Boundary sinh-Gordon model and its supersymmetric extension

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Abstract

Three different aspects of the sinh-Gordon model are explored in this thesis. We begin, in chapter one, with a summary of the model and the necessary background. Chapter two studies the model with two boundary conditions. Two approaches are presented to investigate the reflection factors off the boundaries and the energy of the theory. In chapter three, perturbation theory is developed to study the theory with one general boundary condition. A contribution to the quantum reflection factor is obtained and compared with the result obtained for the special boundary condition. Chapters four and five investigate the supersymmetric extension of the model in the presence of a single boundary. Firstly, the classical limits of the supersymmetric reflection matrices are checked. The exact reflection factors are studied perturbatively up to the second order of the coupling constant. Secondly, the perturbation theory and the path integral formalism are employed in the supersymmetric model to study the quantum reflection factors. We conclude with a brief sixth chapter describing the outlook for further investigations.

Declaration

This thesis is the result of research carried out between March 1996 and June 1999, under the supervision of Professor Ed Corrigan. The work presented in this thesis has not been submitted in fulfilment of any other degree or professional qualification. Chapter one, chapter three on the construction of the Green's function and the computation of the reflection factor for the special boundary condition and the first half part of chapter four are devoted to reviewing the necessary background material, and no claims are made to any originality. Chapters two, the remaining part in chapter three, the second half part of the chapter four and five are based on work with my supervisor, which has not yet been published.

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To my husband Vidat and to little Vilmir, I owe more than I can ever express.

To my parents
with love and gratitude

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Chapter 1

Introduction to Affine Toda Theory

1.1 Introduction

Affine Toda field theory is a theory of massive scalar fields characterised by Lie algebras g of rank r (for a review see [1]). The classical field theory is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(\phi), \tag{1.1}$$

where $\phi^a(x,t)$, $a = 1, ..., r$ are real scalar fields in two-dimensional Minkowski space-time. The potential V distinguishes between the different theories by its relation to different Lie algebras and takes the form

$$V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}. \tag{1.2}$$

where m is the mass parameter, β is dimensionless real coupling constant. The vector α_1,...,α_r are a set of simple roots for the Lie algebra g, meaning that they are linearly independent and any other root may be expressed as linear combination of them. In particular, the special root α_0 is a linear combination of the simple roots

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i \tag{1.3}$$

corresponds to the extra spot on an extended Dynkin diagram for g. The integer n_0 is always chosen to be one. The sum of the integers $h = \sum_{i=0}^{r} n_i$ is called the Coxeter number. This number is characteristic for each type of theory, for example see the table below [2].

g	$a_n^{(1)}$	$b_n^{(1)}$	$c_n^{(1)}$	$d_n^{(1)}$	$e_8^{(1)}$	$f_4^{(1)}$	$g_2^{(1)}$	$a_{2n}^{(2)}$	$a_{2n-1}^{(2)}$	$d_{n+1}^{(2)}$	$e_6^{(2)}$	$d_4^{(3)}$
h	$r+1$	$2r$	$2r$	$2r-2$	30	12	6	$2r+1$	$2r-1$	$r+1$	9	6

Table 1.1: The correspondence between Lie algebra and the Coxeter number

The field equation of motion for affine Toda field theory can be obtained from the Lagrangian (1.1)

$$\partial^2 \phi = -\frac{m^2}{\beta} \sum_{i=1}^{r} n_i \alpha_i e^{\beta \alpha_i \cdot \phi}. \tag{1.4}$$

Through the thesis we will refer the Lagrangian density as the Lagrangian. In principle we know that the Lagrangian is the spatial integral of a Lagrangian density.

The affine Toda theories fall into two classes under the transformation of the root

$$\alpha_i \rightarrow \frac{2\alpha_i}{|\alpha_i|^2}. \tag{1.5}$$

The first class is called the self dual, in which the theories are mapped onto themselves by this transformation. The self dual set are $a_n^{(1)}$, $d_n^{(1)}$, $e_n^{(1)}$, which have roots of equal length and $a_{2n}^{(2)}$ which has roots of three different lengths. The other class contains dual pairs in which the theories are mapped into each other. They are $\left(b_n^{(1)}, a_{2n-1}^{(2)}\right)$, $\left(c_n^{(1)}, d_{n+1}^{(2)}\right)$, $\left(g_2^{(1)}, d_4^{(3)}\right)$ and $\left(f_4^{(1)}, e_6^{(2)}\right)$. The Dynkin diagrams for the self dual and dual pairs can be found in [3].

The mass matrix and couplings of the theory can be obtained by perturbative expansion of the potential (1.2)

$$V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i + \frac{m^2}{2} \sum_{i=0}^{r} n_i \ \alpha_i^a \ \alpha_i^b \ \phi^a \phi^b + \frac{m^2 \beta}{6} \sum_{i=0}^{r} n_i \ \alpha_i^a \ \alpha_i^b \ \alpha_i^c \ \phi^a \ \phi^b \ \phi^c + \ldots, \tag{1.6}$$

where the linear term in ϕ vanishes due to (1.3). The second and third terms display the mass matrix and three-point coupling

$$\left(M^2\right)^{ab} = m^2 \sum_{i=0}^{r} n_i \, \alpha_i^a \, \alpha_i^b,$$
$$C^{abc} = m^2 \beta \sum_{i=0}^{r} n_i \, \alpha_i^a \, \alpha_i^b \, \alpha_i^c, \tag{1.7}$$

A complete calculation of the mass spectrum and the coupling constant for all affine Toda theories was presented in [4].

1.2 The classical Affine Toda theory

Affine Toda theory is an integrable field theory, in the sense that there are infinitely many conserved charges in involution. There are two ways to establish the classical integrability of the theory. One is to consider the densities which integrate to yield the conserved charges labelled by their spins. More specifically, consider the density T in terms of light-cone coordinates and for spin s it satisfies

$$\partial_{\mp} T_{\pm(s-1)} = \partial_{\pm} \Theta_{\pm(s-1)} \tag{1.8}$$

for some $\Theta_{\pm(s-1)}$. The conserved charges is then obtained by the integral of motion

$$Q_s + Q_{-s} = \int_{-\infty}^{\infty} dx \left(T_{\pm(s-1)} - \Theta_{\pm(s-1)}\right). \tag{1.9}$$

Another way is to examine the Lax pair or the zero curvature condition. The basic idea of a Lax pair requires to introduce a two components gauge field , those zero curvature vanishes if and only the affine Toda equation is satisfied.

The zero curvature condition is stated as

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0. \tag{1.10}$$

Here the two components of the two dimensional vector potential A_μ are given by

$$A_0 = \frac{1}{2} H \cdot \partial_0 \phi + \sum_0^r m_i \left(\lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i}\right) e^{\alpha_i \cdot \phi/2},$$
$$A_1 = \frac{1}{2} H \cdot \partial_1 \phi + \sum_0^r m_i \left(\lambda E_{\alpha_i} + \frac{1}{\lambda} E_{-\alpha_i}\right) e^{\alpha_i \cdot \phi/2}, \tag{1.11}$$

where the notations are chosen as $\partial_0 = \partial_t$ and $\partial_1 = \partial_x$, λ is a parameter, H_i are the generators of the Cartan subalgebra and E_{α_i}, $E_{-\alpha_i}$ are step operators corresponding to the simple roots. They satisfy

$$\begin{aligned} [H, E_{\alpha_i}] &= \alpha_i E_{\alpha_i} \\ [E_{\alpha_i}, E_{-\alpha_i}] &= \delta_{ij} \frac{2\alpha_j \cdot H}{|\alpha_j|^2}. \end{aligned} \tag{1.12}$$

The zero curvature condition (1.10) leads to the affine Toda field equation (1.4), once the coefficients m_i are chosen to satisfy

$$m_i^2 = \frac{1}{8} n_i \alpha_i^2. \tag{1.13}$$

While showing the equivalence between the zero curvature condition and the equation of motion, the parameters m and β in field equation have been scaled away for convenience, since they are classically unimportant. The conserved quantities can then be constructed by using the path-ordered exponential of the gauge potential A_1 [5–7].

1.3 The quantum Affine Toda theory

In the quantum field theory, the matrix of the transition amplitudes

$$S_{fi} = {}_{out} < f \mid i >_{in}$$

is known as the S-matrix, which describes a physical scattering process, where $\mid i >$ and $\mid f >$ denote the initial and final state respectively.

In general, the S-matrix of the two-dimensional theory is a very complicated object. However for affine Toda field theories the matter can be simplified considerably. Affine Toda field theory describes a set of distinguishable particles, each particle being distinguished by conserved charges of non-zero spin. We choose to write the two-momentum $p_a = (\omega_a, k_a)$ of a particle a in terms of its rapidity θ_a,

$$\omega_a = m_a \cosh\theta_a, \qquad k_a = m_a \sinh\theta_a, \tag{1.14}$$

where ω_a and k_a are the energy and momentum of the particle a. The single-particle states $| \, p_a >$ are eigenstates of the conserved charges Q_s. Lorentz invariance requires the action of the charges on the eigenstates to be

$$Q_s | \, p_a >= q_s^a e^{s\theta} | \, p_a > . \tag{1.15}$$

Zamolodchikov and Zamolodchikov [8] showed that a multiparticle S-matrix factorises into the two-particle S-matrix, which means the multiparticle S-matrix elements can be regarded as products of a number of two-particle S-matrix elements. This based on two selection rules:

1. The number of the particles remains unchanged after interaction, this implies that there is no particle production.

2. The initial and final momenta are individually the same.

Lorentz invariance requires that the S-matrix elements depend only on the rapidity difference of the particles. A two-particle state consisting of two incoming particles eventually evolves into a state containing the two outgoing particles. The S-matrix elements of the two particle states $p_a(\theta_a)$ and $p_b(\theta_b)$ are described by

$$|p_a(\theta_a) \, p_b(\theta_b) >_{in}= S_{ab} \, (\theta_{ab}) \, |p_a(\theta_a) \; _b(\theta_b >_{out}, \tag{1.16}$$

where the element $S_{ab}(\theta_{ab})$ is interpreted as the two-particle scattering amplitude for $p_a(\theta_a) \, p_b(\theta_b) \to p_a(\theta_a) \, p_b(\theta_b)$, in which the rapidity difference of the two particles is written as $\theta_{ab} = \theta_a - \theta_b$. The S-matrix element is shown pictorially in Figure 1.1.

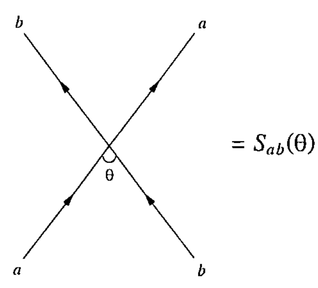

Figure 1.1: Two particle S-matrix element.

The S-matrix is required to satisfy the unitarity, crossing symmetry, Yang-Baxter equation and bootstrap conditions.

1. Unitarity condition: It can be understood as saying that for any two particle process the probability to go to any final state is equal to one.

$$S_{ab}(\theta_{ab})\ S_{ab}(-\theta_{ab}) = 1. \tag{1.17}$$

This is shown in Figure 1.2.

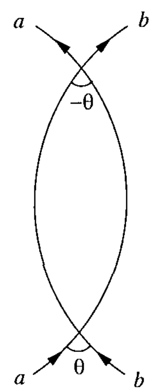

Figure 1.2: The unitarity condition.

2. Crossing symmetry:

For a two particle process there are two channels corresponding to the Mandelstam variables s and t to describe the scattering. The s-channel describes the scattering process $p_a\ p_b \rightarrow p_a\ p_b$, whereas the t-channel describes $p_a\ p_{\bar{b}} \rightarrow p_a\ p_{\bar{b}}$. The crossing condition corresponds to saying that the S-matrix is invariant under a change from the s to t channel, and is shown in Figure1.3.

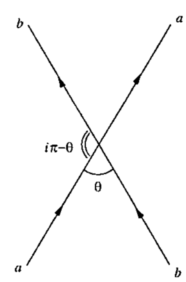

Figure 1.3: The crossing symmetry.

$$S_{ab}(\theta_{ab}) = S_{a\bar{b}}(i\pi - \theta_{ab}) \tag{1.18}$$

These two conditions imply that the S-matrix is a $2\pi i$ periodic function in the rapidity θ_{ab}.

3. Yang-Baxter equation:

In general, the Yang-Baxter equation implies that the two possible orderings for factorisation of the three-particle S-matrix are equivalent. It is shown pictorially in Figure 1.4.

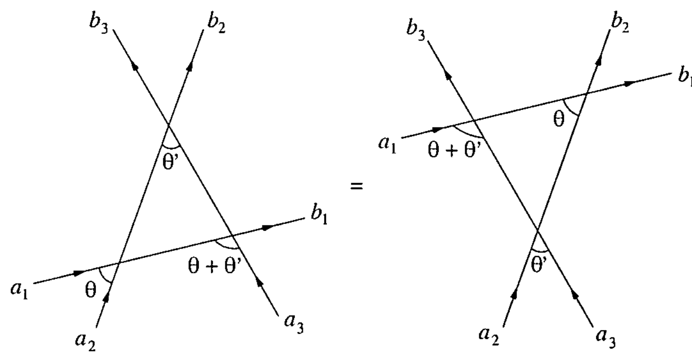

Figure 1.4: The Yang-Baxter equation.

$$S_{a_1a_2}^{c_1c_2}(\theta)\ S_{c_1a_3}^{b_1c_3}(\theta+\theta')\ S_{c_1c_3}^{b_2b_3}(\theta') = S_{a_2a_3}^{c_2c_3}(\theta')\ S_{a_1c_3}^{c_1b_3}(\theta+\theta')\ S_{c_1c_2}^{b_1b_2}(\theta') \qquad (1.19)$$

However, in the case of the affine Toda field theory, it is equivalent to the trivial identity

$$S_{a_1a_2}(\theta)\ S_{a_1a_3}(\theta+\theta')\ S_{a_2a_3}(\theta') = S_{a_2a_3}(\theta')\ S_{a_1a_3}(\theta+\theta')\ S_{a_1a_2}(\theta'). \qquad (1.20)$$

This is because of the fact that the S-matrices are diagonal.

4. Bootstrap equation:

The S-matrix may have bound state poles at purely imaginary values of θ_{ab} in the range $0 < Im\theta < \pi$, which has been given the name 'the physical strip'. Two particles a, b can fuse to a third particle c, when their S-matrix contains a pole at θ_{ab} . The pole occurs when the particle c is on shell. Its momentum is given by $p_c^2 = (p_a + p_b)^2$. This implies

$$m_c^2 = m_a^2 + m_b^2 + 2m_am_b\cos\theta_{ab}^c, \qquad (1.21)$$

where $\theta_{ab} \equiv i\theta^c_{ab}$ is known as the fusing angle for the fusing process $ab \to c$. Through the equation (1.21), it can be seen that the fusing angle has a geometrical interpretation as outside angle of a mass triangle of sides m_a, m_b and m_c. This has been shown in Figure 1.5.

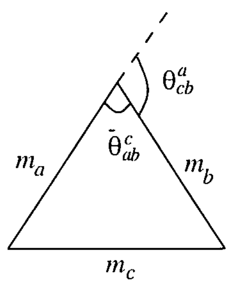

Figure 1.5: The mass triangle.

From the analysis it follows that

$$\theta^c_{ab} + \theta^b_{ac} + \theta^a_{bc} = 2\pi. \qquad (1.22)$$

When a bound state exists, the S-matrix satisfies the bootstrap equation

$$S_{dc}(\theta_{dc}) = S_{da}(\theta_{dc} + i\bar{\theta}^b_{ac})\, S_{db}(\theta_{dc} - i\bar{\theta}^a_{bc}). \qquad (1.23)$$

where $\bar{\theta} = \pi - \theta$. This implies that there are two ways in which a fourth particle d can scatter with these three particles. It is shown pictorially in Figure 1.6.

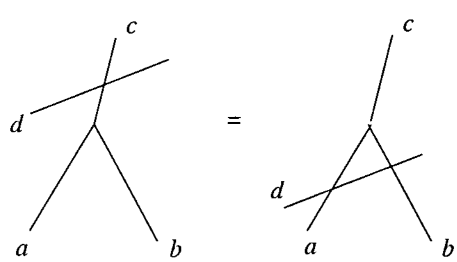

Figure 1.6: The bootstrap relation.

1.4 Affine Toda theory on the half line

Studies of the particle scattering of affine Toda theories have been formulated for those defined on the full line [9, 4, 10–14]. These theories are often called bulk theories. Since many physical systems are finite in their spatial dimension, it is interesting to study theories which are defined on a finite line or on a half line. Such a theory should take boundary effects into account.

In the past few years, there has been considerable progress in understanding affine Toda field theory on a half line. If the affine Toda field theory is restricted to a half line, then there must be a boundary condition at the origin. In this case, the Lagrangian (1.1) might be modified as

$$\mathcal{L} = \theta(-x)\left[\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - V(\phi)\right] - \delta(x)\mathcal{B}, \tag{1.24}$$

where $\mathcal{B}$ is a boundary term and the step function $\theta\left(-x\right)$ may be defined as

$$\theta\left(-x\right) = \begin{cases} 0 & x \geq 0, \\ 1 & x < 0. \end{cases} \tag{1.25}$$

By analysing low spin conserved quantities, integrable boundary conditions for $a_n^{(1)}$, $d_n^{(1)}$ series have been conjectured in [15,16]. It was concluded that the boundary potential has the generic form

$$\mathcal{B} = \frac{m}{\beta^2}\sum_{i=0}^{r} A_i e^{\frac{\beta}{2}\alpha_i\cdot\phi}, \tag{1.26}$$

where it was assumed that the boundary term depends only on fields but not their derivatives. Except for $a_1^{(1)}$, the coefficients A_i are constrained, namely either every coefficient vanishes for Neumann condition or is equal to $2\sqrt{n_i}$. For case $a_1^{(1)}$, the two coefficients A_0 and A_1 are arbitrary.

The conjecture (1.26) was subsequently proved [17] by a Lax pair representation of the boundary problem. It was also shown that a more general boundary condition that includes time derivatives leads to even stricter condition on the boundary potential [18].

The equation of motion (1.4) is now restricted to the region $x < 0$, and it is supplemented by a boundary condition at $x = 0$

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \mathcal{B}}{\partial \phi}. \tag{1.27}$$

Adding a boundary to the theory removes the translation invariance, and therefore the momentum is not longer conserved. However, the energy is given by

$$\mathcal{E} = \int_{-\infty}^{0} dx \left[\frac{1}{2}\left(\partial_t \phi\right)^2 + \frac{1}{2}\left(\partial_x \phi\right)^2 + V(x)\right] + \mathcal{B}, \tag{1.28}$$

and it is always conserved for any choice of $\mathcal{B}$.

Let us now to understand how particles scatter off that boundary. In integrable field theory one expects that the scattering is one to one. An initial state containing a single particle moving towards the boundary will be evolve into a final state with a single particle moving away from the boundary. In a two-particle state, each of the particles will not only scatter from the boundary, but also inevitably from each other. However the order of the individual scattering and reflection should not matter because they depend on the initial condition setting up the two-particle state [8].

For affine Toda field theory, as we have mentioned before, the particles are all distinguishable, therefore there should be a set of reflection factors for one for each particle, for each integrable boundary condition. The rapidity of the particle is reversed on reflection. Thus for the single particle a we can write down

$$\mid p_a(\theta) >_{in} = R_a(\theta) \mid p_a(-\theta) >_{out}, \tag{1.29}$$

where $R_a(\theta)$ represents the reflection factor of the particle on the boundary. This can be seen in Figure 1.7.

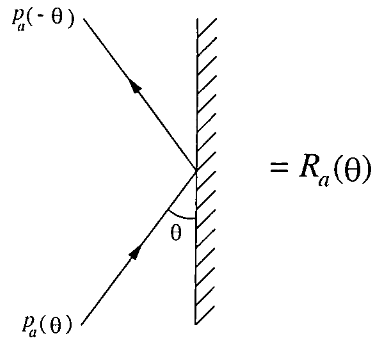

Figure 1.7: Reflection matrix.

In an analogous way as for the bulk theories, one now has the conditions for the boundary S-matrix.

1. Boundary unitarity: This condition follows that the probability of the reflection is equal to one.

$$R_a(\theta)\ R_a(-\theta) = 1. \tag{1.30}$$

It is pictorially shown by the following diagram

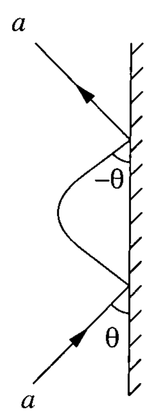

Figure 1.8: The boundary unitarity.

2. The boundary Yang-Baxter equation: Cherednik [19] found the generalisation of the Yang-Baxter equation. This says that the particles scatter factorisably, independent of the order of factorisations on the boundary. For the affine Toda field

theory, it can be expressed

$$S_{ab}(\theta_b - \theta_a)\ R_b(\theta_b)\ S_{ab}(\theta_b + \theta_a)\ R_a(\theta_a) \\ = R_a(\theta_a)\ S_{ab}(\theta_b + \theta_a)\ R_b(\theta_b)\ S_{ab}(\theta_b - \theta_a). \quad (1.31)$$

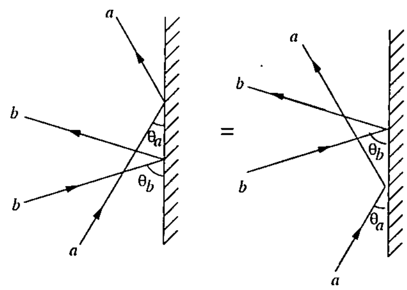

Figure 1.9: The boundary Yang-Baxter relation.

3. Boundary bootstrap equation: the reflection bootstrap equation is found by Fring and Köberle [20, 21]

$$R_c(\theta_c) = R_a(\theta_a) S_{ab}(\theta_a + \theta_b)\ R_b(\theta_b) \quad (1.32)$$

where $\theta_a = \theta_c + i\bar{\theta}_{ac}^b$ and $\theta_b = \theta_c - i\bar{\theta}_{bc}^a$, in which $\bar{\theta} = \pi - \theta$. The boundary bootstrap

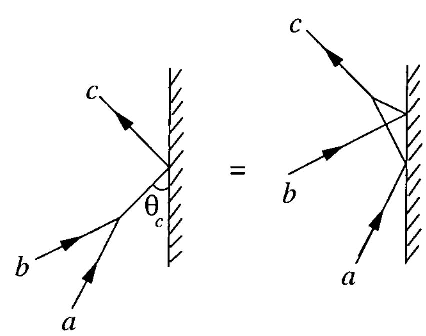

Figure 1.10: The boundary bootstrap relation.

implies the relations between the various reflection factors.

4. Boundary crossing unitarity: This condition is found by Ghoshal and Zamolodchikov [22]. It can be built as the consistency condition of the reflection bootstrap

equation [23]. Consider the particle a and its antiparticle $\bar{a}$ moving toward to each other with the rapidity difference $i\pi$. In this case, the quantum state of the two particles is same as the quantum state of the vacuum. Correspondingly the boundary bootstrap equation (1.32) can be written as

$$1 = R_a(\theta - \frac{i\pi}{2})\; S_{a\bar{a}}(2\theta)\; R_{\bar{a}}(\theta + \frac{i\pi}{2}). \tag{1.33}$$

Using the crossing symmetry and the unitarity conditions for the S-matrix, the consequence of the boundary bootstrap equation gives the boundary crossing equation

$$S_{aa}(2\theta) = R_a(\theta)\; R_{\bar{a}}(\theta + i\pi). \tag{1.34}$$

1.5 The $a_1^{(1)}$ or sinh-Gordon model

The sinh-Gordon model is the simplest model of the affine Toda field theory, based on the root data of the Lie algebra a_1. By setting $i = 1$ and scaling the mass parameter in the potential (1.2) by $m/2$, we can write down the Lagrangian for the sinh-Gordon theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi, \tag{1.35}$$

where m is the mass of the sinh-Gordon particle and β is a real coupling constant. In expressing the potential, we notice that there is only one simple root $\alpha_1 = \sqrt{2}$ corresponding to the sinh-Gordon model and its Coxeter number is $h = 2$ as we can see from the Table 1.1.

As we have mentioned in the previous section, the boundary term in affine Toda field theory takes the form (1.26). For the sinh-Gordon model it can be expressed as

$$\mathcal{B} = \frac{m}{\beta^2}\left(\sigma_1 e^{\beta\phi/\sqrt{2}} + \sigma_0 e^{-\beta\phi/\sqrt{2}}\right). \tag{1.36}$$

Here we redefine the boundary parameters as $\sigma_0 = A_0/2$ and $\sigma_1 = A_1/2$. If we choose the boundary to be fixed at $x = 0$, then the theory on a half line is described by the Lagrangian

$$\mathcal{L} = \theta(-x)\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi\right] - \delta(x)\mathcal{B}, \tag{1.37}$$

where $\mathcal{B}$ is given by (1.36).

From the Lagrangian (1.37), one can easily obtain the equation motion of the field

$$\partial^2\phi = -\frac{m^2}{2\sqrt{2}\beta}\left(e^{\sqrt{2}\beta\phi} - e^{-\sqrt{2}\beta\phi}\right) \qquad x < 0, \tag{1.38}$$

and the boundary condition

$$\partial_x\phi = -\frac{m}{\sqrt{2}\beta}\left(\sigma_1 e^{\beta\phi/\sqrt{2}} - \sigma_0 e^{-\beta\phi/\sqrt{2}}\right) \qquad x = 0. \tag{1.39}$$

The static equation of motion is described by

$$\partial_x^2\phi_0 = \frac{m^2}{\sqrt{2}\beta}\sinh\sqrt{2}\beta\phi_0, \tag{1.40}$$

and the the boundary condition is given by (1.39) in terms of ϕ_0.

Integrating the static equation of motion once, and comparing it with the boundary equation, one obtains

$$\begin{aligned}\partial_x\phi_0 &= \frac{m}{\sqrt{2}\beta}\left(e^{\beta\phi_0/\sqrt{2}} - e^{-\beta\phi_0/\sqrt{2}}\right),\\ e^{\sqrt{2}\beta\phi_0} &= \frac{1+\sigma_0}{1+\sigma_1}.\end{aligned} \tag{1.41}$$

Furthermore, the ground state solution is found to be

$$e^{\beta\phi_0/\sqrt{2}} = \frac{1+e^{m(x-x_0)}}{1-e^{m(x-x_0)}}, \tag{1.42}$$

where the integration constant x_0 can be determined by the boundary condition

$$\coth\frac{m}{2}x_0 = \sqrt{\frac{1+\sigma_0}{1+\sigma_1}}. \tag{1.43}$$

1.6 Classical reflection factor

The classical reflection factor for the sinh-Gordon theory can be obtained by linearising the field ϕ around the static background field ϕ_0. This means we replace ϕ

by $\phi_0 + \phi_1$, where ϕ_0 has the least energy (vacuum). The perturbation ϕ_1 has both kinetic and potential energy. By linearising the equation of motion and the boundary condition for the first order correction to the background field and using (1.43), one deduces that

$$\partial^2 \phi_1 + m^2 \left(1 + \frac{2}{\sinh^2 2(x - x_0)}\right) \phi_1 = 0 \qquad x < 0, \tag{1.44}$$

$$\partial_x \phi_1 + \frac{m}{2} \left(\sigma_1 \sqrt{\frac{1+\sigma_0}{1+\sigma_1}} + \sigma_0 \sqrt{\frac{1+\sigma_1}{1+\sigma_0}}\right) \phi_1 = 0 \qquad x = 0. \tag{1.45}$$

We now redefine $\phi_1 = \phi$ for convenience. The exact solution for these equations has been found in [15]. The eigenfunctions of the second order differential operator in (1.44) corresponding to the eigenvalue $\omega^2 - k^2 - m^2$ can be written as [24]

$$\phi(x,t) = ie^{-i\omega t} r(k) \left[F(k,x)e^{ikx} + F(-k,x)e^{-ikx}\right], \tag{1.46}$$

where $r(k)$ is a real, even function of k, which will be chosen when we consider the Green's function. We will discuss this in detail in chapter three. The function $F(k,x)$ is given by

$$F(k,x) = P(k)(ik - m \coth 2(x - x_0)) \tag{1.47}$$

together with $P(k)=(ik)^2 - 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1)$.

We now can determine the classical reflection factor. It can be calculated in the following way. We send a particle from minus infinity. It reflects off the boundary x=0, and is then caught in the same place from which it was sent. The ratio of the phase difference between the outgoing and ingoing particle is the reflection factor

$$K(\theta) = \frac{F(-k,-\infty)}{F(k,-\infty)}. \tag{1.48}$$

The classical reflection factor can be computed by using (1.47)

$$K = \frac{(-ik+m)\ \left((-ik)^2 + 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0+\sigma_1)\right)}{(ik+m)\ \left((-ik)^2 - 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0+\sigma_1)\right)}. \tag{1.49}$$

By setting k=$m \sinh\theta$ and using the parametrisation

$$\sigma_i = \cos a_i \pi, \qquad i = 0, 1, \tag{1.50}$$

in which a_i are the two constants, we deduce that K is

$$-\left[\frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)}\right]^2 \frac{\sinh\left[\frac{\theta}{2}-\frac{i\pi}{4}\left(1+a_0+a_1\right)\right]\,\sinh\left[\frac{\theta}{2}-\frac{i\pi}{4}\left(1-a_0-a_1\right)\right]}{\sinh\left[\frac{\theta}{2}+\frac{i\pi}{4}\left(1+a_0+a_1\right)\right]\,\sinh\left[\frac{\theta}{2}+\frac{i\pi}{4}\left(1-a_0-a_1\right)\right]} \frac{\sinh\left[\frac{\theta}{2}-\frac{i\pi}{4}\left(1+a_0-a_1\right)\right]\,\sinh\left[\frac{\theta}{2}-\frac{i\pi}{4}\left(1-a_0+a_1\right)\right]}{\sinh\left[\frac{\theta}{2}+\frac{i\pi}{4}\left(1+a_0-a_1\right)\right]\,\sinh\left[\frac{\theta}{2}+\frac{i\pi}{4}\left(1-a_0+a_1\right)\right]}. \tag{1.51}$$

In term of the block notation introduced in [4]

$$(x) = \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi x}{2h}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi x}{2h}\right)}, \tag{1.52}$$

the classical reflection factor can be written

$$K = -\frac{(1)^2}{(1+a_0+a_1)(1-a_0-a_1)(1+a_0-a_1)(1-a_0+a_1)}. \tag{1.53}$$

1.7 S-matrix

The S-matrix describing the elastic scattering of a pair of sinh-Gordon particles is known [9]. In terms of the block notation, it can be written as

$$S(\theta) = -\frac{1}{(B)\ (2-B)}. \tag{1.54}$$

Here the parameter B depends on the sinh-Gordon coupling constant, it has been conjectured to have the form

$$B = \frac{1}{2\pi}\,\frac{\beta^2}{1+\beta^2/4\pi}, \tag{1.55}$$

for which $B\left(4\pi/\beta\right) = 2 - B(\beta)$. This implies that S-matrix is invariant when $\beta \to 4\pi/\beta$ and this is referred to as the weak-strong duality [4].

1.8 Relation with the sine-Gordon model

The $a_1^{(1)}$ theory for imaginary coupling is known as the sine-Gordon model.

The sine-Gordon model in the presence of a reflecting boundary has been studied by Ghoshal and Zamolodchikov [22]. Based on an explicit computation of the first nontrivial integral of motion, they conjectured that the integrable boundary term has the form

$$\mathcal{B} = M \cos \frac{\beta_{sG}(\phi - \phi_0)}{2}, \tag{1.56}$$

where M and ϕ_0 are free parameters, and β_{sG} is the coupling constant for the sine-Gordon model. A proof of this conjecture has been given in [25, 26].

The coupling constant in the sine-Gordon theory is related to the sinh-Gordon coupling constant by $\beta_{sG} = i\sqrt{2}\beta$. We now want to find the relations between the free parameters σ_0 and σ_1 in the sinh-Gordon theory with M and ϕ_0 in the sine-Gordon model. By comparing (1.56) with the boundary action (1.36), one can obtain the following relationships for these two sets of parameters

$$\begin{aligned} \sigma_0 &= \frac{\beta^2 M}{2m} e^{\beta\phi_0/\sqrt{2}}, \\ \sigma_1 &= \frac{\beta^2 M}{2m} e^{-\beta\phi_0/\sqrt{2}}. \end{aligned} \tag{1.57}$$

1.9 Quantum reflection factor

Based on the work [22], Ghoshal [27] suggested a formula for the reflection matrix for the breather states of the sine-Gordon model. For the jth breather it can be expressed as

$$R^j_{sG}(\theta|\eta, \vartheta) = R^j_0(\theta) R^j_1(\theta), \tag{1.58}$$

where

$$\begin{aligned} R^j_0(\theta) = (-1)^{j+1} &\frac{\cosh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right) \cosh\left(\frac{\theta}{2} - \frac{i\pi}{4} - j\frac{i\pi}{4\lambda}\right) \sinh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right)}{\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right) \cosh\left(\frac{\theta}{2} + \frac{i\pi}{4} + j\frac{i\pi}{4\lambda}\right) \sinh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)} \\ &\times \prod_{l}^{j-1} \frac{\sinh\left(\theta + l\frac{i\pi}{2\lambda}\right) \cosh^2\left(\frac{\theta}{2} - \frac{i\pi}{4} - l\frac{i\pi}{4\lambda}\right)}{\sinh\left(\theta - l\frac{i\pi}{2\lambda}\right) \cosh^2\left(\frac{\theta}{2} + \frac{i\pi}{4} + l\frac{i\pi}{4\lambda}\right)}. \end{aligned} \tag{1.59}$$

The parameter λ in here is related to the coupling constant, and given by

$$\lambda = \frac{8\pi}{\beta^2_{sG}} - 1. \tag{1.60}$$

The other part $R^j_1(\theta)$ in the reflection factor contains the parameters η and ϑ and can be written as

$$R^j_1(\theta) = S^j(\eta, \theta)\ S^j(i\vartheta, \theta), \tag{1.61}$$

where the factor $S^j(\eta, \theta)$ depends upon whether j is odd or even. For $j = 2k$, $k = 1, 2, ... < \frac{\lambda}{2}$, we have

$$S^{2k}(x,\theta) = \prod_{l=1}^{k} \frac{\sinh\theta - i\cos\left[\frac{x}{\lambda} - \left(l - \frac{1}{2}\right)\frac{\pi}{\lambda}\right]}{\sinh\theta + i\cos\left[\frac{x}{\lambda} - \left(l - \frac{1}{2}\right)\frac{\pi}{\lambda}\right]} \frac{\sinh\theta - i\cos\left[\frac{x}{\lambda} - \left(l - \frac{1}{2}\right)\frac{\pi}{\lambda}\right]}{\sinh\theta + i\cos\left[\frac{x}{l} + \left(l - \frac{1}{2}\right)\frac{\pi}{\lambda}\right]}. \quad (1.62)$$

For $j = 2k - 1$, $k = 1, 2, ... < \frac{\lambda+1}{2}$, we have

$$S^{2k-1}(x,\theta) = \frac{i\cos\frac{x}{\lambda} - \sinh\theta}{i\cos\frac{x}{\lambda} + \sinh\theta} \prod_{l=1}^{k-1} \frac{\sinh\theta - i\cos\left[\left(\frac{x}{\lambda} - l\frac{\pi}{\lambda}\right)\right.}{\sinh\theta + i\cos\left(\frac{x}{\lambda} - l\frac{\pi}{\lambda}\right)} \frac{\sinh\theta - i\cos\left(\frac{x}{\lambda} - l\frac{\pi}{\lambda}\right)}{\sinh\theta + i\cos\left(\frac{x}{l} + l\frac{\pi}{\lambda}\right)}. \quad (1.63)$$

Once the reflection factor to the sine-Gordon theory is known, we can deduce the quantum reflection factor for the sinh-Gordon model from the lightest breather reflection factor in the sine-Gordon theory by analytic continuation in the coupling constant. The first thing is to write down Ghoshal's reflection factor for the lightest breather $j = 1$. In this case, the equation (1.59) becomes

$$R_0^1(\theta) = \frac{\sinh\left[\frac{\theta}{2} + \frac{i\pi}{4}\left(2 + \frac{1}{\lambda}\right)\right]}{\sinh\left[\frac{\theta}{2} - \frac{i\pi}{4}\left(2 + \frac{1}{\lambda}\right)\right]} \frac{\sinh\left[\frac{\theta}{2} + \frac{i\pi}{4}\left(1 - \frac{1}{\lambda}\right)\right]}{\sinh\left[\frac{\theta}{2} - \frac{i\pi}{4}\left(1 - \frac{1}{\lambda}\right)\right]} \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)}. \quad (1.64)$$

Using the block notation (1.52), this can further be written as

$$R_0^1(\theta) = \left(2 + \frac{1}{\lambda}\right)\left(1 - \frac{1}{\lambda}\right)(1). \quad (1.65)$$

Similarly, the equation (1.61) for $j = 1$ can also be written in terms of block notation as

$$R_1^1(\theta) = \left[\left(1 + \frac{2\eta}{\pi\lambda}\right)\left(1 - \frac{2\eta}{\pi\lambda}\right)\left(1 + \frac{2i\vartheta}{\pi\lambda}\right)\left(1 - \frac{2i\vartheta}{\pi\lambda}\right)\right]^{-1}. \quad (1.66)$$

We hence obtain the reflection factor for the lightest breather of the sine-Gordon model

$$R_{sG}(\theta|\eta,\vartheta) = \frac{\left(2 + \frac{1}{\lambda}\right)\left(1 - \frac{1}{\lambda}\right)(1)}{\left(1 + \frac{2\eta}{\pi\lambda}\right)\left(1 - \frac{2\eta}{\pi\lambda}\right)\left(1 + \frac{2i\vartheta}{\pi\lambda}\right)\left(1 - \frac{2i\vartheta}{\pi\lambda}\right)}. \quad (1.67)$$

Here for the simplicity we write $R_{sG}^1(\theta|\eta,\vartheta)$ as $R_{sG}(\theta|\eta,\vartheta)$. Once we write this factor in terms of B, we will obtain the reflection factor for the sinh-Gordon model.

The two functions λ and B are related to each other via the relation

$$-\frac{1}{\lambda} = \frac{B}{2}, \tag{1.68}$$

which implies the analytic continuation of the coupling. Placing this into (1.67) we finally deduce the quantum reflection factor for sinh-Gordon model

$$R_{shG}(\theta) = \frac{(2-B/2)(1)(1+B/2)}{(1-E)(1+E)(1-F)(1+F)}, \tag{1.69}$$

where E and F are also the functions of the coupling constant and related to the parameters η and ϑ in Ghoshal's reflection factor by

$$E = \frac{\eta}{\pi}B, \qquad F = \frac{i\vartheta}{\pi}B. \tag{1.70}$$

In the limit $\beta \to 0$, the quantum reflection factor (1.69) reduces to the classical one (1.53), where $E(0) = a_0 + a_1$ and $F(0) = a_0 - a_1$.

1.10 The layout of the thesis

In this thesis, we are particularly interested in studying the sinh-Gordon theory. The sinh-Gordon model is studied from three different aspects. The first is to study the model within two boundaries. The second studies the theory with a general one boundary condition. The last studies the supersymmetric extension of the model.

The remaining chapters of the thesis are organised as follows:

In the second chapter, we discuss the sinh-Gordon model on an interval. The field equation of the model is not linear and can therefore not be solved exactly. Thus we must firstly consider a simpler theory. As we know, the free scalar field theory obeys a linear equation. It is therefore exactly solvable. Once we obtain this solution, it will enable us to study the reflection factors and the energy levels. We will then investigate the sinh-Gordon model on an interval. Two different approaches shall be presented to calculate the classical reflection factors and the spectrum of particle energies.

In chapter three we will study the one boundary problem of the sinh-Gordon model. The boundary condition of the model has a more general form. We will develop the perturbation theory and the path integral method to deduce the Feynman diagrams at the one loop order. From the loop calculations the quantum reflection factor can be extracted. However a correction that comes from the boundary contribution of the bubble diagram has been calculated.

Chapters four and five deal with the supersymmetric extension of the sinh-Gordon model with a one boundary condition.

In chapter four, we will perturbatively check the classical limits of the exact reflection factors proposed by Moriconi and Schoutens for the lightest breather multiplets of the sine-Gordon theory. We find that their supersymmetric reflection factors do not have the correct classical limits. A correction has been made to their result. By doing that, we obtain the supersymmetric reflection factors which have the correct classical limits.

In chapter five, we firstly construct the supersymmetric Lagrangian for the sinh-Gordon model and the fermionic propagator in the presence of the boundary. Using the perturbation theory and the path integral formalism, we derive the one-loop Feynman diagrams for the supersymmetric model. We then carry on the calculation of the loop diagrams. From the results we extract the boson and fermion reflection factors of the sinh-Gordon model. The results obtained in this chapter will be compared with the results obtained in the previous chapter. We also discuss the renormalisation of the theory.

The final chapter draws several conclusions from the work and speculates on further possible investigations.

Chapter 2

Sinh-Gordon Model on an Interval

2.1 Introduction

Recently there has been an attempt to investigate the affine Toda field theory within two boundary conditions [3, 28]. In this chapter, we will study the two boundary problem for the simplest model described by a single field in affine Toda theory.

The action of a single field in the presence of the two boundaries at $x = \pm L$ can be described by

$$S = \int_{-\infty}^{\infty} dt \int_{-L}^{L} dx\, \mathcal{L}_0 \;-\; \int_{-\infty}^{\infty} dt\, (\mathcal{B}_+ + \mathcal{B}_-)\,. \tag{2.1}$$

Here $\mathcal{L}_0$ is the Lagrangian in the bulk theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{2.2}$$

and $\mathcal{B}_\pm$ correspond respectively to the boundaries at each end, $x = \pm L$.

The corresponding classical equation of motion and boundary conditions can be written

$$\begin{aligned} \partial^2\phi &= -\frac{\partial V}{\partial\phi}, \qquad |x| < L, \\ \partial_x\phi &= \mp\frac{\partial \mathcal{B}_\pm}{\partial\phi}, \qquad x = \pm L. \end{aligned} \tag{2.3}$$

To begin with we will consider a free scalar field on an interval. As we have mentioned in the last section of the previous chapter, the solution of the free scalar theory can be obtained exactly. From this we will be able to find out which reflection factor corresponds to each boundary and how the energy of the theory is formulated on an interval. We then turn our attention to the sinh-Gordon model. The reflection factors and its energy will be investigated.

2.2 The free scalar field theory

In this section we shall study the free scalar field theory confined on the interval $[-L, L]$. As we know the free scalar theory is the theory of a single field and described by the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{2.4}$$

Let us consider the general quadratic boundary conditions

$$\mathcal{B}_+ = \frac{\lambda}{2}\phi^2, \qquad \mathcal{B}_- = \frac{\mu}{2}\phi^2. \tag{2.5}$$

where λ and μ are two boundary parameters.

For the free scalar field theory, one obtains the equation of motion

$$\left(\partial^2 + m^2\right)\phi = 0, \qquad |x| < L, \tag{2.6}$$

and the boundary equations

$$\begin{aligned} \partial_x\phi &= -\lambda\phi, \qquad & x &= L, \\ \partial_x\phi &= \mu\phi, \qquad & x &= -L. \end{aligned} \tag{2.7}$$

The ground state or the vacuum solution of the theory is zero with these boundaries.

The equation of motion has the plane-wave solution

$$\phi(x,t) = e^{-i\omega t}\left(Ae^{ikx} + Be^{-ikx}\right) + c.c. \tag{2.8}$$

where A and B are coefficients, $c.c.$ implies the complex conjugation of the first term. When this solution is substituted into the boundary equations, we obtain

$$\begin{aligned} (ik+\lambda)e^{-i\omega t+ikL}A - (ik-\lambda)e^{-i\omega t-ikL}B + c.c. &= 0, \\ (ik-\mu)e^{-i\omega t-ikL}A - (ik+\mu)e^{-i\omega t+ikL}B + c.c. &= 0, \end{aligned} \tag{2.9}$$

The set of the boundary equations has a solution, if and only if the determinant of the coefficients is equal to zero. In other word, the relation

$$K_\lambda(k)K_\mu(k)e^{4ikL} = 1 \tag{2.10}$$

is satisfied. Here, $K_\lambda(k)$ and $K_\mu(k)$ are the two independent reflection factors at each boundary, and defined by

$$K_\lambda(k) = \frac{ik+\lambda}{ik-\lambda}, \qquad K_\mu(k) = \frac{ik-\mu}{ik+\mu}. \tag{2.11}$$

The coefficients A and B relate to each other by the reflection factors

$$B = K_\lambda(k)e^{2ikL}A = K_\mu(-k)e^{-2ikL}A, \tag{2.12}$$

which follow from the boundary equations.

The equation of motion for the free scalar field is a linear equation, so the general solution is a superposition of plane wave solutions

$$\phi(x,t) = \sum_{n=1}^{\infty} \phi_n(x,t), \tag{2.13}$$

where $\phi_n(x,t)$ is the generalisation of the solution (2.8), namely

$$\phi_n(x,t) = e^{-i\omega_n t}\left(A_n e^{ik_n x} + B_n e^{-ik_n x}\right) + c.c. \tag{2.14}$$

Correspondingly, the coefficients A_n and B_n, the reflection factors $K_\lambda(k_n)$ and $K_\mu(k_n)$ are the modifications of A and B, $K_\lambda(k)$ and $K_\mu(k)$.

The solution (2.14) can be written as

$$\phi_n(x,t) = e^{-i\omega_n t}A_n\left[e^{ik_n x} + K_\lambda(k_n)e^{ik_n(2L-x)}\right] + c.c. \tag{2.15}$$

Furthermore, we can express it as

$$\phi_n(x,t) = e^{-i\omega_n t}A_n e^{ik_n\delta_n}\left[e^{ik_n(x-\delta)} + e^{-ik_n(x-\delta)}\right] + c.c. \tag{2.16}$$

by defining

$$K_\lambda(k_n) = e^{2ik_n(\delta_n - L)} \quad or \quad K_\mu(k_n) = e^{-2ik_n(\delta_n + L)} \tag{2.17}$$

In order to write down the solution (2.16) in a simple form we need to scale $2A_n e^{ik_n\delta_n} \equiv A_n$, then the general solution becomes

$$\phi(x,t) = \sum_{n=1}^{\infty} \left(A_n e^{-i\omega t} + A_n^* e^{i\omega t}\right) \phi_n, \tag{2.18}$$

where we have redefined $\phi_n = \cos k_n(x - \delta_n)$.

The possible momenta for a particle confined to the interval $[-L, L]$ are expected to be given by the solutions to (2.10). We take the logarithm of the equation first, by expanding the logarithmic function, we have

$$4ik_nL - 2i\pi n = -\frac{2(\lambda - \mu)}{ik_n} - \frac{2}{3}\frac{(\lambda^3 - \mu^3)}{(ik_n)^3} - \frac{2}{5}\frac{(\lambda^5 - \mu^5)}{(ik_n)^5} \cdots. \tag{2.19}$$

Suppose the momentum is quantised in the following way $k_n = \frac{n\pi}{2L} + \epsilon_n$, where ϵ_n is a small correction to the momentum. Equation (2.19) then reduces to

$$\epsilon_n = \frac{\lambda - \mu}{n\pi + 2L\epsilon_n} - \frac{4L^2(\lambda^3 - \mu^3)}{3(n\pi - 2L\epsilon_n)^3} + \frac{16L^4(\lambda^5 - \mu^5)}{5(n\pi + 2L\epsilon_n)^5} + \cdots. \tag{2.20}$$

By expanding the right hand of the expression up to the order of ϵ_n, we obtain the correction of the momentum

$$\begin{aligned} \epsilon_n = & \frac{\lambda - \mu}{n\pi} - \frac{2L}{(n\pi)^3}\left[\frac{2L(\lambda^3 - \mu^3)}{3} + (\lambda - \mu)^2\right] \\ & + \frac{4L^2}{(n\pi)^5}\left[\frac{4L^2}{5}(\lambda^5 - \mu^5) + \frac{6L + 2}{3}(\lambda - \mu)(\lambda^3 - \mu^3) - (\lambda - \mu)^3\right] + \ldots \end{aligned} \tag{2.21}$$

We thus have the approximation to the momentum

$$\begin{aligned} k_n = & \frac{n\pi}{2L} + \frac{\lambda - \mu}{n\pi} - \frac{2L}{(n\pi)^3}\left[\frac{2L(\lambda^3 - \mu^3)}{3} + (\lambda - \mu)^2\right] \\ & + \frac{4L^2}{(n\pi)^5}\left[\frac{4L^2}{5}(\lambda^5 - \mu^5) + \frac{6L + 2}{3}(\lambda - \mu)(\lambda^3 - \mu^3) - (\lambda - \mu)^3\right] + \ldots \end{aligned} \tag{2.22}$$

If the two boundary parameters are equal, there is no correction. In this case, the momentum does not depend on the boundary parameters.

Including the boundary contributions, the energy is given by

$$E = \int_{-L}^{L} dx \left(\dot{\phi}^2 - \mathcal{L}_0 \right) + \mathcal{B}_+ + \mathcal{B}_- . \tag{2.23}$$

We shall now compute the energy for the free particle in the interval $[-L, L]$,

$$E = \int_{-L}^{L} dx \left(\dot{\phi}^2 - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \right) + \frac{1}{2}\lambda \phi^2|_L + \frac{1}{2}\mu \phi^2|_{-L}. \tag{2.24}$$

The energy is independent of time, so we can calculate it at $t = 0$. Using the solution (2.18), the energy can be expressed

$$\begin{aligned} E = \frac{1}{2}\sum_{m,n} \Bigg\{ \int_{-L}^{L} dx \Big[& - \omega_m \omega_n \left(A_m - A_m^*\right)\left(A_n - A_n^*\right) \phi_m \phi_n \\ & + \left(A_m + A_m^*\right)\left(A_n + A_n^*\right)\left(\phi_m' \phi_n' + m^2 \phi_m \phi_n\right) \Big] \\ & + \left(A_m + A_m^*\right)\left(A_n + A_n^*\right) \left[\lambda \phi_m \phi_n \Big|_L + \mu \phi_m \phi_n \Big|_{-L} \right] \Bigg\}. \end{aligned} \tag{2.25}$$

We need to analyse the equal modes ($m = n$) and non equal modes ($m \neq n$) separately in the above equation. In fact the contribution of non equal modes cancels each other. Furthermore it leads to

$$E = \sum_{n=1}^{\infty} \omega_n a_n a_n^*, \tag{2.26}$$

where

$$a_n = A_n \sqrt{2\omega_n} \sqrt{L + \frac{1}{2}\left(\frac{\lambda}{k_n^2 + \lambda^2} - \frac{\mu}{k_n^2 + \mu^2} \right)} \equiv A_n \sqrt{2\omega_n \Omega_n}, \tag{2.27}$$

with

$$\Omega_n = \sqrt{L + \frac{1}{2}\left(\frac{\lambda}{k_n^2 + \lambda^2} - \frac{\mu}{k_n^2 + \mu^2} \right)} \tag{2.28}$$

As expected, the energy has a standard form as a free field theory in the bulk theory. When the two boundary parameters λ and μ are equal, the energy does not depend on the boundary parameters, but the size of the interval

$$E = \sum_{n=1}^{\infty} 2L\omega_n^2 A_n A_n^*. \tag{2.29}$$

The energy ω_n and momentum k_n satisfy the mass-shell condition

$$\omega_n^2 - k_n^2 = m^2.$$

When we quantise the field, the coefficients a_n and a_n^* become operators. They correspond to the annihilation and creation operators respectively and their commutation relations are given by

$$[a_n,\ a_m^*] = \delta_{nm}, \quad [a_n,\ a_m] = [a_n^*,\ a_m^*] = 0. \tag{2.30}$$

The field $\phi(x,t)$ in (2.18) can now be written in terms of these operators as

$$\phi(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left(a_n e^{-i\omega_n t} + a_n^* e^{i\omega_n t}\right) \phi_n, \tag{2.31}$$

where we have redefined $\phi_n \equiv \phi_n/\sqrt{\Omega_n}$.

Using (2.30) the commutator of $\phi(x,t)$ and its time derivative $\dot{\phi}(x,t)$ can be computed. The calculations lead to

$$\left[\phi(x,t),\ \dot{\phi}(x,t)\right] = i\sum_n \phi_n(x,t)\ \phi_n(y,t). \tag{2.32}$$

Here $\{\phi_n\}$ are the complete set of the functions for fields satisfying the boundary conditions. Therefore we have

$$\left[\phi(x,t),\ \dot{\phi}(x,t)\right] = i\delta(x-y), \qquad -L < x,y < L. \tag{2.33}$$

We can see that the quantised field on an interval satisfies the equal time commutation relation as the bulk theory.

Let us now consider the zero mode. This corresponds $k=0$ and $\omega = m$. In this case, the equation of motion (2.6) becomes

$$\left(\partial_t^2 + m^2\right)\phi = 0. \tag{2.34}$$

The solution to the equation is

$$\phi = (Ax+B)\, e^{-i\omega t} + c.c. \tag{2.35}$$

The boundary equations become

$$\begin{aligned} Ae^{-i\omega t} + c.c &= -\lambda\,(AL+B)\, e^{-i\omega t} + c.c, \qquad && x = L, \\ Ae^{-i\omega t} + c.c &= \mu\,(-AL+B)\, e^{-i\omega t} + c.c. && x = -L. \end{aligned} \tag{2.36}$$

The set of these equations provides the relation

$$2L = -\frac{1}{\lambda} - \frac{1}{\mu}, \tag{2.37}$$

otherwise the coefficients A and B are zero, there is no zero mode. We make the parametrisation

$$\lambda = -\frac{1}{L-a}, \qquad \mu = -\frac{1}{L+a}, \tag{2.38}$$

such that two coefficients A and B related by $B = -aA$, where a is a constant. The solution (2.35) can then be written

$$\phi = \phi_0 \left(A_0 \, e^{-i\omega_0 t} + A_0^* \, e^{i\omega_0 t} \right) \tag{2.39}$$

where we have redefined $\phi_0 = x - a$, $A = A_0$ and $\omega = \omega_0$.

Let us denote the ground state energy by E_0

$$E_0 = \int_{-L}^{L} dx \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{1}{2}m^2\phi^2 \right) + \frac{\lambda}{2}\phi^2\Big|_L + \frac{\mu}{2}\phi^2\Big|_{-L} . \tag{2.40}$$

From this we can calculate that

$$E = m a_0 a_0^*, \tag{2.41}$$

where

$$a_0 = 2\sqrt{\frac{1}{3} m L (L^2 + 3a^2) A}. \tag{2.42}$$

In general, the energy level of the free particle can be expressed as

$$E = \sum_{n=0}^{\infty} \omega_n a_n a_n^* \tag{2.43}$$

where $\omega_0 = m$ and $\omega_i = k_i^2 + m^2$ for $i = 1, 2.....$; a_0 and a_i are given by (2.42) and (2.27) respectively. We conclude that the energy of the free particle restricted within two boundary conditions depends on the boundary parameters as well as the size of the interval.

Let us now consider the boundaries different from (2.5). We may choose

$$\begin{aligned} \mathcal{B}_+ &= \frac{\lambda}{2}\phi^2 + \lambda_1\phi + \lambda_2, \\ \mathcal{B}_- &= \frac{\mu}{2}\phi^2 + \mu_1\phi + \mu_2, \end{aligned} \tag{2.44}$$

where λ, λ_1, λ_2 and μ, μ_1 μ_2 are the boundary parameters corresponding to each boundary, respectively. This set of the boundaries generally breaks the reflection symmetry $\phi \to -\phi$, and the boundary equations can be written

$$\begin{aligned} \partial_x\phi &= -\lambda\phi - \lambda_1, \qquad x = L, \\ \partial_x\phi &= \mu\phi + \mu_1, \qquad x = -L. \end{aligned} \tag{2.45}$$

In this case the vacuum solution is not zero. We do the perturbation around the static background ϕ_0 or in other words around the vacuum solution. This means we replace ϕ by $\phi_0 + \phi_1$, where ϕ_1 is the perturbation of the background field. By perturbing the field, we found that ϕ_1 satisfies the same equation of motion and boundary conditions as ϕ does in (2.6) and (2.7). For the static background field ϕ_0 we have

$$\begin{aligned} &\left(\partial_x^2 - m^2\right)\phi_0 = 0, \\ &\partial_x\phi_0 = -\lambda\phi_0 - \lambda_1, \qquad x = -L, \\ &\partial_x\phi_0 = \mu\phi_0 + \mu_1 \qquad x = -L. \end{aligned} \tag{2.46}$$

The solution of ϕ_0 can be directly obtained by solving the first equation in (2.46). It is

$$\phi_0(x) = \alpha_1 e^{mx} + \alpha_2 e^{-mx}. \tag{2.47}$$

The coefficients α_1 and α_2 are fixed by the boundary equations and we find

$$\begin{aligned} \alpha_1 &= -\frac{\lambda_1(m+\mu)e^{mL} + \mu_1(m-\lambda)e^{-mL}}{(m+\lambda)(m+\mu)e^{2mL} - (m-\lambda)(m-\mu)e^{-2mL}}, \\ \alpha_2 &= -\frac{\mu_1(m+\lambda)e^{mL} + \lambda_1(m-\mu)e^{-mL}}{(m+\lambda)(m+\mu)e^{2mL} - (m-\lambda)(m-\mu)e^{-2mL}}. \end{aligned} \tag{2.48}$$

The general solution that corresponds to the boundaries in (2.44) is the combination of the solutions ϕ_0 and ϕ_1. This can be expressed as

$$\phi(x,t) = \alpha_1 e^{mx} + \alpha_2 e^{-mx} + \sum_{n=1}^{\infty}\left(A_n e^{-i\omega t} + A_n^* e^{i\omega t}\right)\phi_n. \tag{2.49}$$

We now calculate the energy. It can be written as

$$\begin{aligned}
E = & \int_{-L}^{L} dx \Big(\frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\phi_1'^2 + \frac{1}{2}m^2\phi_1^2 \Big) + \frac{\lambda}{2}\phi_1^2 \Big|_{L} + \frac{\mu}{2}\phi_1^2 \Big|_{-L} \\
& + \frac{1}{2}\int_{-L}^{L} dx \Big(\frac{1}{2}\phi_0'^2 + +2\phi_0'\phi_1' + m^2\phi_0^2 + 2m^2\phi_0\phi_1 \Big) \\
& + \Big[\frac{\lambda}{2}(\phi_0^2 + 2\phi_0\phi_1) + \lambda_1(\phi_0 + \phi_1) + \lambda_2 \Big]_{L} \\
& + \Big[\frac{\mu}{2}(\phi_0^2 + 2\phi_0\phi_1) + \mu_1(\phi_0 + \phi_1) + \mu_2 \Big]_{-L}
\end{aligned} \tag{2.50}$$

We note that we already have the energy contribution corresponding to the first line in the expression. This is given by (2.26). The rest of the equation can then be calculated. The energy can therefore be written as

$$\begin{aligned}
E = & \sum_{n=1}^{\infty} \omega_n a_n a_n^* + \frac{1}{2}(\alpha_1^2 + \alpha_2^2)\left(e^{2mL} + e^{-2mL}\right) + \lambda_2 + \mu_2 \\
& + \sum_{n=1}^{\infty} mk_n\left(A_n + A_n^*\right)\left[\frac{\left(\alpha_1 e^{mL} - \alpha_2 e^{-mL}\right)}{\sqrt{k_n^2 + \lambda^2}} + \frac{\left(\alpha_2 e^{mL} - \alpha_1 e^{-mL}\right)}{\sqrt{k_n^2 + \mu^2}} \right] \\
& + \frac{1}{2}\left[\lambda\left(\alpha_1 e^{mL} + \alpha_2 e^{-mL}\right) + 2\lambda_1\right]\left[\alpha_1 e^{mL} + \alpha_2 e^{-mL} + 2\sum_{n=1}^{\infty} \frac{k_n(A_n + A_n^*)}{\sqrt{k_n^2 + \lambda^2}} \right] \\
& + \frac{1}{2}\left[\mu\left(\alpha_2 e^{mL} + \alpha_1 e^{-mL}\right) + 2\mu_1\right]\left[\alpha_2 e^{mL} + \alpha_1 e^{-mL} + 2\sum_{n=1}^{\infty} \frac{k_n(A_n + A_n^*)}{\sqrt{k_n^2 + \mu^2}} \right].
\end{aligned} \tag{2.51}$$

2.3 The sinh-Gordon field theory

In the previous section, we have studied the classical reflection factors and the energy of the free particle on an interval. In this and the following sections we consider the same matters for the sinh-Gordon model.

The sinh-Gordon model with one boundary has been introduced in the first chapter, where the boundary term is given by (1.36). We now generalise the one-boundary to the two-boundary case. We choose the boundaries to be at $x = L$ and $x = -L$. For simplicity, we take the boundary parameters in (1.36) to be

$\sigma_0 = \sigma_1 \equiv \sigma$. While $\sigma = P_+$ is the parameter corresponding to the one end, $\sigma = P_-$ is the one at the other end. As such, we have the two boundary terms

$$\mathcal{B}_\pm = \frac{mP_\pm}{\beta^2}\left(e^{\beta\phi/\sqrt{2}} + e^{-\beta\phi/\sqrt{2}}\right), \qquad x = \pm L. \tag{2.52}$$

In addition to the equation of motion (1.38) which is now defined on the interval $-L < x < L$, we have the two boundary equations

$$\partial_x\phi = \mp\frac{mP_\pm}{\sqrt{2}\beta}\left(e^{\beta\phi/\sqrt{2}} - e^{-\beta\phi/\sqrt{2}}\right), \qquad x = \pm L. \tag{2.53}$$

Two different approaches shall be used to calculate the classical reflection factors and the spectrum of particle energies for the sinh-Gordon theory. The first one is to solve directly the equation of motion using an ansatz. Another one is to consider the linear perturbation around the static background solution to the equation of motion and boundary conditions. Once the static background is known, the classical reflection factors are sought by linearising the field equation and boundary condition, and calculating the reflection of a plane wave in the effective potential due to the static background. In the rest of the chapter, we will consider these approaches.

2.4 A solution to the equations of motion

In this section, we will use the first method as mentioned at the end of the previous section to study the reflection factor and the energy of the sinh-Gordon model with two boundaries. We first need to find the solution to the equation of motion.

Let us suppose the sinh-Gordon equation

$$\partial^2\phi = -m^2\sinh\phi, \qquad |x| < L. \tag{2.54}$$

has the solution of the special form

$$e^{\phi/2} = \frac{X - T}{X + T}, \tag{2.55}$$

where X is the function of x and T depends only on t. Since the coupling is classically unimportant, we have taken $\beta = 1/\sqrt{2}$. Substituting the special solution into the equation of motion, we have

$$\left(\frac{\ddot{T}}{T}+\frac{X''}{X}\right)\left(X^2-T^2\right)+\left(\dot{T}^2-X'^2\right)=-m^2\left(X^2+T^2\right), \tag{2.56}$$

here $X'=dX/dx$, $\dot{T}=dT/dt$, $X''=d^2X/dx^2$ and $\ddot{T}=d^2T/dt^2$. In order to fully separate the variables, we choose

$$\begin{aligned} X'' &= \lambda X^3+\mu X^2+\nu X, \\ \ddot{T} &= \bar{\lambda}T^3+\bar{\mu}T^2+\bar{\nu}T. \end{aligned} \tag{2.57}$$

These equations together with (2.56) provide the relationship between the parameters in (2.57), namely $\bar{\lambda}=\lambda$, $\bar{\mu}=\mu=0$ and $\bar{\nu}-\nu=m^2$. By integrating these equations once, we obtain

$$\begin{aligned} X'^2 &= \frac{\lambda}{2}X^4+\nu X^2+2\sigma, \\ \dot{T}^2 &= \frac{\lambda}{2}T^4+(\nu-m^2)T^2+2\sigma, \end{aligned} \tag{2.58}$$

where λ, ν, σ are constant parameters. Solving these equations by setting $\sigma=0$ we deduce that

$$\begin{aligned} X &= -\sqrt{\frac{\nu}{\lambda}}\left(\sinh\sqrt{\nu}(x-x_0)\right)^{-1} \\ T &= -\sqrt{\frac{\nu-m^2}{\lambda}}\left(\sinh\sqrt{\nu-m^2}(t-t_0)\right)^{-1}, \end{aligned} \tag{2.59}$$

where x_0 and t_0 are the integration constants. The solution to the equation of motion can therefore be written

$$e^{\phi/2}=\frac{\sqrt{\nu}\sinh\sqrt{\nu-m^2}(t-t_0)-\sqrt{\nu-m^2}\sinh\sqrt{\nu}(x-x_0)}{\sqrt{\nu}\sinh\sqrt{\nu-m^2}(t-t_0)+\sqrt{\nu-m^2}\sinh\sqrt{\nu}(x-x_0)}. \tag{2.60}$$

When $t=t_0$, we then have $e^{\phi/2}=-1$. It is obvious to see that the solution is complex. This solution is neither a real solution of the sinh-Gordon model nor of the sine-Gordon theory.

The parameter x_0 can be fixed by the boundary conditions

$$X'=\mp mP_{\pm}X, \qquad x=\pm L, \tag{2.61}$$

which follows (2.53). They can be expressed explicitly as

$$e^{2\sqrt{\nu}(L-x_0)}=\frac{mP_+-\sqrt{\nu}}{mP_++\sqrt{\nu}}, \qquad x=L, \tag{2.62}$$

$$e^{2\sqrt{\nu}(L+x_0)} = \frac{mP_- - \sqrt{\nu}}{mP_- + \sqrt{\nu}}, \qquad x = -L. \tag{2.63}$$

Combining these equations, we can write down the spectrum

$$e^{4\sqrt{\nu}L}\, K_L(\nu)\, K_{-L}(\nu) = 1, \tag{2.64}$$

where the reflection factors at each end are defined by

$$K_L(\nu) = -\frac{mP_+ + \sqrt{\nu}}{mP_+ - \sqrt{\nu}}, \qquad K_{-L}(\nu) = -\frac{mP_- + \sqrt{\nu}}{mP_- - \sqrt{\nu}}. \tag{2.65}$$

In order to simplify our study, we set $P_\pm = \cos a_\pm \pi$ and $\sqrt{\nu} = im \sinh\theta$, the reflection factors then turn out to be

$$K_L = -\frac{1}{(1+2a_+)\,(1-2a_+)}, \tag{2.66}$$

$$K_{-L} = -\frac{1}{(1+2a_-)\,(1-2a_-)}, \tag{2.67}$$

where $a_\pm$ are the positive parameters and we have used the block notation (1.52) to write down these factors.

The reflection coefficient of the sinh-Gordon model on a half line was obtained in [15] and it is given in the expression (1.53). We now compare it with the reflection factors in the two boundaries case.

If we take a limit $a_0 = a_1 = a_+$ in (1.53), it is obvious that it will be reduced to the boundary reflection factor on the end $x = L$. Similarly, it corresponds to the other reflection factor in the limit $a_0 = a_1 = a_-$. We can see that the boundary reflection matrices obtained in (2.66) and (2.67) have the form of as one boundary reflection factor, and they are independent of each other.

We will now check whether there is any pole in (2.60). For convenience we take $t_0 = 0$ and $\sqrt{\nu} = m\cos\theta$ in this expression. If the numerator is less than zero, then there is no singularity in the solution. In other words, no singularity can occur when

$$\left|\tan\theta \sinh\Big(m(x-x_0)\cos\theta\Big)\right| > 1. \tag{2.68}$$

We deduce that at the boundaries (2.62) and (2.63) this expression is equivalent to the restrictions

$$\cos^2 a_{\pm}\pi < 1,$$

where the parameters $P_{\pm}$ again are chosen to be $\cos a_{\pm}\pi$. We can thus claim that there is no pole in the solution (2.60).

As we have seen the solution is not singular, it is interesting to investigate its energy. Including the boundary contributions, it is given by

$$E = \int_{-L}^{L} dx \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + m^2 \cosh\phi\right) + \mathcal{B}_+ + \mathcal{B}_-. \tag{2.69}$$

The calculations show that

$$E = 2Lm^2 + 4m(P_+ + P_-). \tag{2.70}$$

The energy depends on the boundary parameters $P_{\pm}$ and the size of the interval, as does the free field theory.

We also find that the ground state ($\phi = 0$) energy is same as (2.70).

It is interesting to obtain that the energy of this particular solution of the sinh-Gordon theory is equal to its vacuum energy. If fact, we can choose the vacuum energy is zero by adding a suitable constant term into the bulk and boundary potentials. Correspondingly, the energy of the theory is zero. This indicates that the solution we have obtained in (2.60) is not a real solution. In order to have non-zero energy, we must look for a real solution to the sinh-Gordon equation. One might obtain it using Hirota' method [29].

The following assumption

$$e^{4ikL}\, K_L(\theta)\, K_{-L}(\theta) = 1 \tag{2.71}$$

was made in [3] regarding the relation of the momenta and the reflection factors for the theory confined on an interval. In this section we have verified that the assumption is true for the sinh-Gordon model, where the classical reflection factors are obtained by a particular solution. This is described by (2.64). The next question is to ask whether this assumption holds in the quantum case. In the following sections, we will attempt to check it by doing the perturbation around the static background.

2.5 The static background solution

In the previous section, we have solved directly the sinh-Gordon equation with a particular ansatz. However the solution was not real. In this section, we will look for the static background solution.

In the static background ϕ_0, the sinh-Gordon equation (2.54) becomes

$$\partial_x^2 \phi_0 = m^2 \sinh \phi_0. \tag{2.72}$$

Integrating the equation once, we have

$$\partial_x \phi_0 = \sqrt{2} m \sqrt{\cosh \phi_0 + \alpha}, \tag{2.73}$$

where α is an integration constant. Furthermore, this can be written

$$\partial_x \phi_0 = \frac{2m}{k} \sqrt{1 + k^2 \sinh^2 \frac{\phi_0}{2}}, \tag{2.74}$$

where $k = \sqrt{2/(1+\alpha)}$.

The solution to (2.74) is the elliptic function

$$\phi_0 = 2 \tanh^{-1} \left[\mathrm{sn} \left(\frac{m(x - x_0)}{k}, \sqrt{1 - k^2} \right) \right], \tag{2.75}$$

where x_0 is another integration constant and the number k is called the modulus of the elliptic function. Using the relationships

$$\begin{aligned} &\cosh^2 \frac{\phi_0}{2} - \sinh^2 \frac{\phi_0}{2} = 1, \\ &\mathrm{cn}^2 \left(\frac{m(x - x_0)}{k}, k' \right) + \mathrm{sn}^2 \left(\frac{m(x - x_0)}{k}, k' \right) = 1, \end{aligned} \tag{2.76}$$

the solution (2.75) can be written in the different forms, where $k' = \sqrt{1 - k^2}$. For examples

$$\sinh^2 \frac{\phi_0}{2} = \frac{\mathrm{sn}^2 \left(\frac{m(x - x_0)}{k}, k' \right)}{\mathrm{cn}^2 \left(\frac{m(x - x_0)}{k}, k' \right)}, \tag{2.77}$$

$$\cosh^2 \frac{\phi_0}{2} = \frac{1}{\mathrm{cn}^2\left(\frac{m(x-x_0)}{k},\ k'\right)}. \tag{2.78}$$

Below, we show that the solution (2.75) satisfies the equation (2.74). Differentiating ϕ_0 in (2.75) with respect to x, we obtain

$$\frac{\partial_x \phi_0}{2\cosh^2(\phi_0/2)} = \frac{m}{k}\,\mathrm{cn}\left(\frac{m(x-x_0)}{k},\ k'\right)\,\mathrm{dn}\left(\frac{m(x-x_0)}{k},\ k'\right), \tag{2.79}$$

Using the relation (2.78), we can simplify the above expression to

$$\partial_x \phi_0 = \frac{2m}{k}\,\frac{\mathrm{dn}\left(\frac{m(x-x_0)}{k},\ k'\right)}{\mathrm{cn}\left(\frac{m(x-x_0)}{k},\ k'\right)}. \tag{2.80}$$

On the other hand, let us look at the equation (2.74) itself. Using another relation (2.77), we can reduce it to (2.80). We thus claim that (2.75) is the solution to the static sinh-Gordon equation.

The boundary equations

$$\partial_x \phi_0 = \mp m P_\pm \left(e^{\phi_0/2} - e^{-\phi_0/2}\right), \qquad x = \pm L, \tag{2.81}$$

can be written in terms of the solution (2.75) as

$$\mathrm{dn}\left(\frac{m(\pm L - x_0)}{k},\ k'\right) = \mp\, k P_\pm\, \mathrm{sn}\left(\frac{m(\pm L - x_0)}{k},\ k'\right). \tag{2.82}$$

The parameters x_0 and k should be fixed by these equations.

2.6 The perturbation around the background solution

Once the static background solution ϕ_0 is known, we can perturb the field ϕ around that background, this means we can replace ϕ by $\phi_0 + \phi_1$. The sinh-Gordon equation of motion and the boundary conditions then become

$$\partial^2 \phi_1 = -\phi_1 \cosh\phi_0, \tag{2.83}$$

$$\partial_x \phi_1 = \mp m P_\pm \phi_1 \cosh\frac{\phi_0}{2}, \qquad x = \pm L, \tag{2.84}$$

where ϕ_1 is the perturbation of the background field. We now need to find the solution to ϕ_1. Suppose the solution has the form

$$\phi_1 = e^{i\omega t} N(x). \tag{2.85}$$

The equations (2.83) and (2.84) can be written in terms of the solution as

$$N'' = \left(-\omega^2 + m^2 \cosh \phi_0\right) N, \tag{2.86}$$

$$N' = \mp m P_\pm \cosh \frac{\phi_0}{2} N, \qquad x = \pm L. \tag{2.87}$$

where $N' = dN/dx$ and $N'' = d^2N/dx^2$. Furthermore, we substitute the background solution (2.75) into the above equations, they then become

$$N'' = \left[-\omega^2 - m^2 + \frac{2m^2}{\mathrm{cn}^2\left(\frac{m(x-x_0)}{k}, k'\right)}\right] N, \tag{2.88}$$

$$N' = \mp \frac{mP_\pm}{\mathrm{cn}\left(\frac{m(x-x_0)}{k}, k'\right)} N, \qquad x = \pm L. \tag{2.89}$$

If the last term in (2.88) can be written in terms of the Weierstrass' function, then the equation can be reduced to the Lamé equation.

In order to see this point, let us recall the relation of the elliptic function and the Weierstrass' $\mathcal{P}(z)$ -function. It is given in [30]

$$\mathcal{P}(z) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(u,k)}, \tag{2.90}$$

where $e_1 = \lambda(2 - k^2)$ and $e_3 = -\lambda(1 + k^2)$ are the bases, the variables z and u are related to each other by $u = \sqrt{e_1 - e_3}\, z$. The parameter λ is the ratio of these bases, namely $\lambda = (k^2 - 2)/(k^2 + 1)$.

On the other hand, we have

$$\mathrm{sn}(u - K, k) = -\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}, \tag{2.91}$$

where K is the period of the elliptic function. By using this relation, we can rewrite (2.90) as

$$\mathcal{P}(z) = \lambda\left[-1 + 2k^2 + \frac{3(1-k^2)}{\mathrm{cn}^2(\sqrt{3\lambda}z + K, k)}\right]. \tag{2.92}$$

From (2.92), we express the elliptic function in terms of the Weierstrass' function

$$\frac{1}{\mathrm{cn}^2(\sqrt{3\lambda}\, z+K,k)} = \frac{1}{3\lambda(1-k^2)}\mathcal{P}(z) + \frac{1-2k^2}{3(1-k^2)}. \qquad (2.93)$$

Let us now come back to (2.88). In this equation k and k' can be interchanged when the relation $k'^2 = 1-k^2$ is satisfied. By changing variable $\sqrt{3\lambda}z + K = m(x-x_0)/k'$ in the equation, we obtain the Lamé equation

$$N'' = \left[-\Omega^2 + 2\mathcal{P}(z)\right] N, \qquad (2.94)$$

where $N'' = d^2N/dz^2$ and $\Omega^2 = 3\lambda k'^2\omega^2/m^2 - \lambda(k^2+1)$.

In general, the solution the solution to the Lamé equation

$$N'' = \Big[\mathcal{P}\,(y) + 2\mathcal{P}\,(z)\Big] N \qquad (2.95)$$

is given by [30]

$$\frac{\sigma(z\pm y)}{\sigma(z)\,\sigma(y)} e^{\mp z\zeta(y)}. \qquad (2.96)$$

where $\sigma(z)$ and $\zeta(y)$ are called Weierstrass' sigma and zeta functions. The function $\mathcal{P}(y)$ corresponds to $-\Omega^2$ in (2.94).

The combined solution

$$N = A\frac{\sigma(z+y)}{\sigma(z)\,\sigma(y)}e^{-z\zeta(y)} + B\frac{\sigma(z-y)}{\sigma(z)\,\sigma(y)}e^{z\zeta(y)} \qquad (2.97)$$

is the general solution to the Lamé equation, where A and B are constants.

We now consider the boundary equation (2.89). It can be written with z variable as

$$N' = \mp\frac{k'P_\pm}{\mathrm{cn}(\sqrt{3\lambda}z+K,k)}N, \qquad z = (m(\pm L - x_0)/k' - K)/\sqrt{3\lambda}. \qquad (2.98)$$

If we replace the general solution into the boundary we have

$$\begin{aligned} &A\left[\frac{\sigma'(\pm\alpha+y)}{\sigma(\pm\alpha)\,\sigma(y)} - \Big(\zeta(\pm\alpha)+\zeta(y)\Big)\frac{\sigma(\pm\alpha+y)}{\sigma(\pm\alpha)\,\sigma(y)}\right]e^{\mp\alpha\zeta(y)} \\ &\qquad + B\left[\frac{\sigma'(\pm\alpha-y)}{\sigma(\pm\alpha)\,\sigma(y)} - \Big(\zeta(\pm\alpha)-\zeta(y)\Big)\frac{\sigma(\pm\alpha-y)}{\sigma(\pm\alpha)\,\sigma(y)}\right]e^{\pm\alpha\zeta(y)} \\ &= \mp\frac{k'P_\pm}{\mathrm{cn}\,(\pm\alpha,\ k)}\left[A\frac{\sigma(\pm\alpha+y)}{\sigma(\pm\alpha)\,\sigma(y)}e^{\mp\alpha\zeta(y)} + B\frac{\sigma(\pm\alpha-y)}{\sigma(\pm\alpha)\sigma(y)}e^{\pm\alpha\,\zeta(y)}\right], \end{aligned} \qquad (2.99)$$

where $\pm\alpha = (m(\pm L - x_0)/k' - K)/\sqrt{3\lambda}$.

The boundary equations are homogenous. They have a solution if and only if the determinant of the coefficients A and B is equal to zero. This provides an equation that allows us to solve the $\sigma(y)$. Once we have $\sigma(y)$, we expect that we could deduce $\mathcal{P}(y)$ using the relations between the Weierstrass' sigma and $\mathcal{P}$-functions. Furthermore the energy can be obtained from $\mathcal{P}(y)$. The work on this calculation is still in progress. We might solve them numerically.

One may obtain the approximation to the energy by directly using the assumption (2.71) for the quantum reflection factors. For each end we suppose the quantum reflection factors are given in (1.69). We need to specify the boundary parameters corresponding to the each boundary in this case. We then can compare the energy obtained here with the one obtained from the solution of the Lamé equation. If they agree, we can conclude that the assumption (2.71) is true for the quantum case.

2.7 Conclusion

In this chapter, two different approaches have been used to calculate the classical reflection factors and the energy of the sinh-Gordon theory.

The first one was to solve directly the equation of motion using the ansatz (2.55). We have found that the solution of the equation has a complex form. It is neither a real solution of the sinh-Gordon model nor of the sine-Gordon equation. Combining this particular solution to the boundary conditions, we deduced the reflection factors (2.66) and (2.67) for the theory. When we chose a special limit for the boundary parameters, the reflection factor was the same as on a half-line as expected. However, we have obtained the energy from this complex solution and found that it depends on the boundary parameters and the size of the interval. We also found the ground state energy is the same as the energy of the theory corresponding to the particular solution. We have checked that the assumption (2.71) holds for the classical reflection factors obtained from a particular solution (2.60).

The other method was to consider the linear perturbation around the static background solution to the equation of motion and the boundary conditions. The purpose of taking this approach is to check whether the assumption (2.71) is true in the quantum case. However the results have not yet come out. This approach might be done by the numerical method. We have only found that the background solution is described by the elliptic function. The energy levels are expected to be obtained by solving the Lamé's equation in this background.

Chapter 3

Sinh-Gordon model on a half-line

3.1 Introduction

In recent years, perturbation theory has been developed to study the reflection matrix in the affine Toda field theory defined on the half line. It can be extracted from the calculation of the two-point function. For the *ade* theories, the reflection factors have been obtained at one loop order for the Neumann condition [31–33].

For the sine-Gordon model the boundary S-matrix of the first breather was studied in [34], where the result agreed with the exact expression given by Ghoshal [27].

Corrigan [24] studied the problem for the sinh-Gordon model with the boundary condition (1.39), in which the boundary parameters $\sigma_0 = \sigma_1 = \sigma$. He calculated reflection factor to $O(\beta^2)$. The result was agreed with the result obtained by Ghoshal [27] for the sine-Gordon breather reflection factors, analytically continued in the coupling constant. He also suggested an interesting dual relationship between models with different boundary conditions. The similar study has been considered for $a_2^{(1)}$ theory [35], in which the perturbative result is in agreement with the exact reflection factor found in [36].

In this chapter, we will study the quantum reflection factor of the sinh-Gordon model for the general boundary condition (1.39) by developing the perturbation theory and the path integral method. The motivation for the work presented in this chapter follows the idea in [24]. To begin with we recall the construction of the Green's function on the half line and the calculation of the reflection factor for the case $\sigma_0 = \sigma_1 = \sigma$. The propagator is determined by making use of the classical lowest energy static background, and the reflection factor can be defined as the coefficient of the reflection term of the exact two point Green's function in the asymptotic region far away from the boundary. We then study the reflection factors for the case $\sigma_0 \neq \sigma_1$ using Corrigan's method. The computations are more intricate in the latter case, since three-point couplings appear at one-loop order in addition to the four-point coupling. To this order, there are three types of Feynman diagram contributing to the two-point correlation function.

3.2 The configuration space propagator

The configuration space propagator or Green's function in the presence of the boundary was constructed in [24]. It was obtained by firstly linearising the field equation and the boundary condition around the static background, then determining the eigenfunction of the second order differential operator in the linearised field equation. The boundary equation should be satisfied by the eigenfunction. Below we summarise how one gets the propagator which is constructed from the eigenfunction. The Green's function or the propagator can be constructed from the non-zero eigenvalues. For the whole line, we know that the Green's function is a symmetric function of x and x'. It is customary to denote this by $G(x - x')$. In the half line, we have to be more careful. Taking into account the reflection off the boundary, we denote it by $G(x, t; x', t')$ and it can be defined as

$$G(x,t;x',t') = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{\phi(x,t)\ \phi^*(x',t')}{\omega^2 - k^2 - m^2 + i\epsilon}, \tag{3.1}$$

where $\phi(x, t)$ is given in (1.46). Substituting it into the Green's function, we have

$$\int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{2r^2(k)P(k)P(-k)(ik+m)(ik-m)}{\omega^2 - k^2 - m^2 + i\epsilon} e^{-i\omega(t-t')} \\ \left[f(k,x)f(-k,x')e^{ik(x-x')} + K_0 f(-k,x)f(-k,x')e^{-ik(x+x')}\right]. \tag{3.2}$$

where the function $f(k,x)$ and the classical reflection factor K_0 are defined by

$$f(k,x) = \frac{ik - m\coth 2(x - x_0)}{ik + m} \tag{3.3}$$

and

$$K_0 = \frac{P(-k)}{P(k)}\frac{ik - m}{ik + m}. \tag{3.4}$$

If we replace k by $m\sinh\theta$ into the last expression, it is reduced to the equation (1.53). If we set $\sigma_0 = \sigma_1 = 1$, then the classical reflection factor becomes

$$K(k) = \frac{ik + m}{ik - m} = \frac{i\sinh\theta + 1}{i\sinh\theta - 1}. \tag{3.5}$$

The factor $r(k)$ appeared in $\phi(x,t)$ can be fixed using the normalisation of the Green's function in the limit $x, x' \to \infty$. At this limit, we have

$$G(x,t;x',t') = \int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\, e^{i\omega(t-t')}\frac{i2r^2(k)P(-k)P(k)(ik+m)(ik-m)}{\omega^2 - k^2 - m^2 + i\epsilon}e^{ik(x-x')}. \tag{3.6}$$

It is also known that in this limit the Green's function has a standard form

$$\int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\, \frac{i}{\omega^2 - k^2 - m^2 + i\epsilon}e^{i\omega(t-t')}e^{ik(x-x')}. \tag{3.7}$$

This allows us to find

$$r^2(k) = \frac{1}{2\,P(k)\,P(-k)\,(ik+m)\,(ik-m)}. \tag{3.8}$$

We can therefore write down the Green's function on the half line for the general boundary condition (1.39) as

$$\begin{aligned} G(x,t;x',t') = &\int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\, \frac{i}{\omega^2 - k^2 - m^2 + i\epsilon}e^{i\omega(t-t')} \\ &\left[f(k,x)\,f(-k,x')e^{ik(x-x')} + K_0 f(-k,x)\,f(-k,x')e^{-ik(x+x')}\right]. \end{aligned} \tag{3.9}$$

For the special case $\sigma_0 = \sigma_1 \equiv \sigma$, it becomes

$$G(x,t;x',t') = \int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\, \frac{ie^{i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon}\left[e^{ik(x-x')} + K_\sigma(k)e^{-ik(x+x')}\right], \tag{3.10}$$

where the classical reflection factor is

$$K_\sigma(k) = \frac{ik + 2\sigma}{ik - 2\sigma}. \tag{3.11}$$

The Green's function (3.9) reduces to

$$G(x,t;x',t') = i\int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\, e^{i\omega(t-t')}\frac{e^{ik(x-x')}+e^{-ik(x+x')}}{\omega^2-k^2-m^2+i\epsilon} \tag{3.12}$$

for the Neumann boundary condition. i.e. vanishing space derivative of field at the boundary.

3.3 Generating functional

In this section, we study the generating functional for the sinh-Gordon theory defined on the half line.

In quantum field theory [37–39] path integral quantisation is based on the construction of a generating functional for the Green's function. The generating functional is the vacuum to vacuum transition amplitude in the presence of the source J and is traditionally denoted as $Z[J]$. In two dimensions the generating functional is defined as

$$Z[J] = N\int \mathcal{D}\phi \exp\left\{i\int d^2x\,[\mathcal{L}+\phi(x)J(x)]\right\}, \tag{3.13}$$

where $J(x)$ is an external source associated to the field $\phi(x)$, N is the $J(x)$-independent normalisation factor. It can be fixed by imposing the normalisation condition $Z[0]=1$.

Before writing down the generating functional for the sinh-Gordon model on the half line, let us study the perturbation expansion of the Lagrangian (1.37) around the static background ϕ_0. For convenience, we take the mass $m=2$. The perturbation of the Lagrangian is

$$\mathcal{L} = \mathcal{L}_c + \mathcal{L}_p + \mathcal{L}_3 + \mathcal{L}_4 \tag{3.14}$$

where

$$\begin{aligned}\mathcal{L}_c = {}& \frac{\theta(-x)}{\beta}\left[\sqrt{2}\phi_0\sinh\sqrt{2}\beta\phi_0 - \frac{2}{\beta}\cosh\sqrt{2}\beta\phi_0\right] \\ & + \frac{\delta(x)}{\beta}\left[\sigma_1 e^{\beta\phi_0/\sqrt{2}}\left(\frac{\phi_0}{\sqrt{2}}-\frac{2}{\beta}\right) - \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\left(\frac{\phi_0}{\sqrt{2}}+\frac{2}{\beta}\right)\right],\end{aligned} \tag{3.15}$$

$$\mathcal{L}_p = -\frac{1}{2}\phi\left[\theta(-x)\left(\partial^2 + 4\cosh\sqrt{2}\beta\phi_0\right) + \delta(x)\left(\partial + \sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right)\right]\phi, \tag{3.16}$$

$$\mathcal{L}_3 = -\frac{\sqrt{2}\beta}{3}\left[\theta(-x)2\sinh\sqrt{2}\beta\phi_0 + \delta(x)\frac{1}{4}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} - \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right)\right]\phi^3, \tag{3.17}$$

$$\mathcal{L}_4 = -\frac{\beta^2}{3}\left[\theta(-x)2\cosh\sqrt{2}\beta\phi_0 - \delta(x)\frac{1}{16}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right)\right]\phi^4. \tag{3.18}$$

They correspond respectively to the constant term, the propagating term, the three coupling term and the four coupling term in the Lagrangian. Here we have integrated out the total derivative terms, using the property of the step function as well as $\phi_0(-\infty) = 0$.

The generating functional can be written as

$$\begin{aligned} Z[J] = N\int \mathcal{D}\phi\ &\exp\left\{-i\beta\int d^2x\left[\theta(-x)\frac{2\sqrt{2}}{3}\sinh\sqrt{2}\beta\phi_0 + \delta(x)\frac{\sqrt{2}}{12}\sigma_-\right]\phi^3\right\} \\ &\exp\left\{-i\beta^2\int d^2x\left[\theta(-x)\frac{1}{3}\cosh\sqrt{2}\beta\phi_0 + \delta(x)\frac{1}{48}\sigma_+\right]\phi^4\right\} \\ &\exp\left\{i\int d^2x\left[-\frac{1}{2}\phi M\phi + J\phi\right]\right\}, \end{aligned} \tag{3.19}$$

where we define

$$\sigma_- \equiv \sigma_1 e^{\beta\phi_0/\sqrt{2}} - \sigma_0 e^{-\beta\phi_0/\sqrt{2}} = \sigma_1\sqrt{\frac{1+\sigma_0}{1+\sigma_1}} - \sigma_0\sqrt{\frac{1+\sigma_0}{1+\sigma_1}}, \tag{3.20}$$

$$\sigma_+ \equiv \sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}} = \sigma_1\sqrt{\frac{1+\sigma_0}{1+\sigma_1}} + \sigma_0\sqrt{\frac{1+\sigma_0}{1+\sigma_1}}, \tag{3.21}$$

and

$$M(x) = \theta(-x)\left(\partial^2 + 4\cosh\sqrt{2}\beta\phi_0\right) + \delta(x)\left(\partial + \sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right). \tag{3.22}$$

We should notice that $\mathcal{L}_c$ has been absorbed into N.

Assuming β is small, we expand the first two exponents in (3.19) in power series in β, namely

$$\begin{aligned} \exp\left[-i\beta\int d^2x\ A(x)\phi^3\right] &= \sum_{n=0}^{\infty}\frac{(-i\beta)^n}{n!}\int d^2x_1...d^2x_n\ A(x_1)\phi^3(x_1)...A(x_n)\phi^3(x_n), \\ \exp\left[-i\beta^2\int d^2x\ B(x)\phi^4\right] &= \sum_{m=0}^{\infty}\frac{(-i\beta^2)^m}{m!}\int d^2x_1...d^2x_m\ B(x_1)\phi^4(x_1)...B(x_m)\phi^4(x_m). \end{aligned} \tag{3.23}$$

For simplicity, we have taken

$$A(x) = \frac{\sqrt{2}}{3}\left[2\theta(-x)\sinh\sqrt{2}\beta\phi_0 + \frac{1}{4}\delta(x)\sigma_-\right]$$
$$B(x) = \frac{1}{3}\left[\theta(-x)\cosh\sqrt{2}\beta\phi_0 + \frac{1}{16}\delta(x)\sigma_+\right]. \tag{3.24}$$

Inserting (3.23) in (3.19), we can replace each power of ϕ in the exponential factors by a functional derivative with respect to J acting on the third exponential factor, namely

$$\phi(x)\exp\left[i\int d^2x\ J(x)\ \phi(x)\right] = \frac{\delta}{i\delta J(x)}\exp\left[i\int d^2y\ J(y)\ \phi(y)\right]. \tag{3.25}$$

Thus the generating functional (3.19) can be written in the following form

$$Z[J] = N\int \mathcal{D}\phi \sum_{n=0}^{\infty}\frac{(-i\beta)^n}{n!}\int d^2x_1...d^2x_n A(x_1)\left(\frac{\delta}{i\delta J(x_1)}\right)^3 ...A(x_n)\left(\frac{\delta}{i\delta J(x_n)}\right)^3$$
$$\sum_{m=0}^{\infty}\frac{(-i\beta)^m}{m!}\int d^2x_1...d^2x_m B(x_1)\left(\frac{\delta}{i\delta J(x_1)}\right)^4 ...B(x_m)\left(\frac{\delta}{i\delta J(x_m)}\right)^4$$
$$\exp\left[i\int d^2x\left[-\frac{1}{2}\phi M\phi + J\phi\right]. \tag{3.26}$$

In fact, the functional derivatives with respect to J can be taken outside the path integral and can then be written in the exponential forms. The remaining integration over ϕ is

$$\int \mathcal{D}\phi\ \exp\left[i\int d^2x\left(-\frac{1}{2}\phi M\phi + J\phi\right)\right]. \tag{3.27}$$

In order to evaluate this functional integral, we shift ϕ

$$\phi(x) \to \phi(x) + \varphi(x). \tag{3.28}$$

Under the transformation, the integral in the exponents becomes

$$\int d^2x\left[-\frac{1}{2}\phi M\phi - \phi M\varphi - \frac{1}{2}\varphi M\varphi + J\phi + J\varphi\right]. \tag{3.29}$$

If φ is now chosen to satisfy

$$M\varphi(x) = J(x), \tag{3.30}$$

then (3.29) becomes

$$\int d^2x \left[-\frac{1}{2}\phi M\phi + \frac{1}{2}\varphi J\right]. \tag{3.31}$$

The solution to (3.30) is

$$\varphi(x) = i\int d^2y\, G(x,y)J(y), \tag{3.32}$$

where $G(x,y)$ is the propagator, obeying

$$MG(x,y) = -i\delta^2(x,y) \tag{3.33}$$

Substituting (3.32) into (3.31), we have

$$-\frac{1}{2}\int d^2x\ \phi M\phi + \frac{i}{2}\int d^2x \int d^2y\ J(x)G(x,y)J(y). \tag{3.34}$$

Therefore (3.27) takes the form

$$\exp\left[-\frac{1}{2}\int d^2x \int d^2y\ J(x)G(x,y)J(y)\right]\int \mathcal{D}\phi\ \exp\left[-\frac{i}{2}\int d^2x\ \phi M\phi\right]. \tag{3.35}$$

This expression has been separated into two parts, one depending on ϕ only, and the other on J only. The result of the Gaussian path integral is a constant number, which we will denote by N'. The first exponent

$$Z_0[J] = \exp\left\{-\frac{1}{2}\left[\int d^2x \int d^2y\ J(x)\ G(x,y)\ J(y)\right]\right\}, \tag{3.36}$$

is called the free field generating functional.

We can thus write down generating functional $Z[J]$ as

$$\begin{aligned} Z[J] = N\exp&\left\{-i\beta\int d^2x\ A(x)\left(\frac{\delta}{i\delta J(x)}\right)^3\right\} \\ &\exp\left\{-i\beta^2\int d^2x\ B(x)\left(\frac{\delta}{i\delta J(x)}\right)^4\right\} Z_0[J] \end{aligned} \tag{3.37}$$

where N includes N'.

In order to calculate $Z[J,\eta,\bar{\eta}]$ up to $O(\beta^2)$, we need to expand the first exponent up to the second order in β and the second exponent up to the first order in β^2, namely

$$\begin{aligned} Z[J] = N\Big[1 + \beta \int d^2x\, A(x) \left(\frac{\delta}{\delta J(x)}\right)^3 - i\beta^2 \int d^2x\, B(x) \left(\frac{\delta}{\delta J(x)}\right)^4 \\ - \frac{1}{2}\beta^2 \int d^2x \int d^2y\, A(x)A(y) \left(\frac{\delta}{\delta J(x)}\right)^3 \left(\frac{\delta}{\delta J(y)}\right)^3\Big] Z_0[J]. \end{aligned} \tag{3.38}$$

By performing the functional derivatives of $Z_0[J]$ with respect to J as required orders, we obtain

$$\begin{aligned} Z[J] = N\Big\{1 &+ \beta \int d^2x\, A(x) \left[3G(x,x) \int d^2y\, J(y)G(x,y) - \left(\int d^2y\, J(y)G(x,y)\right)^3\right] \\ &- i\beta^2 \int d^2x B(x)\Big[3G(x,x)^2 - 6G(x,x)\left(\int d^2y\, J(y)G(x,y)\right)^2 \\ &\qquad + \left(\int d^2y\, J(y)G(x,y)\right)^4\Big] \\ &+ \frac{1}{2}\beta^2 \int d^2x \int d^2y\, A(x)A(y)\Big[-9G(x,x)G(x,y)G(y,y) - 6G(x,y)^3 \\ &\quad +9G(x,x)G(x,y)\left(\int d^2z\, J(z)G(z,y)\right)^2 \\ &\quad +9G(x,x)G(y,y)\int d^2z\, J(z)G(z,y) \int d^2w\, J(w)G(w,y) \\ &\quad -3G(x,x)\int d^2z\, J(z)G(z,y) \left(\int d^2w\, J(w)G(w,y)\right)^3 \\ &\quad +9G(x,y)G(y,y)\left(\int d^2z\, J(z)G(z,y)\right)^2 \\ &\quad +18G(x,y)^2 \int d^2z\, J(z)G(z,x) \int d^2w\, J(w)G(w,y) \\ &\quad -9G(x,y)\left(\int d^2z\, J(z)G(z,x)\right)^2 \left(\int d^2w\, J(w)G(w,y)\right)^2 \\ &\quad -3G(y,y)\left(\int d^2z\, J(z)G(z,x)\right)^3 \int d^2w\, J(w)G(z,) \\ &\quad + \left(\int d^2z\, J(z)G(z,x)\right)^3 \left(\int d^2w\, J(w)G(w,y)\right)^3\Big]\Big\} Z_0[J]. \end{aligned} \tag{3.39}$$

Using the normalisation condition $Z[0] = 1$, we can now fix the constant N up to the order of β^2. We find this to be

$$\begin{aligned} N = 1 &+ 3i\beta^2 \int d^2x\, B(x)G(x,x)^2 \\ &+ \frac{1}{2}\beta^2 \int d^2x \int d^2y\, A(x)A(y)\left[9G(x,x)G(x,y)G(y,y) + 6G(x,y)^2\right]. \end{aligned} \tag{3.40}$$

The role of the normalisation factor N is simply to cancel the disconnected diagrams in each order of the perturbation theory. After placing the normalisation factor into the expression obtained in (3.39), we obtain the generating functional up to the order β^2

$$
\begin{aligned}
Z[J] = \Big\{ 1 + \beta \int d^2x\, A(x) \Big[3G(x,x) \int d^2y\, J(y)G(x,y) - \Big(\int d^2y\, J(y)G(x,y) \Big)^3 \Big] \\
- i\beta^2 \int d^2xB(x) \Big[-6G(x,x) \Big(\int d^2y\, J(y)G(x,y) \Big)^2 + \Big(\int d^2y\, J(y)G(x,y) \Big)^4 \Big] \\
+ \frac{1}{2}\beta^2 \int d^2x \int d^2y\, A(x)A(y) \Big[+ 9G(x,x)G(x,y) \Big(\int d^2z\, J(z)G(z,y) \Big)^2 \\
+9G(x,x)G(y,y) \int d^2z\, J(z)G(z,y) \int d^2w\, J(w)G(w,y) \\
-3G(x,x) \int d^2z\, J(z)G(z,y) \Big(\int d^2w\, J(w)G(w,y) \Big)^3 \\
+9G(x,y)G(y,y) \Big(\int d^2z\, J(z)G(z,y) \Big)^2 \\
+18G(x,y)^2 \int d^2z\, J(z)G(z,x) \int d^2w\, J(w)G(w,y) \\
-9G(x,y) \Big(\int d^2z\, J(z)G(z,x) \Big)^2 \Big(\int d^2w\, J(w)G(w,y) \Big)^2 \\
-3G(y,y) \Big(\int d^2z\, J(z)G(z,x) \Big)^3 \int d^2w\, J(w)G(z,) \\
+ \Big(\int d^2z\, J(z)G(z,x) \Big)^3 \Big(\int d^2w\, J(w)G(w,y) \Big)^3 \Big] \Big\} Z_0[J].
\end{aligned}
\tag{3.41}
$$

The result of the perturbative derivation on the generating functional is ultimately expressed in terms of the integrals over the products of the propagator.

3.4 Two-point Function

In the quantum field theory, the vacuum expectation values of the time-ordered products of field operators are called n-point functions. They can be obtained by taking the functional derivative of the generating functional with respect to the source at $J = 0$, and are defined by

$$
\begin{aligned}
\mathcal{G}(x_1, ..., x_n) &=< 0|T(\phi(x_1), ..., \phi(x_n))|0 > \\
&= \frac{1}{i^n} \frac{\delta^n Z}{\delta J(x_1).....\delta J(x_n)} \Big|_{J=0} .
\end{aligned}
\tag{3.42}
$$

Having obtained the generating functional Z up to the order in β^2, we can proceed to evaluate the two-point functions up to the same order. For two-point functions, the expression (3.42) can be simplified as

$$\mathcal{G}(x_1, x_2) = \frac{1}{i^2} \frac{\delta^2 Z}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} \tag{3.43}$$

By differentiating the generating functional (3.41) successively with respect to sources $J(x_2)$ and $J(x_1)$, we obtain

$$\begin{aligned}\mathcal{G}(x_1, x_2) =& G(x_1, x_2) - 12i\beta^2 \int d^2x\ B(x)\ G(x,x)\ G(x,x_1)\ G(x,x_2) \\ & -12\sqrt{2}\beta^2 \int d^2x \int d^2y\ A(x)\ A(y) \\ & \Big[G(x,y)^2\ G(x,x_1)\ G(y,x_2) + G(x,x)\ G(x,y)\ G(x_1,y)\ G(x_2,y)\Big] .\end{aligned} \tag{3.44}$$

Here we have written $G(x,t;x',t')$ as $G(x,x')$ for simplicity. If we represent the two-point function as a straight line connecting x_1 to x_2, then equation (3.44) may be drawn diagrammatically as in figure 3.1.

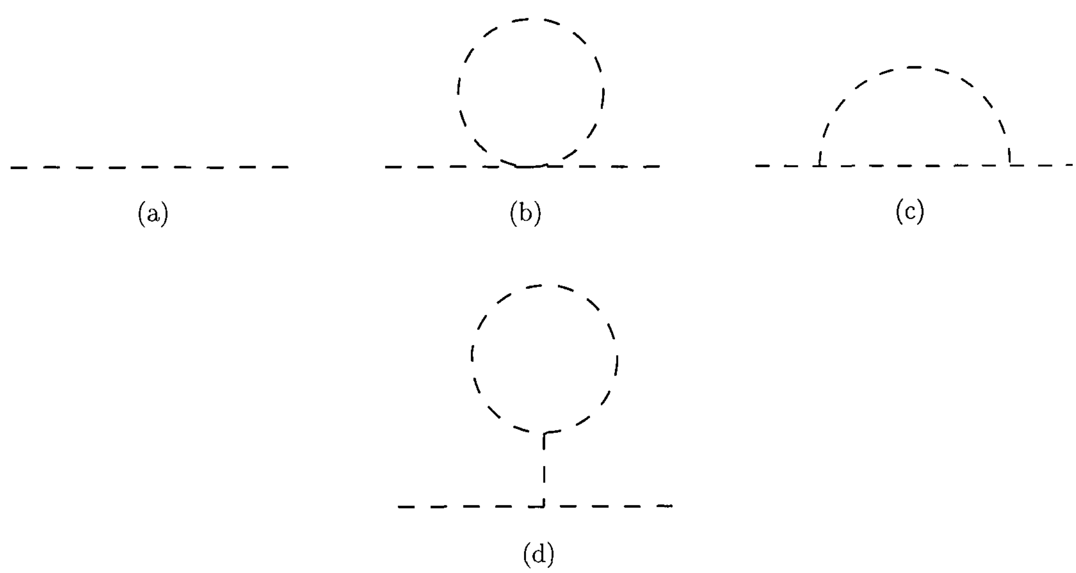

Figure 3.1: Four Feynman diagrams.

There are three types of the one-loop diagram contributing to correction of the two-point Green function. In the rest of this chapter, we shall continue with the computation of these loop diagrams.

3.5 Quantum reflection factor in the $\sigma_0 = \sigma_1$ case

The quantum reflection factor can be obtained by calculating the two-point function. It is defined as the coefficient of the reflected term $e^{-ik(x+x')}$, in the asymptotic region far away from the boundary.

In this section we study the quantum reflection factor that corresponds to special cases of the boundary parameters. If the boundary parameters σ_0 and σ_1 are equal to each other, lets us call them σ, then the classical background solution is $\phi_0 = 0$. In this case $A(x) = 0$ and $B(x) = \frac{1}{3}\theta(-x) + \frac{1}{24}\delta(x)\sigma$ in (3.44). Therefore only a single loop diagram appears in the perturbation theory, as in diagram (b) in Figure 3.1. The calculation of this diagram has been performed in [24] where the quantum correction of the reflection factor up to the order of β^2 is obtained. Below we review the calculation.

Two contributions come from this diagram. One is the vertex situated in the on the boundary and other in the bulk region. The boundary contribution can be described by

$$-\frac{i}{2}\sigma\beta^2 \int_{-\infty}^{+\infty} dt''\; G\left(x, t;\; 0, t''\right)\; G\left(0, t'';\; 0, t''\right)\; G\left(0, t'';\; x', t'\right), \tag{3.45}$$

where the Green's functions are given by (3.10).

The integration over t'' generates a delta function and using this delta function we can integrate out ω' for the last propagator.

Let us first consider the middle propagator. It can be written as

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - 4 + i\epsilon} \left(2 + \frac{4\sigma}{ik'' - 2\sigma}\right). \tag{3.46}$$

The first term in the expression is divergent and a minimal subtraction can be made by adding a suitable counter term. The integrations over ω'' and k'' on the finite parts yield the result

$$-a\,\frac{\cos a\pi}{\sin a\pi}, \tag{3.47}$$

where σ has been replaced by $\cos a\pi$.

For the first propagator, we set $k \to -k$ in the first term, then the boundary contribution (3.45) can be written as

$$\int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{-e^{-i\omega(t-t')}\ e^{-i(kx+k'x')}}{(\omega^2 - k^2 - 4 + i\epsilon)\ (\omega^2 - k'^2 - 4 + i\epsilon)} \left(\frac{2ik}{ik - 2\sigma}\right) \left(\frac{2ik'}{ik' - 2\sigma}\right). \tag{3.48}$$

The momentum integrations k and k' can be done by closing the contours in the upper half plane. By performing the integrals, we obtain

$$\frac{ia\beta^2}{16 \tan^2 a\pi} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\ e^{-i\hat{k}(x+x')} K_\sigma(\hat{k}) \left(\frac{1}{\cosh\theta - \sin a\pi} - \frac{1}{\cosh\theta + \sin a\pi}\right), \tag{3.49}$$

where we have set $\hat{k} = \sqrt{\omega^2 - 4} = 2\sinh\theta$. The reflection factor can be extracted and it is defined as the coefficients of the reflected term in the contribution. It is found to be

$$\frac{ia\beta^2}{4\tan^2 a\pi} \sinh\theta K_\sigma(\hat{k}) \left(\frac{1}{\cosh\theta - \sin a\pi} - \frac{1}{\cosh\theta + \sin a\pi}\right). \tag{3.50}$$

We now study the contribution coming from the bulk, it is given by

$$-4i\beta^2 \int_{-\infty}^{+\infty} dt'' \int_{-\infty}^{0} dx''\ G(x,t;\ 0,t'')\ G(x'',t'';\ x'',t'')\ G(x'',t'';\ x',t'). \tag{3.51}$$

The integration on t'' again provides a delta function allowing us to integrate out the ω' integral in the last propagator. After performing the integral ω'' for the middle propagator, we left are with

$$\begin{aligned} 4i\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \int \frac{dk''}{2\pi} \frac{1}{2\sqrt{k''^2+4}} \frac{e^{-i\omega(t-t')}\ e^{-i(kx+k'x')}}{(\omega^2 - k^2 - 4 + i\epsilon)(\omega^2 - k'^2 - 4 + i\epsilon)} \\ \int_{-\infty}^{0} dx'' K_\sigma(k'') \left[e^{ix''(k+k'-2k'')} + K_\sigma(k') e^{ix''(k-k'-2k'')} \right. \\ \left. + K_\sigma(k) e^{ix''(-k+k'-2k'')} + K_\sigma(k) K_\sigma(k') e^{ix''(-k-k'-2k'')}\right]. \end{aligned} \tag{3.52}$$

To compute the x'' integral, we can use

$$\int_{-\infty}^{0} dx'' e^{(ik+\rho)x''} = \frac{-i}{k - i\rho}, \tag{3.53}$$

where ρ is a small positive constant and will be taken to zero at the end of the calculation. By integrating out x'', the remaining part in (3.52) will be

$$4i\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{e^{-i\omega(t-t')}\, e^{-i(kx+k'x')}}{(\omega^2 - k^2 - 4 + i\epsilon)(\omega^2 - k'^2 - 4 + i\epsilon)}$$
$$\int \frac{dk''}{2\pi} \frac{K_\sigma(k'')}{2\sqrt{k''^2+4}} \left[\frac{-i}{k + k' - 2k'' - i\rho} + \frac{-iK_\sigma(k')}{k - k' - 2k'' - i\rho} \right. \tag{3.54}$$
$$\left. + \frac{-iK_\sigma(k)}{-k + k' - 2k'' - i\rho} + \frac{-iK_\sigma(k)K_\sigma(k')}{-k - k' - 2k'' - i\rho} \right].$$

The next task is to perform the k'' integrals. These can be achieved by closing contours in the upper half plane and letting the branch cuts to run from $2i$ to the infinity along the imaginary axis. If $\sigma > 0$, then there is no pole contribution coming from $K_\sigma(k'')$. If $\sigma < 0$, there will be a pole. However its residue integrating over the integral k and k' gives the exponentially decay to zero when we take the limits $x, x' \to -\infty$. All other poles can be avoided due to the effect of $i\rho$. Let us begin by considering the first k'' integral in the above expression

$$\int_{-\infty}^{\infty} \frac{dk''}{2\pi} \frac{K_\sigma(k'')}{\sqrt{k''^2+4}} \frac{1}{k + k' - 2k'' - i\rho} = \frac{i}{2\pi} \int_2^\infty dy \frac{1}{\sqrt{y^2-4}} \frac{y - 2\sigma}{(y + 2\sigma)\left(y + i\frac{k+k'}{2}\right)}, \tag{3.55}$$

where we have made the change $ik'' = y$. By decomposing into partial fractions and using the result that

$$\int_2^\infty dy \frac{1}{\sqrt{y^2-4}} \frac{1}{y + 2\alpha} = \frac{1}{\sqrt{1-\alpha^2}} \left(\frac{\pi}{2} - \tan^{-1} \sqrt{\frac{1+\alpha}{1-\alpha}} \right), \tag{3.56}$$

the following result can be achieved for (3.55)

$$\frac{i}{2\pi} \left[\frac{2K_\sigma\left(\frac{k+k'}{2}\right)}{\sqrt{4 + (k+k')^2/4}} \left(\frac{\pi}{2} - \tanh^{-1} \sqrt{\frac{4 + i(k+k')}{4 - i(k+k')}} \right) \right.$$
$$\left. + \frac{1 - K_\sigma\left(\frac{k+k'}{2}\right)}{\sqrt{1-\sigma^2}} \left(\frac{\pi}{2} - \tanh^{-1} \sqrt{\frac{1+\sigma}{1-\sigma}} \right) \right]. \tag{3.57}$$

We now perform the k and k' integrals. The contours can be closed in the upper half plane and taking $\hat{k} = \sqrt{\omega^2 - 4} = 2\sinh\theta$, we deduce

$$\frac{i}{2\pi} e^{-i\hat{k}(x+x')} \frac{1}{4\hat{k}^2} \left[\frac{K_\sigma(\hat{k})}{\cosh\theta} \left(\frac{\pi}{4} - \frac{i\theta}{2} \right) + \frac{a\pi(1 - K_\sigma(\hat{k}))}{2\sin a\pi} \right]. \tag{3.58}$$

The other three terms in (3.54) can be integrated in the same way, except that $k+k'$ is replaced by $k-k'$, $-k+k'$ and $-k-k'$ respectively in each term. Combining the results obtained from four terms, we finally obtain the reflection factor coming from the bulk contribution

$$-\frac{i\beta^2}{4}K_\sigma(\hat{k})\sinh\theta\left[\frac{1}{2}\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right.$$
$$\left.+\frac{a}{\sin^2 a\pi}\left(\frac{1}{\cosh\theta-\sin a\pi}-\frac{1}{\cosh\theta+\sin a\pi}\right)\right]. \tag{3.59}$$

If we combine the contributions from the boundary and the bulk, we have the quantum reflection factor up to the order β^2

$$R_\sigma(\hat{k}) = K_\sigma(\hat{k})\left\{1-\frac{i\beta^2}{8}\sinh\theta\left[\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right.\right.$$
$$\left.\left.+\,2a\left(\frac{1}{\cosh\theta-\sin a\pi}-\frac{1}{\cosh\theta+\sin a\pi}\right)\right]\right\}. \tag{3.60}$$

3.6 Quantum reflection factor in the $\sigma_0 \neq \sigma_1$ case

In this section we calculate the reflection factor for the more general case $\sigma_0 \neq \sigma_1$. The computations are more intricate in this case since the three-point coupling appears at one-loop order in addition to the four -point coupling. As we have seen in Figure 3.1, there are three types of Feynman diagram contributing to the two-point function.

Let us begin by considering diagram (b). There are two contributions which need to be calculated in this diagram. One comes from the boundary and another one from the bulk potential. More precisely, in one case the vertex is situated on the boundary at $x = 0$, whilst in the other the vertex is in the bulk region.

The boundary contribution may be described by

$$-\frac{i}{4}\sigma_+\beta^2\int_{-\infty}^{+\infty}dt''\;G\left(x,t;\;0,t''\right)\;G\left(0,t'';\;0,t''\right)\;G\left(0,t'';\;x',t'\right), \tag{3.61}$$

where $G\left(x,t;\;0,t''\right)$, $G\left(0,t'';\;x',t'\right)$ are the two propagators respectively corresponding to the two external lines, and $G\left(0,t'';\;0,t''\right)$ is the middle propagator representing the internal loop-line in (b). The integral over t'' generates a delta function which allows us to perform the frequency integral, let us say for the last propagator. The middle propagator is

$$G(0, t';\ 0, t') = i \int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{f(k'', 0) f(-k'', 0) + K_0 f(-k'', 0) f(-k'', 0)}{\omega''^2 - k''^2 - 4 + i\epsilon}. \tag{3.62}$$

After substituting $f(\pm k'', 0)$ and K_0 from (3.3) and (3.4) into this expression it becomes

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \frac{2ik''(ik'' - 2\coth 2x_0)}{P(k'')} \tag{3.63}$$

Using

$$\coth 2x_0 = \frac{2 + \sigma_0 + \sigma_1}{2\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}}, \tag{3.64}$$

which follows from (1.43), the middle propagator can be written as

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{2i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \left[1 + \frac{1}{P(k'')} \left(ik'' \frac{\sigma_0 + \sigma_1 + 2\sigma_0\sigma_1}{\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}} - 2\sigma_0 - 2\sigma_1 \right) \right]. \tag{3.65}$$

It is clear that the first term is the logarithmic divergent, which can be subtracted by mass renormalisation. The second term is a finite term. After integrating over ω'', the finite part of the middle propagator becomes

$$\frac{1}{2} \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \left[\frac{\frac{\sigma_0+\sigma_1-2\sigma_0\sigma_1}{\sqrt{1-\sigma_0}\sqrt{1-\sigma_1}} + \frac{\sigma_0+\sigma_1+2\sigma_0\sigma_1}{\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}}}{ik'' - \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + \sqrt{1\sigma_0}\sqrt{1-\sigma_1}} \right.$$
$$\left. - \frac{\frac{\sigma_0+\sigma_1-2\sigma_0\sigma_1}{\sqrt{1-\sigma_0}\sqrt{1-\sigma_1}} - \frac{\sigma_0+\sigma_1+2\sigma_0\sigma_1}{\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}}}{ik'' - \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}} \right]. \tag{3.66}$$

Let us first evaluate the first term in the above expression, namely,

$$\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \frac{1}{ik'' - \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}} \tag{3.67}$$

The integration can be done by closing the contour in the upper-half plane, and along the branch cut running from $k'' = 2i$ to infinity. However, we should be careful with the pole at $k'' = -i\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}\right)$. If $\sigma_i > 0$, this pole does have a contribution. If $\sigma_i < 0$ there is an extra pole, but its contribution can be eliminated in the limit $x, x' \to -\infty$, since the exponential decays to zero. By changing $k'' = iy$, the integral is equal to

$$\int_0^\infty \frac{dy}{2\pi} \frac{1}{\sqrt{y^2-4}} \frac{1}{y + \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}}. \tag{3.68}$$

This can be evaluated using (3.56). As the result of the calculation, we obtain for (3.67)

$$-\frac{1}{\pi} \frac{\sqrt{2}}{\sqrt{1-\sigma_0\sigma_1 + \sqrt{1-\sigma_0^2}\sqrt{1-\sigma_1^2}}}$$
$$\times \left[\frac{\pi}{2} - \tan^{-1} \sqrt{\frac{2 + \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}}{2 - \sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + \sqrt{1-\sigma_0}\sqrt{1-\sigma_1}}} \right]. \tag{3.69}$$

Similarly, we can evaluate the second term in (3.66). Using the parametrisation (1.50), the result for the middle propagator simplifies as

$$-\frac{1}{2}\left(a_0 \frac{\cos a_0\pi}{\sin a_0\pi} + a_1 \frac{\cos a_1\pi}{\sin a_1\pi} \right). \tag{3.70}$$

Now lets look at the propagator $G(x,t;0,t'')$ in (3.61),

$$i \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} e^{-i\omega t} \frac{f(k,x)f(-k,0)e^{ikx} + K_0(k)f(-k,0)f(-k,0)e^{-ikx}}{\omega^2 - k^2 - 4 + i\epsilon} \tag{3.71}$$

Using (3.3) and (3.4), setting $k \to -k$ in the first term of the propagator, we have

$$i \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{ie^{-i\omega t}}{\omega^2 - k^2 - 4 + i\epsilon} \frac{2ik\,(ik + 2\coth 2(x-x_0))}{P(k)} e^{-ikx}. \tag{3.72}$$

The momenta may be integrated out by closing the contour in the upper half plane and picking up the pole at $k = \sqrt{k^2-4} \equiv \hat{k}$. The poles coming from $P(k)$ do not contribute if $\sigma_i > 0$. However, there is a pole contribution when $\sigma_i < 0$. We can avoid it since its residue integrated over k yields the exponentially decreasing term as $x \to -\infty$. As a result we have

$$\int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{2\hat{k}} \frac{2i\hat{k}\left(i\hat{k} + 2\coth 2(x-x_0)\right)}{P(\hat{k})} e^{-i\hat{k}x} \tag{3.73}$$

The integral over k' for the propagator $G(0,t'';x',t')$ in (3.61) can be evaluated in a similar way to the k integral. If we combine the three propogators, then the boundary contribution can be written as

$$-\frac{i}{8}\sigma_+\beta^2\left(a_0\frac{\cos a_0\pi}{\sin a_0\pi}+a_1\frac{\cos a_1\pi}{\sin a_1\pi}\right)\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\frac{\left(i\hat{k}+2\coth 2(x-x_0)\right)}{P(\hat{k})}$$
$$e^{-i\hat{k}(x+x')}\frac{\left(i\hat{k}+2\coth 2(x'-x_0)\right)}{P(\hat{k})} \tag{3.74}$$

Here $\coth 2(x-x_0)$ and $\coth 2(x'-x_0)$ tend to minus one in the limits $x, x' \to -\infty$. If we insert $K_0(\hat{k})K_0(-\hat{k})$ into the above expression, then the integrand simplifies to

$$\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}e^{-i\hat{k}(x+x')}K_0(\hat{k})\frac{(i\hat{k}+2)(i\hat{k}-2)}{P(\hat{k})P(-\hat{k})}. \tag{3.75}$$

By making the substitution $\hat{k}=2\sinh\theta$ into the expression, we have

$$-\frac{1}{8}\int\frac{d\omega}{2\pi}e^{-i\hat{k}(x+x')}e^{-i\omega(t-t')}\frac{K_0(\theta)}{\sin^2(e_0\pi/2)-\sin^2(f_0\pi/2)}$$
$$\left[\frac{\sin(e_0\pi/2)}{\cosh\theta-\sin(e_0\pi/2)}-\frac{\sin(e_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}\right.$$
$$\left.+\frac{\sin(f_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}-\frac{\sin(f_0\pi/2)}{\cosh\theta-\sin(f_0\pi/2)}\right] \tag{3.76}$$

where

$$e_0 = a_0 + a_1, \qquad f_0 = a_0 - a_1. \tag{3.77}$$

Substituting the integral into the equation (3.74), the boundary contribution becomes

$$\frac{i\beta^2}{64}\sigma_+\int\frac{d\omega}{2\pi}e^{-i\hat{k}(x+x')}e^{-i\omega(t-t')}\frac{K_0(\theta)}{\sin^2(e_0\pi/2)-\sin^2(f_0\pi/2)}$$
$$\left(a_0\frac{\cos a_0\pi}{\sin a_0\pi}+a_1\frac{\cos a_1\pi}{\sin a_1\pi}\right)\left[\frac{\sin(e_0\pi/2)}{\cosh\theta-\sin(e_0\pi/2)}-\frac{\sin(e_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}\right.$$
$$\left.+\frac{\sin(f_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}-\frac{\sin(f_0\pi/2)}{\cosh\theta-\sin(f_0\pi/2)}\right]. \tag{3.78}$$

From this, we can extract the correction to the reflection factor

$$\frac{i\beta^2}{16}\frac{\sigma_+K_0(\theta)\sinh\theta}{\sin^2(e_0\pi/2)-\sin^2(f_0\pi/2)}\left(a_0\frac{\cos a_0\pi}{\sin a_0\pi}+a_1\frac{\cos a_1\pi}{\sin a_1\pi}\right)$$
$$\left[\frac{\sin(e_0\pi/2)}{\cosh\theta-\sin(e_0\pi/2)}-\frac{\sin(e_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}\right.$$
$$\left.+\frac{\sin(f_0\pi/2)}{\cosh\theta+\sin(f_0\pi/2)}-\frac{\sin(f_0\pi/2)}{\cosh\theta-\sin(f_0\pi/2)}\right]. \tag{3.79}$$

We can compare this result with the contribution to the reflection factor obtained in [24]. If we take $\sigma_0 = \sigma_1 = \sigma$, in other words $a_0 = a_1 = a$, then the equation (3.79) reduces to (3.59).

Now let us consider the case where the vertex is located in the bulk region for diagram (b). The contribution has the integral form

$$-4i\beta^2 \int_{-\infty}^{+\infty} dt'' \int_{-\infty}^{0} dx'' \cosh\sqrt{2}\beta\phi_0 \, G\left(x,t;\; x'',t'\right) G\left(x'',t'';\; x'',t''\right) \; G\left(x'',t'';\; x',t'\right), \tag{3.80}$$

where $\cosh\sqrt{2}\beta\phi_0 = 2\coth^2(x''-x_0)-1$. The integration over t'' is straightforward again and provides the delta function $\delta(\omega-\omega')$ allowing the integral ω or ω' to be performed. The middle propagator is

$$\int \frac{d\omega''}{2\pi}\int\frac{dk''}{2\pi}\,\frac{i}{\omega''^2-k''^2-m^2+i\epsilon}\Bigg[\frac{(ik'')^2-4\coth 2(x''-x_0)}{ik''+2} \\ +K_0(k'')\left(\frac{ik''+2\coth 2(x''-x_0)}{ik''-2}\right)^2 e^{-2ik''x''}\Bigg]. \tag{3.81}$$

Rewriting the first term inside the bracket as

$$1+\frac{4-4\coth^2(x''-x_0)}{(ik'')^2-4}, \tag{3.82}$$

we can separate the finite and divergent parts. The divergency again can be removed by infinite mass renormalisation. The propagator $G(x,t;x'',t')$ can be expressed as

$$\int \frac{d\omega}{2\pi}\int\frac{dk}{2\pi}\,\frac{ie^{-i\omega t}}{\omega^2-k^2-m^2+i\epsilon}\Bigg[\frac{ik-2\coth 2(x''-x_0)}{ik''+2}e^{ikx''} \\ +K_0(k)\frac{ik+2\coth 2(x''-x_0)}{ik-2}e^{-ikx''}\Bigg]e^{-ikx}. \tag{3.83}$$

Here in order to write the exponential dependence on x outside the bracket, we have replaced k by $-k$ in the first term. We have also made the substitution $\coth 2(x''-x_0)=-1$ in the limit $x''\to-\infty$. The third propagator can also be written in the limit $x'\to-\infty$. Combining the three propagators, the bulk contribution can be expressed as

$$-4i\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \int \frac{dk''}{2\pi} \frac{1}{2\sqrt{k''^2+4}} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - 4 + i\epsilon} \frac{ie^{-i(kx+k'x')}}{\omega^2 - k'^2 - 4 + i\epsilon}$$
$$\int_{-\infty}^{0} dx'' \left(2\cosh^2 2(x''-x_0) - 1\right)$$
$$\left[\frac{4 - 4\coth^2 2(x''-x_0)}{(ik+2)(ik-2)} + K_0(k'')\left(\frac{ik'' + 2\coth 2(x''-x_0)}{ik''-2}\right)^2 e^{-2ik''x''}\right]$$
$$\left[\frac{ik' - 2\coth 2(x''-x_0)}{ik'+2} e^{ik'x''} + K_0(k')\frac{ik' + 2\coth 2(x''-x_0)}{ik'-2} e^{-ik'x''}\right]$$
$$\left[\frac{ik - 2\coth 2(x''-x_0)}{ik+2} e^{ik'x''} + K_0(k)\frac{ik + 2\coth 2(x''-x_0)}{ik-2} e^{-ikx''}\right]. \tag{3.84}$$

However the computation is intricate on the bulk contribution. The work on this calculation is still in progress.

We can see that the three point couplings appear in the diagrams c and d.

For the diagram c, there are three contributions need to be calculated. One is to calculate the case where the both vertices sit on the boundary. The second contribution comes from the calculation in which one vertex is situated on the boundary and other vertex is in the bulk region. The third one is to calculate the both vertices in the bulk region. In this diagram the two vertices are symmetrical.

For the diagram d, the two vertices are not symmetrical. In this case we need to calculate four contributions. The first two are to calculate the cases where the both vertices are situated at the boundary or on the bulk region. The third one is to calculate the case where the first vertex is located on the boundary and the second one is in the bulk region. The fourth contribution comes from the calculation where the first vertex is in the bulk region and second one on the boundary.

The computation of these two diagrams may be found elsewhere [40].

The perturbation expansion of the exact quantum reflection factor (1.69) up to

$O(\beta^2)$ is obtained in [24]

$$R_{shG}(\theta) \sim K_0(\theta) \left[1 - \frac{i\beta^2}{8} \sinh\theta \left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta} \right.\right.$$
$$+ \frac{e_1}{\cosh\theta + \sin(e_0\pi/2)} - \frac{e_1}{\cosh\theta - \sin(e_0\pi/2)} \tag{3.85}$$
$$\left.\left. + \frac{f_1}{\cosh\theta + \sin(f_0\pi/2)} - \frac{f_1}{\cosh\theta - \sin(f_0\pi/2)}\right)\right],$$

where e_0 and f_0 are given by (3.77).

If the loop calculations are completed, we will be able to obtain the quantum reflection factors up to the order of the β^2 and it can be compared with (3.85). The coefficients e_1 and f_1 will be then known, once the loop calculations are done.

Chapter 4

Supersymmetric Extension of the sinh-Gordon Model with one Boundary

4.1 Introduction

Over the past few years, there has been an interest in supersymmetric aspects of the integrable two-dimensional quantum field theory. The bosonic Toda field theories have been studied as important examples of integrable models. The problem of incorporating fermions into the bosonic model has been considered by many authors [41–47] with much attention focused on finding a supersymmetric model. It turns out that the bosonic Toda models based on simple Lie algebra cannot be supersymmetrised, except in the simplest case of the Liouville theory [48, 49] and the sine-Gordon model. More precisely, it is believed that the supersymmetry is broken, where the bosonic part of the model is based on a simple Lie algebra of rank bigger than one.

The supersymmetric version of the sine-Gordon equation was first introduced by Hruby [50] and di Vecchia and Ferrara [51] independently. Shankar and Witten constructed the exact S-matrix for the supersymmetric sine-Gordon equation in

1978, while they were studying the supersymmetric nonlinear sigma model [52]. It was further investigated in [53, 54]. All these works considered the $N = 1$ model. The existence of an infinite set of conservation laws had been proved at the classical level [55, 56] as well as at the quantum level [57–60]. For N=2 supersymmetry, the S-matrix was studied in [61], where the theory was approached as a deformation of a superconformal model.

Recently, the supersymmetric aspect of the boundary integrable field theory has been investigated. Warner [62, 63] studied the quantum integrable model that possesses $N = 2$ supersymmetry and argued that one can retain only half the supersymmetry in the presence of the boundary. It has been pointed out by Inami, Odake and Zhang [64] that only a few boundary conditions are compatible with both supersymmetry and integrability for the sine-Gordon model. Their observation is based on the study of conserved charges at the classical level. Subsequently, Moriconi and Schoutens conjectured the exact reflection matrices for the breather multiplets of the $N = 1$ supersymmetric sine-Gordon theory [65]. They also pointed out the connection between their reflection matrices and the classical boundary action as constructed in [64]. Following the idea in [64], similar considerations have been applied to the super-Liouville theory [66]. More recently, Mussardo has calculated the exact form factor for the supersymmetric sinh-Gordon model [67].

In this chapter and the following one, we will study supersymmetric aspect of the sinh-Gordon theory with one boundary condition.

The motivation for studying the boundary sinh-Gordon model in the framework of $N = 1$ supersymmetry arises out of the work in papers [64, 65]. Once the exact reflection factors are known, once could check it by developing the perturbation theory at certain order.

In this chapter, we check the exact reflection matrices proposed by Moriconi and Schoutens perturbatively up to the second order of the coupling constant. We found that the classical limit of their reflection matrices is incorrect. Therefore, a correction has been made to their formula. By doing that, the reflection matrices obtained perturbatively have the correct classical limit. To begin with, let us review the work on the supersymmetric S-matrix and reflection factors.

4.2 S-matrix for the supersymmetric theory

The $N = 1$ supersymmetric theory contains a conserved Majorana supercharge. In terms of the chiral components $Q_\pm$ of the supercharge, we can write the supersymmetry algebra as

$$Q_\pm^2 = \omega \pm k, \qquad \{Q_+, Q_-\} = 0, \qquad \{Q_L, Q_\pm\} = 0. \tag{4.1}$$

where the operator Q_L has eigenvalue $+1$ on bosonic states and -1 on fermionic states. The one-particle state of a massive supersymmetric theory has one bosonic and one fermionic particle of equal mass m and denoted by $|b(\theta)>$ and $|f(\theta)>$.

It follows from algebra (4.1) that, the action of the supercharges $Q_\pm$ on the one particle states can be represented by

$$\begin{aligned} Q_+|b(\theta)> &= \sqrt{m}e^{\theta/2}|f(\theta)>, \quad Q_+|f(\theta)>= \sqrt{m}e^{\theta/2}|b(\theta)>, \\ Q_-|b(\theta)> &= i\sqrt{m}e^{-\theta/2}|f(\theta)>, \quad Q_-|f(\theta)>= -i\sqrt{m}e^{-\theta/2}|b(\theta)> . \end{aligned} \tag{4.2}$$

These relations correspond to the following realization of the algebra

$$Q_+ = \sqrt{m}e^{\theta/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_- = \sqrt{m}e^{-\theta/2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Q_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

Now let us study the S-matrix in the supersymmetric theory. Scattering theories have been discussed in detail in Schoutens's paper [53]. The requirement that supersymmetry commutes with the scattering leads to the conditions on the matrix S. Schoutens looked for a solution of the factorised form

$$S = S_B\, S_{BF}, \tag{4.4}$$

where S_B is the bosonic S-matrix and S_{BF} is the Bose-Fermi S-matrix, which is the supersymmetric piece, responsible for mixing bosons and fermions.

By assuming S_B is a diagonal matrix, he solved the conditions and found S_{BF} for breather multiplets (labelled by i and j indices) of the supersymmetric theory. On the two-particle basis $|b_i b_j>$, $|b_i f_j>$, $|f_i b_j>$, $|f_i f_j>$, it is given in the form

$$S_{BF}^{[ij]}(\theta) = f^{[ij]}(\theta) \begin{pmatrix} 1-r\tilde{r} & 0 & 0 & -i(r+\tilde{r}) \\ 0 & -r+\tilde{r} & 1+r\tilde{r} & 0 \\ 0 & 1+r\tilde{r} & r-\tilde{r} & 0 \\ -i(r+\tilde{r}) & 0 & 0 & 1-r\tilde{r} \end{pmatrix} + g^{[ij]}(\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.5}$$

where $r = (\tanh([\theta + \log(m_i/mj)]/)\,4$ and $\tilde{r} = \tanh([\theta - \log(m_i/mj)]/)4$. The two functions $f^{[ij]}(\theta)$ and $g^{[ij]}(\theta)$ are related to each other and are fixed by the Yang-Baxter equation up to one constant α,

$$f^{[ij]}(\theta) = \frac{\alpha}{4i}\sqrt{m_i m_j}\left[\frac{2\cosh(\theta/2) + (\rho^2+\rho^{-2})}{\cosh(\theta/2)\,\sinh(\theta/2)}\right] g^{[ij]}, \tag{4.6}$$

where $\rho = (m_i/m_j)^{1/4}$.

Using analyticity, crossing symmetry and unitarity conditions of the scattering matrix, Shoutens fixed the function $g^{[ij]}(\theta)$,

$$g^{[ij]}(\theta) = \frac{g_{\Delta_1}(\theta)\, g_{\Delta_2}(\theta)}{g_{\Delta_3}(\theta)} \tag{4.7}$$

where

$$g_\Delta = \frac{\sinh\frac{\theta}{2}}{\sinh\frac{\theta}{2} + i\sin\Delta\pi}\, \exp\left[i\int_0^\infty \frac{dt}{t}\, \frac{\sinh\Delta t\, \sinh(1-\Delta)t}{\cosh^2\frac{t}{2}}\, \sin\frac{\theta t}{\pi}\right] \tag{4.8}$$

with $\Delta_1 = \frac{1}{2}(i+j)\beta, \Delta_2 = \frac{1}{2}(1-(i-j)\beta)$ and, $\Delta_3 = \frac{1}{2}$. We should notice that the parameter β in here is different from the coupling constant we have been using in the sinh-Gordon theory. We shall point out their relation later.

Schoutens also fixed the parameter α by looking at the bootstrap relations. It was showed that for the particles b_i and b_j in a bound state b_k, the following is true

$$\alpha = -\frac{\left(2m_i^2 m_j^2 + 2m_i^2 m_k^2 + 2m_j^2 m_k^2 - m_i^4 - m_j^4 - m_k^4\right)^{1/2}}{2m_i\, m_j\, m_k}. \tag{4.9}$$

The masses m_j of the particle b_j put in

$$m_j = \frac{\sin(j\beta\pi)}{\sin(\beta\pi)}, \quad j = 1, 2, ...n. \tag{4.10}$$

4.3 The construction of the supersymmetric reflection factors

In this section, we review how to obtain the boundary reflection matrix for the N=1 supersymmetric theory. It was assumed in [65] that the reflection matrix can be factorised in a similar way to the bulk S-matrix,

$$R(\theta) = R_B(\theta)\ R_{BF}(\theta), \tag{4.11}$$

where $R_B(\theta)$ is the reflection matrix for the bosonic part of the theory, and R_{BF} is the supersymmetric part of the reflection matrix. In a basis $|b>, |f>$, it can be represented as

$$R_{BF}(\theta) = \begin{pmatrix} R_{bb} & R_{bf} \\ R_{fb} & R_{ff} \end{pmatrix}. \tag{4.12}$$

In order to preserve the supersymmetry, the reflection matrix must commute with a linear combination $Q(\theta)$ of the supercharges. That means, we should be able to act with a supersymmetry before and after reflection, and obtain the same result,

$$Q(\theta)\ R(\theta) = R(\theta)\ Q(-\theta), \tag{4.13}$$

where the linear combination is $Q(\theta) = aQ_+(\theta) + bQ_-(\theta)$, and a and b are real constants. In trying to find the solutions to (4.13), we find that the parameters a

and b are related to each other by $b = \pm a$. We can choose the parameter a as equal to one, and rewrite the linear combinations as

$$Q^{(\pm)}(\theta) = Q_+(\theta) \pm Q_-(\theta). \tag{4.14}$$

By solving the commutation relation (4.13), we obtain the reflection matrices

$$R^{\pm}_{BF}(\theta) = Z^{\pm}(\theta) \begin{pmatrix} \cosh(\frac{\theta}{2} \pm \frac{i\pi}{4}) & e^{\mp i\pi/4}\, Y(\theta) \\ e^{\pm i\pi/4}\, Y(\theta) & \cosh(\frac{\theta}{2} \mp \frac{i\pi}{4}) \end{pmatrix}. \tag{4.15}$$

These are the most general reflection matrices compactible with the realization (4.3).

If one imposes that the boundary has no structure, which means the boundary can not change the fermion number of incoming particles ($Y(\theta) = 0$), then the reflection matrices can be simplified to the following

$$R^{\pm}_{BF}(\theta) = Z^{\pm}(\theta) \begin{pmatrix} \cosh(\frac{\theta}{2} \pm \frac{i\pi}{4}) & 0 \\ 0 & \cosh(\frac{\theta}{2} \mp \frac{i\pi}{4}) \end{pmatrix}. \tag{4.16}$$

Moriconi and Schoutens [65] have found that $Y(\theta) = 0$ by solving the boundary Yang-Baxter equation.

However the following combination of supercharges

$$Q^{(\pm)}(\theta) = Q_+(\theta) \mp Q_-(\theta). \tag{4.17}$$

has been used in order to obtain the result (4.16) in [65]. We should mention that, one could not derive (4.16) using (4.17) as Moriconi and Schoutens suggested, but from (4.14). The ratio of the boson and fermion reflection factor can be read as

$$\begin{aligned} \frac{R^{\pm}_b(\theta)}{R^{\pm}_f(\theta)} &= \frac{\cosh(\frac{\theta}{2} \pm \frac{i\pi}{4})}{\cosh(\frac{\theta}{2} \mp \frac{i\pi}{4})} \\ &= \frac{1 \pm i \sinh\theta}{\cosh\theta}. \end{aligned} \tag{4.18}$$

The ratio does not depend on the coupling constants and boundary parameters. The amplitudes for a particle and its superpartner scattering off the boundary are related by a universal function, independent of the masses.

The factor $Z^{\pm}(\theta)$ in the reflection factors is determined by imposing the boundary unitarity condition $R^{\pm}(\theta)R^{\pm}(-\theta) = 1$ as well as the boundary crossing unitarity condition $R_a\left(\frac{i\pi}{2} - \theta\right) = S_{cc}^{aa}(2\theta)R_c\left(\frac{i\pi}{2} + \theta\right)$. These conditions lead to

$$Z_j^{\pm}(\theta)Z_j^{\pm}(-\theta) = 2/\cosh\theta,$$
$$\frac{Z_j^{\pm}\left(\frac{i\pi}{2} - \theta\right)}{Z_j^{\pm}\left(\frac{i\pi}{2} + \theta\right)} = \mp S_{b_jb_j}^{b_jb_j}(2\theta) + if^{\pm}(\theta)\, S_{f_jf_j}^{b_jb_j}(2\theta), \tag{4.19}$$

where $f^{+}(\theta) = \coth\frac{\theta}{2}$ and $f^{-}(\theta) = \tanh\frac{\theta}{2}$, depending on which sign we choose in $R_{BF}^{(\pm)}(\theta)$. The index j in the factor $Z_j^{(\pm)}(\theta)$ expresses the dependence of these functions on the mass m_j of the reflecting particle.

Moriconi and Schoutens solved (4.19) for $Z_j^{\pm}(\theta)$. For convenience, they first wrote $Z_j^{\pm}(\theta)$ as

$$Z_j^{\pm}(\theta) = \frac{\tilde{Z}_j^{\pm}(\theta)}{\cosh\left(\frac{\theta}{2} \pm \frac{i\pi}{4}\right)}. \tag{4.20}$$

Hence the first condition in (4.19) becomes

$$\tilde{Z}_j^{\pm}(\theta)\tilde{Z}_j^{\pm}(-\theta) = 1. \tag{4.21}$$

Next, they considered the two possibilities $(+)$ and $(-)$ separately for the second condition in (4.20). In the $(+)$ case, the boundary crossing-unitarity condition becomes,

$$\frac{\tilde{Z}_j^{+}\left(\frac{i\pi}{2} - \theta\right)}{\tilde{Z}_j^{+}\left(\frac{i\pi}{2} + \theta\right)} = S_{b_jb_j}^{b_jb_j}(2\theta) - i\coth\frac{\theta}{2}S_{f_jf_j}^{b_jb_j}(2\theta), \tag{4.22}$$

where the scattering elements $S_{b_jb_j}^{b_jb_j}(2\theta)$ and $S_{f_jf_j}^{b_jb_j}(2\theta)$ can be found explicitly from (4.5). Furthermore, one can substitute them into (4.22) and write it explicitly as

$$\frac{\tilde{Z}_j^{+}\left(\frac{i\pi}{2} - \theta\right)}{\tilde{Z}_j^{+}\left(\frac{i\pi}{2} + \theta\right)} = \frac{\sinh\theta - i\sin j\beta t}{\sinh\theta + i\sin j\beta t}$$
$$\times\exp\left[i\int_0^{\infty}\frac{dt}{t}\,\frac{\sinh j\beta t\,\sinh(1-j\beta)t}{\cosh^2\frac{t}{2}\,\cosh t}\,\sin\frac{2\theta t}{\pi}\right], \tag{4.23}$$

They also solved $\tilde{Z}_j^{+}$ for this equation together with the unitarity condition (4.21) that corresponds to the $(+)$ case, using the following integral representation

$$\frac{\sinh\theta - i\sin\alpha\pi}{\sinh\theta + i\sin\alpha\pi} = \exp\left[4i\int_0^{\infty}\frac{dt}{t}\,\frac{\cosh\frac{\alpha}{2}t\,\cosh\frac{1-\alpha}{2}t}{\cosh\frac{t}{2}}\,\sin\frac{\theta t}{\pi}\right], \tag{4.24}$$

and placing the solution back into (4.20) they finally obtained for $Z_j^+(\theta)$,

$$Z_j^+(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)} \exp\left[-2i\int_0^\infty \frac{dt}{t} \frac{\cosh\frac{j\beta t}{2}\cosh\frac{1}{2}(1-j\beta)t}{\cosh^2\frac{t}{2}} \sin\frac{\theta t}{\pi}\right]$$
$$\exp\left[-\frac{i}{2}\int_0^\infty \frac{dt}{t}\frac{\sinh j\beta t\ \sinh(1-\beta j)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \quad (4.25)$$

For the second case $(-)$, the boundary crossing-unitarity condition is relatively simple and can be written as

$$\frac{\tilde{Z}_j^-\left(\frac{i\pi}{2}-\theta\right)}{\tilde{Z}_j^-\left(\frac{i\pi}{2}+\theta\right)} = -S_{b_jb_j}^{b_jb_j}(2\theta) + i\tanh\frac{\theta}{2}S_{f_jf_j}^{b_jb_j}(2\theta) \quad (4.26)$$

After placing the scattering elements into this expression, it becomes

$$\frac{\tilde{Z}_j^-\left(\frac{i\pi}{2}-\theta\right)}{\tilde{Z}_j^-\left(\frac{i\pi}{2}+\theta\right)} = \exp\left[i\int_0^\infty \frac{dt}{t}\frac{\sinh j\beta t\ \sinh(1-j\beta)t}{\cosh^2\frac{t}{2}\ \cosh t}\sin\frac{2\theta t}{\pi}\right]. \quad (4.27)$$

They solved the equation for $\tilde{Z}_j^{(-)}(\theta)$ and obtained

$$Z_j^-(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)}\exp\left[-\frac{i}{2}\int_0^\infty \frac{dt}{t}\frac{\sinh j\beta t\ \sinh(1-j\beta)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \quad (4.28)$$

They claimed that the results obtained in (4.25) and (4.28) are unique solutions to (4.20). By placing them back into (4.20), we have seen that they are indeed solutions to (4.20). However these solutions are not unique. We shall see in section 4.5 that the supersymmetric reflection factors obtained from these solutions do not have the correct classical limits. In order to obtain correct results, we must look for another solution.

Let us look at the integral representation (4.24). It is easy to verify that this expression is equivalent to the following one,

$$\frac{\sinh\theta - i\sin\alpha\pi}{\sinh\theta + i\sin\alpha\pi} = \exp\left[-4i\int_0^\infty \frac{dt}{t}\ \frac{\sinh\frac{\alpha}{2}t\ \sinh\frac{1-\alpha}{2}t}{\cosh\frac{t}{2}}\ \sin\frac{\theta t}{\pi}\right]. \quad (4.29)$$

In fact, one can also find this relation in Mussardo's paper [67]. The equation (4.23) can now be written in terms of this integral representation as

$$\frac{\tilde{Z}_j^+\left(\frac{i\pi}{2}-\theta\right)}{\tilde{Z}_j^+\left(\frac{i\pi}{2}+\theta\right)} = \exp\left[-4i\int_0^\infty \frac{dt}{t}\ \frac{\sinh\frac{j\beta t}{2}\ \sinh\frac{1-j\beta}{2}t}{\cosh^2\frac{t}{2}}\ \sin\frac{2\theta t}{\pi}\right],$$
$$\times\exp\left[i\int_0^\infty \frac{dt}{t}\ \frac{\sinh j\beta t\ \sinh(1-j\beta)t}{\cosh^2\frac{t}{2}\ \cosh t}\ \sin\frac{2\theta t}{\pi}\right]. \quad (4.30)$$

The solution to this equation can be constructed, and furthermore one can obtain the solution for $Z_j^+(\theta)$ as follows,

$$Z_j^+(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)} \exp\left[2i\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{j\beta t}{2}\sinh\frac{1-j\beta}{2}t}{\cosh^2\frac{t}{2}}\sin\frac{\theta t}{\pi}\right]$$
$$\exp\left[-\frac{i}{2}\int_0^\infty \frac{dt}{t}\frac{\sinh j\beta t\ \sinh(1-\beta j)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right].\qquad(4.31)$$

We have found that this solution obtained using (4.29) together with one (4.28) are also solutions to (4.20). Therefore the statement made by Moriconi and Schoutens to the solutions of (4.20) was incorrect. Once we fix the factor $Z_j^{\pm}(\theta)$, the supersymmetric reflection matrices can be obtained.

In reference [64], Inami, Odake and Zhang found that there are two special choices for boundary conditions in the supersymmetric sine-Gordon theory, such that the theory is both supersymmetric and integrable. These conditions at $x=0$ are given by

$$\partial_x\phi \pm \frac{2m}{\beta_{sG}}\sin\frac{\beta_{sG}\phi}{2} = 0. \qquad \psi \mp \bar{\psi} = 0. \qquad (4.32)$$

The parameter in the bosonic part of the conditions is related to the notation introduced by Ghoshal and Zamolodchikov. Their relations are $M = M_{\pm} = \pm\frac{4m}{\beta^2}$, $\phi_0 = 0$.

Moriconi and Schoutens referred to these boundary conditions as $\mathcal{BC}^{\pm}$. They conjectured that at two special points $M_{\pm}$, the reflection factor for the bound state supermultiplets will be the form of $R_B(\theta)\, R_{BF}(\theta)$.

In the supersymmetric theory, we consider the boundary conditions in which ϕ_0 is zero. This implies that $\vartheta = 0$ in the reflection matrix [22]. The relationship of the boundary parameter ϕ_0 and the reflection parameter ϑ is given in [22]. In this case, the sine Gordon reflection factor for the lightest breather (1.67) becomes

$$R_{sG}(\theta|\eta,0) = \frac{\left(2+\frac{1}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)}{\left(1+\frac{2\eta}{\pi\lambda}\right)\left(1-\frac{2\eta}{\pi\lambda}\right)(1)}. \qquad (4.33)$$

The reflection factor is zero when θ takes one of the following values $\frac{i\pi}{2}$, $\frac{i\pi(1-\lambda)}{2\lambda}$ and $\frac{i\pi(2\lambda-1)}{2\lambda}$. However, we can check that one of these zeros is cancelled when $\eta = \lambda\pi, \frac{\pi}{2}, \frac{\pi}{2}(\lambda+1)$ respectively. Ghoshal and Zamolodchikov have identified the point $\frac{\pi}{2}(\lambda+1)$ that corresponds to the free boundary condition which means $M = 0$. It therefore is irrelevant to the discussion on the supersymmetric sine-Gordon theory.

Moriconi and Schoutens specified two things. Firstly, they conjectured that the two special points that corresponds to the integrable and supersymmetric boundary potential are the remaining two points $\eta = \frac{\pi}{2}$ and $\eta = \lambda\pi$. Secondly, they pointed out the correspondence between the two values of η with the reflection factors. They also matched the choice of sign in the reflection matrices $R_{BF}^{(\pm)}$ with the choice of sign in the boundary conditions $\mathcal{BC}^{\pm}$. The exact reflection matrices they proposed for the supersymmetric sine-Gordon theory are

$$\begin{aligned} &R_{sG}(\theta|\pi/2,0)\ R_{BF}^{+}(\theta) \qquad for \qquad \mathcal{BC}^{+}, \\ &R_{sG}\,(\theta|\lambda\pi,0)\ \ R_{BF}^{-}(\theta) \qquad for \qquad \mathcal{BC}^{-}, \end{aligned} \tag{4.34}$$

where $R_{sG}(\theta|\pi/2,0)$ and $R_{sG}(\theta|\lambda\pi,0)$ are the reflection factors of the ordinary sine-Gordon model [27] at two special points respectively.

4.4 The reflection factors for the supersymmetric sinh-Gordon model

According Moriconi and Schoutens' conjecture, we can write the supersymmetric reflection factors for the sinh-Gordon model as follows

$$\begin{aligned} &R_{shG}(\theta|\pi/2,0)\ R_{BF}^{+}(\theta) \qquad for \qquad \mathcal{BC}^{+}, \\ &R_{shG}\,(\theta|\lambda\pi,0)\ \ R_{BF}^{-}(\theta) \qquad for \qquad \mathcal{BC}^{-}, \end{aligned} \tag{4.35}$$

where $R_{shG}(\theta)$ is the reflection factor for sinh-Gordon model and given by (1.69). The supersymmetric part of the reflection matrices $R_{BF}^{(\pm)}(\theta)$ can be obtained from the lightest breather in (4.16) by analytic continuation in the coupling constant which is contained in the factor $Z^{\pm}$.

The parameter β in $Z^{\pm}$ is related to λ in Ghoshal's formula by $\beta = 1/2\lambda$. We can therefore easily find the relation between the parameter β with the sine-Gordon coupling constant β_{sG} up to order $O(\beta_{sG}^2)$

$$\beta = \frac{1}{16\pi}\beta_{sG}^2. \tag{4.36}$$

In the first chapter, we have given the relation of the sine-Gordon and sinh-Gordon coupling constants, that is $\beta_{sG}^2 = -2\beta^2$. However we notice that the notation used by Moriconi and Schoutens for the coupling β is the same as we have been using for the sinh-Gordon model. One should bear in mind that they are two different couplings, one is the coupling of the sine-Gordon theory and another one is the coupling of the sinh-Gordon theory.

By analytic continuation in the coupling constant we mean

$$\beta \to -\frac{\beta^2}{8\pi}. \tag{4.37}$$

4.5 Supersymmetric reflection factors for $\mathcal{BC}^+$ case

In both this and the following section, we study the supersymmetric reflection factors of the sinh-Gordon theory.

According to (4.35), the supersymmetric reflection factors of the sinh-Gordon model corresponding to the boundary condition $\mathcal{BC}^+$ can be written as

$$\begin{aligned} R_b^+(\theta) &= R(\theta|\pi/2,0)\; R_b^{(+)}(\theta) \\ R_f^+(\theta) &= R(\theta|\pi/2,0)\; R_f^{(+)}(\theta), \end{aligned} \tag{4.38}$$

where $R_b^+(\theta)$ and $R_f^+(\theta)$ are the supersymmetric bosonic and fermionic reflection factors respectively. The bosonic part of the reflection factor $R(\theta|\pi/2,0)$ can be obtained using (1.69), and is

$$R(\theta|\pi/2,0) = \frac{(2-B/2)}{(1-B/2)\,(1)}. \tag{4.39}$$

We can write it down more explicitly as

$$\frac{\left(\sinh\theta + \sinh\frac{i\beta}{8}\right) - i\left(\cosh\theta + \cosh\frac{i\beta^2}{8}\right)}{\left(\sinh\theta - \sinh\frac{i\beta}{8}\right) + i\left(\cosh\theta + \cosh\frac{i\beta^2}{8}\right)}\;\frac{\sinh\frac{\theta}{2} - i\cosh\frac{\theta}{2}}{\sinh\frac{\theta}{2} + i\cosh\frac{\theta}{2}}, \tag{4.40}$$

here we have replaced B in terms of its expansion up to the order of β^2. Furthermore, we can obtain the perturbation expansion of this expansion up to $O(\beta^2)$, and find that is

$$R(\theta|\pi/2,0) \sim \frac{i\sinh\theta+1}{i\sinh\theta-1}\left[1-\frac{i\beta^2}{8}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right]. \tag{4.41}$$

In (4.38), $R_b^{(+)}(\theta)$ and $R_f^{(+)}(\theta)$ are given by

$$R_b^{(+)}(\theta) = Z^+(\theta)\cosh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right), \tag{4.42}$$

$$R_f^{(+)}(\theta) = Z^+(\theta)\cosh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right). \tag{4.43}$$

Firstly we perform the calculation using the $Z^+(\theta)$ obtained by Moriconi and Schoutens in (4.25) for the lightest breather ($j=1$). We replace β by $-\beta^2/8\pi$, then the expression becomes

$$Z^+(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}\exp\left[-2i\int_0^\infty\frac{dt}{t}\frac{\cosh\frac{\beta^2 t}{16\pi}\cosh\frac{1}{2}\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}}\sin\frac{\theta t}{\pi}\right]$$
$$\exp\left[-\frac{i}{2}\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{\beta^2 t}{8\pi}\sinh\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \tag{4.44}$$

Substituting this into (4.42), we have

$$R_b^{(+)}(\theta) = \exp\left[-2i\int_0^\infty\frac{dt}{t}\frac{\cosh\frac{\beta^2 t}{16\pi}\cosh\frac{1}{2}\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}}\sin\frac{\theta t}{\pi}\right]$$
$$\exp\left[\frac{i}{2}\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{\beta^2 t}{8\pi}\sinh\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \tag{4.45}$$

Let us look at the first exponential first. It can be written as

$$\exp\left[-2\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{it\theta}{\pi}}{\cosh\frac{t}{2}}-\frac{\beta^2}{8\pi}\int_0^\infty dt\frac{\sinh\frac{t}{2}\sinh\frac{it\theta}{\pi}}{\cosh^2\frac{t}{2}}\right]. \tag{4.46}$$

After performing integrals [68] and then expanding the exponential up to the order of β^2, we obtain

$$\left(1-\frac{i\beta^2}{4\pi}\frac{\theta}{\cosh\theta}\right)\cot^2\left(\frac{i\theta}{2}+\frac{i\pi}{4}\right). \tag{4.47}$$

Now let us look at the second exponential in (4.45). When we also expand it to order $O(\beta^2)$, it becomes

$$1+\frac{i\beta^2}{16\pi}\int_0^\infty dt\frac{\sinh t}{\cosh^2\frac{t}{2}\,\cosh^2 t}\sinh\frac{2t\theta}{\pi}. \tag{4.48}$$

The integral can be evaluated, the expression then is

$$1+\frac{i\beta^2}{16\pi}\left[\frac{2\theta}{\cosh\theta}+\pi\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)\right]. \tag{4.49}$$

Combining the results obtained in two exponentials in (4.45), $R_b^{(+)}(\theta)$ up to order β^2 can be written as

$$-\frac{i\sinh\theta-1}{i\sinh\theta+1}\left[1-\frac{i\beta^2}{16\pi}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)-\frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.50}$$

Finally, substituting this result together with (4.41) into the first expression in (4.38), we can obtain the supersymmetric reflection factor of the boson corresponding to the boundary condition $\mathcal{BC}^+$

$$R_b^+(\theta)=-\left[1-\frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)-\frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.51}$$

In the classical limit $\beta\to 0$, the bosonic reflection factor becomes

$$R_b^+(\theta)\sim -1=K_b^+(\theta). \tag{4.52}$$

However this limit is not correct. From (3.5), we already know that the classical reflection factor in terms of the rapidity is

$$\frac{i\sinh\theta+1}{i\sinh\theta-1}.$$

With similar calculations to those performed for the boson reflection factor, we can obtain the fermionic reflection factor. We found that it is

$$R_f^+(\theta)=-\frac{\cosh\theta}{i\sinh\theta+1}\left[1-\frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\right)-\frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.53}$$

The classical limit of the fermion reflection factor is

$$R_f^+(\theta)\sim -\frac{\cosh\theta}{i\sinh\theta+1}=K_f^+(\theta). \tag{4.54}$$

We will see in the next chapter that this limit is not correct. The ratio of the supersymmetric reflection factors of boson and fermion can be read as

$$\frac{R_b^+(\theta)}{R_f^+(\theta)} = \frac{i\sinh\theta + 1}{\cosh\theta}. \tag{4.55}$$

It is obvious that the ratio of the reflection factors appears correctly, which agrees with (4.18). However the classical limits of the individual boson and fermion reflection factors are not correct.

In order to find the correct supersymmetric reflection factors, we use the expression (4.31) for $Z^+(\theta)$. For the sinh-Gordon theory, this expression becomes

$$Z^+(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2} + \frac{1\pi}{4}\right)} \exp\left[-2i\int_0^\infty \frac{dt}{t} \frac{\sinh\frac{\beta^2 t}{16\pi}\sinh\frac{1}{2}\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}} \sin\frac{t\theta}{\pi}\right]$$
$$\exp\left[\frac{i}{2}\int_0^\infty \frac{dt}{t} \frac{\sinh\frac{\beta^2 t}{8\pi}\sinh\left(1+\frac{\beta^2}{8\pi}\right)t}{\cosh^2\frac{t}{2}\cosh^2 t} \sin\frac{2t\theta}{\pi}\right]. \tag{4.56}$$

After some calculations similar to those performed before, we obtain the supersymmetric reflection of boson up to order β^2

$$R_b^+(\theta) = \frac{i\sinh\theta+1}{i\sinh\theta-1}\left[1 - \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) - \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.57}$$

In the limit $\beta \to 0$, this reflection factor becomes

$$R_b^+(\theta) \sim \frac{i\sinh\theta+1}{i\sinh\theta-1} = K_b^+(\theta), \tag{4.58}$$

which has a correct classical limit.

We also found that the supersymmetric reflection factor for fermion is

$$R_f^+(\theta) = \frac{\cosh\theta}{i\sinh\theta-1}\left[1 - \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) - \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.59}$$

The classical limit of the reflection factor is

$$R_f^+(\theta) \sim \frac{\cosh\theta}{i\sinh\theta-1} = K_f^+(\theta). \tag{4.60}$$

The ratio of the supersymmetric reflection factors for boson and fermion is

$$\frac{R_b^+}{R_f^+} = \frac{i\sinh\theta+1}{\cosh\theta}, \tag{4.61}$$

which agrees with (4.18).

The quantum reflection factors of boson (4.57) and fermion (4.59) are the same except for the classical reflection factors $K_b^+(\theta)$ and $K_f^+(\theta)$ sitting in the formulas. This is what we expect from sypersymmetric theory.

4.6 Supersymmetric reflection factors for $\mathcal{BC}^-$ case

In this section, we will calculate the supersymmetric reflection factors up to order $O(\beta^2)$ for the boundary conditions corresponding to the case $\mathcal{BC}^-$, applying an analysis similar to that presented in the previous section. From (4.35), one can write down the supersymmetric bosonic and fermionic reflection factors corresponding to the boundary condition $\mathcal{BC}^-$ respectively as

$$\begin{aligned} R_b^-(\theta) &= R(\theta|\lambda\pi, 0)R_b^{(-)}(\theta), \\ R_f^-(\theta) &= R(\theta|\lambda\pi, 0)R_f^{(-)}(\theta). \end{aligned} \tag{4.62}$$

Here the factors $R_b^{(-)}(\theta)$ and $R_f^{(-)}(\theta)$ are given by

$$R_b^{(-)}(\theta) = Z^-(\theta)\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right), \tag{4.63}$$

$$R_f^{(-)}(\theta) = Z^-(\theta)\cosh\left(\frac{\theta}{2} + \frac{i\pi}{4}\right). \tag{4.64}$$

We can obtain $R_b^{(-)}(\theta)$ and $R_f^{(-)}(\theta)$ using

$$Z^-(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)}\exp\left[-\frac{i}{2}\int_0^\infty \frac{dt}{t}\frac{\sinh\beta t\ \sinh(1-\beta)t}{\cosh^2\frac{t}{2}\cosh^2 t}\sin\frac{2\theta t}{\pi}\right]. \tag{4.65}$$

We first need to replace β by $-\beta^2/8\pi$ in this expression, and then expand the expression up to the order of β^2. We thus obtain

$$Z^-(\theta) = \frac{1}{\cosh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)}\left[1 + \frac{i\beta^2}{16\pi}\int_0^\infty dt\frac{\sinh t}{\cosh^2\frac{t}{2}\ \cosh^2 t}\sinh\frac{2t\theta}{\pi}\right]. \tag{4.66}$$

The integral can be evaluated. After obtaining the solution for $Z^-(\theta)$ up to the order $O(\beta^2)$, we can respectively write (4.63) and (4.64) in terms of the perturbation solution of $Z^-(\theta)$ as

$$R_b^{(-)}(\theta) = 1 + \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) + \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}, \tag{4.67}$$

$$R_f^{(-)}(\theta) = \frac{i\sinh\theta+1}{\cosh\theta}\left[1 + \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) + \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.68}$$

The bosonic sector of the reflection factor $R(\theta|\lambda\pi, 0)$ in (4.62) can also be obtained from (1.69), that is

$$R(\theta|\lambda\pi, 0) = (1)\,(2 - B/2)\,(1 + B/2)\,. \tag{4.69}$$

Here we first write B in term of its perturbation expansion up to the order $O(\beta^2)$, that means replacing by $\beta^2/2\pi$, it becomes

$$(1)\left(2 - \frac{\beta^2}{4\pi}\right)\left(1 + \frac{\beta^2}{4\pi}\right). \tag{4.70}$$

More precisely, it can be written down as

$$\frac{1 - i\sinh\theta}{\cosh\theta}\;\frac{\sinh\theta + \sinh\frac{i\beta^2}{8} + i\left(\cosh\theta + \cosh\frac{i\beta^2}{8}\right)}{\sinh\theta - \sinh\frac{i\beta^2}{8} - i\left(\cosh\theta + \cosh\frac{i\beta^2}{8}\right)}. \tag{4.71}$$

After the perturbation expansion up to the order of β^2, we obtain

$$\frac{i\sinh\theta - 1}{i\sinh\theta + 1}\left[1 - \frac{i\beta^2}{8}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right)\right]. \tag{4.72}$$

By combining this result with (4.67) and (4.68), we deduce the supersymmetric reflection factors corresponding to the boundary condition $\mathcal{BC}^-$

$$R_b^-(\theta) = \frac{i\sinh\theta - 1}{i\sinh\theta + 1}\left[1 - \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) + \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right], \tag{4.73}$$

$$R_f^-(\theta) = \frac{i\sinh\theta - 1}{\cosh\theta}\left[1 - \frac{i\beta^2}{16}\sinh\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) + \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \tag{4.74}$$

It is easy to find the ratio of the supersymmetric reflection factors

$$\frac{R_b^-(\theta)}{R_f^-(\theta)} = \frac{\cosh\theta}{1 + i\sinh\theta} = \frac{1 - i\sinh\theta}{\cosh\theta}, \tag{4.75}$$

which also agrees with (4.18). The classical limit of the boson reflection factor is

$$K_b^-(\theta) = \frac{i \sinh\theta - 1}{i \sinh\theta + 1}, \tag{4.76}$$

and the fermion reflection factor is

$$K_f^-(\theta) = \frac{i \sinh\theta - 1}{\cosh\theta}. \tag{4.77}$$

We will compare the results obtained here with the supersymmetric reflection factors which will be extracted from the loop calculations in the following chapter.

4.7 Conclusion

In this chapter, we have studied the exact reflection factors proposed by Moriconi and Schoutens for the breather multiplets of the supersymmetric sine-Gordon theory. By looking at the lightest breather solution, we checked the classical limits of the supersymmetric reflection factors for the sinh-Gordon theory up to $O(\beta^2)$. We found that the boson and the fermion reflection factors corresponding to the boundary condition $\mathcal{BC}^+$ do not have the correct classical limits. We therefore have made a correction to their exact reflection factors. As a result of this correction, we obtained the supersymmetric reflection factors which have the correct classical limits. For the boundary condition $\mathcal{BC}^-$, we checked that the supersymmetric reflection factors obtained up to $O(\beta^2)$ do have the corrected classical limits. In the next chapter, we will obtain the supersymmetric reflection factors by another approach. We will extract the reflection factors from the calculation of one-loop Feynman diagrams. We shall compare the boson and the fermion reflection factors obtained from the two different approaches.

Chapter 5

The Quantum Correction to the Supersymmetric Reflection Factor

In the previous chapter, we have checked the classical limits of the exact reflection factors for the supersymmetric sinh-Gordon model up to $O(\beta^2)$. In this chapter, we will develop the perturbation theory in the supersymmetric sinh-Gordon model and will consider the reflection factors from the calculations of the one-loop Feynman diagrams. First of all, we construct the Lagrangian for the supersymmetric sinh-Gordon theory with one boundary condition and discuss the fermion propagator in the presence of the boundary. Using the perturbation theory and path integral formalism, we derive the one-loop Feynman diagrams which will be evaluated. We then perform the calculations for these diagrams in order to obtain the supersymmetric boson and fermion reflection factors. The results obtained here will be compared with the supersymmetric reflection factors obtained in the previous chapter. Finally, a discussion on the renormalisation will be presented.

5.1 The construction of the Lagrangian

Let us first construct the Lagrangian for the supersymmetric (SUSY) sinh-Gordon model on the bulk. It can be described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi - \bar{\psi} i\gamma^\mu \partial_\mu \psi + U(\phi)\bar{\psi}\psi, \tag{5.1}$$

where ϕ is a real scalar field and ψ is a Majorana fermion, its adjoint $\bar{\psi}$ is obtained from ψ by hermitian conjugation, $\bar{\psi}=\psi^\dagger\gamma^0$. We choose the representation for 2-dimensional γ-matrices to be

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{5.2}$$

In this representation, the charge conjugation is simply a complex conjugation, i.e. $\psi^C = \psi^*$, so that a Majorana spinor satisfies $\psi_i = \psi_i^*$, for i = 1, 2, where ψ_1 and ψ_2 are its two components.

By requiring the Lagrangian to be invariant under the SUSY transformation, we can fix the functions $U(\phi)$ in (5.1) as well as $W(\phi)$ in the following susy transformation

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi + \bar{\psi}\epsilon, \\ \delta\psi &= (i\gamma^\mu \partial_\mu \phi + W(\phi))\,\epsilon, \\ \delta\bar{\psi} &= \bar{\epsilon}\,(-i\gamma^\mu\partial_\mu\phi + W(\phi))\,, \end{aligned} \tag{5.3}$$

where ϵ is a constant anti-commuting Majorana spinor.

Under the infinitesimal variations, $\delta\phi$, $\delta\psi$ and $\delta\bar{\psi}$, the Lagrangian (5.1) gives

$$\begin{aligned} \delta\mathcal{L} = {} & \partial_\mu\left(\bar{\epsilon}\psi\partial^\mu\phi + \bar{\psi}\epsilon\partial^\mu\phi\right) + \partial_\nu\left(\bar{\gamma}^\mu\gamma^\nu\psi\partial_\mu\phi + iW\bar{\epsilon}\gamma^\nu\psi\right) \\ & + i\partial_\mu W\left(\bar{\psi}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu\psi\right) - iU\left(\bar{\psi}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu\psi\right)\partial_\mu\phi \\ & - \delta\phi\left(\frac{2\sqrt{2}}{\beta}m^2\sinh\sqrt{2}\beta\phi - UW\right). \end{aligned} \tag{5.4}$$

By setting $\delta\mathcal{L} = 0$, we have two conditions

$$\begin{aligned} UW &= \frac{2\sqrt{2}}{\beta}m^2\sinh\sqrt{2}\beta\phi, \\ (\partial_\mu W - U\partial_\mu\phi)&\left(\bar{\psi}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu\psi\right) = 0. \end{aligned} \tag{5.5}$$

The total derivative terms in (5.4) vanish on the whole line. Solving the conditions (5.5), we obtain

$$W = \pm\frac{\sqrt{2}}{\beta} m \sinh \frac{\sqrt{2}}{2}\beta\phi, \qquad U = \pm m \cosh \frac{\sqrt{2}}{2}\beta\phi. \tag{5.6}$$

We choose the plus sign for W and U, then the supersymmetric Lagrangian for the sinh-Gordon model in the bulk is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi - \bar{\psi}i\gamma^\mu\partial_\mu\psi + 2m\bar{\psi}\psi\cosh\frac{\beta\phi}{\sqrt{2}}. \tag{5.7}$$

It is obvious that this Lagrangian is supersymmetric under the SUSY transformation.

The field equations of motion can be found from (5.7)

$$\begin{aligned} \partial^2\phi &= -\frac{\sqrt{2}m^2}{2\beta}\sinh\sqrt{2}\beta\phi - \frac{\sqrt{2}m}{2}\bar{\psi}\psi\sinh\frac{\sqrt{2}\beta\phi}{2}, \\ i\gamma^\mu\partial_\mu\psi &= m\psi\cosh\frac{\sqrt{2}\beta\phi}{2}, \\ i\partial_\mu\bar{\psi}\gamma^\mu &= -m\bar{\psi}\cosh\frac{\sqrt{2}\beta\phi}{2}. \end{aligned} \tag{5.8}$$

In the presence of the boundary, the theory can be defined by adding the boundary term to the bulk part. Inami, Odake and Zhang [64] constructed the supersymmetric and integrable action for the sine-Gordon model on a half-line by checking the first non-trivial conserved charge. As we know, under analytic continuation, the sine-Gordon model goes into the sinh-Gordon model. We can therefore write down the action for the sinh-Gordon model on a half-line as

$$\begin{aligned} S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi - \bar{\psi}i\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi\cosh\frac{\sqrt{2}\beta\phi}{2}\right] \\ - \int_{-\infty}^{\infty} dt \left[\pm\frac{2m}{\beta^2}\cosh\frac{\sqrt{2}\beta\phi}{2} \mp \frac{1}{2}\bar{\psi}\psi\right]. \end{aligned} \tag{5.9}$$

Let us check whether the action is invariant under the susy transformation. The variation in the action gives

$$\delta S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \Big[\partial_\mu \left(\delta\phi\, \partial^\mu \phi\right) - \partial_\mu \left(i\bar{\psi}\gamma^\mu\, \delta\psi\right)$$
$$- \delta\phi \left(\partial^2\phi + \frac{\sqrt{2}m^2}{2\beta}\sinh\sqrt{2}\beta\phi - \frac{\sqrt{2}}{2}m\beta\bar{\psi}\psi\sinh\frac{\sqrt{2}}{2}\beta\phi\right)$$
$$- \delta\bar{\psi}\left(i\gamma^\mu\partial_\mu\psi - m\psi\cosh\frac{\sqrt{2}}{2}\beta\phi\right) + \left(i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}\cosh\frac{\sqrt{2}}{2}\beta\phi\right)\delta\psi\Big]$$
$$- \int_{-\infty}^{\infty} dt \left[\pm\delta\phi\frac{\sqrt{2}m}{\beta}\sinh\frac{\sqrt{2}}{2}\beta\phi \mp \frac{1}{2}\delta\bar{\psi}\psi \mp \frac{1}{2}\bar{\psi}\delta\psi\right]. \tag{5.10}$$

Integrating out the first and second terms, the remaining part will be

$$\delta S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \Big[\delta\phi\left(-\partial^2\phi - \frac{\sqrt{2}m^2}{2\beta}\sinh\sqrt{2}\beta\phi + \frac{\sqrt{2}}{2}m\beta\bar{\psi}\psi\sinh\frac{\sqrt{2}}{2}\beta\phi\right)$$
$$+ \delta\bar{\psi}\left(-i\gamma^\mu\partial_\mu\psi + m\psi\cosh\frac{\sqrt{2}}{2}\beta\phi\right) + \left(i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}\cosh\frac{\sqrt{2}}{2}\beta\phi\right)\delta\psi\Big]$$
$$+ \int_{-\infty}^{\infty} dt \left[\delta\phi\left(-\partial_x\phi \mp \frac{\sqrt{2}m}{\beta}\sinh\frac{\sqrt{2}}{2}\beta\phi\right) + \left(-i\bar{\psi}\gamma^1 \pm \frac{1}{2}\bar{\psi}\right)\delta\psi \pm \frac{1}{2}\delta\bar{\psi}\psi\right]. \tag{5.11}$$

We now apply the SUSY transformation in the above expression. After some calculations, we deduce that

$$\delta S = i\int_{-\infty}^{\infty} dt \left\{\partial_t\phi\left[\left(-\epsilon_1 \mp \epsilon_2\right)\psi_2 + \left(-\epsilon_2 \mp \epsilon_1\right)\psi_1\right] + \partial_x\phi\left[\left(\epsilon_1 \pm \epsilon_2\right)\psi_2 + \left(-\epsilon_2 \mp \epsilon_1\right)\psi_1\right]\right.$$
$$\left. + \frac{\sqrt{2}m}{\beta}\sinh\frac{\sqrt{2}}{2}\beta\phi\left[\left(-\epsilon_1 \mp \epsilon_2\right)\psi_1 + \left(\epsilon_2 \pm \epsilon_1\right)\psi_2\right]\right\}. \tag{5.12}$$

At this stage, we can see that the action is supersymmetric if and only if the SUSY parameters satisfy the constraints

$$\epsilon_1 = \mp\epsilon_2. \tag{5.13}$$

This implies that only half of the supersymmetry on the bulk theory is preserved in the presence of the boundary.

In addition to the bulk equations of motion (5.8), we now have the boundary conditions for the boson field as

$$\partial_x \phi = \mp \frac{\sqrt{2}m}{\beta} \sinh \frac{\sqrt{2}\beta\phi}{2}, \tag{5.14}$$

and for the fermion field as

$$\psi_1 = \pm \psi_2 \tag{5.15}$$

at the boundary $x = 0$.

When we compare the bosonic boundary conditions with (1.39), we see $\sigma_0 = \sigma_1 = \pm 1$ in the supersymmetric theory. These two points correspond to the SUSY two points $M_\pm = \pm\frac{4m}{\beta^2}$ mentioned in [65].

In terms of component fields, the the equations of motion can be expressed as

$$\begin{aligned} \partial^2 \phi &= -\frac{m^2}{\sqrt{2}\beta} \sinh \sqrt{2}\beta\phi + i\sqrt{2}m\psi_1\psi_2 \sinh \frac{\beta\phi}{\sqrt{2}}, \\ (\partial_t - \partial_x)\,\psi_2 &= m\psi_1 \cosh \frac{\beta\phi}{\sqrt{2}}, \\ (\partial_t + \partial_x)\,\psi_1 &= -m\bar{\psi}_2 \cosh \frac{\beta\phi}{\sqrt{2}}. \end{aligned} \tag{5.16}$$

From the component forms, it is easy to see that the vacuum solution is zero. In other words, $\phi_0 = \psi_{01} = \psi_{02} = 0$ are the solutions to the background fields. Here ϕ_0 is the static background solution corresponding to the boson field and ψ_{01} and ψ_{02} corresponding to the two components of the fermion field.

5.2 Boson Propagator

The construction of the boson propagator in the presence of the boundary has been discussed in detail in chapter three. In the supersymmetric case, we have two kinds of boundary which preserve both supersymmetry and integrability. The boson propagator corresponding to the first boundary condition in(5.14) is given by

$$G(x,t;x',t') = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \left[e^{ik(x-x')} + K_b^+(k)\, e^{-ik(x+x')}\right], \tag{5.17}$$

and for the second boundary condition is

$$G(x,t;x',t') = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \left[e^{ik(x-x')} + K_b^-(k)\, e^{-ik(x+x')} \right]. \tag{5.18}$$

We have seen that the coefficient of the reflected term corresponds to the classical reflection factor in chapter three. In the SUSY case, we have two reflection factors and they are given by

$$K_b^+(k) = \frac{ik+m}{ik-m}, \qquad K_b^-(k) = \frac{ik-m}{ik+m} \tag{5.19}$$

for each boundary. When we set $k = m \sinh\theta$, it is straightforward to see that these classical reflection factors are same as (4.58) and (4.76).

5.3 Fermion Propagator

We are familiar with the fermion propagator on the whole line. In two dimensions, it is usually written as

$$S_F(x-x') = \int \frac{d^2p}{(2\pi)^2} \frac{i}{\gamma^\mu p_\mu - m + i\epsilon}\, e^{-ip(x-x')}, \tag{5.20}$$

where p stands for two-momentum. Here we shall use the notation

$$p_0 = p^0 = \omega, \qquad p_1 = -p^1 = -k.$$

The propagator $S_F(x-x')$ satisfies the differential equation

$$(i\gamma^\mu \partial_\mu - m)\, S_F(x-x') = i\delta_2(x-x'). \tag{5.21}$$

In the representation (5.2), the above condition can be written explicitly as

$$S_F(x-x') = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \begin{pmatrix} m & -i(\omega+k) \\ i(\omega+k) & m \end{pmatrix} e^{ik(x-x')}. \tag{5.22}$$

In the presence of the boundary, we need to modify the fermion propagator, such that it not only satisfies the equation (5.21), but also the boundary conditions (5.15).

By requiring this we have found that the fermion propagator corresponding to the boundary condition $\psi_1 = \psi_2$ is

$$S_F(x,t;x',t') = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t-t')}}{\omega^2-k^2-m^2+i\epsilon}\left[\begin{pmatrix} m & -i(\omega+k) \\ i(\omega-k) & m \end{pmatrix} e^{ik(x-x')} + \frac{\omega}{ik-m}\begin{pmatrix} \omega-k & -im \\ im & \omega+k \end{pmatrix} e^{-ik(x+x')}\right], \quad (5.23)$$

and to the boundary condition $\psi_1 = -\psi_2$ is

$$S_F(x,t;x',t') = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t-t')}}{\omega^2-k^2-m^2+i\epsilon}\left[\begin{pmatrix} m & -i(\omega+k) \\ i(\omega-k) & m \end{pmatrix} e^{ik(x-x')} - \frac{\omega}{ik+m}\begin{pmatrix} \omega-k & -im \\ im & \omega+k \end{pmatrix} e^{-ik(x+x')}\right]. \quad (5.24)$$

Here the classical fermion reflection factors are

$$K_f^+ = \frac{\omega}{ik-m}, \qquad K_f^- = -\frac{\omega}{ik+m}. \quad (5.25)$$

When we place $\omega = m\cosh\theta$ and $k = m\sinh\theta$ into these expressions, the results agree with the classical limits of the fermion reflection factors obtained in (4.60) and (4.77).

One can check that (5.23) and (5.24) satisfy the boundary conditions (5.15) respectively. This can be achieved in the following way. At the boundary $x = 0$, the condition $\psi_1 = \psi_2$ requires that the two elements in the same colum of $S_F(0,t;x',t')$ should be equal to each other. This reduces to the mass shell condition. In other words, we can say that if the mass shell condition is satisfied by the requirement, we then conclude that the propagator satisfies the boundary condition. For $x' = 0$, the two elements in the same row of $S_F(x,t;0,t')$ should be equal to each other subject to the mass shell condition. A similar analysis can be applied to the boundary $\psi_1 = -\psi_2$.

5.4 Generating Functional

In chapter three, we discussed the generating functional for the boson field theory. That approach has a natural extension which leads to a generating functional for the Green's function of the boson and fermion fields in a supersymmetric theory. The generating functional is defined in the usual way by introducing anticommuting external sources $\eta(x)$ and $\bar{\eta}(x)$ in addition to the source $J(x)$, and defining $Z[J,\eta,\bar{\eta}]$ as

$$Z[J,\eta,\bar{\eta}] = N\int \mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left\{i\int d^2x\left[\mathcal{L} + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\}, \tag{5.26}$$

where $\mathcal{L}$ can be written from (5.9)

$$\begin{aligned}\mathcal{L} = \theta(-x)&\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2\beta^2}\cosh\sqrt{2}\beta\phi - \bar{\psi}i\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi\cosh\frac{\sqrt{2}\beta\phi}{2}\right]\\ &-\delta(x)\left[\pm\frac{2m}{\beta^2}\cosh\frac{\sqrt{2}\beta\phi}{2} \mp \frac{1}{2}\bar{\psi}\psi\right].\end{aligned} \tag{5.27}$$

When we expand the second, fourth and fifth terms up to the order of β^2, then the Lagrangian becomes,

$$\mathcal{L} = \mathcal{L}_p + \mathcal{L}_c + \mathcal{L}_4, \tag{5.28}$$

where $\mathcal{L}_c$ corresponds to the constant term, $\mathcal{L}_4$ corresponds to the four point coupling term in the Lagrangian and $\mathcal{L}_p$ is the term that generates the propagators. They are respectively

$$\begin{aligned}\mathcal{L}_c &= \mp\frac{2m}{\beta^2}\delta(x) - \frac{m^2}{2\beta^2}\theta(-x),\\ \mathcal{L}_4 &= \frac{1}{4}\theta(-x)m\beta^2\bar{\psi}\psi\phi^2 \mp \frac{m}{48}\delta(x)\beta^2\phi^4 - \frac{m^2}{12}\theta(-x)\beta^2\phi^4,\\ \mathcal{L}_p &= -\frac{1}{2}\phi\left[\theta(-x)\partial^2 + m^2\theta(-x) + \delta(x)\partial_x \pm m\delta(x)\right]\phi,\\ &\qquad - \bar{\psi}\left[\theta(-x)i\gamma^\mu\partial_\mu - m\theta(-x) \mp \frac{1}{2}\delta(x)\right]\psi.\end{aligned} \tag{5.29}$$

We rewrite the last expression in the form

$$\mathcal{L}_p = -\frac{1}{2}\phi\,\mathcal{M}\phi - \bar{\psi}\,\mathcal{N}\psi. \tag{5.30}$$

Here we have defined the differential operators as

$$\begin{aligned} \mathcal{M} &= \theta(-x)\left(\partial^2 + m^2\right) + \delta(x)\left(\partial_x \pm m\right), \\ \mathcal{N} &= \theta(-x)\left(i\,\not{\partial} - m\right) \mp \frac{1}{2}\delta(x). \end{aligned} \tag{5.31}$$

The generating functional can now be written as

$$\begin{aligned} Z[J,\eta,\bar{\eta}] &= N\int \mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left\{i\int d^2x\left[\mathcal{L}_p + \mathcal{L}_c + \mathcal{L}_4 + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\} \\ &= N\int \mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \\ &\qquad \exp\left\{i\int d^2x\left[\frac{m}{4}\theta(-x)\beta^2\bar{\psi}\psi\phi^2 \mp \frac{m}{48}\beta^2\delta(x)\phi^4 - \frac{m}{12}\theta(-x)\beta^2\phi^4\right]\right\} \\ &\qquad \exp\left(i\int d^2x\left[-\frac{1}{2}\phi\mathcal{M}\phi - \bar{\psi}\mathcal{N}\psi + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right). \end{aligned} \tag{5.32}$$

If we replace each power of ϕ, ψ and $\bar{\psi}$ in the first exponential factor by the functional derivatives with respect to the sources, then the functional integrals can only act on the second exponent

$$\begin{aligned} Z[J,\eta,\bar{\eta}] = N\exp&\left\{i\int d^2x\left[-\frac{m^2}{12}\theta(-x)\beta^2 \mp \frac{m}{48}\beta^2\delta(x)m\right]\left(\frac{\delta}{i\delta J(x)}\right)^4\right\} \\ \exp&\left\{i\int d^2x\frac{m}{4}\theta(-x)\beta^2\left(\frac{\delta}{i\delta J(x)}\right)^2\frac{\delta}{\delta\eta(x)}\frac{\delta}{\bar{\eta}(x)}\right\} \\ \int&\mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\ \exp\left\{i\int d^2x\left[-\frac{1}{2}\phi\,\mathcal{M}\,\phi + J\phi - \bar{\psi}\,\mathcal{N}\,\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\}. \end{aligned} \tag{5.33}$$

By performing the functional integrals, we have

$$\begin{aligned} Z[J,\eta,\bar{\eta}] = N\exp&\left\{i\int d^2x\left[-\frac{m^2}{12}\theta(-x)\beta^2 \mp \frac{m}{48}\beta^2\delta(x)m\right]\left(\frac{\delta}{i\delta J(x)}\right)^4\right\} \\ \exp&\left\{i\int d^2x\frac{m}{4}\theta(-x)\beta^2\left(\frac{\delta}{i\delta J(x)}\right)^2\frac{\delta}{\delta\eta(x)}\frac{\delta}{\bar{\eta}(x)}\right\} Z_0[J,\eta,\bar{\eta}], \end{aligned} \tag{5.34}$$

where $Z_0[J,\eta,\bar{\eta}]$ is the free generating functional of the theory

$$Z_0[J,\eta,\bar{\eta}] = \exp\left\{\int d^2x d^2y\left[-\frac{1}{2}J(x)G(x,y)J(y) - \bar{\eta}(x)S_F(x,y)\eta(y)\right]\right\}. \tag{5.35}$$

In order to calculate $Z[J,\eta,\bar{\eta}]$ up to order β^2, we need to expand the exponents up to the second terms and then perform the functional derivatives of $Z_0[J,\eta,\bar{\eta}]$

$$\begin{aligned} Z = N\Bigg\{ 1 - \frac{i}{12}\beta^2 \int d^2x \left[\theta(-x)m^2 \pm \frac{1}{4}\delta(x)m\right] \\ \left[3G(x,x)^2 - 6G(x,x)\left(\int d^2y\, G(x,y)J(y)\right)^2 \right. \\ \left. + \left(\int d^2y\, G(x,y)\eta(y)J(y)\right)^4\right] \\ + \frac{i}{4}m\beta^2 \int d^2x\theta(-x)\Big[G(x,x)S_F(x,x) \\ - G(x,x)\int d^2y\, S_F(x,y)\eta(y)\int d^2z\bar{\eta}(z)\, S_F(z,x) \\ -S_F(x,x))\left(\int d^2y\; G(x,y)J(y)\right)^2 \\ + \left(\int d^2y\, G(x,y)J(y)\right)^2 \int d^2z\, S_F(x,z)\eta(z)\int d^2w\,\bar{\eta}(z)S_F(w,x)\Big]\Bigg\} Z_0. \end{aligned} \tag{5.36}$$

We now fix the normalisation constant N by imposing the normalisation condition $Z[0] = 1$. This gives us

$$N = 1+\frac{i}{4}\beta^2 \int d^2x \left[\theta(-x)m^2 \pm \frac{1}{4}\delta(x)m\right] G(x,x)^2 - \frac{i}{4}m\beta^2 \int d^2x\theta(-x)G(x,x)S(x,x). \tag{5.37}$$

Substituting this back into (5.36) yields the final expression for the generating function Z up to the desired order

$$\begin{aligned} Z = \Bigg\{ 1 + \frac{i}{12}\beta^2 \int d^2x \left[\theta(-x)m^2 \pm \frac{1}{4}\delta(x)m\right] \left[6G(x,x)\left(\int d^2y\, G(x,y)J(y)\right)^2 \right. \\ \left. - \left(\int d^2y\, G(x,y)J(y)\right)^4\right] \\ + \frac{i}{4}m\beta^2 \int d^2x\theta(-x)\Big[-G(x,x)\int d^2y\, S_F(x,y)\eta(y)\int d^2z\; \bar{\eta}(z)\, S_F(z,x) \\ - S_F(x,x)\left(\int d^2y\; G(x,y)J(y)\right)^2 \\ + \left(\int d^2y\, G(x,y)J(y)\right)^2 \int d^2z\, S_F(x,z)\eta(z)\int d^2w\,\bar{\eta}(z)S_F(w,x)\Big]\Bigg\} Z_0. \end{aligned} \tag{5.38}$$

5.5 The two-point functions

Having obtained the generating functional Z up to $O(\beta^2)$, we can proceed to evaluate the boson and fermion two-point functions up to the same order. These functions are defined by

$$\mathcal{G}(x_1, x_2) = \frac{1}{i^2} \frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=\eta=\bar{\eta}=0}, \tag{5.39}$$

$$\mathcal{S}(x_1, x_2) = \frac{1}{i^2} \frac{\delta^2 Z}{\delta \eta(x_1)\delta \bar{\eta}(x_2)}\bigg|_{J=\eta=\bar{\eta}=0}. \tag{5.40}$$

Using the result obtained in (5.38), it can be easily calculated that the boson two-point function is

$$\mathcal{G}(x_1, x_2) = G(x_1, x_2) - i\beta^2 \left[\theta(-x)m^2 \pm \frac{1}{4}\delta(x)m\right] G(x,x)\ G(x,x_1)\ G(x,x_2)$$
$$+ \frac{im\beta^2}{2} \int d^2x\ \theta(-x)S(x,x)\ G(x,x_1)\ G(x,x_2). \tag{5.41}$$

The result in (5.41) can be represented by means of the following Feynman diagrams.

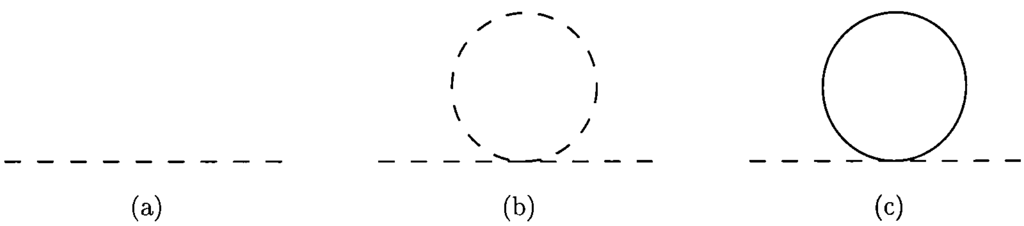

Figure 5.1: Correction to the boson propagator.

Diagram (a) is the tree level boson propagator. The second and third diagrams represent the boson and fermion one loop correction to the boson two point function respectively.

We find that the fermion two-point function is

$$\mathcal{S}(x_1, x_2) = S_F(x_1, x_2) - \frac{i}{4} m\beta^2 \int d^2x\ \theta(-x)G(x,x)\ S_F(x_1,x)\ S_F(x,x_2). \tag{5.42}$$

Similarly, we can represent the result in (5.42) by means of the following Feynman diagrams.

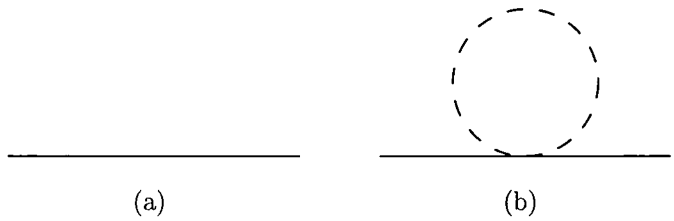

Figure 5.2: Correction to the fermion propagator.

Diagram (a) is the tree level fermion propagator and (b) is the boson one loop correction to the fermion propagator.

For convenience we have denoted the propagators $G(x_1, t_1; x_2, t_2)$ and $S_F(x_1, t_1; x_2, t_2)$ as $G(x_1, x_2)$ and $S_F(x_1, x_2)$.

In the following sections we will carry out the calculation of Feynman diagrams, from which we extract the boson and fermion reflection factors.

5.6 The fermion reflection factor for $\mathcal{BC}^+$ case

In this section we calculate the fermion reflection factor that corresponds to the case when the boundary condition is $\psi_1 = \psi_2$.

Since there is no four Fermi coupling in the correction to the fermion propagator, we need only to calculate the diagram which comes from the bulk potential. That corresponds to the diagram (b) in Figure 5.2. It can be interpreted as

$$-\frac{i}{4}m\beta^2 \int_{-\infty}^{+\infty} dt'' \int_{-\infty}^{0} dx'' \; S_F\left(x, t;\; x', t'\right) \, G\left(x'', t'';\; x'', t''\right) \, S_F\left(x'', t'';\; x', t'\right), \tag{5.43}$$

where the fermion propagator is given by (5.23). The loop propagator is

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \left[1 + K_b^+(k'') e^{-2ik''x''}\right]. \tag{5.44}$$

A counter term is needed to remove the divergence in the first term. We will discuss how to eliminate it in the end of this chapter. The energy integral can be integrated for the second term, and then the finite part of the loop integral will be

$$\int \frac{dk''}{2\pi} \frac{1}{2\sqrt{k''^2 + m^2}} K_b^+(k'') e^{-2ik''x''}. \tag{5.45}$$

Putting it back in expression together with the fermion propagators in expression (5.43), we have

$$-\frac{i}{4}m\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \frac{ie^{-ikx-ik'x'}}{\omega^2 - k'^2 - m^2 + i\epsilon}$$
$$\int_{-\infty}^{0} dx'' \int \frac{dk''}{2\pi} \frac{K_b^+(k'')}{2\sqrt{k''^2+m^2}}$$
$$\left[\begin{pmatrix} m^2 + (\omega-k)(\omega-k') & -2im\omega + im(k-k') \\ 2im\omega + im(k-k') & m^2 + (\omega+k)(\omega+k') \end{pmatrix} e^{ix''(k+k'-2k'')}\right.$$
$$+ \begin{pmatrix} 2m\omega - m(k-k') & -m^2 i - i(\omega-k)(\omega+k') \\ m^2 i + i(\omega+k)(\omega-k') & 2m\omega + 2(k+k') \end{pmatrix} K_f^+(k') e^{ix''(k-k'-2k'')}$$
$$+ \begin{pmatrix} m^2\omega - m(k+k') & -im^2 - i(\omega-k)(\omega+k') \\ im^2 + i(\omega+k)(\omega-k') & 2m\omega + m(k+k') \end{pmatrix} K_f^+(k) e^{ix''(-k+k'-2k'')}$$
$$+ K_f^+(k) K_f^+(k') e^{ix''(-k-k'-2k'')} \times$$
$$\left.\begin{pmatrix} m^2 + (\omega-k)(\omega-k') & -2im\omega + im(k-k') \\ 2im\omega + im(k-k') & m^2 + (\omega+k)(\omega+k') \end{pmatrix}\right]. \tag{5.46}$$

The next step is to perform the x'' integrals by using (3.53). This gives

$$-\frac{i}{4}m\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \frac{ie^{-ikx-ik'x'}}{\omega^2 - k'^2 - m^2 + i\epsilon} \int \frac{dk''}{2\pi} \frac{K_b^+(k'')}{2\sqrt{k''^2+m^2}}$$
$$\left[\begin{pmatrix} m^2 + (\omega-k)(\omega-k') & -2im\omega + im(k-k') \\ 2im\omega + im(k-k') & m^2 + (\omega+k)(\omega+k') \end{pmatrix} \frac{-i}{k+k'-2k''-i\rho}\right.$$
$$+ \begin{pmatrix} 2m\omega - m(k-k') & -m^2 i - i(\omega-k)(\omega+k') \\ m^2 i + i(\omega+k)(\omega-k') & 2m\omega + 2(k+k') \end{pmatrix} \frac{-iK_f^+(k')}{k-k'-2k''-i\rho}$$
$$+ \begin{pmatrix} m^2\omega - m(k+k') & -im^2 - i(\omega-k)(\omega+k') \\ im^2 + i(\omega+k)(\omega-k') & 2m\omega + m(k+k') \end{pmatrix} \frac{-iK_f^+(k)}{-k+k'-2k''-i\rho}$$
$$\left.+ \begin{pmatrix} m^2 + (\omega-k)(\omega-k') & -2im\omega + im(k-k') \\ 2im\omega + im(k-k') & m^2 + (\omega+k)(\omega+k') \end{pmatrix} \frac{-iK_f^+(k)K_f^+(k')}{-k-k'-2k''-i\rho}\right]. \tag{5.47}$$

We can integrate out every k'' integral by closing the contour in the upper half plane and let the branch cuts run from im to $i\infty$, so that we can avoid the pole contribution from $K(k'')$. The other pole can also be avoided due to $i\rho$. Let us begin with the first term for the k'' integral. It can be decomposed into partial fractions

$$\begin{aligned}\int \frac{dk''}{2\pi} \frac{K_b^+(k'')}{2\sqrt{k''^2+m^2}} \frac{-i}{k+k'-2k''-i\rho} \\ = \frac{1}{4\pi}\int_m^\infty dy \frac{1}{\sqrt{y^2+m^2}}\left[\frac{1-K_b^+\left(\frac{k+k'}{2}\right)}{y+m} + \frac{K_b^+\left(\frac{k+k'}{2}\right)}{y+i(k+k')/2}\right],\end{aligned} \tag{5.48}$$

where we make the change $k'' = iy$. The integrals can be evaluated by using

$$\int_m^{+\infty} dy \frac{1}{\sqrt{y^2-m^2}} \frac{1}{y+2a} = \frac{2}{\sqrt{m^2-4a^2}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{m+2a}{m-2a}}\right). \tag{5.49}$$

By changing the variable $y = m\cosh\theta$, we obtain the result for the first k'', which is

$$\frac{1}{4\pi}\left[\frac{1-K_b^+\left(\frac{k+k'}{2}\right)}{m} + \frac{2K_b^+\left(\frac{k+k'}{2}\right)}{\sqrt{m^2+(k+k')^2/4}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{2m+i(k+k')}{2m-i(k+k')}}\right)\right]. \tag{5.50}$$

Having obtained the first k'' integral in (5.47), we now perform the integrals over k and k'. The contours should be closed in the upper half plane due do $x, x' < 0$. These produce

$$\frac{1}{4\pi}e^{-i\hat{k}(x+x')}\frac{\omega}{2\hat{k}^2}\begin{pmatrix} \omega-\hat{k} & -im \\ im & \omega+\hat{k}\end{pmatrix}\left[\frac{1-K_b^+(\hat{k})}{m} + \frac{K_b^+(\hat{k})}{\cosh\theta}\left(\frac{\pi}{4} - \frac{i\theta}{2}\right)\right], \tag{5.51}$$

where $\hat{k} = \sqrt{\omega^2 - m^2}$.

The remaining three terms in (5.47) can be completed in the same way, except that $k+k'$ is replaced by one of $k-k'$, $-k+k'$ and $-k-k'$. Combining the results of the calculations we have

$$\begin{aligned}-\frac{i}{16\pi}m\beta^2\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}e^{-ik(x-x')}\frac{1}{2\hat{k}}\begin{pmatrix} \omega-\hat{k} & -2i \\ 2i & \omega+\hat{k}\end{pmatrix}\frac{K_f^+(\hat{k})}{2m\sinh\theta} \\ \left[2\pi\sinh^2\theta\left(\frac{1}{\cosh\theta+1} - \frac{1}{\cosh\theta}\right) + \frac{4\theta\sinh\theta}{\cosh\theta}\right].\end{aligned} \tag{5.52}$$

From this we can extract the fermion reflection factor. It is defined as the coefficient of the reflected term of the exact two-point correlation function in the asymptotic region far away from the boundary. We therefore obtain the correction to the fermion reflection factor up to order β^2 and it is given by

$$R_f^+(\hat{k}) = K_f^+(\hat{k})\left[1 - \frac{i\beta^2}{16\pi}\sinh\theta\left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta}\right) - \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \quad (5.53)$$

The result obtained here exactly agrees with the one in (4.59).

5.7 The fermion reflection factor for $\mathcal{BC}^-$ case

The other fermion reflection factor corresponding to the boundary condition $\psi_1 = -\psi_2$ can be obtained in the same way as presented in the previous section. In this case, we need to use (5.18) and (5.24) as the boson and fermion propagators respectively. Our calculations show that the fermion reflection factor is

$$R_f^-(\hat{k}) = K_f^-(\hat{k})\left[1 - \frac{i\beta^2}{16\pi}\sinh\theta\left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta}\right) + \frac{i\beta^2}{8\pi}\frac{\theta}{\cosh\theta}\right]. \quad (5.54)$$

This result also agrees with the fermionic reflection factor corresponding to the boundary $\mathcal{BC}^-$ obtained in (4.74)

5.8 The boson reflection factor for $\mathcal{BC}^+$ case

In order to obtain the correction to the boson reflection factor, we need to calculate the diagrams (b) and (c) in Figure 5.1.

We begin by calculating diagram (b). There are two contributions coming from this graph, one from the boundary and the other one from the bulk potential. The first contribution is described by

$$-\frac{i}{4}\beta^2 m\int_{-\infty}^{+\infty} dt''\; G(x,t;\;0,t')\;\; G(0,t'';\;0,t'')\;\; G(0,t'';\;x',t')\,. \quad (5.55)$$

We first use the propagators which correspond to the first set of the boundary conditions in (5.14) and (5.15). The integral over ω'' generates a delta function which allows us to integrate over ω'. The middle propagator is

$$G(0,t'';\ 0,t'') = \int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \left[1 + K_b^+(k'')\right], \tag{5.56}$$

which is the divergent integral. The issue of such divergences will be discussed in some detail in the end of this chapter. For the time being we assume that this divergence has somehow been regularised. It can be replaced by the finite integral

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \frac{2m}{ik'' - m}. \tag{5.57}$$

The ω'' integral can be easily evaluated to obtain

$$\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + m^2}} \frac{m}{ik'' - m}. \tag{5.58}$$

We can avoid the pole contribution by closing the contour in the upper-half plane for the k'' integral. Therefore we only need to consider the branch cut contribution. By replacing $k'' = iy$, we obtain

$$-\frac{2}{\pi} \int_m^\infty dy \frac{1}{\sqrt{k''^2 + m^2}} \frac{1}{y + m} = -\frac{1}{\pi}. \tag{5.59}$$

By placing this in (5.55), the remaining integral will be

$$\frac{im\beta^2}{4\pi} \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{e^{-i\omega(t-t')}}{(\omega^2 - k^2 - m^2 + i\epsilon)} \frac{e^{-ikx - ik'x'}}{(\omega^2 - k'^2 - m^2 + i\epsilon)} \frac{2ik}{(ik - m)} \frac{2ik'}{(ik' - m)}. \tag{5.60}$$

After performing the k and k' integrations by closing the contours in the upper half plane, we obtain

$$-\frac{im\beta^2}{4\pi} \int \frac{d\omega}{2\pi} \frac{K_b^+(\hat{k})}{(i\hat{k} + m)(i\hat{k} - m)} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')}, \tag{5.61}$$

where $\hat{k} = \sqrt{\omega^2 - m^2}$. We can set $\hat{k} = m \sinh\theta$ in (5.61) and extract a contribution to the boundary reflection factor which is

$$\frac{i\beta^2}{2\pi} K_b^+(\hat{k}) \frac{\sinh\theta}{\cosh^2\theta}. \tag{5.62}$$

We now consider the contribution which comes from the bulk potential. From the boson two-point function (5.41), we can write it down as

$$-im^2\beta^2 \int_{-\infty}^{+\infty} dt'' \int_{-\infty}^{0} dx'' G(x,t;\ x',t')\ G(x'',t'';\ x'',t'')\ G(x'',t'';\ x',t'). \tag{5.63}$$

The integral over t'' again provides a delta function which allows us to perform the ω' integral. The middle propagator is

$$G(x'', t''; \; x'', t'') = \int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \left[1 + K_b^+(k'') \, e^{-2ik''x''} \right]. \tag{5.64}$$

Again the first term is divergent. We suppose a minimal subtraction can be made to yield a finite part. We leave the discussion for the end of this chapter. Integrating over ω'' in the second term produces

$$\int \frac{dk''}{2\pi} \frac{K_b^+(k'')}{2\sqrt{k''^2 + m^2}} e^{-2ik''x''}. \tag{5.65}$$

We substitute it with the other two propagators into (5.64) to get

$$\begin{aligned} -im^2\beta^2 \int_{-\infty}^{0} dx'' \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \frac{ie^{-ikx - ik'x'}}{\omega^2 - k'^2 - m^2 + i\epsilon} \\ \int \frac{dk''}{2\pi} \frac{1}{2\sqrt{k''^2 + m^2}} K_b^+(k'') \left[e^{i(k+k'-2k'')x''} + K_b^+(k') e^{i(k-k'-2k'')x''} \right. \\ \left. + K_b^+(k) e^{i(-k+k'-2k'')x''} + K_b^+(k) K_b^+(k') e^{i(-k-k'-2k'')x''} \right]. \end{aligned} \tag{5.66}$$

The integration over x'' can again be obtained by using (3.53). The k'' integrations can be achieved in the same way presented in (5.47).

The result of our calculation for (5.66) is

$$\begin{aligned} -\frac{i\beta^2}{\pi} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{i\hat{k}(x+x')} \left(\frac{1}{2\hat{k}} \right)^2 K_b^+(\hat{k}) \\ \left[\frac{2\sinh^2\theta}{\cosh^2\theta} + \frac{\pi}{2} \sinh^2\theta \left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta} \right) \right]. \end{aligned} \tag{5.67}$$

We can extract the reflection factor from this integral as

$$-\frac{i\beta^2}{2\pi} K_b^+(\hat{k}) \sinh\theta \left[\frac{1}{\cosh^2\theta} + \frac{\pi}{4} \left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta} \right) \right]. \tag{5.68}$$

Combining this factor with (5.62), we obtain the contribution of the boson reflection factor coming from diagram (b) in Figure 5.1

$$-\frac{i\beta^2}{8} K_b^+(\hat{k}) \sinh\theta \left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta} \right). \tag{5.69}$$

The next task is to calculate the fermion loop correction to the boson propagator. The only contribution comes from the bulk potential for diagram (c) in Figure 5.1. We may write down the contribution in the following form

$$\frac{i}{2}m\beta^2 \int_{-\infty}^{+\infty} dt'' \int_{-\infty}^{+\infty} dx'' \, G\left(x,t;\; x',t'\right) Tr S_F\left(x'',t'';\; x'',t''\right) \, G\left(x'',t'';\; x',t'\right). \tag{5.70}$$

The trace of the fermion propagator is

$$Tr S_F\left(x'',t'';\; x'',t''\right) = \int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon} \left[2m + \frac{2\omega''^2}{ik'' - m} \, e^{-2ik''x''} \right], \tag{5.71}$$

where the first term is logarithmically divergent. It may be eliminated by mass renormalisation. The second term also contains a divergence that looks as if it has to do with wave function renormalisation since an exponential term is attached. For the moment, we will not discuss the renormalisation. By rewriting ω''^2 in the second term, we might choose the finite part as

$$\int \frac{d\omega''}{2\pi} \int \frac{dk''}{2\pi} \frac{2im}{\omega''^2 - k''^2 - m^2 + i\epsilon} \, K_b^+(k'') \, e^{-2ik''x''}, \tag{5.72}$$

which can be integrated out to obtain

$$\int \frac{dk''}{2\pi} \, K_b^+(k'') \frac{m}{\sqrt{k''^2 + m^2}} \, e^{-2ik''x''}. \tag{5.73}$$

Placing this back into expression (5.70) together with two boson propagators, we have

$$\begin{aligned} &\frac{i}{2}m^2\beta^2 \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - m^2 + i\epsilon} \frac{ie^{-ikx-ik'x'}}{\omega^2 - k'^2 - m^2 + i\epsilon} \\ &\int \frac{dk''}{2\pi} \frac{K_b^+(k'')}{\sqrt{k''^2 + m^2}} \int_{-\infty}^{0} dx'' \left[e^{i(k+k'-2k'')x''} + K_b^+(k') e^{i(k-k'-2k'')x''} \right. \\ &\left. + K_b^+(k) e^{i(-k+k'-2k'')x''} + K_b^+(k) K_b^+(k') e^{i(-k-k'-2k'')x''} \right]. \end{aligned} \tag{5.74}$$

The remaining integrations are similar to the one that we performed for diagram (b) in Figure 5.1. The calculation gives the following contribution to the reflection factor

$$\frac{i\beta^2}{2\pi} \, K_b^+(\hat{k}) \frac{\sinh\theta}{\cosh^2\theta} + \frac{i\beta^2}{8} \, K_b^+(\hat{k}) \sinh\theta \left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta} \right). \tag{5.75}$$

Combining the results (5.69) and (5.75), we finally obtain the quantum correction of the boson reflection factor up to order β^2. This is given by

$$R_b^+(\hat{k}) = K_b^+(\hat{k})\left[1 + \frac{i\beta^2}{2\pi}\,\frac{\sinh\theta}{\cosh^2\theta}\right]. \tag{5.76}$$

This reflection factor does not agree with the result obtained in (4.57).

There are two other ways to choose the finite part. One way is to choose the finite part as

$$\int \frac{d\omega''}{2\pi}\int \frac{dk''}{2\pi}\frac{-2im}{\omega''^2 - k''^2 - m^2 + i\epsilon}\,e^{-2ik''x''}. \tag{5.77}$$

With the similar integral procedure, we can extract a contribution to the reflection factor

$$-\frac{i\beta^2}{2\sinh\theta}\,K_b^+(\hat{k})\left[\frac{\sinh^2\theta - 1}{4\cosh^3\theta} + \frac{\theta\sinh\theta}{\pi\cosh^3\theta} + \frac{1}{4}\right]. \tag{5.78}$$

This enables us to determine the following reflection factor

$$R_b^+(\hat{k}) = K_b^+(\hat{k})\left[1 - \frac{i\beta^2}{2\pi}\,\frac{\sinh\theta}{\cosh^2\theta} - \frac{i\beta^2}{4}\,\sinh\theta\left(\frac{1}{\cosh\theta + 1} - \frac{1}{\cosh\theta}\right)\right]. \tag{5.79}$$

The finite part chosen by another method is

$$\int \frac{d\omega''}{2\pi}\int \frac{dk''}{2\pi}\frac{i}{\omega''^2 - k''^2 - m^2 + i\epsilon}\,\frac{2m^2}{ik'' - m}e^{-2ik''x''}, \tag{5.80}$$

which correspondingly gives the reflection factor

$$R_b^+(\hat{k}) = K_b^+(\hat{k})\left[1 - \frac{i\beta^2}{2\pi}\left(-\frac{\sinh\theta}{2\cosh^2\theta} + \frac{\pi}{4\sinh\theta}\left(1 - \frac{1}{\cosh^3\theta}\right) + \frac{\theta}{2\cosh^3\theta}\right)\right]. \tag{5.81}$$

However none of these results give the right correction to the boson reflection factor. At the present, we are unable to find a unique way in which to choose the finite part in (5.71). The boson reflection factors obtained here do not agree with the perturbative result which we have obtained in the previous chapter. Principally, the supersymmetry suggests that the boson reflection factor should be same as the fermion one except for the classical factors. This provides enough information to establish the boson reflection factor. It should be the one given in (4.57). However, this leaves the question, how can one get this result from the loop calculation?

5.9 The boson reflection factor for $\mathcal{BC}^-$ case

With the same analysis presented in the previous section, we obtain the bosonic reflection factor that corresponds to the boundary $\mathcal{BC}^-$. We have shown that there are three ways to choose the finite part in the boson loop propagator. The first method produces the result to be

$$R_b^-(\hat{k}) = K_b^-(\hat{k}) \left[1 - \frac{i\beta^2}{2\pi} \frac{\sinh\theta}{\cosh^2\theta}\right]. \tag{5.82}$$

This factor as well as the results obtained in the other two methods discussed in the previous section do not give us the correct answer. As far as supersymmetry is concerned , we can guess the correct result should be (4.73).

On the other hand, the renormalisation may enable us to choose the finite part in the theory.

5.10 Renormalisation

We have seen in previous sections that integrations over one-loop integrals in Feynman diagrams contain divergent terms. In order to eliminate this divergence, we need to consider the renormalisation of the theory.

Since the theory has ultraviolet divergences, one should add renormalisation counter terms to ensure ultraviolet finiteness.

We consider the mass renormalisation at the one-loop level

$$m^2 = m_0^2 + \delta m^2, \tag{5.83}$$

where m_0 is a bare mass and δm^2 contains the divergences.

By substituting (5.83) into the supersymmetric Lagrangian (5.27), we can deduce the two point functions for the boson and fermion with the same analogy to sections 5.4 and 5.5. In this case, we have the following two diagrams in addition to the diagrams given in Figures 5.1 and 5.2.

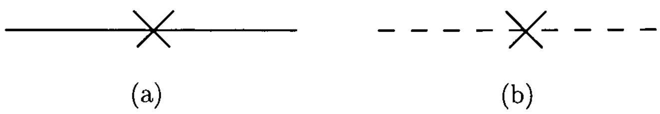

Figure 5.3: Mass counter terms to the boson and fermion propagators.

The diagram (a) in Figure 5.3 represents the mass counter term to the fermion two point function. It is algebraically given by

$$i\frac{\delta m^2}{2m_0}\int d^2x\, \theta(-x)S_F(x_1,x)\; S_F(x,x_2). \tag{5.84}$$

Correspondingly, the diagram (b) in Figure 5.3 represents the mass counter term to the boson two point function. It is found to be

$$i\delta m^2 \int d^2x \left[\theta(-x) \pm \frac{\delta(x)}{2m_0}\right] G(x_1,x)\; G(x,x_2), \tag{5.85}$$

where the first term corresponds to the bulk contribution and the second term to the boundary contribution.

We need to fix the counter mass δm^2 next. Let us begin with the first diagram. The mass counter term given by this diagram is

$$i\frac{\delta m^2}{2m_0}. \tag{5.86}$$

We now come back to study the divergent term in (5.44)

$$-\frac{i}{4}m_0\beta^2\int\frac{d\omega''}{2\pi}\int\frac{dk''}{2\pi}\frac{i}{\omega''^2-k''^2-m^2+i\epsilon}. \tag{5.87}$$

After integrating over the energy integral and then the momentum integral for large Λ, we obtain

$$-\frac{i}{8\pi}m_0\beta^2\ln\frac{2\Lambda}{m_0}, \tag{5.88}$$

where Λ is the ultraviolet cut off. As $\Lambda \to +\infty$, it is obvious that this is logarithmically divergent. The divergence should be eliminated by (5.86). This fixes the mass counter term as

$$\delta m^2 = \frac{m_0^2\beta^2}{4\pi}\ln\frac{2\Lambda}{m_0}. \tag{5.89}$$

For the diagram (b) in Figure 5.3, we need to consider the boundary and bulk contributions separately. For the boundary case, we have seen in (5.56), there is the divergent term including the factor $-im_0\beta^2/4$

$$-\frac{i}{4}m_0\beta^2\int\frac{d\omega''}{2\pi}\int\frac{dk''}{2\pi}\frac{2i}{\omega''^2-k''^2-m^2+i\epsilon}. \tag{5.90}$$

Correspondingly it gives the mass counter term

$$\delta m^2=\frac{m_0^2\beta^2}{2\pi}\ln\frac{2\Lambda}{m_0}. \tag{5.91}$$

However we notice that the mass counter term is fixed differently in the bulk contribution of the boson loop correction to the fermion propagator and in the boundary contribution of the boson loop correction to the boson propagator. In principle, if the mass renormalisation works, the mass counter term should be fixed uniquely. In other words (5.89) and (5.91) should be same.

Let us now to study the bulk contribution. For this case we have the two divergences one comes from the boson loop (b) and the other one from the fermion loop (c) in Figure 5.1. The divergent part of the boson loop can be read from (5.64) including the factor $-im_0^2\beta^2$ as

$$-im_0^2\beta^2\int\frac{d\omega''}{2\pi}\int\frac{dk''}{2\pi}\frac{i}{\omega''^2-k''^2-m^2+i\epsilon}, \tag{5.92}$$

and the fermion loop divergence from (5.71) including the factor $im_0\beta^2/2$ as

$$\frac{i}{2}m_0\beta^2\int\frac{d\omega''}{2\pi}\int\frac{dk''}{2\pi}\frac{i2m_0}{\omega''^2-k''^2-m^2+i\epsilon}. \tag{5.93}$$

We find that (5.92) and (5.93) are the same upto a minus sign. Therefore the divergences coming from the boson loop and the fermion loop exactly cancel. This implies that

$$\delta m^2=0, \tag{5.94}$$

and we do not need to perform mass renormalisation.

As we have seen in the section 5.8, there is another divergence in the second term of (5.71) and it is multiplied by the factor $e^{-2ik''x''}$. This divergence might be eliminated by the coupling or wave function renormalisations.

Let us consider the SUSY transformation (5.3)

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi + \bar{\psi}\epsilon, \\ \delta\psi &= \left(i\gamma^\mu \partial_\mu \phi + \frac{\sqrt{2}}{\beta} \sinh \frac{\beta\phi}{\sqrt{2}} \right) \epsilon, \\ \delta\bar{\psi} &= \bar{\epsilon} \left(-i\gamma^\mu \partial_\mu \phi + \frac{\sqrt{2}}{\beta} \sinh \frac{\beta\phi}{\sqrt{2}} \right), \end{aligned} \tag{5.95}$$

If we substitute the bare quantities

$$\begin{aligned} \phi_0 &= Z_1^{1/2}\phi, & \psi_0 &= Z_2^{1/2}\psi, \\ m_0 &= Z_m m, & \beta_0 &= Z_\beta \beta, \end{aligned} \tag{5.96}$$

into the SUSY transformation, we obtain the following relations between the renormalisation constants

$$Z_1 = Z_2, \qquad Z_m = 1, \qquad Z_\beta = Z_1^{-1/2}. \tag{5.97}$$

Here the mass renormalisation factor $Z_m = 1$ again implies that we do not need to do the mass renormalisation. If this is the case, then it is not clear how we will eliminate the divergent terms (5.87) and (5.90).

5.11 Conclusion

In this chapter, we have computed the supersymmetric fermion and boson reflection factors by calculating the one loop diagrams. Having neglected the divergence in the boson loop correction to the fermion propagator, the result obtained by calculating the one-loop diagrams agrees with the result obtained in the previous chapter. However, the boson reflection factors do not. This motivated us to study the renormalisation. The question is how can we renormalise the theory consistently?

Chapter 6

Conclusion and Outlook

This thesis studies the sinh-Gordon theory known as the simplest model in the affine Toda theory.

We have attempted to investigate the theory with two boundaries. By looking at a particular solution to the equation of motion, we have obtained the reflection factors and the energy. However this solution is not a real solution. When we linearised the field we found that the static background solution was described by an elliptic function and the equation of motion was formulated to the Lamé equation.

We then studied the theory with one boundary condition. By developing perturbation theory, we deduced the three types of one-loop Feynman diagrams in order to calculate the quantum correction of the reflection factor. A correction of the boundary contribution from the bubble diagram has been obtained. In the limit $\sigma_0 = \sigma_1 = \sigma$, this correction agrees with the one obtained in [24].

We also studied the supersymmetric extension of the model with one boundary condition. We have found that the classical limits of the SUSY reflection factors [65] corresponding to the boundary condition $\mathcal{BC}^+$ appear to be incorrect. On the other hand, by calculating the Feynman diagrams at the one loop order, we obtained the fermion reflection factors. These agree with the results obtained in chapter four. Although we were unable to renormalise the theory properly, the SUSY however provides us with an extra ingredient to enable us to determine the boson reflection factors, once we know the fermion reflection factors.

There are several interesting problems which have been suggested for investigation in the literature regarding affine Toda theories.

Firstly, an investigation into the renormalisability of the theory for the sinh-Gordon model with one boundary condition, but with different boundary parameters σ_0 and σ_1. In particular how one could renormalise the wave function, mass, boundary parameters and coupling consistently. This is an important question to study in detail. For the a_n $(n \geq 2)$ theory, the classical integrability fixes the boundary parameters up to signs. The relation between different boundary parameters for the other affine Toda field theories may not be compatible with the classical integrability and renormalisability [16]. How does the renormalisation work in these cases?

Secondly, the need to study the duality of the affine Toda models. There are two types of duality. One is the weak-strong coupling duality, in which the S-matrix is invariant under the transformation $\beta \longrightarrow 4\pi/\beta$. The S-matrix of the affine Toda models based on *ade* series is self-dual. This type of duality has been observed for the sinh-Gordon model [24] and the $a_2^{(1)}$ theory [35]. Another is the particle-soliton duality: the Thirring model [69] exhibits this kind of duality with the sine-Gordon model. One can investigate what kind of duality property is possessed by reflection factors. What happens to these dualities in the presence of boundary conditions [70]?

Thirdly, in the supersymmetric case, it would be interesting to understand the effect of the fermions on the duality transformation, since one would guess the two allowed points are either self-dual or dual partners [24].

Besides these, there remain some problems which need further investigation.

1) In the second chapter, we have attempted to find the solution to the sinh-Gordon equation by two different methods. One method assumed that the solution has the form(2.55). However, the solution obtained with this ansatz is not a real solution. An appropriate assumption can be made as the solution to the sinh-Gordon equation or the equation may be solved using the Hirota's method. The new solution

should allow us to investigate the energy spectrum. The second method was to solve the sinh-Gordon equation around the static background field. We found that the background solution is described by an elliptic function. Lamé's equation needs to be solved in order to obtain an approximate energy spectrum in the two-boundary sinh-Gordon theory.

2) The calculation on the bulk contribution in diagram (b) in Figure 3.1 could be finished. Once the second order quantum correction of the reflection factor for the sinh-Gordon model with the general one-boundary condition is completed, one might try to calculate the correction to a higher order. Calculating the quantum reflection factor is one way of approaching the quantum integrability of the theory.

3) How does the renormalizability of the supersymmetric sinh-Gordon theory work? By looking at the supersymmetric transformations themselves, we deduce that the mass renormalisation need not be performed in order to maintain the supersymmetry. If this is the case, then how can one eliminate the divergences of the loop integral in order to obtain the quantum boson and fermion reflection matrices?

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