,B_{22}\ln^2\frac{q^2}{\Lambda^2}+\cdots\right) \\
&\qquad \times\,\frac{1}{q^4}\left\{4(k\cdot p)^2+2\,k^2p^2-3\,k\cdot p\,(k^2+p^2)\right\}\Bigg\}\,,
\end{aligned}
\tag{3.2.16}
$$

$$
\begin{aligned}
L_B^f &= \frac{1}{2\,\pi^2\,p^2}\int_0^{\Lambda^2} dk^2\,F(k^2)\,B \\
&\times \Bigg\{ (\alpha+\alpha^2\,B_{10})\left\{2\,(k^2+p^2)\,(k^2p^2\,I_{0,2}-I_{2,2})\right\} \\
&\quad + \int_0^{\pi} d\psi\,\sin^2\psi\,\left((\alpha^2\,B_{11}+\alpha^3\,B_{21})\,\ln\frac{q^2}{\Lambda^2}+\alpha^3\,B_{22}\ln^2\frac{q^2}{\Lambda^2}+\cdots\right) \\
&\qquad \times\frac{1}{q^4}\left\{2\,(k^2+p^2)\left(k^2p^2-(k\cdot p)^2\right)\right\}\Bigg\}\,,
\end{aligned}
\tag{3.2.17}
$$

$$
\begin{aligned}
T^f &= \frac{1}{2\,\pi^2\,p^2}\int_0^{\Lambda^2} dk^2 \\
&\times\Bigg\{ F(k^2)\,(\alpha+\alpha^2\,B_{10})\Big(\tau_2\,(k^2+p^2)\left\{k^2p^2\,I_{0,1}-I_{2,1}\right\} \\
&\qquad -\tau_3\left\{4\,I_{2,1}+2\,k^2p^2\,I_{0,1}-3\,(k^2+p^2)\,I_{1,1}\right\} \\
&\qquad -\tau_6\left\{3\,(k^2-p^2)\,I_{1,1}\right\} \\
&\qquad -\tau_8\left\{2\,k^2p^2\,I_{0,1}-2\,I_{2,1}\right\} \Big) \\
&\quad + \int_0^{\pi} d\psi\,\sin^2\psi\,F(k^2)\left((\alpha^2\,B_{11}+\alpha^3\,B_{21})\,\ln\frac{q^2}{\Lambda^2}+\alpha^3\,B_{22}\ln^2\frac{q^2}{\Lambda^2}+\cdots\right) \\
&\qquad \times\Big(\frac{\tau_2}{q^2}(k^2+p^2)\left\{k^2p^2-(k\cdot p)^2\right\} \\
&\qquad -\frac{\tau_3}{q^2}\left\{4\,(k\cdot p)^2+2k^2p^2-3(k^2+p^2)\,k\cdot p\right\} \\
&\qquad -\frac{\tau_6}{q^2}\left\{3\,(k^2-p^2)\,k\cdot p\right\} \\
&\qquad -\frac{\tau_8}{q^2}\left\{2\,k^2p^2-2(k\cdot p)^2\right\} \Big)\Bigg\}.
\end{aligned}
\tag{3.2.18}
$$

Now we are going to solve each integral separately starting with L^f_ξ.

3.2.1 L^f_ξ Calculated

In this calculation there are three different combinations of the angular integrals which we shall denote by different letters (X_i, Y_i, Z_i). As a first step, recalling the combination in Eqn. (3.2.15) as X_1 and referring to Appendix B for I_{nm},

$$\begin{aligned} X_1 &\equiv k^2 p^2 I_{0,2} - p^2 I_{1,2} \\ &= 0 \times \theta^f_- + \frac{\pi}{2}\frac{p^2}{k^2}\theta^f_+ \quad . \end{aligned}$$

where $\theta^f_- = \theta(p^2 - k^2)\,, \qquad \theta^f_+ = \theta(k^2 - p^2) \quad .$ (3.2.19)

We can then rewrite Eqn. (3.2.15) as

$$L^f_\xi = -\frac{\alpha\xi}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)} \quad . \qquad \text{(eq:thk)))}$$

The ratio of the fermion wave-function renormalisations can be expressed as follows by using Eqn. (3.2.10) :

$$\begin{aligned} \frac{F(k^2)}{F(p^2)} = 1 \; &+ \; \alpha\, A_{11}\left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right) \\ &+ \; \alpha^2\Bigg[\; A_{22}\left(\ln^2\frac{k^2}{\Lambda^2} - \ln^2\frac{p^2}{\Lambda^2}\right) + A_{11}^2\left(\ln^2\frac{p^2}{\Lambda^2} - \ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right) \\ &+ (A_{21} - A_{10}A_{11})\left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right) + \cdots\Bigg] . \end{aligned} \qquad (3.2.20)$$

We are interested in the terms which give a contribution to leading and next-to-leading logarithms in the fermion wave-function renormalisation. Keeping this in mind, we concentrate on the integration region between p^2 and Λ^2 in L_ξ,

$$L^f_\xi = -\frac{\alpha\,\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)} \quad . \qquad (3.2.21)$$

Taking into account Eqn. (3.2.20), we can rewrite L^f_ξ as,

$$\begin{aligned} L^f_\xi &= -\frac{\alpha\,\xi}{4\pi}\Bigg\{ \ell^K_1 + \alpha\, A_{11}\left(\ell^K_2 - \ln\frac{p^2}{\Lambda^2}\,\ell^K_1\right) \\ &\quad + \alpha^2\Bigg[A_{22}\left(\ln^2\frac{p^2}{\Lambda^2}\,\ell^K_1 - \ln^2\frac{p^2}{\Lambda^2}\,\ell^K_1\right) + A^2_{11}\left(\ell^K_3 - \ln\frac{p^2}{\Lambda^2}\,\ell^K_2\right) \\ &\quad + (A_{21} - A_{10}\,A_{11})\left(\ell^K_2 - \ln\frac{p^2}{\Lambda^2}\,\ell^K_1\right)\Bigg] + \cdots\Bigg\}, \end{aligned} \tag{3.2.22}$$

where ℓ^K_i are given in Appendix D. On evaluating these integrals, (refer to Appendix D) we obtain :

$$\boxed{\begin{aligned} L^f_\xi &= \frac{\xi}{4\pi}\Bigg[\ \alpha\ln\left(\frac{p^2}{\Lambda^2}\right) \\ &\quad - \alpha^2\,\frac{A_{11}}{2}\ln^2\left(\frac{p^2}{\Lambda^2}\right) \\ &\quad - \frac{\alpha^3}{2}\left(-A^2_{11} + \frac{4}{3}A_{22}\right)\ln^3\left(\frac{p^2}{\Lambda^2}\right) - \frac{\alpha^3}{2}\left(A_{21} - A_{10}A_{11}\right)\ln^2\left(\frac{p^2}{\Lambda^2}\right)\Bigg] \end{aligned}}\ . \tag{3.2.23}$$

We shall now calculate L^f_A.

3.2.2 L^f_A Calculated

As we can see in the Eqn. (3.2.16) there are some terms for which we cannot perform the angular integrations immediately, for example the ones containing $\ln(q^2/\Lambda^2)$ and $\ln^2(q^2/\Lambda^2)$. The angle dependence is through the q^2 term, since $q^2 = k^2 + p^2 + 2\,k\cdot p$. In order to perform the angular integrals, we have to separate out the angular dependence, using

$$\begin{aligned} \ln\frac{q^2}{\Lambda^2} &= \ln\frac{(k^2+p^2-2\,k\cdot p)}{\Lambda^2} = \ln\frac{(k^2+p^2)}{\Lambda^2} + \ln\left(1 - 2\frac{k\cdot p}{(k^2+p^2)}\right), \\ \ln^2\frac{q^2}{\Lambda^2} &= \ln^2\frac{(k^2+p^2)}{\Lambda^2} + 2\,\ln\frac{(k^2+p^2)}{\Lambda^2}\,\ln\left(1-2\frac{k\cdot p}{(k^2+p^2)}\right) + \ln^2\left(1-2\frac{k\cdot p}{(k^2+p^2)}\right). \end{aligned} \tag{3.2.24}$$

We now use the following series representations :

$$
\begin{aligned}
\ln\left(1 - 2\,\frac{k\cdot p}{(k^2+p^2)}\right) &= -\sum_{s=1}^{\infty} \frac{2^s}{s}\,\frac{(k\cdot p)^s}{(k^2+p^2)^s}\,, \\
\ln^2\left(1 - 2\,\frac{k\cdot p}{(k^2+p^2)}\right) &= 2\sum_{n=1}^{\infty} \frac{2^{n+1}}{(n+1)}\,\frac{(k\cdot p)^{n+1}}{(k^2+p^2)^{n+1}}\sum_{t=1}^{n}\frac{1}{t}\,.
\end{aligned}
\tag{3.2.25}
$$

Then :

$$
\begin{aligned}
\ln\frac{q^2}{\Lambda^2} &= \ln\frac{(k^2+p^2)}{\Lambda^2} - 2\,\frac{k\cdot p}{(k^2+p^2)} - 2\,\frac{(k\cdot p)^2}{(k^2+p^2)^2} - \cdots\quad, \\
\ln^2\frac{q^2}{\Lambda^2} &= \ln^2\frac{(k^2+p^2)}{\Lambda^2} \\
&+ 2\,\ln\frac{(k^2+p^2)}{\Lambda^2}\left(-2\frac{k\cdot p}{(k^2+p^2)} - 2\,\frac{(k\cdot p)^2}{(k^2+p^2)^2} - \cdots\right) \\
&+ 4\,\frac{(k\cdot p)^2}{(k^2+p^2)^2} + \cdots\quad.
\end{aligned}
\tag{3.2.26}
$$

Now, these quantities can be written in terms of I_{nm}. Therefore, L_A^f is,

$$
\begin{aligned}
L_A^f &= \frac{1}{2\,\pi^2\,p^2}\int_0^{\Lambda^2} dk^2\,F(k^2)\,A \\
&\times\Bigg\{\left(\alpha + \alpha^2\,B_{10} + \left(\alpha^2\,B_{11} + \alpha^3\,B_{21}\right)\ln\frac{(k^2+p^2)}{\Lambda^2} + \alpha^3\,B_{22}\,\ln^2\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad\times\left(4\,I_{2,2} + 2k^2p^2\,I_{0,2} - 3\,(k^2+p^2)\,I_{1,2}\right) \\
&+\left(\alpha^2\,B_{11} + \alpha^3\,B_{21} + 2\,\alpha^3\,B_{22}\,\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad\times\Bigg[-\frac{2}{(k^2+p^2)}\left(4\,I_{3,2} + 2\,k^2p^2\,I_{1,2} - 3\,(k^2+p^2)\,I_{2,2}\right) \\
&\qquad\quad-\frac{2}{(k^2+p^2)^2}\left(4\,I_{4,2} + 2\,k^2p^2\,I_{2,2} - 3\,(k^2+p^2)\,I_{3,2}\right) \\
&\qquad\quad-\frac{8}{3\,(k^2+p^2)^3}\left(4\,I_{5,2} + 2\,k^2p^2\,I_{3,2} - 3\,(k^2+p^2)\,I_{4,2}\right) \\
&\qquad\quad-\qquad\cdots\cdots\qquad\Bigg] \\
&+\alpha^3\,B_{22}\Bigg[\frac{4}{(k^2+p^2)^2}\left(4\,I_{4,2} + 2\,k^2p^2\,I_{2,2} - 3\,(k^2+p^2)\,I_{3,2}\right) \\
&\qquad\quad+\frac{8}{(k^2+p^2)^3}\left(4\,I_{5,2} + 2\,k^2p^2\,I_{3,2} - 3\,(k^2+p^2)\,I_{4,2}\right) \\
&\qquad\quad-\qquad\cdots\cdots\qquad\Bigg]\Bigg\}\,.
\end{aligned}
\tag{3.2.27}
$$

Referring to Appendix B for the evalution of

$$
\begin{aligned}
Y_1 &\equiv 4\, I_{2,2} + 2\, k^2 p^2\, I_{0,2} - 3(k^2+p^2)\, I_{1,2} \\
&= 0\,, \\
Y_2 &\equiv 4\, I_{3,2} + 2\, k^2 p^2\, I_{1,2} - 3(k^2+p^2)\, I_{2,2} \\
&= -\frac{\pi}{8}\frac{k^2}{p^2}(3p^2-k^2)\,\theta^f_- - \frac{\pi}{8}\frac{p^2}{k^2}(3k^2-p^2)\,\theta^f_+\,, \\
Y_3 &\equiv 4\, I_{4,2} + 2\, k^2 p^2\, I_{2,2} - 3(k^2+p^2)\, I_{3,2} \\
&= -\frac{\pi}{8}\frac{k^4}{p^2}(2p^2-k^2)\,\theta^f_- - \frac{\pi}{8}\frac{p^4}{k^2}(2k^2-p^2)\,\theta^f_+\,, \\
Y_4 &\equiv 4\, I_{5,2} + 2\, k^2 p^2\, I_{3,2} - 3(k^2+p^2)\, I_{4,2} \\
&= -\frac{3\pi}{32}\frac{k^4}{p^2}(2p^2-k^2)(k^2+p^2)\,\theta^f_- - \frac{3\pi}{32}\frac{p^4}{k^2}(2k^2-p^2)(k^2+p^2)\,\theta^f_+\,, \qquad (3.2.28)
\end{aligned}
$$

leads us to write Eqn. (3.2.27) as :

$$
\begin{aligned}
L^f_A &= \frac{1}{4\pi\, p^2}\int_0^{\Lambda^2} dk^2 \left(1+\frac{F(k^2)}{F(p^2)}\right) \\
&\times \Bigg\{ \left(\alpha^2 B_{11} + \alpha^3 B_{21} + 2\,\alpha^3 B_{22}\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad \times \Bigg(\frac{1}{4}\frac{k^2}{p^2}\frac{(3p^2-k^2)}{(k^2+p^2)}\,\theta^f_- + \frac{1}{4}\frac{p^2}{k^2}\frac{(3k^2-p^2)}{(k^2+p^2)}\,\theta^f_+ \\
&\qquad\quad + \frac{1}{2}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}\,\theta^f_- + \frac{1}{2}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}\,\theta^f_+ \\
&\qquad\quad + \cdots\cdots\cdots \Bigg) \\
&+ \alpha^3 B_{22}\left(-\frac{5}{4}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}\,\theta^f_- - \frac{5}{4}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}\,\theta^f_+ + \cdots\right) \\
&+ \mathcal{O}(\alpha^4)\Bigg\}\,. \qquad (3.2.29)
\end{aligned}
$$

Together with

$$
\begin{aligned}
F(k^2)\, A &= \frac{1}{2}\left(1+\frac{F(k^2)}{F(p^2)}\right) \\
&= \frac{1}{2}\left[2 + \alpha\, A_{11}\left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right) + \cdots\cdots\right]. \qquad (3.2.30)
\end{aligned}
$$

Eqn. (3.2.29) can be displayed as :

$$
\begin{aligned}
L_A^f &= \frac{1}{4\pi p^2}\int_0^{\Lambda^2} dk^2 \left[2+\alpha A_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)+\cdots\right] \\
&\times \Bigg\{\left(\alpha^2 B_{11}+\alpha^3 B_{21}+2\alpha^3 B_{22}\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad\times\Bigg[\ \frac{1}{4}\frac{k^2}{p^2}\frac{(3p^2-k^2)}{(k^2+p^2)}\theta_-^f+\frac{1}{4}\frac{p^2}{k^2}\frac{(3k^2-p^2)}{(k^2+p^2)}\theta_+^f \\
&\qquad+\frac{1}{2}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}\theta_-^f+\frac{1}{2}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}\theta_+^f \\
&\qquad+\quad\cdots\cdots\cdots\Bigg] \\
&+\alpha^3 B_{22}\Bigg[-\frac{5}{4}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}\theta_-^f-\frac{5}{4}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}\theta_+^f \\
&\qquad+\quad\cdots\cdots\cdots\Bigg]\Bigg\}. \qquad (3.2.31)
\end{aligned}
$$

Splitting this integral into two regions and keeping the terms to order α^3 gives,

$$
\begin{aligned}
L_A^f &= \frac{1}{4\pi p^2}\int_0^{p^2} dk^2 \\
&\times\Bigg\{\Bigg[\ 2\alpha^2 B_{11}+2\alpha^3 B_{21}+4\alpha^3 B_{22}\ln\frac{p^2}{\Lambda^2}+\alpha^3 A_{11}B_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\Bigg] \\
&\qquad\times\left(\frac{1}{4}\frac{k^2}{p^2}\frac{(3p^2-k^2)}{(k^2+p^2)}+\frac{1}{2}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}+\cdots\right) \\
&\quad+2\alpha^3 B_{22}\left(-\frac{5}{4}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)^2}+\cdots\right)+\mathcal{O}(\alpha^4)\Bigg\} \\
&+\frac{1}{4\pi p^2}\int_{p^2}^{\Lambda^2} dk^2 \\
&\times\Bigg\{\Bigg[\ 2\alpha^2 B_{11}+2\alpha^3 B_{21}+4\alpha^3 B_{22}\ln\frac{k^2}{\Lambda^2}+\alpha^3 A_{11}B_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\Bigg] \\
&\qquad\times\left(\frac{1}{4}\frac{p^2}{k^2}\frac{(3k^2-p^2)}{(k^2+p^2)}+\frac{1}{2}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}+\cdots\right) \\
&\quad+2\alpha^3 B_{22}\left(-\frac{5}{4}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)^2}+\cdots\right)+\mathcal{O}(\alpha^4)\Bigg\}. \qquad (3.2.32)
\end{aligned}
$$

Referring to Appendix D for these integrals, the above expression yields

$$
\begin{aligned}
L_A^f &= \frac{1}{4\pi}\Bigg\{\left(2\,\alpha^2\,B_{11} + 2\,\alpha^3\,B_{21} + \alpha^3\,(4\,B_{22} - A_{11}\,B_{11})\,\ln\frac{p^2}{\Lambda^2}\right)\left(\frac{\ell_1^P}{4} + \frac{\ell_3^P}{2}\right) \\
&\quad + \alpha^3\,A_{11}\,B_{11}\left(\frac{\ell_2^P}{4} + \frac{\ell_4^P}{2}\right) + 2\,\alpha^3\,B_{22}\left(\frac{-5\,\ell_3^P}{4}\right) \\
&\quad + \left(2\,\alpha^2\,B_{11} + 2\,\alpha^3\,B_{21} - \alpha^3\,A_{11}\,B_{11}\,\ln\frac{p^2}{\Lambda^2}\right)\left(\frac{\ell_4^K}{4} + \frac{\ell_6^K}{2}\right) \\
&\quad + 2\,\alpha^3\,(4\,B_{22} + A_{11}\,B_{11})\left(\frac{\ell_5^K}{4} + \frac{\ell_7^K}{2}\right) + \alpha^3\,B_{22}\left(\frac{-5\,\ell_6^K}{4}\right)\Bigg\}\quad ,
\end{aligned}
\tag{3.2.33}
$$

Therefore, the expression for L_A^f is found to be :

$$
\boxed{L_A^f = \frac{3}{8\,\pi}\left[-\alpha^2\,B_{11}\,\ln\left(\frac{p^2}{\Lambda^2}\right) + \alpha^3\,\frac{A_{11}B_{11}}{4}\,\ln^2\left(\frac{p^2}{\Lambda^2}\right) - \alpha^3\,B_{22}\,\ln^2\left(\frac{p^2}{\Lambda^2}\right)\right]}\ .
\tag{3.2.34}
$$

3.2.3 L_B^f Calculated

Now we shall follow the same steps as L_A^f for the L_B^f part, starting from,

$$
\begin{aligned}
L_B^f &= -\frac{1}{2\,\pi^2\,p^2}\int_0^{\Lambda^2} dk^2\,F(k^2)\,B \\
&\times\Bigg\{\left(\alpha + \alpha^2\,B_{10} + \alpha^2\,B_{11}\,\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad\qquad \times\left(2\,(k^2+p^2)\,(k^2p^2\,I_{0,2} - I_{2,2})\right) \\
&\quad + 2\,\alpha^2\,B_{11}\,(k^2+p^2)\Bigg[-\frac{2}{(k^2+p^2)}\left(k^2p^2\,I_{1,2} - I_{3,2}\right) \\
&\qquad - \frac{2}{(k^2+p^2)^2}\left(k^2p^2\,I_{2,2} - I_{4,2}\right) \\
&\qquad - \frac{8}{3\,(k^2+p^2)^3}\left(k^2p^2\,I_{3,2} - I_{5,2}\right) \\
&\qquad - \quad \cdots\Bigg]\Bigg\},
\end{aligned}
\tag{3.2.35}
$$

with the related angular integrals being (see Appendix B)

$$
\begin{aligned}
X_2 &\equiv k^2p^2\, I_{0,2} - I_{2,2}\ , \\
&= \frac{3\,\pi}{8}\,\frac{k^2}{p^2}\,\theta_-^f + \frac{3\,\pi}{8}\,\frac{p^2}{k^2}\,\theta_+^f\ , \\
X_3 &\equiv k^2p^2\, I_{1,2} - I_{3,2} \\
&= \frac{\pi}{4}\,\frac{k^4}{p^2}\,\theta_-^f + \frac{\pi}{4}\,\frac{p^4}{k^2}\,\theta_+^f\ , \\
X_4 &\equiv k^2p^2\, I_{2,2} - I_{4,2} \\
&= \frac{\pi}{32}\,\frac{k^4}{p^2}\,(5k^2+2p^2)\,\theta_-^f + \frac{\pi}{32}\,\frac{p^4}{k^2}\,(5p^2+2k^2)\,\theta_+^f\ , \\
X_5 &\equiv k^2p^2\, I_{3,2} - I_{5,2} \\
&= \frac{3\pi}{32}\,\frac{k^6}{p^4}\,(k^2+p^2)\,\theta_-^f + \frac{3\pi}{32}\,\frac{p^6}{k^4}\,(p^2+k^2)\,\theta_+^f\ ,
\end{aligned}
\tag{3.2.36}
$$

Eqn. (3.2.35) can be written as :

$$
\begin{aligned}
L_B^f &= \frac{1}{4\pi\,p^2}\int_0^{\Lambda^2} dk^2\,\frac{1}{(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right) \\
&\times \left\{\left(\alpha+\alpha^2\,B_{10}+\alpha^2\,B_{11}\,\ln\frac{(k^2+p^2)}{\Lambda^2}\right)\left(\frac{3}{4}\,\frac{k^2}{p^2}\,(k^2+p^2)\,\theta_-^f + \frac{3}{4}\,\frac{p^2}{k^2}\,(k^2+p^2)\,\theta_+^f\right)\right. \\
&\quad +\alpha^2\,B_{11}\left(-\frac{k^4}{p^2}\,\theta_-^f - \frac{p^4}{k^2}\,\theta_+^f\right. \\
&\quad -\frac{1}{8}\,\frac{k^4}{p^2}\,\frac{(5k^2+2p^2)}{(k^2+p^2)}\,\theta_-^f - \frac{1}{8}\,\frac{p^4}{k^2}\,\frac{(5p^2+2k^2)}{(k^2+p^2)}\,\theta_+^f \\
&\quad \left.\left. -\frac{1}{2}\,\frac{k^6}{p^2}\,\frac{1}{(k^2+p^2)}\,\theta_-^f - \frac{1}{2}\,\frac{p^6}{k^2}\,\frac{1}{(k^2+p^2)}\,\theta_+^f + \cdots\right)\right\}.
\end{aligned}
\tag{3.2.37}
$$

Making use of the Eqn. (3.2.20),

$$
\begin{aligned}
F(k^2)\,B &= \frac{1}{2\,(k^2-p^2)}\left(1-\frac{F(k^2)}{F(p^2)}\right) \\
&= \frac{1}{2\,(k^2-p^2)}\Bigg\{-\alpha\,A_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right) \\
&\qquad -\alpha^2\left[A_{22}\left(\ln^2\frac{k^2}{\Lambda^2}-\ln^2\frac{p^2}{\Lambda^2}\right)+A_{11}^2\,\ln\frac{p^2}{\Lambda^2}\left(\ln\frac{p^2}{\Lambda^2}-\ln\frac{k^2}{\Lambda^2}\right)\right. \\
&\qquad \left.+(A_{21}-A_{10}\,A_{11})\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\right]-\cdots\cdots\Bigg\}.
\end{aligned}
\tag{3.2.38}
$$

So L_B^f becomes :

$$
\begin{aligned}
L_B^f &= \frac{1}{4\pi\,p^2}\int_0^{\Lambda^2} dk^2\,\frac{1}{(k^2-p^2)} \\
&\times\Bigg\{A_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)+\alpha\left[A_{22}\left(\ln^2\frac{k^2}{\Lambda^2}-\ln^2\frac{p^2}{\Lambda^2}\right)\right. \\
&\quad \left.+A_{11}^2\,\ln\frac{p^2}{\Lambda^2}\left(\ln\frac{p^2}{\Lambda^2}-\ln\frac{k^2}{\Lambda^2}\right)+(A_{21}-A_{10}\,A_{11})\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\right]+\cdots\Bigg\} \\
&\times\Bigg\{\left(\alpha^2+\alpha^3\,B_{10}+\alpha^3\,B_{11}\,\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad\times\left(-\frac{3}{4}\frac{k^2}{p^2}(k^2+p^2)\,\theta_-^f-\frac{3}{4}\frac{p^2}{k^2}(k^2+p^2)\,\theta_+^f\right) \\
&\quad+\alpha^3\,B_{11}\quad\times\left(\frac{k^4}{p^2}\,\theta_-^f+\frac{p^4}{k^2}\,\theta_+^f\right. \\
&\qquad+\frac{1}{8}\frac{k^4}{p^2}\frac{(5k^2+2p^2)}{(k^2+p^2)}\,\theta_-^f+\frac{1}{8}\frac{p^4}{k^2}\frac{(5p^2+2k^2)}{(k^2+p^2)}\,\theta_+^f \\
&\qquad\left.+\frac{1}{2}\frac{k^6}{p^2}\frac{1}{(k^2+p^2)}\,\theta_-^f+\frac{1}{2}\frac{p^6}{k^2}\frac{1}{(k^2+p^2)}\,\theta_+^f+\cdots\right)\Bigg\}.
\end{aligned}
\tag{3.2.39}
$$

After separating the integration regions, we see that the only contribution to the leading and next-to-leading log terms comes from $p^2 \to \Lambda^2$ region. Hence,

$$
\begin{aligned}
L_B^f &= \frac{1}{4\pi p^2}\int_{p^2}^{\Lambda^2} dk^2 \\
&\times \left\{ A_{11}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)+\alpha\left[A_{22}\left(\ln^2\frac{k^2}{\Lambda^2}-\ln^2\frac{p^2}{\Lambda^2}\right)\right.\right. \\
&\quad \left.\left. +A_{11}^2\ln\frac{p^2}{\Lambda^2}\left(\ln\frac{p^2}{\Lambda^2}-\ln\frac{k^2}{\Lambda^2}\right)+\left(A_{21}-A_{10}A_{11}\right)\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\right]-\cdots\right\} \\
&\times \left\{ \left(\alpha^2+\alpha^3 B_{10}+\alpha^3 B_{11}\ln\frac{k^2}{\Lambda^2}\right)\left(-\frac{3}{4}\frac{p^2}{k^2}\frac{(k^2+p^2)}{(k^2-p^2)}\right)\right. \\
&\quad +\alpha^3 B_{11}\left(\frac{p^4}{k^2(k^2-p^2)}+\frac{1}{8}\frac{p^4}{k^2}\frac{(5p^2+2k^2)}{(k^2+p^2)(k^2-p^2)}\right. \\
&\quad \left.\left. +\frac{1}{2}\frac{p^6}{k^2}\frac{1}{(k^2+p^2)(k^2-p^2)}+\cdots\right)\right\}.
\end{aligned}
\tag{3.2.40}
$$

Referring to Appendix D for the k-integrals, the above expression gives,

$$
\begin{aligned}
L_B^f &= \frac{1}{4\pi}\left\{\alpha^2 A_{11}\frac{-3\,\ell_8^K}{4}+\alpha^3\left(A_{22}\frac{-3\,\ell_{10}^K}{4}-A_{11}^2\ln\frac{p^2}{\Lambda^2}\frac{-3\,\ell_8^K}{4}+\left(A_{21}-A_{10}A_{11}\right)\frac{-3\,\ell_8^K}{4}\right)\right. \\
&\quad +\alpha^3 B_{10}A_{11}\frac{-3\,\ell_8^K}{4} \\
&\quad \left. +\alpha^3 B_{11}A_{11}\left(\frac{-3\,\ell_9^K}{4}+\ell_8^K+\frac{1}{8}\ell_9^K+\frac{1}{2}\ell_{13}^K+\cdots\right)\right\}.
\end{aligned}
\tag{3.2.41}
$$

see Appendix D for the ℓ_i^K. After evaluating these, we finally find the solution for L_B^f :

$$
\boxed{
\begin{aligned}
L_B^f = \frac{1}{4\pi}\left\{\right. &- \alpha^2\frac{3}{8}A_{11}\ln^2\left(\frac{p^2}{\Lambda^2}\right) \\
&+ \alpha^3\left[-\frac{1}{8}A_{11}B_{11}+\frac{3}{4}\left(\frac{A_{11}^2}{2}-\frac{2}{3}A_{22}\right)\right]\ln^3\frac{p^2}{\Lambda^2} \\
&+ \left.\alpha^3\left[-\frac{3}{8}A_{11}B_{10}-\frac{3}{8}\left(A_{21}-A_{10}A_{11}\right)\right]\ln^2\frac{p^2}{\Lambda^2}\right\}
\end{aligned}
}.
\tag{3.2.42}
$$

3.2.4 Transverse Piece Calculated

The last piece we have to calculate is the transverse part, which can be written as :

$$
\begin{aligned}
\frac{1}{F(p^2)} &= 1 - \frac{1}{2\pi p^2}\int_0^{\Lambda^2} dk^2 \\
&\times \Bigg\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11} \ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\qquad \times\Big[\tau_2 (k^2+p^2)\left(k^2p^2 I_{0,1} - I_{2,1}\right) \\
&\qquad\quad - \tau_3 \left(4 I_{2,1} + 2k^2p^2 I_{0,1} - 3(k^2+p^2) I_{1,1}\right) \\
&\qquad\quad - \tau_6 \left(3(k^2-p^2) I_{1,1}\right) \\
&\qquad\quad - \tau_8 \left(2k^2p^2 I_{0,1} - 2 I_{2,1}\right) \Big] \\
&\quad + \alpha^2 B_{11} \Big[\tau_2 (k^2+p^2) \left(-\frac{2}{(k^2+p^2)}\left(k^2p^2 I_{1,1} - I_{3,1}\right) - \frac{2}{(k^2+p^2)^2}\left(k^2p^2 I_{2,1} - I_{4,1}\right) - \cdots \right) \\
&\qquad\quad - \tau_3 \left(-\frac{2}{(k^2+p^2)}\left(4 I_{3,1} + 2k^2p^2 I_{1,1} - 3(k^2+p^2) I_{2,1}\right) - \frac{2}{(k^2+p^2)^2}\left(4 I_{4,1} + 2k^2p^2 I_{2,1} - 3(k^2+p^2) I_{3,1}\right) - \cdots \right) \\
&\qquad\quad - \tau_6\, 3(k^2-p^2) \left(-\frac{2}{(k^2+p^2)} I_{2,1} - \frac{2}{(k^2+p^2)^2} I_{3,1} - \frac{8}{3(k^2+p^2)^3} I_{4,1} - \cdots \right) \\
&\qquad\quad - \tau_8 \left(-\frac{2}{(k^2+p^2)}\left(2k^2p^2 I_{1,1} - 2 I_{3,1}\right) - \frac{2}{(k^2+p^2)^2}\left(2k^2p^2 I_{2,1} - 2 I_{4,1}\right) - \cdots \right) \Big] \\
&\quad + \mathcal{O}(\alpha^4) \Bigg\}.
\end{aligned}
\qquad (3.2.43)
$$

To calculate this expression, we have to input the coefficients of the basis tensors, i.e. the τ_i in the transverse vertex, Eqn. (2.3.64). Let us see what we can say about these coefficients? The transverse vertex is dimensionless. So knowing the dimensions of the basis vectors from Eqn. (2.3.65) would tell us what the dimensions of the τ_i's are. Therefore, if $d \equiv$ momentum^2;

$$\begin{aligned}
\text{dim. of } T_2^\mu &: d^2 \longrightarrow & \text{dim. of } \tau_2 &: \frac{1}{d^2}\,, \\
\text{dim. of } T_3^\mu &: d \longrightarrow & \text{dim. of } \tau_3 &: \frac{1}{d}\,, \\
\text{dim. of } T_6^\mu &: d \longrightarrow & \text{dim. of } \tau_6 &: \frac{1}{d}\,, \\
\text{dim. of } T_8^\mu &: d \longrightarrow & \text{dim. of } \tau_8 &: \frac{1}{d}\,.
\end{aligned} \tag{3.2.44}$$

As was mentioned before, the C-parity operation [27, 25] of Eq. (2.2.3) requires

$$\begin{aligned}
\tau_2(k^2,p^2,q^2) &= \tau_2(p^2,k^2,q^2) \quad , \quad \text{symmetric in k and p} \quad , \\
\tau_3(k^2,p^2,q^2) &= \tau_3(p^2,k^2,q^2) \quad , \quad \text{symmetric in k and p} \quad , \\
\tau_6(k^2,p^2,q^2) &= -\,\tau_6(p^2,k^2,q^2) \quad , \quad \text{antisymmetric in k and p} \quad , \\
\tau_8(k^2,p^2,q^2) &= \tau_8(p^2,k^2,q^2) \quad , \quad \text{symmetric in k and p} \quad .
\end{aligned} \tag{3.2.45}$$

In general, the τ_i can be written as a sum of terms, each with the correct dimensions and symmetry properties, as :

$$\tau = \sum_j f_j\,(k^2,p^2)\,\overline{\tau}_j\,(F,G)\,. \tag{3.2.46}$$

Each of these terms can be divided in two parts : (1) the part responsible for giving the right dimensions which only depends on momenta squared. Although each of the τ_i would have q^2 dependence in general, this would severely complicate the situation. We therefore introduce effective τ_i's which are only functions of p^2 and k^2. This is because the leading and next-to-leading logarithms are generated by regions where either $p^2 >> k^2$ or $k^2 << p^2$, and then $q^2 \simeq \max(k^2,p^2)$. The forms of the τ_i's are guessed such that the integrals are soluble. (2) the part which depends on the fermion and photon wave-function renormalisations at momenta p^2 and k^2. Its structure has to ensure the correct symmetry of τ_i when multiplied with part (1). Hence, we can write :

$$\begin{aligned}
\tau_2 &= \frac{2}{(k^2+p^2)(k^2-p^2)}\,\tau_2' + \frac{2}{(k^2+p^2)^2}\,\tau_2'' , \\
\tau_3 &= \frac{1}{(k^2-p^2)}\,\tau_3' + \frac{1}{(k^2+p^2)}\,\tau_3'' , \\
\tau_6 &= \frac{1}{(k^2+p^2)}\,\tau_6' + \frac{(k^2-p^2)}{(k^2+p^2)^2}\,\tau_6'' , \\
\tau_8 &= \frac{1}{(k^2-p^2)}\,\tau_8' + \frac{1}{(k^2+p^2)}\,\tau_8'' .
\end{aligned} \tag{3.2.47}$$

The factor 2 in the numerator of τ_2 is merely for later convenience. The τ_i' and τ_i'' are antisymmetric and symmetric under $k^2 \to p^2$, respectively. In general, antisymmetric combinations of F and G, i.e. the τ_i', can be written as :

$$\begin{aligned}
\tau_i' &= \alpha\, K_i \left(\ln\frac{p^2}{\Lambda^2} - \ln\frac{k^2}{\Lambda^2}\right) + \alpha^2 \left[J_i \left(\ln^2\frac{k^2}{\Lambda^2} - \ln^2\frac{p^2}{\Lambda^2}\right) + M_i \left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right)\right] \\
&\quad + \mathcal{O}(\alpha^3) , \qquad \{i = 2,3,6,8\} ,
\end{aligned} \tag{3.2.48}$$

and symmetric combinations, τ_i'',as :

$$\begin{aligned}
\tau_i'' &= \alpha \left[K_i' \left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2}\right) + H_i' \right] \\
&\quad + \alpha^2 \left[J_i' \left(\ln^2\frac{k^2}{\Lambda^2} + \ln^2\frac{p^2}{\Lambda^2}\right) + M_i' \left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2}\right) + Q_i' \ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right] \\
&\quad + \mathcal{O}(\alpha^3) , \qquad \{i = 2,3,6,8\} ,
\end{aligned} \tag{3.2.49}$$

K, J, M, K', J', M', H' and Q' are the constants which will be found later. The angular integrals we need are given by :

$$\begin{aligned}
X_6 &\equiv k^2 p^2 I_{0,1} - I_{2,1} \\
&= \frac{\pi}{8}\frac{k^2}{p^2}(3p^2 - k^2)\,\theta_-^f + \frac{\pi}{8}\frac{p^2}{k^2}(3k^2 - p^2)\,\theta_+^f ,
\end{aligned}$$

$$
\begin{aligned}
X_7 &\equiv k^2p^2\, I_{1,1} - I_{3,1} \\
&= \frac{\pi}{16}\frac{k^4}{p^2}\,(2p^2 - k^2)\,\theta^f_- + \frac{\pi}{16}\frac{p^4}{k^2}\,(2k^2 - p^2)\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
X_8 &\equiv k^2p^2\, I_{2,1} - I_{4,1} \\
&= \frac{\pi}{32}\frac{k^4}{p^2}\,(2p^2 - k^2)(k^2+p^2)\,\theta^f_- + \frac{\pi}{32}\frac{p^4}{k^2}\,(2k^2 - p^2)(k^2+p^2)\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Y_5 &\equiv 4\, I_{2,1} + 2\,k^2p^2\, I_{0,1} - 3(k^2+p^2)\, I_{1,1} \\
&= \frac{\pi}{4}\frac{k^2}{p^2}\,(3p^2 - k^2)\,\theta^f_- + \frac{\pi}{4}\frac{p^2}{k^2}\,(3k^2 - p^2)\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Y_6 &\equiv 4\, I_{3,1} + 2\,k^2p^2\, I_{1,1} - 3(k^2+p^2)\, I_{2,1} \\
&= \frac{\pi}{8}\frac{k^2}{p^2}\,\left[k^2(2p^2 - k^2) - 3p^4\right]\,\theta^f_- + \frac{\pi}{8}\frac{p^2}{k^2}\,\left[p^2(2k^2 - p^2) - 3k^4\right]\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Y_7 &\equiv 4\, I_{4,1} + 2\,k^2p^2\, I_{2,1} - 3(k^2+p^2)\, I_{3,1} \\
&= \frac{\pi}{16}\frac{k^4}{p^2}\,(2p^2 - k^2)\,(k^2+p^2)\,\theta^f_- + \frac{\pi}{16}\frac{p^4}{k^2}\,(2k^2 - p^2)\,(k^2+p^2)\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Z_1 &\equiv I_{1,1} \\
&= \frac{\pi}{4}\frac{k^2}{p^2}\,\theta^f_- + \frac{\pi}{4}\frac{p^2}{k^2}\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Z_2 &\equiv I_{2,1} \\
&= \frac{\pi}{8}\frac{k^2}{p^2}\,(k^2+p^2)\,\theta^f_- + \frac{\pi}{8}\frac{p^2}{k^2}\,(k^2+p^2)\,\theta^f_-\;,
\end{aligned}
$$

$$
\begin{aligned}
Z_3 &\equiv I_{3,1} \\
&= \frac{\pi}{16}\frac{k^4}{p^2}\,(k^2+2p^2)\,\theta^f_- + \frac{\pi}{16}\frac{p^4}{k^2}\,(p^2+2k^2)\,\theta^f_+\;,
\end{aligned}
$$

$$
\begin{aligned}
Z_4 &\equiv I_{4,1} \\
&= \frac{\pi}{32}\frac{k^4}{p^2}\,(k^2+2p^2)\,(k^2+p^2)\,\theta^f_- + \frac{\pi}{32}\frac{p^4}{k^2}\,(p^2+2k^2)\,(k^2+p^2)\,\theta^f_+\,. \qquad (3.2.50)
\end{aligned}
$$

We can now rewrite the transverse piece as :

$$
\begin{aligned}
T^f \; = \; & \frac{1}{4\pi\, p^2} \int_0^{\Lambda^2} dk^2\, F(k^2) \\
\times & \left\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11} \ln \frac{(k^2+p^2)}{\Lambda^2} \right) \right. \\
& \times \Bigg[\left(-\frac{\tau_2}{2}(k^2+p^2) + \tau_3 + \tau_8 \right) \left(\frac{k^2}{2p^2}(k^2-3p^2)\,\theta_-^f + \frac{p^2}{2k^2}(p^2-3k^2)\,\theta_+^f \right) \\
& \quad -\tau_6\,(k^2-p^2) \qquad \left(\frac{3}{2}\frac{k^2}{p^2}\theta_-^f + \frac{3}{2}\frac{p^2}{k^2}\theta_+^f \right) \Bigg] \\
& + \alpha^2 B_{11} \\
& \times \Bigg[\left(-\frac{\tau_2}{2}(k^2+p^2) + \tau_3 + \tau_8 \right) \left(\frac{3}{4}\frac{k^4}{p^2}\frac{(2p^2-k^2)}{(k^2+p^2)}\,\theta_-^f + \frac{3}{4}\frac{p^4}{k^2}\frac{(2k^2-p^2)}{(k^2+p^2)}\,\theta_+^f + \cdots \right) \\
& \quad -\tau_3 \left(\frac{3}{2}\frac{k^2p^2}{(k^2+p^2)}\,\theta_-^f + \frac{3}{2}\frac{k^2p^2}{(k^2+p^2)}\,\theta_+^f + \cdots \right) \\
& \quad -\tau_6 \Bigg(-\frac{3}{2}\frac{k^2}{p^2}(k^2-p^2)\,\theta_-^f - \frac{3}{2}\frac{p^2}{k^2}(k^2-p^2)\,\theta_+^f \\
& \qquad -\frac{5}{4}\frac{k^4}{p^2}\frac{(k^2-p^2)(k^2+2p^2)}{(k^2+p^2)^2}\,\theta_-^f \\
& \qquad -\frac{5}{4}\frac{p^4}{k^2}\frac{(k^2-p^2)(p^2+2k^2)}{(k^2+p^2)^2}\,\theta_-^f + \cdots \Bigg) \Bigg] \\
& \left. + \mathcal{O}(\alpha^4) \right\} . \qquad (3.2.51)
\end{aligned}
$$

We would also like to define here two quantities, Z_i' and Z_i'', which we are going to use later :

$$
\begin{aligned}
Z_i' \; = \; & F(k^2)\,\tau_i' \\
= \; & \alpha\, K_i \left(\ln \frac{p^2}{\Lambda^2} - \ln \frac{k^2}{\Lambda^2} \right) \\
& + \alpha^2 \Bigg[J_i \left(\ln^2 \frac{k^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2} \right) + (M_i - K_i A_{10}) \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} \right) \\
& + K_i A_{11} \left(\ln \frac{k^2}{\Lambda^2} \ln \frac{p^2}{\Lambda^2} - \ln^2 \frac{k^2}{\Lambda^2} \right) \Bigg] \quad , \qquad (3.2.52)
\end{aligned}
$$

$$\begin{aligned}
Z''_i &= F(k^2)\,\tau''_i \\
&= \alpha\,K'_i\left[\left(\ln\frac{p^2}{\Lambda^2}+\ln\frac{k^2}{\Lambda^2}\right)+H'_i\right] \\
&+ \alpha^2\Bigg[J'_i\left(\ln^2\frac{k^2}{\Lambda^2}+\ln^2\frac{p^2}{\Lambda^2}\right)+(M'_i+K'_i\,A_{10})\left(\ln\frac{k^2}{\Lambda^2}+\ln\frac{p^2}{\Lambda^2}\right) \\
&+ K'_i\,A_{11}\left(\ln^2\frac{k^2}{\Lambda^2}+\ln\frac{k^2}{\Lambda^2}\,\ln\frac{p^2}{\Lambda^2}\right)+H'_i\,A_{11}\,\ln\frac{k^2}{\Lambda^2}+Q'_i\,\ln\frac{k^2}{\Lambda^2}\,\ln\frac{p^2}{\Lambda^2}\Bigg]\,.
\end{aligned} \tag{3.2.53}$$

As a next step, the effective τ_i will be substituted into Eqn. (3.2.51) and then to avoid the expression being too long, we divide T^f into two parts the symmetric and antisymmetric combinations of $F(p^2)$ and $G(p^2)$. The antisymmetric part is

$$\begin{aligned}
T^{f\,\prime} &= \frac{1}{4\pi\,p^2}\int_0^{\Lambda^2} dk^2\,F(k^2) \\
&\times\Bigg\{\left(\alpha+\alpha^2\,B_{10}+\alpha^2\,B_{11}\,\ln\frac{(k^2+p^2)}{\Lambda^2}\right) \\
&\quad\times\Bigg[(-\tau'_2+\tau'_3+\tau'_8)\left(\frac{k^2}{2p^2}\,\frac{(k^2-3p^2)}{(k^2-p^2)}\,\theta^f_-+\frac{p^2}{2k^2}\,\frac{(p^2-3k^2)}{(k^2-p^2)}\,\theta^f_+\right) \\
&\qquad-\tau'_6\,\frac{(k^2+p^2)}{(k^2-p^2)}\left(\frac{3}{2}\,\frac{k^2}{p^2}\theta^f_-+\frac{3}{2}\,\frac{p^2}{k^2}\theta^f_+\right)\Bigg] \\
&+ \alpha^2\,B_{11} \\
&\quad\times\Bigg[(-\tau'_2+\tau'_3+\tau'_8)\Bigg(\frac{3}{4}\,\frac{k^4}{p^2}\,\frac{(2p^2-k^2)}{(k^2+p^2)\,(k^2-p^2)}\,\theta^f_- \\
&\qquad+\frac{3}{4}\,\frac{p^4}{k^2}\,\frac{(2k^2-p^2)}{(k^2+p^2)\,(k^2-p^2)}\,\theta^f_++\cdots\Bigg) \\
&\qquad-\tau'_3\left(\frac{3}{2}\,\frac{k^2p^2}{(k^2+p^2)\,(k^2-p^2)}\,\theta^f_-+\frac{3}{2}\,\frac{k^2p^2}{(k^2+p^2)\,(k^2-p^2)}\,\theta^f_++\cdots\right) \\
&\qquad+\tau'_6\Bigg(+\frac{3}{2}\,\frac{k^2}{p^2}\,\frac{(k^2-p^2)}{(k^2+p^2)}\,\theta^f_-+\frac{3}{2}\,\frac{p^2}{k^2}\,\frac{(k^2-p^2)}{(k^2+p^2)}\,\theta^f_+ \\
&\qquad-\frac{5}{4}\,\frac{k^4}{p^2}\,\frac{(k^2-p^2)(k^2+2p^2)}{(k^2+p^2)^3}\,\theta^f_--\frac{5}{4}\,\frac{p^4}{k^2}\,\frac{(k^2-p^2)(p^2+2k^2)}{(k^2+p^2)^3}\,\theta^f_-+\cdots\Bigg)\Bigg] \\
&+ \mathcal{O}(\alpha^4)\Bigg\},
\end{aligned} \tag{3.2.54}$$

and the symmetric part is

$$
\begin{aligned}
T^{f\,''} = \; & \frac{1}{4\pi\, p^2} \int_0^{\Lambda^2} dk^2\, F(k^2) \\
\times & \left\{ \left(\alpha + \alpha^2\, B_{10} + \alpha^2\, B_{11}\, \ln \frac{(k^2+p^2)}{\Lambda^2} \right) \right. \\
& \times \left[(-\tau_2'' + \tau_3'' + \tau_8'') \left(\frac{k^2}{2p^2}\, \frac{(k^2-3p^2)}{(k^2+p^2)}\, \theta_-^f + \frac{p^2}{2k^2}\, \frac{(p^2-3k^2)}{(k^2+p^2)}\, \theta_+^f \right) \right. \\
& \left. -\tau_6''\, \frac{(k^2-p^2)^2}{(k^2+p^2)^2} \left(\frac{3}{2}\frac{k^2}{p^2}\theta_-^f + \frac{3}{2}\frac{p^2}{k^2}\theta_+^f \right) \right] \\
& + \alpha^2\, B_{11} \\
& \times \left[(-\tau_2'' + \tau_3'' + \tau_8'') \left(\frac{3}{4}\, \frac{k^4}{p^2}\, \frac{(2p^2-k^2)}{(k^2+p^2)^2}\, \theta_-^f + \frac{3}{4}\, \frac{p^4}{k^2}\, \frac{(2k^2-p^2)}{(k^2+p^2)^2}\, \theta_+^f + \cdots \right) \right. \\
& - \tau_3'' \left(+ \frac{3}{2}\, \frac{k^2 p^2}{(k^2+p^2)^2}\, \theta_-^f + \frac{3}{2}\, \frac{k^2 p^2}{(k^2+p^2)^2}\, \theta_+^f + \cdots \right) \\
& - \tau_6'' \left(- \frac{3}{2}\, \frac{k^2}{p^2}\, \frac{(k^2-p^2)^2}{(k^2+p^2)^2}\, \theta_-^f - \frac{3}{2}\, \frac{p^2}{k^2}\, \frac{(k^2-p^2)^2}{(k^2+p^2)^2}\, \theta_+^f \right. \\
& \quad - \frac{5}{4}\, \frac{k^4}{p^2}\, \frac{(k^2+2p^2)\,(k^2-p^2)^2}{(k^2+p^2)^4}\, \theta_-^f \\
& \quad \left.\left. - \frac{5}{4}\, \frac{p^4}{k^2}\, \frac{(p^2+2k^2)\,(k^2-p^2)^2}{(k^2+p^2)^4}\, \theta_-^f + \cdots \right) \right] \\
& \left. + \; \mathcal{O}(\alpha^4) \right\} .
\end{aligned}
\tag{3.2.55}
$$

To evaluate $T^{f'}$ by using Eqn. (3.2.52), we first split up the integral as;

$$
\begin{aligned}
T^{f\,\prime} &= \frac{1}{4\pi\, p^2} \int_0^{p^2} dk^2 \\
&\times \Bigg\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11} \ln \frac{p^2}{\Lambda^2}\right) \\
&\qquad \times\Bigg[(-Z_2' + Z_3' + Z_8') \left(\frac{1}{2} \frac{k^2}{p^2} \frac{(k^2-3p^2)}{(k^2-p^2)} \right) \\
&\qquad\quad - Z_6' \left(\frac{3}{2} \frac{k^2}{p^2} \frac{(k^2-p^2)}{(k^2+p^2)} \right) \Bigg] \\
&+ \alpha^2 B_{11} \\
&\qquad \times\Bigg[(-Z_2' + Z_3' + Z_8') \left(\frac{3}{4} \frac{k^4}{p^2} \frac{(2p^2-k^2)}{(k^2+p^2)(k^2-p^2)} + \cdots \right) \\
&\qquad\quad - Z_3' \left(\frac{3}{2} \frac{k^2 p^2}{(k^2+p^2)(k^2-p^2)} + \cdots \right) \\
&\qquad\quad - Z_6' \Bigg(\frac{3}{2} \frac{k^2}{p^2} \frac{(k^2+p^2)}{(k^2-p^2)} \\
&\qquad\qquad + \frac{5}{4} \frac{k^4}{p^2} \frac{(k^2-p^2)(k^2+2p^2)}{(k^2+p^2)^3} + \cdots \Bigg) \Bigg] + \cdots\cdots \Bigg\} \\
&+ \frac{1}{4\pi\, p^2} \int_{p^2}^{\Lambda^2} dk^2 \\
&\times \Bigg\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11} \ln \frac{k^2}{\Lambda^2}\right) \\
&\qquad \times\Bigg[(-Z_2' + Z_3' + Z_8') \left(\frac{1}{2} \frac{p^2}{k^2} \frac{(p^2-3k^2)}{(k^2-p^2)} \right) \\
&\qquad\quad - Z_6' \left(\frac{3}{2} \frac{p^2}{k^2} \frac{(k^2+p^2)}{(k^2-p^2)} \right) \Bigg] \\
&+ \alpha^2 B_{11} \\
&\qquad \Bigg\{ (-Z_2' + Z_3' + Z_8') \left(\frac{3}{4} \frac{p^4}{k^2} \frac{(2k^2-p^2)}{(k^2+p^2)(k^2-p^2)} + \cdots \right) \\
&\qquad\quad - Z_3' \left(\frac{3}{2} \frac{k^2 p^2}{(k^2+p^2)(k^2-p^2)} + \cdots \right) \\
&\qquad\quad + Z_6' \Bigg(\frac{3}{2} \frac{p^2}{k^2} \frac{(k^2-p^2)}{(k^2+p^2)} \\
&\qquad\qquad + \frac{5}{4} \frac{p^4}{k^2} \frac{(k^2-p^2)(p^2+2k^2)}{(k^2-p^2)^3} + \cdots \Bigg) + \mathcal{O}(\alpha^4) \Bigg\} .
\end{aligned}
\tag{3.2.56}
$$

We consider the $p^2 \to \Lambda^2$ integration region only which gives the contribution to leading and next-to-leading logarithmic terms. Then referring to Appendix D for the k-integration,

$$
\begin{aligned}
T^{f\,\prime} = \frac{1}{8\pi}\Big\{ & \alpha^2 \left(- K_2 + K_3 + K_8\right) t_1^K + \alpha^3 \left(- J_2 + J_3 + J_8\right) t_2^K \\
& - \alpha^3 \left(- M_2 + M_3 + M_8\right) t_1^K + \alpha^3 A_{10} \left(-K_2 + K_3 + K_8\right) t_1^K \\
& + \alpha^3 A_{11} \left(- K_2 + K_3 + K_8\right) t_3^K \\
& - 3\,\alpha^2 K_6\, t_4^K - 3\,\alpha^3 J_6\, t_5^K + 3\,\alpha^3 \left(M_6 - K_6 A_{10}\right) t_4^K - 3\,\alpha^3 K_6 A_{11}\, t_6^K \\
& + \alpha^3 B_{10} \Big(\left(- K_2 + K_3 + K_8\right) t_1^K - 3\, K_6\, t_4^K \Big) \\
& + \alpha^3 B_{11} \Big(\left(- K_2 + K_3 + K_8\right) t_3^K - 3\, K_6\, t_6^K \\
& \qquad - 3\, K_3\, t_7^K + 3\, K_6\, t_4^K \Big) \\
& + \mathcal{O}(\alpha^4) \Big\} \; .
\end{aligned}
\tag{3.2.57}
$$

This result can be written as :

$$
\boxed{
\begin{aligned}
T^{f\,\prime} = \frac{1}{4\pi}\Big\{ & \alpha^2 \ln^2 \frac{p^2}{\Lambda^2} \left\{ \frac{3}{4} \left(- K_2 + K_3 + K_6 + K_8\right) \right\} \\
& + \alpha^3 \ln^3 \frac{p^2}{\Lambda^2} \Big\{ - \left(- J_2 + J_3 + J_6 + J_8\right) \\
& \qquad + \frac{\left(A_{11} + B_{11}\right)}{4} \left(- K_2 + K_3 + K_6 + K_8\right) \Big\} \\
& + \alpha^3 \ln^2 \frac{p^2}{\Lambda^2} \Big\{ - \frac{3}{4} \left(- M_2 + M_3 + M_6 + M_8\right) \\
& \qquad + \frac{3}{4} \left(A_{10} + B_{10}\right) \left(- K_2 + K_3 + K_6 + K_8\right) \\
& \qquad + \frac{3}{4} B_{11} \left(K_3 - K_6\right) \Big\} \\
& + \mathcal{O}(\alpha^4) \Big\}
\end{aligned}
} \; .
\tag{3.2.58}
$$

Taking Eqn. (3.2.53) into account to evaluate $T^{f\,''}$, we get :

$$
\begin{aligned}
T^{f\,''} &= \frac{1}{4\pi\,p^2}\int_0^{p^2} dk^2 \\
&\times \Bigg\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11}\,\ln\frac{p^2}{\Lambda^2}\right) \\
&\qquad \times \Bigg[(-Z_2'' + Z_3'' + Z_8'') \left(\frac{k^2}{2p^2}\,\frac{(k^2-3p^2)}{(k^2+p^2)}\right) \\
&\qquad\quad - Z_6'' \left(\frac{3}{2}\,\frac{k^2}{p^2}\,\frac{(k^2-p^2)^2}{(k^2+p^2)^2}\right)\Bigg] \\
&+ \alpha^2 B_{11} \\
&\qquad \times \Bigg[(-Z_2'' + Z_3'' + Z_8'') \left(\frac{3}{4}\,\frac{k^4}{p^2}\,\frac{(2p^2-k^2)}{(k^2+p^2)^2} + \cdots\right) \\
&\qquad\quad - Z_3'' \left(\frac{3}{2}\,\frac{k^2p^2}{(k^2+p^2)^2} + \cdots\right) \\
&\qquad\quad - Z_6'' \left(-\frac{3}{2}\,\frac{k^2}{p^2}\,\frac{(k^2-p^2)^2}{(k^2+p^2)^2} \right. \\
&\qquad\qquad \left. -\frac{5}{4}\,\frac{k^4}{p^2}\,\frac{(k^2+2p^2)\,(k^2-p^2)^2}{(k^2+p^2)^4} + \cdots\right)\Bigg] + \cdots\cdots \Bigg\} \\
&+ \frac{1}{4\pi\,p^2}\int_{p^2}^{\Lambda^2} dk^2 \\
&\times \Bigg\{ \left(\alpha + \alpha^2 B_{10} + \alpha^2 B_{11}\,\ln\frac{k^2}{\Lambda^2}\right) \\
&\qquad \times \Bigg[(-Z_2'' + Z_3'' + Z_8'') \left(\frac{p^2}{2k^2}\,\frac{(p^2-3k^2)}{(k^2+p^2)}\right) \\
&\qquad\quad - Z_6'' \left(\frac{3}{2}\,\frac{p^2}{k^2}\,\frac{(k^2-p^2)^2}{(k^2+p^2)^2}\right)\Bigg] \\
&+ \alpha^2 B_{11} \\
&\qquad \Bigg[(-Z_2'' + Z_3'' + Z_8'') \left(\frac{3}{4}\,\frac{p^4}{k^2}\,\frac{(2k^2-p^2)}{(k^2+p^2)^2} + \cdots\right) \\
&\qquad\quad - Z_3'' \left(\frac{3}{2}\,\frac{k^2p^2}{(k^2+p^2)^2} + \cdots\right) \\
&\qquad\quad - Z_6'' \left(-\frac{3}{2}\,\frac{p^2}{k^2}\,\frac{(k^2-p^2)^2}{(k^2+p^2)^2} \right. \\
&\qquad\qquad \left. -\frac{5}{4}\,\frac{p^4}{k^2}\,\frac{(p^2+2k^2)\,(k^2-p^2)^2}{(k^2+p^2)^4} + \cdots\right)\Bigg] \\
&+ \mathcal{O}(\alpha^3) \Bigg\}. \qquad (3.2.59)
\end{aligned}
$$

Referring again to Appendix D for the related k-integrals,

$$
\begin{aligned}
T^{f\,\prime\prime} \;=\; \frac{1}{8\pi}\Bigg\{ & \alpha^2 \Big((-K_2' + K_3' + K_8')\, t_1^P - 3\, K_6'\, t_5^P \Big) \\
& + \alpha^3 \Big((-J_2' + J_3' + J_8')\, t_2^P - 3\, J_6'\, t_6^P \\
& \quad + A_{11}\, (-K_2' + K_3' + K_8')\, t_3^P - 3\, A_{11}\, K_6'\, t_7^P \\
& \quad + (-Q_2' + Q_3' + Q_8')\, \ln \frac{p^2}{\Lambda^2}\, t_4^P - 3\, Q_6'\, \ln \frac{p^2}{\Lambda^2} t_8^P \\
& \quad + B_{11}\, \ln \frac{p^2}{\Lambda^2} \big((-K_2' + K_3' + K_8')\, t_1^P - K_6'\, 3\, t_5^P \big) \Big) \\
& + \alpha^2 \Big((-K_2' + K_3' + K_8')\, t_8^K - 3\, K_6'\, t_{14}^K \\
& \quad + (-H_2' + H_3' + H_8')\, t_9^K - 3\, \alpha^2\, H_6'\, t_{15}^K \Big) \\
& + \alpha^3 \Big((-J_2' + J_3' + J_8')\, t_{10}^K - 3\, J_6'\, t_{16}^K \\
& \quad + (-M_2' + M_3' + M_6')'\, t_8^K - 3\, M_6'\, t_{14}^K \\
& \quad + A_{10}\, (K_2' + K_6' + K_8')\, t_8^K - 3\, A_{10}\, K_6'\, t_{14}^K \\
& \quad + A_{11}\, (-K_2' + K_3' + K_8')\, t_{11}^K - A_{11}\, K_6'\, 3\, t_{17}^K \\
& \quad + A_{11}\, (-H_2' + H_3' + H_8')\, t_{12}^K - 3\, A_{11}\, H_6'\, t_{18}^K \\
& \quad + (-Q_2' + Q_3' + Q_8')\, \ln \frac{p^2}{\Lambda^2}\, t_{12}^K - 3\, Q_6'\, \ln \frac{p^2}{\Lambda^2}\, t_{18}^K \\
& \quad + B_{10}\, (-K_2' + K_3' + K_8')\, t_8^K - 3\, B_{10}\, K_6'\, t_{14}^K \\
& \quad + B_{10}\, (-H_2' + H_3' + H_8')\, t_9^K - 3\, B_{10}\, H_6'\, t_{15}^K \\
& \quad + B_{11}\, (-K_2' + K_3' + K_8')\, t_{11}^K - B_{11}\, K_6'\, 3\, t_{17}^K \\
& \quad + B_{11}\, (-H_6' + H_3' + H_8')\, t_{12}^K - 3\, B_{11}\, H_6'\, t_{18}^K \\
& \quad - 3\, B_{11}\, K_3'\, t_{20}^K + 3\, B_{11}\, K_6'\, t_{14}^K - 3\, B_{11}\, H_3'\, t_{21}^K + 3\, B_{11}\, H_6'\, t_{15}^K \Big) \Bigg\}\,.
\end{aligned}
\tag{3.2.60}
$$

After evaluating these integrals, we can now write the result as :

$$
\boxed{
\begin{aligned}
T^{f''} = \frac{1}{4\pi}\Bigg\{ & \alpha^2 \ln^2\frac{p^2}{\Lambda^2}\left(\frac{9}{4}\left(-K_2'+K_3'+K_6'+K_8'\right)\right) \\
& +\alpha^2 \ln\frac{p^2}{\Lambda^2}\left(\frac{3}{2}\left(-H_2'+H_3'+H_6'+H_8'\right)+2\left(13-16\ln 2\right)K_6'\right. \\
& \qquad \left. +\left(\frac{-7}{2}+8\ln 2\right)\left(-K_2'+K_3'+K_6'+K_8'\right)\right) \\
& +\alpha^3 \ln^3\frac{p^2}{\Lambda^2}\left(\frac{5}{4}\left(A_{11}+B_{11}\right)\left(-K_2'+K_3'+K_6'+K_8'\right)\right. \\
& \qquad \left. +\frac{3}{4}\left(-Q_2'+Q_3'+Q_6'+Q_8'\right)+2\left(-J_2'+J_3'+J_6'+J_8'\right)\right) \\
& +\alpha^3 \ln^2\frac{p^2}{\Lambda^2}\left(\left(\frac{-7}{2}+8\ln 2\right)\left(-J_2'+J_3'+J_6'+J_8'\right)+2\left(13-16\ln 2\right)J_6'\right. \\
& \qquad +\left(\frac{-7}{2}+8\ln 2\right)\left(A_{11}+B_{11}\right)\left(-K_2'+K_3'+K_6'+K_8'\right) \\
& \qquad +2\left(13-16\ln 2\right)\left(A_{11}+B_{11}\right)K_6' \\
& \qquad +\frac{9}{4}\left(-M_2'+M_3'+M_6'+M_8'\right) \\
& \qquad +\frac{9}{4}\left(A_{10}+B_{10}\right)\left(-K_2'+K_3'+K_6'+K_8'\right) \\
& \qquad +\frac{3}{4}\left(A_{11}+B_{11}\right)\left(-H_2'+H_3'+H_6'+H_8'\right) \\
& \qquad +\left(\frac{-7}{2}+8\ln 2\right)\left(-Q_2'+Q_3'+Q_6'+Q_8'\right)+\left(13-16\ln 2\right)Q_6' \\
& \qquad \left. +\frac{9}{4}B_{11}\left(K_3'-K_6'\right)\right)\Bigg\}
\end{aligned}
}
\qquad (3.2.61)
$$

Recall Eqn. (3.2.14),

$$
\frac{1}{F(p^2)} = 1 - \left(L_\xi^f + L_A^f + L_B^f + T^{f\,\prime} + T^{f\,\prime\prime}\right) . \qquad (3.2.14)
$$

Consequently, we at last arrive at the result :

$$
\begin{aligned}
\frac{1}{F(p^2)} \;=\;& 1+\frac{1}{4\pi}\Bigg\{ -\alpha\,\xi\ln\frac{\mathbf{p}^2}{\Lambda^2} \\
&- \alpha^2\ln^2\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[-\left(\frac{\xi}{2}+\frac{3}{8}\right)A_{11} \\
&\qquad +\frac{3}{4}\left(-K_2+K_3+K_6+K_8\right)+\frac{9}{4}\left(-K_2'+K_3'+K_6'+K_8'\right)\Bigg] \\
&- \alpha^2\ln\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[-\frac{3}{2}B_{11} \\
&\qquad +\frac{1}{2}(16\ln 2-7)\left(-K_2'+K_3'+K_6'+K_8'\right) \\
&\qquad +\frac{3}{2}\left(-H_2'+H_3'+H_6'+H_8'\right)+2(13-16\ln 2)K_6'\Bigg] \\
&- \alpha^3\ln^3\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[\left(\frac{\xi}{2}+\frac{3}{8}\right)A_{11}^2-\left(\frac{2\xi}{3}+\frac{1}{2}\right)A_{22}-\frac{A_{11}B_{11}}{8} \\
&\qquad +\frac{1}{4}(A_{11}+B_{11})\left(-K_2+K_3+K_6+K_8\right)-\left(-J_2+J_3+J_6+J_8\right) \\
&\qquad +\frac{5}{4}(A_{11}+B_{11})\left(-K_2'+K_3'+K_6'+K_8'\right)+2\left(-J_2'+J_3'+J_6'+J_8'\right) \\
&\qquad +\frac{3}{4}\left(-Q_2'+Q_3'+Q_6'+Q_8'\right)\Bigg] \\
&- \alpha^3\ln^2\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[(A_{10}A_{11}-A_{21})\left(\frac{\xi}{2}+\frac{3}{8}\right)-\frac{3}{8}A_{11}B_{10}+\frac{3}{8}A_{11}B_{11}-\frac{3}{2}B_{22} \\
&\qquad +\frac{3}{4}(A_{10}+B_{10})\left(-K_2+K_3+K_6+K_8\right) \\
&\qquad -\frac{3}{4}\left(-M_2+M_3+M_6+M_8\right)+\frac{3}{4}B_{11}(K_3-K_6) \\
&\qquad +\frac{1}{2}(16\ln 2-7)\left(-J_2'+J_3'+J_6'+J_8'\right)+2(13-16\ln 2)J_6' \\
&\qquad +\frac{9}{4}\left(-M_2'+M_3'+M_6'+M_8'\right) \\
&\qquad +\frac{1}{2}(16\ln 2-7)(A_{11}+B_{11})\left(-K_2'+K_3'+K_6'+K_8'\right) \\
&\qquad +\frac{9}{4}(A_{10}+B_{10})\left(-K_2'+K_3'+K_6'+K_8'\right) \\
&\qquad +\frac{9}{4}B_{11}(K_3'-K_6')+2(13-16\ln 2)(A_{11}+B_{11})K_6' \\
&\qquad +\frac{1}{4}(16\ln 2-7)\left(-Q_2'+Q_3'+Q_6'+Q_8'\right)+(13-16\ln 2)Q_6' \\
&\qquad +\frac{3}{4}(A_{11}+B_{11})\left(-H_2'+H_3'+H_6'+H_8'\right)\Bigg]+\mathcal{O}(\alpha^4)\Bigg\}
\end{aligned}
\tag{3.2.62}
$$

3.2.5 Quenched Schwinger-Dyson Equation

We shall now consider quenched SD-equation as a simple example. In order to do this (see Chapter 2), the photon wave-function renormalisation should be taken as $G(p^2) = 1$ in the photon propagator, Eqn (2.3.3), or equivalently all B_{ij} terms can be set to zero in Eqn (3.2.11). We can, therefore, write the result replacing all B_{11} and B_{10} terms by zero in the Eqn. (3.2.62) as follows :

$$
\begin{aligned}
\frac{1}{F(p^2)} = 1 + \frac{1}{4\pi}\Bigg\{ & - \alpha\,\xi \ln\frac{\mathbf{p}^2}{\Lambda^2} \\
& - \alpha^2 \ln^2\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[-\left(\frac{\xi}{2}+\frac{3}{8}\right) A_{11} \\
& \qquad + \frac{3}{4}\left(-K_2 + K_3 + K_6 + K_8\right) + \frac{9}{4}\left(-K_2' + K_3' + K_6' + K_8'\right)\Bigg] \\
& - \alpha^2 \ln\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[\frac{1}{2}(16\ln 2 - 7)\left(-K_2' + K_3' + K_6' + K_8'\right) \\
& \qquad + \frac{3}{2}\left(-H_2' + H_3' + H_6' + H_8'\right) + 2\,(13 - 16\ln 2)\,K_6'\Bigg] \\
& - \alpha^3 \ln^3\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[\left(\frac{\xi}{2}+\frac{3}{8}\right) A_{11}^2 - \left(\frac{2\xi}{3}+\frac{1}{2}\right) A_{22} \\
& \qquad + \frac{1}{4} A_{11}\left(-K_2 + K_3 + K_6 + K_8\right) - \left(-J_2 + J_3 + J_6 + J_8\right) \\
& \qquad + \frac{5}{4} A_{11}\left(-K_2' + K_3' + K_6' + K_8'\right) + 2\left(-J_2' + J_3' + J_6' + J_8'\right) \\
& \qquad + \frac{3}{4}\left(-Q_2' + Q_3' + Q_6' + Q_8'\right)\Bigg] \\
& - \alpha^3 \ln^2\frac{\mathbf{p}^2}{\Lambda^2}\Bigg[\left(A_{10}A_{11} - A_{21}\right)\left(\frac{\xi}{2}+\frac{3}{8}\right) \\
& \qquad + \frac{3}{4} A_{10}\left(-K_2 + K_3 + K_6 + K_8\right) - \frac{3}{4}\left(-M_2 + M_3 + M_6 + M_8\right) \\
& \qquad + \frac{1}{2}(16\ln 2 - 7)\left(-J_2' + J_3' + J_6' + J_8'\right) + 2\,(13 - 16\ln 2)\,J_6' \\
& \qquad + \frac{9}{4}\left(-M_2' + M_3' + M_6' + M_8'\right) + \frac{9}{4} A_{10}\left(-K_2' + K_3' + K_6' + K_8'\right) \\
& \qquad + \frac{1}{2}(16\ln 2 - 7) A_{11}\left(-K_2' + K_3' + K_6' + K_8'\right) + 2\,(13 - 16\ln 2)\,A_{11} K_6' \\
& \qquad + \frac{1}{4}(16\ln 2 - 7)\left(-Q_2' + Q_3' + Q_6' + Q_8'\right) + (13 - 16\ln 2)\,Q_6' \\
& \qquad + \frac{3}{4} A_{11}\left(-H_2' + H_3' + H_6' + H_8'\right)\Bigg] + \mathcal{O}(\alpha^4)\Bigg\}
\end{aligned}
\tag{3.2.63}
$$

3.3 Unquenched Schwinger-Dyson Equation for the Photon in massless QED

As a second main topic, in this section, we are going to discuss the unquenched SD-equation for the gauge-boson. We shall try to solve it for the photon wave-function renormalisation in order to find multiplicative renormalisability constraints on the electron-photon vertex. This equation has some different features from the fermion equation. For instance, the two fermion legs have to be treated equally. We can ensure this symmetry property by dividing the external momenta into two equivalent pieces as shown in the following figure.

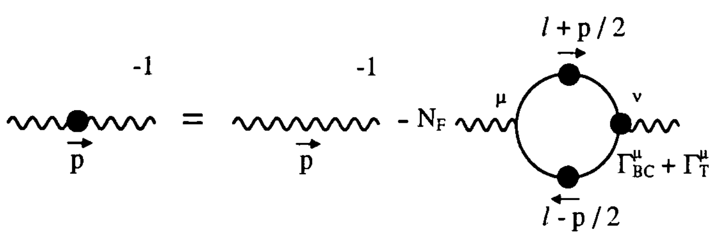


Figure 3.3: Unquenched Schwinger-Dyson equations for photon

Using the Feynman rules, Fig. 3.3 can be expressed as,

$$-i\Delta^{-1}_{\mu\nu}(p) \;=\; -i{\Delta^{0}_{\mu\nu}}^{-1}(p) - (-1)\,N_F \int_M \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\,(-ie\gamma^\mu)\,iS_F(\ell_+)\,(-ie\Gamma^\nu)\,iS_F(\ell_-)\right] , \tag{3.3.1}$$

where

$$\ell_+ \;\equiv\; (\ell + p/2) \quad ,$$
$$\ell_- \;\equiv\; (\ell - p/2) \quad .$$

The definition of the fermion * and photon propagators are given already in Chapter 2,

$$iS_F(\ell_+) \;=\; i\,\frac{F(\ell_+)}{\not\ell_+} \quad ,$$
$$i\Delta^{\mu\nu}(p) \;=\; -\frac{i}{p^2}\left[G(p)\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \xi\,\frac{p^\mu p^\nu}{p^2}\right] .$$

*From now on for convenience, we shall display the fermion and photon wave-function renormalisation functions as follows : $F(p^2) \equiv F(p)$ and $G(p^2) \equiv G(p)$.

We now introduce a projection tensor [10, 11] in order to get a scalar equation for the photon wave-function renormalisation. We can remove the quadratically divergent term and project out the ultraviolet logarithmically divergent terms, by using the following tensor

$$P_{\mu\nu} = \frac{1}{3p^4}\left(4p_\mu p_\nu - p^2 g_{\mu\nu}\right) \quad , \tag{3.3.2}$$

which satisfies the following properties :

$$\begin{aligned} P^{\mu\nu} g_{\mu\nu} &= 0 \quad , \\ P^{\mu\nu} p_\mu p_\nu &= 1 \quad . \end{aligned} \tag{3.3.3}$$

When it acts on the inverse photon propagator, it gives

$$\begin{aligned} P^{\mu\nu}\,\Delta^{-1}_{\mu\nu} &= \frac{1}{G(p)} - \frac{1}{\xi} \quad , \\ P^{\mu\nu}\,\Delta^{0\;-1}_{\mu\nu} &= 1 - \frac{1}{\xi} \quad . \end{aligned} \tag{3.3.4}$$

So, we can write Eqn. (3.3.1) by using $P^{\mu\nu}$ as :

$$\frac{1}{G(p^2)} = 1 + N_F \frac{i\,\alpha}{4\pi^3} \int_M \frac{d^4\ell}{\ell_+^2\,\ell_-^2}\, F(\ell_-)\,F(\ell_+)\,P_{\mu\nu}\,\mathrm{Tr}\left[\gamma^\mu\, \not{\ell}_+\, \Gamma^\nu_{(BC+T)}\, \not{\ell}_-\right] . \tag{3.3.5}$$

Employing the Ball-Chiu definition, Eqn (2.3.34), from Chapter 2, the longitudinal piece of the vertex can be written as follows with the momenta indicated in Fig. 3.3 :

$$\begin{aligned} \Gamma^\nu_{BC} &= A\,\gamma^\nu + B\,(\not{\ell}_+ + \not{\ell}_-)\,(\ell_+ + \ell_-)^\nu \\ &= A\,\gamma^\nu + 4\,B\,\not{\ell}\,\ell^\nu \,, \end{aligned} \tag{3.3.6}$$

where

$$\begin{aligned} A &= \frac{1}{2}\left(\frac{1}{F(\ell_+)} + \frac{1}{F(\ell_-)}\right) \\ &= \frac{1}{2}\,\frac{(F(\ell_-) + F(\ell_+))}{F(\ell_+)\,F(\ell_-)} \quad , \end{aligned}$$

$$
\begin{aligned}
B &= \frac{1}{4\,\ell\cdot p}\left(\frac{1}{F(\ell_+)}-\frac{1}{F(\ell_-)}\right) \\
&= \frac{1}{4\,\ell\cdot p}\,\frac{[F(\ell_-)-F(\ell_+)]}{F(\ell_+)\,F(\ell_-)}\,.
\end{aligned}
\tag{3.3.7}
$$

The transverse piece is then :

$$
\Gamma_T^\nu = \sum_{i=2,3,6,8} \tau_i(p^2,\ell_+^2,\ell_-^2)\,T_i^\nu \quad , \tag{3.3.8}
$$

with the basis tensors of Eqn. (2.3.65) are now

$$
\begin{aligned}
T_2^\nu &= 2\left(\ell^\mu\,p^2 - p^\mu\,\ell\cdot p\right)\,\not\ell\,, \\
T_3^\nu &= \gamma^\nu\,p^2 - p^\nu\,\not p\,, \\
T_6^\nu &= -2\,\gamma^\nu\,\ell\cdot p + 2\,\ell^\nu\,\not p\,, \\
T_8^\nu &= -\gamma^\nu\,p^\mu\,\ell^\lambda\sigma_{\mu\lambda} + p^\nu\,\not\ell - \ell^\nu\,\not p\,.
\end{aligned}
\tag{3.3.9}
$$

Making use of the full vertex, $\Gamma_F^\nu = \Gamma_{BC}^\nu + \Gamma_T^\nu$ of Eqns. (3.3.6–3.3.8), we get :

$$
\begin{aligned}
\frac{1}{G(p^2)} = 1 + N_F\,\frac{i\alpha}{4\pi^3}\int_M \frac{d^4\ell}{\ell_+^2\,\ell_-^2}\,F(\ell_-)\,F(\ell_+)\,P_{\mu\nu}
&\times\Big\{ \quad A\,\mathrm{Tr}\,(\gamma^\mu\,\not\ell_+\,\gamma^\nu\,\not\ell_-) \\
&+4\,B\,\mathrm{Tr}\,(\gamma^\mu\,\not\ell_+\,\not\ell\,\ell^\nu\,\not\ell_-) \\
&+ \quad \mathrm{Tr}\,(\gamma^\mu\,\not\ell_+\,\Gamma_T^\nu\,\not\ell_-) \quad \Big\}\,.
\end{aligned}
\tag{3.3.10}
$$

The traces needed are listed below :

$$
\begin{aligned}
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, A\, \gamma^{\nu}\, \not{\ell}_{-}\right] &= 4\,A \quad \left\{ 2\left(\ell^{\mu}\ell^{\nu} - \frac{p^{\mu}p^{\nu}}{4}\right) - g^{\mu\nu}\left(\ell^{2} - \frac{p^{2}}{4}\right)\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, A\, \gamma^{\nu}\, \not{\ell}_{-}\right] &= \frac{4}{3p^{2}}\,A \quad \left\{ 8\,\frac{(\ell\cdot p)^{2}}{p^{2}} - 2\,\ell^{2} - \frac{3}{2}\,p^{2}\right\} ,\\
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, B\, \not{\ell}\, \ell^{\nu}\, \not{\ell}_{-}\right] &= B \quad \left\{ 4\,\ell^{\mu}\ell^{\nu}\left(\ell^{2} + \frac{p^{2}}{4}\right) - p^{\mu}\ell^{\nu}\,\ell\cdot p\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, B\, \not{\ell}\ell^{\nu}\, \not{\ell}_{-}\right] &= \frac{1}{3p^{2}}\,B \quad \left\{ \left(16\,\frac{\ell^{2}}{p^{2}} - 2\right)(\ell\cdot p)^{2} - 4\ell^{4} - p^{2}\ell^{2}\right\} ,\\
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{2}^{\mu}\, \not{\ell}_{-}\right] &= 4\,\tau_{2} \quad \left\{ 2\left(\ell^{2} + \frac{p^{2}}{4}\right)\left(p^{2}\,\ell^{\mu}\ell^{\nu} - \ell\cdot p\,\ell^{\mu}p^{\nu}\right) \right.\\
&\qquad \left. + \frac{\ell\cdot p}{2}\left(\ell\cdot p\, p^{\mu}p^{\nu} - p^{2}\ell^{\nu}p^{\mu}\right)\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{2}^{\mu}\, \not{\ell}_{-}\right] &= \frac{4\,\tau_{2}}{3p^{2}} \quad \left\{ 2\left(\ell^{2} + \frac{p^{2}}{4}\right)\left((\ell\cdot p)^{2} - \ell^{2}p^{2}\right)\right\} ,\\
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{3}^{\mu}\, \not{\ell}_{-}\right] &= 4\,\tau_{3} \quad \left\{ 2\ell^{\mu}\ell^{\nu}p^{2} + p^{\mu}p^{\nu}\,\ell^{2} - g_{\mu\nu}\left(\ell^{2} - \frac{p^{2}}{4}\right) - 2\,\ell^{\mu}p^{\nu}\,\ell\cdot p\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{3}^{\mu}\, \not{\ell}_{-}\right] &= \frac{4\,\tau_{3}}{3p^{2}} \quad \left\{ 2\left((\ell\cdot p)^{2} - \ell^{2}p^{2}\right) + 3\,p^{2}\left(\ell^{2} - \frac{p^{2}}{4}\right)\right\} ,\\
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{6}^{\mu}\, \not{\ell}_{-}\right] &= 4\,\tau_{6} \quad \left\{ -2\,\ell^{\nu}p^{\mu}\left(\ell^{2} + \frac{p^{2}}{4}\right) - 4\,p^{\mu}p^{\nu}\,\ell\cdot p \right.\\
&\qquad \left. + 2\,g_{\mu\nu}\,\ell\cdot p\left(\ell^{2} - \frac{p^{2}}{4}\right)\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{6}^{\mu}\, \not{\ell}_{-}\right] &= \frac{4\,\tau_{6}}{3p^{2}} \quad \left\{ -6\,\ell\cdot p\left(\ell^{2} - \frac{p^{2}}{4}\right)\right\} ,\\
\mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{8}^{\mu}\, \not{\ell}_{-}\right] &= 4\,\tau_{8} \quad \left\{ -p^{\mu}p^{\nu}\,\ell^{2} - \ell^{\mu}\ell^{\nu}\,p^{2} + \ell^{\mu}p^{\nu}\,\ell\cdot p + p^{\mu}\ell^{\nu}\,\ell\cdot p\right\} ,\\
P_{\mu\nu}\quad \mathrm{Tr}\left[\gamma^{\mu}\, \not{\ell}_{+}\, T_{8}^{\mu}\, \not{\ell}_{-}\right] &= \frac{4\,\tau_{8}}{3p^{2}} \quad \left\{ 2\left((\ell\cdot p)^{2} - \ell^{2}p^{2}\right)\right\} ,
\end{aligned}
\tag{3.3.11}
$$

using these we can re-express Eqn. (3.3.10) as

$$\begin{aligned}
\frac{1}{G(p^2)} &= 1 + N_F \frac{i\alpha}{3\pi^3 p^2} \int_M \frac{d^4\ell}{\ell_+^2\,\ell_-^2} F(\ell_+)\,F(\ell_-) \\
&\quad \times \Bigg\{ A \left[8\frac{(\ell\cdot p)^2}{p^2} - 2\ell^2 - \frac{3}{2}p^2 \right] \\
&\quad + B \left[\left(16\frac{\ell^2}{p^2} - 2\right) (\ell\cdot p)^2 - 4\ell^4 - p^2\ell^2 \right] \\
&\quad + \Bigg[\tau_2 \left\{ 2\left(\ell^2 + \frac{p^2}{4}\right)\left((\ell\cdot p)^2 - \ell^2 p^2\right)\right\} \\
&\quad + \tau_3 \left\{ 2\left((\ell\cdot p)^2 - \ell^2p^2\right) + 3p^2\left(\ell^2 - \frac{p^2}{4}\right)\right\} \\
&\quad + \tau_6 \left\{ -6\,\ell\cdot p \left(\ell^2 - \frac{p^2}{4}\right)\right\} \\
&\quad + \tau_8 \left\{ 2\left((\ell\cdot p)^2 - \ell^2p^2\right)\right\} \Bigg]\Bigg\} .
\end{aligned} \tag{3.3.12}$$

Now, we move to Euclidean space by performing a Wick rotation. Substituting the definition of A and B from Eqn. (3.3.7) leads to :

$$\begin{aligned}
\frac{1}{G(p^2)} &= 1 - \frac{\alpha N_F}{3\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2\,\ell_-^2} \\
&\quad \times \Bigg\{ \frac{1}{2}\,(F(\ell_-) + F(\ell_+)) \left[8\frac{(\ell\cdot p)^2}{p^2} - 2\ell^2 - \frac{3}{2}p^2 \right] \\
&\quad + \frac{1}{4}\,\frac{(F(\ell_-) - F(\ell_+))}{\ell\cdot p} \left[\left(16\frac{\ell^2}{p^2} - 2\right)(\ell\cdot p)^2 - 4\ell^4 - p^2\ell^2 \right] \\
&\quad + F(\ell_+)\,F(\ell_-) \Bigg[\tau_2 \left\{ 2\left(\ell^2 + \frac{p^2}{4}\right)\left((\ell\cdot p)^2 - \ell^2p^2\right)\right\} \\
&\quad - \tau_3 \left\{ 2\left((\ell\cdot p)^2 - \ell^2p^2\right) + 3p^2\left(\ell^2 - \frac{p^2}{4}\right)\right\} \\
&\quad - \tau_6 \left\{ (-6\,\ell\cdot p)\left(\ell^2 - \frac{p^2}{4}\right)\right\} \\
&\quad - \tau_8 \left\{ 2\left((\ell\cdot p)^2 - \ell^2p^2\right)\right\} \Bigg]\Bigg\} .
\end{aligned} \tag{3.3.13}$$

We would like to carry out this calculation in three steps, writing

$$\frac{1}{G(p^2)} = 1 - (\,L_A^\gamma + L_B^\gamma + T^\gamma) \quad , \tag{3.3.14}$$

just as we analogously did for the fermion function in Sect. 3.2, where

$$L_A^\gamma = \frac{\alpha N_F}{6\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+ \ell_-} \left(F(\ell_+) + F(\ell_-) \right) \left(8 \frac{(\ell \cdot p)^2}{p^2} - 2\ell^2 - \frac{3}{2} p^2 \right) , \tag{3.3.15}$$

$$L_B^\gamma = \frac{\alpha N_F}{12\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2 \ell_-^2} \frac{(F(\ell_-) - F(\ell_+))}{\ell \cdot p} \left[\left(16 \frac{\ell^2}{p^2} - 2 \right) (\ell \cdot p)^2 - 4\ell^4 - p^2 \ell^2 \right] , \tag{3.3.16}$$

$$\begin{aligned} T^\gamma = \frac{\alpha N_F}{3\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2 \ell_-^2} F(\ell_+) F(\ell_-) \Bigg\{ & \tau_2 \left[2 \left(\ell^2 + \frac{p^2}{4} \right) \left((\ell \cdot p)^2 - p^2 \ell^2 \right) \right] \\ & - \tau_3 \left[2 \left((\ell \cdot p)^2 - p^2 \ell^2 \right) + 3p^2 \left(\ell^2 - \frac{p^2}{4} \right) \right] \\ & - \tau_6 \left[-6\, \ell \cdot p \left(\ell^2 - \frac{p^2}{4} \right) \right] \\ & - \tau_8 \left[2 \left((\ell \cdot p)^2 - p^2 \ell^2 \right) \right] \Bigg\} . \end{aligned} \tag{3.3.17}$$

We first evaluate L_A^γ.

3.3.1 L_A^γ Calculated

To compute L_A^γ, we need to know the following quantity :

$$\begin{aligned} F(\ell_+) + F(\ell_-) &= 2 + \alpha A_{11} \left(\ln \frac{\ell_+^2}{\Lambda^2} + \ln \frac{\ell_-^2}{\Lambda^2} \right) + 2\alpha A_{10} \\ &+ \alpha^2 A_{22} \left(\ln^2 \frac{\ell_+^2}{\Lambda^2} + \ln^2 \frac{\ell_-^2}{\Lambda^2} \right) + \alpha^2 A_{21} \left(\ln \frac{\ell_+^2}{\Lambda^2} + \ln \frac{\ell_-^2}{\Lambda^2} \right) \\ &+ \mathcal{O}(\alpha^3) \quad . \end{aligned} \tag{3.3.18}$$

As we came across before, ℓ_+^2 and ℓ_-^2 depend on angle. This dependence must be separated out in order to perform the angular integrals. This we do by noting :

$$\begin{aligned} \ln \frac{\ell_+^2}{\Lambda^2} + \ln \frac{\ell_-^2}{\Lambda^2} &= 2 \ln \frac{(\ell^2 + p^2/4)}{\Lambda^2} + \ln \left(1 - \frac{(\ell \cdot p)^2}{(\ell^2 + p^2/4)^2} \right) \\ &= 2 \ln \frac{(\ell^2 + p^2/4)}{\Lambda^2} - \frac{(\ell \cdot p)^2}{(\ell^2 + p^2/4)^2} - \cdots \quad , \end{aligned} \tag{3.3.19}$$

$$
\begin{aligned}
\ln^2\frac{\ell_+^2}{\Lambda^2}+\ln^2\frac{\ell_-^2}{\Lambda^2} &= 2\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}+2\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\ln\left(1-\frac{(\ell\cdot p)^2}{(\ell^2+p^2/4)^2}\right) \\
&\quad + \ln^2\left(1+\frac{\ell\cdot p}{(\ell^2+p^2/4)}\right)+\ln^2\left(1-\frac{\ell\cdot p}{(\ell^2+p^2/4)}\right), \\
&= 2\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}+2\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\left(-\frac{(\ell\cdot p)^2}{(\ell^2+p^2/4)^2}-\cdots\right) \\
&\quad + \left(2\frac{(\ell\cdot p)^2}{(\ell^2+p^2/4)^2}+\cdots\right).
\end{aligned}
\tag{3.3.20}
$$

Substituting the above expressions into Eqn. (3.3.18) yields,

$$
\begin{aligned}
F(\ell_+)+F(\ell_-) &= 2\left[1+\alpha\left(A_{11}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{10}\right)\right. \\
&\quad \left. +\alpha^2\left(A_{22}\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{21}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right)\right] \\
&+ \alpha\left(A_{11}+\alpha A_{21}+2\alpha A_{22}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right)\left(-\frac{(\ell\cdot p)^2}{(\ell^2+p^2/4)^2}-\cdots\right) \\
&+ 2\alpha^2 A_{22}\left(\frac{\ell\cdot p}{(\ell^2+p^2/4)}+\cdots\right)+\mathcal{O}(\alpha^3) .
\end{aligned}
\tag{3.3.21}
$$

Introducing the following definition for the angular integrals :

$$
K_{n,1}=\int_0^\pi d\psi\,\sin\psi\,\frac{(\ell\cdot p)^n}{\ell_+^2\,\ell_-^2}, \tag{3.3.22}
$$

and making use of this together with Eqn. (3.3.21), we obtain :

$$
\begin{aligned}
L_A^\gamma \;=\; & \frac{\alpha N_F}{3\pi^2 p^2}\int_0^{\Lambda^2} d\ell^2\,\ell^2 \\
\times\Bigg\{ & 2\Bigg[1+\;\alpha\left(A_{11}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{10}\right) \\
& +\alpha^2\left(A_{22}\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{21}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right)\Bigg] \\
& \qquad\qquad \times\left(\frac{8}{p^2}K_{2,1}-2\ell^2K_{0,1}-\frac{3}{2}p^2K_{0,1}\right) \\
& -\left(\alpha A_{11}+\alpha^2A_{21}+2\alpha^2A_{22}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right) \\
& \qquad\times\Bigg[\frac{1}{(\ell^2+p^2/4)^2}\left(\frac{8}{p^2}K_{4,1}-2\ell^2K_{2,1}-\frac{3}{2}p^2K_{2,1}\right) \\
& \qquad\quad+\frac{1}{2\,(\ell^2+p^2/4)^4}\left(\frac{8}{p^2}K_{6,1}-2\ell^2K_{4,1}-\frac{3}{2}p^2K_{4,1}\right)+\cdots\Bigg] \\
& +2\,\alpha^2A_{22}\qquad\times\Bigg[\frac{1}{(\ell^2+p^2/4)^2}\left(\frac{8}{p^2}K_{4,1}-2\ell^2K_{2,1}-\frac{3}{2}p^2K_{2,1}\right) \\
& \qquad\quad+\frac{11}{12\,(\ell^2+p^2/4)^4}\left(\frac{8}{p^2}K_{6,1}-2\ell^2K_{4,1}-\frac{3}{2}p^2K_{4,1}\right)+\cdots\Bigg] \\
& +\mathcal{O}(\alpha^3)\Bigg\}\,.
\end{aligned}
\qquad (3.3.23)
$$

Referring to Appendix C for these angular integrals,

$$
\begin{aligned}
X_1^\gamma &\equiv \frac{8}{p^2}K_{2,1}-2\ell^2K_{0,1}-\frac{3}{2}p^2K_{0,1} \\
&= \pi\frac{1}{p^4\,(\ell^2+p^2/4)}\left(16\,\ell^4-3\,p^4\right)\theta_-^\gamma-\frac{\pi}{2}\frac{p^2}{\ell^2\,(\ell^2+p^2/4)}\theta_+^\gamma\,, \\
X_2^\gamma &\equiv \frac{8}{p^2}K_{4,1}-2\ell^2K_{2,1}-\frac{3}{2}p^2K_{2,1} \\
&= \pi\frac{\ell^2}{p^4}\left(16\,\ell^4+4\,\ell^2p^2-3\,p^4\right)\theta_-^\gamma+\frac{\pi}{8}\frac{p^2}{\ell^2}(2\ell^2-p^2)\,\theta_+^\gamma\,, \\
X_3^\gamma &\equiv \frac{8}{p^2}K_{6,1}-2\ell^2K_{4,1}-\frac{3}{2}p^2K_{4,1} \\
&= \frac{\pi}{2}\frac{\ell^4}{p^4}\left(32\,\ell^6+24\,p^2\,\ell^4+12\,p^4\ell^2+3p^6\right)\theta_-^\gamma \\
&+ \frac{\pi\,p^4}{2^7\,\ell^2}\left(-p^4-6\,p^2\,\ell^2+24\,\ell^4\right)\theta_+^\gamma\,.
\end{aligned}
\qquad (3.3.24)
$$

where

$$\begin{aligned}\theta_-^\gamma &\equiv \theta(p^2/4-\ell^2)\,,\\ \theta_+^\gamma &\equiv \theta(\ell^2-p^2/4)\,,\end{aligned}\tag{3.3.25}$$

L_A^γ now takes the form,

$$\begin{aligned}L_A^\gamma &= \frac{\alpha\,N_F}{3\pi\,p^2}\int_0^{\Lambda^2} d\ell^2\,\ell^2\\
&\times\Bigg\{2\left[1+\alpha\left(A_{11}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{10}\right)\right.\\
&\qquad\left.+\alpha^2\left(A_{22}\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}+A_{21}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right)\right]\\
&\qquad\times\left[\frac{1}{p^4\,(\ell^2+p^2/4)}\left(16\,\ell^4-3\,p^4\right)\theta_-^\gamma-\frac{1}{2}\frac{p^2}{\ell^2\,(\ell^2+p^2/4)}\theta_+^\gamma\right]\\
&-\left(\alpha\,A_{11}+\alpha^2\,A_{21}+2\alpha^2\,A_{22}\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\right)\\
&\qquad\times\left[\frac{1}{(\ell^2+p^2/4)^2}\left(\frac{\ell^2}{p^4}\left(16\,\ell^4+4\,\ell^2p^2-3\,p^4\right)\theta_-^\gamma+\frac{1}{8}\frac{p^2}{\ell^2}(2\ell^2-p^2)\,\theta_+^\gamma\right)+\cdots\right]\\
&+2\,\alpha^2\,A_{22}\left[\frac{1}{(\ell^2+p^2/4)^2}\left(\frac{\ell^2}{p^4}\left(16\,\ell^4+4\,\ell^2p^2-3\,p^4\right)\theta_-^\gamma+\frac{1}{8}\frac{p^2}{\ell^2}(2\ell^2-p^2)\,\theta_+^\gamma\right)+\cdots\right]\\
&+\mathcal{O}(\alpha^3)\Bigg\}\ .\end{aligned}\tag{3.3.26}$$

After separating this expression into two regions of integration and collecting the terms which give contributions to the leading and next-to-leading logarithms,

$$
\begin{aligned}
L_A^\gamma &= \frac{2\,\alpha\,N_F}{3\pi p^2} \\
&\times \left\{ \int_0^{p^2/4} d\ell^2 \left[\left(1 + \alpha\,A_{11}\,\ln\frac{p^2}{4\,\Lambda^2} + \alpha^2\,A_{22}\,\ln^2\frac{p^2}{4\,\Lambda^2}\right) \right.\right. \\
&\qquad\qquad \left. \times \frac{\ell^2\,(16\,\ell^4 - 3p^2)}{p^4\,(\ell^2 + p^2/4)} + \mathcal{O}(\alpha^3) \right] \\
&\quad - \int_{p^2/4}^{\Lambda^2} d\ell^2 \left[\left(1 + \alpha\left(A_{11}\,\ln\frac{\ell^2}{\Lambda^2} + A_{10}\right) + \alpha^2\left(A_{22}\,\ln^2\frac{\ell^2}{\Lambda^2} + A_{21}\,\ln\frac{\ell^2}{\Lambda^2}\right)\right) \right. \\
&\qquad\qquad \times \frac{p^2}{2\,(\ell^2 + p^2/4)} \\
&\qquad \left.\left. + \left(\alpha\,A_{11} + 2\,\alpha^2\,A_{22}\,\ln\frac{\ell^2}{\Lambda^2}\right)\frac{p^2\,(2\ell^2 - p^2)}{16\,(\ell^2 + p^2/4)^2} \right] + \mathcal{O}(\alpha^3) \right\},
\end{aligned}
\tag{3.3.27}
$$

and using the results in Appendix E for the ℓ-integrals, we have

$$
\begin{aligned}
L_A^\gamma &= \frac{N_F}{3\pi\,p^2} \left\{ 2\left(\alpha + \alpha^2\,A_{11}\,\ln\frac{p^2}{4\,\Lambda^2} + \alpha^3\,A_{22}\,\ln^2\frac{p^2}{4\,\Lambda^2}\right)\ell_1^P \right. \\
&\quad - 2\left(\alpha + \alpha^2\,A_{10}\right)\ell_1^L - 2\left(\alpha^2\,A_{11} + \alpha^3 A_{21}\right)\ell_2^L \\
&\quad - 2\,\alpha^3\,A_{22}\,\ell_3^L - \alpha^2\,A_{11}\,\ell_4^L - 2\alpha^3\,A_{22}\,\ell_5^L \\
&\quad + \mathcal{O}(\alpha^4)\,,
\end{aligned}
\tag{3.3.28}
$$

where ℓ_i^P and ℓ_i^L are given in Appendix E. Substituting the explicit expression for the ℓ_i^P, ℓ_i^L, we finally obtain the result :

$$
\boxed{
\begin{aligned}
L_A^\gamma &= \frac{N_F}{3\pi} \left\{ \alpha\left[\ln\frac{p^2}{\Lambda^2} - \frac{13}{12}\right] \right. \\
&\quad + \alpha^2\left[\frac{A_{11}}{2}\,\ln^2\frac{p^2}{\Lambda^2} + \left(-\frac{10}{12}\,A_{11} + A_{10}\right)\ln\frac{p^2}{\Lambda^2}\right] \\
&\quad + \alpha^3\left[\frac{A_{22}}{3}\,\ln^3\frac{p^2}{\Lambda^2} + \left(-\frac{10}{12}\,A_{22} + \frac{A_{21}}{2}\right)\ln^2\frac{p^2}{\Lambda^2}\right] \\
&\quad \left. + \mathcal{O}(\alpha^4) \right\}
\end{aligned}
}\,.
\tag{3.3.29}
$$

3.3.2 L_B^γ Calculated

Now we compute L_B^γ, Eqn. (3.3.17). We use the following expression,

$$\begin{aligned} F(\ell_-) &- F(\ell_+) \\ &= (\alpha\, A_{11} + \alpha^2\, A_{21}) \left[\ln\left(1 - \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) - \ln\left((1 + \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right)\right] \\ &+ \alpha^2\, A_{22} \left\{ 2\,\ln\frac{(\ell^2 + p^2/4)}{\Lambda^2} \left[\ln\left(1 - \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) - \ln\left((1 + \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right)\right] \right. \\ &+ \left. \ln^2\left(1 - \frac{\ell\cdot p}{\ell^2 + p^2/4}\right) - \ln^2\left(1 + \frac{\ell\cdot p}{\ell^2 + p^2/4}\right) \right\} \\ &+ \mathcal{O}(\alpha^3) \quad . \end{aligned}$$

Note that

$$\begin{aligned} &\ln\left(1 - \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) - \ln\left(1 + \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) \\ &\qquad = -2\,\frac{\ell\cdot p}{(\ell^2 + p^2/4)} - \frac{2}{3}\,\frac{(\ell\cdot p)^3}{(\ell^2 + p^2/4)^3} + \cdots \; , \\ &\ln^2\left(1 - \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) - \ln^2\left(1 + \frac{\ell\cdot p}{(\ell^2 + p^2/4)}\right) \\ &\qquad = 2\,\frac{(\ell\cdot p)^3}{(\ell^2 + p^2/4)^3} + \frac{5}{3}\,\frac{(\ell\cdot p)^5}{(\ell^2 + p^2/4)^5} - \cdots \; . \end{aligned} \tag{3.3.30}$$

After substituting these quantities into Eqn. (3.3.17), we get

$$\begin{aligned} L_B^\gamma &= \frac{\alpha\, N_F}{12\pi\, p^2} \int_0^{\Lambda^2} \frac{d^4\ell}{\ell_+^2\, \ell_-^2} \left[\left(16\,\frac{\ell^2}{p^2} - 2\right) (\ell\cdot p)^2 - 4\ell^4 - p^2\ell^2 \right] \\ &\times \left\{ \left(\alpha\, A_{11} + \alpha^2\, A_{21} + 2\,\alpha^2\, A_{22}\, \frac{(\ell^2 + p^2/4)}{\Lambda^2} \right) \right. \\ &\qquad \times \left(-2\,\frac{2}{(\ell^2 + p^2/4)} - \frac{2}{3}\,\frac{(\ell\cdot p)^2}{(\ell^2 + p^2/4)^3} + \cdots \right) \\ &\quad + 2\,\alpha^2\, A_{22} \left(2\,\frac{(\ell\cdot p)^2}{(\ell^2 + p^2/4)^3} + \frac{5}{3}\,\frac{(\ell\cdot p)^4}{(\ell^2 + p^2/4)^5} - \cdots \right) \\ &\quad \left. + \mathcal{O}(\alpha^3) \right\} \quad . \end{aligned} \tag{3.3.31}$$

On splitting radial and angular parts and by using the definition of the angular integral, Eqn. (3.3.22), we can display the above equation as,

$$
\begin{aligned}
L_B^\gamma &= \frac{\alpha N_F}{6\pi^2 p^2} \int_0^{\Lambda^2} d\ell^2\, \ell^2 \\
&\times \left\{ \left(\alpha A_{11} + \alpha^2 A_{21} + 2\alpha^2 A_{22} \frac{(\ell^2 + p^2/4)}{\Lambda^2} \right) \right. \\
&\qquad \times \left[-\frac{2}{(\ell^2+p^2/4)} \left\{ \left(16\frac{\ell^2}{p^2} - 2\right) K_{2,1} - (4\ell^4 + p^2\ell^2) K_{0,1} \right\} \right. \\
&\qquad \left. -\frac{2}{3}\frac{1}{(\ell^2+p^2/4)^3} \left\{ \left(16\frac{\ell^2}{p^2} - 2\right) K_{4,1} - (4\ell^4 + p^2\ell^2) K_{2,1} \right\} + \cdots \right] \\
&\quad + \alpha^2 A_{22} \left[\frac{2}{(\ell^2+p^2/4)^3} \left\{ \left(16\frac{\ell^2}{p^2} - 2\right) K_{4,1} - (4\ell^4 + p^2\ell^2) K_{2,1} \right\} \right. \\
&\qquad \left. + \frac{5}{3}\frac{(\ell\cdot p)^4}{(\ell^2+p^2/4)^5} \left\{ \left(16\frac{\ell^2}{p^2} - 2\right) K_{6,1} - (4\ell^4 + p^2\ell^2) K_{4,1} \right\} + \cdots \right] \\
&\quad \left. + \mathcal{O}(\alpha^3) \right\} . \qquad (3.3.32)
\end{aligned}
$$

On performing the angular integrals (refer to Appendix C) :

$$
\begin{aligned}
Y_1^\gamma &\equiv \left(16\frac{\ell^2}{p^2} - 2\right) K_{2,1} - (4\ell^4 + p^2\ell^2) K_{0,1} \\
&= 4\pi \frac{\ell^2}{p^2} (8\ell^2 - 3p^2)\, \theta_-^\gamma - \frac{\pi}{4}\frac{p^2}{\ell^2}\, \theta_+^\gamma , \\
Y_2^\gamma &\equiv \left(16\frac{\ell^2}{p^2} - 2\right) K_{4,1} - (4\ell^4 + p^2\ell^2) K_{2,1} \\
&= 4\pi \frac{\ell^4}{p^4} (8\ell^4 + p^2\ell^2 - p^4)\, \theta_-^\gamma + \frac{\pi}{64}\frac{p^2}{\ell^2} (32\ell^4 - 8p^2\ell^2 - p^4)\, \theta_+^\gamma , \\
Y_3^\gamma &\equiv \left(16\frac{\ell^2}{p^2} - 2\right) K_{6,1} - (4\ell^4 + p^2\ell^2) K_{4,1} \\
&= -\pi \frac{\ell^6}{p^4} (-32\ell^6 - 20\ell^4 p^2 - 150 p^4 \ell^2 + 21 p^6)\, \theta_-^\gamma \\
&\quad - \frac{\pi}{1024}\frac{p^4}{\ell^2} (16 p^4 \ell^2 + 48 p^2 \ell^4 - 384\ell^6 + p^6)\, \theta_+^\gamma , \qquad (3.3.33)
\end{aligned}
$$

we arrive at the following expression,

$$
\begin{aligned}
L_B^\gamma &= \frac{\alpha N_F}{6\pi p^2} \int_{p^2/4}^{\Lambda^2} d\ell^2\, \ell^2 \\
&\times \left\{ \left(\alpha A_{11} + \alpha^2 A_{21} + 2\alpha^2 A_{22} \frac{\ell^2}{\Lambda^2} \right) \right. \\
&\qquad \times \left[-2 \frac{1}{(\ell^2 + p^2/4)} \left(-\frac{p^2}{4\ell^2} \right) \right. \\
&\qquad \left. - \frac{2}{3} \frac{1}{(\ell^2 + p^2/4)^3} \left(\frac{p^2}{64\ell^2} (32\ell^4 - 8p^2\ell^2 - p^4) \right) + \cdots \right] \\
&\quad + \alpha^2 A_{22} \left[\frac{2}{(\ell^2 + p^2/4)^3} \left(\frac{p^2}{64\ell^2} (32\ell^4 - 8p^2\ell^2 - p^4) \right) + \cdots \right] \\
&\quad \left. + \mathcal{O}(\alpha^3) \right\} .
\end{aligned} \tag{3.3.34}
$$

We only keep the $p^2/4 \longrightarrow \Lambda^2$ region in the ℓ-integration as this the leading and next-to-leading contribution to $1/G(p)$. Hence, collecting the related terms and making use of the integrals listed in Appendix E, we have

$$
\begin{aligned}
L_B^\gamma &= \frac{N_F}{6\pi p^2} \int_{p^2/4}^{\Lambda^2} \left\{ \left(\alpha^2 A_{11} + \alpha^3 A_{21} \right) \left(-\ell_1^L + \frac{2}{3} \ell_6^L \right) \right. \\
&\qquad + 2\alpha^3 A_{22} \left(\ell_2^L - \frac{2}{3} \ell_7^L \right) + \alpha^3 A_{22} (2\ell_6^L) \\
&\qquad \left. + \mathcal{O}(\alpha^4) \right\} ,
\end{aligned} \tag{3.3.35}
$$

which takes us to the result :

$$
\boxed{ L_B^\gamma = \frac{N_F}{6\pi} \left[-\alpha^2 \frac{A_{11}}{6} \ln \frac{p^2}{\Lambda^2} - \alpha^3 \frac{A_{22}}{6} \ln^2 \frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha^4) \right] } \quad . \tag{3.3.36}
$$

3.3.3 Transverse Part Calculated

The last step is to calculate the contribution of the transverse piece, Eqn. (3.3.17), to $1/G(p)$,

$$T^\gamma = \frac{\alpha N_F}{3\pi^3 p^2}\int_E \frac{d^4\ell}{\ell_+^2\,\ell_-^2} F(\ell_+)\,F(\ell_-)\Big\{ \quad \tau_2\left[2\left(\ell^2+\frac{p^2}{4}\right)\left((\ell\cdot p)^2 - p^2\ell^2\right)\right]$$
$$- \tau_3\left[2\left((\ell\cdot p)^2 - p^2\ell^2\right) + 3p^2\left(\ell^2 - \frac{p^2}{4}\right)\right]$$
$$- \tau_6\left[-6\,\ell\cdot p\left(\ell^2-\frac{p^2}{4}\right)\right]$$
$$- \tau_8\left[2\left((\ell\cdot p)^2 - p^2\ell^2\right)\quad\right]\Big\}\,. \qquad (3.3.37)$$

To proceed, we shall use the coefficient functions of basis tensors, τ_i, which were discussed in the last section, Eqn. (3.2.47). They can be written down for the momenta relevant to Fig. 3.3 in the form

$$\begin{aligned}
\tau_2 &= \frac{1}{2\,(\ell\cdot p)\,(\ell^2+p^2/4)}\;\tau_2' + \frac{1}{2\,(\ell^2+p^2/4)^2}\;\tau_2''\,,\\
\tau_3 &= \frac{1}{2\,\ell\cdot p}\;\tau_3' + \frac{1}{2\,(\ell^2+p^2/4)}\;\tau_3''\,,\\
\tau_6 &= \frac{1}{2\,(\ell^2+p^2/4)}\;\tau_6' + \frac{\ell\cdot p}{2\,(\ell^2+p^2/4)^2}\;\tau_6''\,,\\
\tau_8 &= \frac{1}{2\,\ell\cdot p}\;\tau_8' + \frac{1}{2\,(\ell^2+p^2/4)}\;\tau_8''\,.
\end{aligned} \qquad (3.3.38)$$

The difference that arises in this case is that these coefficients are now angle dependent. Now, we shall follow the same method to calculate T^γ as we did for the fermion equation. Hence separating T^γ into two pieces according to two different groups of combinations of the $F(\ell)$ and $G(\ell)$,

$$T^\gamma = T^{\gamma\,\prime}(\tau_i') + T^{\gamma\,\prime\prime}(\tau_i'')\,, \qquad (3.3.39)$$

where

$$\begin{aligned}
T^{\gamma\,\prime} &= \frac{\alpha N_F}{3\,\pi^3 p^2}\int_E \frac{d^4\ell}{\ell_+^2\,\ell_-^2} F(\ell_+)\,F(\ell_-)\\
&\quad\times\Bigg[\,\frac{1}{\ell\cdot p}\left((\ell\cdot p)^2 - p^2\ell^2\right)\left(\tau_2' - \tau_3' - \tau_8'\right)\\
&\qquad - \frac{3}{2}\,\frac{p^2}{\ell\cdot p}\,(\ell^2 - p^2/4)\,\tau_3' + 3\,\ell\cdot p\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)}\,\tau_6'\;\Bigg],
\end{aligned} \qquad (3.3.40)$$

$$
\begin{aligned}
T^{\gamma\,''} \;=\;& \frac{\alpha\, N_F}{3\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2\, \ell_-^2}\, F(\ell_+)\, F(\ell_-) \\
& \times \left\{ \frac{1}{(\ell^2 + p^2/4)} \left((\ell\cdot p)^2 - p^2 \ell^2 \right) \left(\tau_2'' - \tau_3'' - \tau_8'' \right) \right. \\
& \left. - \frac{3}{2\,(\ell^2 + p^2/4)} \left(\ell^2 - \frac{p^2}{4} \right) \left[p^2\, \tau_3'' - \frac{2(\ell\cdot p)^2}{(\ell^2 + p^2/4)}\, \tau_6'' \right] \right\} .
\end{aligned}
\tag{3.3.41}
$$

We first consider T' by referring to τ_i' in Eqn. (3.2.48).

$$
\tau_i' \;=\; -\,\alpha\, K_i \left(\ln \frac{\ell_+^2}{\Lambda^2} - \ln \frac{\ell_-^2}{\Lambda^2} \right) + \alpha^2 \left[J_i \left(\ln^2 \frac{\ell_+^2}{\Lambda^2} - \ln^2 \frac{\ell_-^2}{\Lambda^2} \right) + M_i \left(\ln \frac{\ell_+^2}{\Lambda^2} - \ln \frac{\ell_-^2}{\Lambda^2} \right) \right] .
\tag{3.3.42}
$$

Substituting it in $T^{\gamma\prime}$, we find

$$
\begin{aligned}
T^{\gamma\prime} \;=\;& \frac{\alpha\, N_F}{3\,\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2\, \ell_-^2}\, F(\ell_+^2)\, F(\ell_-^2) \\
\times\;& \left\{ \frac{1}{\ell\cdot p} \left((\ell\cdot p)^2 - p^2 \ell^2 \right) \right. \\
& \times \left[\left(-\alpha\, (K_2 - K_3 - K_8) + \alpha^2\, (M_2 - M_3 - M_8) \right) \left(\ln \frac{\ell_+^2}{\Lambda^2} - \ln \frac{\ell_-^2}{\Lambda^2} \right) \right. \\
& \left. + \alpha^2\, (J_2 - J_3 - J_8) \left(\ln^2 \frac{\ell_+^2}{\Lambda^2} - \ln^2 \frac{\ell_-^2}{\Lambda^2} \right) + \mathcal{O}(\alpha^3) \right] \\
& + \frac{3}{2\,\ell\cdot p} \left(\ell^2 - p^2/4 \right) \\
& \times \left[p^2\, (\alpha\, K_3 - \alpha^2\, M_3) \left(\ln \frac{\ell_+^2}{\Lambda^2} - \ln \frac{\ell_-^2}{\Lambda^2} \right) \right. \\
& \left. - \alpha^2\, p^2\, J_3 \left(\ln^2 \frac{\ell_+^2}{\Lambda^2} - \ln^2 \frac{\ell_-^2}{\Lambda^2} \right) + \mathcal{O}(\alpha^3) \right] \\
& + 3\,\ell\cdot p\, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)} \\
& \times \left[(-\alpha\, K_6 + \alpha^2\, M_6) \left(\ln \frac{\ell_+^2}{\Lambda^2} - \ln \frac{\ell_-^2}{\Lambda^2} \right) \right. \\
& \left. \left. + \alpha^2\, J_6 \left(\ln^2 \frac{\ell_+^2}{\Lambda^2} - \ln^2 \frac{\ell_-^2}{\Lambda^2} \right) + \mathcal{O}(\alpha^3) \right] \right\} .
\end{aligned}
\tag{3.3.43}
$$

Now making use of Eqn. (3.3.30), we can write T'_γ as :

$$
\begin{aligned}
T^{\gamma'} &= \frac{\alpha\, N_F}{3\,\pi^3 p^2} \int_E \frac{d^4\ell}{\ell_+^2\,\ell_-^2} F(\ell_+)\,F(\ell_-) \\
&\times \left\{ \frac{1}{\ell\cdot p} \left((\ell\cdot p)^2 - p^2\ell^2 \right) \right. \\
&\qquad \times \left[-\alpha\ (K_2 - K_3 - K_8) \left(2\,\frac{\ell\cdot p}{(\ell^2+p^2/4)} + \frac{2}{3}\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} + \cdots \right) \right. \\
&\qquad\quad + \alpha^2\,(M_2 - M_3 - M_8) \left(2\,\frac{\ell\cdot p}{(\ell^2+p^2/4)} + \frac{2}{3}\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} + \cdots \right) \\
&\qquad\quad \left. + \alpha^2\,(J_2 - J_3 - J_8) \left(-2\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} - \frac{5}{3}\,\frac{(\ell\cdot p)^5}{(\ell^2+p^2/4)^5} + \cdots \right) + \mathcal{O}(\alpha^3) \right] \\
&\quad + \frac{3}{2\,\ell\cdot p}(\ell^2 - p^2/4) \\
&\qquad \times \left[p^2\,(\alpha\,K_3 - \alpha^2\,M_3) \left(2\,\frac{\ell\cdot p}{(\ell^2+p^2/4)} + \frac{2}{3}\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} + \cdots \right) \right. \\
&\qquad\quad \left. - \alpha^2\,p^2\,J_3 \left(-2\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} - \frac{5}{3}\,\frac{(\ell\cdot p)^5}{(\ell^2+p^2/4)^5} + \cdots \right) + \mathcal{O}(\alpha^3) \right] \\
&\quad + 3\,\ell\cdot p\,\frac{(\ell^2 - p^2/4)}{(\ell^2+p^2/4)} \\
&\qquad \times \left[p^2\,(-\alpha\,K_6 + \alpha^2\,M_6) \left(2\,\frac{\ell\cdot p}{(\ell^2+p^2/4)} + \frac{2}{3}\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} + \cdots \right) \right. \\
&\qquad\quad \left.\left. + \alpha^2\,J_6 \left(-2\,\frac{(\ell\cdot p)^3}{(\ell^2+p^2/4)^3} - \frac{5}{3}\,\frac{(\ell\cdot p)^5}{(\ell^2+p^2/4)^5} + \cdots \right) \right] + \mathcal{O}(\alpha^3) \right\}. \qquad (3.3.44)
\end{aligned}
$$

To continue this process, we calculate the product of F's

$$
\begin{aligned}
& F(\ell_+)\,F(\ell_-) \\
&= (1 + 2\,\alpha\,A_{10} + \alpha^2\,A_{10}^2) + 2\,\left(\alpha\,A_{11} + \alpha^2\,A_{21} + \alpha^2\,A_{11}\,A_{10} \right) \ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
&\quad + \alpha^2\,(\,2\,A_{22} + A_{11}^2)\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
&\quad + \left(\alpha\,A_{11} + \alpha^2\,A_{21} + \alpha^2\,A_{11}\,A_{10} + 2\,\alpha^2\,A_{22}\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} + \alpha^2\,A_{11}^2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \right) \\
&\qquad\qquad \times \ln\left(1 - \frac{(\ell\cdot p)^2}{(\ell^2+p^2/4)^2} \right) \\
&\quad + \alpha^2\,A_{22}\left[\ln^2\left(1 + \frac{\ell\cdot p}{(\ell^2+p^2/4)} \right) + \ln^2\left(1 - \frac{\ell\cdot p}{(\ell^2+p^2/4)} \right) \right] \\
&\quad + \alpha^2\,A_{11}^2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\ \ln\left(1 + \frac{\ell\cdot p}{\ell^2+p^2/4} \right) \\
&\quad + \alpha^2\,A_{11}^2\,\ln\left(1 + \frac{\ell\cdot p}{\ell^2+p^2/4} \right) \ln\left(1 - \frac{\ell\cdot p}{\ell^2+p^2/4} \right) + \mathcal{O}(\alpha^3) \Big\}. \qquad (3.3.45)
\end{aligned}
$$

Using the series representation for $\ln\left(1 \pm (\ell \cdot p)/(\ell^2 + p^2/4)\right)$, Eqn. (3.2.25), and performing the angular integrals by using Appendix C , we can represent $T^{\gamma\prime}$ by

$$
\begin{aligned}
T^{\gamma\prime} \;=\; & \frac{2\,N_F}{3\,\pi^2 p^2} \int_0^{\Lambda^2} d\ell^2\, \ell^2 \\
\times & \Bigg\{ -\,2\,\alpha^2 \,(K_2 - K_3 - K_8) \left(1 + \,2\,\alpha\, A_{10} + \,2\,\alpha\, A_{11}\, \ln \frac{(\ell^2 + p^2/4)}{\Lambda^2} \right) \\
& \quad \times \left(\frac{K_{2,1}}{(\ell^2 + p^2/4)} - \frac{\ell^2 p^2\, K_{0,1}}{(\ell^2 + p^2/4)} + \frac{K_{4,1}}{3\,(\ell^2 + p^2/4)^3} - \frac{\ell^2 p^2\, K_{2,1}}{3\,(\ell^2 + p^2/4)^3} + \cdots \right) \\
& -\,2\,\alpha^3 \,A_{11}\,(K_2 - K_3 - K_8) \\
& \quad \times \Bigg(-\,\frac{K_{4,1}}{(\ell^2 + p^2/4)^3} + \frac{\ell^2 p^2\, K_{2,1}}{(\ell^2 + p^2/4)^3} - \frac{K_{6,1}}{3\,(\ell^2 + p^2/4)^5} + \frac{\ell^2 p^2\, K_{4,1}}{3\,(\ell^2 + p^2/4)^5} \\
& \qquad - \frac{K_{6,1}}{2\,(\ell^2 + p^2/4)^5} + \frac{\ell^2 p^2\, K_{4,1}}{2\,(\ell^2 + p^2/4)^5} - \frac{K_{8,1}}{6\,(\ell^2 + p^2/4)^7} + \frac{\ell^2 p^2\, K_{6,1}}{6\,(\ell^2 + p^2/4)^7} + \cdots \Bigg) \\
& +\,2\,\alpha^3 \,(M_2 - M_3 - M_8) \\
& \quad \times \left(\frac{K_{2,1}}{(\ell^2 + p^2/4)} - \frac{\ell^2 p^2\, K_{0,1}}{(\ell^2 + p^2/4)} + \frac{K_{4,1}}{3\,(\ell^2 + p^2/4)^3} - \frac{\ell^2 p^2\, K_{2,1}}{3\,(\ell^2 + p^2/4)^3} + \cdots \right) \\
& +\,2\,\alpha^3 \,(J_2 - J_3 - J_8) \\
& \quad \times \left(-\,\frac{K_{4,1}}{(\ell^2 + p^2/4)^3} + \frac{\ell^2 p^2\, K_{2,1}}{(\ell^2 + p^2/4)^3} - \frac{5\,K_{6,1}}{6\,(\ell^2 + p^2/4)^5} + \frac{5\ell^2 p^2\, K_{4,1}}{6\,(\ell^2 + p^2/4)^5} + \cdots \right) \\
& +\,p^2 (\alpha^2\, K_3 - \alpha^3\, M_3) \\
& \quad \times \left(3\,\frac{(\ell^2 - p^2/4)\, K_{0,1}}{(\ell^2 + p^2/4)} + \frac{(\ell^2 - p^2/4)\, K_{2,1}}{(\ell^2 + p^2/4)^3} + \cdots \right) \\
& +\,2\,(-\,\alpha^2\, K_6 + \,\alpha^3\, M_6) \\
& \quad \times \left(3\,\frac{(\ell^2 - p^2/4)\, K_{2,1}}{(\ell^2 + p^2/4)^2} + \frac{(\ell^2 - p^2/4)\, K_{4,1}}{(\ell^2 + p^2/4)^4} + \cdots \right) \\
& +\,\alpha^3\, p^2\, K_3 \left(\,2\, A_{10} + \,2\, A_{11}\, \ln \frac{\ell^2 + p^2/4}{\Lambda^2} \right) \\
& \quad \times \left(3\,\frac{(\ell^2 - p^2/4)\, K_{0,1}}{(\ell^2 + p^2/4)} + \frac{(\ell^2 - p^2/4)\, K_{2,1}}{(\ell^2 + p^2/4)^3} + \cdots \right) \\
& -\,2\,\alpha^3\, K_6 \left(\,2\, A_{10} + \,2\, A_{11}\, \ln \frac{\ell^2 + p^2/4}{\Lambda^2} \right) \\
& \quad \times \left(3\,\frac{(\ell^2 - p^2/4)\, K_{2,1}}{(\ell^2 + p^2/4)^2} + \frac{(\ell^2 - p^2/4)\, K_{4,1}}{(\ell^2 + p^2/4)^4} + \cdots \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha^3 A_{11}\, p^2\, K_3 \\
&\quad\times\Bigg(-3\,\frac{(\ell^2-p^2/4)\,K_{2,1}}{(\ell^2+p^2/4)^3} - \frac{5}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^5}\,K_{4,1} + \frac{1}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^5}\,K_{6,1} + \cdots\Bigg) \\
&-2\,\alpha^3 A_{11}\, K_6 \\
&\quad\times\Bigg(-\,\frac{(\ell^2-p^2/4)\,K_{4,1}}{(\ell^2+p^2/4)^3} - \frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^5}\,K_{5,1} + \cdots\Bigg) \\
&-\alpha^3\, p^2\, J_3 \\
&\quad\times\Bigg(-3\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^3}\,K_{2,1} - \frac{5}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^5}\,K_{4,1} + \cdots\Bigg) \\
&+\alpha^3\, J_6 \\
&\quad\times\Bigg(-6\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^4}\,K_{4,1} - 5\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^6}\,K_{6,1} + \cdots\Bigg) + \mathcal{O}(\alpha^4)\Bigg\} \quad .
\end{aligned}
\tag{3.3.46}
$$

Evaluating these angular integrals and concentrating on the $p^2 \to \Lambda^2$ region for the ℓ-integration, we obtain,

$$
\begin{aligned}
T^{\gamma\prime} \;=\;& \frac{2\,N_F}{3\,\pi p^2}\int_{p^2/4}^{\Lambda^2} d\ell^2 \\
\times\;& \Bigg\{ -\,2\,\alpha^2\,(K_2-K_3-K_8)\,\left(1+\,2\,\alpha\,A_{10}+\,2\,\alpha\,A_{11}\,\ln\frac{\ell^2}{\Lambda^2}\right) \\
&\qquad\qquad\times\left(\frac{1}{8}\,\frac{p^2}{(\ell^2+p^2/4)} - \frac{1}{2}\,\frac{p^2\,\ell^2}{(\ell^2+p^2/4)^2)}\right) \\
&+2\,\alpha^3\,(M_2-M_3-M_8)\,\left(\frac{1}{8}\,\frac{p^2}{(\ell^2+p^2/4)} - \frac{1}{2}\,\frac{p^2\,\ell^2}{(\ell^2+p^2/4)^2)}\right) \\
&+p^2\,\left(\alpha^2\,K_3-\alpha^3\,M_3\right)\quad\left(1+\,2\,\alpha\,A_{10}+\,2\,\alpha\,A_{11}\,\ln\frac{\ell^2}{\Lambda^2}\right) \\
&\qquad\qquad\times\left(\frac{3}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^2}\right) \\
&+2\,(-\alpha^2\,K_6+\,\alpha^3\,M_6)\quad\left(1+\,2\,\alpha\,A_{10}+\,2\,\alpha\,A_{11}\,\ln\frac{\ell^2}{\Lambda^2}\right) \\
&\qquad\qquad\times\left(\frac{3}{8}\,\frac{(\ell^2-p^2/4)\,p^2}{(\ell^2+p^2/4)^2}\right) + \mathcal{O}(\alpha^4)\Bigg\}\,.
\end{aligned}
\tag{3.3.47}
$$

Now we are going to evaluate the ℓ-integrals by using Appendix E,

$$T^{\gamma\prime} = \frac{2\,N_F}{3\pi p^2}\left\{ -\; 2\,\alpha^2\,(K_2 - K_3 - K_8)\,(t_1^L + t_4^L) - 4\,\alpha^3\,A_{11}\,(K_2 - K_3 - K_8)\,(t_2^L + t_5^L) + \frac{3\,p^2}{2}\,\alpha^2\,K_3\,t_7^L + 3\,p^2\,\alpha^3\,A_{11}\,K_3\,t_8^L - \frac{3\,p^2}{4}\,\alpha^2\,K_6\,t_7^L - \frac{3}{2}\,\alpha^3\,A_{11}\,K_6\,t_8^L + \mathcal{O}(\alpha^4)\right\}. \tag{3.3.48}$$

We eventually arrive at the result :

$$T^{\gamma\,\prime} = \frac{N_F}{3\pi}\left\{ \alpha^2\left[\left(-\frac{3}{2}\,(K_2 - K_3 - K_8) - 3\,K_3 + \frac{3}{2}\,K_6\right)\,\ln\frac{p^2}{\Lambda^2}\right] - \alpha^3\left[\left(\frac{3}{2}\,A_{11}\,(K_2 - K_3 - K_8) - 3\,A_{11}\,K_3 + \frac{3}{2}\,A_{11}\,K_6\right)\,\ln^2\frac{p^2}{\Lambda^2}\right] + \mathcal{O}(\alpha^4)\right\}. \tag{3.3.49}$$

We can start to deal with $T^{\gamma\prime\prime}$ which is,

$$T^{\gamma\,\prime\prime} = \frac{\alpha\,N_F}{3\pi^3 p^2}\int_E \frac{d^4\ell}{\ell_+^2\,\ell_-^2}\,F(\ell_+)\,F(\ell_-) \times\left\{ \frac{1}{(\ell^2 + p^2/4)}\,\left((\ell\cdot p)^2 - p^2\ell^2\right)\,(\tau_2'' - \tau_3'' - \tau_8'') - \frac{3}{2\,(\ell^2 + p^2/4)}\,\left(\ell^2 - \frac{p^2}{4}\right)\,\left[p^2\,\tau_3'' - \frac{2(\ell\cdot p)^2}{(\ell^2 + p^2/4)}\,\tau_6''\right]\right\}. \tag{3.3.50}$$

Recalling the definition of τ_i'' from Eqn. (3.2.49),

$$\tau_i'' = \alpha\,\left[K_i\,\left(\ln\frac{\ell_-^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) + H_i\right] + \alpha^2\left[J_i\,\left(\ln^2\frac{\ell_-^2}{\Lambda^2} + \ln^2\frac{\ell_+^2}{\Lambda^2}\right) + M_i\,\left(\ln\frac{\ell_-^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) + Q_i\,\ln\frac{\ell_-^2}{\Lambda^2}\,\ln\frac{\ell_+^2}{\Lambda^2}\right] + \mathcal{O}(\alpha^3), \tag{3.3.51}$$

and inserting this into Eqn. (3.3.50), we find :

$$
\begin{aligned}
T^{\gamma\prime\prime} &= \frac{\alpha N_F}{3\pi^3 p^2} \int \frac{d^4\ell}{\ell_+^2\,\ell_-^2}\, F(\ell_+)\, F(\ell_-) \\
&\times \Bigg\{ \frac{1}{(\ell^2+p^2/4)} \left((\ell\cdot p)^2 - p^2\ell^2\right) \\
&\qquad \times \Bigg[\alpha\,(K_2' - K_3' - K_8') \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) + \alpha\,(H_2' - H_3' - H_8') \\
&\qquad + \alpha^2\,(J_2' - J_3' - J_8') \left(\ln^2\frac{\ell_+^2}{\Lambda^2} + \ln^2\frac{\ell_+^2}{\Lambda^2}\right) \\
&\qquad + \alpha^2\,(M_2' - M_3' - M_8') \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) \\
&\qquad + \alpha^2\,(Q_2' - Q_3' - Q_8')\, \ln\frac{\ell_+^2}{\Lambda^2}\, \ln\frac{\ell_+^2}{\Lambda^2} \Bigg] \\
&\quad - \frac{3\,p^2}{2}\, \frac{(\ell^2 - p^2/4)}{(\ell^2+p^2/4)} \\
&\qquad \times \Bigg[\alpha\, K_3' \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) + \alpha\, H_3' \\
&\qquad + \alpha^2\, J_3' \left(\ln^2\frac{\ell_+^2}{\Lambda^2} + \ln^2\frac{\ell_+^2}{\Lambda^2}\right) + \alpha^2\, M_3' \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) \\
&\qquad + \alpha^2\, Q_3' \,\ln\frac{\ell_+^2}{\Lambda^2}\, \ln\frac{\ell_+^2}{\Lambda^2} \Bigg] \\
&\quad + 3\, \frac{(\ell^2 - p^2/4)}{(\ell^2+p^2/4)^2}\, (\ell\cdot p)^2 \\
&\qquad \times \Bigg[\alpha\, K_6' \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) + \alpha\, H_6' \\
&\qquad + \alpha^2\, J_6' \left(\ln^2\frac{\ell_+^2}{\Lambda^2} + \ln^2\frac{\ell_+^2}{\Lambda^2}\right) + \alpha^2\, M_6' \left(\ln\frac{\ell_+^2}{\Lambda^2} + \ln\frac{\ell_+^2}{\Lambda^2}\right) \\
&\qquad + \alpha^2\, Q_6' \,\ln\frac{\ell_+^2}{\Lambda^2}\, \ln\frac{\ell_+^2}{\Lambda^2} \Bigg] \Bigg\} . \qquad (3.3.52)
\end{aligned}
$$

Making use of Eqn (3.3.20) and referring to Appendix C for the angular integration, we can rewrite the above equation as,

$$
\begin{aligned}
T^{\gamma''} &= \frac{2\,N_F}{3\pi^2 p^2}\int_0^{\Lambda^2} d\ell^2\,\ell^2 \\
&\times\Bigg\{\frac{1}{(\ell^2+p^2/4)}\,(1+2\,\alpha\,A_{10}) \\
&\quad\times\Bigg[\;\alpha^2\,(H_2'-\,H_3'-\,H_8')\,(K_{2,1}-p^2\ell^2\,K_{0,1}) \\
&\quad\quad+\Big(\alpha^2\,(K_2'-\,K_3'-\,K_8')+\,\alpha^3\,(M_3'-\,M_3'-\,M_8')\Big) \\
&\quad\quad\quad\times\left\{2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\,(K_{2,1}-p^2\ell^2\,K_{0,1})-\cdots\right\} \\
&\quad\quad+\alpha^3\,(J_3'-\,J_3'-\,J_8')\left\{2\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,(K_{2,1}-p^2\ell^2\,K_{0,1})+\cdots\right\} \\
&\quad\quad+\alpha^3\,(Q_3'-\,Q_3'-\,Q_8')\left\{\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,(K_{2,1}-p^2\ell^2\,K_{0,1})+\cdots\right\}+\,\mathcal{O}(\alpha^4)\Bigg] \\
&\quad+\frac{1}{(\ell^2+p^2/4)}\,\alpha\,A_{11} \\
&\quad\times\Bigg[\alpha^2\,(H_2'-\,H_3'-\,H_8')\left\{2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}(K_{2,1}-p^2\ell^2\,K_{0,1})-\cdots\right\} \\
&\quad\quad+\alpha^2\,(K_2'-\,K_3'-\,K_8')\left\{4\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}(K_{2,1}-p^2\ell^2\,K_{0,1})+\cdots\right\}+\,\mathcal{O}(\alpha^3)\Bigg] \\
&\quad-\frac{3\,p^2}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)}\,(1+2\,\alpha\,A_{10}) \\
&\quad\times\Bigg[\alpha^2\,H_3'\,K_{0,1}+\Big(\alpha^2\,K_3'+\,\alpha^3\,M_3'\Big)\left\{2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{0,1}-\cdots\right\}+\cdots\Bigg] \\
&\quad\quad+\alpha^3\,J_3'\,\left\{2\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{0,1}+\cdots\right\}+\,\alpha^3\,Q_3'\left\{\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{0,1}+\cdots\right\}+\,\mathcal{O}(\alpha^4)\Bigg] \\
&\quad-\frac{3\,p^2}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)}\,\alpha\,A_{11}\left[2\,\alpha^2\,H_3'\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{0,1}+\,4\,\alpha^2\,K_3'\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{0,1}+\,\mathcal{O}(\alpha^3)\right] \\
&\quad+3\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^2}\,(1+2\,\alpha\,A_{10}) \\
&\quad\times\Bigg[\alpha^2\,H_6'\,K_{2,1}+\Big(\alpha^2\,K_6'+\,\alpha^3\,M_6'\Big)\left\{2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{2,1}-\cdots\right\} \\
&\quad\quad+\alpha^3\,J_6'\left\{2\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{2,1}+\cdots\right\}+\,\alpha^3\,Q_6'\left\{\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{2,1}+\cdots\right\}+\,\mathcal{O}(\alpha^4)\Bigg] \\
&\quad+3\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^2}\,\alpha\,A_{11}\left[2\,\alpha^2\,H_6'\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{2,1}+\,4\,\alpha^2\,K_6'\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2}\,K_{2,1}-\,\mathcal{O}(\alpha^3)\right]\Bigg\}.
\end{aligned}
\qquad (3.3.53)
$$

On evaluating the integrals, we obtain :

$$
\begin{aligned}
T^{\gamma\,\prime\prime} \;=\;& \frac{2\,N_F}{3\pi p^2}\int_0^{\Lambda^2} d\ell^2\,\ell^2 \\
\times & \Bigg\{ \frac{1}{(\ell^2+p^2/4)} \\
& \times \Bigg[\alpha^2\,(H_2'-H_3'-H_8') + 2\,\alpha^2\,(K_2'-K_3'-K_8')\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
& + 2\,\alpha^3\,A_{10}\,(H_2'-H_3'-H_8') \\
& + \alpha^3\Big((M_3'-M_3'-M_8') + 2\,A_{10}\,(K_3'-K_3'-K_8') + A_{11}\,(H_2'-H_3'-H_8')\Big) \\
& \qquad\qquad \times 2\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
& + \alpha^3\,\big(2\,(J_2'-J_3'-J_8') + (Q_2'-Q_3'-Q_8') + 4\,A_{11}\,(K_2'-K_3'-K_8')\big) \\
& \qquad\qquad \times \ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2} + \mathcal{O}(\alpha^4)\Bigg] \\
& \qquad \times \left\{ \frac{p^2}{8\,\ell^2}\,\theta_+^\gamma + \frac{2\,\ell^2}{p^2}\,\theta_-^\gamma - \frac{p^2}{2\,(\ell^2+p^2/4)}\,\theta_+^\gamma - \frac{2\,\ell^2}{(\ell^2+p^2/4)}\theta_-^\gamma \right\} \\
& - \frac{3\,p^2}{2}\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)} \\
& \times \Bigg[\alpha^2\,H_3' + 2\,\alpha^2\,K_3'\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} + 2\,\alpha^3\,A_{10}\,H_3' \\
& + \Big(2\,\alpha^3\,M_3' + 4\,A_{10}\,K_3' + 2\,\alpha^3\,A_{11}\,H_3'\Big)\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
& + \Big(2\,\alpha^3\,J_3' + \alpha^3\,Q_3' + 4\,\alpha^3\,A_{11}\,K_3'\Big)\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2} + \mathcal{O}(\alpha^4)\Bigg] \\
& \qquad \times \left\{ \frac{1}{2\,\ell^2\,(\ell^2+p^2/4)}\,\theta_+^\gamma + \frac{2}{p^2\,(\ell^2+p^2/4)}\,\theta_-^\gamma \right\} \\
& + 3\,\frac{(\ell^2-p^2/4)}{(\ell^2+p^2/4)^2} \\
& \times \Bigg[\alpha^2\,H_6' + 2\,\alpha^2\,K_6'\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} + 2\,\alpha^3\,A_{10}\,H_6' \\
& + \Big(2\,\alpha^3\,M_6' + 4\,A_{10}\,K_6' + 2\,\alpha^3\,A_{11}\,H_6'\Big)\,\ln\frac{(\ell^2+p^2/4)}{\Lambda^2} \\
& + \Big(2\,\alpha^3\,J_6' + \alpha^3\,Q_6' + 4\,\alpha^3\,A_{11}\,K_6'\Big)\,\ln^2\frac{(\ell^2+p^2/4)}{\Lambda^2} + \mathcal{O}(\alpha^4)\Bigg] \\
& \qquad \times \left\{ \frac{p^2}{8\,\ell^2}\,\theta_+^\gamma + \frac{2\,\ell^2}{p^2}\,\theta_-^\gamma \right\} \quad . \qquad\qquad (3.3.54)
\end{aligned}
$$

Now we perform the ℓ-integral, referring to Appendix E ,

$$
\begin{aligned}
T^{\gamma\prime\prime} \;=\; & \frac{2\,N_F}{3\pi p^2} \\
\times\Bigg\{ & \Bigg[\ln\frac{p^2}{4\,\Lambda^2}\,\Big(2\,\alpha^2\,(K_2' - K_3' - K_8')\Big) \\
& + \ln^2\frac{p^2}{4\,\Lambda^2}\Big(2\,\alpha^3\,(J_2' - J_3' - J_8') \\
& \qquad + \alpha^3\,(Q_2' - Q_3' - Q_8') + 4\,\alpha^3\,A_{11}\,(K_2' - K_3' - K_8')\Big)\Bigg]\,(t_1^P + t_2^P) \\
& -\frac{3\,p^2}{2}\Bigg[\ln\frac{p^2}{4\,\Lambda^2}\,\Big(2\,\alpha^2\,K_3'\Big) \\
& + \ln^2\frac{p^2}{4\,\Lambda^2}\Big(2\,\alpha^3\,J_3' + \alpha^3 Q_3' + 4\,\alpha^3\,A_{11}\,K_3'\Big)\Bigg]\,t_3^P \\
& +3\,\Bigg[\ln\frac{p^2}{4\,\Lambda^2}\,\Big(2\,\alpha^2\,K_6'\Big) \\
& + \ln^2\frac{p^2}{4\,\Lambda^2}\,\Big(2\,\alpha^3\,J_6' + \alpha^3\,Q_6' + 4\,\alpha^3\,A_{11}\,K_6'\Big)\Bigg]\,t_4^P \\
& + \Bigg[\alpha^2\,(H_2' - H_3' - H_8')\,(t_1^L + t_4^L) \\
& \quad + \Big(2\,(\alpha^2 + 2\,\alpha^3\,A_{10})\,(K_2' - K_3' - K_8') + 2\,\alpha^3\,(M_2' - M_3' - M_8') \\
& \quad + 2\,\alpha^3\,A_{11}\,(H_2' - H_3' - H_8')\Big)\,(t_2^L + t_5^L) \\
& \quad + \alpha^3\Big(2\,(J_2' - J_3' - J_8') + (Q_2' - Q_3' - Q_8') \\
& \quad + 4\,A_{11}\,(K_2' - K_3' - K_8')\Big)\,(t_3^L + t_6^L)\Bigg] \\
& -\frac{3\,p^2}{4}\Bigg[\alpha^2\,H_3'\,t_7^L + \Big(2\,(\alpha^2 + 2\,\alpha^3\,A_{10})\,K_3' + 2\,\alpha^3\,M_3' + 2\,\alpha^3\,A_{11}\,H_3'\Big)\,t_8^L \\
& \quad + \alpha^3\,\big(2\,J_3' + Q_3' + 4\,A_{11}\,K_3'\big)\,t_9^L\Bigg] \\
& +\frac{3\,p^2}{8}\Bigg[\alpha^2\,H_6'\,t_7^L + \Big(2\,(\alpha^2 + 2\,\alpha^3\,A_{10})\,K_6' + 2\,\alpha^3\,M_6' + 2\,\alpha^3\,A_{11}\,H_6'\Big)\,t_8^L \\
& \quad + \alpha^3\,\big(2\,J_6' + Q_6' + 4\,A_{11}\,K_6'\big)\,t_9^L\Bigg] \\
& + \mathcal{O}(\alpha^4)\Bigg\} \quad . \qquad (3.3.55)
\end{aligned}
$$

Evaluating the above integrals, we arrive at the end of the calculation :

$$
\begin{aligned}
T^{\gamma\,\prime\prime} \;=\; & \frac{N_F}{3\pi} \\
\times\Bigg\{ \quad & \alpha^2 \ln\frac{p^2}{\Lambda^2} \\
& \times \Bigg[(K_2' - K_3' - K_8')\left(-\frac{9}{4} + 3\ln 2\right) + K_3'(-3 + 6\ln 2) + K_6'\left(-\frac{27}{4} + 9\ln 2\right) \\
& \quad + \frac{3}{4}(H_2' - H_3' - H_8') + \frac{3}{2}H_3' - \frac{3}{4}H_6' \Bigg] \\
& + \alpha^2 \ln^2\frac{p^2}{\Lambda^2} \\
& \times \left[\frac{3}{4}(K_2' - K_3' - K_8') + \frac{3}{2}K_3' - \frac{3}{4}K_6' \right] \\
& + \alpha^3 \ln^3\frac{p^2}{\Lambda^2} \\
& \times \Bigg[\frac{1}{2}(J_2' - J_3' - J_8') + J_3' - \frac{1}{2}J_6' \\
& \quad + A_{11}(K_2' - K_3' - K_8') + 2A_{11}K_3' - A_{11}K_6' \\
& \quad + \frac{1}{4}(Q_2' - Q_3' - Q_8') + \frac{1}{2}Q_3' - \frac{1}{4}Q_6' \Bigg] \\
& + \alpha^3 \ln^2\frac{p^2}{\Lambda^2} \\
& \times \Bigg[(J_2' - J_3' - J_8')\left(-\frac{9}{4} + 3\ln 2\right) + J_3'(-3 + 6\ln 2) + J_6'\left(-\frac{27}{4} + 9\ln 2\right) \\
& \quad + A_{11}(K_2' - K_3' - K_8')\left(-\frac{9}{2} + 6\ln 2\right) \\
& \quad + A_{11}K_3'(-6 + 12\ln 2) + A_{11}K_6'\left(-\frac{27}{2} + 18\ln 2\right) \\
& \quad + \frac{3}{4}(M_2' - M_3' - M_8') + \frac{3}{2}M_3' - \frac{3}{4}M_6' \\
& \quad + A_{10}\frac{3}{2}(K_2' - K_3' - 2K_8') + A_{10}3K_3' - A_{10}\frac{3}{2}K_6' \\
& \quad + (Q_2' - Q_3' - Q_8')\left(-\frac{9}{8} + \frac{3}{2}\ln 2\right) + Q_3'\left(-\frac{3}{2} + 3\ln 2\right) + Q_6'\left(-\frac{27}{8} + \frac{9}{2}\ln 2\right) \\
& \quad + \frac{3}{4}A_{11}(H_2' - H_3' - H_8') + \frac{3}{2}A_{11}H_3' - \frac{3}{4}A_{11}H_6' \Bigg] + \mathcal{O}(\alpha^4) \Bigg\}
\end{aligned}
\tag{3.3.56}
$$

To bring these four results together, we recall the formula below, Eqn. (3.3.14),

$$
\frac{1}{G(p^2)} \;=\; 1 - (L_A^\gamma + L_B^\gamma + T^{\gamma\,\prime} + T^{\gamma\,\prime\prime}) \quad . \tag{3.3.14}
$$

We can then write $1/G(p)$ in the following form, analogous to the result for $1/F(p)$ in the fermion case, Eqn. (3.2.62),

$$
\begin{aligned}
\frac{1}{G(p^2)} &= 1 + \frac{N_F}{3\,\pi} \\
\times \Bigg\{ & - \alpha \ln \frac{\mathbf{p}^2}{\Lambda^2} + \alpha\,\frac{13}{12} \\
& - \alpha^2 \ln^2 \frac{\mathbf{p}^2}{\Lambda^2} \\
& \qquad \times \left[\frac{A_{11}}{2} + \frac{3}{4}\left(K'_2 + K'_3 - K'_6 - K'_8\right) \right] \\
& - \alpha^2 \ln \frac{\mathbf{p}^2}{\Lambda^2} \\
& \qquad \times \Bigg[\left(A_{10} - \frac{11}{12} A_{11} \right) + \frac{3}{2}\left(K_2 + K_3 - K_6 - K_8\right) \\
& \qquad + \left(\frac{-9}{4} + 3 \ln 2 \right) \left(K'_2 + K'_3 - K'_6 - K'_8\right) + \frac{3}{2} K'_3 + \left(-9 + 12 \ln 2\right) K'_6 \\
& \qquad + \frac{3}{4}\left(H'_2 + H'_3 - H_6 - H'_8\right) \Bigg] \\
& - \alpha^3 \ln^3 \frac{\mathbf{p}^2}{\Lambda^2} \\
& \qquad \times \Bigg[\frac{A_{22}}{3} + \frac{1}{2}\left(J'_2 + J'_3 - J'_6 - J'_8\right) \\
& \qquad + A_{11}\left(K'_2 + K'_3 - K'_6 - K'_8\right) + \frac{1}{4}\left(Q'_2 + Q'_3 - Q'_6 - Q'_8\right) \Bigg] \\
& - \alpha^3 \ln^2 \frac{\mathbf{p}^2}{\Lambda^2} \\
& \qquad \times \Bigg[\left(\frac{A_{21}}{2} - \frac{11}{12} A_{22} \right) - \frac{3}{2} A_{11}\left(K_2 + K_3 - K_6 - K_8\right) \\
& \qquad + \left(\frac{-9}{4} + 3 \ln 2 \right) \left(J'_2 + J'_3 - J'_6 - J'_8\right) + \frac{3}{2} J'_3 + \left(-9 + 12 \ln 2\right) J'_6 \\
& \qquad + \left(\frac{-9}{2} + 6 \ln 2 \right) A_{11}\left(K'_2 + K'_3 - K'_6 - K'_8\right) \\
& \qquad + 3\,A_{11}\,K'_3 + A_{11}\left(-18 + 24 \ln 2\right) K'_6 \\
& \qquad + \frac{3}{4}\left(M'_2 + M'_3 - M'_6 - M'_8\right) + \frac{3}{2} A_{10}\left(K'_2 + K'_3 - K'_6 - K'_8\right) \\
& \qquad + \frac{3}{4} A_{11}\left(H'_2 + H'_3 - H'_6 - H'_8\right) + \left(\frac{-9}{8} + \frac{3}{2} \ln 2 \right) \left(Q'_2 + Q'_3 - Q'_6 - Q'_8\right) \\
& \qquad + \frac{3}{4} Q'_3 + \left(\frac{-9}{2} + 6 \ln 2 \right) Q'_6 \Bigg] - \mathcal{O}(\alpha^4) \Bigg\}
\end{aligned}
\tag{3.3.57}
$$

Until now we have considered the SD-equation for fermion and photon wave-function renormalisations. As seen in Eqns. (3.2.62. 3.3.57), these equations depend on the constants which appear in the coefficient functions, τ's of the basis vectors. These constants determine which combinations of F and G we are talking about in the transverse vertex construction. Multiplicative renormalisability imposes conditions on the fermion-boson vertex through these constants. In order to find these constraints, we shall first work out in the next chapter what the general multiplicatively renormalisable forms of the fermion and photon functions are.

Chapter 4

How Multiplicative Renormalisation Puts Conditions on the Fermion and Photon Wave-function Renormalisations

If you wish to advance into the infinite,
explore the finite in all directions.
-Goethe-

4.1 Introduction

The divergent nature of QFT requires renormalisation [40, 7, 4, 39]. This is because in almost every field theory, the corresponding Feynman graphs have loops which give ultraviolet divergent integrals. In order to have a meaningful theory, these infinities have to be removed, so that we are left with finite quantities which are observable. This renormalisation can be carried out in a number of different ways. The divergences seen in Feynman diagrams are thought to be caused by the infinite nature of certain bare (unphysical and thus unmeasurable) quantities such as the coupling constant, mass and Green's functions etc. A redefinition of these bare quantities in terms of physical, measurable quantities removes the divergences. After this redefinition, we have physical parameters with finite measurable values. If this process is successful for the relevant field theory by using any renormalisation scheme, then we say this theory is renormalisable.

The cornerstone of the renormalisation is regularisation [40, 4, 5, 39]. This means we can manipulate the divergent integrals to give a finite answer in terms of the regularisation parameters which depend on the regularisation method used. There are several regularisation techniques, for instance, dimensional regularisation, cut-off regularisation etc :
Cut-off regularisation [5, 7] means we replace the infinite upper limit of the divergent momentum integral with some ultraviolet finite cut-off parameter, Λ. The result is finite but will depend on this parameter.
Dimensional Regularisation [4, 7] means we evaluate the divergent integrals in a d-dimensional space. This introduces an infinitesimal parameter $\epsilon = d - 4$. This often results in a pole at $d = 4$.

After regularising the theory, it must be renormalised. Whichever regularisation scheme is used the renormalised quantities must be independent of the regularisation technique, i.e. the parameters, Λ, ϵ. As was mentioned above there are several techniques to make the theory renormalisable to all orders in perturbation theory. We mention two equivalent schemes: the counterterm and multiplicative renormalisations. For the **counterterm renormalisation** we begin with the physical Lagrangian defined in terms of physical quantities. New terms proportional to the original ones (counter terms) are added directly to the Lagrangian in order to cancel the divergent parts. After this cancellation, we have a finite theory.

The renormalisation procedure considered in this thesis is multiplicative renormalisation [4, 28, 22]. In **multiplicative renormalisation** the divergent sum over all the relevant Feynman diagrams are absorbed into the redefinition of physical quantities and Green's functions. We shall consider multiplicative renormalisation in the case of QED since this is main interest of the thesis.

4.2 Multiplicative Renormalisability of QED

In QED, the full-propagators and the vertex function are all divergent. However, by introducing the following definition of the finite (renormalised) propagators and the vertex function, we can absorb these divergences into functions, Z_1, Z_2 and Z_3 which are infinite sums of the corresponding, divergent Feynman graphs. We remove the divergences at all orders by introducing fields [40, 4, 7] :

$$\Psi_R = Z_2^{-1/2}\,\Psi_0\,, \qquad A_R^\mu = Z_3^{-1/2}\,A_0^\mu\,, \qquad \Gamma_R = Z_1^{-1}\,\Gamma_0\,, \tag{4.2.1}$$

where Z_1, Z_2 and Z_3 are the vertex, fermion and photon renormalisation constants, respectively. Subscripts $_R$ and 0 denote renormalised and bare quantities. The divergence of the fermion propagator is absorbed into Z_2 :

$$S_R(p,\mu) \;=\; Z_2^{-1}(\mu,\Lambda)\,S_0(p,\Lambda) \quad , \tag{4.2.2}$$

and similarly for the photon function :

$$\Delta^R_{\mu\nu}(p) \;=\; Z_3^{-1}\,\Delta^0_{\mu\nu}(p)\,. \tag{4.2.3}$$

The gauge covariance of the photon propagator requires that the covariant gauge parameter is similarly renormalised.

$$\xi_R \;=\; Z_3^{-1}\,\xi\,. \tag{4.2.4}$$

Analogously to the others, the divergence of the vertex function is canceled by the factor Z_1 :

$$\Gamma^R_\mu = Z_1^{-1}\,\Gamma^0_\mu\,, \tag{4.2.5}$$

with the above definitions, the coupling constant is renormalised according to,

$$e_R = \frac{Z_2}{Z_1}\sqrt{Z_3}\, e\,. \tag{4.2.6}$$

Making use of the Ward-Takahashi identity [7, 40] :

$$Z_1 = Z_2\,, \tag{4.2.7}$$

the coupling constant renormalisation becomes,

$$e_R = Z_3^{1/2}\, e\,. \tag{4.2.8}$$

4.3 The General Multiplicatively Renormalisable $G(p^2)$

In this section, we shall look for the most general form of the multiplicatively renormalisable photon wave-function renormalisation. In order to do so, the renormalised G_R can be written in the following form by using Eqn. (4.2.3) :

$$G_R(p,\mu) \;=\; Z_3^{-1}(\mu,\Lambda)\, G_0(p,\Lambda)\,. \tag{4.3.1}$$

Now, introducing the following quantities, for convenience,

$$P \;=\; \ln\frac{p^2}{\mu^2}\,, \qquad M \;=\; \ln\frac{\mu^2}{\Lambda^2}\quad,$$

and

$$P+M \;=\; \ln\frac{p^2}{\Lambda^2}\quad, \tag{4.3.2}$$

we define the most general perturbative expansion of the unrenormalised photon wave-function renormalisation up to next-to-leading logarithms by * :

$$\begin{aligned} G_0(p,\Lambda) \;=\;& 1+\alpha_0\left(B_{11}\,(P+M)+B_{10}\right)+\,\alpha_0^2\left(B_{22}\,(P+M)^2+B_{21}\,(P+M)\right) \\ &+\;\alpha_0^3\left(B_{33}\,(P+M)^3+B_{32}\,(P+M)^2\right)+\alpha_0^4\left(B_{44}\,(P+M)^4+B_{43}\,(P+M)^3\right) \\ &+\;\mathcal{O}(\alpha_0^5)\,. \end{aligned} \tag{4.3.3}$$

*The notation α_0 has been adopted instead of α, which was used in previous chapters, in order to stress that the coupling is bare

The renormalised photon wave-function constant is :

$$Z_3^{-1} = 1 + \alpha_0 (z_{11} M + z_{10}) + \alpha_0^2 (z_{22} M^2 + z_{21} M) + \alpha_0^3 (z_{33} M^3 + z_{32} M^2) + \alpha_0^4 (z_{44} M^4 + z_{43} M^3) + \mathcal{O}(\alpha^5) , \qquad (4.3.4)$$

and since $\alpha = e^2/4\pi$, using Eqn. (4.2.8), the bare coupling constant, in terms of renormalised quantities, is :

$$\alpha_0 = \alpha_R \left(1 + \alpha_R(\chi_{11} M + \chi_{10}) + \alpha_R^2 (\chi_{22} M^2 + \chi_{21} M) + \alpha_R^3 (\chi_{33} M^3 + \chi_{32} M^2) + \cdots\cdots\right) , \qquad (4.3.5)$$

where B's, Z's and χ's are constants to be determined. After having prepared all the necessary ingredients, we can renormalise the photon wave-function renormalisation at $\mu^2 = \Lambda^2$,

$$\begin{aligned} \mu^2 = \Lambda^2 \quad &\Longrightarrow \quad M = \ln\frac{\mu^2}{\Lambda^2} = 0 \quad , \\ &\Downarrow \\ Z_3(\Lambda^2, \Lambda^2) \quad &= \quad constant \quad , \\ &\Downarrow \\ G_R(p,\mu) \quad &= \quad G_0(p,\Lambda)\Big|_{\mu^2=\Lambda^2} \quad . \end{aligned} \qquad (4.3.6)$$

Therefore, as a left hand side of the Eqn. (4.2.3), we can take the following form for the renormalised photon wave-function renormalisation :

$$G_R(p,\mu) = 1 + \alpha_R (B_{11} P + B_{10}) + \alpha_R^2 \left(B_{22} P^2 + B_{21} P\right) + \alpha_R^3 \left(B_{33} P^3 + B_{32} P^2\right) + \alpha_R^4 \left(B_{44} P^4 + B_{43} P^3\right) + \mathcal{O}(\alpha_R^5) . \qquad (4.3.7)$$

To find the right hand side of Eqn. (4.2.3), we multiply G_0, Eqn. (4.3.3), by Z_3^{-1}, Eqn. (4.3.4). Then to convert the bare coupling constant to the renormalised one, we insert Eqn. (4.3.5) into it. Order by order comparison of this result for different powers of M with Eqn. (4.3.7) yields :

$$\alpha_R\, M \quad : \quad B_{11} + z_{11} = 0\ ,$$

$$\alpha_R \,\mathrm{const} \quad : \quad B_{10} + Z_{10} = 0\ ,$$

$$\alpha_R^2\, M^2 \quad : \quad B_{11}\, z_{11} + B_{11}\, \chi_{11} + B_{22} + z_{11}\, \chi_{11} + z_{22} = 0\ ,$$

$$\alpha_R^2\, M\, P \quad : \quad B_{11}\, \chi_{11} + 2\, B_{22} + B_{11}\, z_{11} = 0\ ,$$

$$\alpha_R^2\, M \quad : \quad B_{11}\, z_{10} + B_{10}\, \chi_{11} + z_{10}\, \chi_{11} + B_{10}\, z_{11} z_{21} + B_{21} + B_{11}\, \chi_{10} + z_{11}\, \chi_{10} = 0\ ,$$

$$\begin{aligned}\alpha_R^3\, M^3 \quad : \quad & 2\, z_{22}\, \chi_{11} + B_{33} + z_{33} + B_{11}\, \chi_{22} + B_{22}\, z_{11} + B_{11}\, z_{22} + 2\, B_{11}\, z_{11}\, \chi_{11} \\ & + 2\, B_{22}\, \chi_{11} + z_{11}\, \chi_{22} = 0\ ,\end{aligned}$$

$$\alpha_R^3\, M^2\, P \quad : \quad 2\, B_{22}\, z_{11} + B_{11}\, \chi_{22} + 4\, B_{22}\, \chi_{11} + B_{11}\, z_{22} + 2\, B_{11}\, z_{11}\, \chi_{11} + 3\, B_{33} = 0\ ,$$

$$\alpha_R^3\, M\, P^2 \quad : \quad 3\, B_{33} + 2\, B_{22}\, \chi_{11} + B_{22}\, z_{11} = 0\ ,$$

$$\begin{aligned}\alpha_R^3\, M^2 \quad : \quad & B_{11}\, \chi_{21} + B_{21}\, Z_{11} + 2\, B_{21}\, \chi_{11} + B_{10} \chi_{22} + 2\, B_{11}\, z_{10}\, \chi_{11} + 2\, z_{21} \chi_{11} \\ & + z_{11}\, \chi_{21} + B_{10}\, z_{22} + 2\, B_{11}\, z_{11}\, \chi_{10} + B_{32} + 2\, B_{10}\, z_{11}\, \chi_{11} + z_{10}\, \chi_{22} + z_{32} \\ & + 2\, B_{22}\, \chi_{10} + B_{11}\, z_{21} + B_{22}\, z_{10} + 2\, z_{22}\, \chi_{10} = 0\ ,\end{aligned}$$

$$\begin{aligned}\alpha_R^3\, M\, P \quad : \quad & 2\, B_{32} + B_{11}\, z_{21} + 2\, B_{22}\, z_{10} + 2\, B_{11}\, z_{10}\, \chi_{11} + 2\, B_{21}\, z_{11} \\ & + 2\, B_{11}\, z_{11}\, \chi_{10} + 4\, B_{22}\, \chi_{10} + B_{11}\, \chi_{21} = 0\ .\end{aligned} \tag{4.3.8}$$

Solving the above equations for Z's, B's and χ's gives the following relations :

$$\begin{aligned} z_{11} &= -B_{11}\ , \\ z_{10} &= -B_{10}\ , \\ z_{22} &= \frac{B_{11}}{2}\,(B_{11} + \chi_{11})\ , \\ z_{21} &= 2\, B_{10}\, B_{11} - B_{21}\ , \\ z_{32} &= -\frac{3}{2}\, B_{10}\, B_{11}\, \chi_{11} - \frac{3}{2}\, B_{10}\, B_{11}^2 + B_{21}\, B_{11} - B_{11}\, \chi_{10}\, \chi_{11} + B_{21}\, \chi_{11} + \frac{1}{2}\, \chi_{21}\ , \\ z_{33} &= -\frac{B_{11}^3}{6} - \frac{B_{11}^2\, \chi_{11}}{2} - \frac{B_{11}\, \chi_{11}^2}{2}\ , \end{aligned}$$

$$
\begin{aligned}
B_{22} &= \frac{B_{11}}{2}(B_{11} - \chi_{11}) , \\
B_{33} &= \frac{1}{3}\left(\frac{B_{11}^2}{2} - \frac{3}{2} B_{11}^2 \chi_{11} + B_{11} \chi_{11}^2\right) , \\
B_{32} &= B_{11} \chi_{10} \chi_{11} + B_{21} B_{11} + \frac{1}{2} B_{10} B_{11} \chi_{11} - \frac{1}{2} B_{10} B_{11}^2 - B_{21} \chi_{11} - \frac{1}{2} B_{11} \chi_{21} , \\
\\
\chi_{22} &= \chi_{11}^2 , \\
\chi_{33} &= \chi_{11}^3 , \\
\chi_{32} &= -2 \chi_{11}^2 \chi_{10} + \frac{5}{2} \chi_{11} \chi_{21} .
\end{aligned} \tag{4.3.9}
$$

These relations are between z's–B's and B's–χ's. We also need to know the relation between z's and χ's. Hence, this time we compare Eqn (4.3.5), order by order, with the definition of the coupling constant given in Eqn. (4.2.8). Then we find :

$$
\begin{aligned}
\alpha_R M &: \quad \chi_{11} = z_{11} , \\
\alpha_R &: \quad \chi_{10} = z_{10} , \\
\alpha_R^2 M^2 &: \quad \chi_{22} = z_{11} \chi_{11} + z_{22} , \\
\alpha_R^2 M &: \quad \chi_{21} = z_{11} \chi_{10} + z_{10} \chi_{11} + z_{21} , \\
\alpha_R^3 M^3 &: \quad \chi_{33} = z_{11} \chi_{22} + 2 z_{22} \chi_{11} + z_{33} , \\
\alpha_R^3 M^2 &: \quad \chi_{32} = z_{11} \chi_{21} + z_{10} \chi_{22} + 2 z_{22} \chi_{10} + 2 z_{21} \chi_{11} + z_{32} .
\end{aligned} \tag{4.3.10}
$$

On collecting all the above relations, we can display them as :

$$
\begin{aligned}
\chi_{11} &= z_{11} = -B_{11} , \\
\chi_{10} &= z_{10} = -B_{10} , \\
\chi_{22} &= \chi_{11}^2 = B_{11}^2 , \\
\chi_{21} &= 4 B_{10} B_{11} - B_{21} , \\
\chi_{33} &= \chi_{11}^3 = -B_{11}^3 , \\
\chi_{32} &= -8 B_{10} B_{11}^2 + \tfrac{5}{2} B_{11} B_{21} .
\end{aligned}
$$

$$
\begin{aligned}
B_{22} &= B_{11}^2 \,, \\
B_{33} &= B_{11}^3 \,, \\
B_{32} &= \frac{5}{2} B_{11} B_{21} - 2 B_{10} B_{11}^2 \,, \\
z_{22} &= 0 \,, \\
z_{33} &= 0 \,, \\
z_{21} &= 2 B_{10} B_{11} - B_{21} \,, \\
z_{32} &= B_{11}^2 B_{10} - \tfrac{1}{2} B_{11} B_{21} \,.
\end{aligned}
$$

Finally, making use of these results for the photon wave-function renormalisation constant,

$$
\begin{aligned}
Z_3 = 1 \; &+ \; \alpha_0 \left(B_{11} \ln \frac{\mu^2}{\Lambda^2} + B_{10} \right) + \alpha_0^2 \left(B_{11}^2 \ln^2 \frac{\mu^2}{\Lambda^2} + B_{21} \ln \frac{\mu^2}{\Lambda^2} \right) \\
&+ \; \alpha_0^3 \left(B_{11}^3 \ln^3 \frac{\mu^2}{\Lambda^2} + \left(\frac{5}{2} B_{11} B_{21} - 2 B_{10} B_{11}^2 \right) \ln^2 \frac{\mu^2}{\Lambda^2} \right) \\
&+ \; \mathcal{O}(\alpha^4) \,, \qquad (4.3.11)
\end{aligned}
$$

the coupling constant,

$$
\begin{aligned}
\alpha_0 \; = \; \alpha_R \Bigg[& 1 + \alpha_R \left(-B_{11} \ln \frac{\mu^2}{\Lambda^2} - B_{10} \right) \\
& + \alpha_R^2 \left(B_{11}^2 \ln^2 \frac{\mu^2}{\Lambda^2} + (4 B_{10} B_{11} - B_{21}) \ln \frac{\mu^2}{\Lambda^2} \right) \\
& + \alpha_R^3 \left(-B_{11}^3 \ln^3 \frac{\mu^2}{\Lambda^2} + \left(-8 B_{10} B_{11}^2 + \frac{5}{2} B_{11} B_{21} \right) \ln^2 \frac{\mu^2}{\Lambda^2} \right) + \mathcal{O}(\alpha^4) \Bigg] \,,
\end{aligned}
$$

(4.3.12)

and the multiplicatively renormalisable bare photon wave-function renormalisation,

$$
\boxed{
\begin{aligned}
G_0(p,\Lambda) = 1 \; &+ \; \alpha_0 \left(B_{11} \ln \frac{p^2}{\Lambda^2} + B_{10} \right) + \alpha_0^2 \left(B_{11}^2 \ln^2 \frac{p^2}{\Lambda^2} + B_{21} \ln \frac{p^2}{\Lambda^2} \right) \\
&+ \; \alpha_0^3 \left(B_{11}^3 \ln^3 \frac{p^2}{\Lambda^2} + \left(\frac{5}{2} B_{11} B_{21} - 2 B_{10} B_{11}^2 \right) \ln^2 \frac{p^2}{\Lambda^2} \right) \\
&+ \; \mathcal{O}(\alpha_0^4)
\end{aligned}
} \,,
$$

(4.3.13)

the inverse of G_0 :

$$
\begin{aligned}
\frac{1}{G_0(p^2,\Lambda^2)} = 1 \quad &+ \quad \alpha_0 \left(-B_{11} \ln \frac{p^2}{\Lambda^2} - B_{10} \right) \\
&+ \quad \alpha_0^2 \left((-B_{21} + 2\, B_{11}\, B_{10}) \ln \frac{p^2}{\Lambda^2} + B_{10}^2 \right) \\
&+ \quad \alpha_0^3 \left[\left(\frac{B_{21}\, B_{11}}{2} + B_{10}\, B_{11}^2 \right) \ln^2 \frac{p^2}{\Lambda^2} + \left(2\, B_{10}\, B_{21} - 3\, B_{11}\, B_{10}^2 \right) \ln \frac{p^2}{\Lambda^2} \right] + \cdots
\end{aligned}
\tag{4.3.14}
$$

Hence the renormalised photon wave-function renormalisation can be written as :

$$
\boxed{
\begin{aligned}
G_R(p,\mu) = 1 \quad &+ \quad \alpha_R\, B_{11} \ln \frac{p^2}{\mu^2} + \alpha_R^2 \left(B_{11}^2 \ln^2 \frac{p^2}{\mu^2} + (B_{21} - 2\, B_{10}\, B_{11}) \ln \frac{p^2}{\mu^2} \right) \\
&+ \quad \alpha_R^3 \left(B_{11}^3 \ln^3 \frac{p^2}{\mu^2} + \left(\frac{5}{2}\, B_{21}\, B_{11} - 5\, B_{10}\, B_{11}^2 \right) \ln^2 \frac{p^2}{\mu^2} \right) \\
&+ \quad \mathcal{O}(\alpha_R^4)
\end{aligned}
}
\tag{4.3.15}
$$

4.4 The General Multiplicatively Renormalisable $F(p^2)$

Analogously to the previous section, we deal with the fermion wave-function renormalisation. We similarly define the following general perturbative expansion of the unrenormalised F_0 to next-to-leading order in powers of logarithms as :

$$
\begin{aligned}
F_0(p,\Lambda) = 1 \quad &+ \quad \alpha_0 \left(A_{11} \ln \frac{p^2}{\Lambda^2} + A_{10} \right) \\
&+ \quad \alpha_0^2 \left(A_{22} \ln^2 \frac{p^2}{\Lambda^2} + A_{21} \ln \frac{p^2}{\Lambda^2} \right) \\
&+ \quad \alpha_0^3 \left(A_{33} \ln^3 \frac{p^2}{\Lambda^2} + A_{32} \ln^2 \frac{p^2}{\Lambda^2} \right) .
\end{aligned}
\tag{4.4.1}
$$

We again introduce P and M of Eqn. (4.3.2) and note that the gauge dependence of the parameters A_{ij} can be usefully displayed. Since gauge dependence arises from photon propagators, any A_{ij} cannot have a higher power of ξ than ξ^i. Then $F_0(p,\Lambda)$ can be written as :

$$\begin{aligned} F_0(p,\Lambda) = 1 &+ \alpha_0\,(a_1\xi + b_1)\,(P+M) + \alpha_0\,(q_1\xi + r_1) \\ &+ \alpha_0^2(a_2\,\xi^2 + b_2\,\xi + c_2)\,(P+M)^2 + \alpha_0\,(q_2\,\xi^2 + r_2\,\xi + s_2)\,(P+M) \\ &+ \alpha_0^3(a_3\,\xi^3 + b_3\,\xi^2 + c_3\,\xi + d_3)\,(P+M)^3 \\ &+ \alpha_0^2(q_3\,\xi^3 + r_3\,\xi^2 + s_3\,\xi + t_3)\,(P+M)^2 \\ &+ \mathcal{O}(\alpha_0^4)\,, \end{aligned} \tag{4.4.2}$$

and the fermion wave-function renormalisation constant becomes :

$$Z_2^{-1}(\mu/\Lambda) = 1 \;+\; \alpha_R\,(z_1\,M + y_1) + \alpha_R^2\,(z_2\,M^2 + y_2\,M) + \alpha_R^3\,(z_3\,M^3 + y_3)\,M^2 + \mathcal{O}(\alpha^4)\,. \tag{4.4.3}$$

Recalling Eqn. (4.2.4),

$$\xi_0 = Z_3\,\xi_R \;, \tag{4.2.4}$$

using Eqn. (4.3.4) for Z_3, we have

$$\begin{aligned} \xi_0 &= \xi_R \\ &\times \left[1 - \alpha_0\,(B_{11}\,M + B_{10}) - \alpha_0^2\,(B_{21} - 2\,B_{10}B_{11})\;M + \alpha_0^3\left(B_{10}B_{11}^2 - \frac{B_{21}B_{11}}{2}\right)\,M^2 + \cdots\right]^{-1} \end{aligned} \tag{4.4.4}$$

and Eqn. (4.2.8),

$$\alpha_0 = Z_3^{-1}\,\alpha_R\,, \tag{4.2.8}$$

we obtain the following relations between bare and renormalised quantities,

$$\begin{aligned} \alpha_0\,\xi &= \alpha_R\,\xi_R \;, \\ \alpha_0^2\,\xi &= \alpha_R^2\,\xi_R\,Z_3^{-1} \;, \\ \alpha_0^3\,\xi^2 &= \alpha_R^3\,\xi_R^2 Z_3^{-1} \;. \end{aligned} \tag{4.4.5}$$

Making use of all the above quantities, the unrenormalised wave-function renormalisation can be written as :

$$\begin{aligned}
F_0(p,\Lambda) \;&=\; 1+\alpha_R\,\xi_R\,a_1\,(P+M) \\
&+\alpha_R\,b_1\left\{1-(B_{11}\,M+B_{10})\,\alpha_R+\Big(B_{11}^2\,M^2+(4B_{10}B_{11}-B_{21})\,M\Big)\,\alpha_R^2\right\}(P+M) \\
&+\alpha_R\,\xi_R\,q_1 \\
&+\alpha_R\,r_1\left\{1-(B_{11}\,M+B_{10})\,\alpha_R+\Big(B_{11}^2\,M^2+(4B_{10}B_{11}-B_{21})\,M\Big)\,\alpha_R^2+\cdots\right\} \\
&+\alpha_R^2\,\left\{a_2\,\xi_R^2+b_2\,\xi_R\,(1-\alpha_R B_{11}\,M-B_{10})+c_2\,(1-2\alpha_R\,(B_{11}\,M+B_{10}))\right\}(P+M)^2 \\
&+\alpha_R^2\,\left\{q_2\,\xi_R^2+r_2\,\xi_R\,(1-\alpha_R B_{11}\,M-B_{10})+s_2\,(1-2\alpha_R\,(B_{11}\,M+B_{10}))\right\}(P+M) \\
&+\alpha_R^3\,\left\{a_3\,\xi_R^3+b_3\,\xi_R^2+c_3\,\xi_R+d_3\right\}(P+M)^3 \\
&+\alpha_R^3\,\left\{q_3\,\xi_R^3+r_3\,\xi_R^2+s_3\,\xi_R+t_3\right\}(P+M)^2 \\
&+\mathcal{O}(\alpha_R^4)\,.
\end{aligned} \tag{4.4.6}$$

Renormalising F (as for G) at

$$\boxed{\mu^2=\Lambda^2} \qquad \Longrightarrow \qquad \boxed{F_R(p,\mu)\;=\;F_0(p,\Lambda)\Big|_{\mu^2=\Lambda^2}}\,, \tag{4.4.7}$$

recalling Eqn. (4.2.2),

$$F_R(p,\mu)=Z_2^{-1}(\mu/\Lambda)\,F_0(p,\Lambda)\;, \tag{4.2.2}$$

we then can display the left hand side of this equation as :

$$\begin{aligned}
F_R(p,\mu)=1\;&+\;\alpha_R\,(a_1\,\xi+b_1)\,P+\alpha_R\,(q_1\,\xi+r_1) \\
&+\;\alpha_R^2\,(a_2\,\xi^2+b_2\,\xi+c_2)\,P^2+\alpha_R^2\,(q_2\,\xi^2+r_2\,\xi+s_2)\,P \\
&+\;\alpha_R^3\,(a_3\,\xi^3+b_3\,\xi^2+c_3\,\xi+d_3)\,P^3 \\
&+\;\alpha_R^3(q_3\,\xi^3+r_3\,\xi^2+s_3\,\xi+t_3)\,P^2+\;\mathcal{O}(\alpha^4)\,.
\end{aligned} \tag{4.4.8}$$

After multiplying F_0, Eqn. (4.4.6), by Z_2^{-1}, Eqn. (4.4.3), in order to obtain the right hand side of the Eqn. (4.2.2), we can compare two sides order by order in powers of M. We find for the terms listed as $\alpha_R^n\,M^m\,P^p$ to be :

$$\alpha_R M \quad : \quad a_1\,\xi_R + z_1 + b_1 = 0 \quad ,$$
$$\alpha_R \quad : \quad q_1\,\xi_R + r_1 + y_1 = 0 \quad ,$$

$$\alpha_R^2 M^2 \quad : \quad a_2\,\xi_R^2 + (b_2 + z_1\,a_1)\,\xi_R + z_2 - z_1\,B_{11} + c_2 - b_1\,B_{11} + z_1\,b_1 = 0 \quad ,$$
$$\alpha_R^2 P M \quad : \quad 2\,a_2\,\xi_R^2 + (z_1\,a_1 + 2\,b_2)\,\xi_R + 2\,c_2 - b_1\,B_{11} + z_1\,b_1 = 0 \quad ,$$
$$\alpha_R^2 M \quad : \quad q_2\,\xi_R^2 + (y_1\,a_1 + z_1\,q_1 + r_2)\,\xi_R$$
$$+\,y_1\,b_1 + z_1\,r_1 + y_2 + s_2 - b_1\,B_{10} - z_1\,B_{10} - y_1\,b_1 - r_1\,B_{11} = 0\;,$$

$$\alpha_R^3 M^3 \quad : \quad a_3\,\xi_R^3 + (z_1\,a_2 + b_3)\,\xi_R^2$$
$$+\,(c_3 + z_2\,a_1 - z_1\,a_1\,B_{11} + z_1\,b_2 - b_2\,B_{11})\,\xi_R \quad ,$$
$$\alpha_R^3 P M^2 \quad : \quad 3\,a_3\,\xi_R^3 + (3\,b_3 + 2\,z_1\,a_2)\,\xi_R^2$$
$$+\,(3\,c_3 + 2\,a_2\,z_1 - 2\,b_2\,B_{11} - z_1\,B_{11}\,a_1)\,\xi_R$$
$$+\,2\,z_1\,c_2 + z_2\,b_1 - 4\,c_2\,B_{11} + 3\,d_3 + b_1\,B_{11}^2 - 2\,z_1\,b_1\,B_{11} = 0 \quad ,$$
$$\alpha_R^3 P^2 M \quad : \quad 3\,a_3\,\xi_R^3 + (z_1\,a_2 + 3\,b_3)\,\xi_R^2$$
$$+\left(z_1\,a_1\,b_1 + \frac{3}{2}\,a_1\,b_1^2 + \frac{1}{2}\,a_1\,b_1\,B_{11}\right)\xi_R$$
$$+\,d_3 - 2\,c_2\,B_{11} + z_1\,c_2 \quad ,$$
$$\alpha_R^3 M^2 \quad : \quad q_3\,\xi_R^3 + (r_3 + z_1\,q_2 + y_1\,a_2)\,\xi_R^2$$
$$+\Big(s_3 + y_2\,a_1 + z_1\,r_2 - b_2\,B_{10} - r_2\,B_{11} + y_1\,b_2 + z_2\,q_1 - z_1\,a_1\,B_{10}$$
$$-\,y_1\,a_1\,B_{11} - z_1\,q_1\,B_{11}\Big)\,\xi_R$$
$$+\,y_3 + t_3 + z_2\,r_1 + y_2\,b_1 + 4\,z_1\,B_{10}\,B_{11} - z_1\,B_{21} + y_1\,B_{11}^2$$
$$-\,2\,z_2\,B_{10} - 2\,y_2\,B_{11} - b_1\,B_{21} + r_1\,B_{11}^2 - 2\,c_2\,B_{10} - 2\,s_2\,B_{11} + z_1\,s_2 + y_1\,c_2$$
$$-\,2\,z_1\,b_1\,B_{10} - 2\,y - 1\,b_1\,b_{11} + 4\,b_1\,B_{10}\,B_{11} - 2\,z_1\,r_1\,B_{11} \quad ,$$
$$\alpha_R^3 P M \quad : \quad 2\,q_3\,\xi_R^3 + (2\,r_3 + z_1\,q_2 + 2\,y_1 a_2)\,\xi_R$$
$$+\Big(z_1\,r_2 - y_1\,a_1\,B_{11} - z_1\,a_1\,B_{10} - 2\,b_1\,B_{10} + 2\,y_1\,b_2 - r_2\,B_{11}$$
$$+\,y_2\,a_1 + 2\,s_3\Big)\,\xi_R$$
$$-\,2\,s_2\,B_{11} + 4\,b_1\,B_{11}\,B_{10} + z_1\,s_2 + 2\,t_3 + 2\,y_1\,c_2$$
$$-\,b_1\,B_{21} - 4\,c_2\,B_{10} - 2\,z_1\,b_1\,B_{10} - 2\,y_1\,b_1\,B_{11}\;. \qquad (4.4.9)$$

Solving the above equations leads to :

$$
\begin{aligned}
z_1 &= -a_1\,\xi_R - b_1\;, \\
z_2 &= \frac{a_1^2}{2}\,\xi_R^2 + (a_1\,b_1 - a_1\,B_{11})\;\xi_R + \frac{b_1}{2}\,(b_1 - B_{11})\;, \\
z_3 &= -\frac{a_1^3}{6}\,\xi_R^3 + \left(-\frac{1}{2}\,a_1^2\,b_1 + a_1^2\,B_{11}\right)\,\xi_R^2 - \frac{1}{2}\,\left(a_1\,b_1^2 + 3\,a_1\,b_1\,B_{11} - a_1\,B_{11}^2\right)\xi_R \\
&\quad -\frac{1}{6}b_1^3 - \frac{1}{2}b_1^2\,B_{11} - \frac{1}{3}\,b_1\,B_{11}^2\;, \\
y_1 &= -q_1\,\xi_R - r_1\;, \\
y_2 &= (2\,a_1\,q_1 - q_2)\,\xi_R^2 + (2\,a_1\,r_1 - r_2 + 2\,b_1\,q_1 - a_1\,B_{10} - q_1\,B_{11})\;\xi_R \\
&\quad +2\,r_1\,b_1 - s_2\;, \\
y_3 &= \left(q_2\,a_1 - \frac{3}{2}a_1^2\,q_1\right)\,\xi_R^3 \\
&\quad + \left(-3\,a_1\,b_1\,q_1 + q_2\,b_1 - \frac{3}{2}a_1^2\,r_1 + r_2\,a_1 - 2\,q_2\,B_{11} + a_1^2\,B_{10} + 4\,a_1\,q_1\,B_{11}\right)\xi_R^2 \\
&\quad +(a_1\,b_1\,B_{10} + r_2\,b_1 - \frac{3}{2}q_1\,b_1^2 - \frac{5}{2}a_1\,r_1\,B_{11} - 3\,a_1\,b_1\,r_1 + a_1\,s_2 + 3\,q_1\,b_1\,B_{11} \\
&\quad -\frac{3}{2}r_2\,B_{11} - q_1\,B_{11}^2 - a_1\,B_{21})\,\xi_R \\
&\quad -\frac{1}{2}b_1\,B_{21} + s_2\,b_1 - \frac{3}{2}\,r_1\,b_1^2 + \frac{3}{2}\,r_1\,b_1\,B_{11} - s_2\,B_{11} + b_1\,B_{11}\,B_{10}\;, \\
a_2 &= \frac{a_1^2}{2}\;, \qquad\qquad a_3 = \frac{a_1^3}{6}\;, \\
b_2 &= a_1\,b_1\;, \qquad\qquad b_3 = \frac{1}{2}a_1^2\,b_1\;, \\
c_2 &= \frac{1}{2}\left(b_1\,B_{11} + b_1^2\right)\;, \qquad\qquad c_3 = \frac{1}{2}\left(a_1\,b_1^2 + a_1\,b_1\,B_{11}\right)\;, \\
d_3 &= \frac{1}{6}b_1^3 + \frac{1}{2}b_1^2\,B_{11} + \frac{1}{3}b_1\,B_{11}^2\;, \\
q_3 &= a_1\,q_2 - \frac{1}{2}a_1^2 q_1\;, \\
r_3 &= -\frac{1}{2}a_1^2\,r_1 + a_1\,r_2 - a_1\,b_1\,q_1 + b_1\,q_2\;, \\
s_3 &= a_1\,s_2 + \frac{1}{2}r_2\,B_{11} + b_1\,r_2 - \frac{1}{2}q_1\,b_1^2 - a_1\,b_1\,r_1 - \frac{1}{2}\,a_1\,r_1\,B_{11}\;, \\
t_3 &= -\frac{1}{2}r_1\,b_1\,B_{11} + b_1\,s_2 - b_1\,B_{11}\,B_{10} + \frac{1}{2}b_1\,B_{21} + s_2\,B_{11} - \frac{1}{2}r_1\,b_1^2\;.
\end{aligned}
\qquad (4.4.10)
$$

Substituting these relations into the unrenormalised fermion function , we eventually arrive at :

$$\begin{aligned}
F_0(p,\Lambda) &= 1+\alpha_0\,(a_1\,\xi+b_1)\,\ln\frac{p^2}{\Lambda^2}+\alpha_0(q_1\,\xi+r_1)\\
&+ \alpha_0^2\left(\frac{a_1^2}{2}\,\xi^2+a_1\,b_1\,\xi+\frac{b_1}{2}\,(B_{11}+b_1)\right)\,\ln^2\frac{p^2}{\Lambda^2}+\alpha_0^2\left(q_2\xi^2+r_2\,\xi+s_2\right)\,\ln\frac{p^2}{\Lambda^2}\\
&+ \alpha_0^3\left(\frac{a_1^3}{6}\,\xi^3+\frac{a_1^2\,b_1}{2}\,\xi^2+\frac{a_1\,b_1}{2}\,(b_1+B_{11})\,\xi+\frac{b_1^3}{6}+\frac{b_1\,B_{11}^2}{3}+\frac{b_1^2\,B_{11}}{2}\right)\,\ln^3\frac{p^2}{\Lambda^2}\\
&+ \alpha_0^3\left[\left(q_2\,a_1-\frac{a_1^2\,q_1}{2}\right)\,\xi^3+\left(-\frac{a_1^2\,r_1}{2}+r_2\,a_1+q_2\,b_1-a_1\,b_1\,q_1\right)\xi^2\right.\\
&\qquad+\left(\frac{r_2\,B_{11}}{2}+a_1\,s_2+r_2\,b_1-\frac{q_1\,b_1^2}{2}-\frac{a_1\,r_1\,B_{11}}{2}-a_1\,b_1\,r_1\right)\,\xi\\
&\qquad\left.+\left(\frac{b_1\,B_{21}}{2}-b_1\,B_{11}\,B_{10}+s_2\,B_{11}-\frac{r_1\,b_1^2}{2}+s_2\,b_1-\frac{r_1\,b_1\,B_{11}}{2}\right)\right]\,\ln^2\frac{p^2}{\Lambda^2}\,,
\end{aligned}\tag{4.4.11}$$

and its inverse :

$$\begin{aligned}
\frac{1}{F_0(p,\Lambda)}=1 &+ \alpha_0\,(-a_1\,\xi-b_1)\,\ln\frac{p^2}{\Lambda^2}+\alpha_0(-q_1\,\xi-r_1)\\
&+ \alpha_0^2\left(\frac{a_1^2}{2}\,\xi^2+a_1\,b_1\,\xi+\frac{b_1}{2}\,(b_1-B_{11})\right)\,\ln^2\frac{p^2}{\Lambda^2}+\alpha_0^2\Big((2\,a_1\,q_1-q_2)\xi^2\\
&+ (2\,a_1\,r_1+2\,b_1\,q_1-r_2)\,\xi+2\,b_1\,r_1-s_2\Big)\,\ln\frac{p^2}{\Lambda^2}\\
&+ \alpha_0^3\left(-\frac{a_1^3}{6}\,\xi^3-\frac{a_1^2\,b_1}{2}\,\xi^2+\frac{a_1\,b_1}{2}\,(-b_1+B_{11})\,\xi-\frac{b_1^3}{6}-\frac{b_1\,B_{11}^2}{3}+\frac{b_1^2\,B_{11}}{2}\right)\,\ln^3\frac{p^2}{\Lambda^2}\\
&+ \alpha_0^3\left[\left(q_2\,a_1-\frac{3}{2}a_1^2\,q_1\right)\,\xi^3+\left(-\frac{3}{2}a_1^2\,r_1+r_2\,a_1+q_2\,b_1-3\,a_1\,b_1\,q_1\right)\xi^2\right.\\
&+ \left(-\frac{r_2\,B_{11}}{2}+a_1\,s_2+r_2\,b_1-\frac{3}{2}q_1\,b_1^2+\frac{a_1\,r_1\,B_{11}}{2}-3\,a_1\,b_1\,r_1+q_1\,b_1\,B_{11}\right)\,\xi\\
&+ \left.\left(-\frac{b_1\,B_{21}}{2}+b_1\,B_{11}\,B_{10}+s_2\,B_{11}-\frac{3}{2}r_1\,b_1^2-s_2\,b_1+\frac{3}{2}r_1\,b_1\,B_{11}\right)\right]\,\ln^2\frac{p^2}{\Lambda^2}\,,
\end{aligned}\tag{4.4.12}$$

and the renormalised fermion wave-function renormalisation becomes :

$$
\begin{aligned}
F_R(p,\mu) = 1 \;&+\; \alpha_R\,(a_1\,\xi_R + b_1)\,\ln\frac{p^2}{\mu^2} \\
&+\; \alpha_R^2\left(\frac{a_1^2}{2}\,\xi_R^2 + a_1\,b_1\,\xi_R + \frac{b_1}{2}(b_1 + B_{11})\right)\ln^2\frac{p^2}{\mu^2} \\
&+\; \alpha_R^2\Big((q_2 - a_1\,q_1)\,\xi_R^2 + (r_2 - b_1\,q_1 - a_1\,r_1)\,\xi_R - b_1\,B_{10} + s_2 - r_1\,b_1\Big)\ln\frac{p^2}{\mu^2} \\
&+\; \alpha_R^3\left(\frac{a_1^3}{6}\,\xi_R^3 + \frac{a_1^2\,b_1}{2}\,\xi_R^2 + \frac{a_1\,b_1}{2}(b_1 + B_{11})\,\xi_R + \frac{b_1^3}{6} + \frac{b_1\,B_{11}^2}{3} + \frac{b_1^2\,B_{11}}{2}\right)\ln^3\frac{p^2}{\mu^2} \\
&+\; \alpha_R^3\bigg((a_1\,q_2 - a_1^2\,q_1)\,\xi_R^3 + (-a_1^2\,r_1 - 2\,a_1\,b_1\,q_1 + r_2\,a_1 + q_2\,b_1)\,\xi_R^2 \\
&\qquad + \Big(r_2\,b_1 - q_1\,b_1^2 - a_1\,b_1\,B_{10} - \frac{1}{2}b_1\,q_1\,B_{11} + \frac{1}{2}\,r_2\,B_{11} + a_1\,s_2 \\
&\qquad - \frac{1}{2}\,a_1\,r_1\,B_{11} - 2\,a_1\,b_1\,r_1\Big)\,\xi_R \\
&\qquad - b_1^2\,B_{10} + s_2\,b_1 - 2\,b_1\,B_{11}\,B_{10} + \frac{1}{2}\,b_1\,B_{21} \\
&\qquad + s_2\,B_{11} - r_1\,b_1\,B_{11} - r_1\,b_1^2\bigg)\ln^2\frac{p^2}{\mu}
\end{aligned}
\qquad (4.4.13)
$$

Then the A_{ij} of Eqn. (4.4.1) are :

$$
\begin{aligned}
A_{11} &= a_1\,\xi + b_1 \quad , \\
A_{10} &= q_1\xi + r_1 \quad , \\
A_{22} &= \frac{a_1^2}{2}\,\xi^2 + a_1\,b_1\,\xi + \frac{b_1}{2}(B_{11} + b_1) \quad , \\
A_{21} &= q_2\xi^2 + r_2\,\xi + s_2 \quad , \\
A_{33} &= \frac{a_1^3}{6}\,\xi^3 + \frac{1}{2}\,a_1^2\,b_1\,\xi^2 + \frac{a_1\,b_1}{2}(B_{11} + b_1) + \frac{b_1^3}{6} + \frac{b_1\,B_{11}^2}{3} + \frac{1}{2}\,b_1^2\,B_{11} \quad ,
\end{aligned}
$$

$$A_{32} = \left(q_2\, a_1 - \frac{a_1^2\, q_1}{2}\right) \xi^3 + \left(-\frac{a_1^2\, r_1}{2} + a_1\, r_2 + b_1\, q_2 - a_1\, b_1\, q_1\right) \xi^2$$
$$+ \left(\frac{r_2\, B_{11}}{2} + a_1\, s_2 + r_2\, b_1 - \frac{1}{2} a_1\, r_1\, B_{11} - a_1\, b_1\, r_1\right) \xi$$
$$+\frac{1}{2} b_1\, B_{21} - b_1\, B_{11}\, B_{10} + s_2\, B_{11} - \frac{1}{2} r_1\, b_1^2 + s_2\, b_1 - \frac{1}{2} r_1\, b_1\, B_{11} \quad . \quad (4.4.14)$$

4.5 Non-Perturbative Applications of MR

Multiplicative renormalisability renormalises the theory to all orders in perturbation theory. We assume that it works non-perturbatively too [22, 23, 28]. The following examples demonstrate the application of the above result to non-perturbative QED.

4.5.1 Quenched Example

In the case of quenched QED, $G = 1$, means

$$\boxed{Z_3 = 1} \quad \Longrightarrow \boxed{\alpha_0 = \alpha_R} \quad . \quad (4.5.1)$$

Collecting only leading terms, the renormalised fermion wave-function renormalisation takes the form :

$$F_R(p,\mu) = 1 + \alpha_R\,(a_1\,\xi + b_1)\,\ln\frac{p^2}{\mu^2} + \alpha_R^2\,\frac{(a_1\,\xi + b_1)^2}{2}\,\ln^2\frac{p^2}{\mu^2} + \alpha_R^3\,\frac{(a_1\,\xi + b_1)^3}{6}\,\ln^3\frac{p^2}{\mu^2} + \mathcal{O}(\alpha_R^4)\,. \quad (4.5.2)$$

We can easily sum the above expression to give

$$F_R(p,\mu) = \exp\left(\alpha_R\,(a_1\,\xi + b_1)\,\ln\frac{p^2}{\mu^2}\right) \quad .$$

After a very trivial step we reach the non-perturbative form of the fermion wave-function renormalisation :

$$\boxed{F_R(p,\mu) \;=\; \left(\tfrac{p^2}{\mu^2}\right)^{\alpha_R\,(a_1\,\xi + b_1)}} \, . \tag{4.5.3}$$

4.5.2 Unquenched Example

To illustrate a simple example in this case, we consider only leading logarithms and $\xi = 0$. Then the renormalised wave-function renormalisation becomes :

$$\begin{aligned} F_R(p,\mu) = 1 \;&+\; \alpha_R\, b_1\, \ln\frac{p^2}{\mu^2} \\ &+\; \alpha_R^2\, \frac{b_1}{2}\,(b_1 + B_{11})\,\ln^2\frac{p^2}{\mu^2} \\ &+\; \alpha_R^3\, \frac{b_1}{6}\,(b_1 + B_{11})\,(b_1 + 2\,B_{11})\,\ln^3\frac{p^2}{\mu^2} + \mathcal{O}(\alpha_R^4)\,. \end{aligned} \tag{4.5.4}$$

This expression can be written as

$$\begin{aligned} F_R(p,\mu) = 1 \;&+\; \alpha_R\,\frac{b_1}{B_{11}}\left(B_{11}\ln\frac{p^2}{\mu^2}\right) \\ &+\; \frac{\alpha_R^2}{2}\,\frac{b_1}{B_{11}}\left(\frac{b_1}{B_{11}}+1\right)\left(B_{11}^2\,\ln^2\frac{p^2}{\mu^2}\right) \\ &+\; \frac{\alpha_R^3}{6}\,\frac{b_1}{B_{11}}\left(\frac{b_1}{B_{11}}+1\right)\left(\frac{b_1}{B_{11}}+2\right)\left(B_{11}^3\,\ln^3\frac{p^2}{\mu^2}\right) + \mathcal{O}(\alpha_R^4) \\ =\;& \left(1 - \alpha_R\, B_{11}\,\ln\frac{p^2}{\mu^2}\right)^{-b_1/B_{11}} \,. \end{aligned} \tag{4.5.5}$$

Recalling the following relations for the coupling constant,

$$\begin{aligned} \alpha_0 \;&\equiv\; \alpha(\Lambda) = \alpha_R\left(1 - \alpha_R\, B_{11}\, M + \alpha_R^2\, B_{11}^2\, M^2 + \cdots\right) \\ &=\; \alpha(\mu)\left(1 - \alpha(\mu)\, B_{11}\,\ln\frac{\mu^2}{\Lambda^2} + \alpha^2(\mu)\, B_{11}^2\,\ln^2\frac{\mu^2}{\Lambda^2} + \cdots\right) \\ &=\; \frac{\alpha(\mu)}{(1 + \alpha(\mu)\, B_{11}\,\ln\mu^2/\Lambda^2)}\,, \end{aligned} \tag{4.5.6}$$

where $\alpha(\mu) \equiv \alpha_R$. This leads to,

$$\left(1 - \alpha(\mu)\, B_{11}\, \ln \frac{\Lambda^2}{\mu^2}\right) \;=\; \frac{\alpha(\mu)}{\alpha(\Lambda)} \quad , \tag{4.5.7}$$

similarly,

$$\left(1 - \alpha(\mu)\, B_{11}\, \ln \frac{p^2}{\mu^2}\right) \;=\; \frac{\alpha(\mu)}{\alpha(p)} \,. \tag{4.5.8}$$

Substituting the above expression into Eqn. (4.5.5), we arrive at the result

$$\boxed{F_R = \left[\frac{\alpha(p)}{\alpha(\mu)}\right]^{b_1/B_{11}}} \quad , \tag{4.5.9}$$

in terms of the running coupling, as the renormalisation group requires.

In this chapter we established a most general multiplicative renormalisable form for the fermion, Eqn. (4.4.12), and photon wave-function Eqn. (4.3.14) renormalisations. We have also demonstrated how this method works non-perturbatively. Now as a natural step we will make use of these multiplicatively renormalisable functions F and G from Eqns. (3.2.62, 3.3.57) to constrain the vertex function. This is what we shall do in the next chapter.

Chapter 5

MR Constraints on the Vertex

Physicists like to think that
all you have to do is say,
these are the conditions,
now what happens next?
-R.P. Feynman-

5.1 Introduction

In this Chapter, we shall combine the results of the previous two chapters to find the constraints on the fermion-photon vertex imposed by multiplicative renormalisability. Having calculated the multiplicatively renormalisable fermion, Eqn. (4.4.12), and photon, Eqn. (4.3.14) wave-function renormalisations, in Chapter 3, it becomes possible to compare these functions order by order with those computed by solving the SD-equations directly, Eqn. (3.2.62, 3.3.57). As a second step, we also take into consideration the other information detailed in Chapter 6, and conditions listed in Chapter 2 on the vertex function to show how to construct a nonperturbative vertex ansatz for unquenched QED.

We first take the quenched case as a simpler example to demonstrate the procedure given above. Hopefully this will be a helpful example to understand the unquenched case which is relatively more complicated.

Before we start to give this example, we want to make a point clear which is important for the following procedures, that is : $K_i, J_i, M_i, K_i', J_i', H_i', Q_i'$ and M_i' are constants in Eqns. (3.2.48, 3.2.49). Nevertheless, they may depend on ξ and N_F. However, this can only be in a particular way : the K_i, K_i', M_i, M_i' and H_i'can at most be linear, J_i, J_i' and Q_i' quadratic in ξ or N_F according to the multiplication factors (i.e. logarithms) appearing in Eqns. (3.2.48, 3.2.49). Consequently, we should keep this point in mind when the procedure of comparison is performed in the rest of the chapter.

5.2 M.R. constraints in Quenched QED

As was mentioned earlier, in the quenched approximation, the photon wave-function renormalisation, $G = 1$. We therefore only have the fermion wave-function renormalisation to deal with. On comparing Eqn. (3.2.63) and Eqn. (4.4.12) in each order for the ξ term, we obtain :

$\alpha_0 \ln\frac{\mathbf{p}^2}{\Lambda^2}$ **comparison:**

$$-a_1\,\xi - b_1 = -\frac{\xi}{4\,\pi} \quad ,$$

$$\Downarrow$$

$$\boxed{a_1 = \frac{1}{4\,\pi}} \quad , \quad \boxed{b_1 = 0} \quad . \tag{5.2.1}$$

Using this information together with Eqn. (4.4.14), we find,

$$\boxed{A_{11} = \frac{\xi}{4\,\pi}} \quad , \quad \boxed{A_{22} = \frac{A_{11}^2}{2}} \quad . \tag{5.2.2}$$

α_0 comparison:

$$-q_1\,\xi - r_1 \;\; = \;\; 0 \;\; ,$$

$$\Downarrow$$

$$\boxed{q_1 = r_1 = 0} \quad . \tag{5.2.3}$$

Recalling A_{10} from the previous chapter and making use of the values of q_1 and r_1, the first next-to-leading term in F becomes

$$\boxed{A_{10} = 0} \quad . \tag{5.2.4}$$

$\alpha_0^2 \ln^2 \frac{p^2}{\Lambda^2}$ comparison:

$$\frac{a_1^2}{2}\,\xi^2 \;\; = \;\; -\frac{1}{4\,\pi}\left[-\left(\frac{\xi}{2} + \frac{3}{8}\right) A_{11} \right.$$
$$\left. + \frac{3}{4}\,(-K_2 + K_3 + K_6 + K_8) + \frac{9}{4}\,(-K_2' + K_3' + K_6' + K_8') \right] ,$$

$$\Downarrow$$

$$\boxed{\frac{A_{11}}{2} = (-K_2 + K_3 + K_6 + K_8) + 3\,(-K_2' + K_3' + K_6' + K_8')} \, . \tag{5.2.5}$$

$\alpha_0^2 \ln \frac{p^2}{\Lambda^2}$ comparison:

$$\begin{aligned} - q_2^2\,\xi^2 - r_2\,\xi - s_2 \\ = -\frac{1}{4\,\pi}\Big[\frac{1}{2}(16\,\ln 2 - 7)\,(-K_2' + K_3' + K_6' + K_8') + 2\,K_6'\,(13 - 16\ln 2) \\ + \frac{3}{2}\,(-H_2' + H_3' + H_6' + H_8')\Big] \;, \end{aligned}$$

$$\Downarrow$$

$$\boxed{q_2 = 0\;, \qquad s_2 = 0} \;. \qquad (5.2.6)$$

Referring to Table 5.2, we can see that the K_i' only depend on A_{11}, i.e. ξ, and the H_i' depend on A_{10}. This is where Eqn (5.2.6) comes from. Moreover the study in quenched QED of the fermion wave-function renormalisation using the Landau Khalatnikov[30] transformation shows that, to order α^2, the gauge dependence merely occurs in the leading logarithm term. Having $q_2 = s_2 = 0$, the r_2 term is the only one left as the next-to-leading contribution in F at order α^2 and makes the fermion wave-function renormalisation gauge dependent unless $r_2 = 0$. Therefore, we take $r_2 = 0$ and A_{21} is then simply :

$$\boxed{A_{21} = 0} \;. \qquad (5.2.7)$$

and

$$\boxed{\begin{aligned} 0 \;=\; & \frac{1}{2}(16\,\ln 2 - 7)\,(-K_2' + K_3' + K_6' + K_8') + 2\,K_6'\,(13 - 16\ln 2) \\ & + \frac{3}{2}\,(-H_2' + H_3' + H_6' + H_8') \end{aligned}} \;. \qquad (5.2.8)$$

$\alpha_0^3 \ln^3 \frac{p^2}{\Lambda^2}$ comparison:

$$
\begin{aligned}
-\frac{a_1^3}{6}\xi^3 &= -\frac{1}{4\pi}\Bigg[\left(\frac{\xi}{2}+\frac{3}{8}\right)A_{11}^2 - \left(\frac{2\xi}{3}+\frac{1}{2}\right)A_{22} \\
&\quad + \frac{1}{4}A_{11}\left(-K_2+K_3+K_6+K_8\right) - \left(-J_2+J_3+J_6+J_8\right) \\
&\quad + \frac{5}{4}A_{11}\left(-K_2'+K_3'+K_6'+K_8'\right) + 2\left(-J_2'+J_3'+J_6'+J_8'\right) \\
&\quad + \frac{3}{4}\left(-Q_2'+Q_3'+Q_6'+Q_8'\right)\Bigg] \quad ,
\end{aligned}
$$

$$\Downarrow$$

$$
\boxed{
\begin{aligned}
\frac{A_{11}^2}{4} &= \left(-J_2+J_3+J_6+J_8\right) - 2\left(-J_2'+J_3'+J_6'+J_8'\right) \\
&\quad -\frac{3}{4}\left(-Q_2'+Q_3'+Q_6'+Q_8'\right) - \frac{1}{2}A_{11}\left(-K_2'+K_3'+K_6'+K_8'\right)
\end{aligned}
} \quad . \tag{5.2.9}
$$

$\alpha_0^3 \ln^2 \frac{p^2}{\Lambda^2}$ comparison:

$$
\begin{aligned}
0 &= \left(A_{10}A_{11}-A_{21}\right)\left(\frac{\xi}{2}+\frac{3}{8}\right) \\
&+ \frac{3}{4}A_{10}\left(-K_2+K_3+K_6+K_8\right) - \frac{3}{4}\left(-M_2+M_3+M_6+M_8\right) \\
&+ \frac{1}{2}(16\ln 2-7)A_{11}\left(-K_2'+K_3'+K_6'+K_8'\right) + 2(13-16\ln 2)A_{11}K_6' \\
&+ \frac{1}{2}(16\ln 2-7)\left(-J_2'+J_3'+J_6'+J_8'\right) + 2(13-16\ln 2)J_6' \\
&+ \frac{9}{4}\left(-M_2'+M_3'+M_6'+M_8'\right) + \frac{9}{4}A_{10}\left(-K_2'+K_3'+K_6'+K_8'\right) \\
&+ \frac{1}{4}(16\ln 2-7)\left(-Q_2'+Q_3'+Q_6'+Q_8'\right) + (13-16\ln 2)Q_6' \\
&+ \frac{3}{4}A_{11}\left(-H_2'+H_3'+H_6'+H_8'\right) \quad ,
\end{aligned}
$$

$$\Downarrow$$

$$\boxed{\begin{aligned} 0 = & \frac{3}{4}\Big[(-M_2 + M_3 + M_6 + M_8) - 3\,(-M_2' + M_3' + M_6' + M_8')\Big] \\ & + \frac{1}{2}(-7 + 16\ln 2)\,(-J_2' + J_3' + J_6' + J_8') + 2\,(13 - 16\ln 2)\,J_6' \\ & + \frac{1}{4}(-7 + 16\ln 2)\,(-Q_2' + Q_3' + Q_6' + Q_8') + (13 - 16\ln 2)\,Q_6 \\ & - \frac{3}{4}A_{11}\,(-H_2' + H_3' + H_6' + H_8')\,. \end{aligned}} \qquad (5.2.10)$$

We see that finding a multiplicatively renormalisable fermion wave-function renormalisation is entirely related to the structure of the transverse vertex. Having a multiplicatively renormalisable F requires that the transverse component must have a particular form to obey the above constraints.

5.2.1 Non-Perturbative Vertex Ansatz in Quenched QED

Obviously, all these constraints depend on the combinations of various constants yet to be determined according to different combinations of the wave-function renormalisation. We take the Ball-Chiu [26] construction, which determines the longitudinal piece, as a guide. This suggests that we consider symmetric and antisymmetric combinations of the function F. To be explicit, we use the following forms for the non-perturbative fermion-photon vertex :

1	$\frac{1}{F(k^2)} - \frac{1}{F(p^2)}$	$K = A_{11}$ $J = A_{11}^2 - A_{22}$ $M = -A_{21} + 2\,A_{11}\,A_{10}$
2	$F(k^2) - F(p^2)$	$K = -\,A_{11}$ $J = A_{22}$ $M = A_{21}$

Table 5.1: Antisymmetric combinations of F in quenched QED

1'	$\frac{1}{F(k^2)} + \frac{1}{F(p^2)} - 2$	$K' = -A_{11}$ $J' = A_{11}^2 - A_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10}$ $Q' = 0$, $H' = -2\,A_{10}$
2'	$F(k^2) + F(p^2) - 2$	$K' = A_{11}$ $J' = A_{22}$, $M' = A_{21}$ $Q' = 0$, $H' = -2\,A_{10}$

Table 5.2: Symmetric combinations of F in quenched QED

To use these tables, consider the first entry. If the τ_i involve an antisymmetric combination of F, e.g. $\frac{1}{F(k^2)} - \frac{1}{F(p^2)}$, then the constants K_i, J_i, M_i are as given on the right hand side of the Table 5.1. The next step is to write down the coefficient functions τ_i of the basis vectors as a combination of the above choices. Then :

$$\begin{aligned} \tau_i &= x_i * 1 + y_i * 2 \\ &+ x_i' * 1' + y_i' * 2' , \qquad i = 2, 3, 6, 8 \end{aligned} \tag{5.2.11}$$

where x_i, y_i, x_i' and y_i' are constants. The numbers $1, 2, 1'$ and $2'$ denote the different combinations of the wave-function renormalisation in the left hand column of Tables 5.1, 5.2. Which combination of these forms $(1,1',2,2')$ can be taken to construct the simplest non-perturbative transverse vertex that satisfies the given constraints? In order to find this out, let us insert Eqn. (5.2.11) into Eqns. (5.2.5-5.2.10). Starting from the former, Eqn (5.2.5), we have :

$$\begin{aligned} \frac{A_{11}}{2} &= A_{11}\,(-x_2+x_3+x_6+x_8) - A_{11}\,(-y_2+y_3+y_6+y_8) \\ &- 3\,A_{11}\,(-x_2'+x_3'+x_6'+x_8') + 3\,A_{11}\,(-y_2'+y_3'+y_6'+y_8')\;. \end{aligned} \tag{5.2.12}$$

From now on we shall use the following nomenclature, since the same combinations of the different constants in Eqn. (5.2.12) appear everywhere,

$$\begin{aligned} X^i &= -x_2^i + x_3^i + x_6^i + x_8^i\,, & \left\{X^i = X, Y\cdots,\;\; x^i = x, y\cdots\right\} \\ X^{i'} &= -x_2^{i'} + x_3^{i'} + x_6^{i'} + x_8^{i'}\,, & \left\{X^{i'} = X', Y'\cdots,\; x^{i'} = x', y'\cdots\right\} \end{aligned} \tag{5.2.13}$$

Therefore, Eqn. (5.2.12) can be written as

$$\begin{aligned} \frac{A_{11}}{2} &= A_{11}\,[(X-Y) - 3\,(X'-Y')] \quad, \\ \frac{1}{2} &= (X-Y) - 3\,(X'-Y') \quad. \end{aligned} \tag{5.2.14}$$

Eqn. (5.2.8) yields,

$$\begin{aligned} \frac{A_{11}^2}{4} &= \frac{A_{11}^2}{2}\,[(X-Y) - (X'+3Y')] \quad, \\ \frac{1}{2} &= (X-Y) - (X'+3Y')\;, \end{aligned} \tag{5.2.15}$$

and Eqn. (5.2.9) gives :

$$0 = \left(-\frac{7}{2} + 8\ln 2\right)(X'-Y') + 2\,(13 - 16\ln 2)\,(x_6' - y_6')\;. \tag{5.2.16}$$

Solving these equations gives,

$$\begin{aligned} X' &= 3\,Y' \quad, \\ \frac{1}{2} &= X - Y - 6\,Y' \quad, \\ 0 &= -\left(-\frac{7}{2} + 8\,\ln 2\right) Y_f' - 4\,(13 - 16\ln 2)\,(x_6' - y_6')\;. \end{aligned} \tag{5.2.17}$$

In order to find the simplest solution satisfying these equations, Eqn. (5.2.17), it is sufficient to choose :

$$X = -x_2 + x_3 + x_6 + x_8 = \frac{1}{2},$$

and the rest will be

$$Y = X' = Y' = 0\,. \tag{5.2.18}$$

Obviously, every x_i, which satisfies this equation, can be a solution, for instance,

$$x_6 = \frac{1}{2}, \qquad -x_2 + x_3 + x_8 = 0 \quad . \tag{5.2.19}$$

The simplest example is to take the coefficient functions,

$$\tau_2 = \tau_3 = \tau_8 = 0 \quad ,$$

and then τ_6 becomes :

$$\tau_6 = \frac{1}{2\,(k^2+p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\,. \tag{5.2.20}$$

Hence, an ansatz for the non-perturbative transverse vertex could have the form,

$$\Gamma_T^\mu = \frac{1}{2\,(k^2+p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\left[\gamma^\mu(p^2-k^2) + (p+k)^\mu \;\not{q}\right]\,. \tag{5.2.21}$$

This is in fact very similar to the CP-vertex, except that there is no kinematic singularities in the $k^2 \to p^2$ limit.

Taking Eqns. (5.2.2, 5.2.4, 5.2.7) into account together with Eqn. (4.4.11), we can write down the leading logarithm terms for the wave-function renormalisation in the quenched case as :

$$F = 1 + \frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2} + \frac{1}{2}\left(\frac{\alpha\xi}{4\pi}\right)^2 \ln^2\frac{p^2}{\Lambda^2} + \cdots\,. \tag{5.2.22}$$

By summing this, we can rewrite it as :

$$F(p^2) = \left(\frac{p^2}{\Lambda^2}\right)^{\frac{\alpha\xi}{4\pi}}, \tag{5.2.23}$$

which is clearly multiplicatively renormalisable.

5.3 M.R. constraints on Γ_T via the Fermion Wave-function Renormalisation

In this and the next section, we shall repeat the above strategy, to find constraints for unquenched QED,

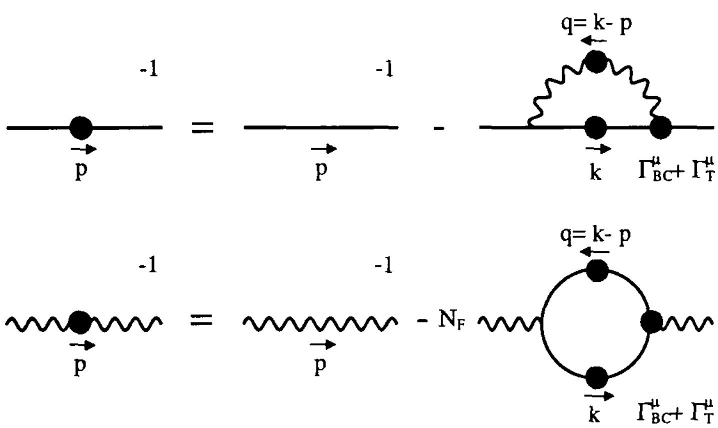
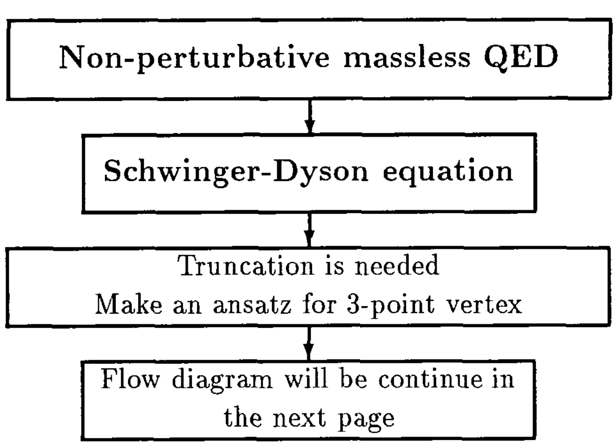


Figure 5.1: Unquenched Schwinger-Dyson equations for fermion and gauge boson

Now we display the layout of whole calculation again below, since it is such a lengthy procedure.



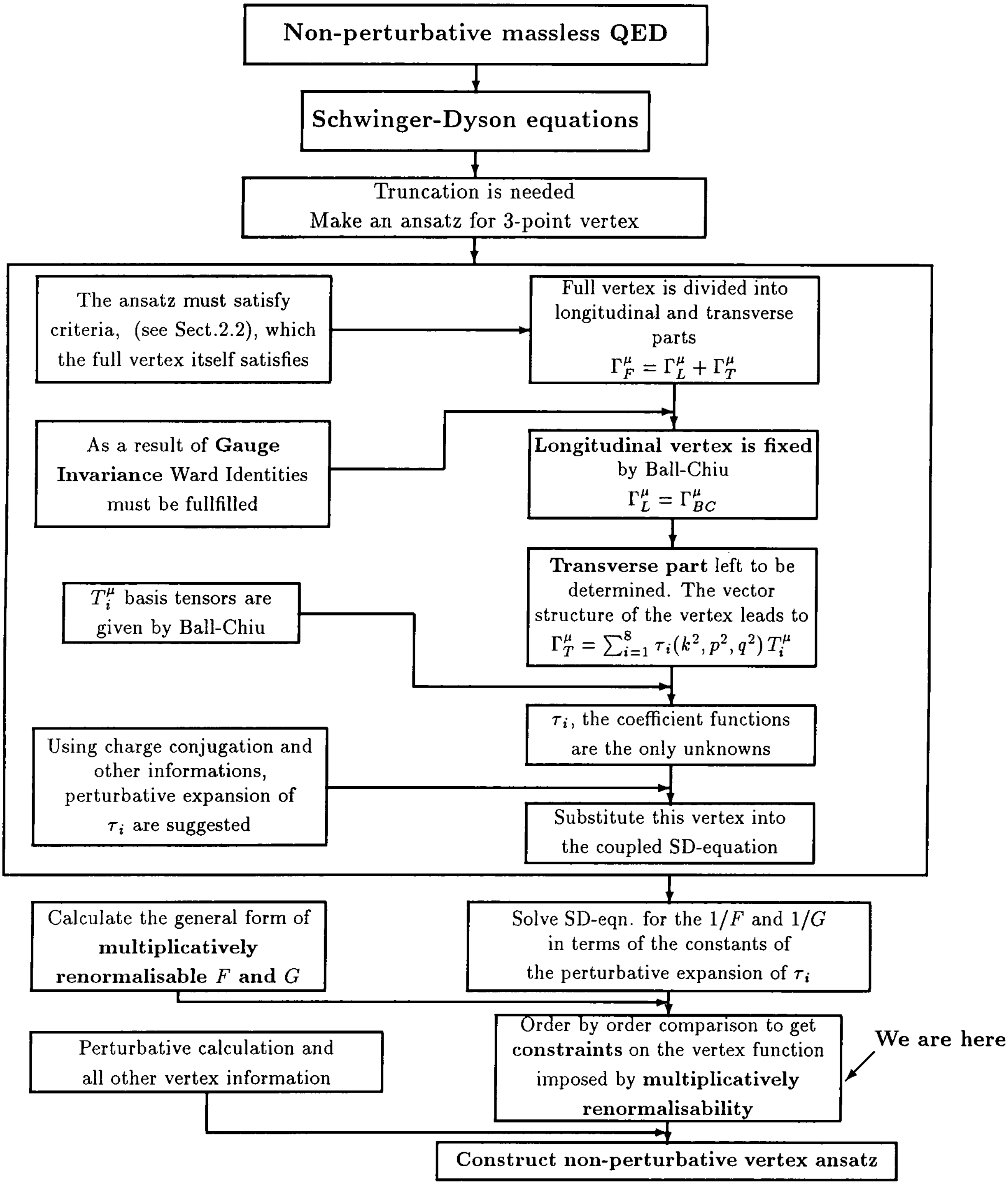


Figure 5.2: Flow diagram of the Schwinger-Dyson calculation

Here, we first deal with the wave-function renormalisation. To do this, we start by comparing ξ terms order by order between multiplicatively renormalisable F, Eqn. (4.4.12), with those found by solving the SD-equation, Eqn. (3.2.62) :

$\alpha_0 \ln\frac{p^2}{\Lambda^2}$ comparison:

$$-a_1\,\xi - b_1 \;=\; -\frac{\xi}{4\,\pi} \;,$$

$$\Downarrow$$

$$\boxed{a_1 = \frac{1}{4\,\pi}} \quad , \quad \boxed{b_1 = 0} \quad . \tag{5.3.1}$$

Then A_{11}, Eqn. (4.4.14), becomes

$$\boxed{A_{11} = \frac{\xi}{4\pi}} \quad . \tag{5.3.2}$$

α_0 comparison:

$$-q_1\,\xi - r_1 \;=\; 0 \;,$$

$$\Downarrow$$

$$\boxed{q_1 = r_1 = 0} \quad . \tag{5.3.3}$$

This requires A_{10} to be

$$\boxed{A_{10} = 0} \quad . \tag{5.3.4}$$

$\alpha_0^2 \ln^2\frac{p^2}{\Lambda^2}$ comparison:

$$\begin{aligned}\frac{a_1^2}{2}\,\xi^2 \;=\; -\frac{1}{4\,\pi}\Bigg[&-\left(\frac{\xi}{2}+\frac{3}{8}\right)A_{11} \\ &+\frac{3}{4}\left(-K_2+K_3+K_6+K_8\right)+\frac{9}{4}\left(-K_2'+K_3'+K_6'+K_8'\right)\Bigg] \;,\end{aligned}$$

$$\Downarrow$$

$$\boxed{\frac{A_{11}}{2} = \left(-K_2+K_3+K_6+K_8\right)+3\left(-K_2'+K_3'+K_6'+K_8'\right)} \;. \tag{5.3.5}$$

$\boldsymbol{\alpha_0^2 \ln \frac{p^2}{\Lambda^2}}$ **comparison:**

$$-\frac{q_2^2}{2}\xi^2 - r_2\,\xi - s_2$$

$$= -\frac{1}{4\pi}\left[-\frac{3}{2}B_{11} + \frac{1}{2}(-7 + 16\ln 2)\,(-K_2' + K_3' + K_6' + K_8') + 2\,(13 - 16\ln 2)\,K_6' + \frac{3}{2}\,(-H_2' + H_3' + H_6' + H_8') \right] ,$$

$$\Downarrow$$

$$\boxed{q_2 = 0} \tag{5.3.6}$$

and

$$\boxed{r_2\,\xi + s_2 = \frac{1}{4\pi}\left[-\frac{3}{2}B_{11} + \frac{1}{2}(16\ln 2 - 7)\,(-K_2' + K_3' + K_6' + K_8') + 2\,(13 - 16\ln 2)\,K_6' + \frac{3}{2}\,(-H_2' + H_3' + H_6' + H_8') \right]} . \tag{5.3.7}$$

$\boldsymbol{\alpha_0^3 \ln^3 \frac{p^2}{\Lambda^2}}$ **comparison:**

$$-\frac{a_1^3}{6}\xi^3 = -\frac{1}{4\pi}\Bigg[\left(\frac{\xi}{2} + \frac{3}{8}\right) A_{11}^2 - \left(\frac{2\xi}{3} + \frac{1}{2}\right) A_{22} - \frac{A_{11}\,B_{11}}{8} + \frac{1}{4}(A_{11} + B_{11})\,(-K_2 + K_3 + K_6 + K_8) - (-J_2 + J_3 + J_6 + J_8) + \frac{5}{4}(A_{11} + B_{11})\,(-K_2' + K_3' + K_6' + K_8') + 2\,(-J_2' + J_3' + J_6' + J_8') + \frac{3}{4}\,(-Q_2' + Q_3' + Q_6' + Q_8') \Bigg] ,$$

$$\Downarrow$$

$$\boxed{\frac{A_{11}^2}{4} = (-J_2 + J_3 + J_6 + J_8) - 2\,(-J_2' + J_3' + J_6' + J_8') - \frac{3}{4}\,(-Q_2' + Q_3' + Q_6' + Q_8') - \frac{1}{2}(A_{11} + B_{11})\,(-K_2' + K_3' + K_6' + K_8')} . \tag{5.3.8}$$

$\alpha_0^3 \ln^2 \frac{p^2}{\Lambda^2}$ comparison:

$$\begin{aligned}
a_1\,\xi\,(r_2\,\xi + s_2) \; - \; B_{11}\left(\frac{r_2}{2}\xi + s_2\right) \\
= -\frac{1}{4\,\pi}\Bigg[& (A_{10}\,A_{11} - A_{21})\left(\frac{\xi}{2} + \frac{3}{8}\right) - \frac{3}{8}A_{11}\,B_{10} + \frac{3}{8}A_{11}\,B_{11} - \frac{3}{2}B_{22} \\
& + \frac{3}{4}(A_{10} + B_{10})\,(-K_2 + K_3 + K_6 + K_8) - \frac{3}{4}\,(-M_2 + M_3 + M_6 + M_8) \\
& + \frac{3}{4}B_{11}\,(K_3 - K_6) + \frac{1}{2}(16\,\ln 2 - 7)\,(A_{11} + B_{11})\,(-K_2' + K_3' + K_6' + K_8') \\
& + \frac{1}{2}(16\,\ln 2 - 7)\,(-J_2' + J_3' + J_6' + J_8') + 2\,(13 - 16\ln 2)\,J_6' \\
& + \frac{9}{4}\,(-M_2' + M_3' + M_6' + M_8') + \frac{9}{4}(A_{10} + B_{10})\,(-K_2' + K_3' + K_6' + K_8') \\
& + \frac{9}{4}B_{11}\,(K_3' - K_6') + 2\,(13 - 16\ln 2)\,(A_{11} + B_{11})\,K_6' \\
& + \frac{1}{4}(16\,\ln 2 - 7)\,(-Q_2' + Q_3' + Q_6' + Q_8') + (13 - 16\ln 2)\,Q_6' \\
& + \frac{3}{4}(A_{11} + B_{11})\,(-H_2' + H_3' + H_6' + H_8')\Bigg] \quad ,
\end{aligned}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}
\frac{3}{4\,(4\pi)} & \left[(-M_2 + M_3 + M_6 + M_8) - 3\,(-M_2' + M_3' + M_6' + M_8')\right] \\
& = \frac{3}{2}A_{11}\,A_{21} + \frac{\xi}{2}r_2\,B_{11} \\
& - \frac{3}{8\,(4\pi)}A_{21} + \frac{15}{8\,(4\pi)}A_{11}\,B_{11} \\
& + \frac{3}{4\,(4\pi)}B_{11}\,\left[(K_3 - K_6) + 3\,(K_3' - K_6')\right] \\
& + \frac{1}{2\,(4\pi)}(-7 + 16\ln 2)\,(-J_2' + J_3' + J_6' + J_8') \\
& + \frac{2}{4\pi}(13 - 16\ln 2)\,J_6' - \frac{3}{4\,(4\pi)}(A_{11} + B_{11})\,(-H_2' + H_3' + H_6' + H_8') \\
& + \frac{1}{4\,(4\pi)}(16\,\ln 2 - 7)\,(-Q_2' + Q_3' + Q_6' + Q_8') + \frac{1}{4\pi}(13 - 16\,\ln 2)\,Q_6'
\end{aligned}} \; .$$

(5.3.9)

The conditions in the boxes, Eqns. (5.3.5–5.3.9), are the constraints multiplicative renormalisability of the fermion wave-function renormalisation imposes on the vertex function.

5.4 M.R. constraints on the Γ_T via the Photon Wave-function Renormalisation

To construct a suitable ansatz for the transverse vertex, we need as much information as possible coming from the coupled SD system. Solving this system for the fermion wave-function renormalisation only gives part of the constraints. The rest is extracted from the photon function. To do this, we repeat the same steps for the photon function as we have carried out for F. Comparison takes place between Eqn (4.3.14) and Eqn (3.3.57) order by order for G. Obviously, this time instead of looking at the way terms depend on the gauge parameter ξ, we compare the dependence of N_F, the number of flavors hidden in the B terms. Then :

$\alpha_0 \ln\frac{p^2}{\Lambda^2}$ comparison:

$$\boxed{B_{11} = \frac{N_F}{3\,\pi}} \quad . \tag{5.4.1}$$

α_0 comparison:

$$\boxed{B_{10} = -\frac{13}{12}\frac{N_F}{(3\pi)}} \quad . \tag{5.4.2}$$

$\alpha_0^2 \ln^2\frac{p^2}{\Lambda^2}$ comparison:

$$0 \;=\; \frac{A_{11}}{2} + \frac{3}{4}(K_2' + K_3' - K_6' - K_8') \quad ,$$

$$\Downarrow$$

$$\boxed{\frac{2}{3}A_{11} = (-K_2' - K_3' + K_6' + K_8')} \quad . \tag{5.4.3}$$

$\alpha_0^2 \ln\frac{\mathbf{p}^2}{\Lambda^2}$ **comparison:**

$$
\begin{aligned}
-B_{21} + 2\,B_{11}\,B_{10} \;=\; & -\frac{N_F}{3\pi}\Bigg[\left(A_{10} - \frac{11}{12}A_{11}\right) - \frac{3}{2}(K_2 + K_3 - K_6 - K_8) \\
& + \left(-\frac{9}{4} + 3\ln 2\right)(K_2' + K_3' - K_6' - K_8') \\
& + \frac{3}{2}K_3' + (-9 + 12\ln 2)\,K_6' \\
& + \frac{3}{4}(H_2' + H_3' - H_6' - H_8')\Bigg] \;,
\end{aligned}
$$

$$\Downarrow$$

$$
\boxed{
\begin{aligned}
-B_{21} + 2\,B_{11}\,B_{10} \;=\; & -\frac{N_F}{3\pi}\Bigg[\left(\frac{7}{12} - 2\ln 2\right)A_{11} + \frac{3}{2}(-K_2 - K_3 + K_6 + K_8) \\
& -\frac{3}{4}(-H_2' - H_3' + H_6' + H_8') \\
& +\frac{3}{2}K_3' + (-9 + 12\ln 2)\,K_6'\Bigg]
\end{aligned}
}\;. \tag{5.4.4}
$$

$\alpha_0^3 \ln^3\frac{\mathbf{p}^2}{\Lambda^2}$ **comparison:**

$$
\begin{aligned}
0 \;=\; & \frac{A_{22}}{3} + \frac{1}{2}(J_2' + J_3' - J_6' - J_8') \\
& + A_{11}\,(K_2' + K_3' - K_6' - K_8') + \frac{1}{4}(Q_2' + Q_3' - Q_6' - Q_8') \;,
\end{aligned}
$$

$$\Downarrow$$

$$
\boxed{A_{11}^2 = -\,(-J_2' - J_3' + J_6' + J_8') - \frac{1}{2}(-Q_2' - Q_3' + Q_6' + Q_8')}\;. \tag{5.4.5}
$$

$\alpha_0^3 \ln^2 \frac{\mathbf{p}^2}{\Lambda^2}$ **comparison:**

$$\begin{aligned}
&\frac{B_{11}}{2}\left(-B_{21}+2\,B_{10}\,B_{11}\right)\\
&\quad = -\frac{N_F}{3\pi}\Bigg[\;\frac{A_{21}}{2}-\frac{11}{12}A_{22}-\frac{3}{2}A_{11}\left(K_2+K_3-K_6-K_8\right)\\
&\qquad +\left(-\frac{9}{4}+3\,\ln 2\right)\left(J_2'+J_3'-J_6'-J_8'\right)+\frac{3}{2}J_3'\\
&\qquad +\left(-9+12\ln 2\right)J_6'+\left(-\frac{9}{2}+6\ln 2\right)A_{11}\left(K_2'+K_3'-K_6'-K_8'\right)\\
&\qquad +3\,A_{11}\,K_3'+2\left(-9+12\,\ln 2\right)A_{11}\,K_6'\\
&\qquad +\frac{3}{4}\left(M_2'+M_3'-M_6'-M_8'\right)+\frac{3}{2}A_{10}\left(K_2'+K_3'-K_6'-K_8'\right)\\
&\qquad +\frac{3}{4}A_{11}\left(H_2'+H_3'-H_6'-H_8'\right)+\left(\frac{-9}{8}+\frac{3}{2}\,\ln 2\right)\left(Q_2'+Q_3'-Q_6-Q_8'\right)\\
&\qquad +\frac{3}{4}Q_3'+\left(-\frac{9}{2}+6\,\ln 2\right)Q_6'\Bigg]\quad,
\end{aligned}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}
\frac{3}{4}\left(-M_2'-M_3'+M_6'+M_8'\right) \;=\;& \frac{A_{21}}{2}+\left(\frac{7}{24}-\ln 2\right)A_{11}^2+\left(-\frac{7}{24}+\ln 2\right)A_{11}\,B_{11}\\
&+\frac{3}{4}\left(-K_2-K_3+K_6+K_8\right)\left(2\,A_{11}-B_{11}\right)\\
&-\frac{3}{8}\left(-H_2'-H_3'+H_6'+H_8'\right)\left(2\,A_{11}-B_{11}\right)\\
&+\frac{3}{4}K_3'\left(4\,A_{11}-B_{11}\right)+\frac{1}{2}\left(-9+12\,\ln 2\right)K_6'\left(4\,A_{11}-B_{11}\right)\\
&+\frac{3}{2}J_3'+\left(-9+12\,\ln 2\right)J_6'+\frac{3}{4}Q_3'+\frac{1}{2}(-9+12\,\ln 2)\,Q_6'
\end{aligned}}\;.$$

(5.4.6)

Again the conditions on the vertex have been enclosed in boxes, Eqns. (5.4.3–5.4.6). The general multiplicative renormalisability constraints on the 3-point vertex function are expressed in terms of the constants $K, M, J, K', J', M', H', Q'$. These appear in the definition of the coefficient functions of the basis vectors in the transverse vertex, τ_i', τ_i'', in Eqns. (3.2.48, 3.2.49),

$$\tau_i' = \alpha K_i \left(\ln\frac{p^2}{\Lambda^2} - \ln\frac{k^2}{\Lambda^2}\right) + \alpha^2\left[J_i\left(\ln^2\frac{k^2}{\Lambda^2} - \ln^2\frac{p^2}{\Lambda^2}\right) + M_i\left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right)\right] + \mathcal{O}(\alpha^3)\,, \qquad \{i = 2,3,6,8\} \qquad (3.2.48)$$

$$\tau_i'' = \alpha\left[K_i'\left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2}\right) + H_i'\right] + \alpha^2\left[J_i'\left(\ln^2\frac{k^2}{\Lambda^2} + \ln^2\frac{p^2}{\Lambda^2}\right) + M_i'\left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2}\right) + Q_i'\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right] + \mathcal{O}(\alpha^3)\,, \qquad \{i = 2,3,6,8\} \qquad (3.2.49)$$

and they also appear in the combination of the non-perturbative F's and G's, Tables 5.3, 5.4. Basically in the above expressions the τ_i' and τ_i'' have been written as a general perturbative expansion of the different combinations of non-perturbative F and G. Each symmetric and antisymmetric combination of the F and G gives similar perturbative expansions but with different constants. Thus, as we shall see below, each individual non-perturbative form of the fermion and photon wave-function renormalisations has its own set of constants. Here some examples are listed in analogy with the quenched case of Tables 5.3, 5.4. Again the right hand side of the tables below gives the constants, e.g. $K, J, M, H, Q, K', J', M', H', Q'$, for the τ_i's constructed from the combinations of F and G given on the left.

Antisymmetric combinations of F and G are :

1	$\frac{1}{F(k^2)} - \frac{1}{F(p^2)}$	K$= A_{11}$ J $= A_{11}^2 - A_{22}$ M$= -A_{21} + 2\,A_{11}\,A_{10}$
2	$\frac{1}{G(k^2)} - \frac{1}{G(p^2)}$	K$= B_{11}$ J $= B_{11}^2 - B_{22}$ M$= -B_{21} + 2\,B_{11}\,B_{10}$
3	$\frac{G(k^2)}{F(k^2)} - \frac{G(p^2)}{F(p^2)}$	K$= A_{11} - B_{11}$ J $= A_{11}^2 - A_{22} - A_{11}\,B_{11} + B_{22}$ M$= -A_{21} + 2\,A_{11}\,A_{10} - A_{10}\,B_{11} - A_{11}\,B_{10} + B_{21}$
4	$\frac{G(p^2)}{F(k^2)} - \frac{G(k^2)}{F(p^2)}$	K$= A_{11} + B_{11}$ J $= A_{11}^2 - A_{22} - B_{22}$ M$= -A_{21} + 2\,A_{11}\,A_{10} + A_{10}\,B_{11} - A_{11}\,B_{10} - B_{21}$
5	$\frac{1}{F(k^2)\,G(k^2)} - \frac{1}{F(p^2)\,F(p^2)}$	K$= A_{11} + B_{11}$ J $= A_{11}^2 - A_{22} + A_{11}\,B_{11} + B_{11}^2 - B_{22}$ M$= -A_{21} + 2\,A_{11}\,A_{10} + A_{10}\,B_{11} + A_{11}\,B_{10} - B_{21} + 2\,B_{10}\,B_{11}$
6	$\frac{1}{F(k^2)\,G(p^2)} - \frac{1}{F(p^2)\,G(k^2)}$	K$= A_{11} - B_{11}$ J $= A_{11}^2 - A_{22} - B_{11}^2 + B_{22}$ M$= -A_{21} + 2\,A_{11}\,A_{10} - A_{10}\,B_{11} + A_{11}\,B_{10} + B_{21} - 2\,B_{10}\,B_{11}$

Table 5.3: Possible antisymmetric combinations of F and G

$1'$	$\frac{1}{F(k^2)} + \frac{1}{F(p^2)} - 2$	$K' = -A_{11}$ $J' = A_{11}^2 - A_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10}$ $Q' = 0$, $H' = -2\,A_{10}$
$2'$	$\frac{1}{G(k^2)} + \frac{1}{G(p^2)} - 2$	$K' = -B_{11}$ $J' = B_{11}^2 - B_{22}$ $M' = -B_{21} + 2\,B_{11}\,B_{10}$ $Q' = 0$, $H' = -2\,B_{10}$
$3'$	$\frac{G(k^2)}{F(k^2)} + \frac{G(p^2)}{F(p^2)} - 2$	$K' = -A_{11} + B_{11}$ $J' = A_{11}^2 - A_{22} - A_{11}\,B_{11} + B_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10} - A_{10}\,B_{11} - A_{11}\,B_{10} + B_{21}$ $Q' = 0$, $H' = -2\,(A_{10} - B_{10})$
$4'$	$\frac{G(p^2)}{F(k^2)} + \frac{G(k^2)}{F(p^2)} - 2$	$K' = -A_{11} + B_{11}$ $J' = A_{11}^2 - A_{22} + B_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10} - A_{10}\,B_{11} - A_{11}\,B_{10} + B_{21}$ $Q' = -2\,A_{11}\,B_{11}$, $H' = -2\,(A_{10} - B_{10})$
$5'$	$\frac{1}{F(k^2)\,G(k^2)} + \frac{1}{F(p^2)\,G(p^2)} - 2$	$K' = -A_{11} - B_{11}$ $J' = A_{11}^2 - A_{22} + A_{11}\,B_{11} + B_{11}^2 - B_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10} + A_{10}\,B_{11} + A_{11}\,B_{10} - B_{21} + 2\,B_{10}\,B_{11}$ $Q' = 0$, $H' = -2\,(A_{10} + B_{10})$
$6'$	$\frac{1}{F(k^2)\,G(p^2)} + \frac{1}{F(p^2)\,G(k^2)} - 2$	$K' = -A_{11} - B_{11}$ $J' = A_{11}^2 - A_{22} + B_{11}^2 - B_{22}$ $M' = -A_{21} + 2\,A_{11}\,A_{10} + A_{10}\,B_{11} + A_{11}\,B_{10} - B_{21} + 2\,B_{10}\,B_{11}$ $Q' = 2\,A_{11}\,B_{11}$, $H' = -2\,(A_{10} + B_{10})$
$7'$	$G(k^2) + G(p^2) - 2$	$K' = B_{11}$, $J' = B_{22}$, $M' = B_{21}$ $Q' = 0$, $H' = -2\,B_{10}$
$8'$	$F(k^2) + F(p^2) - 2$	$K' = A_{11}$, $J' = A_{22}$, $M' = A_{21}$ $Q' = 0$, $H' = -2\,A_{10}$

Table 5.4: Possible symmetric combinations of F and G

5.5 Applications

The next step is to make use of all these examples for the multiplicative renormalisability constraints. In order to satisfy these, we have a set of equations to solve. As a first step the coefficient functions, τ_i, can be written as a sum of different non-perturbative forms of F and G with the above examples. Hence, τ_i' and τ_i'' become :

$$\tau_i = \tau_i' + \tau_i'' \quad , \tag{5.5.1}$$

In general, a symmetric combination of F and G is :

$$\tau_i' \;=\; x_i * 1 + y_i * 2 + z_i * 3 + t_i * 4 + s_i * 5 + r_i * 6 \;, \tag{5.5.2}$$

while an antisymmetric combination of F and G is :

$$\tau_i'' \;=\; x_i' * 1' + y_i' * 2' + z_i' * 3' + t_i' * 4' + s_i' * 5' + r_i' * 6' + v_i' * 7 + w_i' * 8 \,, \tag{5.5.3}$$

where $x, x', y, y' \cdots$ are constants. The number of constants needed to solve these equations is proportional to the number of various combinations of the F and G. These combinations will appear in the ansatz for the non-perturbative transverse vertex. We then try to solve these equations by choosing a minimal number of combinations, in order to find the simplest possible vertex ansatz. However, at this moment we cannot decide which of these forms are necessary and which not, so we keep them all. Hence, the next step is to insert Eqn. (5.5.2) and Eqn. (5.5.3) in the constraints to find these equations, starting with the ones which come from the fermion wave-function renormalisation comparisons, Eqns (5.3.5-5.3.9).

5.5.1 Application to the Fermion Function

Eqn. (5.3.5) gives :

$$\frac{A_{11}}{2} = A_{11}\left[(X_f + Z_f + T_f + S_f + R_f) - 3\left(X'_f + Z'_f + T'_f + S'_f + R'_f - W'_f\right)\right]$$
$$+ B_{11}\left[(Y_f - Z_f + T_f + S_f - R_f) + 3\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f + V'_f\right)\right],$$

comparing ξ and N_F terms on both sides of the equation

$$\boxed{\begin{array}{lcl} (1) : \frac{1}{2} & = & (X_f + Z_f + T_f + S_f + R_f) - 3\left(X'_f + Z'_f + T'_f + S'_f + R'_f - W'_f\right) \\ (2) : 0 & = & (Y_f - Z_f + T_f + S_f - R_f) + 3\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f + V'_f\right) \end{array}} . \tag{5.5.4}$$

Using Eqn. (5.3.8) :

$$\frac{A_{11}^2}{4} = \frac{A_{11}^2}{2}\left[(X_f + Z_f + T_f + S_f + R_f) - \left(X'_f + Z'_f + T'_f + S'_f + R'_f + 3\,W'_f\right)\right]$$
$$+ A_{11}\,B_{11}\left[-Z_f - S_f + \frac{1}{2}\left(X'_f + Y'_f + 4\,Z'_f + 3\,T'_f - 2\,S'_f - R'_f - V'_f - W'_f\right)\right]$$
$$+ B_{11}^2\left[Z_f - T_f + \frac{1}{2}\left(Y'_f - 5\,Z'_f - 5\,T'_f + S'_f + R'_f - 5\,V'_f\right)\right],$$

and comparing ξ^2, $\xi\,N_F$ and N_F^2 terms leads to :

$$\boxed{\begin{array}{lcl} (3) : \frac{1}{2} & = & (X_f + Z_f + T_f + S_f + R_f) - \left(X'_f + Z'_f + T'_f + S'_f + R'_f + 3\,W'_f\right) \\ \hline (4) : 0 & = & -Z_f - S_f + \frac{1}{2}\left(X'_f + Y'_f + 4\,Z'_f + 3\,T'_f - 2\,S'_f - R'_f - V'_f - W'_f\right) \\ \hline (5) : 0 & = & Z_f - T_f + \frac{1}{2}\left(Y'_f - 5\,Z'_f - 5\,T'_f + S'_f + R'_f - 5\,V'_f\right) \end{array}} . \tag{5.5.5}$$

Eqn. (5.3.7) yields :

$$\begin{aligned}
A_{21} &= r_2\,\xi + s_2 \\
&= \frac{1}{4\pi}\Bigg\{ A_{11}\Bigg[\left(-\frac{7}{2} + 8\ln 2\right)\left(-X'_f - Z'_f - T'_f - S'_f - R'_f + W'_f\right) \\
&\qquad + 2\,(13 - 16\ln 2)\,(-x'_6 - z'_6 - t'_6 - s'_6 - r'_6 + w'_6)\Bigg] \\
&\quad - \frac{3}{2}B_{11} + \left(-\frac{7}{2} + 8\ln 2\right) B_{11}\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f + V'_f\right) \\
&\quad + 2\,(13 - 16\ln 2)\,B_{11}\,(-y'_6 + z'_6 + t'_6 - s'_6 - r'_6 + v'_6) \\
&\quad + 3\,B_{10}\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f - V'_f\right) \,,
\end{aligned}$$

and looking at the coefficients of the ξ and N_F terms, we obtain :

$$\boxed{\begin{aligned}
(6)\ :\ r_2 &= -\frac{1}{(4\pi)^2}\Bigg[\left(-\frac{7}{2} + 8\ln 2\right)\left(X'_f + Z'_f + T'_f + S'_f + R'_f - W'_f\right) \\
&\qquad + 2\,(13 - 16\ln 2)\,(x'_6 + z'_6 + t'_6 + s'_6 + r'_6 - w'_6)\Bigg] \\
(7)\ :\ s_2 &= \frac{1}{4\pi}\Bigg[-\frac{3}{2}B_{11} + \left(-\frac{7}{2} + 8\ln 2\right) B_{11}\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f + V'_f\right) \\
&\qquad + 2\,(13 - 16\ln 2)\,B_{11}\,(-y'_6 + z'_6 + t'_6 - s'_6 - r'_6 + v'_6) \\
&\qquad + 3\,B_{10}\left(-Y'_f + Z'_f + T'_f - S'_f - R'_f - V'_f\right)\Bigg]
\end{aligned}} \; . \tag{5.5.6}$$

The last Eqn. (5.3.9) gives :

$$
\begin{aligned}
\frac{3}{8} A_{11} B_{11} \;=\; & A_{11}^2 \Big[-\Big(-\frac{7}{2} + 8 \ln 2\Big) \Big(X'_f + Z'_f + T'_f + S'_f + R'_f - 2 W'_f \Big) \\
& - 2 (13 - 16 \ln 2) \, (x'_6 + z'_6 + t'_6 + s'_6 + r'_6 - 2 w'_6) \Big] \\
+ \; & A_{11} B_{11} \Big[-\frac{1}{2} \Big(-\frac{7}{2} + 8 \ln 2\Big) \Big(X'_f + 3 Y'_f + 2 S'_f + 2 R'_f - 3 V'_f - W'_f \Big) \\
& + (13 - 16 \ln 2) \, (- x'_6 - 3 y'_6 - 2 s'_6 - 2 r'_6 + 3 v'_6 + w'_6) \\
& + \frac{3}{4} (x_3 + z_3 + t_3 + s_3 + r_3) - \frac{3}{4} (x_6 + z_6 + t_6 + s_6 + r_6) \\
& - \frac{9}{4} (x'_3 + z'_3 + t'_3 + s'_3 + r'_3 - w'_3) \\
& + \frac{9}{4} (x'_6 + z'_6 + t'_6 + s'_6 + r'_6 - w'_6) \Big] \\
+ \; & A_{11} B_{10} \Big[- \frac{3}{4} (- Z_f - T_f + S_f + R_f) \\
& + \frac{3}{4} \Big(-4 Y'_f + Z_f + T_f - S_f - R_f - 4 V'_f \Big) \Big] \\
+ \; & B_{11}^2 \Big[\frac{3}{4} (y_3 - z_3 + t_3 + s_3 - r_3) + \frac{3}{4} (-y_6 + z_6 - t_6 - s_6 + r_6) \\
& - \frac{9}{4} (y'_3 - z'_3 - t'_3 + s'_3 + r'_3 - v'_3) \\
& - \frac{9}{4} (-y'_6 + z'_6 + t'_6 - s'_6 - r'_6 + v'_6) \\
& + \Big(-\frac{7}{2} + 8 \ln 2\Big) (Z'_f + T'_f + V'_f) + 2 (13 - 16 \ln 2) (z'_6 + t'_6 + v'_6) \Big] \\
+ \; & B_{11} B_{10} \Big[- \frac{3}{2} (Y_f + S_f - R_f) \\
& - \frac{3}{2} \Big(- 4 Y'_f + Z_f + T'_f - 4 S'_f - 4 R'_f - V'_f \Big) \Big] ,
\end{aligned}
\tag{5.5.7}
$$

again comparing the ξ and N_F terms, we get :

$$\begin{aligned}
(8) : \frac{3}{8} &= -\frac{1}{2}\left(-\frac{7}{2} + 8\ln 2\right)\left(X'_f + 3Y'_f + 2S'_f + 2R'_f - 3V'_f - W'_f\right) \\
&+ (13 - 16\ln 2)\,(-x'_6 - 3y'_6 - 2s'_6 - 2r'_6 + 3v'_6 + w'_6) \\
&+\frac{3}{4}\,(x_3 + z_3 + t_3 + s_3 + r_3) - \frac{3}{4}\,(x_6 + z_6 + t_6 + s_6 + r_6) \\
&-\frac{9}{4}\,(x'_3 + z'_3 + t'_3 + s'_3 + r'_3 - w'_3) + \frac{9}{4}\,(x'_6 + z'_6 + t'_6 + s'_6 + r'_6 - w'_6) \\
&-\frac{13}{16}\,(Z_f + T_f - S_f - R_f) - \frac{13}{16}\left(-4Y'_f + Z'_f + T'_f - S'_f - R'_f - 4V'_f\right) \\
(8) : 0 &= -\left(-\frac{7}{2} + 8\ln 2\right)\left(X'_f + Z'_f + T'_f + S'_f + R'_f - 2W'_f\right) \\
&- 2(13 - 16\ln 2)\,(x'_6 + z'_6 + t'_6 + s'_6 + r'_6 - 2w'_6) \\
(9) : 0 &= \frac{3}{4}\,(y_3 - z_3 + t_3 + s_3 - r_3) + \frac{3}{4}\,(-y_6 + z_6 - t_6 - s_6 + r_6) \\
&-\frac{9}{4}\,(y'_3 - z'_3 - t'_3 + s'_3 + r'_3 - v'_3) - \frac{9}{4}\,(-y'_6 + z'_6 + t'_6 - s'_6 - r'_6 + v'_6) \\
&+ \left(-\frac{7}{2} + 8\ln 2\right)\left(Z'_f + T'_f + V'_f\right) + 2(13 - 16\ln 2)\,(z'_6 + t'_6 + v'_6) \\
&+\frac{13}{8}(Y_f + S_f - R_f) + \frac{13}{8}\left(-4Y'_f + Z_f + T'_f - 4S'_f - 4R'_f - V'_f\right)
\end{aligned} \quad . \tag{5.5.8}$$

5.5.2 Application to the Photon Function

We now substitute τ' and τ'' into the constraints which the comparison with the photon wave-function renormalisation has yielded, Eqns. (5.4.3-5.4.6). Using Eqn. (5.4.3), we get

$$\begin{aligned}
\frac{2}{3}A_{11} &= -A_{11}\left(X'_\gamma + Z'_\gamma + T'_\gamma + S'_\gamma + R'_\gamma - W'_\gamma\right) \\
&+ B_{11}\left(-Y'_\gamma + Z'_\gamma + T'_\gamma - S'_\gamma - R'_\gamma + V'_\gamma\right) \quad ,
\end{aligned}$$

and comparing ξ and N_F in this equation gives :

$$\boxed{\begin{aligned} (10)\ :\ -\frac{2}{3} &= \left(X'_\gamma + Z'_\gamma + T'_\gamma + S'_\gamma + R'_\gamma - W'_\gamma\right) \\ (11)\ :\ 0 &= \left(-Y'_\gamma + Z'_\gamma + T'_\gamma - S'_\gamma - R'_\gamma + V'_\gamma\right) \end{aligned}} \quad . \qquad (5.5.9)$$

Now Eqn. (5.4.5) yields :

$$\begin{aligned} A_{11}^2 &= -\frac{A_{11}^2}{2}\left(X'_\gamma + Z'_\gamma + T'_\gamma + S'_\gamma + R'_\gamma + W'_\gamma\right) \\ &\quad + A_{11}\,B_{11}\left(Z'_\gamma + T'_\gamma - S'_\gamma - R'_\gamma\right) \\ &\quad - B_{22}\left(Z'_\gamma + T'_\gamma + V'_\gamma\right) \quad , \end{aligned}$$

the following equations :

$$\boxed{\begin{aligned} (14)\ :\ -2 &= \left(X'_\gamma + Z'_\gamma + T'_\gamma + S'_\gamma + R'_\gamma + W'_\gamma\right) \\ (15)\ :\ 0 &= \left(Z'_\gamma + T'_\gamma - S'_\gamma - R'_\gamma\right) \\ (16)\ :\ 0 &= \left(Z'_\gamma + T'_\gamma + V'_\gamma\right) \end{aligned}} \quad . \qquad (5.5.10)$$

Eqn. (5.4.4) leads to :

$$\boxed{\begin{aligned} B_{21} &= A_{11}\,B_{11}\Bigg[\left(\frac{7}{12} - 2\ln 2\right) + \frac{3}{2}\left(X'_\gamma + Z'_\gamma + T'_\gamma + S'_\gamma + R'_\gamma\right) \\ &\qquad + \frac{3}{2}\left(-x'_3 - z'_3 - t'_3 - s'_3 - r'_3 + w'_3\right) \\ &\qquad + (-9 + 12\ln 2)\left(-x'_6 - z'_6 - t'_6 - s'_6 - r'_6 + w'_6\right)\Bigg] \\ &+ B_{11}^2\Bigg[\frac{3}{2}\left(Y_p - Z_p + T_p + S_p - R_p\right) + \frac{3}{2}\left(-y'_3 + z'_3 + t'_3 - s'_3 - r'_3 + v'_3\right) \\ &\qquad + (-9 + 12\ln 2)\left(-y'_6 + z'_6 + t'_6 - s'_6 - r'_6 + v'_6\right)\Bigg] \\ &+ 2\,B_{11}\,B_{10} \end{aligned}} \quad .$$

(5.5.11)

The last constraint, Eqn. (5.4.6), gives :

$$\begin{aligned}
-\left(\frac{7}{24}-\ln 2\right) A_{11}^2 \;+\; & \left(\frac{7}{24}-\ln 2\right) A_{11}\, B_{11} \\
= \;& A_{11}^2 \Bigg[\frac{3}{2}\,(X_p + Z_p + T_p + S_p + R_p) \\
& \quad -3\,(x_3' + z_3' + t_3' + s_3' + r_3' - w_3') \\
& \quad -2(-9+12\ln 2)\,(x_6' + z_6' + t_6' + s_6' + r_6' - w_6') \\
& \quad +\frac{3}{4}\,(x_3' + z_3' + t_3' + s_3' + r_3' + w_3') \\
& \quad +\frac{1}{2}(-9+12\ln 2)\,(x_6' + z_6' + t_6' + s_6' + r_6' + w_6') \Bigg] \\
+ \;& A_{11}\, B_{11} \Bigg[\frac{3}{4}\,(-X_p + 2Y_p - 3Z_p + T_p + S_p - 3R_p) \\
& \quad +3\,(-y_3' + z_3' + t_3' - s_3' - r_3' + v_3') \\
& \quad +\frac{3}{4}\,(x_3' + z_3' + t_3' + s_3' + r_3' - w_3') \\
& \quad +2(-9+12\ln 2)\,(-y_6' + z_6' + t_6' - s_6' - r_6' + v_6') \\
& \quad +\frac{1}{2}(-9+12\ln 2)\,(x_6' + z_6' + t_6' + s_6' + r_6' - w_6') \\
& \quad +\frac{3}{2}\,(- z_3' - t_3' + s_3' + r_3') \\
& \quad +(-9+12\ln 2)\,(- z_6' - t_6' + s_6' + r_6') \Bigg] \\
+ \;& B_{11}^2 \Bigg[-\frac{3}{4}\,(Y_p - Z_p + T_p + S_p - R_p) \\
& \quad -\frac{3}{4}\,(-y_3' + z_3' + t_3' - s_3' - r_3' + v_3') \\
& \quad -\frac{1}{2}(-9+12\ln 2)\,(-y_6' + z_6' + t_6' - s_6' - r_6' + v_6') \\
& \quad +\frac{3}{2}\,(z_3' + v_3' + t_3') \\
& \quad +(-9+12\ln 2)\,(z_6' + v_6' + t_6') \Bigg] \\
+ \;& B_{11}\, B_{10} \Bigg[-\frac{3}{2}\left(Y_p' + S_p' + R_p' \right) \Bigg] \quad .
\end{aligned} \tag{5.5.12}$$

comparing ξ^2, N_F^2 and $\xi\, N_F$ terms, we obtain the equations below :

$$
\begin{aligned}
(17):\ -\left(\frac{7}{24}-\ln 2\right) \;=\;& \frac{3}{2}\,(X_p+Z_p+T_p+S_p+R_p)\\
&-3\,(x_3'+z_3'+t_3'+s_3'+r_3'-w_3')\\
&-2(-9+12\ln 2)\,(x_6'+z_6'+t_6'+s_6'+r_6'-w_6')\\
&+\frac{3}{4}\,(x_3'+z_3'+t_3'+s_3'+r_3'+w_3')\\
&+\frac{1}{2}(-9+12\ln 2)\,(x_6'+z_6'+t_6'+s_6'+r_6'+w_6')\\[2ex]
(18):\ \left(\frac{7}{24}-\ln 2\right) \;=\;& \frac{3}{4}\,(-X_p+2Y_p-3Z_p+T_p+S_p-3R_p)\\
&+3\,(-y_3'+z_3'+t_3'-s_3'-r_3'+v_3')\\
&+\frac{3}{4}\,(x_3'+z_3'+t_3'+s_3'+r_3'-w_3')\\
&+2(-9+12\ln 2)\,(-y_6'+z_6'+t_6'-s_6'-r_6'+v_6')\\
&+\frac{1}{2}(-9+12\ln 2)\,(x_6'+z_6'+t_6'+s_6'+r_6'-w_6')\\
&+\frac{3}{2}\,(-z_3'-t_3'+s_3'+r_3')\\
&+(-9+12\ln 2)\,(-z_6'-t_6'+s_6'+r_6')\\[2ex]
(19):\ 0 \;=\;& -\frac{3}{4}\,(Y_\gamma-Z_\gamma+T_\gamma+S_\gamma-R_\gamma)\\
&-\frac{3}{4}\,(-y_3'+z_3'+t_3'-s_3'-r_3'+v_3')\\
&-\frac{1}{2}(-9+12\ln 2)\,(-y_6'+z_6'+t_6'-s_6'-r_6'+v_6')\\
&+\frac{3}{2}\,(z_3'+v_3'+t_3')\\
&+(-9+12\ln 2)\,(z_6'+v_6'+t_6')+\frac{13}{8}\left(Y_\gamma'+S_\gamma'+R_\gamma'\right)
\end{aligned}
\qquad (5.5.13)
$$

As far as leading and next-to-leading terms are concerned, the above constraints ensure that both fermion and photon propagators are multiplicatively renormalisable in massless unquenched QED. Having these constraints imposes conditions on the transverse part of the vertex. Our intention in future work is to find simple solutions to these constraints and so obtain a non-perturbative form for the fermion-photon vertex. To aid this we need as much information as possible about the vertex function. Part of this comes from the perturbative calculation which we are going to present in the next chapter, where we compute the 3-point vertex and the coefficient constants, τ_i, at one loop order. Surprisingly this perturbative calculation has not been done before. This is because it involves rather complicated and lengthly algebra, as we shall see.

Chapter 6

Complete Analytic Form of One-Loop QED Vertex in Any Covariant Gauge

Contradiction is not a sign of falsity,
nor the lack of contradiction a sign of truth.
-Pascal-

6.1 Introduction

In this chapter, the one loop vertex in QED is calculated in arbitrary covariant gauges as an analytic function of its momenta. As mentioned before, the vertex is decomposed into a longitudinal part, that is fully responsible for ensuring the Ward and Ward-Takahashi identities are satisfied, and a transverse part. Furthermore, in this chapter the transverse part is decomposed into 8 independent components each being separately free of kinematic singularities in **any** covariant gauge in a basis that modifies that proposed by Ball and Chiu [26]. Analytic expressions for all 11 components of the $O(\alpha)$ vertex are given explicitly in terms of elementary functions and one Spence function. These results greatly simplify in particular kinematic regimes.

The only truncation of the complete set of Schwinger-Dyson equations, that we know of, that maintains the gauge invariance and multiplicative renormalizability of a gauge theory at every level of approximation is perturbation theory. Physically meaningful solutions of the Schwinger-Dyson equations must agree with perturbative results in the weak coupling regime. Perturbation theory can thus serve as a guide to allowed non-perturbative forms. As mentioned earlier, the wave-function renormalisation, $F(p^2)$, and the mass function, $M(p^2)$, are the constituents of the full fermion propagator and they can be calculated at each order in perturbation theory. Now these two functions must occur in the fermion-boson vertex, since the Ward-Takahashi identity relates the 3-point Green's function to the fermion propagator in a well-known way. This is satisfied at every order of perturbation theory. Indeed, such identities are true non-perturbatively. Thanks to the works of Ball and Chiu [26] we know how to express the non-perturbative structure of the part of the vertex (a part conventionally called the longitudinal component) that fulfills the Ward-Takahashi identity in terms of the two non-perturbative functions describing the fermion propagator, Eqn. (2.3.34). We have also learnt that multiplicative renormalizability of the fermion propagator imposes further constraints on the vertex, as extensively discussed in Chapters 2, 5, but these have yet to be fully exploited. While the bare fermion-boson vertex in a minimal coupling gauge theory is simply γ^μ, in general the vertex involves twelve spin amplitudes that can be constructed from γ^μ and the two independent 4-momenta at the vertex as elucidated by Bernstein [41]. This would suggest that the complete fermion-boson vertex involved a large number of independent functions. However, some of these at least must be related to the fermion functions $F(p^2)$, $M(p^2)$, not to mention the analogous boson renormalization function $G(p^2)$. It is to the nature of these forms that perturbation theory can be a guide, but only if we calculate in an arbitrary gauge. For instance, if we calculated the vertex in massless QED merely in the Landau gauge we would find the γ^μ component was like its bare form just γ^μ. This would serve little as a pointer to the form $\frac{1}{2}\left[F^{-1}(k^2) + F^{-1}(p^2)\right]\gamma^\mu$ as its non-perturbative structure. Only by calculating the vertex in an arbitrary gauge does this result become clearer. Ball and Chiu have performed this $O(\alpha)$ calculation of the vertex in the Feynman gauge and we will be able to check their result and correct a couple of minor misprints in their published work.

Thus our aim is to compute the fermion-boson vertex to one loop in perturbation theory in any covariant gauge and to decompose it into its 12 spin components, of these all but 1 is zero. This full vertex is by its very nature free of kinematic singularities. We then divide the vertex into two parts : the longitudinal and transverse pieces. The longitudinal component alone fulfills the Ward-Takahashi and Ward identities. The way to ensure this without introducing kinematic singularities was fully described by Ball and Chiu as discussed in Chapter 2. We then investigate the transverse part and decompose it into the basis of 8 vectors proposed by Ball and Chiu [26]. We examine each coefficient of these and find that two have singularities in arbitrary gauges. These are not present in the Feynman gauge in which Ball and Chiu work. We propose a straightforward modification of their basis that ensures each transverse component is separately free of kinematic singularities in any covariant gauge. This makes this basis a natural one for future non-perturbative studies.

We divide the discussion into 6 parts:

- the one-loop calculation of the vertex in asymptotic limit is presented in Sect. 6.2.4 and extraction of the one-loop transverse vertex in the same limit is in Sect. 6.2.5;
- the one loop calculation of the vertex, its decomposition into spin amplitudes and the expression of these in terms of known functions, including one Spence function with 10 different arguments are all presented in Sect. 6.3.6;

- the one loop calculation of the fermion propagator to determine the functions $F(p^2)$, $M(p^2)$, which fix the $O(\alpha)$ longitudinal part of the vertex is in Sect. 6.4.1;
- the extraction of the transverse part of the one loop vertex and its decomposition into 8 independent components in the Ball-Chiu basis are described in the rest of Sect. 6.4.2;
- checking the singularity structure of each of the components of the vertex is given in Sect. 6.5. This leads to the proposal of a new basis for the transverse vertex, which has coefficients that have only the singularities of the full vertex;
- deducing the form of the vertex in specific kinematic regimes.

6.2 Getting Started

6.2.1 Definitions: Feynman rules and basis vectors

For the most part the definitions given here are standard, but they are stated here to make this chapter self contained. The perturbative calculation involves the use of bare quantities defined as follows in Minkowski space :

$$\text{bare vertex :} \quad -ie\Gamma^0_\mu = -ie\gamma^0_\mu \quad , \tag{6.2.1}$$

$$\text{fermion propagator :} \quad iS^0_F(p) = i(\not p + m)/(p^2 - m^2) \quad , \tag{6.2.2}$$

$$\text{photon propagator :} \quad i\Delta^0_{\mu\nu}(p) = -i\left[p^2 g_{\mu\nu} + (\xi - 1)p_\mu p_\nu\right]/p^4 \quad , \tag{6.2.3}$$

where e is the usual QED coupling and the parameter ξ specifies the covariant gauge.

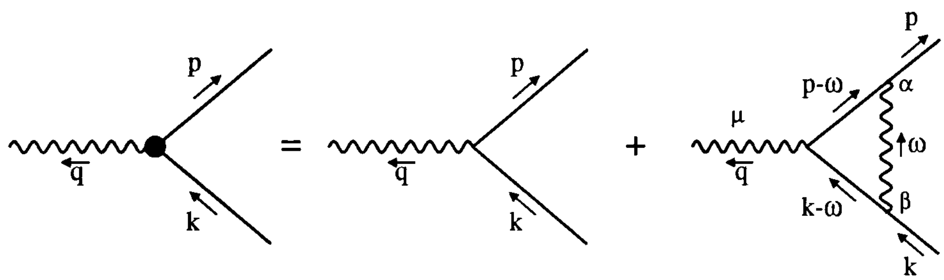


Figure 6.1: The fermion-boson vertex to one loop order showing the definition of momenta and Lorentz indices

The vertex, Fig. 6.1, $\Gamma^\mu(k,p)$ can be expressed in terms of 12 spin amplitudes formed from the vectors γ^μ, k^μ, p^μ and the spin scalars $1, \not{k}$, $\not{p}$ and $\not{k}\ \not{p}$ [41]. Thus we can write

$$\Gamma^\mu = \sum_{i=1}^{12} P^i V_i^\mu , \tag{6.2.4}$$

where we number the V_i^μ as follows

$$\begin{aligned} V_1^\mu &= k^\mu \not{k} \ , \quad V_2^\mu = p^\mu \not{p} \ , \quad V_3^\mu = k^\mu \not{p}, \quad V_4^\mu = p^\mu \not{k} \\ V_5^\mu &= \gamma^\mu \not{k}\not{p}, \quad V_6^\mu = \gamma^\mu \ , \quad V_7^\mu = k^\mu \ , \quad V_8^\mu = p^\mu \\ V_9^\mu &= p^\mu \not{k}\not{p}, \quad V_{10}^\mu = k^\mu \not{k}\not{p}, \quad V_{11}^\mu = \gamma^\mu \not{k}, \quad V_{12}^\mu = \gamma^\mu \not{p} \ . \end{aligned} \tag{6.2.5}$$

The vertex satisfies the Ward-Takahashi identity

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p) \quad , \tag{2.2.1}$$

where $q = k - p$, and the Ward identity

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p^\mu} S_F^{-1}(p) \quad , \tag{2.2.2}$$

as the non-singular $k \to p$ limit of Eqn. (2.2.1). With the fermion propagator given to any order by Eqn. (2.3.1), we follow Ball and Chiu and define the longitudinal component of the vertex by

$$\begin{aligned}\Gamma_L^\mu &= \frac{\gamma^\mu}{2}\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\\ &+ \frac{1}{2}\frac{(\not p+\not k)\,(k+p)^\mu}{(k^2-p^2)}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)\\ &- \frac{(p+k)^\mu}{(k^2-p^2)}\left(\frac{M(k^2)}{F(k^2)}-\frac{M(p^2)}{F(p^2)}\right) \quad . \end{aligned} \tag{2.3.34}$$

Γ_L^μ alone then satisfies the Ward-Takahashi identity, Eq. (2.2.1) and being free of kinematic singularities the Ward identity, Eq.(2.2.2), too. The full vertex can then be written as

$$\Gamma^\mu(k,p) = \Gamma_T^\mu(k,p) + \Gamma_L^\mu(k,p) \quad , \tag{2.3.25}$$

where the transverse part satisfies

$$q_\mu \Gamma_T^\mu(k,p) = 0 \quad \text{and} \quad \Gamma_T^\mu(p,p) = 0 \quad . \tag{6.2.6}$$

The Ward-Takahashi identity fixes 4 coefficients of the 12 spin amplitudes in terms of the fermion functions — the 3 combinations explicitly given in Eq. (2.3.34), while the coefficient of $\sigma_{\mu\nu} k^\mu p^\nu$ must be zero [26]. The transverse component $\Gamma_T^\mu(k,p)$ thus involves 8 vectors, which can be expressed in Ball-Chiu form

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau^i(k^2,p^2,q^2)\, T_i^\mu(k,p) \quad , \tag{2.3.64}$$

where

$$\begin{aligned} T_1^\mu &= p^\mu(k\cdot q) - k^\mu(p\cdot q)\\ T_2^\mu &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)]\,(\not k + \not p)\\ T_3^\mu &= q^2\gamma^\mu - q^\mu \not q\\ T_4^\mu &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)]\, k^\lambda p^\nu \sigma_{\lambda\nu}\\ T_5^\mu &= q_\nu \sigma^{\nu\mu}\\ T_6^\mu &= \gamma^\mu(p^2-k^2) + (p+k)^\mu \not q\\ T_7^\mu &= \frac{1}{2}(p^2-k^2)\,[\gamma^\mu(\not p+\not k) - p^\mu - k^\mu] + (k+p)^\mu\, k^\lambda p^\nu \sigma_{\lambda\nu}\\ T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not p - p^\mu \not k \end{aligned} \tag{6.2.7}$$

with

$$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad . \tag{6.2.8}$$

The coefficients τ_i are Lorentz scalar functions of k and p, i.e. functions of k^2, p^2, q^2.

A general constraint on the eight τ_i's comes from C-parity transformations. The full vertex must transform under charge conjugation, C, in the same way as the bare vertex [41, 25, 12], so that

$$C\,\Gamma_\mu(k,p)\,C^{-1} = -\Gamma^T_\mu(-p,-k) \quad . \qquad (2.2.3)$$

From the Ward-Takahashi identity, Eq. (2.2.1), it is clear that $\Gamma^\mu_L(k,p)$ must be symmetric under $k \leftrightarrow p$ interchange. The symmetry of the transverse part depends on its γ-matrix structure. Thus from Eq. (2.2.3) together with

$$C\gamma_\mu C^{-1} = -\gamma^T_\mu \quad , \qquad (6.2.9)$$

yields the following transformation properties for $\tau_i(k^2, p^2, q^2)$:

$$\begin{aligned} \tau_i(k^2,p^2,q^2) &= \tau_i(p^2,k^2,q^2) \qquad \text{for} \qquad i = 1,2,3,4,5,7,8 \\ \tau_6(k^2,p^2,q^2) &= -\tau_6(p^2,k^2,q^2) \quad . \end{aligned} \qquad (6.2.10)$$

6.2.2 Calculation of One-Loop QED Vertex in the Asymptotic Limit

In this section the one-loop 3-point vertex function shown in Fig. 6.1 is calculated in arbitrary covariant gauge in the asymptotic limit [22], which means
$k^2 >> k \cdot p >> (p^2, m^2)$. This calculation is extremely simple when compared with vertex calculation at all momenta, which is going to be presented later in this chapter. Hovever, this limit can be very helpful for two reasons : (1) Later, it will provide a check to ensure the correctness of the calculation of the vertex in this limit. (2) Even without a lot of complicated algebra being involved, it makes us realize that the Feynman parametrization method does not lead to a reasonably simple answer, since it results in a number of unrelated integrals that cannot be expressed in terms of elementary functions.

The vertex of Fig. 6.1 is naturally written as

$$\Gamma^\mu(k,p) = \gamma^\mu + \Lambda^\mu(k,p) \quad . \qquad (6.2.11)$$

From the Feynman rules specified in previous section, Λ^μ to $O(\alpha)$ is simply given by :

$$-ie\Lambda^\mu = \int_M \frac{d^4w}{(2\pi)^4}(-ie\gamma^\alpha)\, iS_F^0(p-w)\,(-ie\gamma^\mu)iS_F^0(k-w)\,(-ie\gamma^\beta)i\Delta^0_{\alpha\beta}(w)\,, \tag{6.2.12}$$

where M denotes the loop integral is to be performed in Minkowski space. Substituting Eqns. (6.2.3) for $S_F^0(p)$ and $\Delta^0_{\mu\nu}(p)$, we have with $\alpha \equiv e^2/4\pi$:

$$\begin{aligned}
\Lambda^\mu &= \frac{-ie^2}{(2\pi)^4}\int_M d^4w\, \gamma^\alpha \frac{(\not p - \not w + m)}{[(p-w)^2 - m^2]}\gamma^\mu \frac{(\not k - \not w + m)}{[(k-w)^2-m^2]}\gamma^\beta \left[\frac{g_{\alpha\beta}}{w^2} + (\xi - 1)\frac{w_\alpha w_\beta}{w^4}\right], \\
\Lambda^\mu &= -\frac{i\alpha}{4\,\pi^3}\int_M d^4w\, \frac{\gamma^\alpha\,(\not p - \not w + m)\,\gamma^\mu\,(\not k - \not w + m)\,\gamma_\alpha}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\
&\quad - \frac{i\,\alpha}{4\pi^3}(\xi - 1)\int_M d^4w\, \frac{\not w\,(\not p - \not w + m)\,\gamma^\mu\,(\not k - \not w + m)\,\not w}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,,
\end{aligned} \tag{6.2.13}$$

on separating the $g_{\alpha\beta}$ and $w_\alpha w_\beta$ parts of the photon propagator. By making use of the Feynman parametrization, the two integrals in Eqn. (6.2.13) can be solved in the asymptotic limit. We will do this calculation in two parts, one which vanishes in Feynman gauge called Λ_2^μ and the other which does not called Λ_1^μ, as follows :

6.2.3 Λ_1^μ Calculated

In this section we handle the first integral in Eqn. (6.2.13) and solve it by using a cut-off regularization method,

$$\Lambda_1^\mu \equiv -\frac{i\alpha}{4\pi^3}\int_M d^4w\, \frac{\gamma^\alpha(\not p - \not w + m)\,\gamma^\mu\,(\not k - \not w + m)\,\gamma_\alpha}{w^2\,[(k-w)^2-m^2]\,[(p-w)^2-m^2]}\ . \tag{6.2.14}$$

We start by applying a convenient Feynman parametrization to this integral which is :

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy\, \frac{1}{[a+(b-a)x+(c-a)y]^3}\ . \tag{6.2.15}$$

With the choice of

$$a = w^2 \quad , \quad b = (k-w)^2 - m^2 \quad , \quad c = (p-w)^2 - m^2\,, \tag{6.2.16}$$

we get the following expression for the Λ_1^μ :

$$\Lambda_1^\mu = \frac{-i\alpha}{2\pi^3} \int d^4w \int_0^1 dx \int_0^{1-x} dy \frac{\gamma^\alpha (\not p - \not w + m) \gamma^\mu (\not k - \not w + m) \gamma_\alpha}{[w^2 + (k^2 - 2k \cdot w - m^2) x + (p^2 - 2p \cdot w - m^2) y]^3} \quad . \tag{6.2.17}$$

As a standard procedure to avoid performing a tricky angular integration, we change the variable of integration to w', where

$$w' = w - xk - yp \quad . \tag{6.2.18}$$

Consequently, with this new variable w', Eqn. (6.2.17) becomes

$$\begin{aligned} \Lambda_1^\mu \;=\; & -\frac{i\alpha}{2\pi^3} \int_0^1 dx \int_0^{1-x} dy \int_M \frac{d^4w'}{\left(w'^2 - D\right)^3} \\ & \times \left[\gamma^\alpha (\not p - \not w' - x \not k - y \not p + m) \gamma^\mu (\not k - \not w' - x \not k - y \not p + m) \gamma_\alpha\right] \quad , \end{aligned} \tag{6.2.19}$$

where

$$D = -k^2 x (1-x) - p^2 y (1-y) + 2xyk \cdot p + m^2 x + m^2 y \quad . \tag{6.2.20}$$

Referring to Eqn. (F.2), integrals of odd powers of w' give zero. Hence, after discarding these odd terms, we can separate $w^{0'}$ and $w^{2'}$ part of the numerator in Eqn. (6.2.19) to simplify it :

$$\Lambda_1^\mu = -\frac{i\alpha}{2\pi^3} \int_0^1 dx \int_0^{1-x} dy \int_M d^4w' \frac{\left(A_{10} + w^{2'} A_{12}\right) + m\, A_{m0}}{\left(w'^2 - D\right)^3} \quad . \tag{6.2.21}$$

So, now we will evaluate the w'-integration in Euclidean space by employing a Wick rotation, but we will leave external momenta and the mass still in Minkowski space,

$$\Lambda_1^\mu \;=\; \frac{\alpha}{2\pi^3} \int_0^1 dx \int_0^{1-x} dy \int_E d^4w_E' \frac{\left(-A_{10} + w_E^{2'} A_{12}\right) - m\, A_{m0}}{\left(w_E'^2 + D\right)^3} \quad , \tag{6.2.22}$$

where

$$\begin{aligned} A_{10} &= -2\Big[(1-x)(1-y) \not k \gamma_\mu \not p + xy \not p \gamma_\mu \not k \\ &\qquad -x(1-x) \not k \gamma_\mu \not k - y(1-y) \not p \gamma_\mu \not p + m^2 \gamma_\mu\Big] , \\ A_{12} &= \gamma_\mu , \\ A_{m0} &= 4(p^\mu + k^\mu) \quad . \end{aligned} \tag{6.2.23}$$

Making use of the two w'-integrals

$$\int_0^{\Lambda^2} d^4w' \frac{1}{(w'^2+D)^3} = -\frac{\pi^2}{2}\left[\frac{2\,w'^2+D}{(w'^2+D)^2}\right]_0^{\Lambda^2} ,$$
$$\underset{\Lambda^2\to\infty}{=} \frac{\pi^2}{2\,D} \qquad (6.2.24)$$

and

$$\int_0^{\Lambda^2} d^4w' \frac{w'^2}{(w^2+D)^3} = \frac{\pi^2}{2}\left[\frac{1}{2}\frac{w'^4}{(w'^2+D)^2}+\frac{w'^2}{w'^2+D}-\ln(w'^2+D)\right]_0^{\Lambda^2}$$

which gives

$$\int_0^{\Lambda^2} d^4w' \frac{w'^2}{(w^{2\prime}+D)^3} \underset{\Lambda^2\to\infty}{=} \pi^2\left(\ln\frac{\Lambda^2}{D}-\frac{3}{2}\right) , \qquad (6.2.25)$$

Eqn. (6.2.22) can be written as :

$$\Lambda_1^\mu = \frac{\alpha}{2\pi}\int_0^1 dx \int_0^{1-x} dy\left\{-A_{10}\frac{1}{2D}+A_{12}\left(\ln\frac{\Lambda^2}{D}-\frac{3}{2}\right)-A_{m0}\frac{m}{2D}\right\} . \qquad (6.2.26)$$

Obviously, A_{10}, A_{12} and A_{m0} are symmetric and quadratic in x and y. We want to make the integrand linear in either of the two variables in Eqn. (6.2.26). We do this by replacing the y-variable by

$$y = z\,(1-x) , \qquad (6.2.27)$$

Consequently, we get :

$$\begin{aligned}\Lambda_1^\mu = &\frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dz\,(1-x)\\ &\times\left\{\gamma_\mu\left(\ln\frac{\Lambda^2}{D}-\frac{3}{2}\right)-4m(p^\mu+k^\mu)\frac{1}{2D}\right.\\ &+\Big((1-x)(1-z+zx)\,\not{k}\gamma_\mu\,\not{p}+xz(1-x)\,\not{p}\gamma_\mu\,\not{k}\\ &\left.-x(1-x)\,\not{k}\gamma_\mu\,\not{k}-z(1-x)(1-z+zx)\,\not{p}\gamma_\mu\,\not{p}+m^2\gamma_\mu\Big)\frac{1}{D}\right\} ,\end{aligned} \qquad (6.2.28)$$

where

$$D = (k^2 + p^2z^2 - 2k\cdot p\,z)\,x^2 + (-k^2 + p^2z - 2p^2z^2 + 2k\cdot pz + m^2)\,x$$
$$+ \; (-p^2z + p^2z^2 + m^2z) \quad . \qquad (6.2.29)$$

Now let us perform the x-integral first and, for that, recall the related integrals from Appendix G :

$$\int_0^1 dx\,\frac{(1-x)}{D} = -\frac{1}{k^2}\ln\frac{k^2}{p^2} + \mathcal{O}(k^{-4}) \quad ,$$
$$\int_0^1 dx\frac{x^n(1-x)}{D} = 0 \quad ,$$
$$\int_0^1 dx(1-x)\ln(D) \equiv \int_0^1 dx(1-x)\ln(k^2) = \frac{\ln(k^2)}{2} \quad . \qquad (6.2.30)$$

After using these integrals in the previous equation, we collect the terms which are proportional to $\mathcal{O}(k^0\ln k^2, k)$ and $\mathcal{O}(k^{-1}\ln k^2)$ in Λ_1^μ as these are the terms we are interested in :

$$\Lambda_1^\mu = \frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dz\,(1-x)\left(\gamma_\mu\ln\frac{\Lambda^2}{k^2} + (1-z)\;\not{k}\gamma_\mu\;\not{p}\frac{1}{D} - 2m\,k^\mu\frac{1}{D}\right) \quad , \qquad (6.2.31)$$

which, on x-integration, gives

$$\Lambda_1^\mu = \frac{\alpha}{2\pi}\int_0^1 dz\left(-\frac{\gamma_\mu}{2}\ln\frac{k^2}{\Lambda^2} - (1-z)\,\frac{\not{k}\gamma_\mu\,\not{p}}{k^2}\ln\frac{k^2}{p^2} + 2m\,\frac{k^\mu}{k^2}\ln\frac{k^2}{p^2}\right) \quad . \qquad (6.2.32)$$

Evaluating the z-integral, we finally have Λ^μ to order α :

$$\Lambda_1^\mu \overset{k^2\to\infty}{=} \frac{\alpha}{4\pi}\left(-\frac{\not{k}\gamma_\mu\,\not{p}}{k^2}\ln\frac{k^2}{p^2} - \gamma_\mu\ln\frac{k^2}{\Lambda^2} + 4m\frac{k^\mu}{k^2}\ln\frac{k^2}{p^2}\right) \quad . \qquad (6.2.33)$$

6.2.4 Λ_2^μ and Λ^μ Calculated

We now turn our attention to the 2nd integral in Eqn. (6.2.13), which is:

$$\Lambda_2^\mu = -\frac{i\alpha}{4\pi^3}(\xi-1)\int_M d^4w\frac{\not{w}\,(\not{p}-\not{w}+m)\,\gamma_\mu\,(\not{k}-\not{w}+m)\,\not{w}}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \quad . \qquad (6.2.34)$$

Once more, by using a suitable Feynman parametrization for the above expression :

$$\frac{1}{a^2\,bc} = 3!\int_0^1 dx\int_0^{1-x} dy\,\frac{(1-x-y)}{[a+(b-a)x+(c-a)y]^4}\ , \tag{6.2.35}$$

with the same choice of a, b, c as in Eqn. (6.2.16) and the same change of variables from w to w' as mentioned there, Λ_2^μ takes the form :

$$\begin{aligned}\Lambda_2^\mu &= -6\frac{i\alpha}{4\pi^3}(\xi-1)\int_M d^4w'\int_0^1 dx\int_0^{1-x} dy\frac{(1-x-y)}{\left(w'^2-D\right)^4}\\ &\times\left[\left(A_{20}(x,y)+w'^2A_{22}(x,y)+w'^4A_{24}(x,y)\right)+m\left(A_{m0}(x,y)+w'^2A_{m2}(x,y)\right)\right]\ .\end{aligned} \tag{6.2.36}$$

In the above expression, the numerator has been written after eliminating odd integrals and collecting even powers of w. Wick rotating to Euclidean space, Eqn. (6.2.36) becomes,

$$\begin{aligned}\Lambda_2^\mu &= \frac{3\alpha}{2\pi^3}(\xi-1)\int_0^1 dx\int_0^{1-x} dy\int_E d^4w_{\rm E}'\frac{(1-x-y)}{\left(w_{\rm E}'^{\,2}-D\right)^4}\\ &\times\left[\left(A_{20}(x,y)-w_{\rm E}'^{\,2}A_{22}(x,y)+w_{\rm E}'^{\,4}A_{24}(x,y)\right)+m\left(A_{m0}(x,y)-w_{\rm E}'^{\,2}A_{m2}(x,y)\right)\right]\ .\end{aligned} \tag{6.2.37}$$

Making use of the integrals :

$$\begin{aligned}\int_0^{\Lambda^2}\frac{d^4w}{(w^2+D)^4} &= -\frac{1}{3}\frac{\partial}{\partial D}\int_0^{\Lambda^2}\frac{d^4w}{(w^2+D)^3}\overset{\Lambda^2\to\infty}{=}\frac{\pi^2}{6D^2}\ ,\\ \int_0^{\Lambda^2}\frac{d^4w\,w^2}{(w^2+D)^4} &= -\frac{1}{3}\frac{\partial}{\partial D}\int_0^{\Lambda^2}\frac{d^4w\,w^2}{(w^2+D)^3}\overset{\Lambda^2\to\infty}{=}\frac{\pi^2}{3D}\ ,\\ \int_0^{\Lambda^2}\frac{d^4w\,w^4}{(w^2+D)^4} &= \int_0^{\Lambda^2}\frac{d^4w\,w^2}{(w^2+D)^3}-D\int_0^{\Lambda^2}\frac{d^4w\,w^2}{(w^2+D)^4}\\ &\overset{\Lambda^2\to\infty}{=}\pi^2\left(\ln\frac{\Lambda^2}{D}-\frac{11}{6}\right)\ ,\end{aligned} \tag{6.2.38}$$

in Eqn. (6.2.37) and changing the y-variable to z as in Eqn. (6.2.27), we get

$$\begin{aligned}\Lambda_2^\mu &= \frac{3\alpha}{2\pi}(\xi-1)\int_0^1 dx\int_0^{1-x} dy\,(1-x)\,(1-z)\\ &\times\left\{A_{20}(x,z)\frac{1}{6D^2}-A_{22}(x,z)\frac{1}{3D}+A_{24}(x,z)\left(\ln\frac{\Lambda^2}{D}-\frac{11}{6}\right)\right.\\ &\left.\qquad+m\left(A_{m0}(x,z)\frac{1}{6D^2}-A_{m2}(x,z)\frac{1}{3D}\right)\right\}\ ,\end{aligned} \tag{6.2.39}$$

where

$$A_{20}(x,z) = \Big[- x^2 z(1-x)^2 k^2 \not{p} \not{k}\gamma_\mu + x^3(1-x)(1-z+zx)k^2 \not{k}\not{p}\gamma_\mu - x^3(1-x)k^4\gamma_\mu - x^2z(1-x)^2k^2\gamma_\mu \not{k}\not{p} + x^3 z(1-x)k^2\gamma_\mu \not{p}\not{k}\Big](1-x) \quad ,$$

$$A_{22}(x,z) = \Big[-\frac{1}{2}(1-x)(1-z+2zx)\not{k}\gamma_\mu \not{p} - xz(1-x)\not{p}\gamma_\mu \not{k} + \frac{1}{2}x(1-2x)\not{k}\gamma_\mu \not{k} - x(1-2z-2zx)p^\mu \not{k} - z(1-x)(1-2x)k^\mu \not{p} - x(1-2x)k^\mu \not{k} - z(1-x)^2\gamma_\mu \not{k}\not{p} - x(1-2x)k^2\gamma_\mu - x(1-z+zx)\not{k}\not{p}\gamma_\mu\Big](1-x) \quad ,$$

$$\begin{aligned} A_{24}(x,z) &= \gamma_\mu(1-x) \quad , \\ A_{m0}(x,z) &= x^2k^2\gamma_\mu \not{k} - 2x^3k^2k^\mu \quad , \\ A_{m2}(x,z) &= k^\mu(1-3x) \quad . \end{aligned} \tag{6.2.40}$$

Recall the related integrals from Appendix G to evaluate x-integral :

$$\begin{aligned} \int_0^1 dx \frac{(1-x)^2}{D^2} &= -\frac{1}{k^2 a}\left(1+\frac{2k\cdot p\, z}{k^2}\right) + \mathcal{O}(k^{-4}, k^{-6}\ln k^2) \quad , \\ \int_0^1 dx \frac{x(1-x)^2}{D^2} &= \frac{1}{k^4}\left(1+\frac{4k\cdot p\, z}{k^2}\right)\ln\frac{k^2}{p^2} + \mathcal{O}(k^{-6}) \quad , \\ \int_0^1 dx \frac{x^n(1-x)^2}{D^2} &= 0 + \mathcal{O}(k^{-4}, k^{-6}\ln k^2) \quad , \qquad n = 2, 3..... \quad . \end{aligned} \tag{6.2.41}$$

Keeping only terms which would not vanish after performing the x-integral, and collecting terms proportional to $\mathcal{O}(k^0 \ln k^2, k\ln k^2)$ in Λ_2^μ, we get:

$$\begin{aligned} \Lambda_2^\mu &= \frac{3\alpha}{2\pi}(\xi-1)\int_0^1 dx \int_0^1 dz(1-x)(1-z) \\ &\times \left[(1-x)\gamma_\mu \ln\frac{\Lambda^2}{D} - \left(-\frac{1}{2}(1-z)\not{k}\gamma_\mu \not{p} - zk^\mu \not{p} - z(1-x)\gamma_\mu \not{k}\not{p}\right)\frac{(1-x)^2}{3D} \right. \\ &\left. - m\, k^\mu \frac{(1-3x)}{3D}\right] \quad . \end{aligned} \tag{6.2.42}$$

Performing the x and z integration, we have :

$$\Lambda_2^\mu \stackrel{k^2\to\infty}{=} \frac{\alpha}{4\pi}(\xi-1)\left(-\gamma_\mu \ln\frac{k^2}{\Lambda^2}+\frac{k^\mu\, \not p}{k^2}\ln\frac{k^2}{p^2}-m\frac{k^\mu}{k^2}\ln\frac{k^2}{p^2}\right) \quad . \tag{6.2.43}$$

As a last step to completing the calculation for one-loop QED vertex in the asymptotic limit, we add Λ_1^μ and Λ_2^μ to arrive at the final answer :

$$\begin{aligned}\Lambda^\mu &= \Lambda_1^\mu+\Lambda_2^\mu \\ &\stackrel{k^2\to\infty}{=} \frac{\alpha}{4\pi}\left\{-\xi\,\gamma_\mu\ln\frac{k^2}{\Lambda^2}+\frac{1}{k^2}(-\not k\gamma_\mu\not p+(\xi-1)\,k^\mu\,\not p)\ln\frac{k^2}{p^2}\right. \\ &\qquad \left. +m\,(\xi+3)\,\frac{k^\mu}{k^2}\ln\frac{k^2}{p^2}\right\} \quad .\end{aligned} \tag{6.2.44}$$

6.2.5 The Transverse Vertex

Having calculated the full one-loop vertex, there is only one more step towards obtaining the transverse vertex which is to compute the longitudinal part of the vertex to order α, asymptotically, and subtracting it from the full vertex. Calculation of the longitudinal part, which is the Ball-Chiu vertex, depends direction the knowledge of $1/F$ and M/F since they appear in Eqn. (2.3.34). Expressions for these quantities can be borrowed from the detailed discussion in Sect. 6.4.1. :

$$\begin{aligned}\frac{1}{F(p^2)} &= 1-\frac{\alpha\xi}{4\pi}\ln\frac{p^2}{\Lambda^2} \quad , \\ \frac{M(p^2)}{F(p^2)} &= -m\left(\frac{\alpha}{4\pi}(\xi+3)\ln\frac{p^2}{\Lambda^2}\right) \quad .\end{aligned} \tag{6.2.45}$$

Inserting these quantities into Eqn. (2.3.34) leads us to :

$$\begin{aligned}\Gamma_L^\mu &= \Gamma_{BC}^\mu \\ &= -\gamma^\mu\frac{\alpha\xi}{8\pi}\ln\frac{k^2p^2}{\Lambda^4}+\frac{\alpha\xi}{8\pi}\frac{(k^\mu\,\not k+k^\mu\,\not p+p^\mu\,\not k)}{k^2}\ln\frac{p^2}{k^2} \\ &\quad +\frac{\alpha}{4\pi}(\xi+3)\,\frac{k^\mu}{k^2}\ln\frac{k^2}{p^2} \quad .\end{aligned} \tag{6.2.46}$$

Since our aim is to find the transverse part of the vertex to order α, in the limit $k^2 >> k \cdot p >> (p^2, m^2)$, we subtract Eqn. (6.2.46) from Eqn. (6.2.44). Consequently :

$$\begin{aligned}\Gamma^\mu_T &= \Lambda^\mu - \Gamma^\mu_L \\ &= -\frac{\alpha\xi}{8\pi}\gamma^\mu \ln\frac{k^2}{p^2} + \frac{\alpha}{4\pi k^2}[-\not{k}\gamma^\mu \not{p} + (\xi-1)\,k^\mu \not{p}]\ln\frac{k^2}{p^2} \\ &+ \frac{\alpha\xi}{8\pi k^2}(k^\mu \not{k} + k^\mu \not{p} + p^\mu \not{k})\ln\frac{k^2}{p^2} \quad , \qquad (6.2.47)\end{aligned}$$

Since the gauge independent part does not give any contribution to leading log. term of the fermion propagator we can drop this term. Now by rearranging the above expression, we find

$$\Gamma^\mu_T \overset{k^2\to\infty}{=} -\frac{\alpha\xi}{8\pi}\gamma^\mu \ln\frac{k^2}{p^2} + \frac{\alpha\xi}{8\pi k^2}(k^\mu \not{k} + p^\mu \not{k} - p^\mu \not{k})\ln\frac{k^2}{p^2}$$

$$\text{for} \qquad k^2 >> k \cdot p >> (p^2, m^2) \quad . \qquad (6.2.48)$$

6.3 Exact calculation of one loop vertex

In this section the 3-point vertex function shown in Fig. 6.1 is calculated to one-loop order in an arbitrary covariant gauge for every range of momenta and its complete analytic form is presented. In order to do this, we can start off from Eqn. (6.2.13), which is

$$\begin{aligned}\Lambda^\mu &= -\frac{i\alpha}{4\,\pi^3}\int_M d^4w\, \frac{\gamma^\alpha\,(\not{p} - \not{w} + m)\,\gamma^\mu\,(\not{k} - \not{w} + m)\,\gamma_\alpha}{w^2\,[(p-w)^2 - m^2]\,[(k-w)^2 - m^2]} \\ &- \frac{i\,\alpha}{4\pi^3}(\xi - 1)\int_M d^4w\, \frac{\not{w}\,(\not{p} - \not{w} + m)\,\gamma^\mu\,(\not{k} - \not{w} + m)\,\not{w}}{w^4\,[(p-w)^2 - m^2]\,[(k-w)^2 - m^2]} \ , \qquad (6.2.13)\end{aligned}$$

What makes the present calculation in an arbitrary covariant gauge significantly longer and more complicated than that of Ball and Chiu in the Feynman gauge ($\xi = 1$) is the form of the photon propagator, see Eqn. (6.2.3). The decomposition of the loop integrals of Eqs. (6.2.12-6.2.13) into scalar forms in the general case brings greater complexity because of the potential appearance of infrared divergences in Eq. (6.2.13). Nevertheless, knowing the Feynman gauge result is a most helpful check on our results.

Our first step is to perform a little γ-matrix algebra where the useful identities of γ-matrices are given in Appendix A and so we rewrite Eq. (6.2.13) as

$$\Lambda^\mu = -\frac{i\,\alpha}{4\,\pi^3}\int_M d^4w \left\{ \frac{A^\mu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} + (\xi-1)\frac{B^\mu}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \right\}, \tag{6.3.1}$$

where

$$A^\mu = \gamma^\alpha\,(\not p - \not w)\,\gamma^\mu\,(\not k - \not w)\,\gamma_\alpha + m\,\gamma^\alpha\,[(\not p - \not w)\,\gamma^\mu + \gamma^\mu(\not k - \not w)]\,\gamma_\alpha + m^2\,\gamma^\alpha\,\gamma^\mu\,\gamma_\alpha\,, \tag{6.3.2}$$

$$B^\mu = \not w\,(\not p - \not w)\,\gamma^\mu\,(\not k - \not w)\,\not w + m\,\not w\,[(\not p - \not w)\,\gamma^\mu + \gamma^\mu(\not k - \not w)]\,\not w + m^2\,\not w\,\gamma^\mu\,\not w\,. \tag{6.3.3}$$

To proceed, we introduce the following seven basic integrals over the loop momentum d^4w : $J^{(0)}, J^{(1)}_\mu, J^{(2)}_{\mu\nu}, I^{(0)}, I^{(1)}_\mu, I^{(2)}_{\mu\nu}$ and $K^{(0)}$.

$$J^{(0)} = \int_M d^4w \frac{1}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.4}$$

$$J^{(1)}_\mu = \int_M d^4w \frac{w_\mu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.5}$$

$$J^{(2)}_{\mu\nu} = \int_M d^4w \frac{w_\mu w_\nu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.6}$$

$$I^{(0)} = \int_M d^4w \frac{1}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.7}$$

$$I^{(1)}_\mu = \int_M d^4w \frac{w_\mu}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.8}$$

$$I^{(2)}_{\mu\nu} = \int_M d^4w \frac{w_\mu w_\nu}{w^4\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,, \tag{6.3.9}$$

$$K^{(0)} = \int_M d^4w \frac{1}{[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\,. \tag{6.3.10}$$

Now, we shall give the outline of this calculation for ease of understanding :

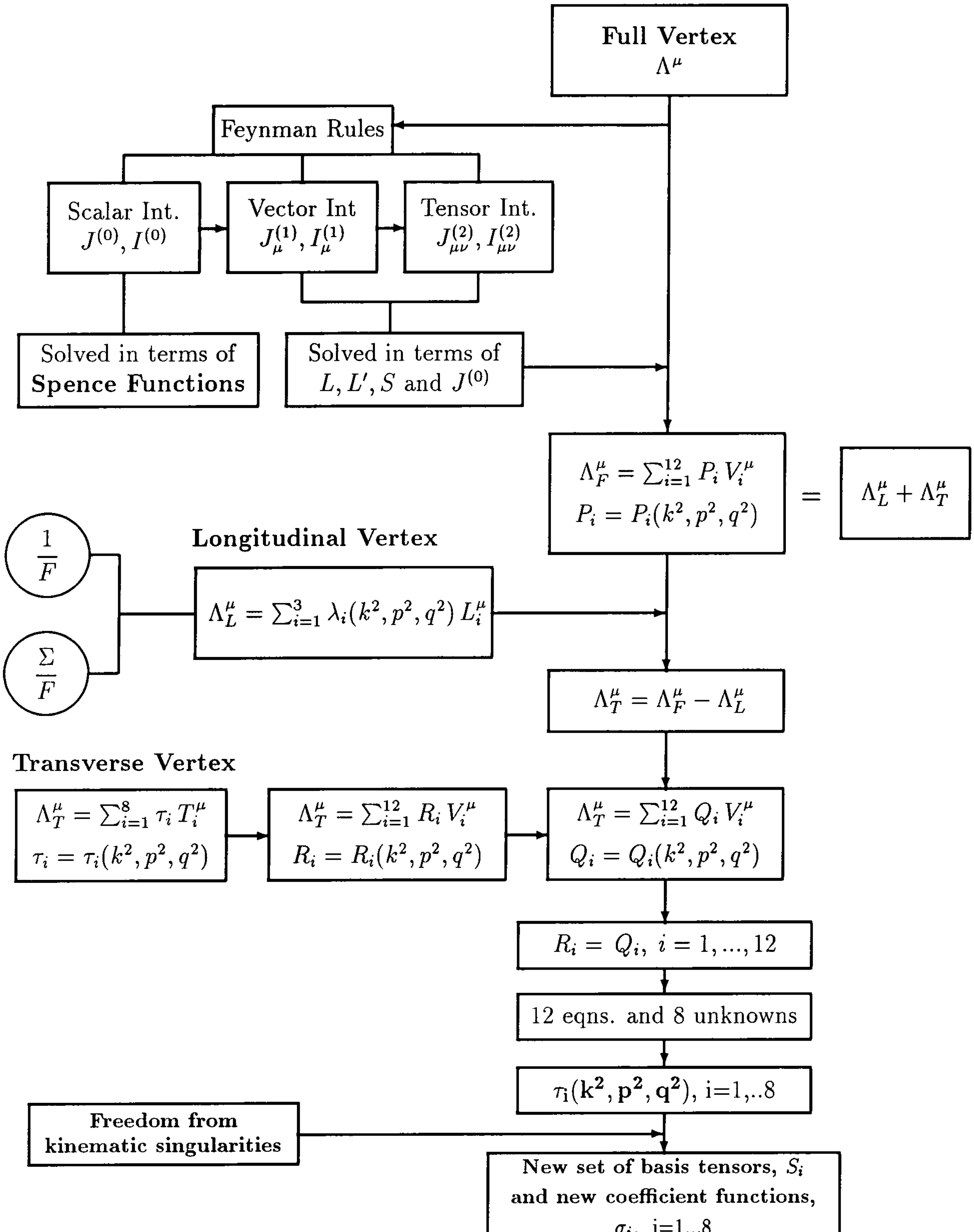


Figure 6.2: Flow diagram of the one-loop vertex calculation

Λ^μ of Eq. (6.3.1) can then be re-expressed in terms of five of these as :

$$\begin{aligned}
\Lambda^\mu &= -\frac{i\,\alpha}{4\,\pi^3}\Bigg\{ \quad \left(\gamma^\alpha\left(\not p\gamma^\mu \not k + m\not p\gamma^\mu + m\gamma^\mu \not k + m^2\gamma^\mu\right)\gamma_\alpha\right) J^{(0)} \\
&\qquad - \left(\gamma^\alpha\left(\not p\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \not k + m\,\gamma^\nu\gamma^\mu + m\gamma^\mu\gamma^\nu\right)\gamma_\alpha\right) J^{(1)}_\nu \\
&\qquad + \left(\gamma^\alpha\gamma^\nu\gamma^\mu\gamma^\lambda\gamma_\alpha\right) J^{(2)}_{\nu\lambda} \\
&+ \quad (\xi-1)\Bigg(\quad \left(-\gamma^\nu\not p\gamma^\mu - \gamma^\mu\not k\gamma^\nu - m\,\gamma^\mu\gamma^\nu - m\,\gamma^\nu\gamma^\mu\right) J^{(1)}_\nu \\
&\qquad + \quad \gamma^\mu K^{(0)} \\
&\qquad + \Big(\gamma^\nu\not p\gamma^\mu\not k\gamma^\lambda + m\gamma^\nu\not p\gamma^\mu\gamma^\lambda + m\gamma^\nu\gamma^\mu\not k\gamma^\lambda \\
&\qquad\qquad + m^2\gamma^\nu\gamma^\mu\gamma^\lambda\Big) I^{(2)}_{\nu\lambda}\Bigg)\Bigg\}. \qquad (6.3.11)
\end{aligned}$$

Our next step is to compute the basic scalar, vector and tensor integrals of Eqs. (6.3.4-6.3.10), [42, 43, 44] each of which is a function of k and p. We relegate to Appendices H and I the tabulation of each of the intermediate integrals.

6.3.1 $J^{(0)}$ Calculated

$J^{(0)}$ is the only scalar integral that appears in the result of the vertex calculation and it can be solved in terms of some special function which is called *Spence function* or *Dilogarithm* [45]. This is a kind of integral that one always comes across when triangular Feynman graph is calculated. In this context, a very useful piece of work comes from t'Hooft and Veltman [42]. They have developed a method to compute the general solution of one-loop one, two, three and four-point functions. We shall follow this method to evaluate the $J^{(0)}$ integral:

$$J^{(0)} = \int_M d^4w \,\frac{1}{w^2\,\left[(p-w)^2 - m^2\right]\,\left[(k-w)^2 - m^2\right]}\;. \qquad (6.3.4)$$

First of all, as a standard procedure, a Feynman parametrization is introduced :

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^x dy \frac{1}{\left[a + (b-a)x + (c-b)y\right]^3} \qquad . \qquad (6.3.12)$$

With the choice

$$\begin{aligned} a &= (k-w)^2 - m^2 \\ b &= (p-w)^2 - m^2 \\ c &= w^2 , \end{aligned} \tag{6.3.13}$$

the denominator of the $J^{(0)}$ integral can be written as

$$D \equiv w^2 + 2w \cdot (kx - px + py - k) + \left(p^2 - k^2\right) x + \left(m^2 - p^2\right) y + k^2 \quad . \tag{6.3.14}$$

To get rid of the scalar product of w, the following substitution is made :

$$w' = w + kx - px + py - k \quad . \tag{6.3.15}$$

Therefore, $J^{(0)}$ can be expressed as

$$J^{(0)} = 2 \int_0^1 dx \int_0^x dy \int_M d^4w' \frac{1}{(w'^2 + N)^3} \quad , \tag{6.3.16}$$

where

$$\begin{aligned} N \;=\; & (k-p)^2 x (1-x) - p^2 y^2 - 2p \cdot (k-p) xy \\ & + y \left(2k \cdot p - p^2 + m^2\right) - m^2 \quad . \end{aligned} \tag{6.3.17}$$

Using dimensional regularization to evaluate the w'-integral (see Appendix F) yields

$$\int d^4w' \frac{1}{(w'^2 + N)^3} = i\pi^2 \frac{\Gamma(1)}{\Gamma(3)} \frac{1}{N} \quad . \tag{6.3.18}$$

Substituting this result into Eqn. (6.3.16) gives

$$J^{(0)} = i\pi^2 \int_0^1 dx \int_0^x dy \left[ax^2 + by^2 + cxy + dx + ey + f\right]^{-1} \quad , \tag{6.3.19}$$

with

$$\begin{aligned} a &= -(k-p)^2, \qquad b = -p^2, \qquad c = -2p \cdot (k-p), \qquad d = (k-p)^2 \quad , \\ e &= 2k \cdot p - p^2 + m^2, \qquad f = -m^2 \quad . \end{aligned} \tag{6.3.20}$$

Changing the y-variable to

$$y' = y - \alpha x \quad , \tag{6.3.21}$$

with the special choice of α such that

$$\left(a + c\alpha + b\alpha^2\right) = 0 \quad , \tag{6.3.22}$$

the integrand of Eqn. (6.3.19) becomes linear in x :

$$\frac{J^{(0)}}{i\pi^2} = \int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy' \left[(Bx + A)\right]^{-1} \quad , \tag{6.3.23}$$

where

$$\begin{aligned} B &= 2b\alpha y' + cy' + d + e\alpha \quad , \\ A &= by'^2 + ey' + f \quad . \end{aligned} \tag{6.3.24}$$

Now it is easy to evaluate the x integral. In order to do this, x and y should be interchanged with some simple transformations. This gives :

$$\frac{J^{(0)}}{i\pi^2} = \int_0^{(1-\alpha)} dy' \int_{y/(1-\alpha)}^1 dx \frac{1}{(Bx + A)} - \int_0^{-\alpha} dy' \int_{-y/\alpha}^1 dx \frac{1}{(Bx + A)} \quad . \tag{6.3.25}$$

Performing the x-integral we have,

$$\begin{aligned} \frac{J^{(0)}}{i\pi^2} &= \int_0^{(1-\alpha)} dy' \frac{1}{B} \left\{ \ln(A + B) - \ln\left(A + B\frac{y'}{1-\alpha}\right) \right\} \\ &- \int_0^{-\alpha} dy' \frac{1}{B} \left\{ \ln(A + B) - \ln\left(A - B\frac{y'}{\alpha}\right) \right\} \quad . \end{aligned} \tag{6.3.26}$$

Rearranging the integrals and adding the same extra term $\ln(by_0{}^2 + ey_0 + f)$ to each of the integrand in a way that the total contribution of this term is zero leads to :

$$\begin{aligned} \frac{J^{(0)}}{i\pi^2} &= \int_{-\alpha}^{(1-\alpha)} dy' \frac{1}{B} \left\{ \ln(A + B) - \ln\left(by_0{}^2 + ey_0 + f\right) \right\}. \\ &- \int_0^{(1-\alpha)} dy' \frac{1}{B} \left\{ \ln\left(A + B\frac{y'}{(1-\alpha)}\right) - \ln\left(by_0{}^2 + ey_0 + f\right) \right\} \\ &+ \int_0^{-\alpha} dy' \frac{1}{B} \left\{ \ln\left(A - B\frac{y'}{\alpha}\right) - \ln\left(by_0{}^2 + ey_0 + f\right) \right\} \quad , \end{aligned} \tag{6.3.27}$$

where

$$B = 0 \quad \Longrightarrow y' = \frac{-d - e\alpha}{2b\alpha + c} = y_0 \quad . \tag{6.3.28}$$

By making further changes of variables

$$y' = y - \alpha \quad , \quad y' = (1-\alpha)\,y \quad , \quad y' = -\alpha y \quad , \tag{6.3.29}$$

in the above three integrals, we get the following expression for $J(0)$:

$$\begin{aligned}
\frac{J^{(0)}}{i\pi^2} = \; & \int_0^1 dy\, \frac{1}{[(c+2b\alpha)\,y + 2a + d + (e+c)\,\alpha]} \\
& \times \left\{ \ln\left[b\,y^2 + (c+e)\,y + a + d + f\right] - \ln\left[b\,y_1^2 + (c+e)\,y_1 + a + d + f\right] \right\} \\
& - \int_0^1 dy\, \frac{(1-\alpha)}{[(c+2b\alpha)\,(1-\alpha)\,y + d + e\,\alpha]} \\
& \times \left\{ \ln\left[(a+b+c)\,y^2 + (d+e)\,y + f\right] - \ln\left[(a+b+c)\,y_2^2 + (d+e)\,y_2 + f\right] \right\} \\
& - \int_0^1 dy\, \frac{\alpha}{[-(c+2b\alpha)\,\alpha\,y + d + e\alpha]} \\
& \times \left\{ \ln\left[a\,y^2 + d\,y + f\right] - \ln\left[a\,y_3^2 + d\,y_3 + f\right] \right\} \quad ,
\end{aligned} \tag{6.3.30}$$

where

$$y_1 = y_0 + \alpha \quad , \quad y_2 = \frac{y_0}{(1-\alpha)} \quad , \quad y_3 = -\frac{y_0}{\alpha} \quad . \tag{6.3.31}$$

Rewriting Eqn. (6.3.30) in a more convenient way for its evaluation in terms of Spence functions, we obtain

$$\begin{aligned}
\frac{J^{(0)}}{i\pi^2} = \frac{1}{(c+2\,b\,\alpha)} \Bigg\{ & \int_0^1 dy \frac{1}{(y-y_1)} \Bigg\{ \ln\left[b\,y^2 + (c+e)\,y + a + d + f\right] \\
& \qquad - \ln\left[b\,y_1^2 + (c+e)\,y_1 + a + d + f\right] \Bigg\} \\
& - \int_0^1 dy \frac{(1-\alpha)}{(y-y_2)} \Bigg\{ \ln\left[(a+b+c)\,y^2 + (d+e)\,y + f\right] \\
& \qquad - \ln\left[(a+b+c)\,y_2^2 + (d+e)\,y_2 + f\right] \Bigg\} \\
& + \int_0^1 dy \frac{\alpha}{(y-y_3)} \left\{ \ln\left[a\,y^2 + d\,y + f\right] - \ln\left[a\,y_3^2 + d\,y_3 + f\right] \right\} \Bigg\}.
\end{aligned} \tag{6.3.32}$$

Three integrands in Eqn. (6.3.32) have a similar form. Thus, if one of these is solved, its solution can be applied to the other two integrals. Now, let us take the third integral in Eqn. (6.3.32) which is the simplest and call it S_3,

$$S_3 \equiv \int_0^1 dy \frac{1}{(y - y_3)} \left[\ln(ay^2 + dy + f) - \ln(a{y_3}^2 + dy_3 + f)\right] \quad . \tag{6.3.33}$$

After splitting up the logarithms, we can write S_3 as follows,

$$S_3 = \int_0^1 dy \frac{1}{(y - y_3)} \left[\ln(y - y_1) + \ln(y - y_2) - \ln(y_3 - y_1) - \ln(y_3 - y_2)\right] \quad , \tag{6.3.34}$$

where y_1 and y_2 are the roots of the second order polynominal $(x^2 + dx/a + c/a)$. Now let us concentrate on the first and third terms in Eqn. (6.3.34) and call their sum R,

$$R \equiv \int_0^1 dy \frac{1}{(y - y_3)} \left[\ln(y - y_1) - \ln(y_3 - y_1)\right] \quad . \tag{6.3.35}$$

We are now approaching the integral form of the Spence function. In order to achieve this, let us replace y by y' :

$$y' = y - y_1 \quad . \tag{6.3.36}$$

The R-integral is then divided into two pieces :

$$\begin{aligned} R &= \int_0^{1-y_1} dy' \frac{1}{y' + y_1 - y_3} \left[\ln y' - \ln(y_3 - y_1)\right] \\ &- \int_0^{-y_1} dy' \frac{1}{y' + y_1 - y_3} \left[\ln y' - \ln(y_3 - y_1)\right] \quad . \end{aligned} \tag{6.3.37}$$

Making further substitutions

$$y' = (1 - y_1)\, y'' \qquad \text{and} \qquad y' = -y_1\, y'' \quad , \tag{6.3.38}$$

respectively in the above two integrals, we find

$$\begin{aligned} R &= \int_0^1 dy'' \left\{ \frac{(1 - y_1)}{(1 - y_1)y'' + y_1 - y_3} \left[\ln\left((1 - y_1)y''\right) - \ln(y_3 - y_1)\right] \right\} \\ &- \int_0^1 dy'' \left\{ \frac{y_1}{y_1 y'' - y_1 + y_3} \left[\ln(-y_1 y'') - \ln(y_3 - y_1)\right] \right\} \quad . \end{aligned} \tag{6.3.39}$$

After integrating by parts, we obtain

$$
\begin{aligned}
R &= \ln\left(1+\frac{1-y_1}{y_1-y_3}\right)\ln\left(\frac{1-y_1}{y_3-y_1}\right) - \int_0^1 dy''\frac{1}{y''}\ln\left(1-\frac{y-1-1}{y_1-y_3}y''\right) \\
&- \ln\left(1-\frac{y_1}{y_1-y_3}\right)\ln\left(\frac{-y_1}{y_3-y_1}\right) - \int_0^1 dy''\frac{1}{y''}\ln\left(1-\frac{y-1}{y_1-y_3}y''\right) \quad . \qquad (6.3.40)
\end{aligned}
$$

Since the definition of the Spence function is :

$$
Sp(x) = -\int_0^1 dt\,\frac{\ln(1-xt)}{t} \quad , \qquad (6.3.41)
$$

R can be written in terms of Spence functions as

$$
\begin{aligned}
R &= Sp\left(\frac{y_1-1}{y_1-y_3}\right) + \ln\left(\frac{1-y_3}{y_1-y_3}\right)\ln\left(\frac{y_1-1}{y_1-y_3}\right) \\
&- Sp\left(\frac{y_1}{y_1-y_3}\right) - \ln\left(\frac{-y_3}{y_1-y_3}\right)\ln\left(\frac{y_1}{y_1-y_3}\right) \quad . \qquad (6.3.42)
\end{aligned}
$$

Using the following property of the Spence functions,

$$
Sp(x) = -Sp(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x) \quad , \qquad (6.3.43)
$$

we can eliminate the explicit logarithms and R becomes simpler :

$$
R = Sp\left(\frac{y_3}{y_3-y_1}\right) - Sp\left(\frac{y_3-1}{y_3-y_1}\right) \quad . \qquad (6.3.44)
$$

Now taking into account the second and fourth terms in Eqn. (6.3.34) as well, the complete S_3 can be written as :

$$
S_3 = Sp\left(\frac{y_3}{y_3-y_1}\right) - Sp\left(\frac{y_3-1}{y_3-y_1}\right) + Sp\left(\frac{y_3}{y_3-y_2}\right) - Sp\left(\frac{y_3-1}{y_3-y_2}\right) \quad . \qquad (6.3.45)
$$

Having computed the third integral in Eqn. (6.3.32), we can apply this solution to the other integrals in Eqn. (6.3.32). But before continuing with this procedure, we first substitute a, b, c, d, e and f from Eqn. (6.3.20) into Eqn. (6.3.32) :

$$
\begin{aligned}
\frac{J^{(0)}}{i\pi^2} &= \frac{1}{-2\,(\pm\Delta)} \\
\times\Bigg\{ & \int_0^1 dy' \frac{1}{y'-y_1}\left(\ln\left[y'^2 - \left(1+\frac{m^2}{p^2}\right) y' + \frac{m^2}{p^2}\right] - \ln\left[{y_1}^2 - \left(1+\frac{m^2}{p^2}\right) y_1 + \frac{m^2}{p^2}\right]\right) \\
- & \int_0^1 dy' \frac{1}{y'-y_2}\left(\ln\left[y'^2 - \left(1+\frac{m^2}{k^2}\right) y' + \frac{m^2}{k^2}\right] - \ln\left[{y_2}^2 - \left(1+\frac{m^2}{k^2}\right) y_2 + \frac{m^2}{k^2}\right]\right) \\
+ & \int_0^1 dy' \frac{1}{y'-y_3}\left(\ln\left[y'^2 - y' + \frac{m^2}{q^2}\right] - \ln\left[{y_3}^2 - y_3 + \frac{m^2}{q^2}\right]\right)\Bigg\},
\end{aligned}
\tag{6.3.46}
$$

where

$$
\Delta^2 = (k\cdot p)^2 - k^2 p^2 \quad . \tag{6.3.47}
$$

Now using Eqn. (6.3.45) with choice of $(+\Delta)$, $J^{(0)}$ acquires the form,

$$
\begin{aligned}
J^{(0)} = \frac{i\pi^2}{-2\,\Delta}\Bigg\{ & Sp\left(\frac{y_1}{y_1-1}\right) + Sp\left(\frac{y_1}{y_1-\frac{m^2}{p^2}}\right) - Sp\left(\frac{y_1-1}{y_1-\frac{m^2}{p^2}}\right) \\
& -Sp\left(\frac{y_2}{y_2-1}\right) - Sp\left(\frac{y_2}{y_2-\frac{m^2}{k^2}}\right) + Sp\left(\frac{y_2-1}{y_2-\frac{m^2}{k^2}}\right) \\
& +Sp\left(\frac{y_3}{y_3-q_1}\right) - Sp\left(\frac{y_3-1}{y_3-q_1}\right) \\
& +Sp\left(\frac{y_3}{y_3-q_2}\right) - Sp\left(\frac{y_3-1}{y_3-q_2}\right)\Bigg\} \quad ,
\end{aligned}
\tag{6.3.48}
$$

where

$$
\begin{aligned}
\alpha &= 1 + \frac{-(k\cdot p)+\Delta}{p^2}, \quad y_1 = y_0 + \alpha, \quad y_2 = \frac{y_0}{(1-\alpha)}, \quad y_3 = -\frac{y_0}{\alpha} \quad , \\
y_0 &= \frac{1}{2\,p^2\,\Delta}\left[k^2p^2 - 2(k\cdot p)^2 + 2(k\cdot p)\Delta - p^2\Delta + p^2(k\cdot p) - m^2(k\cdot p - \Delta)\right] \quad , \\
q_1 &= \frac{1+\sqrt{1-4m^2/q^2}}{2}, \qquad q_2 = \frac{1-\sqrt{1-4m^2/q^2}}{2} \quad .
\end{aligned}
\tag{6.3.49}
$$

Massless case:

It is useful to consider the massless case, when the integral $J^{(0)}$ greatly simplifies. Taking $m \rightarrow 0$, the $J^{(0)}$ becomes

$$J^{(0)} = -\frac{i\pi^2}{2\Delta}\left\{ Sp\left(\frac{y_1}{y_1-1}\right) - Sp\left(\frac{y_1-1}{y_1}\right) - Sp\left(\frac{y_2}{y_2-1}\right) \right.$$
$$\left. + Sp\left(\frac{y_2-1}{y_2}\right) + Sp\left(\frac{y_3}{y_3-1}\right) - Sp\left(\frac{y_3-1}{y_3-1}\right) \right\} \quad . \qquad (6.3.50)$$

Using the following properties of the Spence function:

$$Sp(x) + Sp\left(\frac{x}{x-1}\right) = -\frac{1}{2}\ln^2(1-x), \qquad x < 1 \qquad (6.3.51)$$

$$Sp(x) + Sp\left(\frac{1}{x}\right) = -\frac{1}{2}\ln^2(x) - i\pi \ln x + \frac{\pi^2}{3}, \qquad x > 1 \qquad (6.3.52)$$

$$\begin{aligned} Sp(x+y-xy) &= Sp(x) + Sp(y) - Sp\left(\frac{xy-y}{x}\right) \\ &- Sp\left(\frac{xy-x}{y}\right) + \frac{1}{2}\ln^2\frac{x}{y} - \frac{\pi^2}{6} \\ &- \ln\left(\frac{x+y+xy}{x}\right)\ln\left(\frac{y-xy}{x}\right) \\ &- \ln\left(\frac{x+y+xy}{x}\right)\ln\left(\frac{x-xy}{y}\right) \quad , \end{aligned} \qquad (6.3.53)$$

and writing dilogarithms in terms of the Spence function

$$f(x) = Sp(1-x) \quad , \qquad (6.3.54)$$

straightforward but tedious algebra enables to us to write $J^{(0)}$ as

$$J^{(0)} = \frac{1}{\Delta}\left\{ f\left(\frac{k\cdot p-\Delta}{p^2}\right) - f\left(\frac{k\cdot p+\Delta}{p^2}\right) + \frac{1}{2}\ln\frac{q^2}{p^2}\ln\left(\frac{k\cdot p-\Delta}{k\cdot p+\Delta}\right) \right\} . \qquad (6.3.55)$$

6.3.2 $I^{(0)}$ Calculated

$$I^{(0)} = \int d^d w \frac{1}{[(p-w)^2 - m^2]\,[(k-w)^2 - m^2]\,w^4} \quad . \tag{6.3.8}$$

This integral did not appear in the result of the one-loop vertex calculation but only in an intermediate stage. In any gauge theory, when one deals with vector particles (such as the photon here) in other than Feynman gauge ($\xi = 1$), the above integral appears in the calculation. The w^4 term in the denominator puts the $I^{(0)}$ vertex-type loop integral in a different category compared to the $J^{(0)}$-type integral. In order to evaluate $I^{(0)}$ integral, a recursive algorithm is used proposed by Davydychev for the vertex-type diagram [44] in the massless case. We introduce this technique to solve the massive $I^{(0)}$ integral below.

Methodology:

In general, the Feynman integral corresponding to Fig. 6.1 can be written in the following form :

$$I(\nu_1, \nu_2, \nu_3) = \int d^d w \frac{1}{[(p-w)^2 - m^2]^{\nu_1}\,[(k-w)^2 - m^2]^{\nu_2}\,[w^2]^{\nu_3}} , \tag{6.3.56}$$

where d is the space-time dimension, ν_i $(i = 1, 2, 3)$ are the powers of denominators assumed to be positive integers. If one of the indices ν_i vanishes then this integral takes the form of the one-loop 2-point integral. This method (algorithm) is based on the so-called *integration by parts* technique. By using the following identity, which is true for $i = 1, 2, 3$

$$\int d^d w \frac{\partial}{\partial w_\mu} \left\{ \frac{(q_i - w)}{[(p-w)^2 - m^2]^{\nu_1}\,[(k-w)^2 - m^2]^{\nu_2}\,[w^2]^{\nu_3}} \right\} = 0 , \tag{6.3.57}$$

we can define our q_1, q_2 and q_3 for the $I^{(0)}$-integral as:

$$i = 1, 2, 3, \qquad q_1 = p, \qquad q_2 = k, \qquad q_3 = 0 \quad . \tag{6.3.58}$$

This identity comes from the possibility of throwing away surface terms in the case of dimensionally regularized integrals. Performing the derivative in Eqn. (6.3.57), we find

$$dI(\nu_1, \nu_2, \nu_3) = \int d^d w (q_i - w)_\mu \left\{ \frac{2\nu_1 (p-w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2 (k-w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} + \frac{2\nu_3 w^\mu}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right\} = 0, \tag{6.3.59}$$

where

$$A = (p-w)^2 - m^2, \qquad B = (k-w)^2 - m^2, \qquad C = w^2 \quad . \tag{6.3.60}$$

For $i = 1$, Eqn. (6.3.59) can be written as

$$dI(\nu_1, \nu_2, \nu_3) = \int d^d w \, (p - w)_\mu \left\{ \frac{2\nu_1 (p-w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2 (k-w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} + \frac{2\nu_3 w^\mu}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right\} = 0 \, . \tag{6.3.61}$$

Making use of the following identity [46, 47]

$$2(q_i - w)_\mu (q_j - w)^\mu = (q_i - w)^2 + (q_i - w)^2 - (q_i - q_j)^2 \quad , \tag{6.3.62}$$

for $\underline{i = 1}$ we have,

$$\begin{aligned} 2(p-w)_\mu (p-w)^\mu &= 2(p-w)^2 \quad , \\ 2(p-w)_\mu (k-w)^\mu &= 2(p-w)^2 + (k-w)^2 - (p-k)^2 \quad , \\ 2(p-w)_\mu (-w)^\mu &= 2(p-w)^2 + w^2 - p^2 \quad . \end{aligned} \tag{6.3.63}$$

If we substitute Eqn. (6.3.63) into Eqn. (6.3.61), and also add and subtract m^2 term to each of the integrands in the following way, we find

$$\begin{aligned} dI(\nu_1, \nu_2, \nu_3) = \int d^d w \Bigg\{ & \frac{2\nu_1 \left[(p-w)^2 \pm \mathbf{m^2}\right]}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} \\ & + \frac{2\nu_2 \left[(p-w)^2 \pm \mathbf{m^2} + (k-w)^2 \pm \mathbf{m^2} - q^2\right]}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} \\ & + \frac{2\nu_3 \left[(p-w)^2 \pm \mathbf{m^2} + w^2 - p^2\right]}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \Bigg\} \quad . \end{aligned} \tag{6.3.64}$$

Recalling A, B and C from Eqn. (6.3.60), we can rewrite the above equation in a simpler form :

$$\begin{aligned} dI(\nu_1,\nu_2,\nu_3) &= \int d^d w \left\{ \frac{2\nu_1+\nu_2+\nu_3}{A^{\nu_1}B^{\nu_2}C^{\nu_3}} + \frac{2\nu_1 m^2}{A^{\nu_1+1}B^{\nu_2}C^{\nu_3}} \right. \\ &\quad + \frac{\nu_2}{A^{\nu_1-1}B^{\nu_2+1}C^{\nu_3}} + \frac{2\nu_2 m^2 - \nu_2 q^2}{A^{\nu_1}B^{\nu_2+1}C^{\nu_3}} \\ &\quad \left. + \frac{\nu_3}{A^{\nu_1-1}B^{\nu_2}C^{\nu_3+1}} + \frac{2\nu_3 m^2 - \nu_3 p^2}{A^{\nu_1}B^{\nu_2}C^{\nu_3+1}} \right\} \quad . \qquad (6.3.65) \end{aligned}$$

Refering to the definition of $I(\nu_1,\nu_2,\nu_3)$, Eqn. (6.3.56), we find the following relations between the integrals :

$$\begin{aligned} &-2\nu_1 m^2 I(\nu_1+1,\nu_2,\nu_3) + \left(q^2-2m^2\right)\nu_2 I(\nu_1,\nu_2+1,\nu_3) \\ &\quad + \left(p^2-m^2\right)\nu_3 I(\nu_1,\nu_2,\nu_3+1) = (2\nu_1+\nu_2+\nu_3-d)\, I(\nu_1,\nu_2,\nu_3) \\ &\quad + \nu_2 I(\nu_1-1,\nu_2+1,\nu_3) + \nu_3 I(\nu_1-1,\nu_2,\nu_3+1) \quad . \qquad (6.3.66) \end{aligned}$$

Eqn. (6.3.66) is written in such a form that integrals with the sum of the indices $\sigma = \nu_1+\nu_2+\nu_3$ are collected on the right hand side, and integrals with $\sigma = \nu_1+\nu_2+\nu_3+1$ are on the left. Similar calculations for $i = 2, 3$ yield,

$\underline{i=2}$:

$$\begin{aligned} &\nu_1\left(q^2-2m^2\right) I(\nu_1+1,\nu_2,\nu_3) - 2\nu_2 m^2 I(\nu_1,\nu_2+1,\nu_3) \\ &\quad + \nu_3\left(k^2-m^2\right) I(\nu_1,\nu_2,\nu_3+1) = (\nu_1+2\nu_2+\nu_3-d)\, I(\nu_1,\nu_2,\nu_3) \\ &\quad + \nu_1 I(\nu_1+1,\nu_2-1,\nu_3) + \nu_3 I(\nu_1,\nu_2-1,\nu_3+1), \qquad (6.3.67) \end{aligned}$$

$\underline{i=3}$:

$$\begin{aligned} &\nu_1\left(p^2-m^2\right) I(\nu_1+1,\nu_2,\nu_3) + \nu_2\left(k^2-m^2\right) I(\nu_1,\nu_2+1,\nu_3) \\ &\quad = (\nu_1+\nu_2+2\nu_3-d)\, I(\nu_1,\nu_2,\nu_3) + \nu_1 I(\nu_1+1,\nu_2,\nu_3-1) \\ &\quad + \nu_3 I(\nu_1,\nu_2+1,\nu_3-1) \quad . \qquad (6.3.68) \end{aligned}$$

Eqns. (6.3.66-6.3.68) can be regarded as a system of simultaneous equations to be solved for the integrals $I(\nu_1+1,\nu_2,\nu_3)$, $I(\nu_1,\nu_2+1,\nu_3)$ and $I(\nu_1,\nu_2,\nu_3+1)$ with the determinant

$$2\chi \equiv \begin{vmatrix} -2\nu_1 m^2 & \nu_2(q^2-2m^2) & \nu_3(p^2-m^2) \\ \nu_1(q^2-2m^2) & -2\nu_2 m^2 & \nu_3(k^2-m^2) \\ \nu_1(p^2-m^2) & \nu_2(k^2-m^2) & 0 \end{vmatrix} \quad . \tag{6.3.69}$$

Evaluating the above determinant, we obtain,

$$\begin{aligned} 2\chi &= \nu_1\,\nu_2\,\nu_3 \Big[\ 2\,(p^2-m^2)(q^2-2m^2)(k^2-m^2) \\ &\quad + 2\,m^2(p^2-m^2)^2 + 2\,m^2(k^2-m^2)^2 \Big] \quad . \end{aligned} \tag{6.3.70}$$

If these equations are solved for $J(\nu_1,\nu_2,\nu_3+1)$, we find :

$$I(\nu_1,\nu_2,\nu_3+1) = \frac{1}{2\chi}\times \begin{vmatrix} -2\nu_1 m^2 & \nu_2(q^2-2m^2) & \begin{array}{r}(2\nu_1+\nu_2+\nu_3-d)I(\nu_1,\nu_2,\nu_3)\\ +\nu_2 I(\nu_1-1,\nu_2+1,\nu_3)\\ +\nu_3 I(\nu_1-1,\nu_2,\nu_3+1)\end{array} \\ \nu_1(q^2-2m^2) & -2\nu_2 m^2 & \begin{array}{r}(\nu_1+2\nu_2+\nu_3-d)I(\nu_1,\nu_2,\nu_3)\\ +\nu_1 I(\nu_1+1,\nu_2-1,\nu_3)\\ +\nu_3 I(\nu_1,\nu_2-1,\nu_3+1)\end{array} \\ \nu_1(p^2-m^2) & \nu_2(k^2-m^2) & \begin{array}{r}(\nu_1+\nu_2+2\nu_3-d)I(\nu_1,\nu_2,\nu_3)\\ +\nu_1 I(\nu_1+1,\nu_2,\nu_3-1)\\ +\nu_2 I(\nu_1,\nu_2+1,\nu_3-1)\end{array} \end{vmatrix} . \tag{6.3.71}$$

Having calculated the determinant of this matrix leads us to the following relation :

$$\begin{aligned} I\,(\nu_1,\nu_2,\nu_3+1) = \frac{\nu_1\nu_2}{2\chi}\Bigg\{ & \Big[4m^4-(q^2-2m^2)^2\Big]\times \\ & [(\nu_1+\nu_2+2\,\nu_3-d)\,I\,(\nu_1,\nu_2,\nu_3)+\nu_1 I\,(\nu_1+1,\nu_2,\nu_3-1)+\nu_2 I\,(\nu_1,\nu_2+1,\nu_3-1)] \\ & + \Big[(p^2-m^2)(q^2-2m^2)+2m^2(k^2-m^2)\Big]\times \\ & [(\nu_1+2\nu_2+\nu_3-d)\,I\,(\nu_1,\nu_2,\nu_3)+\nu_1 I\,(\nu_1+1,\nu_2-1,\nu_3)+\nu_3 I\,(\nu_1,\nu_2-1,\nu_3+1)] \\ & + \Big[(k^2-m^2)(q^2-2m^2)+2m^2(p^2-m^2)\Big]\times \\ & [(\nu_1+\nu_2+2\nu_3-d)\,I\,(\nu_1,\nu_2,\nu_3)+\nu_2 I\,(\nu_1-1,\nu_2+1,\nu_3)+\nu_3 I\,(\nu_1-1,\nu_2,\nu_3+1)] \Bigg\}. \end{aligned} \tag{6.3.72}$$

To solve for the $I^{(0)}$ integral, Eqn. (6.3.8), ν_1, ν_2 and ν_3 have to be chosen as follows in Eqn. (6.3.72) :

$$\nu_1 = \nu_2 = \nu_3 = 1 \quad . \tag{6.3.73}$$

We can then write the $I^{(0)}$ integral as,

$$\begin{aligned} I^{(0)} &= I(1,1,2) \\ &= \frac{1}{2\chi}\Bigg\{ I(1,1,1)\,(4-d)\Big[4m^4 + (p^2-m^2)(q^2-2m^2) \\ &\quad + (k^2-m^2)(q^2-2m^2) + 2m^2(p^2-m^2) + 2m^2(k^2-m^2) - (q^2-2m^2)^2\Big] \\ &\quad + [I(2,1,0)+I(1,2,0)]\left[4m^4 - (q^2-2m^2)^2\right] \\ &\quad + [I(2,0,1)+I(1,0,2)]\left[(p^2-m^2)(q^2-2m^2) + 2m^2(k^2-m^2)\right] \\ &\quad + [I(0,2,1)+I(0,1,2)]\left[(k^2-m^2)(q^2-2m^2) + 2m^2(p^2-m^2)\right]\Bigg\}. \end{aligned} \tag{6.3.74}$$

Apparently the complete solution for $I^{(0)}$ depends on the integrals of two-point functions which are $I(2,1,0), I(1,2,0), I(2,0,1), I(1,0,2), I(0,2,1)$ and $I(0,1,2)$. Since $\epsilon = 4-d$, one can notice that the $\epsilon I(1,1,1)$ term in Eqn. (6.3.74) disappears as $\epsilon \to 0$.

To evaluate $I(2,1,0)$ and $I(1,2,0)$, we can refer to the integral $\int d^d w \left[(p-w)^2 - m_2^2\right]^{-1} \left[(k-w)^2 - m_1^2\right]^{-1}$ whose solution is given in Appendix H. Then we define $I(2,1,0)$ from Eqn. (6.3.56) as :

$$I(2,1,0)(k,p) = \int \frac{d^d w}{[(p-w)^2 - m^2]^2\,[(k-w)^2 - m^2]} \quad . \tag{6.3.75}$$

The relation between $I(2,1,0)$ and Eqn. (H.9) is given below

$$\begin{aligned} I(2,1,0)(k,p) &= \frac{1}{2m^2}\frac{\partial}{\partial m_2}\int \frac{d^d w}{[(p-w)^2 - m_2^2]\,[(k-w)^2 - m_1^2]}\Bigg|_{m_1=m_2=m} \\ &= 2i\pi^2 \frac{1}{(q^2-4m^2)} S \quad . \end{aligned} \tag{6.3.76}$$

Similarly, we find for $I(1,2,0)$:

$$\begin{aligned}
I(1,2,0)(k,p) &= I(2,1,0)(k,p) = 2i\pi^2\frac{1}{(q^2-4m^2)}S \quad , \\
I(2,0,1) &= i\pi^2\frac{1}{(p^2-m^2)}L \quad , \\
I(1,0,2) &= \frac{i\mu^\epsilon}{(m^2-p^2)}\left(C-\frac{p^2+m^2}{p^2-m^2}L\right) \quad , \\
I(0,1,2) &= I(1,0,2)\,(k^2\leftrightarrow p^2) \quad , \\
I(0,2,1) &= I(2,0,1)\,(k^2\leftrightarrow p^2) \quad .
\end{aligned} \tag{6.3.77}$$

Substituting Eqn. (6.3.77) into Eqn. (6.3.74), we finally complete the solution of the $I^{(0)}$-integral :

$$\begin{aligned}
I^{(0)} &= I(1,1,2) \\
&= i\pi^2\Bigg\{\frac{1}{\chi}\Bigg[-2q^2S + p^2\,\frac{[(p^2-m^2)\,q^2+2\,m^2(k^2-p^2)]}{(p^2-m^2)^2}\,L \\
&\qquad + k^2\,\frac{[(k^2-m^2)\,q^2-2\,m^2(k^2-p^2)]}{(k^2-m^2)^2}\,L'\Bigg] \\
&\qquad - \frac{\mu^\epsilon C}{(p^2-m^2)\,(k^2-m^2)}\Bigg\} \quad .
\end{aligned} \tag{6.3.78}$$

In the massless case:

Letting mass m, go to zero, χ becomes

$$\chi = k^2p^2q^2 \quad . \tag{6.3.79}$$

and $I^{(0)}$ becomes appreciably simpler

$$I^{(0)} = I(1,1,2) = \frac{i\pi^2}{k^2p^2}\left[-C+\ln\left(\frac{k^2p^2}{q^2\mu^\epsilon}\right)\right] \quad . \tag{6.3.80}$$

6.3.3 $J_\mu^{(1)}$ Calculated

The method of relating Lorentz vector and tensor integrals to scalar integrals is by now standard [26]. Recall Eqn. (6.3.6)

$$J_\mu^{(1)} = \int_M d^4w\,\frac{w_\mu}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \quad . \tag{6.3.6}$$

As a Lorentz vector $J_\mu^{(1)}$ can only have components in the directions of the 4-momenta k_μ and p_μ. Thus, we can write :

$$J_\mu^{(1)} = \frac{i\pi^2}{2}\left[k_\mu J_A(k,p) + p_\mu J_B(k,p)\right] \quad , \tag{6.3.81}$$

where J_A, J_B must be scalar functions of k and p. The factor of $i\pi^2/2$ is taken out purely for later convenience. By forming scalar products as

$$\begin{aligned} k^\mu J_\mu^{(1)} &= \frac{i\pi^2}{2}\left(k^2 J_B + k\cdot p J_B\right) \quad , \\ p^\mu J_\mu^{(1)} &= \frac{i\pi^2}{2}\left(k\cdot p J_A + p^2 J_B\right) \quad , \end{aligned} \tag{6.3.82}$$

and solving Eqn. (6.3.82) for J_A and J_B leads us to

$$\begin{aligned} J_B(k,p) &= \frac{1}{i\pi^2\Delta^2}\left[\ 2\,k\cdot p\,k^\mu J_\mu^{(1)} - 2\,k^2 p^\mu J_\mu^{(1)}\right] \quad , \\ J_A(k,p) &= J_B(p,k) \quad , \end{aligned} \tag{6.3.83}$$

where

$$\Delta^2 = (k\cdot p)^2 - k^2p^2 \quad , \tag{6.3.84}$$

is the ubiquitous triangle function of k, p and q. Then substituting Eqn. (6.3.6) into Eqn. (6.3.83), we obtain

$$\begin{aligned} J_B(k,p) = \frac{1}{i\pi^2\Delta^2}\Bigg[&\ 2\,k\cdot p\int_M d^4w\,\frac{k\cdot w}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]} \\ &-2\,k^2\int_M d^4w\,\frac{p\cdot w}{w^2\,[(p-w)^2-m^2]\,[(k-w)^2-m^2]}\Bigg] . \end{aligned} \tag{6.3.85}$$

One then rewrites the numerators appearing in the integrands using the identities

$$\begin{aligned} 2k\cdot w &= k^2 + w^2 - m^2 - \left[(k-w)^2 - m^2\right] \quad , \\ 2p\cdot w &= p^2 + w^2 - m^2 - \left[(p-w)^2 - m^2\right] \quad , \end{aligned} \tag{6.3.86}$$

to obtain the following expression for the J_B in d-dimension :

$$\begin{aligned} J_B(k,p) &= \frac{1}{i\pi^2\Delta^2}\Bigg\{ \left(k\cdot p - k^2\right)\int \frac{d^d w}{[(k-w)^2-m^2][(p-w)^2-m^2]} \\ &+ \left[k\cdot p\left(k^2-m^2\right) - k^2\left(p^2-m^2\right)\right]\int \frac{d^d w}{w^2[(k-w)^2-m^2][(p-w)^2-m^2]} \\ &- k\cdot p\int \frac{d^d w}{w^2[(p-w)^2-m^2]} + k^2\int \frac{d^d w}{w^2[(k-w)^2-m^2]}\Bigg\} . \end{aligned} \tag{6.3.87}$$

The 16 basic scalar integrals, of which 4 appear in this equation, namely, $Q_7(k,p), Q_8(k,p)$, $Q_{14}(k,p)$ and $Q_{14}(p,k)$, are given in the Appendix I. We thus deduce

$$\begin{aligned} J_B(k,p) &= \frac{1}{\Delta^2}\Bigg\{ \left(k\cdot p - k^2\right)\mu^\epsilon\left[C+2-2S\right] \\ &+ \left[k\cdot p\left(k^2-m^2\right) - k^2\left(p^2-m^2\right)\right] J_0 \\ &- k\cdot p\mu^\epsilon\left(C+2-L\right) + k^2\mu^\epsilon\left(C+2-L'\right)\Bigg\} . \end{aligned} \tag{6.3.88}$$

Tiding up this expression, we find,

$$J_B(k,p) = \frac{1}{\Delta^2}\left\{\frac{J_0}{2}\left(m^2\, k\cdot q + p^2\, k\cdot q\right) + k\cdot p\, L - k^2 L' + 2\, k\cdot q\, S\right\} , \tag{6.3.89}$$

J_A and J_B are related to each other by the following simple relation which enables us to write J_A immediately :

$$J_A(k,p) = J_B(p,k) ,$$

$$J_A(k,p) = \frac{1}{\Delta^2}\left\{\frac{J_0}{2}\left(-m^2\, k\cdot q - k^2\, k\cdot q\right) + k\cdot p\, L' - p^2 L - 2\, k\cdot q\, S\right\}, \tag{6.3.90}$$

where

$$J^{(0)} = \frac{i\,\pi^2}{2} J_0 , \tag{6.3.91}$$

$$L = \left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right) , \tag{6.3.92}$$

$$L' = L(p\leftrightarrow k) , \tag{6.3.93}$$

$$S = \frac{1}{2}\left(1-4\frac{m^2}{q^2}\right)^{1/2}\ln\frac{\left[(1-4m^2/q^2)^{1/2}+1\right]}{\left[(1-4m^2/q^2)^{1/2}-1\right]} . \tag{6.3.94}$$

As we can see from these expressions L, L' and S are all elementary functions and $J^{(0)}$ is expressed in terms of Spence functions. Substituting the solutions for J_A and J_B in Eqn. (6.3.81), we find

$$\begin{aligned} J^{(1)}_{\mu} &= \frac{1}{\Delta^2}\Big\{ \quad k_\mu \left[\frac{J_0}{2}\left(-m^2\, k\cdot q - k^2\, k\cdot q\right) + k\cdot p\, L' - p^2 L - 2\, k\cdot q\, S\right] \\ &\qquad + p_\mu \left[\frac{J_0}{2}\left(m^2\, k\cdot q + p^2\, k\cdot q\right) + k\cdot p\, L - k^2 L' + 2\, k\cdot q\, S\right]\Big\} . \end{aligned} \tag{6.3.95}$$

6.3.4 $J^{(2)}_{\mu\nu}$ Calculated

$$J^{(2)}_{\mu\nu} = \int_M d^4 w \, \frac{w_\mu w_\nu}{w^2\, [(p-w)^2 - m^2]\, [(k-w)^2 - m^2]} \quad . \tag{6.3.7}$$

In an analogous fashion, the tensor integral $J^{(2)}_{\mu\nu}$ of Eqn. (6.3.7) can be expressed in terms of scalar integrals K,J_C, J_D and J_E by

$$\begin{aligned} J^{(2)}_{\mu\nu} &= \frac{i\pi^2}{2}\Big\{ \frac{g_{\mu\nu}}{d} K_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{4}\right) J_C \\ &\quad + \left(p_\mu k_\nu + k_\mu p_\nu - 2\, g_{\mu\nu}\frac{k\cdot p}{4}\right) J_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{4}\right) J_E \Big\} . \end{aligned} \tag{6.3.96}$$

All but $K(k, p)$ are ultraviolet finite and so the number of dimensions d has been set equal to 4. In $d \equiv 4 + \epsilon$ dimensions, with μ the usual scale parameter introduced to ensure that the coupling α remains dimensionless for any d. So we rewrite $J_{\mu\nu}$ in d dimension :

$$\begin{aligned} J^{(2)}_{\mu\nu} &= \frac{i\pi^2}{2}\Big\{ \frac{g_{\mu\nu}}{d} K_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{d}\right) J_C \\ &\quad + \left(p_\mu k_\nu + k_\mu p_\nu - 2\, g_{\mu\nu}\frac{(k\cdot p)}{d}\right) J_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{d}\right) J_E \Big\} . \end{aligned} \tag{6.3.97}$$

Contracting the indices with $g^{\mu\nu}$, Eqn. (6.3.97) yields

$$\begin{aligned} K_0 &= \frac{2}{i\pi^2} g^{\mu\nu} J^{(2)}_{\mu\nu} \\ &= \frac{2}{i\pi^2}\int_M d^4 w \, \frac{1}{[(p-w)^2 - m^2]\, [(k-w)^2 - m^2]} \\ &= \frac{2}{i\pi^2} K^{(0)} \quad . \end{aligned} \tag{6.3.98}$$

Recalling the $K^{(0)}$-integral from Appendix I, we can write,

$$\begin{aligned} K_0 &= 2\mu^\epsilon (C - 2S + 2) \quad , \\ \text{where} \qquad C &= -\frac{2}{\epsilon} - \gamma - \ln(\pi) - \ln(m^2/\mu^2) \,. \end{aligned} \tag{6.3.99}$$

We also have to evaluate J_C, J_D and J_E. The first step towards this is to saturate the indices with k^ν and p^ν :

$$\begin{aligned} &\frac{2}{i\pi^2} k^\nu \left(k^\mu J^{(2)}_{\mu\nu}\right) = \\ &\left[\frac{k^2}{d} K_0 + \left(k^4 J_C + 2k^2\, k\cdot p J_D\right)\left(1 - \frac{1}{d}\right) + \left((k\cdot p)^2 - \frac{k^2p^2}{d}\right) J_E\right] \\ &\frac{2}{i\pi^2} p^\nu \left(p^\mu J^{(2)}_{\mu\nu}\right) = \\ &\left[\frac{p^2}{d} K_0 + \left((k\cdot p)^2 - \frac{k^2p^2}{d}\right) J_C + \left(2p^2\, k\cdot p\, J_D + p^4\, J_E\right)\left(1 - \frac{1}{d}\right)\right] \\ &\frac{1}{i\pi^2}\left(p^\nu \left(k^\mu J^{(2)}_{\mu\nu}\right) + k^\nu \left(p^\mu J^{(2)}_{\mu\nu}\right)\right) = \\ &\left[\frac{k\cdot p}{d} K_0 + \left(k^2 J_C + p^2 J_E\right) k\cdot p \left(1 - \frac{1}{d}\right) + \left[(k\cdot p)^2\left(1 - \frac{2}{d}\right) + k^2p^2\right] J_D\right]. \end{aligned} \tag{6.3.100}$$

Rearranging Eqn. (6.3.100),

$$\begin{aligned} &\frac{2}{i\pi^2} k^\nu \left(k^\mu J^{(2)}_{\mu\nu}\right) - \frac{k^2}{d} K_0 = \\ &\left[k^4 J_C + 2k^2 k\cdot p J_D\right]\left(\frac{3}{4} + \frac{\epsilon}{16}\right) + \left[(k\cdot p)^2 - k^2p^2\left(\frac{1}{4} - \frac{\epsilon}{16}\right)\right] \\ &\frac{2}{i\pi^2} p^\nu \left(p^\mu J^{(2)}_{\mu\nu}\right) - \frac{p^2}{d} K_0 = \\ &\left[(k\cdot p)^2 - k^2p^2\left(\frac{1}{4} - \frac{\epsilon}{16}\right)\right] J_C + \left[2p^2 k\cdot p J_D + p^4 J_E\right]\left(\frac{3}{4} + \frac{\epsilon}{16}\right) \\ &\frac{1}{i\pi^2}\left(p^\nu \left(k^\mu J^{(2)}_{\mu\nu}\right) + k^\nu \left(p^\mu J^{(2)}_{\mu\nu}\right)\right) - \frac{k\cdot p}{d} K_0 = \\ &2k^2 k\cdot p\left(\frac{3}{4} + \frac{\epsilon}{16}\right) J_C + 2\left[k^2p^2 + \left(\frac{1}{2} + \frac{\epsilon}{8}\right)(k\cdot p)^2\right] J_D + 2p^2 k\cdot p\left(\frac{3}{4} + \frac{\epsilon}{16}\right) J_E. \end{aligned} \tag{6.3.101}$$

To solve this system for J_C, J_D and J_E, we use the matrix method :

$$J_C = \frac{Y}{X} \quad , \tag{6.3.102}$$

where

$$X \equiv \begin{vmatrix} k^4\epsilon' & 2k^2 k\cdot p\epsilon' & (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- \\ (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- & 2p^2 k\cdot p\epsilon' & p^4\epsilon' \\ 2k^2 k\cdot p\epsilon' & (k\cdot p)^2\epsilon^+ + 2k^2p^2 & 2p^2 k\cdot p\epsilon' \end{vmatrix} \tag{6.3.103}$$

with $\epsilon^+ = 1 + \epsilon/4, \ \ \epsilon^- = 1 - \epsilon/4$ and $\epsilon' = 3/4 + \epsilon/16$.

$$Y \equiv \begin{vmatrix} \frac{2}{i\pi^2}k^\nu k^\mu J^{(2)}_{\mu\nu} - \frac{k^2}{d}K_0 & 2k^2 k\cdot p\epsilon' & (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- \\ \frac{2}{i\pi^2}p^\nu p^\mu J^{(2)}_{\mu\nu} - \frac{p^2}{d}K_0 & (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- & 2p^2 k\cdot p\epsilon' \\ \frac{1}{i\pi^2}\Big(p^\nu k^\mu J^{(2)}_{\mu\nu} + k^\nu p^\mu J^{(2)}_{\mu\nu} & (k\cdot p)^2\epsilon^+ + 2k^2p^2 & 2p^2 k\cdot p\epsilon' \\ \quad - \ \frac{k\cdot p}{d}K_0\Big) & & \end{vmatrix} . \tag{6.3.104}$$

Substituting Eqns. (6.3.103, 6.3.104) into Eqn. (6.3.102) we obtain J_C in the form :

$$\begin{aligned} J_C(k,p) \;&=\; \frac{2}{d\,\Delta^2}p^2 K_0\left(1 - \frac{\epsilon}{4}\right) \\ &+\; \frac{2}{i\pi^2\Delta^4}\Bigg\{\; p^\nu\left(p^\mu J^{(2)}_{\mu\nu}\right)(k\cdot p)^2 + k^2p^2\left(\frac{1}{2} - \frac{\epsilon}{4}\right) \\ &\quad + k^\nu\left(k^\mu J^{(2)}_{\mu\nu}\right)p^4\left(-\frac{3}{2} + \frac{\epsilon}{4}\right) \\ &\quad + \left[p^\nu\left(k^\mu J^{(2)}_{\mu\nu}\right) + k^\nu\left(p^\mu J^{(2)}_{\mu\nu}\right)\right]p^2 k\cdot p\left(-\frac{3}{2} + \frac{\epsilon}{4}\right)\Bigg\} \quad . \end{aligned} \tag{6.3.105}$$

Similarly J_D is

$$J_D = \frac{Z}{X} \quad , \tag{6.3.106}$$

where

$$Z \equiv \left| \begin{array}{ccc} k^4\epsilon' & \frac{2}{i\pi^2}k^\nu k^\mu J^{(2)}_{\mu\nu} - \frac{p^2}{d}K_0 & (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- \\ (k\cdot p)^2 - \frac{k^2p^2}{4}\epsilon^- & \frac{2}{i\pi^2}p^\nu p^\mu J^{(2)}_{\mu\nu} - \frac{p^2}{d}K_0 & p^4\epsilon' \\ 2k^2k\cdot p\epsilon' & \frac{1}{i\pi^2}\Big(p^\nu k^\mu J^{(2)}_{\mu\nu} + k^\nu p^\mu J^{(2)}_{\mu\nu} \\ & \quad -\frac{k\cdot p}{d}K_0\Big) & 2p^2k\cdot p\epsilon' \end{array} \right| . \tag{6.3.107}$$

Replacing Z and X in Eqn. (6.3.106) with Eqns.(6.3.107,6.3.103) we obtain J_D as,

$$\begin{aligned} J_D(k,p) &= -\frac{2}{d\,\Delta^2}p^2K_0\left(1-\frac{\epsilon}{4}\right) \\ &+ \frac{2}{i\pi^2\Delta^4}\Bigg\{p^\nu\left(p^\mu J^{(2)}_{\mu\nu}\right)\left[k^2k\cdot p\left(-\frac{3}{2}-\frac{\epsilon}{4}\right)\right] \\ &\quad +k^\nu\left(k^\mu J^{(2)}_{\mu\nu}\right)p^2k\cdot p\left(-\frac{3}{2}+\frac{\epsilon}{4}\right) \\ &\quad +\left[p^\nu\left(k^\mu J^{(2)}_{\mu\nu}\right)+k^\nu\left(p^\mu J^{(2)}_{\mu\nu}\right)\right]\left[(k\cdot p)^2\left(1-\frac{\epsilon}{4}\right)+\frac{k^2p^2}{2}\right]\Bigg\}, \end{aligned} \tag{6.3.108}$$

$J_C(k,p)$ and $J_E(k,p)$ are symmetric functions in k and p, hence J_E can be written as,

$$J_E(k,p) \;=\; J_C(k,p) \quad . \tag{6.3.109}$$

Now, we contract the indices of $J^{(2)}_{\mu\nu}$ with p^μ and make use of Eqn. (6.3.86) to write the tensor integral of Eqn. (6.3.7) in terms of some vector integrals

$$\begin{aligned} p^\mu J^{(2)}_{\mu\nu}(k,p) &= -\frac{1}{2}\int d^dw\frac{w_\nu}{w^2\left[(k-w)^2-m^2\right]} \\ &\quad +\frac{1}{2}\int d^dw\frac{w_\nu}{\left[(k-w)^2-m^2\right]\left[(p-w)^2-m^2\right]} \\ &\quad +\frac{(p^2-m^2)}{2}\int d^dw\frac{w_\nu}{w^2\left[(k-w)^2-m^2\right]\left[(p-w)^2-m^2\right]} \\ &= -Q_{\nu 10}(p,k) + (p^2-m^2)\,J^{(1)}_\nu(k,p) + Q^\nu{}_9(k,p)\,. \end{aligned} \tag{6.3.110}$$

Recalling Q^{ν}_{10} and Q^{ν}_{9} from Appendix. E the above expression takes the following form

$$
\begin{aligned}
p^{\mu} J^{(2)}_{\mu\nu}(k,p) &= -\frac{i\pi^2\mu^{\epsilon}}{4} k_{\nu}\left(C+2-\frac{m^2}{k^2}-L'\right)+\frac{(p^2-m^2)}{2}J^{(1)}_{\nu} \\
&+\frac{i\pi^2\mu^{\epsilon}}{4}(k+p)_{\nu}\left(C+2-\frac{m^2}{p^2}-2S\right) \quad , \qquad (6.3.111)
\end{aligned}
$$

In a similar fashion, $k^{\mu} J^{(2)}_{\mu\nu}$ yields

$$
\begin{aligned}
k^{\mu} J^{(2)}_{\mu\nu}(k,p) &= p^{\mu} J^{(2)}_{\mu\nu}(p,k) \\
&= -\frac{i\pi^2\mu^{\epsilon}}{4} p_{\nu}\left(C+2-\frac{m^2}{p^2}-L\right)+\frac{(k^2-m^2)}{2}J^{(1)}_{\nu} \\
&+\frac{i\pi^2\mu^{\epsilon}}{4}(k+p)_{\nu}\left(C+2-\frac{m^2}{p^2}-2S\right) \quad . \qquad (6.3.112)
\end{aligned}
$$

Therefore, by substituting the two Eqns. (6.3.111,6.3.112) into the Eqns.(6.3.105,6.3.108), we find

$$
\begin{aligned}
J_C(k,p) &= \frac{1}{4\,\Delta^2}\Bigg\{\left(2\,p^2+2\,k\cdot p\,\frac{m^2}{k^2}\right)-4\,k\cdot p\,S+2\,k\cdot p\left(1-\frac{m^2}{k^2}\right)L' \\
&+\left(2\,k\cdot p\,(p^2-m^2)+3\,p^2\,(m^2-k^2)\right)J_A+p^2(m^2-p^2)J_B\Bigg\}\,, \\
J_D(k,p) &= \frac{1}{4\Delta^2}\Bigg\{\; 2\,k\cdot p\,\left[(k^2-m^2)J_A+(p^2-m^2)J_B-1\right] \\
&-k^2\left[2\,\frac{m^2}{k^2}-2S+\left(1-\frac{m^2}{k^2}\right)L'+(p^2-m^2)J_A\right] \\
&-p^2\left[-2\,S+\left(1-\frac{m^2}{p^2}\right)L+(k^2-m^2)J_B\right]\Bigg\}\,,
\end{aligned}
$$

and

$$
J_E(k,p) = J_C(p,k) \quad , \qquad (6.3.113)
$$

all of which involve the previously defined J_A, J_B, L, L' and S of Eqs. (6.3.94).

6.3.5 $I^{(1)}_{\mu}$ and $I^{(2)}_{\mu\nu}$ Calculated

$$
I^{(1)}_{\mu} = \int_M d^4w\,\frac{w_{\mu}}{w^4\,\left[(p-w)^2-m^2\right]\,\left[(k-w)^2-m^2\right]} \quad . \qquad (6.3.9)
$$

In a way analogous to the computation of $J_\mu^{(1)}$ and $J_{\mu\nu}^{(2)}$, the ultraviolet finite integrals $I_\mu^{(1)}$ and $I_{\mu\nu}^{(2)}$ [44] of Eqns. (6.3.9, 6.3.10) can be re-expressed in terms of scalar integrals, I_A, I_B, I_C, I_D, I_E, that in turn involve the same functions we have already computed. Thus

$$I_\mu^{(1)} = \frac{i\pi^2}{2}\left[k_\mu I_A(k,p) + p_\mu I_B(k,p)\right] \quad , \tag{6.3.114}$$

where

$$\begin{aligned} I_A(k,p) = \frac{1}{\Delta^2}\Bigg\{ &-\frac{k\cdot q}{2}J_0 - \frac{2q^2}{\chi}\Big\{(m^2-p^2)k^2 - (m^2-k^2)\,k\cdot p\Big\}\,S \\ &+\frac{1}{(m^2-p^2)}\left[p^2 - k\cdot p + \frac{p^2q^2}{\chi}(k^2-m^2)\Big(m^2+k\cdot p\Big)\right]L \\ &+\frac{k^2q^2}{\chi}\Big(m^2+k\cdot p\Big)L'\Bigg\} \quad , \end{aligned} \tag{6.3.115}$$

and

$$I_B(k,p) = I_A(p,k) \quad . \tag{6.3.116}$$

χ appearing in the denominator is the same as in Eqn. (6.3.70)

$$\begin{aligned} \chi &= (q^2-2m^2)(p^2-m^2)(k^2-m^2) + m^2(p^2-m^2)^2 + m^2(k^2-m^2)^2 \\ &= p^2k^2q^2 + 2\left[(p^2+k^2)\,k\cdot p - 2p^2k^2\right]m^2 + m^4q^2 \,, \end{aligned} \tag{6.3.117}$$

$$I_{\mu\nu}^{(2)} = \int_M d^4w \, \frac{w_\mu w_\nu}{w^4\left[(p-w)^2-m^2\right]\left[(k-w)^2-m^2\right]} \quad . \tag{6.3.118}$$

$$\begin{aligned} I_{\mu\nu}^{(2)} = \frac{i\pi^2}{2}\Bigg\{ &\frac{g_{\mu\nu}}{4}J_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{4}\right)I_C \\ &+\left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}\frac{(k\cdot p)}{2}\right)I_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{4}\right)I_E\Bigg\}, \end{aligned} \tag{6.3.119}$$

$$
\begin{aligned}
I_C(k,p) &= \frac{1}{4\Delta^2}\Bigg\{ 2p^2 J_0 - 4\,\frac{k\cdot p}{k^2}\left(1+\frac{m^2}{(k^2-m^2)}L'\right) \\
&\quad + \left(2\,k\cdot p - 3p^2\right) J_A - p^2 J_B \\
&\quad + \left(-2\,k\cdot p\,(m^2-p^2) + 3p^2(m^2-k^2)\right) I_A + p^2(m^2-p^2) I_B \Bigg\} \quad ,
\end{aligned} \tag{6.3.120}
$$

$$
\begin{aligned}
I_D(k,p) &= \frac{1}{4\,\Delta^2}\Bigg\{ -2(k\cdot p)J_0 + 2\left(1+\frac{m^2}{(k^2-m^2)}L'\right) + 2\left(1+\frac{m^2}{(p^2-m^2)}L\right) \\
&\quad + \left(2\,k\cdot p - k^2\right) J_A + \left(2\,k\cdot p - p^2\right) J_B \\
&\quad + \left(k^2(m^2-p^2) - 2\,k\cdot p\,(m^2-k^2)\right) I_A \\
&\quad + \left(p^2(m^2-k^2) - 2\,k\cdot p\,(m^2-p^2)\right) I_B \Bigg\} \quad ,
\end{aligned} \tag{6.3.121}
$$

$$
I_E(k,p) = I_C(p,k) \quad . \tag{6.3.122}
$$

The $1/\chi$ term in I_A, I_B, I_C and I_D arises from the extra $1/w^2$ factor that occurs in the second integral of Eq. (6.2.13). Notice that the $1/\chi$ term arises in all but the Feynman gauge. The possibility of singularities at $\chi = 0$ has consequences as we shall see later.

6.3.6 The Complete vertex Λ^μ

In terms of the basic functions $J_0, J_A, J_B, J_C, J_D, J_E, I_A, I_B, I_C, I_D, I_E$ and the ultraviolet divergent K_0, all of which depend on the momenta k and p, i.e. are functions of the Lorentz scalars k^2, p^2 and q^2, Λ^μ can be written completely as :

$$
\begin{aligned}
\Lambda_\mu(k,p) \;=\; \frac{\alpha}{4\pi}\Bigg\{\Bigg[&\Big(-4k_\mu \not p + 2\gamma_\mu \not k \not p + 4m(p_\mu + k_\mu) - 2m^2\gamma_\mu\Big) J_0 \\
&+ \Big(4k_\mu \not p - 2\gamma_\mu \not k \not p + 4k_\mu \not k - 2k^2\gamma_\mu - 8mk_\mu\Big) J_A \\
&+ \Big(4k_\mu \not p - 2\gamma_\mu \not k \not p + 4p_\mu \not p - 2p^2\gamma_\mu - 8mp_\mu\Big) J_B \\
&+ \Big(-4k_\mu \not k + k^2\gamma_\mu\Big) J_C \\
&+ (-4k_\mu \not p - 4p_\mu \not k + 2\gamma_\mu k\cdot p - 2\gamma_\mu \not k \not p) J_D \\
&+ \Big(-4p_\mu \not p + p^2\gamma_\mu\Big) J_E \\
&+ \gamma_\mu(1 + \frac{3}{4}\epsilon)\mu^\epsilon K_0 \Bigg] \\
+ \quad (\xi - 1)\Bigg[&\Big(-4k_\mu \not p + 2\gamma_\mu \not k \not p + 4m(p_\mu + k_\mu) - 2m^2\gamma_\mu\Big) \frac{J_0}{4} \\
&+ \Bigg(2k^2 p_\mu \not k - k^2 k_\mu \not p + \frac{k^2}{2}\gamma_\mu \not k \not p + mk^2 p_\mu \\
&\quad + (mk^2 - 4mk\cdot p)k_\mu + 2mk_\mu \not k \not p + m(2k\cdot p - k^2)\gamma_\mu \not k \\
&\quad - mk^2\gamma_\mu \not p + 2m^2 k_\mu \not k - \frac{m^2k^2}{2}\gamma_\mu\Bigg) J_C \\
&+ \Bigg(2k^2 p_\mu \not p - 2(k\cdot p - m^2)k_\mu \not p + 2p^2 k_\mu \not k + k\cdot p\,\gamma_\mu \not k \not p \\
&\quad + 2mp^2\gamma_\mu \not k + 2mp_\mu \not k \not p - 2m(p^2 + k\cdot p)k_\mu \\
&\quad + 2(k^2 - k\cdot p)p_\mu - 2mk^2\gamma_\mu \not p \\
&\quad + 2mk_\mu \not k \not p + 2m^2 p_\mu \not k - m^2 k\cdot p\gamma_\mu\Bigg) I_D \\
&+ \Bigg(\frac{p^2}{2}\gamma_\mu \not k \not p + p^2 k_\mu \not p + m(p^2 - 2k\cdot p)\gamma_\mu \not p - mp^2 p_\mu \\
&\quad + 2mp_\mu \not k \not p + mp^2\gamma_\mu \not k - mp^2 k_\mu \\
&\quad + 2m^2 p_\mu \not p - \frac{m^2}{2}p^2\gamma_\mu\Bigg) I_E \\
&+ \Big(-2p_\mu \not k + 2k_\mu \not p - \gamma_\mu \not k \not p - k^2\gamma_\mu - 2mk_\mu\Big) J_A \\
&+ \Big(-p^2\gamma_\mu - \gamma_\mu \not k \not p - 2mp_\mu\Big) J_B \\
&+ 2\gamma_\mu\mu^\epsilon(C + 2 - 2S)\Bigg]\Bigg\} \quad . \qquad (6.3.123)
\end{aligned}
$$

Now we will express Λ^μ, analogously to Eqn. (6.2.4), as

$$\Lambda^\mu(k,p) = \sum_{i=1}^{12} \bar{P}_1^i \, V_i^\mu \quad , \tag{6.2.4}$$

with

$$\bar{P}_1^i \;=\; \frac{\alpha}{4\pi} P_1^i \quad , \tag{6.3.124}$$

where the subscript on the P^i indicates this calculation is only to first order in α. Rearranging Eqn. (6.3.1), P_1^i can be explicitly displayed as follows :

$$\begin{aligned}
P_1^1 &= 2J_A - 2J_C + (\xi-1)\left(m^2 I_C + p^2 I_D\right) \;, \\
P_1^2 &= 2J_B - 2J_E + (\xi-1)\left(k^2 I_D + m^2 I_E\right) \;, \\
P_1^3 &= -2J_0 + 2J_A + 2J_B - 2J_D \\
&\quad + (\xi-1)\left(-\frac{J_0}{2} - \frac{k^2}{2} I_C - \; k\cdot p\, I_D + m^2 I_D + \frac{p^2}{2} I_E + J_A\right) \;, \\
P_1^4 &= -2J_D + (\xi-1)\left(k^2 I_C + m^2 I_D - J_A\right) \;, \\
P_1^5 &= J_0 - J_A - J_B + (\xi-1)\left(\frac{J_0}{4} + \frac{k^2}{4} I_C + \frac{k\cdot p}{2} I_D + \frac{p^2}{4} I_E - \frac{1}{2} J_A - \frac{1}{2} J_B\right) \;, \\
P_1^6 &= \left(-m^2 J_0 - k^2 J_A - p^2 J_B + \frac{k^2}{2} J_C + k\cdot p\, J_D + \frac{p^2}{2} J_E + \frac{1}{2}\left(1 + \frac{3\epsilon}{4}\mu^\epsilon\right) K_0\right) \\
&\quad + (\xi-1)\bigg(- m^2 \frac{J_0}{4} - \frac{m^2}{4} k^2 I_C - \frac{m^2}{2}\, k\cdot p\, I_D - \frac{m^2}{4} p^2 I_E - \frac{k^2}{2} J_A - \frac{p^2}{2} J_B \\
&\quad + \mu^\epsilon \left[C + 2 - 2S\right]\bigg) \;, \\
P_1^7 &= 2mJ_0 - 4mJ_A \\
&\quad + (\xi-1)m\bigg(\frac{J_0}{2} - 2\, k\cdot p\, I_C + \frac{k^2}{2} I_C - p^2 I_D - \; k\cdot p\, I_D - \frac{p^2}{2} I_E - J_A\bigg) \;, \\
P_1^8 &= 2mJ_0 - 4mJ_B \\
&\quad + (\xi-1)m\left(\frac{J_0}{2} + \frac{k^2}{2} I_C - \; k\cdot p\, I_D + k^2 I_D - \frac{p^2}{2} I_E - J_B\right) \;, \\
P_1^9 &= (\xi-1)m\,(I_D + I_E) \;, \\
P_1^{10} &= (\xi-1)m\,(I_D + I_C) \;, \\
P_1^{11} &= (\xi-1)m\left(p^2 I_D + \; k\cdot p\, I_C + \frac{p^2}{2} I_E - \frac{k^2}{2} I_C\right) \;, \\
P_1^{12} &= (\xi-1)m\left(-k^2 I_D - \; k\cdot p\, I_E + \frac{p^2}{2} I_E - \frac{k^2}{2} I_C\right) \quad .
\end{aligned} \tag{6.3.125}$$

Notice that both the integrals I_A, I_B cancel out in this result. Moreover, this result has an ultraviolet divergent term in P_1^6 because the fermion propagator is UV divergent. As we shall see in the next section, the presence of the same kind of divergence in the longitudinal part ensures that they cancel out and leave the transverse vertex ultraviolet finite. Though this expression appears to involve all 12 spin vectors, one of their coefficients is not independent. The Ward-Takahashi identity, Eq. (2.2.1), only involves $\not k, \not p, 1$ as spin structure on the right hand side. This means that $\not k\ \not p$ and $\not p\ \not k$ terms that occur in $q_\mu \Gamma^\mu$ must occur in the form of the anticommutator, $\{\not k, \not p\} = 2k \cdot p$. Consequently, the coefficients P_i of Eq. (6.2.4) are related by :

$$P_1^{12} = P_1^9(p^2 - k \cdot p) + P_1^{10}(k \cdot p - k^2) - P_1^{11} \quad . \tag{6.3.126}$$

Formally, this completes our calculation of the one loop corrections to the QED vertex in any covariant gauge for arbitrary momenta.

6.4 Analytic Structure of the Vertex

6.4.1 Longitudinal Part

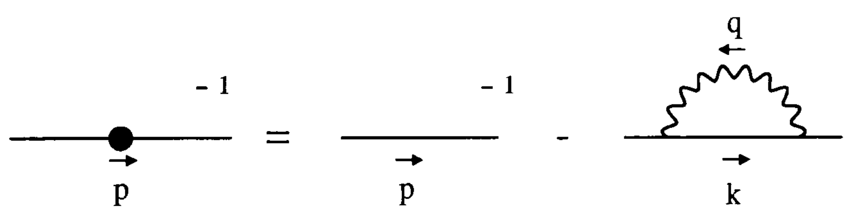


Figure 6.3: The inverse fermion propagator to one loop order in perturbation theory

As explained in Sect. 6.2.1 (and in Sect. 2.3.2), the longitudinal component of the vertex is determined by the fermion functions, $F(p^2), M(p^2)$, thanks to the Ward-Takahashi identity. In this section we compute these functions to $\mathcal{O}(\alpha)$ by calculating the one loop corrections to the fermion propagator, Fig. 6.3, which can be written as

$$iS_F^{-1} = iS_F^{0\,-1} + \Sigma(p^2) \quad . \tag{6.4.1}$$

We now replace the inverse fermion propagator by Eqn. (2.3.1)

$$i\frac{\not p - M(p^2)}{F(p^2)} = i(\not p - m) + \Sigma(p^2) \quad , \tag{6.4.2}$$

where

$$\Sigma(p^2) = \int_M \frac{d^4k}{(2\pi)^4}\,(-ie\gamma^\mu)\, i\frac{(\not k + m)}{k^2 - m^2}\,(-ie\gamma^\nu)\,\frac{(-1)}{q^2}\left(g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right) \quad . \tag{6.4.3}$$

We can split Eqn. (6.4.2) into two pieces according to whether they involve odd or even numbers of γ-matrices :

$$\frac{i\,\not p}{F(p^2)} = i\,\not p + \Sigma_1 \quad , \tag{6.4.4}$$

$$-\frac{iM(p^2)}{F(p^2)} = -im + \Sigma_2 \quad . \tag{6.4.5}$$

The odd number of γ-matrices contribute to Σ_1,

$$\Sigma_1 = -\frac{\alpha}{4\pi^3}\int_M \frac{d^4k}{q^2(k^2 - m^2)}\left[\gamma^\mu\,\not k\gamma_\mu + \frac{(\xi - 1)}{q^2}\,\not q\,\not k\,\not q\right] \quad , \tag{6.4.6}$$

and the even number of γ-matrices give Σ_2 :

$$\Sigma_2 = -\frac{\alpha m}{4\pi^3}\int_M \frac{d^4k}{q^2(k^2 - m^2)}\left[\gamma_\mu\gamma^\mu + (\xi - 1)\right] \quad . \tag{6.4.7}$$

Obviously, knowing Σ_1 would tell us what $1/F$ is. We first concentrate on evaluating this term. After performing some γ-matrix algebra in d-dimensions, we can write Eqn. (6.4.6) as

$$\Sigma_1 = -\frac{\alpha}{4\pi^3}\left\{(2 - d)\gamma_\mu\int_M d^dk\frac{k^\mu}{q^2(k^2 - m^2)} + (\xi - 1)\int_M d^dk\frac{(k^2\,\not k - 2k^2\,\not p + \not p\,\not k\,\not p)}{(k - p)^4(k^2 - m^2)}\right\} \quad . \tag{6.4.8}$$

Solving the above integrals using dimensional regularisation (see Appendix H) leads to

$$\begin{aligned}\Sigma_1 &= -\frac{i\alpha\mu^\epsilon}{4\pi}\Bigg\{ (\epsilon-2)\,\gamma_\mu\frac{p^\mu}{2}\left[C+\frac{m^2}{p^2}+2+\left(1+\frac{m^2}{p^2}\right)L\right] \\ &+(\xi-1)\frac{\not p}{(m^2-p^2)}\Bigg[\quad m^2\left\{C+\left(1-\frac{m^2}{p^2}\right)-\left(1+\frac{m^2}{p^2}\right)L\right\} \\ &\qquad -2m^2\left\{C-\left(1+\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\right\} \\ &\qquad +p^2\quad\left\{C+\left(1-\frac{m^2}{p^2}\right)-\left(1+\frac{m^2}{p^2}\right)L\right\}\Bigg]\Bigg\}\quad,\end{aligned} \tag{6.4.9}$$

where C has been defined before in Eqn. (6.3.99). After some tidying up, Σ_1 acquires the form

$$\begin{aligned}\Sigma_1 &= \frac{\alpha\mu^\epsilon}{4\pi}\not p\Bigg\{\ C+1+\frac{m^2}{p^2}+\left(1+\frac{m^2}{p^2}\right)L \\ &+(\xi-1)\left[C+\left(1+\frac{m^2}{p^2}\right)(1-L)\right]\Bigg\}\quad.\end{aligned} \tag{6.4.10}$$

If we substitute this result in Eqn. (6.4.5) and multiply it by $i\not p$, we then obtain the inverse fermion wavefunction renormalization to $\mathcal{O}(\alpha)$ as,

$$F^{-1}(p^2) \;=\; 1+\frac{\alpha\xi}{4\pi}\left[C\mu^\epsilon+\left(1+\frac{m^2}{p^2}\right)(1-L)\right]\quad. \tag{6.4.11}$$

To find the ratio M/F we shall evaluate Σ_2, Eqn. (6.4.7). This task can be carried out by referring to Appendix H. It eventually yields,

$$\Sigma_2 \;=\; -m\,\frac{i\alpha\mu^\epsilon}{4\pi}\Bigg\{(\epsilon+4)\,(C+2-L)+(\xi-1)\,(C+2-L)\Bigg\}\quad. \tag{6.4.12}$$

After substituting this expression into the Eqn. (6.4.5) and multiplying by the factor $(-i)$, we obtain,

$$\frac{M(p^2)}{F(p^2)} = m+\frac{\alpha m\mu^\epsilon}{\pi}\Bigg\{C+\frac{3}{2}-L+\frac{(\xi-1)}{4}(C+2-L)\Bigg\}\quad. \tag{6.4.13}$$

Having calculated two cornerstones of Ball-Chiu vertex $1/F$ and M/F up to $\mathcal{O}(\alpha)$ allows us to write what the longitudinal vertex is to this order. Now let us call Sect.6.3.6 where we defined the full vertex in tems of 12 spin amplitudes and their coefficients. Four of these 12 components define what is called the longitudinal vertex. This is related by the Ward-Takahashi identity to the fermion propagator. This fact allows three of these components to be expressed in terms of the fermion wave-function renormalisation $F(p^2)$, and its mass function $M(p^2)$ and forces a fourth to be zero. As we decribed in Sect. 2.3.2 Ball and Chiu have shown how to construct this longitudinal vertex in a way free of kinematic singularities. As mentioned several times, this freedom is essential in ensuring that the Ward identity is the $q \rightarrow 0$ limit of the Ward-Takahashi identity. Therefore the longitudinal component of the vertex is :

$$\Gamma_L^\mu \;=\; \lambda_1 L_1^\mu + \lambda_2 L_2^\mu + \lambda_3 L_3^\mu + \lambda_4 L_4^\mu \quad , \tag{6.4.14}$$

in terms of 4 tensors :

$$\begin{aligned}
L_1^\mu &= V_6^\mu \quad , \\
L_2^\mu &= V_1^\mu + V_2^\mu + V_3^\mu + V_4^\mu \quad , \\
L_3^\mu &= V_7^\mu + V_8^\mu \quad , \\
L_4^\mu &= 2k\cdot p\, V_6^\mu - V_5^\mu \quad ,
\end{aligned} \tag{6.4.15}$$

and the 4 coefficients

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \quad , \\
\lambda_2 &= \frac{1}{2\,(k^2-p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) \quad , \\
\lambda_3 &= -\frac{1}{(k^2-p^2)}\left(\frac{\Sigma(k^2)}{F(k^2)} - \frac{\Sigma(p^2)}{F(p^2)}\right) \quad , \\
\lambda_4 &= 0 \quad .
\end{aligned} \tag{6.4.16}$$

Simple substitution of Eqns. (6.4.15, 6.4.16) into the Eqn. (6.4.14) gives the longitudinal vertex, which we write out as :

$$\begin{aligned}
\Gamma_L^\mu &= \frac{\alpha\xi}{8\pi}\gamma^\mu\left[2\,C\mu^\epsilon + \left(1+\frac{m^2}{k^2}\right)(1-L') + \left(1+\frac{m^2}{p^2}\right)(1-L)\right] \\
&+ \frac{\alpha\xi}{4\pi}\left(k^\mu \not p + k^\mu \not k + p^\mu \not p + p^\mu \not k\right)\frac{1}{2\,(k^2-p^2)} \\
&\qquad \times\left[m^2\left(\frac{1}{k^2}-\frac{1}{p^2}\right) - \left(1+\frac{m^2}{k^2}\right)L' + \left(1+\frac{m^2}{p^2}\right)L\right] \\
&- \frac{\alpha\, m}{4\pi}(3+\xi)\frac{(p+k)^\mu}{(k^2-p^2)}\left(L-L'\right) \quad . \qquad (6.4.17)
\end{aligned}$$

6.4.2 The Transverse Vertex

Having calculated the vertex to $O(\alpha)$, Eqn. (6.3.125), we can subtract from it the longitudinal vertex of Sect. 6.4.1, Eqn. (6.4.17), and obtain Eqn. (2.3.64) for the transverse vertex to $O(\alpha)$. This is given by a rather lengthy expression,

$$\begin{aligned}
\Gamma_T^\mu(k,p) &= \frac{\alpha}{4\pi}\Bigg\{\sum_{i=1}^{12} V_i^\mu\Bigg(\frac{1}{2\,\Delta^2}\left[a_1^{(i)}+(\xi-1)a_2^{(i)}\right]J_A \\
&\quad + \frac{1}{2\,\Delta^2}\left[b_1^{(i)}+(\xi-1)b_2^{(i)}\right]J_B \\
&\quad + \frac{1}{2\,\Delta^2}\left[c_1^{(i)}+(\xi-1)c_2^{(i)}\right]I_A \\
&\quad + \frac{1}{2\,\Delta^2}\left[d_1^{(i)}+(\xi-1)d_2^{(i)}\right]I_B \\
&\quad + \frac{1}{2\,p^2\,(k^2-p^2)\,(p^2-m^2)\,\Delta^2}\left[e_1^{(i)}+(\xi-1)e_2^{(i)}\right]L \\
&\quad + \frac{1}{2\,k^2\,(k^2-p^2)\,(k^2-m^2)\,\Delta^2}\left[f_1^{(i)}+(\xi-1)f_2^{(i)}\right]L' \\
&\quad + \frac{1}{\Delta^2}\left[g_1^{(i)}+(\xi-1)g_2^{(i)}\right]S \\
&\quad + \frac{1}{2\Delta^2}\left[h_1^{(i)}+(\xi-1)h_2^{(i)}\right]J_0 \\
&\quad + \frac{1}{\Delta^2}\left[l_1^{(i)}+(\xi-1)l_2^{(i)}\right]\Bigg)\Bigg\} \quad , \qquad (6.4.18)
\end{aligned}$$

in terms of the 12 vectors V_i^μ of Eqn. (6.2.5) with the coefficients which are listed in the Appendix J.

Our task is now to represent this result in terms of the eight basis vectors, orthogonal to the boson momentum, each unconstrained by the Ward-Takahashi identity, defining $\Gamma_T^\mu(k,p)$, Eq. (2.3.64). Thus from Eqn. (6.2.7) we can alternatively write out

$$
\begin{aligned}
\Gamma_T^\mu \;=\;& k^\mu \not{k} \left[\tau_2(p^2 - k\cdot p) - \tau_3 + \tau_6\right] \\
+\;& p^\mu \not{p} \left[\tau_2(k^2 - k\cdot p) - \tau_3 - \tau_6\right] \\
+\;& k^\mu \not{p} \left[\tau_2(p^2 - k\cdot p) + \tau_3 - \tau_6 + \tau_8\right] \\
+\;& p^\mu \not{k} \left[\tau_2(k^2 - k\cdot p) + \tau_3 + \tau_6 - \tau_8\right] \\
+\;& \gamma^\mu \left[\tau_3 q^2 + \tau_6(p^2 - k^2) + \tau_8(k\cdot p)\right] \\
+\;& \gamma^\mu \not{k}\not{p} \left[-\tau_8\right] \\
+\;& p^\mu \left[\tau_1(k^2 - k\cdot p) - \tau_4(k^2 - k\cdot p)(k\cdot p) - \tau_5 + \frac{\tau_7}{2}(k^2 - p^2 - 2k\cdot p)\right] \\
+\;& k^\mu \left[\tau_1(p^2 - k\cdot p) - \tau_4(p^2 - k\cdot p)(k\cdot p) + \tau_5 + \frac{\tau_7}{2}(k^2 - p^2 - 2k\cdot p)\right] \\
+\;& p^\mu \not{k}\not{p} \left[\tau_4(k^2 - k\cdot p) + \tau_7\right] \\
+\;& k^\mu \not{k}\not{p} \left[\tau_4(p^2 - k\cdot p) + \tau_7\right] \\
+\;& \gamma^\mu \not{k} \left[-\tau_5 + \frac{\tau_7}{2}(p^2 - k^2)\right] \\
+\;& \gamma^\mu \not{p} \left[\tau_5 + \frac{\tau_7}{2}(p^2 - k^2)\right] \quad .
\end{aligned}
$$

Comparing Eqns. (6.4.18) and (6.4.19), we have 12 equations for the 8 unknown τ_i. Since Γ_T^μ is transverse to the vector q_μ, Eq. (6.2.6), only 8 of these equations are independent. Here are the 12 equations we need to solve for the τ_i's arising from the comparison of the coefficients of the various tensors :

k^μ $\not k$ comparison :

$$\frac{\alpha}{4\pi}\Bigg(\frac{\mathbf{J_0}}{4\pi^4}\Big\{-\left(2\Delta^2+3p^2q^2\right)m^4-\left[4\left(p^2+k\cdot p\right)\Delta^2+6p^2q^2k\cdot p\right]m^2-p^2\left[2(k^2+q^2)\Delta^2+3p^2k^2q^2\right]$$
$$+(\xi-1)\Big[-\left(2\Delta^2+3p^2q^2\right)m^4+p^2\left[-2(p^2+k^2-k\cdot p)\Delta^2-3p^2k^2q^2\right]\Big]\Big\}$$
$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{-\left(2\Delta^2-3p^2q^2\right)-2k\cdot p\Delta^2-3p^2q^2k\cdot p$$
$$+\frac{(\xi-1)}{\chi}\Big[\quad -q^2\left(2\Delta^2-3p^2q^2\right)m^6+\Big[\left(2(p^2-k^2)p\cdot q+6p^2(k^2+p^2)\right)\Delta^2$$
$$+3p^2\left(4p^2k^2q^2-k\cdot p(p^2-k^2)^2\right)\Big]m^4$$
$$-\Big[\Big(8(p^2-k^2)^2+6p^2p\cdot q-6k^2k\cdot q-6(p^4+k^4)\Big)\Delta^2$$
$$+3p^2k^2\left(4k^2p\cdot q-4p^2k\cdot q-(p^2-k^2)^2\right)\Big]m^2+p^4k^2q^2\left(2\Delta^2+3q^2k\cdot p\right)\Big]\Big\}$$
$$+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{\left(\Delta^2-2p^2k\cdot p\right)m^2-2k^2p^4+\frac{(\xi-1)}{\chi}\Big[\left(\Delta^2-2p^2k\cdot p\right)m^2+2p^4k^2\Big]\Big\}$$
$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\left[\Delta^4-3(p^2-k^2)p^4p\cdot q\right]m^2+p^2p\cdot q\left[(k\cdot p+p^2)\Delta^2-3p^2k\cdot p(p^2-k^2)\right]$$
$$+\frac{(\xi-1)}{\chi}\Big[\quad q^2\left[\Delta^4-3p^4(p^2-k^2)p\cdot q\right]m^6$$
$$+\Big[\left(p^2(p^2-k^2)+2k^2p\cdot q\right)\Delta^4-2p^4(p^2-k^2)(p^2+2k^2)\Delta^2$$
$$+3p^6(p^2-k^2)\left(k\cdot p(p^2+3k^2)-k^2(3p^2+k^2)\right)\Big]m^4$$
$$+p^2\Big[\left(k^2(k^2-p^2)-2p^2k\cdot q\right)\Delta^4$$
$$+2p^2(p^2-k^2)\left((p^2+k^2)(-2p^2+k\cdot p)-p^2k^2\right)\Delta^2$$
$$+3p^4k^2(p^2-k^2)\left((p^2+k^2)p\cdot q-2p^2k\cdot q\right)\Big]m^2$$
$$+p^4k^2q^2p\cdot q\left[(p^2+k\cdot p)\Delta^2+3p^2k\cdot p(p^2-k^2)\right]\Big]\Big\}$$
$$+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\left[-\Delta^4+2(p^2-k^2)(k^2+k\cdot p)\Delta^2+3k^2p^2(p^2-k^2)k\cdot q\right]m^2$$
$$-k^2\Delta^4+2k^2(p^2-k^2)p\cdot q\Delta^2-3p^2k^4(p^2-k^2)p\cdot q$$
$$+\frac{(\xi-1)}{\chi}\Big[\quad q^2\left[\Delta^4-2(p^2-k^2)(k^2+k\cdot p)\Delta^2-3p^2k^2(p^2-k^2)k\cdot q\right]m^6$$
$$+\Big[\left[2p^2(k^2-p^2)+(k^4-p^4)-p^2q^2\right]\Delta^4+2k^2(p^2-k^2)\Big[p^2(p^2+2k^2)$$
$$+k\cdot p(p^2-k^2)\Big]\Delta^2-3k^4p^2(p^2-k^2)\left[p\cdot q(k^2+p^2)-2p^2k\cdot q\right]\Big]m^4$$
$$+k^2\Big[\left[p^2(p^2-k^2)+2k^2p\cdot q\right]\Delta^4$$
$$-2p^2(p^2-k^2)\left[(k^2+p^2)(k\cdot p-2k^2)-p^2k^2\right]\Delta^2$$
$$-3p^4k^2(p^2-k^2)\left[2k^2p\cdot q-2(k^2+p^2)k\cdot q\right]\Big]m^2$$
$$+k^4p^2q^2\left[\Delta^4+2p^2(k^2-p^2)\Delta^2+3p^2k^2(p^2-k^2)p\cdot q\right]\Big]\Big\}\Bigg)$$
$$=\tau_2\left(\mathbf{p}^2-\mathbf{k}\cdot\mathbf{p}\right)-\tau_3+\tau_6 \qquad (6.4.19)$$

p^μ $\not p$ comparison :

$$
\begin{aligned}
&\frac{\alpha}{4\pi}\Big(\frac{\mathbf{J_0}}{4\pi^4}\Big\{-\left(2\Delta^2+3k^2q^2\right)m^4-\left[4\left(k^2+k\cdot p\right)\Delta^2+6k^2q^2k\cdot p\right]m^2-k^2\left[2(p^2+q^2)\Delta^2+3p^2k^2q^2\right]\\
&\qquad+(\xi-1)\Big[-\left(2\Delta^2+3k^2q^2\right)m^4+k^2\left[2(p^2+k^2-k\cdot p)\Delta^2+3p^2k^2q^2\right]\Big]\Big\}\\
&\quad+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{-\left(2\Delta^2+3k^2q^2\right)-2k\cdot p\Delta^2-3k^2q^2k\cdot p\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[\ -q^2\left(2\Delta^2-3k^2q^2\right)m^6+\Big[\left(2(p^2-k^2)k\cdot q+6k^2(k^2+p^2)\right)\Delta^2\\
&\qquad\qquad+3k^2\left(4p^2k^2q^2-k\cdot p(p^2-k^2)^2\right)\Big]m^4\\
&\qquad\qquad+\Big[\Big(4k^2(p^2-k^2)^2+6k^4p\cdot q-6k^2p^2k\cdot q-12p^2k^4\Big)\Delta^2\\
&\qquad\qquad+3p^2k^4\left(-4k^2p\cdot q+4p^2k\cdot q+(p^2-k^2)^2\right)\Big]m^2+k^4p^2q^2\left(2\Delta^2+3q^2k\cdot p\right)\Big]\Big\}\\
&\quad+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{\left(\Delta^2-2k^2k\cdot p\right)m^2-2p^2k^4+\frac{(\xi-1)}{\chi}\Big[\left(\Delta^2-2k^2k\cdot p\right)m^2+2k^4p^2\Big]\Big\}\\
&\quad+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\left[\Delta^4+2(p^2-k^2)(p^2+k\cdot p)\Delta^2-3k^2p^2(p^2-k^2)p\cdot q\right]m^2\\
&\qquad\qquad+p^2\Delta^4-2p^2(p^2-k^2)k\cdot q\Delta^2-3k^2p^4(p^2-k^2)k\cdot q\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[\ -q^2\left[\Delta^4+2(p^2-k^2)(p^2+k\cdot p)\Delta^2-3p^2k^2(p^2-k^2)p\cdot q\right]m^6\\
&\qquad\qquad+\Big[\left[2k^2(k^2-p^2)+(k^4-p^4)+k^2q^2\right]\Delta^4+2p^2(p^2-k^2)\Big[k^2(k^2+2p^2)\\
&\qquad\qquad-k\cdot p(p^2-k^2)\Big]\Delta^2-3p^4k^2(p^2-k^2)\left[-k\cdot q(k^2+p^2)+2k^2p\cdot q\right]\Big]m^4\\
&\qquad\qquad-p^2\Big[\left[-k^2(p^2-k^2)-2p^2k\cdot q\right]\Delta^4\\
&\qquad\qquad+2k^2(p^2-k^2)\left[(k^2+p^2)(k\cdot p-2p^2)-p^2k^2\right]\Delta^2\\
&\qquad\qquad+3k^4p^2(p^2-k^2)\left[-2p^2k\cdot q+(k^2+p^2)p\cdot q\right]\Big]m^2\\
&\qquad\qquad+p^4k^2q^2\left[\Delta^4-2k^2(k^2-p^2)\Delta^2+3p^2k^2(p^2-k^2)k\cdot q\right]\Big]\Big\}\\
&\quad+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\left[-\Delta^4+3(p^2-k^2)k^4k\cdot q\right]m^2+k^2k\cdot q\left[(k\cdot p+k^2)\Delta^2+3k^2k\cdot p(p^2-k^2)\right]\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[\ -q^2\left[\Delta^4-3k^4(p^2-k^2)k\cdot q\right]m^6\\
&\qquad\qquad+\Big[\left(k^2(p^2-k^2)+2p^2k\cdot q\right)\Delta^4-2k^4(p^2-k^2)(k^2+2p^2)\Delta^2\\
&\qquad\qquad+3k^6(p^2-k^2)\left(k\cdot p\,(k^2+3p^2)+p^2(3k^2+p^2)\right)\Big]m^4\\
&\qquad\qquad+k^2\Big[\left(p^2(k^2-p^2)-2p^2k\cdot q\right)\Delta^4\\
&\qquad\qquad+2k^2(p^2-k^2)\left((p^2+k^2)(-2k^2+k\cdot p)-p^2k^2\right)\Delta^2\\
&\qquad\qquad+3k^4p^2(p^2-k^2)\left(-(p^2+k^2)k\cdot q+2k^2p\cdot q\right)\Big]m^2\\
&\qquad\qquad+k^4p^2q^2k\cdot q\left[(k^2+k\cdot p)\Delta^2-3k^2k\cdot p(p^2-k^2)\right]\Big]\Big\}\Big)\\
&=\tau_2(\mathbf{k^2-k\cdot p})-\tau_3-\tau_6
\end{aligned}
\qquad (6.4.20)
$$

k^μ $\not p$ comparison :

$$
\begin{aligned}
&\frac{\alpha}{4\pi}\Big(\frac{\mathbf{J}_0}{4\pi^4}\Big\{\,(2\Delta^2+3q^2k\cdot p)\,m^4+\big(8(q^2+k\cdot p)\Delta^2+6k^2p^2q^2\big)\,m^2\\
&\qquad +8\Delta^4+\big(4p^2k\cdot q-4k^2p\cdot q-2p^2p^2\big)\,\Delta^2-3p^2k^2q^2k\cdot p\\
&\qquad +(\xi-1)\Big[\,(2\Delta^2+3q^2k\cdot p)\,m^4-2k^2k\cdot p\Delta^2-3p^2k^2q^2k\cdot p\Big]\Big\}\\
&\quad+\frac{\mathbf{S}}{\boldsymbol{\Delta}^4}\Big\{\,(2\Delta^2+3q^2k\cdot p)\,m^2+(5q^2+2k\cdot p)\Delta^2+3p^2k^2q^2\\
&\qquad +\frac{(\xi-1)}{\chi}\Big[\quad q^2\,(2\Delta^2+3q^2k\cdot p)\,m^6+\Big[-4\Delta^4+\big(3(p^2+k^2)^2+2p^2k\cdot q-2k^2p\cdot q\big)\,\Delta^2\\
&\qquad\qquad +3p^2k^2\,\big((p^2-k^2)^2+4p^2k\cdot q-4k^2p\cdot q\big)\,\Big]m^4+\Big[4(k^2+p^2)\Delta^4+\Big(12p^2k^2(p^2+k^2)\\
&\qquad\qquad -2k\cdot p(p^2+k^2)^2-2k^2k\cdot p(p^2+k^2)+4k^2p^2k\cdot q\Big)\Delta^2-3p^2k^2\Big(k\cdot p(p^2-k^2)^2\\
&\qquad\qquad -4p^2k^2q^2\Big)\Big]m^2-4p^2k^2\Delta^4-p^2k^2\Big((p^2-k^2)^2+4p^2k\cdot q-8k^2p\cdot q\\
&\qquad\qquad +2k^2(p^2+k^2)+8p^2k^2\Big)-3p^4k^4\,\big((p^2-k^2)^2-4k^2p\cdot q+4p^2k\cdot q\big)\,\Big]\Big\}\\
&\quad+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{\,\big(p^2k^2+(k\cdot p)^2\big)\,m^2+2p^2k^2k\cdot p+\frac{(\xi-1)}{\chi}\Big[\,\big(p^2k^2+(k\cdot p)^2\big)\,m^2-2p^2k^2k\cdot p\Big]\Big\}\\
&\quad+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\,[\Delta^4+p^2(p^2-k^2)\Delta^2+3p^4(p^2-k^2)k\cdot q]\,m^2\\
&\qquad\qquad +p^2\Delta^4-p^2(p^2-k^2)(5p^2-4k\cdot p)\Delta^2+3p^4k^2(p^2-k^2)p\cdot q\\
&\qquad +\frac{(\xi-1)}{\chi}\Big[\quad q^2\,[\Delta^4+p^2(p^2-k^2)\Delta^2+3p^4(p^2-k^2)k\cdot q]\,m^6+\Big[\,\big(p^2(p^2-k^2)+2k^2p\cdot q\big)\,\Delta^4\\
&\qquad\qquad -p^2(p^2-k^2)\,\big(3p^2(k^2+p^2)-2k^2p\cdot q\big)\,\Delta^2-3p^4k^2(p^2-k^2)\Big(q^2(p^2+k^2)\\
&\qquad\qquad +(p^2-k^2)k\cdot q\Big)\Big]m^4-p^2\Big[\,\big(p^2q^2+(p^2-k^2)(3p^2+k^2)\big)\,\Delta^4\\
&\qquad\qquad +p^2(p^2-k^2)\,\big(5(k^2+p^2)k\cdot q+3k\cdot p(k^2+p^2)-2k^2p\cdot q\big)\,\Delta^2\\
&\qquad\qquad +3p^4k^2(p^2-k^2)\,\big(q^2(k^2+p^2)+(p^2-k^2)p\cdot q\big)\,\Big]m^2\\
&\qquad\qquad +p^4k^2q^2p\cdot q\,\big[(2k^2+p\cdot q)\Delta^2+3p^2k^2(k^2-p^2)\big]\,\Big]\Big\}\\
&\quad+\frac{\mathbf{L}'}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\,[-\Delta^4+k^2(p^2-k^2)\Delta^2-3k^4(p^2-k^2)p\cdot q]\,m^2\\
&\qquad\qquad -k^2\Delta^4-k^2(p^2-k^2)(5k^2-4k\cdot p)\Delta^2-3k^4p^2(p^2-k^2)k\cdot q\\
&\qquad +\frac{(\xi-1)}{\chi}\Big[\quad q^2\,[-\Delta^4+k^2(p^2-k^2)\Delta^2-3k^4(p^2-k^2)p\cdot q]\,m^6+\Big[\,\big(k^2(p^2-k^2)+2p^2k\cdot q\big)\,\Delta^4\\
&\qquad\qquad -k^2(p^2-k^2)\,\big(3k^2(k^2+p^2)+2p^2(k^2-k\cdot q)\big)\,\Delta^2-3k^4p^2(p^2-k^2)\Big(q^2(p^2+k^2)\\
&\qquad\qquad +(p^2-k^2)p\cdot q\Big)\Big]m^4+\Big[\,\big(p^2k^2(k^2-p^2)-2k^4p\cdot q\big)\,\Delta^4\\
&\qquad\qquad -k^4(p^2-k^2)\,\big(6p^2k^2+3p^2(p^2+k^2)-2k^2k\cdot p-2(p^2+k^2)k\cdot p\big)\,\Delta^2\\
&\qquad\qquad +3p^2k^6(p^2-k^2)\,\big((k^2+p^2)p\cdot q-2p^2k\cdot q\big)\,\Big]m^2-p^2k^4q^2\Delta^4\\
&\qquad\qquad +3p^2k^6(p^2-k^2)(p^2+k^2-2p\cdot q)\Delta^2\\
&\qquad\qquad +3p^4k^6(p^2-k^2)\,\big((p^2-k^2)k\cdot q-4k^2k\cdot p+2k^2(p^2+k^2)\big)\,\Big]\Big\}\Big)\\
&=\tau_2(\mathbf{p}^2-\mathbf{k}\cdot\mathbf{p})+\tau_3-\tau_6+\tau_8
\end{aligned}
\tag{6.4.21}
$$

$p^\mu\ \not{k}$ comparison :

$$
\begin{aligned}
&\frac{\alpha}{4\pi}\Big(\frac{\mathbf{J_0}}{4\pi^4}\Big\{\ (2\Delta^2+3q^2k\cdot p)\,m^4+\big(4(p^2+k^2)\Delta^2+6p^2k^2q^2\big)\,m^2p^2k^2\,(2\Delta^2+3q^2k\cdot p)\\
&\qquad +(\xi-1)\Big[\ (2\Delta^2+3q^2k\cdot p)\,m^4-2p^2k\cdot p\Delta^2-3p^2k^2q^2k\cdot p\Big]\Big\}\\
&\quad+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{\ (2\Delta^2+3q^2k\cdot p)\,m^2+(p^2+k^2)\Delta^2+3p^2k^2q^2\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[q^2\,(2\Delta^2+3q^2k\cdot p)\,m^6+\Big[-4\Delta^4+\big(3(p^2+k^2)^2+2p^2k\cdot q-2k^2p\cdot q\big)\,\Delta^2\\
&\qquad\qquad+3p^2k^2\,\big((p^2-k^2)^2-4k^2p\cdot q+4p^2k\cdot q\big)\Big]m^4+\Big[4(k^2+p^2)\Delta^4+\Big(12p^2k^2(p^2+k^2)\\
&\qquad\qquad-2k\cdot p(p^2+k^2)^2-4p^2k^2p\cdot q(p^2+k^2)\Big)\Delta^2-3p^2k^2\Big(k\cdot p(p^2-k^2)^2\\
&\qquad\qquad-4p^2k^2q^2\Big)\Big]m^2-4p^2k^2\Delta^4-p^2k^2\Big((p^2-k^2)^2-4k^2p\cdot q\\
&\qquad\qquad+8p^2k\cdot q+2p^2(p^2+k^2)+8p^2k^2\Big)-3p^4k^4\,\big((p^2-k^2)^2+4p^2k\cdot q-4k^2p\cdot q\big)\Big]\Big\}\\
&\quad+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{\ \big(p^2k^2+(k\cdot p)^2\big)\,m^2+2p^2k^2k\cdot p+\frac{(\xi-1)}{\chi}\Big[\ \big(p^2k^2+(k\cdot p)^2\big)\,m^2-2p^2k^2k\cdot p\Big]\Big\}\\
&\quad+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\ \big(\Delta^4+p^2(p^2-k^2)\Delta^2+3p^4(p^2-k^2)k\cdot q\big)\,m^2\\
&\qquad\qquad p^2\Delta^4-p^4(p^2-k^2)\Delta^2+3k^2p^4(p^2-k^2)p\cdot q\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[\ \ q^2\,\big[\Delta^4+p^2(p^2-k^2)\Delta^2+3p^4(p^2-k^2)k\cdot q\big]\,m^6+\Big[\big(-p^2(p^2-k^2)-2k^2p\cdot q\big)\,\Delta^4\\
&\qquad\qquad-p^2(p^2-k^2)\,\big(3p^2(k^2+p^2)-2p^2k\cdot q\big)\,\Delta^2-3p^4k^2(p^2-k^2)\Big(q^2(p^2+k^2)\\
&\qquad\qquad+(p^2-k^2)k\cdot q\Big)\Big]m^4+\Big[\big(p^2k^2(k^2-p^2)-2p^4k\cdot q\big)\,\Delta^4\\
&\qquad\qquad-p^4(p^2-k^2)\,\big(6p^2k^2+3k^2(p^2+k^2)-2p^2k\cdot p-2(p^2+k^2)k\cdot p\big)\,\Delta^2\\
&\qquad\qquad-3k^2p^6(p^2-k^2)\,\big((k^2+p^2)k\cdot q-2k^2p\cdot q\big)\Big]m^2-k^2p^4q^2\Delta^4\\
&\qquad\qquad-3k^2p^6(p^2-k^2)(p^2+k^2+2k\cdot q)\Delta^2\\
&\qquad\qquad-3k^4p^6(p^2-k^2)\,\big((p^2-k^2)p\cdot q-4p^2k\cdot p+2p^2(p^2+k^2)\big)\Big]\Big\}\\
&\quad+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\ \big(-\Delta^4+k^2(p^2-k^2)\Delta^2-3k^4(p^2-k^2)p\cdot q\big)\,m^2\\
&\qquad\qquad-k^2\Delta^4-k^4(p^2-k^2)\Delta^2-3p^2k^4(p^2-k^2)k\cdot q\\
&\qquad+\frac{(\xi-1)}{\chi}\Big[\ \ q^2\,\big[-\Delta^4+k^2(p^2-k^2)\Delta^2-3k^4(p^2-k^2)p\cdot q\big]\,m^6+\Big[\big(k^2(p^2-k^2)+2p^2k\cdot q\big)\,\Delta^4\\
&\qquad\qquad-p^2(p^2-k^2)\,\big(3p^2(k^2+p^2)-2k^2p\cdot q\big)\,\Delta^2-3k^4p^2(p^2-k^2)\Big(q^2(p^2+k^2)\\
&\qquad\qquad+(p^2-k^2)p\cdot q\Big)\Big]m^4+k^2\Big[\big(k^2q^2-(p^2-k^2)(3k^2+p^2)\big)\,\Delta^4\\
&\qquad\qquad+k^2(p^2-k^2)\,\big(5(k^2+p^2)p\cdot q-3k\cdot p(k^2+p^2)-2p^2k\cdot q\big)\,\Delta^2\\
&\qquad\qquad-3k^4p^2(p^2-k^2)\,\big(q^2(k^2+p^2)+(p^2-k^2)k\cdot q\big)\Big]m^2\\
&\qquad\qquad+k^4p^2q^2k\cdot q\,\big[(2p^2-k\cdot q)\Delta^2-3p^2k^2(k^2-p^2)\big]\Big]\Big\}\Big)\\
&=\tau_2(\mathbf{k^2-k\cdot p})+\tau_3+\tau_6-\tau_8 \qquad (6.4.22)
\end{aligned}
$$

$\gamma^\mu_\mu\ \not{k}\ \not{p}$ **comparison :**

$$\frac{\alpha}{4\pi}\Big(\frac{\mathbf{J_0}}{4\pi^4}\Big\{-2q^2m^2 - 2k\cdot pq^2\Big\}$$
$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{-2\Delta^2q^2\Big\}$$
$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{-2p^2\Delta^2(p^2-k^2)p\cdot q\Big\}$$
$$+\frac{\mathbf{L}'}{\mathbf{2p^2\Delta^4(p^2-k^{2)}}}\Big\{2k^2\Delta^2(p^2-k^2)k\cdot q\Big\}\Big)$$
$$= -\tau_8 \tag{6.4.23}$$

γ_μ **comparison :**

$$\frac{\alpha}{4\pi}\Big(\frac{\mathbf{J_0}}{4\pi^4}\Big\{q^2\Delta^2m^4 + 4\Delta^4m^2 + p^2k^2q^2\Delta^2$$
$$+(\xi-1)\Big[-q^2\Delta^2 + p^2k^2q^2\Delta^2\Big]\Big\}$$
$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{-\Delta^2\left(q^2m^2 - q^2k\cdot p\right)\Big\}$$
$$+(\xi-1)\Big[-\Delta^2\left(q^2m^2 - q^2k\cdot p\right)\Big]\Big\}$$
$$+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{-\Delta^2\left((k^2+p^2)m^2 + 2p^2k^2\right)$$
$$+\frac{(\xi-1)}{\chi}\Big[-\Delta^2\left((k^2+p^2)m^2 + 2p^2k^2\right)\Big]\Big\}$$
$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\Delta^2(p^2-k^2)\left((\Delta^2 - p^2p\cdot q)m^2 + p^2\Delta^2 + p^4k\cdot q\right)\Big\}$$
$$+(\xi-1)\Big[\Delta^2(p^2-k^2)\left((\Delta^2 - p^2p\cdot q)m^2 + p^2\Delta^2 + p^4k\cdot q\right)\Big]\Big\}$$
$$+\frac{\mathbf{L}'}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{-\Delta^2(p^2-k^2)\left((\Delta^2 + l^2p\cdot q)m^2 + k^2\Delta^2 - k^4p\cdot q\right)\Big\}\Big)$$
$$+(\xi-1)\Big[-\Delta^2(p^2-k^2)\left((\Delta^2 + l^2p\cdot q)m^2 + k^2\Delta^2 - k^4p\cdot q\right)\Big]\Big\}$$
$$= \tau_3\,\mathbf{q}^2 + \tau_6\,(\mathbf{p}^2 - \mathbf{k}^2) + \tau_8\mathbf{k}\cdot\mathbf{p} \tag{6.4.24}$$

k^μ comparison :

$$\frac{\alpha m}{4\pi}\Bigg(\frac{\mathbf{J_0}}{4\pi^4}\Big\{8\Delta^2\left(p\cdot q m^2+(\Delta^2+p^2k\cdot q)\right)$$

$$(\xi-1)\Big[\left[(-4q^2+8p\cdot q-6k\cdot p)\,\Delta^2+3p^2\left(2k^2p\cdot q-(p^2+k^2)k\cdot q\right)\right]m^2$$

$$-q^2\left[2(p\cdot q-p^2)\Delta^2+3p^2k^2p\cdot q\right]\Big]\Big\}$$

$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{8\Delta^2p\cdot q$$

$$+\frac{(\xi-1)}{\chi}\Big\{\;q^2\left[(-4q^2+8p\cdot q-6k\cdot p)\Delta^2+3p^2(2k^2p\cdot q-(p^2+k^2)k\cdot q)\right]m^4$$

$$+\Big[\Big(-49p^2k^2q^2+7p^4p\cdot q+6k^2k\cdot p(k^2-p^2)+21p^2k\cdot p(p^2-k^2)$$

$$-p^2k^2(p^2+k\cdot p)-46p^4k^2\Big)\Delta^2+6p^2k^2\Big(p^4+k^4)k\cdot q+4p^2k^2(p^2-k^2)$$

$$-4p^4k\cdot q+10p^2k^2k\cdot q\Big)\Big]m^2-p^2q^2\Big[\left(3k^2(p^2+k^2)-8k^2p\cdot q+2p^2k\cdot q\right)\Delta^2$$

$$+3p^2k^2\left(k^2k\cdot q+p^2k\cdot q-2k^2p\cdot q\right)\Big]\Big\}$$

$$+\frac{(\xi-1)}{\chi}\Big[2p^2\left(2\Delta^2-k^2p\cdot q\right)\Big]\Big\}$$

$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{8p^2\Delta^2\left[p^2(p^2-k^2)-\Delta^2\right]$$

$$+\frac{(\xi-1)}{\chi}\Big\{p^2\Big[-q^2\left[2\Delta^4+(p^2-k^2)(2p\cdot q-p^2)\Delta^2+3p^2q^2(p^2-k^2)k\cdot p\right]m^4$$

$$+\Big[\left(-8(p^2-k^2)^2+8k^2k\cdot q-4(p^2-k^2)(k^2+k\cdot p)\right)\Delta^4$$

$$+p^2(p^2-k^2)\left(-7(p^2+k^2)^2+26k\cdot p(p^2+k^2)+4p^2k\cdot q+2k^2(p^2-k^2)\right)\Delta^2$$

$$+6p^4k^2(p^2-k^2)\left(4q^2k\cdot p-(p^2-k^2)^2\right)\Big]m^2$$

$$-p^2q^2\Big[2k^2\Delta^4+(p^2-k^2)\left(2k\cdot p(p^2+k^2)-9p^2k^2\right)\Delta^2$$

$$3p^2k^2(p^2-k^2)\left(k^2p\cdot q-p^2k\cdot q\right)\Big]\Big\}$$

$$+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{8k^2\Delta^2\left(\Delta^2-(p^2-k^2)k\cdot p\right)$$

$$+\frac{(\xi-1)}{\chi}\Big\{q^2\Big[(6k^2-4p^2)\Delta^4-k^2(p^2-k^2)(p\cdot q-4k\cdot q+k\cdot p)\Delta^2+3k^4p^2q^2(p^2-k^2)\Big]m^4$$

$$-\Big[\left(12k^4(p^2-k^2)+12p^2k\cdot p(p^2-k^2)-4k^4p\cdot q+4p^4k\cdot q\right)\Delta^4$$

$$-2k^2(p^2-k^2)\left(-19p^2k^2k\cdot q+3k\cdot p(k^4-p^4)-5p^4k\cdot q-12p^4k^2)\right)\Delta^2$$

$$-6k^4p^2(p^2-k^2)\left((p^2-k^2)^2k\cdot p-4p^2k^2q^2\right)\Big]m^2$$

$$-k^4p^2\Big[\left(10(k^2-p^2)+4p\cdot q\right)\Delta^4$$

$$-(p^2-k^2)\left(-12k^2p\cdot q+14p^2k\cdot q+3(p^2-k^2)^2+p^2(p^2+k^2)+10p^2k^2\right)\Delta^2$$

$$-3p^2k^2(p^2-k^2)\left((p^2-k^2)^2+4p^2k\cdot q-4k^2p\cdot q\right)\Big\}\Bigg)$$

$$=\tau_1\,(\mathbf{p^2-k\cdot p})-\tau_4\,(\mathbf{p^2-k\cdot p})(\mathbf{k\cdot p})+\tau_5+\frac{\tau_7}{}2(\mathbf{k^2-p^2-2k\cdot p}) \tag{6.4.25}$$

p$^\mu$ comparison :

$$\frac{\alpha m}{4\pi}\Bigg(\frac{\mathbf{J_0}}{4\pi^4}\Big\{8\Delta^2\left(-k\cdot q m^2+\Delta^2-k^2 p\cdot q\right)$$
$$+(\xi-1)\Big[\Big\{\Big[\left(2(k^2-p^2)+6(k\cdot p-2k^2)\right)\Delta^2+3k^2(p^2+k^2)p\cdot q-6p^2k^2k\cdot q\Big]m^2$$
$$+q^2\left(2\Delta^2+3p^2k^2\right)\Big\}\Big]\Big\}$$

$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{-8\Delta^2 k\cdot q$$
$$+\frac{(\xi-1)}{\chi}\Big[\Big\{q^2\Big[\left(6k\cdot p-10k^2-2p^2\right)\Delta^2+3k^2\left(p\cdot q(p^2+k^2)-2p^2k\cdot q\right)\Big]m^4$$
$$+\Big[12(p^2+k^2)\Delta^4$$
$$+\Big(39p^2k^2q^2+5k^4k\cdot q-13k^4k\cdot q-4p^2k\cdot p(p^2-k^2)+24p^2k^2(k^2+k\cdot p)\Big)\Delta^2$$
$$+6k^2p^2\Big((p^4+k^4)k\cdot q-4k^4p\cdot q+6p^2k^2k\cdot q-4p^2k^2p\cdot q\Big)\Big]m^2$$
$$+3p^2k^2q^4k\cdot q\Big\}\Big]\Big\}$$

$$+\frac{\mathbf{1}}{\mathbf{4k^2p^2\Delta^2}}\Big\{$$
$$+\frac{(\xi-1)}{\chi}\Big[2mp^2k^2k\cdot q\Big]\Big\}$$

$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{-8p^2\Delta^2\left(\Delta^2+k\cdot p(p^2-k^2)\right)$$
$$+\frac{(\xi-1)}{\chi}\Big[\quad mp^2\Big\{-q^2\left[2\Delta^4-(p^2-k^2(2q^2-k^2)\Delta^2-3p^2k^2q^2(p^2-k^2))\right]m^4$$
$$+\Big[\left(8(k^4-p^4)-4k\cdot p(p^2+k^2)+8p^2k^2\right)\Delta^4$$
$$+2(p^2-k^2)\left(-13p^2k^2k\cdot q-k\cdot p(p^4-k^4)-3p^4k\cdot q-12p^2k^2\right)\Delta^2$$
$$+6p^2k^2(p^2-k^2)\left[(p^2-k^2)^2k\cdot p-4p^2k^2q^2\right]\Big]m^2$$
$$+p^2k^2q^2k\cdot p\left[(-2k\cdot q+4p^2)\Delta^2+3p^2k\cdot q(p^2-k^2)\right]\Big]\Big\}$$

$$+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{8k^2\Delta^2\left[k^2(p^2-k^2)+\Delta^2\right]$$
$$+\frac{(\xi-1)}{\chi}\Big[\quad k^2\Big\{q^2\left[2\Delta^4+7k^2(p^2-k^2)\Delta^2-3k^2(p^2-k^2)(k^2p\cdot q-p^2k\cdot q)\right]m^4$$
$$-\Big[\left(4k^2(p^2-k^2)-4k^2p\cdot q+4p^2k\cdot q\right)\Delta^4$$
$$-k^2(p^2-k^2)\left(-5(p^2-k^2)^2+14k\cdot p(p^2+k^2)-2p^2(p^2+k^2)-4p^2k\cdot q\right)\Delta^2$$
$$-6k^4p^2(p^2-k^2)\left(4q^2k\cdot p-(p^2-k^2)^2\right)\Big]m^2$$
$$-p^2k^2q^2k\cdot q\left[(2p\cdot q+4k^2)\Delta^2+3k^2(p^2-k^2)p\cdot q\right]\Big]\Big\}\Bigg)$$

$$=\tau_1\left(\mathbf{k^2-k\cdot p}\right)-\tau_4\left(\mathbf{k^2-k\cdot p}\right)(\mathbf{k\cdot p})-\tau_5+\frac{\tau_7}{\mathbf{2}}\left(\mathbf{k^2-p^2-2k\cdot p}\right) \qquad (6.4.26)$$

p^μ $\not k$ $\not p$ comparison :

$$\frac{\alpha m}{4\pi}\Bigg(\frac{\mathbf{J_0}}{4\pi^4}\Big\{(\xi-1)\Big[-\big[3q^2k\cdot qm^2+p\cdot q(2\Delta^2+3k^2q^2)\big]\Big]\Big\}$$
$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{\frac{(\xi-1)}{\chi}\Big[q^2\Big\{-3q^2k\cdot qm^4\Big\}$$
$$+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[(p^2-k^2)\Big\{q^2\left[(3p^2+2k\cdot p)\Delta^2-3p^2(k^2p\cdot q-p^2k\cdot q)\right]m^4$$
$$+\Big[4k^2\Delta^4-\left(3p^2k^2k\cdot q-p^2k\cdot p(p^2-k^2)-5p^4p\cdot q-24p^2k^2\right)\Delta^2$$
$$+6p^4k^2\left(4q^2k\cdot p-(p^2-k^2)^2\right)\Big]m^2$$
$$-3p^4k^2q^2k\cdot qp\cdot q\Big\}\Big]\Big\}$$
$$+\frac{\mathbf{L'}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[mk^2(p^2-k^2)\Big\{q^2(\Delta^2+3k^2q^2)m^4$$
$$+\Big[\left(-14k^2(p^2+k^2)+4k^2p\cdot q+2k\cdot p(p^2+k^2)\right)\Delta^2$$
$$+k^2\left(6k\cdot p(p^2-k^2)^2-24p^2k^2q^2\right)\Big]m^2$$
$$+k^2q^2\left[(2p^2+k^2)\Delta^2+3p^2k^2q^2\right]\Big\}\Big]\Big\}\Bigg)$$
$$=\tau_4\,(\mathbf{p^2-k\cdot p})+\tau_7 \qquad (6.4.27)$$

k^μ $\not k$ $\not p$ comparison :

$$\frac{\alpha m}{4\pi}\Bigg(\frac{\mathbf{J_0}}{4\pi^4}\Big\{(\xi-1)\Big[-\big[-3q^2p\cdot qm^2-k\cdot q(2\Delta^2+3p^2q^2)\big]\Big]\Big\}$$
$$+\frac{\mathbf{S}}{\mathbf{\Delta^4}}\Big\{\frac{(\xi-1)}{\chi}\Big[q^2\Big\{3q^2p\cdot qm^4\Big\}$$
$$+\frac{\mathbf{L}}{\mathbf{-2k^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[-(p^2-k^2)\Big\{q^2\left[(3k^2+2k\cdot p)\Delta^2-3k^2(-p^2k\cdot q+k^2p\cdot q)\right]m^4$$
$$+\Big[4p^2\Delta^4-\left(-3p^2k^2k\cdot q+k^2k\cdot p(p^2-k^2)+5k^4k\cdot q-24p^2k^2\right)\Delta^2 nn (6.4.28)$$
$$+6k^4p^2\left(4q^2k\cdot p-(p^2-k^2)^2\right)\Big]m^2$$
$$-3k^4p^2q^2p\cdot qk\cdot q\Big\}\Big]\Big\}$$
$$+\frac{\mathbf{L'}}{\mathbf{-2k^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[-k^2(p^2-k^2)\Big\{q^2(\Delta^2+3p^2q^2)m^4$$
$$+\Big[\left(-14p^2(p^2+k^2)+4p^2k\cdot q+2k\cdot p(p^2+k^2)\right)\Delta^2$$
$$+p^2\left(-6k\cdot p(p^2-k^2)^2-24p^2k^2q^2\right)\Big]m^2$$
$$+p^2q^2\left[(2k^2+p^2)\Delta^2+3p^2k^2q^2\right]\Big\}\Big]\Big\}\Bigg)$$
$$=\tau_4(\mathbf{k^2-k\cdot p})+\tau_7 \qquad (6.4.29)$$

$\gamma_\mu\ \not{k}$ **comparison :**

$$
\begin{aligned}
&\frac{\alpha\, m}{4\pi}\Big(\frac{\mathbf{J}_0}{4\pi^4}\Big\{(\xi-1)\Big[-mq^2(p^2-m^2)\Big]\Big\}\\
&\quad+\frac{\mathbf{S}}{\boldsymbol{\Delta}^4}\Big\{\frac{(\xi-1)}{\chi}\Big[-mq^2(p^2-m^2)\,(q^2m^2+2\Delta^2+q^2k\cdot p)\Big]\Big\}\\
&\quad+\frac{1}{\mathbf{4k^2p^2\Delta^2}}\Big\{+\frac{(\xi-1)}{\chi}\Big[-2mp^2\Delta^2\Big]\Big\}\\
&\quad+\frac{\mathbf{L}}{\mathbf{-2k^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[-p^2(p^2-k^2)\Big\{\,[2\Delta^2-(p^2+k^2)p\cdot q+2p^2k\cdot q]\,m^4\\
&\qquad\qquad+[-2(p^2+k^2)\Delta^2-p^2(p^2-k^2)^2-4p^4k\cdot q+4p^2k^2p\cdot q]\,m^2\\
&\qquad\qquad-p^4q^2k\cdot p+p^4k^2q^2\Big\}\Big]\Big\}\\
&\quad+\frac{L'}{\mathbf{-2k^2\Delta^4(p^2-k^2)}}\Big\{\,m(p^2-k^2)\Big\{q^2(2\Delta^2-k^2k\cdot q)m^4+\Big[(2k^2(k^2-p^2)+4p^2k\cdot p)\Delta^2\\
&\qquad\qquad+4p^2k^4q^2-k^2k\cdot p(p^2-k^2)^2\Big]m^2+k^4p^2q^2p\cdot q\Big\}\Big]\Big\}\Big)\\
&\quad=-\tau_5+\frac{\tau_7}{\mathbf{2}}\,(\mathbf{p^2-k^2})
\end{aligned}
\tag{6.4.30}
$$

$\gamma_\mu\ \not{p}$ **comparison :**

$$
\begin{aligned}
&\frac{\alpha\, m}{4\pi}\Big(\frac{\mathbf{J}_0}{4\pi^4}\Big\{(\xi-1)\Big[q^2(k^2-m^2)\Big]\Big\}\\
&\quad+\frac{\mathbf{S}}{\boldsymbol{\Delta}^4}\Big\{\frac{(\xi-1)}{\chi}\Big[q^2(k^2+m^2)\,(q^2m^2+2\Delta^2+q^2k\cdot p)\,\Big\}\\
&\quad+\frac{1}{\mathbf{4k^2p^2\Delta^2}}\Big\{+\frac{(\xi-1)}{\chi}\Big[2mk^2\Delta^2[\Big]\Big\}\\
&\quad+\frac{\mathbf{L}}{\mathbf{2p^2\Delta^4(p^2-k^2)}}\Big\{\frac{(\xi-1)}{\chi}\Big[k^2(p^2-k^2)\Big\{\,[2\Delta^2+(p^2+k^2)k\cdot q-2k^2p\cdot q]\,m^4\\
&\qquad\qquad+[2(p^2+k^2)\Delta^2+k^2(p^2-k^2)^2-4k^4p\cdot q+4p^2k^2k\cdot q]\,m^2\\
&\qquad\qquad-p^4q^2k\cdot p+p^4k^2q^2\Big\}\Big]\Big\}\\
&\quad+\frac{\mathbf{L}'}{\mathbf{-2p^2\Delta^4(p^2-k^2)}}\Big\{(p^2-k^2)\Big\{q^2(2\Delta^2-p^2p\cdot q)m^4+\Big[(2p^2(k^2-p^2)+4k^2k\cdot p)\Delta^2\\
&\qquad\qquad+4k^2p^4q^2-k^2k\cdot p(p^2-k^2)^2\Big]m^2+p^4k^2q^2p\cdot q\Big\}\Big]\Big\}\Big)\\
&\quad=\tau_5+\frac{\tau_7}{\mathbf{2}}(\mathbf{p^2-k^2})
\end{aligned}
\tag{6.4.31}
$$

Laborious solution of these 12 equations yields expressions for the 8 transverse coefficients τ_i. Each is a function of k^2, p^2, q^2 and ξ. The results are as follows :

$$\tau_1 = \frac{\alpha}{4\pi}\frac{(3+\xi)}{\Delta^2}m\Bigg\{ - \frac{1}{2}\left(m^2 + k\cdot p\right) J_0 - 2S + \frac{(p^2 + k\cdot p)\,L}{(p^2-k^2)} - \frac{(k^2 + k\cdot p)\,L'}{(p^2-k^2)}\Bigg\} \quad , \tag{6.4.32}$$

$$\begin{aligned}
\tau_2 = & -\frac{3}{4\Delta^2}\left(m^2 + k\cdot p\right)\tau_8 \\
& + \frac{\alpha}{4\pi\Delta^2}\Bigg\{\frac{1}{8}\left(q^2 - 4m^2\right)J_0 + \frac{(L+L')}{4} - \frac{1}{2} - \frac{m^2}{2\,p^2k^2}\,k\cdot p \\
& \qquad + \frac{1}{2\,(p^2-k^2)}\left[(k^2 + k\cdot p)\left(1+\frac{m^2}{k^2}\right)L' - (p^2 + k\cdot p)\left(1+\frac{m^2}{p^2}\right)L\right]\Bigg\} \\
& + \frac{\alpha}{8\pi\Delta^2}(\xi - 1)\Bigg\{\left[\frac{1}{2}\left(p^2 + k^2\right) + \frac{3q^2}{4\Delta^2}\left(p^2k^2 - m^4\right)\right]J_0 + 1 - \frac{m^2}{p^2\,k^2}\,k\cdot p \\
& \qquad + \frac{1}{\chi}\Bigg[\frac{-2m^2}{(p^2-k^2)}\left\{\left(p^2 + m^2\,\frac{k^2}{p^2}\right)L - \left(k^2 + m^2\,\frac{p^2}{k^2}\right)L'\right\}\Delta^2 \\
& \qquad - q^2\,\frac{[p^2 + 2\,k\cdot p + k^2]}{2\,(p^2-k^2)}\left\{m^6\left(\frac{L}{p^2} - \frac{L'}{k^2}\right) + p^2k^2\,(L - L')\right\} \\
& \qquad + \frac{q^2}{2}m^4\left\{\left(1 - \frac{m^2}{p^2}\right)L + \left(1 - \frac{m^2}{k^2}\right)L'\right\} - \frac{m^2}{2}(p^4 - k^4)(L - L') \\
& \qquad + \frac{q^2}{2}\left[k^2p^2\left(L + L'\right) - m^2\left(k^2L + p^2L'\right)\right] \\
& \qquad + \frac{3\,m^2}{2}(p^2 - k^2)\left[(m^2 - p^2)\,L - (m^2 - k^2)\,L'\right] \\
& \qquad + \frac{3q^4}{4\Delta^2}(m^4 - p^2k^2)\left[(m^2 - p^2)L + (m^2 - k^2)L'\right] \\
& \qquad + \frac{3\,q^2}{4\,\Delta^2}(p^2 - k^2)\,(m^4 - p^2\,k^2)\left[(m^2 + p^2)\,L - (m^2 + k^2)\,L'\right] \\
& \qquad + \Bigg[- 8\,q^2(m^4 + k^2p^2) + 12p^2k^2(q^2 - m^2) + 2q^2(k^2 + p^2)m^2 \\
& \qquad - \frac{3}{\Delta^2}m^4q^4(m^2 + k\cdot p) + \frac{3}{\Delta^2}k^2p^2q^2\,k\cdot p\,(q^2 - m^2) \\
& \qquad + 2(p^2 - k^2)^2m^2 - \frac{9}{\Delta^2}m^2p^2k^2q^2\,k\cdot p + \frac{3}{\Delta^2}m^2p^2k^2(p^2 - k^2)^2\Bigg]\,S\Bigg]\Bigg\},
\end{aligned}$$

(6.4.33)

$$
\begin{aligned}
\tau_3 = & - \frac{3}{32m\Delta^2}\,(\,m^2 + k\cdot p\,)(p^2-k^2)^2\,\tau_1(\xi=1) \\
& + \frac{\alpha}{8\pi\Delta^2}\Bigg\{\left(-\Delta^2 - \frac{(p^2+k^2-2\,m^2)^2}{8}\right) J_0 - 2\,(\,m^2 + k\cdot p)S \\
& \qquad + \frac{k\cdot p}{2}\left(\left(1-\frac{m^2}{p^2}\right)L + \left(1-\frac{m^2}{k^2}\right)L'\right) + \frac{1}{4}(p^2-k^2)\,(L-L') \\
& \qquad + (\,m^2 + k\cdot p) + \frac{1}{2p^2k^2}\,(p^2+k^2)\,(\,p^2k^2 + m^2\,k\cdot p\,)\Bigg\} \\
& + \frac{\alpha}{8\pi\Delta^2}\,(\xi-1)\Bigg\{\left[\frac{(p^2k^2-m^4)}{2} - \frac{(p^2-k^2)^2}{4} - \frac{3}{8\Delta^2}(p^2-k^2)^2\,(p^2k^2-m^4)\right] J_0 \\
& \qquad + \frac{m^2}{2p^2k^2}\,k\cdot p\,(\,k^2+p^2) - \frac{(p^2+k^2)}{2} - \left(k\cdot p - m^2\right) \\
& \qquad + \frac{1}{\chi}\Bigg[-m^2\left\{\left(p^2 + m^2\,\frac{k^2}{p^2}\right)L + \left(k^2 + m^2\,\frac{p^2}{k^2}\right)L'\right\}\Delta^2 \\
& \qquad - \frac{m^6}{2}q^2\,k\cdot p\left(\frac{L}{p^2} + \frac{L'}{k^2}\right) - \frac{1}{2}p^2k^2q^2\,k\cdot p\,\left(L + L'\right) \\
& \qquad + \frac{1}{4}q^2m^2\left(p^2L - k^2L'\right) - \frac{1}{2}q^2m^4\left(p^2L + k^2L'\right) \\
& \qquad + \frac{1}{2}(p^2-k^2)q^2(m^4 + p^2k^2)(L-L') \\
& \qquad + \frac{m^2}{4}(p^4-k^4)\left[(p^2-m^2)L - (k^2-m^2)L'\right] \\
& \qquad + \frac{m^2}{2}(p^4-k^4)[k\cdot p - m^2](L-L') + m^2(p^2-k^2)(p^4L - k^4L') \\
& \qquad - \frac{m^4}{4}(p^2-k^2)[p^2 + 2\,k\cdot p + k^2](L-L') \\
& \qquad + \frac{3}{8\Delta^2}q^2(p^2-k^2)[p^2 + 2\,k\cdot p + k^2]\left[(p^4k^2-m^6)L - (k^4p^2-m^6)L'\right] \\
& \qquad - \frac{3m^2}{8\Delta^2}(p^2-k^2)^3\left[p^2(m^2-k^2)L - k^2(m^2-p^2)L'\right] \\
& \qquad + \frac{3m^4}{8\Delta^2}q^2(p^2-k^2)^2\left[(p^2-m^2)L + (k^2-m^2)L'\right] \\
& \qquad - \frac{3}{8\Delta^2}p^2k^2q^2(p^2-k^2)^2\left[(p^2-m^2)L + (k^2-m^2)L'\right] \\
& \qquad + \Bigg[-8m^4\Delta^2 + 4m^2(p^2+k^2)\Delta^2 + 2m^2(p^2-k^2)^2[m^2-(p^2+k^2)] \\
& \qquad - 2m^4q^2[m^2 + k\cdot p] + 2p^2k^2q^2[m^2 + k\cdot p] + 2m^4(p^2-k^2)^2 \\
& \qquad + \frac{3}{2\Delta^2}q^2(p^2-k^2)^2(m^4-p^2k^2)\left[m^2 + k\cdot p\right]\Bigg]S\Bigg]\Bigg\},
\end{aligned}
\tag{6.4.34}
$$

$$\begin{aligned}
\tau_4 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\Bigg\{ & - \left(1+\frac{3q^2}{2\Delta^2}\left(m^2+k\cdot p\right)\right)J_0 - \frac{2}{p^2k^2}\,k\cdot p \\
+\frac{1}{\chi}\Bigg[& - \frac{4m^2}{(p^2-k^2)}\left(\frac{k^2}{p^2}L-\frac{p^2}{k^2}L'\right)\Delta^2 + 3m^2(p^2-k^2)(L-L') \\
& - \frac{m^4q^2}{(p^2-k^2)}[k^2+2\,k\cdot p\,+p^2]\left(\frac{L}{p^2}-\frac{L'}{k^2}\right) \\
& - q^2\left\{p^2\left(\frac{m^4}{p^4}-1\right)L+k^2\left(\frac{m^4}{k^4}-1\right)L'\right\} \\
& + \frac{3m^2q^2}{\Delta^2}(p^2-k^2)\Big\{(m^2+p^2)L-(m^2+k^2)L'\Big\} \\
& - \frac{3q^2}{2\Delta^2}(p^2-k^2)(m^4-p^2k^2)(L-L') - \frac{3q^4}{2\Delta^2}(m^4-k^2p^2)(L+L') \\
& - \frac{3m^2q^4}{\Delta^2}\Big\{(p^2-m^2)L+(k^2-m^2)L'\Big\} \\
& + \Big\{-20m^2q^2-2q^2(p^2+k^2)-\frac{12m^2q^4}{\Delta^2}[m^2+k\cdot p] \\
& + \frac{6q^4}{\Delta^2}(m^4-p^2k^2)\Big\}S\Bigg]\Bigg\} \qquad , \qquad (6.4.35)
\end{aligned}$$

$$\begin{aligned}
\tau_5 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\Bigg\{ & - \left[\Delta^2-\frac{1}{4}(p^2-k^2)^2+\frac{q^2}{2}[m^2+k\cdot p]\right]J_0 + \frac{(p^2+k^2)}{p^2k^2}\Delta^2 \\
+\frac{1}{\chi}\Bigg[& - m^2(p^2-k^2)\left(\frac{p^2}{k^2}L'-\frac{k^2}{p^2}L\right)\Delta^2 + 2m^2(p^2-k^2)(L-L')\Delta^2 \\
& + 2m^2q^2\left\{\left(1-\frac{m^2}{p^2}\right)L+\left(1-\frac{m^2}{k^2}\right)L'\right\}\Delta^2 \\
& + m^2q^2\left\{(m^2+k^2)\frac{L}{p^2}+(m^2+p^2)\frac{L'}{k^2}\right\}\Delta^2 \\
& + \frac{m^2q^2}{2}\left\{q^2[m^2+\,k\cdot p]+q^2\,k\cdot p\,-\frac{1}{2}(p^2-k^2)^2\right\}(L+L') \\
& + \frac{m^2}{2}(p^2-k^2)\left\{q^2m^2+2q^2\,k\cdot p\,-\frac{1}{2}(p^2-k^2)^2\right\}(L-L') \\
& + \frac{1}{4}q^4(p^2+k^2)(p^2L+k^2L') - \frac{1}{4}q^2(p^4-k^4)(p^2L-k^2L') \\
& + q^2(p^2+k^2-2m^2)\left(q^2m^2+q^2k\cdot p+2\Delta^2\right)S\Bigg]\Bigg\} \qquad , \qquad (6.4.36)
\end{aligned}$$

$$
\begin{aligned}
\tau_6 \;=\; & \frac{(p^2-k^2)}{2}\tau_2(\xi=1) \\
+\; & \frac{\alpha}{8\pi\Delta^2}(\xi-1)\Bigg\{(p^2-k^2)\left[\frac{q^2}{4}-\frac{3q^2}{8\Delta^2}(m^4-p^2k^2)\right]J_0-\frac{(p^2-k^2)}{2p^2k^2}\left[m^2\,k\cdot p\,-p^2k^2\right] \\
& +\frac{1}{\chi}\Bigg[\;m^2\Delta^2\left\{\left(p^2-m^2\frac{k^2}{p^2}\right)L-\left(k^2-m^2\frac{p^2}{k^2}\right)L'\right\} \\
& \quad +\frac{1}{2}m^2(k^4-p^4)[m^2-k\cdot p](L+L')-2m^2(k\cdot p)(p^4L-k^4L') \\
& \quad -\frac{m^6q^2}{2}\left\{\left(1+\frac{k\cdot p}{p^2}\right)L-\left(1+\frac{k\cdot p}{k^2}\right)L'\right\} \\
& \quad -\frac{m^2}{2}\,k\cdot p\,(p^2-k^2)\left[(m^2-p^2)L+(m^2-k^2)L'\right] \\
& \quad -\frac{1}{2}p^2k^2q^2\left\{\left[k^2-k\cdot p\right]L-\left[p^2-k\cdot p\right]L'\right\} \\
& \quad -q^2\left[p^4(m^2-k^2)L-k^4(m^2-p^2)L'\right]+2m^2k^2p^2\left(k^2L-p^2L'\right) \\
& \quad +\frac{3m^2}{8\Delta^2}(p^4-k^4)(p^2-k^2)\left[p^2(m^2-k^2)L-k^2(m^2-p^2)L'\right] \\
& \quad -\frac{3m^2}{4\Delta^2}p^2k^2q^2(p^2-k^2)\left[(m^2+p^2)L+(m^2+k^2)L'\right] \\
& \quad -\frac{3m^2}{4\Delta^2}p^2k^2(p^2-k^2)^2\left[(m^2-p^2)L-(m^2-k^2)L'\right] \\
& \quad -\frac{3m^2}{8\Delta^2}p^2q^2(p^4-k^4)\left[(m^2+k^2)L-(m^2+p^2)L'\right] \\
& \quad +\frac{3}{8\Delta^2}q^4(p^2-k^2)\left[(m^6+p^4k^2)L+(m^6+p^2k^4)L'\right] \\
& \quad +\frac{3q^2}{8\Delta^2}(p^2-k^2)^2\left[(m^6-p^4k^2)L-(m^6-p^2k^4)L'\right] \\
& \quad +(p^2-k^2)\Bigg[-\frac{3m^4}{2\Delta^2}q^4\left[m^2+\,k\cdot p\right]+\frac{3}{2\Delta^2}k^2p^2q^4\left[m^2+\,k\cdot p\right] \\
& \qquad -4m^4q^2+2m^2q^2(p^2+k^2)\Bigg]\;S\;\Bigg]\Bigg\} \qquad , \qquad (6.4.37)
\end{aligned}
$$

$$\tau_7 = \frac{\alpha m}{8\pi\Delta^2}(\xi - 1)\Bigg\{ \; - \; \frac{q^2}{2}J_0 - \frac{2}{p^2k^2}\Delta^2$$

$$+ \; \frac{1}{\chi}\Bigg[- \frac{2m^2q^2}{(p^2-k^2)}\left\{\frac{(m^2-k^2)}{p^2}L - \frac{(m^2-p^2)}{k^2}L'\right\}\Delta^2$$

$$-2m^2\left\{\left(1 - \frac{k^2}{p^2}\right)L + \left(1 - \frac{p^2}{k^2}\right)L'\right\}\Delta^2$$

$$-q^2(k\cdot p)\left[(m^2-p^2)L + (m^2-k^2)L'\right]$$

$$+q^2\left[p^2(m^2-k^2)L + k^2(m^2-p^2)L'\right]$$

$$-2q^2\left\{q^2\left[m^2 + k\cdot p\right] + 2\Delta^2\right\} S \Bigg]\Bigg\} \quad , \qquad (6.4.38)$$

and

$$\tau_8 = \frac{\alpha}{4\pi\Delta^2}\left\{\frac{q^2}{2}\left(k\cdot p + m^2\right)J_0 + 2q^2S + \left(-p^2 + k\cdot p\right)L + \left(-k^2 + k\cdot p\right)L'\right\} \quad . \qquad (6.4.39)$$

These τ_i's are given in an arbitrary covariant gauge specified by ξ. As promised, all the τ's have been expressed in terms of elementary functions and one scalar integral.

By letting $m \longrightarrow 0$, the τ_i's simplify enormously and four of them, corresponding to $i = 1, 4, 5, 7$, actually vanish :

Massless case:

$m \to 0$:

$$\begin{aligned}
\tau_2 \;=\; & \frac{\alpha}{8\pi\Delta^2}\Bigg\{ J_0\Bigg[\left(\frac{k^2+p^2}{2} + \frac{3}{4\Delta^2}\, p^2 k^2 q^2\right)(\xi-2) + k\cdot p\Bigg] \\
& + \ln\frac{k^2}{p^2}\Bigg[\left(\frac{(k+p)^2}{2\,(p^2-k^2)} + \frac{3}{4\Delta^2}\, k\cdot p\,(p^2-k^2)\right)(\xi-2) + \frac{(p+k)^2}{(p^2-k^2)}\Bigg] \\
& + \ln\frac{q^4}{k^2p^2}\Bigg[\left(\frac{3}{4\Delta^2}\, k\cdot p\, q^2 + 1\right)(\xi-2) + 1\Bigg] \\
& + (\xi-2)\Bigg\}
\end{aligned} \tag{6.4.40}$$

$$\begin{aligned}
\tau_3 \;=\; & \frac{\alpha}{8\pi\Delta^2}\Bigg\{ J_0\Bigg[\left(\frac{(k^2+p^2)^2}{8} - \frac{3}{8\Delta^2}(k\cdot p)^2(k^2-p^2)^2\right)(\xi-2) - \Delta^2\Bigg] \\
& + \ln\frac{k^2}{p^2}\Bigg[\frac{(k^2-p^2)}{4}\left(-1 + \frac{3}{2\Delta^2} k\cdot p\,(k+p)^2\right)(\xi-2)\Bigg] \\
& + \ln\frac{q^4}{k^2p^2}\Bigg[\frac{k\cdot p}{2}\left(1 - \frac{3}{4\Delta^2}(k^2-p^2)^2\right)(\xi-2)\Bigg] \\
& - \frac{(k+p)^2}{2}(\xi-2)\Bigg\}
\end{aligned} \tag{6.4.41}$$

$$\begin{aligned}
\tau_6 \;=\; & \frac{\alpha}{8\pi\Delta^2}\frac{(p^2-k^2)}{2}\Bigg\{ J_0\Bigg[\left(-\frac{q^2}{4} + \frac{3}{4\Delta^2} q^2 (k\cdot p)^2\right)(\xi-2)\Bigg] \\
& + \ln\frac{k^2}{p^2}\Bigg[\left(\frac{3}{4\Delta^2} k\cdot p\,(p^2-k^2) - \frac{(p+k)^2}{2(p^2-k^2)}\right)(\xi-2)\Bigg] \\
& + \ln\frac{q^4}{k^2p^2}\Bigg[\frac{3}{4\Delta^2} k\cdot p\, q^2\,(\xi-2)\Bigg] \\
& + (\xi-2)\Bigg\}
\end{aligned} \tag{6.4.42}$$

$$\tau_8 \;=\; \frac{\alpha}{8\pi\Delta^2}\left\{q^2\left[k\cdot p\,J_0+\frac{q^4}{k^2p^2}\right]+(p^2-k^2)\ln\frac{k^2}{p^2}\right\} \quad . \tag{6.4.43}$$

Our next step is to explore various limits which we now undertake.

6.5 Freedom from Kinematic Singularities

Clearly the full vertex, $\Gamma^\mu(k,p)$, is free of kinematic singularities. The Ball-Chiu construction ensures that the longitudinal part is free of them, so the transverse part must be. However, after decomposing this transverse part into 8 components, it is not necessary that the individual components will each be free of kinematic singularities. Ball and Chiu showed that with their choice of eight basis vectors Eqn. (6.2.7), the transverse vertex in the Feynman gauge possessed this property of being singularity free. Here we explicitly consider whether this is true in an arbitrary covariant gauge. Indeed such checks are far longer than the initial calculation reported above. We consider several limits in turn.

6.5.1 $\Delta^2 \to 0$:

This limit arises when two external fermion legs are parallel or anti-parallel to each other. The proof depends crucially on the behaviour of the combination of Spence functions forming the integral J_0 that appears in every τ_i. Thus, for instance, when we consider the limit $\Delta^2 \to 0$, i.e. $(k\cdot p)^2 \to k^2p^2$, we can deduce from Eqs. (6.3.4, I.15-I.18) that J_0 can be expanded in powers of Δ^2 as :

$$J_0 = J_0^0 + J_0^1\Delta^2 + \mathcal{O}(\Delta^4) \quad , \tag{6.5.1}$$

where

$$J_0^0 \;=\; -\frac{1}{m^2+\sqrt{k^2p^2}}\left[4S-2\frac{(k^2+\sqrt{k^2p^2})}{k^2-p^2}L'+2\frac{(p^2+\sqrt{k^2p^2})}{k^2-p^2}L\right] ,$$

$$J_0^1 \;=\; \left[\frac{2}{3q_0^2(m^2+\sqrt{k^2p^2})\sqrt{k^2p^2}}-Y_1(k^2,p^2)L'-Y_1(p^2,k^2)L-Z_1(k^2,p^2)S\right] ,$$

and Y_1 and Z_1 are defined as

$$Y_1(k^2,p^2) = \frac{(k^2-m^2)}{3\left(\sqrt{k^2}-\sqrt{p^2}\right)^3\left(m^2+\sqrt{k^2p^2}\right)^3\sqrt{k^2p^2}} \times\left(3\sqrt{k^2}\left(m^2-p^2\right)+\sqrt{p^2}\left(k^2-m^2\right)\right),$$

$$Z_1(k^2,p^2) = -\frac{1}{q_0^2\left(m^2+\sqrt{k^2p^2}\right)^3(q_0^2-4m^2)\sqrt{k^2p^2}}$$
$$\times\left(8m^6-8m^4\left(k^2+p^2-\frac{4}{3}\sqrt{k^2p^2}\right)\right.$$
$$\left.+\ m^2\left(2q_0^4+\frac{8}{3}\sqrt{k^2p^2}\left(k^2+p^2+\sqrt{k^2p^2}\right)\right)+\frac{2}{3}\sqrt{k^2p^2q_0^4}\right),$$

$$q_0^2 = k^2+p^2-2\sqrt{k^2p^2}.$$

Together with the known behaviour of all the other functions, such as L, L' and S, it is a lengthy but straightforward calculation to deduce that each τ_i is finite in the limit $\Delta^2\to 0$, despite the appearance of explicit $1/\Delta^2$ and $1/\Delta^4$ terms. As we have seen above in the massive case, the expressions become very long and complicated. therefore for the sake of simplifying matters and for better understanding, we shall evaluate the limits for the massless case rather than massive, unless it is necessary. We can start with the same limit discussed above, $\Delta^2\to 0$. We start with Eqn. (6.3.55) :

$$J_0=\frac{2}{\Delta}\left[f\left(\frac{k\cdot p-\Delta}{p^2}\right)-f\left(\frac{k\cdot p+\Delta}{p^2}\right)+\frac{1}{2}\ln\frac{q^2}{p^2}\ln\left(\frac{k\cdot p-\Delta}{k\cdot p+\Delta}\right)\right], \quad (6.3.55)$$

We expand this expression in powers Δ^2, recalling Eqn. (6.5.1) :

$$J_0\to J_0^0+J_0^1\Delta^2+J_0^2\Delta^4+\mathcal{O}(\Delta^6), \quad (6.5.1)$$

where

$$J_0^0 = \frac{2}{\sqrt{p^2{q_0}^2}}\ln\frac{k^2}{p^2}-\frac{2}{\sqrt{k^2p^2}}\ln\frac{q_0^2}{p^2},$$

$$J_0^1 = \frac{2}{3k^2p^2q_0^2}-\frac{(\sqrt{k^2}-3\sqrt{p^2})}{3\sqrt{k^2p^6q_0^6}}\ln\frac{k^2}{p^2}+\frac{1}{3\sqrt{k^6p^6}}\ln\frac{q_0^2}{p^2},$$

$$J_0^2 = -\frac{1}{10k^4p^4q_0^4}(3k^2+3p^2-8\sqrt{k^2p^2}) - \frac{3}{20\sqrt{k^{10}p^{10}}}\ln\frac{q_0^2}{p^2} + \frac{1}{20\sqrt{k^6p^{10}q_0^{10}}}\left(3\sqrt{k^6}-15\sqrt{k^4p^2}+25\sqrt{k^2p^4}-5\sqrt{p^6}\right)\ln\frac{k^2}{p^2}\,. \tag{6.5.2}$$

Some other quantities acquire the following form in this limit :

$$\begin{aligned} k\cdot p &\rightarrow \sqrt{k^2p^2} + \frac{\Delta^2}{2\sqrt{k^2p^2}} - \frac{\Delta^4}{8\sqrt{k^6p^6}} + \cdots \\ q_0^2 &= k^2+p^2-2\sqrt{k^2p^2} \\ q^2 &\rightarrow q_0^2 - \frac{\Delta^2}{\sqrt{k^2p^2}} + \frac{\Delta^4}{4\sqrt{k^6p^6}} + \cdots \\ \ln\frac{q^2}{p^2} &= \ln\frac{q_0^2}{p^2} - \frac{\Delta^2}{\sqrt{k^2p^2q^4}} + \frac{\Delta^4}{4\sqrt{k^6p^6q_0^4}} - \frac{\Delta^4}{2k^2p^2q_0^4}\cdots\,. \end{aligned} \tag{6.5.3}$$

The τ_i's then can be written as :

$$\begin{aligned} \tau_2 &= \frac{\alpha}{24\pi p^4}\left[1+(\xi-1)\left(\frac{1}{3}-\ln\frac{q_0^4}{p^4}\right)\right] \quad , \\ \tau_3 &= \frac{\alpha}{24\pi p^2}\left[-\frac{29}{3}+2\ln\frac{q_0^4}{p^4}+(\xi-1)\left(-\frac{7}{3}+\ln\frac{q_0^4}{p^4}\right)\right] \quad , \\ \tau_6 &= 0 \quad , \\ \tau_8 &= -\frac{\alpha}{2\pi p^2} \quad . \end{aligned} \tag{6.5.4}$$

6.5.2 $\chi \rightarrow 0$:

As seen from Eqn. (6.2.12) the full vertex, and hence the transverse part, has no pole singularities when $\chi \rightarrow 0$. However, the expressions for $\tau_2, \tau_3, \tau_4, \tau_5, \tau_6$ and τ_7,
Eqns. (6.4.32, 6.4.39), have explicit factors of $1/\chi$ in all but the Feynman gauge. As can be seen from Eqn. (6.3.70) χ only vanishes if **both** p^2 and k^2 tend to m^2, i.e. when both of the fermion legs, Fig. 6.1, are on-mass-shell. Then as $k^2 \rightarrow p^2$, $\chi = q^2(p^2-m^2)^2$. In this limit the full vertex only has logarithmic singularities, like $\ln(1-m^2/p^2)$: these arise when the external legs are on-shell or when the internal fermions can be real. Con-

sequently, an acceptable basis for the transverse vertex is one in which only these logarithmic singularities occur. Explicit calculation shows that τ_2, τ_3, τ_5 and τ_6, given by Eqs. (6.4.33, 6.4.34, 6.4.36, 6.4.37), do have only these logarithmic terms when $\chi \to 0$. However, both τ_4 and τ_7 have poles in $1/(p^2 - m^2)$ term. In this limit the two vectors T_4^μ and T_7^μ point in the same direction :

$$\begin{aligned} T_4^\mu &= (p^\mu(k\cdot q) - k^\mu(p\cdot q))\, k^\lambda p^\nu \sigma_{\lambda\nu} \\ &\underset{k^2\to p^2}{=} (k^2 - k\cdot p)\,(p+k)^\mu\, k^\lambda\, p^\nu\, \sigma_{\lambda\nu} \quad , \\ T_7^\mu &= \frac{1}{2}(p^2-k^2)\,[\gamma^\mu(\not p + \not k) - p^\mu - k^\mu] + (k+p)^\mu\, k^\lambda p^\nu \sigma_{\lambda\nu} \\ &\underset{k^2\to p^2}{=} (p+k)^\mu\, k^\lambda\, p^\nu\, \sigma_{\lambda\nu} \quad . \end{aligned} \tag{6.5.5}$$

Consequently these two coefficients, τ_4, τ_7, can have a singularity, which cancels in Γ_T^μ. These singularities are readily removed by choosing a new basis for the transverse vertex, the S_i^μ $(i = 1, ..., 8)$. Clearly this only involves changes to T_4^μ and T_7^μ. Note that these singularities do not arise in the Feynman gauge ($\xi = 1$), and so Ball and Chiu were not aware of this constraint.

We write

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \sigma^i S_i^\mu \ , \tag{6.5.6}$$

where

$$\begin{aligned} S_1^\mu &= p^\mu(k\cdot q) - k^\mu(p\cdot q) \quad , \\ S_2^\mu &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)]\,(\not k + \not p) \quad , \\ S_3^\mu &= q^2\gamma^\mu - q^\mu \not q \quad , \\ S_4^\mu &= q^2\,[\gamma^\mu(\not p + \not k) - p^\mu - k^\mu] + 2(p-k)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \quad , \\ S_5^\mu &= q_\nu \sigma^{\nu\mu} \quad , \\ S_6^\mu &= \gamma^\mu(p^2 - k^2) + (p+k)^\mu \not q \quad , \\ S_7^\mu &= \frac{1}{2}(p^2-k^2)\,[\gamma^\mu(\not p + \not k) - p^\mu - k^\mu] + (k+p)^\mu\, k^\lambda p^\nu \sigma_{\lambda\nu} \quad , \\ S_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not p - p^\mu \not k \quad . \end{aligned} \tag{6.5.7}$$

Then

$$\sigma_i \ \equiv \ \tau_i \quad \text{for} \quad i = 1,2,3,5,6,8 \ , \tag{6.5.8}$$

and

$$\sigma_4 = \frac{(k^2 - p^2)}{4}\tau_4 \quad , \tag{6.5.9}$$

$$\sigma_7 = \tau_7 + \frac{q^2}{2}\tau_4 \quad . \tag{6.5.10}$$

σ_7 is then given explicitly by :

$$\begin{aligned}
\sigma_7 = \; & \frac{\alpha m}{8\pi\Delta^2}(\xi - 1)\Bigg\{ \left(-q^2 - \frac{3q^4}{4\Delta^2}(m^2 + k\cdot p)\right) J_0 - \frac{q^2}{p^2k^2}\, k\cdot p - 2\frac{\Delta^2}{p^2k^2} \\
& + \frac{1}{\chi}\Bigg[\Delta^2\left\{-2\frac{m^4q^2}{(p^2-k^2)}\left(\frac{L}{p^2} - \frac{L'}{k^2}\right) - 2m^2(p^2-k^2)\left(\frac{L}{p^2} - \frac{L'}{k^2}\right)\right\} \\
& - \frac{m^4q^4}{2(p^2-k^2)}[k^2 + 2(k\cdot p) + p^2]\left(\frac{L}{p^2} - \frac{L'}{k^2}\right) \\
& - \frac{q^4}{2}\left\{p^2\left(\frac{m^4}{p^4} - 1\right)L + k^2\left(\frac{m^4}{k^4} - 1\right)L'\right\} \\
& + \frac{3}{2}\frac{m^2q^4}{\Delta^2}(p^2 - k^2)\left\{(m^2 + p^2)L - (m^2 + k^2)L'\right\} \\
& - \frac{3q^4}{4\Delta^2}(p^2 - k^2)(m^4 - p^2k^2)(L - L') \\
& - \frac{3q^6}{4\Delta^2}(m^4 - p^2k^2)(L + L') + \frac{3m^2q^2}{2}(p^2 - k^2)\left(L - L'\right) \\
& - \frac{3m^2q^6}{2\Delta^2}\left[(p^2 - m^2)L + (k^2 - m^2)L'\right] \\
& + q^2k\cdot p\left[(p^2 - m^2)L + (k^2 - m^2)L'\right] \\
& - q^2\left[p^2(k^2 - m^2)L + k^2(p^2 - m^2)L'\right] \\
& + \Big\{ - 10m^2q^4 - q^4(p^2 + k^2) - 6\frac{m^2q^6}{\Delta^2}(m^2 + k\cdot p) + \frac{3q^6}{\Delta^2}(m^4 - p^2k^2) \\
& - 2q^4(m^2 + k\cdot p) - 4q^2\Delta^2\Big\}S\Bigg]\Bigg\} \; .
\end{aligned} \tag{6.5.11}$$

In this new basis, all the σ_i's $(i = 1, ..., 8)$ have no singularities other than the expected logarithmic ones. Note that in this new basis, the C-parity operation of Eq. (2.2.3) requires

$$\sigma_4(k^2, p^2, q^2) = -\sigma_4(p^2, k^2, q^2) \; , \tag{6.5.12}$$

which Eq. (6.5.7) ensures.

6.5.3 Asymptotic limit:

It is convenient to give here the simple asymptotic limit for the transverse vertex. In the limit that either of the fermion momenta are large, e.g. $k^2 >> k\cdot p >> (p^2, m^2)$,

$$\ln\frac{q^4}{k^2p^2} = \ln\frac{k^4}{p^4} + \frac{2}{3k^6}\left[6k^2p^2k\cdot p - 8(k\cdot p)^3 + 3k^4p^2 - 6k^2(k\cdot p)^2 - 6k^4k\cdot p\right] ,$$

$$\begin{aligned} J_0 &= \frac{2}{k^2}\left[\left(1+\frac{(k\cdot p)}{k^2} - \frac{p^2}{3k^2} + \frac{4(k\cdot p)^2}{3k^4}\right)\ln\frac{k^2}{p^2} \right. \\ &\left. + \left(2+\frac{(k\cdot p)}{k^2} - \frac{2p^2}{9k^2} + \frac{8(k\cdot p)^2}{9k^4}\right)\right] + \mathcal{O}(1/k^5) , \end{aligned} \tag{6.5.13}$$

then the τ_i's come out as :

$$\begin{aligned} \tau_2 &= \frac{\alpha}{12\pi k^4}(2\xi-1)\ln\frac{k^2}{p^2} - \frac{\alpha}{36\pi k^4}(5\xi-1) , \\ \tau_3 &= -\frac{\alpha}{12\pi k^2}(\xi+1)\ln\frac{k^2}{p^2} - \frac{\alpha}{18\pi k^2}(\xi+7) , \\ \tau_6 &= -\frac{\alpha}{24\pi k^2}(\xi-2)\ln\frac{k^2}{p^2} - \frac{\alpha}{18\pi k^2}(\xi-2) , \\ \tau_8 &= -\frac{\alpha}{4\pi k^2}\ln\frac{k^2}{p^2} - \frac{\alpha}{4\pi k^2} . \end{aligned} \tag{6.5.14}$$

Consequently, these τ's lead to the following transverse vertex, which is the well known limit derived in Sect. 6.2.5 Eqn. (6.2.48) :

$$\Gamma_T^\mu = \frac{\alpha\xi}{8\pi}\left[\frac{k^\mu \not k}{k^2} - \gamma^\mu\right]\ln\left(\frac{k^2}{p^2}\right) . \tag{6.5.15}$$

6.5.4 Photon Mass Shell Limit, $q^2 \to 0$ and $k^2 \to p^2$:

In the photon mass-shell limit, $q^2 \to 0$:

$$\begin{aligned} k\cdot p &\to \frac{p^2+k^2}{2} , \\ \Delta^2 &\to \frac{(p^2-k^2)^2}{4} , \\ J_0 &\to \frac{2}{(p^2-k^2)}\left[f\left(\frac{k^2}{p^2}\right) - f\left(\frac{p^2}{k^2}\right) + \frac{1}{2}\ln\frac{k^2}{p^2}\ln\frac{q^4}{k^2p^2}\right] , \end{aligned} \tag{6.5.16}$$

where

$$f\left(\frac{k^2}{p^2}\right) - f\left(\frac{p^2}{k^2}\right) = F$$
$$= 2\,\frac{p^2-k^2}{p^2} + \frac{(p^2-k^2)^2}{p^4} + \frac{13}{18}\frac{(p^2-k^2)^3}{p^6} + \cdots\cdots . \tag{6.5.17}$$

In this limit, the τ's are :

$$\begin{aligned}
\tau_2\underset{q^2\to 0}{} &= \frac{\alpha}{2\pi(p^2-k^2)^2}\Bigg\{ -1 - \frac{1}{2}\frac{p^2+k^2}{p^2-k^2}\ln\frac{k^2}{p^2} \\
&\quad + (\xi-1)\Bigg[\; 1 + \frac{p^2+k^2}{p^2-k^2}F + \frac{5}{2}\frac{p^2+k^2}{p^2-k^2}\ln\frac{k^2}{p^2} \\
&\quad + \left(1+\frac{1}{2}\frac{p^2+k^2}{p^2-k^2}\ln\frac{k^2}{p^2}\right)\ln\frac{q^4}{k^2p^2}\Bigg]\Bigg\} ,
\end{aligned}$$

$$\tau_2\underset{\substack{q^2\to 0\\ k^2\to p^2}}{} = \frac{\alpha}{24\pi p^4}\left[1+(\xi-1)\left(\frac{1}{3}-\ln\frac{q_0^4}{p^4}\right)\right] , \tag{6.5.18}$$

$$\begin{aligned}
\tau_3\underset{q^2\to 0}{} &= \frac{1}{4\pi(p^2-k^2)^2} \\
&\times \Bigg\{ \;(1-\xi)\,\frac{\left(k^2+p^2\right)^2}{p^2-k^2}\,F \\
&\quad + 2\,(2-\xi)\left(k^2+p^2\right) + (2-\xi)\,\frac{\left(k^2-p^2\right)}{2}\left(1-6\frac{(k^2+p^2)^2}{(p^2-k^2)^2}\right)\ln\frac{k^2}{p^2} \\
&\quad + \left[\left(\frac{2k^2p^2}{p^2-k^2} + (1-\xi)\,\frac{(k^2+p^2)^2}{2(p^2-k^2)}\right)\ln\frac{k^2}{p^2} + (2-\xi)\left(p^2+k^2\right)\right]\ln\frac{q^4}{k^2p^2}\Bigg\} ,
\end{aligned}$$

$$\tau_3\underset{\substack{q^2\to 0\\ k^2\to p^2}}{} = \frac{\alpha}{24\pi p^2}\left[-\frac{29}{3} + 2\ln\frac{q_0^4}{p^4} + (\xi-1)\left(-\frac{7}{3}+\ln\frac{q_0^4}{p^4}\right)\right] , \tag{6.5.19}$$

$$\begin{aligned}
\tau_6\underset{q^2\to 0}{} &= \frac{\alpha}{8\pi\Delta^2(p^2-k^2)}(\xi-2)\left[\frac{p^2+k^2}{p^2-k^2}\ln\frac{k^2}{p^2}+2\right] , \\
\tau_6\underset{\substack{q^2\to 0\\ k^2\to p^2}}{} &= 0 , \qquad (6.5.20) \\
\tau_8\underset{q^2\to 0}{} &= \frac{\alpha}{8\pi\Delta^2}(p^2-k^2)\ln\frac{k^2}{p^2} , \\
\tau_8\underset{\substack{q^2\to 0\\ k^2\to p^2}}{} &= -\frac{\alpha}{2\pi p^2} . \qquad (6.5.21)
\end{aligned}$$

We finally came to the end of this chapter and to the end of this thesis having presented the complete one loop calculation of the fermion-boson vertex in QED in an arbitrary covariant gauge. In this perturbative calculation we have computed the coefficient functions, τ_i, of the basis tensors to order α. These functions are the only unknowns in the transverse vertex. From the beginning of this thesis all our effort has been learn about the non-perturbative structure of the vertex. However, as we saw it is not easy to determine this. Perturbation theory is simpler, but even there their form is very complicated. Nevertheless, non-perturbative coefficient functions must agree with this calculation in the weak coupling limit. Therefore, these perturbative τ_i hopefully will guide us to construct a non-perturbative ansatz together with the constraints from multiplicative renormalisability found in Chapters 4, 5. This construction is for future work.

Chapter 7

Conclusions

With everything that we do,
we desire more or less the end;
we are impatient to be done with it
and glad when it is finished.
It is only the end in general,
the end of all ends,
that we wish,
as a rule, to put off as long as possible.
-Schopenhauer-

We began this thesis with the aim of investigating QED from a non-perturbative point of view. The SD-equations are introduced as the field equations of the relevant Quantum Field Theory. Because these equations are an infinite nested set that relates one Green's function to another, they are not solvable unless they are truncated at some level. The solution of the SD-equations have to be multiplicatively renormalisable and gauge covariant since all full n-point functions must have these features. Though we need to truncate these equations in order to find their solution, we know that even the solution for the 2-point functions must know about the higher point functions. So, a reasonable truncation seems to be to approximate the 3-point vertex in a way that incorporates some how all the necessary information from the higher point functions. Then, the SD-equations can be solved for the 2-point functions, i.e. for the fermion and boson wave-function renormalisations and the fermion mass function. To do this, various 3-point vertex ansatz are proposed in the case of quenched QED, each of them were an improvment on the previous one and have their own features and deficiencies. In general, the aim of all ansatz for the 3-point vertex should be to ensure the solutions of the SD-equation for the 2-point functions respect gauge invariance and multiplicative renormalisation.

In this thesis, we extend these studies to massless unquenched QED. The purpose of this study was to understand the structure of the fermion-boson vertex and construct a non-perturbative 3-point function in the case of unquenched QED. We have carried out this investigation along two different directions : (1) the SD-equations are studied in order to deduce the constraints needed to ensure both the fermion and photon propagators are multiplicatively renormalisable, (2) the one loop perturbative calculation of the 3-point vertex is performed in an arbitrary covariant gauge.

In both directions, the vertex function, being a Lorentz vector, involves twelve independent spin and Lorentz vectors. Each of these vectors has a coefficient that is an analytic function of the three Lorentz scalars, k^2, p^2 and q^2, that can be formed from the two independent 4-momenta flowing through the vertex in the case of the coupling of two spin-1/2 particles with a vector boson.

Four of the 12 components define what is called the longitudinal vertex. This is related by the Ward-Takahashi identity to the fermion propagator. This fact allows three of these components to be expressed in terms of the fermion wave-function renormalisation $F(p^2)$, and its mass function $M(p^2)$ and forces a fourth to be zero. Ball and Chiu have shown how to construct this longitudinal vertex in a way free of kinematic singularities. This freedom is essential in ensuring the Ward identity is the $q \to 0$ limit of the Ward-Takahashi identity. Since the vertex can be written in terms of the longitudinal and transverse parts, the rest of these 12 component (8 of them) give the transverse piece. In the massless case, four of them are zero and so we are left with only four tensors multiplied with four coefficient functions, τ_i. The only unknowns in the vertex functions are these coefficient functions. Their symmetry properties are determined by the parity operation and their dimension is fixed by the dimensionless of the transverse vertex. Making use of these conditions the logarithmic expansion of the coefficient functions are substituted in the fermion and boson SD-equations in order to solve for the fermion and photon wave-function renormalisations. After calculating the general multiplicative forms of F and G, these are compared with the ones coming from the solution of the SD-equations. This comparison gives the constraints on the vertex function imposed by multiplicative renormalisability. Attempts to find a simple solution to these constraints will be the basis of future work.

To gain more knowledge about the vertex function, we have also calculated the coefficient functions for one-loop fermion-boson vertex in arbitrary covariant gauges since a perturbative calculation provides a very useful check on the non-perturbative one. Surprisingly this one loop calculation had only been previously performed in the Feynman gauge by Ball and Chiu [26]. Our results correct some typographical errors in their publication in that gauge. The vertex has only logarithmic singularities: these arise either when the external legs are on-shell or when the internal fermions can be real.

Having calculated the complete one-loop 3-point vertex at order α allows us to subtract the longitudinal vertex from our one loop answer to find the transverse vertex to $\mathcal{O}(\alpha)$. This result can be represented in terms of a basis of eight vectors orthogonal to the boson momentum, each unconstrained by the Ward-Takahashi identity.

We propose a new transverse basis S_i^μ $(i = 1...,8)$, Eqns. (6.5.6, 6.5.7), which has components with only the logarithmic singularities of the full vertex. This basis modifies the $T_i^\mu\,(i = 1,....,8)$ of Eqn. (6.2.7) proposed by Ball and Chiu [26]. Though their basis has no additional singularities in the Feynman gauge, this is not the case in any other gauge. Eqns. (6.2.4, 6.4.18–6.4.39, 6.5.6–6.5.10) constitute our new result in QED to one loop. The same and/or related integrals arise in QCD, and so this calculation could, in principle, be extended to non-Abelian theories in any covariant gauge too. However, the length of this calculation, as presented in Chapter 6, is what has doubtlessly deterred previous computations.

Though our perturbative calculations are self-evidently only true to $\mathcal{O}(\alpha)$, our aim has been wider. The hope is that the coefficients of each of the transverse vectors, S_i^μ, like those of the longitudinal component, are free of kinematic singularities at all orders in perturbation theory and even non-perturbatively. Just as use of the Ward-Takahashi identity specifies non-perturbatively the longitudinal vertex in terms of the fermion propagator, Eqn. (2.3.34), multiplicative renormalisability too imposes relationships between the vertex and the fermion propagator. These constrain the transverse vertex. A start has been made in analysing these powerful conditions. Ignoring such requirements and use of, for instance, a bare vertex (*the rainbow approximation*) in studies of chiral symmetry breaking leads to the generation of highly gauge dependent masses. In contrast non-perturbative enforcement of the Ward-Takahashi identity and the constraints of multiplicative renormalisability dramatically reduces or even eliminates [23, 38] this unphysical gauge dependence. Indeed, knowing the vertex in any covariant gauge may give us an understanding of how the essential gauge dependence of the vertex demanded by its Landau-Khalatnikov transformation [30] is satisfied non-perturbatively. Moreover, having a basis for the transverse vertex with coefficients free of non-dynamical singularities is a key step in further investigations of a meaningful non-perturbative truncation of Schwinger-Dyson equations.

Obviously, the natural end of this study will be to bring all the results together to construct simple ansatze for the possible non-perturbative fermion-photon vertex. An extension of the multiplicative renormalisability constraints to massive QED may be on a list of future work.

An unquenched vertex ansatz should be particularly useful in numerical studies of dynamical mass generation in QED and other gauge theories. Previous work [48, 49, 34, 35] has highlighted the sensitivity of the results to the structure of the full vertex. An unquenched vertex ansatz can also be adopted for the quark-gluon vertex in studies of QCD, of quark confinement and of chiral symmetry breaking. This should prove more realistic than the *rainbow approximation* often used.

Appendix A

A γ-matrix algebra in 4-dimensions

$$i) \quad \not{p} = \gamma^\mu p_\mu \quad , \tag{A.1}$$

$$ii) \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad , \tag{A.2}$$

$$iii) \quad \gamma^\mu \gamma_\mu = 4 \quad , \tag{A.3}$$

$$iv) \quad \gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha \quad , \tag{A.4}$$

$$v) \quad \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4g^{\alpha\beta} \quad , \tag{A.5}$$

$$vi) \quad \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\mu = -2\gamma^\delta \gamma^\beta \gamma^\alpha \quad ,$$

γ-matrix algebra in d-dimensions : ,

$$vii) \quad \gamma^\mu \gamma_\mu = d \quad , \tag{A.6}$$

$$viii) \quad \gamma_\mu \gamma^\alpha \gamma^\mu = -(d-2)\gamma^\mu \quad , \tag{A.7}$$

$$ix) \quad \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = (d-4)\gamma^\alpha \gamma^\beta + 4g^{\mu\nu} \quad , \tag{A.8}$$

$$x) \quad \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\lambda = g^{\alpha\mu}\gamma^\beta(-8+2d) \quad , \\ +g^{\alpha\beta}\gamma^\mu(8-2d) - (6-d)\gamma^\mu \gamma^\beta \gamma^\alpha \quad , \tag{A.9}$$

$$xi) \quad \gamma_\beta \gamma_\nu \gamma_\mu \gamma^\nu \gamma^\beta = (2-d)\gamma_\beta \gamma_\mu \gamma^\beta - (2-d)^2 \gamma_\mu \quad . \tag{A.10}$$

Appendix B

B Angular Integration for the Fermion SD-Equation

This appendix is related to the fermion SD-equation of Chapter 2,3 [50] :

Recalling the definition of the angular integration, Eqn. (3.2.13);

$$I_{n,m} = \int_0^\pi d\psi \, \sin^2\psi \, \frac{(k\cdot p)^n}{(q^2)^m} \, , \tag{B.1}$$

where

$$k\cdot p = |k|\,|p|\,\cos\psi \qquad \text{and} \qquad q = k - p \, . \tag{B.2}$$

It can be in the form of

$$I_{n,m} = |k|^n\,|p|^n \int_0^\pi d\psi \, \sin^2\psi \, \frac{\cos^n\psi}{(a - b\cos\psi)^m} \, , \tag{B.3}$$

with

$$a = k^2 + p^2 \quad , \quad b = 2\,|k|\,|p| \, . \tag{B.4}$$

Now, we shall calculate $I_{n,m}$ for different n and m values starting from $n = 0, m = 1$:

$$I_{0,1} = \int_0^\pi d\psi \, \frac{\sin^2\psi}{(a - b\cos\psi)} \, . \tag{B.5}$$

Changing the ψ variable as

$$z = \cos\psi \, , \tag{B.6}$$

integral becomes

$$I_{0,1} = \int_{-1}^{1} dz \, \frac{\sqrt{1 - z^2}}{(a - b\,z)} \, , \tag{B.7}$$

by making further changes of variable

$$y = a - b\,z \, , \tag{B.8}$$

we get

$$I_{0,1} \;=\; \frac{1}{b^2}\int_{a-b}^{a+b}\frac{dy}{y}\,\frac{R}{\sqrt{R}}\,, \tag{B.9}$$

where

$$R \equiv b^2-(a-y)^2 = b^2-a^2+2\,a\,y-y^2\,.$$

We divide this integral into three pieces

$$I_{0,1} = \frac{(b^2-a^2)}{b^2}\int_{a-b}^{a+b}\frac{dy}{y\sqrt{R}} + \frac{a}{b}\int_{a-b}^{a+b}\frac{dy}{\sqrt{R}} + \frac{1}{b^2}\int_{a-b}^{a+b}dy\,\frac{(a-y)}{\sqrt{R}}\,, \tag{B.10}$$

start to deal with the first one :

$$I^1_{0,1} \;=\; \int_{a-b}^{a+b}\frac{dy}{y\sqrt{R}}\,. \tag{B.11}$$

By making following changes of variables three times one after the other :

$$\begin{aligned}
z &= \frac{1}{y} &&: I^1_{0,1} = \frac{1}{\sqrt{a^2-b^2}}\int_{\frac{1}{a+b}}^{\frac{1}{a-b}} dz\left[\frac{b^2}{(a^2-b^2)^2}-\left(\frac{a}{a^2-b^2}-z\right)^2\right]^{-1/2}, \\
w &= z-\frac{a}{(a^2-b^2)} &&: I^1_{0,1} = \frac{1}{\sqrt{a^2-b^2}}\int_{\frac{-b}{(a^2-b^2)}}^{\frac{b}{(a^2-b^2)}} dw\left[\frac{b^2}{(a^2-b^2)}-w^2\right]^{-1/2}, \\
w &= \frac{b\,\sin\theta}{(a^2-b^2)} &&: I^1_{0,1} = \frac{1}{\sqrt{a^2-b^2}}\int_{-\pi/2}^{\pi/2} d\theta\,.
\end{aligned} \tag{B.12}$$

Finally, the result is :

$$I^1_{0,1} \;=\; \frac{\pi}{\sqrt{a^2-b^2}}\,. \tag{B.13}$$

We now evaluate the second integral in Eqn. (B.10) :

$$I^2_{0,1} \;=\; \int_{a-b}^{a+b}\frac{dy}{\sqrt{R}}\,. \tag{B.14}$$

Again changing the integration variables twice :

$$w = y - a \,, \qquad I^2_{0,1} = \int_{-b}^{b} \frac{dw}{\sqrt{b^2 - w^2}} \,, \tag{B.15}$$

and

$$w = b \sin\theta \,, \qquad I^2_{0,1} = \int_{-\pi/2}^{\pi/2} d\theta \,. \tag{B.16}$$

It results in

$$I^2_{0,1} = \pi \,. \tag{B.17}$$

The third and last integral to evaluate is :

$$I^3_{0,1} = \int_{a-b}^{a+b} dy \, \frac{(a-y)}{\sqrt{R}} \,. \tag{B.18}$$

Making changes of y-variable

$$w = y - a \,, \qquad I^3_{0,1} = \int_{-b}^{b} \frac{w\, dw}{\sqrt{b^2 - w^2}} \,, \tag{B.19}$$

and furthermore

$$w = b \sin\theta \,, \qquad I^3_{0,1} = b \int_{\pi/2}^{-\pi/2} d\theta \, \sin\theta \,. \tag{B.20}$$

This integral gives

$$I^3_{0,1} = 0 \,. \tag{B.21}$$

Therefore, adding these three integrals together leads to :

$$I_{0,1} = \frac{\pi}{b^2} \left(a - \sqrt{a^2 - b^2} \right) \,, \tag{B.22}$$

where

$$\sqrt{a^2 - b^2} = \left[(p^2 + k^2)^2 - 4\, k^2\, p^2 \right]^{1/2} = [(p^2 - k^2)^2]^{1/2} = |p^2 - k^2| \,. \tag{B.23}$$

We now introduce the following quantity for convenience to use from now on,

$$h(x) = \frac{1}{2} (1 + x - |1 - x|) = \begin{cases} x & x < 1 \\ 1 & x \geq 1 \end{cases} \,.$$

Inserting the a and b quantities from Eqn. (B.4), $I_{0,1}$ can be written as

$$
\begin{aligned}
I_{0,1} &= \frac{\pi}{4\,k^2\,p^2}\left((p^2+k^2) - |p^2-k^2|\right) \\
&= \frac{\pi}{4\,k^2}\left(1+\frac{k^2}{p^2} - \left|1-\frac{k^2}{p^2}\right|\right) \; .
\end{aligned}
\tag{B.24}
$$

Eventually, we find

$$
I_{0,1} = \frac{\pi}{2\,k^2}\,h\left(\frac{k^2}{p^2}\right) \; . \tag{B.25}
$$

The angular integration for $n = m = 0$:

$$
I_{0,0} = \int_0^\pi d\psi\,\sin^2\psi = \frac{\pi}{2} \; , \tag{B.26}
$$

$n = 2, m = 0$:

$$
\begin{aligned}
I_{2,0} &= \int_0^\pi d\psi\,\sin^2\psi\,(k\cdot p)^2 \\
&= |k^2|^2\,|p^2|^2 \int d\psi\,\sin^2\psi\,\cos^2\psi \\
&= \frac{\pi}{8}\,k^2\,p^2 \; ,
\end{aligned}
\tag{B.27}
$$

$n = 4, m = 0$:

$$
\begin{aligned}
I_{4,0} &= \int_0^\pi d\psi\,\sin^4\psi\,(k\cdot p)^2 \\
&= \frac{\pi}{16}\,k^4\,p^4 \; ,
\end{aligned}
\tag{B.28}
$$

$n = 1, m = 0$:

$$
\begin{aligned}
I_{1,0} &= \int_0^\pi d\psi\,k\cdot p \\
&= |k^2|\,|p^2| \int d\psi\,\sin^2\psi\,\cos\psi \\
&= 0 \; ,
\end{aligned}
\tag{B.29}
$$

$n = n, m = 0$:

$$
I_{n,0} = 0 \qquad , \qquad n : \text{odd number} \; . \tag{B.30}
$$

Now we shall write Eqn. (B.3) for $m = 1$ as :

$$\begin{aligned} I_{n,1} &= |k|^n\,|p|^n \int_0^\pi d\psi\, \sin^2\psi \frac{\cos^n\psi}{(a - b\,\cos\psi)} \\ &= -\frac{1}{2}\, I_{n-1,0} + \frac{a}{2}\, I_{n-1,1}\,, \end{aligned} \tag{B.31}$$

we then can calculate this expression for different values of n starting from $n = 1$:

$$\begin{aligned} I_{1,1} &= -\frac{1}{2}\, I_{0,0} + \frac{a}{2}\, I_{0,1} \\ &= -\frac{\pi}{4} + \frac{a\,\pi}{2\,b^2}\,(a - \sqrt{a^2 - b^2}) \quad , \end{aligned}$$

recalling a and b from Eqn. (B.4)

$$\begin{aligned} I_{1,1} &= -\frac{\pi}{4} + \frac{(p^2 + k^2)\,\pi}{8\,k^2\,p^2}\left[(p^2 + k^2) - |p^2 - k^2|\right] \\ &= \frac{\pi}{8\,k^2\,p^2}\left[p^4 + k^4 - |p^4 - k^4|\right] \\ &= \frac{\pi\,p^2}{8\,k^2}\left[1 + \frac{k^4}{p^4} - \left|1 - \frac{k^4}{p^4}\right|\right] \\ I_{1,1} &= \frac{\pi p^2}{4\,k^2}\, h\left(\frac{k^4}{p^4}\right)\,. \end{aligned} \tag{B.32}$$

$n = 2$:

$$\begin{aligned} I_{2,1} &= -\frac{1}{2}\, I_{1,0} + \frac{a}{2}\, I_{1,1} \\ &= \frac{\pi\,a^2}{4\,b^2}\,(a - \sqrt{a^2 - b^2}) - \frac{\pi\,a}{8} \\ &= \frac{\pi\,p^2}{8\,k^2}\,(p^2 + k^2)\, h\left(\frac{k^4}{p^4}\right)\,. \end{aligned} \tag{B.33}$$

$n = 3$:

$$\begin{aligned} I_{3,1} &= -\frac{1}{2}\, I_{2,0} + \frac{a}{2}\, I_{2,1} \\ &= \frac{\pi\,a^3}{8\,b^2}\,(a - \sqrt{a^2 - b^2}) - \frac{\pi\,a^2}{16} - \frac{\pi\,b^2}{64} \\ &= \frac{\pi\,p^6}{16\,k^2}\, h\left(\frac{k^8}{p^8}\right) + \frac{\pi\,p^4}{8}\, h\left(\frac{k^4}{p^4}\right)\,. \end{aligned} \tag{B.34}$$

$n = 4$:

$$
\begin{aligned}
I_{4,1} &= -\frac{1}{2}\, I_{3,0} + \frac{a}{2}\, I_{3,1} \\
&= \frac{\pi\, a^4}{16\, b^2}\,(a - \sqrt{a^2 - b^2}) - \frac{\pi\, a^3}{32} - \frac{\pi\, a\, b^2}{128} \\
&= \frac{\pi\, p^6\,(p^2 + k^2)}{32\, k^2}\, h\left(\frac{k^8}{p^8}\right) + \frac{\pi\, p^4\,(p^2 + k^2)}{16}\, h\left(\frac{k^4}{p^4}\right) \, . \qquad \text{(B.35)}
\end{aligned}
$$

$n = 5$:

$$
\begin{aligned}
I_{5,1} &= -\frac{1}{2}\, I_{4,0} + \frac{a}{2}\, I_{4,1} \\
&= \frac{\pi\, a^5}{32\, b^2}\,(a - \sqrt{a^2 - b^2}) - \frac{\pi\, a^4}{64} - \frac{\pi\, a^2\, b^2}{256} - \frac{\pi\, b^4}{512} \\
&= \frac{\pi\, p^{10}}{64\, k^2}\, h\left(\frac{k^{12}}{p^{12}}\right) \\
&+ \frac{\pi\, p^8}{16}\, h\left(\frac{k^8}{p^8}\right) + \frac{5\,\pi}{64}\, k^2\, p^6\, h\left(\frac{k^4}{p^4}\right) \, . \qquad \text{(B.36)}
\end{aligned}
$$

Now we shall calculate the angular integrals for the $n = n, m = 2$. In order to do this, we take the derivative of Eqn. (B.31) with respect to a;

$$
\begin{aligned}
\frac{\partial}{\partial\, a}\, I_{n,1} &= \frac{\partial}{\partial\, a}\left[\frac{b^n}{2^n}\int_0^\pi d\psi\, \sin^2\theta\, \frac{\cos^n\psi}{(a - b\,\cos\psi)}\right] , \\
&= -\frac{b^n}{2^n}\int_0^\pi d\psi\, \sin^2\psi\, \frac{\cos^n\psi}{(a - b\,\cos\psi)} , \\
\frac{\partial}{\partial\, a}\, I_{n,1} &= -I_{n,2} \, . \qquad \text{(B.37)}
\end{aligned}
$$

Let us use this equality to evaluate the angular integration for the different values of n.

$n = 0$:

$$
\begin{aligned}
I_{0,2} &= -\frac{\partial}{\partial\, a}\, I_{0,1} \\
&+ \frac{\pi}{b^2}\left(\frac{a}{\sqrt{a^2 - b^2}} - 1\right) \\
&+ \frac{\pi}{2\, k^2}\,\frac{1}{|p^2 - k^2|}\, h\left(\frac{k^2}{p^2}\right) \, . \qquad \text{(B.38)}
\end{aligned}
$$

$n = 1:$

$$\begin{aligned} I_{1,2} &= -\frac{\partial}{\partial a} I_{1,1} \\ &+ \frac{\pi}{2\,b^2}\left(\frac{a^2}{\sqrt{a^2-b^2}} - 2\,a + \sqrt{a^2-b^2}\right) \\ &= \frac{\pi\,p^2}{2\,k^2}\,\frac{1}{|p^2-k^2|}\,h\left(\frac{k^4}{p^4}\right)\;. \end{aligned} \tag{B.39}$$

$n = 2:$

$$\begin{aligned} I_{2,2} &= -\frac{\partial}{\partial a} I_{2,1} \\ &= \frac{\pi}{4\,b^2}\left(\frac{a^3}{\sqrt{a^2-b^2}} - 3\,a^2 + 2\,a\sqrt{a^2-b^2} + \frac{b^2}{2}\right) \\ &= \frac{3\,\pi\,p^4}{8\,k^2\,|p^2-k^2|}\,h\left(\frac{k^6}{p^6}\right) + \frac{\pi\,p^2}{8\,|p^2-k^2|}\,h\left(\frac{k^2}{p^2}\right)\;. \end{aligned} \tag{B.40}$$

$n = 3:$

$$\begin{aligned} I_{3,2} &= -\frac{\partial}{\partial a} I_{3,1} \\ &= \frac{\pi}{8\,b^2}\left(-4\,a^3 + 3\,a^2\sqrt{a^2-b^2} + \frac{a^4}{\sqrt{a^2-b^2}} + a\,b^2\right) \\ &= \frac{\pi\,p^6}{4\,k^2\,|p^2-k^2|}\,h\left(\frac{k^8}{p^8}\right) + \frac{\pi\,p^4}{4\,|p^2-k^2|}\,h\left(\frac{k^4}{p^4}\right)\;. \end{aligned} \tag{B.41}$$

$n = 4:$

$$\begin{aligned} I_{4,2} &= -\frac{\partial}{\partial a} I_{4,1} \\ &= \frac{\pi}{16\,b^2}\left(\frac{a^5}{\sqrt{a^2-b^2}}\right) - 5\,a^4 + 4\,a^3\sqrt{a^2-b^2} + \frac{3\,a^2\,b^2}{2} + \frac{b^4}{8} \\ &= \frac{5\,\pi}{32}\,\frac{p^8}{k^2\,|p^2-k^2|}\,h\left(\frac{k^{10}}{p^{10}}\right) + \frac{9\,\pi}{32}\,\frac{p^6}{|p^2-k^2|}\,h\left(\frac{k^6}{p^6}\right) + \frac{\pi}{16}\,\frac{k^2\,p^4}{|p^2-k^2|}\,h\left(\frac{k^2}{p^2}\right)\;. \end{aligned} \tag{B.42}$$

$n = 5$:

$$
\begin{aligned}
I_{5,2} &= -\frac{\partial}{\partial a} I_{5,1} \\
&= \frac{\pi}{32\,b^2}\left[-5\,a^5 + 5\,a^4\sqrt{a^2-b^2} - a^5 + \frac{a^6}{\sqrt{a^2-b^2}} + 2\,a^3\,b^2 + \frac{a\,b^4}{4}\right] \\
&= \frac{3\,\pi\,p^{10}}{32\,k^2\,|p^2-k^2|}\,h\left(\frac{k^{12}}{p^{12}}\right) + \frac{\pi\,p^8}{4\,|p^2-k^2|}\,h\left(\frac{k^8}{p^8}\right) + \frac{5\,\pi\,k^2\,p^6}{32\,|p^2-k^2|}\,h\left(\frac{k^4}{p^4}\right)\ .
\end{aligned}
\tag{B.43}
$$

Appendix C

C Angular Integrations for the Photon SD-Equation

This appendix is related to the fermion SD-equation of Chapter 4 [50, 43].

Recalling the definition of the angular integral for the photon SD-equation, Eqn. (3.3.22),

$$K_{n,1} = \int_0^\pi d\psi \, \sin^2\psi \frac{(\ell\cdot p)^n}{[(\ell^2+p^2/4)^2-(\ell\cdot p)^2]} \, . \tag{C.1}$$

Splitting up the denominator of this integral as

$$\begin{aligned} \frac{1}{[(\ell^2+p^2/4)^2-)\ell\cdot p)^2]} &= \frac{1}{a^2-b^2\cos^2\psi} \\ &= \frac{1}{2a}\left(\frac{1}{a-b\cos\psi}+\frac{1}{a+b\cos\psi}\right) , \end{aligned} \tag{C.2}$$

where

$$a = \ell^2 + \frac{p^2}{4} , \qquad b = |\ell|\,|p| , \tag{C.3}$$

$K_{n,1}$ then can be rewritten

$$K_{n,1} = \frac{1}{2a} I'_{n,1} + \frac{1}{2a} J_{n,1} , \tag{C.4}$$

where

$$\begin{aligned} I'_{n,1} &= \int_0^\pi d\psi \, \sin^2\psi \frac{1}{2a}\left(\frac{(\ell\cdot p)^n}{a-b\cos\psi}\right) , \\ J_{n,1} &= \int_0^\pi d\psi \, \sin^2\psi \frac{1}{2a}\left(\frac{(\ell\cdot p)^n}{a+b\cos\psi}\right) . \end{aligned} \tag{C.5}$$

Making use of Eqn. (B.25) for J and I' integrals, we find that

$$J_{0,1} = I'_{0,1} = \frac{\pi}{2\,\ell^2} h\left(\frac{4\ell^2}{p^2}\right) . \tag{C.6}$$

Therefore $K_{0,1}$ becomes

$$K_{0,1} = \frac{I'_{0,1}}{a} = \frac{1}{(\ell^2+p^2/4)} \frac{\pi}{2\,\ell^2} h\left(\frac{4\ell^2}{p^2}\right) . \tag{C.7}$$

Analogously to the angular integral calculation in fermion case, we can derive the following relations :

$$
\begin{aligned}
J_{n,1} &= J_{n-1,0} - a\, J_{n-1,1} \quad , \\
I'_{n,1} &= -I'_{n-1,0} + a\, I'_{n-1,1} \quad ,
\end{aligned}
\tag{C.8}
$$

for $n = 1$ these give

$$
J_{1,1} = J_{0,0} - a\, J_{0,1} = -\, I'_{1,1} \, ,
$$

then $K_{1,1}$ gives

$$
K_{1,1} = \frac{1}{2\,a} \left(I'_{1,1} + J_{1,1} \right) = 0 \, . \tag{C.9}
$$

For $n = 2$:

$$
J_{2,1} = J_{1,0} - a\, J_{1,1} = -a\, J_{1,1} = a\, I'_{1,1} \, , \tag{C.10}
$$

thus at the end $K_{2,1}$ can be written in terms of $I_{1,1}$, Eqn. (B.32)

$$
\begin{aligned}
K_{2,1} &= I'_{1,1} = 2\, I_{1,1} \quad , \\
&= \frac{\pi\, p^2}{8\, \ell^2}\, h \left(\frac{16\, \ell^4}{p^4} \right) \, .
\end{aligned}
\tag{C.11}
$$

$n = 3$:

$$
\begin{aligned}
J_{3,1} &= J_{2,0} - a\, J_{2,1} = -\, I'_{3,1} \quad , \\
K_{3,1} &= 0 \quad .
\end{aligned}
\tag{C.12}
$$

$n = 4$:

$$
J_{4,1} = J_{3,0} - a\, J_{3,1} = -a\, J_{3,1} = a\, I'_{3,1} \quad .
$$

$K_{4,1}$ can be expressed in terms of $I_{3,1}$, Eqn. (B.34),

$$
\begin{aligned}
K_{4,1} &= I'_{3,1} = 2\, I_{3,1} \quad , \\
&= \frac{\pi\, p^6}{2^7\, \ell^2}\, h \left(\frac{2^8\, \ell^8}{p^8} \right) + \frac{\pi}{16}\, p^4\, h \left(\frac{16\, \ell^4}{p^4} \right) \, ,
\end{aligned}
\tag{C.13}
$$

and $n = 5$:

$$
K_{5,1} = 0 \, . \tag{C.14}
$$

Appendix D

D Integrals Used for the Fermion Equation [50]

$$\begin{aligned} \ell_1^P &= \int_0^{p^2} dk^2 \, \frac{k^2}{p^4} \, \frac{(3p^2 - k^2)}{(k^2 + p^2)} \\ &= \frac{7}{2} - 4 \, \ln 2 \quad , \end{aligned} \tag{D.1}$$

$$\begin{aligned} \ell_2^P &= \int_0^{p^2} dk^2 \, \frac{k^2}{p^4} \, \frac{(3p^2 - k^2)}{(k^2 + p^2)} \ln \frac{k^2}{\Lambda^2} \\ &= \left(\frac{7}{2} - 4 \, \ln 2 \right) \, \ln \frac{p^2}{\Lambda^2} + 2 \, \zeta(2) - \frac{15}{4} \quad , \end{aligned} \tag{D.2}$$

$$\begin{aligned} \ell_3^P &= \int_0^{p^2} dk^2 \, \frac{k^4}{p^4} \, \frac{(2p^2 - k^2)}{(k^2 + p^2)^2} \\ &= -7 \, \ln 2 + 5 \quad , \end{aligned} \tag{D.3}$$

$$\begin{aligned} \ell_4^P &= \int_0^{p^2} dk^2 \, \frac{k^4}{p^4} \, \frac{(2p^2 - k^2)}{(k^2 + p^2)^2} \ln \frac{k^2}{\Lambda^2} \\ &= (-7 \, \ln 2 + 5) \, \ln \frac{p^2}{\Lambda^2} - 3 \, \ln 2 + \frac{7}{2} \, \zeta(2) - \frac{15}{4} \quad , \end{aligned} \tag{D.4}$$

$$\begin{aligned} \ell_5^P &= \int_0^{p^2} dk^2 \, \frac{k^2}{p^4} \, \frac{(k^2 + p^2)}{(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} \right) \\ &= -\frac{9}{4} + 2 \, \zeta(2) \quad , \end{aligned} \tag{D.5}$$

$$\begin{aligned} \ell_6^P &= \int_0^{p^2} dk^2 \, \frac{k^2}{p^4} \, \frac{(k^2 + p^2)}{(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} \, \ln \frac{p^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2} \right) \\ &= \left(-\frac{9}{4} + 2 \, \zeta(2) \right) \, \ln \frac{p^2}{\Lambda^2} \quad , \end{aligned} \tag{D.6}$$

$$\begin{aligned} \ell_7^P &= \int_0^{p^2} dk^2 \, \frac{k^4}{p^4} \, \frac{1}{(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} \right) \\ &= -\frac{5}{4} + \zeta(2) \quad , \end{aligned} \tag{D.7}$$

$$\begin{aligned}\ell_8^P &= \int_0^{p^2} dk^2\, \frac{k^4}{p^4}\, \frac{(5k^2+2p^2)}{(k^2+p^2)\,(k^2-p^2)} \left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right) \\ &= -\frac{13}{4} + \frac{11}{4}\,\zeta(2) \quad ,\end{aligned} \tag{D.8}$$

$$\begin{aligned}\ell_9^P &= \int_0^{p^2} dk^2\, \frac{k^6}{p^4}\, \frac{1}{(k^2+p^2)\,(k^2-p^2)} \left(\ln\frac{k^2}{\Lambda^2} - \ln\frac{p^2}{\Lambda^2}\right) \\ &= -\frac{1}{4} + \frac{1}{4}\,\zeta(2) \quad .\end{aligned} \tag{D.9}$$

$$\begin{aligned}\ell_1^K &= \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} \\ &= -\ln\frac{p^2}{\Lambda^2} \quad ,\end{aligned} \tag{D.10}$$

$$\begin{aligned}\ell_2^K &= \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2}\, \ln\frac{k^2}{\Lambda^2} \\ &= -\frac{1}{2}\, \ln\frac{p^2}{\Lambda^2} \quad ,\end{aligned} \tag{D.11}$$

$$\begin{aligned}\ell_3^K &= \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2}\, \ln^2\frac{k^2}{\Lambda^2} \\ &= -\frac{1}{3}\, \ln^2\frac{p^2}{\Lambda^2} \quad ,\end{aligned} \tag{D.12}$$

$$\begin{aligned}\ell_4^K &= \int_0^{p^2} dk^2\, \frac{1}{k^2}\, \frac{(3k^2-p^2)}{(k^2+p^2)} \\ &= -3\,\ln\frac{p^2}{\Lambda^2} + 4\,\ln 2 \quad ,\end{aligned} \tag{D.13}$$

$$\begin{aligned}\ell_5^K &= \int_0^{p^2} dk^2\, \frac{1}{k^2}\, \frac{(3k^2-p^2)}{(k^2+p^2)}\, \ln\frac{k^2}{\Lambda^2} \\ &= -\frac{3}{2}\, \ln^2\frac{p^2}{\Lambda^2} - 2\,\zeta(2) \quad ,\end{aligned} \tag{D.14}$$

$$\begin{aligned}\ell_6^K &= \int_0^{p^2} dk^2\, \frac{p^2}{k^2}\, \frac{(2k^2-p^2)}{(k^2+p^2)^2} \\ &= \frac{3}{2} + \ln 2 \quad ,\end{aligned} \tag{D.15}$$

$$\begin{aligned} \ell_7^K &= \int_0^{p^2} dk^2\, \frac{p^2}{k^2}\, \frac{(2k^2 - p^2)}{(k^2 + p^2)^2} \ln \frac{k^2}{\Lambda^2} \\ &= \left(\frac{3}{2} + \ln 2\right) \ln \frac{p^2}{\Lambda^2} - \frac{1}{2}\zeta(2) + 3\,ln2 \quad , \end{aligned} \tag{D.16}$$

$$\begin{aligned} \ell_8^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{1}{k^2}\, \frac{(k^2 + p^2)}{(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2}\right) \\ &= \frac{1}{2} \ln^2 \frac{p^2}{\Lambda^2} + 2\,\zeta(2) \quad , \end{aligned} \tag{D.17}$$

$$\begin{aligned} \ell_9^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{1}{k^2}\, \frac{(k^2 + p^2)}{(k^2 - p^2)} \left(\ln^2 \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} \ln \frac{k^2}{\Lambda^2}\right) \\ &= \frac{1}{6} \ln^3 \frac{p^2}{\Lambda^2} + 2\,\zeta(2) \ln \frac{p^2}{\Lambda^2} \quad , \end{aligned} \tag{D.18}$$

$$\begin{aligned} \ell_{10}^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{1}{k^2}\, \frac{(k^2 + p^2)}{(k^2 - p^2)} \left(\ln^2 \frac{k^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2}\right) \\ &= \frac{2}{3} \ln^3 \frac{p^2}{\Lambda^2} + 2\,\zeta(2) \quad , \end{aligned} \tag{D.19}$$

$$\begin{aligned} \ell_{11}^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{p^2}{k^2}\, \frac{1}{(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2}\right) \\ &= \zeta(2) \quad , \end{aligned} \tag{D.20}$$

$$\begin{aligned} \ell_{12}^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{p^2}{k^2}\, \frac{(5p^2 + 2k^2)}{(k^2 + p^2)\,(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2}\right) \\ &= \frac{11}{4}\zeta(2) \quad , \end{aligned} \tag{D.21}$$

$$\begin{aligned} \ell_{13}^K &= \int_{p^2}^{\Lambda^2} dk^2\, \frac{p^4}{k^2}\, \frac{1}{(k^2 + p^2)\,(k^2 - p^2)} \left(\ln \frac{k^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2}\right) \\ &= \frac{1}{4} - \frac{\zeta(2)}{4} \quad . \end{aligned} \tag{D.22}$$

$$\begin{aligned} t_1^P &= \int_0^{p^2} dk^2 \frac{k^2}{p^4}\, \frac{(k^2 - 3p^2)}{(k^2 + p^2)} \left(\ln \frac{p^2}{\Lambda^2} + \ln \frac{k^2}{\Lambda^2}\right) \\ &= (-7 + 8\,\ln 2) \ln \frac{p^2}{\Lambda^2} + c \quad , \end{aligned} \tag{D.23}$$

$$t_2^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-3p^2)}{(k^2+p^2)} \left(\ln^2\frac{p^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2} \right)$$
$$= (-7+8\ \ln 2)\ \ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.24}$$

$$t_3^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-3p^2)}{(k^2+p^2)} \left(\ln\frac{p^2}{\Lambda^2}\ \ln\frac{k^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2} \right)$$
$$= (-7+8\ \ln 2)\ \ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.25}$$

$$t_4^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-3p^2)}{(k^2+p^2)} \ln\frac{k^2}{\Lambda^2}$$
$$= \frac{1}{2}(-7+8\ \ln 2)\ \ln\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.26}$$

$$t_5^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-p^2)^2}{(k^2+p^2)^2} \left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2} \right)$$
$$= (-11+16, \ln 2)\ \ln\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.27}$$

$$t_6^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-p^2)^2}{(k^2+p^2)^2} \left(\ln^2\frac{p^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2} \right)$$
$$= (-11+16\ \ln 2)\ \ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.28}$$

$$t_7^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-p^2)^2}{(k^2+p^2)^2} \left(\ln\frac{p^2}{\Lambda^2}\ \ln\frac{k^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2} \right)$$
$$= (-11+16\ \ln 2)\ \ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.29}$$

$$t_8^P = \int_0^{p^2} dk^2 \frac{k^2}{p^4} \frac{(k^2-p^2)^2}{(k^2+p^2)^2} \ln\frac{k^2}{\Lambda^2}$$
$$= (-\frac{11}{2}+8\ \ln 2)\ \ln\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \tag{D.30}$$

$$t_1^K = \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-3k^2)}{(k^2-p^2)} \left(\ln\frac{p^2}{\Lambda^2} - \ln\frac{k^2}{\Lambda^2} \right)$$
$$= \frac{3}{2}\ \ln^2\frac{p^2}{\Lambda^2} + 2\zeta(2) \quad , \tag{D.31}$$

$$t_2^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{(p^2 - 3k^2)}{(k^2 - p^2)} \left(\ln^2 \frac{k^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2} \right)$$
$$= -2 \ln^3 \frac{p^2}{\Lambda^2} - 4\zeta(3) - 4\,\zeta(2) \quad , \tag{D.32}$$

$$t_3^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{(p^2 - 3k^2)}{(k^2 - p^2)} \left(\ln \frac{p^2}{\Lambda^2} \ln \frac{k^2}{\Lambda^2} - \ln^2 \frac{k^2}{\Lambda^2} \right)$$
$$= \frac{1}{2} \ln^3 \frac{p^2}{\Lambda^2} + 2\,\zeta(2) \ln \frac{p^2}{\Lambda^2} + 4\zeta(3) \quad , \tag{D.33}$$

$$t_4^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{(p^2 - k^2)}{(k^2 + p^2)} \left(\ln \frac{p^2}{\Lambda^2} - \ln \frac{k^2}{\Lambda^2} \right)$$
$$= -\frac{1}{2} \ln^2 \frac{p^2}{\Lambda^2} - 2\zeta(2) \quad , \tag{D.34}$$

$$t_5^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{(p^2 - k^2)}{(k^2 + p^2)} \left(\ln^2 \frac{k^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2} \right)$$
$$= \frac{2}{3} \ln^3 \frac{p^2}{\Lambda^2} + 4\zeta(3) + 4\,\zeta(2) \quad , \tag{D.35}$$

$$t_6^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{(p^2 - k^2)}{(k^2 + p^2)} \left(\ln \frac{p^2}{\Lambda^2} \ln \frac{k^2}{\Lambda^2} - \ln^2 \frac{k^2}{\Lambda^2} \right)$$
$$= -\frac{1}{6} \ln^3 \frac{p^2}{\Lambda^2} - 2\,\zeta(2) \ln \frac{p^2}{\Lambda^2} + 4\zeta(3) \quad , \tag{D.36}$$

$$t_7^K = \int_{p^2}^{\Lambda^2} dk^2 \, \frac{1}{k^2} \, \frac{k^2 p^2}{(p^2 - k^2)(k^2 + p^2)} \left(\ln \frac{p^2}{\Lambda^2} - \ln \frac{k^2}{\Lambda^2} \right)$$
$$= -\frac{1}{2} \ln^2 \frac{p^2}{\Lambda^2} - \frac{1}{4} \zeta(2) \quad , \tag{D.37}$$

$$t_8^K = \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \, \frac{(p^2 - 3k^2)}{(k^2 + p^2)} \left(\ln \frac{p^2}{\Lambda^2} + \ln \frac{k^2}{\Lambda^2} \right)$$
$$= \frac{9}{2} \ln^2 \frac{p^2}{\Lambda^2} + 8 \ln 2 \, \ln \frac{p^2}{\Lambda^2} + c \quad , \tag{D.38}$$

$$t_9^K = \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \, \frac{(p^2 - 3k^2)}{(k^2 + p^2)}$$
$$= 3 \ln \frac{p^2}{\Lambda^2} + 4 \ln 2 \quad , \tag{D.39}$$

$$\begin{aligned} t^K_{10} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-3k^2)}{(k^2+p^2)} \left(\ln^2\frac{p^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2}\right) \\ &= 4\ln^3\frac{p^2}{\Lambda^2} + 8\ln 2\ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \end{aligned} \tag{D.40}$$

$$\begin{aligned} t^K_{11} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-3k^2)}{(k^2+p^2)} \left(\ln\frac{p^2}{\Lambda^2}\ln\frac{k^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2}\right) \\ &= \frac{5}{2}\ln^3\frac{p^2}{\Lambda^2} + 8\ln 2\ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \end{aligned} \tag{D.41}$$

$$\begin{aligned} t^K_{12} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-3k^2)}{(k^2+p^2)} \ln\frac{k^2}{\Lambda^2} \\ &= \frac{3}{2}\ln^2\frac{p^2}{\Lambda^2} + 4\ln 2\ln\frac{p^2}{\Lambda^2} + c \quad , \end{aligned} \tag{D.42}$$

$$\begin{aligned} t^K_{13} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-3k^2)}{(k^2+p^2)} \ln^2\frac{k^2}{\Lambda^2} \\ &= \ln^3\frac{p^2}{\Lambda^2} + 4\ln 2\ln^2\frac{p^2}{\Lambda^2} + \mathcal{O}(\ln\frac{p^2}{\Lambda^2}) \quad , \end{aligned} \tag{D.43}$$

$$\begin{aligned} t^K_{14} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-k^2)^2}{(k^2+p^2)^2} \left(\ln\frac{p^2}{\Lambda^2} + \ln\frac{k^2}{\Lambda^2}\right) \\ &= -\frac{3}{2}\ln^2\frac{p^2}{\Lambda^2} - 4\ln\frac{p^2}{\Lambda^2} - 4\ln 2 \quad , \end{aligned} \tag{D.44}$$

$$\begin{aligned} t^K_{15} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-k^2)^2}{(k^2+p^2)^2} \\ &= -\ln\frac{p^2}{\Lambda^2} - 2 \quad , \end{aligned} \tag{D.45}$$

$$\begin{aligned} t^K_{16} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-k^2)^2}{(k^2+p^2)^2} \left(\ln^2\frac{p^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2}\right) \\ &= -\frac{4}{3}\ln^3\frac{p^2}{\Lambda^2} - 4\ln^2\frac{p^2}{\Lambda^2} - 8\ln 2\ln\frac{p^2}{\Lambda^2} + c \quad , \end{aligned} \tag{D.46}$$

$$\begin{aligned} t^K_{17} &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2-k^2)^2}{(k^2+p^2)^2} \left(\ln\frac{p^2}{\Lambda^2}\ln\frac{k^2}{\Lambda^2} + \ln^2\frac{k^2}{\Lambda^2}\right) \\ &= -\frac{5}{6}\ln^3\frac{p^2}{\Lambda^2} - 4\ln^2\frac{p^2}{\Lambda^2} - 12\ln 2\ln\frac{p^2}{\Lambda^2} + c \quad , \end{aligned} \tag{D.47}$$

$$
\begin{aligned}
t_{18}^K &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{p^2 - k^2)^2}{(k^2+p^2)^2} \ln \frac{k^2}{\Lambda^2} \\
&= -\frac{1}{2} \ln^2 \frac{p^2}{\Lambda^2} - 2 \ln \frac{p^2}{\Lambda^2} - 4 \ln 2 \quad ,
\end{aligned}
\tag{D.48}
$$

$$
\begin{aligned}
t_{19}^K &= \int_{p^2}^{\Lambda^2} dk^2 \frac{1}{k^2} \frac{(p^2 - k^2)^2}{(k^2+p^2)^2} \ln^2 \frac{k^2}{\Lambda^2} \\
&= -\frac{1}{3} \ln^3 \frac{p^2}{\Lambda^2} + 2 \ln^2 \frac{p^2}{\Lambda^2} + 8 \ln 2 \ln \frac{p^2}{\Lambda^2} + c \quad ,
\end{aligned}
\tag{D.49}
$$

$$
\begin{aligned}
t_{20}^K &= \int_{p^2}^{\Lambda^2} dk^2 \frac{k^2}{(k^2+p^2)^2} \left(\ln \frac{p^2}{\Lambda^2} + \ln \frac{k^2}{\Lambda^2} \right) \\
&= -\frac{3}{2} \ln^2 \frac{p^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} - 2 \ln 2 \ln \frac{p^2}{\Lambda^2} + 3 \ln 2 + 2\zeta(2) \quad ,
\end{aligned}
\tag{D.50}
$$

$$
\begin{aligned}
t_{21}^K &= \int_{p^2}^{\Lambda^2} dk^2 \frac{k^2}{(k^2+p^2)^2} \\
&= -\ln \frac{p^2}{\Lambda^2} - \frac{1}{2} - 2 \ln 2 \quad ,
\end{aligned}
\tag{D.51}
$$

where c is some constant.

Appendix E

E Integrals Used for the Photon Equation [50]

$$\ell_1^P = \int_0^{p^2/4} d\ell^2 \, \frac{\ell^2 \, (16\ell^4 - 3p^4)}{p^4 \, (\ell^2 + p^2/4)}$$

$$= p^2 \left(-\frac{13}{24} + \frac{1}{2} \ln 2 \right) \quad , \tag{E.1}$$

$$\ell_1^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{2\,(\ell^2 + p^2/4)}$$

$$= p^2 \left(-\frac{1}{2} \ln \frac{p^2}{\Lambda^2} + \frac{1}{2} \ln 2 \right) \quad , \tag{E.2}$$

$$\ell_2^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{2\,(\ell^2 + p^2/4)} \ln \frac{\ell^2}{\Lambda^2}$$

$$= p^2 \left(-\frac{1}{4} \ln^2 \frac{p^2}{\Lambda^2} + \frac{1}{2} \ln 2 \, \ln \frac{p^2}{\Lambda^2} \right) \quad , \tag{E.3}$$

$$\ell_3^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{2\,(\ell^2 + p^2/4)} \ln^2 \frac{\ell^2}{\Lambda^2}$$

$$= p^2 \left(-\frac{1}{6} \ln^3 \frac{p^2}{\Lambda^2} + \frac{1}{2} \ln 2 \, \ln^2 \frac{p^2}{\Lambda^2} \right) \quad , \tag{E.4}$$

$$\ell_4^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2\,(2\,\ell^2 - p^2)}{8\,(\ell^2 + p^2/4)^2}$$

$$= p^2 \left(-\frac{1}{4} \ln \frac{p^2}{\Lambda^2} + c \right) \quad , \tag{E.5}$$

$$\ell_5^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2\,(2\,\ell^2 - p^2)}{8\,(\ell^2 + p^2/4)^2} \ln \frac{\ell^2}{\Lambda^2}$$

$$= p^2 \left(-\frac{1}{8} \ln^2 \frac{p^2}{\Lambda^2} + \mathcal{O}\left(\ln \frac{p^2}{\Lambda^2} \right) \right) \quad , \tag{E.6}$$

$$\ell_6^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{64\,(\ell^2 + p^2/4)^3}\,(\,32\,\ell^4 - 8\,p^2\,\ell^2 - p^4) = p^2\left(-\frac{1}{2}\ln\frac{p^2}{\Lambda^2}\right) \quad , \tag{E.7}$$

$$\ell_7^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{64\,(\ell^2 + p^2/4)^3}\,(\,32\,\ell^4 - 8\,p^2\,\ell^2 - p^4)\,\ln\frac{\ell^2}{\Lambda^2} = p^2\left(-\frac{1}{4}\ln^2\frac{p^2}{\Lambda^2}\right) \quad , \tag{E.8}$$

$$t_1^P = \int_0^{p^2/4} d\ell^2 \, \frac{2\,\ell^4}{p^2\,(\ell^2 + p^2/4)} = p^2\left(-\frac{1}{16} + \frac{\ln 2}{8}\right) \quad , \tag{E.9}$$

$$t_2^P = \int_0^{p^2/4} d\ell^2 \, \frac{-2\,\ell^4}{p^2\,(\ell^2 + p^2/4)^2} = p^2\left(-\frac{3}{4} + \ln 2\right) \quad , \tag{E.10}$$

$$t_3^P = \int_0^{p^2/4} d\ell^2 \, \frac{2\,\ell^2\,(\ell^2 - p^2/4)}{p^2\,(\ell^2 + p^2/4)^2} = \left(1 - \frac{3\,\ln 2}{2}\right) \quad , \tag{E.11}$$

$$t_4^P = \int_0^{p^2/4} d\ell^2 \, \frac{2\,\ell^4\,(\ell^2 - p^2/4)}{p^2\,(\ell^2 + p^2/4)^2} = p^2\left(-\frac{7}{16} + \frac{5\,\ln 2}{8}\right) \quad , \tag{E.12}$$

$$t_1^L = \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{8\,(\ell^2 + p^2/4)} = p^2\left(-\frac{1}{8}\ln\frac{p^2}{\Lambda^2} + c\right) \quad , \tag{E.13}$$

$$
\begin{aligned}
t_2^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{8\,(\ell^2 + p^2/4)} \, \ln \frac{\ell^2}{\Lambda^2} \\
&= p^2 \left(-\frac{1}{16} \, \ln^2 \frac{p^2}{\Lambda^2} + \frac{1}{8} \, \ln 2 \, \ln \frac{p^2}{\Lambda^2} + c \right) \quad , \qquad \text{(E.14)}
\end{aligned}
$$

$$
\begin{aligned}
t_3^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{p^2}{8\,(\ell^2 + p^2/4)} \, \ln^2 \frac{\ell^2}{\Lambda^2} \\
&= p^2 \left(-\frac{1}{24} \, \ln^3 \frac{p^2}{\Lambda^2} + \frac{1}{8} \ln 2 \, \ln^2 \frac{p^2}{\Lambda^2} \right) \quad , \qquad \text{(E.15)}
\end{aligned}
$$

$$
\begin{aligned}
t_4^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{-p^2\,\ell^2}{2\,(\ell^2 + p^2/4)^2} \\
&= p^2 \left(\frac{1}{2} \, \ln \frac{p^2}{\Lambda^2} + c \right) \quad , \qquad \text{(E.16)}
\end{aligned}
$$

$$
\begin{aligned}
t_5^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{-p^2\,\ell^2}{2\,(\ell^2 + p^2/4)^2} \, \ln \frac{\ell^2}{\Lambda^2} \\
&= p^2 \left(\frac{1}{4} \, \ln^2 \frac{p^2}{\Lambda^2} + \frac{1}{4} \, \ln \frac{p^2}{\Lambda^2} - \frac{1}{2} \ln 2 \, \ln \frac{p^2}{\Lambda^2} + c \right) \quad , \qquad \text{(E.17)}
\end{aligned}
$$

$$
\begin{aligned}
t_6^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{-p^2\,\ell^2}{2\,(\ell^2 + p^2/4)^2} \, \ln^2 \frac{\ell^2}{\Lambda^2} \\
&= p^2 \left(\frac{1}{6} \, \ln^3 \frac{p^2}{\Lambda^2} - \frac{1}{2} \, \ln 2 \, \ln^2 \frac{p^2}{\Lambda^2} + \frac{1}{4} \, \ln^2 \frac{p^2}{\Lambda^2} + \mathcal{O}(\ln) \right) \quad , \qquad \text{(E.18)}
\end{aligned}
$$

$$
\begin{aligned}
t_7^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)^2} \\
&= -\ln \frac{p^2}{\Lambda^2} + c \quad , \qquad \text{(E.19)}
\end{aligned}
$$

$$
\begin{aligned}
t_8^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)^2} \, \ln \frac{\ell^2}{\Lambda^2} \\
&= -\frac{1}{2} \, \ln^2 \frac{p^2}{\Lambda^2} + \ln 2 \, \ln \frac{p^2}{\Lambda^2} - \ln \frac{p^2}{\Lambda^2} + c \quad , \qquad \text{(E.20)}
\end{aligned}
$$

$$\begin{aligned} t_9^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)^2} \ln^2 \frac{\ell^2}{\Lambda^2} \\ &= -\frac{1}{3} \ln^3 \frac{p^2}{\Lambda^2} + \ln 2 \ln^2 \frac{p^2}{\Lambda^2} - \ln^2 \frac{p^2}{\Lambda^2} \quad , \end{aligned} \tag{E.21}$$

$$\begin{aligned} t_{10}^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)} \\ &= p^2 \left(\frac{\Lambda^2}{p^2} + \frac{1}{2} \ln \frac{p^2}{\Lambda^2} \right) \quad , \end{aligned} \tag{E.22}$$

$$\begin{aligned} t_{11}^L &= \int_{p^2/4}^{\Lambda^2} d\ell^2 \, \frac{(\ell^2 - p^2/4)}{(\ell^2 + p^2/4)} \ln \frac{\ell^2}{\Lambda^2} \\ &= p^2 \left(\frac{\Lambda^2}{p^2} \ln \frac{p^2}{\Lambda^2} - \frac{\Lambda^2}{p^2} + \frac{1}{2} \ln \frac{p^2}{\Lambda^2} \right) \quad , \end{aligned} \tag{E.23}$$

where c is some constant.

Appendix F

F Some standard integrals in d-dimensions

$$i) \quad \int \frac{d^d k}{(k^2+s)^n} = i\pi^{d/2} \frac{\Gamma(n-d/2)}{\Gamma(n)} \frac{1}{s^{(n-d/2)}} \quad , \tag{F.1}$$

$$ii) \quad \int d^d k \, \frac{k^\mu}{(k^2+s)^n} = 0 \quad , \tag{F.2}$$

$$iii) \quad \int d^d k \, \frac{k^\mu k^\nu}{(k^2+s)^n} = i\pi^{d/2} \frac{\Gamma(n-d/2-1)}{2\Gamma(n)} \frac{g^{\mu\nu}}{s^{(n-d/2-1)}} \quad , \tag{F.3}$$

$$iv) \quad \int d^d k \, \frac{k^2}{(k^2+s)^n} = i\pi^{d/2} \frac{\Gamma(n-d/2-1)}{2\Gamma(n)} \frac{d}{s^{(n-d/2-1)}} \quad . \tag{F.4}$$

Appendix G

G Related Integrals to Vertex calculation in Asymptotic Limit [50]

This appendix is related to the one-loop vertex calculation in the asymptotic limit of Sect. 6.3 :

Definitions:

$$D = a + bx + cx^2 \quad ,$$

where

$$\begin{aligned} a &= -p^2z + p^2z^2 + m^2z \quad , \\ b &= -k^2 + p^2z - 2p^2z^2 + 2k\cdot pz + m^2 - m^2z \quad , \\ c &= k^2 + p^2z^2 - 2k\cdot pz \quad , \\ \Delta &= 4ac - b^2 \\ &= -k^4 - 2k^2p^2z + 2m^2k^2z + 4k^2k\cdot pz + 2k^2m^2 \quad , \end{aligned}$$

and

$$\sqrt{-\Delta} = k^2 + p^2z - m^2z - 2k\cdot pz - m^2 \quad . \tag{G.1}$$

$$i)\int_0^1 \frac{dx}{D} = \frac{1}{\sqrt{-\Delta}}\ln\frac{b+2cx-\sqrt{-\Delta}}{b+2cx+\sqrt{-\Delta}}\bigg|_0^1 = \ln\left(\frac{b+2c-\sqrt{-\Delta}}{b+2c+\sqrt{-\Delta}}\,\frac{b+\sqrt{-\Delta}}{b-\sqrt{-\Delta}}\right) \, , \tag{G.2}$$

where

$$\begin{aligned} b+\sqrt{-\Delta} &= 2\left(p^2z - p^2z^2 - m^2z\right) \quad , \\ b-\sqrt{-\Delta} &= 2\left(-k^2 - p^2z^2 + 2k\cdot pz + m^2\right) \quad , \\ b+2c+\sqrt{-\Delta} &= 2\left(k^2 + p^2z - m^2z - 2k\cdot pz\right) \quad , \\ b+2c-\sqrt{-\Delta} &= 2m^2 \quad . \end{aligned}$$

Therefore we find for Eqn. (C.2) :

$$i) \quad \int_0^1 \frac{dx}{D} = \frac{1}{k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right)\left[\ln\left(\frac{m^2}{p^2}z - z + z^2\right) - \ln\frac{k^2}{m^2} - \ln\frac{k^2}{p^2}\right] + \mathcal{O}(k^{-4}) \,, \tag{G.3}$$

$$ii) \quad \int_0^1 dx\,\frac{x}{D} = \frac{1}{2c}\ln D\Big|_0^1 - \frac{b}{2c}\int_0^1 \frac{dx}{D} \,, \tag{G.4}$$

where

$$\ln D\Big|_0^1 = \ln\frac{m^2}{p^2} - \ln\left(\frac{m^2}{p^2}z - z + z^2\right) \quad ,$$

$$iii) \quad \int_0^1 dx\,\frac{x}{D} = \frac{1}{2k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right)\left(\ln\frac{m^2}{p^2} - \ln\frac{k^2}{m^2} - \ln\frac{k^2}{p^2}\right) \,, \tag{G.5}$$

$$\begin{aligned} iv) \quad \int_0^1 dx\,\frac{x^2}{D} &= \frac{x}{c}\Big|_0^1 - \frac{b}{2c^2}\ln D\Big|_0^1 + \frac{b^2-2ac}{2c^2}\int_0^1\frac{dx}{D} \\ &= \frac{1}{k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right) + \frac{1}{2k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right)\left(\ln\frac{m^2}{p^2} - \ln\frac{k^2}{m^2} - \ln\frac{k^2}{p^2}\right) \\ &\quad + \mathcal{O}(k^{-4}) \,, \end{aligned} \tag{G.6}$$

$$\begin{aligned} v) \quad \int_0^1 dx\,\frac{x^3}{D} &= \frac{x^2}{2c}\Big|_0^1 - \frac{bx}{c^2}\Big|_0^1 + \frac{b^2-2ac}{2c^3}\ln D\Big|_0^1 - \frac{b\,(b^2-3ac)}{2c^3}\int_0^1\frac{dx}{D} \\ &= \frac{3}{2k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right) - \frac{1}{k^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right)\left(\ln\frac{k^2}{m^2} + \ln\frac{k^2}{p^2} + \ln\frac{p^2}{m^2}\right) \\ &\quad + \mathcal{O}(k^{-4}) \,, \end{aligned} \tag{G.7}$$

$$vi) \quad \int_0^1 dx\,\frac{x^n(1-x)}{D} = 0 + \mathcal{O}(k^{-2}, k^{-4}\ln k^2) \quad , \qquad n = 1, 2, \cdots \,,$$

$$vii) \quad \int_0^1 dx \frac{1}{D^2} = \frac{b+2cx}{\Delta D}\bigg|_0^1 + \frac{2c}{\Delta}\int_0^1 \frac{dx}{D}$$

$$= -\frac{1}{k^2}\left(\frac{1}{m^2}+\frac{1}{a}\right)\left(1+\frac{2k\cdot p\, z}{k^2}\right)$$

$$+\frac{2}{k^4}\left(1+\frac{4k\cdot p\, z}{k^2}\right)\left[\ln\frac{k^2}{m^2}+\ln\frac{k^2}{p^2}-\ln\left(\frac{m^2}{p^2}z-z+z^2\right)\right]$$

$$+\mathcal{O}(k^{-4})\,, \tag{G.8}$$

$$viii) \quad \int_0^1 dx \frac{x}{D^2} = -\frac{2a+bx}{\Delta D}\bigg|_0^1 - \frac{b}{\Delta}\int_0^1 dx\frac{1}{D}$$

$$= -\frac{1}{k^2m^2}\left(1+\frac{2k\cdot p\, z}{k^2}\right)$$

$$+\frac{1}{k^4}\left(1+\frac{4k\cdot p\, z}{k^2}\right)\left[\ln\frac{k^2}{m^2}+\ln\frac{k^2}{p^2}-\ln\left(\frac{m^2}{p^2}z-z+z^2\right)\right]$$

$$+\mathcal{O}(k^{-4})\,, \tag{G.9}$$

$$ix) \quad \int_0^1 dx\, \frac{x^2}{D^2} = \frac{ab+(b^2-2ac)x}{c\,\Delta\, D}\bigg|_0^1 + \frac{2a}{\Delta}\int_0^1\frac{dx}{D}$$

$$= \frac{-1}{k^2m^2}\left(1+\frac{2k\cdot p\, z}{k^2}\right) + \mathcal{O}(k^{-4})\,, \tag{G.10}$$

$$x) \quad \int_0^1 dx\frac{(1-x)^2}{D^2} = -\frac{1}{k^2a}\left(1+\frac{2k\cdot p\, z}{k^2}\right) + \mathcal{O}(k^{-4}, k^{-6}\ln k^2)\,, \tag{G.11}$$

$$xi) \quad \int_0^1 dx \frac{x^3}{D^2} = \frac{1}{2c^2} \ln D \Big|_0^1 + \frac{a(2ac-b^2)+b(3ac-b^2)x}{c^2 \Delta D} \Big|_0^1 - \frac{b(6ac-b^2)}{2c^2\Delta} \quad ,$$

$$= \frac{-1}{k^2m^2}\left(1+\frac{2k\cdot p\,z}{k^2}\right) - \frac{1}{2\,k^4}\left(1+\frac{4k\cdot pz}{k^2}\right)\left[\ln\frac{k^2}{m^2}+\ln\frac{k^2}{p^2}+\ln\frac{p^2}{m^2}\right]$$

$$+\mathcal{O}(k^{-4}) \quad , \tag{G.12}$$

$$xii) \quad \int_0^1 dx \frac{x(1-x)^2}{D^2} = \frac{1}{k^4}\left(1+\frac{4k\cdot p\,z}{k^2}\right)\ln\frac{k^2}{p^2} + \mathcal{O}(k^{-6})\,, \tag{G.13}$$

$$xiii) \quad \int_0^1 dx \frac{x^n(1-x)^2}{D^2} = 0 + \mathcal{O}(k^{-4}) \quad , \qquad n = 2, 3\cdots \quad . \tag{G.14}$$

Appendix H

H d-dimensional Integrals Corresponding to the Vertex Calculation

This appendix briefly outlines the evalution of the related integrals in Chapter 6. We first deal with general 2-point scalar integral as follows :

$$I_{pn}(k,p,m_1,m_2) = \int d^d w \frac{1}{\left[(k-w)^2 - m_1^2\right]^p \left[(p-w)^2 - m_2^2\right]^n} \, . \tag{H.1}$$

Using Feynman parametrization :

$$\frac{1}{A^n \, B^p} = \frac{\Gamma\,(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx \, x^{(n-1)} (1-x)^{p-1} \frac{1}{[\,xA + (1-x)\,B\,]^{n+p}} \, , \tag{H.2}$$

where

$$\begin{aligned} A &= (p-w)^2 - m_2^2 \quad , \\ B &= (k-w)^2 - m_1^2 \quad , \end{aligned}$$

the denominator of the I_{pn}-integral becomes :

$$K \equiv w^2 - 2w \cdot (\,px + k(1-x)\,) + p^2 x + k^2\,(1-x) - m_2^2 x - m_1^2\,(1-x) \; .$$

We now change the variable of integration as :

$$w^{'} = w - (\,p\,x + k\,(1-x)\,) \; . \tag{H.3}$$

Consequently, K is,

$$K = w^{'2} + (k-p)^2\, x\,(1-x) - m_2^2\, x - m_1^2\,(1-x) \quad . \tag{H.4}$$

Substituting K into the Eqn. (H.2), we find the I_1-integral to be :

$$I_{pn}(k,p,m_1,m_2) = \frac{\Gamma\,(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx \, x^{n-1} (1-x)^{p-1} \int d^d w \frac{1}{\left(w'^2 - L\right)^{n+p}} \, , \tag{H.5}$$

where

$$L = -(k-p)^2 x(1-x) + m_2^2 x + m_1^2 (1-x) \quad . \tag{H.6}$$

After making a Wick rotation and using Eqn. (F.1), we find,

$$I_{pn}(k,p,m_1,m_2) = i(-1)^{n+p} \frac{\Gamma(n+p-d/2)}{\Gamma(n)\Gamma(p)} \int_0^1 dx\, x^{n-1}(1-x)^{p-1} L^{d/2-n-p} . \tag{H.7}$$

i) $\underline{n = p = 1, m_1 = m_2 = m :}$

$$I_{11}(k,p,m,m) = i\, p^{d/2} \frac{\Gamma(2-d/2)}{\Gamma(1)\,\Gamma(1)} \int d^4 w \frac{1}{\left[(k-w)^2 - m^2\right]\left[(p-w)^2 - m^2\right]} . \tag{H.8}$$

If we take in Eqn. (H.7), we get

$$I_{11}(k,p,m,m) = i\pi^{d/2} \frac{\Gamma(2-d/2)}{\Gamma(1)\Gamma(1)} \int_0^1 dx\, L^{d/2-2} \quad , \tag{H.9}$$

where

$$L = -q^2 x(1-x) + m^2 \quad , \quad q = k - p ,$$

and

$$d = \epsilon + 4 .$$

We then find I_{11} to be :

$$I_{11}(k,p,m,m) = i\pi^2 \mu^\epsilon \int_0^1 dx \left[C - \ln\left(1 - \frac{q^2}{m^2} x(1-x)\right)\right] ,$$

where

$$C = -\frac{2}{\epsilon} - \gamma - \ln\pi - \ln\frac{m^2}{\mu^2} \quad .$$

Performing the x-integral gives

$$I_{11}(k,p,m,m) = i\pi^2 \mu^\epsilon \left(C + 2 - 2S \right) , \tag{H.10}$$

where

$$S = \frac{1}{2}\left(1 - 4\frac{m^2}{q^2}\right)^{1/2} \ln \frac{\left(1 - 4m^2/q^2\right)^{1/2} + 1}{\left(1 - 4m^2/q^2\right)^{1/2} - 1} \quad . \tag{H.11}$$

ii) $\underline{n = p = 1; m_1 = 0\,,\ m_2 = m, k = 0:}$

$$I_{11}(0, p, 0, m) \quad = \quad \int d^d w \, \frac{1}{\left[(p - w)^2 - m^2\right] w^2} \quad . \tag{H.12}$$

Applying Eqn. (H.2) to this case, we get,

$$I_{11}(0, p, 0, m) = i\pi^{d/2} \frac{\Gamma(2 - d/2)}{\Gamma(1)\,\Gamma(1)} \int_0^1 dx \, L^{d/2-2} \quad ,$$

where

$$L = -p^2 x\,(1 - x) + m^2 x \quad .$$

Performing the x-integral we have,

$$I_{11}(0, p, 0, m) = i\pi^2 \mu^\epsilon \,(\,C + 2 - L\,) \quad , \tag{H.13}$$

where

$$L = \left(1 - \frac{m^2}{p^2}\right) \ln \left(1 - \frac{p^2}{m^2}\right) \quad . \tag{H.14}$$

iii) $\underline{n = p = 1; p = 0, m_2 = 0, m_1 = m:}$

Moreover,

$$\begin{aligned} I_{11}(k, 0, m, 0) \quad &= \quad \int d^d w \, \frac{1}{w^2\left[(k - w)^2 - m^2\right]} \quad , \\ &= \quad i\pi^2 \mu^\epsilon \left(C + 2 - L'\right) \quad , \end{aligned} \tag{H.15}$$

where

$$L' = \left(1 - \frac{m^2}{k^2}\right) \ln \left(1 - \frac{k^2}{m^2}\right) \quad . \tag{H.16}$$

Now we take 2-point vector integral :

$$I^{\nu}_{pn}(k,p,m_1,m_2) \;=\; \int d^4w \,\frac{w^{\nu}}{\left[(k-w)^2-m_1^2\right]^p\left[(p-w)^2-m_2^2\right]^n}\,. \tag{H.17}$$

After using the Feynman parametrization, changing the variable of integration and performing the integral in d-dimensions, we get :

$$I^{\nu}_{pn}(k,p,m_1,m_2) = \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)}\int_0^1 dx\, x^{n-1}(1-x)^{p-1}\int d^dw'\,\frac{w'^{\nu}+p^{\nu}x+k^{\nu}(1-x)}{(w'-L)^{n+p}}\,.$$

On tidying up we arrive at :

$$\begin{aligned}I^{\nu}_{pn}(k,p,m_1,m_2) \;=\;& i\pi^{d/2}(-1)^{n+p}\frac{\Gamma(n+p-d/2)}{\Gamma(n)\Gamma(p)}\\ &\times\Bigg[\;p^{\nu}\int_0^1 dx x^{n}\,(1-x)^{p-1}\,L^{d/2-n-p}\\ &+k^{\nu}\int_0^1 dx x^{n-1}\,(1-x)^{p}\,L^{d/2-n-p}\Bigg]\;,\end{aligned} \tag{H.18}$$

where

$$L = -q^2\,x\,(1-x) + m_2^2\,x + m_1^2\,(1-x) \qquad . \tag{H.19}$$

iv) $\underline{n=p=1; m_1=m_2=m:}$

$$I^{\nu}_{11}(k,p,m,m) \;=\; \int d^dw\,\frac{w^{\nu}}{\left[(k-w)^2-m^2\right]\left[(k-w)^2-m^2\right]}\,. \tag{H.20}$$

On using the general solution which is $I^{\nu}_{11}(k,p,m_1,m_2)$, this is :

$$I^{\nu}_{11}(k,p,m,m) \;=\; i\pi^{d/2}\Gamma\left(2-d/2\right)\left(p^{\nu}\int_0^1 dx\, xL^{d/2-2}+k^{\nu}\int_0^1 dx\,(1-x)\,L^{d/2-2}\right)\,,$$

where

$$L \;=\; -q^2x(1-x)+m^2\quad ,$$

and tidyding up, one gets :

$$I^{\nu}_{11}(k,p,m,m) = i\pi^2\mu^{\epsilon}\int_0^1 dx\ (p^{\nu}x + k^{\nu}(1-x))\left[C - \ln\left(1 - \frac{q^2}{m^2}x(1-x)\right)\right] .$$

Now evaluating the x-integral, we obtain,

$$I^{\nu}_{11}(k,p,m,m) = i\pi^2\mu^{\epsilon}\frac{(p^{\nu}+k^{\nu})}{2}(C + 2 - 2S) \quad . \tag{H.21}$$

v) $\underline{n = p = 1; k = 0, m_1 = 0, m_2 = m :}$

$$I^{\nu}_{11}(0,p,0,m) = \int_0^1 d^d w \frac{w^{\nu}}{w^2\left[(p-w)^2 - m^2\right]} . \tag{H.22}$$

Again making use of Eqn. (H.20), we get

$$\begin{aligned} I^{\nu}_{11}(0,p,0,m) &= i\pi^{d/2}\Gamma(2-d/2)\,p^{\nu}\int_0^1 dx\, xL^{d/2-2} \quad , \\ &= i\pi^2\mu^{\epsilon}p^{\nu}\int_0^1 dx\, x\left[-C - \ln\left(1 - \frac{p^2}{m^2}(1-x)\right)\right] \quad . \end{aligned}$$

Evaluating x-integral, we find :

$$I^{\nu}_{11}(0,p,0,m) = i\pi^2\mu^{\epsilon}\frac{p^{\nu}}{2}\left[C + 2 - \frac{m^2}{p^2} - \left(1 - \frac{m^2}{p^2}\right)^2 \ln\left(1 - \frac{p^2}{m^2}\right)\right] \quad . \tag{H.23}$$

vi) $\underline{n = p = 1; p = 0, m_2 = 0, m_1 = m :}$

$$\begin{aligned} I^{\nu}_{11}(k,0,m,0) &= \int d^d w \frac{w^{\nu}}{w^2\left[(k-w)^2 - m^2\right]}, \\ &= i\pi^2\mu^{\epsilon}\frac{k^{\nu}}{2}\left[C + 2 - \frac{m^2}{k^2} - \left(1 - \frac{m^2}{k^2}\right)^2 \ln\left(1 - \frac{k^2}{m^2}\right)\right] \quad , \end{aligned} \tag{H.24}$$

$$J_{np} = \int d^d k \frac{1}{\left[(k-p)^2\right]^n \left[(k^2 - m^2)\right]^p} \quad . \tag{H.25}$$

The obvious thing to do here is to use Feynman parametrization :

$$\frac{1}{A^n B^p} = \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)}\int_0^1 dx\, x^{n-1}(1-x)^{p-1}\frac{1}{\left[xA + (1-x)B\right]^{n+p}} , \tag{H.26}$$

where

$$\begin{aligned} A &= (k-p)^2 \quad , \\ B &= k^2 - m^2 \quad , \end{aligned}$$

with the denominator :

$$K = k^2 - 2\,k\cdot p\,x + p^2 x - m^2(1-x) \quad .$$

The next step is to change the variable to k' where

$$k' = k - px \quad ,$$

J_{np} then becomes :

$$J_{np} = \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx\, x^{n-1}\,(1-x)^{p-1} \int d^d k \, \frac{1}{(k^2 - L)^{n+p}} \quad ,$$

where

$$L = -p^2 x(1-x) + m^2(1-x) \quad .$$

Then for this general case, we find :

$$\begin{aligned} J_{np} &= i\pi^{d/2}\,(-1)^{n+p}\,\frac{\Gamma(n+p-d/2)}{\Gamma(n)\Gamma(p)}\,(m^2)^{d/2-n-p} \\ &\quad \times \int_0^1 dx, x^{n-1}(1-x)^{d/2-n-1}\left(1 - \frac{p^2}{m^2}x\right)^{d/2-n-p} \quad . \end{aligned} \tag{H.27}$$

i) <u>$n = p = 1$:</u>

$$\begin{aligned} J_{11} &= \int d^d k \, \frac{1}{[(k-p)^2]\,(k^2 - m^2)} \\ &= i\pi^2\mu^\epsilon \int_0^1 dx \left[C - \ln(1-x) - \ln\left(1 - \frac{p^2}{m^2}\,x\right) \right] \\ &= i\pi^2\mu^\epsilon \left[C + 2 - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 + \frac{p^2}{m^2}\right) \right] \quad . \end{aligned} \tag{H.28}$$

ii) <u>$n=2, p=1$:</u>

$$J_{21}(p) = \int d^dk \frac{1}{(k-p)^4 (k^2-m^2)} \quad , \tag{H.29}$$

$$\begin{aligned} J_{21}(p) &= \frac{1}{(m^2-p^2)} \left(p_\alpha \frac{\partial}{\partial p_\alpha} \int d^dk \frac{1}{(k-p)^2(k^2-m^2)} + \int d^d \frac{1}{(k-p)^2 (k^2-m^2)} \right) \\ &= \frac{i\pi^2\mu^\epsilon}{(m^2-p^2)} \left[C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right] \quad , \end{aligned} \tag{H.30}$$

$$J'_{21}(p) = \int d^dw \frac{1}{w^4 \; [(p-w)^2 - m^2]} \quad . \tag{H.31}$$

Letting $p \to -p$:

$$J'_{21}(p) = \int d^dw \frac{1}{w^4 \; [(p+w)^2 - m^2]} \quad .$$

Changing the w-variable as $w = k - p$, J'_{21} becomes :

$$\begin{aligned} J'_{21}(p) &= J_{21}(-p) \quad , \\ &= \frac{i\pi^2\mu^\epsilon}{(m^2-p^2)} \left[C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right] \quad . \end{aligned} \tag{H.32}$$

$$\begin{aligned} J^\nu_{np} &= \int d^dk \frac{k^\nu}{[(k-p)^2]^n \, [k^2-m^2]^p} \quad , \\ &= i\pi^{d/2}(-1)^{n+p} \frac{\Gamma(n+p-d/2)}{\Gamma(n)\Gamma(p)} p_\mu \int_0^1 dx \, n^n (1-x)^{p-1} L^{d/2-n-p} \quad , \end{aligned} \tag{H.33}$$

where

$$L = -p^2 x(1-x) + m^2(1-x) \; .$$

iii) <u>$n=p=1$:</u>

$$J^\nu_{11} = \int d^dk \frac{k^\nu}{(k-p)^2(k^2-m^2)} \quad , \tag{H.34}$$

Making use of Eqn. (H.33) :

$$J^{\nu}_{11} = i\pi^2 p_\mu \mu^\epsilon \int_0^1 dx \left[C\,x - x\ln(1-x) - x\ln\left(1 - \frac{p^2}{m^2}\,x\right)\right] ,$$

$$= i\pi^2 \frac{p_\mu}{2}\mu^\epsilon \left[C + \frac{m^2}{p^2} + 2 - \left(1 - \frac{m^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right] \quad . \qquad \text{(H.35)}$$

iv <u>$n = 2, p = 1$:</u>

$$J^{\nu}_{21}(p) = \int d^d k \frac{k^\nu}{(k-p)^4\,(k^2 - m^2)} \quad , \qquad \text{(H.36)}$$

$$J^{\nu}_{21}(p) = \frac{i\pi^2 \mu^\epsilon p^\nu}{(m^2 - p^2)} \left[C + \left(1 - \frac{m^2}{p^2}\right) - \left(1 + \frac{m^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right] , \qquad \text{(H.37)}$$

$$J^{\nu'}_{21}(p) = \int d^d w \frac{w^\nu}{w^4\,[(p-w)^2 - m^2]} \quad . \qquad \text{(H.38)}$$

If we change the k-variable to $w = k - p$ in $J^{\nu}_{21}(k, p)$, and let $p \to -p$, we get :

$$J^{\nu'}_{21}(p) = \int d^d w \frac{(w-p)^\nu}{w^4\,[(w-p)^2 - m^2]} \quad .$$

The first term in the expression below is the one we want to evaluate, $J'_{21}(p)$:

$$J'_{21}(p) = p^\nu \int d^d w \, \frac{1}{w^4\,[(w-p)^2 - m^2]} + \int d^d k \, \frac{k^\nu}{(k-p)^4\,(k^2 - m^2)}\Bigg|_{p\to -p}$$

$$= i\pi^2 \frac{p^\nu}{p^2}\left[1 + \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right)\right] \quad . \qquad \text{(H.39)}$$

Appendix I

I More Integrals for the Vertex Calculation [50]

$$Q_1 = \int_M d^d k \frac{1}{(k-p)^2 \, [k^2 - m^2]}$$
$$= i\pi^2 \mu^\epsilon \left\{ C - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \qquad \text{(I.1)}$$

$$Q_2^\nu = \int_M d^d k \frac{k^\nu}{(k-p)^2 \, [k^2 - m^2]}$$
$$= i\pi^2 \mu^\epsilon \left\{ \frac{p^\nu}{2} \left[C + 2 + \frac{m^2}{p^2} - \left(1 - \frac{m^4}{p^4}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right] \right\} \quad , \qquad \text{(I.2)}$$

$$Q_3 = \int_M d^d k \frac{1}{(k-p)^4 \, [k^2 - m^2]}$$
$$= \mu^\epsilon \frac{i\pi^2}{(m^2 - p^2)} \left\{ C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \qquad \text{(I.3)}$$

$$Q_4^\nu = \int_M d^d k \frac{k^\nu}{(k-p)^4 \, [k^2 - m^2]}$$
$$= \mu^\epsilon \frac{i\pi^2 p^\nu}{(m^2 - p^2)} \left\{ C + \left(1 - \frac{m^2}{p^2}\right) - \left(1 + \frac{m^4}{p^4}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \qquad \text{(I.4)}$$

$$Q_5 = \int_M d^d k \frac{k^2}{(k-p)^4 \, [k^2 - m^2]}$$
$$= m^2 Q_3 \quad , \qquad \text{(I.5)}$$

$$Q_6^\nu = \int_M d^d k \frac{k^2 \, k^\nu}{(k-p)^4 \, [k^2 - m^2]}$$
$$= m^2 Q_4 \quad , \qquad \text{(I.6)}$$

$$Q_7 = \int_M d^d w \frac{1}{[(k-w)^2 - m^2] \, [(p-w)^2 - m^2]}$$
$$= i\pi^2 \mu^\epsilon \left[C + 2 - 2S \right] \quad , \qquad \text{(I.7)}$$

$$\begin{aligned} Q_8 &= \int_M d^d w \, \frac{1}{[(p-w)^2 - m^2] \, w^2} \\ &= i\pi^2 \mu^\epsilon \, [C + 2 - L] \quad , \end{aligned} \tag{I.8}$$

$$\begin{aligned} Q_9^\nu &= \int_M d^d w \, \frac{w^\nu}{[(k-w)^2 - m^2] \, [(p-w)^2 - m^2]} \\ &= \frac{i\pi^2}{2} \mu^\epsilon (p^\nu + k^\nu) \, [C + 2 - 2S] \quad , \end{aligned} \tag{I.9}$$

$$\begin{aligned} Q_{10}^\nu &= \int_M d^d w \, \frac{w^\nu}{[(p-w)^2 - m^2] \, w^2} \\ &= \frac{i\pi^2}{2} \mu^\epsilon p^\nu \left\{ C + 2 - \frac{m^2}{p^2} - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \end{aligned} \tag{I.10}$$

$$\begin{aligned} Q_{11} &= \int_M d^d w \, \frac{1}{[(p-w)^2 - m^2] \, w^4} \\ &= \frac{i\pi^2}{(m^2 - p^2)} \mu^\epsilon \left\{ C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \end{aligned} \tag{I.11}$$

$$\begin{aligned} Q_{12}^\nu &= \int_M d^d w \, \frac{w^\nu}{[(p-w)^2 - m^2] \, w^4} \\ &= i\pi^2 \frac{p^\nu}{p^2} \left\{ 1 + \frac{m^2}{p^2} \ln\left(1 - \frac{p^2}{m^2}\right) \right\} \quad , \end{aligned} \tag{I.12}$$

$$\begin{aligned} Q_{13} &= \int_M d^d w \, \frac{1}{[(p-w)^2 - m^2] \, [(k-w)^2 - m^2] \, w^4} = I^{(0)} \\ &= i\pi^2 \mu^\epsilon \left\{ \frac{1}{\chi} \left[-2q^2 S + p^2 \frac{(p^2 - m^2) q^2 + 2m^2 (k^2 - p^2)}{(p^2 - m^2)^2} L \right. \right. \\ &\qquad \left. \left. + k^2 \frac{(k^2 - m^2) q^2 - 2m^2 (k^2 - p^2)}{(k^2 - m^2)^2} L' \right] - \frac{C}{(p^2 - m^2)(k^2 - m^2)} \right\} , \end{aligned} \tag{I.13}$$

where we recall

$$C = -\frac{2}{\epsilon} - \gamma - \ln(\pi) - \ln \frac{m^2}{\mu^2} \quad ,$$

$$\begin{aligned}
L &= \left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right) \quad , \\
L' &= L(p \leftrightarrow k) \quad , \\
S &= \frac{1}{2}\left(1-4\frac{m^2}{q^2}\right)^{1/2} \ln\frac{\left[(1-4m^2/q^2)^{1/2}+1\right]}{\left[(1-4m^2/q^2)^{1/2}-1\right]} \quad .
\end{aligned} \tag{I.14}$$

$$Q_{14} = \int_M d^d w \frac{1}{[(p-w)^2-m^2]\,[(k-w)^2-m^2]\,w^2} = J^{(0)} \quad ,$$

$J^{(0)}$ is naturally expressed in terms of the Spence function $Sp(x)$:

$$Sp(x) = -\int_0^x dy \frac{\ln(1-y)}{y} \quad , \tag{I.15}$$

so that

$$\begin{aligned}
J^{(0)} &= \frac{i\pi^2}{-2\,\Delta}\left\{ Sp\left(\frac{y_1}{y_1-1}\right) + Sp\left(\frac{y_1}{y_1-\frac{m^2}{p^2}}\right) - Sp\left(\frac{y_1-1}{y_1-\frac{m^2}{p^2}}\right)\right. \\
&- Sp\left(\frac{y_2}{y_2-1}\right) - Sp\left(\frac{y_2}{y_2-\frac{m^2}{k^2}}\right) + Sp\left(\frac{y_2-1}{y_2-\frac{m^2}{k^2}}\right) \\
&+ \left. Sp\left(\frac{y_3}{y_3-q_1}\right) - Sp\left(\frac{y_3-1}{y_3-q_1}\right) + Sp\left(\frac{y_3}{y_3-q_2}\right) - Sp\left(\frac{y_3-1}{y_3-q_2}\right)\right\},
\end{aligned} \tag{I.16}$$

where

$$\begin{aligned}
\alpha &= 1+\frac{-(k\cdot p)+\Delta}{p^2}, \quad y_1 = y_0+\alpha, \quad y_2 = \frac{y_0}{(1-\alpha)}, \quad y_3 = -\frac{y_0}{\alpha} \quad , \\
y_0 &= \frac{1}{2\,p^2\,\Delta}\left[k^2p^2 - 2(k\cdot p)^2 + 2(k\cdot p)\Delta - p^2\Delta + p^2(k\cdot p) - m^2(k\cdot p-\Delta)\right] \quad , \\
q_1 &= \frac{1+\sqrt{1-4m^2/q^2}}{2}, \quad q_2 = \frac{1-\sqrt{1-4m^2/q^2}}{2} \quad .
\end{aligned} \tag{I.17}$$

In the massless case, J_0 simplifies to

$$J_0 = \frac{2}{\Delta}\left[Sp\left(\frac{p^2 - k\cdot p + \Delta}{p^2}\right) - Sp\left(\frac{p^2 - k\cdot p - \Delta}{p^2}\right) + \frac{1}{2}\ln\left(\frac{k\cdot p - \Delta}{k\cdot p + \Delta}\right)\ln\left(\frac{q^2}{p^2}\right)\right] \quad . \tag{I.18}$$

Appendix J

J Coefficients Related to the Vertex Calculations

In this appendix the coefficients of the 12 vectors V_i^μ in Eqn. (6.4.18) are explicitly tabulated.

$$
\begin{aligned}
a_1^{(1)} &= 3p^2(k^2-m^2) - 2\,k\cdot p\,(p^2-m^2) + 4\Delta^2, \\
a_2^{(1)} &= k\cdot p\,(p^2+m^2) - \frac{3m^2p^2}{2} - \frac{p^2k^2}{2}, \\
a_1^{(2)} &= k^2(k^2-m^2), \\
a_2^{(2)} &= k^2\,k\cdot p - \frac{k^2}{2}(m^2+k^2), \\
a_1^{(3)} &= k^2(p^2-m^2) - 2\,k\cdot p\,(k^2-m^2) + 4\Delta^2, \\
a_2^{(3)} &= m^2\,k\cdot p + \Delta^2 - \frac{k^2}{2}(m^2+p^2), \\
a_1^{(4)} &= k^2(p^2-m^2) - 2\,k\cdot p\,(k^2-m^2), \\
a_2^{(4)} &= -\frac{3k^2p^2}{2} - \frac{m^2k^2}{2} + k\cdot p\,(m^2+k^2) - 2\,\Delta^2, \\
a_1^{(5)} &= -2\Delta^2, \\
a_2^{(5)} &= -\frac{\Delta^2}{2}, \\
a_1^{(6)} &= -(k^2+m^2)\,\Delta^2, \\
a_2^{(6)} &= -\left(\frac{m^2}{2}+k^2\right)\Delta^2, \\
a_1^{(7)} &= -8m\Delta^2, \\
a_2^{(7)} &= m\left[-3\,(k\cdot p)^2 + k^2\,k\cdot p + 2p^2\,k\cdot p - 2\Delta^2\right], \\
a_2^{(8)} &= m\left[2k^2\,k\cdot p - \frac{k^2p^2}{2} - (k\cdot p)^2 - \frac{k^4}{2}\right], \\
a_2^{(9)} &= m\left[-k^2 + k\cdot p\right], \\
a_2^{(10)} &= m\left[-\frac{3p^2}{2} - \frac{k^2}{2} + 2\,k\cdot p\right], \\
a_2^{(11)} &= -m\frac{q^2}{2}\,k\cdot p \\
a_2^{(12)} &= m\frac{q^2}{2}\,k^2, \\
a_1^{(i)} &= 0, \qquad i = 8,9,10,11,12 \quad .
\end{aligned}
\tag{J.1}
$$

$$
\begin{aligned}
b_1^{(1)} &= a_1^{(2)}(k \leftrightarrow p), & b_1^{(2)} &= a_1^{(1)}(k \leftrightarrow p), \\
b_1^{(i)} &= a_1^{(i)}(k \leftrightarrow p), & i &= 3,4,5,6 \\
b_1^{(7)} &= a_1^{(8)}(k \leftrightarrow p), & b_1^{(8)} &= a_1^{(7)}(k \leftrightarrow p), \\
b_1^{(i)} &= 0, & i &= 9,10,11,12
\end{aligned}
$$

$$
\begin{aligned}
b_2^{(1)} &= a_2^{(2)}(k \leftrightarrow p), & b_2^{(2)} &= a_2^{(1)}(k \leftrightarrow p), \\
b_2^{(i)} &= a_2^{(i)}(k \leftrightarrow p), & i &= 5,6 \\
b_2^{(9)} &= a_2^{(10)}(k \leftrightarrow p), & b_2^{(10)} &= a_2^{(9)}(k \leftrightarrow p), \\
b_2^{(11)} &= -a_2^{(12)}(k \leftrightarrow p), & b_2^{(12)} &= -a_2^{(11)}(k \leftrightarrow p), \\
b_2^{(3)} &= (m^2+p^2)(k\cdot p) - (k\cdot p)^2 - \frac{p^2}{2}(m^2+k^2), \\
b_2^{(4)} &= m^2(k\cdot p) - \frac{p^2}{2}(m^2+k^2), \\
b_2^{(7)} &= m\left[-\Delta^2 + \frac{p^2}{2}(p^2-k^2)\right], \\
b_2^{(8)} &= m\left[k^2(k\cdot p) - (k\cdot p)^2 - 2\Delta^2\right] \quad .
\end{aligned}
\tag{J.2}
$$

$$
\begin{aligned}
c_1^{(i)} &= 0, \qquad\qquad i = 1,..,12 \\
c_2^{(1)} &= -\frac{p^4k^2}{2} + \frac{3m^4p^2}{2} - m^2p^2k^2 + p^2k^2\, k\cdot p - m^4\, k\cdot p, \\
c_2^{(2)} &= -\frac{k^4p^2}{2} - m^2k^2\, k\cdot p + \frac{m^4k^2}{2} + k^4\, k\cdot p, \\
c_2^{(3)} &= (m^2 - k^2)\left(\Delta^2 + \frac{k^2}{2}(k^2+m^2) - m^2\, k\cdot p\right), \\
c_2^{(4)} &= \frac{m^4k^2}{2} + k^2p^2\, k\cdot p - m^4\, k\cdot p - \frac{3k^4p^2}{2} + m^2p^2k^2, \\
c_2^{(5)} &= \frac{(k^2-m^2)}{2}\Delta^2, \\
c_2^{(6)} &= \frac{m^2(m^2-k^2)}{2}\Delta^2, \\
c_2^{(7)} &= m\Big[\left(3(k\cdot p)^2 - 2p^2\, k\cdot p - k^2\, k\cdot p\right)m^2 + \frac{p^4k^2}{2} + 3p^2k^2\, k\cdot p \\
&\quad - 2p^2(k\cdot p)^2 - \frac{p^2k^4}{2} - k^2(k\cdot p)^2\Big], \\
c_2^{(8)} &= m\Big[\left((k\cdot p)^2 - 2k^2\, k\cdot p + \frac{k^2p^2}{2} + \frac{k^4}{2}\right)m^2 \\
&\quad -p^2k^4 + p^2k^2\, k\cdot p - k^2(k\cdot p)^2 + k^4\, k\cdot p\Big], \\
c_2^{(9)} &= m\left[\left(k^2 - k\cdot p\right)m^2 - \frac{q^2}{2}k^2\right], \\
c_2^{(10)} &= m\left[\left(\frac{3p^2}{2} - 2\,k\cdot p + \frac{k^2}{2}\right)m^2 + q^2\, k\cdot p + 2\Delta^2\right], \\
c_2^{(11)} &= m\left(m^2q^2\frac{k\cdot p}{2} + \frac{k^2p^2}{2}(k^2-p^2) + p^2\, k\cdot p\,(k\cdot p - k^2)\right), \\
c_2^{(12)} &= m\left[-m^2\frac{q^2}{2}k^2 - \frac{k^2\, k\cdot p}{2}(p^2+k^2)\right] \quad . \qquad\qquad (J.3)
\end{aligned}
$$

$$
\begin{aligned}
d_1^{(i)} &= 0, \qquad\qquad i = 1,..,12 \\
d_2^{(1)} &= c_2^{(2)}(k \leftrightarrow p), \qquad\qquad d_2^{(2)} = c_2^{(1)}(k \leftrightarrow p), \\
d_2^{(i)} &= c_2^{(i)}, \qquad\qquad i = 5,6 \\
d_2^{(9)} &= c_2^{(10)}(k \leftrightarrow p), \qquad\qquad d_2^{(10)} = c_2^{(9)}(k \leftrightarrow p), \\
d_2^{(11)} &= -c_2^{(12)}(k \leftrightarrow p), \qquad\qquad d_2^{(12)} = -c_2^{(11)}(k \leftrightarrow p), \\
d_2^{(3)} &= m^2(k\cdot p)^2 - \frac{p^4k^2}{2} + \frac{m^4p^2}{2} - p^2(k\cdot p)^2 + p^2k^2\,k\cdot p - m^4\,k\cdot p \\
d_2^{(4)} &= \frac{m^4p^2}{2} + m^2p^2\,k\cdot p - m^4\,k\cdot p - \frac{k^2p^4}{2}, \\
d_2^{(7)} &= m\left[\left(\Delta^2 + \frac{p^2}{2}(k^2-p^2)\right)m^2 - p^2\Delta^2\right], \\
d_2^{(8)} &= m\left[\left(-k^2\,k\cdot p + (k\cdot p)^2\right)m^2 - p^2\Delta^2 - \frac{q^2}{2}k^2p^2\right] \quad .
\end{aligned}
\qquad (J.4)
$$

$$
\begin{aligned}
e_1^{(1)} &= (m^4 - p^4)\Delta^2, \\
e_2^{(1)} &= (m^4 - p^4)\Delta^2 + m^2 p^4 (k^2 - p^2), \\
e_1^{(2)} &= -2(p^2 - m^2)^2 (k^2 - p^2)\, k \cdot p - (p^4 - m^4)\Delta^2, \\
e_2^{(2)} &= (m^4 - p^4)\Delta^2 + m^2 p^2 k^2 (k^2 - p^2) - 2m^4\, k \cdot p\, (k^2 - p^2), \\
e_1^{(3)} &= -(p^4 - m^4)\Delta^2 + p^2 (k^2 - p^2)(p^2 - m^2)^2, \\
e_2^{(3)} &= (m^4 - p^4)\Delta^2 - 2m^2 p^2\, k \cdot p\, (k^2 - p^2) + m^4 p^2 (k^2 - p^2), \\
e_1^{(4)} &= -(p^4 - m^4)\Delta^2 + p^2 (k^2 - p^2)(p^2 - m^2)^2, \\
e_2^{(4)} &= (m^4 - p^4)\Delta^2 + m^4 p^2 (k^2 - p^2), \\
e_2^{(5)} &= 0, \\
e_1^{(6)} &= (p^4 - m^4)(k^2 - p^2)\Delta^2, \\
e_2^{(6)} &= (p^4 - m^4)(k^2 - p^2)\Delta^2, \\
e_1^{(7)} &= 8mp^2 (p^2 - m^2)\Delta^2, \\
e_2^{(7)} &= m \left[2p^2 (p^2 - m^2)\Delta^2 - m^2 p^4 (k^2 - p^2) \right], \\
e_1^{(8)} &= 8mp^2 (p^2 - m^2)\Delta^2, \\
e_2^{(8)} &= m \left[2p^2 (p^2 - m^2)\Delta^2 + m^2 p^2 k^2 (k^2 - p^2) \right], \\
e_2^{(9)} &= m^3 (k^2 - p^2) \left[p^2 - 2\, k \cdot p \right], \\
e_2^{(10)} &= m^3 p^2 (k^2 - p^2), \\
e_2^{(11)} &= m^3 p^2 (k^2 - p^2) \left[p^2 - k \cdot p \right], \\
e_2^{(12)} &= m^3 (k^2 - p^2) \left[2(k \cdot p)^2 - p^2 (k \cdot p) - p^2 k^2 \right], \\
e_1^{i} &= 0, \qquad\qquad i = 5, 9, 10, 11, 12 \quad .
\end{aligned}
\tag{J.5}
$$

$$
\begin{aligned}
f_1^{(1)} &= -e_1^{(2)}(k \leftrightarrow p), & f_1^{(2)} &= -e_1^{(1)}(k \leftrightarrow p),\\
f_1^{(3)} &= -e_1^{(4)}(k \leftrightarrow p), & f_1^{(4)} &= -e_1^{(3)}(k \leftrightarrow p),\\
f_1^{(i)} &= 0, & i &= 5,9,10,11,12\\
f_1^{(i)} &= -e_1^{(i)}(k \leftrightarrow p), & i &= 6,7,8\\
f_2^{(1)} &= -e_2^{(2)}(k \leftrightarrow p), & f_2^{(2)} &= -e_2^{(1)}(k \leftrightarrow p),\\
f_2^{(3)} &= -e_2^{(4)}(k \leftrightarrow p), & f_2^{(4)} &= -e_2^{(3)}(k \leftrightarrow p),\\
f_2^{(9)} &= -e_2^{(10)}(k \leftrightarrow p), & f_2^{(10)} &= -e_2^{(9)}(k \leftrightarrow p),\\
f_2^{(11)} &= e_2^{(12)}(k \leftrightarrow p), & f_2^{(12)} &= e_2^{(11)}(k \leftrightarrow p),\\
f_2^{(i)} &= -e_2^{(i)}(k \leftrightarrow p), & i &= 5,6\\
f_2^{(7)} &= m\left[(k^2-p^2)\left(-k^2p^2 - 2k^2\, k\cdot p + 4(k\cdot p)^2\right)m^2 - 2k^2(k^2-m^2)\Delta^2\right],\\
f_2^{(8)} &= m\left[\left(-2k^2m^2(k^2-p^2)\, k\cdot p - k^4m^2(k^2-p^2)\right) - 2k^2(k^2-m^2)\Delta^2\right] \quad .
\end{aligned}
\tag{J.6}
$$

$$
\begin{aligned}
g_1^{(i)} &= 0, & i &= 5,7,8,9,10,11,12\\
g_1^{(i)} &= 2\,k\cdot p, & i &= 1,2\\
g_1^{(i)} &= -(k^2+p^2), & i &= 3,4\\
g_1^{(6)} &= -2\,\Delta^2,\\
g_2^{(i)} &= 0, & i &= 1,...,12 \quad .
\end{aligned}
\tag{J.7}
$$

$$
\begin{aligned}
h_1^{(i)} &= 0, & i &= 1,2,4,9,10,11,12\\
h_1^{(3)} &= -4\Delta^2,\\
h_1^{(5)} &= 2\Delta^2,\\
h_1^{(6)} &= -2m^2\Delta^2,\\
h_1^{(i)} &= 4m\Delta^2, & i &= 7,8\\
h_2^{(i)} &= 0, & i &= 5,6,11,12\\
h_2^{(1)} &= h_2^{(2)}(k \leftrightarrow p) = p^2\left[m^2 - k\cdot p\right]\Delta^2,\\
h_2^{(3)} &= h_2^{(4)}(k \leftrightarrow p) = k^2p^2 - m^2\, k\cdot p,\\
h_2^{(7)} &= h_2^{(8)}(k \leftrightarrow p) = m\left[-p^2\, k\cdot p + \Delta^2 + (k\cdot p)^2\right],\\
h_2^{(9)} &= h_2^{(10)}(k \leftrightarrow p) = m\left[k^2 - k\cdot p\right] \quad .
\end{aligned}
\tag{J.8}
$$

$$
\begin{aligned}
l_1^{(1)} &= l_1^{(2)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2k^2p^2} - \left(m^2\frac{(k\cdot p)}{k^2} + p^2\right), \\
l_1^{(3)} &= l_1^{(4)}(k \leftrightarrow p) = (m^2 + k\cdot p) + \frac{m^2\,\Delta^2}{2k^2p^2}, \\
l_1^{(i)} &= 0, \qquad i = 5,7,8,9,10,11,12 \\
l_1^{(6)} &= \left[-1 - \frac{m^2}{2}\left(\frac{1}{p^2} + \frac{1}{k^2}\right)\right]\Delta^2, \\
l_2^{(1)} &= l_2^{(2)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2k^2p^2} - m^2\frac{k\cdot p}{k^2} + p^2, \\
l_2^{(3)} &= l_2^{(4)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2\,k^2p^2} + m^2 - k\cdot p, \\
l_2^{(5)} &= 0, \\
l_2^{(6)} &= \left[1 - \frac{m^2}{2}\left(\frac{1}{p^2} + \frac{1}{k^2}\right)\right]\Delta^2, \\
l_2^{(7)} &= m\left[-k\cdot p + 2\frac{(k\cdot p)^2}{k^2} - p^2\right], \\
l_2^{(8)} &= m\left[-k\cdot p + k^2\right], \\
l_2^{(9)} &= l_2^{(10)}(k \leftrightarrow p) = m\left[-\frac{k\cdot p}{p^2} + 1\right], \\
l_2^{(11)} &= -l_2^{(12)}(k \leftrightarrow p) = m\left[\frac{(k\cdot p)^2}{k^2} - p^2\right] \quad .
\end{aligned} \tag{J.9}
$$

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