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بسم الله الرحمن الرحيم

On Monopoles in Low Energy String Theory and Non-abelian Particle Trajectories

by

Azizollah Azizi

A thesis submitted for the degree of

Doctor of Philosophy

July 1997

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UNIV DUNELM
2 3 JAN 1998

To *my Wife* and *Children*

Abstract

On Monopoles in Low Energy String Theory and Non-abelian Particle Trajectories

Azizollah Azizi

This thesis is mainly concerned with monopoles. First, the existence of monopoles and their behaviour in the Yang–Mills-Higgs theories, and in parallel, the instanton solutions of the Yang-Mills fields are explained.

One part of this work is about monopoles and instantons in low-energy string theory. A general instanton solution for the heterotic string theory is obtained by using the ADHM construction for the classical subgroups of the string gauge group. In this direction, the embedding of subgroups and a general formula for the dilaton are explained. In the next topic of this part, the H-monopole and its generalisation to different subgroups of the string gauge group are discussed.

In the second part, the motion of the Yang–Mills particles in the Yang–Mills–Higgs fields are studied. Planar orbits are observed for a particle in a monopole field when the Higgs field contribution is neglected. The planar orbits are studied further with some numerical analysis of the equations of motion. By regarding the Higgs field contribution, a complete set of equations are worked out for the particle and fields. In this scenario, the planar motions as well as three-dimensional bounded motions are studied. At the end, the force exerted by the non-abelian Yang–Mills-Higgs fields on a particle with non-abelian charge is explored.

Preface

I declare that no part of my works in this thesis has been submitted for any degree at any university, or no published as an article in any journal. The work of other authors in this thesis is referred to at the appropriate point in the text.

I would like to acknowledge the Ministry of Culture and Higher Education of the Islamic Republic of Iran for providing the financial support for the duration of my Ph. D. studies in the University of Durham.

I wish to thank my supervisor Professor Edward Corrigan for his guidance, encouragement and help in preparing this thesis. Thanks are also due to the others who helped me in understanding and learning some concepts, in particular Dr Anne Taormina and Dr Peter Bowcock.

I express my highest appreciation to my wife for her supporting and encouragement, in addition to her tolerance and patience during this long time away from Iran. And thanks to my Iranian friends and my friends in the Department of Mathematical Sciences who made our stay in the beautiful city of Durham more enjoyable.

The poems in the first page of each chapter are from the gnostic poets, *Jalaluddin Rumi* (1207-1273 AD) and *Mahmud Shabestari* (1250-1320 AD). Thanks to Dr Colin Turner (CMEIS) for English translations.

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Chapter 1

Introduction

جهان را سر بسر آئینه می دان

به هر یك ذره ای صد مهر تابان

اگر یك قطره را دل بر شکافی

برون آید از آن صد بحر صافی

« شبستری »

Know that the cosmos is but one vast mirror,

and that each atom contains a hundred blazing suns;

Split open the heart of a single drop of water,

and see a hundred pure seas flow forth.

"Shabestari"

During the last two and a half decades, monopoles have been studied extensively. Monopoles are solutions to the Yang–Mills–Higgs fields. The most significant topological property of monopoles is the quantisation of magnetic charge in the classical limit. Monopoles are mostly studied in a time-invariant perspective, which gives a three spatial dimensional soliton solutions to the theories. A BPS monopole is a solution that minimises energy of the Yang–Mills–Higgs fields.

Instantons are solutions to the four-dimensional Euclidean Yang–Mills theories, which satisfy some conservation laws, and allow classical quantised quantity referred to as instanton number. Instantons are the fields solutions that minimise the Yang–Mills action.

The BPS monopoles are solutions to the Bogomol'nyi equations, and the instantons are solutions of the (anti)self-dual equations. It is relatively easy to write down the Bogomol'nyi equations as (anti)self-dual relations. In this scenario the Higgs field is considered to be the fourth spatial component of a five-dimensional pure Yang–Mills field and the fourth spatial dimension has no contribution in the fields. This simulation makes a motivation to modify the instanton rules and use them for monopoles. For example the Nahm modification of ADHM construction is adapted to find the general monopole solution of the four-dimensional Yang–Mills–Higgs fields, so that ADHM construction gives the rules of finding the general instanton solution to the four-dimensional Euclidean Yang–Mills fields. We have reviewed monopoles and instantons in chapter 2.

It is more obvious to apply the above mentioned procedure for monopoles and instantons in superstring theory which are originally built in a 10-dimensional space-time and then compactified down to 4-dimensional space-time. The idea is that five spatial dimensions are compactified in a surface and the remaining 5-dimensional space-time is prepared for the instanton and/or monopole solutions. This has been done, and some authors have successfully used the instanton approach to form some monopole solutions for the heterotic superstring theory.

In chapter 3 the equations of motion of the fields are explained, and solutions of an $SU(2)$ subgroup of the heterotic superstring gauge field ($E_8 \times E_8$ or $SO(32)$) in both instanton and monopole cases are discussed. Then a general instanton solution is obtained by applying the ADHM construction which gives a general solution for any classical subgroup of the main gauge group. We have explained the embedding of subgroups by introducing the Dynkin index of embedding. So for any subgroup we may construct the solution in the minimal embedding, and use the index of embedding when we are asked to generalise solution to any embedding.

The H-monopoles are new objects that behave like a monopole, and appear when the anti-symmetric tensor field is compactified down from higher dimensions to 4-dimensions. The BPS solution of the string field equations is also a H-monopole solution. We then explain the relation between the total mass of a H-monopole with its charge, from which we may see another Bogomol'nyi bound analogous to the relation between the mass and charge of a BPS monopole.

In the last part of chapter 3 monopoles in the $SU(N)$ subgroups of the heterotic superstring gauge group are studied. We discuss the monopole behaviour of the $SU(3)$ subgroup in some details and show that the monopole's magnetic charge and the charge associated to the anti-symmetric tensor field are of opposite sign. The chapter is ended by reviewing the different kinds of monopole and instanton solutions in the low-energy superstring theory.

In chapter 4 we turn to a different usage of monopoles. In the Yang–Mills quantum field theories, such as electroweak and QCD, particles are assigned with some charges like hypercharge and colour. In QED (an abelian gauge theory) the charge is a conserved quantity that is related to the gauge-invariant property of the Lagrangian and expressed by a unique real number. In the non-abelian theories the charge can no longer be expressed by a unique real number and might be displayed as a vector in the space of the gauge group (space of symmetry).

Wong extracted the classical equations of a particle with a non-abelian charge in a classical Yang–Mills field. The equation of motion of a (non-abelian) particle in a (non-abelian) field is the modification of the Lorentz force in the usual electrodynamics. Now the charge is a vector (in the isospace) and therefore may evolves in the time. Wong has given the equation of evolution of the charge isovector, while the length of the charge vector remains constant.

As an application of the Wong equations of motion we consider the 't Hooft or BPS monopoles and launch a Yang–Mills test particle in the field of the monopole. The speed and the total angular momentum of the particle and the field are constants of motion. We explain the equations of motion and enumerate some results. An interesting consequence of this motion is planar orbits. When a test particle is launched in the field of the monopole, while the direction of its charge isovector is normal to its position and velocity vectors, it will move in the plane normal to the charge isovector forever. We have explained the planar motions and the conditions for bounded orbits and their stability, and performed some numerical analysis of the equations of motion.

Next, we introduce the Higgs field to exert a force on the particle in addition to the monopole force. We have introduced Wong's equations in five dimensions, and modified the equations of motion for a particle in the Yang–Mills–Higgs fields. We have chosen the extra fifth-dimension to have no contribution in the fields, leaving it as a dynamical variable. A close relation between the evolution of the fifth dimension and the isospace vectors, the Higgs and non-abelian charge, are observed. Motion of a test particle, now in presence of a force from the Higgs field, is studied. Three-dimensional bounded orbits are observed, and stable planar orbits are allowed .

Finally we investigate motion of a particle in the Yang–Mills–Higgs field, and introduce the generalised Lorentz force by interpreting the components of the fields and particle interaction. We show the generalised force shrinks to the usual Lorentz force at large distances where we may interpret the non-abelian fields and particle as the usual fields and particle. The thesis is concluded by summarising the results in chapter 5.

Chapter 2

Monopoles In Gauge Theories

هر یکی قولیست ضد هم دگر

چون یکی باشد یکی زهر و شکر

تا ز زهر و از شکر درنگذری

کی تو از گُلزار وحدت بو بَری

« مولوی »

Everything seems to have an opposite,

one appears to be poison the other sweet;

While you are stuck in the illusion of the sugar and the poison,

how would you be able to detect the fragrance of the rose-garden of unity?

"Rumi"

Magnetic monopoles were first introduced by Dirac as a part of modified classical and quantum electrodynamics [1]. Magnetic property of a field can be considered as dual of its electrical property, *ie.* they transform under a duality transformation, and in the same sense magnetic monopoles can be supposed as electric monopoles companion. However, lack of experimental observation of magnetic monopoles convinced many people such as Dirac himself[1] to doubt the existence of monopoles. Although with the assumption of presence of monopoles some nice properties[2] arise, if there is no evidence, it is no problem to lay them aside and return to the original theory which existed before Dirac's modification.

In late 60's to early 70's some new non-abelian gauge field theories appeared such as $SU(2) \times U(1)$ theory of electroweak interactions, $SU(3)$ theory of strong interactions, and many unified theories. For the first time 't Hooft [3] and Polyakov [4] showed monopole solutions arise in some of these theories. However, these monopoles are different from Dirac's. Contrary to Dirac monopoles, these monopoles come from theories without modification, and not believing in monopoles would cause the whole theory to be turned upside down (*ie.* they are predictions and their absence requires an explanation). On the other hand, the 't Hooft–Polyakov solution has a natural interpretation as an extended object, which at large distances provides an explicit model of a Dirac monopole, but whose short-distance structure has been modified so that it has finite energy [5].

We shall start from the 't Hooft–Polyakov model of a monopole and then go farther to explain the BPS monopole, and monopoles in other non-abelian gauge

[1]Dirac in a letter to Monopoles Meeting (1981) had mentioned "I am inclined now to believe that monopoles do not exist." [2, page iii].

[2]For example: Existence of magnetic monopoles causes quantisation of electric and magnetic charges at the same time.

theories. We will explain one of the monopole companions, instanton, to complete this introduction for later needs.

Convention: In this thesis we use the "Natural Units", that is "rationalised Gaussian cgs" or "Heaviside-Lorentz" system, with $c = \hbar = 1$.

2.1 The 't Hooft–Polyakov Model of a Monopole

To show how magnetic monopoles arise in non-abelian gauge field theories, we explain the simplest case in which $SO(3)$, the three dimensional rotation group, is the gauge group. Based on dependency of gauge fields on the gauge group, an "internal space" can be attributed to this gauge group, which we may refer to as "isospace". In $SO(3)$ case isospace is a three dimensional space, and we point to these dimensions with indices $a, b, c, \ldots$. We choose $\mu, \nu, \ldots$ for normal 4-dimensional space-time indices, and $i, j, k, \ldots$ just for spatial coordinate indices. The isospace is an Euclidean space while space-time is a Minkowskian space with the metric

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{2.1}$$

The Lagrangian of a $SO(3)$ gauge field interacting with a Higgs field $\mathbf{\Phi}$ is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} (\mathbf{D}^\mu \mathbf{\Phi})^a (\mathbf{D}_\mu \mathbf{\Phi})^a - V(\mathbf{\Phi}), \tag{2.2}$$

where $F^a_{\mu\nu}$ are the gauge field strengths:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc} A^b_\mu A^c_\nu. \tag{2.3}$$

In the above equation A^a_μ is gauge potential and e is coupling constant. The tensors $\mathbf{F}_{\mu\nu}$ and $\mathbf{\Phi}$ are defined as $F^a_{\mu\nu}\mathbf{T}^a$ and $\Phi^a\mathbf{T}^a$ [3] where $\mathbf{T}^a$, with $a = 1, 2, 3$, are generators of the gauge group $SO(3)$ (or equivalently $SU(2)$). The covariant derivative $(\mathbf{D}_\mu\mathbf{\Phi})^a$

[3] It is convenient to make clear some notations here. We use normal vector $\vec{S} = (S_1, S_2, S_3)$ that is a (gauge independent) vector in real spatial space, and isovector $\vec{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$ that is a vector in isospace (and each component is a scalar in real space). So in the case of gauge tensor, $F^a_{\mu\nu}$ with $a = 1, 2, 3$, $\vec{F}_{\mu\nu} = (F^1_{\mu\nu}, F^2_{\mu\nu}, F^3_{\mu\nu})$ is an isovector. We can also have a "bi-vector" that means, it is a vector in real space in some sense and an isovector in isospace in some sense. An example is gauge vector potential A^a_i, that $\vec{A}_i = (A^1_i, A^2_i, A^3_i)$ is an isovector and $\vec{A}^a = (A^a_1, A^a_2, A^a_3)$ is a vector. We keep the bold characters like $\mathbf{A}_i$ to be a matrix (unless we specify the case).

is defined as:

$$(\mathbf{D}_\mu \mathbf{\Phi})^a \equiv \partial_\mu \Phi^a + e\epsilon^{abc} A^b_\mu \Phi^c. \tag{2.4}$$

The last term in the Lagrangian is gauge independent potential $V(\mathbf{\Phi})$,

$$V(\mathbf{\Phi}) = \frac{\lambda}{4}(\Phi^b \Phi^b - a^2)^2, \tag{2.5}$$

where λ and a are arbitrary constants. By using the Euler–Lagrange equation and the Lagrangian (2.2) the equations of motion are

$$(\mathbf{D}_\nu \mathbf{F}^{\mu\nu})^a = -e\epsilon^{abc}\Phi^b(\mathbf{D}^\mu \mathbf{\Phi})^c \tag{2.6}$$

$$(\mathbf{D}_\mu \mathbf{D}^\mu \mathbf{\Phi})^a = \lambda \Phi^a (\Phi^b \Phi^b - a^2) \tag{2.7}$$

which are supplemented by the Bianchi identity

$$(\mathbf{D}^\mu\, {}^*\mathbf{F}_{\mu\nu})^a = 0, \qquad {}^*F^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F^a_{\rho\sigma}, \tag{2.8}$$

where tensor ${}^*F^a_{\mu\nu}$ is known as "dual" of the gauge field tensor $F^a_{\mu\nu}$. By analogy with the electric and magnetic fields, $\mathcal{E}_i$ and $\mathcal{B}_i$, one can define

$$E^a_i \equiv F^a_{0i} = -F^{a0i} \tag{2.9}$$

$$B^a_i \equiv \frac{1}{2}\epsilon_{ijk} F^{ajk}. \tag{2.10}$$

The energy density corresponding to the Lagrangian of eq(2.2) is

$$\mathcal{E} = \frac{1}{2}\left[(E^a_i)^2 + (B^a_i)^2 + ((\mathbf{D}_0\mathbf{\Phi})^a)^2 + ((\mathbf{D}_i\mathbf{\Phi})^a)^2\right] + V(\mathbf{\Phi}). \tag{2.11}$$

Notice that $\mathcal{E} \geq 0$, and vanishes if, and only if:

$$F^a_{\mu\nu} = (\mathbf{D}_\mu \mathbf{\Phi})^a = V(\mathbf{\Phi}) = 0. \tag{2.12}$$

A field configuration which these equations allow, is called a "vacuum" configuration. An example is:

$$\Phi^b = a\delta^{b3}, \qquad A^b_\mu = 0. \tag{2.13}$$

Since $\mathcal{E} = 0$ is a gauge-invariant condition, any gauge transformation of eq(2.12) will also provide a vacuum configuration.

A field configuration with $(\mathbf{D}^\mu \mathbf{\Phi})^a = V(\mathbf{\Phi}) = 0$, but $F^{a\mu\nu}$ not necessarily vanishing is called "Higgs vacuum". Actually the finite energy condition enforces eqs(2.12) to be satisfied asymptotically at large distances. In particular, this requires $\Phi^b\Phi^b = a^2$ when we consider eq(2.5), which states the Higgs field sweeps the surface of a two dimensional sphere of radius a in the isospace.

The finite-energy non-singular classic solution of eqs(2.6) and (2.7) can be of the form [3] (see also [7]):

$$\begin{aligned} A^{a0}(\vec{r}) &= J(r)\frac{x^a}{er^2}, && (2.14)\\ A^{ai}(\vec{r}) &= \epsilon^{aij}\frac{x^j}{er^2}[1-K(r)], && (2.15)\\ \Phi^a(\vec{r}) &= \frac{x^a}{er^2}H(r), && (2.16)\end{aligned}$$

where $H(r)$, $J(r)$, and $K(r)$ are certain functions of the radius r. For a finite energy solution, energy density $\mathcal{E}$ asymptotically vanishes, then the condition $V(\mathbf{\Phi}) = 0$ at large distances implies

$$\lim_{r\to\infty} \Phi^b(\vec{r}) = a\frac{x^b}{r}. \tag{2.17}$$

Replacing eqs(2.14)–(2.16) with $J = 0$ (for a pure magnetic field) in eqs(2.6) and (2.7), after some algebraic calculations, by applying the appropriate boundary conditions for a finite energy solution, one may find at large distances (see [5])[4]:

$$F^{aij} \sim -\frac{1}{er^4}\epsilon^{ijk}x^a x^k. \tag{2.18}$$

't Hooft introduced a gauge-invariant electromagnetic tensor which reduces to the usual electromagnetic field tensor when the scalar field $\mathbf{\Phi}$ has only third component, *ie.* $\Phi^b = a\delta^{b3}$, (see also [15])

$$\mathcal{F}_{\mu\nu} = F^a_{\mu\nu}\hat{\Phi}^a - \frac{1}{e}\epsilon^{abc}\hat{\Phi}^a(\mathbf{D}_\mu\hat{\mathbf{\Phi}})^b(\mathbf{D}_\nu\hat{\mathbf{\Phi}})^c. \tag{2.19}$$

It is more convenient to write $\mathcal{F}_{\mu\nu}$ as

$$\mathcal{F}_{\mu\nu} = \partial_\mu(\hat{\Phi}^a A^a_\nu) - \partial_\nu(\hat{\Phi}^a A^a_\mu) - \frac{1}{e}\epsilon^{abc}\hat{\Phi}^a(\partial_\mu\hat{\Phi}^b)(\partial_\nu\hat{\Phi}^c), \tag{2.20}$$

[4]Equivalently one may use eqs(2.15) and (2.16) to find F^a_{ij} and $(\mathbf{D}_i\mathbf{\Phi})^a$ directly from their definitions (note that $E^a_i = V(\mathbf{\Phi}) = 0$ but $(\Phi^a)^2 \to a^2$ as $r \to \infty$), and then apply the finite energy condition on eq(2.11) to find the limiting behaviours of the unknowns $H(r)$ and $K(r)$ (and their derivatives). Then using these asymptotic behaviours, the result (2.18) is straightforward [6, section 23.3].

where we have used eqs(2.3) and (2.4), and

$$\hat{\Phi}^a = \frac{\Phi^a}{(\Phi^a\Phi^a)^{1/2}} \,. \tag{2.21}$$

In a gauge field which $\hat{\Phi}^a$ is a fixed unit vector in three-dimensions, as it had been promised, $\mathcal{F}_{\mu\nu}$ is the ordinary electromagnetic field tensor, *ie.* $\mathcal{F}_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$. Since $(\mathbf{D}_\nu\mathbf{\Phi})^a$ vanishes rapidly as $r \to \infty$, the magnetic part of the field tensor in this gauge is given for $r \to \infty$ by the first term in eq(2.19). So at large distances from the centre of the monopole (*ie.* in the Higgs vacuum) from eq(2.18) and (2.19), or equivalently replacing eqs(2.15) and (2.16) in eq(2.20), the magnetic monopole property of the solution can be derived:

$$\mathcal{B}^i \equiv \frac{1}{2}\epsilon^{ijk}\mathcal{F}_{jk} \to \frac{1}{2}\epsilon^{ijk}F^a_{jk}\hat{\Phi}^a \sim -\frac{x^i}{er^3}, \tag{2.22}$$

which looks like magnetic field around a point magnetic monopole with magnetic charge $g = -\frac{4\pi}{e}$:

$$\mathcal{B}^i = \frac{g}{4\pi}\frac{x^i}{r^3}. \tag{2.23}$$

The Dirac's quantisation condition is satisfied:

$$\frac{q_0 g}{4\pi} = -\frac{1}{2}, \tag{2.24}$$

where $g = -\frac{4\pi}{e}$, and $q_0 = \frac{1}{2}q = \pm\frac{1}{2}e$ is the smallest possible charge which might enter the theory (q is electric charge of the charged vector bosons). One of the outstanding properties of monopoles comes in the simple relation of Dirac's quantisation condition:

$$\frac{q_0 g}{4\pi} = \frac{N}{2}, \tag{2.25}$$

where N is an integer number. This relation says if there are monopoles in the nature (even one) and they follow existing physical theories, then the quantisation of both magnetic and electric charges is guaranteed.

Magnetic charges do not come out of theories similar to their electric companions. Two noted differences are: Electric currents are Noether's currents that means electrical property of fields can be concluded from the symmetry of theory, while magnetic property does not come from such kind of symmetries. Actually such as solitons, monopoles are topological objects (next topic) such that their conservation laws are automatically exploited from their constructions, and not affected by equations of motion. The existence and quantisation of magnetic charges depends on topological characteristics of the space (or vacuum) that is appointed by boundary conditions. The other point is: electrical distribution of particles (eg. charged vector bosons) is singular and localised in a point, while in the 't Hooft–Polyakov monopole, the magnetic distribution is distributed in a portion of space and not singular.

Dyons

The monopole solution of 't Hooft and Polyakov obtained here is electrically neutral because of the condition $J = 0$ we entered in the equation (2.14). This is not a necessary consequence of the spherical symmetry which was used to obtain a solution possessing a monopole behaviour at large distances. Julia and Zee [11] obtained spherically symmetric solutions with $J \neq 0$. The same as the 't Hooft–Polyakov monopole, the magnetic charge exists and quantised. The electrical charge no more vanishes, and arbitrary (at least classically) which can be quantised by some proper quantum-mechanical treatment. These solutions with both electric and magnetic charges are called "dyons" [12]. A generalised Dirac's quantisation condition is established when two dyons of charges (q_1, g_1) and (q_2, g_2) are supposed to be in interaction

$$\frac{q_1 g_2 - q_2 g_1}{4\pi} = \frac{1}{2} n_{1,2} \qquad n_{1,2} \text{ an integer.} \tag{2.26}$$

In this work we will treat monopoles in our arguments.

2.2 Topological Property of Magnetic Monopoles

In the Higgs vacuum $(\mathbf{D}_\mu \Phi)^a = 0$, therefore from eq(2.19) and eqs(2.6) and (2.8) one may show in the Higgs vacuum the Maxwell equations are satisfied:

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0, \qquad \text{and} \qquad \partial_\mu {}^*\mathcal{F}^{\mu\nu} = 0, \tag{2.27}$$

where ${}^*\mathcal{F}^{\mu\nu}$ is the dual tensor of $\mathcal{F}^{\mu\nu}$. This is an important conclusion, that is in the Higgs vacuum the only non-vanishing component of the gauge field tensor is the component associated with the $U(1)$ group of rotations about $\vec{\Phi}$, $\mathcal{F}_{\mu\nu}$, which satisfies the Maxwell' s equations. In this sense, outside the region of monopole, the $SO(3)$ gauge theory is locally indistinguishable from conventional electromagnetic theory.

Now by considering the global attributes of the Higgs vacuum, we study the magnetic flux, g, through the closed surface $S_2^{(\mathrm{phy})}$. By Maxwell's equations g will be non-zero only if $S_2^{(\mathrm{phy})}$ surrounds a region which the conditions of the Higgs vacuum fails. Then

$$\begin{aligned} g &= \oint_{S_2^{(\mathrm{phy})}} \vec{\mathcal{B}} \cdot d\vec{S} \\ &= -\frac{1}{2e} \oint_{S_2^{(\mathrm{phy})}} \epsilon^{ijk} \epsilon^{abc} \hat{\Phi}^a (\partial^j \hat{\Phi}^b)(\partial^k \hat{\Phi}^c)\, dS^i, \end{aligned} \tag{2.28}$$

using eq(2.20) and the fact that the contribution of A_μ vanishes by Stokes' theorem. Notice that the derivatives $\partial^i \vec{\Phi}$ occurring in eq(2.28) are those tangential to $S_2^{(\mathrm{phy})}$, so that the magnetic charge within $S_2^{(\mathrm{phy})}$ depends only on the value of the Higgs field on $S_2^{(\mathrm{phy})}$. For a small increment on the Higgs field $\mathbf{\Phi}$:

$$\vec{\Phi}' = \vec{\Phi} + \delta\vec{\Phi}, \qquad \vec{\Phi} \cdot \delta\vec{\Phi} = 0. \tag{2.29}$$

Replacing $\vec{\Phi}'$ in eq(2.28), keeping the first order terms the integrand of eq(2.28) will have an extra term

$$\delta \left[\epsilon^{abc} \hat{\Phi}^a \partial^j \hat{\Phi}^b \partial^k \hat{\Phi}^c\right] = \epsilon^{abc} \left\{3 \left[\delta\hat{\Phi}^a \partial^j \hat{\Phi}^b \partial^k \hat{\Phi}^c\right] + \partial^j \left[\hat{\Phi}^a \delta\hat{\Phi}^b \partial^k \hat{\Phi}^c\right] - \partial^k \left[\hat{\Phi}^a \delta\hat{\Phi}^b \partial^j \hat{\Phi}^c\right]\right\},$$

which vanishes. Obtaining this result is straightforward. The integral of the last two terms in this expression vanishes by Stokes' theorem, and the first term vanishes by using the fact that $\partial^j \vec{\Phi} \times \partial^k \vec{\Phi}$ is parallel to $\vec{\Phi}$, and therefore perpendicular to $\delta\vec{\Phi}$ (by using the second relation in eq(2.29)). Consequently a small variation in the Higgs field $\mathbf{\Phi}$, subject to asymptotic Higgs vacuum condition, produces no change in the flux, g. This extends to any change in $\mathbf{\Phi}$ which can be built up by small deformations. Such a deformation is called a "homotopy". Examples of homotopies in the physical context under discussion are: the time development of $\mathbf{\Phi}$, the change in Φ under a continuous gauge transformation, and the change induced by altering $S_2^{(\mathrm{phy})}$ continuously in the Higgs vacuum. Consequently g is time-independent, gauge-invariant and unchanged under any continuous deformation of the surface $S_2^{(\mathrm{phy})}$ containing the monopole or monopoles.

Let us define a current (say magnetic current or a topological current), based on the conservative quantity we discussed

$$k^\mu = -\frac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\epsilon^{abc}\partial_\nu\hat{\Phi}^a\partial_\rho\hat{\Phi}^b\partial_\sigma\hat{\Phi}^c. \tag{2.30}$$

Using eq(2.20), it is simple to see the magnetic current comes in the right-hand side of a Maxwell equation that is null in the ordinary electromagnetism, *ie.* $\partial_\nu{}^*\mathcal{F}^{\mu\nu} = k^\mu$ (see eq(2.27). This comes in the analogy with the ordinary Maxwell equation $\partial_\nu\mathcal{F}^{\mu\nu} = j^\mu$ where j^μ is the electric current. The magnetic charge density $k^0 = -\frac{1}{2e}\epsilon^{ijk}\epsilon^{abc}\partial^i\hat{\Phi}^a\partial^j\hat{\Phi}^b\partial^k\hat{\Phi}^c$, and the magnetic charge $g = \int k^0\, d^3x$. We see the magnetic current depends on the Higgs field only, and moreover, this current is identically conserved:

$$\partial_\mu k^\mu = 0. \tag{2.31}$$

It is clear that the conservation of the current does not follows from the dynamics (or a symmetry of Lagrangian), and this is the fact we mentioned earlier: monopoles are topological objects *ie.* these objects are coming from topological behaviour of the fields where are appointed by the non-trivial boundary conditions.

To see g is quantised, note that we may write $g = -4\pi N/e$ where:

$$N = \frac{1}{8\pi}\oint_{S_2^{(\mathrm{phy})}} dS^i\,\epsilon^{ijk}\epsilon^{abc}\Phi^a\partial_j\Phi^b\partial_k\Phi^c. \tag{2.32}$$

The number N has the geometrical interpretation of being the number of times $\vec{\Phi}(\vec{r})$ covers the sphere $S_2^{(\mathrm{int})} = \{\vec{\Phi} : (\Phi^1)^2 + (\Phi^2)^2 + (\Phi^3)^2 = a^2\}$ as $\vec{r}$ covers $S_2^{(\mathrm{phy})}$ once (*ie.* the number of times $S_2^{(\mathrm{phy})}$ is wrapped about $S_2^{(\mathrm{int})}$ by the map $\vec{\Phi} : S_2^{(\mathrm{phy})} \to S_2^{(\mathrm{int})}$). Thus N must be an integer; it is called by mathematicians the Brouwer degree or Poincaré–Hopf index of map (some authors refer to N as 'winding' number). To show that every integer may realised for suitable $\vec{\Phi}$ consider

$$\vec{\Phi}_N(\vec{r}) = a(\cos N\varphi\sin\theta,\, \sin N\varphi\sin\theta,\, \cos\theta), \tag{2.33}$$

where (r, θ, φ) are spherical polar coordinates. This covers $S_2^{(\text{int})}$ N times as $\hat{\vec{r}}$ covers $S_2^{(\text{phy})}$ once, and yields N in eq(2.32).

We have seen that magnetic charge is topologically conserved and quantised in units of $4\pi/e$ for topological reasons. Further, since the smallest electric charge that we expect on quantising the theory is $q_0 = \frac{1}{2}e$ we have obtained Dirac's quantisation condition (2.25) by topological methods.

2.3 The Bogomol'nyi Bound and BPS monopole

An important feature of the 't Hooft–Polyakov monopole solution is that the mass is calculable. The mass of these solutions has a lower bound in terms of its electric and magnetic charges, first found by Bogomol'nyi [8]. In the Higgs vacuum $\Phi^b\Phi^b = a^2$ and $(\mathbf{D}_\mu\mathbf{\Phi})^a = 0$, so from eq(2.19)

$$\mathcal{F}_{\mu\nu} = \frac{1}{a}F^a_{\mu\nu}\Phi^a. \tag{2.34}$$

For any solution the magnetic charge is

$$g = \oint_S \vec{\mathcal{B}}\cdot d\vec{S} = \frac{1}{a}\oint_S B^a_i\Phi^a dS^i = \frac{1}{a}\int_V B^a_i(\mathbf{D}^i\mathbf{\Phi})^a d^3r, \tag{2.35}$$

where the surface integral is taken in a sphere at infinity. We have used definition of $\mathcal{B}^i$ from eq(2.22), eqs(2.10) and (2.34) to conclude second surface integral from first one. To conclude the volume integral from the surface integral we have used divergence theorem, and $(\mathbf{D}^i\mathbf{B}_i)^a = 0$ which comes from the Bianchi identity (2.8). Similarly using $\mathcal{E}^i = \mathcal{F}^{0i}$, the definition of E^a_i from (2.9) and equations of motion (2.6), the electric charge is:

$$q = \oint_S \vec{\mathcal{E}}\cdot d\vec{S} = \frac{1}{a}\int_V E^a_i(\mathbf{D}^i\mathbf{\Phi})^a d^3r. \tag{2.36}$$

To see how the covariant derivative has appeared in the volume integrals in eqs(2.35) and (2.36), one may suppose a normalised basis $\mathbf{T}^a$ for the gauge group such that $\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b) = \zeta\delta^{ab}$, and consequently

$$\mathrm{tr}(\mathbf{D}_\mu(\mathbf{PQ})) = \partial_\mu(\mathrm{tr}(\mathbf{PQ})) = \partial_\mu(\mathrm{tr}(P^a\mathbf{T}^a\,Q^b\mathbf{T}^b)) = \zeta\,\partial_\mu(P^aQ^a). \tag{2.37}$$

Consider the centre of mass frame of the monopole. The mass of the monopole is given by:

$$\begin{aligned} M &= \int \mathcal{E} d^3r = \int d^3r \left\{ \frac{1}{2} \left[(E_i^a)^2 + (B_i^a)^2 + ((\mathbf{D}_0\mathbf{\Phi})^a)^2 + ((\mathbf{D}_i\mathbf{\Phi})^a)^2 \right] + V(\mathbf{\Phi}) \right\} \\ &\geq \frac{1}{2} \int d^3r \left[(E_i^a)^2 + (B_i^a)^2 + ((\mathbf{D}_i\mathbf{\Phi})^a)^2 \right] \\ &= \frac{1}{2} \int d^3r \, [E_i^a - (\mathbf{D}_i\mathbf{\Phi})^a \sin\theta]^2 + \frac{1}{2} \int d^3r \, [B_i^a - (\mathbf{D}_i\mathbf{\Phi})^a \cos\theta]^2 \\ &\qquad\qquad\qquad + a(q\sin\theta + g\cos\theta) \\ &\geq a(q\sin\theta + g\cos\theta), \end{aligned} \tag{2.38}$$

where we have used eqs(2.35) and (2.36) in the third line. The parameter θ is an arbitrary real angle, and the inequality (2.38) is correct for any angle θ. Then we can choose θ such that optimise the above inequality:

$$M \geq a\sqrt{q^2 + g^2}. \tag{2.39}$$

To obtain the above inequality, simply we can find the maximum value of the function $f(\theta) = q\sin\theta + g\cos\theta$.

Inequality (2.39) which is known as the "Bogomol'nyi Bound" shows that there is a lower bound for the mass of any monopole solution of non-abelian theories [8]. In the case of 't Hooft–Polyakov monopole, that is a pure magnetic charge ($q = 0$)

$$M \geq a|g|. \tag{2.40}$$

The above procedure has been brought from Goddard and Olive [5] (after Coleman *et al* [9]). Using the value of the magnetic charge $|g| = \frac{4\pi}{e}$ one can relate the monopole mass M to the mass[5] of heavy gauge boson $M_q = qa = ae$:

$$M \geq a\frac{4\pi}{e} = \frac{4\pi}{e^2}ae = \frac{4\pi}{q^2}M_q = \frac{\nu}{\alpha}M_q, \tag{2.41}$$

where α (=1/137 for electron) is the fine-structure constant and $\nu = 1$ or $\frac{1}{4}$ depending on whether the charge on the electron is q or $\frac{1}{2}q$. It is seen the mass of monopole

[5]The masses of particles that would be expected in the theory specified by Lagrangian (2.2), are calculated from the Lagrangian in the usual way by recognising that when we expand about the vacuum the coefficient of the quadratic term in the boson fields is the square of mass divided by 2.

is much larger than the mass of the heavy gauge boson, which itself would be very large when its value is estimated in some unified theories. We mentioned in the introduction of this chapter there is no evidence yet for existence of monopoles. The above restriction in the mass of monopoles makes the observation of monopoles outside of today's experimental power (*ie.* the required energy for magnetic monopoles pair production in laboratories is more than the power of present particle accelerators). On the other hand still there is no evidence to say the hypothesis of monopoles is wrong, and therefore monopoles are an important and rich concept, and an integral part of non-abelian field theories.

The BPS Monopole

There is an exact solution which saturates the Bogomol'nyi bound (2.40), constructed by Prasad and Sommerfield [10]. In the BPS[6] limit we are seeking a solution with pure magnetic charge ($q = 0$) and the mass $M = a|g|$. Therefore from inequalities (2.38) these equations are required:

$$(\mathbf{D}_0\mathbf{\Phi})^a = 0, \qquad E_i^a = 0, \tag{2.42}$$

$$V(\mathbf{\Phi}) = 0, \tag{2.43}$$

$$B_i^a = \pm(\mathbf{D}_i\mathbf{\Phi})^a, \qquad + \text{ for } g > 0, \ - \text{ for } g < 0. \tag{2.44}$$

The equation (2.43) is realised if the coupling constant λ vanishes, although consistently the boundary condition $|\Phi| \to a$ as $r \to \infty$ is retained as a remnant of V. This condition guarantees charges are well-defined and quantised and the mass is finite as we described them earlier in this chapter. In this scenario equations of motion (2.6) and (2.7) become

$$(\mathbf{D}_\nu\mathbf{F}^{\mu\nu})^a = -e\epsilon^{abc}\Phi^b(\mathbf{D}^\mu\mathbf{\Phi})^c \tag{2.45}$$

$$(\mathbf{D}_\mu\mathbf{D}^\mu\mathbf{\Phi})^a = 0. \tag{2.46}$$

[6]BPS stands for Bogomol'nyi–Prasad–Sommerfield.

The zero component of eqs(2.45) is satisfied with eqs(2.42), and eq(2.46) is equivalent to the Bianchi identity $(\mathbf{D}^i\mathbf{B}_i)^a = 0$ using eqs(2.44). Therefore remains only to show

$$(\mathbf{D}_j\mathbf{F}^{ij})^a = -e\epsilon^{abc}\Phi^b(\mathbf{D}^i\mathbf{\Phi})^c. \tag{2.47}$$

To show this, we can follow this procedure. Put $F^{aij} = \epsilon^{ijk}B_k^a = \epsilon^{ijk}(\mathbf{D}_k\mathbf{\Phi})^a$ in the right-hand side of eq(2.47):

$$\begin{aligned}
(\mathbf{D}_j\mathbf{F}^{ij})^a &= \epsilon^{ijk}(\mathbf{D}_j\mathbf{D}_k\mathbf{\Phi})^a = \frac{1}{2}\epsilon^{ijk}\left([\mathbf{D}_j,\mathbf{D}_k]\mathbf{\Phi}\right)^a \\
&= \frac{1}{2}\epsilon^{ijk}\left([\mathbf{F}_{jk},\mathbf{\Phi}]\right)^a = \left[\frac{1}{2}\epsilon^{ijk}\mathbf{F}_{jk},\mathbf{\Phi}\right]^a \\
&= [\mathbf{B}_i,\mathbf{\Phi}]^a = [\mathbf{D}_i\mathbf{\Phi},\mathbf{\Phi}]^a \\
&= -e\epsilon^{abc}\Phi^b(\mathbf{D}_i\mathbf{\Phi})^c,
\end{aligned} \tag{2.48}$$

where we have used the identity $[\mathbf{D}_i,\mathbf{D}_j]\mathbf{R} = [\mathbf{F}_{ij},\mathbf{R}]$, which $\mathbf{R}$ is any isovector in the same isospace as $\mathbf{F}_{ij}$.

Equation (2.44) is a first order differential equation version of the Yang–Mills field equations (2.6). Replacing the ansatz of eqs(2.15) and (2.16) in eq(2.44) after some algebraic calculation one can find

$$H(r) = aer\coth(aer) - 1 \tag{2.49}$$

$$K(r) = \frac{aer}{\sinh(aer)}, \tag{2.50}$$

first obtained by Prasad and Sommerfield [10]. The BPS monopole is not localised in the same way of the 't Hooft–Polyakov monopole. For the 't Hooft–Polyakov monopole we can consider a finite radius R_0 (determined by the Compton wavelength, $\hbar/M$, of the heavy particles of the theory) such that outside the radius R_0 the fields configuration become a Higgs vacuum. But in the BPS monopole the fields configuration can not touch the Higgs vacuum, while the Higgs field is now massless (because we chose $\lambda = 0$) and long-range. In the BPS monopole, both the Yang–Mills and Higgs fields contribute in mass density equally (see eq(2.44) and mass density equation (2.11)), while in the 't Hooft–Polyakov monopole ($\lambda > 0$) the Higgs field has no contribution in far distances, where we called Higgs vacuum. For the BPS monopole the mass density is:

$$\mathcal{E} = \frac{1}{2}\left[(B_i^a)^2 + ((\mathbf{D}_i\mathbf{\Phi})^a)^2\right] = ((\mathbf{D}_i\mathbf{\Phi})^a)^2 = \frac{1}{2}\partial^2(\Phi^a)^2. \tag{2.51}$$

To show the last step, the same as in eq(2.37) we use a normalised basis $\mathbf{T}^a$ for the gauge group such that $\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b) = \zeta\delta^{ab}$,

$$\mathrm{tr}(\mathbf{D}_i\mathbf{\Phi})^2 = (\mathbf{D}_i\mathbf{\Phi})^a(\mathbf{D}_i\mathbf{\Phi})^b\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b) = \zeta((\mathbf{D}_i\mathbf{\Phi})^a)^2,$$

and then

$$\begin{aligned}
((\mathbf{D}_i\mathbf{\Phi})^a)^2 &= \frac{1}{\zeta}\mathrm{tr}(\mathbf{D}_i\mathbf{\Phi})^2 = \frac{1}{\zeta}\mathrm{tr}(\mathbf{D}_i(\mathbf{\Phi}\mathbf{D}_i\mathbf{\Phi}))\\
&= \frac{1}{\zeta}\partial_i(\mathrm{tr}(\mathbf{\Phi}\mathbf{D}_i\mathbf{\Phi})) = \frac{1}{\zeta}\partial_i\left(\frac{1}{2}\mathrm{tr}\mathbf{D}_i(\mathbf{\Phi})^2\right)\\
&= \frac{1}{\zeta}\partial_i\left(\frac{1}{2}\partial^i(\mathrm{tr}\mathbf{\Phi}^2)\right) = \frac{1}{\zeta}\frac{1}{2}\partial^2\mathrm{tr}(\mathbf{\Phi}^2)\\
&= \frac{1}{2}\partial^2(\Phi^a)^2,
\end{aligned}$$

where we have used eq(2.46) in the second equality. Using eq(2.16) and the spherical coordinates

$$\mathcal{E} = \frac{1}{2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}(\Phi^a)^2\right) = \frac{1}{2}\frac{1}{r}\frac{d^2}{dr^2}\left(\frac{H^2}{e^2 r}\right). \tag{2.52}$$

From the above equation and eq(2.49) one can show mass density is finite in the origin ($= \frac{1}{3}a^4e^2$), and exponentially decreasing at large distances. The solution we have described here is charge one monopole[7] that is clearly time-independent. For the sectors with $N \neq 1$ presumably there are solutions that depend on time, similar to a two-soliton solution that separate to become two separate solitons while energy decreases in this process. The BPS solutions with $N \neq 1$ static monopoles are constructed [17, 18].

An useful reformulation of the Bogomol'nyi equations is described in this paragraph. Imagine an Euclidean 4-dimensional space that has three dimensions as normal spatial dimensions and the fourth is also as a spatial dimension but different from normal spatial dimensions in nature. Indeed the theories in this space are time-independent, therefore we may use some formal properties of this space if we are

[7]From eq(2.51) or eq(2.52) one can see mass $= \int \mathcal{E}d^3r = \frac{4\pi}{e}a$ that shows $|g| = \frac{4\pi}{e}$ or $N = 1$.

dealing with time-independent solutions. If we make the identification $\Phi^a \equiv A_4^a$, and $F_{i4}^a \equiv (\mathbf{D}_i\mathbf{\Phi})^a$ we may write

$$F_{\alpha\beta}^a = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}^a, \qquad \alpha, \beta, \gamma, \delta = 1, \ldots, 4 \tag{2.53}$$

and recognise the Bogomol'nyi equations, (2.44), as self-dual[8] relations in four dimensions reduced to three, since no field depends on x^4, the fourth spatial dimension coordinate. In the next section we will discuss field solutions to four-dimensional Euclidean Yang–Mills equations, the Yang–Mills instantons, which satisfy (anti)self-dual relation (2.53).

2.4 The Yang–Mills Instantons

The term "instantons" refer to localised finite solutions of the classical Euclidean field equations of a theory (instead of Minkowskian versions we discussed in previous sections). Therefore instantons may have some similar concepts as monopoles have. In this section we discard the Higgs field, $\mathbf{\Phi}$, and just consider the $SU(2)$ gauge fields, A_α^a, with $a = 1, 2, 3$ and $\alpha = 1, 2, 3, 4$. To make the notation simpler, it is helpful to represent three vector-fields A_α^a, with a matrix-valued vector-field $\mathbf{A}_\alpha$ defined by

$$\mathbf{A}_\alpha(x) \equiv e\chi^a A_\alpha^a(x), \qquad \chi^a = \frac{\sigma^a}{2i}, \tag{2.54}$$

where e is coupling constant such as given in eq(2.3), and σ^a are the Pauli spin matrices. Here x stands for (x_1, x_2, x_3, x_4), as a four-dimensional spatial vector in an Euclidean space with metric

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3, 4. \tag{2.55}$$

The three matrices χ^a in eq(2.54) form the generators of the two-dimensional representation of the group $SU(2)$, and satisfy the Lie algebra

$$[\chi^a, \chi^b] = \epsilon^{abc}\chi^c. \tag{2.56}$$

[8]This means the tensor $F_{\alpha\beta}^a$ is equal to its dual tensor ${}^*F_{\alpha\beta}^a$ defined in eq(2.8). An "anti-self-dual" relation is defined the same as self-dual relation (2.53), but with a minus sign in front of right-hand side expression.

Correspondingly, we define a matrix valued field tensor:

$$\mathbf{F}_{\alpha\beta} \equiv e\chi^a F^a_{\alpha\beta}, \tag{2.57}$$

where

$$\mathbf{F}_{\alpha\beta} = \partial_\alpha \mathbf{A}_\beta - \partial_\beta \mathbf{A}_\alpha + [\mathbf{A}_\alpha, \mathbf{A}_\beta]. \tag{2.58}$$

Actually with this treatment the coefficient e has been removed from back of the bracket in the above equation.

The Euclidean action is obtained from Minkowskian Lagrangian (2.2), after omitting the scalar field Φ^a, and using the matrix notation for $\mathbf{A}_\alpha$. The action is

$$S = -\frac{1}{2e^2}\int d^4x \,\mathrm{tr}\left[\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}\right], \tag{2.59}$$

and the Euclidean Yang–Mills equations of motion are

$$\mathbf{D}_\alpha \mathbf{F}^{\alpha\beta} \equiv \partial_\alpha \mathbf{F}^{\alpha\beta} + [\mathbf{A}_\alpha, \mathbf{F}^{\alpha\beta}] = 0, \tag{2.60}$$

where both equations are invariant under gauge transformation.

The Yang–Mills instantons are finite-action (in analogous to finite-energy in monopoles case) solutions of eq(2.60). To find them one may proceed just as what was done in the case of static monopoles in the earlier sections. First of all a boundary condition is needed to be satisfied by any finite-action field configurations. As a first step towards this goal, we consider zero-action configuration (In analogous to zero-energy configuration in monopole case eq(2.12)). From eq(2.59) we see $S = 0$ if and only if $\mathbf{F}_{\alpha\beta} = 0$. This allows an infinity of possibilities for the vector-field $\mathbf{A}_\alpha$. Actually $\mathbf{F}_{\alpha\beta} = 0$ is a gauge-invariant equation, and therefore not only satisfied by $\mathbf{A}_\alpha = 0$, but also by any gauge-transformed field obtained from $\mathbf{A}_\alpha = 0$. These fields are called "pure gauges", and given by

$$\mathbf{A}_\alpha(x) = \mathbf{U}(x)\partial_\alpha[\mathbf{U}^{-1}(x)], \tag{2.61}$$

where $\mathbf{U}(x)$, at each point x, is any element of the gauge group $SU(2)$ in its 2×2 representation. Turning to finite-action configuration, it is clear from eq(2.59) that $\mathbf{F}_{\alpha\beta}$ must vanish on the boundary of Euclidean space, *ie.* on the three-dimensional sphere surface S_3 at $r = \infty$ where $r = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$ is the radius of the sphere in four dimensions. In fact $\mathbf{F}_{\alpha\beta}$ must vanish faster than $1/r^2$ as $r \to \infty$, and this can be obtained if we choose the following boundary condition on $\mathbf{A}_\alpha$:

$$\lim_{r\to\infty} \mathbf{A}_\alpha = \lim_{r\to\infty} \mathbf{U}\partial_\alpha \mathbf{U}^{-1}, \tag{2.62}$$

for some $\mathbf{U}$ in gauge group $SU(2)$. Comparing with the 't Hooft–Polyakov monopoles; Far from the monopole centre, the fields configuration should become the Higgs vacuum, while in instantons, far from the instanton location, fields configuration become pure gauge fields.

A question is what is similar to the BPS monopoles in instantons scenario. This is a good point to think about an exact instanton solution. We borrow Rajaraman [14] to find a relation similar to eq(2.40) for instantons. We begin with the trivial identity

$$-\int d^4x\, \mathrm{tr}[(\mathbf{F}_{\alpha\beta} \pm {}^*\mathbf{F}_{\alpha\beta})^2] \geq 0, \tag{2.63}$$

where ${}^*\mathbf{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}\mathbf{F}_{\gamma\delta}$ is the dual tensor. Using $\mathrm{tr}(\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}) = \mathrm{tr}({}^*\mathbf{F}_{\alpha\beta}\,{}^*\mathbf{F}^{\alpha\beta})$, this gives

$$2\,e^2 S = -\int d^4x\, \mathrm{tr}(\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}) \geq \pm \int d^4x\, \mathrm{tr}(\mathbf{F}_{\alpha\beta}\,{}^*\mathbf{F}^{\alpha\beta}) \tag{2.64}$$

From these equations we see the absolute minima value of action S occur when

$$\mathbf{F}_{\alpha\beta} = \mp\, {}^*\mathbf{F}_{\alpha\beta}. \tag{2.65}$$

Thus, self-dual and anti-self-dual configurations extremise S, and solve the field equations (2.60). Of course the absolute minima of S need not be its only extrema. Therefore this derivation does not prove that all solutions of eq(2.60) are (anti)self-dual, but conversely shows the (anti)self-dual configurations are solutions of the equations of motion.

In the BPS monopole, the Bianchi identity was automatically satisfied by assumption of the Bogomol'nyi equations and the equations of motion. Here the similar thing happens, *ie.* by choosing (anti)self-dual solutions, both equations of motion and Bianchi identity become the same equations. Before going to find a solution to (anti)self-dual equation, it is instructive to comment (anti)self-dual solutions in

Minkowskian field theories. Looking at "dualising" (*) as an operator, one may write the dual of dual tensor

$$\begin{aligned} {}^{**}\mathbf{F}_{\alpha\beta} &\equiv \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}\,{}^{*}\mathbf{F}_{\gamma\delta} \\ &= \frac{1}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}\epsilon_{\gamma\delta}{}^{\kappa\lambda}\,\mathbf{F}_{\kappa\lambda} \\ &= -\mathbf{F}_{\alpha\beta}. \end{aligned} \tag{2.66}$$

Therefore in Minkowskian space the eigenvalue of operator $({}^*)^2$ is -1, while in Euclidean space, ${}^{**}\mathbf{F}_{\alpha\beta} = \mathbf{F}_{\alpha\beta}$ that shows, eigenvalue of $({}^*)^2$ is $+1$. Hence the eigenvalues of dualising operator are $\pm i$ for Minkowskian field configurations, and ± 1 for Euclidean case, that is summarised as: Real (anti)self-dual solutions can not exist for Minkowskian field theories (of course in four dimensions).

Let us now look for some solutions of self-duality (or anti-self-duality) condition (2.65). The first attempt was done by Belavin *et al* [13], and many people have contributed in evolution of solutions to (anti)self-dual relation. Following Rajaraman [14], the 't Hoof ansatz for the gauge field is:

$$\mathbf{A}_\alpha = i\bar{\Sigma}_{\alpha\beta}\,\partial_\beta(\ln\phi(x)) \tag{2.67}$$

where $\phi(x)$ is a scalar function to be obtained, and $\bar{\Sigma}_{\alpha\beta}$ are the components of an anti-self-dual matrix built from Pauli matrices

$$\bar{\Sigma}_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} 0 & \sigma_3 & -\sigma_2 & -\sigma_1 \\ -\sigma_3 & 0 & \sigma_1 & -\sigma_2 \\ \sigma_2 & -\sigma_1 & 0 & -\sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 & 0 \end{bmatrix} \tag{2.68}$$

which can be written compactly in the form

$$\bar{\Sigma}_{\alpha\beta} = \bar{\eta}^{a\alpha\beta}\sigma^a/2 \quad a = 1,2,3 \tag{2.69}$$

where

$$\bar{\eta}^{a\alpha\beta} = -\bar{\eta}^{a\beta\alpha} = \begin{cases} \epsilon^{a\alpha\beta} & \text{for } \alpha,\beta = 1,2,3 \\ -\delta^{a\alpha} & \text{for } \beta = 4. \end{cases} \tag{2.70}$$

Replacing the ansatz (2.67) in eq(2.58) to find $\mathbf{F}_{\alpha\beta}$, and then dualising $\mathbf{F}_{\alpha\beta}$, duality equation (2.65) can be found as a compact single relation

$$\frac{\partial^2\phi}{\phi} = 0, \tag{2.71}$$

where $\partial^2 = \partial_\alpha \partial^\alpha \phi$. When ϕ is non-singular, eq(2.71) reduces to $\partial^2 \phi = 0$ which permits only the trivial solution $\phi =$ constant, leading to $\mathbf{A}_\alpha = 0$; But when singular $\phi(x)$ are considered we get a non-singular solution for the gauge field[9]. A general form of solution to eq(2.71) is

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|x_\alpha - a_{i\alpha}|^2}, \tag{2.72}$$

where $a_{i\alpha}$ and λ_i are any real constants[10]. These solutions when inserted in eq(2.67) will yield (after some gauge transformations) N-instanton solutions. For the instanton solution, $N = 1$, the gauge field is found

$$\mathbf{A}_\alpha(x) = -2i\Sigma_{\alpha\beta} \frac{(x_\beta - a_\beta)}{|x - a|^2 + \lambda^2}, \tag{2.73}$$

that is of course a suitable gauge transformation of original $\mathbf{A}_\alpha$ found in eq(2.67). In the equation (2.73) $\Sigma_{\alpha\beta}$ is an element of self-dual matrix

$$\Sigma_{\alpha\beta} = \eta^{a\alpha\beta}\sigma^a/2 \quad \text{with} \quad \eta^{a\alpha\beta} = -\eta^{a\beta\alpha} = \begin{cases} \epsilon^{a\alpha\beta} & \text{for} \quad \alpha, \beta = 1, 2, 3 \\ \delta^{a\alpha} & \text{for} \quad \beta = 4. \end{cases} \tag{2.74}$$

The solution (2.73) is non-singular at any point x for any given $\lambda \neq 0$. One may interpret a $(= (a_1, a_2, a_3, a_4))$ as 'location' of the instanton, and can be chosen arbitrarily because of the translational invariance of the Yang–Mills equation. Similarly constant λ represents the 'size' of the instanton. This freedom is related to the scale-invariance of the Yang–Mills system under the scale transformation $x_\alpha \to \lambda x_\alpha$ and $\mathbf{A}_\alpha \to \lambda \mathbf{A}_\alpha$ for any $\lambda \neq 0$. The one-instanton solution (2.73) is essentially spherically symmetric about the point a, in analogous to the BPS monopole solution in previous section.

When $\mathbf{A}_\alpha(x)$ from eq(2.73) substituted into eq(2.58) it leads to field tensor

$$\mathbf{F}_{\alpha\beta} = 4i\Sigma_{\alpha\beta} \frac{\lambda^2}{[|x - a|^2 + \lambda^2]^2}. \tag{2.75}$$

Since $\Sigma_{\alpha\beta}$ is self-dual, so is $\mathbf{F}_{\alpha\beta}$, and clearly localised and finite. Obtaining anti-self-dual solution is straightforward by interchanging $\bar{\Sigma}_{\alpha\beta}$ with $\Sigma_{\alpha\beta}$ in ansatz (2.67), and therefore in eqs(2.73) and (2.75). This solution may be called the anti-instanton.

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[9] One can examine, as an example, $\phi(x) = \frac{1}{|x|^2}$, and shows at $x \neq 0$ and $x = 0$ (singular point of ϕ), the function $\frac{\partial^2 \phi}{\phi}$ vanishes.

[10] A more general solution can be written in the form $\sum_{i=1}^{N+1} \frac{\lambda_i}{|x_\alpha - a_{i\alpha}|^2}$, which reduces to eq(2.72) where $a_{N+1} \to \infty$, $\lambda_{N+1} \to \infty$ with $(\lambda_{N+1}/a_{N+1})^2 = 1$.

Instanton Number

In topological aspect of monopoles in previous sections we addressed the quantisation of monopole charge to the context of winding number or "homotopy index". The same concept arises in instanton case. Let us define the quantity k:

$$k = -\frac{1}{16\,\pi^2}\int d^4x \,\mathrm{tr}\left[{}^*\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}\right], \tag{2.76}$$

where ${}^*\mathbf{F}_{\alpha\beta}$ is the dual tensor. The integral in eq(2.76) does indeed give the homotopy index or winding number of $S_3^{(\mathrm{phy})}$ into $S_3^{(\mathrm{int})}$. To show this, the first step is to write the integral as a surface integral over $S_3^{(\mathrm{phy})}$. From the defining equation (2.58) one may simply find the identity

$$\mathbf{D}_\alpha{}^*\mathbf{F}^{\alpha\beta} \equiv \partial_\alpha{}^*\mathbf{F}^{\alpha\beta} + \left[\mathbf{A}_\alpha, {}^*\mathbf{F}^{\alpha\beta}\right] = 0. \tag{2.77}$$

Now defining a density correspondence to k

$$\begin{aligned}
-16\,\pi^2\,k(x) &= \mathrm{tr}\left[{}^*\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}\right] \\
&= \mathrm{tr}\left[(\partial_\alpha\mathbf{A}_\beta - \partial_\beta\mathbf{A}_\alpha){}^*\mathbf{F}^{\alpha\beta} + (\mathbf{A}_\alpha\mathbf{A}_\beta - \mathbf{A}_\beta\mathbf{A}_\alpha){}^*\mathbf{F}^{\alpha\beta}\right] \\
&= \mathrm{tr}\left\{(\partial_\alpha\mathbf{A}_\beta - \partial_\beta\mathbf{A}_\alpha){}^*\mathbf{F}^{\alpha\beta} + \mathbf{A}_\alpha\left[\mathbf{A}_\beta, {}^*\mathbf{F}^{\alpha\beta}\right]\right\}, \\
&= \mathrm{tr}\left[(\partial_\alpha\mathbf{A}_\beta - \partial_\beta\mathbf{A}_\alpha){}^*\mathbf{F}^{\alpha\beta} - \mathbf{A}_\alpha\partial_\beta{}^*\mathbf{F}^{\alpha\beta}\right] \\
&= \mathrm{tr}\left[\partial_\alpha\mathbf{A}_\beta{}^*\mathbf{F}^{\alpha\beta} - \partial_\beta\left(\mathbf{A}_\alpha{}^*\mathbf{F}^{\alpha\beta}\right)\right],
\end{aligned} \tag{2.78}$$

where the cyclic property of trace as well as eq(2.77) have been used. Next mixing the definition of dual tensor ${}^*\mathbf{F}_{\alpha\beta}$ and eq(2.58), and replacing in eq(2.78)

$$\begin{aligned}
-16\,\pi^2\,k(x) &= \mathrm{tr}\left\{\epsilon^{\alpha\beta\gamma\delta}\left[(\partial_\alpha\mathbf{A}_\beta)(\partial_\gamma\mathbf{A}_\delta + \mathbf{A}_\gamma\mathbf{A}_\delta) - \partial_\beta(\mathbf{A}_\alpha\partial_\gamma\mathbf{A}_\delta + \mathbf{A}_\alpha\mathbf{A}_\gamma\mathbf{A}_\delta)\right]\right\} \\
&= \mathrm{tr}\left\{\epsilon^{\alpha\beta\gamma\delta}\left[2\,\partial_\alpha\left(\mathbf{A}_\beta\partial_\gamma\mathbf{A}_\delta + \frac{2}{3}\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{A}_\delta\right)\right]\right\},
\end{aligned} \tag{2.79}$$

where we have used

$$\mathrm{tr}\left[\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha\mathbf{A}_\beta)\mathbf{A}_\gamma\mathbf{A}_\delta\right] = \frac{1}{3}\,\mathrm{tr}\left[\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha(\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{A}_\delta)\right],$$

which again can be obtained by using the cyclicity of trace and the antisymmetry of $\epsilon^{\alpha\beta\gamma\delta}$. Thus

$$k(x) = \partial_\alpha\,j^\alpha \tag{2.80}$$

where

$$j^\alpha \equiv -\frac{1}{8\,\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\left[\mathbf{A}_\beta\partial_\gamma\mathbf{A}_\delta + \frac{2}{3}\,\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{A}_\delta\right]. \tag{2.81}$$

Hence

$$k = \int k(x)d^4x = \oint_{S_3^{(\text{phy})}} j_\alpha\, dS^\alpha. \tag{2.82}$$

Further, on the surface at infinity $S_3^{(\text{phy})}$, the finite-action configurations have $\mathbf{F}_{\alpha\beta} = 0$ and hence $\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma\mathbf{A}_\delta = -\epsilon^{\alpha\beta\gamma\delta}\mathbf{A}_\gamma\mathbf{A}_\delta$. thus

$$k = \frac{1}{24\,\pi^2}\oint_{S_3^{(\text{phy})}} \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}[\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{A}_\delta]\, dS_\alpha. \tag{2.83}$$

Finally, inserting the asymptotic behaviour (2.61) of the fields $\mathbf{A}_\alpha$ we have

$$k = -\frac{1}{24\,\pi^2}\oint dS_\alpha\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\left[(\partial_\beta\mathbf{U})\mathbf{U}^{-1}(\partial_\gamma\mathbf{U})\mathbf{U}^{-1}(\partial_\delta\mathbf{U})\mathbf{U}^{-1}\right]. \tag{2.84}$$

Thus, we have written the volume integral in (2.76) as a surface integral, with the integrand directly in terms of the group-element-valued function $\mathbf{U}$ on $S_3^{(\text{phy})}$, corresponding to any given finite-action configuration.

By definition of $SU(2)$, the matrices $\mathbf{U}$ are the set of all 2×2 unitary unimodular matrices. Such matrices can be written uniquely in the form:

$$\mathbf{U} = \sum_{\alpha=1}^{4} a_\alpha\,\mathrm{s}_\alpha \tag{2.85}$$

where $\mathrm{s}_4 = \mathbf{I}$, the unit 2×2 matrix, and

$$\mathrm{s}_j = i\,\sigma_j, \qquad j = 1, 2, 3, \tag{2.86}$$

and a_α are any four real numbers satisfying

$$\sum_\alpha a_\alpha\, a_\alpha = 1. \tag{2.87}$$

The group is thus parametrised by these four real variables a_α, subject to the constraint (2.87). The group space is therefore the three dimensional surface of a unit sphere in four dimensions, which we call it $S_3^{(\text{int})}$. The function $\mathbf{U}$ is therefore a mapping of $S_3^{(\text{phy})}$ into $S_3^{(\text{int})}$.

Notice again the similarity to the discussion of the monopole system in previous sections. There, the boundary conditions involved mapping of S_2 into S_2. Here, we have the corresponding situation for three dimensional spherical surfaces. Still a little bit work is needed to show when integration is done over $S_3^{(\mathrm{phy})}$ once, the group space $S_3^{(\mathrm{int})}$ may be spanned an integral number of times (the proof can be found in literature such as [14]). One may explain the subject according to the homotopy groups:

$$\pi_n(S_n) = \mathbb{Z}, \tag{2.88}$$

where $\pi_n(S_n)$ refers to the homotopy group for the mapping of S_n into S_n and $\mathbb{Z}$ refers to the group of integers. That is, mappings of $S_n \to S_n$ come in a discrete infinity of homotopy classes, each characterised by an integer. This is true for all positive integers. For the monopole case $n = 2$ and for instanton case $n = 3$. Finally, the constant $-1/(16\,\pi^2)$ in eq(2.76) has been so arranged that k will be in fact equal to this integer. This can be checked by the prototype example

$$\mathbf{U}_1(x) = \frac{(x_4 + i\,x_j\,\sigma_j)}{|x|} = \sum_\alpha \hat{x}_\alpha\, s_\alpha. \tag{2.89}$$

On comparing with the general representation (2.85), we see that this gauge function corresponds to $a_\alpha = \hat{x}_\alpha$. That is, every point on $S_3^{(\mathrm{phy})}$ is mapped on the corresponding point (at the same polar angles) on $S_3^{(\mathrm{int})}$. Thus the homotopy index k must be equal to unity. On inserting eq(2.89) into eq(2.84), a little algebra yields

$$\begin{aligned} k &= -\frac{1}{24\,\pi^2}\int_{S_3^{(\mathrm{phy})}} \left(\frac{-12\,x^\alpha}{|x|^4}\right) dS^\alpha, \\ &= \frac{1}{2\,\pi^2}\int d\Omega\, x^\alpha\, |x|^2\, \frac{x^\alpha}{|x|^4} \\ &= \frac{1}{2\,\pi^2}\int d\Omega = 1, \end{aligned} \tag{2.90}$$

where we have used: $dS^\alpha = |x|^2\, x^\alpha\, d\Omega$, which $d\Omega$ is the solid angle element in Euclidean four-space, and the total solid angle $\int d\Omega$ in four-space is $2\,\pi^2$.

The integer number k that is often called the "Pontryagin index" by mathematicians, is normally known as "instanton number" by physicists. The instanton number for self-dual solution we found in eq(2.73) is

$$k = -\frac{1}{16\,\pi^2}\int d^4x\, \mathrm{tr}\left[{}^*\mathbf{F}_{\alpha\beta}\,\mathbf{F}^{\alpha\beta}\right],$$

$$\begin{aligned}
&= -\frac{1}{16\,\pi^2}\int d^4x\,(-16)\,\frac{\lambda^4}{[|x-a|^2+\lambda^2]^4}\mathrm{tr}(\Sigma_{\alpha\beta})^2 \\
&= \frac{\lambda^4}{\pi^2}\int\frac{y^3\,dy\,d\Omega}{[y^2+\lambda^2]^4}\,\frac{12\cdot 2}{4} \\
&= 1.
\end{aligned} \tag{2.91}$$

This answer is obviously equivalent to the result of eq(2.90), since the gauge field (2.73) behaves as a pure gauge as $x \to \infty$,

$$\mathbf{A}_\alpha \to \mathbf{U}_1(x)\partial_\alpha\left[\mathbf{U}_1(x)\right]^{-1}, \tag{2.92}$$

where $\mathbf{U}_1$ is the prototype example of eq(2.89). For anti-self-dual solution which is known as "anti-instanton" the instanton number is -1.

2.5 Monopoles in Arbitrary Gauge Groups

In previous sections we found a clear picture of a magnetic monopole associated with some $SO(3)$ (and equivalently $SU(2)$) gauge theories, with a definite internal structure and calculable mass. The 't Hooft–Polyakov and the BPS monopoles were charge one spherically symmetric solutions. Still the problem is established for more general solutions other than spherical ones; solutions with more than one monopole,*ie.* separated monopoles, monopole solutions with charge greater than one Dirac's unit, $\frac{4\pi}{e}$, or monopoles in the other gauge groups. Much attempts have been done to find and discuss magnetic monopoles in different kinds of gauge groups. Actually much has been learned about exact (superimposed and separated) multi-monopoles in arbitrary gauge theories. The first attack to the problem was to construct some exact static finite energy solutions of the equations of motion, in terms of elementary functions. The BPS monopole that is a charge one spherically symmetric $SU(2)$ monopole, has been generalised to obtain spherically symmetric solutions for larger gauge groups, $SU(N)$ [15, 16]. Finding solutions with charge two or more are not as simple as charge one sectors, but has been done in principle using the twister approach [17, 18].

Atiyah, Drinfeld, Hitchin and Manin [19] constructed a method (ADHM) that gives "instanton" solution of any self-dual Euclidean Yang–Mills fields. The ADHM construction was originally obtained using algebraic geometry, but can be derived and explained in terms of matrix algebra [20, 21]. Four years later Nahm [22] adapted the ADHM formalism to self-dual (*ie.* the BPS monopoles, see eq(2.53)) multi-monopoles for arbitrary charge and arbitrary gauge group, which has been referred as "ADHMN" construction. In the ADHMN construction the regularity of the solution is automatic, and is generalised to gauge groups beyond $SU(2)$. Using the ADHMN approach, some solutions have been constructed (or re-constructed). Some examples are: Charge two $SU(2)$ monopole solutions with axial symmetry [23], $SU(N)$ axially symmetric solutions [24], SU(N) spherically symmetric solutions [25] and general solutions of $SU(2)$ for 2-monopoles [26].

In large gauge groups like the $E_8 \times E_8$ of "heterotic superstring theory" finding analytic solutions is impractical. One can think about monopoles in smaller subgroups of these large groups (eg. $SU(2)$ subgroups of $E_8 \times E_8$). We will discuss magnetic monopoles in string theory in next chapter.

In this section we describe $SU(N)$ gauge field monopoles in some details. We use a natural generalisation of the $SU(2)$ BPS monopoles. Therefore we seek some time-independent spherically symmetric solution that saturate the Bogomol'nyi bound, and satisfy the BPS equations, (2.42)–(2.44).

2.5.1 Monopoles in $SU(N)$ Gauge fields

For some convenience we use $SU(N+1)$, for $N \geq 1$. The model we consider here is an $SU(N+1)$ gauge field coupled to a massless scalar field in the adjoint representation. The Lagrangian for this theory is

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2}(\mathbf{D}^\mu\mathbf{\Phi})^a(\mathbf{D}_\mu\mathbf{\Phi})^a, \tag{2.93}$$

where the field strength $F^a_{\mu\nu}$ and the covariant derivative $\mathbf{D}_\mu$ have similar definitions as in section (2.1):

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc}A^b_\mu A^c_\nu \tag{2.94}$$

$$(D_\mu\mathbf{\Phi})^a \equiv \partial_\mu\Phi^a + ef^{abc}A^b_\mu\Phi^c, \tag{2.95}$$

where f^{abc} are structure constants of the gauge group $SU(N+1)$ with respect to a basis $\mathbf{T}^a$ and $a, b, c = 1, 2, \ldots, N, N+1$. Wilkinson and Goldhaber [27] have given a generalised ansatz for the gauge and Higgs fields:

$$\mathbf{A}_i = \epsilon_{ijb}\, x^j \, \frac{(\mathbf{T}^b - \mathbf{M}^b(r))}{er^2}, \qquad (b = 1, 2, 3), \tag{2.96}$$

$$\mathbf{\Phi} = \frac{\tilde{\mathbf{\Phi}}(r)}{e}, \tag{2.97}$$

where $\mathbf{A}_i$, $\mathbf{M}^b$ and $\mathbf{\Phi}$ are (three, three and one) $(N+1) \times (N+1)$ matrices. Here $\mathbf{M}^b(r)$ and $\tilde{\mathbf{\Phi}}(r)$ are unknown matrix functions, and $\mathbf{T}^b$ are three $(N+1) \times (N+1)$ matrices which generate the maximal embedding of $SU(2)$ in $SU(N+1)$. Nothing depends on time, and $\mathbf{A}_0 = 0$. The solution is prepared to be spherical symmetric and therefore one can look at eqs(2.96) and (2.97) along any axis, say z axis [16]. The Bogomol'nyi eqs(2.44) then become

$$B_3^a = (\mathbf{D}_3\mathbf{\Phi})^a, \tag{2.98}$$

$$B_\pm^a = (\mathbf{D}_\pm\mathbf{\Phi})^a. \tag{2.99}$$

Using the ansatz (2.96) and (2.97) the above equations become

$$r^2 \frac{d\tilde{\mathbf{\Phi}}}{dr} = [\mathbf{M}_+, \mathbf{M}_-] - \mathbf{T}^3, \tag{2.100}$$

$$\frac{d\mathbf{M}_\pm}{dr} = \mp[\mathbf{M}_\pm, \tilde{\mathbf{\Phi}}], \tag{2.101}$$

where $\mathbf{T}^a$ ($a = 1, 2$ and 3) the generators of the maximal embedding of $SU(2)$ in $SU(N+1)$ are chosen such that

$$\mathbf{T}^3 = \mathrm{diag}\left[\frac{1}{2}N, \frac{1}{2}N - 1, \ldots, -\frac{1}{2}N + 1, -\frac{1}{2}N\right]. \tag{2.102}$$

It has been shown [27] that the matrix functions, $\mathbf{M}_+(r)$ and $\tilde{\mathbf{\Phi}}(r)$, can be taken as[11]

$$\tilde{\mathbf{\Phi}} = \frac{1}{2}\begin{pmatrix} \phi_1 & & & & \\ & \phi_2 - \phi_1 & & & \\ & & \ddots & & \\ & & & \phi_N - \phi_{N-1} & \\ & & & & -\phi_N \end{pmatrix}, \tag{2.103}$$

[11] There are some motivation for this selection. In eq(2.100), $\mathbf{T}^3$ is diagonal, and taking an idea from ladder operators in quantum mechanics ($[J_+, J_-] = \hbar J_z$ with J_z diagonal), but remind we are in a completely classical framework, M_+ and consequently M_- can chosen in the same way to have a diagonal result from the bracket in right-hand side of eq(2.100). Immediately one can choose $\mathbf{\Phi}$ as a diagonal matrix.

$$M_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & a_1 & & & \\ & 0 & a_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & a_N \\ & & & & 0 \end{pmatrix}, \tag{2.104}$$

where ϕ_m and a_m are real functions of the radius r, and $\mathbf{M}_- = (\mathbf{M}_+)^T$. Substituting eqs(2.103) and (2.104) into the first order eqs(2.100) and (2.101) the field equations become [27, 16]

$$r^2 \frac{d\phi_m}{dr} = (a_m)^2 - m\bar{m}, \tag{2.105}$$

$$\frac{da_m}{dr} = \left(-\frac{1}{2}\phi_{m-1} + \phi_m - \frac{1}{2}\phi_{m+1}\right) a_m, \tag{2.106}$$

where $1 \leq m \leq N$, $\bar{m} = N + 1 - m$ and $\phi_0 = \phi_{N+1} = 0$. The equation (2.106) can be solved [28, 16] by introducing N new functions $Q_1, Q_2 \ldots, Q_N$ with relations

$$a_m = \frac{r}{Q_m}(m\bar{m}Q_{m-1}Q_{m+1})^{1/2}, \tag{2.107}$$

$$\phi_m = -\frac{d \ln Q_m}{dr} + \frac{m\bar{m}}{r}, \tag{2.108}$$

where $Q_0 = Q_{N+1} \equiv 1$, and Q_m never vanishes except at the origin. The remaining equation (2.105) now becomes a homogeneous differential equation in the Q_m:

$$Q'_m Q'_m - Q_m Q''_m = m\bar{m} Q_{m+1} Q_{m-1}, \tag{2.109}$$

for $m = 1, 2, \ldots, N$. For a useful interpretation of the the solution, one can observe the radial magnetic field $\mathbf{B}_3$ is to be written in the form[12]

$$\mathbf{B}_3 = \frac{1}{2e}\mathrm{diag}(B_1, B_2 - B_1, \ldots, B_N - B_{N-1}, -B_N), \tag{2.110}$$

then B_m are given in terms of the Q_m by the relation

$$B_m = -\frac{d^2 \ln Q_m}{dr^2} - \frac{m\bar{m}}{r^2}. \tag{2.111}$$

Let us look for solutions of eq(2.109). Before going forward, it is convenient to clear the situation of the previously solved problem (*ie.* $SU(2)$) when $N = 1$. In

[12] In fact we are looking along z-direction, therefore we have to know about $\mathbf{B}_3$ as the radial magnetic field. From eq(2.96) one can see $\mathbf{A}_3$ is diagonal and therefore $[\mathbf{A}_3, \mathbf{\Phi}] = 0$. Now $\mathbf{B}_3 = \mathbf{D}_3\mathbf{\Phi} = \partial_3\mathbf{\Phi} = (d/dr)\mathbf{\Phi}$, which shows magnetic field is diagonal and eq(2.111) is concluded.

this case the equation (2.109) becomes $Q'^2 - QQ'' = 1$ which clearly has a solution $Q(r) = \frac{1}{\alpha}\sinh(\alpha r)$, where α is an arbitrary. If this value is substituted in eq(2.108) and then eq(2.103), the solution of eq(2.16) with eq(2.49) can be concluded in the z-direction with $\alpha = ae$. In the rest of this section we suggest $N > 1$. In eq(2.109) with $Q_0 = Q_{N+1} = 1$, for $m = 1, 2, \ldots, N$ there are N mixed second order differential equations with $2N$ parameters. From the exponential behaviour of "sinh" in the $N = 1$ case, the ansatz for general case can be of the form [16]

$$Q_1 = N! \sum_{i=1}^{N+1} A_i e^{\alpha_i r}, \tag{2.112}$$

where the $2N + 2$ parameters α_i and A_i are arbitrary. Actually we included Q_{N+1} here in the ansatz, and from its unit value two extra constraints will appear which reduces the arbitrary constants to $2N$ for the moment. Once $Q_0 \equiv 1$ and Q_1 are given, the remaining Q_m (including Q_{N+1}) may be determined uniquely by repeated use of eq(2.109). One finds the explicit form of Q_m as:

$$Q_m = (-1)^{m(m-1)/2} \beta_m \sum_{D_m} \left[\prod_{i \in D_m} (A_i e^{\alpha_i r}) \right] \left[\prod_{\substack{i,j \in D_m \\ j>i}} (\alpha_i - \alpha_j)^2 \right], \tag{2.113}$$

where the constants β_m are given by

$$\beta_m = \left(\prod_{n=1}^{N} n! \right) \Big/ \left(\prod_{k=1}^{m-1} k! \prod_{l=1}^{\bar{m}-1} l! \right), \tag{2.114}$$

and the sum in eq(2.113) is over the $\binom{N+1}{m}$ distinct ways that the integers $1, 2, \ldots, N+1$ may be defined into two groups D_m and $D_{\bar{m}}$ with m elements in D_m and $\bar{m}$ elements in $D_{\bar{m}}$. The constraints are

$$\sum_{i=1}^{N+1} \alpha_i = 0, \tag{2.115}$$

$$(-1)^{(1/2)N(N+1)} \left(\prod_{i=1}^{N+1} A_i \right) \left[\prod_{j>i} (\alpha_i - \alpha_j)^2 \right] = 1. \tag{2.116}$$

We are only interested in those solutions which are regular at the origin, *ie.* those for which $Q_m \sim r^{m\bar{m}}$ as $r \to 0$. If we impose this condition for $m = 1$, it then follows for all m by virtue of the differential equation (2.109). In order that $Q_1 \sim r^N$ at the origin, from eqs(2.112) we have

$$\sum_{i=1}^{N+1} A_i \alpha_i^n = 0, \quad n = 0, 1, 2, \ldots, N-1 \tag{2.117}$$

$$\sum_{i=1}^{N+1} A_i \alpha_i^N = 1. \tag{2.118}$$

Regarding the α_i as given, these linear equations have unique solution

$$A_i = \prod_{j \neq i} (\alpha_i - \alpha_j)^{-1}. \tag{2.119}$$

Note that this choice automatically satisfies the constraint (2.116), and in addition to eqs(2.117) and (2.118), $\sum_{i=1}^{N+1} A_i \alpha_i^{N+1} = \sum_{i=1}^{N+1} \alpha_i = 0$. So we have an N-parameter solution depending on $\alpha_1, \ldots, \alpha_{N+1}$ with $\sum \alpha_i = 0$. Inserting eq(2.119) into eq(2.113) the solution becomes

$$Q_m = \beta_m \sum_{D_m} \left(\prod_{i \in D_m} e^{\alpha_i r} \right) \left[\prod_{\substack{i \in D_m \\ j \in D_{\bar{m}}}} (\alpha_i - \alpha_j)^{-1} \right]. \tag{2.120}$$

Since $\beta_m = \beta_{\bar{m}}$ and $\sum \alpha_i = 0$, these solutions have the property

$$Q_m(-r) = (-1)^{m\bar{m}} Q_{\bar{m}}(r). \tag{2.121}$$

The physically interesting solutions are those for real α_i. To see the behaviour at infinity let us first consider distinct α_i with $\alpha_1 > \alpha_2 > \cdots > \alpha_{N+1}$. Then the asymptotic behaviour of the Q_m is given by

$$\ln Q_m \sim \sum_{i=1}^{m} \alpha_i r + O(1). \tag{2.122}$$

Using eqs(2.108) and (2.111) one can find the asymptotic form of the Higgs and magnetic fields

$$\mathbf{\Phi} \sim -\frac{1}{2e} \mathrm{diag}(\alpha_1, \ldots, \alpha_{N+1}), \tag{2.123}$$

$$\mathbf{B}_3 \sim -\frac{1}{er^2} \mathbf{T}^3, \tag{2.124}$$

where $\mathbf{T}^3$ is defined in eq(2.102).

2.5.2 Monopoles in $SU(3)$ Gauge fields

In this short section we discuss the BPS monopoles in $SU(3)$ gauge fields. Using the system of equations (2.109) there are just two coupled equations:

$$Q_1'Q_1' - Q_1Q_1'' = 2Q_2, \tag{2.125}$$

$$Q_2'Q_2' - Q_2Q_2'' = 2Q_1. \tag{2.126}$$

The solution is

$$\begin{aligned} Q_1 &= 2(A_1e^{\alpha_1 r} + A_2e^{\alpha_2 r} + A_3e^{\alpha_3 r}), \\ Q_2 &= 2(A_1e^{-\alpha_1 r} + A_2e^{-\alpha_2 r} + A_3e^{-\alpha_3 r}), \end{aligned} \tag{2.127}$$

with

$$A_i = \prod_{j\neq i}(\alpha_i - \alpha_j)^{-1}, \tag{2.128}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{2.129}$$

We see $Q_1(-r) = Q_2(r)$ and $Q_i \sim r^2$ at the origin. Here there are only two free parameters, say α_1 and α_2. To see the behaviour at infinity we choose values of α_i such that $\alpha_1 > \alpha_2 > \alpha_3$, then

$$\ln Q_1 = \alpha_1 r + O(1), \tag{2.130}$$

$$\ln Q_2 = -\alpha_3 r + O(1), \tag{2.131}$$

and the asymptotic behaviours of the Higgs and magnetic fields are

$$\mathbf{\Phi} \sim -\frac{1}{2e}\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3), \tag{2.132}$$

$$\mathbf{B}_3 \sim -\frac{1}{er^2}\mathbf{T}^3, \tag{2.133}$$

where $\mathbf{T}^3 = \mathrm{diag}(1, 0, -1)$. For real α_i, the Q_i can never vanish except at the origin, and we have a meaningful solution with distinct values of α_i (or equivalently distinct eigenvalues of the Higgs field $\mathbf{\Phi}$ at infinity). A familiar example is the embedding of Prasad-Sommerfield solution in $SU(3)$ with the values

$$\alpha_1 = 2, \quad \alpha_2 = 0, \quad \alpha_3 = -2$$

which gives $Q_1 = Q_2 = \sinh^2 r$.

It remains to think about repeated eigenvalues of the Higgs field, α_i. If three α_i are equal, they must be zero and that is not interesting (no monopoles if the Higgs field vanishes in vacuum). If two of three α_i are equal then the coefficients A_i diverge from eq(2.128). But a tricky way [16] shows in fact the solutions have a finite limit. For example for $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, -2)$, we can use $(\alpha_1, \alpha_2, \alpha_3) = (1+\delta, 1-\delta, -2)$, for small auxiliary parameter δ that finally goes to zero. Inserting these in eqs(2.127), and after the limit $\delta \to 0$

$$Q_1 = \frac{2}{9}\left[(3r-1)e^r + e^{-2r}\right], \tag{2.134}$$

and $Q_2(r) = Q_1(-r)$. This is a rescaled version of the $SU(3)$ solution of reference [28]. Here the asymptotic behaviour of Q_1 and Q_2 are different

$$\ln Q_1 = r + \ln(r) + O(1), \qquad \ln Q_2 = 2r + O(1), \tag{2.135}$$

so that asymptotic behaviour of the Higgs and magnetic fields are

$$\mathbf{\Phi} \sim \frac{1}{e}\mathrm{diag}(-\frac{1}{2}, -\frac{1}{2}, 1), \quad \mathbf{B}_3 \sim \frac{1}{er^2}\mathrm{diag}(-\frac{1}{2}, -\frac{1}{2}, 1). \tag{2.136}$$

A similar solution arises with $\alpha_2 = \alpha_3$.

This chapter was an introduction to instantons and monopoles (BPS monopole in particular) on some classical gauge groups. In next chapter we will discuss the instantons and monopoles in "heterotic string theory". The gauge group of this theory is the semi-simple exceptional group $E_8 \times E_8$ (or $SO(32)$). These groups are too large and complicated to obtain a general solution to the whole gauge group. Therefore people make solutions for this theory by picking up subgroups of the main gauge group. Once a subgroup of the main gauge group is chosen, one can use the results of the solved materials in this subgroup. By the way this is called embedding of a subgroup inside the main gauge group.

Chapter 3

Monopoles In String Theory

شد آن وحدت ازین کثرت پدیدار

یکی را چون شُمردی گشت بسیار

عدد گر چه یکی دارد بدایت

ولیکن نبودش هرگز نهایت

« شبستری »

Oneness becomes manifest in multiplicity,

just as the number one, when counted up, becomes many;

Although every number has its beginning in the number one,

there is never an end to the succession of numbers.

"Shabestari"

String theory is known as a candidate for unifying the fundamental interactions, and a promising approach to gather general relativity with quantum field theories in a unique formalism. This theory with these ideal goals is a non-abelian theory and might have some soliton solutions. Some sort of instanton and monopole solutions have been constructed to the coupled $N = 4$ super Yang–Mills supergravity equations of motion which arise in the low energy approximation to the heterotic superstring compactified down to four dimensions. In this chapter we explain these solutions and will find some generalisations of them.

3.1 Solitons in String Theory

A "five-brane" is an extended soliton solution to ten-dimensional string theory with $(5 + 1)$-dimensional translational symmetry. Explicit five-brane solutions have been constructed from a generalisation of the Yang–Mills instantons in which the four-dimensional instanton sits in the directions transverse to the five-brane. When such objects are compactified to four dimensions, they can be classified by the embedding of the core instanton in spacetime and internal space [29]. First we explain the supersymmetric solution to the low-energy string theory, from which five-brane is constructed.

3.1.1 Supersymmetric Solution to Low-Energy String Theory

Our starting point is "low-energy effective action" of heterotic string theory compactified on a six-dimensional torus.[1] The soliton solutions (five-brane and magnetic

[1]For importance and derivation of low-energy effective action a good reference is chapters 13–16 of reference [37]. A newer review can be found in [38].

monopoles) arise from the low-energy effective action for the massless fields of the heterotic string. At lowest order in α', the effective action is given by $N = 1$ super Yang–Mills coupled to supergravity theory in ten dimensions[2]. In "sigma model" variables, the bosonic part is

$$S_{10} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{g}\,e^{-2\phi}[R + 4(\partial\phi)^2 - \frac{1}{3}H^2 - \frac{1}{30}\alpha'\,\mathrm{Tr}\mathbf{F}^2], \tag{3.1}$$

where κ is ten-dimensional gravitational coupling constant. The metric g_{MN} with $M, N = 0, 1, \ldots, 9$ is related to standard Einstein metric, the metric of space-time which is considered as a ten-dimensional manifold $\mathcal{M}$, by $\hat{g}_{MN} = e^{-\phi/2}g_{MN}$, and $g = |\det(g_{MN})|$. The other quantities are: Scalar dilaton, ϕ; scalar curvature, R; antisymmetric tensor field, H_{MNP}; and the Yang–Mills field, F^a_{MN}. The Yang–Mills gauge fields are in the adjoint representation of $E_8 \times E_8$, or $SO(32)$, with the trace conventionally normalised such that the Cartan invariant inner product $<\mathbf{T}^a, \mathbf{T}^b> = \frac{1}{30}\mathrm{Tr}(\mathbf{T}^a\mathbf{T}^b) = \delta^{ab}$. The Bianchi identity that supplements eq(3.1) is written as

$$\mathrm{d}H = \alpha'(\mathrm{tr}\mathbf{R}\wedge\mathbf{R} - \frac{1}{30}\mathrm{Tr}\mathbf{F}\wedge\mathbf{F}). \tag{3.2}$$

The above equation has been written in differential form language. The operator d which is called "exterior derivative", is a generally covariant operator that is independent of the choice of metric. The three-form field H is

$$H = \mathrm{d}B + \alpha'(\omega_3^L - \frac{1}{30}\omega_3^{YM}), \tag{3.3}$$

where B is two-form antisymmetric tensor field, and ω_3^{YM} and ω_3^L are the Yang–Mills and the Lorentz Chern-Simons three-forms respectively

$$\omega_3^{YM} = \mathrm{Tr}(\mathbf{A}\wedge\mathrm{d}\mathbf{A}) + \frac{2}{3}e\,\mathrm{Tr}(\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}), \tag{3.4}$$

$$\omega_3^L = \mathrm{tr}(\omega\wedge\mathrm{d}\omega) + \frac{2}{3}\mathrm{tr}(\omega\wedge\omega\wedge\omega), \tag{3.5}$$

where $\mathbf{A}$ and ω are matrix-valued gauge field and spin connection one-forms, respectively, and $\wedge$ is 'wedge' product. The Bianchi equation (3.2) is concluded from eq(3.3)

[2]This theory can be dimensionally reduced to give the $N = 4$ theory in $3 + 1$ dimensions [43]. Equivalently, this corresponds to the massless sector after compactifying six internal dimensions of the string on a torus.

and relations $d\omega_3^{YM} = \mathrm{Tr}(\mathbf{F}\wedge\mathbf{F})$ and $d\omega_3^L = \mathrm{tr}(\mathbf{R}\wedge\mathbf{R})$ where are obtained from eqs(3.4) and (3.5). Some care must be taken of getting trace from gauge field tensor and curvature tensor. By definition $\mathrm{Tr}(\mathbf{F}\wedge\mathbf{F}) = F^a\wedge F^b\,\mathrm{Tr}(\mathbf{T}^a\mathbf{T}^b)$ where a and b are indices in adjoint representation of $E_8\times E_8$ or $SO(32)$, and $\mathrm{tr}(\mathbf{R}\wedge\mathbf{R}) = R^{M'N'}\wedge R_{N'M'}$ where M' and N' are "tangent space" indices and running from 0 to 9. By definition $F^a = \frac{1}{2}F^a_{MN}dx^M\wedge dx^N$ and $R^{M'N'} = \frac{1}{2}R_{MN}{}^{M'N'}dx^M\wedge dx^N$, where

$$
\begin{aligned}
F^a_{MN} &= \partial_M A^a_N - \partial_N A^a_M + ef^{abc}A^b_M A^c_N, \qquad (3.6)\\
R_{MN}{}^{M'N'} &= \partial_M\omega_N^{M'N'} - \partial_N\omega_M^{M'N'} + [\omega_M,\omega_N]^{M'N'}\\
&= R_{MNPQ}e^{PM'}e^{QN'}. \qquad (3.7)
\end{aligned}
$$

Using matrix-valued forms, eqs(3.6) and (3.7) can be abbreviated as

$$\mathbf{F} = d\mathbf{A} + e\mathbf{A}\wedge\mathbf{A}, \qquad \mathbf{R} = d\omega + \omega\wedge\omega. \qquad (3.8)$$

In eq(3.7) R_{MNPQ} is the Riemann curvature tensor and $\omega_M^{M'N'}$ are spin connections

$$
\begin{aligned}
\omega_M^{M'N'} &= \frac{1}{2}e^{NM'}(\partial_M e_N^{N'} - \partial_N e_M^{N'}) - \frac{1}{2}e^{NN'}(\partial_M e_N^{M'} - \partial_N e_M^{M'})\\
&\quad -\frac{1}{2}e^{NM'}e^{PN'}(\partial_N e_{PP'} - \partial_P e_{NP'})e_M^{P'}. \qquad (3.9)
\end{aligned}
$$

At each point x the set of vielbein $e^M_{M'}(x)$, $M' = 0,\ldots,9$, is an orthonormal basis for the tangent space at that point. Orthonormality means that

$$e^{MM'}e^N_{M'} = g^{MN}, \quad \text{or equivalently} \quad e^M_{M'}e_{MN'} = \eta_{M'N'}, \qquad (3.10)$$

with $\eta_{M'N'}$ being the flat space metric. The indices $M,N,\ldots$ are raised and lowered with g_{MN}, the metric tensor of manifold $\mathcal{M}$, while the indices $M',N',\ldots$ are raised and lowered with $\eta_{M'N'}$, the flat space metric tensor.

The supersymmetry transformations for the fermion fields, to the lowest order, are given by

$$
\begin{aligned}
\delta\chi &= \mathbf{F}_{MN}\gamma^{MN}\varepsilon, \qquad (3.11)\\
\delta\lambda &= (\gamma^M\partial_M\phi - \frac{1}{6}H_{MNP}\gamma^{MNP})\varepsilon, \qquad (3.12)\\
\delta\psi_M &= (\partial_M + \frac{1}{4}\Omega_{-M}^{M'N'}\gamma_{M'N'})\varepsilon, \qquad (3.13)
\end{aligned}
$$

where χ, λ and ψ_M are the gaugino, dilatino and gravitino respectively, ε is a Majorana-Weyl spinor, and $\Omega_{\pm M}^{M'N'} \equiv \omega_M^{M'N'} \pm H_M^{M'N'}$ are generalised spin connections. In the above equations $\gamma^{M_1 M_2 \cdots M_n} = \frac{1}{n!}\gamma^{[M_1}\gamma^{M_2}\cdots\gamma^{M_n]}$, when $[M_1, M_2, \ldots, M_n]$ represents complete permutations of M_1, M_2, ..., and M_n with '+' sign for even permutations and '−' sign for odd permutations, $H_M^{N'P'} = H_{MNP}e^{NN'}e^{PP'}$, and $\gamma^{M'N'} = e_M^{M'}e_N^{N'}\gamma^{MN}$.

A supersymmetric solution is one for which there is at least one positive chirality spinor ε satisfying

$$\delta\chi = \delta\lambda = \delta\psi_M = 0. \tag{3.14}$$

3.1.2 Five-Brane Solution

The five-brane ansatz preserves a chiral half of the supersymmetries and is given by [29, 31]

$$\begin{aligned} F^a_{\alpha\beta} &= \pm\frac{1}{2}(\sqrt{g})^{-1}\epsilon_{\alpha\beta}{}^{\gamma\delta}F^a_{\gamma\delta}, & (3.15)\\ H_{\alpha\beta\gamma} &= \mp(\sqrt{g})^{-1}\epsilon_{\alpha\beta\gamma}{}^{\delta}\,\partial_\delta\phi, & (3.16)\\ g_{\alpha\beta} &= e^{2\phi}\delta_{\alpha\beta}, \qquad g_{\alpha'\beta'} = \eta_{\alpha'\beta'}, & (3.17)\end{aligned}$$

where $\alpha, \beta, \ldots = 1, 2, 3, 4$ denote transverse space and $\alpha', \beta', \ldots = 0, 5, 6, 7, 8, 9$ show orthogonal indices and $\epsilon^{1234} = +1$. The other components of the gauge field, $\mathbf{F}_{MN}$, and the antisymmetric field, H_{MNP}, are set to be zero. The (anti)self-dual equation (3.15) (remember eq(2.53)) is the core of the soliton solutions — five-branes (instantons) and monopoles.

Let us return to the five-brane ansatz and check how this ansatz satisfy the equations (3.14). In addition to the ansatz (3.15)–(3.17) we define chiral spinors $\varepsilon_\pm$ obeying

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta}\,\gamma^{\alpha\beta\gamma\delta}\varepsilon_\pm &= \pm\,4!\,\sqrt{g}\,\varepsilon_\pm & (3.18)\\ \epsilon_{\alpha'\beta'\gamma'\delta'\kappa'\lambda'}\,\gamma^{\alpha'\beta'\gamma'\delta'\kappa'\lambda'}\varepsilon_\pm &= \pm\,6!\,\varepsilon_\pm, & (3.19)\end{aligned}$$

or equivalently $\gamma^{1234}\varepsilon_\pm = \pm(\sqrt{g})^{-1}\varepsilon_\pm$ and $\gamma^{056789}\varepsilon_\pm = \pm\varepsilon_\pm$. Here and hereafter g is measured in the four-dimensional space denoted by $1, 2, 3$ and 4. Using eq(3.18), the definition of $\gamma^{\alpha\beta\gamma\delta}$ and the anti-commutator relation $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$ one can derive

$$\epsilon_{\alpha\beta\gamma}{}^{\delta}\,\gamma^{\alpha\beta\gamma}\varepsilon_{\pm} = \mp 3!\,\sqrt{g}\,\gamma^{\delta}\varepsilon_{\pm} \tag{3.20}$$

$$\epsilon_{\alpha\beta}{}^{\gamma\delta}\,\gamma^{\alpha\beta}\varepsilon_{\pm} = \mp 2!\,\sqrt{g}\,\gamma^{\gamma\delta}\varepsilon_{\pm}. \tag{3.21}$$

The equations (3.18), (3.19) and (3.20) with the ansatz (3.16) are enough for dilatino equation (3.12) to vanish. The gravitino equation (3.13) can be expanded as

$$\psi_M = \partial_M\varepsilon + \frac{1}{4}\omega_M^{M'N'}\gamma_{M'N'}\varepsilon - \frac{1}{4}H_{MNP}\gamma^{NP}\varepsilon = 0. \tag{3.22}$$

Considering $M = \alpha'$ we get $\partial_{\alpha'}\varepsilon_{\pm} = 0$, and $M = \alpha$, using eq(3.21) and eq(3.16)

$$\partial_\alpha\varepsilon_{\pm} + \frac{1}{4}\omega_\alpha^{mn}\gamma_{mn}\varepsilon_{\pm} + \frac{1}{2}\partial_\beta\phi\,\gamma_\alpha{}^{\beta}\,\varepsilon_{\pm} = 0, \tag{3.23}$$

where $m, n, \ldots$ represents $1,2,3,4$ from the vielbein indices. Choosing $e_\alpha^m(x) = e^{-\phi(x)}\hat{e}_\alpha^m$ with $\hat{e}_\alpha^m$ are constants such that satisfy $\hat{e}_\alpha^m\hat{e}_{\beta m} = \delta_{\alpha\beta}$ and $\hat{e}_\alpha^m\hat{e}^{\alpha n} = \eta^{mn}$, conditions (3.10) are satisfied by regarding eq(3.17). Now using these new basis the spin connections ω_α^{mn} can be found simply

$$\omega_\alpha^{mn} = \partial_\beta\phi\,\hat{e}^{\beta m}\hat{e}_\alpha^n - \partial_\beta\phi\,\hat{e}_\alpha^m\hat{e}^{\beta n}, \tag{3.24}$$

and $\gamma_\alpha{}^\beta = \gamma_{mn}\hat{e}_\alpha^m\hat{e}^{\beta n}$. Replacing these equations in eq(3.23), the result becomes very simple

$$\partial_\alpha\varepsilon_{\pm} = 0. \tag{3.25}$$

Therefore by choosing constant $\varepsilon_{\pm}$, the gravitino equation (3.13) vanish. the next step is the gaugino equation (3.11). This equation can be solved by choosing a (anti)self-dual gauge field configurations. Replacing eq(3.15) in eq(3.11) and using eq(3.21) one can find $\delta\chi = -\delta\chi$ for $\varepsilon_{\pm}$, and therefore $\delta\chi = 0$ identically. In the above derivations, we should know the self-dual fields come with the positive chiral spinor, and the anti-self-dual fields come with the negative chiral spinor, *ie.* the equations are consistent only with these choices. In the ansatz (3.16) the '$-$' sign come with the self-dual (and positive chiral spinor) case, and '+' sign come with the other one.

Using the metric of eq(3.17), $\mathrm{tr}(\mathbf{R}\wedge\mathbf{R})$ vanishes, and the Bianchi identity shrinks to

$$\mathrm{d}H = -\frac{\alpha'}{30}\mathrm{Tr}(\mathbf{F}\wedge\mathbf{F}). \tag{3.26}$$

Using definition of the exterior derivative 'd' and the ansatz (3.15)–(3.17)

$$\begin{aligned} \mathrm{d}H &= \frac{1}{3!}\partial_\delta H_{\alpha\beta\gamma} dx^\delta \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ &= \frac{1}{6}\partial_\delta H_{\alpha\beta\gamma}\epsilon^{\delta\alpha\beta\gamma} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &= \pm\frac{1}{2}\partial^2(e^{2\phi}) dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \quad \left\{ \begin{array}{ll} + & \text{for } \varepsilon_+ \\ - & \text{for } \varepsilon_-, \end{array} \right. \end{aligned} \tag{3.27}$$

$$\begin{aligned} \mathbf{F} \wedge \mathbf{F} &= \frac{1}{2 \cdot 2}\mathbf{F}_{\alpha\beta}\mathbf{F}_{\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \\ &= \frac{1}{4}\mathbf{F}_{\alpha\beta}\mathbf{F}_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &= \pm\frac{1}{2}\sqrt{g}\,\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &= \pm\frac{1}{2}\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \quad \left\{ \begin{array}{ll} + & \text{for self-dual} \\ - & \text{for anti-self-dual.} \end{array} \right. \end{aligned} \tag{3.28}$$

In eqs(3.27) and (3.28) we have used sum over indices in $\partial^2 = \frac{\partial^2}{\partial x^\lambda \partial x^\lambda}$ and $\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta}$, that are now calculated in Euclidean flat space. The Bianchi identity becomes

$$\partial^2 e^{2\phi} = -\frac{\alpha'}{30}\left[\mathrm{Tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta})\right]. \tag{3.29}$$

Now we have to find solutions of the (anti)self-dual gauge field and the dilaton from eqs(3.15) and (3.29). Equation (3.29) shows the dilaton ϕ directly depends to the choice of the Yang–Mills fields, and the Yang–Mills fields must satisfy the (anti)self-duality relations. As we know the (anti)self-dual solutions depend on the choice of the gauge group. Therefore from now on our construction depends on the choice of a subgroup of the main gauge group, as we know working with the large groups like $E_8 \times E_8$ and $SO(32)$ is not provided. From the instanton subject in section (2.4) we know the solution of (anti)self-dual gauge field for an $SU(2)$ gauge group. Therefore, to use this solution we place the gauge field $\mathbf{F}_{\alpha\beta}$ in an $SU(2)$ subgroup of the gauge group. Now using the self-dual solution of section (2.4) one may find

$$\mathbf{A}_\alpha = 2\Sigma_{\alpha\beta}\frac{x^\beta}{|x|^2 + \lambda^2} \tag{3.30}$$

$$\mathbf{F}_{\alpha\beta} = -4\Sigma_{\alpha\beta}\frac{\lambda^2}{(|x|^2 + \lambda^2)^2} \tag{3.31}$$

where

$$(\Sigma_{\alpha\beta}) = \begin{bmatrix} 0 & \tau_3 & -\tau_2 & \tau_1 \\ -\tau_3 & 0 & \tau_1 & \tau_2 \\ \tau_2 & -\tau_1 & 0 & \tau_3 \\ -\tau_1 & -\tau_2 & -\tau_3 & 0 \end{bmatrix} \tag{3.32}$$

which τ_a with $a = 1, 2$ and 3 are generators of an $SU(2)$ subgroup of the gauge group. For the anti-self-dual solution $\Sigma_{\alpha\beta}$ is replaced by $\bar{\Sigma}_{\alpha\beta}$. Different choices of the $SU(2)$ subgroups embedded in $E_8 \times E_8$ (or $SO(32)$), affect the calculation. For the minimal embedding of $SU(2)$ subgroup [44][3] in $E_8 \times E_8$ (*ie.* the $SU(2)$ part in $SU(2) \times E_7 \subset E_8$),

$$\mathrm{Tr}(\Sigma_{\alpha\beta}\Sigma_{\alpha\beta}) = 12 \cdot 30 = 360 = \mathrm{Tr}(\bar{\Sigma}_{\alpha\beta}\bar{\Sigma}_{\alpha\beta}).$$

Hence the Bianchi identity becomes

$$\partial^2 e^{2\phi} = -192\alpha' \frac{\lambda^4}{(|x|^2 + \lambda^2)^4}. \tag{3.33}$$

[3] The minimal embedding of $SU(2)$ in any other simple Lie group $\bar{G}$ has been studied by Bernard *et al* [44]. They have introduced the $SU(2)$ minimal embedding, is to be the embedding that minimise the topological charge

$$k = \frac{1}{32\pi^2} \int {}^*F^a_{\beta\gamma} F^a_{\beta\gamma} d^4x, \tag{I}$$

where $F^a_{\beta\gamma}$ are the components of the field $\mathbf{F}_{\beta\gamma} = F^a_{\beta\gamma}\mathbf{T}^a$. The $\mathbf{T}^a$'s form a basis in adjoint representation of the Lie algebra associated to the group $\bar{G}$ with the normalisation condition $< \mathbf{T}^a, \mathbf{T}^b >= \frac{1}{C(\bar{G})}\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b) = \delta^{ab}$, where $C(\bar{G})$ is the quadratic Casimir operator. The $SU(2)$ solution they have considered is the 't Hooft solution

$$A^a_\beta = -\bar{\eta}^a_{\beta\gamma}\partial_\gamma \left[1 + \sum_{i=1}^{k} \frac{\lambda_i^2}{(x - x_i)^2}\right], \tag{II}$$

which $\bar{\eta}^a_{\beta\gamma}$ are defined in eq(2.70) (see eqs(2.67) and (2.72)). Given three matrices $\{J^a\}$, $a = 1, 2, 3$ in the adjoint representation of $\bar{G}$ which obey

$$[\mathbf{J}^a, \mathbf{J}^b] = i\,\epsilon^{abc}\mathbf{J}^c, \tag{III}$$

or equivalently $[\mathbf{J}^3, \mathbf{J}^\pm] = \pm\mathbf{J}^\pm$, $[\mathbf{J}^+, \mathbf{J}^-] = \mathbf{J}^3$ with $\mathbf{J}^\pm = (\mathbf{J}^1 \pm i\mathbf{J}^2)/\sqrt{2}$, we can easily offer a solution in $\bar{G}$, $\mathbf{F}_{\beta\gamma} = F^a_{\beta\gamma}\mathbf{J}^a$, where $F^a_{\beta\gamma}$, $a = 1, 2, 3$ are the $SU(2)$ components of the single instanton solution obtained by setting $k = 1$ in equation (II). The topological charge k' of this solution is obtained by the length of any one of the matrices $\mathbf{J}^a$

$$k' =< \mathbf{J}^a, \mathbf{J}^a >= \frac{1}{C(\bar{G})}\mathrm{tr}(\mathbf{J}^a\mathbf{J}^a) \quad (\text{no sum on } a). \tag{IV}$$

The minimal $SU(2)$ embedding is the $SU(2)$ subalgebra whose generators $\mathbf{J}^a$ of $\bar{G}$ have minimum length. They have shown the minimum valve of k' is 1, for the $SU(2)$ subgroup generated by E_α, $E_{-\alpha}$ and $[E_\alpha, E_{-\alpha}]$, where α is a root of maximum length.

Finally the dilaton can be found as

$$e^{2\phi} = e^{2\phi_0} + 8\alpha' \frac{|x|^2 + 2\lambda^2}{(|x|^2 + \lambda^2)^2}, \tag{3.34}$$

where ϕ_0 is a constant and equal to the value of the dilaton ϕ at infinity.

3.1.3 Five-Brane's Charges

There are two charges associated with the five-brane solution. These are the "instanton winding number"

$$k = \frac{1}{480\,\pi^2} \int \mathrm{Tr}\mathbf{F} \wedge \mathbf{F}, \tag{3.35}$$

where the integral is over a four-dimensional cross section, and the "axion charge"

$$Q = -\frac{1}{2\,\pi^2} \int H, \tag{3.36}$$

where the integral is over an asymptotic S^3 surrounding the five-brane. These charges are both quantised [36] with the minimal allowed values given by $k = 1$ and $Q = \alpha'$. For the solution we discussed here these take the values

$$k = 1, \qquad Q = 8\alpha', \tag{3.37}$$

where $k = 1$ is in agreement with the choice of the minimal $SU(2)$ embedding. These five-brane solutions are referred to as "gauge five-branes". The solutions which are obtained from the upper/lower sign in ansatz (3.15) and (3.16) are known as five-brane and anti-five-brane solutions respectively. The form of the dilaton is the same in both cases, while they have opposite H-charges.

3.2 General Instanton Solution

A general solution can be inserted here, *ie.* soliton solutions for the other subgroups of the gauge group are considered, associated to the (anti)self-dual relation (3.15). When we provide a general (anti)self-dual solution, the only equation remains to be solved is the Bianchi identity (3.29). We briefly, without going further to describe ADHM construction in details, explain how we can find out a general formula for instantons (five-branes) in heterotic string theory. To do this, instead of the simplest case of $SU(2) \subset E_8 \times E_8$ we can choose any classical compact subgroups of the large gauge group $E_8 \times E_8$ (or $SO(32)$).

3.2.1 ADHM Construction, Briefly

Suppose G is a simple compact gauge group. Then we can suggest a gauge vector potential $\mathbf{A}_\alpha$ and the field strength $\mathbf{F}_{\alpha\beta}$ associated to it, the action S, the Yang–Mills field equations, and the (anti)self-dual field configurations that minimise the action (and satisfy the Yang–Mills equations) as we had in section (2.4). As we said these solutions are known as instantons.

The ADHM[4], construction leads to an expression for the gauge potential of the form

$$\mathbf{A}_\alpha = \mathbf{v}^\dagger \partial_\alpha \mathbf{v}, \tag{3.38}$$

where $\mathbf{v}$ ia a matrix function of the spatial coordinates x (with real, complex or quaternionic entries depending on the group considered). The conditions that $\mathbf{v}$ has to satisfy are normalisation condition, $\mathbf{v}^\dagger\mathbf{v} = \mathbf{1}$, and certain linear conditions of the form $\mathbf{v}^\dagger\mathbf{\Delta}(x) = 0$, where $\mathbf{\Delta}(x)$ is a linear function of $x = x^\alpha e^\alpha$ where $e^4 = 1$ and $e^j = i\sigma^j$ (σ^j, with $j = 1,2,3$ are 2×2 Pauli matrices). The choice of matrices $\mathbf{v}$ and $\mathbf{\Delta}$ depends on the gauge group G and the instanton number k. For $Sp(N)$, $\mathbf{v}$ is a $(N+k) \times N$ matrix with quaternion entries depending upon x and $\mathbf{\Delta}(x) = \mathbf{a} + \mathbf{b}x$ where $\mathbf{a}$ and $\mathbf{b}$ are matrices of dimension $(N+k) \times k$, with constant quaternion entries (not depending upon x), which carry all the information about the instantons. Then $\mathbf{A}_\alpha$ is an $N \times N$ matrix of quaternions or $2N \times 2N$ matrix with complex entries (with a special condition for the definition of $Sp(N)$ groups, see eg. [46, page 392]). For $SU(N)$, $\mathbf{v}$ is a $(N+2k) \times N$ complex matrix and $\mathbf{\Delta}(x) = (\mathbf{\Delta}_1(x), \mathbf{\Delta}_2(x))$ where $\mathbf{\Delta}_i(x) = \mathbf{a}_i + \mathbf{b}_j x_{ji}$ (for $i,j = 1$ and 2) and $\mathbf{a}_i$ and $\mathbf{b}_i$ are complex matrices of dimension $(N+2k) \times k$. In this case $\mathbf{A}_\alpha$ is a complex $N \times N$ matrix. For the orthogonal groups a similar formalism may be found in literature. The constructions is done in the fundamental representation of the gauge group G.

[4]We have expressed the problem very briefly, just to introduce the quantities. For a review of ADHM construction in matrix algebra see [47, 20]

The two conditions we mentioned in the previous paragraph, are sufficient to ensure that $\mathbf{F}_{\alpha\beta} = \partial_\alpha \mathbf{A}_\beta - \partial_\beta \mathbf{A}_\alpha + [\mathbf{A}_\alpha, \mathbf{A}_\beta]$ satisfies the self-duality equation, and convince $\mathbf{\Delta}(x)^\dagger \mathbf{\Delta}(x)$, that is a $2k \times 2k$ matrix, is viewed as the tensor product of a $k \times k$ hermitian matrix and the unit matrix $\mathbf{1}_2$. Now in ADHM solution a fundamental rôle is played by the $k \times k$ matrix $\mathbf{f}$, which is defined as

$$\mathbf{\Delta}^\dagger \mathbf{\Delta} = \mathbf{1}_2 \mathbf{f}^{-1}, \tag{3.39}$$

and a not quite obvious fact states a direct relation between this function $\mathbf{f}$ and the density $\mathrm{tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta})$ [48]

$$\mathrm{tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta}) = -\partial^2\partial^2 \ln(\det \mathbf{f}). \tag{3.40}$$

Using eqs(3.40) and (3.39) it is simple to see $\mathrm{tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta}) = \frac{1}{2}\partial^2\partial^2 \ln(\det \mathbf{\Delta}^\dagger \mathbf{\Delta})$, then

$$\begin{aligned} \int \mathrm{tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta})\, d^4x &= \int \frac{1}{2}\partial^2\partial^2 \ln(\det \mathbf{\Delta}^\dagger \mathbf{\Delta})\, d^4x \\ &= \int_{S^3_\infty} \frac{1}{2} d\Omega\, |x|^2 x^\alpha \partial_\alpha \partial^2 \ln(\det \mathbf{\Delta}^\dagger \mathbf{\Delta}). \end{aligned} \tag{3.41}$$

We have used divergence theorem and the area element of three-sphere, $dS^\alpha = d\Omega\, |x|^2 x^\alpha$, in the last equality. As $|x| \to \infty$, in the right-hand side of the above equation $\mathbf{\Delta}^\dagger \mathbf{\Delta} \to \mathbf{b}^\dagger \mathbf{b}\, |x|^2$ and so

$$\ln(\det \mathbf{\Delta}^\dagger \mathbf{\Delta}) \to \ln \det(\mathbf{b}^\dagger \mathbf{b}) + 4k \ln |x|,$$

as $\mathbf{b}^\dagger \mathbf{b}$ is really a $2k \times 2k$ matrix. Using this result in eq(3.41) we obtain $-16\,\pi^2 k$ for the integral of the right-hand side. Then the instanton number k can be written as

$$k = -\frac{1}{16\,\pi^2} \int \mathrm{tr}({}^*\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta}) d^4x, \tag{3.42}$$

when the self-dual field ${}^*\mathbf{F}_{\alpha\beta} = \mathbf{F}_{\alpha\beta}$ is considered (see eq(2.76)). Replacing x by $x^\dagger$ everywhere in the above rules leads to anti-self-dual solution.

A simple set of equations for $SU(2)$ ($\equiv Sp(1)$), solutions of type of section (2.4) (with $5k$ degrees of freedom), is obtained by taking

$$\begin{aligned} [\mathbf{\Delta}(x)]_{ij} &= \delta_{ij}(x - a_i), \qquad i,j = 1,\ldots,k \\ [\mathbf{\Delta}(x)]_{k+1,i} &= \lambda_i, \qquad\qquad\quad i = 1,\ldots,k \end{aligned} \tag{3.43}$$

where λ's are all real, and a's are quaternions, *ie.* $a_i = a_i{}^\alpha e^\alpha$. For $k = 1$ ($a \equiv a_1$ and $\lambda \equiv \lambda_1$), $\mathbf{\Delta}^\dagger \mathbf{\Delta} = (|x^\alpha - a^\alpha|^2 + \lambda^2)\mathbf{1}_2$, therefore the one-dimensional $\mathbf{f}$ is

$$\mathbf{f} = \left(|x^\alpha - a^\alpha|^2 + \lambda^2\right)^{-1}, \tag{3.44}$$

where λ is the size and a^α are the location of the instanton. Replacing eq(3.44) in eq(3.40) we obtain

$$\mathrm{tr}(\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta}) = -96 \frac{\lambda^4}{\left(|x-a|^2 + \lambda^2\right)^4}, \tag{3.45}$$

that is of course gives $k = 1$ when the integral of eq(3.42) is done.[5] Comparing this result with the result from eq(2.75), we may choose the basis $\tau^a = \sigma^a/2i$, $a = 1, 2, 3$, with the commutation relation $[\tau^a, \tau^b] = \epsilon^{abc}\tau^c$, therefore convention on trace will be $\mathrm{tr}(\tau^a\tau^b) = -\frac{1}{2}\delta^{ab}$.

3.2.2 Embedding of Subgroups

Embedding of a group G inside group $\bar{G}$ means, finding the subgroups of group $\bar{G}$ which are isomorphic to the group G. When embedding of the group G inside the gauge group $E_8 \times E_8$ is considered, we need to know how the group G is embedded inside the group $E_8 \times E_8$. The gauge group $E_8 \times E_8$ is too large and can contain many non-conjugate smaller groups as its subgroups. The choice of an embedding may affect the calculations, from which it is necessary to know the subgroups' "indices of embedding".

Let us see how the choice of a subgroup can alter the result of calculations. Suppose the $SU(2)$ subgroups of a larger gauge group $\bar{G}$. Now we may pick up three elements of the gauge group $\bar{G}$ such that satisfy the relation $[\tau^a, \tau^b] = \epsilon^{abc}\tau^c$, $a = 1, 2$ and 3 as the definition of $SU(2)$ algebra. For different conjugacy classes we can choose three elements of the gauge group $\bar{G}$ such that they satisfy this certain definition for the $SU(2)$ algebra. Now the point is: When we take the trace from these different sets of basis (which any set of them satisfy the same commutation

[5] In this construction $\mathbf{v}$ is a 2×1 matrix of quaternions, $\mathbf{v} = \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right]$, $p = p^\alpha e^\alpha$, $q = q^\alpha e^\alpha$ such that p^α and q^α are functions of x^β. Applying the two conditions on $\mathbf{v}$ imply five equations with eight unknowns. Actually these ambiguities in $\mathbf{v}$ correspond to gauge transformations on $\mathbf{A}_\alpha$, therefore this is nothing to be worried about it. Setting $p^1 = p^2 = p^3 = 0$ leads to the solutions (2.73) and (2.75) of section (2.4).

relation), we may get different results. For example, the $SU(3)$ group contains two different classes of conjugate $SU(2)$ subgroups. When we use a certain definition for the $SU(2)$ subalgebras, the trace of one of the classes is four times of trace of the other one. We may label these two different classes by 1 and 4. These numbers are indices of embedding.

Let us explain the concept of index of embedding in some details. The concept of "index of embedding" invented first by Dynkin [50]. Dynkin works are purely in abstract algebra. We try here to explain the index of embedding in the language of matrix representations. Consider an orthonormal basis $\{\mathbf{T}^a\}$, $a = 1, \dots, \dim(G)$, which $\dim(G)$ is the dimension of group G, such that satisfy the commutation relation

$$[\mathbf{T}^a, \mathbf{T}^b] = f^{abc}\mathbf{T}^c, \tag{3.46}$$

where f^{abc} are structure constants. Suppose the sets of orthonormal basis $\{\mathbf{T}^a_i\}$, $a = 1, \dots, \dim(G)$, $i = 1, \dots, n$ where n is number of distinct conjugate classes of embedding of G in $\bar{G}$. Now we consider the trace of each basis

$$\mathrm{tr}(\mathbf{T}^a_i\mathbf{T}^b_i) = c_i\delta^{ab} \qquad \text{no sum over } i, \tag{3.47}$$

and introduce the set $S = \{c_i\}$. Let us call the minimum value of this set $c_m = c$. The Dynkin index of embedding for embedding i is the ratio c_i/c, which is an integer by considering the Dynkin theorems in the subject of index of embedding [50, pages 130-131].

To go closer to the original definition of index of embedding, we may consider a normalisation condition on the basis of the main group $\bar{G}$,

$$\mathrm{tr}(\mathbf{M}^A\mathbf{M}^B) = c'\delta^{AB}, \tag{3.48}$$

where $\mathbf{M}^A$ are the basis of group $\bar{G}$, and $A, B = 1, \dots, \dim(\bar{G})$. Now we renormalise the minimum value c to become c', $\mathrm{tr}(\mathbf{T}'^a_m\mathbf{T}'^b_m) = c'\delta^{ab}$. Because of this convention on the basis of the minimal embedding, we have to change the structure constants

$$[\mathbf{T}'^a_m, \mathbf{T}'^b_m] = f'^{abc}\mathbf{T}'^c_m. \tag{3.49}$$

Now we rearrange the older basis $\{\mathbf{T}_i^a\}$ of each embedding i, to $\{\mathbf{T}_i'^a\}$, such that the new basis of each embedding satisfy the new commutation relation (3.49). The index of embedding, j_i, of the embedding i is defined as[6]

$$j_i = \frac{c_i'}{c'}, \tag{3.50}$$

where $c_i' = \mathrm{tr}(\mathbf{T}_i'^a \mathbf{T}_i'^a)$ (no sum over a and i).

How can we relate the concept of minimal embedding to the our physical subjects? We may define a minimal embedding of a subgroup G in a larger group $\bar{G}$ is the embedding which gives the norm of quantity

$$\int \mathrm{tr}({}^*\mathbf{F}_{\alpha\beta}\mathbf{F}_{\alpha\beta})d^4x, \tag{3.51}$$

minimum, when we use a definite commutation relation for the Lie algebra associated to the subgroup G. We may normalise this quantity for the minimal embedding to gain the instanton number k (see eq(3.42)). So for the embedding i, the instanton number will be $j_i k$, where j_i is the index of embedding.

As we saw, the minimal embedding of each subgroup has the minimum trace when we use a unique definition for the algebra associated to the subgroup. In principle the minimal embedding of every subgroup G of $\bar{G}$ are regular subgroups of $\bar{G}$, and therefore we may choose the basis of minimal embedding of a subgroup from the basis of the main group. Therefore the trace of the basis of other embeddings will be a multiple of the trace of the main group $\bar{G}$.

3.2.3 A General Solution for Dilaton

The ADHM construction we discussed in this section gives us the gauge fields in the fundamental representation of the gauge group G. In the previous section the theory of low energy superstring was done in the adjoint representation (which is the same as the fundamental representation) of the gauge group, $E_8 \times E_8$. Now when we consider embedding of G inside the heterotic superstring gauge group, we may have more than one choice of conjugate classes. Each of these conjugate classes are

[6]The index of embedding in defined as the ratio of the bilinear form on G obtained by lifting the value of Killing form on $\bar{G}$ to the Killing form on G itself [51, pages 140-141].

labeled with an index of embedding which are usually different integers. (But two distinct conjugate classes may have the same index of embedding.)

We normalised the basis of $E_8 \times E_8$ such that $\mathrm{Tr}(\mathbf{T}^a\mathbf{T}^b) = 30\,\delta^{ab}$. From the previous discussion on the Dynkin's index of embedding, for any subgroup G of $E_8 \times E_8$ always we find a class of conjugacy with the minimum index of embedding, 1. Each of these minimal embeddings as an independent group, also has a minimal $SU(2)$ embedding with the index of embedding 1. The index of embedding of the minimal $SU(2)$ of any minimal subgroup of $E_8 \times E_8$ has the index 1 in $E_8 \times E_8$, *ie.* a minimal embedding of a minimal embedding is a minimal embedding. Now when we clear the situation of a minimal $SU(2)$ subgroup with the unit instanton number, it will be straightforward to accept the generalisation to any embedding.

For a unit instanton number solution the answer of ADHM construction will be the same as the 't Hooft solution we explained in section (2.4) (see footnote 3 on page 43). To pass from the solution in an individual $SU(2)$ to a minimal $SU(2)$ subgroup, we might lean on components of the fields. Choosing the conventions we have explained after eq(3.45), we may define the instanton number as eq(I) in footnote 3. Now we consider the minimal embedding of $SU(2)$, and write down the instanton number as

$$\begin{aligned} k &= \frac{1}{32\pi^2}\int {}^*F^a_{\alpha\beta}\,F^a_{\alpha\beta}\,d^4x \\ &= \frac{1}{32\pi^2\,C(G)}\int \mathrm{Tr}({}^*\mathbf{F}^{\alpha\beta}\mathbf{F}^{\alpha\beta})d^4x \\ &= \frac{1}{16\pi^2\,C(G)}\int \mathrm{Tr}\mathbf{F}\wedge\mathbf{F}, \end{aligned} \tag{3.52}$$

where $C(G)$ is the quadratic Casimir operator, which is used to normalise the basis in the adjoint representation. The equation (3.35) is obtained from eq(3.52) when we consider $E_8 \times E_8$ with $C(G) = 30$.

With these treatments we have to bring a factor $-1/60$ in front of eq(3.42), to obtain eq(3.35) and with the same reason a factor -60 in front of the right-hand side of eq(3.40). Replacing on the Bianchi identity (3.29) we get

$$\partial^2 e^{2\phi} = -2\,\alpha'\,\partial^2\partial^2 \ln\det\mathbf{f}, \tag{3.53}$$

which gives

$$e^{2\phi} = e^{2\phi_0} - 2\,\alpha'\,\partial^2 \ln \det \mathbf{f}, \tag{3.54}$$

that is a compact result for dilaton ϕ, when $\mathbf{f}$ is known by ADHM constructions.

3.3 Monopoles in String Theory

In previous section we described instantons in string theory. In this section we discuss monopoles in the low-energy superstring theory. Our starting point for constructing monopoles is the $N = 1$ super Yang–Mills coupled to supergravity in 10 dimensions. This theory can be dimensionally reduced to give the $N = 4$ theory in $3 + 1$ dimensions. The action is the ten-dimensional action (3.1). There are several kinds of monopoles that arise from this theory, which have come in the literature[7]. In this chapter we will work on the BPS monopoles. In this section we will discuss a charge one BPS monopole in an $SU(2)$ subgroup of the main gauge group $E_8 \times E_8$ or $SO(32)$, but we try to explain the theory and find the formulas as far as possible without regarding a special subgroup.

3.3.1 BPS Monopoles in String Theory

As we know a BPS monopole is a static spherical solution of the Yang–Mills–Higgs fields. To build the BPS monopoles in string theory, some treatment beyond the method we explained in the previous chapter is applied. As we pointed out in section (2.3), we can reconstruct the field equations in a five-dimensional space-time configuration, such that the Higgs field plays the rôle of the fifth component of the Yang–Mills field, and the spatial fifth dimension has no contribution in the evolution of the fields. In eq(2.53) we saw the BPS monopole satisfies the self-dual relation, and this motivates us to find a possible way for finding such solutions in the string theory. In this construction we suppose five dimensions of the total ten dimensions are compactified in a five-torus, and the remaining $1+4$ dimensions play the rôle for constructing the BPS monopoles in the string theory.

[7]We have brought a short review of these monopoles with some references at the end of this chapter.

As in the previous section, but with a different insight, we have a four-dimensional spatial space in addition to one time dimension to write down the similar equations. Here we adapt the fourth component of the Yang–Mills field to be the Higgs field, *ie.* $A_4^a \equiv \Phi^a$, and the fourth dimension x^4 does not contribute in the evolution of the fields—*ie.* non of the field variables is a function of x^4. This extra dimension that is sometimes called "internal dimension", is compactified in a circle *ie.* S^1, and x^4 is the coordinate along S^1. The radius of this circle can be small but non-zero, and $1/R$ is interpreted as the vacuum expectation value of the Higgs field $\mathbf{\Phi}$, *ie.* $1/R = a$ [45]. In this set-up, calculations are the same as in the previous section, but instead of the assumption of the (anti)self-dual instanton configuration in eq(3.31), we assume a BPS monopole configuration to solve the self-dual ansatz (3.15), and then replace in the Bianchi identity (3.29).

The BPS monopoles are static monopoles, *ie.* fields are not time-varying, and expressed by the Bogomol'nyi equations (2.42)–(2.44),

$$\mathbf{A}^0 = 0, \quad \mathbf{F}_{ij} = \pm\epsilon^{ijk}\,\mathbf{D}_k\mathbf{\Phi}, \quad V(\mathbf{\Phi}) = 0. \tag{3.55}$$

Looking at the equations (2.10), (2.44) and (2.53) it is easy to see the minus sign relates anti-self-dual fields to the negative magnetic charge, while the plus sign relates the self-dual fields to the positive magnetic charge. Therefore in ansatz (3.15)–(3.17), a positive chiral spinor ε_+ comes with a positive charge magnetic monopole, and negative chiral spinor ε_- comes with a negative charge magnetic monopole. Replacing from eqs(3.55) into the Bianchi identity (3.29) we obtain

$$\begin{aligned}
\partial^2 e^{2\phi} &= -\frac{\alpha'}{30}\mathrm{Tr}\left(\mathbf{F}^{ij}\mathbf{F}^{ij} + 2\,\mathbf{F}_{i4}\mathbf{F}_{i4}\right) \\
&= -\frac{\alpha'}{30}\left[4\,\mathrm{Tr}\,(\mathbf{D}_k\mathbf{\Phi})^2\right] \\
&= -\frac{\alpha'}{15}\partial^2\left(\mathrm{Tr}\mathbf{\Phi}^2\right),
\end{aligned} \tag{3.56}$$

where we have used $(\mathbf{D}_k\mathbf{\Phi})^2 = \frac{1}{2}(\mathbf{D}^2\mathbf{\Phi}^2 - \mathbf{\Phi}\mathbf{D}^2\mathbf{\Phi})$, which $\mathbf{D}^2\mathbf{\Phi} = 0$ (equation (2.46)), and the identity $\mathrm{Tr}(\mathbf{D}_k\mathbf{A}) = \partial_k(\mathrm{Tr}\mathbf{A})$ for any valid $\mathbf{A}$ by using the definition of covariant derivative $\mathbf{D}_k$.

Now we may write the general solution of dilaton for the BPS monopoles in any arbitrary subgroup embedded in the main gauge group. The solution relates the dilaton, ϕ, to the Higgs field, $\mathbf{\Phi}$, via the relation

$$e^{2\phi} = e^{2\phi_0} - \frac{\alpha'}{15}\mathrm{Tr}(\mathbf{\Phi}^2 - \mathbf{\Phi}_0^2), \tag{3.57}$$

where the constants is set such that ϕ_0 is the value of the dilaton ϕ at infinity where $\mathbf{\Phi}_0$ is the value of the scalar Higgs field (remember $\Phi \to a$ as $r \to \infty$ for the $SU(2)$ case). For asymptotically flat space ϕ_0 is set to be zero. For the minimal embedding (of any subgroup $G \subset E_8 \times E_8$) we may write the above relation in a simpler form

$$e^{2\phi} = e^{2\phi_0} - 2\alpha'\,(\Phi^2 - \Phi_0^2), \tag{3.58}$$

where $\Phi^2 = \Phi^a\Phi^a$. So for the embedding i we may write

$$e^{2\phi} = e^{2\phi_0} - 2\alpha'\,j_i\,(\Phi^2 - \Phi_0^2). \tag{3.59}$$

where j_i indicates the index of embedding.

3.3.2 H-Monopoles

The only non vanishing components of antisymmetric field tensor are H_{ij4} which is sometimes called $H_{(4)}$ field. For the minimal embedding

$$\begin{aligned} H_{4ij} &\equiv H_{(4)ij} \\ &= \mp(\sqrt{g})^{-1}\,\epsilon_{ij4}{}^{k}\,\partial_k\phi = \pm\frac{1}{2}\epsilon^{ijk}\,\partial_k e^{2\phi} && (3.60) \\ &= \mp 2\alpha'\,\epsilon^{ijk}\,\Phi^a\partial_k\Phi^a. && (3.61) \end{aligned}$$

For a minimal $SU(2)$ subgroup of $E_8 \times E_8$ the BPS solutions for the gauge and the Higgs fields are (see eqs(2.14)–(2.16) and (2.49)–(2.50)):

$$A^{b0} = 0, \quad A^{bi} = \epsilon^{bij}\frac{x^j}{r^2}[K(r) - 1], \quad A^{b4} \equiv \Phi^b(\mathbf{r}) = \frac{x^b}{r^2}H(r), \tag{3.62}$$

where

$$H(r) = ar\coth(ar) - 1, \qquad K(r) = \frac{ar}{\sinh(ar)}, \tag{3.63}$$

which $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, indices $i, j, k = 1, 2, 3$ and $b = 1, 2, 3$. In the above equations the Yang–Mills coupling constant, e, is set to the unit. In this case the dilaton is

$$e^{2\phi} = e^{2\phi_0} - 2\alpha'(\frac{H^2}{r^2} - a^2), \tag{3.64}$$

and the antisymmetric field tensor is

$$H_{4ij} \equiv H_{(4)ij} = \mp 2\alpha' \epsilon^{ijk} \frac{x^k}{r^4} H(1 - K^2). \tag{3.65}$$

Asymptotic behaviour of this only non-vanishing component of antisymmetric field tensor is

$$H_{4ij} \sim \mp 2\alpha' \, a \, \epsilon^{ijk} \frac{x^k}{r^3}, \quad \text{as } r \to \infty, \tag{3.66}$$

Where in comparison with the Yang–Mills field in eq(2.18) exhibits a monopole field, and therefore it is called "H-monopole" (and sometimes $H_{(4)}$-monopole). Since H_{4ij} is a gauge invariant field strength of the $U(1)$ field coming from B_{4i} in the compactification, we see that the BPS gauge monopole is also an $H_{(4)}$-monopole with magnetic charge $\mp 8\pi\alpha' a$. Therefore from eqs(3.65) and (3.66) the negative $H_{(4)}$-charge is relevant to the positive magnetic charge for the self-dual field, and the positive $H_{(4)}$-charge is relevant to the negative magnetic charge for the anti-self-dual field.

3.3.3 Mass of H-Monopoles

Mass of the monopoles have been discussed in this section, are calculated by taking into account the total energy of the fields, *ie.* Yang–Mills–Higgs, Antisymmetric tensor field, dilaton and gravitation fields. This given by the '00' component of the stress-energy tensor in an orthogonal basis, $T_{00}(r)$, which might be calculated directly by computing the fields [39, 32]. We calculate the total energy-momentum tensor by using the metric (3.17) directly, when the metric is assumed to be asymptotically Minkowskian, *ie.* $e^{2\phi} \to 1$ as $r \to \infty$. We do the calculations in the sigma model variables. The four-dimensional metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.67}$$

so that $h_{\mu\nu}$ vanishes at infinity (however, $h_{\mu\nu}$ is not assumed to be small everywhere). Then the exact Einstein equation can be written as

$$R^{(1)}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}{}_{\lambda} = -\kappa_4^2 T_{\mu\nu}, \tag{3.68}$$

where κ_4 is four-dimensional gravitational coupling constant, and $R^{(1)}{}_{\mu\nu}$ is the part of the Ricci tensor linear in $h_{\mu\nu}$ [49]

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 h^{\lambda}{}_{\mu}}{\partial x^{\lambda}\partial x^{\nu}} - \frac{\partial^2 h^{\lambda}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\lambda}\partial x_{\lambda}}\right). \tag{3.69}$$

The tensor $T_{\mu\nu}$ in eq(3.68) is the total energy-momentum tensor of matter and gravitation (for a comprehensive explanation see Weinberg [49]). The indices are raised and lowered with η's.

For the monopoles we discussed in this section

$$h_{00} = 0, \qquad h_{ij} = (-1 + e^{2\phi})\delta_{ij}, \tag{3.70}$$

and then

$$R^{(1)}{}_{ij} = \frac{1}{2}\left(\frac{\partial^2 e^{2\phi}}{\partial x^i \partial x^j} + \frac{\partial^2 e^{2\phi}}{\partial x^k \partial x^k}\delta_{ij}\right), \tag{3.71}$$

$$\begin{aligned} R^{(1)\lambda}{}_{\lambda} &\equiv \eta^{\lambda\nu}R^{(1)}{}_{\nu\lambda} \\ &= 2\frac{\partial^2 e^{2\phi}}{\partial x^i \partial x^i}, \end{aligned} \tag{3.72}$$

where the repeated indices are consumed for summation, and the rest of the components vanish. So the 00 component of the energy-momentum tensor, *ie.* the energy (or mass) density, is

$$T_{00} = -\frac{1}{\kappa_4^2}\,\partial^2 e^{2\phi}. \tag{3.73}$$

As we saw a direct relation between the dilaton ϕ and the Higgs field $\mathbf{\Phi}$ in eq(3.56), one can explain the energy density completely based in the Higgs field.

The mass as a function of r, the distance from the origin, is $M(r) = \int_0^r T_{00}(r)\, d^3r$. Therefore

$$\begin{aligned} M(r) &= -\int \frac{1}{\kappa_4^2}\,\partial^2 e^{2\phi}\, d^3r \\ &= -\frac{1}{\kappa_4^2}\epsilon^{ijk}\int \partial_i H_{4jk}\, d^3r, \end{aligned} \tag{3.74}$$

where the integral is done over the volume of the sphere of radius r, and in derivation we have used the self-dual solution from eq(3.60) to find

$$\epsilon^{ijk}H_{4jk} = \epsilon^{ijk}\left(\frac{1}{2}\epsilon^{jkl}\partial_l e^{2\phi}\right) = \partial_i e^{2\phi}.$$

The equation (3.74) can be written as

$$M(r) = -\frac{2}{\kappa_4^2}\int \mathrm{d}H_{(4)} = -\frac{2}{\kappa_4^2}\int_{S^2} H_{(4)}, \tag{3.75}$$

where S^2 is a two-dimensional sphere of radius r, and $H_{(4)}$ is the two form antisymmetric field. The total mass of the monopole is calculated when the integration in eq(3.74) is done over whole space. In the other hand the $H_{(4)}$-charge, $g_{(4)}$, is defined as

$$g_{(4)} \equiv \int_{S^2_\infty} H_{(4)}, \tag{3.76}$$

where S^2_∞ denotes the two-dimensional sphere at infinity. Therefore the total mass is

$$M = -\frac{2}{\kappa_4^2}\, g_{(4)}. \tag{3.77}$$

For the BPS monopole of eq(3.65)

$$\begin{aligned} M &= -\frac{1}{\kappa_4^2}\epsilon^{ijk}\int_{\mathbb{R}^3} \partial_i H_{4jk}\, d^3r \\ &= -\frac{1}{\kappa_4^2}\epsilon^{ijk}\int_{S^2_\infty} H_{4jk}\, r x^i d\Omega, \\ &= \frac{16\pi\alpha' a}{\kappa_4^2}, \end{aligned} \tag{3.78}$$

where the last result is concluded from the previous line by providing eq(3.66). And

$$\begin{aligned} g_{(4)} &\equiv \int_{S^2} H_{(4)} = \frac{1}{2\pi R}\int_{S^2\times S^1} H = \frac{1}{2\pi R}\int_{\mathbb{R}^3\times S^1} \mathrm{d}H, \\ &= -\frac{\alpha'}{30}\frac{1}{2\pi R}\int_{\mathbb{R}^3\times S^1} \mathrm{Tr}\mathbf{F}\wedge\mathbf{F}, \\ &= -8\pi\alpha' a, \end{aligned} \tag{3.79}$$

where in the last line we have used $\int \mathrm{Tr}\mathbf{F}\wedge\mathbf{F} = 480\,\pi^2$ for the self-dual solution[8], and replaced R by $a = 1/R$. For the anti-self-dual case , the equation (3.77) comes with positive sign, where the charge is positive.

[8]One point is needed to be made clear between the instantons in previous section and here in monopole solutions. For the instanton case the integral is taken over $\mathbb{R}^4$, and therefore a calculation

The $H_{(4)}$-monopole we have discussed, saturate the lower bound of a Bogomol'nyi bound [39]. The equation (3.77) is analogous to the lower bound in eq(2.40) in previous chapter.

3.4 Monopoles in $SU(N)$ Subgroups

In section (2.5) we described the BPS monopole in $SU(N+1)$ $(N>1)$ gauge fields. In the construction there, we chose $\mathbf{T}^a$(a=1,2,3) to be the generators of the maximal $SU(2)$ subgroup embedded in the gauge group $SU(N+1)$, such that $\mathbf{T}^3$ be a diagonal $(N+1)\times(N+1)$ matrix in the fundamental representation of $SU(N+1)$ (eq(2.102)). In this basis the Higgs field $\mathbf{\Phi}$ is a diagonal matrix with null trace. In this section we want to describe the situation of H-monopoles in an $SU(N+1)$ subgroup of the heterotic superstring gauge group. We explain the simple case, $SU(3)\subset E_8\times E_8$ in some details to show the overall specifications. The other $SU(N)$ subgroups follow the rules we will explain for $SU(3)$.

When the Gell-Mann basis[9] is assumed in the fundamental representation of $SU(3)$, the matrices λ_2, λ_5 and λ_7 make the maximal $SU(2)$ subgroup of $SU(3)$. To see what is happening, we diagonalise λ_7 such that

$$\lambda_7 \longrightarrow \lambda_7' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.80}$$

then

$$\begin{aligned} \mathbf{\Phi} &= \frac{1}{2}\begin{pmatrix} \phi_1 & & \\ & \phi_2-\phi_1 & \\ & & -\phi_2 \end{pmatrix} \\ &= \frac{3}{8}(\phi_2-\phi_1)\lambda_3' + \frac{1}{4}(\phi_2+\phi_1)\lambda_7' + \frac{\sqrt{3}}{8}(\phi_2-\phi_1)\lambda_8', \end{aligned} \tag{3.81}$$

for minimal embedding of $SU(2)$ subgroup shows $\int \mathrm{Tr}\mathbf{F}\wedge\mathbf{F} = 480\,\pi^2$ when we use eq(3.34). In monopole case the situation is slightly different, *ie.* the fourth spatial dimension is compactified in a circle of radius R, and the fields do not depend on the coordinate x^4. Therefore in integration we should know this "internal" dimension is bounded while the other three spatial dimensions are unbounded. Then the integration over a four dimensional space is broken down to an integral over three dimensions. Using minimal embedding of $SU(2)$ and eq(3.65), the result has the same magnitude as in instanton case.

[9]Gell-Mann introduced eight traceless Hermitian matrices $\lambda_1, \lambda_2, \ldots \lambda_8$ which normally can be found in literature. As an example see: [46, page 502].

where λ'_l ($l = 1, \ldots, 8$) are the basis of $SU(3)$ in fundamental representation such that λ'_7 is diagonal as in eq(3.80). We should mention although we use an $SU(2)$ subgroup of $SU(3)$, but this does not mean that we work only in this subgroup. In eq(3.81) we see the Higgs field is in the entire $SU(3)$ group (not only in the maximal $SU(2)$ subgroup where we have assumed), and this is the same for the magnetic field $\mathbf{B}$ as it came in eq(2.110). Here the procedure has been done in the fundamental representation, but finally the quantities can be written in any representation.

Let us go back to the heterotic string gauge group $E_8 \times E_8$. In principle, any of the distinct $SU(3)$ subgroups of the gauge group can be chosen, but for practical calculation we choose the minimal embedding, *ie.* the $SU(3)$ part in $SU(3)\times E_6 \subset E_8$. Here $SU(3)$ is in the adjoint representation of $E_8 \times E_8$ which the matrices have the dimensions of 496. The thing we need is the trace of $\mathbf{\Phi}^2$ to calculate the H-monopole. By choosing the $SU(3)$ minimal embedding

$$\mathrm{Tr}\mathbf{\Phi}^2 = 30\,\Phi^2 = 30\,\left((\phi_1)^2 - \phi_1\phi_2 + (\phi_2)^2\right), \tag{3.82}$$

where the coefficient 30 comes from the normalisation condition of the main gauge group. The quantities ϕ_1 and ϕ_2 were found in section (2.5)

$$\phi_1 \;=\; -\frac{\alpha_1 A_1 e^{\alpha_1 r} + \alpha_2 A_2 e^{\alpha_2 r} + \alpha_3 A_3 e^{\alpha_3 r}}{A_1 e^{\alpha_1 r} + A_2 e^{\alpha_2 r} + A_3 e^{\alpha_3 r}} + \frac{2}{r}, \tag{3.83}$$

$$\phi_2 \;=\; \frac{\alpha_1 A_1 e^{-\alpha_1 r} + \alpha_2 A_2 e^{-\alpha_2 r} + \alpha_3 A_3 e^{-\alpha_3 r}}{A_1 e^{-\alpha_1 r} + A_2 e^{-\alpha_2 r} + A_3 e^{-\alpha_3 r}} + \frac{2}{r}, \tag{3.84}$$

where A_i are functions of α_i, and α_i are free parameters with a constraint:

$$\begin{aligned} A_1 &= \frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}, & A_2 &= \frac{1}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}, \\ A_3 &= \frac{1}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}, & \alpha_1+\alpha_2+\alpha_3 &= 0. \end{aligned} \tag{3.85}$$

Now from eq(3.58)

$$e^{2\phi} = e^{2\phi_0} - 2\alpha'\left[\left((\phi_1)^2 - \phi_1\phi_2 + (\phi_2)^2\right) - a^2\right], \tag{3.86}$$

where a^2 is the valve of $(\phi_1)^2 - \phi_1\phi_2 + (\phi_2)^2$ at infinity. For $\alpha_1 > \alpha_2 > \alpha_3$, $a^2 = (\alpha_1)^2 + \alpha_1\alpha_2 + (\alpha_2)^2$. With $\alpha_1 = 2a$ and $\alpha_2 = 0$ we have embedding of the Prasad–Sommerfield solution in $SU(3)$ (*ie.* solution is entirely in the maximal $SU(2)$ subgroup of $SU(3)$[10]), which is in agreement with our result in eq(3.64)[11].

The antisymmetric field strength H_{ij4} is

$$\begin{aligned} H_{4ij} &= \pm\frac{1}{2}\epsilon^{ijk}\,\partial_k e^{2\phi} \\ &= \mp\alpha'\,\epsilon^{ijk}\,\frac{x^k}{r}\left(2\phi_1\phi_1{}' - \phi_1{}'\phi_2 - \phi_1\phi_2{}' + 2\phi_2\phi_2{}'\right), \end{aligned} \tag{3.87}$$

where $\partial_k = \partial/\partial x^k$, prime indicates d/dr and

$$\frac{d\phi_1}{dr} = \frac{A_1 e^{-\alpha_1 r} + A_2 e^{-\alpha_2 r} + A_3 e^{-\alpha_3 r}}{\left(A_1 e^{\alpha_1 r} + A_2 e^{\alpha_2 r} + A_3 e^{\alpha_3 r}\right)^2} - \frac{2}{r^2}, \tag{3.88}$$

$$\frac{d\phi_2}{dr} = \frac{A_1 e^{\alpha_1 r} + A_2 e^{\alpha_2 r} + A_3 e^{\alpha_3 r}}{\left(A_1 e^{-\alpha_1 r} + A_2 e^{-\alpha_2 r} + A_3 e^{-\alpha_3 r}\right)^2} - \frac{2}{r^2}. \tag{3.89}$$

Let us see the behaviour of the $H_{(4)}$ field when $r \to \infty$. To show this we need a convention in order of α's. First assume distinct values for α's (*ie.* no repeated eigenvalues for the Higgs field when treated in the fundamental representation). Suppose the order $\alpha_1 > \alpha_2 > \alpha_3$, then

$$\phi_1 \sim -\alpha_1, \qquad \frac{d\phi_1}{dr} \sim \frac{-2}{r^2}, \tag{3.90}$$

$$\phi_2 \sim \alpha_3, \qquad \frac{d\phi_2}{dr} \sim \frac{-2}{r^2}. \tag{3.91}$$

In derivation of the above equations the terms $\frac{A_3}{(A_1)^2}\exp(\alpha_2-\alpha_1)r$ and $\frac{A_1}{(A_3)^2}\exp(\alpha_3-\alpha_2)r$ appears in the two right equations respectively, that vanish faster than $-2/r^2$ when $r \to \infty$. Therefore

$$H_{4ij} \sim \mp\,2\alpha'\,(\alpha_1-\alpha_3)\,\epsilon^{ijk}\,\frac{x^k}{r^3}. \tag{3.92}$$

Now remains to think about repeated values for α's. The case in which three α's are have the same value, ie '0', is not interesting. Then suppose two of three α's are

[10] With this choice $\alpha_3 = -2a$, and $\phi_1 = \phi_2 = -2(ar\coth(ar) - 1)/r$ and in the equation (3.81) $\mathbf{\Phi}$ has only λ_7' component.

[11] The charge of this $SU(2)$ embedding is four times of the minimal solution eq(3.64). This agrees with the maximal $SU(2)$ embedding of a minimal $SU(3)$ embedded in $E_8 \times E_8$, which makes an index of embedding 4.

identical, eg. $\alpha_1 = \alpha_2 = \alpha$ and $\alpha_3 = -2\alpha$. If we follow the tricky way we mentioned in section (2.5)[12] we find ϕ_1 and ϕ_2 in less symmetrical form

$$\phi_1 = -\frac{\alpha(3\alpha r+2)e^{\alpha r} - 2\alpha e^{-2\alpha r}}{(3\alpha r-1)e^{\alpha r} + e^{-2\alpha r}} + \frac{2}{r}, \tag{3.93}$$

$$\phi_2 = \frac{\alpha(3\alpha r-2)e^{-\alpha r} + 2\alpha e^{2\alpha r}}{(3\alpha r+1)e^{-\alpha r} - e^{2\alpha r}} + \frac{2}{r}. \tag{3.94}$$

Following the previous case it is not difficult to show as $r \to \infty$,

$$H_{4ij} \sim \mp 6\,\alpha'\,|\alpha|\,\epsilon^{ijk}\,\frac{x^k}{r^3}. \tag{3.95}$$

Both possible cases eq(3.92) and (3.95) show, the magnetic charge of the monopole and the H_4-charge are in opposite signs, and this result is independent of the eigenvalues of Higgs field $\mathbf{\Phi}$ at infinity, where we expected the fields behave as monopole.

Monopoles in the Other Subgroups

In addition to the $SU(2)$ and $SU(3)$ subgroups, the group E_8 contains other $SU(N)$ subgroups of rank eight and less, and the larger group $E_8 \times E_8$ might have some bigger special unitary groups. Similar to the $SU(3)$ subgroup, we can use the results of section (2.5) to explain monopoles in the other $SU(N)$ subgroups.

For the other subgroups, one may use the Nahm construction, which is a modification of the ADHM construction for the monopole case, to explain the situation for a more general solution, as we did with the instantons in section (3.2).

3.5 epilogue

At the end of this chapter we enumerate some other kinds of soliton solutions to the low-energy string theory that have been come in the literature.

Several kinds of the five-branes (instantons solutions) arise from the low-energy effective action. The neutral five-brane solutions [30, 31] can be obtained from the

[12]We select $(\alpha_1, \alpha_2, \alpha_3) = (\alpha+\delta, \alpha-\delta, -2\alpha)$, as δ is a small arbitrary variable that finally is impelled to zero. Inserting these quantities in eq(2.127), using eq(2.128), and after the limit $\delta \to 0$ one can see

$$Q_1 = \frac{2}{9\alpha^2}\left[(3\alpha r-1)e^{\alpha r} + e^{-2\alpha r}\right],$$

and $Q_2(r) = Q_1(-r)$.

gauge five-branes by taking the limit of the size of the instanton goes to zero [32]. For $\lambda = 0$ eq(3.30) is gauge equivalent to $\mathbf{A}_\alpha = 0$, and the Bianchi identity (3.26) is solved as $e^{2\phi} = e^{2\phi_0} + n\alpha'/x^2$, which gives $k = 0$ and $Q = n\alpha'$. There is also an exact solution to the string theory without higher order corrections in α' known as the symmetric five-brane solutions [30]. This solution can be obtained by equating the connection $\Omega_{+\alpha}$ embedded in an $SU(2)$ subgroup of the gauge group, with the $SU(2)$ gauge connection A_α so that $\mathrm{d}H = 0$. This solution allows $k = 1$ with $Q = n\alpha'$.

Such as the instantons, in the monopole case also several kinds of solutions have been assessed: The 't Hooft-Polyakov or BPS monopoles, the Kaluza-Klein or metric monopoles [33, 34, 35], and the H-monopoles [36, 35, 32]. A BPS monopole could be derived for string theory compactified to four dimensions on a six torus [39] as we explained in this chapter. Based on symmetric five-brane solutions it is possible to find monopole solutions which are exact solutions of string theory [40, 41]. To see how monopoles arise from instantons, it is considered that one of the four transverse coordinates (eg. x^4) has been singled out, and all the field dependence on x^4 projected out. By considering this fourth coordinate to be periodic with some period and looking for solutions to the self-dual equation on the space $\mathbb{R}^3 \times S^1$, monopoles can arise from periodic instantons [42], and are known as periodic monopoles. In this scenario the BPS gauge monopoles, neutral H-monopoles and symmetric monopoles are constructed [32].

Monopoles in general and monopoles in string theory in particular have been studied from different points of view. Today, one of the most popular subjects is the duality in field theories, and in particular in the string theories. (As an example, the above mentioned monopoles are predicted by S-duality [38, 53, 52].) We did not deal with duality in this thesis. For a new review in monopoles and electromagnetic duality in supersymmetric gauge theories see [54]. In instanton case also some new progress have been provided. A supersymmetric linear sigma model of ADHM construction has been prescribed in two dimensions [55, 56]. This is a stringy way of constructing the Yang–Mills instantons. In this chapter we dealt only with the 10-dimensional low-energy superstring theory, so there was no chance to test the mentioned stringy way for the sigma-model of string theory in two-dimensions.

Chapter 4

Coloured Particle in Monopole Field

در هر فلکی مردمکی می بینم

هر مردمکش را فلکی می بینم

ای اَحول اگر یکی دو می بینی تو

بر عکس تو من دو را یکی می بینم

« مولوی »

Each world is as the pupil of an eye,

every pupil is itself a whole world;

You, O cross-eyed one, may see two where there is one,

but I see one where there appears to be two.

"Rumi"

Coloured particles which may be called non-abelian or Yang–Mills particles, are particles which carry scalar non-abelian charge (isospin, colour, and such) instead of gauge-invariant scalar charge. In this chapter we discuss equations of motion of a classical Yang–Mills particle in the Yang–Mills fields. First, we review Wong's equations of motion for a system of non-abelian particles and fields. Then, we discuss Wong's equations for a particle in a monopole field and will describe planar orbits which arise from the equations. Next we generalise Wong's equations in five dimensions and explain a particle's motion in a monopole field when the Higgs field is counted in the equations. In this reinvestigation of the earlier work, planar orbits and non-planar bounded orbits are allowed for a test particle in a BPS monopole field. At the end of chapter we explore a generalisation of the Lorentz force that is valid for the Yang–Mills particles and fields.

4.1 Equations of Motion of Yang–Mills Particles in Yang–Mills Fields

In analogy with an abelian system (a classical point charged particle interacting with the electromagnetic field), one can see a rich range of phenomena that occurs in non-abelian systems. Wong proposed a system of equations to describe the classical dynamics of such systems (consisting of coloured particles in non-abelian fields) by generalising the Lorentz force and Maxwell equations of electrodynamics. The non-abelian particle is characterised by an isovector $\mathbf{I}$ (in analogy with the gauge invariant scalar charge q of an electrically charged point particle) which transforms under the adjoint representation of the internal symmetry group of the field—*ie.* the gauge group. Consider the interaction between the Yang–Mills field

$\mathbf{A}_\mu(x)$ ($x \equiv (x_0, x_1, x_2, x_3)$ with x_0 standing for time component and the rest for spatial components) and a spin-$\frac{1}{2}$ field $\Psi(x)$ which transforms under the fundamental representation of $SU(2)$ with generators $\chi^a = \sigma^a/2i$ (σ^a with $a = 1,2,3$ are Pauli matrices) satisfying eq(2.56). The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \bar{\Psi}\gamma^\mu(\partial_\mu + eA^a_\mu\chi^a)\Psi - m\bar{\Psi}\Psi, \tag{4.1}$$

where e is coupling constant, m is the mass of particle and γ-matrices are Hermitian and satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. The field strengths $F^a_{\mu\nu}$ are defined in eq(2.3). The field equations are

$$(\mathbf{D}^\nu\mathbf{F}_{\mu\nu})^a = -e\bar{\Psi}\gamma_\mu\chi^a\Psi, \tag{4.2}$$

$$\gamma^\mu(\partial_\mu + e\chi^a A^a_\mu)\Psi + m\Psi = 0. \tag{4.3}$$

This is the usual Dirac-like treatment to find classical equations of motion from quantum recipe, by regarding the equation (4.3) as a one-particle Dirac equation for a coloured particle in a given external field $\mathbf{A}_\mu$. This has been done by Wong [57], and the following equations have been formulated[1]:

$$m\ddot{x}_\mu = eF^a_{\mu\nu}I^a\dot{x}^\nu, \tag{4.4}$$

$$\dot{I}^a + e\epsilon^{abc}A^b_\mu I^c\dot{x}^\mu = 0, \tag{4.5}$$

where $x(\tau)$ is the world line of the particle in space-time, τ is proper time and "dot" denotes differentiation with respect to the proper time. The right-hand side

[1]The original Wong's discussion has taken place within a Hamiltonian formalism by considering the spin-$\frac{1}{2}$ fermion field. Instead of spin-$\frac{1}{2}$, one may consider a boson field in the presence of an external gauge field and the Hamiltonian formalism to find the Wong equations. Or a classical Lagrangian formalism may be used to extract out the Wong equations directly. (For the early works see [58] and [59]; For a short review and references see [60]). A recent work in the Lagrangian is done by authors of reference [60] using both the spin-$\frac{1}{2}$ and spinless particles. For the spinless particles the Lagrangian is

$$L = \frac{1}{2}\left(m\,\dot{x}_\mu\dot{x}^\mu + i\lambda^a\dot{\lambda}^a - ie\,\epsilon^{abc}A^a_\mu\lambda^b\lambda^c\dot{x}^\mu\right),$$

where $x(\tau)$ is the particle's path, m is mass of the particle, A^a_μ is the Yang-Mills gauge vector, ϵ^{abc} are structure constants of the gauge group and λ^a are dynamical variables from which the colour charge I^a are constructed

$$I^a = -\frac{i}{2}f^{abc}\lambda^b\lambda^c,$$

and satisfy $\{\lambda^a, \lambda^b\} = 0$. The Wong's equations (4.4) and (4.5) are directly found from the above Lagrangian by using the Euler–Lagrange equations for x^μ and λ^a.

of eq(4.4) obviously represents a generalisation of the Lorentz force. In the limit we are considering a particle is thus described by an internal isovector $\vec{I}$ as well as its space-time coordinates.

A Yang–Mills (coloured or non-abelian) current is carried by a point particle, analogous to the electric current in Maxwell electrodynamics, is[2]

$$J^a_\mu(y) = e \int I^a(\tau)\dot{x}_\mu(\tau)\delta^{(4)}\left(y - x(\tau)\right) d\tau. \tag{4.6}$$

where y is an arbitrary point in the space-time, $x(\tau)$ is the particle's path and $\delta^{(4)}$ is four-dimensional Dirac's Delta function. Hence the field equation (4.2) in classical limit is

$$(\mathbf{D}^\nu \mathbf{F}_{\mu\nu})^a = J^a_\mu. \tag{4.7}$$

The conclusion of this section is: In the classical limit, eqs(4.4) and (4.5) together with Equation (4.7) completely describe the interaction among a system of non-abelian particles and non-abelian electromagnetic (*ie.* Yang–Mills) fields. Taking the covariant derivative of eq(4.7) produces[3] $(\mathbf{D}_\mu \mathbf{J}^\mu)^a = 0$ that is consistent with the definition (4.6) and eq(4.5). In matrix-valued notation

$$\begin{aligned}
0 &= \mathbf{D}_\mu \mathbf{J}^\mu, \\
&= e\int d\tau\, \frac{dx^\mu(\tau)}{d\tau}\, \mathbf{D}_\mu\left(\mathbf{I}(\tau)\,\delta^4(y - x(\tau))\right), \\
&= e\int d\tau\, \frac{dx^\mu(\tau)}{d\tau} \left\{\mathbf{I}(\tau)\,\frac{\partial}{\partial y^\mu}\,\delta^4(y - x(\tau)) + e\,[\mathbf{A}_\mu\,,\mathbf{I}]\,\delta^4(y - x(\tau))\right\}, \\
&= e\int d\tau \left\{-\mathbf{I}\frac{d}{d\tau}\,\delta^4(y - x(\tau)) + e\,\frac{dx^\mu(\tau)}{d\tau}[\mathbf{A}_\mu\,,\mathbf{I}]\,\delta^4(y - x(\tau))\right\}, \\
&= e\int d\tau \left\{\frac{d\mathbf{I}}{d\tau}\,\delta^4(y - x(\tau)) + e\,\frac{dx^\mu(\tau)}{d\tau}[\mathbf{A}_\mu\,,\mathbf{I}]\,\delta^4(y - x(\tau))\right\}, \\
&= e\int d\tau \left\{\frac{d\mathbf{I}}{d\tau} + e\,\frac{dx^\mu(\tau)}{d\tau}[\mathbf{A}_\mu\,,\mathbf{I}]\right\}\,\delta^4(y - x(\tau)),
\end{aligned} \tag{4.8}$$

where in the derivation we have used the definition of covariant derivative, symmetry property of Dirac-delta function with respect to its both variables and a total

[2]In quantum field theory the right-hand side of eq(4.2) is interpreted as current.

[3]From Left-hand side of eq(4.7) we have $\mathbf{D}^\mu\mathbf{D}^\nu\mathbf{F}_{\mu\nu} = \frac{1}{2}[\mathbf{D}^\mu, \mathbf{D}^\nu]\mathbf{F}_{\mu\nu} = \frac{1}{2}[\mathbf{F}^{\mu\nu}, \mathbf{F}_{\mu\nu}] = 0$, where we have used the identity we used in page (18).

derivative integration. The equation (4.8) vanishes identically everywhere except on the path of the particle $x(\tau)$, so the quantity between curly brackets vanishes to maintain the validity of the equation everywhere, yielding eq(4.5). Therefore the consistency between the current definition and the equations of motion is valid.

The last point is

$$\frac{d}{d\tau}I^2 = 0, \qquad \text{where } I^2 \equiv I^a I^a, \tag{4.9}$$

which is an immediate consequence of eq(4.5). The equation(4.9) shows the isovector I^a performs a precessional motion in isospace, *ie.* the vector I^a sweeps the surface of a sphere in isospace such that the radius of this sphere could be understood as norm (absolute value) of charge isovector. Equivalently one can interpret eq(4.9) as a conservation law of scalar charge for a non-abelian point particle.

4.2 Yang–Mills Particles in a Monopole Field

As an interesting problem, one can consider the interaction between a coloured particle and a monopole. Soon after discovery of the first monopole in the Yang–Mills–Higgs theories [3, 4], Schechter [61] investigated the classical motion of a coloured test particle in an external field given by the BPS monopole. First we describe the Schechter formalism and explain the results, then we study the planar motions which are solutions of the problem with some numerical results describing orbits at the end.

4.2.1 Equations of Motion of a Yang–Mills Particle in a Monopole Field

The formalism of Wong can be applied here by adding the Yang–Mills test particle to the theory by including the additional term

$$-\bar{\Psi}(\gamma^\mu \partial_\mu + e\gamma^\mu A^a_\mu \chi^a + m)\Psi, \tag{4.10}$$

to the Lagrangian (2.2). Applying the same treatment indicated in the previous subsection, the equations of motion of particle are the same as eqs(4.4) and (4.5) with the field equations being those of equations (2.6) and (2.7). Here, a colour test particle with mass m is subjected to an external magnetic monopole field, and the contribution of the particle to the evolution of the field is ignored.

For the 't Hooft-Polyakov monopoles, replacing the ansatz (2.14)–(2.16) with $J(r) = 0$ in eqs(2.3) and (2.4), a direct calculation shows

$$\begin{aligned} F^a_{0i} &= 0, \qquad (4.11) \\ F^a_{ij} &= \frac{1}{er^2}\Big\{ -2\,\epsilon^{aij}(1-K) + \epsilon^{ijk}\frac{x^a x^k}{r^2}(1-K)^2 \\ &\quad + (\epsilon^{aik}\,x^j x^k + \epsilon^{akj}\,x^i x^k)\frac{1}{r^2}(rK' + 2(1-K))\Big\}, \qquad (4.12) \\ (\mathbf{D}_i\Phi)^a &= \frac{1}{er^2}\left\{ \frac{x^a x^i}{r^2}(rH' - H - HK) + HK\,\delta^{ai}\right\}, \qquad (4.13) \end{aligned}$$

where prime indicates d/dr. Using the identity

$$\epsilon^{ijl}\,x^l x^k + \epsilon^{ilk}\,x^l x^j + \epsilon^{ljk}\,x^l x^i = r^2\,\epsilon^{ijk}, \qquad (4.14)$$

eq(4.12) is simplified to

$$F^a_{ij} = \epsilon^{ijk}\frac{1}{er^2}\left\{ \frac{x^a x^k}{r^2}(K^2 - rK' - 1) + rK'\,\delta^{ak}\right\}. \qquad (4.15)$$

For the BPS monopole of section (2.3)

$$\begin{aligned} A^{a0} &= 0, & \partial_0 A^{ia} &= \partial_0 \Phi^a = 0, \\ A^{ai} &= \epsilon^{aij}\frac{x^j}{er^2}(1-K), & K &= \frac{aer}{\sinh(aer)}, \\ \Phi^a &= \frac{x^a}{er^2}H, & H &= aer\coth(aer) - 1, \qquad (4.16) \end{aligned}$$

which satisfy the Bogomol'nyi equation (2.44).

At large distances, K and K' vanish exponentially; from 't Hooft tensor (2.19), the field has the form of a pure magnetic monopole: $\mathcal{B}^i = \frac{1}{2}\epsilon^{ijk}\mathcal{F}_{jk} = -\frac{x^i}{r^3}$. Thus an electrically charged particle coupled to an abelian vector potential corresponding to $\mathcal{F}_{\mu\nu}$ would move exactly as a charged particle in a pure magnetic field.

In a non-relativistic frame, $\tau = x_0 = t$, using eqs(4.15), eqs(4.4) and (4.5) can be written in vector notation

$$\begin{aligned} m\dot{\vec{v}} &= \frac{K^2 - rK' - 1}{r^4}(\vec{v}\times\vec{r})\left(\vec{r}\cdot\vec{I}\right) + \frac{K'}{r}\left(\vec{v}\times\vec{I}\right) \qquad (4.17) \\ \dot{\vec{I}} &= \frac{1-K}{r^2}(\vec{r}\times\vec{v})\times\vec{I}. \qquad (4.18) \end{aligned}$$

Define at each point along the trajectory of the particle an orthogonal set of vectors,

$$\vec{r}, \quad \vec{w} = \vec{r} \times \vec{v}, \quad \vec{z} = \vec{r} \times \vec{w}, \qquad \text{where } \vec{v} = \frac{d\vec{r}}{dt}. \tag{4.19}$$

Then without loss of generality, non-abelian charge I^a may be written as

$$\vec{I} = \alpha\hat{r} + \beta\hat{w} + \gamma\hat{z}, \tag{4.20}$$

where $\hat{r}$, $\hat{w}$ and $\hat{z}$ are unit vectors along the three directions. The coefficients α, β and γ satisfy

$$\alpha^2 + \beta^2 + \gamma^2 = \text{constant}, \tag{4.21}$$

that is a direct consequence of eq(4.9). Substituting eq(4.20) into eq(4.18) gives three equations for α, β and γ:

$$\begin{aligned} \dot{\alpha} &= -\gamma|\vec{w}|K/r^2, && (4.22)\\ \dot{\beta} &= \frac{r\gamma}{|\vec{w}|^2}\dot{\vec{v}}\cdot(\vec{r}\times\vec{v}), && (4.23)\\ \dot{\gamma} &= -\frac{r\beta}{|\vec{w}|^2}\dot{\vec{v}}\cdot(\vec{r}\times\vec{v}) + \frac{\alpha|\vec{w}|K}{r^2}. && (4.24) \end{aligned}$$

Next, using the moving frame (4.19) in eq(4.17) we obtain the ordinary equation of motion:

$$\begin{aligned} m\dot{\vec{v}} &= \frac{\beta}{r^2}|\vec{w}|K'\,\hat{r} + \frac{\beta}{r^2}(\vec{r}\cdot\vec{v})K'\,\hat{z} \\ &\qquad -\frac{1}{r^3}\left[\alpha|\vec{w}|(K^2-1) + \gamma r(\vec{r}\cdot\vec{v})K'\right]\hat{w}, \end{aligned} \tag{4.25}$$

where we have used

$$\begin{aligned} \vec{w}\times\vec{z} &= |\vec{w}|^2\vec{r}, \qquad \vec{z}\times\vec{r} = r^2\vec{w}, \qquad \vec{v}\cdot\vec{z} = -|\vec{w}|^2 \\ \vec{r}\times\vec{I} &= -\gamma r\,\hat{w} + \beta r\,\hat{z}, \\ \vec{w}\times\vec{I} &= \gamma|\vec{w}|\,\hat{r} - \alpha|\vec{w}|\,\hat{z}, \\ \vec{z}\times\vec{I} &= -\beta|\vec{w}|\vec{r} + \alpha r\vec{w}, \\ \vec{v}\times\vec{I} &= \frac{1}{r}\left\{\beta|\vec{w}|\,\hat{r} - [\alpha|\vec{w}| + \gamma(\vec{r}\cdot\vec{v})]\,\hat{w} + \beta(\vec{r}\cdot\vec{v})\,\hat{z}\right\}. \end{aligned} \tag{4.26}$$

In addition to $|\vec{I}|$, the length of the charge isovector, two other constants of motion are obtained from the equations of motion (4.17) and (4.18). Multiplying both sides of eq(4.17) by $\vec{v}$, one finds the kinetic energy is constant, so the speed $|\vec{v}|$, length of the velocity $\vec{v}$, is a constant of motion. The other constant is the vector

$$\vec{J} = m(\vec{r} \times \vec{v}) + K\vec{I} + \frac{(1-K)\left(\vec{r} \cdot \vec{I}\right)}{r^2}\vec{r}. \tag{4.27}$$

This quantity has been verified to be the total angular momentum of the particle and the fields. The second and third terms of $\vec{J}$ correspond to the angular momentums associated with the fields of the monopole and the test charge in analogy with the $\vec{E} \times \vec{B}$ contribution in the usual monopole case [62]. To show $\vec{J}$ is constant, one may show $d\vec{J}/dt$ vanishes by using the equations of motion:

$$\begin{aligned}\dot{\vec{J}} \;=\; & m\left(\vec{r} \times \dot{\vec{v}}\right) + \frac{(\vec{r} \cdot \vec{v})\,K'}{r}\vec{I} + K\dot{\vec{I}} \\ & + \frac{\left(\vec{v} \cdot \vec{I}\right)(1-K)}{r^2}\vec{r} + \frac{\left(\vec{r} \cdot \dot{\vec{I}}\right)(1-K)}{r^2}\vec{r} - \frac{\left(\vec{r} \cdot \vec{I}\right)(\vec{r} \cdot \vec{v})K'}{r^3}\vec{r} \\ & + \frac{\left(\vec{r} \cdot \vec{I}\right)(1-K)}{r^2}\vec{v} - 2\frac{\left(\vec{r} \cdot \vec{I}\right)(\vec{r} \cdot \vec{v})(1-K)}{r^4}\vec{r},\end{aligned} \tag{4.28}$$

where we have used $(dK/dt) = (dr/dt)K'$, and $(dr/dt) = (\vec{r} \cdot \vec{v})/r$. Replacing $m\dot{\vec{v}}$ and $\dot{\vec{I}}$ from eqs(4.17) and (4.18) and using the identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$, all terms in the right-hand side of eq(4.28) cancel each other, so the result is concluded.

For small distances near the centre of the BPS monopole, $K = aer/\sinh(aer) \sim 1$ and $K' \sim 0$. So from eqs(4.17) and (4.18), $m\dot{\vec{v}} = 0$ and $\dot{\vec{I}} = 0$, which show a free motion around the center of the BPS monopole.

Now, we consider the equations for large r. At large distances K and K' fall off exponentially (see eqs(4.16)), so $K(r)$ and $K'(r)$ are effectively zero for large r. This simplifies eq(4.22) to give $\alpha = \alpha_0 =$ constant, and eq(4.25) to give

$$m\dot{\vec{v}} = -\frac{\alpha_0}{r^3}\vec{v} \times \vec{r}. \tag{4.29}$$

Equation (4.29) is identical to the equation of a charged particle moving in the field of a pure magnetic monopole (*ie.* Dirac point monopole). It is clear that the motion cannot be given by an equation like (4.29) everywhere.

Some results may be derived from the equations of motion [61]:

- In order to have no terms which are not in the direction of $\vec{w}$ in the right-hand side of equation of motion (4.25), we require $\beta = 0$. From eq(4.23) and then from eq(4.24) it is required $\gamma = \alpha = 0$. Therefore $\vec{I} = 0$ and this is a contradiction.

- At large distances, from eq(4.29) one can verify that in addition to the speed v, the magnitude of angular momentum $l = |\vec{l}\,| = |m(\vec{r} \times \vec{v})|$, the vector $\vec{j} = \vec{l} + \alpha_0\,\hat{r}$ and the scalar $\vec{j} \cdot \hat{r} = \alpha_0$ are constants of motion.

- The above item leads to the well-known result [63] that the particle moves on the surface of a cone whose axis (through the origin) is parallel to $\vec{j}$ and whose half-angle is $\cos^{-1}(\alpha_0/j)$. the particle moves towards the origin on the surface of the cone until it reaches the minimum distance $r_{\text{min}} = l/mv$, and then winds its way back out. Actually for distances close to the origin the particle may leave the surface of the cone, and does not follow the predicted trajectory. The r motion is $r = (r^2_{\text{min}} + v^2t^2)^{1/2}$.

- While α, the component of the isospin vector in the particle's radius vector, is constant, eq(4.21) presents a precession of the isospin vector (in isospace) around the direction of the particle's radius vector. As $t \to \pm\infty$, β and γ drive to constant values *ie.* β_0 and γ_0 (can be seen easily from equations of motion). From the standpoint of giving a physical interpretation, it is encouraging that asymptotically the particle's charge isospin vector has a fixed orientation as the "identity" of the particle. If at a time t_0, $\beta = \gamma = 0$, eqs(4.23) and (4.24) (remember $K = 0$ at large distances) show they will stay zero at all times (when the condition $K = 0$ is confirmed). Similar to the Higgs field that is radial in large distances, and by a gauge transformation can be deformed to a constant field say in z-direction [5], here also the same gauge transformation sets the isospin vector in z-direction [61]. Therefore eq(4.29) indicates a particle with electrical charge $q = e\alpha_0$, in the field of a point monopole with charge $-4\pi/e$. As we told already, at distances far from origin the motion is identical to a motion of an electrical particle in a pure magnetic monopole (*ie.* non-abelian property disappears).

Now we describe some solutions of the equations of motion (4.17) and (4.18). Considering a general solution for equations of motion (4.17) and (4.18) is not easy. So we explain some specific solutions in some detail.

If the particle is launched in a radial direction, while the charge isovector is also initially radial, the particle will move uniformly in the radial direction. This is the only choice for radial motion. This can be simply seen from the equations of motion (4.17) and (4.18), when $\vec{r} \times \vec{v}$ vanishes as the condition for radial motion. So $m\dot{\vec{v}} = \frac{K'}{r}\left(\vec{v} \times \vec{I}\right)$ and $\dot{\vec{I}} = 0$. For a radial motion, the acceleration $\dot{\vec{v}}$ must be in the radial direction, while we see $\vec{v} \times \vec{I}$ is normal to the radial direction (as we know $\vec{v}$ is radial). So the only possible case is a constant charge isovector $\vec{I}$ along the radial axis, and therefore a uniform radial motion occurs because $m\dot{\vec{v}} = 0$. If the initial velocity points to the origin, the particle passes through the origin.

If the particle is launched in the field (in any direction) while the charge isovector is normal to the both initial particle's radial direction and velocity vector, then the particle will move on a plane normal to the charge isovector and the charge isovector remains constant (next topic). Under these circumstances radial motion is not allowed. We will show bounded orbits are allowed in the planar motion sector, while we have not observed bounded motions in the general three-dimensional theory.

In any case other than the two cases mentioned, the particle will move on a spatial curve. At the end of this section a numerical analysis of the general three-dimensional equations is described. In the next topic we explain the planar motion of a particle in the BPS monopole.

4.2.2 Planar Orbits

A planar motion is identified by a conserved vector normal to the plane of motion. For non-zero values of position and velocity of a non-uniform motion, the plane of motion is normal to $\vec{r} \times \vec{v}$ at each time. Therefore, if in eq(4.25) the component of the force in the direction $\vec{r} \times \vec{v}$ vanishes, a planar motion takes place, provided that the equation of evolution for the charge isovector is valid.

A quick look at equations (4.22) to (4.25) shows if we put $\alpha = \gamma = 0$ we get

$$\begin{aligned} \dot{\alpha} &= 0, && (4.30) \\ \dot{\beta} &= 0, && (4.31) \\ \dot{\gamma} &= -\frac{r\beta}{|\vec{w}|^2}\dot{\vec{v}} \cdot (\vec{r} \times \vec{v}), && (4.32) \\ m\dot{\vec{v}} &= \frac{\beta}{r^2}|\vec{w}|K'\hat{r} + \frac{\beta}{r^2}(\vec{r}\cdot\vec{v})K'\hat{z}, && (4.33) \end{aligned}$$

where in the last equation there is no component along $\vec{r} \times \vec{v}$ (*ie.* $\dot{\vec{v}} \cdot (\vec{r} \times \vec{v}) = 0$) in the right-hand side, and therefore the condition for a planar motion is obtained. Using eq(4.33) the third equation becomes

$$\dot{\gamma} = 0. \tag{4.34}$$

The equations (4.30) and (4.34) are consistent with the assumption $\alpha = \gamma = 0$, and eq(4.31) becomes

$$\beta = \text{constant} \neq 0, \tag{4.35}$$

which is required for the non-vanishing charge isovector. So the equations of motion (4.22)–(4.25) transformed to a new set of consistent equations which has only one equation, eq(4.33), to be solved. This equation as we said has no component in the direction normal to the position and velocity vectors, and this was the condition for the planar motion we mentioned at the beginning of this subsection. So $\alpha = \gamma = 0$ implies planar motion. We will show also that the planar motion condition necessitates $\alpha = \gamma = 0$.

To have no term in the direction normal to the plane of motion (as the condition for planar motion), we need to equate the coefficient of $\hat{w}$ in eq(4.25) to zero. So eq(4.25) breaks into two separate equations

$$m\dot{\vec{v}} = \frac{\beta}{r^2}|\vec{w}|K'\hat{r} + \frac{\beta}{r^2}(\vec{r}\cdot\vec{v})K'\hat{z} \tag{4.36}$$

$$\alpha|\vec{w}|(K^2 - 1) + \gamma r(\vec{r}\cdot\vec{v})K' = 0, \tag{4.37}$$

where the first equation is the same as eq(4.33). With this treatment we are imposing an extra constraint on the original equations of motion, and this is not necessarily consistent with the other equations. To show the consistency, we may solve six equations out of seven (six original equations of motion plus one constraint because of the planar motion condition), and examine the validity of the last one with the resulting solution. From eq(4.36), $\dot{\vec{v}}$ has no term in the direction $\vec{r} \times \vec{v}$, so from equation (4.23), $\dot{\beta} = 0$, which gives $\beta = \beta_0 =$ constant. Therefore the equation of motion (4.36) does not depend on α and β. But α and β through their relations depend on $\vec{r}$ and $\vec{v}$

$$\dot{\alpha} = -\frac{\gamma|\vec{w}|K}{r^2}, \tag{4.38}$$

$$\dot{\gamma} = \frac{\alpha|\vec{w}|K}{r^2}, \tag{4.39}$$

where $|\vec{w}| = |\vec{r} \times \vec{v}|$. So with a solution for $\vec{r}$ (and $\vec{v}$) we have to show the consistency of eqs(4.37), (4.38) and (4.39) all together. Replacing α from eq(4.37) into eq(4.39) we obtain

$$\dot{\gamma} = -\frac{KK'}{K^2-1}\,\dot{r}, \tag{4.40}$$

where we have used $\vec{r} \cdot \vec{v} = r\,\dot{r}$. Equation (4.40) is solvable and the solution is

$$\gamma = c_1\, e^{1/[2(K^2-1)]}, \tag{4.41}$$

where c_1 is a constant. Using eqs(4.37) and (4.41)

$$\alpha = -c_1 \frac{r^2\, K'\, \dot{r}}{|\vec{w}|(K^2-1)}\, e^{1/[2(K^2-1)]}. \tag{4.42}$$

From eqs(4.38) and (4.39) (or equivalently from eq(4.21) and the fact that β is a constant) one may simply find

$$\alpha^2 + \gamma^2 = {c_2}^2, \tag{4.43}$$

where ${c_2}^2 = I^2 - \beta_0^2$ is another constant. So, finally we have to show the solutions (4.41) and (4.42) for γ and α satisfy eq(4.43). As we see, a solution for $\vec{r}$ and $\vec{v}$ (in fact $\vec{v}$ is enough) is needed to replace for appropriate quantities in (4.42).

The equation (4.36) stands alone and may be solved independently from the other equations. Taking the normal direction to the plane of motion as the z-direction in a cylindrical coordinate, one can write down the equations of motion in the polar plane. We can simply replace $\{\hat{r}, \hat{w}, \hat{z}\}$ by $\{\hat{r}, \hat{k}, -\hat{\theta}\}$. In the polar plane

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}, \tag{4.44}$$

$$\dot{\vec{v}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}, \tag{4.45}$$

so eq(4.36) in the cylindrical coordinates is

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\,\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\,\hat{\theta}\right] = \beta\dot{\theta}K'\,\hat{r} - \beta\,\frac{\dot{r}}{r}\,K'\,\hat{\theta}. \tag{4.46}$$

A set of two nonlinear differential equations appears

$$\begin{cases} m(\ddot{r} - r\dot{\theta}^2) = \beta\dot{\theta}K', \\ \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -\beta\,\dfrac{\dot{r}}{r}\,K', \end{cases} \tag{4.47}$$

which governs the motion of the particle in the polar plane. From the last equation one finds

$$mr^2\dot{\theta} + \beta K \equiv j = \text{ constant}, \tag{4.48}$$

Obtaining K' from the former equation of eq(4.47) and replacing in the later one, we find

$$\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} = 0,$$

which produce another constant of motion

$$\dot{r}^2 + r^2\dot{\theta}^2 \equiv v^2 = \text{ constant}. \tag{4.49}$$

Both of the constants are in agreement with the overall discussion we had earlier about constants of motion.

Now we can replace $\dot{r}$ and $|\vec{w}| = |r^2\dot{\theta}|$ from eqs(4.48) and (4.49)

$$|\vec{w}| = |r^2\dot{\theta}| = \left|\frac{j - \beta K}{m}\right|, \qquad \dot{r}^2 = v^2 - \left(\frac{j - \beta K}{mr}\right)^2, \tag{4.50}$$

into eq(4.43)

$$c_1\, e^{[1/(K^2-1)]}\left\{1 + \frac{r^4(K')^2\left[v^2 - \left(\frac{j-\beta K}{mr}\right)^2\right]}{\left(\frac{j-\beta K}{m}\right)^2(K^2-1)^2}\right\} = {c_2}^2, \tag{4.51}$$

to check validity of the extra constraint that was imposed from the planar motion condition. In eq(4.51) $K = aer/\sinh(aer)$, and c_1, c_2, m, β, v, j, a and e are constants. This means the coefficient of the constant c_1 that is an explicit function of r, should be a constant. This can not happen in general. At least numerically, we may show the above function of r is not a constant for the solutions we will discuss in the next subsection. The only clear possibility is for $r =$ constant, *ie.* for a circular motion. For $r =$ constant, $\dot{r} = 0$, so from eq(4.42) $\alpha = 0$ and from eq(4.43) (or eq(4.41)) $\gamma =$ constant, where from eq(4.38) this constant must be zero. Therefore $c_1 = c_2 = 0$. For the other cases (*ie.* not necessarily circular motions), eq(4.51) necessitates $c_1 = c_2 = 0$. So a planar orbit is possible if and only if the charge isovector has no component in the plane of motion, that means in the planar motion the charge isovector identifies the plane and remains constant.

4.2.3 Analytic Description of Planar Orbits

A Lagrangian for the two-dimensional motion discussed above is offered as

$$L = T - U = \frac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2) + \beta\,\dot{\theta}\,K(r). \tag{4.52}$$

The form of the "potential" $U = -\beta\,\dot{\theta}\,K(r)$ (with the presence of $\dot{\theta}$) apparently shows the force due to it, is not a central force. For a two-dimensional central force, the angular momentum $l = mr^2\dot{\theta}$ is a constant of motion, while here this quantity alone is not a constant. As we mentioned before, the total angular momentum of the particle and the fields is constant, and expressed in eq(4.48). Another difference is: In the central force problem the total energy of the system including the kinetic energy of the particle and the potential, $T + U$, is a constant of motion, while here (in contrast to the angular momentum case) only the kinetic part alone is a constant of motion (eq(4.49)). Likewise, it is shown in the central force that only the "inverse square law" and "Hooke's law" can make closed stable orbits (known as Bertrand's Theorem) [65, section 3-6], while in the present case which is different to these, closed orbits are allowed, however, their stability will need to be investigated. Using the Euler-Lagrange equations, the equations of motion (4.47) are simply derived from the Lagrangian (4.52).

To describe the orbits in the plane of motion we may study the one-dimensional equivalent of eqs(4.47). As we see from eq(4.49), the kinetic energy ($mv^2/2$) is a constant of motion. In the usual central forces in fact the mechanical energy ($mv^2/2+V(r)$, $V(r)$ is potential energy) is a constant of motion. We may reorganise the equations of motion (4.47) to obtain a one-dimensional equation in a standard central form. To do this, we may use the original set (4.47) (as is done in classical mechanics) or use the first order equations (4.48) and (4.49). Replacing $\dot{\theta}$ from eq(4.48) into the first equation of eq(4.47)

$$\begin{aligned} m\ddot{r} &= \beta\left(\frac{j-\beta K}{mr^2}\right)K' + mr\left(\frac{j-\beta K}{mr^2}\right)^2 \\ &= -\frac{m}{2\dot{r}}\frac{d}{dt}\left(\frac{j-\beta K}{mr}\right)^2, \end{aligned} \tag{4.53}$$

which gives

$$m\ddot{r} = -\frac{dV(r)}{dr}, \tag{4.54}$$

where

$$V(r) = \frac{m}{2}\left(\frac{j-\beta K}{mr}\right)^2. \tag{4.55}$$

The equation (4.54) is a one-dimensional equation of motion with a force related to the potential $V(r)$ on the right-hand side. The first integration of eq(4.54) gives

$$E = \frac{1}{2}m\dot{r}^2 + V(r), \tag{4.56}$$

which is indeed equivalent to eq(4.49), *ie.* $E = mv^2/2$ and $V(r) = mr^2\dot{\theta}^2/2$.

The potential $V(r)$ is a function of r, and through j depends on the initial conditions of the motion. To see the shape of the potential $V(r)$ we may simply analyse its derivative with respect to r

$$V'(r) = \frac{-1}{2\,m\,r^3}(j-\beta K)(j-\beta K+\beta rK'). \tag{4.57}$$

The derivative $V'(r)$ vanishes if one of its two factors vanishes, *ie.* $K(r) = j/\beta$ or $K(r) - rK'(r) = j/\beta$. To find the solutions of these equations we may find the points at which the constant function j/β coincides with the functions $K(r) = r/\sinh(r)$ ($a = e = 1$) or $K(r) - rK'(r) = r^2\cosh(r)/\sinh^2(r)$. Looking at Fig(4.1), for

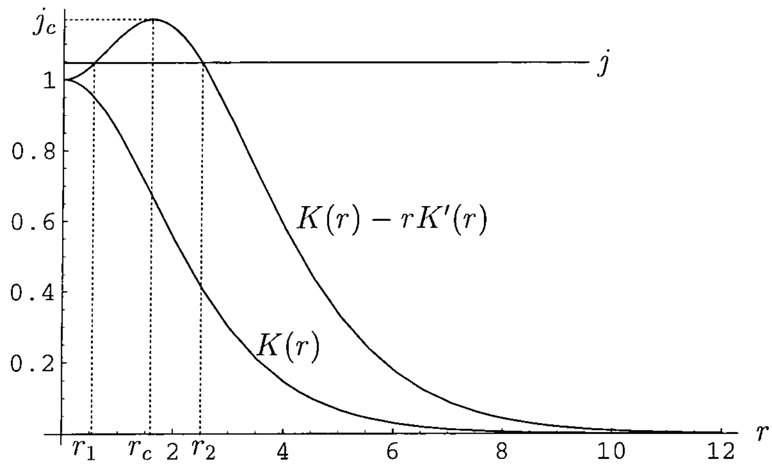

Figure 4.1: Solutions of $V'(r) = 0$.

$0 < j < j_c$ (say $\beta = 1$) there are two solutions for $V'(r) = 0$, therefore $V(r)$ has two extrema at r_1 and r_2. The potential $V(r)$ is a non-negative function and $V(r) \to 0$ as $r \to \infty$. So the two extrema must be a minimum and a maximum respectively, *ie.* r_1 is the minimum and r_2 is the maximum. For $0 < j \leq 1$, $V(r)$ is tangent to the r-axis at the minimum point r_1 (because $V(r)$ is also vanishes for $j = \beta K$), and this is the only point that V(r) touches the r-axis. For $1 < j < j_c$, $V(r)$ does not coincide with the r-axis. For $j \neq 1$, $V(r) \to \infty$ as $r \to 0$, but for $j = 1$ this limit is finite and $V(r) \to 0$ as $r \to 0$ (so $r_1 = 0$). For $j = j_c$ ($r = r_c$) there is only a saddle point, and for $j \leq 0$ and $j > j_c$ there is no extremum and $V(r)$ is monotonically decreasing. Figure (4.2) shows the possible shapes of $V(r)$.

For a given j we may discuss the orbit of the particle, subject to the initial conditions. The limit value j_c may depend on the other constants of theory such as a, e and β. To see how j_c depends on a and e (instead of setting $a = e = 1$ in last steps), we may consider a and e in K as it came in (4.16). Therefore, we have $K = aer/\sinh(aer)$ and $K - rK' = a^2e^2r^2\cosh(aer)/\sinh^2(aer)$. If we plot K and $K - rK'$ in Fig(4.1) versus aer, we see r_c is rescaled, but j_c remains unaltered (up to the constant factor β). It is not difficult to find r_c and j_c (with $a = e = \beta = 1$)

$$r_c = 1.606115299, \qquad j_c = 1.169230089. \tag{4.58}$$

Let see the situation of a motion when j and E (or v) are given. If $0 < j < j_c$ but $j \neq 1$, we have the top-left plot in Fig(4.2). The altitude and latitude of the

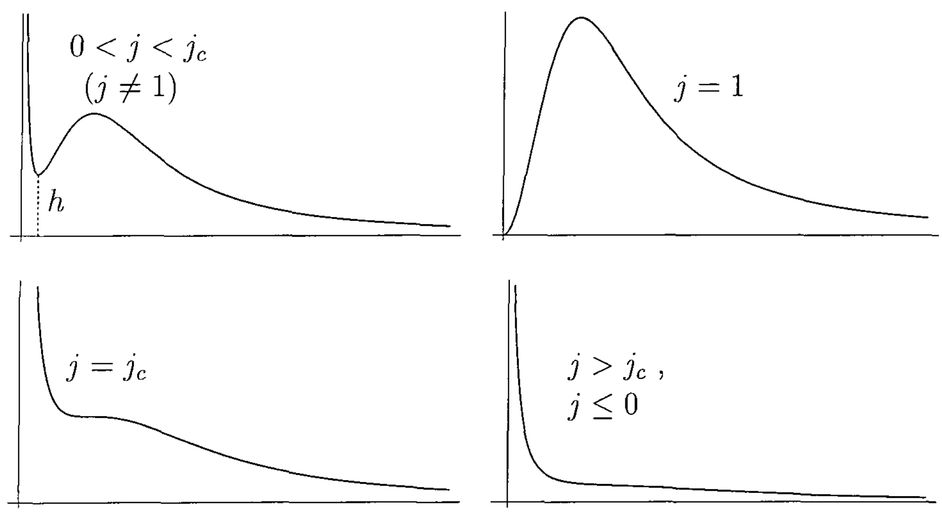

Figure 4.2: One dimensional potential $V(r)$ (vertical axis) versus radius r (horizontal axis). In the top left plot, the height h is zero for $1 < j < j_c$.

extrema from the horizontal and vertical axes are related to the constant j. Figure (4.3) shows the different possibilities of motion subject to energy E.

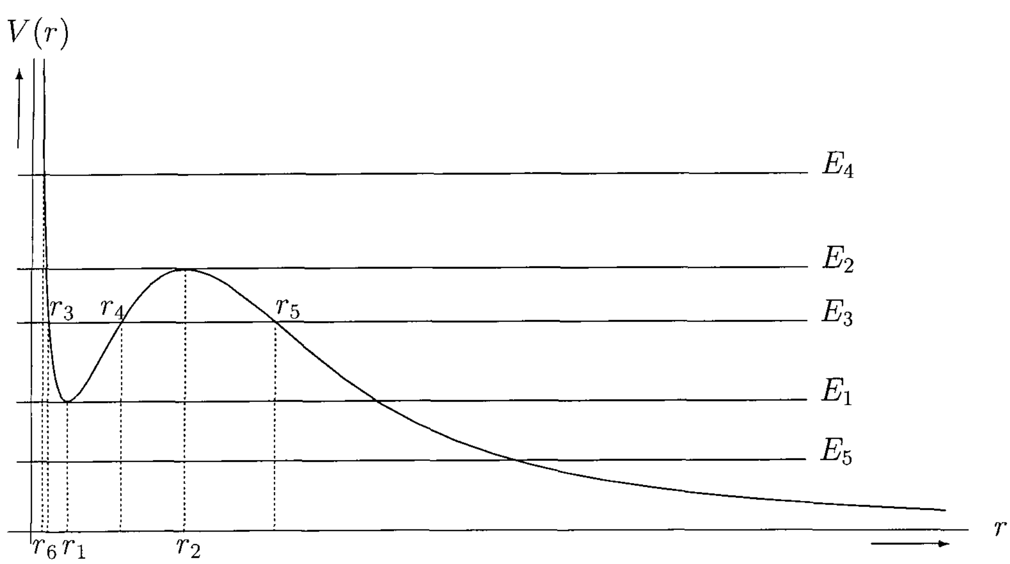

Figure 4.3: Different possibilities of motion for $0 < j < j_c$ $(j \neq 1)$.

If the particle starts its motion with the energy E_4, it will be scattered to infinity and can never come closer to the origin than r_6 (see Fig(4.7) in the next subsection). A motion with the energy E_2 will move on a circle of radius r_2 even it starts its motion from inside or outside of the radius r_2. This motion is unstable *ie.* a small perturbation banishes the particle from the radius r_2. A small perturbation to the left may make a bounded orbit if the perturbed energy is less than E_2, while a small perturbation to the right sends the particle to infinity even if the energy is less than E_2 (see Fig(4.8) in the next subsection). A particle with the energy E_3 is bounded and moves between two radii r_3 and r_4 if it starts the motion in between the two radii (see Fig(4.7) in the next subsection), but it will be scattered to infinity if starts the motion from $r \geq r_5$. A particle with the energy E_1 moves on a circle of radius r_1. This is a stable motion (see Fig(4.7) in the next subsection). The situation of a particle with the energy E_5 is similar to the particle with the energy E_4. In Fig(4.3) r_1 and r_2 are given by r_1 and r_2 in Fig(4.1) when j is specified.

For $0 < j < 1$, $V(r)$ is tangent to the r-axis at r_1, therefore the case with energy E_5 is inappropriate and E_1 lies on the r-axis (so $E_1 = 0$). A stationary particle (a particle with $E = 0$ *ie.* $\dot{r}_0 = \dot{\theta}_0 = 0$) settles in this category and remains at r_1 without moving. So a stable circular motion is possible only for $1 < j < j_c$.

For $0 < j < j_c$, a bounded motion is possible if $E_1 \leq E \leq E_2$. Using Fig(4.1), $j = K(r_2) - r_2K'(r_2) = K(r_2) + r_2^2\dot{\theta}_2^2$, which gives $\dot{\theta}_2 = -K'(r_2)/r_2$. So $E_2 = (\dot{r}_2^2 + r_2^2\dot{\theta}_2^2)/2 = r_2^2\dot{\theta}_2^2/2 = (K'(r_2))^2/2$, where index 2 shows the value of each quantity at r_2. For the minimum value we can do the same. For $1 \leq j \leq j_c$ we have $(K'(r_1))^2/2 \leq E \leq (K'(r_2))^2/2$, and for $0 < j \leq 1$, $0 \leq E \leq (K'(r_2))^2/2$. For each case radii r_1 and r_2 are determined by j, so with j we can describe the overall properties of the motion. As a problem we may determine the maximum value that E can take for a bounded motion. By equating the derivative of function $(K'(r))^2/2$ to zero, we find $r = 1.606115299$ that is equal to r_c (see eq(4.58)) and so the maximum value of $(K'(r))^2/2$ is $E_c = .04799323563$. This is not an accident, because in fact the energy E is equal to the potential $V(r)$ at the turning points, so at the turning points $E = V(r) = (j - K)^2/2r^2$. Clearly, the maximum value of E for a bounded motion occurs when $j = j_c$ (see Fig(4.1)). So if $E > E_c$ we can immediately conclude that the motion is not bounded.

Another point that is worth mentioning is: If the particle is launched into the field from $r > r_m$, (for some r_m) we remain in the category $0 < j < 1$ for bounded motions. From Fig(4.1) one can simply see the maximum distance from the origin for a maximum turning point for the category $1 \leq j \leq j_c$ occurs in $j = 1$. Then r_m is the non-zero root of $K(r) - rK'(r) = 1$, which gives $r_m = 2.676073965$. Clearly if a particle is launched in the field from $r > r_m$ it will not be bounded if $j \geq 1$. So the only possibility for a bounded orbit for this particle is $0 < j < 1$ (not any motion is necessarily bounded).

Motions with $j \neq 1$ do not pass the origin at all. For the bounded orbits, a particle moves in its path between two circles of radii r_3 and r_4. In the turning points $V(r) = E$, so $\dot{r} = 0$. For $0 < j < 1$, in the radius r_1 (the point which locates between the turning points and minimises $V(r)$) $V(r) = 0$ (see the top-left plot in Fig(4.2) with $h = 0$), so in this radius $\dot{\theta} = 0$ and the orbit must be tangent to a radius at this point (see Fig(4.4)). In the radius r_1 ($= OP_1 = OP_2$ in Fig(4.4)),

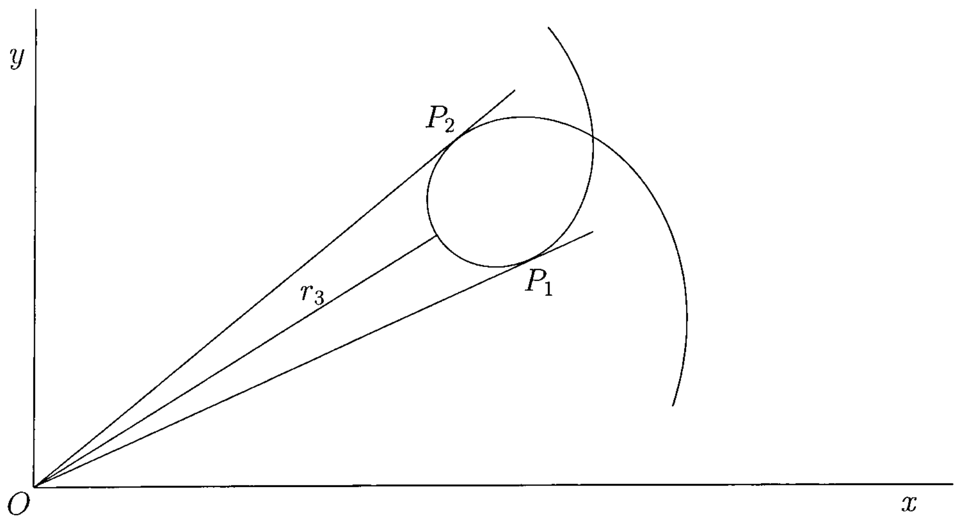

Figure 4.4: Internal loops are possible for $0 < j < 1$.

$\dot{\theta} = 0$, but $\dot{r} \neq 0$ (because $E \neq 0$), so $d\theta/dr = 0$. This means at this point the direction of variations of r remains unchanged, while the direction of variations of θ is changed. Therefore, each time that the particle completes a motion between two radii (for example starting from the upper turning point and returning back to the point after traveling to the lower turning point), it passes two times from the desired point (r_1). So the particle makes an internal loop outside the origin (*ie.* the loop does not surround the origin) in each travel. So in a 2π rotation the particle may construct several inner loops which are lying on the main orbit around the origin.

This case does not happen for $1 < j < J_c$ because always $V(r) > 0$.

This is an important result. For $1 < j < J_c$ the particle rotates around the origin once in every 2π rotation (but not necessarily in a closed orbit). This is similar to the Kepler problem in gravitation. This motion definitely can happen only inside the region $r \leq r_m$ as we explained earlier. But for $0 < j < 1$ the situation is different. In a 2π rotation, a particle trajectory may form several loops outside the origin (see Fig(4.5)). This case may happen anywhere in the plane subject to suitable initial values.

For $j = 1$ (the top-right plot in Fig(4.2)), the situation is the same as $0 < j < 1$, but here the lower turning point is fixed, $r_3 = r_1 = 0$ (in contrast to the case of $j \neq 1$ where the lower turning point depends on E). In this case, the upper turning point, $r_4 \leq r_m = 2.676073965$. So in a bounded motion the particle passes the origin periodically. In fact this case is settled between the two parts of the previously studied case. For $1 < j < j_c$ a bounded orbit turns around the origin once in each 2π period, and the particle's orbit comes closer to the origin on a point of its trajectory when j takes a value closer to one (*ie.* perihelion becomes shorter). In the limit, for $j = 1$ the trajectory crosses the origin, and in each 2π rotation, the particle passes the origin once. When j takes a (positive) value less than 1, the orbit leaves the origin and makes a loop in the opposite side (see Fig(4.5)).

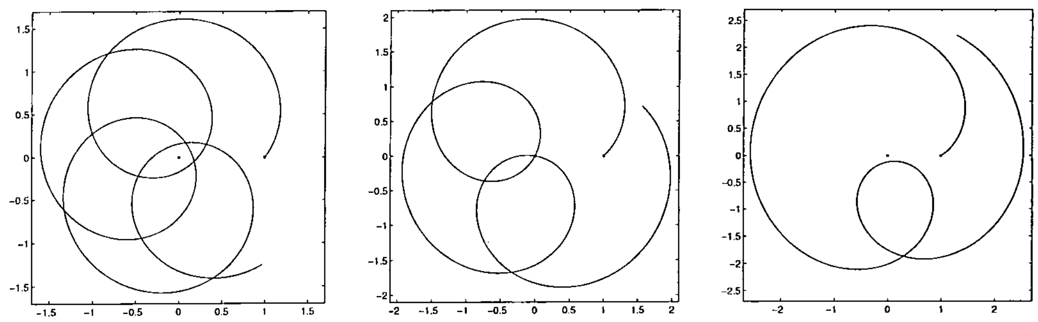

Figure 4.5: In the left plot $j > 1$, In the middle one $j = 1$ and in the right plot $j < 1$. In the three cases particle has started the motion from $(0, 1)$ with the same energy, $E = 0.025$. The total angular momentum j are 1.03, 1 and 0.97 respectively. The starting point and the origin are marked by black dots.

For $j = 1$, $r = 0$ is an extremum point of the potential $V(r)$, and $V(r)$ is tangent to the r-axis in $r = 0$. So it is important to study any probable ambiguity at the origin. The origin is a turning point, and also the orbit is tangent to a radius at the origin. So it is a question to know about the velocity components at the origin. It is not difficult to show neither $\dot{r}$ nor $\dot{\theta}$ vanishes at $r = 0$. One may show

$$\begin{aligned}\lim_{r\to 0}\dot{\theta} &= \lim_{r\to 0}\left\{\frac{1-K(r)}{r^2}\right\} = \frac{1}{6}\\ \lim_{r\to 0}\dot{r}^2 &= \lim_{r\to 0}\left\{v^2 - \left(\frac{1-K(r)}{r}\right)^2\right\} = v^2.\end{aligned} \tag{4.59}$$

The above limits may look strange, but still $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ is valid.

For $j = j_c$ (bottom-left plot in Fig(4.2)) a particle with the exact energy E_c has a non-stable circular motion in the saddle point, otherwise the particle is scattered to infinity. For $j \le 0$ and $j > j_c$ the particle is scattered to infinity and there is no chance for bounded orbits.

Finally, if a particle starts its motion from the point (r_0, θ_0) with the velocity $(\dot{r}_0, \dot{\theta}_0)$, we may calculate the two important constants j and E (or equivalently v) and very soon recognise the motion to be bounded or not, and to which of the preceding cases it belongs.

In principle, we have discussed the overall characters and properties of the orbits, but the actual equation of the orbit must be obtained by integrating the differential equations of motion (4.48) and (4.49). Replacing $\dot{\theta}$ from eq(4.48) in eq(4.49) we get

$$t = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{v^2 - \left(\frac{j-\beta K}{mr}\right)^2}}\,, \tag{4.60}$$

where the motion is supposed to be started from the initial value r_0 at time $t = 0$. As it stands eq(4.60) gives t as a function of r and the constants of integration E (or v), j and r_0. However it may be inverted, at least formally, to give r as a function of t and the constants. Once the solution for r is found, the solution for θ follows immediately from eq(4.48). At large distances from the centre, K and K' vanish and the particle moves in a straight line. Clearly in the areas too close to the center of the monopole, K' vanishes as well (but not K), and a free motion is valid. However, in the other areas K and K' are important and cannot be ignored. Again using $\dot{\theta}$ from eq(4.48) and $\dot{r}$ form eq(4.49) (after some replacement of $\dot{\theta}$), after some rearrangement of variables an integral equation for the orbit of the particle is found as:

$$\theta - \theta_0 = \pm \int_{r_0}^{r} \frac{dr}{r\sqrt{\left(\frac{mvr}{j-\beta K}\right)^2 - 1}}. \tag{4.61}$$

With the presence of the hyperbolic function in the integral, it seems difficult to solve it by changing the variables. Instead of the above integral equation for the orbit, we may replace dt from eq(4.48) into eq(4.54) and find a second order differential equation for the orbit. But non of these help us to find analytic solutions for the orbits. So in the next section we present some numerical solutions of eqs(4.47) and observe the results we found in the previous pages.

4.2.4 Numerical Observations

In this subsection we present some numerical solutions for the set of equations (4.47) for planar motions and eqs(4.17) and (4.18) for non-planar motions. We have used the Runge–Kutta method (of fourth order) for solving first order differential equations. The required programmes are written in the "MATLAB" programming package and are explained in the appendix at the end of this thesis. In a planar motion let us suppose a particle of unit mass, $m = 1$, and unit charge, $\vec{I} = \beta\hat{k}$, $\beta = +1$, has been launched in the field of a BPS monopole with $a = e = 1$, from a point $(r_0\,,\,\theta_0)$, with an initial velocity $(\dot{r}_0\,,\,\dot{\theta}_0)$. The following results are concluded for the different initial values for which we have tested the equations.

The monopole forces the particle to move on a curve in the plane such that if the thumb of right hand stands in the direction of the charge isovector, then the sense of closing the rest of fingers shows the direction of rotation of particle. In Fig(4.6) particle is launched in four different directions (with the same energy), and the particle moves counter-clockwise in each case (the charge isovector is normal to the paper plain and outward). This observation is in a great difference with the usual scattering of an electric particle in a Maxwell field. If the behaviour of this system was like a usual electrodynamics system, the two bottom plots of Fig(4.6) would be the mirror images with respect to the x-axis, while here we see one motion is

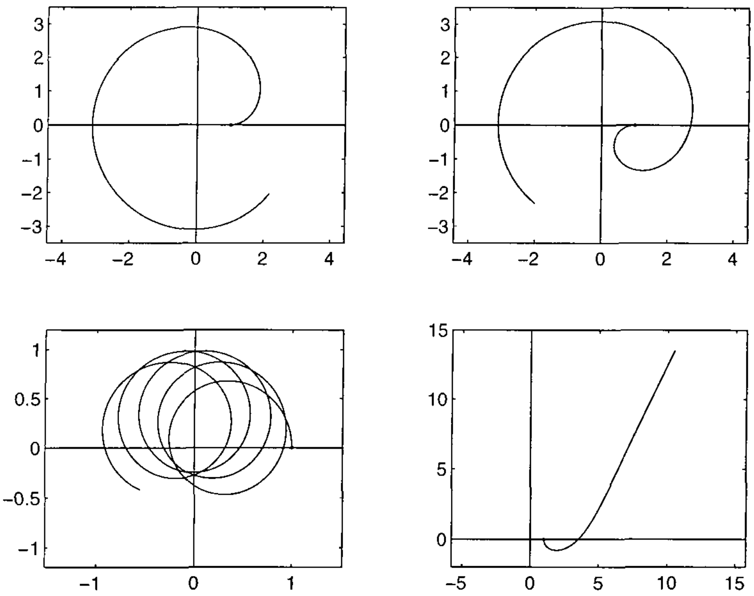

Figure 4.6: Particle is launched from point (1,0) (the dark point in the x-axis) with the same speed (0.1842556) in each case. (The number of significant digits in the velocity is not crucial; it has been chosen for a better illustration of the plots.)

bounded while the other one is scattered to the infinity. (Note in the last two plots, in spite of the fact that the energy is the same in both cases, j is different so they follow the two different models of j–E graphs we explained in subsection (4.2.3).)

This point can be explained analytically when the condition for the planar motion is used in the original equation (4.17). In eq(4.17) $\vec{r} \cdot \vec{I} = 0$ (for the planar motion), so the equations of motion in a compact form is

$$m\dot{\vec{r}} = \frac{K'}{r}\left(\vec{v} \times \vec{I}\right). \tag{4.62}$$

The term in the right-hand side of eq(4.62) is the force that is exerted from the fields onto the particle. The $\vec{I} = \beta\hat{k}$ is a vector normal to the plane of motion and in the upward direction. So the cross-product of $\vec{v}$ and $\vec{I}$ is a vector in the plane of motion and always normal to the velocity vector and in counter-clockwise direction. Therefore the particle is forced to move counter-clockwise.

Bounded and unbounded orbits are allowed depending on the choice of initial values. Closed bounded orbits may exist for each point in the plane, depending on the initial velocity. There are lots of various orbits, circles, Limaçon-shapes, curves with many loops (loops may surround the centre or not), the simple scattered

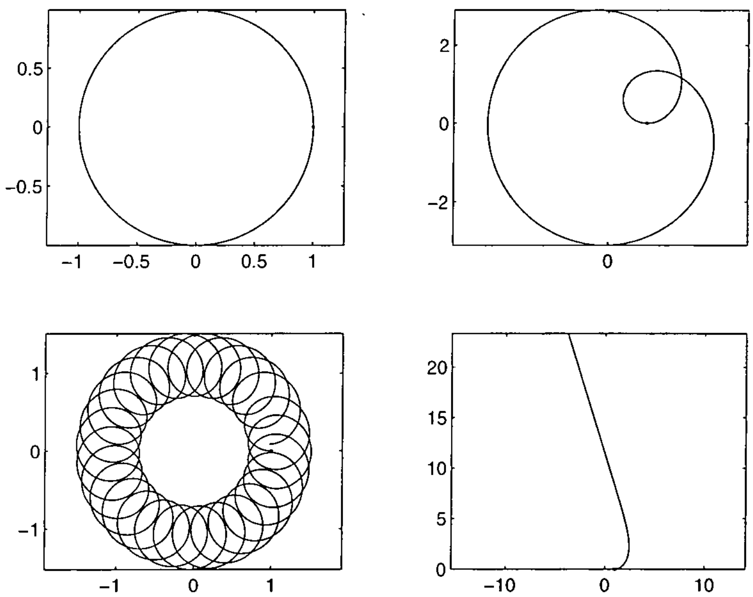

Figure 4.7: For each case motion is started from point (1,0) (the dark point), but the initial energy is different for each case. For the circle, initial velocity is 0.2663673990, and for the Limaçon-shape, the initial velocity is $(\dot{r}_0, \dot{\theta}_0) = (0.1842556, 0)$.

curves and many other complicated curves. A variety of possible orbits are shown in Figs(4.7). For closed orbits the initial conditions are specified. For example, in the two top plots of Fig(4.7) that the initial position of the particle is known (*ie.* $(r, \theta) = (1, 0)$), the velocities are specified up to some significant digits.

Although the circular motions are closed orbits, in general we can not find a prescription for closed orbits. By numerical methods and trial and error we may find some closed orbits. The top-left plot in Fig(4.7) is an example.

The figure (4.8) shows an unstable motion around the maximum point for $0 < j < j_c$. In the top-left plot, the particle is launched from point $(1, 0)$ with $(\dot{r}_0, \dot{\theta}_0) = (0.18427354763893, 0)$, so it will rotate in a radial direction at $r_2 = 3.16299356716209$. This case is calculated only to be compared with the closed top-left plot in Fig(4.7). So for any $0 < j < j_c$ (*ie.* r_0 and $\dot{\theta}_0$ are specified), we can find a value $\dot{r}_0$ and send the particle to rotate on the critical radius r_2. For smaller or bigger energies, the motion will be bounded between two circles, or scattered to infinity, as was explained earlier. The bottom-left and top-right plots show these this point. The bottom-right plot is another example of a unstable motion. Particle starts the motion from the

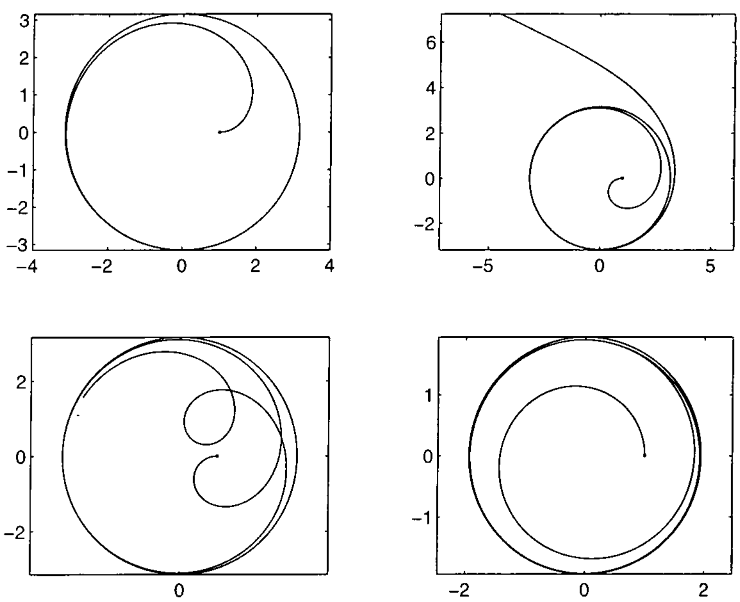

Figure 4.8: For each case motion is started from point (1,0) (the dark point). The plots show stability of the orbits around r_2.

same point as the other three, but j and energy are different. We can compare this case with the circular motion in Fig(4.7).

Numerical Observations For Non-Planar Motions

Now we explain non-planar motions which are indeed follow the general equations of motion (4.17) and (4.18). The computation programmes have been explained in the appendix. Suppose the particle is launched in the field from the point $(0, 0, 10)$ with an initial velocity $(0, 0, -0.1)$ while the charge isovector is initially $(0, 0, 1)$. As we explained before the particle moves on the z-direction and passes through the origin on an enough time. Now suppose instead of launching the particle in the z-direction, launch it from the point $(0, 1, 10)$. So the initial conditions are as before unless an impact parameter is taken into consideration. Of course with these initial conditions the particle moves on a curve which is no longer planar. The top two plots in Fig(4.9) show the orbits for the above mentioned two cases.

Instead of considering an impact parameter in the above case, that leads to a non-planar motion, any small deviation in the initial velocity in the normal direction to the plane or in the charge isovector in the plane causes non-planar orbits as

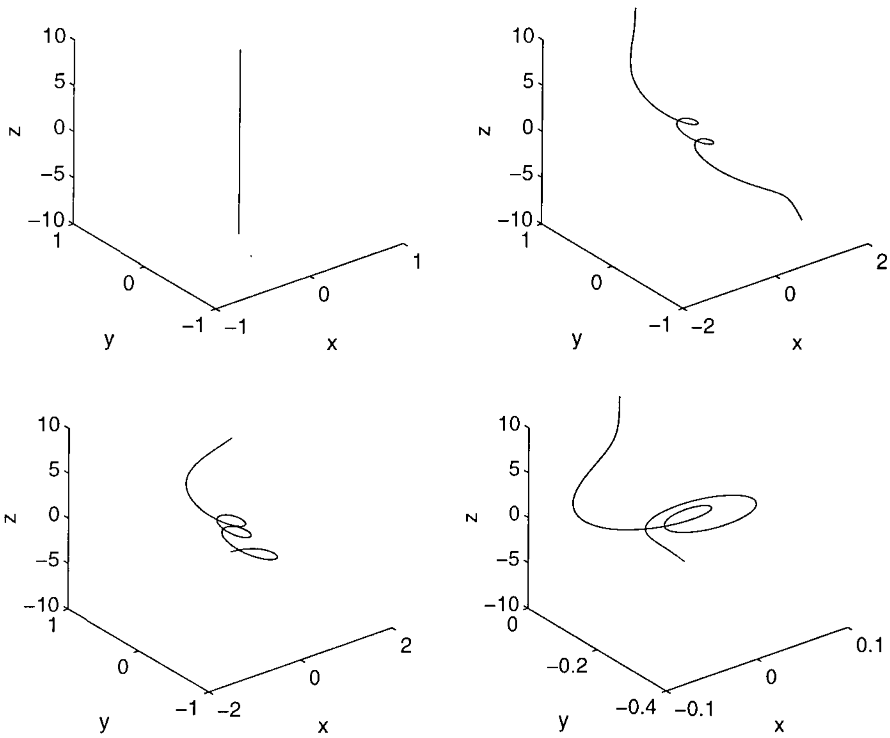

Figure 4.9: Spatial orbits.

well. Examples for these two cases are the bottom plots in Fig(4.9). In the right plot particle is launched from $(0, 0, 10)$ with the velocity $(0, 0.01, -0.1)$ and charge isovector $(0, 0, 1)$; and in the left plot from $(0, 0, 10)$ with the velocity $(0, 0, -0.1)$ and the charge isovector $(0, 0.1, 1)$.

By changing the initial values we may collect a wide range of spatial orbits. Organising these plots to get some useful results are not straightforward, and in addition needs a long time for each computation. As an example, the following observation may lead us to the idea of standard one-dimensional potential we explained in the section on planar motion. In the continuation of the top two plots in Fig(4.9) we can increase the impact parameter. Note that the particle reflects back in the z-direction, when we put $(0, 2, 10)$ for the location of particle and the other initial values are unchanged. Now we can decrease the impact parameter and then play with it by adding and subtracting the earlier values to find a plot such that the particle stays around the xy-plane (at least for a while). We may continue this procedure to get a better and better result. In Fig(4.10), the left plot is the

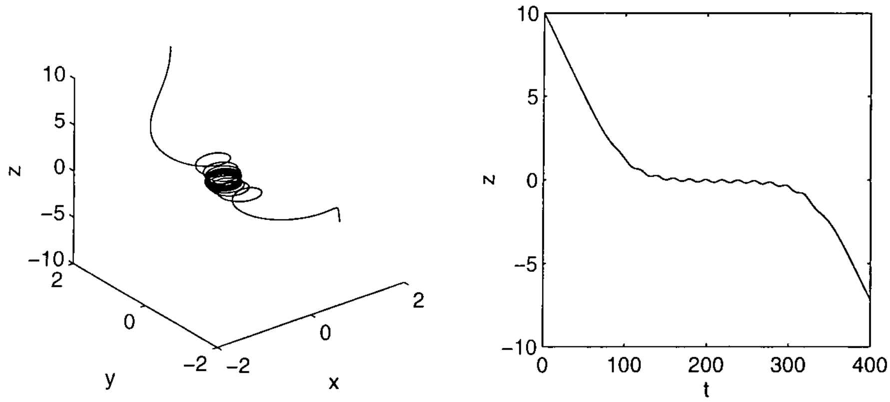

Figure 4.10: Non-stable motion.

three-dimensional orbit and the right one shows the time variation of z component. This plot is resulting from the same initial values as the two top plots in Fig(4.9), but with impact parameter 1.95265 (*ie.* $(0, 1.95265, 10)$ for the initial location). The particle has a small oscillatory motion along the z-axis close to the xy-plane for a while. We may increase duration of the delay around the xy-plane for a longer time by changing the value of impact parameter to a better value.

Based on the observation we explained in the last paragraph, one may compare this situation with the unstable extremum points in the one-dimensional potential model we explained for the planar motions (see Fig(4.8)), but in a three-dimensional context. So if this is like an unstable extremum, there might exist a stable analogue of the one-dimensional potential in the three-dimensional context. If a minimum exists for the general potential model, a stable planar motion would be allowed. This means, stability of planar motion might be possible *ie*: if some small normal perturbations disturb the planar motion, the orbit should stay bounded around the plane. In the next topic we analyse this problem in some detail.

It is proper here to say a word about the scattering problem. With the above results from the numerical works, we see the scattering depends on many parameters is not as simple as in the two-dimensional central force problems. We may keep the initial conditions of the problem unchanged but alter the impact parameter. As the impact parameter changes, the plane spanned by the initial and final velocity vectors changes. So it is needed to introduce three scattering angles instead of one which is used in the usual two-dimensional scattering problems. Even in two-dimensional planar motions the scattering of the particle is not symmetric with respect to the positive and negative values of the impact parameter (see the two bottom plots in Fig(4.6)). The problem of scattering in three-dimensional motions is a separate problem, so we skip it here.

4.2.5 Stability of Planar Motions

The conditions for planar motion look too strong and therefore the stability of planar motions may be very weak. This means, if a little deviation in the quantities perturbs the planar motion in the normal direction to the plane, the particle will leave the plane and be scattered to infinity. Because the perturbations are generally in three dimensions, we should use the general equations of motion (4.17) and (4.18). So, let us first find the required equations for perturbations in three dimensions.

Suppose small perturbations in $\vec{r}$ and $\vec{I}$ in the form

$$\vec{r} \longrightarrow \vec{r} + \vec{\epsilon}, \qquad \vec{I} \longrightarrow \vec{I} + \vec{\delta}, \tag{4.63}$$

where $\vec{\epsilon}$ and $\vec{\delta}$ are small quantities. For the other quantities we will have

$$\vec{v} \longrightarrow \vec{v} + \dot{\vec{\epsilon}}, \quad \dot{\vec{v}} \longrightarrow \dot{\vec{v}} + \ddot{\vec{\epsilon}}, \quad \dot{\vec{I}} \longrightarrow \dot{\vec{I}} + \dot{\vec{\delta}}, \quad r \longrightarrow r + \frac{\vec{\epsilon} \cdot \vec{r}}{r}, \tag{4.64}$$

where in the last one we have kept only the first order approximation. Replacing the unperturbed quantities in equation of motion (4.17) and (4.18) with the perturbed quantities from the right-hand side of the above relations, we find

$$\begin{aligned}
m\ddot{\vec{\epsilon}} &= \frac{K^2 - rK' - 1}{r^4}\Big\{(\vec{v} \times \vec{\epsilon})(\vec{r} \cdot \vec{I}) + (\dot{\vec{\epsilon}} \times \vec{r})(\vec{r} \cdot \vec{I}) \\
&+ (\vec{v} \times \vec{r})(\vec{r} \cdot \vec{\delta}) + (\vec{v} \times \vec{r})(\vec{\epsilon} \cdot \vec{I})\Big\} \\
&- \frac{4K^2 - 2rKK' + r^2K'' - 3rK' - 4}{r^3}(\vec{\epsilon} \cdot \vec{r})(\vec{v} \times \vec{r})(\vec{r} \cdot \vec{I}) \\
&+ \frac{K'}{r}\Big\{(\vec{v} \times \vec{\delta}) + (\dot{\vec{\epsilon}} \times \vec{I})\Big\} + \frac{rK'' - K'}{r^3}(\vec{\epsilon} \cdot \vec{r})(\vec{v} \times \vec{I}), \qquad (4.65) \\
\dot{\vec{\delta}} &= \frac{1-K}{r^2}\Big\{(\vec{\epsilon} \times \vec{v}) \times \vec{I} + (\vec{r} \times \dot{\vec{\epsilon}}) \times \vec{I} + (\vec{r} \times \vec{v}) \times \vec{\delta}\Big\} \\
&- \frac{2(1-K) + rK'}{r^4}(\vec{\epsilon} \cdot \vec{r})\Big[(\vec{r} \times \vec{v}) \times \vec{I}\Big], \qquad (4.66)
\end{aligned}$$

where in derivation we have ignored the second order perturbations and used the unperturbed equations (4.17) and (4.18).

Now for the planar motion case, where we have suggested to make a small perturbation in the space, we have

$$\begin{aligned} m\ddot{\vec{\epsilon}} &= \frac{K^2 - rK' - 1}{r^4}\left\{(\vec{v}\times\vec{r})(\vec{r}\cdot\vec{\delta}) + (\vec{v}\times\vec{r})(\vec{\epsilon}\cdot\vec{I})\right\} \\ &\quad + \frac{K'}{r}\left\{(\vec{v}\times\vec{\delta}) + (\dot{\vec{\epsilon}}\times\vec{I})\right\} + \frac{rK'' - K'}{r^3}(\vec{v}\times\vec{I})(\vec{\epsilon}\cdot\vec{r}), \qquad (4.67) \\ \dot{\vec{\delta}} &= \frac{1-K}{r^2}\left\{(\vec{r}\times\vec{v})\times\vec{\delta} + (\vec{I}\cdot\vec{\epsilon})\,\vec{v} - (\vec{I}\cdot\dot{\vec{\epsilon}})\,\vec{r}\right\}, \qquad (4.68) \end{aligned}$$

and we have used $\vec{r}\cdot\vec{I} = \vec{v}\cdot\vec{I} = 0$. Only the terms in the second row of eq(4.67) can be derived from the planar equation of motion (4.62), and the remaining terms in both equations have appeared by considering the general three-dimensional equations of motion.

Cylindrical coordinates are suitable to write down the equations for each component separately. We may write

$$\begin{aligned} \vec{\epsilon} &= \epsilon_r\,\hat{r} + \epsilon_\theta\,\hat{\theta} + \epsilon_z\,\hat{z}, \\ \vec{\delta} &= \delta_r\,\hat{r} + \delta_\theta\,\hat{\theta} + \delta_z\,\hat{z}. \qquad (4.69) \end{aligned}$$

So

$$\begin{aligned} \dot{\vec{\epsilon}} &= (\dot{\epsilon}_r - \dot{\theta}\epsilon_\theta)\hat{r} + (\dot{\epsilon}_\theta + \dot{\theta}\epsilon_r)\hat{\theta} + \dot{\epsilon}_z\hat{k}, \\ \dot{\vec{\delta}} &= (\dot{\delta}_r - \dot{\theta}\delta_\theta)\hat{r} + (\dot{\delta}_\theta + \dot{\theta}\delta_r)\hat{\theta} + \dot{\delta}_z\hat{k}, \qquad (4.70) \end{aligned}$$

and

$$\ddot{\vec{\epsilon}} = (\ddot{\epsilon}_r - \ddot{\theta}\epsilon_\theta - 2\dot{\theta}\dot{\epsilon}_\theta - \dot{\theta}^2\epsilon_r)\hat{r} + (\ddot{\epsilon}_\theta + \ddot{\theta}\epsilon_r + 2\dot{\theta}\dot{\epsilon}_r - \dot{\theta}^2\epsilon_\theta)\hat{\theta} + \ddot{\epsilon}_z\hat{k}. \qquad (4.71)$$

Replacing from eqs(4.69)–(4.71) in eqs(4.67) and (4.68), and using $\vec{I} = \beta\hat{k}$ ($\beta =$ constant), $\vec{r}$ and $\vec{v}$ in the polar plane (plane of motion) one may find a complete set of six linear differential equations for six unknown perturbation components

$$m\ddot{\vec{\epsilon}} = \left\{\beta\frac{K'}{r}\dot{\epsilon}_\theta + \beta K''\dot{\theta}\epsilon_r + K'\dot{\theta}\delta_z\right\}\hat{r}$$

$$
\begin{aligned}
&+\left\{-\beta\frac{K'}{r}\dot{\epsilon}_r-\beta\,\frac{rK''-K'}{r^2}\dot{r}\epsilon_r+\beta\frac{K'}{r}\dot{\theta}\epsilon_\theta-\frac{K'}{r}\dot{r}\delta_z\right\}\hat{\theta}\\
&+\left\{-\beta\,\frac{K^2-rK'-1}{r^2}\dot{\theta}\epsilon_z+\frac{1-K^2}{r}\dot{\theta}\delta_r+\frac{K'}{r}\dot{r}\delta_\theta\right\}\hat{k},
\end{aligned}
\tag{4.72}
$$

$$
\begin{aligned}
\dot{\vec{\delta}} &= \left\{\frac{1-K}{r^2}(\,-\beta r\dot{\epsilon}_z+\beta\dot{r}\epsilon_z-r^2\dot{\theta}\delta_\theta\,)\right\}\hat{r}\\
&\quad\left\{\frac{1-K}{r}(\,\beta\dot{\theta}\epsilon_z+r\dot{\theta}\delta_r)\right\}\hat{\theta},
\end{aligned}
\tag{4.73}
$$

where in the left-hand side we may replace $\ddot{\vec{\epsilon}}$ and $\dot{\vec{\delta}}$ from eqs(4.71) and (4.70), and write down equations of motion for each component. In the first instance we find $\dot{\delta}_z=0$, so

$$\delta_z=\text{constant}. \tag{4.74}$$

Therefore for small perturbations the total component of the charge isovector in the normal direction to the plane of motion, $\beta+\delta_z$, remains constant. For example, if initially the perturbation of the charge isovector takes place in the plane of motion, this perturbation remains in the plane when the perturbation is small.

Let us choose an auxiliary variable $\vec{\sigma}\equiv\dot{\vec{\epsilon}}$

$$
\begin{aligned}
\vec{\sigma} &= \sigma_r\hat{r}+\sigma_\theta\hat{\theta}+\sigma_z\hat{k},\\
\dot{\vec{\sigma}} &= (\dot{\sigma}_r-\dot{\theta}\sigma_\theta\,)\hat{r}+(\dot{\sigma}_\theta+\dot{\theta}\sigma_r)\hat{\theta}+\dot{\sigma}_z\hat{k}.
\end{aligned}
\tag{4.75}
$$

Now we can write nine first-order differential equations for the perturbations (let us set $m=1$)

$$
\begin{aligned}
\dot{\epsilon}_r &= \dot{\theta}\,\epsilon_\theta+\sigma_r,\\
\dot{\epsilon}_\theta &= -\dot{\theta}\,\epsilon_r+\sigma_\theta\\
\dot{\epsilon}_z &= \sigma_z\\
\dot{\sigma}_r &= \beta\,\frac{rK''-k'}{r}\,\dot{\theta}\,\epsilon_r+\left(\beta\frac{K'}{r}+\dot{\theta}\right)\sigma_\theta+K'\dot{\theta}\,\delta_z,\\
\dot{\sigma}_\theta &= -\beta\,\frac{rK''-k'}{r^2}\,\dot{r}\,\epsilon_r-\left(\beta\frac{K'}{r}+\dot{\theta}\right)\sigma_r+\frac{K'}{r}\dot{r}\,\delta_z,\\
\dot{\sigma}_z &= -\beta\,\frac{K^2-rK'-1}{r^2}\,\dot{\theta}\,\epsilon_z+\frac{1-K^2}{r}\,\dot{\theta}\,\delta_r+\frac{K'}{r}\dot{r}\,\delta_\theta\\
\dot{\delta}_r &= \beta\,\frac{1-K}{r^2}\dot{r}\,\epsilon_z-\beta\,\frac{1-K}{r}\,\sigma_z+K\dot{\theta}\,\delta_\theta,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\delta}_\theta &= \beta \frac{1-K}{r} \dot{\theta}\, \epsilon_z - K\dot{\theta}\, \delta_r, \\
\dot{\delta}_z &= 0.
\end{aligned} \tag{4.76}
$$

Equivalently we may write the above set of equations in matrix form

$$\dot{\vec{S}} = \mathbf{M}\vec{S} \tag{4.77}$$

where

$$\vec{S} = \begin{bmatrix} \epsilon_r & \epsilon_\theta & \epsilon_z & \sigma_r & \sigma_\theta & \sigma_z & \delta_r & \delta_\theta & \delta_z \end{bmatrix}^T, \tag{4.78}$$

(T stands for the transpose), and

$$
\mathbf{M}(\vec{r}, \dot{\vec{r}}) =
\begin{bmatrix}
0 & \dot{\theta} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\dot{\theta} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{\beta(rK''-K')\dot{\theta}}{r} & 0 & 0 & 0 & \frac{\beta K'}{r} + \dot{\theta} & 0 & 0 & 0 & K'\dot{\theta} \\
-\frac{\beta(rK''-K')\dot{r}}{r^2} & 0 & 0 & -\frac{\beta K'}{r} - \dot{\theta} & 0 & 0 & 0 & 0 & \frac{K'\dot{r}}{r} \\
0 & 0 & \frac{\beta(1-K^2+rK')\dot{\theta}}{r^2} & 0 & 0 & 0 & \frac{\left(1-K^2\right)\dot{\theta}}{r} & \frac{K'\dot{r}}{r} & 0 \\
0 & 0 & \frac{\beta\left(1-K^2\right)\dot{r}}{r^2} & 0 & 0 & -\frac{\beta(1-K)}{r} & 0 & K\dot{\theta} & 0 \\
0 & 0 & \frac{\beta(1-K)\dot{\theta}}{r} & 0 & 0 & 0 & -K\dot{\theta} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{4.79}
$$

The matrix $\mathbf{M}$ which is a function of the variables $\vec{r}$ and $\dot{\vec{r}}$ is implicitly a function of time t. If $\mathbf{M}$ was a constant matrix with eigenvalues λ_i and eigenvectors $\vec{R}_i$, the solution of the linear differential equation (4.77) would be $\sum_{i=1}^{9} C_i \exp(\lambda_i t)\vec{R}_i$, where C_i are constants. But now the solution cannot be as simple as this, because the eigenvalues of the matrix $\mathbf{M}$ are functions of time. The only chance for solving the equations in this way is for circular motions when r and $\dot{\theta}$ are constants and $\dot{r}$ vanishes. For example for the circular motion of Fig(4.7), $r = 1$, $\theta_0 = 0$, $\dot{r} = 0$, $\dot{\theta} = 0.2663673990$ and $\beta = 1$ the eigenvalues of $\mathbf{M}$ are

$$0.4420586021,\ -0.4420586021,\ 0.5092646668,\ -0.5092646667,$$

$$0.2351894807i,\ -0.2351894807i,\ 0,\ 0,\ 0.$$

We have used the "MAPLE" programming package to calculate the above values. Using the MAPLE package we can find the eigenvectors as well. Now We need initial values of the perturbation quantities to determine the constants C_i. Suppose in the circular motion, instead of starting the motion at $t = 0$ exactly from the point $(1, 0, 0)$ in the plane of motion, start the motion from $(1, 0, 0.001)$ and the other initial values of the circular motion do not alter. So we have set an small perturbation only in the z-direction (normal to the plane of motion) *ie.* $S(t = 0) = [0, 0, 0.001, 0, 0, 0, 0, 0, 0]^T$. Equating $S(t = 0) = \sum_{i=1}^{9} C_i \vec{R}_i$, the constants C_i are found. Let us study the result for one of the perturbation's components, eg. ϵ_z

$$\begin{aligned}\epsilon_z &= 0.0005111498065\, e^{0.5092646668\, t} + 0.0005111498062\, e^{-0.5092646667\, t} \\ &\quad - 0.0000222996127\, e^{0.2351894807\, i\, t}.\end{aligned}$$

Clearly the first exponential term in the right-hand side of the above equation in spite of its small coefficient, diverges as $t \to \infty$. Therefore we can judge the circular motion under study is not stable.

An analytic solution of the set of equation (4.76) (or equivalently (4.77)) are not available, so we may examine the equations of perturbation by numerical methods. We can study any solution in the plane, with some small values for perturbation quantities. Suppose a list of data of position and velocity of a planar motion is available. So we may use the data and the set of equations (4.76) and a numerical method such as the Runge–Kutta method (or even simpler methods) for computing the differential equations. It is more convenient to calculate the data of planar motion in a procedure and at the same time compute the perturbation quantities for each set of $(r, \theta, \dot{r}, \dot{\theta})$. The required programme is given in the appendix (see parts **P0** and **P5**). We have studied the problem with different choices of the perturbation quantities for different planar solutions, and the results are the same as above. Indeed if the initial perturbation in the charge isovector being in the z-direction, or the initial perturbation in $\vec{r}$ and $\vec{v}$ being in the plane of motion, the motion will stay planar. The stable and unstable planar motions (for perturbations in the plane) were discussed earlier.

4.3 General Equations of Motion in Five Dimensions

In the previous section we described the motion of a coloured test particle in which the colour was coupled only to the Yang–Mills field, and the non-relativistic real space motion at large distances was stated to be the same as for an electric point particle in a Dirac point monopole field. Fehér has given a reinvestigation for a classical motion of a coloured test particle in the Prasad-Sommerfield monopole field. He has considered coupling the particle to both the Yang–Mills and the Higgs fields, and proved the existence of bounded orbits at non-relativistic limit at large distances. He has used the Wong equations and regarded the Higgs field as the fifth component of a Yang–Mills vector field in five dimensions [64, page 46] (see also the conventions before eq(2.53)), interpreting the motion in the fifth direction as providing an effective mass of the particle. In Fehér's article [66] the limit of non-relativistic motion is unclear, so the claim to have found bounded orbits at far distances is questionable.

Fehér has supposed an affine parameter on the path of the particle in the five–dimensional manifold to write down the Wong's equations. Then he has reformulated the equations by using the proper time parameter as the projection of the affine parameter in the four-dimensional space-time path of the particle, and interpreted the mass as the derivative of the proper time by this affine parameter. The formalism of Fehér is not too clear and not a natural generalisation of the four-dimensional Wong's equations. We have searched the literature but we have not found any other work in this direction. Therefore we have reformulated the problem again.

In this section we want to rebuild the equations of motion of a coloured particle in a non-abelian Yang–Mills–Higgs field in a five dimensional space-time in a natural way. The Lagrangian regarding to the Kinetic part of the fields (with the usual definition) is:

$$-\frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu}-\frac{1}{2}(\mathbf{D}^{\mu}\Phi)^{a}(\mathbf{D}_{\mu}\Phi)^{a}. \tag{4.80}$$

This field system can be regarded as a pure Yang–Mills system over a five dimensional flat space-time $\mathcal{M}^5$, for which the corresponding connection is invariant with respect to translations of the fifth coordinate. We suggest the fifth coordinate, say x^5 here, as an internal but dynamical coordinate so that no field variables depend on this internal coordinate. So the Lagrangian (4.80) can formally be written in pure Yang–Mills fields in five-dimensions in a compact form:

$$\mathcal{L}_f = -\frac{1}{4}F^{aAB}F^a_{AB}, \tag{4.81}$$

where $A, B, \ldots = 0, 1, 2, 3$ and 5 denote indices in the five dimensional space-time, and

$$F^a_{AB} = \partial_A A^a_B - \partial_B A^a_A + e\, f^{abc} A^b_A A^c_B \tag{4.82}$$

is the gauge field strength which f^{abc} are the structure constants, and e is the coupling constant of the particle with the Yang–Mills–Higgs field. We have defined the Higgs field as the fifth component of the Yang–Mills field:

$$\Phi^a \equiv A^a_5, \tag{4.83}$$

$$F^a_{\mu 5} = \partial_\mu A^a_5 - \partial_5 A^a_\mu + e\, f^{abc} A^b_\mu A^c_5 = (\mathbf{D}_\mu \Phi)^a, \tag{4.84}$$

where $\mu = 0, 1, 2, 3$ shows the usual four-dimensional space-time indices, and a shows the isospace indices — that is $= 1, 2, 3$ for $SU(2)$ as the gauge group. Now we enter the particle into the field equations by adding the term

$$\mathcal{L}_p = -\bar{\Psi}(\gamma^A \partial_A + e\,\gamma^A A^a_A \chi^a + m)\Psi, \tag{4.85}$$

to the Lagrangian (4.81):

$$\mathcal{L} = -\frac{1}{4}F^{aAB}F^a_{AB} - \bar{\Psi}(\gamma^A \partial_A + e\,\gamma^A A^a_A \chi^a + m)\Psi. \tag{4.86}$$

In the above equations γ^A are Dirac gamma matrices, with $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. The equation (4.85) is the five-dimensional analogue of the equation (4.10). Comparing these two equations with each other, the extra term

$$-\bar{\Psi}(\gamma^5 \partial_5 + e\,\gamma^5 \Phi^a \chi^a)\Psi = -e\,\bar{\Psi}\gamma^5 \Phi^a \chi^a \Psi,$$

appears in our generalisation, which indicates the interaction between the particle and the Higgs field [67, 68, 69]. The Lagrangian (4.86) in five-dimensions is analogous to the Lagrangian (4.1) in four-dimensions. The equations of motion are essentially similar to Wong's equations in four-dimensions. The five dimensional motion of a particle in this pure Yang–Mills field is governed by the following Wong equations:

$$m\frac{d^2x^A}{d\tau^2} = e\,\frac{dx^B}{d\tau}F^{aA}{}_B I^a, \tag{4.87}$$

$$\frac{dI^a}{d\tau} + e\,f^{abc}\,\frac{dx^A}{d\tau}A^b_A I^c = 0, \tag{4.88}$$

where I^a is the charge isovector.

The fifth dimension is a dynamical variable, so the evolution of this internal coordinate (x^5) is governed by

$$\begin{aligned} m\frac{d^2x^5}{d\tau^2} &= e\,\frac{dx^B}{d\tau}F^{a5}{}_B I^a \\ &= -e\,\frac{dx^\mu}{d\tau}(\mathbf{D}_\mu\mathbf{\Phi})^a I^a. \end{aligned} \tag{4.89}$$

For the components of the real four dimensional space-time the equations of motion are

$$\begin{aligned} m\frac{d^2x^\mu}{d\tau^2} &= e\,\frac{dx^B}{d\tau}F^{a\mu}{}_B I^a \\ &= e\,\frac{dx^5}{d\tau}F^{a\mu}{}_5 I^a + e\,\frac{dx^\nu}{d\tau}F^{a\mu}{}_\nu I^a \\ &= e\,\frac{dx^\nu}{d\tau}F^{a\mu}{}_\nu I^a + e\,\frac{dx^5}{d\tau}(\mathbf{D}^\mu\mathbf{\Phi})^a I^a. \end{aligned} \tag{4.90}$$

In Fehér's work an extra term appears in the right-hand side of eq(4.90) which comes from the difference between the affine parameter in the five dimensions and four dimensions. The Wong equation for non-abelian charge **I** in five dimensions, (4.88), can be expanded as

$$\frac{dI^a}{d\tau} + e\,f^{abc}\frac{dx^\mu}{d\tau}A^b_\mu I^c + e\,f^{abc}\frac{dx^5}{d\tau}\Phi^b I^c = 0, \tag{4.91}$$

which in comparison with eq(4.5) in four-dimensions, contains an extra term regarding to the Higgs field. Multiplying both sides of eq(4.91) by I^a, we obtain the same result as in eq(4.9):

$$\frac{d(I^aI^a)}{d\tau} = 0, \quad \text{so } I^aI^a = \text{constant}, \tag{4.92}$$

which indicates conservation of the length of charge isovector. It is in this sense that non-abelian charge is conserved.

The field equations which arise from the Lagrangian (4.86) are:

$$\mathbf{D}_B\mathbf{F}^{AB} = \mathbf{J}^A, \tag{4.93}$$

where $\mathbf{J}^A$ are current due to the coloured particle(s). In the matrix representation

$$\mathbf{A}_B \equiv A^a_B T^a, \quad \mathbf{\Phi} \equiv \Phi^a T^a, \quad \mathbf{F}_{AB} \equiv F^a_{AB} T^a, \quad \mathbf{D}_B \equiv \mathbf{1}\partial_B + e\,[\mathbf{A}_B\,,\ ] \tag{4.94}$$

where T^a 's are generators of the gauge group and $\mathbf{1}$ is the unit matrix of the same dimension as the T^a 's. For $\mathbf{F}_{AB}$ (or equivalently $\mathbf{\Phi}$ and $\mathbf{A}_B$) one may simply show[4] the identities

$$\mathbf{D}_A\mathbf{D}_B\mathbf{F}^{AB} = 0, \tag{4.95}$$

$$\mathbf{D}_A\mathbf{F}_{BC} + \mathbf{D}_C\mathbf{F}_{AB} + \mathbf{D}_B\mathbf{F}_{CA} = 0. \tag{4.96}$$

From eq(4.95) and eq(4.93) the conservation of the coloured (non-abelian) current $\mathbf{J}^A$ is given

$$\mathbf{D}_A\mathbf{J}^A = 0. \tag{4.97}$$

Expanding eq(4.93) the fifth[5] and the space-time components of the current are:

$$\mathbf{J}^5 = \mathbf{D}_B\mathbf{F}^{5B} = \mathbf{D}_\mu\mathbf{F}^{5\mu} = -\mathbf{D}_\mu\mathbf{D}^\mu\mathbf{\Phi}, \tag{4.98}$$

$$\mathbf{J}^\mu = \mathbf{D}_B\mathbf{F}^{\mu B} = \mathbf{D}_\nu\mathbf{F}^{\mu\nu} + \mathbf{D}_5\mathbf{F}^{\mu 5}. \tag{4.99}$$

The last term in the right-hand side of eq(4.99) is simplified by our initial principles:

$$\mathbf{D}_5\mathbf{F}^{\mu 5} = [\mathbf{A}_5\,,\mathbf{F}^{\mu 5}], \qquad (\partial_5\mathbf{F}^{\mu 5} = 0)$$

then

$$\mathbf{J}^\mu = \mathbf{D}_\nu\mathbf{F}^{\mu\nu} + [\mathbf{\Phi}\,,\mathbf{D}^\mu\mathbf{\Phi}], \tag{4.100}$$

[4]See footnote 3 on page 66, but for indices run in five-dimensions to prove eq(4.95). The Bianchi identity (4.96) is a direct consequence of the definitions (4.82), $\mathbf{F}_{AB} = \partial_A\mathbf{A}_B - \partial_B\mathbf{A}_A + e[\mathbf{A}_A, \mathbf{A}_B]$, and covariant derivative (4.94).

[5]When the potential $V(\mathbf{A}^5) \equiv V(\Phi^a\Phi^a) = \frac{\lambda}{4}(\Phi^b\Phi^b - a^2)^2$ is considered in the Lagrangian (4.86), the fifth component of current will be:

$$\mathbf{J}^5 = -\mathbf{D}_\mu\mathbf{D}^\mu\mathbf{\Phi} + \lambda\mathbf{\Phi}(\Phi^a\Phi^a - a^2).$$

that shows the usual current in the four-dimensional space-time has a contribution from the Higgs field.

For a non-abelian point particle with charge $\mathbf{I}$, the current can be defined in five dimensions in the normal way as in Wong's work in four dimensions (see eq(4.6))

$$\mathbf{J}^A(y) = e \int d\tau \, \mathbf{I}(\tau) \, \frac{dx^A(\tau)}{d\tau} \, \delta^5(y - x(\tau)), \tag{4.101}$$

where $x(\tau)$ is the location (world-line) of the charged particle in the five-dimensional space-time, y is an arbitrary point in the space-time, and δ is the Dirac delta-function. The consistency between the above definition and the equations of motion is valid. This was explained in eq(4.8) for four dimensions, but the same argument could apply in five dimensions.

The equations (4.89), (4.90), (4.91) and eqs(4.98) and (4.100) completely describe the motion of a coloured particle in the non-abelian Yang–Mills–Higgs field. It is possible to find a first integral of eq(4.89). The right-hand side of eq(4.89) can be expanded as

$$-e \, \frac{dx^\mu}{d\tau} \frac{d\Phi^a}{dx^\mu} I^a - e^2 \, \frac{dx^\mu}{d\tau} f^{abc} A^b_\mu \Phi^c I^a,$$

and then eq(4.89) can be written

$$m \frac{d^2 x^5}{d\tau^2} = -e \, \frac{d\Phi^a}{d\tau} I^a - e^2 \, \frac{dx^\mu}{d\tau} f^{abc} I^a A^b_\mu \Phi^c. \tag{4.102}$$

Multiplying both sides of eq(4.91) by Φ^a, one can show

$$\Phi^a \frac{dI^a}{d\tau} = e \, \frac{dx^\mu}{d\tau} f^{abc} I^a A^b_\mu \Phi^c, \tag{4.103}$$

which if substituted in eq(4.102), we have

$$m \frac{d^2 x^5}{d\tau^2} = -e \, \frac{d\Phi^a}{d\tau} I^a - e \, \Phi^a \frac{dI^a}{d\tau} = -e \, \frac{d}{d\tau} (\Phi^a I^a). \tag{4.104}$$

From this equation we obtain a first order differential equation for the internal coordinate

$$m \frac{dx^5}{d\tau} = -e \, \Phi^a I^a + h, \tag{4.105}$$

where h is the constant of integration, which is indeed a constant of motion[6]. This equation presents a relation for the internal component of the particle's momentum.

The constant, h, in eq(4.105) depends on the initial orientation of the two isovectors and the initial value of $dx^5/d\tau$. For example, in the field of the 't Hooft–Polyakov monopole if the particle starts the motion from the rest, $dx^A/d\tau = 0$, in the Higgs vacuum ($\vec{\Phi} = a\hat{r}$), and if the particle's charge isovector lies in the radial direction, the constant will be eaI, where I is the norm of the charge isovector which is always constant. If the particle starts the motion while the charge isovector lies in a tangential direction, then the constant will be zero. Regardless of the constant, the equation (4.105) shows the internal component of momentum is proportional to the projection of the Higgs field on the direction of the particle's charge isovector, and the proportionality factor is $-eI$, *ie.* the particle's charge. One can replace $dx^5/d\tau$ from eq(4.105) in eqs(4.90) and (4.91) to obtain a complete set of equations independent of the internal coordinate.

The equations of motion become simpler with some interesting consequences if we use a BPS magnetic monopole as the source of the Yang–Mills–Higgs field. In the next section we will turn to that.

4.4 Particle in the Field of a BPS Monopole

In this section we use the procedure described in the previous section and apply the conditions of the BPS monopole for a test particle. With a test particle we mean: relative to the monopole, the particle is so small in mass and charge such that the resulting perturbation due to the particle can be ignored. Therefore the particle has no contribution in the evolution of the fields, and we ignore the current **J** in the left-hand side of field equations (4.98) and (4.100). Thus the BPS monopole conditions (2.42)–(2.44) satisfy the field equations, and we can use solution (4.16) for the fields.

[6] As the fields are independent of the fifth dimension x^5, in a Lagrangian approach, as we explained in footnote 1 page 65, in five dimensions $\partial L/\partial x^5$ vanishes. Therefore $\partial L/\partial \dot{x}^5 = m\dot{x}^5 + e\Phi^a I^a$ is a constant, in agreement with eq(4.105). So the momentum conjugate to x^5, h, is a constant of motion.

4.4.1 Motion of a Test Particle in the BPS monopole Field

The equations (4.16) satisfy the field equations, and in a non-relativistic framework the equations of motion (4.90) and (4.105) become:

$$\begin{aligned} m\frac{d^2x^i}{dt^2} &= e\frac{dx^j}{dt}F^a_{ij}I^a + e\,p\,(\mathbf{D}_i\Phi)^a I^a, &(4.106)\\ m\,p &= -e\,\Phi^a I^a + h, &(4.107)\end{aligned}$$

where we have substituted τ by $t = x^0$, dropped the equation for x^0 and defined

$$p = \frac{dx^5}{dt}. \tag{4.108}$$

The constant h depends on the initial conditions as we explained after its introduction in eq(4.105). A force due to the Higgs field has appeared in the equation of motion, eq(4.106), beyond the usual Lorentz force. The equation of evolution of the charge isovector (4.91) becomes:

$$\frac{dI^a}{dt} + e\,\epsilon^{abc}\,\frac{dx^i}{dt}A^b_i I^c + e\,p\,\epsilon^{abc}\,\Phi^b I^c = 0. \tag{4.109}$$

By replacing F^a_{ij}, $(\mathbf{D}_i\Phi)^a$, A^a_i and Φ^a from eqs(4.15) and (4.16) into eqs(4.106)–(4.109), the equations in a convenient form are[7]

$$\begin{aligned} m\dot{\vec{v}} &= e\,\vec{v}\times\vec{B}^a I^a - e\,p\,\vec{B}^a I^a \\ &= \frac{\vec{r}\cdot\vec{I}}{r^4}\left(K^2 - rK' - 1\right)\left[(\vec{v}\times\vec{r}) - p\,\vec{r}\right] + \frac{K'}{r}\left[(\vec{v}\times\vec{I}) - p\vec{I}\right], &(4.110)\\ m\,p &= -\frac{H}{r^2}\left(\vec{r}\cdot\vec{I}\right) + h, &(4.111)\\ \dot{\vec{I}} &= \frac{1-K}{r^2}(\vec{r}\times\vec{v})\times\vec{I} - \frac{pH}{r^2}\left(\vec{r}\times\vec{I}\right), &(4.112)\end{aligned}$$

where the magnetic field is

$$B^a_i = \frac{1}{2}\epsilon^{ijk}F^a_{jk} = -(\mathbf{D}_i\Phi)^a = \frac{1}{er^2}\left\{\frac{x^a x^i}{r^2}\left(K^2 - rK' - 1\right) + rK'\,\delta^{ai}\right\}. \tag{4.113}$$

As before, the energy and the total angular momentum are constants of motion. Using the general equation in five dimensions (4.87), multiplying both sides by

[7] For conventions on notation see footnote 3 page 8.

$dx^A/d\tau$, one can simply find

$$\frac{d}{d\tau}\left[\frac{1}{2}m\left(\frac{dx^A}{d\tau}\right)^2\right]=0$$

which implies

$$E \equiv \frac{1}{2}mv^2+\frac{1}{2}mp^2 = \text{ constant.} \tag{4.114}$$

Here, $\vec{v}$ is velocity of the particle ($v=|\vec{v}|$), and p is defined in eq(4.108). The validity of relation (4.114) can be checked directly by using the equations of motion (4.107), (4.110) and (4.112) to show

$$m\dot{\vec{v}}\cdot\vec{v}+m\dot{p}p=0.$$

Another constant is $\vec{J}$, the total angular momentum of particle and fields defined in eq(4.27)

$$\vec{J}=m(\vec{r}\times\vec{v})+K\vec{I}+\frac{(1-K)(\vec{I}\cdot\vec{r})}{r^2}\vec{r}. \tag{4.115}$$

The same as before replacing from eqs(4.110) and (4.112) in eq(4.28), after some algebra, all terms cancel each other to imply $\dot{\vec{J}}=0$ and so

$$\vec{J}= \text{ constant.} \tag{4.116}$$

As we are working in the classical framework, one may think about some hidden conserved quantity such as the Lenz vector in the Kepler problem (inverse square law of force). However none has been found.

In the first view, we find the equations (4.110) and (4.112) are too complicated to be solved. Therefore we consider the asymptotic behaviour of the equations at large distances. Because of the different behaviour of K and K' with H at large distances we may consider two cases. At large distances $K(r)$ and $K'(r)$ vanish exponentially, and $\vec{B}^a=-(x^a/er^4)\vec{r}$. If r is not too much bigger than 1, $H(r)\to aer-1$, and if $r>>1$, then 1 might be ignored and so $H=aer$. So at large distances (but not too far) eqs(4.110) and (4.111) become

$$m\dot{\vec{v}} = \frac{\alpha}{r^3}\left[\vec{r}\times\vec{v}+p\vec{r}\right], \tag{4.117}$$

$$p = \left(-e\,a\,\alpha+h+\frac{\alpha}{r}\right)/m. \tag{4.118}$$

where as before we have defined the charge isovector as

$$\vec{I} = \alpha\hat{r} + \beta\hat{w} + \gamma\hat{z}, \tag{4.119}$$

in an orthogonal moving frame along the particle trajectory:

$$\vec{r}, \quad \vec{w} = \vec{r}\times\vec{v}\ \left(\vec{v} = \frac{d\vec{r}}{dt}\right), \quad \vec{z} = \vec{r}\times\vec{w}, \tag{4.120}$$

where hatted letters denote the unit vectors along each axis. Evidently the coefficients α, β, and γ satisfy

$$I \equiv (I^a I^a)^{1/2} = (\alpha^2+\beta^2+\gamma^2)^{1/2} = \text{ constant.} \tag{4.121}$$

From eq(4.112) after a little algebra we find

$$\begin{aligned}
\dot{\alpha} &= -\frac{K|\vec{w}|}{r^2}\gamma, && (4.122)\\
\dot{\beta} &= \left\{r\frac{\dot{\vec{v}}\cdot\vec{w}}{|\vec{w}|^2} + \frac{pH}{r}\right\}\gamma, \qquad p = \frac{1}{m}\left(-\frac{\alpha H}{r}+h\right), && (4.123)\\
\dot{\gamma} &= \left\{-r\frac{\dot{\vec{v}}\cdot\vec{w}}{|\vec{w}|^2} - \frac{pH}{r}\right\}\beta + \frac{K|\vec{w}|}{r^2}\alpha. && (4.124)
\end{aligned}$$

These equations, by using the asymptotic behaviours of K and H at large distances, and using the asymptotic equation (4.117), become

$$\begin{aligned}
\dot{\alpha} &= 0, && (4.125)\\
\dot{\beta} &= \left[\frac{\alpha}{mr^2} + p\frac{aer-1}{r}\right]\gamma, && (4.126)\\
\dot{\gamma} &= -\left[\frac{\alpha}{mr^2} + p\frac{aer-1}{r}\right]\beta, && (4.127)
\end{aligned}$$

where p has the asymptotic value in eq(4.118). From eq(4.125), obviously $\alpha = \alpha_0$ is a constant.

For this asymptotic case, using eqs(4.117) and (4.125), the length of the angular momentum $l = |m(\vec{r}\times\vec{v})|$, the total angular momentum vector $\vec{j} = \vec{l} + \alpha_0\hat{r}$, in addition to $\vec{j}\cdot\hat{r} = \alpha_0$ are constants of motion (in agreement with eq(4.115)). From eqs(4.117) and (4.118) one may simply find $m(v^2+p^2)/2$ is also a constant of motion (see eq(4.114)). If $j = |\vec{j}\,| = 0$ so $l = \alpha_0 = 0$, then particle moves uniformly in a radial direction (or stays at rest). For $j \neq 0$, the motion will take place on a cone with the axis $\vec{j}$ and the half-angle $\cos^{-1}(\alpha_0/j)$.

We may compare the forces on the right-hand side of the asymptotic equation (4.117) with forces due to certain point objects sitting at the origin. From the components of the isovector charge $\vec{I}$, only α (that is a constant) has appeared in the equation of motion, eq(4.117). Therefore we may assume only $e\alpha$ portion of the particle's charge, eI, participates in the motion at large distances

$$m\dot{\vec{v}} = -\frac{\alpha}{r^3}\left(\vec{v}\times\vec{r}\right) + \frac{\alpha(h - e\,a\,\alpha)}{mr^3}\,\vec{r} + \frac{\alpha^2}{mr^4}\,\vec{r}. \tag{4.128}$$

The first term in the right-hand side of eq(4.128) is a force due to a point magnetic monopole. Comparing to eq(4.29) the second and the third terms are forces exerted from the scalar Higgs field on the coloured test particle. The second term corresponds to the force due to an electric point charge on the test particle and the third term has the characteristic of a spherical charge distribution of total charge zero. At close distances the force from the fields on the particle are so complicated, and at far distances the dominant terms are those in the right-hand side of eq(4.128). For $r >> 1$ the dominant forces are the first and second terms in the right-hand side of eq(4.128).

For too large distances, *ie.* $r >> 1$, where the Higgs field asymptotically becomes

$$\vec{\Phi} = a\hat{r}, \tag{4.129}$$

we may neglect 1 in the term $aer - 1$, and rewrite the equations of motion (4.117), (4.118) and (4.125)-(4.127)

$$\begin{aligned} m\dot{\vec{v}} &= \frac{\alpha}{r^3}\left[\vec{r}\times\vec{v} + p\,\vec{r}\right], &(4.130)\\ p &= \frac{(-e\,a\,\alpha + h)}{m}, &(4.131)\end{aligned}$$

$$\begin{aligned} \dot{\alpha} &= 0, &(4.132)\\ \dot{\beta} &= eap\,\gamma, &(4.133)\\ \dot{\gamma} &= -eap\,\beta. &(4.134)\end{aligned}$$

which now p is a constant and $\dot{\beta}$ and $\dot{\gamma}$ have a simpler forms. In this approximation (too large distances) l, α and $\vec{j}$ (so $\hat{j} \cdot \hat{r}$) are constants of motion (the same as in the large distances approximation). Now p is a constant, so from eq(4.130) $mv^2/2 + \alpha p/r =$ constant (that can be obtained by expanding $E = m(v^2 + p^2)/2 =$ constant at large distances using eq(4.118) and then dropping the order of $1/r^2$ and redefining p as in eq(4.131)).

The equations (4.133) and (4.134) provide a precession motion for the charge isovector $\vec{I}$, around the radial direction of the particle in the isospace

$$\beta(t) = \sqrt{I^2 - \alpha^2}\,\sin(aep\,t + \Omega_0), \quad \gamma(t) = \sqrt{I^2 - \alpha^2}\,\cos(aep\,t + \Omega_0), \qquad (4.135)$$

where Ω_0 is a constant. The charge isovector moves around a circle of radius $(I^2 - \alpha^2)^{1/2}$ with a constant angular frequency $\omega = aep$. Therefore p measures how fast the charge isovector moves around in the isospace, when the particle travels its path in the real space. So p which was defined as the velocity in the fifth-spatial direction (eq(4.108)) appears as the velocity of the charge isovector in its precession around the particle's radial direction (note, ae has the dimension of $(\text{length})^{-1}$).

4.4.2 Solutions of the Equations of Motion

In section (4.2) we observed planar motions and bounded and we presented numerical works in two and three dimensions. There, the only force on the particle was the force from the monopole and a force from the Higgs field on the particle was not considered. In the first subsection of this section we explained the equations of motion containing the Higgs and the particle interaction as well as the monopole force, and in this subsection we search for the solutions.

Planar Motions

It is interesting to know if the planar motions occur here exactly in the same context as before. As we said before, in a planar motion $\vec{r} \times \vec{v}$ is always normal to the plane of motion, so in the equations of motion the coefficient of $\vec{w} = \vec{r} \times \vec{v}$ was set to zero and we found some consistent solutions. Using the moving frame (4.120) the component of eq(4.110) in $\hat{w}$-direction is

$$-\frac{\alpha(K^2 - rK' - 1)}{r^3}|\vec{w}| - \frac{K'}{r^2}(\alpha|\vec{w}| + \gamma(\vec{r}\cdot\vec{v})) - \frac{K'}{r}p\,\beta. \tag{4.136}$$

For a planar motion this coefficient must identically be equal to zero, so

$$(K^2 - 1)\,|\vec{w}|\,\alpha + r^2K'\,p\,\beta + rK'(\vec{r}\cdot\vec{v})\,\gamma = 0. \tag{4.137}$$

Under these considerations the equations of motions (4.110) and (4.122)–(4.124) become

$$\begin{aligned} m\dot{\vec{v}} &= \left[-\frac{K^2 - rK' - 1}{r^2}p\,\alpha + \frac{K'}{r^2}(|\vec{w}|\,\beta - r\,p\,\alpha)\right]\hat{r} \\ &\quad + \frac{K'}{r^2}[(\vec{r}\cdot\vec{v})\beta - r\,p\,\gamma]\hat{z}, && (4.138)\\ \dot{\alpha} &= -\frac{K\,w}{r^2}\gamma, && (4.139)\\ \dot{\beta} &= \frac{p\,H}{r}\gamma, && (4.140)\\ \dot{\gamma} &= \frac{K\,w}{r^2}\alpha - \frac{p\,H}{r}\beta, && (4.141)\end{aligned}$$

and p is unchanged

$$mp = -\frac{H}{r}\alpha + h. \tag{4.142}$$

Let us first examine the above equations at large distances. At large distances where K and K' vanish, eq(4.137) necessitate $\alpha = 0$. Replacing this result in eq(4.138) shows the particle move on a straight line at large distances (see also eq(4.117)). Also at large distances p is a constant and β and γ have a precessional motion (if $p \neq 0$), which are compatible with the asymptotic behaviour of equations we studied before.

In fact eq(4.137) is an extra equation and might not be consistent with the equations of motion (4.138)–(4.141) in general. But it might be consistent with equations of motion under some circumstances. Finding the conditions where this extra equation might be consistent with the others does not seem to be easy. Looking at eq(4.137), one may choose $\alpha = \beta = \gamma = 0$ which is of course a contradiction (while $\vec{I}$ is a non-zero vector). One acceptable possibility is $\alpha = \gamma = p = 0$ which causes eq(4.137) to vanish identically. From the equation (4.142) the condition $p = 0$ is equivalent to $h = 0$. So

$$\alpha = \gamma = h = 0, \tag{4.143}$$

are conditions for planar motion subject to the validity of equations of motion. Replacing from eq(4.143) in eqs(4.138)–(4.141) we obtain exactly the planar equations of section (4.2). So the planar motion and indeed the bounded orbits are allowed in this regime as well. Because the equations of planar motion are the same as before we skip their solutions in this section. The stability of planar motion must be studied independently. In the previous case we stated the planar motions are not stable, but in this case the Higgs field might play a rôle to keep the particle close to the plane and does not let it scatter to infinity. I have not checked this problem.

A proper question is, under what circumstances the force from the Higgs field on the particle fails, *ie.* the generalised equations of motion we found in this section shrink to the equations of motion we found for the particle in section (4.2) (which were feeling only a force from the monopole and not from the Higgs field). In fact it is not possible to ignore the Higgs and the particle interaction in general. It is clear that equations (4.17) and (4.18) are obtained from the equations of motion (4.110) and (4.112) if p set to zero which exerts an additional constraint. Setting $p = 0$ occasions $\alpha = hr/H$, then from eq(4.22) we find γ and from eq(4.24) β. So we may replace α, β and γ in eqs(4.23) and (4.25) to find two parallel equations which are too complicated (and I think they are not consistent in general). A possible case (may be the only one) is the mentioned planar motion, which means in the specified planar motions the Higgs interaction has no contribution.

Radial Motions

A radial motion is possible if initially the charge isovector and the particle velocity are radial. In this case from eq(4.112) the charge isovector remains constant, $\alpha = I$ and $\beta = \gamma = 0$ and from eq(4.110)

$$m\ddot{r} = \frac{I}{r^2}\left(K^2 - 1\right)p, \tag{4.144}$$

where $p = -IH/r + h$. In the previous case (the monopole interaction only), the right-hand side of eq(4.144) was vanishing ($p = 0$) and the particle had a uniform radial motion, and could pass through the origin. The equation (4.144) shows a different situation.

Assume the particle is moving along a radial direction say z-axis. From eq(4.114) we may write

$$E = \frac{1}{2}m\dot{r}^2 + V(r), \tag{4.145}$$

where

$$V(r) = \frac{1}{2}mp^2 = \frac{1}{2}m\left(\frac{-IH(r)}{r} + h\right)^2, \tag{4.146}$$

is the one-dimensional potential, and r (is the variable along the z-axis and) takes both negative and positive values. The time-derivative of eq(4.145) leads to eq(4.144). Regardless of mass, a, e and I, the potential $V(r)$ depends on the constant h. The figure (4.11) shows the different shapes of $V(r)$ with respect to the different values of h. (To see these results one may equate $V'(r)$ to zero and find the roots, that are

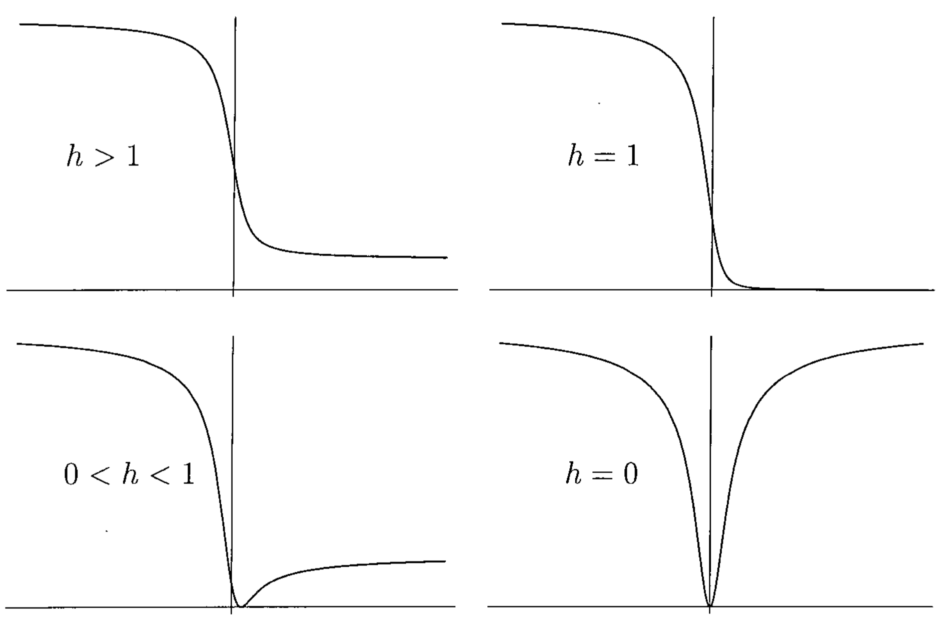

Figure 4.11: One dimensional potential $V(r)$ (vertical axis) versus r (horizontal axis). For a given h, the mirror image of $V(r)$ with respect to the vertical axis gives $V(r)$ for $-h$.

indeed the roots of $p = 0$, and follow the instructions after eq(4.57).) For a negative value of h, $V(r)$ is the mirror image of the the potential for $-h$, with respect to the vertical axis.

From Fig(4.11), if $|h| \geq 1$, the particle may be repelled back (before reaching the origin, just at the origin or after passing the origin), or just passes through the origin depending on the energy E. For $0 < |h| < 1$, in addition to the mentioned possibilities for $|h| \leq 1$, bounded motions are also possible (in the $h = 0$ case, the particle just passes or is bound).

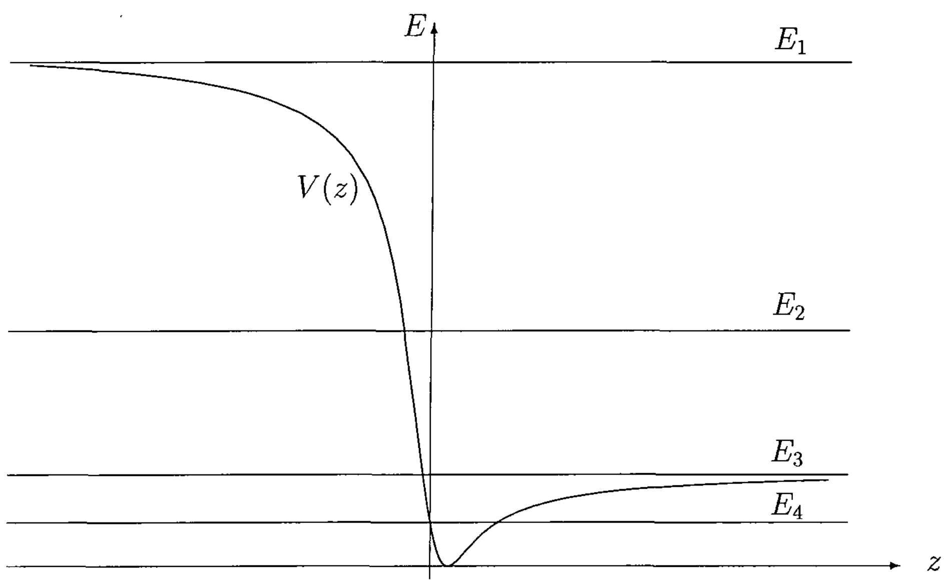

Figure 4.12: Different possibilities of motion for $0 < h < 1$.

Referring to Fig(4.12), For $E \geq E_1$ the particle passes through the origin and travels to infinity. For $E_1 < E \leq E_3$, such as E_2, the particle is repelled back (even before reaching the origin or after passing through the origin) in its trajectory and travels to infinity. For $0 < E < E_3$ the orbit is bounded and the particle oscillates along the z-axis. For $h = 0$ the oscillation is symmetric with respect to the origin (the origin is the equilibrium point). But in the other cases the origin is not the equilibrium point (center of the oscillatory motion) and the amplitude of the motion on the two sides of the origin are not equal (even not equal either side of the equilibrium point). And, on top of all, for the energies less than E_4, the particle oscillates only on one side of the origin. At the equilibrium point $p = 0$, the position of equilibrium (with respect to the origin) depends on the constant h. For larger $|h|$, the location of the equilibrium point is more distant.

The rôle of h in the radial motion is similar to the rôle of j in the planar motion. As the orientation of the charge isovector and the Higgs field are fixed in the radial motion, the constant h depends only on the initial starting point, r_0, and the initial value of p, p_0. The initial value of p, p_0, is a free parameter in the radial motion and it must be determined by the overall theory of motion. From eq(4.145) one may write an integral equation for t and r, similar to eq(4.60).

$$t = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{2E/m - [-IH(r)/r + h]^2}} \tag{4.147}$$

Three-Dimensional Bounded Orbits

We studied the planar and radial motions in the last pages. In both cases, bounded orbits were allowed. Suppose a particle is moving in a bounded orbit in a plane, say xy-plane, so that the charge isovector is normal to the plane of motion along the z-axis. Regardless of the motion in the plane, suppose the particle has also a motion in the z-direction such that the particle can oscillate in the z-direction. This is a motivation to believe, if we mix the initial condition of the both motions, we may get a bounded motion in three dimensions. Of course, we do not say that the result motion is superposition of the two mentioned motions. It is clear that the equations governing the motion (*ie.* eqs(4.110) and (4.112)) are not linear, therefore the superposition of the solutions is not necessarily a solution. The above motivation is correct only for the starting point, and for the other instants we must follow the equations of motion. Let us examine an example by numerical solution. Under these circumstances, the planar motion condition requires $h = 0$. For example, with $h = 0$, if the particle is launched into the fields with the initial values $[\vec{r}_0, \vec{v}_0, \vec{I}_0] = [[1,0,0],[0.1,0,0],[0,0,1]]$, the result is a bounded planar motion in the xy-plane (see plot bottom-left in Fig(4.7)). If the particle is launched in the field with the initial values $[[0,0,0],[0,0,0.1],[0,0,1]]$, the result is a symmetric bounded oscillation along the z-axis around the origin. So we expect, if the particle is launched with the initial values $[[1,0,0],[0.1,0,0.1],[0,0,1]]$, the result being a bounded orbit in three dimensions. By chance it is right. Using the three dimensional equations of motion (4.110) and (4.112), a numerical analysis as we have explained in the appendix (see parts **P0** and **P3**) confirm the claim as it is plotted in Fig(4.13). We have tested

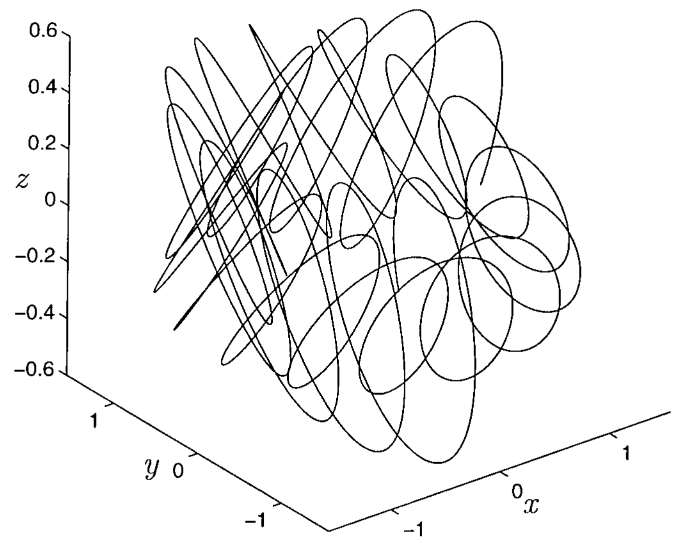

Figure 4.13: Bounded orbit in three dimensions.

the motion for a remarkable amount of time (10000 units of time), and numerically result is obtained.

One result which we may get quickly from the above discussion is the stability of the planar motions. In fact it is sensible to understand the stability of the planar motions in the new scenario (compare with the results of section (4.2)). When small perturbations normal to the plane of motion disturb the motion, there is a vertical force to keep the motion oscillating close to the plane so that the pattern of planar orbit stays unchanged. So involving the Higgs field interaction makes a significant difference. In section (4.2) the Higgs field interaction was ignored, so the planar motions were not stable (in the sense of vertical perturbations). (Note in Fig(4.13) the changes in the planar motion is not small, but still the projection of the three-dimensional orbit in the xy-plane follows the same pattern as the pure planar motion has (see plot bottom-left in Fig(4.7)).) We have checked the stability of the planar motions by considering small perturbations in any of the motion's parameters, numerically by using the general equations of motion (see the appendix, part **P0** and **P3** for programmes.)

In general the closed orbits are unknown in the non-planar motions. For example if we start the motion from the initial values $[[1,0,0],[0,0.2663673990,0],[0,0,1]]$ and $h = 0$, we obtain a closed circular motion, and if we start the motion from $[[0,0,0],[0,0,0.1],[0,0,1]]$ we obtain a symmetric radial oscillation. The motion with initial values $[[1,0,0],[0,0.2663673990,0.1],[0,0,1]]$ forms a bounded orbit such that the intersection of the orbits with xy-axis is bounded between two circles, and the z-direction has an oscillatory motion along the z-axis with the domain changes between a minimum and a maximum periodically (see Fig(4.14). If the equations of motions

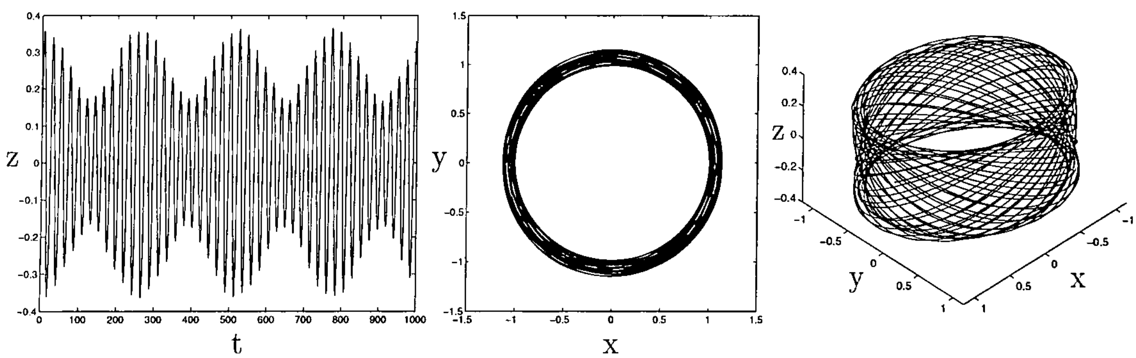

Figure 4.14: The left plot shows the variation of z-direction versus time, the middle one shows the intersection of the motion in the xy-plane, and the right plot shows the three-dimensional orbit.

were linear we might say, the superposition of two closed orbits are closed if the ratio of periods of two motions being a rational number. But in our case the equations of motion are not linear, so we are not able to use this theorem. We might play with the parameters to gain a closed orbit in three dimensions. Studying the bounded and closed orbits needs an analytic description of the equations of motion, which is not available here.

With $h = 0$ we may choose any combination of the initial values (not only a combination of the planar and radial motions initial values) and test the equations of motion by numerical computations (see the appendix for programmes). The bounded orbits are observed for different kinds of combinations of the initial values. Of course for many initial values we cannot expect a bounded orbit. For non-zero h's the combination of initial values is too sensitive and for most of them the orbit is unbounded. But still for some initial values bounded orbits are observed. An example is $[[1,0,0],[0.1,0,0],[1,0,1]]$ with $h = 0.5$. Now the initial values $[[1,0,0],[0.1,0,0],[0,0,1]]$ with $h = 0.5$ is neither a planar motion nor a non-planar bounded motion, but when it mixed with an oscillatory motion in the x-direction, the resulting three-dimensional motion will be bounded.

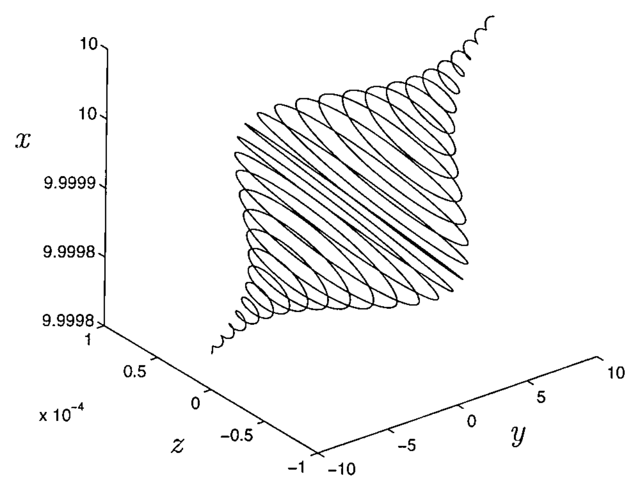

Figure 4.15: Three-dimensional motion at large distance.

At far distances particle moves on a surface of a cone (see page 104). The figure (4.15) shows the orbit for an initial values $[[10,10,0],[0,-0.1,0],[0,0,1]]$ with $h = 1$ in 200 units of time. The variations on x and z are small and of the order 10^{-4} (the unusual ticks in the vertical axis is badly managed by 'MATLAB', and means the variations in this axis is of order 10^{-4}). The particle which has started the motion with an initial velocity in the negative y-direction, moves along the y-direction almost uniformly.

4.4.3 The Force Law

In this subsection we consider further the force on a non-abelian particle in a BPS monopole field configuration ($\mathbf{B}_i = \pm\mathbf{D}_i\Phi$, $\mathbf{A}_0 = 0$, $V(\Phi) = 0$) which we studied in the previous subsection. We shall show the force has the form of a 'generalised'

Lorentz force. In the next section we explore a generalised force law for the complete Yang–Mills–Higgs fields.

Looking at the first equality in eq(4.110), if we suppose eI^a as electric charge of the particle (in non-abelian sense), then $eI^a \vec{v} \times \vec{B}^a$ is a magnetic force due to a magnetic field $\vec{B}^a$ on the particle, and $eI^a p\vec{B}^a$ may be interpreted as an electric force due to an electric field $p\,\vec{B}^a$ on the particle (regardless of the notation $\vec{B}^a$ that is used for magnetic field).

We may explain the above concept of generalisation and make it clearer. In the usual electrodynamics, the Coulomb law states that the force of an electric field $\vec{E}$ on an electric charged particle with the charge q is $q\vec{E}$, and the Lorentz law states that the force of a magnetic field $\vec{B}$ on an electric charged particle with the charge q is $q\vec{v} \times \vec{B}$, where $\vec{v}$ is the velocity of particle. Now we change the mode to the non-abelian fields and particles. Assume a field B_i^a in the space and a non-abelian test particle with charge isovector Y^a in this field. The particle feels only the component of the magnetic field that is projected along its charge isovector. Let us choose a unit vector $\hat{n}$ along the charge isovector (remember in the isospace)

$$\vec{Y} \doteq q\hat{n}, \tag{4.148}$$

where $q = |\vec{Y}| =$ constant is the charge of the particle. Now the effective magnetic field the particle feels is

$$\mathcal{H}_i = \vec{B}_i \cdot \hat{n}, \tag{4.149}$$

or equivalently $\vec{\mathcal{H}} = \vec{B}^a \hat{n}^a$. Now we can write down the equation of motion of the particle, eq(4.110), in a familiar form (for positive magnetic charge):

$$m\dot{\vec{v}} = q\,\vec{v} \times \vec{\mathcal{H}} + p\,q\,\vec{\mathcal{H}}, \tag{4.150}$$

where here $q = eI$.

The equation (4.150) ia a generalisation of the Lorentz force. Comparing with the usual electrodynamics two major differences show themselves in eq(4.150). The first difference is: In eq(4.150), rather than a term analogous to the usual Lorentz force (first term in the right-hand side), there is another term that is similar to the

Coulomb force in the usual electrodynamics. We should remember that this new term is originally different from the Coulomb force. The Coulomb force is regarded as the zeroth component of the Yang-Mills fields, which in our discussions has been ignored (remember we are working in stationary fields with $\mathbf{A}_0 = 0$), but here the origin of the Coulomb-like force is the Higgs field. The coefficient p in the front of the Coulomb-like force contains some information about the interaction of the Higgs field and the charge isovector. We will show how this generalised equation reduces to the normal Lorentz force when the field and particle can be regarded as non-abelian field and particle. The second difference is: In the usual Lorentz force, the force vanishes if and only if the non-vanishing magnetic field and the particle's velocity become parallel. But in the non-abelian case it might happen that neither of the particle's velocity or the non-abelian magnetic field vanish, nor the particle's velocity and the magnetic field (in any sense, either in the non-abelian form B_i^a or in the Higgs gauge-invariant form $\mathcal{B}_i$ (see eqs(2.19) and (2.20))) are parallel, but the force vanishes. This happens when the projection of the magnetic field along the charge isovector vanishes *ie.* $\vec{\mathcal{H}} = 0$.

We can define a usual (say abelian) particle in general as a non-abelian particle whose charge isovector is fixed in the isospace *ie.*

$$Y^a = q\delta^{a3}. \tag{4.151}$$

Now when the charge isovector $\vec{Y}$ takes a radial direction in the isospace, we can transform its direction to the 3-direction by a proper gauge transformation[8]. This happens for both the charge isovector and the Higgs field simultaneously when both the isovectors take radial direction in a part of the space. A good example is the regions too far from the core of the fields, where we found the equations of motion asymptotically.

[8]At each point (r, θ, ϕ) in the real space, the direction $\hat{r}$ in the isospace rotates to the 3-direction with the gauge transformation

$$U^{-1} = \cos\frac{\theta}{2} + i\vec{z}\cdot\vec{\sigma}\sin\frac{\theta}{2}, \qquad z^i = \frac{\epsilon^{ij3}\hat{r}^j}{\sin\theta},$$

where σ^a are Pauli sigma matrices.

Suppose $\vec{Y} = q\hat{r}$ $(q = eI)$ and $\vec{\Phi} = a\hat{r}$ in the asymptotic case. So charge isovector has no components in the directions normal to the radial direction, and therefore $\beta = \gamma = 0$. In this case our definitions of $\mathcal{B}$ and $\mathcal{H}$ overlap, and the equation of motion (4.150) transforms to eq(4.130) (for self-dual case):

$$m\dot{\vec{v}} = \frac{I}{r^3}\left[(\vec{v}\times\vec{r}) + p\,\vec{r}\right], \qquad p = (-eaI + h)/m. \tag{4.152}$$

Both of the isovectors can be rotated by a gauge transformation to lie in the 3-direction by a gauge transformation, where we expect to have the usual abelian electrodynamics laws. By the rotation, the particle will be the usual particle that defined in eq(4.151), and the electromagnetic field becomes the usual one (see explanations after formula (2.21) in page 11). In addition, in the usual space that we are talking about, there should be no trace of the free parameter p which is related to the extra dimension. So we may have $p = 0$, and set the constant h

$$h = eaI. \tag{4.153}$$

Now eq(4.152) is reverted to the proper usual Lorentz force. The differences between the generalised force and the usual Lorentz force we enumerated before automatically disappear, because the factor p vanishes and the extra term in the generalised force is gone. Also the Higgs field and the charge isovector have become parallel, therefore the second difference we mentioned can no more happen.

In general we need 13 initial values ($x^A(t_0)$, $\dot{x}^A(t_0)$ and $I^a(t_0)$) to determine solutions of the equations of motion. In the non-relativistic framework these are 11 values which 9 of them are normal ones (position, velocity and charge isovector initial values in the three-spatial dimension) that is indeed needed to solve the equations (4.106) and (4.109). The other two initial values are connected to the internal fifth dimension x^5. Because neither the fields nor the equations of motions depend on the internal direction, therefore the initial starting point in the x^5-direction is not important. But dx^5/dt (the time-variation of internal direction) which was called p is very important. The constant h appeared as a constant of integration and includes the information about the initial orientation of charge isovector and the Higgs field, and the initial value of $\dot{x}^5$ *ie.* p_0. To obtain the usual electrodynamics law from the generalised ones, we need the condition (4.153) to be valid.

4.5 General Force Law for a Coloured Particle in the Yang–Mills–Higgs Fields

Now we can consider a general force law by considering the general Yang–Mills–Higgs fields in four dimensions. In fact the general equation in a Lorentz invariant form is eq(4.90) when we replace $dx^5/d\tau \equiv p$ from eq(4.105). This general equation has a companion that describes the evolution of (say) the generalised charge (charge isovector). The generalised force of a Yang–Mills–Higgs field on a coloured particle (in a non-relativistic framework) is offered as:

$$m\dot{\vec{v}} = q\vec{\mathcal{E}} + q\,\vec{v} \times \vec{\mathcal{B}} + p\,q\,\vec{\mathcal{G}}, \tag{4.154}$$

where

$$\begin{aligned}
q &= |\vec{Y}|, && (4.155)\\
p &= (-\vec{\Phi}\cdot\vec{Y} + h)/m, && (4.156)\\
\mathcal{E}^i &= \frac{1}{q}\,F^{a0i}\,Y^a = \frac{1}{q}\,\vec{E}^i\cdot\vec{Y}, && (4.157)\\
\mathcal{B}^i &= \frac{1}{2q}\,\epsilon^{ijk}\,F^{ajk}\,Y^a = \frac{1}{q}\,\vec{B}^i\cdot\vec{Y}, && (4.158)\\
\mathcal{G}^i &= \frac{1}{q}\,(\mathbf{D}^i\mathbf{\Phi})^a\,Y^a = \frac{1}{q}\,\vec{G}^i\cdot\vec{Y}, && (4.159)
\end{aligned}$$

which $\vec{Y}$ is the particle's charge isovector, $\vec{v}$ is the particle's velocity, F^a_{ij} are the Yang–Mills field tensors, Φ^a is the Higgs scalar field and h is a free parameter. We have defined

$$\mathbf{G}_i = \mathbf{D}_i\mathbf{\Phi} \tag{4.160}$$

where $\mathbf{D}_i$ is the covariant derivative. The indices $i, j = 1, 2, 3$ and $a = 1, \ldots, N$, where N is the dimension of the gauge group. Therefore the vectors like $\vec{\Phi}$, $\vec{Y}$ and $\vec{E}^i$ are vectors in N-dimensional isospace, while vectors like $\vec{\mathcal{E}}$ are vectors in three-dimensional real space. The fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the effective electric and magnetic fields that particle feels from the Yang–Mills fields, and $\vec{\mathcal{G}}$ is another effective field that particle feels from the scalar Higgs field. In the 't Hooft–Polyakov monopole field configuration, $\vec{\mathcal{G}}$ vanishes in the Higgs vacuum (outside of the monopole radius), while in the BPS monopole $\vec{\mathcal{G}}$ does not vanish in the finite distances.

To find out a similarity between $\vec{\mathcal{E}}$ and $\vec{\mathcal{G}}$ we suppose a time-invariant Yang–Mills–Higgs field. With this condition we may write

$$\begin{aligned}\mathbf{E}_i &= \mathbf{F}_{0i} = \partial_0 \mathbf{A}_i - \partial_i \mathbf{A}_0 + [\mathbf{A}_0, \mathbf{A}_i] \\ &= -\partial_i \mathbf{A}_0 - [\mathbf{A}_i, \mathbf{A}_0] \\ &= -\mathbf{D}_i \mathbf{A}_0. \end{aligned} \tag{4.161}$$

In the non-relativistic framework, we may decide to call $\mathbf{A}_0$ as a scalar field $\boldsymbol{\Psi}$,

$$\mathbf{E}_i = -\mathbf{D}_i \boldsymbol{\Psi}, \tag{4.162}$$

in analogy with $\vec{E} = -\vec{\nabla}\phi$ in the usual electrostatics. The equations (4.160) and (4.162) show the similar definitions for two components of the Yang–Mills–Higgs field. But the rôle of these two fields are not the same in the motion of a particle when exposed to the fields. This is because the rôle of two fields in spite of similarity in the equations, are not the same in origin. The Yang–Mills field is assumed to be generated by matter while the Higgs field that is a scalar field is attributed to the vacuum. The connection between the particle and the scalar Yang–Mills field is the particle's charge q. But in the case of the Higgs field, this connection is pq. In absence of $\mathbf{A}_0$, and for the case of $\vec{v} = 0$, there still might be a force from the Higgs field on the particle *ie.* $pq\vec{G}$. For the 't Hooft-Polyakov monopole at large distances this force is negligible. But, for the BPS monopole at large distances, this force is zero only if the Higgs field and the charge isovectors are parallel.

Chapter 5

Summary

ز هر یك نقطه دوری گشته دایر
همو مرکز هم او در دور سایر
اگر یك ذره را بر گیری از جای
خِلَل یابد همه عالم سرا پای
« شبستری »

The centre of every circle itself becomes a circle -
see as it changes, now centre, now circle;
Were you to remove a single atom from its place,
the whole cosmos would collapse and fall into ruin.

"Shabestari"

In chapter 2 we explained two topological objects of the Yang–Mills fields, monopoles and instantons. Each of these solutions carries a conserved current which is not obtainable from symmetries of the fields Lagrangian. The charge associated to these solutions are classically quantised. Monopoles are solutions of the four-dimensional Minkowskian Yang–Mills fields. The importance of the Higgs field in monopole solutions is essential. Instantons are solutions of the four-dimensional Euclidean Yang–Mills fields. In monopole solutions, the finiteness of the fields' energy and in the instantons, finiteness of the action are desired. Both solutions are consequences of the boundary conditions and satisfy a lower bound. The lower bound for the monopole case gives the Bogomol'nyi equations from which we get the BPS monopole, and in the instanton case gives the (anti)self-dual equations. Monopoles and instantons have been studied by many authors and generalised to arbitrary Lie groups.

String theory is a well-known non-abelian theory. The heterotic superstring theory has a very large symmetry group ($E_8 \times E_8$ or $SO(32)$) and is originally formulated in a ten-dimensional space-time. We studied the low-energy heterotic superstring theory in ten-dimensions that is compactified in a six-torus, and become a four-dimensional theory.

A supersymmetric solution to the ten-dimensional theory is known. The five-brane solution is a supersymmetric solution of the fields. In this solution an (anti)self-dual equation plays a central rôle, from which the five-brane solution is known as instanton solution. We used the known $SU(2)$ instanton solution in the five-brane ansatz and solved the equations for the dilaton, and introduced two charges associated to the five-branes, the instanton number and the axion charge — that former is associated to the Yang–Mills fields and later to the anti-symmetric tensor field, H.

The heterotic superstring gauge group, $E_8 \times E_8$, is a semi-simple exceptional group. A general instanton solution for this large group is not provided. The well-known ADHM construction gives a rule to find the general instanton solution for classical groups. Although we are not able to provide a general instanton solution to $E_8 \times E_8$, we are able to study the instanton solution in the classical subgroups of $E_8 \times E_8$. The solution we mentioned in the last paragraph was in fact an $SU(2)$ instanton solution. There we supposed the instanton to lie in one of the $SU(2)$ subgroups of $E_8 \times E_8$, *ie.* in the minimal one.

Any larger group may contain some non-conjugate versions of a subgroup. The instanton solution of a group when is embedded in $E_8 \times E_8$, may give different results. This means if we calculate the instanton number for different embedding of an instanton solution of a subgroup, we may receive different answers. For this reason we studied the embedding of subgroups, using the Dynkin index of embedding. We are able (at least theoretically) to use ADHM construction to formulate a general solution for the minimal embedding of each classical subgroup of $E_8 \times E_8$ and then, using the index of embedding, generalise the solution for any arbitrary embedding of the subgroup.

Any subgroup of $E_8 \times E_8$ has an embedding of index 1, which we call it minimal embedding. Based on our discussions in chapter 3, for any minimal subgroup G of $E_8 \times E_8$ we may choose a collection of elements of the basis of $E_8 \times E_8$ and form a basis for G. So the normalisation condition for the basis of $E_8 \times E_8$ and a minimal embedding are the same — which gives the index 1. For any subgroup we may define commutation relations of these chosen bases. Based on these commutation relations we may pick up some arbitrary collections of the elements of $E_8 \times E_8$ to make an orthonormal basis for any arbitrary embedding of subgroup G. The ratio of the normalisation constants of any arbitrary embedding and the minimal embedding is the desired index of that embedding. A general solution for the dilaton were presented.

Monopoles are solutions to the ten-dimensional low-energy string theory compactified on a six-dimensional torus. The same as instantons we presented a general solution of the dilaton for a BPS monopole for any embedding of any arbitrary subgroup of $E_8 \times E_8$.

The H-monopole that is a consequence of compactification of the ten-dimensional theory to the four dimensions, is a solution with the monopole behaviour at infinity. Using a method from the general relativity, instead of calculating any individual component of the the stress-energy tensor, we calculated the total energy (mass) of the H-monopole, and found a relation between the mass and the charge of H-monopole (in analogy to the BPS monopole).

We did not go further to use ADHMN construction to explain the generalised monopole solution of any arbitrary subgroup of $E_8 \times E_8$. We explained the spherical symmetric BPS monopole in any $SU(N)$ subgroup of $E_8 \times E_8$. We examined an $SU(3)$ solution and calculated the H-monopole and observed the BPS-charge and H-charge are always in opposite signs.

The next topic dealt with was that of a coloured particle in a monopole field. After explaining the Wong equations of motion, we explained motion of a test particle in a monopole field. We proposed the force due to a monopole on a charged particle, and enumerated the behaviour of the motion at large distances. The speed, v, and the total angular momentum of the field and particle, $\vec{J}$, were global constants of motion. At large distances particle moves on a cone and if it moves toward the origin, it will turn backward on the cone surface at a minimum distance, if particle does not come too close to the origin to violate the asymptotic behaviour. Clearly, the particle leaves the surface of the cone if it enters the areas close to the centre of the monopole.

In contrast to the point monopoles which support only the conical orbits, the planar orbits are allowed for the BPS monopole. Simply, when the particle starts its motion in a plane normal to the charge isovector, it will stay in the plane forever. We obtained the equations of motion of the particle in the plane and studied the orbits in some details. We described the orbits of a particle in the planar motion by regarding the standard one-dimensional potential model. So we may explain the orbit of a particle if we know the particle's energy and the total angular momentum j. The bounded orbits as well as unbounded orbits are allowed. We have shown for energies larger than a specified energy, the particle does not bind. Also the bounded orbits are allowed only if $0 < j < j_c$. Because of presence of a hyperbolic term in the equations, an analytic solution for the equations is not provided, instead some numerical solutions of the equations presented. Various orbits (two and three dimensional) are illustrated in the figures of section (4.2).

As an interesting result, the particle follows a direction like the right-hand laws in the usual electrodynamics. When the particle is launched in the field, in a plane normal to the particle's charge isovector (say plane of the paper), it moves on a curve counterclockwise if the charge isovector is normal to the plane of paper and upward. This is the force law in the planar motions. A result of this force law is the loops which do not circulate the origin. A circular motion is a good example of an attractive force such as gravity. When particle moves away from the origin the attractive forces appear, and the orbits show this point clearly. But when the particle moves toward the origin, a repulsive (anti-gravity) force appears and the particle moves away from the origin and perhaps create a loop outside the origin. The speed stays constant during the motion.

The stability of the planar motion in three-dimensional context was another subject we studied. Although a bounded planar motion might be stable in the plane when a small perturbation disturb the motion in the plane, but if such a perturbation disturb the motion normal to the plane, is the planar motion stays planar or quasi-planar? We showed in fact the planar orbits can not be stable in this manner. So if even a small perturbation occur normal to the planar motion, the perturbation will grow and finally the particle scatters to infinity.

When a particle is launched in a monopole field, the Higgs field is already has taken into account. But we did not consider a force from the Higgs field on the particle individually. Actually, existence of a monopole is due to both the Yang–

Mills and the Higgs fields. So in fact the Higgs field has had its contribution in the force on the particle via the monopole. The main problem is, if the Higgs field itself can interact directly with the particle. But how we can enter the force due to the Higgs field on the particle. We mentioned a work done by another author. His procedure was not a natural generalisation of the Yang-Mills forces in a particle, and contains a term associated to the mass of particle which is not constant even in a non-relativistic classical limit. The mass depends on the other dynamical components of motion. It is quite complicated to have a clear interpretation of the motion at large distances.

To solve this problem, we started by generalising the Wong equations in a five-dimensional space-time. Our motivation was the frequently repeated point: simulation of the Higgs field as the fifth component of the Yang–Mills field, to make a pure five-dimensional Yang–Mills theory. We did not use the four-dimensional classical Wong equations to write the five-dimensional ones, but we instead used the generalised five-dimensional quantum model, based on the generalised five-dimensional Dirac equation (to be studied in direction of the Klein–Gordon theories), to extract out the classical equations of motion. The fifth dimension is suggested to be a dynamical variable, while its contribution to the fields are not considered. Therefore some extra terms associated to the dx^5/dt were added to the equations of motion. The equations and variables are independent of x^5 itself.

We observed, dx^5/dt is in fact an internal variable, *ie.* this variable is only a function of isovectors, the Higgs field and charge isovectors. We had called the extra dimension, x^5, an internal dimension, which is now sensible in the same way as the space of symmetry (isospace) is called internal space. These definitions have different meanings. But at this stage we see a close relations between these concepts. The time-variation of the internal direction, x^5, is related to the field and particle internal (isospace) vectors. If we replace dx^5/dt in the equations of motion, it gives a completely four-dimensional equations, which contain terms corresponding to the Higgs interaction with the particle. The only remnant of the fifth-dimension in the equations is the free parameter h, that depends on the initial value of dx^5/dt.

The asymptotic behaviour of the equations of motion of a test particle in the BPS monopole field shows two extra terms corresponding to an electric point charge and an electric charge distribution, due to the Higgs field interaction, in addition to a point monopole interaction. At very large distances, speed of the particle in the fifth direction (internal direction), is related to the precession frequency of the charge isovector in the isospace. This result confirms the close connection of the internal direction and the isospace (internal space) we stated before.

As before a particle was launched into the fields and the orbits were studied. Planar motions can happen the same as before. The radial motion is in a big difference from the previous case. In the monopole force only, the particle may travel along a radius uniformly, but in this case (monopole + Higgs interactions) the particle no more travels uniformly and more interesting that it can oscillate along an axis even in one side on the origin. We have explained this problem with some details. Also in contrast to the previous case the planar motions are stable in this regime, and the bounded spatial orbits which are not observed in the previous case, have been observed here. Some orbits are illustrated in section (4.4).

As before, if the particle is far enough from the origin of the fields, the fields and the particle behave as the usual non-abelian fields and particle. At this stage we should set $h = 0$ to resolve the trace of the extra (fifth) dimension in the theory (or vice versa). So, asymptotically with $h = 0$ and radial charge isovector the charge and the Higgs field transfer to the usual particle and electromagnetic field by a suitable gauge transformation.

Exploring the force from the Yang–Mills–Higgs field on a coloured particle was our last subject. When a coloured particle is launched in the Yang–Mills–Higgs field, the particle feels the fields by its charge isovector as its sense of smell. So only the projection of the fields in the direction of the charge isovector comes into account. For example, if the direction of the charge isovector lies in the direction normal to the force field due to the Higgs field, the particle feels no force from the Higgs field. Therefore we may measure the components of the fields in direction of the charge isovector and write down the equations of motion independent of the internal space (isospace).

The scalar Higgs field plays a rôle like a scalar field due to an electric charge. A difference is; the coupling constant of the particle and the electric scalar field is the particle's charge (that is a constant), while in the Higgs field the coupling constant is a function of the Higgs field itself (this reminds us of the Brans–Dicke theory in the general relativity [49, page 157]). This is a generalised form of the Lorentz force, for a Yang–Mills–Higgs field in the classical limit.

Appendix

This appendix includes the programmes in the "MATLAB" programming package (version 4.2). For solving the differential equations we have used the Runge–Kutta Method (of fourth order), which is found in literature (eg. [70, page 1040]). For a set of three second-order differential equations we may define six first-order differential equations. So for three-dimensional motions such as equations of motion (4.17) and (4.18) for the Yang–Mills case (monopole only), and (4.110) and (4.112) for the Yang–Mills–Higgs case (monopole + Higgs field contribution), nine first-order differential equations are considered. For the set of two equations (4.47) in the planar motion we need only four first-order differential equations.

P0

The main programme "`comput.m`" contains the initial values and the loop procedure, and must be run in the "MATLAB Command Window"

```
clear, echo off, hold off, format compact, format long
%
dim = 9;                         % number of equations.  *
yi = [0 1 10 0 0 -.2 1 0 0];     % initial values.       **
ti = 0;                          % initial time value.
tf = 200;                        % final time value.
nsteps = 200000;                 % number of steps.
x = zeros(nsteps+1,dim);         % x is an array with "nsteps+1" rows
x(1,:) = yi;                     % and "dim" columns for storing data.
h = (tf - ti)/nsteps;            % steps' size.
t = ti + h*[0:nsteps]';
%
for i = 1:nsteps
  y = x(i,:);             k1 = diffeq(y);
  y = x(i,:) + 0.5*h*k1;  k2 = diffeq(y);
```

```
  y = x(i,:) + 0.5*h*k2;  k3 = diffeq(y);
  y = x(i,:) + h*k3;      k4 = diffeq(y);
  x(i+1,:) = x(i,:) + h*(k1 + 2*(k2 + k3) + k4)/6;
end
plot3(x(:,1),x(:,2),x(:,3));    %                       ***
```

The above procedure is written for the general three-dimensional equations. In the above programme `yi` is used for initial values $[x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}, I_{x0}, I_{y0}, I_{z0}]$, and `dim` shows the number of first-order equations. The function `diffeq` which is called in the main programme, contains the differential equations of motion.

P1

For the Yang–Mills case we use eqs(4.17) and (4.18), so `diffeq.m` is:

```
function[vec] = diffeq(y)
% m = e = a = 1
%
R  = y(1:3);
V  = y(4:6);
I  = y(7:9);
%
r  = norm(R);
K  = r/sinh(r);
KK = (sinh(r) - r*cosh(r))/(sinh(r))^2;    % = K'
%
VV = (K^2 - r*KK - 1)*cross(V,R)*dot(R,I)/r^4 + KK*cross(V,I)/r;
II = (1 - K)*cross(cross(R,V),I)/r^2;
vec = [V,VV,II];
```

P2

For the planar motion there are only few changes to the programme `comput.m` (see **P0**). In the line *, the number of the first-order equations is `dim = 4;` . In the line **, the initial values `yi` are four values for $[r_0, \theta_0, \dot{r}_0, \dot{\theta}_0]$. Instead of plotting in a three-dimensional Cartesian frame, `plot(x(:,1).*cos(x(:,2)),x(:,1).*sin(x(:,2)));` is replaced in the line ***.

For the planar motion eq(4.47)) is used and the function `diffeq.m` is:

```
function[vec] = diffeq(y)
% m = e = a = 1, beta = 1
%
r = y(1);
KK = (sinh(r) - r*cosh(r))/(sinh(r))^2;     % = K', (K = r/sinh(r))
%
rr = y(4)*KK + y(1)*y(4)^2;
tt = ( -y(3)*KK/y(1) - 2*y(3)*y(4) )/y(1);
vec = [y(3),y(4),rr,tt];
```

P3

For the Yang–Mills–Higgs case eqs(4.110) and (4.112) are used and the function `diffeq.m` is:

```
function[vec] = diffeq(y)
% m = e = a = 1,
h = 0.5;  % we should specify a number. h = I_r(t0)H(r0)/r0 + p0.
          % h = 0 is used for planar motions.
%
R  = y(1:3);
V  = y(4:6);
I  = y(7:9);
%
r  = norm(R);
K  = r/sinh(r);
KK = (sinh(r) - r*cosh(r))/(sinh(r))^2;   % = K'
H  = r*coth(r) - 1;
p = -H*dot(R,I)/r^2 + h;
%
VV = (K^2 - r*KK - 1)*dot(R,I)*(cross(V,R) - p*R)/r^4 + ...
     KK*(cross(V,I) - p*I)/r;
II = (1 - K)*cross(cross(R,V),I)/r^2 - p*H*cross(R,I)/r^2;
vec = [V,VV,II];
```

P4

For a better understanding and checking of the equations of motion, these plots are useful: `plot(t,x(:,i))` for i from 1 to dim, `plot(x(:,i),x(:,j))` and `plot3(x(:,i),x(:,j),x(:,k))` for i, j and k from any set of {1,2,3}, {4,5,6} and {7,8,9}. For the two-dimensional motion which is written in the polar plane, appropriate substitution of coordinates might be used.

To check the correctness of constants of motion such as $|\vec{I}|$, $|\vec{v}|$, $\vec{J}$, $m(v^2+p^2)/2$, and so on, for the appropriate set of equations, we may run a programme subsequent to the main programme `comput.m` in the MATLAB Command Window. For $|\vec{I}|$, $|\vec{v}|$ *it is quit simple* to run a line in the command prompt:
`plot(t,sqrt(x(:,7).^2 + x(:,8).^2 + x(:,9).^2))` for the charge isovector and `plot(t,sqrt(x(:,4).^2 + x(:,5).^2 + x(:,6).^2))` for the speed. For $\vec{J}$ we may run this programme:

```
R = x(:,1:3)';
V = x(:,4:6)';
I = x(:,7:9)';
r = sqrt(dot(R,R));
J = zeros(3,nsteps+1);
%
K  = r./sinh(r);
L = cross(R,V);
alpha = dot(I,R)./r;
%
for i=1:3
  J(i,:) = L(i,:) + K.*I(i,:) + alpha.*(1-K).*R(i,:)./r;
end
plot(t,sqrt(dot(J,J)))      % must be a straight line
```

For $m(v^2+p^2)/2$ the below programme might be used (note `m = 1;`):

```
R  = x(:,1:3)';
V  = x(:,4:6)';
I  = x(:,7:9)';
%
h  = 0,5;             % the value of h must be specified here.
r  = sqrt(dot(R,R));
H  = r.*coth(r) - 1;
p = -H.*dot(R,I)./r.^2 + h;
%
E = p.*p/2 + dot(V,V)/2;   % m = 1
plot(t,E)                  % must be a straight line
```

For the constants at large distances such as $|m(\vec{r}\times\vec{v})|$, α_0 and so on, some appropriate programmes like the above ones can help. For the planar motion we need to substitute the Cartesian coordinates with the polar ones.

To show the accuracy of computations, in addition to plot the constants, we may compare the maximum and the minimum of the computed values for each constant. For example we may execute `max(E)/min(E)` after we run the above programme for `E`. A better result (1 for this example) is gained when we increase the value of `nsteps` in the main programme.

P5

To test the first-order perturbation in the planar motion, the main procedure is the same as `comput.m` in part **P0** with only few changes. The equations of perturbation should come along the equations of the planar motion in the function `diffeq.m`. In each step that a value for the planar motion variables are calculated, the corresponding perturbation quantities are calculated. So we may add the differential equations of perturbation to the procedure of planar motion in part **P2**. The required changes to `comput.m` are: `dim = 13` in line *, and initial values $[r_0, \theta_0, \dot{r}_0, \dot{\theta}_0, \epsilon_{r0}, \epsilon_{\theta 0}, \epsilon_{z0}, \sigma_{r0}, \sigma_{\theta 0}, \sigma_{z0}, \delta_{r0}, \delta_{\theta 0}, \delta_{z0}]$ in line **.

```
function[vec] = diffeq(y)
% m = e = a = 1; beta = 1;
r = y(1);
K = r/sinh(r);
KK = (sinh(r) - r*cosh(r))/(sinh(r))^2;    % K'
KKK = -2*cosh(r)/sinh(r)^2 + 2*r*cosh(r)^2/sinh(r)^3 - r/sinh(r); % K''
%
rr = y(4)*KK + y(1)*y(4)^2;
tt = ( -y(3)*KK/y(1) - 2*y(3)*y(4) )/y(1);
%
er = y(4)*y(6) + y(8);
et = -y(4)*y(5) + y(9);
ez = y(10);
sr = (r*KKK-KK)*y(4)*y(5)/r + (KK/r+y(4))*y(9) + KK*y(4)*y(13);
st = -(r*KKK-KK)*y(3)*y(5)/r^2 - (KK/r+y(4))*y(8) + KK*y(3)*y(13)/r;
sz = -(K^2-r*KK-1)*y(4)*y(7)/r^2 + (1-K^2)*y(4)*y(11)/r + KK*y(3)*y(12)/r;
dr = (1-K)*y(3)*y(7)/r^2 - (1-K)*y(10)/r + K*y(4)*y(12);
dt = (1-K)*y(4)*y(7)/r - K*y(4)*y(11);
dz = 0;
```

```
%
vec = [y(3),y(4),rr,tt,er,et,ez,sr,st,sz,dr,dt,dz];
```

Instead of the line *** in `comput.m` we plot any appropriate two or three-dimensional plots. Note the values computed in this procedure are in the cylindrical coordinates. After running the programme we may plot any of the perturbation components with respect to the time to see the result. We may increase the time when the plots do not show a divergence.

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