Twist or theory of immersions of surfaces in four-dimensional spheres and hyperbolic spaces

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Twistor Theory of Immersions of Surfaces in Four-dimensional Spheres and Hyperbolic Spaces

by

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Abstract.

Twistor Theory of Immersions of Surfaces
in Four-dimensional Spheres and Hyperbolic Spaces

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Let $f : S \to S^4$ be an immersion of a Riemann surface in the 4-sphere. The thesis begins with a study of the adapted moving frame of f in order to produce conditions for certain naturally defined lifts to $\mathrm{SO}(5)/\mathrm{U}(2)$ and $\mathrm{SO}(5)/T^2$ to be conformal, harmonic and holomorphic with respect to two different but naturally occuring almost complex structures. This approach brings together the results of a number of authors regarding lifts of conformal, minimal immersions including the link with solutions of the Toda equations. Moreover it is shown that parallel mean curvature immersions have harmonic lifts into $\mathrm{SO}(5)/\mathrm{U}(2)$.

A certain natural lift of f into $\mathbb{C}\mathrm{P}^3$, the twistor space of S^4, is studied more carefully via an explicit description and in the case of f being a conformal immersion this gives a beautiful and simple formula for the lift in terms of a stereographic co-ordinate associated to f. This involves establishing explicitly the two-to-one correspondence between elements of the matrix groups $\mathrm{Sp}(2)$ and $\mathrm{SO}(5)$ and working with quaternions. The formula enables properties of such lifts to be explored and in particular it is shown that the harmonic sequence of a harmonic lift is either finite or satisfies a certain symmetry property. Uniqueness properties of harmonic lifts are also proved.

Finally, the ideas are extended to the hyperbolic space H^4 and after an exposition of the twistor fibration for this case, a method for constructing superminimal immersions of surfaces into H^4 from those in S^4 is given.

Preface.

Twistor Theory of Immersions of Surfaces

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Helen Linda Fawley

This work has been sponsored by the U.K. Engineering and Physical Sciences Research Council.

The thesis is based on research carried out between October 1993 and July 1997 under the supervision of Dr L M Woodward. It has not been submitted for any other degree either at Durham or at any other University.

No claim of originality is made for the material presented in chapter 1 or chapter 2. The material in chapters 3, 4 and 5 is original apart from the review sections at the start of chapter 3 and the background material on the twistor fibration which appears in sections 4.2 to 4.6. The work in chapter 6 has been done in collaboration with Dr J Bolton and Dr L M Woodward and to the best of my knowledge has not previously appeared in print. It has also been necessary throughout the work to use the results of other authors and these are clearly accredited where they occur.

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Finally, thankyou to Mum and to Dad, whose unfailing love, support and encouragement has brought me this far. This thesis is dedicated to them.

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CHAPTER 1

Introduction

The recent intensive investigation of minimal immersions of surfaces in spheres was initiated by the work of Calabi in the late 1960's [13]. He studied minimal immersions of the 2-sphere in S^{2m} by associating to each immersion a holomorphic curve in the homogeneous Kähler manifold $SO(2m+1)/U(m)$. Using complex analysis on these holomorphic curves gives rise to a complete classification of all minimal 2-spheres in an n-sphere in terms of holomorphic 2-spheres in complex projective space. Calabi was also able to produce a formula for the area of such minimal surfaces.

This work was extended by Chern [14, 15], who associated to a minimal immersion $f : S^2 \to S^{2m}$ a holomorphic curve $\Phi : S^2 \to \mathbb{C}P^{2m}$ called the directrix curve. This curve is rational and real isotropic. He exploited properties of this curve to study minimal immersions of S^2 in S^4 and spaces of constant curvature in general. Using these ideas, Barbosa [2](1975) was able to improve Calabi's result to show that the area of f is a multiple of 4π and together with invariants of the directrix curve this gave rise to examples with a prescribed area. Barbosa also obtained a rigidity theorem showing that isometric minimal immersions $f_1, f_2 : S^2 \to S^{2m}$ differ by a rigid motion of the ambient space S^{2m}.

Throughout the 1980's a number of mathematicians and physicists worked more generally on a similar analysis of harmonic maps (branched minimal immersions) of surfaces into Riemannian symmetric spaces. In each case, this was done by associating to each harmonic map a holomorphic curve in a suitable homogeneous Kähler manifold, and progress was made on harmonic maps of S^2 into $\mathbb{C}P^n$ (Eells-Wood) [18], complex Grassmannians (Wolfson) [30]. This was later extended by Uhlenbeck through the study of harmonic 2-spheres in $U(n)$ and compact Lie Groups in general [28].

By studying almost complex curves in S^6 and associated holomorphic maps into the complex hyperquadric Q_5, Bryant [10] showed that every Riemann surface occurs as a minimal surface in S^6 (with a finite number of branch points). Further, by applying Calabi's techniques to the twistor fibration $\pi : \mathbb{C}P^3 \to S^4$ he was able to prove that any compact Riemann surface may be conformally and harmonically immersed in the 4-sphere [11]. The proof of this result has two important components: first of all, for a Riemann surface M^2, one shows that if $\phi : M^2 \to \mathbb{C}P^3$ is a horizontal, holomorphic curve then $\pi\phi : M^2 \to S^4$ is a superminimal immersion and conversely, every superminimal immersion $\Phi : M^2 \to S^4$ is of the form $\pi\phi$ where ϕ is an essentially unique horizontal, holomorphic curve in $\mathbb{C}P^3$. Bryant then derives a 'Weierstrass' formula showing how to produce $\phi(f, g)$ from any pair of meromorphic functions (f, g) on M^2, to conclude that $\pi\phi(f, g)$ is conformal and minimal (indeed superminimal).

A useful and powerful tool in the study of harmonic maps into $\mathbb{C}P^n$ is the harmonic sequence of harmonic maps derived in a particular way from the given one. Wolfson introduced this concept in his study of harmonic maps into complex Grassmannians, but the original idea goes back to Laplace. For a Riemann surface M, the harmonic sequence of a harmonic map $\phi : M \to \mathbb{C}P^n$ neatly characterises the properties of that map, and this approach includes the case of harmonic maps into S^n via stereographic projection to $\mathbb{R}P^n$ and inclusion in $\mathbb{C}P^n$. For conformal minimal immersions of S^2 into $\mathbb{C}P^n$ the harmonic sequence is finite and is essentially the Frenet frame of an associated holomorphic curve. In the case of minimal immersions of S^2 into S^n, this curve is the directrix curve discussed above. This approach features strongly in the work of Bolton and Woodward [4, 5, 6, 8].

Eells and Salamon [17] (1984) studied conformal and harmonic maps of a Riemann surface M into an oriented Riemannian 4-manifold N via the twistor space of N. Instead of considering the natural almost complex structure J_1 (integrable if N is $\pm$-self-dual) they used a different (never integrable) almost complex structure J_2 obtained from J_1 by reversing the orientation along the fibres. This provided a parametrisation of conformal and harmonic maps and showed that there is a bijective correspondence between such maps and (non-vertical) J_2-holomorphic curves $\psi : M \to S_\pm$, where $S_\pm$ are fibre bundles over N of unit eigenvectors of the Hodge $*$-operator acting on $\bigwedge^2 TN$. This gives a twistorial description for all conformal and harmonic maps into N^4 and leads to the distinguishing of special classes. Although the J_2-holomorphic curves are somewhat more difficult to deal with, this approach encompasses many of the previous results. For example, if ψ is J_1- and J_2-holomorphic then it is horizontal and projects to a real isotropic harmonic map, such as Bryant's superminimal immersions mentioned above. Further, Eells and Salamon used the twistor bundle $\mathbb{C}P^3 \to S^4$ to produce examples of harmonic maps into $\mathbb{C}P^3$.

Much of the above work has been unified in the monograph of Burstall and Rawnsley [12], who study harmonic maps of S^2 into an inner symmetric space N. It is shown that such maps correspond to holomorphic curves in a certain flag manifold which is holomorphically embedded in the twistor space of N and this enables stable harmonic 2-spheres to be completely classified.

The connection between harmonic maps of surfaces into $\mathbb{C}P^n$ and S^n and solutions of the Toda equations has been the subject of increasing attention since a Lie algebra formulation of the Toda equations was given almost simultaneously by Adler, Kostant and Symes in the late 1970's [1, 24, 27]. They showed that for each simple Lie algebra there is a corresponding Toda system and that knowing the solutions of this system is equivalent to knowing the weight structure of the fundamental representations of the Lie algebra. Two particular forms of the Toda equations, open and affine, can be formulated for each Lie algebra and the solutions to both may be interpreted in terms of special types of harmonic maps into G/T. These have been investigated and classified by Bolton, Pedit and Woodward [5, 9]. This theory forms part of a larger programme to study the relation between integrable systems and harmonic maps into symmetric and related spaces (see for example [20]).

This thesis is one step on the way to a unified description of these ideas and includes extensions to immersions other than minimal ones into S^4. The work is organised as follows:

Chapter 2 gathers some background material on defining natural lifts of immersions $f : S \to S^4$, the relevant harmonic sequence theory and some remarks on complex curves. Chapter 3 defines lifts of f to the homogeneous spaces $\mathrm{SO}(5)/K$ in terms of the adapted moving frame of f. This approach enables the lifts and their properties to be studied in a unified way and conditions for such lifts to be conformal and to be harmonic are derived. For $K = \mathrm{U}(2)$, this gives rise to holomorphicity results and in particular two theorems of Eells-Salamon are proved directly. For $K = T$, it is shown that this framework is an excellent one in which to see the link between harmonic maps f and solutions of the $\mathfrak{so}(5)$-Toda equations. Further the lift of an immersion with parallel mean curvature in S^4 into $\mathrm{SO}(5)/\mathrm{U}(2)$ is shown to be harmonic.

In chapter 4, twistor lifts of f to $\mathbb{CP}^3$ are studied and an explicit formula for the lift is produced, which has a simple form in the case where f is conformal. This is achieved using the twistor fibration $\pi : \mathbb{CP}^3 \to S^4$, finding the correspondence between $\mathrm{Sp}(2)$ and $\mathrm{SO}(5)$, identifying $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{Sp}(1)$ and working with quaternions.

Chapter 5 uses the formula for the lift to examine some properties of lifts of conformal immersions. Some examples are given which demonstrate the equivariance of the lifts and the condition for harmonicity is derived. Holomorphic lifts are studied and found to be unique and harmonic lifts which are not holomorphic are found to have harmonic sequences with a particular symmetry property. In particular this shows that all conformal harmonic lifts are either superminimal or superconformal maps into $\mathbb{C}P^3$. The 'positive' and 'negative' lifts of [17] are easily seen using the methods of chapter 4.

In chapter 6, these ideas are extended to immersions of surfaces into H^4, a four-dimensional hyperbolic space. It is shown that there is an entirely analogous description of the twistor fibration for H^4 and the lifts into the homogeneous spaces of $\mathrm{SO}(1,4)$ to that of the S^4-case. Further, superminimal immersions in H^4 may be constructed from those in S^4 via a 'twistor transform' and an algorithm analogous to that of Bryant.

Finally, Chapter 7 is an appendix which contains the conventions regarding the quaternions, $\mathbb{H}^n$ and the groups $\mathrm{Sp}(n)$ used in the thesis.

CHAPTER 2

Background Material

This chapter sets out the background material which will be required in the sequel. For a Riemann surface S, section 2.1 gives the natural way in which lifts of immersions $f : S \to S^4$ to homogeneous spaces $SO(5)/K$ are described in terms of the adapted frame associated to f, the spaces $SO(5)/K$ being viewed as different generalised flag manifolds. The equivariance of such lifts under the actions of isometries and conformal transformations is discussed in section 2.1.2. The definition of a harmonic map appears in section 2.2 and in more detail in 2.2.1. The theory of harmonic sequences is given briefly in 2.2.2 and contains many definitions and results which will be required in the later chapters. Section 2.3 discusses J_1- and J_2-holomorphic curves and in particular shows how to recognise such curves in $\mathbb{C}P^3$. Let us begin with the following definition:

Definition 2.1. *A differentiable mapping f of a manifold M into another manifold N is called an immersion if $df_p : T_pM \to T_{f(p)}N$ is injective for every point p of M.*

2.1. Lifts of Immersions $f : S \to S^4$

2.1.1. Flag Manifolds. At each point $p \in S$, the tangent bundle to S^4 restricted to S (considered at the point p) splits as

$$T_{f(p)}S^4 = T_pS \oplus N_pS$$

where T_pS is the tangent space and N_pS the normal space to S in S^4 and both T_pS and N_pS are 2-planes. Since f is normal to S^4 in $\mathbb{R}^5$ we write

$$\mathbb{R}^5 = \{f(p)\} \oplus T_pS \oplus N_pS$$

where $\{f\}$ is the line in $\mathbb{R}^5$ determined by f.

Now choose oriented orthonormal bases e_1, e_2 for T_pS and e_3, e_4 for N_pS so that e_1, e_2, e_3, e_4 gives the standard orientation on S^4, i.e.

$$\mathbb{R}^5 = \{f\} \oplus \mathrm{span}\{e_1, e_2\} \oplus \mathrm{span}\{e_3, e_4\}.$$

Thus given any $f : S \to S^4$, there is an adapted orthonormal frame

$$\tilde{F} = (f \mid e_1 \mid e_2 \mid e_3 \mid e_4) \in \mathrm{SO}(5)$$

and this frame defines a local lift of f to SO(5). In general, the frame cannot be chosen globally since a basis has been nominated on each of the 2-planes TS and NS. However, the frame $\tilde{F}$ is unique up to rotations in these planes, giving a global lift of f into $\mathrm{SO}(5)/T^2$. Hence, there is a naturally defined global lift of f into each of the homogeneous spaces $\mathrm{SO}(5)/K$, where $K \subset \mathrm{SO}(5)$ is a subgroup of maximal rank. This means that K is one of $T^2, \mathrm{U}(2), \mathrm{SO}(2) \times \mathrm{SO}(3), \mathrm{SO}(4)$. For any such K, the inclusion $T \subset K$ induces a projection $\sigma_K : \mathrm{SO}(5)/T^2 \to \mathrm{SO}(5)/K$. Thus given a lift $\tilde{f} : S \to \mathrm{SO}(5)/T^2$ we have a map $\hat{f} = \sigma_K \tilde{f} : S \to \mathrm{SO}(5)/K$. Also, $\mathrm{SO}(5)/K$ for $K = T^2$, $\mathrm{SO}(2) \times \mathrm{SO}(3)$ or $\mathrm{U}(2)$ is a complex manifold and σ_K is complex analytic for $K = \mathrm{SO}(2) \times \mathrm{SO}(3)$ or $\mathrm{U}(2)$ (see [23]).

The homogeneous spaces $\mathrm{SO}(5)/K$ may be identified as different types of generalised flag manifolds. Their elements are given in terms of different descriptions of $\mathbb{R}^5$ as orthogonal direct sum decompositions of oriented subspaces. In each case SO(5) acts on the flags and K is the stabiliser of a typical flag.

For example, in the case of $\mathrm{SO}(5)/T^2$ the elements are direct sums of oriented subspaces

$$\mathbb{R}^5 = L \oplus V_1 \oplus V_2, \qquad \dim L = 1,\ \dim V_i = 2 \qquad (i = 1, 2)$$

with the orientation induced on $\mathbb{R}^5$ by those on L, V_1 and V_2 agreeing with the standard one. For $\mathrm{SO}(5)/\mathrm{U}(2)$ the decompositions are of the form

$$\mathbb{R}^5 = L \oplus V, \qquad \dim L = 1,\ \dim V = 4$$

with V having an orthogonal complex structure compatible with the metric and orientation. This is a particularly interesting case since, as will be shown in Chapter 4, $\mathrm{SO}(5)/\mathrm{U}(2)$ may be identified with the total space of the bundle of orthogonal almost complex structures on S^4 and also with the space of maximal isotropic subspaces of $\mathbb{C}^5$ (see [2]).

For $\mathrm{SO}(5)/\mathrm{SO}(2) \times \mathrm{SO}(3)$ the decompositions are of the form

$$\mathbb{R}^5 = W \oplus W^{\perp}, \qquad \dim W = 2.$$

Finally, for $\mathrm{SO}(5)/\mathrm{SO}(4)$ the decompositions are of the form

$$\mathbb{R}^5 = L \oplus V, \qquad \dim L = 1,\ \dim V = 4$$

and $\mathrm{SO}(5)/\mathrm{SO}(4) = S^4$.

Note that the projection maps σ_K may now be understood in terms of 'forgetting' certain properties. For example, if $(L, V_1, V_2) \in \mathrm{SO}(5)/T^2$ then both V_1 and V_2 have a natural orthogonal complex structure and hence so does $V = V_1 \oplus V_2$. The projection $\sigma_{\mathrm{U}(2)} : \mathrm{SO}(5)/T^2 \to \mathrm{SO}(5)/\mathrm{U}(2)$ sending (L, V_1, V_2) to (L, V) 'forgets' the decomposition of V.

From the discussion above it is now clear how the lifts may be identified. For $\tilde{f} : S \to \mathrm{SO}(5)/T^2$ we have

$$\tilde{f}(p) = (\{f(p)\}, T_p(S), N_p(S))$$

and the others can be immediately written down from this. In particular note that the map $g : S \to \mathrm{SO}(5)/\mathrm{SO}(2) \times \mathrm{SO}(3)$ which is given by $g(p) = (T_p(S), T_p(S)^\perp)$ is just the Gauss map of $f : S \to S^4$.

Since Sp(2) is the universal cover of SO(5), the above discussion may be rewritten in terms of homogeneous spaces of Sp(2). In particular, it will be shown in Chapter 4 that $\mathrm{SO}(5)/\mathrm{U}(2) = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{Sp}(1)$ and it is also not hard to show that $\mathrm{SO}(5)/T^2 = \mathrm{Sp}(2)/T^2$ and $\mathrm{SO}(5)/\mathrm{SO}(4) = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

There are a number of different ways of describing these lifts, and in the sequel, Chapter 3 will consider the description in terms of the moving frame while Chapter 4 will focus on the use of quaternions in determining an explicit formula for the lift.

2.1.2. Equivariance of Lifts. The group SO(5) acts on $\mathbb{R}^5$ and hence on S^4 as a group of isometries. Also ([7]) SO(5) acts on $\mathbb{CP}^3$ as the group of holomorphic isometries which preserve the horizontal distribution.

Let $f : S \to S^4$ and let $\tilde{f}$ denote the lift of f to $\mathrm{SO}(5)/T^2$. Suppose $g \in \mathrm{SO}(5)$ and look at the lift of gf. This is the flag given by

$$(\{gf\} \oplus T^{gf}S \oplus N^{gf}S)$$

where $T^{gf}S \oplus N^{gf}S$ denotes the decomposition of $(gf)^{-1}TS^4$ with respect to the map gf. But this is just g applied to the flag

$$(\{f\} \oplus T^f S \oplus N^f S)$$

or, in other words, $g\tilde{f}$. So the new lift is obtained by applying g to the original lift and this is expressed neatly as $\widetilde{gf} = g\tilde{f}$ for $g \in \mathrm{SO}(5)$.

Now suppose that there exists an isometry h of S such that $f(h(p)) = gf(p)$ for $g \in \mathrm{SO}(5)$. Then $\widetilde{gf} = g\tilde{f}$ implies that the same symmetry is exhibited by the lift and vice-versa. For example, f has a 2-fold symmetry if and only if $\tilde{f}$ has a 2-fold symmetry. Examples of immersions and their lifts which have SO(3)-symmetry and which have S^1-symmetry are given in Chapter 5.

Now, the group $\mathrm{SO}(1,n)$ is defined as

$$\mathrm{SO}(1,n) = \{A \in \mathrm{GL}(n+1,\mathbb{R}) \mid A^t I_{1,n} A = I_{1,n}\}$$

where $I_{1,n} = \left(\begin{array}{c|c} -1 & \\ \hline & I_n \end{array}\right)$. It is the conformal group of S^{n-1}, taking $(n-2)$-spheres to $(n-2)$-spheres. $\mathrm{SO}(1,n)$ acts as

$$\left(\begin{array}{c|c} t & v^t \\ \hline u & A \end{array}\right)\begin{pmatrix} 1 \\ x \end{pmatrix},$$

where $t \in \mathbb{R}$, $x, u, v \in \mathbb{R}^n$, taking

$$x \longmapsto \frac{Ax+u}{v.x+t}.$$

This is a conformal transformation and all conformal transformations are of this form. [26]

If $g \in \mathrm{SO}(1,5)$ then it is again the case that $g\tilde{f} = \widetilde{gf}$ i.e. applying a conformal transformation and then lifting gives the same result as applying the conformal transformation to the lift.

2.2. Harmonic Maps and Minimal Immersions

Consider a smooth map $\phi : M \to N$ between the manifolds M and N and suppose M is compact. There are two interesting variational problems for such maps:

(1) If (N,h) is a Riemannian manifold with metric h then under each immersion ϕ, M inherits a metric $\phi^* h$ and we ask when M is a minimal submanifold of N

- that is, when ϕ is a minimal immersion. This is the case when ϕ is a critical point of

$$V(\phi) = \int_M dV(\phi^* h)$$

($dV(\phi^* h)$ is the volume element for $\phi^* h$) for which the condition is given by the Euler-Lagrange equations, $\text{trace}(\Pi(\phi)) = 0$.

(2) If M and N are each equipped with their own metric, we can ask when ϕ is a harmonic map between these Riemannian manifolds. This is the case when ϕ is a critical point of the energy functional

$$E(\phi) = \int_M |d\phi|^2 dvol_M.$$

The condition is again given by the Euler-Lagrange equations, which are equivalent to $\text{trace}\nabla d\phi = 0$.

These questions are about two quite different properties of ϕ. However, if $\phi : (M, g) \to (N, h)$ is an isometric immersion then ϕ is harmonic if and only if it is minimal [16]. Moreover, if $\dim M = 2$ and ϕ is a conformal immersion then ϕ is harmonic if and only if it is minimal [8].

2.2.1. Understanding $\nabla d\phi$. Suppose $\phi : M \to N$ is a smooth map between Riemannian manifolds, so that ϕ is harmonic if and only if $\text{trace}\nabla d\phi = 0$. Now, $\nabla d\phi : TM \otimes TM \to TN$ with

$$(\nabla d\phi)(X, Y) = (\hat{\nabla}_X d\phi)Y, \qquad X, Y \in TM$$

and $\hat{\nabla}$ represents the connection on the bundle $\text{Hom}(TM, \phi^* TN)$. The differential $d\phi$ may be thought of as a section of this bundle and since there are connections on TM and $\phi^* TN$ (the latter being induced by the connection on TN), $\hat{\nabla}$ is the connection induced by these. Indeed,

$$(\hat{\nabla}_X d\phi)Y = \nabla^N_X d\phi(Y) - d\phi(\nabla^M_X Y),$$

where ∇^M, ∇^N denote the Levi-Civita connections on M and N respectively. Also, if ϕ is isometric then $\nabla d\phi$ is precisely the second fundamental form of ϕ.

2.2.2. Harmonic Sequences. A harmonic map $\phi : S \to \mathbb{C}P^n$ possesses a series of related harmonic maps $\ldots \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2 \ldots$ called the harmonic sequence of $\phi = \phi_0$. It is by studying this sequence that many other interesting properties of ϕ may be uncovered. A brief overview of this theory is given below. For further details the reader is directed to the papers of Bolton and Woodward [4, 6, 8].

Let $\phi : S \to \mathbb{C}P^n$ be a map of a Riemann surface S into $\mathbb{C}P^n$. With respect to a local complex co-ordinate z on S, and writing ∂, $\bar{\partial}$ for the partial derivatives $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$ respectively, the condition for ϕ to be harmonic may be written as

$$\left(\nabla_{\bar{\partial}} d\phi\right)(\partial) = 0$$

i.e.

$$\nabla_{\bar{\partial}}^{\mathbb{C}P^n}\left(d\phi(\partial)\right) - d\phi\left(\nabla_{\bar{\partial}}^{S}\partial\right) = 0.$$

But since S is a Riemann surface, $\nabla_{\bar{\partial}}^{S}\partial = 0$ and the condition is

$$\nabla_{\bar{\partial}}^{\mathbb{C}P^n}\partial\phi = 0. \tag{2.1}$$

The key lies in interpreting harmonic maps in terms of line bundles as follows: let $L \to \mathbb{C}P^n$ denote the tautological line bundle whose fibre over $x \in \mathbb{C}P^n$ is the line $x \subset \mathbb{C}^{n+1}$. Then there is a bijective correspondence between maps $\phi : S \to \mathbb{C}P^n$ and smooth complex line subbundles of $S \times \mathbb{C}^{n+1}$ given by $\phi \leftrightarrow \phi^* L$. The other basic ideas involved are the expression of (2.1) as a holomorphicity condition and the use of some complex variable theory. Then via particular holomorphic and anti-holomorphic bundle maps one builds a sequence of line bundles $\{L_i\}$ and each L_i is given by $L_i = \phi_i^* L$ for some uniquely determined $\phi_i : S \to \mathbb{C}P^n$. Moreover each ϕ_i is harmonic.

The theory of harmonic maps $\phi : S \longrightarrow \mathbb{C}P^n$ includes that of harmonic maps $f : S \longrightarrow S^n$ of surfaces into S^n. For, if $\pi : S^n \longrightarrow \mathbb{R}P^n$ is stereographic projection and $i : \mathbb{R}P^n \longrightarrow \mathbb{C}P^n$ is the inclusion map, then $\phi = i\pi f$ is harmonic if and only if f is harmonic.

The local description goes as follows: Suppose that $\phi : S \longrightarrow \mathbb{C}P^n$ is a harmonic map and suppose that ϕ is linearly full, which means that the image of ϕ is not contained in any proper projective complex linear subspace of $\mathbb{C}P^n$. Locally, write $\phi = \phi_0 = [f_0]$ where f_0 is a $\mathbb{C}^{n+1}\backslash\{0\}$-valued function and suppose that f_0 is chosen to be a holomorphic section of $L_0 = \phi_0^* L$. Then maps ϕ_p are defined inductively by $\phi_p = [f_p]$ where

$$\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{\partial}{\partial z}\log|f_p|^2 f_p$$
$$\frac{\partial f_p}{\partial \bar{z}} = -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}.$$

Each of the maps ϕ_p is harmonic, ϕ_p is related to ϕ_{p+1} in a simple way and by construction, $\langle f_{p+1}, f_p\rangle = 0$.

Note: it can be shown that if $\phi_0 = [f_0]$ with f_0 not a holomorphic section of L_0 then the condition for ϕ_0 to be harmonic is given locally by

$$\frac{\partial^2 f_0}{\partial z \partial \bar{z}} - \frac{1}{|f_0|^2}\langle\frac{\partial f_0}{\partial z}, f_0\rangle\frac{\partial f_0}{\partial \bar{z}} - \frac{1}{|f_0|^2}\langle\frac{\partial f_0}{\partial \bar{z}}, f_0\rangle\frac{\partial f_0}{\partial z} = \mu f_0$$

for some $\mu \in \mathbb{C}$. However, it is true that there always exists a function $\lambda(z,\bar{z})$ such that λf_0 is a holomorphic section of the bundle.

Define $\gamma_p = \dfrac{|f_{p+1}|^2}{|f_p|^2}$ (if ϕ_p is $\pm$-holomorphic then set $\gamma_{p\mp1} = 0$). Then for a conformal harmonic map, (see [6] for the non-conformal version) the metric induced by ϕ_p is given by

$$ds_p^2 = (\gamma_{p-1} + \gamma_p)|dz|^2.$$

The curvature of ϕ_p is

$$K(\phi_p) = K_p = -\frac{1}{F_p}\frac{\partial^2}{\partial z \partial \bar{z}} \log F_p, \qquad F_p = \frac{1}{2}(\gamma_{p-1} + \gamma_p). \tag{2.2}$$

The Kähler angle is a function $\theta : S \longrightarrow [0, \pi]$ defined by

$$\phi^* \omega = \cos \theta dA$$

where ω is the Kähler form on $\mathbb{C}P^n$ and θ essentially measures by how much $\phi : S \longrightarrow \mathbb{C}P^n$ fails to be holomorphic. The Kähler angle θ_p corresponding to ϕ_p is given by

$$(\tan \frac{1}{2}\theta_p)^2 = \frac{\gamma_{p-1}}{\gamma_p}. \tag{2.3}$$

There are globally defined forms

$$\Gamma_p = \gamma_p |dz|^2 \qquad \text{and} \qquad U_{p+k,p} = u_{p+k,p} dz^k$$

with

$$u_{p+k,p} = \frac{\langle f_{p+k}, f_p \rangle}{|f_p|^2}.$$

Then the fact that $\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial \bar{z} \partial z}$ gives rise to the unintegrated Plücker formulae

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_p = \gamma_{p+1} - 2\gamma_p + \gamma_{p-1}$$

so that any two consecutive Γ-invariants of a harmonic map determine all the Γ-invariants for that map and for fixed $k \in \mathbb{N}$, the Γ-invariants together with the set $\{U_{1,0}, \dots, U_{k,0}\}$ determine $\{U_{q+1,q}, \dots, U_{q+k,q}\}$ for all $q \in \mathbb{Z}$ [6].

Congruence Theorem [6]. *Let S be a connected Riemann surface. Let ψ, $\tilde{\psi} : S \longrightarrow \mathbb{C}P^n$ be harmonic maps with $\Gamma_{-1} = \tilde{\Gamma}_{-1}$, $\Gamma_0 = \tilde{\Gamma}_0$. If $U_{p,0} = \tilde{U}_{p,0}$ for $p = 2, \dots, n+1$ then there exists a holomorphic isometry g of $\mathbb{C}P^n$ such that $\tilde{\psi} = g\psi$. If ψ is linearly full then g is unique.*

This theorem says that up to a holomorphic isometry of $\mathbb{C}P^n$ a harmonic map is determined by the invariants Γ_{-1}, Γ_0 and $U_{2,0}, \dots, U_{n+1,0}$.

Definition 2.2. *A harmonic map $\phi : S \to \mathbb{C}P^n$ is k-orthogonal if k consecutive elements of the harmonic sequence are mutually orthogonal.*

Therefore, a harmonic map is conformal if and only if it is 3-orthogonal.

Proposition 2.3. [5] *If some k consecutive elements in a harmonic sequence are mutually orthogonal then every k consecutive elements are mutually orthogonal.*

In particular, if $\langle f_{p+2}, f_p \rangle = 0$ for some p (i.e. ϕ_p is conformal), then $\langle f_{p+2}, f_p \rangle = 0$ for all p and every element ϕ_p in the sequence is conformal.

In all cases, ϕ is at most $(n+1)$-orthogonal and there are two ways in which this case can arise:

The simplest case is that in which the harmonic sequence has finite length and thus reduces to the Frenet frame of a holomorphic curve. This is the case where the $u_{p+k,p}$ are zero for all p. A harmonic map with such a harmonic sequence is said to be *superminimal* (or pseudo-holomorphic [6], [13] or complex isotropic [18]) and the harmonic sequence has exactly $n+1$ elements if ϕ is full. These are mutually orthogonal with the first a holomorphic curve and the last an anti-holomorphic curve. For example, every harmonic map of S^2 into $\mathbb{C}\mathrm{P}^n$ is superminimal.

Definition 2.4. *A map $\phi : S \to \mathbb{C}P^n$ is called real isotropic if $\dfrac{\partial^k f}{\partial z^k} \cdot \dfrac{\partial^k f}{\partial z^k} = 0$ for all k.*

Then ϕ is superminimal if and only if ϕ is harmonic and real isotropic.

If the harmonic map ϕ is $(n+1)$-orthogonal but is not superminimal then ϕ is said to be *superconformal.*

Note that every harmonic map $\phi : S \to \mathbb{C}\mathrm{P}^1$ and every conformal harmonic map

$\phi : S \to \mathbb{CP}^2$ is either superminimal or superconformal. However, in general the sequence is infinite in both directions and, apart from consecutive elements being orthogonal by construction, there are no orthogonality properties.

S^n Case. Suppose $f : S \to S^n$ and let $\phi = i\pi f : S \to \mathbb{CP}^n$. Then ϕ is harmonic and linearly full if and only if f is harmonic and linearly full. The harmonic sequence $\{f_p\}$ of ϕ is constructed in the manner described above and this is said to be the harmonic sequence of f. Also, f is called k-orthogonal if ϕ is k-orthogonal in the sense of definition 2.2.

Since $\langle f, f\rangle = 1$ and f is real, the sequence of sections $\{f_p\}$ may be chosen in a particularly nice way. For,

$$0 = \frac{\partial}{\partial \bar{z}}\langle f, f\rangle = 2\langle \frac{\partial f}{\partial \bar{z}}, f\rangle$$

so that taking $f_0 = f$ gives f_0 as a global holomorphic section of L_0. Induction using equations (2.2) shows that

$$f_{-p} = (-1)^p \frac{\bar{f}_p}{|f_p|^2}.$$

Proposition 2.5. [6] *If f is $(2k-1)$-orthogonal for some $k \leq \frac{1}{2}n + 1$ then f is $2k$-orthogonal.*

Thus for $f : S \to S^{2m}$, if f is $(2m+1)$-orthogonal then f is superminimal. A harmonic map into S^{2m} is said to be superconformal if it is $2m$-orthogonal but not superminimal. Notice that this is not just a trivial modification of the $\mathbb{CP}^n$ case since, for $n = 2m$, the harmonic sequences which arise from linearly full f are not orthogonally periodic.

There are two types of superconformal harmonic maps into S^{2m}, namely those which are linearly full in S^{2m} and those which are linearly full in a totally geodesic S^{2m-1} in S^{2m}. In the latter case, the harmonic sequence is orthogonally periodic. Notice that any conformal harmonic map $f : S \to S^3$ or S^4 is either superconformal or superminimal and so is every almost complex curve $f : S \to S^6$ [5]. For example, the Veronese surface in S^4 is superminimal and the Clifford torus is a superconformal surface in S^3. These examples are studied in detail in Chapter 5.

2.3. Complex Curves

2.3.1. Horizontal and Vertical Subspaces. Let Z be a Kähler manifold equipped with an orthogonal complex structure J (orthogonal means that J preserves lengths) and consider a fibration

$$\begin{array}{ccc} Z & \longleftarrow & Y \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array}$$

over a real manifold X with fibre Y, a complex submanifold so that π is a Riemannian submersion. Then at each point $p \in Z$, the tangent space T_pZ (a complex vector space) decomposes as the orthogonal direct sum of two pieces:

$$T_pZ = H_p \oplus V_p.$$

H_p is called the horizontal subspace and V_p the vertical subspace. In particular, assuming each Y_p is a complex submanifold, $V_p = T_pY_p$ where $Y_p = \pi^{-1}\pi(p)$ is the fibre over $\pi(p)$.

Now $d\pi : T_pZ \to T_{\pi(p)}X$ maps H_p isomorphically, indeed isometrically, onto $T_{\pi(p)}X$. (H_p is a complex vector subspace of T_pZ and $V_p = \ker(d\pi_p)$). Also if $J_H = J|_{H_p}$ and $J_V = J|_{V_p}$ then the complex structure J splits as

$$J = J_H \oplus J_V.$$

2.3.2. Complex Curves. A map ϕ between a Riemann surface (S, J^S) and an almost complex manifold (Z, J^Z) is complex analytic if its differential $d\phi : T_xS \to$

$T_{\phi(x)}Z$ is a complex linear map, that is ([17])

$$d\phi \circ J^S = J^Z \circ d\phi. \tag{2.4}$$

Given any complex structure J on Z, ϕ is said to be a complex curve if (2.4) holds.

Recall that there is an orthogonal complex structure $J = J_1 = J_H \oplus J_V$ on Z and that J_1 is integrable on Z. Now define $J_2 = J_H \oplus (-J_V)$. Then J_2 is an orthogonal almost complex structure on Z but J_2 is not integrable [16]. When ϕ is a complex curve with respect to the almost complex structure J_i we say that ϕ is J_i-holomorphic. Note that ϕ is J_1- and J_2-holomorphic if and only if ϕ is a horizontal, holomorphic curve [17].

2.3.3. $Z = \mathbb{C}\mathrm{P}^3$. Let $[v]$ be a point in $\mathbb{C}\mathrm{P}^3$. Tangent vectors to $\mathbb{C}\mathrm{P}^3$ at $[v]$ correspond to linear maps from span$\{v\}$ to span$\{v\}^\perp$. Suppose we have a curve $\alpha : I \to \mathbb{C}\mathrm{P}^3$, $\alpha(t) = [z(t)]$ with $\alpha(t_0) = [z(t_0)] = [v]$. Then

$$\alpha'(t_0) \leftrightarrow z(t_0) \mapsto z'(t_0) - \frac{\langle z'(t_0), z(t_0)\rangle}{|z(t_0)|^2} z(t_0).$$

Thus given $\psi : S \to \mathbb{C}\mathrm{P}^3$ written locally as $\psi(x,y) = [\phi(x,y)]$ then

$$d\psi\left(\frac{\partial}{\partial x}\right) = \Phi_x : \phi(x,y) \mapsto \phi_x(x,y) - \frac{\langle \phi_x(x,y), \phi(x,y)\rangle}{|\phi(x,y)|^2}\phi(x,y) \tag{2.5}$$

and so on.

In this context it is useful to think of a complex manifold as a real manifold with an automorphism J on each tangent space. For example, $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C} = V^+ \oplus V^-$ where V^+ and V^- are the $\pm i$-eigenspaces respectively and identifying V^+ with $\mathbb{C}^n$ gives $Jv = iv$, that is, J is equivalent to multiplication by i. It is usual to identify $T\mathbb{C}\mathrm{P}^n \cong T^{(1,0)}\mathbb{C}\mathrm{P}^n$ ($T^{(1,0)}$ is the $+i$-eigenspace of J_1 (or J_2)).

Finally, note that the complex curve condition (2.4) is $d\pi(T^{(1,0)}S) \subset T^{(1,0)}Z$. ($T^{(1,0)}S \cong TS$).

2.3.4. How to Recognise a J_1-holomorphic Curve in $\mathbb{C}P^3$. Let $z = x + iy$ be a local complex co-ordinate on S and let $J^S : TS \to TS$ act as

$$J^S\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \qquad J^S\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}.$$

Suppose $\phi : S \to \mathbb{C}P^3$ and write $d\phi\left(\frac{\partial}{\partial t}\right) = \Phi_t$ where Φ_t is the linear map described in (2.5) above. Then the left and right hand sides of (2.4) are given by

$$d\phi \circ J^S\left(\frac{\partial}{\partial x}\right) = d\phi\left(\frac{\partial}{\partial y}\right) = \Phi_y$$

and

$$J_1 \circ d\phi\left(\frac{\partial}{\partial x}\right) = J_1(\Phi_x) = i\Phi_x$$

respectively, so that ϕ is J_1-holomorphic if and only if

$$\Phi_x + i\Phi_y = 0$$

or, alternatively,

$$f_{\bar{z}} - \frac{\langle f_{\bar{z}}, f\rangle}{|f|^2} f = 0, \qquad f = [\phi].$$

2.3.5. How to Recognise a J_2-holomorphic Curve in $\mathbb{C}P^3$. Since $T\mathbb{C}P^3 = H \oplus V$ splits as a direct sum of horizontal and vertical subspaces, a vector $u \in T\mathbb{C}P^3$ may be decomposed into horizontal and vertical components as $u = u_H + u_V$. Then

$$J_1(u) = J_1(u_H + u_V) = J_H(u_H) + J_V(u_V),$$

$$J_2(u) = J_2(u_H + u_V) = J_H(u_H) - J_V(u_V) = J_1(u_H - u_V) = J_1(\hat{u}),$$

say, so that knowing how to obtain $\hat{u}$ from u means that it is only necessary to use the complex structure J_1 and the projections onto the horizontal and vertical subspaces in calculations. To this end, note that at the point $[p] \in \mathbb{C}P^3$, the vertical subspace $V_p = \mathrm{Hom}\{[p], j[p]\}$ and so

$$u_V = \frac{\langle u, p\rangle}{|p|^2} p + \frac{\langle u, jp\rangle}{|p|^2} jp = \frac{\langle u, jp\rangle}{|p|^2} jp.$$

This gives

$$u_H = u - u_V = u - \frac{\langle u, jp \rangle}{|p|^2} jp$$

and

$$\hat{u} = u_H - u_V = u - 2\frac{\langle u, jp \rangle}{|p|^2} jp.$$

Notice that $\hat{u}$ corresponds to the linear map $\hat{\Phi}_t$ where

$$\hat{\Phi}_t : \phi \mapsto \phi_t - \frac{\langle \phi_t, \phi \rangle}{|\phi|^2}\phi - 2\frac{\langle \phi_t, j\phi \rangle}{|\phi|^2} j\phi.$$

Let z, J^S, ϕ, $d\phi$ be as in section 2.3.4 above. Then the left and right hand sides of (2.4) with $J^Z = J_2$ are given by

$$d\phi \circ J^S \left(\frac{\partial}{\partial x} \right) = d\phi \left(\frac{\partial}{\partial y} \right) = \Phi_y$$

and

$$J_2 \circ d\phi \left(\frac{\partial}{\partial x} \right) = J_2(\Phi_x) = J_1(\hat{\Phi}_x) = i\hat{\Phi}_x$$

respectively. Also,

$$d\phi \circ J^S \left(\frac{\partial}{\partial y} \right) = -d\phi \left(\frac{\partial}{\partial x} \right) = -\Phi_x$$

and

$$J_2 \circ d\phi \left(\frac{\partial}{\partial y} \right) = J_2(\Phi_y) = J_1(\hat{\Phi}_y) = i\hat{\Phi}_y$$

so that ϕ is J_2-holomorphic if and only if

$$\hat{\Phi}_x + i\Phi_y = 0$$

$$\Phi_x + i\hat{\Phi}_y = 0.$$

These conditions expand to give

$$\begin{aligned} \phi_{\bar{z}} - \frac{\langle \phi_{\bar{z}}, \phi \rangle}{|\phi|^2}\phi - \frac{\langle \phi_{\bar{z}}, j\phi \rangle}{|\phi|^2} j\phi = 0 \\ \langle \phi_z, j\phi \rangle = 0. \end{aligned} \tag{2.6}$$

Note that if $\phi = [1, z_1, z_2, z_3]$, the equations (2.6) hold if and only if

$$z_{1_z} - z_{2_z} z_3 + z_{3_z} z_2 = 0$$
$$z_{2_{\bar{z}}} = \frac{z_{1_{\bar{z}}}}{1 + |z_1|^2} (\bar{z}_1 z_2 - \bar{z}_3),$$
$$z_{3_{\bar{z}}} = \frac{z_{1_{\bar{z}}}}{1 + |z_1|^2} (\bar{z}_1 z_3 + \bar{z}_2).$$

In particular, if ϕ is J_1-holomorphic ($z_{j_{\bar{z}}} = 0$, $j = 1, 2, 3$) and J_2-holomorphic then ϕ is horizontal. Moreover, any two of these properties imply the third.

J_2-holomorphic curves arise as the lifts of conformal, harmonic maps (minimal immersions). Indeed, a result of Eells-Salamon (theorem 3.12) shows that there is an essentially bijective correspondence between J_2-holomorphic curves and conformal, minimal immersions. Also, proposition 3.4 gives the conditions on the moving frame of $f : S \to S^4$ for the lift to $\mathbb{CP}^3$ to be J_2-holomorphic. The results in section 5.5.1 give the implications of J_2-holomorphicity for the harmonic sequence.

CHAPTER 3

Lifts by Moving Frames

In this chapter, the lifts of $f : S \to S^4$ to different homogeneous spaces of SO(5) are studied. The chapter begins with the theory of moving frames followed by some discussion on maps into homogeneous spaces of Lie groups. The focus then falls particularly on the case of SO(5), where it is discovered that, owing to metric considerations, it is in fact better to consider this in the wider context of SO(6). In section 3.4, the moving frame is used to define lifts to $SO(6)/K$ and to explore the harmonicity and conformality conditions for these lifts. Section 3.5 illuminates these conditions for lifts to SO(5)/U(2) and also derives the J_1- and J_2-holomorphicity conditions for such lifts. These afford straightforward proofs of two results of Eells-Salamon (theorems 3.12, 3.14). Section 3.6 focuses on the above conditions for lifts to $SO(5)/T^2$ and 3.7 shows that the moving frame description is a useful context in which to demonstrate the link with solutions of the $\mathfrak{so}(5)$-Toda equations. Further results appearing in section 3.8 include the remark that harmonic $f : S \to S^4$ has harmonic lifts to SO(5)/U(2) and $SO(5)/T^2$. Moreover, the lift into SO(5)/U(2) of an immersion with parallel mean curvature is harmonic.

3.1. Moving Frames

First note that the following conventions on indices will be used

$$0 \leq a, b, \ldots \leq n$$

$$1 \leq i, j, \ldots \leq 2$$

$$r, s, \ldots \in \{0, 3, \ldots, n\}.$$

Let e_a and θ^b be smooth fields of dual orthonormal frames and co-frames (one-forms). Let $x : S \to N^n$ be an immersion of the surface S in an n-dimensional manifold $N \subseteq \mathbb{R}^{n+1}$ and choose the frame field $e_a(p)$, $p \in S$ such that $e_1(p), e_2(p)$ are the tangent vectors and $e_0(p), e_3(p), \ldots, e_n(p)$ are the normal vectors with $e_0(p)$ normal to N in $\mathbb{R}^{n+1}$. Then, restricting to S, $\theta^r = 0$ and

$$dx = \theta^1 e_1 + \theta^2 e_2. \tag{3.1}$$

The first fundamental form, or metric, is given by

$$I = ds^2 = dx.dx = (\theta^1)^2 + (\theta^2)^2$$

and the area form is $\theta^1 \wedge \theta^2$.

The rate of change of the moving frame in the directions e_a gives rise to 1-forms w_a^b such that

$$de_a = w_a^b e_b. \tag{3.2}$$

Let $F = (e_0 | \ldots | e_n)$, with $e_0 = \dfrac{x}{|x|}$, be an orthonormal frame for the immersion $x : S \to N$ and let $A = F^{-1}dF$ be the matrix of 1-forms (w_a^b). Notice that

$$w_a^b = e_b.de_a = d(e_a.e_b) - de_b.e_a = -w_b^a$$

so that in particular $w_a^a = 0$ for all a and A is a skew-symmetric matrix. Also, since $\text{span}\{e_3, \ldots, e_n\} = N(S)$ and $de_0 \in TS$, $w_0^r = 0$.

Applying the exterior differential operator d to equation (3.1) and using the fact that $d^2 = 0$ gives

$$0 = d(dx) = d(\theta^a e_a) = d\theta^a e_a - \theta^a \wedge de_a$$

$$= d\theta^a e_a - \theta^a \wedge w_a^b e_b \qquad \text{by (3.2)}$$

$$= \{d\theta^b - \theta^a \wedge w_a^b\} e_b.$$

Similarly,

$$0 = d(de_a) = d(w_a^b e_b) = dw_a^b e_b - w_a^b \wedge de_b$$

$$= dw_a^b e_b - w_a^c \wedge de_c$$

$$= \{dw_a^b - w_a^c \wedge w_c^b\} e_b$$

and the resulting equations

$$d\theta^b + w_a^b \wedge \theta^a = 0 \tag{3.3}$$

$$dw_a^b + w_c^b \wedge w_a^c = 0 \tag{3.4}$$

are called the structure equations for the submanifold $x(S)$ of N. Some of these equations are given particular names. Equations (3.3) with the value of b restricted to $\{0, 3, \ldots, n\}$ are called the symmetry equations;

$$0 = d\theta^r + w_a^r \wedge \theta^a = w_a^r \wedge \theta^a.$$

Equations (3.4) are categorised depending as (a, b) is restricted to (i, j), (r, s) or a mixture (r, i) and

$$dw_i^j + w_c^j \wedge w_i^c = 0 \qquad \text{Gauss equation on } S$$

$$dw_s^r + w_c^r \wedge w_s^c = 0 \qquad \text{Gauss equations on } NS$$

$$dw_i^r + w_c^r \wedge w_i^c = 0 \qquad \text{Codazzi-Mainardi equations.}$$

Futhermore, the equations (3.4) are precisely the integrability conditions required to ensure the existence of an orthonormal frame field $e_0, \ldots, e_n$ such that $de_a = w_a^b e_b$

and equations (3.3) are the conditions which ensure that $dx = \theta^a e_a$ can be integrated up to give a co-ordinate neighbourhood for a surface with metric $(\theta^1)^2 + (\theta^2)^2$.

Now suppose that $w_a^b = h^b_{a\,j}\theta^j$ for real coefficients $h^b_{a\,j}$ so that $w_a^b(e_i) = h^b_{a\,i}$ and in particular the $A_i = A(e_i)$ are real matrices given by

$$A_i = F^{-1}dF(e_i) = F^{-1}e_i(F) = (h^b_{a\,i}).$$

Since

$$\begin{aligned}(w_a^b \wedge \theta^a)(e_i, e_j) &= w_a^b(e_i)\theta^a(e_j) - w_a^b(e_j)\theta^a(e_i) \\ &= h^b_{a\,i}\delta_{aj} - h^b_{a\,j}\delta_{ai} \qquad (3.5) \\ &= h^b_{j\,i} - h^b_{i\,j},\end{aligned}$$

the symmetry equations imply

$$h^0_{1\,2} = h^0_{2\,1},\ h^3_{1\,2} = h^3_{2\,1},\ h^4_{1\,2} = h^4_{2\,1}, \ldots \qquad (3.6)$$

Also, $h^b_{a\,i} = e_b.e_i(e_a)$, so that

$$h^b_{0\,i} = e_b.e_i(e_0) = e_b.e_i = \delta_{bi}. \qquad (3.7)$$

In the same way that for a curve on a surface in $\mathbb{R}^3$ particular 1-forms are related to the notions of geodesic curvature, normal curvature and geodesic torsion, the 1-forms (w_a^b) in higher dimensional cases also have a geometrical significance. Indeed, the Levi-Civita connection for the target manifold N is given by the matrix of 1-forms (w_a^b) where equations (3.3) hold. Restriction to (w_j^i) over S gives a connection ∇^{tan} over S - in fact, $\nabla^{tan} = \nabla^S$, the uniquely defined Levi-Civita connection corresponding to the metric on S induced from that on N. Then the submatrix (w_s^r), $3 \leq r, s \leq n$ gives a connection ∇^{nor} on $N(S)$. Recall that for surfaces immersed in $\mathbb{R}^3$, the second fundamental form $I\!I$ is defined by

$$I\!I = -de_3.dx$$

where e_3 is normal to the surface in $\mathbb{R}^3$. By analogy, here we take the scalar product of $-de_r$ with dx;

$$\begin{aligned}\langle -de_r, dx\rangle &= \langle -w_r^b e_b, \theta^i e_i\rangle \\ &= -w_r^b \theta^i \delta_{bi} \\ &= w_i^r \theta^i \\ &= h^r_{i\,j}\theta^i\theta^j.\end{aligned} \tag{3.8}$$

Thus the $h^r_{i\,j}$ are none other than the components of $I\!I$ in the direction e_r. In view of this discussion, the matrix of 1-forms has a geometrical interpretation as:

$$\left(\begin{array}{c|c|c} & I & \\ \hline I & \nabla^{tan} & I\!I \\ \hline & I\!I & \nabla^{nor} \end{array}\right).$$

Finally, it is useful to calculate $\nabla^S_{e_i} e_j$ in terms of the coefficients $h^a_{b\,i}$. Since

$$\nabla^S_{e_i} e_j = a_{ij}e_1 + b_{ij}e_2 \qquad \text{for } i,j = 1,2$$

then

$$a_{ij} = \langle \nabla_{e_i} e_j, e_1\rangle = \langle w_j^k(e_i)e_k, e_1\rangle = w_j^1(e_i) = h^1_{j\,i},$$

$$b_{ij} = \langle \nabla_{e_i} e_j, e_2\rangle = \langle w_j^k(e_i)e_k, e_2\rangle = w_j^2(e_i) = h^2_{j\,i}$$

and

$$\nabla^S_{e_i} e_j = h^1_{j\,i}e_1 + h^2_{j\,i}e_2. \tag{3.9}$$

3.2. Maps into Homogeneous Spaces

Let G be a semi-simple, compact Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$ and consider two subspaces of $\mathfrak{g}$:

$$\mathfrak{g}_X = \{Y \in \mathfrak{g} : [X, Y] = 0\}$$

and

$$\mathfrak{g}^X = \{Z \in \mathfrak{g} : Z = ad(X)Y \text{ for some } Y \in \mathfrak{g}\}.$$

Then $\mathfrak{g}_X$ is the kernel of $ad(X)$ and $\mathfrak{g}^X = \mathrm{Im}ad(X) = ad(X)\mathfrak{g}$.

CLAIM. *Let $\mathfrak{g}$ be semi-simple. Then with respect to a positive definite Ad-invariant inner product, the spaces $\mathfrak{g}_X$ and $\mathfrak{g}^X$ are orthogonal and $\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X$.*

PROOF. See [19].

There is a subgroup G_X corresponding to $\mathfrak{g}_X$,

$$G_X = \{g \in G \mid Ad(g)X = X\}$$

and $\mathfrak{g}_X$ is the Lie algebra of G_X. Then, by the Orbit-Stabiliser Theorem,

$$G/G_X = Ad(G).X = \{Ad(g).X | g \in G\}.$$

In particular if $Y \in G$, the projection $G \to G/G_X$ takes Y to $Ad(Y).X \in G/G_X$.

It is natural to embed $G/G_X \hookrightarrow \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, via $gG_X \longmapsto Ad(g)X$ and carry out all calculations in the Lie algebra. (Thus, since $\mathfrak{g}$ is a vector space, we can think of $G/G_X \subseteq \mathbb{R}^k$ for some k.) So, for a map of S into G/K, if $K = G_X$ for some X the inclusion of $G/K = G/G_X$ in $\mathfrak{g}$ simplifies all the calculations. So the question is, given K, how to choose X so that $G_X = K$? The requirement is to choose an X for which the stabiliser $\mathfrak{g}_X$ is $\mathfrak{k}$, the Lie algebra of K. The different conjugacy classes $< X >$ give rise to different subgroups $K = G_X$ and, clearly, different representatives X give different embeddings of G/G_X in $\mathfrak{g}$.

As an example, take $G = \mathrm{SO}(3)$, and choose $X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$. Then

$$\begin{aligned} \mathfrak{so}(3)_X &= \{Y \in \mathfrak{so}(3) \mid [X, Y] = 0\} \\ &= \left\{ \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} = \mathfrak{so}(2). \end{aligned}$$

Thus $\mathrm{SO}(3)_X = \mathrm{SO}(2)$. Then

$$\mathrm{SO}(3)/\mathrm{SO}(3)_X = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} \subseteq \mathfrak{so}(3).$$

The Lie algebra $\mathfrak{so}(3)$ is identified with $\mathbb{R}^3$ via the correspondence

$$\begin{pmatrix} 0 & z & y \\ -z & 0 & x \\ -y & -x & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and $[A, \tilde{A}]$ corresponds to $\underline{a} \wedge \underline{\tilde{a}}$. In particular

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The $\mathrm{SO}(3)$-orbit of $(0, 0, 1)$ in $\mathbb{R}^3$ is S^2 (notice the stabiliser of $(0, 0, 1)$ is $\mathrm{SO}(2)$). Therefore $\mathrm{SO}(3)/\mathrm{SO}(3)_X = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2$. Further examples follow in section 3.3.

In order to look at the properties of a map $\phi : M \to N = G/K$ it is necessary to use a suitable metric on G/K. It is natural to use a G-invariant metric, of which there are many choices. Perhaps the most natural is that given by the Killing form on the Lie algebra $\mathfrak{g}$. In the case of $K = T$, the maximal torus of G, this metric is not Kähler, although there are plenty of choices of G-invariant metrics which are. To obtain a G-invariant metric on G/T, choose an $Ad(T)$-invariant metric $\langle\ ,\ \rangle$ on $\mathfrak{g}$ and define a metric on $M = G/T$ by

$$\langle Z_1, Z_2 \rangle_X = \langle Ad(g^{-1})Z_1, Ad(g^{-1})Z_2 \rangle$$

where $Z_1, Z_2 \in T_X M$, $X = Ad(g)H \in \mathfrak{g}$ for some particular $g \in G$ and a regular element $H \in \mathfrak{t}$. Every G-invariant metric on G/T is so obtained [9].

3.3. The $G = \mathrm{SO}(5)$ Case

Recall (section 2.1) that an immersion $f : S \to S^4$ gives rise to an orthonormal frame $\tilde{F} \in \mathrm{SO}(5)$ and that there are naturally defined lifts of f into the homogeneous spaces $\mathrm{SO}(5)/K$. The thesis is concerned in particular with the maximal subgroups $K = \mathrm{U}(2)$ and T^2.

It is well-known (and will be shown in Chapter 4) that the homogeneous space $\mathrm{SO}(5)/\mathrm{U}(2)$ may be identified with $\mathbb{C}\mathrm{P}^3$. Despite the fact that $\mathbb{C}\mathrm{P}^3$ is a symmetric space, its representation as the quotient $\mathrm{SO}(5)/\mathrm{U}(2)$ is not, in the sense that $(\mathrm{SO}(5), \mathrm{U}(2))$ is not a 'symmetric pair' (there is no involutive automorphism of $\mathrm{SO}(5)$ which has $\mathrm{U}(2)$ as fixed subalgebra [22]). However, recall that the inclusion of $\mathrm{SO}(2m-1)$ in $\mathrm{SO}(2m)$ induces a map $\mathrm{SO}(2m-1)/\mathrm{U}(2m-1) \hookrightarrow \mathrm{SO}(2m)/\mathrm{U}(2m)$ and that this is in fact a bijection. Then

$$\mathrm{SO}(5)/\mathrm{U}(2) \hookrightarrow \mathrm{SO}(6)/\mathrm{U}(3)$$

and $(\mathrm{SO}(6), \mathrm{U}(3))$ is a symmetric pair. Thus it is more convenient to work in the context of $G = \mathrm{SO}(6)$, and indeed the $\mathrm{SO}(6)$-invariant metric induced by the inclusion of $\mathrm{SO}(6)/\mathrm{U}(3)$ in the Lie algebra $\mathfrak{so}(6)$ is then the standard Fubini-Study metric on $\mathbb{C}\mathrm{P}^3$.

The Lie algebra $\mathfrak{so}(6)$ splits as $\mathfrak{so}(6) = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$

is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{so}(6)$. Now,

$$\mathfrak{so}(6) = \{X \in \mathrm{M}(6;\mathbb{R}) | X + X^t = 0\}$$

and thinking of such matrices in terms of 2×2 blocks gives straightforward descriptions of the decompositions of $\mathfrak{so}(6)$ for different K. To this end, describe an $\mathfrak{so}(6)$-matrix as (B_{pq}) for $p, q \in \{1,2,3\}$, where each B_{pq} is a 2×2 block in which p labels rows and q columns.

Let $K = T$. Then $\mathfrak{so}(6)$ decomposes as $\mathfrak{so}(6) = \mathfrak{t} \oplus \mathfrak{m}$ where

- the diagonal blocks B_{pp} of $A \in \mathfrak{so}(6)$ are each of the form (λJ), $\lambda \in \mathbb{R}$, $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and so these blocks belong to $\mathfrak{t}$, the Lie algebra of the maximal torus.
- the remaining 2×2 blocks B_{pq}, $p \neq q$, make up the complement $\mathfrak{m}$. (Notice that this gives $\dim \mathfrak{m} = 12$.)

An element $H = \mathrm{diag}\{\lambda_1 J, \lambda_2 J, \lambda_3 J\} \in \mathfrak{t}$ is regular when $\lambda_1, \lambda_2, \lambda_3$ are distinct real numbers and for such H, $\mathfrak{so}(6)_H = \mathfrak{t}$.

Let $K = \mathrm{U}(3)$. Then $\mathfrak{so}(6)$ decomposes as $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}$ where

- since $\mathfrak{t} \subseteq \mathfrak{u}(3)$, the diagonal blocks B_{pp} (see above) form part of the $\mathfrak{u}(3)$-component of $X \in \mathfrak{so}(6)$.
- the off-diagonal blocks B_{pq}, $p \neq q$, of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ split as

$$\frac{1}{2}\begin{pmatrix} a+d & -b+c \\ b-c & a+d \end{pmatrix} \quad - \quad \mathfrak{u}(3)\text{-component}$$

and

$$\frac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & -a+d \end{pmatrix} \quad - \quad \mathfrak{m}\text{-component.}$$

Counting independent entries shows that here $\dim \mathfrak{m} = 6$.

Notice that $H = \mathrm{diag}\{J, J, J\}$ is a regular element of $\mathfrak{u}(3)$ since the stabiliser of such an H in $\mathfrak{so}(6)$ is $\mathfrak{u}(3)$.

3.4. Description of Lifts into $\mathbf{SO(5)}/K$

Let $F : U \to \mathrm{SO}(5)$ be an adapted orthonormal frame for the immersion $f : S \to S^4$. By the remarks in section 3.3 and an abuse of the notation, let us also write F for $\left(\begin{array}{c|c} 1 & \\ \hline & \mathrm{F} \end{array}\right)$, the composition of F with the inclusion of SO(5) in SO(6).

Then the lifts $\phi : S \to \mathrm{SO}(6)/K$ are given by

$$\phi = FHF^{-1} = Ad(F)H \subseteq \mathfrak{so}(6)$$

$$\text{where} \begin{cases} H = \mathrm{diag}\{\lambda_1 J, \lambda_2 J, \lambda_3 J\} & \text{when } K = T, \\ H = \mathrm{diag}\{J, J, J\} & \text{when } K = \mathrm{U}(3). \end{cases}$$

Moreover, tangent vectors to $\mathrm{SO}(6)/K$ at the point F are described by $d\phi$ and

$$d\phi = Ad(F)([F^{-1}dF, H]) \in Ad(F)\mathfrak{m}. \tag{3.10}$$

Write $d\phi = Ad(F)[A, H]$ where, as in section 3.1, A is the matrix of 1-forms $F^{-1}dF$ and $A_i = A(e_i) = F^{-1}dF(e_i)$. Now, A has the form

$$\left(\begin{array}{c|c|c} 0 & -P^t & 0 \\ \hline P & aJ & -Q^t \\ \hline 0 & Q & bJ \end{array}\right)$$

with $P, Q \in M_2(\mathbb{R})$ and $a, b \in \mathbb{R}$ so that for $i = 1, 2$,

$$A_i = \left(\begin{array}{c|c|c} 0 & -P_i^t & 0 \\ \hline P_i & a_i J & -Q_i^t \\ \hline 0 & Q_i & b_i J \end{array}\right). \tag{3.11}$$

In fact it is clear that

$$P = \begin{pmatrix} 0 & w_0^1 \\ 0 & w_0^2 \end{pmatrix} \quad \text{so that} \quad P_i = \begin{pmatrix} 0 & h_{0\,i}^1 \\ 0 & h_{0\,i}^2 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{1i} \\ 0 & \delta_{2i} \end{pmatrix} \ \text{(by (3.7))}$$

$$\text{and } Q = \begin{pmatrix} w_1^3 & w_2^3 \\ w_1^4 & w_2^4 \end{pmatrix} \quad \text{so that} \quad Q_i = \begin{pmatrix} h_{1\,i}^3 & h_{2\,i}^3 \\ h_{1\,i}^4 & h_{2\,i}^4 \end{pmatrix}.$$

Also $a = w_1^2$ and $b = w_3^4$ give $a_i = h_{1\,i}^2$ and $b_i = h_{3\,i}^4$ respectively.

Let $H = \operatorname{diag}\{\lambda_1 J, \lambda_2 J, \lambda_3 J\}$ for some $\lambda_k \in \mathbb{R}$. Then

$$[A_i, H] = \left(\begin{array}{c|c|c} 0 & -X_i^t & 0 \\ \hline X_i & 0 & -Y_i^t \\ \hline 0 & Y_i & 0 \end{array}\right), \tag{3.12}$$

where

$$X_i = \lambda_1 P_i J - \lambda_2 J P_i \quad \text{and} \quad Y_i = \lambda_2 Q_i J - \lambda_3 J Q_i. \tag{3.13}$$

3.4.1. Harmonic Lifts. Recall that a map ϕ is harmonic if and only if $\operatorname{trace}\nabla d\phi = 0$. By section 2.2.1, ϕ is harmonic if and only if

$$Ad(F)\sum_{i=1}^{2}\left\{[A_i,[A_i,H]] + [e_i(A_i),H] - [A(\nabla^S_{e_i}e_i),H]\right\}^{tan} = 0. \tag{3.14}$$

Here *tan* denotes the projection onto $\mathfrak{m} = T_I\mathrm{SO}(6)/K$. Calculations using (3.11), (3.12) and (3.9) show that

$$[A_i,[A_i,H]] = \left(\begin{array}{c|c|c} X_i^t P_i - P_i^t X_i & a_i X_i^t J & P_i^t Y_i^t - X_i^t Q_i^t \\ \hline a_i J X_i & X_i P_i^t - P_i X_i^t + Y_i^t Q_i - Q_i^t Y_i & b_i Y_i^t J - a_i J Y_i^t \\ \hline Q_i X_i - Y_i P_i & b_i J Y_i - a_i Y_i J & Y_i Q_i^t - Q_i Y_i^t \end{array}\right),$$

$$[e_i(A_i), H] = \left(\begin{array}{c|c|c} 0 & -e_i(X_i)^t & 0 \\ \hline e_i(X_i) & 0 & -e_i(Y_i)^t \\ \hline 0 & e_i(Y_i) & 0 \end{array}\right)$$

and

$$\begin{aligned}[A(\nabla^S_{e_i} e_i), H] &= \sum_{j=1}^{2} [h^j_{i\,i} A_j, H] \\ &= \left(\begin{array}{c|c|c} 0 & -h^1_{i\,i} X_1^t - h^2_{i\,i} X_2^t & 0 \\ \hline h^1_{i\,i} X_1 + h^2_{i\,i} X_2 & 0 & -h^1_{i\,i} Y_1^t - h^2_{i\,i} Y_2^t \\ \hline 0 & h^1_{i\,i} Y_1 + h^2_{i\,i} Y_2 & 0 \end{array}\right).\end{aligned}$$

Applying these results to (3.14) produces the conditions on the frame for ϕ to be harmonic.

Theorem 3.1. *The map $\phi : S \to SO(6)/K$ given by $\phi = FHF^{-1}$ is harmonic if and only if all of the following hold:*

(1) $\sum_{i=1}^{2} \{h^2_{1\,i} J X_i + e_i(X_i) - h^1_{i\,i} X_1 - h^2_{i\,i} X_2\}^{\mathfrak{m}} = 0$
(2) $\sum_{i=1}^{2} \{Q_i X_i - Y_i P_i\}^{\mathfrak{m}} = 0$
(3) $\sum_{i=1}^{2} \{h^4_{3\,i} J Y_i - h^2_{1\,i} Y_i J + e_i(Y_i) - h^1_{i\,i} Y_1 - h^2_{i\,i} Y_2\}^{\mathfrak{m}} = 0$

where $\mathfrak{so}(6) = \mathfrak{k} \oplus \mathfrak{m}$ and X_i, Y_i, P_i, Q_i are as in section 3.4.

3.4.2. Conformal Lifts. It is also interesting to compute the conditions on the coefficients $h^b_{a\,i}$ for the lifts to SO(5)/T and SO(5)/U(2) to be conformal maps. First, note that $\phi : S \to G/K$ is conformal if for all $p \in S$ and all $v_1, v_2 \in T_pS$,

$$\langle d\phi_p(v_1), d\phi_p(v_2)\rangle_{\phi(p)} = \lambda \langle v_1, v_2\rangle_p \tag{3.15}$$

where λ is a scalar function and $\langle\ ,\ \rangle$ denotes the inner product induced on G/K from the Killing form on $\mathfrak{g}$, i.e.

$$\langle A, B\rangle = -\frac{1}{2} \text{ trace } (\text{ad}(A) \text{ ad}(B))$$

and for the classical groups this is equal to $-\text{trace}(AB)$, up to multiplication by a scalar.

Let $TS = \text{span}\{e^w e_1, e^w e_2\}$ and recall (3.10) that $d\phi = Ad(F)[F^{-1}dF, H]$ for a suitable H. Then the left hand side of (3.15) becomes

$$\begin{aligned}\langle d\phi(e_i), d\phi(e_j)\rangle &= e^{2w}\langle Ad(F)[A_i, H], Ad(F)[A_j, H]\rangle \\ &= -\frac{1}{2}e^{2w} \text{ trace } ([A_i, H][A_j, H])\end{aligned}$$

and the right hand side reads

$$\lambda\langle e^w e_1, e^w e_2\rangle = \lambda e^{2w}\delta_{ij}.$$

By (3.12),

$$[A_i, H][A_j, H] = \left(\begin{array}{c|c|c} -X_i^t X_j & 0 & X_i^t Y_j^t \\ \hline 0 & -X_i X_j^t - Y_i^t Y_j & 0 \\ \hline Y_i X_j & 0 & -Y_i Y_j^t \end{array}\right)$$

and

$$- \text{ trace } ([A_i, H][A_j, H]) = \text{ trace } \{X_i^t X_j + X_i X_j^t + Y_i^t Y_j + Y_i Y_j^t\}$$

from which straightforward calculations give

$$\begin{aligned}\text{trace } ([A_1, H][A_1, H]) &= \lambda_1^2 + \lambda_2^2 + (\lambda_2 h^3_{21} + \lambda_3 h^4_{11})^2 + (\lambda_2 h^4_{21} - \lambda_3 h^3_{11})^2 \\ &\quad + (\lambda_2 h^3_{11} - \lambda_3 h^4_{21})^2 + (\lambda_2 h^4_{11} + \lambda_3 h^3_{21})^2, \\ \text{trace } ([A_2, H][A_2, H]) &= \lambda_1^2 + \lambda_2^2 + (\lambda_2 h^3_{22} + \lambda_3 h^4_{12})^2 + (\lambda_2 h^4_{22} - \lambda_3 h^3_{12})^2 \\ &\quad + (\lambda_2 h^3_{12} - \lambda_3 h^4_{22})^2 + (\lambda_2 h^4_{12} + \lambda_3 h^3_{22})^2, \\ \text{trace } ([A_1, H][A_2, H]) &= (\lambda_2^2 + \lambda_3^2)(h^3_{12}(h^3_{11} + h^3_{22}) + h^4_{12}(h^4_{11} + h^4_{22})) \\ &\quad - 2\lambda_2\lambda_3(h^3_{11}h^4_{22} - h^3_{22}h^4_{11}).\end{aligned}$$

Then ϕ is conformal if and only if both

$$\text{trace}([A_1, H]^2) = \lambda = \text{ trace}([A_2, H]^2)$$

$$\text{trace}([A_1, H][A_2, H]) = 0$$

are satisfied, which gives the following

Theorem 3.2. *Let $\phi : S \to SO(6)/K$ for $K = T, U(3)$ be given by $\phi = FHF^{-1}$ with $H = diag\{\lambda_1 J, \lambda_2 J, \lambda_3 J\}$. Then ϕ is conformal if and only if*

$$((h^3_{11})^2 - (h^3_{22})^2 + (h^4_{11})^2 - (h^4_{22})^2) = \frac{4\lambda_2\lambda_3}{(\lambda_2^2 + \lambda_3^2)}(h^4_{12}(h^3_{11} + h^3_{22}) - h^3_{12}(h^4_{11} + h^4_{22})),$$

$$(h^3_{12}(h^3_{11} + h^3_{22}) + h^4_{12}(h^4_{11} + h^4_{22})) = \frac{2\lambda_2\lambda_3}{(\lambda_2^2 + \lambda_3^2)}(h^3_{11}h^4_{22} - h^3_{22}h^4_{11}).$$

3.5. Lifts to SO(5)/U(2)

These ideas will now be studied in the particular case of lifts into the homogeneous space SO(5)/U(2).

3.5.1. Harmonic Lifts to SO(5)/U(2). First suppose that $K = \mathrm{U}(3)$, so that $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $X_i = [P_i, J]$, $Y_i = [Q_i, J]$. For ease of notation, we will study the entries of the matrix trace$\nabla d\hat{f}$ in terms of the 2×2 blocks B_{pq}:

$$\begin{aligned} B_{21} &= \sum_{i=1}^{2}\{a_i J[P_i, J] + e_i([P_i, J]) - \sum_{k=1}^{2} h^k_{ii}[P_k, J]\}^{\mathrm{m}} \\ &= h^2_{11}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + h^2_{12}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - h^2_{11}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - h^1_{22}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= 0. \end{aligned}$$

$$\begin{aligned} B_{31} &= \sum_{i=1}^{2}\{Q_i[P_i, J] - [Q_i, J]P_i\}^{\mathrm{m}} \\ &= \sum_{i=1}^{2}\left\{\begin{pmatrix} h^3_{1i}h^1_{0i} + h^3_{2i}h^2_{0i} & (2h^3_{1i} - h^4_{2i})h^2_{0i} - (2h^3_{2i} + h^4_{1i})h^1_{0i} \\ h^4_{1i}h^1_{0i} + h^4_{2i}h^2_{0i} & (2h^4_{1i} + h^3_{2i})h^2_{0i} - (2h^4_{2i} - h^3_{1i})h^1_{0i} \end{pmatrix}\right\}^{\mathrm{m}} \\ &= \sum_{i=1}^{2}\begin{pmatrix} h^4_{2i}h^1_{0i} - h^4_{1i}h^2_{0i} & -h^3_{2i}h^1_{0i} + h^3_{1i}h^2_{0i} \\ -h^3_{2i}h^1_{0i} + h^3_{1i}h^2_{0i} & -h^4_{2i}h^1_{0i} + h^4_{1i}h^2_{0i} \end{pmatrix}. \end{aligned}$$

So for this block in the matrix to have zero $\mathfrak{m}$-component we require

$$\begin{aligned} h^1_{02}(h^4_{11} - h^4_{22}) - h^4_{12}(h^1_{01} - h^2_{02}) &= 0 \\ h^1_{02}(h^3_{11} - h^3_{22}) - h^3_{12}(h^1_{01} - h^2_{02}) &= 0. \end{aligned} \tag{3.16}$$

But these are precisely two of the Gauss equations on $N(S)$ and so always hold. Finally,

$$\begin{aligned} B_{32} &= \sum_{i=1}^{2}\{b_i J[Q_i, J] - a_i[Q_i, J]J + e_i([Q_i, J]) - \sum_{k=1}^{2} h^k_{ii}[Q_k, J]\}^{\mathfrak{m}} \\ &= \sum_{i=1}^{2}\left\{(a_i + b_i)\begin{pmatrix} -s_i & r_i \\ r_i & s_i \end{pmatrix} + e_i\begin{pmatrix} r_i & s_i \\ s_i & -r_i \end{pmatrix} - \sum_{k=1}^{2} h^k_{ii}\begin{pmatrix} r_k & s_k \\ s_k & -r_k \end{pmatrix}\right\}^{\mathfrak{m}} \end{aligned}$$

where $r_i = h^3_{2i} + h^4_{1i}$ and $s_i = h^4_{2i} - h^3_{1i}$. So for this block in the matrix to have zero $\mathfrak{m}$-component we require

$$\begin{aligned} e_1(r_1) + e_2(r_2) - h^2_{11}(s_1 + r_2) - h^2_{12}(s_2 - r_1) - h^4_{31}s_1 - h^4_{32}s_2 &= 0 \\ e_1(s_1) + e_2(s_2) - h^2_{11}(s_2 - r_1) + h^2_{12}(s_1 + r_2) + h^4_{31}r_1 + h^4_{32}r_2 &= 0. \end{aligned} \tag{3.17}$$

By the Codazzi equations, these conditions simplify somewhat, giving rise to the following Theorem:

Theorem 3.3. *The lift to $SO(5)/U(2)$ of an immersion $f : S \to S^4$ is harmonic if and only if the equations*

$$\begin{aligned} e_1(h^3_{11} + h^3_{22}) - e_2(h^4_{11} + h^4_{22}) - h^4_{31}(h^4_{11} + h^4_{22}) - h^4_{32}(h^3_{11} + h^3_{22}) &= 0 \\ e_1(h^4_{11} + h^4_{22}) + e_2(h^3_{11} + h^3_{22}) + h^4_{31}(h^3_{11} + h^3_{22}) - h^4_{32}(h^4_{11} + h^4_{22}) &= 0 \end{aligned} \tag{3.18}$$

hold.

3.5.2. Holomorphic Lifts to SO(5)/U(2).

By the decomposition given at the end of section 3.3, observe that an element of $\mathfrak{m}$ is of the form

$$\left(\begin{array}{c|c|c} & -X^t & -Z^t \\ \hline X & & -R^t \\ \hline Z & R & \end{array}\right),$$

where

$$X = \begin{pmatrix} -y & x \\ x & y \end{pmatrix}, \qquad Z = \begin{pmatrix} -w & z \\ z & w \end{pmatrix}, \qquad R = \begin{pmatrix} r & s \\ s & -r \end{pmatrix} \tag{3.19}$$

and so an element of $\mathfrak{m}$ is determined by a six-tuple (r, s, x, y, z, w). Also, (r, s) describes the part of $\mathfrak{m}$ which is tangential to the fibres whereas (x, y, z, w) belongs to the horizontal subspace. The almost complex structure J_1 acts on $\mathfrak{m}$ as

$$J_1(r, s, x, y, z, w) = (-s, r, -y, x, -w, z) \tag{3.20}$$

and the almost complex structure J_2 acts as

$$J_2(r, s, x, y, z, w) = (s, -r, -y, x, -w, z). \tag{3.21}$$

Let J^S be the complex structure on S acting as $J^S(e_1) = e_2$, $J^S(e_2) = -e_1$ on an orthonormal basis e_1, e_2 for TS.

Proposition 3.4. (1) *The lift $\hat{f} : S \to SO(5)/U(2)$ is J_1-holomorphic if and only if*

$$r_2 = -s_1 \text{ and } s_2 = r_1. \tag{3.22}$$

(2) *The lift $\hat{f} : S \to SO(5)/U(2)$ is J_2-holomorphic if and only if*

$$r_2 = s_1 \text{ and } s_2 = -r_1, \tag{3.23}$$

with $r_i = h^3_{2i} + h^4_{1i}$ and $s_i = h^4_{2i} - h^3_{1i}$.

Proof. By (2.4), $\hat{f}$ is J_k-holomorphic if and only if

$$d\hat{f}(e_2) = J_k(d\hat{f}(e_1))$$

or, equivalently in matrix notation,

$$[A_2, H] = J_k[A_1, H].$$

Now, in terms of the characterisation in (3.19),

$$[A_1, H] \leftrightarrow (r_1, s_1, 0, -1, 0, 0)$$
$$[A_2, H] \leftrightarrow (r_2, s_2, 1, 0, 0, 0)$$

so that by (3.20), (3.21),

$$J_1[A_1, H] = (-s_1, r_1, 1, 0, 0, 0) \qquad \text{and} \qquad J_2[A_1, H] = (s_1, -r_1, 1, 0, 0, 0).$$

Hence the results.

□

These results will be used to give straightforward proofs of theorems 3.12 and 3.14 which are due to Eells-Salamon [17].

3.5.3. Conformal Lifts to SO(5)/U(2). In the SO(5)/U(2) case, the conditions on the frame for ϕ to be conformal are expressed concisely in terms of the coefficients r_i and s_i. Indeed, setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in theorem 3.2 gives the result of corollary 3.5.

Corollary 3.5. *The lift $\hat{f} : S \to SO(5)/U(2)$ is conformal if and only if*

$$(h^3_{11} - h^4_{12})^2 + (h^3_{12} + h^4_{11})^2 = (h^3_{12} - h^4_{22})^2 + (h^3_{22} + h^4_{12})^2$$
$$(h^3_{12} + h^4_{11})(h^3_{22} + h^4_{12}) + (h^3_{11} - h^4_{12})(h^3_{12} - h^4_{22}) = 0$$

or, in the notation of section 3.5.1,

$$r_1^2 + s_1^2 = r_2^2 + s_2^2, \qquad r_1 s_1 + r_2 s_2 = 0.$$

3.6. Lifts to SO(5)/T^2

Now let $K = T^3$ be the maximal torus of SO(6). Then H is a regular element of $\mathfrak{t}$ if and only if $\lambda_1, \lambda_2, \lambda_3$ are distinct real numbers. In this case, $\mathfrak{so}(6)$ splits as $\mathfrak{so}(6) = \mathfrak{t} \oplus \mathfrak{m}$, where $\mathfrak{m} = T_I SO(6)/T^3$. Thus, by the discussion at the end of section

3.3, for each $X \in \mathfrak{so}(6)$ the $\mathfrak{t}$-component of X consists of the three 2×2 blocks along the diagonal and the remaining (off-diagonal) blocks form the $\mathfrak{m}$-component.

Recall (section 3.4.1) that $\tilde{f} : S \to \mathrm{SO}(6)/T^3$ is harmonic if and only if (3.14) holds. We now study the $\mathfrak{m}$-component of the matrix trace$\nabla d\tilde{f}$ in 2×2 blocks:

$$\begin{aligned} B_{21} &= \sum_{i=1}^{2}\{h^2_{1\,i} JX_i + e_i(X_i) - \sum_{k=1}^{2} h^k_{i\,i} X_k\} \\ &= \sum_{i=1}^{2}\{h^2_{1\,1} X_2 - h^2_{1\,2} X_1 + e_i(X_i) - h^1_{i\,i} X_1 - h^2_{i\,i} X_2\} \\ &= 0 \end{aligned}$$

since the X_i are constant matrices.

$$\begin{aligned} B_{31} &= \sum_{i=1}^{2}\{Q_i X_i - Y_i P_i\} \\ &= \sum_{i=1}^{2} \begin{pmatrix} \lambda_1(h^3_{1\,i} h^1_{0\,i} + h^3_{2\,i} h^2_{0\,i}) & 2\lambda_2(h^3_{1\,i} h^2_{0\,i} - h^3_{2\,i} h^1_{0\,i}) + \lambda_3(h^4_{1\,i} h^1_{0\,i} + h^4_{2\,i} h^2_{0\,i}) \\ \lambda_1(h^4_{1\,i} h^1_{0\,i} + h^4_{2\,i} h^2_{0\,i}) & 2\lambda_2(h^4_{1\,i} h^2_{0\,i} - h^4_{2\,i} h^1_{0\,i}) + \lambda_3(h^3_{1\,i} h^1_{0\,i} + h^3_{2\,i} h^2_{0\,i}) \end{pmatrix} \\ &= \sum_{i=1}^{2} \begin{pmatrix} \lambda_1(h^3_{1\,i}\delta_{1i} + h^3_{2\,i}\delta_{2i}) & 2\lambda_2(h^3_{1\,i}\delta_{2i} - h^3_{2\,i}\delta_{1i}) + \lambda_3(h^4_{1\,i}\delta_{1i} + h^4_{2\,i}\delta_{2i}) \\ \lambda_1(h^4_{1\,i}\delta_{1i} + h^4_{2\,i}\delta_{2i}) & 2\lambda_2(h^4_{1\,i}\delta_{2i} - h^4_{2\,i}\delta_{1i}) + \lambda_3(h^3_{1\,i}\delta_{1i} + h^3_{2\,i}\delta_{2i}) \end{pmatrix}. \end{aligned}$$

Applying the Gauss equations and using the fact that $\lambda_1, \lambda_2, \lambda_3$ are distinct real numbers shows that this component is zero if and only if

$$\begin{aligned} h^3_{1\,1} + h^3_{2\,2} &= 0 \\ h^4_{1\,1} + h^4_{2\,2} &= 0. \end{aligned} \tag{3.24}$$

Finally,

$$B_{32} = \sum_{i=1}^{2}\{h^4_{3\,i} JY_i - h^2_{1\,i} Y_i J + e_i(Y_i) - \sum_{k=1}^{2} h^k_{i\,i} Y_k\}.$$

This block gives rise to four equations which simplify via the Gauss equations to

$$\lambda_2\{e_2(h^3_{1\,1} + h^3_{2\,2}) - h^4_{3\,2}(h^4_{1\,1} + h^4_{2\,2})\} + \lambda_3\{e_1(h^4_{1\,1} + h^4_{2\,2}) + h^4_{3\,1}(h^3_{1\,1} + h^3_{2\,2})\} = 0$$

$$\lambda_3\{e_2(h^3_{1\,1} + h^3_{2\,2}) - h^4_{3\,2}(h^4_{1\,1} + h^4_{2\,2})\} + \lambda_2\{e_1(h^4_{1\,1} + h^4_{2\,2}) + h^4_{3\,1}(h^3_{1\,1} + h^3_{2\,2})\} = 0$$

$$\lambda_2\{e_2(h^4_{1\,1} + h^4_{2\,2}) + h^4_{3\,2}(h^3_{1\,1} + h^3_{2\,2})\} - \lambda_3\{e_1(h^3_{1\,1} + h^3_{2\,2}) - h^4_{3\,1}(h^4_{1\,1} + h^4_{2\,2})\} = 0$$

$$\lambda_3\{e_2(h^4_{1\,1} + h^4_{2\,2}) + h^4_{3\,2}(h^3_{1\,1} + h^3_{2\,2})\} - \lambda_2\{e_1(h^3_{1\,1} + h^3_{2\,2}) - h^4_{3\,1}(h^4_{1\,1} + h^4_{2\,2})\} = 0.$$

These calculations and the fact that $\lambda_1, \lambda_2, \lambda_3$ are considered to be distinct real numbers, establish the following result:

Theorem 3.6. *The lift to $SO(6)/T^3 = SO(5)/T^2$ of an immersion $f : S \to S^4$ is harmonic if and only if*

$$h^3_{11} + h^3_{22} = 0 = h^4_{11} + h^4_{22},$$

that is, if and only if $f : S \to S^4$ is itself a harmonic map.

3.7. Link with the Toda Equations

An important and much studied case is that in which $f : S \to S^4$ is conformal and harmonic. Such maps give rise, via their lifts into $\mathrm{SO}(5)/T^2$, to solutions of the Toda field equations for SO(5). The salient features of this theory will now be presented and it will be shown that the formulation of the lift in terms of the moving frame is ideal as it enables the key results to be read off from the matrices. The work in this section follows that of Bolton, Pedit and Woodward [5] and of Bolton and Woodward [9] and the reader is referred to these papers for details and proofs.

For any simple compact Lie group G, G/T is a Kähler manifold and an m-symmetric space for some m. This means that at each point p of G/T there is a geodesic symmetry of G/T of order m having p as an isolated fixed point. The symmetry comes from a canonically constructed automorphism τ (the Coxeter automorphism) of the Lie algebra and m is one plus the height of the highest root of $\mathfrak{g}$. Under the action of τ the complexification $\mathfrak{g}^{\mathbb{C}}$ splits as a direct sum

$$\mathfrak{g}^{\mathbb{C}} = \mathcal{M}_0 \oplus \ldots \oplus \mathcal{M}_{m-1}$$

where each $\mathcal{M}_k$ is the ξ_k-eigenspace of τ, $\xi = \exp(2\pi i/m)$. Then $\psi : S \to G/T$ is said to be τ-*adapted* if $d\psi(T^{1,0}S)$ lies in the subspace $\mathcal{M}_1$ of $\mathfrak{g}^{\mathbb{C}}$ and satisfies the non-degeneracy condition that for each $p \in S$ and all non-zero $v \in T_p^{1,0}S$, $d\psi_p(v)$ is cyclic ([5]), except perhaps for a discrete set of points p.

Any τ-adapted map $\psi : S \to G/T$ has an essentially unique local framing $F : S \to G$ whose Maurer-Cartan equations can be written as the Toda field equations for G. Thus, there is a bijective correspondence between solutions of the affine $\mathfrak{g}$-Toda equations and τ-adapted maps $\psi : S \to G/T$.

The key result [5] is that superconformal, harmonic maps $f : S \to S^{2m}$ have τ-adapted lifts into $\mathrm{SO}(2m+1)/T$ and conversely, if $\tilde{f} : S \to \mathrm{SO}(2m+1)/T$ is τ-adapted then $\pi \circ \tilde{f} : S \to S^{2m}$ is superconformal harmonic. This will now be illustrated in the SO(5)-case.

The Lie group SO(5) has rank 2 and the Lie algebra $\mathfrak{so}(5)$ has positive roots

$$\sigma_1, \quad \sigma_2, \quad \sigma_2 - \sigma_1, \quad \sigma_1 + \sigma_2$$

of which σ_1 and $\sigma_2 - \sigma_1$ are simple. The highest root is $\sigma_1 + \sigma_2$ with height 3. Let us express the matrix $X \in \mathfrak{so}(5)^{\mathbb{C}}$ as B_{pq} for $1 \le p, q \le 3$ where $B_{21}, B_{31} \in \mathbb{C}^2$, $B_{pq} \in \mathrm{M}_2(\mathbb{C})$ for $2 \le p, q \le 3$ and $B_{pq} = -B_{qp}^t$. Then the root spaces corresponding to the positive roots are described by the following basis elements, all other blocks B_{pq} consisting entirely of zeros in each case:

$$\mathfrak{g}^{\sigma_1} : B_{21} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = -B_{12}^t, \qquad \mathfrak{g}^{\sigma_2} : B_{31} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = -B_{13}^t,$$

$$\mathfrak{g}^{\sigma_2-\sigma_1} : B_{32} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = -B_{23}^t, \qquad \mathfrak{g}^{\sigma_1+\sigma_2} : B_{32} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = -B_{23}^t.$$

Then the direct sum $\mathfrak{g}_+$ of the positive root spaces consists of matrices of the form

$$\left(\begin{array}{cc|cc|cc} & & -a & ia & -b & ib \\ \hline a & & & & -c & ic \\ -ia & & & & -d & id \\ \hline b & & c & d & & \\ -ib & & -ic & -id & & \end{array}\right).$$

Now, $d\psi = Ad(F)[A,H] \in Ad(F)\mathfrak{m}$ and $T^{1,0}S = \text{span}\left\{\frac{\partial}{\partial z}\right\} = \text{span}\{e_1 - ie_2\}$. So

$$d\psi(T^{1,0}S) = d\psi(e_1 - ie_2) = [A_1 - iA_2, H].$$

By (3.12),

$$([A_1 - iA_2, H])_{21} = \begin{pmatrix} \lambda_1 & -i\lambda_2 \\ -i\lambda_1 & -\lambda_2 \end{pmatrix}$$

and $([A_1 - iA_2, H])_{32}$ has the form

$$\begin{pmatrix} \lambda_2(h^3_{12} - ih^3_{22}) + \lambda_3(h^4_{11} - ih^4_{12}) & -\lambda_2(h^3_{11} - ih^3_{12}) + \lambda_3(h^4_{12} - ih^4_{22}) \\ \lambda_2(h^4_{12} - ih^4_{22}) - \lambda_3(h^3_{11} - ih^3_{12}) & -\lambda_2(h^4_{11} - ih^4_{12}) - \lambda_3(h^3_{12} - ih^3_{22}) \end{pmatrix}. \quad (3.25)$$

Let f be a superminimal immersion. Then f is harmonic and real isotropic (section 2.2.2) so that

$$h^3_{11} = -h^3_{22} = h^4_{12}$$
$$h^4_{11} = -h^4_{22} = -h^3_{12}.$$

Thus f being superminimal implies that B_{32} has the form

$$\begin{pmatrix} \lambda & i\lambda \\ -i\lambda & \lambda \end{pmatrix},$$

that is, $d\phi(T^{1,0}S) \subseteq [\mathfrak{g}_+]$ which means that ψ is holomorphic. Conversely, if ψ is holomorphic, f is real isotropic and harmonic, that is to say, superminimal.

The automorphism τ is conjugation by the matrix $\operatorname{diag}(1, J, -I_2)$ and $\xi = \exp(\frac{\pi i}{2}) = i$. The i-eigenspace $\mathcal{M}_1$ is given by

$$\mathcal{M}_1 = \mathfrak{g}^{-(\sigma_1+\sigma_2)} \oplus \mathfrak{g}^{\sigma_1} \oplus \mathfrak{g}^{\sigma_2-\sigma_1}$$

and $\mathcal{M}_1$ consists of matrices of the form

$$\left(\begin{array}{c|cc|cc} & -a & ia & & \\ \hline a & & & -b & -c \\ -ia & & & -ib & -ic \\ \hline & b & ib & & \\ & c & ic & & \end{array}\right).$$

Now $\psi : S \to G/T$ is τ-adapted when $d\phi(T^{1,0}S) \subseteq [\mathcal{M}_1]$ and satisfies the non-degeneracy condition. Using (3.25), this is the case if and only if f is conformal and harmonic but not superminimal (i.e. f is superconformal) that is, such f has a τ-adapted lift to $\mathrm{SO}(5)/T^2$.

The following theorem is due to Black [3]:

Theorem 3.7. *Let $\psi : S \to G/T$ be τ-adapted. Then*

(1) *ψ is harmonic.*
(2) *For every closed subgroup K, $T \subset K \subset G$ the projection $\sigma_K \psi : S \to G/K$ is harmonic.*

Moreover, ψ and $\sigma_K \psi$ are equiharmonic, that is to say, harmonic with respect to any G-invariant metric on G/K.

By a τ-holomorphic map we mean a τ-adapted holomorphic map and in this case there is a bijective correspondence between solutions of the open $\mathfrak{so}(5)$-Toda equations and τ-holomorphic maps $\psi : S \to \mathrm{SO}(5)/T^2$.

On the other hand, when f is superconformal then ψ is τ-adapted and together with a non-singularity condition (amounting to requiring $d\psi$ to have a non-zero component in each of the root spaces which make up $\mathcal{M}_1$) this case gives a bijection with solutions of the affine $\mathfrak{so}(5)$-Toda equations. (Such ψ is called τ-primitive).

3.8. Some Further Results

Let $\hat{f}$ denote the lift of f to $SO(5)/U(2)$ and $\tilde{f}$ the lift of f to $SO(5)/T^2$.

Corollary 3.8. *If f is a harmonic map then both $\hat{f}$ and $\tilde{f}$ are harmonic maps.*

PROOF. Since f is a harmonic map into S^4 if and only if

$$h^3_{11} + h^3_{22} = 0 = h^4_{11} + h^4_{22},$$

the result follows easily from Theorems 3.3 and 3.6.

□

Corollary 3.9. *If f has parallel mean curvature, then $\hat{f}$ is harmonic.*

PROOF. Recall that the mean curvature vector of f is H where

$$H = \sum_i \frac{1}{2} II(e_i, e_i) = h^r_{ii} e_r$$

(sum over $r = 0, 3, 4$ and $i = 1, 2$). H is said to be parallel if

$$\nabla^\perp_X H = 0 \qquad \text{for all } X$$

($\nabla^\perp$ denotes the connection on the normal bundle). Now,

$$\begin{aligned}
\nabla^\perp_{e_j} h^r_{ii} e_r &= e_j(h^r_{ii}) e_r + h^r_{ii} \nabla^\perp_{e_j} e_r \\
&= e_j(h^r_{ii}) e_r + h^r_{ii} w^s_r(e_j) e_s \\
&= e_j(h^3_{11} + h^3_{22}) e_3 + e_j(h^4_{11} + h^4_{22}) e_4 + (h^3_{11} + h^3_{22}) h^4_{3j} e_4 + (h^4_{11} + h^4_{22}) h^3_{4j} e_3
\end{aligned}$$

so that f has parallel mean curvature vector if and only if

$$e_1(h^3_{1\,1} + h^3_{2\,2}) = h^4_{3\,1}(h^4_{1\,1} + h^4_{2\,2})$$

$$e_2(h^3_{1\,1} + h^3_{2\,2}) = h^4_{3\,2}(h^4_{1\,1} + h^4_{2\,2})$$

$$e_1(h^4_{1\,1} + h^4_{2\,2}) = -h^4_{3\,1}(h^3_{1\,1} + h^3_{2\,2})$$

$$e_2(h^4_{1\,1} + h^4_{2\,2}) = -h^4_{3\,2}(h^3_{1\,1} + h^3_{2\,2}).$$

Applying these conditions to the equations (3.18) gives the result.

□

Example 3.10. *All tori in $S^3 \subseteq S^4$ of the form*

$$f(x,y) = (0, r_1 \cos\frac{x}{r_1}, r_1 \sin\frac{x}{r_1}, r_2 \cos\frac{y}{r_2}, r_2 \sin\frac{y}{r_2})\ ,\ r_1^2 + r_2^2 = 1$$

have constant mean curvature in S^3 and thus have parallel mean curvature vector in S^4.

Now suppose that f is a conformal immersion. Then f_x and f_y have the same length and are orthogonal so that we may take

$$e_1 = e^{-w} f_x, \qquad e_2 = e^{-w} f_y \qquad \text{where } e^w = |f_x| = |f_y|.$$

Also $e_1 \leftrightarrow e^{-w}\dfrac{\partial}{\partial x}$, $e_2 \leftrightarrow e^{-w}\dfrac{\partial}{\partial y}$ and using $h^b_{a\,i} = e_b.e_i(e_a)$ gives the coefficients as:

$$h^2_{1\,1} = -e^{-w} w_y, \qquad h^2_{1\,2} = e^{-w} w_x$$

$$h^4_{3\,1} = e^{-w} e_4.e_{3x}, \qquad h^4_{3\,2} = e^{-w} e_4.e_{3y} \tag{3.26}$$

$$h^j_{1\,1} = -e^{-2w} e_j.f_{xx}, \qquad h^j_{1\,2} = -e^{-2w} e_j.f_{xy}, \qquad h^j_{2\,2} = -e^{-2w} e_j.f_{yy}, \qquad (j = 1, 2).$$

Then, when f is conformal, the lift $\hat{f}$ is harmonic if and only if

$$\begin{aligned} (e_4.\phi)_x + (e_3.\phi)_y &= e^w(Ae_4 - Be_3).\phi \\ (e_3.\phi)_x - (e_4.\phi)_y &= e^w(Ae_3 + Be_4).\phi \end{aligned} \tag{3.27}$$

where $\phi = f_{xx} + f_{yy}$, $A = 2h^2_{1\,2} + h^4_{3\,2}$ and $B = 2h^2_{1\,1} + h^4_{3\,1}$.

Remark 3.11. If $\tilde{J}$ is given by $\tilde{J}(e_1) = e_2$, $\tilde{J}(e_3) = e_4$, then the second equation of (3.27) is obtained from the first by applying $\tilde{J}$.

Conditions (3.23) and the coefficients (3.26) afford a straightforward proof of the following result of Eells-Salamon [17]:

Theorem 3.12. *The lift* $\hat{f} : S \to SO(5)/U(2)$ *is* J_2*-holomorphic if and only if* f *is conformal and harmonic.*

PROOF. Suppose f is conformal. Then conditions (3.23) hold if and only if

$$e_4.f_{xx} + e_3.f_{xy} = e_3.f_{xy} - e_4.f_{yy}$$

$$e_4.f_{xy} - e_3.f_{xx} = e_3.f_{yy} + e_4.f_{xy}$$

if and only if

$$e_3.(f_{xx} + f_{yy}) = 0$$

$$e_4.(f_{xx} + f_{yy}) = 0$$

that is, if and only if f is harmonic.

□

Recall (from section 2.2.2) that a map $f : S \to S^4$ is real isotropic if $f_z.f_z = 0$ (f is conformal) and $f_{zz}.f_{zz} = 0$.

Lemma 3.13. *The following properties of* f *are equivalent:*

(1) $f_{zz}.f_{zz} = 0$,

(2) $I\!I(e_1, e_1) - I\!I(e_2, e_2)$ *and* $2I\!I(e_1, e_2)$ *are a conformal basis for* NS,

(3) *The coefficients* $h^b_{a\,i}$ *satisfy the equations*

$$(h^3_{1\,1} - h^3_{2\,2})^2 + (h^4_{1\,1} - h^4_{2\,2})^2 = 4((h^3_{1\,2})^2 + (h^4_{1\,2})^2), \tag{3.28}$$

$$h^3_{1\,2}(h^3_{1\,1} - h^3_{2\,2}) + h^4_{1\,2}(h^4_{1\,1} - h^4_{2\,2}) = 0. \tag{3.29}$$

PROOF.

(1) $\iff$ (2): If $e_1 = \dfrac{\partial}{\partial x}$, $e_2 = \dfrac{\partial}{\partial y}$ then

$$I\!I(e_1, e_1) = f_{xx} - (f_{xx}.f)f, \qquad I\!I(e_2, e_2) = f_{yy} - (f_{yy}.f)f,$$
$$I\!I(e_1, e_2) = f_{xy}$$

and $f_{zz} = I\!I\left(\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial z}\right)$. Then

$$\begin{aligned} f_{zz} &= \frac{1}{4} I\!I(e_1 - ie_2, e_1 - ie_2) \\ &= \frac{1}{4}(I\!I(e_1, e_1) - I\!I(e_2, e_2) - 2iI\!I(e_1, e_2)) \end{aligned}$$

so that $f_{zz}.f_{zz} = 0$ if and only if both

$$|I\!I(e_1, e_1) - I\!I(e_2, e_2)|^2 = 4|I\!I(e_1, e_2)|^2$$
$$(I\!I(e_1, e_1) - I\!I(e_2, e_2) \perp I\!I(e_1, e_2)$$

hold, that is, if and only if $I\!I(e_1, e_1) - I\!I(e_2, e_2)$ and $2I\!I(e_1, e_2)$ are a conformal basis.

(2) $\iff$ (3): Recall

$$I\!I(e_1, e_1) = h^3_{1\,1} e_3 + h^4_{1\,1} e_4,$$
$$I\!I(e_2, e_2) = h^3_{2\,2} e_3 + h^4_{2\,2} e_4,$$
$$I\!I(e_1, e_2) = h^3_{1\,2} e_3 + h^4_{1\,2} e_4.$$

Then computing the lengths $|I\!I(e_1, e_1) - I\!I(e_2, e_2)|^2$, $|2I\!I(e_1, e_2)|^2$ and the inner product $(I\!I(e_1, e_1) - I\!I(e_2, e_2)).I\!I(e_1, e_2)$ shows easily that (2) and (3) are equivalent.

□

Theorem 3.14. [17]. *Let $f : S \to S^4$ be a conformal map. Then f is real isotropic if and only if the lift $\hat{f} : S \to SO(5)/U(2)$ is J_1-holomorphic.*

PROOF. Suppose first that the lift $\hat{f}$ is J_1-holomorphic, Then, by (3.22),

$$h^3_{22} + h^4_{12} = -h^4_{12} + h^3_{11}$$

$$\text{and} \qquad h^4_{22} - h^3_{12} = h^3_{12} + h^4_{11}.$$

Thus, $h^3_{11} - h^3_{22} = 2h^4_{12}$ and $h^4_{11} - h^4_{22} = -2h^3_{12}$ so that

$$(h^3_{11} - h^3_{22})^2 + (h^4_{11} - h^4_{22})^2 = 4((h^3_{12})^2 + (h^4_{12})^2),$$

$$\text{and} \qquad h^3_{12}(h^3_{11} - h^3_{22}) + h^4_{12}(h^4_{11} - h^4_{22}) = 0.$$

Then by lemma 3.13 $f_{zz}.f_{zz} = 0$ and together with conformality this means that f is real isotropic.

Conversely, suppose that f is real isotropic. Then in particular $f_{zz}.f_{zz} = 0$ so that by lemma 3.13, $\mathit{II}(e_1, e_1) - \mathit{II}(e_2, e_2)$ and $2\mathit{II}(e_1, e_2)$ are a conformal basis for NS and it follows that not both of h^3_{12}, h^4_{12} are zero. Suppose $h^4_{12} \neq 0$. Then in (3.29),

$$\frac{h^3_{12}}{h^4_{12}}(h^3_{11} - h^3_{22}) = -(h^4_{11} - h^4_{22})$$

and substituting into (3.28) gives

$$(h^3_{11} - h^3_{22})^2 = 4(h^4_{12})^2.$$

Thus

$$(h^3_{11} - h^3_{22}) = \pm 2h^4_{12}$$

$$(h^4_{11} - h^4_{22}) = \mp 2h^3_{12}.$$

(The choice of sign here just corresponds to the choice of orientation on S^4.) Comparing with (3.22) shows that the lift of f is J_1-holomorphic.

□

CHAPTER 4

Lifts by Quaternions

In this chapter it is shown that the lift of an immersion $f : S \to S^4$ to the twistor space $\mathbb{C}P^3$ may be obtained explicitly by formulating everything in terms of quaternions. Section 4.1 elucidates the correspondence between elements of SO(5) and elements of the universal cover Sp(2) in that it is shown how $P \in$ Sp(2) gives a matrix $\tilde{P} \in$ SO(5) (proposition 4.1) and, more importantly, vice versa (theorem 4.3). Thus, the adapted frame of f may be thought of as an element of Sp(2). Twistor theory appears in sections 4.2 and 4.3 and the specialisation of this for the S^4-case and in terms of quaternions is found in sections 4.4 and 4.5. The identification of $\mathbb{C}P^3$ with Sp(2)/U(1) $\times$ Sp(1) now makes it a straightforward matter to write down the lift of f into $\mathbb{C}P^3$ (corollary 4.8). In the case that f is a conformal immersion this gives rise to a particularly beautiful and simple formula for the lift, involving a stereographic co-ordinate q associated to f (theorem 4.11).

4.1. Explicit Relationship Between Sp(2) and SO(5)

It is well known that Sp(2) is the universal cover of SO(5). The following sections make this correspondence explicit and a standard epimorphism taking elements of Sp(2) to elements of SO(5) is given. It seems more difficult to discover the correspondence in the opposite direction but it turns out to be simply a matter of setting up a good notation and carrying out some linear algebra using quaternions.

4.1.1. The Epimorphism α : Sp(2) $\longrightarrow$ SO(5). First, recall that a quaternion is naturally expressed in the form $a + bj$, where a and b are complex numbers and that this gives rise to a description of the quaternions as certain matrices in $M_2(\mathbb{C})$ via the correspondence

$$(a + bj) \leftrightarrow \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

Then elements of Sp(2) are thought of as U(4)-matrices by identifying

$$\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \leftrightarrow \left(\begin{array}{c|c} (p_1) & (p_3) \\ \hline (p_2) & (p_4) \end{array}\right)$$

(note the 'transpose' which occurs here). Indeed, Sp(2) is the subgroup of U(4) which preserves the left quaternionic vector space structure on $\mathbb{C}^4 = \mathbb{H}^2$ and defining

$$\hat{J} = \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right) \tag{4.1}$$

this is expressed formally as

$$\mathrm{Sp}(2) = \{A \in \mathrm{U}(4) : \hat{J}\bar{A} = A\hat{J}\}.$$

It is useful to identify the spaces Skew$(4;\mathbb{C})$ and $\bigwedge^2 \mathbb{C}^4$ via the correspondence

$$\sum_{i<j} p_{ij} e_i \wedge e_j \leftrightarrow \begin{pmatrix} 0 & -p_{12} & -p_{13} & -p_{14} \\ p_{12} & 0 & -p_{23} & -p_{24} \\ p_{13} & p_{23} & 0 & -p_{34} \\ p_{14} & p_{24} & p_{34} & 0 \end{pmatrix}.$$

The group U(4) acts on $\bigwedge^2 \mathbb{C}^4$ as a group of isometries and so induces an action of Sp(2) on $\bigwedge^2 \mathbb{C}^4$. Observe that Sp(2) fixes $\hat{J} = e_1 \wedge e_2 + e_3 \wedge e_4$ and as a result takes the orthogonal complement $W = \text{span}\{e_1 \wedge e_2 + e_3 \wedge e_4\}^{\perp}$ into itself.

Now, W has an orthonormal basis given by

$$\begin{aligned} F_1 &= \frac{i}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4) & & \\ F_2 &= \frac{i}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3) & F_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3) \\ F_4 &= \frac{-i}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4) & F_5 &= \frac{-1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4). \end{aligned} \tag{4.2}$$

Let $*$ be the Hodge star operator, with $*^2 = 1$. Then, extending $*$ to be conjugate linear rather than complex linear, it is easy to check that $*F_k = F_k$ for all $k = 1, \dots, 5$, that is, that $\{F_k\}$ is a real basis for W which corresponds to the basis $\{X_k\}$ of $\text{Skew}(4;\mathbb{C})$ with

$$X_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} & -i & & \\ i & & & \\ \hline & & & i \\ & & -i & \end{array}\right), X_2 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} & & & -i \\ & & i & \\ \hline & -i & & \\ i & & & \end{array}\right), X_3 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right)$$

$$X_4 = -\frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} & & -i & \\ & & & -i \\ \hline i & & & \\ & i & & \end{array}\right), X_5 = -\frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} & & -1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right). \tag{4.3}$$

The standard epimorphism $\alpha : \text{Sp}(2) \to \text{SO}(5)$ is given as follows:

If $A \in \text{Sp}(2)$, then the columns of the corresponding matrix $\alpha(A) = \tilde{A} \in \text{SO}(5)$ are given by

$$\tilde{A}_k = A.X_k = AX_kA^t, \qquad \text{for } k = 1, \dots, 5,$$

where $\tilde{A}_k$ is thought of as a vector in $\mathbb{R}^5$ by taking as the entries the coefficients of the expression of $\tilde{A}_k$ as a linear combination of the X_j.

This formulation gives rise to the following proposition:

Proposition 4.1. *Under the standard epimorphism $\alpha : Sp(2) \to SO(5)$, the elements* $P = \pm \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$ *of $Sp(2)$ map to the element $\tilde{P} \in SO(5)$ where*

$$\tilde{P} = \left(\begin{array}{c|c|c|c|c} |p_1|^2 - |p_3|^2 & \multicolumn{4}{c}{2p_1\bar{p}_3} \\ \hline 2\bar{p}_1 p_2 & \begin{array}{c}\bar{p}_1 p_4 \\ +\bar{p}_3 p_2\end{array} & \begin{array}{c}\bar{p}_1 i p_4 \\ -\bar{p}_3 i p_2\end{array} & \begin{array}{c}\bar{p}_1 j p_4 \\ -\bar{p}_3 j p_2\end{array} & \begin{array}{c}\bar{p}_1 k p_4 \\ -\bar{p}_3 k p_2\end{array} \end{array}\right).$$

PROOF. Let $p_k = a_k + b_k j$, so that

$$P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 & -\bar{b}_1 & a_3 & -\bar{b}_3 \\ b_1 & \bar{a}_1 & b_3 & \bar{a}_3 \\ a_2 & -\bar{b}_2 & a_4 & -\bar{b}_4 \\ b_2 & \bar{a}_2 & b_4 & \bar{a}_4 \end{pmatrix} \in \mathrm{U}(4).$$

Then the columns of $\tilde{P} = \alpha(P)$ are given by calculating $\tilde{P}_k = PX_kP^t$ for each $k = 1, \dots, 5$. It is convenient (and more illuminating) to think of $\mathbb{R}^5 = \mathbb{R} \oplus \mathbb{R}^4 = \mathbb{R} \oplus \mathbb{H}$ and to express column vectors in the form $\begin{pmatrix} r \\ q \end{pmatrix}$ where $r \in \mathbb{R}$ and $q \in \mathbb{H}$. Thus, for example,

$$\begin{aligned} \tilde{P}_1 &= \begin{pmatrix} |a_1|^2 + |b_1|^2 - |a_3|^2 - |b_3|^2 \\ 2(\bar{a}_1 a_2 + b_1 \bar{b}_2) + 2(\bar{a}_1 b_2 - \bar{a}_2 b_1) j \end{pmatrix} \\ &= \begin{pmatrix} |p_1|^2 - |p_3|^2 \\ 2\bar{p}_1 p_2 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Similarly, writing the remaining columns in this way gives

$$\tilde{P}_2 = \begin{pmatrix} 2\mathrm{Re}(a_1\bar{a}_3 + b_1\bar{b}_3) \\ \bar{p}_1 p_4 + \bar{p}_3 p_2 \end{pmatrix}, \tilde{P}_3 = \begin{pmatrix} 2\mathrm{Im}(a_1\bar{a}_3 + b_1\bar{b}_3) \\ \bar{p}_1 i p_4 - \bar{p}_3 i p_2 \end{pmatrix}$$

$$\tilde{P}_4 = \begin{pmatrix} 2\mathrm{Re}(b_1 a_3 - a_1 b_3) \\ \bar{p}_1 j p_4 - \bar{p}_3 j p_2 \end{pmatrix}, \tilde{P}_5 = \begin{pmatrix} 2\mathrm{Im}(b_1 a_3 - a_1 b_3) \\ \bar{p}_1 k p_4 - \bar{p}_3 k p_2 \end{pmatrix}$$

so that

$$\tilde{P} = (\tilde{P}_1|\dots|\tilde{P}_5) = \left(\begin{array}{c|cccc} |p_1|^2 - |p_3|^2 & & & 2p_1\bar{p}_3 & \\ \hline 2\bar{p}_1p_2 & \begin{array}{c}\bar{p}_1p_4 \\ +\bar{p}_3p_2\end{array} & \begin{array}{c|}\bar{p}_1ip_4 \\ -\bar{p}_3ip_2\end{array} & \begin{array}{c|}\bar{p}_1jp_4 \\ -\bar{p}_3jp_2\end{array} & \begin{array}{c}\bar{p}_1kp_4 \\ -\bar{p}_3kp_2\end{array} \end{array}\right).$$

□

4.1.2. The Inverse of the Map α. Given a matrix P in Sp(2), the epimorphism α produces the corresponding matrix in SO(5). It will now be shown how to invert this correspondence, that is, how to produce from the element $\tilde{P}$ of SO(5) the related covering element(s) in Sp(2).

Let the given matrix $\tilde{P} \in SO(5)$ be written in the form

$$\tilde{P} = \left(\begin{array}{c|c|c|c|c} X_0 & A_0 & B_0 & C_0 & D_0 \\ \hline X & A & B & C & D \end{array}\right) \tag{4.5}$$

where $X_0, A_0, \dots, D_0 \in \mathbb{R}$, $X, A, \dots, D \in \mathbb{H}$. Then the columns of $\tilde{P}$ are orthonormal vectors with respect to the standard inner product on $\mathbb{R}^5$ so that, for example,

$$0 = \begin{pmatrix} A_0 \\ A \end{pmatrix} \cdot \begin{pmatrix} B_0 \\ B \end{pmatrix} = A_0B_0 + \mathrm{Re}(A\bar{B}). \tag{4.6}$$

If x denotes the quaternion given by $\frac{X}{1+X_0}$, define unit quaternions $a = A - A_0x$, $b = B - B_0x$, $c = C - C_0x$, $d = D - D_0x$. Then $\begin{pmatrix} A_0 \\ A \end{pmatrix} \cdot \begin{pmatrix} B_0 \\ B \end{pmatrix} = \mathrm{Re}(a\bar{b})$ and similarly for all other pairs. It is clear that if any four columns (rows) of an SO(5) matrix are given then the remaining column (row) is uniquely determined by orthogonality, unit length and determinant conditions. This fact is expressed neatly in terms of quaternions as:

Proposition 4.2. $\bar{a}b + \bar{c}d = 0$.

Theorem 4.3. *The matrix $\tilde{P} \in SO(5)$ corresponds to the following matrix in the covering space $Sp(2)$, up to sign:*

$$P = \frac{1}{\Delta}\begin{pmatrix} w & wx \\ -wa\bar{x} & wa \end{pmatrix},$$

where $w = \bar{a} + i\bar{b} + j\bar{c} + k\bar{d}$ and $\Delta^2 = |w|^2(1 + |x|^2)$.

PROOF. First note that P is indeed an Sp(2) matrix, since the rows are of unit length and are orthogonal under the standard $\mathbb{H}^2$ inner product (see appendix). Proposition 4.2 and the fact that a, b, c, d are unit quaternions and $\mathrm{Re}(\bar{a}b) = 0$ give

$$\begin{aligned} iwa &= i(\bar{a} + i\bar{b} + j\bar{c} + k\bar{d})a \\ &= i(1 + i\bar{b}a + j\bar{c}a + k\bar{d}a) \\ &= i\bar{b}b + \bar{a}b + k\bar{d}b + j\bar{c}b \\ &= wb. \end{aligned} \tag{4.7}$$

In a similar way, $jwa = wc$ and $kwa = wd$. In order to prove the theorem, it is enough to show that $\alpha(P) = \tilde{P}$. Set

$$p_1 = \frac{w}{\Delta}\,,\; p_2 = \frac{wx}{\Delta}\,,\; p_3 = -\frac{wa\bar{x}}{\Delta}\,,\; p_4 = \frac{wa}{\Delta}.$$

Then straightforward calculations show

$$\begin{aligned} |p_1|^2 - |p_3|^2 &= \frac{1 - |x|^2}{1 + |x|^2} = X_0, \\ 2\bar{p}_1 p_2 &= X, \\ \bar{p}_1 p_4 + \bar{p}_3 p_2 &= A\;,\; \ldots\ldots\;,\; \bar{p}_1 k p_4 - \bar{p}_3 k p_2 = D. \end{aligned} \tag{4.8}$$

Finally,

$$\begin{aligned}
2p_1\bar{p}_3 &= -\frac{2wx\bar{a}\bar{w}}{\Delta^2} \\
&= -\frac{2}{\Delta^2}(\mathrm{Re}(wx\bar{a}\bar{w}) + (wx\bar{a}\bar{w})_i i + (wx\bar{a}\bar{w})_j j + (wx\bar{a}\bar{w})_k k) \\
&= -\frac{2}{\Delta^2}(\mathrm{Re}(wx\bar{a}\bar{w}) - \mathrm{Re}(wx\bar{a}\bar{w}i)i - \mathrm{Re}(wx\bar{a}\bar{w}j)j - \mathrm{Re}(wx\bar{a}\bar{w}k)k) \quad (4.9) \\
&= -2\frac{|w|^2}{\Delta^2}(\mathrm{Re}(x\bar{a}) + \mathrm{Re}(x\bar{b})i + \mathrm{Re}(x\bar{c})j + \mathrm{Re}(x\bar{d})k) \\
&= (A_0 + B_0 i + C_0 j + D_0 k).
\end{aligned}$$

□

JUSTIFICATION FOR PROPOSITION 4.2. Suppose the matrices

$$\left(\begin{array}{c|c|c|c|c} X_0 & A_0 & B_0 & C_0 & D_0 \\ \hline X & A & B & C & D \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|cccc} |p_1|^2 - |p_3|^2 & \multicolumn{4}{c}{2p_1\bar{p}_3} \\ \hline 2\bar{p}_1 p_2 & \bar{p}_1 p_4 + \bar{p}_3 p_2 & \bar{p}_1 i p_4 - \bar{p}_3 i p_2 & \bar{p}_1 j p_4 - \bar{p}_3 j p_2 & \bar{p}_1 k p_4 - \bar{p}_3 k p_2 \end{array}\right)$$

correspond for some $p_1, \ldots, p_4 \in \mathbb{H}$ satisfying

$$|p_1|^2 + |p_2|^2 = 1 = |p_3|^2 + |p_4|^2, \quad p_1\bar{p}_3 + p_2\bar{p}_4 = 0.$$

Then $\bar{p}_3 = -\frac{\bar{p}_1 p_2 \bar{p}_4}{|p_1|^2}$ implies $|p_3|^2 + |p_4|^2 = \frac{|p_4|^2}{|p_1|^2} = 1$ so that $|p_1|^2 = |p_4|^2$ and $|p_2|^2 = |p_3|^2$. Comparing corresponding matrix entries shows $|p_1|^2 - |p_3|^2 = X_0$ giving $1 + X_0 = 2|p_1|^2$ and $2\bar{p}_1 p_2 = X$ whence $p_2 = \frac{p_1 X}{2|p_1|^2} = p_1 x$. Thus $p_3 = -p_4\bar{x}$.

Now,

$$\begin{aligned}
A &= \bar{p}_1 p_4 + \bar{p}_3 p_2 = \bar{p}_1 p_4 - x(\overline{\bar{p}_1 p_4})x, \\
B &= \bar{p}_1 i p_4 + \bar{p}_3 i p_2 = \bar{p}_1 i p_4 - x(\overline{\bar{p}_1 i p_4})x \quad \text{etc.}
\end{aligned} \tag{4.10}$$

CLAIM: If $Z = Y - x\bar{Y}x$ for quaternions Y, Z then $Y = \frac{Z + x\bar{Z}x}{1 - |x|^4}$.

For,

$$\begin{aligned} Z = Y - x\bar{Y}x &\Rightarrow Y = Z + x\bar{Y}x \\ &\Rightarrow \bar{Y} = \bar{Z} + \bar{x}Y\bar{x} \\ &\Rightarrow Y = Z + x(\bar{Z} + \bar{x}Y\bar{x})x = Z + x\bar{Z}x + |x|^4 Y \\ &\Rightarrow Y(1 - |x|^4) = Z + x\bar{Z}x. \end{aligned} \tag{4.11}$$

Thus,

$$\bar{p}_1 p_4 = \frac{A + x\bar{A}x}{1 - |x|^4} = \frac{a}{1 + |x|^2} \tag{4.12}$$

and similarly, $\bar{p}_1 i p_4 = \dfrac{b}{1+|x|^2}$, $\bar{p}_1 j p_4 = \dfrac{c}{1+|x|^2}$, $\bar{p}_1 k p_4 = \dfrac{d}{1+|x|^2}$. Equation (4.12) implies $\bar{p}_1 = a\bar{p}_4$ and substituting gives

$$\bar{p}_4 i p_4 = \frac{\bar{a}b}{1+|x|^2}\ ,\ \bar{p}_4 j p_4 = \frac{\bar{a}c}{1+|x|^2}\ ,\ \bar{p}_4 k p_4 = \frac{\bar{a}d}{1+|x|^2}.$$

But

$$\frac{\bar{a}b}{1+|x|^2} = \bar{p}_4 i p_4 = \frac{\bar{p}_4 j p_4 \bar{p}_4 k p_4}{|p_4|^2} = \frac{\bar{a}c\bar{a}d}{1+|x|^2} = -\frac{\bar{c}d}{1+|x|^2}$$

so that $\bar{a}b + \bar{c}d = 0$ is a compatibility condition on the columns. □

4.2. Twistor Space

Let N be a $2m$-dimensional Riemannian manifold with a fixed orientation. At each point $x \in N$, let Z_x denote the space of orthogonal complex structures on the tangent space T_xN which are compatible with both metric and orientation. Then $Z = \bigcup_{x\in N} Z_x$ is a fibre bundle over N. The projection $\pi : Z \to N$ is a Riemannian fibration associated to the orthonormal frame bundle of N and the fibre of π is the symmetric space $\mathrm{SO}(2m)/\mathrm{U}(m)$. Z is called the twistor space of N and $\pi : Z \to N$ the twistor fibration [12].

Now suppose that $N = S^{2m}$, a $2m$-dimensional sphere. Then $N = \mathrm{SO}(2m+1)/\mathrm{SO}(2m)$ and since $\pi : Z \to N$ has fibre $\mathrm{SO}(2m)/\mathrm{U}(m)$, the twistor space of S^{2m} is $Z = \mathrm{SO}(2m+$

1)/U(m).

4.3. Harmonic Maps and Horizontal, Holomorphic Curves

This section will illuminate the correspondence between superminimal harmonic maps $S \to S^{2m}$ and horizontal, holomorphic curves in $\mathrm{SO}(2m+1)/\mathrm{U}(m)$.

Definition 4.4. *A subspace $V(x) \subset \mathbb{C}^{2m+1}$ is called a maximal isotropic subspace if $dim V(x) = m$ and $v.v = 0$ for all $v \in V(x)$.*

Given a superminimal harmonic map $\phi : S \to S^{2m}$, form the harmonic sequence $\psi_0, \dots, \psi_{2m}$ with $\psi_m = \psi = i\pi\phi$. Then, for each $x \in S$,

$$V(x) = \mathrm{span}\{\psi_0(x), \dots, \psi_{m-1}(x)\} \subseteq \mathbb{C}^{2m+1}$$

is a maximal isotropic subspace and $V(x) \in I_m(\mathbb{C}^{2m+1})$, the space of maximal isotropic m-planes in $\mathbb{C}^{2m+1}$. Notice also that for each $v \in V$, $[v] \in Q_{2m-1}$ where

$$Q_{2m-1} = \{[X_0, \dots, X_{2m}] \in \mathbb{CP}^{2m} \mid X_0^2 + \dots + X_{2m}^2 = 0\}.$$

CLAIM. *$I_m(\mathbb{C}^{2m+1}) \subseteq Gr_m(\mathbb{C}^{2m+1})$ as a complex submanifold in a natural way.*

CLAIM. *$I_m(\mathbb{C}^{2m+1}) \cong SO(2m+1)/U(m)$.*

PROOF OF CLAIM: Use the Orbit-Stabiliser Theorem. $\mathrm{SO}(2m+1)$ rotates m-planes into m-planes and since $\mathrm{SO}(2m+1)$ is an orthogonal group the action preserves the dot product. Also $\mathrm{SO}(2m+1)$ acts transitively on $I_m(\mathbb{C}^{2m+1})$. Let $V \in I_m(\mathbb{C}^{2m+1})$. Then the elements taking V to itself make up the subgroup $\mathrm{U}(m)$. □

Thus, a superminimal map $\phi : S \to S^{2m}$ gives rise to a map $\tilde{\phi} : S \to \mathrm{SO}(2m+1)/\mathrm{U}(m)$ via

$$\tilde{\phi}(x) = V(x)$$

and with $\rho\tilde{\phi} = \phi$,

$$\rho : \mathrm{SO}(2m+1)/\mathrm{U}(m) \to \mathrm{SO}(2m+1)/\mathrm{SO}(2m) = S^{2m}$$

is the canonical projection which means $\rho(V) = e_V$ where $\mathbb{C}e_V = (V \oplus \bar{V})^{\perp}$ in $\mathbb{C}^{2m+1}$.

Proposition 4.5. *The map $\tilde{\phi}$ is* (1) *holomorphic and* (2) *horizontal.*

Proof.

(1) Let $\psi_p = [f_p]$ for each p. Then the harmonic sequence of ϕ looks (locally) like $f_0, \dots, f_{2m}$ with

$$\begin{aligned} \frac{\partial f_p}{\partial z} &= f_{p+1} + \frac{\partial}{\partial z} \log |f_p|^2 f_p \\ \frac{\partial f_p}{\partial \bar{z}} &= -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}. \end{aligned}$$

As a map into $\mathrm{Gr}_m(\mathbb{C}^{2m+1})$ $\tilde{\phi}$ is given by

$$f_0 \wedge \dots \wedge f_{m-1}.$$

But

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(f_0 \wedge \dots \wedge f_{m-1}) &= \frac{\partial f_0}{\partial \bar{z}} \wedge f_1 \wedge \dots \wedge f_{m-1} + \dots + f_0 \wedge \dots \wedge f_{m-2} \wedge \frac{\partial f_{m-1}}{\partial \bar{z}} \\ &= 0. \end{aligned}$$

(2) To show that $\tilde{\phi}$ is horizontal, check that $\left| d\tilde{\phi}\left(\frac{\partial}{\partial z}\right)\right|^2 = \left| d\phi\left(\frac{\partial}{\partial z}\right)\right|^2$.

$$\begin{aligned} \frac{\partial}{\partial z}(f_0 \wedge \dots \wedge f_{m-1}) &= \frac{\partial f_0}{\partial z} \wedge f_1 \wedge \dots \wedge f_{m-1} + \dots + f_0 \wedge \dots \wedge f_{m-2} \wedge \frac{\partial f_{m-1}}{\partial z} \\ &= \lambda(f_0 \wedge \dots \wedge f_{m-1}) + (f_0 \wedge \dots \wedge f_{m-2} \wedge f_m) \end{aligned}$$

and

$$\left\{ \frac{\partial}{\partial z}(f_0 \wedge \dots \wedge f_{m-1}) \right\}^{(f_0 \wedge \dots \wedge f_{m-1})^{\perp}} = f_0 \wedge \dots \wedge f_{m-2} \wedge f_m.$$

So

$$\left| d\tilde{\phi}\left(\frac{\partial}{\partial z}\right)\right|^2 = \frac{|f_0 \wedge \dots \wedge f_{m-2} \wedge f_m|^2}{|f_0 \wedge \dots \wedge f_{m-1}|^2} = \frac{|f_m|^2}{|f_{m-1}|^2} = \left| d\phi\left(\frac{\partial}{\partial z}\right)\right|^2.$$

□

Conversely, suppose that $\tilde{\phi}$ is holomorphic and horizontal and for each $x \in S$, let $\tilde{\phi}(x) = V(x)$ and $\mathbb{C}^{2m+1} = \{f\} \oplus V \oplus \bar{V}$. Then

$$\partial V \subseteq \{f\} \oplus V \qquad \text{(horizontality)} \tag{4.13}$$

$$\bar{\partial} V \subseteq V \qquad \text{(holomorphicity)} \tag{4.14}$$

Now, $f.v = 0$ for all $v \in V(x)$ so that

$$\bar{\partial}(f.V) = \bar{\partial} f.V = 0$$

and $f.f = 1$ gives $\bar{\partial} f.f = 0$ so that

$$\bar{\partial} f \in V. \tag{4.15}$$

Then acting with ∂ and $\bar{\partial}$ on (4.15) and its conjugate respectively give

$$\partial\bar{\partial} f \in \partial V \quad \subseteq \{f\} \oplus V \qquad \text{by (4.13)} \tag{4.16}$$

$$\bar{\partial}\partial f \in \overline{(\partial V)} \subseteq \{f\} \oplus \bar{V} \qquad \text{by (4.13)} \tag{4.17}$$

so that $\partial\bar{\partial} f \in \{f\}$ and f is harmonic.

Further, $\bar{\partial} f$ is isotropic since $\bar{\partial} f \in V$ (by (4.15)) and using (4.14),

$$\bar{\partial}\bar{\partial} f \in \bar{\partial} V \subseteq V$$

shows that $\bar{\partial}^2 f$ is isotropic, and so on by induction. Thus f satisfies the isotropy requirements to be superminimal.

This is known as the twistor description of superminimal harmonic maps $S \to S^{2m}$.

4.4. The S^4 Case

Setting m equal to 2 throughout the above discussion gives the twistor space of S^4 as $\mathrm{SO}(5)/\mathrm{U}(2)$ and a correspondence between superminimal surfaces in S^4 and horizontal, holomorphic curves in SO(5)/U(2). However, this low dimensional case is especially interesting as the fact that the dimension of S as a surface in S^4 is the same as the codimension of S means that there is a natural way to construct a 'twistor' lift into SO(5)/U(2) of any immersion $f : S \to S^4$, irrespective of it being superminimal or not. This construction is given below.

Suppose $f : S \to S^4$ is any immersion. Then at each point $x \in S$, the tangent space to S^4 decomposes as

$$T_{f(x)}S^4 = T_xS \oplus N_x$$

where T_xS is the tangent space to S at x and N_x the normal space at x. Choose an oriented, orthonormal basis e_1, e_2 for T_xS and e_3, e_4 for N_x so that e_1, e_2, e_3, e_4 gives the standard orientation on S^4. Then (definition 4.4) span$\{e_1 + ie_2, e_3 + ie_4\}$ is a maximal isotropic subspace of the complexification $\mathbb{C}^5$ of $\mathbb{R}^5$. So for any f, this is how to construct

$$\tilde{\phi} : S \to \mathrm{SO}(5)/\mathrm{U}(2).$$

Further, there is a natural identification of SO(5)/U(2) with the complex projective space $\mathbb{C}\mathrm{P}^3$ and there are several (non-trivial!) ways of seeing this. One way to proceed is to obtain both SO(5)/U(2) and $\mathbb{C}\mathrm{P}^3$ as homogeneous spaces of Sp(2) and then show that these are the same. Since Sp(2) is the universal cover of SO(5), $\mathrm{SO}(5)/\mathrm{U}(2) = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{Sp}(1)$. On the other hand, recall $\mathrm{Sp}(2) \subset \mathrm{U}(4)$ and that U(4) acts on $\mathbb{C}\mathrm{P}^3$ as a group of isometries. Then Sp(2) acts transitively on $\mathbb{C}\mathrm{P}^3$ - as the restriction of the U(4) action. Moreover, the stabiliser of $[1, 0, 0, 0]$ is

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & q \end{pmatrix} |z \in \mathbb{C}\ ,\ |z|^2 = 1\ ,\ q \in \mathbb{H}\ ,\ |q|^2 = 1 \right\} = S^1 \times \mathrm{Sp}(1).$$

Thus, the twistor space of S^4 is $\mathbb{C}\mathrm{P}^3$ and the bundle $\pi : \mathbb{C}\mathrm{P}^3 \to S^4$ is precisely the twistor fibration of Penrose. The next section shows how S^4 may be identified with

$\mathbb{HP}^1$ and that as a consequence the twistor fibration is expressed neatly in terms of quaternions.

4.5. The Twistor Fibration for S^4

Recall first the Hopf fibration

$$\pi : \mathbb{H}^2 \setminus \{0\} \to \mathbb{HP}^1$$

given by $(q_1, q_2) \mapsto [q_1, q_2]$. Here $\mathbb{H}^2$ is regarded as a left $\mathbb{H}$-module so that $[q_1, q_2] = [q_1', q_2']$ if and only if there exists $q \in \mathbb{H} \setminus \{0\}$ such that $(q_1', q_2') = q(q_1, q_2)$. (The reader is referred to the appendix for details on $\mathbb{H}^n$, $\mathrm{Sp}(n)$ and the conventions used in this section and subsequently.) Also there is a diffeomorphism of $\mathbb{HP}^1$ with S^4 under which $[q_1, q_2] \in \mathbb{HP}^1$ corresponds to

$$\left(\frac{2\bar{q}_1 q_2, q_1\bar{q}_1 - q_2\bar{q}_2}{q_1\bar{q}_1 + q_2\bar{q}_2}\right) \in S^4 \subseteq \mathbb{H} \oplus \mathbb{R}.$$

Taking co-ordinate neighbourhoods x_+ , $x_- : \mathbb{H} \to S^4$ with

$$x_+(q) = \left(\frac{2q, 1 - q\bar{q}}{1 + q\bar{q}}\right), \qquad x_-(q') = \left(\frac{2\bar{q}', q'\bar{q}' - 1}{q'\bar{q}' + 1}\right), \qquad qq' = 1,$$

the standard metric on S^4 is given by

$$ds^2 = \frac{4}{(q\bar{q}+1)^2} d\bar{q}dq = \frac{4}{(q'\bar{q}'+1)^2} d\bar{q}'dq'.$$

In terms of these local co-ordinates $\pi : \mathbb{H}^2 \setminus \{0\} \to \mathbb{HP}^1$ is

$$\pi(q_1, q_2) = q_1^{-1} q_2 = q \qquad \text{if } q_1 \neq 0$$

$$\text{and} \quad \pi(q_1, q_2) = q_2^{-1} q_1 = q' \qquad \text{if } q_2 \neq 0.$$

Now $dq = q_1^{-1} dq_2 - q_1^{-1} dq_1 q_1^{-1} q_2$ and the differential $d\pi$ at (q_1, q_2) is described by

$$d\pi(p_1, p_2) = q_1^{-1} p_2 - q_1^{-1} p_1 q_1^{-1} q_2.$$

The fibre of π through the point (q_1, q_2) is given by

$$\begin{aligned}\ker d\pi &= \{(p_1, p_2) | q_1^{-1} p_2 - q_1^{-1} p_1 q_1^{-1} q_2 = 0\} \\ &= \mathbb{H}(q_1, q_2).\end{aligned}$$

There is an $\mathbb{H}$-valued bilinear form on $\mathbb{H}^2$ defined by

$$\langle (p_1, p_2), (q_1, q_2) \rangle = 4(p_1 \bar{q}_1 + p_2 \bar{q}_2)$$

and this gives rise to a metric on $\mathbb{H}^2$ defined by

$$|(q_1, q_2)|^2 = 4(|q_1|^2 + |q_2|^2). \tag{4.18}$$

The horizontal subspace at the point (q_1, q_2) is the orthogonal complement, with respect to this inner product, of the fibre $\mathbb{H}(q_1, q_2)$. $(\mathbb{H}(q_1, q_2))^\perp$ is mapped bijectively onto $T_{\pi(q_1, q_2)} S^4$ - indeed more is true:

CLAIM. *With respect to the metric* (4.18) *on* $\mathbb{H}^2$ *and the standard metric on* S^4, *the differential* $d\pi$ *maps* $(\mathbb{H}(q_1, q_2))^\perp$ *isometrically onto* $T_{\pi(q_1, q_2)} S^4$ *when* $|q_1|^2 + |q_2|^2 = 1$.

PROOF OF CLAIM: Suppose $|q_1|^2 + |q_2|^2 = 1$. Then $(\mathbb{H}(q_1, q_2))^\perp = \mathbb{H}\left(\frac{q_1}{|q_1|^2}, -\frac{q_2}{|q_2|^2}\right)$ and

$$\begin{aligned}\left|\lambda\left(\frac{q_1}{|q_1|^2}, -\frac{q_2}{|q_2|^2}\right)\right|^2 &= 4|\lambda|^2\left(\frac{|q_1|^2}{|q_1|^4} + \frac{|q_2|^2}{|q_2|^4}\right) \\ &= \frac{4|\lambda|^2}{|q_1|^2 |q_2|^2}.\end{aligned}$$

Also,

$$\begin{aligned}d\pi\left(\lambda\left(\frac{q_1}{|q_1|^2}, -\frac{q_2}{|q_2|^2}\right)\right) &= -q_1^{-1}\frac{\lambda q_2}{|q_2|^2} - q_1^{-1}\frac{\lambda q_1}{|q_1|^2} q_1^{-1} q_2 \\ &= -\frac{1}{|q_1|^2 |q_2|^2} q_1^{-1} \lambda q_2\end{aligned}$$

so that

$$\left| d\pi\left(\lambda\left(\frac{q_1}{|q_1|^2}, -\frac{q_2}{|q_2|^2}\right)\right)\right|^2_{\pi(q_1,q_2)} = \frac{4}{\left(\frac{|q_1|^2}{|q_2|^2}+1\right)^2}\frac{|q_1|^{-2}|\lambda|^2|q_2|^2}{|q_1|^4|q_2|^4}$$
$$= \frac{4|\lambda|^2}{|q_1|^2|q_2|^2}.$$

□

Now identify $\mathbb{H}^2$ with $\mathbb{C}^4$ via the identification of $\mathbb{H}$ with $\mathbb{C}^2$ given by

$$q \leftrightarrow (z, w) \qquad \text{if } q = z + wj.$$

Then $\pi : \mathbb{H}^2 \setminus \{0\} \to S^4$ and indeed $\pi|_{S^7}$ factors through $\mathbb{C}\mathrm{P}^3$ to give $\pi_{S^4} : \mathbb{C}\mathrm{P}^3 \to S^4 = \mathbb{H}\mathrm{P}^1$ with

$$\pi_{S^4}[z_0, z_1, z_2, z_3] = [z_0 + z_1 j, z_2 + z_3 j]$$

or, equivalently,

$$\pi_{S^4}[z_0, z_1, z_2, z_3] = \left(\frac{2(\bar{z}_0 - z_1 j)(z_2 + z_3 j), |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}\right)$$
$$= \left(\frac{2(\bar{z}_0 z_2 + z_1 \bar{z}_3), 2(\bar{z}_0 z_3 - z_1 \bar{z}_2), |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}\right).$$

Thus if $\rho : \mathbb{H}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^3$ is defined by

$$\rho(z_0 + z_1 j, z_2 + z_3 j) = [z_0, z_1, z_2, z_3]$$

we have a commutative triangle of fibrations

$$\begin{array}{ccc} \mathbb{H}^2 \setminus \{0\} & & \\ & \searrow^{\rho} & \\ \pi \downarrow & & \mathbb{C}\mathrm{P}^3 \\ & \swarrow_{\pi_{S^4}} & \\ S^4 & & \end{array}$$

and if $\mathbb{C}\mathrm{P}^3$ is given the standard Fubini-Study metric then $\rho|_{S^7}$, π_{S^4} and $\pi|_{S^7}$ are Riemannian submersions.

Furthermore the set of complex structures on $T_{\pi(q_1,q_2)}S^4$, $(q_1,q_2) \in S^7$ may be identified with $\pi_{S^4}^{-1}\pi(q_1,q_2)$. For on $(q_1,q_2) \times (\mathbb{H}(q_1,q_2))^\perp$ there is a canonical complex structure derived by considering $(\mathbb{H}(q_1,q_2))^\perp$ as a left $\mathbb{C}$-vector subspace of $\mathbb{C}^4$. Then since the differential $d\pi_{(q_1,q_2)}$ maps this space isometrically onto $T_{\pi(q_1,q_2)}S^4$ it follows that $T_{\pi(q_1,q_2)}S^4$ inherits a complex structure compatible with the metric. It is not hard to show that all such complex structures on $T_{\pi(q_1,q_2)}S^4$ are so obtained and that two are the same if and only if $(q_1', q_2') = \mu(q_1,q_2)$, $\mu \in S^1$ (where $\pi(q_1',q_2') = \pi(q_1,q_2)$). Thus $\pi_{S^4} : \mathbb{C}P^3 \to S^4$ is the bundle of almost complex structures on S^4.

Consider a tangent vector v to $\mathbb{C}P^3$ at the point p. Thus $\langle v, p\rangle = 0$. The fibre at p is given as the projective space of $\text{span}\{p, pj\}$ so the condition for horizontality is $\langle v, pj\rangle = 0$. Thus for $h(z)$ to be a horizontal curve, the condition is

$$\left\langle h' - \frac{\langle h', h\rangle}{|h|^2} h, hj \right\rangle = 0.$$

But $\langle h, hj\rangle = 0$ so the condition is

$$\langle h', hj\rangle = 0$$

and if $h = (z_0, z_1, z_2, z_3)$, horizontality is given explicitly by

$$z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2 = 0.$$

The horizontal distribution on $\mathbb{C}P^3$ is a holomorphic 2-plane bundle. Let us use local co-ordinates again. This time suppose $z_0 \neq 0$ and write $\xi_\alpha = \dfrac{z_\alpha}{z_0}$ $(\alpha = 1, 2, 3)$. Thus we have a co-ordinate neighbourhood on $\mathbb{C}P^3$ given by

$$(\xi_1, \xi_2, \xi_3) \mapsto [z_0, z_1, z_2, z_3] = [1, \xi_1, \xi_2, \xi_3].$$

In terms of this the Fubini-Study metric is given by

$$ds^2 = 4\frac{(1 + \Sigma\xi_\alpha\bar{\xi}_\alpha)\Sigma d\xi_\alpha d\bar{\xi}_\alpha - (\Sigma\xi_\alpha d\bar{\xi}_\alpha)(\Sigma\bar{\xi}_\alpha d\xi_\alpha)}{(1 + \Sigma\xi_\alpha\bar{\xi}_\alpha)^2}.$$

4.6. Results of Bryant

By studying the geometry of the twistor fibration $\pi : \mathbb{C}\mathrm{P}^3 \to S^4$, Bryant [11] proved that if $\phi : M^2 \to \mathbb{C}\mathrm{P}^3$ is a horizontal, holomorphic curve then $\pi\phi : M^2 \to S^4$ is a superminimal immersion (may have branch points) and conversely, that every superminimal $f : M^2 \to S^4$ is of the form $\pi\phi$ where ϕ is an essentially unique horizontal, holomorphic curve $\phi : M^2 \to \mathbb{C}\mathrm{P}^3$. Further, there is a Weierstrass type formula which produces horizontal, holomorphic curves in $\mathbb{C}\mathrm{P}^3$ from meromorphic functions on M^2.

Theorem 4.6. *Let M^2 be a connected Riemann surface and let f, g be meromorphic functions on M^2 with g non-constant. Let $\Phi(f,g) : M^2 \to \mathbb{C}P^3$ be defined by*

$$\Phi(f,g) = [1, f - \frac{1}{2}g\frac{df}{dg}, g, \frac{1}{2}\frac{df}{dg}].$$

Then $\Phi(f,g)$ is a horizontal, holomorphic curve in $\mathbb{C}P^3$. Conversely, any horizontal, holomorphic curve $\Phi : M^2 \to \mathbb{C}P^3$ is either of the form $\Phi(f,g)$ for some unique meromorphic functions f, g on M or Φ has image in some $\mathbb{C}P^1 \subseteq \mathbb{C}P^3$.

Use of the Riemann-Roch theorem shows that for a compact Riemann surface f and g can be found which make $\Phi(f,g)$ an immersion. This implies that any compact Riemann surface can be conformally and minimally (in fact superminimally) immersed in S^4.

4.7. The Lift to $\mathbb{C}\mathrm{P}^3$

Let $f : S \to S^4$ be given by $f(z) = (X_0(z), \ldots, X_4(z))$ with z a local complex coordinate on S and let $\sigma : S^4 \to \mathbb{R}^4$ denote stereographic projection from $(-1,0,0,0,0)$ onto $\mathbb{R}^4 = \mathbb{H}$. Then $\sigma \circ f = \dfrac{X}{1+X_0} = q \in \mathbb{H}$, where $X = X_1 + X_2 i + X_3 j + X_4 k$.

Now, the adapted frame of f is of the form

$$\tilde{F} = \left(\begin{array}{c|c|c|c|c} X_0 & A_0 & B_0 & C_0 & D_0 \\ \hline X & A & B & C & D \end{array}\right)$$

for mutually orthogonal unit vectors $\begin{pmatrix} A_0 \\ A \end{pmatrix}, \ldots, \begin{pmatrix} D_0 \\ D \end{pmatrix}$ with

$$\text{span}\left\{\begin{pmatrix} A_0 \\ A \end{pmatrix}, \begin{pmatrix} B_0 \\ B \end{pmatrix}\right\} = TS \text{ and span}\left\{\begin{pmatrix} C_0 \\ C \end{pmatrix}, \begin{pmatrix} D_0 \\ D \end{pmatrix}\right\} = NS.$$

Then in the notation of section 4.1.2, $x \equiv q$ and theorem 4.3 gives the following:

Theorem 4.7. *An immersion $f : S \to S^4$ lifts to $F \in Sp(2)$ where*

$$F = \frac{1}{\Delta}\begin{pmatrix} w & wq \\ -wa\bar{q} & wa \end{pmatrix},$$

with $w = \bar{a} + i\bar{b} + j\bar{c} + k\bar{d}$ and $\Delta^2 = |w|^2(1 + |q|^2)$.

Using this Sp(2)-description of the frame, it is now a straightforward matter to write down the twistor lift of f into $\mathbb{C}\mathrm{P}^3$. Let the lift be denoted by $\hat{f}$ and by an abuse of the notation, write $[z_0 + z_1 j, z_2 + z_3 j]$ to mean $[z_0, z_1, z_2, z_3] \in \mathbb{C}\mathrm{P}^3$. Then

$$\hat{f} = [1, 0]F = [w, wq] \in \mathbb{C}\mathrm{P}^3.$$

In fact, this expression for the lift of f can be simplified;

Corollary 4.8. *The twistor lift of an immersion $f : S \to S^4$ is given by*

$$\hat{f} = [\bar{a} + i\bar{b}, (\bar{a} + i\bar{b})q] \in \mathbb{C}P^3$$

where $q = \sigma \circ f$, $a = A - A_0 q$, $b = B - B_0 q$ and $\begin{pmatrix} A_0 \\ A \end{pmatrix}, \begin{pmatrix} B_0 \\ B \end{pmatrix} \in \mathbb{R} \oplus \mathbb{H}$ are an orthonormal basis for TS.

PROOF. It is enough to show that $w = \lambda(\bar{a} + i\bar{b})$ for some $\lambda \in \mathbb{C}$, or, in other words, that $\overline{w(\bar{a} + i\bar{b})}$ is a complex number.

Clearly, $w\overline{(\bar{a}+i\bar{b})}$ commutes with all real numbers and by 4.7,

$$\begin{aligned} iw\overline{(\bar{a}+i\bar{b})} = iw(a-bi) &= iwa - iwbi \\ &= (-wbi + wa)i \\ &= w\overline{(\bar{a}+i\bar{b})}i \end{aligned} \tag{4.19}$$

so that $w\overline{(\bar{a}+i\bar{b})}$ commutes with complex numbers and so is itself a complex number.

□

4.8. Lifts of Conformal Immersions $f : S \to S^4$

4.8.1. Conformal Maps. The map $f(z)$ is said to be conformal if and only if $\langle f_z, f_{\bar{z}} \rangle = 0$, that is, if and only if $|f_x|^2 = |f_y|^2$ and $f_x.f_y = 0$, $z = x + iy$. In order to obtain results about the twistor lift of f, it is useful to determine the condition placed on $q = \xi + \eta j$ by the requirement that f is a conformal immersion. Since stereographic projection σ of S^4 onto $\mathbb{R}^4$ is a conformal diffeomorphism, it holds that

$$\langle f_z, f_{\bar{z}} \rangle = 0 \text{ if and only if } \langle (\sigma \circ f)_z, (\sigma \circ f)_{\bar{z}} \rangle = 0.$$

If $f = (X_0, X) \subset \mathbb{R} \oplus \mathbb{H}$, then

$$\sigma \circ f = \left(\frac{X_1 + iX_2}{1 + X_0}, \frac{X_3 + iX_4}{1 + X_0} \right) = (\xi, \eta) \in \mathbb{C}^2 = \mathbb{R}^4$$

and

$$\langle (\sigma \circ f)_z, (\sigma \circ f)_{\bar{z}} \rangle = \langle (\xi_z, \eta_z), (\xi_{\bar{z}}, \eta_{\bar{z}}) \rangle = \xi_z \bar{\xi}_z + \eta_z \bar{\eta}_z$$

giving the following;

Lemma 4.9. *An immersion $f : S \to S^4$ is conformal if and only if $\xi_z \bar{\xi}_z + \eta_z \bar{\eta}_z = 0$, where f corresponds via stereographic projection to $q = \xi + \eta j \in \mathbb{H}$.*

4.8.2. The Moving Frame. In the case where $f : S \to S^4$ is a conformal map, it is natural to choose e_1, e_2 in the frame to be $e_1 = \dfrac{f_x}{|f_x|}$ and $e_2 = \dfrac{f_y}{|f_y|}$ since the tangent vectors f_x and f_y are orthogonal (and also of the same length). This will enable an explicit formula for the lift to be found in the conformal case, and to this end, the vectors $e_1 = \begin{pmatrix} A_0 \\ A \end{pmatrix}$ and $e_2 = \begin{pmatrix} B_0 \\ B \end{pmatrix}$ will now be determined in terms of the quaternionic co-ordinate q.

In the notation of section 4.1.2, identify $f = \left(\dfrac{1-|q|^2, 2q}{1+|q|^2}\right) = (X_0, X)$ so that

$$(A_0, A) = \frac{f_x}{|f_x|} = \frac{1}{|f_x|}(X_{0x}, X_x).$$

Then differentiating gives

$$X_{0x} = \frac{(1-|q|^2)_x}{1+|q|^2} - \frac{(1-|q|^2)|q|_x^2}{(1+|q|^2)^2} = \frac{-2|q|_x^2}{(1+|q|^2)^2},$$

$$X_x = \frac{2q_x}{1+|q|^2} - \frac{2q|q|_x^2}{(1+|q|^2)^2} = \frac{2(q_x - q\bar{q}_x q)}{(1+|q|^2)^2}$$

and

$$\begin{aligned} |f_x|^2 = |X_{0x}|^2 + |X_x|^2 &= \frac{4}{(1+|q|^2)^4}(|q_x - q\bar{q}_x q|^2 + (|q|_x^2)^2) \\ &= 4\frac{|q_x|^2}{(1+|q|^2)^2}, \end{aligned} \tag{4.20}$$

whence

$$A_0 = \frac{-|q|_x^2}{|q_x|(1+|q|^2)}, \qquad A = \frac{(q_x - q\bar{q}_x q)}{|q_x|(1+|q|^2)}. \tag{4.21}$$

Similarly,

$$B_0 = \frac{-|q|_y^2}{|q_y|(1+|q|^2)}, \qquad B = \frac{(q_y - q\bar{q}_y q)}{|q_y|(1+|q|^2)}. \tag{4.22}$$

Notice that $|f_x|^2 = |f_y|^2$ if and only if $|q_x|^2 = |q_y|^2$.

4.8.3. Explicit Description of the Lift. By corollary 4.8, the vectors $e_1 = \begin{pmatrix} A_0 \\ A \end{pmatrix}$ and $e_2 = \begin{pmatrix} B_0 \\ B \end{pmatrix}$ are enough to determine the lift of f to $\mathbb{CP}^3$. Given (4.21) and (4.22),

$$\begin{aligned} a = A - A_0 q &= \frac{(q_x - q\bar{q}_x q)}{|q_x|(1+|q|^2)} - \frac{-|q|_x^2}{|q_x|(1+|q|^2)} q \\ &= \frac{q_x - q\bar{q}_x q + (q_x \bar{q} + q\bar{q}_x) q}{|q_x|(1+|q|^2)} \\ &= \frac{q_x}{|q_x|} \end{aligned} \tag{4.23}$$

or equivalently,

$$a = \frac{(q_z + q_{\bar{z}})}{|q_x|}. \tag{4.24}$$

Similarly,

$$b = B - B_0 q = \frac{q_y}{|q_y|} = \frac{q_y}{|q_x|}. \tag{4.25}$$

Thus

$$\bar{a} + i\bar{b} = \frac{\bar{q}_x + i\bar{q}_y}{|q_x|} = 2\frac{\bar{q}_{\bar{z}}}{|q_x|} \tag{4.26}$$

and the conformal version of theorem 4.7 is

Corollary 4.10. *A conformal immersion $f : S \to S^4$ lifts to $F \in Sp(2)$ where*

$$F = \frac{e^{i\theta}}{\Delta'} \begin{pmatrix} \bar{q}_{\bar{z}} & \bar{q}_{\bar{z}} q \\ -\frac{\bar{q}_{\bar{z}} q_x \bar{q}}{|q_x|} & \frac{\bar{q}_{\bar{z}} q_x}{|q_x|} \end{pmatrix},$$

with $f = \left(\frac{2q, 1 - |q|^2}{1 + |q|^2} \right)$ *and* $\Delta' = |\bar{q}_{\bar{z}}|(1+|q|^2)^{\frac{1}{2}}$ *and* $e^{i\theta} \in \mathbb{C}$.

Taking $[1,0]F$ (or applying (4.26) to corollary 4.8), the explicit formula for the lift to $\mathbb{CP}^3$ in the conformal case is now clear.

Theorem 4.11. *The twistor lift $\hat{f} : S \to \mathbb{CP}^3$ of a conformal immersion $f : S \to S^4$ is given by*

$$\hat{f}(z) = [\bar{\xi}_{\bar{z}}, -\eta_{\bar{z}}, \bar{\xi}_{\bar{z}}\xi + \eta_{\bar{z}}\bar{\eta}, \bar{\xi}_{\bar{z}}\eta - \eta_{\bar{z}}\bar{\xi}] \tag{4.27}$$

where $q = \xi + \eta j$ *is the quaternion obtained by the stereographic projection of* f *to* $\mathbb{R}^4 = \mathbb{H}$.

It is important to check that this beautiful formula for the lift is in fact well-defined. There are two areas of potential difficulty, firstly it is conceivable that both the functions $\bar{\xi}_{\bar{z}}$ and $\eta_{\bar{z}}$ are zero and secondly, that this lift is defined only for the particular choice of quaternionic co-ordinate (stereographic projection).

Although it is true that both $\bar{\xi}_{\bar{z}}$ and $\eta_{\bar{z}}$ could be zero, the functions $\bar{\xi}_{\bar{z}}$, $\eta_{\bar{z}}$, $\xi_{\bar{z}}$, $\bar{\eta}_{\bar{z}}$ cannot all be zero (for then q would have to be constant). Suppose, without loss of generality, that $\xi_{\bar{z}}$ is non-zero. Then by the conformality condition, $\bar{\xi}_{\bar{z}}$ can be replaced wherever it occurs by $\bar{\xi}_{\bar{z}} = -\dfrac{\eta_{\bar{z}}\bar{\eta}_{\bar{z}}}{\xi_{\bar{z}}}$, in which case (4.27) becomes

$$\left[-\frac{\eta_{\bar{z}}\bar{\eta}_{\bar{z}}}{\xi_{\bar{z}}}, -\eta_{\bar{z}}, -\frac{\eta_{\bar{z}}\bar{\eta}_{\bar{z}}}{\xi_{\bar{z}}}\xi + \eta_{\bar{z}}\bar{\eta}, -\frac{\eta_{\bar{z}}\bar{\eta}_{\bar{z}}}{\xi_{\bar{z}}}\eta - \eta_{\bar{z}}\bar{\xi}\right].$$

Thus, the lift may be wriiten equally well in terms of $\xi_{\bar{z}}$, $\bar{\eta}_{\bar{z}}$ as

$$[\hat{f}] = [\bar{\eta}_{\bar{z}}, \xi_{\bar{z}}, \bar{\eta}_{\bar{z}}\xi - \xi_{\bar{z}}\bar{\eta}, \bar{\eta}_{\bar{z}}\eta + \xi_{\bar{z}}\bar{\xi}]$$

and no difficulties are encountered when $\bar{\xi}_{\bar{z}}$ and $\eta_{\bar{z}}$ are zero.

In the notation of section 4.1.2, for the vectors $\begin{pmatrix} |p_1|^2 - |p_2|^2 \\ 2\bar{p}_1 p_2 \end{pmatrix}$ and $\begin{pmatrix} X_0 \\ X \end{pmatrix}$ to correspond, there are two choices;

$$\text{either} \qquad p_2 = p_1 \frac{X}{1 + X_0} \qquad \text{or} \qquad p_1 = p_2 \frac{\bar{X}}{1 - X_0}.$$

Suppose the underlying immersion f is conformal. Then the first case gives the lift $[\hat{f}]$ as

$$[\hat{f}]_q = [\bar{\xi}_{\bar{z}}, -\eta_{\bar{z}}, \bar{\xi}_{\bar{z}}\xi + \eta_{\bar{z}}\bar{\eta}, \bar{\xi}_{\bar{z}}\eta - \eta_{\bar{z}}\bar{\xi}]$$

with $q = \dfrac{X}{1 + X_0} = \xi + \eta j$. The second case gives $[\hat{f}]$ in terms of the quaternionic co-ordinate $\tilde{q} = \dfrac{\bar{X}}{1 - X_0} = \rho + \tau j$ as

$$[\hat{f}]_{\tilde{q}} = [\bar{\rho}_{\bar{z}}\rho + \tau_{\bar{z}}\bar{\tau}, \bar{\rho}_{\bar{z}}\tau - \tau_{\bar{z}}\bar{\rho}, \bar{\rho}_{\bar{z}}, -\tau_{\bar{z}}].$$

The co-ordinates q and $\tilde{q}$ are related in that

$$|\tilde{q}|^2 = \frac{1}{|q|^2} = \lambda, \qquad \text{say, and} \qquad \rho = \lambda\bar{\xi}, \qquad \tau = -\lambda\eta. \tag{4.28}$$

Then it is enough to show that $\dfrac{\bar{\rho}_{\bar{z}}\rho + \tau_{\bar{z}}\bar{\tau}}{\bar{\xi}_{\bar{z}}}$ and $\dfrac{\bar{\rho}_{\bar{z}}\tau - \tau_{\bar{z}}\bar{\rho}}{-\eta_{\bar{z}}}$ are the same. Now,

$$-\eta_{\bar{z}}(\bar{\rho}_{\bar{z}}\rho + \tau_{\bar{z}}\bar{\tau}) = \lambda^2\eta_{\bar{z}}(\bar{\xi}_{\bar{z}}\xi + \eta_{\bar{z}}\bar{\eta}) \qquad \text{by (4.28)}$$

and

$$\begin{aligned} \bar{\xi}_{\bar{z}}(\bar{\rho}_{\bar{z}}\tau - \tau_{\bar{z}}\bar{\rho}) &= -\lambda^2\bar{\xi}_{\bar{z}}(\bar{\xi}_{\bar{z}}\eta - \eta_{\bar{z}}\bar{\xi}) \\ &= \lambda^2\eta_{\bar{z}}(\eta_{\bar{z}}\bar{\eta} + \bar{\xi}_{\bar{z}}\xi) \qquad \text{by (4.28) and conformality.} \end{aligned}$$

So the two choices are equivalent and the lift does not depend on whether stereographic projection is from the North or South pole of S^4.

Applications of the formula (4.27) are discussed in the next chapter. The lifts are produced and investigated for certain examples and, more generally, the formula is used to find the conditions for harmonic lifts. This leads to an interesting observation for the harmonic sequence of a conformal, harmonic lift.

CHAPTER 5

Examples and Applications

The formula obtained in chapter 4 for the lift of an immersion $f : S \to S^4$ into $\mathbb{C}P^3$ will now be put to use in examining some properties of such lifts. Section 5.1 gives some examples which display the equivariance of lifts as discussed in section 2.1.2. In particular, the SO(3)-symmetry of the Veronese surface (5.1.1), the lift of the Clifford torus gives a torus in $\mathbb{C}P^3$ (5.1.2) and an example with S^1-symmetry (5.1.3) are given. After some notation (section 5.2) to describe the differentiation of quaternion products, the harmonicity condition for the lift is derived (section 5.3).

Section 5.4 studies holomorphic lifts in detail, confirming that they project to real isotropic maps into S^4 and moreover showing that such lifts are unique. Section 5.5 studies the harmonic sequence of a harmonic lift $\hat{f}$ and it turns out that when the lift is harmonic but not holomorphic, the harmonic sequence has a particular symmetry (called j-symmetry). Such harmonic maps in $\mathbb{C}P^3$ project to conformal maps into S^4 and j-symmetric lifts are unique. Studying the harmonic sequence reveals that when the lift is conformal it is automatically 4-orthogonal. Also, the comparison of the harmonic sequence of the Clifford torus with that of its lift shows (perhaps surprisingly) that the maps are not congruent, but they are closely related.

Finally, section 5.7 discusses positive and negative lifts to $\mathbb{C}P^3$ ([17]) and shows that these both arise naturally from the Sp(2)-description of the moving frame of f. This gives rise to a straightforward proof of a theorem of Eells-Salamon.

5.1. Examples

5.1.1. The Veronese Immersion. Let $f : S^2 \to S^4$ be given by

$$f(x_1,x_2,x_3) = \sqrt{3}(\frac{1}{2\sqrt{3}}(2x_3^2 - x_1^2 - x_2^2), x_1x_3, x_2x_3, \frac{1}{2}(x_1^2 - x_2^2), x_1x_2).$$

Using the stereographic co-ordinate $z = \dfrac{x_1 + ix_2}{1 + x_3}$ and writing $\mathbb{R}^5 = \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$, f takes the form

$$f(z) = \frac{1}{(1+|z|^2)^2}(1 - 4|z|^2 + |z|^4, 2\sqrt{3}(1-|z|^2)z, 2\sqrt{3}z^2).$$

Stereographic projection from S^4 to $\mathbb{H}$ via $(t,v,w) \mapsto \dfrac{(v+wj)}{1+t}$ produces

$$q = \frac{\sqrt{3}((1-|z|^2)z + z^2j)}{(1-|z|^2+|z|^4)} = \xi + \eta j$$

and

$$\xi_{\bar z} = \frac{\sqrt{3}(1-2|z|^2)}{(1-|z|^2+|z|^4)^2}, \quad \eta_{\bar z} = \frac{\sqrt{3}(1-2|z|^2)z^3}{(1-|z|^2+|z|^4)^2}.$$

Then f is conformal ($\xi_{\bar z}\bar\xi_{\bar z} + \eta_{\bar z}\bar\eta_{\bar z} = 0$) so that the formula for the lift may be used to give

$$\begin{aligned} \hat f(z) &= [\bar\xi_{\bar z}, -\eta_{\bar z}, \bar\xi_{\bar z}\xi + \eta_{\bar z}\bar\eta, \bar\xi_{\bar z}\eta - \eta_{\bar z}\bar\xi] \\ &= [1, -z^3, \sqrt{3}z, \sqrt{3}z^2]. \end{aligned} \tag{5.1}$$

Computing the harmonic sequence of $\hat f$ shows that the invariants γ_p ($p = 0,1,2$) are all constant and by the formulae 2.2, 2.3 this tells us that the lift has constant curvature and constant Kähler angle. Indeed, it has been shown in [4] that, more generally, each ψ_p in the harmonic sequence of the Veronese immersion

$$\psi(z) = [1, \sqrt{\binom{n}{1}}z, \dots, \sqrt{\binom{n}{r}}z^r, \dots, z^n] \in \mathbb{CP}^n$$

has constant curvature and Kähler angle. Also, Calabi's result [13] tells us that (up to holomorphic isometries of $\mathbb{C}P^n$) $\psi(z)$ is the only linearly full holomorphic curve in $\mathbb{C}P^n$ of constant curvature.

There is a particularly nice way to observe these facts which moreover constructs all linearly full $\psi : S^2 \to \mathbb{C}P^n$ of constant curvature and Kähler angle. The method is given below for the $\mathbb{C}P^3$ case, but the general case is entirely similar.

First note that the group of isometries of $\mathbb{C}P^3$ has two components - namely the holomorphic and the anti-holomorphic isometries. The identity component (the holomorphic isometries) is just $\mathrm{PU}(4) = \mathrm{U}(4)/Z(\mathrm{U}(4))$ where $Z(\mathrm{U}(4))$ consists of the scalar matrices λI, $\lambda \in S^1$.

Suppose $\psi : S^2 \to \mathbb{C}P^3$ is linearly full and has constant curvature and constant Kähler angle. Then by the Extension Theorem of Bolton-Woodward [6] there exists for each $h \in \mathrm{SO}(3)$ a corresponding $g \in \mathrm{PU}(4)$ such that

$$\psi h = g \psi$$

and moreover, $\phi : \mathrm{SO}(3) \to \mathrm{PU}(4)$ taking h to g is a homomorphism. This is used to construct all examples as follows:

$\mathrm{SU}(2)$ acts on $\mathbb{C}^2$ and hence on the third symmetric tensor power $S^3(\mathbb{C}^2)$ of $\mathbb{C}^2$ (i.e. the subspace of $(\mathbb{C}^2)^{\otimes 3}$ which is invariant under the action of the symmetric group S_3 given by

$$\sigma(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = e_{\sigma(i_1)} \otimes e_{\sigma(i_2)} \otimes e_{\sigma(i_3)}).$$

$S^3(\mathbb{C}^2)$ has dimension 4 and $S^3(\mathbb{C}^2) \cong \mathbb{C}^4$. These actions of $\mathrm{SU}(2)$ give a homomorphism $S^3_\rho : \mathrm{SU}(2) \to \mathrm{SU}(4)$. Let e_1, e_2 be the standard basis for $\mathbb{C}^2$ and consider the following unitary basis of $S^3(\mathbb{C}^2) \cong \mathbb{C}^4$:

$$\begin{aligned} \tilde{e}_1 &= e_1 \otimes e_1 \otimes e_1, \quad \tilde{e}_2 = -e_2 \otimes e_2 \otimes e_2, \\ \tilde{e}_3 &= \frac{1}{\sqrt{3}}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1), \\ \tilde{e}_4 &= \frac{1}{\sqrt{3}}(e_1 \otimes e_2 \otimes e_2 + e_4 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_1). \end{aligned}$$

Let $A \in \mathrm{SU}(2)$ be given by

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

whence $Ae_1 = \alpha e_1 + \beta e_2$ and $Ae_2 = -\bar{\beta} e_1 + \bar{\alpha} e_2$. So A acts on $S^3(\mathbb{C}^2)$ by acting on the basis $\{\tilde{e}_i\}$. For example,

$$\begin{aligned} A\tilde{e}_1 = A(e_1 \otimes e_1 \otimes e_1) &= (\alpha e_1 + \beta e_2) \otimes (\alpha e_1 + \beta e_2) \otimes (\alpha e_1 + \beta e_2) \\ &= \alpha^3 \tilde{e}_1 - \beta^3 \tilde{e}_2 + \sqrt{3}\alpha^2 \beta \tilde{e}_3 + \sqrt{3}\alpha\beta^2 \tilde{e}_4 \end{aligned}$$

and so on. Then $\tilde{A} = (A\tilde{e}_1 \mid A\tilde{e}_2 \mid A\tilde{e}_3 \mid A\tilde{e}_4)$ is given by

$$\tilde{A} = \begin{pmatrix} \alpha^3 & \bar{\beta}^3 & -\sqrt{3}\alpha^2\bar{\beta} & \sqrt{3}\alpha\bar{\beta}^2 \\ -\beta^3 & \bar{\alpha}^3 & -\sqrt{3}\bar{\alpha}\beta^2 & -\sqrt{3}\bar{\alpha}^2\beta \\ \sqrt{3}\alpha^2\beta & -\sqrt{3}\bar{\alpha}\bar{\beta}^2 & (|\alpha|^2 - 2|\beta|^2)\alpha & -(2|\alpha|^2 - |\beta|^2)\bar{\beta} \\ \sqrt{3}\alpha\beta^2 & \sqrt{3}\bar{\alpha}^2\bar{\beta} & (2|\alpha|^2 - |\beta|^2)\beta & (|\alpha|^2 - 2|\beta|^2)\bar{\alpha} \end{pmatrix}.$$

Then with J as in (4.1), it follows that $JA = \bar{A}J$, so that $A \in \mathrm{Sp}(2)$. Thus S^3_ρ is a homomorphism $S^3_\rho : \mathrm{SU}(2) \to \mathrm{Sp}(2)$, and via the double covers $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ the following diagram commutes,

$$\begin{array}{ccc} \mathrm{SU}(2) & \xrightarrow{S^3_\rho} & \mathrm{Sp}(2) \\ {\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\ \mathrm{SO}(3) & \longrightarrow & \mathrm{SO}(5) \end{array}$$

giving rise to a representation of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$ and hence a (non-standard) action of $\mathrm{SO}(3)$ on S^4.

The orbit of $[S^3(e_1^3)]$ is the Veronese immersion and from this description the SO(3)-symmetry is clear. The remaining columns (the orbits in $\mathbb{C}P^3$ of $\tilde{e}_2$, $\tilde{e}_3$ and $\tilde{e}_4$) give the other elements of the Frenet frame of the Veronese immersion.

Note that the map is obtained in terms of the standard complex co-ordinate z on S^2 by choosing

$$A = \frac{1}{(1+|z|^2)^{\frac{1}{2}}}\begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \in \mathrm{SU}(2) \tag{5.2}$$

and the Veronese surface is then the orbit of the point $[1, 0, 0, 0]$.

Remark 5.1. More generally, for $\psi : S^2 \to \mathbb{C}P^n$, use $S^n(\mathbb{C}^2)$ to describe the map $\phi : \mathrm{SO}(3) \to \mathrm{PU}(n+1)$. The orbits in $\mathbb{C}P^n$ of

$$[S^n(e_1^n)], \dots, [S^n(e_1^{n-k} \otimes e_2^k)], \dots, [S^n(e_2^n)]$$

are the elements of the Frenet frame and taking A as in (5.2) above gives $[S^n(e_1^n)]$ as

$$[1, \sqrt{\binom{n}{1}}z, \dots, \sqrt{\binom{n}{r}}z^r, \dots, z^n] \in \mathbb{C}P^n.$$

5.1.2. The Clifford Torus. Consider the torus $f : \mathbb{R}^2 \to S^3 \subseteq S^4$ given by

$$f(x,y) = \frac{1}{\sqrt{2}}(0, \cos 2\sqrt{2}x, \sin 2\sqrt{2}x, \cos 2\sqrt{2}y, \sin 2\sqrt{2}y).$$

Then

$$q = \frac{1}{\sqrt{2}}\left(e^{\sqrt{2}i(z+\bar{z})} + e^{\sqrt{2}(z-\bar{z})}j\right) = \xi + \eta j$$

and

$$\xi_{\bar{z}} = -\sqrt{2}i\xi, \quad \eta_{\bar{z}} = -\sqrt{2}\eta.$$

Now f is conformal (check $\xi_{\bar z}\bar\xi_{\bar z} + \eta_{\bar z}\bar\eta_{\bar z} = 0$) so that the formula may be used to produce the lift of f to $\mathbb{C}P^3$.

$$\begin{aligned}[\hat f(z,\bar z)] &= [\bar\xi_{\bar z}, -\eta_{\bar z}, \bar\xi_{\bar z}\xi + \eta_{\bar z}\bar\eta, \bar\xi_{\bar z}\eta - \eta_{\bar z}\bar\xi] \\ &= [-\sqrt{2}i\bar\xi, \sqrt{2}\eta, \sqrt{2}(i|\xi|^2 - |\eta|^2), \sqrt{2}(1-i)\bar\xi\eta] \\ &= [\sqrt{2}, \sqrt{2}ie^{\sqrt{2}((1+i)z-(1-i)\bar z)}, (1-i)e^{\sqrt{2}i(z+\bar z)}, (1+i)e^{\sqrt{2}(z-\bar z)}].\end{aligned}$$

In general, $\hat f$ is not a holomorphic section of the bundle but there exists $\lambda \in \mathbb{C}$ such that $\lambda\hat f$ is a holomorphic section. Here, multiplication by

$$\frac{1}{2\sqrt{2}}\exp\left(-\frac{(1+i)}{\sqrt{2}}z + \frac{(1-i)}{\sqrt{2}}\bar z\right)$$

shows that $[\hat f]$ may be written as

$$[\hat f] = [e^{\hat D z - \overline{\hat D z}}\hat u_0],$$

where

$$\hat D = \frac{1}{\sqrt{2}}\mathrm{diag}(-(1+i), (1+i), -(1-i), (1-i)), \qquad \hat u_0 = \frac{1}{2}(1, i, \frac{(1-i)}{\sqrt{2}}, \frac{(1+i)}{\sqrt{2}}).$$

Then $\hat D \in \mathrm{U}(4)$ and $|\hat u_0|^2 = 1$.

With the lift in this form, it is now a straightforward matter to check that $\hat f$ is harmonic and to write down the harmonic sequence of $\hat f$. For,

$$\hat f_z = \hat D e^{\hat D z - \overline{\hat D z}}\hat u_0$$

and

$$\begin{aligned}\langle \hat f_z, \hat f\rangle &= \langle \hat D e^{\hat D z - \overline{\hat D z}}\hat u_0, e^{\hat D z - \overline{\hat D z}}\hat u_0\rangle \\ &= \langle \hat D\hat u_0, \hat u_0\rangle\end{aligned}$$

(since $\hat D$ commutes with $e^{\hat D z - \overline{\hat D z}}$). By computation, $\langle \hat D\hat u_0, \hat u_0\rangle = 0$. Thus

$$\hat f_1 = \hat D e^{\hat D z - \overline{\hat D z}}\hat u_0$$

and

$$\frac{\partial \hat{f}_1}{\partial \bar{z}} = -\bar{\hat{D}}\hat{D}e^{\hat{D}z-\overline{\hat{D}z}}\hat{u}_0 = -\hat{f}$$

so that $\hat{f}$ is harmonic.

The elements of the harmonic sequence $\{\hat{f}_p\}$ are given by

$$\hat{f}_p(z) = \hat{D}^p e^{\hat{D}z-\overline{\hat{D}z}}\hat{u}_0$$

and since $\hat{D}^4 = -I$, it is clear that this sequence is orthogonally periodic of order 4.

5.1.3. S^1-Symmetry. Let $f : S^2 \to S^4$ be given by

$$f(x,y,w) = \frac{1}{\Delta_1+\Delta_2}(2(\bar{a}_0a_1(1+w)^{k_1+k_2} + \bar{a}_2a_3(1-w)^{k_1+k_2})(x+iy)^{k_1},$$
$$2(\bar{a}_0a_2(1+w)^{k_1} - \bar{a}_1a_3(1-w)^{k_1})(x+iy)^{k_1+k_2}, \Delta_1 - \Delta_2)$$

where the a_j are non-zero complex numbers and the k_j are positive integers satisfying $k_2a_1a_2 = -(2k_1+k_2)a_0a_3$ with

$$\Delta_1 = |a_0|^2 + |a_3|^2(1-w)^{2k_1+k_2},$$

$$\Delta_2 = |a_1|^2(1+w)^{k_1+k_2}(1-w)^{k_1} + |a_2|^2(1+w)^{k_1}(1-w)^{k_1+k_2}.$$

With respect to the stereographic co-ordinate $z = \dfrac{x+iy}{1+w}$,

$$f(z) = \frac{1}{\Delta_1+\Delta_2}(2(\bar{a}_0a_1 + \bar{a}_2a_3|z|^{2(k_1+k_2)})z^{k_1},$$
$$2(\bar{a}_0a_2 - \bar{a}_1a_3|z|^{2k_1})z^{k_1+k_2}, \Delta_1 - \Delta_2)$$

with

$$\Delta_1 = |a_0|^2 + |a_3|^2|z|^{2(k_1+k_2)}, \qquad \Delta_2 = (|a_1|^2 + |a_2|^2|z|^{2k_2})|z|^{2k_1}$$
$$k_2a_1a_2 = -(2k_1+k_2)a_0a_3.$$

Stereographically projecting f onto $\mathbb{H}$ gives the co-ordinate q and

$$q = \frac{1}{\Delta_1}((\bar{a}_0a_1 + \bar{a}_2a_3|z|^{2(k_1+k_2)})z^{k_1} + (\bar{a}_0a_2 - \bar{a}_1a_3|z|^{2k_1})z^{k_1+k_2}j) = \xi + \eta j.$$

Then differentiating with respect to z and $\bar{z}$ gives

$$\xi_{\bar{z}} = \mu \bar{a}_2 a_3 |z|^{2(k_1+k_2-1)} z^{k_1+1}$$

$$\bar{\xi}_{\bar{z}} = \lambda a_0 \bar{a}_1 \bar{z}^{k_1-1}$$

$$\eta_{\bar{z}} = -\lambda a_3 \bar{a}_1 |z|^{2(k_1-1)} z^{k_1+k_2+1}$$

$$\bar{\eta}_{\bar{z}} = \mu a_0 \bar{a}_2 \bar{z}^{k_1+k_2-1}$$

where

$$\lambda = k_1|a_0|^2 - k_2|a_2|^2|z|^{2(k_1+k_2)} - (k_1+k_2)|a_3|^2|z|^{2(2k_1+k_2)}$$

$$\mu = (k_1+k_2)|a_0|^2 + k_2|a_1|^2|z|^{2k_1} - k_1|a_3|^2|z|^{2(2k_1+k_2)}$$

and it is clear from these that $\xi_{\bar{z}}\bar{\xi}_{\bar{z}} + \eta_{\bar{z}}\bar{\eta}_{\bar{z}} = 0$ and f is conformal. Thus the formula may be applied to produce the lift of f into $\mathbb{C}P^3$.

Observe that

$$z^{k_1-1}(\bar{\xi}_{\bar{z}}, \eta_{\bar{z}}) = \lambda'(a_0, a_3 z^{2k_1+k_2}), \qquad \lambda' = \lambda \bar{a}_1 |z|^{2(k_1-1)}.$$

Also,

$$\bar{\xi}_{\bar{z}}\xi + \eta_{\bar{z}}\bar{\eta} = \lambda' a_1 z^{k_1},$$

$$\bar{\xi}_{\bar{z}}\eta - \eta_{\bar{z}}\bar{\xi} = \lambda' a_2 z^{k_1+k_2}$$

and

$$\hat{f} = [a_0, a_3 z^{2k_1+k_2}, a_1 z^{k_1}, a_2 z^{k_1+k_2}].$$

Then $\hat{f}$ is holomorphic and is also horizontal. The horizontality and the fact that $\hat{f}(e^{i\theta}z) = \alpha(e^{i\theta})\hat{f}(z)$ for a diagonal matrix α show that $\hat{f}$ has S^1-symmetry. The equivariance of the lift under the SO(5)-action on S^4 and $\mathbb{C}P^3$ shows that the underlying map $f : S \to S^4$ is also an S^1-symmetric immersion. Moreover, since the lift $\hat{f}$ is harmonic, the equivariance extends to the harmonic sequences of f and $\hat{f}$ [7].

5.2. Some Notation

In the work which follows, we will shortly encounter the problem of how to differentiate a product of two quaternions with respect to the complex co-ordinate z (or $\bar{z}$). The difficulty arises because, for complex z, $zj = j\bar{z}$, so that the effect of moving the $\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right)$ through the first factor to apply it to the second is more complicated to describe. In fact, to differentiate the second factor it is necessary to split the first factor into its $\mathbb{C}$ and $\mathbb{C}j$ components.

Let $p_1, p_2 \in \mathbb{H}$ with $p_1 = a + bj$, $a, b \in \mathbb{C}$. Differentiating the product p_1p_2 yields

$$\frac{\partial}{\partial z}(p_1p_2) = \frac{\partial p_1}{\partial z}p_2 + a\frac{\partial p_2}{\partial z} + bj\frac{\partial p_2}{\partial \bar{z}}$$

which will be represented in the sequel in the following way:

NOTATION. $(p_1p_2)_z = p_{1z}p_2 + (p_1p_2)_{[z]}$ where $(p_1p_2)_{[z]} = a(p_{2z}) + bj(p_{2\bar{z}})$, $(p_1 = a+bj)$. Furthermore, the complex components of p_1 will be written as $a = (p_1)^{\mathbb{C}}$ and $b = (p_1)^{\mathbb{C}j}$ so that $p_1 = (p_1)^{\mathbb{C}} + (p_1)^{\mathbb{C}j}j$.

Lemma 5.2. *If f is conformal then $(\bar{q}_{\bar{z}}q)_{[\bar{z}]} = 0$.*

PROOF. $(\bar{q}_{\bar{z}}q)_{[\bar{z}]} = \bar{\xi}_{\bar{z}}q_{\bar{z}} - \eta_{\bar{z}}jq_z = \bar{\xi}_{\bar{z}}\xi_{\bar{z}} + \eta_{\bar{z}}\bar{\eta}_{\bar{z}}$.

□

5.3. Harmonic Lifts of Conformal Immersions

The formula for the lift to $\mathbb{C}P^3$ which was obtained in the previous chapter will now be put to use in studying properties of the lift. In particular, it is interesting to see when the lift is a harmonic map into $\mathbb{C}P^3$ and, further, when it gives rise to a holomorphic curve in $\mathbb{C}P^3$.

Proposition 5.3. *Suppose $f : S \to S^4$ is conformal and let $\mu = \dfrac{\bar{\xi}_{z\bar{z}}\eta_{\bar{z}} - \eta_{z\bar{z}}\bar{\xi}_{\bar{z}}}{|\bar{\xi}_{\bar{z}}|^2 + |\eta_{\bar{z}}|^2}$. Then $\hat{f}$ is harmonic if*

$$\mu_z|\hat{f}|^2 + \mu\{\langle \hat{f}, \hat{f}_{\bar{z}}\rangle - \langle \hat{f}_z, \hat{f}\rangle\} = 0. \tag{5.3}$$

PROOF. Let $\hat{f} = (\bar{q}_{\bar{z}}, \bar{q}_{\bar{z}}q)$. Differentiating gives

$$\hat{f}_{\bar{z}} = (\bar{q}_{z\bar{z}}, \bar{q}_{z\bar{z}}q + (\bar{q}_{\bar{z}}q)_{[\bar{z}]}).$$

But $(\bar{q}_{\bar{z}}q)_{[\bar{z}]} = 0$ (by lemma (5.2)) so that

$$\hat{f}_{\bar{z}} = (\bar{q}_{z\bar{z}}, \bar{q}_{z\bar{z}}q) = \frac{\bar{q}_{z\bar{z}}\overline{\bar{q}_{\bar{z}}}}{|\bar{q}_{\bar{z}}|^2}(\bar{q}_{\bar{z}}, \bar{q}_{\bar{z}}q).$$

Thus, the form of $\hat{f}_{\bar{z}}$ is rather special in that

$$\hat{f}_{\bar{z}} = \nu\hat{f} + \mu j\hat{f} \tag{5.4}$$

where

$$\nu = \frac{(\bar{q}_{z\bar{z}}\overline{\bar{q}_{\bar{z}}})^{\mathbb{C}}}{|\bar{q}_{\bar{z}}|^2} = \frac{\langle \hat{f}_{\bar{z}}, \hat{f}\rangle}{|\hat{f}|^2}$$

and

$$\mu = \frac{(\bar{q}_{z\bar{z}}\overline{\bar{q}_{\bar{z}}})^{\mathbb{C}j}}{|\bar{q}_{\bar{z}}|^2} = \frac{\bar{\xi}_{z\bar{z}}\eta_{\bar{z}} - \eta_{z\bar{z}}\bar{\xi}_{\bar{z}}}{|\bar{q}_{\bar{z}}|^2}.$$

Now, $\hat{f}$ is a harmonic map into $\mathbb{C}P^3$ if and only if

$$\hat{f}_{z\bar{z}} - \frac{\langle \hat{f}_z, \hat{f}\rangle}{|\hat{f}|^2}\hat{f}_{\bar{z}} - \frac{\langle \hat{f}_{\bar{z}}, \hat{f}\rangle}{|\hat{f}|^2}\hat{f}_z = \alpha\hat{f} \tag{5.5}$$

for some $\alpha \in \mathbb{C}$. But differentiating (5.4) with respect to z gives

$$(\hat{f}_{\bar{z}})_z = \mu_z j\hat{f} + \mu j\hat{f}_{\bar{z}} + \left(\frac{\langle \hat{f}_{\bar{z}}, \hat{f}\rangle}{|\hat{f}|^2}\right)_z \hat{f} + \frac{\langle \hat{f}_{\bar{z}}, \hat{f}\rangle}{|\hat{f}|^2}\hat{f}_z$$

so that for (5.5) to hold, the coefficient of $j\hat{f}$ must be zero, that is,

$$\mu_z|\hat{f}|^2 + \mu\{\langle \hat{f}, \hat{f}_{\bar{z}}\rangle - \langle \hat{f}_z, \hat{f}\rangle\} = 0.$$

□

Remark 5.4. In terms of the complex co-ordinates ξ, η the condition (5.3) is given by

$$(1+|q|^2)(\bar{\xi}_{z\bar{z}\bar{z}}\eta_{\bar{z}} - \eta_{z\bar{z}\bar{z}}\bar{\xi}_{\bar{z}}) + \bar{\xi}_{\bar{z}\bar{z}}\left\{\frac{(1+|q|^2)}{|\bar{q}_{\bar{z}}|^2}\left((|\xi_z|^2 - |\eta_{\bar{z}}|^2)\eta_{z\bar{z}} - 2\bar{\xi}_{z\bar{z}}\xi_z\eta_{\bar{z}}\right) - |q|^2_z\eta_{\bar{z}}\right\}$$
$$+\,\eta_{\bar{z}\bar{z}}\left\{\frac{(1+|q|^2)}{|\bar{q}_{\bar{z}}|^2}\left((|\xi_z|^2 - |\eta_{\bar{z}}|^2)\bar{\xi}_{z\bar{z}} + 2\eta_{z\bar{z}}\bar{\eta}_z\bar{\xi}_{\bar{z}}\right) + |q|^2_z\bar{\xi}_{\bar{z}}\right\} = 0.$$

Given $q = \xi + \eta j$, this condition is not too difficult to investigate. However, there is more to say about harmonic lifts of conformal immersions, in particular that such a lift is either a holomorphic curve in $\mathbb{C}\mathrm{P}^3$ or has a harmonic sequence with a particular symmetry property involving the quaternion j. These cases will now be studied separately.

5.4. Holomorphic Lifts

It is not difficult to see that proposition 5.3 has the following corollary:

Corollary 5.5. *If f is conformal then $\hat{f}$ is holomorphic if and only if*

$$\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \eta_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}} = 0.$$

Proof. First note that $\hat{f}$ is holomorphic if $\hat{f}_{\bar{z}} = \lambda\hat{f}$ for some complex multiple λ. But

$$\hat{f}_{\bar{z}} = (\bar{q}_{\bar{z}\bar{z}}, \bar{q}_{\bar{z}\bar{z}}q) = \frac{\bar{q}_{\bar{z}\bar{z}}\overline{\bar{q}_{\bar{z}}}}{|\bar{q}_{\bar{z}}|^2}(\bar{q}_{\bar{z}}, \bar{q}_{\bar{z}}q)$$

and $\bar{q}_{\bar{z}\bar{z}}\overline{\bar{q}_{\bar{z}}} \in \mathbb{C}$ if and only if $(\bar{q}_{\bar{z}\bar{z}}\overline{\bar{q}_{\bar{z}}})^{\mathbb{C}j} = 0$. This is the case when

$$\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \eta_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}} = 0.$$

□

5.4.1. Holomorphic curves in $\mathbb{C}P^3$. Some properties of holomorphic curves in $\mathbb{C}P^3$ will now be investigated. Let $\psi : S \to \mathbb{C}P^3$ be holomorphic and write $\psi = [g]$ with $g = (z_0, z_1, z_2, z_3)$. Then, taking $z_0 = 1$, ψ is holomorphic if and only if z_1, z_2 and z_3 are meromorphic functions.

Proposition 5.6. *A holomorphic curve $[g] : S \to CP^3$ has the following properties:*

(1) *$\pi[g]$ is conformal.*

(2) *Moreover, $\pi[g]$ is real isotropic.*

(3) *The twistor lift of $\pi[g]$ is $[g]$.*

PROOF.

(1) $\pi[g] = \pi[1, z_1, z_2, z_3] = [1 + z_1 j, z_2 + z_3 j] = [1, q] \in \mathbb{H}P^1$ so that

$$\xi = \frac{z_2 + z_1\bar{z}_3}{1 + |z_1|^2}, \qquad \eta = \frac{z_3 - z_1\bar{z}_2}{1 + |z_1|^2}. \tag{5.6}$$

Then,

$$\xi_{\bar{z}} = \frac{z_{2\bar{z}} + z_{1\bar{z}}\bar{z}_3 + z_1\bar{z}_{3\bar{z}}}{1 + |z_1|^2} - \frac{(z_2 + z_1\bar{z}_3)}{(1 + |z_1|^2)^2}|z_1|^2_{\bar{z}}$$

and since $(z_j)_{\bar{z}} = 0$, this gives

$$\xi_{\bar{z}} = \frac{z_1}{(1 + |z_1|^2)^2}((1 + |z_1|^2)\bar{z}_{3\bar{z}} - (z_2 + z_1\bar{z}_3)\bar{z}_{1\bar{z}}).$$

Similarly,

$$\begin{aligned}\bar{\xi}_{\bar{z}} &= \frac{\bar{z}_{2\bar{z}} + \bar{z}_{1\bar{z}}z_3 + \bar{z}_1 z_{3\bar{z}}}{1 + |z_1|^2} - \frac{(\bar{z}_2 + \bar{z}_1 z_3)}{(1 + |z_1|^2)^2}|z_1|^2_{\bar{z}} \\ &= \frac{1}{(1 + |z_1|^2)^2}((1 + |z_1|^2)\bar{z}_{2\bar{z}} + (z_3 - \bar{z}_2 z_1)\bar{z}_{1\bar{z}})\end{aligned}$$

and

$$\begin{aligned}\eta_{\bar{z}} &= \frac{-z_1}{(1 + |z_1|^2)^2}((1 + |z_1|^2)\bar{z}_{2\bar{z}} + (z_3 - z_1\bar{z}_2)\bar{z}_{1\bar{z}}), \\ \bar{\eta}_{\bar{z}} &= \frac{1}{(1 + |z_1|^2)^2}((1 + |z_1|^2)\bar{z}_{3\bar{z}} - (z_2 - z_1\bar{z}_3)\bar{z}_{1\bar{z}}).\end{aligned}$$

Then, by inspection, $\xi_{\bar{z}}\bar{\xi}_{\bar{z}} + \eta_{\bar{z}}\bar{\eta}_{\bar{z}} = 0$.

(2) Recall that f is said to be real isotropic if and only if span$\{f_z, f_{zz}\}$ is an isotropic subspace of $\mathbb{R}^5$, that is, if and only if f is conformal and $\xi_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}\bar{z}} + \eta_{\bar{z}\bar{z}}\bar{\eta}_{\bar{z}\bar{z}} = 0$. Here

$$\begin{aligned}
\xi_{\bar{z}\bar{z}} &= \frac{z_1}{(1+|z_1|^2)}(\bar{z}_{3\bar{z}\bar{z}} - \xi\bar{z}_{1\bar{z}\bar{z}} - 2\xi_{\bar{z}}\bar{z}_{1\bar{z}}),\\
\bar{\xi}_{\bar{z}\bar{z}} &= \frac{1}{(1+|z_1|^2)}(\bar{z}_{2\bar{z}\bar{z}} + \eta\bar{z}_{1\bar{z}\bar{z}} + 2\eta_{\bar{z}}\bar{z}_{1\bar{z}}),\\
\eta_{\bar{z}\bar{z}} &= \frac{-z_1}{(1+|z_1|^2)}(\bar{z}_{2\bar{z}\bar{z}} + \eta\bar{z}_{1\bar{z}\bar{z}} + 2\eta_{\bar{z}}\bar{z}_{1\bar{z}}),\\
\bar{\eta}_{\bar{z}\bar{z}} &= \frac{1}{(1+|z_1|^2)}(\bar{z}_{3\bar{z}\bar{z}} - \xi\bar{z}_{1\bar{z}\bar{z}} - 2\xi_{\bar{z}}\bar{z}_{1\bar{z}}),
\end{aligned}$$

which, by inspection, give the result.

(3) Given $[g]$ holomorphic, πg is conformal (by (1)) and so πg lifts to $[\bar{q}_{\bar{z}}, \bar{q}_{\bar{z}}q] \in \mathbb{C}\mathrm{P}^3$ where $q = \xi + \eta j$ is given by (5.6) above. Suppose $[\hat{f}] = [w_0, w_1, w_2, w_3]$. Then $w_0 = \bar{\xi}_{\bar{z}}$, $w_1 = -\eta_{\bar{z}}$ and so on. Now $\eta_{\bar{z}} = -z_1\bar{\xi}_{\bar{z}}$ so that $w_1 = z_1 w_0$ which implies

$$w_2 = \bar{\xi}_{\bar{z}}\xi + \eta_{\bar{z}}\bar{\eta} = w_0\xi - w_1\bar{\eta} = w_0(\xi - z_1\bar{\eta}) = w_0 z_2.$$

Similarly, $w_3 = w_0 z_3$, and

$$[\hat{f}] = [w_0, w_1, w_2, w_3] = [w_0, w_0 z_1, w_0 z_2, w_0 z_3] = [g].$$

□

Notice that parts (1) and (2) of this proposition demonstrate another proof of the result due to Eells-Salamon which was formulated in terms of the frame in Chapter 3 (see theorem 3.14).

5.5. Harmonic Lifts with j-Symmetry

Suppose now that the lift is harmonic but not holomorphic (the $\mu \neq 0$ case) and study the harmonic sequence of $\hat{f}$. Since $\hat{f}$ is not a holomorphic curve, $\hat{f}_{-1}$ is non-zero and

$$\hat{f}_{-1} = \hat{f}_{\bar{z}} - \frac{\langle \hat{f}_{\bar{z}}, \hat{f} \rangle}{|\hat{f}|^2} \hat{f}.$$

But by (5.4),

$$\hat{f}_{\bar{z}} - \frac{\langle \hat{f}_{\bar{z}}, \hat{f} \rangle}{|\hat{f}|^2} \hat{f} = \mu j \hat{f},$$

so that $\hat{f}_{-1} = \mu j \hat{f}$ for $\mu \in \mathbb{C} \backslash \{0\}$. Indeed, by the construction of the harmonic sequence, this has implications for the whole sequence of harmonic maps in that it implies that span$\{\hat{f}_{-(p+1)}\}$ =span$\{j \hat{f}_p\}$ for all p.

Proposition 5.7. *If $\hat{f}_{-1} = \mu j \hat{f}$ then $\hat{f}_{-(p+1)} = \mu j \hat{f}_p$ for $p \geq 0$.*

PROOF. Proceed by induction. Suppose $\hat{f}_{-k} = \mu j \hat{f}_{k-1}$ for some $k > 1$. Then construct the next element along to the left in the harmonic sequence, $\hat{f}_{-(k+1)}$, by

$$\begin{aligned} \hat{f}_{-(k+1)} &= (\hat{f}_{-k})_{\bar{z}} - \frac{\langle (\hat{f}_{-k})_{\bar{z}}, \hat{f}_{-k} \rangle}{|\hat{f}_{-k}|^2} \hat{f}_{-k} \\ &= \mu_{\bar{z}} j \hat{f}_{k-1} + \mu j (\hat{f}_{k-1})_z - \frac{\langle \mu_{\bar{z}} j \hat{f}_{k-1} + \mu j (\hat{f}_{k-1})_z, \mu j \hat{f}_{k-1} \rangle}{|\mu|^2 |\hat{f}_{k-1}|^2} \mu j \hat{f}_{k-1} \\ &= \mu_{\bar{z}} j \hat{f}_{k-1} + \mu j (\hat{f}_{k-1})_z - \mu_{\bar{z}} j \hat{f}_{k-1} - \frac{\langle j (\hat{f}_{k-1})_z, j \hat{f}_{k-1} \rangle}{|\hat{f}_{k-1}|^2} \mu j \hat{f}_{k-1}. \end{aligned}$$

Now $\langle ja, jb \rangle = (-ja\bar{b}j)^{\mathbb{C}} = -j(a\bar{b})^{\mathbb{C}} j = -j \langle a, b \rangle j$ which means

$$\begin{aligned} \hat{f}_{-(k+1)} &= \mu \left[j (\hat{f}_{k-1})_z - j \frac{\langle (\hat{f}_{k-1})_z, \hat{f}_{k-1} \rangle}{|\hat{f}_{k-1}|^2} \hat{f}_{k-1} \right] \\ &= \mu j \hat{f}_k. \end{aligned}$$

Hence the result.

□

So if $\hat{f}_{-1} = \mu j \hat{f}$, the harmonic sequence of $\hat{f}$ has a type of symmetry in that it has the form

$$\ldots \mu j \hat{f}_2, \mu j \hat{f}_1, \mu j \hat{f}, \hat{f}, \hat{f}_1, \hat{f}_2 \ldots$$

We will call a sequence with this property a *j-symmetric harmonic sequence* and $\hat{f}$ a *j-symmetric harmonic map.*

This discussion shows that proposition 5.3 has the following important corollary;

Corollary 5.8. *A twistor lift to $\mathbb{C}P^3$ of a conformal immersion $f : S \to S^4$ which is harmonic is either a holomorphic curve or has a j-symmetric harmonic sequence.*

5.5.1. Harmonic Maps in $\mathbb{C}\mathrm{P}^3$ with j-Symmetry. Let $[g] : S \to \mathbb{C}\mathrm{P}^3$ be a harmonic map and write $g = (1, z_1, z_2, z_3)$. Then,

$$g_{-1} = g_{\bar{z}} - \frac{\langle g_{\bar{z}}, g\rangle}{|g|^2} g = (-\lambda, z_{1\bar{z}} - \lambda z_1, z_{2\bar{z}} - \lambda z_2, z_{3\bar{z}} - \lambda z_3)$$

with $\lambda = \dfrac{\langle g_{\bar{z}}, g\rangle}{|g|^2}$. Now $jg = \hat{J}\bar{g}$ where $\hat{J}$ is the standard complex structure (4.1) and $g_{-1} = \mu j g$ for some $\mu \in \mathbb{C}$ if and only if

$$\lambda = \mu \bar{z}_1, \qquad z_{1\bar{z}} - \lambda z_1 = \mu,$$

$$z_{2\bar{z}} - \lambda z_2 = -\mu z_3, \qquad z_{3\bar{z}} - \lambda z_3 = \mu \bar{z}_2.$$

Then the j-symmetry of g is expressed by the conditions

$$\begin{aligned} z_{2\bar{z}} &= \frac{z_{1\bar{z}}}{1 + |z_1|^2}(\bar{z}_1 z_2 - \bar{z}_3) \\ z_{3\bar{z}} &= \frac{z_{1\bar{z}}}{1 + |z_1|^2}(\bar{z}_1 z_3 + \bar{z}_2). \end{aligned} \tag{5.7}$$

Proposition 5.9. *A harmonic map $[g] : S \to \mathbb{C}P^3$ with a j-symmetric harmonic sequence has the following properties:*

(1) *πg is conformal.*

(2) *The twistor lift of πg is g.*

PROOF. $\pi[g] = \pi[1, z_1, z_2, z_3] = [1 + z_1 j, z_2 + z_3 j] = [1, q] \in \mathbb{HP}^1$ so that as in (5.6),

$$\xi = \frac{z_2 + z_1 \bar{z}_3}{1 + |z_1|^2}, \qquad \eta = \frac{z_3 - z_1 \bar{z}_2}{1 + |z_1|^2}.$$

Then use of the equations (5.7) shows that $\xi_{\bar{z}}$, $\bar{\xi}_{\bar{z}}$, $\eta_{\bar{z}}$, $\bar{\eta}_{\bar{z}}$ are the same as in the proof of proposition 5.6 so that the conclusions arrived at in (1) and (3) apply in the j-symmetric case as well.

□

Note that here we do not have the result that πg is real isotropic - the conditions (5.7) are weaker than the holomorphicity conditions $z_{j_{\bar{z}}} = 0$ and consequently do not produce $\xi_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}\bar{z}} + \eta_{\bar{z}\bar{z}}\bar{\eta}_{\bar{z}\bar{z}} = 0$.

It is not hard to check that a holomorphic map into $\mathbb{CP}^3$ is always conformal. However, this is not the case for j-symmetric harmonic maps. Let $\hat{f}$ be a j-symmetric harmonic lift of f. Then $\hat{f}$ is conformal if and only if $\langle \hat{f}_1, \hat{f}_{-1} \rangle = 0$ where $\ldots \hat{f}_{-1}, \hat{f}, \hat{f}_1, \ldots$ is the harmonic sequence of $\hat{f}$. But (by (5.4)), $\hat{f}_{-1} = \mu j \hat{f}$ and by construction

$$\hat{f}_1 = \hat{f}_z - \frac{\langle \hat{f}_z, \hat{f} \rangle}{|\hat{f}|^2} \hat{f}$$

so that

$$\langle \hat{f}_1, \hat{f}_{-1} \rangle = \langle \hat{f}_z - \frac{\langle \hat{f}_z, \hat{f} \rangle}{|\hat{f}|^2} \hat{f}, \mu j \hat{f} \rangle = \mu \langle \hat{f}_z, j \hat{f} \rangle$$

and $\hat{f}$ is conformal if and only if

$$\langle \hat{f}_z, j \hat{f} \rangle = 0. \tag{5.8}$$

Let $\hat{f}$ be given by $(1, z_1, z_2, z_3)$. Then (5.8) holds if and only if

$$\langle (0, z_{1z}, z_{2z}, z_{3z}), (-\bar{z}_1, 1, -\bar{z}_3, \bar{z}_2) \rangle = z_{1z} - z_{2z} z_3 + z_{3z} z_2 = 0. \tag{5.9}$$

Thus (5.9), (5.7) and the discussion in 2.3.5 give rise to the following observation:

Proposition 5.10. *If $\hat{f} : S \to \mathbb{C}P^3$ is the lift of a conformal immersion $f : S \to S^4$ given by $\hat{f} = [\bar{q}_{\bar{z}}, \bar{q}_{\bar{z}}q]$ and $\hat{f}$ is conformal and harmonic with a j-symmetric harmonic sequence then $\hat{f}$ is J_2-holomorphic and so f is harmonic.*

5.6. More About Harmonic Sequences

It has been established in corollary 5.8 that a harmonic lift of a conformal immersion has a rather special harmonic sequence. This section contains two further observations, the first of which is a general remark and the second discusses the relationship between the harmonic sequences of the Clifford Torus and its lift.

Theorem 5.11. *Suppose the lift $\hat{f}$ of f is harmonic and linearly full. If the harmonic sequence of $\hat{f}$ is 3-orthogonal then it is 4-orthogonal.*

PROOF. Recall from corollary 5.8 that such an $\hat{f}$ is either holomorphic or has a j-symmetric harmonic sequence. Suppose first that $\hat{f}$ is holomorphic. Then $\hat{f}$ is conformal and all the elements of the harmonic sequence are mutually orthogonal. So $\hat{f}$ is 4-orthogonal. On the other hand, suppose that $\hat{f}$ is a j-symmetric harmonic map. Then the harmonic sequence of $\hat{f}$ has the form

$$\ldots, \mu j \hat{f}_1, \mu j \hat{f}, \hat{f}, \hat{f}_1, \ldots$$

Thus $\hat{f}_{-1} = \mu j \hat{f}$, $\hat{f}_{-2} = \mu j \hat{f}_1$ so that

$$\langle \hat{f}_{-2}, \hat{f}_1 \rangle = \langle \mu j \hat{f}_1, \hat{f}_1 \rangle = 0$$

always. Therefore it follows that if $\langle \hat{f}_{-1}, \hat{f}_1 \rangle = 0$, the sequence is automatically not only 3-orthogonal but 4-orthogonal.

□

Let us now compare the harmonic sequence of the lift of the Clifford Torus with that of the torus itself as a surface in S^3. Let

$$f(x,y) = \frac{1}{\sqrt{2}}(\cos 2x, \sin 2x, \cos 2y, \sin 2y).$$

and let e_1, e_2, e_3, e_4 be the standard basis on $\mathbb{C}^4$. Then with respect to the unitary basis $\tilde{e}_1 = \frac{(e_1 - ie_2)}{\sqrt{2}}$, $\tilde{e}_2 = \frac{(e_1 + ie_2)}{\sqrt{2}}$, $\tilde{e}_3 = \frac{(e_3 - ie_4)}{\sqrt{2}}$, $\tilde{e}_4 = \frac{(e_3 + ie_4)}{\sqrt{2}}$, f may be written

$$\begin{aligned} f(z,\bar{z}) &= \frac{1}{2}\left(e^{i(z+\bar{z})}\tilde{e}_1 + e^{-i(z+\bar{z})}\tilde{e}_2 + e^{(z-\bar{z})}\tilde{e}_3 + e^{-(z-\bar{z})}\tilde{e}_4\right) \\ &= e^{Dz-\overline{Dz}}u_0 \end{aligned}$$

where

$$D = \operatorname{diag}(i,-i,1,-1) \in \mathrm{U}(4), \qquad u_0 = \frac{1}{2}(1,1,1,1).$$

Then the harmonic sequence of f is given by $\{f_p\}$ where

$$f_p = D^p e^{Dz-\overline{Dz}}u_0.$$

Recall from section 5.1.2 that $\hat{f}(z,\bar{z}) = [e^{\hat{D}z-\overline{\hat{D}z}}\hat{u}_0]$, where

$$\hat{D} = \frac{1}{\sqrt{2}}\operatorname{diag}(-(1+i),(1+i),-(1-i),(1-i)), \qquad \hat{u}_0 = \frac{1}{2}(1,i,\frac{(1-i)}{\sqrt{2}},\frac{(1+i)}{\sqrt{2}}).$$

In order to see whether the maps f and $\hat{f}$ are congruent - that is, related by a holomorphic isometry of $\mathbb{CP}^3$ - we look at the invariants Γ, $\hat{\Gamma}$ and U, $\hat{U}$. Recall that by the congruence theorem, a harmonic sequence is uniquely determined by γ_0, γ_1 and $\{u_{k,0}\}_{k=1}^n$. First consider $\phi : S \to \mathbb{CP}^n$ of the form $\phi = [e^{Az-\overline{Az}}u]$ with $A \in \mathrm{U}(n+1)$ a diagonal matrix and $u \in \mathbb{C}^{n+1}\backslash\{0\}$. Then

$$\phi_p = [A^p e^{Az-\overline{Az}}u]$$

and

$$|\phi_p|^2 = \langle A^p e^{Az-\overline{Az}}u, A^p e^{Az-\overline{Az}}u\rangle = |u|^2,$$

that is, $|\phi_p|^2$ is constant for all p. Thus, for the Clifford torus and its lift, $|f_p|^2 = |u_0|^2 = 1$ and $|\hat{f}_p|^2 = |\hat{u}_0|^2 = 1$ respectively and

$$\gamma_p = \frac{|f_{p+1}|^2}{|f_p|^2} = 1 = \hat{\gamma}_p \quad \text{for all } p.$$

Now recall that for a harmonic sequence $\{\phi_p\}$,

$$u_{p,q} = \frac{\langle \phi_p, \phi_q \rangle}{|\phi_q|^2}, \qquad p > q.$$

Then, by construction, $\langle f_{p+1}, f_p \rangle = 0$ and since f is conformal, $\langle f_{p+2}, f_p \rangle = 0$, so that $u_{1,0} = u_{2,0} = 0$. Also,

$$u_{3,0} = \langle f_3, f \rangle = \langle D^3 u_0, u_0 \rangle = 0,$$

$$u_{4,0} = \langle f_4, f \rangle = \langle D^4 u_0, u_0 \rangle = \langle u_0, u_0 \rangle = 1.$$

Now,

$$\hat{D}^2 = \operatorname{diag}(i, i, -i, -i)$$

so that

$$\hat{u}_{2,0} = \langle \hat{f}_2, \hat{f} \rangle = \langle \hat{D}^2 \hat{u}_0, \hat{u}_0 \rangle = 0$$

and $\hat{D}^3 = \frac{1}{\sqrt{2}}\operatorname{diag}((1-i), -(1-i), (1+i), -(1+i))$ so that

$$\hat{u}_{3,0} = \langle \hat{f}_3, \hat{f} \rangle = \langle \hat{D}^3 \hat{u}_0, \hat{u}_0 \rangle = 0.$$

Finally,

$$\hat{u}_{4,0} = \langle \hat{f}_4, \hat{f} \rangle = \langle \hat{D}^4 \hat{u}_0, \hat{u}_0 \rangle = -\langle \hat{u}_0, \hat{u}_0 \rangle = -1,$$

which demonstrates that the invariants $u_{4,0}$ and $\hat{u}_{4,0}$ are not the same. Thus, by the congruence theorem, the maps f and $\hat{f}$ are not congruent - there is no holomorphic isometry of $\mathbb{C}P^3$ which takes one into the other.

However, f and $\hat{f}$ are related by a simple change of co-ordinate on the domain. The change of co-ordinate required is

$$z \longmapsto -\frac{(1+i)}{\sqrt{2}} z$$

and this corresponds to a rotation of the domain through the angle $-\frac{3\pi}{4}$.

5.7. Positive and Negative Lifts

In section 4.2 the twistor space $Z(N)$ of a Riemannian 4-manifold N was defined to be the space of orthogonal complex structures on the tangent spaces of N compatible with the orientation of N. For $N = S^4$, $Z(N) = \mathbb{C}\mathrm{P}^3$ and we obtain a twistor lift $\hat{f}$ to the twistor space of a map $f : S \to S^4$. Now, reversing the orientation on N gives a second twistor space, say $Z_-(N)$, which is compatible with this opposite orientation of N.

When $\dim N = 4$, the special nature of SO(4) gives a nice way to think about $Z(N)$. Consider $\bigwedge^2 T(N)$, the space of 2-vectors on N. Then $\bigwedge^2 T(N)$ splits as

$$\textstyle\bigwedge^2 T(N) = \bigwedge^2_+ T(N) \oplus \bigwedge^2_- T(N)$$

into the direct sum of the ± 1-eigenspaces of the Hodge $*$- operator. Then $Z_\pm(N) := S(\bigwedge^2_\pm T(N))$, the unit sphere bundle of $\bigwedge^2_\pm T(N)$. $Z_+(N) = Z(N)$ as before, but $Z_-(N)$ corresponds to $Z(N)$ under the other choice of orientation on N. Then we can define twistor lifts $\hat{f}_\pm : S \to Z_\pm(N)$. When $N = S^4$, $Z_\pm(N)$ are both isomorphic to $\mathbb{C}\mathrm{P}^3$ and the lifts $\hat{f}_\pm$ are the 'subsidiary Gauss lifts' of Eells-Salamon [17].

In this work so far, only the lift $\hat{f}_+$ has been investigated, that is, by $\hat{f}$ we have really meant $\hat{f}_+$. But the antipodal map $a : S^4 \to S^4$ reverses the orientation on S^4 taking f to $-f$ so that the lift $\hat{f}_-$ is just the lift $\hat{f}$ of the map $a \circ f : S \to S^4$.

For conformal f it is interesting to observe that the lift $\hat{f}_-$ to $\mathbb{C}\mathrm{P}^3$ is closely related to the second row of F, the Sp(2) description of the adapted frame of f (cf. section 4.8.2). Let $f = (X_0, \dots, X_4)$ as before, set $q = \frac{X}{1 + X_0}$. Then $a \circ f = -(X_0, \dots, X_4)$ which corresponds to $\hat{q} = -\frac{X}{1 - X_0} = -\frac{X}{(1 + X_0)|q|^2} = -\frac{q}{|q|^2}$. If f is conformal, then

$a \circ f$ is conformal so that the lift $\hat{f}_-$ is given by

$$
\begin{aligned}
[\hat{f}_-] &= \left[-\left(\frac{q}{|q|^2}\right)_{\bar{z}}, \left(\frac{q}{|q|^2}\right)_{\bar{z}} \frac{q}{|q|^2} \right] \\
&= [(\bar{q}q)_{[\bar{z}]}\bar{q}, -(\bar{q}q)_{[\bar{z}]}] \qquad (5.10) \\
&= [q_{\bar{z}}\bar{q}, -q_{\bar{z}}].
\end{aligned}
$$

But $[q_{\bar{z}}\bar{q}, -q_{\bar{z}}] = [j(\bar{q}_{\bar{z}}q)_{[\bar{z}]}\bar{q}, -j(\bar{q}_{\bar{z}}q)_{[\bar{z}]}]$ so that comparing with corollary 4.10, the second row of F is a complex multiple of $j\hat{f}_-$.

Using (5.10) as the form of $[\hat{f}_-]$ it is a straightforward matter to compute the condition for $\hat{f}_-$ to be (J_1) holomorphic. First observe that

$$(q_{\bar{z}}\bar{q})_{[\bar{z}]} = \xi_{\bar{z}}\bar{q}_{\bar{z}} + \eta_{\bar{z}}j(\bar{q}_z) = 0$$

and

$$(\hat{f}_-)_{\bar{z}} = \frac{(q_{\bar{z}\bar{z}}\overline{q_{\bar{z}}})}{|q_{\bar{z}}|^2}(q_{\bar{z}}\bar{q}, -q_{\bar{z}}) = \lambda\hat{f}_-$$

for $\lambda \in \mathbb{C}$ if and only if

$$(q_{\bar{z}\bar{z}}\overline{q_{\bar{z}}})^{\mathbb{C}j} = \xi_{\bar{z}}\eta_{\bar{z}\bar{z}} - \xi_{\bar{z}\bar{z}}\eta_{\bar{z}} = 0.$$

Comparing with corollary 5.5 shows there is much similarity with the calculation in the $\hat{f}_+$ case. In fact, the conditions for $\hat{f}_-$ to have properties such as holomorphicity, harmonicity and so on may be easily determined from those for $\hat{f}_+$ by sending $q \mapsto \bar{q}$, that is, $\xi \mapsto \bar{\xi}$, $\eta \mapsto -\eta$. Moreover, it is not difficult to see that theorem 5.11, proposition 5.7 and corollary 5.8 hold equally well for $\hat{f}_-$ as they do for $\hat{f}_+$. These results are summarised in theorem 5.12.

Theorem 5.12. *If the twistor lift $\hat{f}_\pm : S \to \mathbb{C}P^3$ of a conformal immersion $f : S \to S^4$ is harmonic then it is either holomorphic or it is harmonic with a j-symmetric harmonic sequence.*

Recall that in theorem 3.14 of Chapter 3, the result of Eells-Salamon was obtained up to a choice of sign, attributed to the choice of orientation on S^4. The use of positive and negative lifts makes this precise as can be seen in the proof of the following theorem:

Theorem 5.13. *f is a real isotropic immersion if and only if one of the lifts $\hat{f}_+$, $\hat{f}_-$ is J_1-holomorphic.*

PROOF. By (2) in proposition 5.6, either of $\hat{f}_\pm$ holomorphic implies that $f = \pi \hat{f}_\pm$ is real isotropic. Conversely, suppose f is real isotropic. Then $f_z.f_z = 0$ and $f_{zz}.f_{zz} = 0$, that is, in terms of the quaternionic co-ordinate q,

$$\xi_{\bar{z}}\bar{\xi}_{\bar{z}} + \eta_{\bar{z}}\bar{\eta}_{\bar{z}} = 0 \tag{5.11}$$

$$\xi_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}\bar{z}} + \eta_{\bar{z}\bar{z}}\bar{\eta}_{\bar{z}\bar{z}} = 0. \tag{5.12}$$

Differentiating (5.11) with respect to $\bar{z}$ gives

$$\xi_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}} + \xi_{\bar{z}}\bar{\xi}_{\bar{z}\bar{z}} + \eta_{\bar{z}\bar{z}}\bar{\eta}_{\bar{z}} + \eta_{\bar{z}}\bar{\eta}_{\bar{z}\bar{z}} = 0. \tag{5.13}$$

If $\xi_{\bar{z}\bar{z}}$, $\bar{\xi}_{\bar{z}\bar{z}}$, $\eta_{\bar{z}\bar{z}}$, $\bar{\eta}_{\bar{z}\bar{z}}$ are all zero then the holomorphicity conditions are trivially satisfied so, without loss of generality, suppose $\eta_{\bar{z}\bar{z}} \neq 0$. Then (5.12) implies

$$\bar{\eta}_{\bar{z}\bar{z}} = -\frac{\xi_{\bar{z}\bar{z}}\bar{\xi}_{\bar{z}\bar{z}}}{\eta_{\bar{z}\bar{z}}}$$

and substituting into (5.13) gives

$$-\frac{\xi_{\bar{z}\bar{z}}}{\eta_{\bar{z}\bar{z}}}(\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \bar{\xi}_{\bar{z}}\eta_{\bar{z}\bar{z}}) + (\xi_{\bar{z}}\bar{\xi}_{\bar{z}\bar{z}} + \bar{\eta}_{\bar{z}}\eta_{\bar{z}\bar{z}}) = 0. \tag{5.14}$$

Recall from the discussion following theorem 4.11 that not all of $\xi_{\bar{z}}$, $\bar{\xi}_{\bar{z}}$, $\eta_{\bar{z}}$, $\bar{\eta}_{\bar{z}}$ can be zero and suppose that $\eta_{\bar{z}} \neq 0$. Then by (5.11),

$$\bar{\eta}_{\bar{z}} = -\frac{\xi_{\bar{z}}\bar{\xi}_{\bar{z}}}{\eta_{\bar{z}}}$$

and (5.14) becomes

$$-\frac{\xi_{\bar{z}\bar{z}}}{\eta_{\bar{z}\bar{z}}}(\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \bar{\xi}_{\bar{z}}\eta_{\bar{z}\bar{z}}) + \frac{\xi_{\bar{z}}}{\eta_{\bar{z}}}(\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \bar{\xi}_{\bar{z}}\eta_{\bar{z}\bar{z}}) = 0,$$

i.e.

$$(\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \bar{\xi}_{\bar{z}}\eta_{\bar{z}\bar{z}})(\eta_{\bar{z}\bar{z}}\xi_{\bar{z}} - \eta_{\bar{z}}\xi_{\bar{z}\bar{z}}) = 0$$

so that either

$$\bar{\xi}_{\bar{z}\bar{z}}\eta_{\bar{z}} - \bar{\xi}_{\bar{z}}\eta_{\bar{z}\bar{z}} = 0 \qquad (\hat{f}_{+}\ J_1\text{-holomorphic})$$

$$\text{or} \quad \xi_{\bar{z}\bar{z}}\eta_{\bar{z}} - \xi_{\bar{z}}\eta_{\bar{z}\bar{z}} = 0 \qquad (\hat{f}_{-}\ J_1\text{-holomorphic}).$$

□

CHAPTER 6

Twistor Lifts for H^4

The ideas of chapter 4 will now be extended to the case of immersions $f : S \to H^4$ of surfaces in four-dimensional hyperbolic space. The chapter begins with a review of semi-Euclidean space (section 6.1), its hyperquadrics and the semi-orthogonal groups (section 6.2) before progressing to a discussion of the homogeneous spaces $\mathrm{SO}(1,4)/K$. As in the S^4-case, these may be identified as flag manifolds and section 6.4 shows how the lifts of f to these spaces may be defined.

Recall that for minimal totally isotropic immersions of surfaces into S^4 one considers the twistor fibration $\pi_{S^4} : \mathbb{C}\mathrm{P}^3 \to S^4$. An immersion $\phi : S \to S^4$ of a Riemann surface defines a lifting $\Phi : S \to \mathbb{C}\mathrm{P}^3$ and ϕ is minimal and totally isotropic if and only if the lift Φ is holomorphic and tangential to the horizontal distribution. Furthermore there is an algorithm due to Bryant for constructing explicitly all 'horizontal' holomorphic curves.

It is then shown how to do the same for minimal immersions into H^4. In section 6.5 the twistor fibration $\pi_{H^4} : Z(H^4) \to H^4$ and the horizontal distribution on the twistor space $Z(H^4)$ are described. Further, section 6.6 relates real isotropic minimal immersions into S^4 with those into H^4 and shows that, via a twistor transform, there is an analogous algorithm to that of Bryant for constructing such immersions into H^4.

The review material in sections 6.1, 6.2 is taken from O'Neill [25].

6.1. Semi-Euclidean Space

To begin with, some definitions;

Definition 6.1. *A metric tensor g on a smooth manifold M is a symmetric non-degenerate $(0,2)$ tensor field on M of constant index.*

Definition 6.2. *A semi-Riemannian manifold is a smooth manifold M equipped with a metric tensor g.*

The index ν of g on a semi-Riemannian manifold is called the index of M : $0 \leq \nu \leq n = \dim M$. If $\nu = 0$, M is a Riemannian manifold and each g_p is a (positive definite) inner product on T_pM. If $\nu = 1$ and $n \geq 2$, M is a Lorentz manifold. Semi-Riemannian manifolds are also called pseudo-Riemannian manifolds in the literature.

It is well-known that for each $p \in \mathbb{R}^n$ there is a canonical linear isomorphism from $\mathbb{R}^n$ to $T_p\mathbb{R}^n$ that sends v to $v_p = \sum v_i \partial_i$ (in terms of natural co-ordinates). So the dot product on $\mathbb{R}^n$ gives rise to a metric tensor on $\mathbb{R}^n$ with

$$\langle v_p, w_p \rangle = v.w = \sum v_i w_i.$$

Now, for an integer ν with $0 \leq \nu \leq n$, changing the first ν plus signs in the above sum to minus signs gives another metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n} v_j w_j$$

of index ν. The resulting semi-Riemannian manifold is the semi-Euclidean space $\mathbb{R}^{\nu,n-\nu}$. (Note that this reduces to $\mathbb{R}^n$ if $\nu = 0$). If $n \geq 1$, $\mathbb{R}^{1,n}$ is called Minkowski $(n+1)$-space and if $n = 3$ it is the simplest example of a relativistic spacetime.

6.2. Hyperquadrics

Now consider the semi-Euclidean space $\mathbb{R}^{1,4}$ (Minkowski 5-space) and let

$$q(v) = \langle v, v\rangle = -v_0^2 + v_1^2 + \ldots + v_4^2$$

be the inner product. Then $Q = q^{-1}(\epsilon r^2)$ is a semi-Riemannian hypersurface of $\mathbb{R}^{1,4}$ where $\epsilon = \pm 1$, $r > 0$. These hypersurfaces are called the (central) hyperquadrics of $\mathbb{R}^{1,4}$. The two families $\{\epsilon = 1\}, \{\epsilon = -1\}$ fill all of $\mathbb{R}^{1,4}$ (except for the null-cone $\Lambda = q^{-1}(0)\backslash 0$ and the origin).

Definition 6.3. (1) *The pseudosphere of radius $r > 0$ in $\mathbb{R}^{1,4}$ is the hyperquadric*

$$S_1^4(r) = q^{-1}(r^2) = \{p \in \mathbb{R}^{1,4} \mid \langle p, p\rangle = r^2\}$$

with dimension 4 *and index* 1.

(2) *The pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}^{1,4}$ is the hyperquadric*

$$H^4(r) = q^{-1}(-r^2) = \{p \in \mathbb{R}^{1,4} \mid \langle p, p\rangle = -r^2\}$$

with dimension 4 *and index* 0.

The hyperquadric $H^4(r)$ is a Riemannian manifold and consists of two connected components, each diffeomorphic to $\mathbb{R}^4$. These components are congruent under the isometry $(p_1, \ldots, p_5) \to (-p_1, \ldots, p_5)$ of $\mathbb{R}^{1,4}$. The component through $(r, 0, \ldots, 0)$ is called the upper and the one through $(-r, 0, \ldots, 0)$ the lower embedding of hyperbolic 4-space $H^4(r)$ in $\mathbb{R}^{1,4}$.

The geodesics of either of these hyperquadrics Q in $\mathbb{R}^{1,4}$ are the curves sliced from Q by planes through the origin of $\mathbb{R}^{1,4}$ and, in general, geodesics can be either spacelike ($\langle v, v\rangle > 0$), null ($\langle v, v\rangle = 0,\ v \neq 0$) or timelike ($\langle v, v\rangle < 0$) vectors. However, as seen above, $H^4(r)$ is a special case in that $\nu = 0$ and all geodesics are spacelike. In fact, geodesics on $H^4(r)$ are branches of hyperbole in $\mathbb{R}^{1,4}$.

It is also true that hyperquadrics have constant curvature [25]. $S_1^4(r)$ is a complete semi-Riemannian manifold with constant positive curvature $K = \frac{1}{r^2}$, while $H^4(r)$ is a complete Riemannian manifold with constant negative curvature $K = -\frac{1}{r^2}$. $H^4(r)$ is non-compact, but on both the hyperquadrics, all points and all directions are geometrically the same.

Let us take $r = 1$ and write $H^4 = H^4(1) = \{x \in \mathbb{R}^{1,4} \mid q(x) = -1\}$. Then S^4 and H^4 are related via stereographic projection onto the disc $D^4 = \{q \in \mathbb{H} \mid |q|^2 < 1\}$. The upper embedding of H^4 projects onto the disc through $(0,0,0,0,1) \in \mathbb{R}^{4,1}$ via $(q,t) \mapsto \left(\frac{q}{t}, 1\right)$. Stereographic projection of the upper hemisphere of S^4 onto the disc $D^4 = \{(q,0) \in \mathbb{R}^5 = \mathbb{H} \oplus \mathbb{R}, |q|^2 < 1\}$ is given by $(q,t) \mapsto \left(\frac{q}{1+t}, 0\right)$. Then a point $(q,t) \in H^4$ corresponds to a point $(x,s) \in S^4$ where

$$(x,s) = \left(\frac{2tq, t^2 - |q|^2}{t^2 + |q|^2}\right) = \left(\frac{2tq, 1}{2t^2 - 1}\right) \tag{6.1}$$

since $|q|^2 - t^2 = -1$.

Recall (section 2.1.2) the group SO(1,4) is defined as

$$\mathrm{SO}(1,4) = \{A \in \mathrm{GL}(5,\mathbb{R}) \mid A^t I_{1,4} A = I_{1,4}\}$$

where $I_{1,4} = \left(\begin{array}{c|c} -1 & \\ \hline & I_4 \end{array}\right)$. This is the group of isometries of H^4 [26] and SO(1,4) acts as

$$\left(\begin{array}{c|c} t & v^t \\ \hline u & A \end{array}\right)\begin{pmatrix} 1 \\ x \end{pmatrix},$$

where $t \in \mathbb{R}$, $x, u, v \in \mathbb{R}^4$, taking

$$x \longmapsto \frac{Ax + u}{v.x + t}.$$

6.3. The Flag Manifolds $\mathrm{SO}(1,4)/K$

In a way which is entirely analogous to that discussed in Chapter 2 for SO(5), the homogeneous spaces $\mathrm{SO}(1,4)/K$ for subgroups $K \subseteq \mathrm{SO}(4)$ may be identified as different types of flag manifolds. Their elements are given in terms of different flags in $\mathbb{R}^{1,4}$ as orthogonal direct sum decompositions of oriented subspaces. In each case $\mathrm{SO}(1,4)$ acts on the flags and K is the stabiliser of a typical flag.

For example, in the case of $\mathrm{SO}(1,4)/T^2$ the elements are direct sums of oriented subspaces

$$\mathbb{R}^{1,4} = L \oplus V_1 \oplus V_2, \qquad \dim L = 1,\ \dim V_i = 2\ (i = 1,2)$$

where L is timelike (and the V_i are spacelike) with the orientation induced on $\mathbb{R}^{1,4}$ by those on L, V_1 and V_2 agreeing with the standard one. For $\mathrm{SO}(1,4)/\mathrm{U}(2)$ the decompositions are of the form

$$\mathbb{R}^{1,4} = L \oplus V, \qquad \dim L = 1,\ \dim V = 4$$

with V having an orthogonal complex structure compatible with the metric and orientation. Indeed, $\mathrm{SO}(1,4)/\mathrm{U}(2)$ may be identified with the total space of the bundle of orthogonal almost complex structures on H^4. Finally, for $\mathrm{SO}(5)/\mathrm{SO}(4)$ the decompositions are of the form

$$\mathbb{R}^{1,4} = L \oplus V, \qquad \dim L = 1,\ \dim V = 4$$

and $\mathrm{SO}(1,4)/\mathrm{SO}(4) = H^4$ (cf. $\mathrm{SO}(5)/\mathrm{SO}(4) = S^4$).

Again, in an analogous way to the SO(5) case, the projection maps σ_K may also be understood as 'forgetful maps'. For example, if $(L, V_1, V_2) \in \mathrm{SO}(1,4)/T^2$ then both V_1 and V_2 have a natural orthogonal complex structure and hence so does $V = V_1 \oplus V_2$. The projection $\sigma_{\mathrm{U}(2)} : \mathrm{SO}(1,4)/T^2 \to \mathrm{SO}(1,4)/\mathrm{U}(2)$ sending (L, V_1, V_2) to (L, V) 'forgets' the decomposition of V.

Let $\phi : H^4 \to S^4$ be given by the correspondence (6.1) which maps the upper embedding of H^4 into the upper hemisphere of S^4 via

$$\phi(t,x) = \left(\frac{2tx, t^2 - |x|^2}{t^2 + |x|^2}\right).$$

Then ϕ is a conformal embedding of H^4 in S^4 ($\phi : H^4 \to \mathbb{H} \to S^4$ is a composition of conformal maps). Thus there exists $\lambda : H^4 \to \mathbb{R}$ such that for all $p \in H^4$,

$$d\phi_p : T_pH^4 \to T_{\phi(p)}S^4$$

is an isomorphism with

$$\lambda(p)^2 \langle X, Y\rangle_p = \langle d\phi_p(X), d\phi_p(Y)\rangle_{\phi(p)}, \qquad X, Y \in T_pH^4.$$

Then

$$\frac{1}{\lambda(p)} d\phi_p : T_pH^4 \to T_{\phi(p)}S^4$$

is an isometry and hence is SO(4)-invariant.

If $\phi(t,x) = (\tilde{t}, \tilde{x})$, $d\phi(a,v) = (\tilde{a}, \tilde{v})$, let $i_\phi : \mathrm{SO}(1,4) \hookrightarrow \mathrm{SO}(5)$ be the embedding which takes

$$\begin{pmatrix} t & a_1 & a_2 & a_3 & a_4 \\ x & v_1 & v_2 & v_3 & v_4 \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{t} & \frac{1}{\lambda}\tilde{a}_1 & \frac{1}{\lambda}\tilde{a}_2 & \frac{1}{\lambda}\tilde{a}_3 & \frac{1}{\lambda}\tilde{a}_4 \\ \tilde{x} & \frac{1}{\lambda}\tilde{v}_1 & \frac{1}{\lambda}\tilde{v}_2 & \frac{1}{\lambda}\tilde{v}_3 & \frac{1}{\lambda}\tilde{v}_4 \end{pmatrix}$$

where $\lambda = \lambda(t,x)$. Then by the above, i_ϕ is SO(4)-invariant. Therefore, if K is any subgroup of SO(4) then there is a corresponding inclusion

$$\mathrm{SO}(1,4)/K \hookrightarrow \mathrm{SO}(5)/K$$

given by $gK \mapsto i_\phi(g)K$. In particular this gives embeddings

$$\mathrm{SO}(1,4)/T^2 \hookrightarrow \mathrm{SO}(5)/T^2 \quad \text{and} \quad \mathrm{SO}(1,4)/\mathrm{U}(2) \hookrightarrow \mathrm{SO}(5)/\mathrm{U}(2).$$

In this way, $\mathrm{SO}(1,4)/\mathrm{U}(2)$ may be identified with the part of $\mathbb{C}\mathrm{P}^3$ which lies over the upper hemisphere of S^4. This is referred to in the sequel as $\mathbb{C}\mathrm{P}^3_+$, the space of almost complex structures on H^4_+ the upper embedding of H^4. The embeddings $\mathrm{SO}(1,4)/K \hookrightarrow \mathrm{SO}(5)/K$ are part of a much wider programme described by Wolf [29].

6.4. Lifts of Immersions $f : S \longrightarrow H^4$

At each point $p \in S$, the tangent bundle to H^4 restricted to S (considered at the point p) splits as

$$T_{f(p)}H^4 = T_pS \oplus N_pS$$

where T_pS is the tangent space and N_pS the normal space to S in H^4 and both TS and NS are (spacelike) 2-planes. Since f is normal to H^4 in $\mathbb{R}^{1,4}$ we write

$$\mathbb{R}^{1,4} = \{f(p)\} \oplus T_pS \oplus N_pS$$

where $\{f\}$ is the line in $\mathbb{R}^{1,4}$ determined by f.

Now choose an oriented orthonormal basis e_1, e_2 for T_pS and e_3, e_4 for N_pS so that e_1, e_2, e_3, e_4 gives the standard orientation on H^4. i.e.

$$\mathbb{R}^{1,4} = \{f\} \oplus \text{span}\{e_1, e_2\} \oplus \text{span}\{e_3, e_4\}.$$

Thus given any $f : S \longrightarrow H^4$, we have an adapted orthonormal frame

$$\tilde{F} = (f \mid e_1 \mid e_2 \mid e_3 \mid e_4) \in \text{SO}(1,4)$$

and this frame defines a local lift of f to $\text{SO}(1,4)$. In general, the frame cannot be chosen globally since a basis has been nominated on each of the 2-planes TS and NS. However, the frame $\tilde{F}$ is unique up to rotations in these planes, giving a global lift of f into $\text{SO}(1,4)/T^2$. Hence, there is a naturally defined global lift of f into each of the homogeneous spaces $\text{SO}(1,4)/K$, where K is a subgroup of $\text{SO}(4)$. For any such K, the inclusion $T \subset K$ induces a projection $\sigma_K : \text{SO}(1,4)/T^2 \longrightarrow \text{SO}(1,4)/K$. Thus given a lift $\tilde{f} : S \longrightarrow \text{SO}(1,4)/T^2$ we have a map $\hat{f} = \sigma_K \tilde{f} : S \longrightarrow \text{SO}(1,4)/K$.

From the discussion above it is now clear how the lifts may be identified. For $\tilde{f} : S \longrightarrow \text{SO}(1,4)/T^2$ we have

$$\tilde{f}(p) = (\{f(p)\}, T_p(S), N_p(S))$$

and the others can be immediately written down from this in the same way as in the S^4 case.

6.5. The Twistor Bundle for H^4

As in section 4.5 consider the Hopf fibration

$$\pi : \mathbb{H}^2\backslash\{0\} \to \mathbb{HP}^1$$

and let $\mathbb{H}^2_*$ be the subspace of $\mathbb{H}^2\backslash\{0\}$ given by

$$\mathbb{H}^2_* = \{(q_1, q_2) \in \mathbb{H}^2 \mid |q_1| \neq |q_2|\}.$$

Define $\mathbb{HP}^1_* = \pi(\mathbb{H}^2_*)$. Then $\mathbb{HP}^1_*$ is the disjoint union of two open 4-discs. Indeed, identifying $\mathbb{HP}^1$ with S^4 as in section 4.5 so that

$$[q_1, q_2] \leftrightarrow \left(\frac{2\bar{q}_1 q_2, q_1\bar{q}_1 - q_2\bar{q}_2}{q_1\bar{q}_1 + q_2\bar{q}_2}\right)$$

we see that $\mathbb{HP}^1_*$ is identified with the complement of the equatorial S^3.

Now let $H^4_* = \{(q,t) \in \mathbb{H} \oplus \mathbb{R} \mid q\bar{q} - t^2 = -1\}^+$ be one (say the upper) component of H^4 and define $\pi_* : \mathbb{H}^2_* \to H^4_*$ by

$$\pi_*(q_1, q_2) = \left(\frac{2\bar{q}_1 q_2, q_1\bar{q}_1 + q_2\bar{q}_2}{q_1\bar{q}_1 - q_2\bar{q}_2}\right),$$

(observe that $\dfrac{4|q_1|^2|q_2|^2 - (|q_1|^2 + |q_2|^2)^2}{(|q_1|^2 - |q_2|^2)^2} = -1$). This is a locally trivial fibre bundle with fibre $\mathbb{H}$. If the components of $\mathbb{H}^2_*$ are denoted by $\mathbb{H}^2_+$, $\mathbb{H}^2_-$ according as $|q_1|^2 - |q_2|^2$ is > 0 or < 0 and H^4_+, H^4_- are the components of H^4_* corresponding to $t > 0$ or $t < 0$ then

$$\pi_\pm : \mathbb{H}^2_\pm \to H^4_\pm$$

where $\pi_\pm = \pi_* |_{\mathbb{H}^2_\pm}$.

As in the S^4-case, there are co-ordinate neighbourhoods $x_\pm : D(\mathbb{H}) \to H^4_*$, where $D(\mathbb{H}) = \{q \in \mathbb{H} \mid |q| < 1\}$, given by

$$x_+(q) = \left(\frac{2q, 1+|q|^2}{1-|q|^2}\right), \qquad x_-(q') = \left(\frac{2\bar{q}', 1+|q'|^2}{|q'|^2-1}\right), \qquad qq' = 1.$$

In terms of these local co-ordinates the metric on H^4 (induced from the metric $|(q,t)|^2 = q\bar{q} - t^2$ on $\mathbb{H} \oplus \mathbb{R}$) is given by

$$ds^2 = \frac{4}{(|q|^2-1)^2} d\bar{q}dq = \frac{4}{(|q'|^2-1)^2} d\bar{q}'dq'$$

and

$$\begin{aligned} \pi_+(q_1, q_2) &= q_1^{-1} q_2 = q \quad \text{for } (q_1, q_2) \in \mathbb{H}^2_+, \\ \pi_-(q_1, q_2) &= q_2^{-1} q_1 = q' \quad \text{for } (q_1, q_2) \in \mathbb{H}^2_-. \end{aligned} \tag{6.2}$$

The differentials $d\pi_\pm$ can be described as follows: Firstly,

$$dq = q_1^{-1} dq_2 - q_1^{-1} dq_1 q_1^{-1} q_2,$$
$$dq' = q_2^{-1} dq_1 - q_2^{-1} dq_2 q_2^{-1} q_1$$

so that

$$d\pi_+(p_1, p_2) = q_1^{-1} p_2 - q_1^{-1} p_1 q_1^{-1} q_2,$$
$$d\pi_-(p_1, p_2) = q_2^{-1} p_1 - q_2^{-1} p_2 q_2^{-1} q_1.$$

As before, the fibre of $\pi_\pm$ through (q_1, q_2) is given by

$$\ker d\pi_\pm = \mathbb{H}(q_1, q_2).$$

Now consider the $\mathbb{H}$-valued bilinear form defined on $\mathbb{H}^2$ by

$$\langle (p_1, p_2), (q_1, q_2) \rangle = 4(p_1\bar{q}_1 - p_2\bar{q}_2).$$

This gives a metric on $\mathbb{H}^2_*$ defined by $|(q_1, q_2)|^2 = 4||q_1|^2 - |q_2|^2|$. The horizontal subspace at (q_1, q_2) is given by the orthogonal complement of the fibre with respect to this inner product i.e.

$$(\mathbb{H}(q_1, q_2))^\perp = \{(p_1, p_2) \in \mathbb{H}^2 \mid p_1\bar{q}_1 - p_2\bar{q}_2 = 0\}.$$

If $q_2 = 0$ we have $(\mathbb{H}(q_1, 0))^\perp = \mathbb{H}(0,1)$, while if $q_2 \neq 0$ it follows that the complement $(\mathbb{H}(q_1, q_2))^\perp = \mathbb{H}\left(\frac{q_1}{|q_1|^2}, \frac{q_2}{|q_2|^2}\right)$.

Now let $Q^7 = \{(q_1, q_2) \in \mathbb{H}^2_+ \mid |q_1|^2 - |q_2|^2 = 1\}$ and consider $d\pi_\pm |_{Q^7}$. If (p_1, p_2) is a point in $(\mathbb{H}(q_1, 0))^\perp$ then

$$\begin{aligned} d\pi_+\left(\lambda\left(\frac{q_1}{|q_1|^2}, \frac{q_2}{|q_2|^2}\right)\right) &= q_1^{-1}\lambda\frac{q_2}{|q_2|^2} - q_1^{-1}\lambda\frac{q_1}{|q_1|^2}q_1^{-1}q_2 \\ &= \frac{1}{|q_1|^2|q_2|^2}q_1^{-1}\lambda q_2 \end{aligned} \tag{6.3}$$

Now,

$$\begin{aligned} \left|\lambda\left(\frac{q_1}{|q_1|^2}, \frac{q_2}{|q_2|^2}\right)\right|^2 &= 4|\lambda|^2\left(\frac{|q_1|^2}{|q_1|^4} - \frac{|q_2|^2}{|q_2|^4}\right) \\ &= \frac{4|\lambda|^2}{|q_1|^2|q_2|^2} \end{aligned}$$

while

$$\begin{aligned} \left|d\pi_+\left(\lambda\left(\frac{q_1}{|q_1|^2}, \frac{q_2}{|q_2|^2}\right)\right)\right|^2 &= \frac{4}{\left(\frac{|q_2|^2}{|q_1|^2} - 1\right)^2}\frac{|q_1|^{-2}|\lambda|^2|q_2|^2}{|q_1|^4|q_2|^4} \\ &= \frac{4|\lambda|^2}{|q_1|^2|q_2|^2}. \end{aligned}$$

Hence $d\pi_+$ maps $(q_1, q_2) \times (\mathbb{H}(q_1, q_2))^\perp$ isometrically onto $T_{\pi_+(q_1,q_2)}H^4_+$.

Arguments similar to those in section 4.5 now show that $\pi_\pm$ may be factored via $\mathbb{CP}^3_\pm$ $(= \rho\mathbb{H}^2_\pm)$ and that $\mathbb{CP}^3_\pm$ is the bundle of almost complex structures on $H^4_\pm$.

Let us consider $\mathbb{CP}^3_+$ more carefully. Use a co-ordinate neighbourhood on $\mathbb{CP}^3_+$ derived from that on $\mathbb{CP}^3$ mentioned in section 4.5 viz.

$$(\xi_1, \xi_2, \xi_3) \mapsto [1, \xi_1, \xi_2, \xi_3]$$

where $1 + |\xi_1|^2 - |\xi_2|^2 - |\xi_3|^2 > 0$. Then $\rho : \mathbb{H}^2_+ \to \mathbb{CP}^3_+$ given by $\rho(z_0 + z_1 j, z_2 + z_3 j) = [z_0, z_1, z_2, z_3] \leftrightarrow \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right)$. Now

$$d\rho_{(z_0,z_1,z_2,z_3)}(p_0, p_1, p_2, p_3) = \frac{1}{z_0^2}(z_0p_1 - z_1p_0, z_0p_2 - z_2p_0, z_0p_3 - z_3p_0)$$

so that $\ker d\rho_{(z_0,z_1,z_2,z_3)} = \mathbb{C}(z_0, z_1, z_2, z_3)$.

Let $(\mathbb{C}(z_0, z_1, z_2, z_3))^{\perp}$ be the orthogonal complement with respect to the $\mathbb{C}$-valued bilinear form. Then $d\rho$ maps $(\mathbb{C}(z_0, z_1, z_2, z_3))^{\perp}$ isomorphically onto $T_{[z_0,z_1,z_2,z_3]}\mathbb{C}P^3_+$. We now give $\mathbb{C}P^3_+$ a metric by making $\rho : Q^7 \to \mathbb{C}P^3_+$ a Riemannian submersion. More precisely suppose $(z_0, z_1, z_2, z_3) \in Q^7$ so that

$$z_0\bar{z}_0 + z_1\bar{z}_1 - z_2\bar{z}_2 - z_3\bar{z}_3 = 1 \tag{6.4}$$

and let $(p_0, p_1, p_2, p_3) \in (\mathbb{C}(z_0, z_1, z_2, z_3))^{\perp}$ so that

$$p_0\bar{z}_0 + p_1\bar{z}_1 - p_2\bar{z}_2 - p_3\bar{z}_3 = 0.$$

Now

$$|(p_0, p_1, p_2, p_3)|^2 = 4(p_0\bar{p}_0 + p_1\bar{p}_1 - p_2\bar{p}_2 - p_3\bar{p}_3).$$

Writing $\xi_k = \dfrac{z_k}{z_0}$, $(k = 1, 2, 3)$ so that

$$dz_k = \xi_k dz_0 + z_0 d\xi_k \tag{6.5}$$

and writing

$$\bar{z}_0 dz_0 + \bar{z}_1 dz_1 - \bar{z}_2 dz_2 - \bar{z}_3 dz_3 = 0 \tag{6.6}$$

we have from (6.5)

$$\bar{z}_k dz_k = \bar{\xi}_k \xi_k \bar{z}_0 dz_0 + \bar{z}_0 z_0 \bar{\xi}_k d\xi_k. \tag{6.7}$$

Thus (by (6.6) and (6.7)),

$$\begin{aligned} \bar{z}_0 dz_0 &= -\bar{z}_1 dz_1 + \bar{z}_2 dz_2 + \bar{z}_3 dz_3 \\ &= (-\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2 + \bar{\xi}_3\xi_3)\bar{z}_0 dz_0 + \bar{z}_0 z_0(-\bar{\xi}_1 d\xi_1 + \bar{\xi}_2 d\xi_2 + \bar{\xi}_3 d\xi_3) \end{aligned}$$

so that

$$\bar{z}_0 dz_0(1 + \bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3) = -\bar{z}_0 z_0(\bar{\xi}_1 d\xi_1 - \bar{\xi}_2 d\xi_2 - \bar{\xi}_3 d\xi_3).$$

Thus using (6.4)

$$\bar{z}_0 dz_0 = -\frac{(\bar{\xi}_1 d\xi_1 - \bar{\xi}_2 d\xi_2 - \bar{\xi}_3 d\xi_3)}{(1 + \bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3)^2}.$$

Now the metric is given by

$$\begin{aligned} ds^2 &= 4(d\bar{z}_0 dz_0 + d\bar{z}_1 dz_1 - d\bar{z}_2 dz_2 - d\bar{z}_3 dz_3) \\ &= 4\{d\bar{z}_0 dz_0(1+\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3) + \bar{z}_0 z_0(d\bar{\xi}_1 d\xi_1 - d\bar{\xi}_2 d\xi_2 - d\bar{\xi}_3 d\xi_3) + \\ &\qquad \bar{z}_0 dz_0(\bar{\xi}_1 d\xi_1 - \bar{\xi}_2 d\xi_2 - \bar{\xi}_3 d\xi_3) + z_0 d\bar{z}_0(\xi_1 d\bar{\xi}_1 - \xi_2 d\bar{\xi}_2 - \xi_3 d\bar{\xi}_3)\} \\ &= 4\frac{(1+\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3)(d\bar{\xi}_1 d\xi_1 - d\bar{\xi}_2 d\xi_2 - d\bar{\xi}_3 d\xi_3) - (\bar{\xi}_1 d\xi_1 - \bar{\xi}_2 d\xi_2 - \bar{\xi}_3 d\xi_3)^2}{(1+\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3)^2}. \end{aligned}$$

This is a Kähler metric on $\mathbb{C}P^3_+$ with Kähler form

$$\alpha = -4i\partial\bar{\partial}\log(1+\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3).$$

However, it is not a metric of constant holomorphic sectional curvature.

Let us now consider the horizontal distribution by first considering the tangents along the fibre and then taking the orthogonal complement with respect to the above metric. As before, use local co-ordinates (ξ_1, ξ_2, ξ_3), taking $z_0 \neq 0$. The fibre through $[1, \xi_1, \xi_2, \xi_3]$ has tangent vector $(1+\bar{\xi}_1\xi_1, \xi_2\bar{\xi}_1 - \bar{\xi}_3, \xi_3\bar{\xi}_1 + \bar{\xi}_2)$ and (η_1, η_2, η_3) is in the orthogonal complement of this with respect to the metric above if and only if

$$\begin{aligned} &(1+\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2 - \bar{\xi}_3\xi_3)\{(1+\bar{\xi}_1\xi_1)\eta_1 - (\bar{\xi}_2\xi_1 - \xi_3)\eta_2 - (\bar{\xi}_3\xi_1 + \xi_2)\eta_3\} - \\ &\qquad \{\xi_1(1+\bar{\xi}_1\xi_1) - \xi_2(\bar{\xi}_2\xi_1 - \xi_3) - \xi_3(\bar{\xi}_3\xi_1 + \xi_2)\}\{\bar{\xi}_1\eta_1 - \bar{\xi}_2\eta_2 - \bar{\xi}_3\eta_3\} = 0 \end{aligned}$$

that is, if and only if

$$\eta_1 + \xi_3\eta_2 - \xi_2\eta_3 = 0.$$

Thus the co-tangent bundle along the fibres has local holomorphic section $d\xi_1 + \xi_3 d\xi_2 - \xi_2 d\xi_3$ and the horizontal bundle has local holomorphic sections $(\xi_3, -1, 0)$, $(\xi_2, 0, 1)$.

6.6. The Twistor Transform

Suppose that $f : S \to S^4$ is a superminimal immersion of a Riemann surface into S^4. Then from f it is possible to construct a superminimal immersion $\phi : S^* \to H^4$ (where $S^* = S\backslash f^{-1}(S^3)$) as follows:

Since f is real isotropic, the complex 2-dimensional subspace V of $\mathbb{C}^5$ spanned by $\partial\phi$, $\partial^2\phi$ is a maximal isotropic subspace. The transform $I : \mathbb{C}^5 \to \mathbb{C}^{1,4}$ sending $(z_0, \ldots, z_4)$ to $(iz_0, z_1, \ldots, z_4)$ maps V into the isotropic subspace $I(V)$ of $\mathbb{C}^{1,4}$. Then taking the real line in $\mathbb{C}^{1,4}$ orthogonal to $I(V) \oplus \overline{I(V)}$ and intersecting with H^4_+ gives the map $\phi : S^* \to H^4$ mentioned above.

Since the space of almost complex structures on S^4 (resp. H^4) is the same as the space of maximal isotropic subspaces of $\mathbb{C}^5$ (resp. $\mathbb{C}^{1,4}$), the transform should be explicable in terms of a map between twistor spaces.

Recall that for holomorphic $[g] = [1, \xi_1, \xi_2, \xi_3] \in \mathbb{CP}^3$, g is horizontal if

$$d\xi_1 + \xi_2 d\xi_3 - \xi_3 d\xi_2 = 0.$$

Multiplying ξ_2, ξ_3 by $\pm i$ changes $d\xi_1 + \xi_2 d\xi_3 - \xi_3 d\xi_2$ into $d\xi_1 - \xi_2 d\xi_3 + \xi_3 d\xi_2$ and the vanishing of this is precisely the condition for holomorphic $[1, \xi_1, \xi_2, \xi_3] \in \mathbb{CP}^3_+$ to be horizontal. Let $I : \mathbb{CP}^3 \to \mathbb{CP}^3_+$ be defined by

$$I([z_0, z_1, z_2, z_3]) = [z_0, z_1, iz_2, iz_3].$$

Then I is holomorphic and maps horizontal subspaces of $\mathbb{CP}^3$ to horizontal subspaces of $\mathbb{CP}^3_+$. Thus if g is holomorphic and horizontal, so is $I([g])$. So the transform gives a one-to-one correspondence between superminimal immersions $\phi : S \to S^4 \backslash S^3$ and superminimal immersions $\psi : S \to H^4_*$.

Then, by the Weierstrass formula of Bryant it is clear how to construct holomorphic and horizontal curves in $\mathbb{CP}^3_+$ (and hence superminimal immersions into H^4). If $f, g : S \to \mathbb{C}$ are meromorphic functions with g non-constant then the map $\Psi(f,g) : M \to \mathbb{CP}^3_+$ defined by

$$\Psi(f,g) = [1, f - \frac{1}{2} g \frac{f'}{g'}, -ig, -\frac{i}{2} \frac{f'}{g'}]$$

is holomorphic and horizontal.

CHAPTER 7

Appendix: The Quaternions, $\mathbb{H}^n$ and Sp(n)

Recall that the quaternions are the set $\mathbb{H}$ of ordered quadruples of real numbers, or equivalently, of ordered pairs of complex numbers

$$a_0 + a_1 i + a_2 j + a_3 k \leftrightarrow (a_0, a_1, a_2, a_3) \leftrightarrow (a_0 + a_1 i, a_2 + a_3 i), \;\; a_r \in \mathbb{R}$$

with

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k,\, jk = i,\, ki = j.$$

Addition and multiplication of quaternions are defined by

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$(\alpha_1, \alpha_2)(\beta_1, \beta_2) = (\alpha_1\beta_1 - \alpha_2\bar{\beta}_2, \alpha_1\beta_2 + \alpha_2\bar{\beta}_1) \qquad \alpha_j, \beta_j \in \mathbb{C}$$

Notice that the operation of multiplication is not commutative, and so the set $\mathbb{H}$ together with the above operations is a skew-field.

The quaternion $q = a_0 + a_1 i + a_2 j + a_3 k = (\alpha_1, \alpha_2)$ has conjugate $\bar{q} = a_0 - a_1 i - a_2 j - a_3 k = (\bar{\alpha_1}, -\alpha_2)$. As we have seen above, $\mathbb{H}$ is identified with $(\mathbb{C}^2)^n = \mathbb{C}^{2n}$ i.e.

$$(q_1, \ldots, q_n) \leftrightarrow (z_1, z_2, \ldots, z_{2n-1}, z_{2n})$$

where $q_r = z_{2r-1} + z_{2r} j$, for $r = 1, \ldots, n$. We make $\mathbb{H}^n$ into an $\mathbb{H}$-module by letting $\mathbb{H}$ act on the left. $\mathbb{H}^n$ is a left $\mathbb{H}$-vector space, with scalar multiplication by elements of $\mathbb{H}$ given by $q(q_1, \ldots, q_n) = (qq_1, \ldots, qq_n)$. Since $\mathbb{C} \subset \mathbb{H}$ as a sub-ring (the elements of the form $a_0 + a_1 i$) it follows that $\mathbb{H}^n$ is also a (left) $\mathbb{C}$-vector space. Moreover, for $\lambda \in \mathbb{C}$,

$$\lambda(q_1, \ldots, q_n) = (\lambda q_1, \ldots, \lambda q_n) \leftrightarrow (\lambda z_1, \lambda z_2, \ldots, \lambda z_{2n-1}, \lambda z_{2n})$$

so that the identification of $\mathbb{H}^n$ with the $\mathbb{C}$-vector space $\mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism.

A map $T : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$-linear if $T(qv) = qT(v)$ for all $q \in \mathbb{H}$ and $v \in \mathbb{H}^n$. The set of invertible $\mathbb{H}$-linear maps forms a group denoted by $GL(n; \mathbb{H})$ and, conversely, any $\mathbb{H}$-linear map is represented by an $n \times n$ matrix with $\mathbb{H}$ coefficients (with respect to the standard basis $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\mathbb{H}^n$) and $GL(n; \mathbb{H})$ is precisely the group of all such invertible matrices.

Since any $\mathbb{H}$-linear map is $\mathbb{C}$-linear, it follows that an $\mathbb{H}$-linear map T can be represented by a complex $2n \times 2n$ matrix. Conversely, a $\mathbb{C}$-linear map T is $\mathbb{H}$-linear if and only if $Tj = jT$. We recall that

$$j(z_{2r-1} + z_{2r} j) = -\bar{z}_{2r} + \bar{z}_{2r-1} j$$

so that if $v \in \mathbb{H}^n = \mathbb{C}^{2n}$ then

$$jv = \overline{Jv} = J\bar{v}, \text{ where } J = \operatorname{diag}\left\{ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \ldots, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\}.$$

Thus the matrix A represents an $\mathbb{H}$-linear map if and only if $Ajv = jAv$ for all $v \in \mathbb{C}^{2n}$ that is, if and only if $AJ\bar{v} = J(\overline{Av})$, $\iff AJ = J\bar{A}$. Thus, in particular, $GL(n; \mathbb{H}) = \{A \in GL(n; \mathbb{C}) | AJ = J\bar{A}\}$.

The condition $AJ = J\bar{A}$ may be understood as follows: Let

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \ldots & A_{nn} \end{pmatrix}$$

where each A_{rs} is a 2×2 matrix. Then $AJ = J\bar{A}$ if and only if $A_{rs}\tilde{J} = \tilde{J}\bar{A}_{rs}$, where $\tilde{J} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, i.e.

$$A_{rs} = \begin{pmatrix} a_{rs} & -\bar{b}_{rs} \\ b_{rs} & \bar{a}_{rs} \end{pmatrix}.$$

Then $A \in M_{2n}(\mathbb{C})$ with $AJ = J\bar{A}$ corresponds to $\hat{A} = (p_{ij}) \in M_n(\mathbb{H})$, $p_{sr} = a_{rs} + brsj$. (Notice there is a 'transpose' effect here.) Then the actions of $M_{2n}(\mathbb{C})$ on $\mathbb{C}^{2n}$ and $M_n(\mathbb{H})$ on $\mathbb{H}^n$ correspond via

$$A \begin{pmatrix} z_1 \\ \vdots \\ z_{2n} \end{pmatrix} \leftrightarrow (q_1, \ldots, q_n)\hat{A}, \text{ where } q_r = z_{2r-1} + z_{2r}j.$$

To obtain an inner product on $\mathbb{H}^n$, we consider the map $<,>: \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ defined by

$$< (q_1, \ldots, q_n), (p_1, \ldots, p_n) >= q_1\bar{p}_1 + \cdots + q_n\bar{p}_n$$

(cf Hermitian inner product on $\mathbb{C}^n$). This form is sesquilinear in the sense that

$$\begin{aligned} &< v, w > = \overline{< w, v >} \\ &< v_1 + v_2, w > =< v_1, w > + < v_2, w > \quad \text{for all } q \in \mathbb{H}, \\ &< qv, w > = q < v, w > \qquad v, v_1, v_2, w \in \mathbb{H}^n \\ &< v, qw > =< v, w > \bar{q} \end{aligned}$$

and determines a norm on $\mathbb{H}^n$ via $||v||^2 =< v, v >$. (Note that $< v, v >\in \mathbb{R}, < v, v >\geq 0$ with equality if and only if $v = 0$, and that this agrees with the usual norm on $\mathbb{C}^{2n}$). We define $\mathrm{Sp}(n)$ to be the subgroup of $GL(n; \mathbb{H})$ which preserves the norm. Thus

$$\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) | AJ = J\bar{A}\}.$$

Finally, we note that right multiplication by elements of $\mathbb{H}$ is $\mathbb{H}$-linear (with respect to left multiplication by $\mathbb{H}$). Thus right multiplication by a unit quaternion is an $\mathbb{H}$-linear isometry of $\mathbb{H}^n$ and so determines an element of $\mathrm{Sp}(n)$ i.e. $\mathrm{Sp}(n)$ acts on $\mathbb{H}^n$ on the right. Since

$$(z_1 + z_2 j)(a + bj) = az_1 - \bar{b}z_2 + (bz_1 + \bar{a}z_2)j$$

we see that right multiplication by $a + bj$ corresponds to

$$\mathrm{diag}\left\{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \dots, \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right\} \in \mathrm{GL}(n; \mathbb{H})$$

and this element lies in $\mathrm{Sp}(n)$ when $a + bj$ is a unit quaternion.

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