# *Conformal field theories on random surfaces and the non-critical string*

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by

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A thesis submitted for the degree
of Doctor of Philosophy

Department of Mathematical Sciences
University of Durham

June 1997

*To São and Artur with love*

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**Abstract**

Recently, it has become increasingly clear that boundaries play a significant role in the understanding of the non-perturbative phase of the dynamics of strings. In this thesis we propose to study the effects of boundaries in non-critical string theory.

We thus analyse boundary conformal field theories on random surfaces using the conformal gauge approach of David, Distler and Kawai. The crucial point is the choice of boundary conditions on the Liouville field. We discuss the Weyl anomaly cancellation for Polyakov's non-critical open bosonic string with Neumann, Dirichlet and free boundary conditions. Dirichlet boundary conditions on the Liouville field imply that the metric is discontinuous as the boundary is approached. We consider the semi-classical limit and argue how it singles out the free boundary conditions for the Liouville field. We define the open string susceptibility, the anomalous gravitational scaling dimensions and a new Yang-Mills Feynman mass critical exponent.

Finally, we consider an application to the theory of non-critical dual membranes. We show that the strength of the leading stringy non-perturbative effects is of the order $e^{-O(1/\beta_{st})}$, a result that mimics those found in critical string theory and in matrix models. We show how this restricts the space of consistent theories. We also identify non-critical one dimensional D-instantons as dynamical objects which exchange closed string states and calculate the order of their size. The extension to the minimal $c \leq 1$ boundary conformal models is also briefly discussed.

# Preface

This thesis is based on research carried out by the author between June 1994 and May 1997. The material presented has not been submitted previously for any degree in either this or any other University.

The first chapter introduces the thesis. In chapter 2 a review of previously well known results on closed surfaces is considered. Chapters 3, 4 and 5 are original work developed in collaboration with Paul Mansfield. A short version of this is now published in [1]. In chapter 6 new original research material [2] is discussed with the exception of section 6.1, a review of T-duality and D-branes. In chapter 7 a summary of the results of the thesis is presented.

I would like to thank my supervisor Paul Mansfield for many helpful and encouraging discussions, for keeping my interest in string theory alive and for suggesting the work on non-critical dual membranes. I would like to thank the Junta Nacional de Investigação Científica e Tecnológica for a PhD fellowship under the PRAXIS XXI program, BD/2828/93-RM. I would like to thank Luís Bento for his friendship and support.

# Contents

# Chapter 1

# Introduction

In 1981 Polyakov [3] showed that when non-critical strings are quantised so as to maintain reparametrisation invariance, the scale of the metric becomes a dynamical degree of freedom even though it decouples classically or when the quantum theory is defined on the critical dimension. The associated action is that of a soluble Liouville quantum field theory but we find an unusual conformally invariant functional measure. This is a deeply non-linear theory which has not yet been completely solved [4, 5, 6, 7].

That the effect of the measure could be accounted for by a simple renormalisation of the action was first realized by David, Distler and Kawai [7]. In this conformal gauge approach explicit reparametrisation invariance can be kept throughout and Weyl invariance is ensured at the quantum level. Furthermore the theory is valid for closed surfaces of arbitrary genus and the coupling of the minimal models [8] to 2D quantum gravity is to be described by two conformally extended Liouville theories [9] which are complementary [10]. Most importantly the results were shown to be in impressive agreement not only with the semi-classical limit [11] and with the light-cone gauge formulation of Knizhnik, Polyakov and Zamolodchikov [6] but also with the theory of dynamical triangulated random surfaces [12, 13]. Finally, the theory can be immediately generalised to the supersymmetric case [14] and to $\mathcal{W}$-gravity [10] and super $\mathcal{W}$-gravity [15].

In this thesis we study the effects of boundaries on the DDK approach, extending their results to the cases of open string theory and to the coupling of boundary conformal field theories to 2D quantum gravity [1]. We will aim to see if the close affinity between the matter and gravitational sectors still holds when the random surfaces have boundaries and the minimal model becomes a boundary conformal field theory. The main issue is the choice of boundary conditions on the gravitational sector. Since the Liouville theory has a natural generalisation in the presence of boundaries, we expect the coupling of the minimal boundary conformal field theories to 2D quantum gravity to be again described by two conformally extended Liouville theories which are complementary. Once this is settled we will consider an application to the theory of non-critical dual membranes where we discuss their existence and also the strength of the leading stringy non-perturbative effects to which they are associated [2].

In Chapter 2 we review the points which are important to keep in mind when we consider boundaries. We start with Polyakov's conformal gauge approach to the non-critical closed bosonic string. We discuss the string partition function introducing the DDK renormalisation ansätse. We use a linear Coulomb gas perturbative expansion [10, 16] to show how Weyl invariance at the quantum level determines the field and coupling renormalisations of the Liouville action. We consider the anomalous effects of the constant zero mode of the covariant Laplacian and show that quantum Weyl invariance is only possible with a charge selection rule. We also consider the scattering amplitudes and define the dressed closed string vertex operators. We then discuss some of the critical exponents associated with the closed string 2D random surfaces. First we analyse the saddle point expansion of the Liouville functional integral and calculate the susceptibility exponent to one loop. Then we introduce the DDK scaling argument [7] to determine the string susceptibility and compare with the saddle point limit. Finally we apply the scaling argument to calculate the anomalous gravitational scaling dimensions of closed string tachyons. To conclude the chapter we extend the results to the coupling of minimal models to closed 2D quantum gravity noting the close affinity between the matter and gravitational sectors of the theory [10].

In Chapter 3 we start by presenting our solution in the simple case of Polyakov's non-critical open bosonic string. Since the key point is the choice of boundary conditions on the Liouville field we discuss the Weyl anomaly cancellation for Neumann, Dirichlet and free boundary conditions. We again use a linear Coulomb gas perturbative expansion to find the renormalised central charge of the conformally extended Liouville theory that describes the gravitational sector. As expected this will be shown to be the same central charge calculated for the coupling on closed surfaces. Since the metric is to be written as a reference metric multiplied by the exponential of the Liouville field, the theory must be independent of a shift in this field together with a compensating Weyl transformation on the reference metric. This leads to the dressing of primary operators that acquire conformal weight $(1, 1)$ on the bulk and conformal weight $(1/2, 1/2)$ on the boundary. Consequently, we show that the Liouville field renormalisation is equal to the one found for closed surfaces both on the bulk and on the boundary of the open surfaces. This only works for Neumann and free boundary conditions on the Liouville field. The Dirichlet boundary conditions freeze the Liouville boundary quantum dynamics so that, it is not possible to cancel all the boundary terms in the Weyl anomaly by a shift in the boundary values of the Liouville field, without leading to a discontinuity in the metric as the boundary is approached. Due to the presence of the boundary we find new renormalised couplings to 2D gravity. Under Weyl invariance at the quantum level we show that they are all determined by the bulk or closed surface couplings as would be expected. We also show how the Coulomb gas screening charge selection rule is a crucial condition for the cancellation of non-local and Weyl anomalous contributions to the correlation functions due to zero modes.

In Chapter 4 we analyse the semi-classical limit which singles out the free boundary conditions on the Liouville field as being the most natural. We define the open string susceptibility, the anomalous gravitational scaling dimensions and a new mass critical exponent. In the context of Yang-Mills theory this mass exponent has an interesting physical interpretation as the critical exponent associated with the Feynman propagator for a test particle which interacts with the gauge fields. Finally we present a connection with the results obtained with matrix models of dynamical triangulated random surfaces.

In Chapter 5 we generalise the open string analysis to a natural Feigin-Fuchs representation of $c \leq 1$ minimal conformal field theories on open random surfaces [17, 18], thus concluding that the close affinity between the matter and gravitational sectors of the theory still holds when boundaries are present.

In chapter 6 we consider an application to the theory of dual membranes. First we review how T-duality leads to the definition of the D-brane, a new extended object which consistently interacts with critical strings [19]. Then we discuss the non-critical D-instanton and show that in the limiting case of a one dimensional target space the strength of its associated leading stringy non-perturbative effects is of the order $e^{-O(1/\beta_{st})}$, where $\beta_{st}$ is the string coupling constant. This naturally mimics the result obtained in the 26 dimensional critical theory [20] and in the matrix models [21] since the weight of holes in the world-sheet only depends on the topology. However we find that not all of the non-critical theories allowed by perturbative Weyl invariance are consistent. These theories are characterised by different positive values of the renormalised Liouville cosmological constants $\lambda_2$ and $\mu_2$, respectively associated with the renormalisation counterterms in the boundary length and in the area of the string world-sheet. We show that only those which satisfy $\lambda_2 \geq \mu_2$ with $\lambda_2 > 0$, $\mu_2 \geq 0$ or those where $\lambda_2 = 0$, $\mu_2 \geq 0$ lead to acceptable non-perturbative effects. We also consider the D-instanton exchange of closed string states and show that the size of the D-instanton is of the order of $\sqrt{\alpha'}/\ln(1/\lambda_2)$ for small $\lambda_2 > 0$, $\mu_2 \geq 0$ or of the order of $\sqrt{\alpha'}/\ln(1/\mu_2^2)$ for $\lambda_2 = 0$ and small $\mu_2 > 0$. Above $\alpha'$ is related to the string tension $\mathrm{T} = 1/(2\pi\alpha')$. Finally we consider the one loop partition function for the non-critical string in one target space dimension compactified on a circle of radius $R$. We calculate for both Neumann and Dirichlet boundary conditions on the matter scalar field and compare with the results obtained on the torus and the corresponding matrix model [22, 23]. The possible but still unfinished extension of our results to the case of boundary conformal models is also discussed.

In Chapter 7 we present our conclusions.

# Chapter 2

# Conformal field theories on closed random surfaces

## 2.1 Closed string 2D quantum gravity

A fluctuating string propagating in a $d$ dimensional target space may be first quantised by a Feynman functional integral over 2D random surfaces [3, 24, 25, 26, 27]. For the scattering of $n$ closed bosonic string states the following connected topological expansion is postulated

$$< W_1 \ldots W_n >= \sum_{\text{topologies}} {\beta_{st}}^{-\chi_c} \int \mathcal{D}_{\tilde{g}}(X,\tilde{g}) W_1 \ldots W_n e^{-S[X,\tilde{g}]} . \tag{2.1}$$

This is a perturbative multiloop expansion on the genus $h$ of all possible closed Riemann surfaces. Each term in the series corresponds to a given surface topology and is weighted by the loop counting parameter ${\beta_{st}}^{-\chi_c}$, where ${\beta_{st}}^2$ plays the role of Planck's constant $\beta^2 \sim \hbar$. Here, $\chi_c$ is the Euler characteristic of the closed Riemann surface. It is related to the world-sheet integral of the scalar curvature, $\tilde{R}$, and to the genus $h$ by the classical Gauss-Bonnet theorem

$$\frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{R} = \chi_c = 2 - 2h , \tag{2.2}$$

where $\tilde{g} = \det\tilde{g}_{ab}$.

The amplitude must then be defined so as to maintain explicit reparametrisation invariance. This means invariance under the group Diff($D$) of the differentiable general coordinate transformations or diffeomorphisms of the parameter domain $D$ in $\mathcal{R}^2$, $\xi^a \to \eta^a(\xi)$. Under these transformations the matter field $X^\mu$ remains unchanged but the world-sheet intrinsic metric transforms according to the law

$$\tilde{g}_{ab} \to g_{ab}(\xi) = \tilde{g}_{rs}(\eta)\frac{\partial\eta^r}{\partial\xi^a}\frac{\partial\eta^s}{\partial\xi^b}\,.$$

Apart from the loop counting parameter, each multiloop contribution to the $n$-point scattering amplitude is also going to be weighted by the exponential of an action $S[X,\tilde{g}]$. In Polyakov's model this action is given by the covariant world-sheet integral

$$S[X,\tilde{g}] = \frac{1}{16\pi}\int d^2\xi\sqrt{\tilde{g}}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2\int d^2\xi\sqrt{\tilde{g}}\,.$$

The matter action of the field $X^\mu$ is just the standard bosonic string action of Brink, Di Vecchia and Howe. It has the same classical dynamics of the Nambu-Goto action. Here $X^\mu$ is a coordinate scalar field taking values in the $d$ dimensional target space, giving the position of a point in the string world-sheet with reference to some frame in the target space. As a first approximation, we assume it to be the flat Euclidean $d$ dimensional space with metric $\eta_{\mu\nu} = \mathrm{diag}(1,1,\ldots,1)$. As a consequence, the metric $\tilde{g}_{ab}$ must also have Euclidean signature. Although at the classical level this metric is related to $X^\mu$ because of a null stress-energy tensor, at the quantum level we take it as an independent dynamical variable. So we actually have a theory of $d$ free bosons coupled to 2D quantum gravity.

Alongside the action for the matter fields, we have introduced a bare cosmological constant in the area of the string world-sheet. It is a covariant renormalisation counterterm needed to cancel ultraviolet divergencies which are going to be generated as a result of the functional integration. This counterterm is naturally associated with the reparametrisation invariant functional integration measures and with the covariant regulator we always need to introduce to define the singular functional integrals.

To define covariant functional integration measures we must consider $\mathcal{L}^2$ inner products which are reparametrisation invariant. Thus, $\mathcal{D}_{\tilde{g}}X$ and $\mathcal{D}_{\tilde{g}}\tilde{g}$ are induced from the following norms

$$||\delta X||_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\delta X\cdot\delta X\,, \quad ||\delta\tilde{g}||_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\left(\tilde{g}^{ac}\tilde{g}^{bd} + u\tilde{g}^{ab}\tilde{g}^{cd}\right)\delta\tilde{g}_{ab}\delta\tilde{g}_{cd}\,,$$

where $u$ is a non-negative numerical constant. Clearly, such functional integration measures are not unique. Ultralocality [24, 26, 27] restricts the ambiguity to be of the form of the renormalisation counterterm. We may then consider the measures as unique pointwise products over the world-sheet provided we normalise the inner products as follows

$$\int \mathcal{D}_{\tilde{g}}\delta X e^{-||\delta X||_{\tilde{g}}^2} = 1, \quad \int \mathcal{D}_{\tilde{g}}\delta\tilde{g} e^{-||\delta\tilde{g}||_{\tilde{g}}^2} = 1\,.$$

Of course, the ambiguity is still implicit in the counterterm parameter $\mu_0^2$. It will be reflected by any residual constant that survives the renormalisation process. In the end, this should be fixed by a physical renormalisation condition.

To describe the initial and final closed bosonic string configurations we have introduced string states defined by reparametrisation invariant integral vertex operators $W_i$, $i = 1,\cdots,n$. For example, the closed string tachyon vertex operator with momentum $p^\mu$ is $\int d^2\xi\sqrt{\tilde{g}}e^{ip\cdot X}$.

If we look at the above written norms and vertex operators, it is easy to check that they violate the Weyl symmetry which produces local rescalings of the metric

$$\tilde{g}_{ab}(\xi) \rightarrow e^{\varphi(\xi)}\tilde{g}_{ab}(\xi)\,.$$

This is the characteristic feature of Polyakov's quantum geometry of strings. In direct contrast with the canonical quantisation formalism [27, 28], we give up explicit Weyl invariance in the construction of the theory to have explicit reparametrisation invariance. Consequently, the theory will develop a Weyl anomaly which we must cancel in order to ensure Weyl invariance at the quantum level.

### 2.1.1 The non-critical string partition function

For a given topology the partition function is defined by the following functional integral

$$Z(\chi_c) = \int \mathcal{D}_{\tilde{g}}(X, \tilde{g}) e^{-S[X,\tilde{g}]} .$$

Any metric $\tilde{g}_{ab}$ can be written as a reparametrisation of the conformal gauge $e^{\varphi}\hat{g}_{ab}$, where the reference metric $\hat{g}_{ab}$ depends on the modular parameters of the surface. This is possible because any arbitrary variation of the metric of a 2D Riemann surface can be given as a sum of variations due to a Weyl transformation, a reparametrisation and a change in the modular parameters [24, 25, 26, 27]

$$\delta\tilde{g}_{ab} = \delta\varphi\tilde{g}_{ab} + \tilde{\nabla}_a\delta\theta_b + \tilde{\nabla}_b\delta\theta_a + \sum_{A=1}^{M}\delta y^A\psi_{A,ab} .$$

Here, $\delta\varphi$ parametrises the infinitesimal Weyl scaling obtained from varying the Liouville field $\varphi$, $\delta\theta_a$ defines the infinitesimal reparametrisation and $\delta y^A$ is the infinitesimal variation of the modular parameter $y^A$. The $\psi_{A,ab}$ are known as the Beltrami differentials.

The variations of $\tilde{g}_{ab}$ induced by the reparametrisations and by the Weyl scalings are not orthogonal because the set of conformal transformations

$$\tilde{\nabla}_a\delta\theta_b + \tilde{\nabla}_b\delta\theta_a = \tilde{\nabla}_c\delta\theta^c\tilde{g}_{ab}$$

are equivalent to a Weyl transformation. These transformations define the infinitesimal conformal group spanned by the conformal Killing vectors, the solutions to the above conformal Killing equation. These are transformations which preserve the conformal gauge.

If we absorb the conformal transformation we can represent the arbitrary change in the metric as follows

$$\delta\tilde{g}_{ab} = \delta\varphi\tilde{g}_{ab} + \tilde{P}_{ab}\left(\delta\theta\right) + \sum_{A=1}^{M}\delta y^A\psi_{A,ab} .$$

Here $\tilde{P}_{ab}(\delta\theta) = \tilde{\nabla}_a \delta\theta_b + \tilde{\nabla}_b \delta\theta_a - \tilde{g}_{ab}\tilde{\nabla}_c \delta\theta^c$ is a differential operator which acts on world-sheet vectors to make symmetric and traceless tensors. Now both variations due to Weyl and coordinate transformations are orthogonal with respect to the measure for metric variations

$$\|\delta\tilde{g}\|_{\tilde{g}}^2 = 2(1+2u)\int d^2\xi\sqrt{\tilde{g}}(\delta\varphi)^2 + \int d^2\xi\sqrt{\tilde{g}}\tilde{g}_{ab}\delta\theta^a\left[\tilde{P}^\dagger\tilde{P}(\delta\theta)\right]^b.$$

Above $\tilde{P}^\dagger$ is the operator which acts on symmetric, traceless tensors to make vectors. It is defined using $\tilde{P}$ following an integration by parts and it has the explicit form, $\tilde{P}^\dagger(h)^a = -2\tilde{\nabla}_b h^{ab}$.

It is the operator $\tilde{P}^\dagger$ which allows us to define the space of the moduli of the surface. Because it is positive definite, the norm of $\delta\tilde{g}_{ab}$ will only vanish if $\delta\tilde{g}_{ab} = 0$. So any traceless zero mode of $\tilde{P}^\dagger$ will be orthogonal to the Weyl scalings and to the infinitesimal diffeomorphisms and, so, corresponds to a change of the moduli. These zero modes are the holomorphic quadratic differentials, $B_j$, and are not the same as the Beltrami differentials. They are Weyl invariant and the Beltrami differentials change according to the law, $\delta\psi_{A,ab} = \delta\varphi\psi_{A,ab}$. It is this fact which makes the modular parameters good coordinates in the space of metrics. They generate derivatives that commute with the derivatives constructed out of $\varphi$ and $\theta^a$ [27]. On the contrary, the holomorphic parameters would lead to a non-zero commutator with the Liouville mode derivative.

The holomorphic quadratic differentials and the Beltrami differentials are nevertheless related by matrix transformations. They form different basis systems for the space of the traceless zero modes of $\tilde{P}^\dagger$. This space has a finite dimension, $M$, which is related to that of the space of conformal Killing vectors, $C$, by the Riemann-Roch theorem, $C - M = 3\chi_c$ [24, 27].

Integrating the matter and reparametrisation ghost fields we find [3, 24, 25, 26, 27]

$$Z(\chi_c) = \int \prod_1^M dy^A \mathcal{D}_{\tilde{g}}\varphi \frac{\det(B_i,\psi_A)}{\sqrt{\det(B_j,B_k)}} \frac{\sqrt{\mathrm{Det}'\tilde{P}^\dagger\tilde{P}}}{\mathrm{Vol(CKV)}} \left(\frac{\mathrm{Det}'\tilde{\Delta}}{\int d^2\xi\sqrt{\tilde{g}}}\right)^{-d/2}.$$

where the prime denotes the omission of the zero modes and the covariant Laplacian $\tilde{\Delta}$ is $-(1/\sqrt{\tilde{g}})\partial_a \tilde{g}^{ab}\sqrt{\tilde{g}}\partial_b$. The area integral is a consequence of the constant zero mode of the Laplacian, leading to the factorisation of the infinite volume of the target space, which we have cancelled out by a normalisation constant. This volume reflects the invariance of the functional integral under target space Poincaré transformations $X^\mu \to \Lambda^\mu_\nu X^\nu + A^\mu$, where $\eta_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta = \eta_{\alpha\beta}$. We have also divided by the volume of the space of conformal Killing vectors Vol(CKV). This reflects the invariance under reparametrisations, which we have also cancelled out by a normalisation constant. Actually, this is a partial cancellation since we only have integrated over those diffeomorphisms which are continuously connected with the identity. The disconnected coordinate transformations have been taken into account by integrating over the moduli space instead of the Teichmüller space [24, 25].

Due to their dependence on the conformal field $\varphi$, these infinite determinants generate a Weyl anomaly [24, 25, 26, 27]. If we use the covariant heat kernel to regularise them, it is easy to see that the Weyl anomaly only depends on the values of the heat kernels for small proper time cutoff $\sqrt{\varepsilon}$. This means that the Weyl anomaly is a local phenomena which only reflects the structure of the world-sheet at short distances. For the infinitesimal change, $\rho$, in the Liouville field we find

$$\delta_\rho \ln\left[\frac{\sqrt{\mathrm{Det}'\tilde{P}^\dagger\tilde{P}}}{\sqrt{\tilde{\det}(B_j,B_k)}\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}'\tilde{\Delta}}{\int d^2\xi\sqrt{\tilde{g}}}\right)^{-d/2}\right] = \frac{d-26}{48\pi}\int d^2\xi\sqrt{\tilde{g}}\tilde{R}\rho$$
$$+\frac{d-2}{8\pi\epsilon}\int d^2\xi\sqrt{\tilde{g}}\rho + O(\varepsilon)\,.$$

This infinitesimal change is valid for any metric $\tilde{g}_{ab}$. It can be integrated to give the well known Liouville action for the conformal field $\varphi$ coupled to 2D quantum gravity as represented by the reference metric $\hat{g}_{ab}$. Since the inner product of the holomorphic quadratic differentials and the Beltrami differentials is Weyl invariant, the moduli integration measure has been chosen to be independent of $\varphi$ and that leads us to cast the partition function in the form

$$Z(\chi_c) = \int\prod_1^M dy^A \det(B_i,\psi_A)\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\sqrt{\hat{\det}(B_j,B_k)}\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-d/2}$$

$$\times \int \mathcal{D}_{\hat{g}}\varphi e^{-S_L[\varphi,\hat{g}]} \,,$$

where the Liouville action is given by

$$S_L[\varphi,\hat{g}] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\varphi\hat{\Delta}\varphi + \hat{R}\varphi\right) + \mu_1^2\int d^2\xi\sqrt{\hat{g}}e^{\varphi} \,.$$

Here we have already used the renormalisation counterterm to eliminate the divergent contribution in $1/\varepsilon$. The area piece in $\mu_1^2$ is the finite term left over from that process.

If $d = 26$ and we choose the renormalisation constant $\mu_1^2$ to be set to zero, we can easily see that the Weyl anomaly disappears. If we then divide off the corresponding infinite volume we get a Weyl invariant quantum partition function. This is precisely the value of the critical dimension of target space, deduced from the canonical no-ghost theorem by imposing the Virasoro gauge conditions to eliminate negative norm states [27, 28].

For non-critical target space dimensions, the conformal mode becomes dynamical and we need to integrate the Liouville field theory for $\varphi$. This is also consistent with the no-ghost theorem [27, 28], being the conformal field the additional degree of freedom when the string moves in a non-critical target space.

### 2.1.2 The DDK renormalisation ansätse

First we note that the natural measure for the Liouville field theory is induced by the following inner product on variations of $\varphi$

$$||\delta\varphi||_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}(\delta\varphi)^2 = \int d^2\xi\sqrt{\hat{g}}e^{\varphi}(\delta\varphi)^2 \,, \quad \int \mathcal{D}_{\tilde{g}}\delta\varphi e^{-||\delta\varphi||_{\tilde{g}}^2} = 1 \,.$$

Due to the presence of $e^{\varphi}$, this measure is not the usual one we find in quantum field theory. According to David, Distler and Kawai [7] we may introduce a canonical measure in the background $\hat{g}_{ab}$, provided we renormalise the Liouville field $\varphi \to \alpha\phi$ and its couplings to 2D quantum gravity

$$S_L[\phi,\hat{g}] = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi\right) + \mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}.$$

Since the functional integration is bound to be divergent, we must also include in the DDK ansätse the introduction of a renormalisation cosmological constant counterterm in the area of the world-sheet calculated with respect to the background metric $\hat{g}_{ab}$. This is also natural when we take into account the need to absorb the ambiguity of the functional integration measure in such a renormalisation counterterm.

From this it follows that the partition function should be equivalent to

$$Z(\chi_c) = \int\prod_1^M dy^A \det(B_i,\psi_A)\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\sqrt{\hat{\det}(B_j,B_k)}\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-d/2}$$
$$\times\int\mathcal{D}_{\hat{g}}\phi e^{-S_L[\phi,\hat{g}]-\mu_3^2\int d^2\xi\sqrt{\hat{g}}}.$$

Since the separation of $\tilde{g}_{ab}$ into the scale $e^{\phi}$ and reference metric $\hat{g}_{ab}$ is arbitrary, the new theory is required to be invariant under simultaneous shifts in $\phi$ and compensating scalings of $\hat{g}_{ab}$. Thus, a form of Weyl invariance must be preserved at the quantum level.

### 2.1.3 Anomaly cancellation for coupling and field renormalisation

To integrate the Liouville mode, we may start by taking a perturbative approach and expand the exponential interaction in powers of the mass scale $\mu_2^2$ [7, 10]

$$e^{-\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}} = \sum_{s=0}^{\infty}\frac{(-1)^s\mu_2^{2s}}{s!}\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}\right)^s.$$

This has the effect of replacing the exponential interaction by terms in the Liouville action which are linear in $\phi$. Then we factor out the contribution from the zero mode of the covariant Laplacian. When integrated over, it yields a delta function which imposes a constraint on the sum of the coefficients of the linear terms, a "charge selection rule" [7, 10, 16]. Only terms with zero total charge contribute to the amplitude

$$\int d^2\xi \sqrt{\hat{g}} \hat{J}^s = 0\,, \tag{2.3}$$

where the $(s+1)^{\text{th}}$ order Liouville gravity current is the coefficient of the term in the action that is linear in $\phi$

$$\hat{J}^s = Q\hat{R} - 8\pi\alpha \sum_{P=1}^{s} \frac{\delta^2(\xi - \xi_P)}{\sqrt{\hat{g}(\xi)}}\,. \tag{2.4}$$

This perturbative expansion in the Liouville mass scale $\mu_2^2$ should be handled with care. When we enforce the charge selection rule, in general $s$ is not an integer. This can be made explicit by integrating over the zero mode of the Liouville field. In the case of the partition function this leads to the following result [29, 30, 31, 32]

$$\frac{\mu_2^{2s}}{\alpha}\Gamma(-s)\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\right)^s,$$

where $\bar{\phi}$ is orthogonal to the zero mode, $\int d^2\xi \sqrt{\hat{g}}\bar{\phi} = 0$, and $\alpha s = Q\chi_c/2$. In this approach it is assumed that we can treat $s$ as an integer to take the calculation forward. Actually, though up to a normalisation constant, for $s$ a positive integer we simply recover the usual perturbative approach. In the end an analytic continuation in $s$ is considered [30, 31, 32]. Consequently, the Liouville gravity current is still given by Eq. (2.4) and the charge selection rule is now automatic with an $s$ which does not have to be an integer.

We then only need to shift the linear pieces associated with the Liouville gravity current. To do so we introduce the covariant Laplacian Green's function satisfying

$$\hat{\Delta}\hat{G}(\xi, \xi') = \frac{\delta^2(\xi - \xi')}{\sqrt{\hat{g}(\xi)}} - \frac{1}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}}\,. \tag{2.5}$$

The last term is necessary for consistency when we integrate Eq. (2.5) with respect to $\xi$. However, Eq. (2.5) does not fix the Green's function uniquely since we can add on an arbitrary constant to $\hat{G}$ and still satisfy it. So, we will further specify that the Green's function is symmetric in its arguments and orthogonal to the constant zero mode

$$\int d^2\xi \sqrt{\hat{g}(\xi)}\hat{G}(\xi,\xi') = 0. \tag{2.6}$$

We then find the non-local functional

$$\hat{\mathcal{F}}^s[\hat{g}] = \frac{1}{16\pi}\int d^2\xi' d^2\xi'' \sqrt{\hat{g}(\xi')}\hat{J}^s(\xi')\hat{G}(\xi',\xi'')\sqrt{\hat{g}(\xi'')}\hat{J}^s(\xi'').$$

So, for the partition function we write

$$Z^s(\chi_c) = \int \prod_1^M dy^A \det(B_i,\psi_A)\frac{\mu_2^{2s}}{\alpha}\Gamma(-s)\left(\int d^2\xi\sqrt{\hat{g}}\right)^s \frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\sqrt{\hat{\det}(B_j,B_k)}\mathrm{Vol(CKV)}}$$
$$\times\left(\frac{\mathrm{Det}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(d+1)/2} e^{\hat{\mathcal{F}}^s[\hat{g}]-\mu_3^2\int d^2\xi\sqrt{\hat{g}}}.$$

We now must ensure that the theory is Weyl invariant at the quantum level. Since we have integrated the Liouville mode that means the partition function must be independent of any scalings of the reference metric. Thus, we need to calculate the background Weyl anomaly and in the end cancel it.

First note that the heat kernel leads us to the following local contributions

$$\delta_\rho \ln\left[\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\sqrt{\hat{\det}(B_j,B_k)}\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(d+1)/2}\right] = \frac{d-25}{48\pi}\int d^2\xi\sqrt{\tilde{g}}\tilde{R}\rho$$
$$+\frac{d-1}{8\pi\epsilon}\int d^2\xi\sqrt{\tilde{g}}\rho + O(\varepsilon)\,. \tag{2.7}$$

On the other hand we have a local contribution coming from the background renormalisation counterterm

$$\delta_\rho\mu_3^2\int d^2\xi\sqrt{\hat{g}} = \mu_3^2\int d^2\xi\sqrt{\hat{g}}\rho\,. \tag{2.8}$$

Moreover, each one of the factors $\int d^2\xi\sqrt{\hat{g}}$ introduces another local piece in the Weyl anomaly by its dependence on the background metric. This is just

$$\delta_\rho \int d^2\xi \sqrt{\hat{g}} = \int d^2\xi \sqrt{\hat{g}}\rho \,. \tag{2.9}$$

The rest of the anomaly comes from the non-local functional $\hat{\mathcal{F}}^s$. Performing the infinitesimal Weyl transformations we find $\delta_\rho \sqrt{\hat{g}}\hat{J}^s = Q\sqrt{\hat{g}}\hat{\Delta}\rho$. Then integrating by parts and using Eq. (2.5) we generate

$$\delta_\rho \hat{\mathcal{F}}^s = \frac{Q^2}{8\pi}\int d^2\xi \sqrt{\hat{g}}\hat{R}\rho - \alpha Q \sum_{P=1}^{s} \rho(\xi_P) - \frac{Q}{8\pi}\int d^2\xi \sqrt{\hat{g}}\hat{J}^s \delta_\rho \ln \int d^2\xi \sqrt{\hat{g}}$$
$$+\frac{1}{16\pi}\int d^2\xi' d^2\xi'' \sqrt{\hat{g}(\xi')}\hat{J}^s(\xi')\delta_\rho \hat{G}(\xi',\xi'')\sqrt{\hat{g}(\xi'')}\hat{J}^s(\xi'') \,.$$

To calculate the anomaly contribution associated with the change in the background Green's function we note that Eq. (2.5) leads to

$$\hat{\Delta}\delta_\rho \hat{G}(\xi,\xi') = -\frac{\rho(\xi)}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}} + \frac{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}\rho(\xi'')}{\left[\int d^2\xi''' \sqrt{\hat{g}(\xi''')}\right]^2} \,. \tag{2.10}$$

When we multiply Eq. (2.10) by the Green's function, integrate and make use of the Weyl transformation of Eq. (2.6),

$$\int d^2\xi \sqrt{\hat{g}(\xi)}\delta_\rho \hat{G}(\xi,\xi') = -\int d^2\xi \sqrt{\hat{g}(\xi)}\rho(\xi)\hat{G}(\xi,\xi') \,,$$

we find the following non-local Weyl anomaly

$$\delta_\rho \hat{G}(\xi,\xi') = -\frac{1}{\int d^2\xi''' \sqrt{\hat{g}(\xi''')}}\int d^2\xi'' \sqrt{\hat{g}(\xi'')}\rho(\xi'')\left[\hat{G}(\xi'',\xi) + \hat{G}(\xi'',\xi')\right] \,.$$

The zero mode is entirely responsible for the non-vanishing of $\delta_\rho \hat{G}$ when the arguments of the Green's function are distinct. Since the zero mode is a constant, it contributes regardless of the distance between the two points $\xi$, $\xi'$. When the points are coincident it still contributes, but there is a further contribution because the Green's function requires regularisation. This may be done in a reparametrisation invariant way using the heat kernel, so we set

$$\hat{G}_\varepsilon(\xi,\xi') = \int_\varepsilon^{+\infty} dt \left[\hat{\mathcal{G}}(t,\xi,\xi') - \frac{1}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}}\right] \,,$$

where $\mathcal{G}$ satisfies the heat equation

$$-\frac{\partial}{\partial t}\hat{\mathcal{G}}(t,\xi,\xi') = \hat{\Delta}\hat{\mathcal{G}}(t,\xi,\xi')\,, \quad \hat{\mathcal{G}}(0,\xi,\xi') = \frac{\delta^2(\xi-\xi')}{\sqrt{\hat{g}(\xi)}}\,.$$

For coincident arguments the regularisation of the Green's function is controlled by the small $t$ behaviour of the heat kernel which is computable in a standard perturbation series, so that we obtain [33]

$$\delta_\rho \hat{G}_\varepsilon(\xi,\xi) = \frac{\rho(\xi)}{4\pi} - \frac{2}{\int d^2\xi''\sqrt{\hat{g}(\xi'')}}\int d^2\xi'\sqrt{\hat{g}(\xi')}\rho(\xi')\hat{G}(\xi',\xi)\,.$$

Consequently, we find

$$\delta_\rho \hat{\mathcal{F}}^s = \frac{Q^2}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{R}\rho - \alpha(Q-\alpha)\sum_{P=1}^{s}\rho(\xi_P) - \frac{Q}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{J}^s\delta_\rho \ln\int d^2\xi\sqrt{\hat{g}}$$
$$-\frac{1}{8\pi\int d^2\xi\sqrt{\hat{g}}}\left(\int d^2\xi\sqrt{\hat{g}}\hat{J}^s\right)\int d^2\xi' d^2\xi''\sqrt{\hat{g}(\xi')}\rho(\xi')\hat{G}(\xi',\xi'')\sqrt{\hat{g}(\xi'')}\hat{J}^s(\xi'')\,. \quad (2.11)$$

The non-local contributions to the Weyl anomaly are cancelled by the charge conservation selection rule given in Eq. (2.3). A look to Eqs. (2.7), (2.8), (2.9) and (2.11) shows us that to eliminate the local terms we need

$$Q = \pm\sqrt{\frac{25-d}{6}}\,, \quad 1 - \alpha Q + \alpha^2 = 0\,, \quad \mu_3^2 = \frac{d-1}{8\pi\varepsilon}\,.$$

This leads to the following two branches

$$\alpha_\pm = \frac{1}{2}\left(Q \pm \sqrt{Q^2-4}\right)\,.$$

Choosing $Q$ to be positive we write

$$\alpha_\pm = \frac{1}{2\sqrt{6}}\left(\sqrt{25-d} \pm \sqrt{1-d}\right)\,.$$

These results for the non-critical closed string show that the gravitational sector can be interpreted as a conformally extended Liouville field theory [9]. In this picture $Q$ defines the central charge of the theory $c_\phi = 1 + 6Q^2$, which has its value fixed by demanding that it cancels the central charges of the matter and ghost systems $c_M + c_{gh} = d - 26$.

We have interpreted the Liouville field as an arbitrary Weyl scaling over the closed Riemann surfaces. Then we found that because $1 - \alpha Q + \alpha^2 = 0$, the value of $\alpha$ is exactly right to define a Liouville vertex operator $\int d^2\xi\sqrt{\tilde{g}}e^{\alpha\phi}$ of zero conformal weight. On the extended field theory it corresponds to a primary field $: e^{\alpha\phi} :$ of weight (1,1).

In the DDK approach we have to restrict the range of validity of the theory to target space dimensions $d \leq 1$. In this way both $Q$ and $\alpha$ are real parameters and $e^{\alpha\phi}$ may be interpreted as a real Weyl scaling for a real scalar field $\phi$. The results are in agreement with the KPZ light-cone analysis on the sphere which found the same range of allowed target space dimensions [6]. If $d > 1$ then $\alpha$ is a complex number. For a real scalar field $\phi$ this leads to a complex Weyl scaling and the DDK approach seems to break down. In the case $d \geq 25$ we could try to consider the transformation $\phi \to i\phi$ to avoid this problem since $\alpha$ is pure imaginary. However we would then find that the Liouville action has a kinetic term of the wrong sign. The conformal mode becomes a "ghost field" and once more the DDK approach seems to break down.

### 2.1.4 Tachyon gravitational dressings

We have seen that the calculation of the partition function under the principle of quantum Weyl invariance has given us the definition of the renormalised parameters $Q$ and $\alpha$ in terms of the original parameter of the theory, the target space dimension $d$. We now use the same principle to analyse the conformal weights of the vertex operators in the non-critical theory.

For simplicity we consider the following closed string tachyon vertex operator

$$W_j = \int d^2\xi_j \sqrt{\tilde{g}} e^{ip_j \cdot X} .$$

The amplitude for the scattering of $n$ such tachyon states is given by Eq. (2.1) where we introduce $n$ of the above closed string tachyon vertex operators. For each topology we have to start by shifting the zero mode of the coordinate field $X^\mu$. This will lead to the tachyon charge conservation selection rule, that is to the conservation of momentum

$$\sum_j p_j^\mu = 0\,.$$

Next, we use the covariant Laplacian's Green's function in the metric $\tilde{g}_{ab}$ to shift the tachyon current

$$\tilde{J}_j^\mu = -8\pi i \sum_j p_j^\mu \frac{\delta^2(\xi-\xi_j)}{\sqrt{\tilde{g}(\xi)}}\,.$$

We then find

$$\begin{aligned}\tilde{\mathcal{F}}_j[\tilde{g}] &= \frac{1}{16\pi}\int d^2\xi' d^2\xi'' \sqrt{\tilde{g}(\xi')}\tilde{J}_j(\xi')\tilde{G}(\xi',\xi'')\sqrt{\tilde{g}(\xi'')}\tilde{J}_j(\xi'') \\ &= -4\pi \sum_{jj'} p_j \cdot p_{j'} \tilde{G}(\xi_j,\xi_{j'})\,.\end{aligned}$$

This is a reparametrisation invariant functional. Its dependence on the Liouville mode $\varphi$ can be calculated by looking at the Weyl anomaly it generates. This is simply determined by the correspondent anomaly of the Green's function. Using the conservation of momentum we find

$$\delta_\rho \tilde{\mathcal{F}}_j = -\sum_j p_j^2 \rho(\xi_j)\,.$$

This result leads to a new contribution to the Liouville action which is simply linear in $\varphi$. The rest of the contributions to this action are exactly as in the case of the partition function

$$S_{Lj}[\varphi,\hat{g}] = \frac{26-d}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\varphi\hat{\Delta}\varphi + \hat{R}\varphi\right) + \sum_j \left(p_j^2 - 1\right)\varphi(\xi_j) + \mu_1^2 \int d^2\xi \sqrt{\hat{g}} e^\varphi\,.$$

On the other hand we should also note that the background functional now includes the reference tachyon functional $\hat{\mathcal{F}}_j[\hat{g}]$ in addition to the usual partition function contributions.

To integrate the Liouville mode we again proceed with the change to a canonical $\phi$ integration measure and introduce the renormalised Liouville action

$$S_{Lj}[\phi,\hat{g}] = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi\right) - \sum_j \gamma_j\phi(\xi_j) + \mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}\,.$$

Above the renormalised parameters $\gamma_j$ define dressed tachyon vertex operators

$$W_j^D = \int d^2\xi_j\sqrt{\hat{g}}e^{\gamma_j\phi + ip_j\cdot X}\,.$$

This Liouville action is then integrated just as in the case of the partition function. The only difference now is that we have introduced additional gravitational charges which must obey the conservation law coming from the integration of the zero mode of $\phi$ to ensure that Weyl invariance at the quantum level is not spoiled by non-local contributions to the background Weyl anomaly. We then conclude that to guarantee Weyl invariance at the quantum level $Q$, $\alpha$ and $\mu_3^2$ must be given as before and each tachyon dressing parameter must satisfy the following equation

$$\Delta_j^0 - \gamma_j(\gamma_j - Q) = 1\,, \quad \Delta_j^0 = p_j^2\,.$$

Here the contribution associated with the free conformal weight $\Delta_j^0$ comes from the reference functional $\hat{\mathcal{F}}_j[\hat{g}]$.

The above equation shows that the dressed tachyon vertex operator corresponds to a dressed conformal field $: e^{ip_j\cdot X}e^{\gamma_j\phi} :$ whose conformal weight is (1,1) as a consequence of the quantum Weyl invariance of the theory. This is to be compared with the Weyl invariance of the critical theory. There we just put $d = 26$ to cancel the anomaly and eliminate the dynamical Liouville mode. When we calculate the tachyon scattering amplitude the contributions from the vertex operators generate new terms to the Weyl anomaly. These are cancelled provided we restrict the mass spectrum of the theory. In the case of the closed string tachyons we find the squared mass, $m_j^2 = -p_j^2 = -1$. The mass spectrum condition says that the tachyons are associated with conformal operators $e^{ip_j\cdot X}$ of weight (1,1). This coincides with what is found from the canonical Virasoro physical state conditions.

If we solve for $\gamma_j$ in terms of $Q$ and $p_j^2$ we get

$$\gamma_j = \frac{1}{2}\left[Q \pm \sqrt{Q^2 + 4(p_j^2 - 1)}\right] .$$

Just as $\alpha$ should be real so should be $\gamma_j$. This implies that $d \leq 1$ and $p_j^2 \geq (d-1)/24$.

This result can be generalised to any string vertex operator and its correspondent conformal field. The dressing parameter satisfies a similar equation with $\Delta_j^0$ standing for its free conformal dimension. The non-critical conformal weight is just equal to the symmetric of the critical invariant squared mass of the state.

## 2.2 Closed string critical exponents

To be able to sum over 2D random surfaces is important for as many reasons as there are physically relevant problems where such a sum plays a significant role [34]. For instance we can find the theory of random surfaces in the study of interfaces and domain walls in condensed matter physics, in the study of crystal growth, in the theory of 2D statistical systems, in the study of gauge theories and of course in the theory of relativistic strings and 2D quantum gravity.

It is clear that Polyakov's functional integral approach to string theory works as a technique to sum over the random Riemann surfaces in the continuous theory. Actually, it is the simplest model we can consider. The closed string partition function path integral is to be interpreted as the definition of the statistical sum over all 2D closed random surfaces embedded into a $d-$ dimensional target space

$$Z = \sum_{\text{surfaces}} e^{-\mu_0^2 A}$$

where $A$ is the area of the surface and the bare mass scale $\mu_0^2$ may be interpreted as an inverse of the temperature.

An exact meaning can also be given to this sum if we consider the lattice discretisation of the continuous surfaces. It is then seen that the sum

$$Z = \sum_{A=0}^{\infty} \Gamma(A) e^{-\mu_0^2 A},$$

never converges because the number of different surfaces with given area $A$, $\Gamma(A)$, just grows too fast as $A \to +\infty$. Introducing the entropy of the random surfaces with area $A$, $\sigma(A)$, it is found that [35]

$$\sigma(A) \equiv \ln \Gamma(A) \sim A \ln A .$$

The sum over the random surfaces may be further decomposed by summing over their topology

$$Z = \sum_{A=0}^{\infty} \sum_{\chi_c} \Gamma(\chi_c, A) e^{-\mu_0^2 A} .$$

Here $\Gamma(\chi_c, A)$ is the number of surfaces of given area $A$ and topology $\chi_c$. This function has a smoother asymptotic behavior since when $A \to +\infty$ it looks like [35]

$$\Gamma(\chi_c, A) \sim A^{\Gamma(\chi_c)-3} e^{\kappa A},$$

where $\kappa$ is a cutoff dependent coefficient and $\Gamma(\chi_c)$ the closed string susceptibility at the second order phase transition critical point, $\mu_0^2 - k \to 0$.

In Polyakov's continuum limit we have [11]

$$Z(\chi_c) = \int_0^{\infty} \Gamma(\chi_c, A) e^{-\mu_1^2 A} dA .$$

After integrating the matter and ghost fields we find

$$\Gamma(\chi_c, A) = \int \prod_1^M dy^A \det(B_i, \psi_A) \frac{\sqrt{\mathrm{Det}' \hat{P}^\dagger \hat{P}}}{\sqrt{\det(\hat{B_j}, B_k) \mathrm{Vol(CKV)}}} \left( \frac{\mathrm{Det}' \hat{\Delta}}{\int d^2\xi \sqrt{\hat{g}}} \right)^{-d/2}$$
$$\times \int \mathcal{D}_{\hat{g}} \varphi e^{-S_L^0[\varphi, \hat{g}]} \delta \left( \int d^2\xi \sqrt{\hat{g}} e^{\varphi} - A \right) .$$

Above we have factored out the renormalisation counterterm. Note that in the renormalisation process the infinite constant $\mu_0^2$ has given way to the finite constant $\mu_1^2$. When $\mu_1^2 \to 0$ we approach the critical point. Thus the Liouville action has lost its cosmological constant piece

$$S_L^0[\varphi, \hat{g}] = \frac{26-d}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left( \frac{1}{2} \varphi \hat{\Delta} \varphi + \hat{R} \varphi \right) .$$

### 2.2.1 The saddle point expansion

We may calculate the closed string susceptibility exponent outside the DDK renormalisation ansätse using the saddle point approximation to the Liouville functional integral [11]. This can be done for an arbitrary topology, showing that the critical exponent only depends on the Euler characteristic. We may ignore the background and modular contributions in the above formulas and just consider

$$\Gamma(\chi_c, A) = \int \mathcal{D}_{\hat{g}}\varphi e^{-S_L^0[\varphi,\hat{g}]}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi} - A\right) .$$

Representing the delta function by an integral over an imaginary Lagrange multiplier $p$ leads to the Euclidean action

$$S[\varphi, \hat{g}, p, A] = \frac{26-d}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\varphi\hat{\Delta}\varphi + \hat{R}\varphi\right) - p\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi} - A\right) .$$

In the saddle point approximation we expand around the solution of the following classical problem

$$\tilde{R} = \eta, , \quad \int d^2\xi\sqrt{\tilde{g}} = A ,$$

where $\eta = p_c\gamma$, $\gamma = 48\pi/(26-d)$. In the conformal gauge $\tilde{g}_{ab} = e^{\varphi}\hat{g}_{ab}$, the classical Liouville field $\varphi_c$ must satisfy the Liouville equation

$$\hat{R} + \hat{\Delta}\varphi_c = \eta e^{\varphi_c} , \quad \int d^2\xi\sqrt{\hat{g}}e^{\varphi_c} = A ,$$

where $\eta$ is constrained by the Gauss-Bonnet theorem and by the condition on the area, $\eta = 4\pi\chi_c/A$.

Let us first follow Zamolodchikov [11] and consider the case of the sphere. The solution to the Liouville equation is then easy to calculate explicitly. Using the whole plane with complex coordinates $z = x + iy$, $\bar{z} = x - iy$ we find

$$\varphi_c = -\ln\frac{\eta}{8}(1 + z\bar{z})^2 ,$$

which is just the conformal factor of the metric of a sphere of area $A$, stereographically projected onto the whole flat Euclidean space, where infinity corresponds to a single point on the sphere.

Then consider the fluctuations $\psi$ and $q$ around the classical values, $\varphi = \varphi_c + \psi$, $p = p_c + q$. If we expand to one loop the action $S[\varphi, \hat{g}, p, A]$ using the Liouville equation we get

$$\Gamma(\chi_c, A) = e^{-S^c_L[\varphi_c, g_c]} \int \mathcal{D}_{g_c}\psi \exp\left(-\frac{1}{2\gamma}\int d^2\xi\sqrt{g_c}\psi\mathcal{O}_c\psi\right)\delta\left(\int d^2\xi\sqrt{g_c}\psi\right),$$

where $\mathcal{O}_c = \Delta_c - \eta$, $g_{cab} = e^{\varphi_c}\hat{g}_{ab}$ and the classical Liouville action in the whole plane is

$$S^c_L[\varphi_c, g_c] = \frac{\eta}{2\gamma}\int d^2 z e^{\varphi_c}\varphi_c .$$

To tree level we only need to calculate the classical action

$$S^c_L[\varphi_c, g_c] = \frac{26-d}{12}\chi_c \ln A .$$

This gives the tree level contribution to the string susceptibility

$$\Gamma^0(\chi_c) = \chi_c\frac{d-26}{12} + 3 .$$

To get the one loop correction we need to integrate the field $\psi$. First we note that the conformal gauge has not been fixed uniquely. On the sphere, the Euclidean action $S[\varphi, \hat{g}, p, A]$ is invariant under the group of $SL(2, C)$ Möbius transformations

$$z \to z' = \frac{az+b}{cz+d}, \quad \varphi(z,\bar{z}) \to \varphi(z', \bar{z}') + \ln\left|\frac{dz'}{dz}\right|^2,$$

where $a$, $b$, $c$ and $d$ are complex numbers satisfying the unit determinant condition $ad - bc = 1$. These are the conformal transformations generated by the infinitesimal conformal Killing vectors $\varepsilon(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2$. The corresponding conformal infinitesimal variations in $\varphi$, $\delta_M\varphi$, are the zero modes of $\mathcal{O}_c$. For the sphere we have three zero modes corresponding to six conformal Killing vectors. Factoring out the Möbius invariance we find

$$\Gamma(2,A) = A^{(d-26)/6} A^{1/2} \int \mathcal{D}_{g_c} \psi_\perp \exp\left[ -\frac{1}{2\gamma} \int d\Omega \psi_\perp \left( L^2 - 2 \right) \psi_\perp \right] \delta\left( \int \frac{d\Omega}{4\pi} \psi_\perp \right),$$

where $\psi_\perp$ is the component of $\psi$ which is orthogonal to the conformal zero modes. Above we have also introduced the element of solid angle $d\Omega = d\cos\theta d\phi$ and the spherical Laplacian

$$L^2 = -\left[ \frac{\partial^2}{\partial\theta^2} + \left( \frac{1}{\sin\theta} - \tan\frac{\theta}{2} \right) \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right].$$

written in the spherical coordinates $z = e^{i\phi}\tan(\theta/2)$, $\bar{z} = e^{-i\phi}\tan(\theta/2)$. The operator $L^2$ is the squared angular momentum. Its eigenvalues are $l(l+1)$, $l \geq 0$ and integer, and its eigenfunctions are the spherical harmonics $Y_{lm}(\theta,\phi)$, $-l \leq m \leq l$ and also integer. The zero modes of $L^2 - 2$ are in fact three and correspond to $l = 1$.

To proceed with the integration we expand $\delta\psi$ in the spherical harmonics. Because of their orthogonality the delta function is simply the delta function of the constant zero mode of the covariant Laplacian $L^2$. Thus we obtain

$$\Gamma(2,A) = A^{(d-26)/6+1} \left[ \mathrm{Det}'' \left( \frac{L^2 - 2}{A} \right) \right]^{-1/2},$$

where in the determinant we only use the eigenvalues corresponding to $l \geq 2$. The determinant can be calculated using the heat kernel, so leading to [11]

$$\Gamma^1(2) = \frac{d - 19}{6} + 2.$$

This is the one loop result of the saddle point approximation controlled by the $\gamma$ parameter. It is only exact asymptotically in the semi-classical limit $d \to -\infty$.

For a general topology we follow similar steps [11]. For the torus the Euler characteristic is just zero. Then the only zero mode coincides with the constant zero mode of the covariant Laplacian. For higher loop orders there are no conformal Killing vectors and so no zero modes in the integrals over $\psi$. That can be seen noting that the constant we subtract from the covariant Laplacian in the operator $\mathcal{O}_c$ becomes negative. For all topologies we have to cancel the zero mode of the covariant Laplacian using the delta function and the orthogonality relations of the relevant eigenfunctions. If that is properly done then the local nature of the anomaly associated with the determinant shows that the only difference between the sphere and the other topologies in what concerns the leading area dependence is just the Euler characteristic [11]

$$\Gamma^1(\chi_c) = \chi_c \frac{d-19}{12} + 2\,. \tag{2.12}$$

### 2.2.2 The DDK scaling argument

To calculate the string susceptibility using the DDK conformal gauge approach we consider the function $\Gamma(\chi_c, A)$ after the renormalisation procedure

$$\Gamma(\chi_c, A) = \int \mathcal{D}_{\hat{g}}\phi e^{-S_L^0[\phi,\hat{g}]} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A\right)\,. \tag{2.13}$$

Here we have already factorised the renormalisation counterterm. Note that now $\mu_1^2$ has been renormalised to $\mu_2^2$. So, the renormalised Liouville action is

$$S_L^0\left[\phi, \hat{g}\right] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi\right)\,.$$

Next we shift the renormalised Liouville field by a constant $\phi \to \phi + \rho/\alpha$. Since we keep $\hat{g}_{ab}$ fixed our functional integral should scale. Recall that the theory is invariant under arbitrary scalings of the reference metric once we have integrated $\phi$. So, it is only invariant under a shift of the integration variable provided this is compensated by a Weyl transformation of the reference metric. Because we consider a translational invariant quantum measure in Eq. (2.13), the scaling behaviour is determined by the change in the action $S_L^0[\phi, \hat{g}]$ and in the delta function which is used to fix the area $A$ of the surface. Thus the shifts in the action and in the delta function are

$$S_L^0[\phi,\hat{g}] \to S_L^0[\phi,\hat{g}] + \frac{Q\chi_c}{2\alpha}\rho\,,$$

$$\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - A\right) \to e^{-\rho}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - e^{-\rho}A\right)\,.$$

Then it is easy to see that $\Gamma(\chi_c, A)$ is going to scale as follows

$$\Gamma(\chi_c, A) = e^{-(\frac{Q\chi_c}{2\alpha}+1)\rho}\Gamma\left(\chi_c, e^{-\rho}A\right)\,.$$

Thus for $A \to +\infty$ we have

$$\Gamma(\chi_c, A) \sim A^{-(\frac{Q\chi_c}{2\alpha}+1)}\,,$$

which means that the closed string susceptibility is

$$\Gamma(\chi_c) = -\frac{Q\chi_c}{2\alpha} + 2\,.$$

Introducing the values of $Q$ and $\alpha$ we write

$$\Gamma(\chi_c) = \frac{\chi_c}{24}\left[d - 25 \pm \sqrt{(25-d)(1-d)}\right] + 2\,.$$

If we take the limit $d \to -\infty$, it is easy to see that we recover the semi-classical one loop result given in Eq. (2.12). On the other hand, if we consider the case of the sphere and choose the branch

$$\alpha_- = \frac{1}{2\sqrt{6}}\left(\sqrt{25-d} - \sqrt{1-d}\right)\,,$$

then we find the KPZ result [6]

$$\Gamma(2) = \frac{d - 1 - \sqrt{(25-d)(1-d)}}{12}\,.$$

In KPZ's light-cone formulation [6], $\Gamma(2)$ is related to the central charge of the $SL(2,\mathcal{R})$ Kač-Moody symmetry used to solve the theory, $c_k = \Gamma(2) - 3$. This leads to a relation between $c_k$ and $\alpha$, $c_k + 2 = -2/\alpha^2$. So, just like $\alpha$, $c_k$ has a branch point at $d = 1$ and becomes a complex number for $d > 1$.

### 2.2.3 The tachyon gravitational scaling dimensions

Although formal the DDK scaling argument is remarkably successful in the calculation of the closed string susceptibility. This encourages us to go further and find out what happens to other critical exponents using the same simple reasoning. As a last example let us calculate the tachyon anomalous gravitational scaling dimensions.

We consider the expectation value of a tachyon vertex operator $\int d^2\xi\sqrt{\tilde{g}}e^{ip_j\cdot X}$ for a given topology $\chi_c$ and area $A$. Having integrated the matter and ghost fields we renormalise the Liouville action to introduce the canonical measure in the background metric $\hat{g}_{ab}$. Next we factor out the renormalised cosmological constant using the integral over the area and the corresponding delta function. Here it is important to note that we have not imposed the conservation of momentum. So, we would need to consider the scaling of the non-local terms associated with the Weyl transformation of the Green's function. However, because the Green's function is orthogonal to the constant zero mode, those terms do not scale if the scaling is a constant. Thus we only need to consider

$$< W > (\chi_c, A) = \frac{1}{\Gamma(\chi_c, A)} \int \mathcal{D}_{\hat{g}}\phi e^{-S_L^0[\phi,\hat{g}]+\gamma_j\phi} \delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} - A\right) .$$

The gravitational scaling dimension $\Delta$ of the vertex operator is defined by the asymptotic behavior of this function as $A \to +\infty$

$$< W > (\chi_c, A) \sim A^{1-\Delta} .$$

Using the DDK scaling argument we shift $\phi$ by the constant $\rho/\alpha$ to find that

$$\Delta_\pm = 1 - \frac{\gamma_{j\pm}}{\alpha} ,$$

This then leads to the KPZ equation for the anomalous gravitational scaling dimension [6]

$$\Delta - \Delta_j^0 = -\alpha^2\Delta(\Delta - 1) .$$

## 2.3 Minimal models on closed random surfaces

The DDK approach to the non-critical closed bosonic string has a serious drawback. It seems that it is only valid for unrealistic target space dimensions $d \leq 1$. To be on the safe side, we can just describe a weakly coupled phase where the free field perturbative approach can be justified. However we have noted that the relativistic string is but one possible field where the theory of random surfaces is relevant. We can also consider the case of 2D critical statistical systems, in particular the case of minimal $c \leq 1$ conformal field theories coupled to 2D quantum gravity.

The Coulomb gas representation of conformal field theories due to Dotsenko and Fateev [10, 16] has a natural Lagrangian interpretation. We introduce the action

$$S_M[\Phi, \tilde{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left[ \frac{1}{2} \tilde{g}^{ab} \partial_a \Phi \partial_b \Phi + i\left(\beta - 1/\beta\right) \tilde{R} \Phi \right] + \mu^2 \int d^2\xi \sqrt{\tilde{g}} \left( e^{i\beta\Phi} + e^{-i/\beta\Phi} \right) , \quad (2.14)$$

to define the minimal unitary series of conformal field theories on closed surfaces. This is a conformally extended Liouville theory [9] with imaginary coupling, $i\beta$, on a surface with metric $\tilde{g}_{ab}$ and curvature $\tilde{R}$. The central charge of the matter theory is $c_M = 1 - 6(\beta - 1/\beta)^2$ which means the minimal models [8] are at the rational points $\beta^2 = (2 + k')/(2 + k)$. The primary fields are vertex operators given by

$$U(jj') = \int d^2\xi \sqrt{\tilde{g}} \exp\left[ -i \left( j\beta - \frac{j'}{\beta} \right) \Phi \right] ,$$

where $j, j' \geq 0$ are half-integer spins labelling pairs of representations of the Lie algebra $A_1$. To couple this theory to gravity we treat $\tilde{g}_{ab}$ as a dynamical variable and add a cosmological constant term $\mu_0^2 \int d^2\xi \sqrt{\tilde{g}}$ to the action. This is the only non-trivial part of the pure gravity action in two dimensions and is necessary as a counterterm. For a given topology a general correlation function is defined by the standard functional integral

$$< \prod_{ii'} U(ii') >= \int \mathcal{D}_{\tilde{g}}(\Phi, \tilde{g}) \prod_{ii'} U(ii') \exp\left\{ -S_M[\Phi, \tilde{g}] - \mu_0^2 \int d^2\xi \sqrt{\tilde{g}} \right\} .$$

To compute this Green's function we expand the matter exponential interactions in powers of the mass scale $\mu^2$

$$\exp\left[\mu^2\int d^2\xi\sqrt{\tilde{g}}\left(e^{i\beta\Phi}+e^{-i/\beta\Phi}\right)\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\mu^2\int d^2\xi\sqrt{\tilde{g}}\left(e^{i\beta\Phi}+e^{-i/\beta\Phi}\right)\right]^n.$$

This is a natural way to implement the Coulomb gas insertion of screening operators. As for the string, the kinetic term has a constant zero mode which leads to a similar charge selection rule. For each correlator the charges of the vertex operators have to be combined with the screening charges to balance the topological background gravity charge.

In the conformal gauge, we decompose $\tilde{g}_{ab}$ as a reparametrisation of $e^{\varphi}\hat{g}_{ab}$. Integrating over the matter field and reparametrisations generates a Weyl anomaly which yields a kinetic term for $\varphi$ if the matter central charge is not balanced by the reparametrisation ghost charge. For this non-critical theory there results a Liouville field theory for $\varphi$

$$S_L\left[\varphi,\hat{g}\right]=\frac{25+6(\beta-1/\beta)^2}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\varphi\hat{\Delta}\varphi+\hat{R}\varphi\right)+\mu_1^2\int d^2\xi\sqrt{\hat{g}}e^{\varphi}.$$

The functional integral volume element for this theory is induced by the inner product on variations of the Liouville field

$$||\delta\varphi||_{\tilde{g}}^2=\int d^2\xi\sqrt{\hat{g}}e^{\varphi}(\delta\varphi)^2.$$

We then apply the DDK ansätse and replace this by a conventional field theory measure, renormalising the Liouville mode and its couplings to 2D quantum gravity

$$S_L\left[\phi,\hat{g}\right]=\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left[\frac{1}{2}\phi\hat{\Delta}\phi+i\left(\gamma-\frac{1}{\gamma}\right)\hat{R}\phi\right]+\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}.$$

Note that these are free field renormalisations. They can be understood in the context of an expansion in powers of the renormalised cosmological constant mass scale $\mu_2^2$. Under this expansion the gravitational sector should be interpreted as another Coulomb gas conformally extended Liouville theory where free field renormalisations can be used. This means the coupling of the minimal model to 2D gravity is described by two Coulomb gas conformally extended Liouville theories in the fixed background defined by the reference metric of the conformal gauge $\hat{g}_{ab}$. The matter theory gets dressed by the gravitational sector in such a way that an independent charge selection rule is imposed on each theory.

Once more we should be aware that the DDK formulation is only expected to be true when the minimal model and gravity interact in a weak coupling phase in which it is legitimate to expand the exponential interactions perturbatively. A strong affinity between the minimal model and the gravitational sector [10] is already clear in this picture. To actually prove it we have to determine the renormalised parameters of the gravitational Liouville theory. As for the string, the principle is quantum Weyl invariance. Recall that the theory is invariant under simultaneous shifts in $\phi$ and compensating scalings of $\hat{g}_{ab}$. When we integrate $\phi$ we generate a background Weyl anomaly which we add to the background anomaly coming from the integration of the matter field and the reparametrisation ghosts. The theory is Weyl invariant at the quantum level if this anomaly is absent and the amplitude is independent of the conformal factor of the reference metric.

### 2.3.1 Weyl anomaly cancellation

The anomaly cancellation sets the total central charge of the system to zero. This gives $\gamma = \pm i\beta$. Also the Liouville field renormalisation parameter $\alpha$ must satisfy $1 - \alpha(\beta + 1/\beta) + \alpha^2 = 0$ if we choose $\gamma = -i\beta$. Then we have two branches $\alpha_+ = \beta$ and $\alpha_- = 1/\beta$, the last one turning out to be of pure quantum mechanical nature. When the cosmological constant is non-zero this means we have two sets of Coulomb gas screening charges which leads to the conformally extended Liouville theory

$$
\begin{aligned}
S_L[\phi, \hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left[ \frac{1}{2} \phi \hat{\Delta} \phi + (\beta + 1/\beta) \hat{R} \phi \right] \\
+ \mu_2^2 \int d^2\xi \sqrt{\hat{g}} \left( e^{\beta\phi} + e^{1/\beta\phi} \right) .
\end{aligned}
$$

The dressed vertex operators of vanishing conformal weight are

$$U_D(jj') = \int d^2\xi\sqrt{\hat{g}}\exp\left[\left(l\beta - \frac{l'}{\beta}\right)\phi\right]\exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\Phi\right],$$

where $l = -j$, $l' = j' + 1$ or $l = j + 1$, $l' = -j'$. Translated to field theory language this means that the matter primary field $: \exp[-i(j\beta - j'/\beta)\Phi] :$ of conformal weight $\Delta_0(jj') = [(j+1/2)\beta - (j'+1/2)/\beta]^2 - 1/4(\beta - 1/\beta)^2$ gets dressed by the gravitational primary field $: \exp[(l\beta - l'/\beta)\phi] :$ of conformal weight $\Delta_0(ll') = -[(l-1/2)\beta + (l'-1/2)/\beta]^2 + 1/4(\beta + 1/\beta)^2$ to define the dressed primary field of the full theory $: \exp[-i(j\beta - j'/\beta)\Phi]\exp[(l\beta - l'/\beta)\phi] :$ of total conformal weight (1,1). For $j = j' = 0$ we recover the screening vertex operators of the gravitational sector.

As in the case of the string, it is important to note that Weyl invariance at the quantum level is only possible because we have imposed an independent charge conservation selection rule on the matter and the gravitational sectors. For each sector the Gaussian integrals over $\Phi$ and $\phi$ yield contributions of the form of the exponential of

$$\mathcal{F}^s[g] = \frac{1}{16\pi}\int d^2\xi' d^2\xi'' \sqrt{g(\xi')}J^s(\xi')G(\xi',\xi'')\sqrt{g(\xi'')}J^s(\xi''),$$

where $g_{ab}$ stands for either $\tilde{g}_{ab}$ or $\hat{g}_{ab}$ and $J^s$ is the coefficient of the term in the action that is linear in the field.

The presence of the Laplacian's zero mode then leads to a non-local Weyl anomaly

$$\delta_\rho\mathcal{F}^s = -\frac{Q}{8\pi}\int d^2\xi\sqrt{g}J^s\delta_\rho \ln\int d^2\xi\sqrt{g}$$
$$-\frac{1}{8\pi\int d^2\xi\sqrt{g}}\int d^2\xi\sqrt{g}J^s\int d^2\xi' d^2\xi''\sqrt{g(\xi')}\rho(\xi')G(\xi',\xi'')\sqrt{g(\xi'')}J^s(\xi''),$$

where $Q$ is either $i(\beta - 1/\beta)$ or $i(\gamma - 1/\gamma)$. When we integrate the zero mode of the fields in each sector the charge selection rule gives $\int d^2\xi\sqrt{g}J^s = 0$ for all non-zero contributions to the amplitude, leading to the cancellation of the non-local anomaly.

It is now clear that in the conformal gauge approach of David, Distler and Kawai the coupling of the minimal model to 2D gravity is to be seen as an interacting Coulomb gas of two conformally extended Liouville field theories in the background $\hat{g}_{ab}$. Each theory satisfies an independent charge conservation selection rule. However, the two theories are not independent but rather they are complementary in that $\gamma = \pm i\beta$ and Weyl invariance at the quantum level requires the dressing of vertex operators yielding $U(jj')V(-j, j'+1)$ or $U(jj')V(j+1, -j')$. We also note that the Liouville field of the gravitational sector is here seen as an arbitrary real Weyl scaling. This sets a bound for the matter theory since for $c > 1$ we would find complex Weyl scalings and the DDK approach seems to break down. For $c > 1$ we might expect to find a strongly coupled non-linear phase which we might not be able to reach using just this perturbative approach.

### 2.3.2 Critical exponents

The random surfaces susceptibility exponent $\Gamma(\chi_c)$ is defined using the expectation value [7, 11]

$$Z(\chi_c, A) = \int \mathcal{D}_{\tilde{g}}(\Phi, \tilde{g}) \exp\left\{-S_M[\Phi, \tilde{g}] - \mu_0^2 \int d^2\xi \sqrt{\tilde{g}}\right\} \delta\left(\int d^2\xi \sqrt{\tilde{g}} - A\right).$$

In the asymptotic limit $A \to +\infty$ we have $Z(\chi_c, A) \sim A^{\Gamma(\chi_c)-3}$. The DDK scaling argument yields $\Gamma(\chi_c) = 2 - \chi_c Q/(2\alpha_\pm)$, where we have written $Q = \beta + 1/\beta$. Here the semi-classical limit corresponds to $\beta \to +\infty$ and as expected it selects the solution $\alpha_+ = \beta$. So in fact the operator $: e^{1/\beta\phi} :$ has no classical analog, being of pure quantum mechanical nature. Within the local DDK approach this branch should also be eliminated from the quantum theory because it leads to a non-local operator with respect to the metric $\tilde{g}_{ab}$ [36]. However we should note that we may keep it provided we only demand locality with respect to the reference metric $\hat{g}_{ab}$. In this point of view $\alpha_- = 1/\beta$ is a reflection of non-perturbative phenomena in the coupling of the minimal models to 2D quantum gravity [37].

The gravitational scaling dimensions of the matter primary fields $\Delta(jj')$ are defined by [7]

$$< U(jj') > (\chi_c, A) = \frac{1}{Z(\chi_c, A)} \int \mathcal{D}_{\tilde{g}}(\Phi, \tilde{g}) U(jj') \exp\left\{-S_M[\Phi, \tilde{g}] - \mu_0^2 \int d^2\xi \sqrt{\tilde{g}}\right\}$$
$$\times \delta\left(\int d^2\xi \sqrt{\tilde{g}} - A\right) .$$

In the asymptotic limit $A \to +\infty$ we define $< U(jj') > (\chi_c, A) \sim A^{1-\Delta(jj')}$. Then the same scaling argument leads to $\Delta(jj') = 1 - \beta(jj')_\pm/\alpha_\pm$. Here $\beta(jj')_\pm$ defines the coefficient of the two possible dressings of the primary field $U(jj')$. When this is combined with the bare conformal weight using the equation which defines $\alpha$ it gives the KPZ equation [6]

$$\Delta - \Delta_0 = -\alpha^2 \Delta(\Delta - 1).$$

These results for the critical exponents of a $c \leq 1$ minimal conformal field theory on closed random surfaces agree with the KPZ light-cone analysis on the sphere [6]. Distler, Hlousek and Kawai [12] also used this conformal gauge approach to calculate the Hausdorff dimension of the random surfaces, $d_H$. According to their calculations $d_H \propto 1/|\Gamma(2)|$, which confirms the branch point at $c = 1$ where $d_H \to +\infty$. For $c > 1$ $d_H$ is a complex number, once more signaling a new strongly coupled phase where the local DDK assumption seems to break down. All the results are in striking agreement with those of the theory of dynamical triangulated random surfaces [12, 13].

# Chapter 3

# Open String 2D Quantum Gravity

## 3.1 Introduction

Polyakov's functional integral approach to the quantisation of the closed bosonic string is easily extended to the open string. We introduce the same covariant topological expansion for the connected amplitude of $n$ string states where now we sum over all open Riemann surfaces

$$< W_1 \ldots W_n >= \sum_{\text{topologies}} \beta_{st}{}^{-\chi_o} \int \mathcal{D}_{\tilde{g}}(X, \tilde{g}) W_1 \ldots W_n e^{-S[\tilde{g},X]} . \tag{3.1}$$

Clearly the matter action of Polyakov's model remains the same and so do the functional integration measures for $X^\mu$ and $\tilde{g}_{ab}$. A similar ultralocal ambiguity is present and we deal with it as in the case of the closed string. In the open string the non-uniqueness of the measures involves two arbitrary constants associated with two renormalisation cosmological constant counterterms, one in the area of the string world-sheet $\mu_0^2 \int d^2\xi \sqrt{\tilde{g}}$ and the other in its boundary length $\lambda_0 \oint d\tilde{s}$. As before the area and length counterterms are also necessary due to short distance singularities and are non-trivial pure gravity contributions to the action in two dimensions. A curious feature of the open string is that they are not alone. Although not associated with ultraviolet divergencies or with an ambiguity in the functional measures, the integral of the geodesic curvature $\nu_0 \oint d\tilde{s} k_{\tilde{g}}$ should be introduced as another pure gravity contribution to the action. Above the geodesic curvature is defined by $k_{\tilde{g}} = -\tilde{n}^a \tilde{t}^b \tilde{\nabla}_b \tilde{t}_a$, where $\tilde{n}^a$, $\tilde{t}^b$ are respectively the outward normal and the tangent vectors to the boundary [33, 38, 39]. We will see in what follows that the integral of the geodesic curvature is absolutely necessary for our solution of the non-critical open string. Here we just note that this term can be written as $(\nu_0/2)\int d^2\xi\sqrt{\tilde{g}}\tilde{R}$ if we use the Gauss-Bonnet theorem

$$\int d^2\xi\sqrt{\tilde{g}}\tilde{R} + 2\oint d\tilde{s}k_{\tilde{g}} = 4\pi\chi_o\,, \tag{3.2}$$

where $\chi_o$ is the Euler characteristic of the open Riemann surface. It is given by $\chi_o = 2-2h-b-c$ where $h$ is the genus of the surface, $b$ is the number of smooth boundaries and $c$ is the number of crosscaps. In the open string the Gauss-Bonnet theorem cannot fix both the integrals of the scalar curvature $\tilde{R}$ and of the geodesic curvature $k_{\tilde{g}}$ so that we should in fact allow one of these as a pure gravity contribution to the action. In the closed string that does not happen because we only have $\tilde{R}$ and so Eq. (2.2) means its world-sheet integral is trivial, leading only to the string coupling constant factor.

Thus the action is now given by

$$S[X,\tilde{g}] = \frac{1}{16\pi}\int d^2\xi\sqrt{\tilde{g}}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2\int d^2\xi\sqrt{\tilde{g}} + \lambda_0\oint d\tilde{s} + \nu_0\oint d\tilde{s}k_{\tilde{g}}\,,$$

and the functional measures are induced by the $\mathcal{L}^2$ norms

$$\|\delta X\|_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\delta X\cdot\delta X\,,\quad \int\mathcal{D}_{\tilde{g}}\delta X e^{-\|\delta X\|_{\tilde{g}}^2} = 1\,,$$

$$\|\delta\tilde{g}\|_{\tilde{g}}^2 = \int d^2\xi\sqrt{\tilde{g}}\left(\tilde{g}^{ac}\tilde{g}^{bd} + u\tilde{g}^{ab}\tilde{g}^{cd}\right)\delta\tilde{g}_{ab}\delta\tilde{g}_{cd},\quad \int\mathcal{D}_{\tilde{g}}\delta\tilde{g}e^{-\|\delta\tilde{g}\|_{\tilde{g}}^2} = 1\,.$$

The theory of closed strings was defined to be explicitly invariant under general reparametrisations of the parameter domain in $\mathcal{R}^2$. So naturally in the open string we still aim to have explicit covariance. However due to the presence of boundary effects associated with the internal metric's $\mathcal{L}^2$ norm, we can only consider an invariance under the diffeomorphisms which preserve the $\mathcal{R}^2$ parameter domain but allow for general reparametrisations along the boundary [38]. This means that the component of the reparametrisation ghosts $\theta^a$ along the outward normal to the boundary must be zero, $\tilde{n} \cdot \theta = 0$, but its component along the tangent $\tilde{t} \cdot \theta$ may take an arbitrary value.

For the open string we also introduce the string states using covariant vertex operators. The open string states correspond to boundary vertex operators such as the tachyon with momentum $p^\mu$, $\oint d\tilde{s} e^{ip \cdot X}$. Closed string states can also be attached to the bulk of the open Riemann surface, so that in general we have a mixed type amplitude with boundary and bulk vertex operators.

Just as in the closed string, the price we pay for explicit covariance is the explicit violation of the Weyl symmetry. Thus a Weyl anomaly will be generated which must be cancelled out at the quantum level. This should again replace the Virasoro physical state conditions and lead to the critical dimension of the target space, $d = 26$, as well as to the spectrum of the open string. For non-critical target space dimensions the Liouville scale of the intrinsic metric $\tilde{g}_{ab}$ becomes once more the non-trivial dynamical degree of freedom we need to take into account.

## 3.2 Free boundary conditions

For simplicity we consider Polyakov's open bosonic string partition function $Z$ for the topology of a disc [38, 39, 33]. We can do so and still get results which are valid for any topology because the analysis will be based on the local nature of the Weyl anomaly and because the functional measure for the integration over the moduli space of the open Riemann surfaces can be defined to be independent of the scalings of the metric.

Let us start by taking free boundary conditions on the string field $X^\mu$, on the Liouville conformal gauge factor $\varphi$ and on the reparametrisation ghosts $\theta^a$. More precisely, we initially require that $X^\mu$, $\varphi$ and $\tilde{t} \cdot \theta$ take prescribed values $Y^\mu$, $\psi$ and $\eta$ on the boundary and then we integrate over these boundary values [33]. The functional $Z[Y, \psi, \eta]$, obtained as an intermediate step, has the physical interpretation of being the tree level (in the sense of string loops) contribution to the wave functional of the vacuum for closed string theory in the Schrödinger representation.

The quantum partition function is thus given by

$$Z = \int \mathcal{D}_{\tilde{g}}(Y, \psi, \eta) Z[Y, \psi, \eta]$$

where the wave functional is

$$Z[Y, \psi, \eta] = \int \mathcal{D}_{\tilde{g}}(X, \tilde{g}) e^{-S[X, \tilde{g}]} \,. \tag{3.3}$$

To calculate $Z$ let us first determine the wave functional $Z[Y, \psi, \eta]$. We start by separating $X^\mu$ into two parts $X^\mu = X_c^\mu + \bar{X}^\mu$. We define $X_c^\mu$ and $\bar{X}^\mu$ in such a way that the string action gets split into two independent pieces, one for $X_c^\mu$ which contains all the dependence on the boundary value $Y^\mu$ and another for $\bar{X}^\mu$. This is easily done if we fix $X_c^\mu$ using $Y^\mu$,

$$\tilde{\Delta} X_c^\mu = 0, \quad X_c^\mu|_B = Y^\mu, \tag{3.4}$$

and impose on $\bar{X}^\mu$ an homogeneous Dirichlet boundary condition $\bar{X}^\mu|_B = 0$. Here we have used the notation $B$ to say that the fields are evaluated at a point $\xi$ of the boundary $B$. As required by the wave equation, on $X_c^\mu$ we must further impose the following consistency condition

$$\oint d\tilde{s} \partial_{\tilde{n}} X_c^\mu = 0 \,, \tag{3.5}$$

where $\partial_{\tilde{n}}$ is the outward normal derivative on the boundary. Eq. (3.4) is solved in terms of $Y^\mu$ using the Green's function for the Laplacian with homogeneous Dirichlet boundary conditions defined for the metric $\tilde{g}_{ab}$. We will separate the boundary value $Y^\mu$ into a constant piece, and a piece that is orthogonal with respect to the natural metric on the boundary, i.e. we write $Y^\mu = Y_0^\mu + \bar{Y}^\mu$ where $\oint d\tilde{s} \bar{Y}^\mu = 0$. Then the solution is

$$X^\mu_c(\xi') = Y^\mu_0 - \oint d\tilde{s}(\xi)\partial_{\tilde{n}}\tilde{G}_D(\xi,\xi')\bar{Y}^\mu(\xi) \tag{3.6}$$

if the point $\xi'$ is not in the boundary and $X^\mu_c|_B = Y^\mu$ if it is. Of course here we have considered

$$\tilde{\Delta}\tilde{G}_D(\xi,\xi') = \frac{\delta^2(\xi-\xi')}{\sqrt{\tilde{g}(\xi)}} \tag{3.7}$$

where $\tilde{G}_D(\xi,\xi') = 0$ if either argument lies on the boundary. In this case we can integrate Eq. (3.7) leading to an integral condition on its outward normal derivative

$$\oint d\tilde{s}(\xi)\partial_{\tilde{n}}\tilde{G}_D(\xi,\xi') = -1, \tag{3.8}$$

which allows the decomposition of $X^\mu_c$ given in Eq. (3.6).

We can prove that Eq. (3.6) is the solution of Eq. (3.4) using Green's theorem

$$-\oint d\tilde{s}\partial_{\tilde{n}}X^\mu_c\tilde{G}_D(\xi,\xi') + \oint d\tilde{s}\partial_{\tilde{n}}\tilde{G}_D(\xi,\xi')X^\mu_c = \int d^2\xi\sqrt{\tilde{g}}\tilde{\Delta}X^\mu_c\tilde{G}_D(\xi,\xi')$$
$$-\int d^2\xi\sqrt{\tilde{g}}\tilde{\Delta}\tilde{G}_D(\xi,\xi')X^\mu_c\,.$$

Then Eqs. (3.4) and (3.7) plus the fact that the Dirichlet Green's function is zero on the boundary lead us to Eq. (3.6). Applying Eq. (3.8) it is clear that this solution satisfies the consistency condition given in Eq. (3.5).

The string action can now be cast in the form

$$S[X,\tilde{g}] = S_c[X_c,\tilde{g}] + S[\bar{X},\tilde{g}]\,.$$

The action for $\bar{X}^\mu$ is just the free bosonic string action where the kinetic kernel is the covariant Laplacian. To find $S_c[X_c,\tilde{g}]$ as a boundary action we take a total derivative and use Eq. (3.4). We may write the result introducing the boundary kinetic kernel

$$\tilde{K}_D(\xi,\xi') = -\frac{1}{8\pi}\partial_{\tilde{n}}\partial_{\tilde{n}'}\tilde{G}_D(\xi,\xi')\,,$$

leading to

$$S_c[X_c, \tilde{g}] = \frac{1}{2} \oint d\tilde{s}(\xi) d\tilde{s}(\xi') Y(\xi) \cdot \tilde{K}_D(\xi, \xi') Y(\xi').$$

When we integrate $X^\mu$ keeping $Y^\mu$ fixed $\mathcal{D}_{\tilde{g}} X$ is actually $\mathcal{D}_{\tilde{g}} \bar{X}$. For the integration over the metric $\mathcal{D}_{\tilde{g}} \tilde{g}$ we proceed like in the closed string, and use the conformal gauge where we decompose the integration over $\tilde{g}_{ab}$ into an integration over $\varphi$ and an integration over $\theta^a$. On the disc an arbitrary infinitesimal variation of $\tilde{g}_{ab}$ is

$$\delta \tilde{g}_{ab} = \delta\varphi \tilde{g}_{ab} + \tilde{\nabla}_a \delta\theta_b + \tilde{\nabla}_b \delta\theta_a \, .$$

As in the closed string the variations of $\tilde{g}_{ab}$ induced by the reparametrisation ghosts and by Weyl transformations are not orthogonal. They intersect in the conformal Killing vectors

$$\tilde{P}_{ab}(\delta\theta) = 0 \, .$$

This set of reparametrisations belongs to the group $SL(2, \mathcal{R})$ of Möbius transformations and map the disc onto itself. So, redefining $\varphi$ we write

$$||\delta\tilde{g}||^2_{\tilde{g}} = 2(1+2u) \int d^2\xi \sqrt{\tilde{g}} (\delta\varphi)^2 + \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ac} \tilde{g}^{bd} \tilde{P}_{ab}(\delta\theta) \tilde{P}_{cd}(\delta\theta) \, .$$

After a total derivative we can see that this hides a boundary contribution

$$||\delta\tilde{g}||^2_{\tilde{g}} = 2(1+2u) \int d^2\xi \sqrt{\tilde{g}} (\delta\varphi)^2 + \int d^2\xi \sqrt{\tilde{g}} \tilde{g}_{ab} \delta\theta^a \left[\tilde{P}^\dagger \tilde{P}(\delta\theta)\right]^b$$
$$+2 \oint d\tilde{s} \tilde{n}_a \delta\theta_b \tilde{P}^{ab}(\delta\theta).$$

Because of this boundary term we cannot have covariance under general diffeomorphisms on the open string. At first sight, it even seems that the boundary condition $\tilde{n} \cdot \delta\theta = 0$ is not enough to solve the problem. Fortunately, it is possible to use the covariant heat kernel to show that the remaining boundary term can only lead to contributions which are absorbed by the renormalisation counterterm in the length of the boundary [38].

We now split $\theta^a$ into a field $\bar{\theta}^a$ vanishing at the boundary and another field $\vartheta^a$ such that at the boundary $\vartheta^a = \eta\tilde{t}^a$. Assuming that $\vartheta^a$ is fixed by its boundary value in some way we obtain

$$\|\delta\tilde{g}\|^2_{\tilde{g}} = 2(1+2u)\int d^2\xi\sqrt{\tilde{g}}(\delta\varphi)^2 + \int d^2\xi\sqrt{\tilde{g}}\delta\bar{\theta}\cdot\tilde{P}^\dagger\tilde{P}(\delta\bar{\theta}).$$

Omiting the renormalisation counterterms we integrate $\bar{X}^\mu$ and $\bar{\theta}$ to find

$$Z[Y,\psi,\eta] = \exp\{-S_c[X_c,\tilde{g}]\}\int \mathcal{D}_{\tilde{g}}\varphi\left(\mathrm{Det}'\tilde{\Delta}\right)^{-d/2}\frac{\sqrt{\mathrm{Det}'\tilde{P}^\dagger\tilde{P}}}{\mathrm{Vol(CKV)}}$$

where as usual the prime denotes the omission of the zero modes and we have divided by the volume of the space of conformal Killing vectors Vol(CKV). As is well known these infinite determinants generate a Weyl anomaly [3, 24, 25, 26, 27, 33, 38, 39]. If we use the covariant heat kernel to regularise them it is easy to see that the Weyl anomaly only depends on the values of the heat kernels for small proper time cutoff $\sqrt{\varepsilon}$. This means that the Weyl anomaly is a local phenomenon which only reflects the structure of the world-sheet at short distances. Since $\sqrt{\varepsilon}$ can be made infinitesimally small, the bulk and boundary contributions to the anomaly must be independent. Using locality, reparametrisation invariance and dimensional analysis we are led to the following expansion in powers of the proper time cutoff $\sqrt{\varepsilon}$

$$\delta_\rho \ln\left[\left(\mathrm{Det}'\tilde{\Delta}\right)^{-d/2}\frac{\sqrt{\mathrm{Det}'\tilde{P}^\dagger\tilde{P}}}{\mathrm{Vol(CKV)}}\right] = \frac{d-26}{48\pi}\int d^2\xi\sqrt{\tilde{g}}\tilde{R}\rho + C\oint d\tilde{s}k_{\tilde{g}}\rho$$
$$+C_1\oint d\tilde{s}\partial_{\tilde{n}}\rho + \frac{C_2}{\varepsilon}\int d^2\xi\sqrt{\tilde{g}}\rho + \frac{C_3}{\sqrt{\varepsilon}}\oint d\tilde{s}\rho + O(\sqrt{\varepsilon})\,, \tag{3.9}$$

where $C$ and the $C_i$, $i = 1,2,3$, are dimensionless constants. The $C_i$ can be determined exactly [38, 33] but we will not worry about them because they all are absorbed in the renormalisation counterterms. We calculate $C$ using the commutativity of Weyl transformations [33]. Consider the gravity currents $\sqrt{\tilde{g}}\tilde{R}$ and $d\tilde{s}k_{\tilde{g}}$. Their transformation laws under an infinitesimal Weyl scaling are

$$\delta_\rho\sqrt{\tilde{g}}\tilde{R} = \sqrt{\tilde{g}}\tilde{\Delta}\rho, \quad \delta_\rho d\tilde{s}k_{\tilde{g}} = \frac{1}{2}d\tilde{s}\partial_{\tilde{n}}\rho\,.$$

Then taking a second Weyl transformation $\rho'$ on Eq. (3.9) and antisymmetrising in $\rho$ and $\rho'$ gives

$$\frac{d-26}{48\pi}\int d^2\xi\sqrt{\tilde{g}}\left(\rho\tilde{\Delta}\rho' - \rho'\tilde{\Delta}\rho\right) + \frac{C}{2}\oint d\tilde{s}\left(\rho\partial_{\tilde{n}}\rho' - \rho'\partial_{\tilde{n}}\rho\right) = 0.$$

Thus we find

$$C = \frac{d-26}{24\pi}.$$

Integrating the infinitesimal variation leads to the usual Liouville action plus background contributions depending on the reference metric of the conformal gauge $\hat{g}_{ab}$

$$Z[Y,\psi,\eta] = \exp\left\{-S_c[X_c,\tilde{g}]\right\}\int \mathcal{D}_{\hat{g}}\varphi\left(\mathrm{Det}'\hat{\Delta}\right)^{-d/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^{\dagger}\hat{P}}}{\mathrm{Vol(CKV)}}\exp\left\{-S_L[\varphi,\hat{g}]\right\},$$

where the Liouville action is given by

$$S_L\left[\varphi,\hat{g}\right] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) - \frac{d-26}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi$$
$$+\mu_1^2\int d^2\xi\sqrt{\hat{g}}e^{\varphi} + \lambda_1\oint d\hat{s}e^{\varphi/2} + \nu_1\oint d\hat{s}\partial_{\hat{n}}\varphi\,.$$

Here $\mu_1^2$, $\lambda_1$ and $\nu_1$ are arbitrary finite constants left over from the renormalisation process.

Once more we find the critical target space dimension $d = 26$ where the Liouville mode decouples from the theory. As in the closed string this is the same result deduced from the canonical no-ghost theorem by imposing the Virasoro gauge conditions.

For the non-critical target space dimensions the conformal scale of the metric is that extra degree of freedom already expected in the canonical formulation. It is to the Liouville mode we turn our attention next. We consider its integration and determine the renormalisation of the couplings to 2D quantum gravity in the context of the DDK approach.

### 3.2.1 Anomaly cancellation for coupling renormalisation

To integrate the Liouville mode we start by taking the Coulomb gas perturbative approach expanding the area cosmological constant counterterm. In each order of perturbation theory we split $\varphi$ in two fields $\varphi_c$, $\bar{\varphi}$ in exactly the same way we split $X^\mu$ previously. As before the Liouville action becomes the sum of two independent pieces, $S_L[\varphi_c, \hat{g}]$, which contains all the dependence on the boundary value $\psi$, and $S_L[\bar{\varphi}, \hat{g}]$. We further split $\psi = \psi_0 + \bar{\psi}$ into a constant $\psi_0$ and an orthogonal piece $\bar{\psi}$. The field $\varphi_c$ is now expressed in terms of $\bar{\psi}$ and $\psi_0$

$$\varphi_c(\xi') = \psi_0 - \oint d\hat{s}(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\bar{\psi}(\xi)\,. \tag{3.10}$$

Let us take the lowest order in the area cosmological constant perturbative expansion. When we integrate $\varphi$ we consider a fixed value of $\psi$. Then $\mathcal{D}_{\hat{g}}\varphi = \mathcal{D}_{\hat{g}}\bar{\varphi}$ and the lowest order contribution to the wave functional is given by

$$Z^{00}[Y,\psi,\eta] = e^{-S_c[X_c,\hat{g}]-S_c^0[\varphi_c,\hat{g}]}\left(\mathrm{Det}'\hat{\Delta}\right)^{-d/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}\bar{Z}^0[Y,\psi,\eta]$$

where

$$\bar{Z}^0[Y,\psi,\eta] = \int \mathcal{D}_{\hat{g}}\bar{\varphi}e^{-\bar{S}^0[\bar{\varphi},\hat{g}]}\,.$$

Above we have introduced the lowest order Liouville actions for $\bar{\varphi}$

$$\bar{S}^0[\bar{\varphi},\hat{g}] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\bar{\varphi}\hat{\Delta}\bar{\varphi} + \hat{R}\bar{\varphi}\right) + \nu_1\oint d\hat{s}\partial_{\hat{n}}\bar{\varphi}$$

and for $\varphi_c$

$$\begin{aligned} S_c^0[\varphi_c,\hat{g}] = &-\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi_c\partial_b\varphi_c + \hat{R}\varphi_c\right) \\ &-\frac{d-26}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi_c + \lambda_1\oint d\hat{s}e^{\varphi_c/2}\,. \end{aligned} \tag{3.11}$$

The functional integration measure for the integral over $\bar{\varphi}$ is conformally invariant but non-linear in the Liouville field

$$\|\delta\bar{\varphi}\|^2_{\hat{g}} = \int d^2\xi \sqrt{\hat{g}} e^{\varphi} (\delta\bar{\varphi})^2.$$

To proceed we follow the closed string analysis and use the DDK renormalisation ansätse. We may consider a canonical measure in the background $\hat{g}_{ab}$,

$$\left\|\delta\bar{\phi}\right\|^2_{\hat{g}} = \int d^2\xi \sqrt{\hat{g}} (\delta\bar{\phi})^2,$$

provided we renormalise the Liouville field and its couplings to 2D gravity. Observe that this renormalisation involves the whole Liouville field. As pointed out by Symanzik, in the presence of the boundary we should expect to take independent bulk and boundary renormalisations [40]. Since the boundary pieces of the Liouville mode are fixed at the moment we do not need to worry about them for the time being. We also note that the canonical measure can only be introduced if a set of background counterterms is included

$$S_R(\hat{g}) = \mu_3^2 \int d^2\xi \sqrt{\hat{g}} + \lambda_3 \oint d\hat{s} + \nu_3 \oint d\hat{s} k_{\hat{g}}.$$

When we renormalise the field $\bar{\varphi} \to \alpha\bar{\phi}$ and its couplings to gravity we get the following renormalised lowest order Liouville action

$$\bar{S}^0[\bar{\phi}, \hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left( \frac{1}{2} \bar{\phi} \hat{\Delta} \bar{\phi} + Q \hat{R} \bar{\phi} \right) + \nu_2 \oint d\hat{s} \partial_{\hat{n}} \bar{\phi}.$$

The renormalised parameters of the theory are determined by requiring invariance under a shift in $\phi$ and a compensating Weyl transformation of the reference metric. Once $\phi$ has been integrated out the result is required to be invariant under Weyl transformations of the metric alone. For the moment we integrate $\bar{\phi}$. To do so we need to follow Alvarez [39] and set $\nu_2$ to zero because the standard way to deal with a term that is linear in the field is to shift the integration variable, in this case by a constant, but this would spoil the homogeneous Dirichlet condition on $\bar{\phi}$. Next we change variables as follows $\sqrt{8\pi}\bar{\phi} \to \bar{\phi} + \hat{O}^0_Q$. Here we have set

$$\hat{O}^0_Q(\xi') = \int d^2\xi \sqrt{\hat{g}(\xi)} \hat{J}^0_Q(\xi) \hat{G}_D(\xi, \xi'), \quad \hat{O}^0_Q|_B = 0$$

and introduced the current $\hat{J}^0_Q = Q\hat{R}$. As a result we get the free field integrand

$$S_F[\phi, \hat{g}] = \frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\bar{\phi}\hat{\Delta}\bar{\phi}$$

plus the non-local functional

$$\hat{\mathcal{F}}^0_D[\hat{g}] = \frac{Q^2}{16\pi}\int d^2\xi d^2\xi'\sqrt{\hat{g}(\xi)}\hat{R}(\xi)\hat{G}_D(\xi,\xi')\sqrt{\hat{g}(\xi')}\hat{R}(\xi'). \tag{3.12}$$

Because there is no zero mode $\hat{G}_D(\xi,\xi')$ is Weyl invariant for distinct values of its arguments (coincident values require regularisation which introduces dependence on the scale of the metric). Thus, the Weyl anomaly associated with Eq. (3.12) is determined by the scaling of the current

$$\delta_\rho\sqrt{\hat{g}}\hat{R} = \sqrt{\hat{g}}\hat{\Delta}\rho\,. \tag{3.13}$$

Integrating by parts we find

$$\delta_\rho\hat{\mathcal{F}}^0_D = \frac{Q^2}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{R}\rho + \frac{Q^2}{8\pi}\oint d\hat{s}(\xi)\int d^2\xi'\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\sqrt{\hat{g}(\xi')}\hat{R}(\xi'). \tag{3.14}$$

The product of functional determinants resulting from the integration over the matter field, the reparametrisations and $\bar{\phi}$ also varies under a Weyl transformation

$$\delta_\rho \ln\left[\left(\mathrm{Det}'\hat{\Delta}\right)^{-(d+1)/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}\right] = \frac{d-25}{48\pi}\int d^2\xi\sqrt{\hat{g}}\hat{R}\rho + \frac{d-25}{24\pi}\oint d\hat{s}k_{\hat{g}}\rho$$

$$+C'_1\oint d\hat{s}\partial_{\hat{n}}\rho + \frac{C'_2}{\varepsilon}\int d^2\xi\sqrt{\hat{g}}\rho + \frac{C'_3}{\sqrt{\varepsilon}}\oint d\hat{s}\rho + O(\sqrt{\varepsilon}), \tag{3.15}$$

where the $C'_i$, $i = 1, 2, 3$, are dimensionless constants which as before can be determined exactly.

Ignoring the counterterms for the moment we cancel the bulk local piece of the Weyl anomaly between Eqs. (3.14) and (3.15) if we set

$$Q = \pm\sqrt{\frac{25-d}{6}}.$$

Since $\rho$ is an arbitrary infinitesimal Weyl scaling in the bulk and on the boundary of the surface we also need to deal with the non-local term and with the local boundary contribution in the geodesic curvature found respectively in Eqs. (3.14) and (3.15). To do so we have to consider the integration over the boundary values of the Liouville field.

First we integrate $Y^\mu$ and $\eta$. The boundary measures for these fields are induced by the natural reparametrisation invariant inner products on variations of the boundary values

$$||\delta Y||^2_{\tilde{g}} = \oint d\tilde{s}\delta Y \cdot \delta Y, \quad ||\delta\eta||^2_{\tilde{g}} = \oint d\tilde{s}(\delta\eta)^2.$$

As the formalism is explicitly reparametrisation invariant the integration over $\eta$ is trivial leading to an overall factor. For the boundary matter field we find

$$\int \mathcal{D}_{\tilde{g}}Y \exp\{-S_c[X_c,\tilde{g}]\} = \left(\frac{\mathrm{Det}'\tilde{K}_D}{\oint d\tilde{s}}\right)^{-d/2} \int \prod_\mu dY_0^\mu. \tag{3.16}$$

Above we took into account the zero mode of the boundary kernel $\hat{K}_D$. Its existence can be seen by considering the eigenvalue problem

$$\oint d\hat{s}(\xi)\hat{K}_D(\xi,\xi')\hat{v}_N(\xi) = \hat{\lambda}_N\hat{v}_N(\xi').$$

These eigenfunctions form a complete

$$\sum_N \hat{v}_N(\xi)\hat{v}_N(\xi') = \hat{\delta}_B(\xi-\xi') \tag{3.17}$$

and orthonormal set

$$\oint d\hat{s}(\xi)\hat{v}_N(\xi)\hat{v}_M(\xi) = \delta_{NM} \tag{3.18}$$

of functions on the boundary. Here the boundary delta function is defined by

$$\oint d\hat{s}(\xi)\hat{\delta}_B(\xi-\xi')f(\xi) = f(\xi').$$

Then the eigenvalues may be expressed as

$$\hat{\lambda}_N = \oint d\hat{s}(\xi)d\hat{s}(\xi')\hat{v}_N(\xi)\hat{K}_D(\xi,\xi')\hat{v}_N(\xi') \tag{3.19}$$
$$= -\frac{1}{8\pi}\oint d\hat{s}(\xi)d\hat{s}(\xi')\hat{v}_N(\xi)\partial_{\hat{n}}\partial_{\hat{n}'}\hat{G}_D(\xi,\xi')\hat{v}_N(\xi').$$

Now define $\hat{V}_N$ to be the solution of Laplace's equation with boundary value $v_N$

$$\hat{\Delta}\hat{V}_N = 0, \quad \hat{V}_N|_B = \hat{v}_N.$$

This has the solution

$$\hat{V}_N(\xi') = -\oint d\hat{s}(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\hat{v}_N(\xi),$$

enabling us to write the eigenvalues as

$$\hat{\lambda}_N = \frac{1}{8\pi}\oint d\hat{s}(\xi)\hat{V}_N(\xi)\partial_{\hat{n}}\hat{V}_N(\xi) = \frac{1}{8\pi}\int d^2\sqrt{\hat{g}}\hat{g}^{ab}\partial_a\hat{V}_N\partial_b\hat{V}_N.$$

Thus $\hat{\lambda}_N \geq 0$ and it is only zero when $\hat{V}_N$ is constant. Denoting this solution by $N = 0$ and using the normalisation condition we conclude that $\hat{K}_D$ has the zero mode

$$\hat{v}_0 = \left(\oint d\hat{s}\right)^{-1/2}.$$

The determinant in Eq. (3.16) will generate a new boundary term for the Liouville action. This is the gluing anomaly found in [33]. The kernel $\tilde{K}_D$ has a boundary heat kernel which can only be sensitive to short distance effects, and since the boundary has no intrinsic geometry it can only be sensitive to the invariant length of the boundary. As a consequence covariance and dimensional analysis lead to a contribution to the Weyl anomaly which can be absorbed into the cosmological constant counterterm in the invariant world-sheet length of the boundary ( see Appendix ).

To cancel the remaining terms in the Weyl anomaly we have to integrate $\psi$. Just as in the case of $\bar{\varphi}$ we have a non-linear inner product on variations of $\psi$

$$||\delta\psi||^2_{\hat{g}} = \oint d\hat{s} e^{\psi/2} (\delta\psi)^2.$$

We will assume, following David, Distler and Kawai, that we can use the inner product that is more usual for a quantum field in the background $\hat{g}_{ab}$,

$$||\delta\Psi||^2_{\hat{g}} = \oint d\hat{s} (\delta\Psi)^2,$$

provided we renormalise $\psi_0 \to \alpha_0 \Psi_0$, and $\bar{\psi} \to \alpha_B \bar{\Psi}$ as well as their couplings to 2D quantum gravity. Note that this means we need to introduce an independent field renormalisation for $\bar{\varphi}_c$, the component of $\varphi_c$ orthogonal to the zero mode $\psi_0$. According to Eq. (3.10), its explicit expression in terms of $\bar{\psi}$ involves a coupling to 2D gravity. Thus we must also consider $\bar{\varphi}_c \to \bar{\alpha}_B \bar{\phi}_c$. This is to be done in each order of the perturbative expansion in the length cosmological constant. Note that we have allowed for a different renormalisation of $\psi_0$ and $\bar{\psi}$. This is because we take independent bulk and boundary renormalisations and $\psi_0$ is related to the zero mode of the Laplacian on closed surfaces that would be generated if we glued together two disc shaped topologies to obtain a sphere, corresponding to the inner product of the closed string vacuum with itself. Thus $\psi_0$ is really associated with the Liouville field in the bulk and should be renormalised accordingly.

Now when we decompose $\psi$ into $\psi_0$ and $\bar{\psi}$ Eq. (3.11) can be rewritten as

$$S^0_c[\bar{\psi}, \psi_0, \hat{g}] = -\frac{d-26}{12} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\psi}(\xi') - \frac{d-26}{24\pi} \oint d\hat{s} k_{\hat{g}} \bar{\psi}$$
$$+\lambda_1 \oint d\hat{s} e^{\bar{\psi}+\psi_0} + \frac{d-26}{48\pi} \int d^2\xi \sqrt{\hat{g}(\xi)} \hat{R}(\xi) \oint d\hat{s}(\xi') \partial_{\hat{n}'} \hat{G}_D(\xi, \xi') \bar{\psi}(\xi')$$
$$-\frac{d-26}{12} \chi_o \psi_0.$$

Introducing the coupling renormalisation parameters $Q_0$, $Q_B$ and $\bar{Q}_B$ we write the renormalised lowest order boundary action

$$S^{00}_c[\bar{\Psi}, \Psi_0, \hat{g}] = \frac{1}{2} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\Psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\Psi}(\xi') + \oint d\hat{s} \hat{H}^{00}_D \bar{\Psi}$$

$$+\frac{Q_0\chi_o}{2}\Psi_0, \tag{3.20}$$

where we have the current

$$\hat{H}_D^{00}(\xi) = -\frac{Q_B}{8\pi}\int d^2\xi'\sqrt{\hat{g}(\xi')}\hat{R}(\xi')\partial_{\hat{n}}\hat{G}_D(\xi,\xi') + \frac{\bar{Q}_B}{8\pi}k_{\hat{g}}(\xi). \tag{3.21}$$

To integrate this we shift out the linear piece in $\bar{\Psi}$. We introduce the Green's function of $\hat{K}_D$ defined by

$$\oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\hat{G}_K(\xi'',\xi') = \hat{\delta}_B(\xi-\xi') - \frac{1}{\oint d\hat{s}(\xi''')}. \tag{3.22}$$

The last term on the right-hand side of Eq. (3.22) is necessary to ensure consistency when the equation is integrated with respect to $\hat{s}(\xi)$, since

$$\oint d\hat{s}(\xi)\hat{K}_D(\xi,\xi') = 0. \tag{3.23}$$

Its value is fixed by the zero mode of $\hat{K}_D$ we have calculated before. Also $\hat{G}_K(\xi,\xi')$ is symmetric in its arguments and is orthogonal to the constant zero mode

$$\oint d\hat{s}(\xi)\hat{G}_K(\xi,\xi') = 0. \tag{3.24}$$

Then we can consider the shift $\bar{\Psi} \to \bar{\Psi} + \hat{\mathcal{F}}_K^{00}$ where

$$\hat{\mathcal{F}}_K^{00}(\xi') = \oint d\hat{s}(\xi)\hat{H}_D^{00}(\xi)\hat{G}_K(\xi,\xi')$$

is also orthogonal to the zero mode. Thus the integration leads to

$$\int \mathcal{D}_{\hat{g}}(\bar{\Psi},\Psi_0)\exp\{-S_c^{00}[\bar{\Psi},\Psi_0,\hat{g}]\} = e^{\hat{\mathcal{F}}_B^{00}}\left(\frac{\mathrm{Det}'\hat{K}_D}{\oint d\hat{s}}\right)^{-d/2}\int d\Psi_0 e^{-Q_0\chi_o\Psi_0/2}$$

where

$$\hat{\mathcal{F}}_B^{00} = \frac{1}{2}\oint d\hat{s}(\xi)d\hat{s}(\xi')\hat{H}_D^{00}(\xi)\hat{G}_K(\xi,\xi')\hat{H}_D^{00}(\xi'). \tag{3.25}$$

The determinant only changes the background renormalisation counterterm in the world-sheet length. The important contribution to the Weyl anomaly comes from Eq. (3.25). To calculate it we first need the Weyl transformation associated with Eq. (3.21). Using Eq. (3.13) and the corresponding transformation of the geodesic curvature

$$\delta_\rho d\hat{s}k_{\hat{g}} = \frac{1}{2}d\hat{s}\partial_{\hat{n}}\rho,$$

we take a total derivative and introduce the boundary kernel $\hat{K}_D$ to find

$$\delta_\rho[d\hat{s}(\xi')\hat{H}_D^{00}(\xi')] = Q_B \oint d\hat{s}(\xi)\rho(\xi)d\hat{s}(\xi')\hat{K}_D(\xi,\xi') + \frac{1}{16\pi}(\bar{Q}_B - 2Q_B)d\hat{s}(\xi')\partial_{\hat{n}'}\rho(\xi'). \tag{3.26}$$

Then Eqs. (3.21) and (3.22) lead us to

$$\delta_\rho\hat{\mathcal{F}}_B^{00} = -\frac{Q_B^2}{8\pi}\oint d\hat{s}(\xi)\int d^2\xi'\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\sqrt{\hat{g}(\xi)}\hat{R}(\xi') + \frac{Q_B^2}{4\pi}\oint d\hat{s}k_{\hat{g}}\rho. \tag{3.27}$$

Above we have taken $\bar{Q}_B = 2Q_B$ which is a condition needed to eliminate the contribution associated with the outward normal derivative of $\rho$

$$\frac{\bar{Q}_B - 2Q_B}{16\pi}\oint d\hat{s}(\xi)d\hat{s}(\xi')\partial_{\hat{n}}\rho(\xi)\hat{G}_K(\xi,\xi')\hat{H}_D^{00}(\xi').$$

Also we note that the zero mode integration defines a net charge selection rule for the gravitational sector just like in the closed string. This allow us to ignore the non-local contributions to Eq. (3.27) coming from the zero mode of the kernel $\hat{K}_D$. They will be generated by Eq. (3.22) and by the non-local Weyl anomaly associated with $\hat{G}_K(\xi,\xi')$. We find ( see Appendix )

$$\delta_\rho\hat{G}_K(\xi,\xi') = -\frac{1}{2\oint d\hat{s}(\xi''')}\oint d\hat{s}(\xi'')\rho(\xi'')\left[\hat{G}_K(\xi'',\xi) + \hat{G}_K(\xi'',\xi')\right]. \tag{3.28}$$

This will also contribute when the two points approach each other. In this case we must also include the contribution coming from the regularisation of $\hat{G}_K$ at coincident points. We use the reparametrisation invariant heat kernel

$$\hat{G}_{K\varepsilon}(\xi,\xi') = \int_{\epsilon}^{\infty} dt \left[\hat{\mathcal{G}}_K(t,\xi,\xi') - \frac{1}{\oint d\hat{s}(\xi'')}\right]$$

where $\hat{\mathcal{G}}_K$ satisfies the generalised heat equation

$$-\frac{\partial}{\partial t}\hat{\mathcal{G}}_K(t,\xi,\xi') = \oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\hat{\mathcal{G}}_K(t,\xi'',\xi'), \quad \hat{\mathcal{G}}_K(0,\xi,\xi') = \hat{\delta}_B(\xi-\xi').$$

For coincident arguments the regularisation of the Green's function is controlled by the small-$t$ behaviour of the heat kernel which is computable in a standard perturbation series [33]. This thus leads to ( see Appendix )

$$\delta_\rho \hat{G}_{K\varepsilon}(\xi,\xi) = 4\rho(\xi) - \frac{1}{\oint d\hat{s}(\xi'')}\oint d\hat{s}(\xi')\rho(\xi')\hat{G}_K(\xi',\xi). \tag{3.29}$$

All these non-local contributions always decouple one of the variables, so they will generate terms in Eq. (3.27) which will all be proportional to the net charge on the whole surface.

To eliminate remaining terms between Eqs. (3.14), (3.15) and (3.27) we need $Q_B = Q$. If we finally tune the background cosmological counterterm contributions to zero we get a Weyl invariant lowest order partition function. This shows that we need to include the counterterm in the geodesic curvature because otherwise the finite contribution coming from the reparametrisation ghosts cannot be eliminated. Of course in this particular lowest order case we have a null contribution to the partition function because the net charge, $\oint d\hat{s}(\xi)\hat{H}_D^{00}(\xi)$, is the topological background gravity charge which for the disc is never zero due to the Gauss-Bonnet theorem. However all the terms we have discussed will persist in the more complicated expressions that satisfy the charge selection rule.

This analysis still leaves the parameter $Q_0$ undetermined. To find it we make the connection with the closed string partition function. As explained earlier this is obtained by identifying the arguments of two copies of $Z[Y,\psi,\eta]$ and integrating over these boundary values. This corresponds to gluing together two discs along their boundaries to produce a sphere. The closed string partition function is

$$Z_{\text{closed}} = \int \mathcal{D}_{\bar{g}}(Y,\psi,\eta) Z^1_{\text{open}}[Y,\psi,\eta] Z^2_{\text{open}}[Y,\psi,\eta].$$

When we integrate the string field $X^\mu$ and the reparametrisation ghosts in each open string wave functional we find

$$Z_{\text{closed}} = \int \mathcal{D}_{\bar{g}}(\psi,\varphi_1,\varphi_2) \exp\{-S_L[\varphi_1,\hat{g}_1] - S_L[\varphi_2,\hat{g}_2]\}$$

$$\times \left(\text{Det}'\hat{\Delta}_1\right)^{-d/2} \frac{\sqrt{\text{Det}'\hat{P}_1^\dagger \hat{P}_1}}{\text{Vol}_1(\text{CKV})} \left(\text{Det}'\hat{\Delta}_2\right)^{-d/2} \frac{\sqrt{\text{Det}'\hat{P}_2^\dagger \hat{P}_2}}{\text{Vol}_2(\text{CKV})}.$$

Here the boundary fields $Y^\mu$, $\eta$ have already been integrated and absorbed in the length renormalisation counterterm. The next step is to perturb in each area renormalisation counterterm and in the common length cosmological constant. Just like before we split each field $\varphi_i$, $i = 1, 2$ in two independent fields $\varphi_{ci}$, $\bar{\varphi}_i$. In the present case we only need to consider the lowest order in the perturbative expansion. Then we have the following decomposition

$$Z_{\text{closed}} = Z_B^{00} \bar{Z}^1_{\text{open}} \bar{Z}^2_{\text{open}}$$

$$\times \left(\text{Det}'\hat{\Delta}_1\right)^{-d/2} \frac{\sqrt{\text{Det}'\hat{P}_1^\dagger \hat{P}_1}}{\text{Vol}_1(\text{CKV})} \left(\text{Det}'\hat{\Delta}_2\right)^{-d/2} \frac{\sqrt{\text{Det}'\hat{P}_2^\dagger \hat{P}_2}}{\text{Vol}_2(\text{CKV})}$$

where the boundary partition function is

$$Z_B^{00} = \int \mathcal{D}_{\bar{g}}(\psi,\psi_0) \exp\{-S_B^{00}[\psi,\psi_0,\hat{g}]\}.$$

Above we have used the simple property that the outward normal derivative of one of the open surfaces is just the inward normal derivative of the other at the common boundary, plus the Gauss-Bonnet theorems given in Eqs. (2.2) and (3.2) to find the boundary action

$$S_B^{00}(\bar{\psi},\psi_0,\hat{g}) = -\frac{d-26}{6} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\psi}(\xi) \hat{K}_D(\xi,\xi') \bar{\psi}(\xi') - \frac{d-26}{12} \chi_c \psi_0.$$

Next we renormalise the fields and their couplings to 2D gravity to consider canonical measures in the background $\hat{g}_{ab}$. When we integrate $\bar{\phi}_i$ we get the same Weyl anomaly for each field and again using the property of the normal derivative at the common boundary we can easily see that the boundary contributions cancel up to the usual length renormalisation counterterm, leading to $Q_i = Q$, $i = 1, 2$, where the $Q_i$ define the renormalisation of the coupling of the $\varphi_i$ to the scalar curvature $\hat{R}_i$. The boundary integration is just equal to the zero mode charge selection rule. If $Q_0 = Q$ that is exactly the selection rule we get for the closed string.

### 3.2.2 Anomaly cancellation for Liouville field renormalisation

So far we have only been able to determine parameters associated with the renormalisation of the couplings to 2D quantum gravity. To go further and calculate the Liouville field renormalisation we need to consider higher orders in the Coulomb gas perturbative expansion. In the case of the couplings we have seen that the renormalised central charge of the conformally extended Liouville field theory is exactly the same as the corresponding central charge of the same theory on a closed surface. We have also proved that the boundary couplings are fixed by this value of the central charge. We have seen that this is all a consequence of the quantum Weyl invariance of the theory. Now we want to find out if the bulk field renormalisation is equal to the corresponding closed string parameter and if the boundary field renormalisation is actually the same as its bulk counterpart as it should happen when we interpret the Liouville field as an arbitrary Weyl scaling defined everywhere on the surface including its boundary. As Symanzik's work makes it clear, this is not something we should take for granted. We will now show that this is also a consequence of the quantum Weyl invariance assumed for the theory.

We start with the case where we have a single Liouville vertex operator on the bulk

$$\int d^2\xi \sqrt{\hat{g}} e^{\alpha_0 \Psi_0 + \alpha \bar{\phi} + \bar{\alpha}_B \bar{\phi}_c}. \tag{3.30}$$

In this case we find the following action for $\bar{\phi}$

$$S_L^1\left[\bar{\phi},\hat{g}\right]=\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\bar{\phi}\hat{\Delta}\bar{\phi}+\hat{J}_Q^1\bar{\phi}\right)$$

where we need the current

$$\hat{J}_Q^1(\xi)=Q\hat{R}(\xi)-8\pi\alpha\frac{\delta^2(\xi-\xi')}{\sqrt{\hat{g}(\xi)}}.$$

By shifting $\bar{\phi}$ we generate the functional

$$\hat{\mathcal{F}}_D^1[\hat{g}]=\hat{\mathcal{F}}_D^0[\hat{g}]-\alpha Q\int d^2\xi\sqrt{\hat{g}(\xi)}\hat{R}(\xi)\hat{G}_D(\xi,\xi')+4\pi\alpha^2\hat{G}_D(\xi',\xi'). \tag{3.31}$$

On the other hand we also find the following renormalised boundary action

$$S_c^{10}[\bar{\Psi},\Psi_0,\hat{g}]=\frac{1}{2}\oint d\hat{s}(\xi)d\hat{s}(\xi')\bar{\Psi}(\xi)\hat{K}_D(\xi,\xi')\bar{\Psi}(\xi')+\oint d\hat{s}\hat{H}_D^{10}\bar{\Psi}$$
$$+\left(\frac{Q_0\chi_o}{2}-\alpha_0\right)\Psi_0$$

where we have introduced the current

$$\hat{H}_D^{10}(\xi)=\hat{H}_D^{00}(\xi)+\bar{\alpha}_B\partial_{\hat{n}}\hat{G}_D(\xi,\xi').$$

In this case we get

$$\hat{\mathcal{F}}_B^{10}=\hat{\mathcal{F}}_B^{00}+\bar{\alpha}_B\oint d\hat{s}(\xi)d\hat{s}(\xi'')\hat{H}_D^{00}(\xi)\hat{G}_K(\xi,\xi'')\partial_{\hat{n}''}\hat{G}_D(\xi'',\xi')$$
$$+\frac{1}{2}\bar{\alpha}_B^2\oint d\hat{s}(\xi)d\hat{s}(\xi'')\partial_{\hat{n}}\hat{G}_D(\xi,\xi')\hat{G}_K(\xi,\xi'')\partial_{\hat{n}''}\hat{G}_D(\xi'',\xi'). \tag{3.32}$$

To analyse the anomaly cancellation in this order of the perturbative expansion we first recall that although $\hat{G}_D(\xi,\xi')$ is Weyl invariant for distinct values of its arguments, at coincident points it requires regularisation which introduces dependence on the scale of the metric. To calculate the correspondent Weyl transformation we represent $\hat{G}_D(\xi,\xi')$ in terms of the Green's function $\hat{G}(\xi,\xi')$ considered on the whole plane

$$\hat{G}_D(\xi,\xi')=\hat{G}(\xi,\xi')-\hat{H}_D(\xi,\xi'),$$

where $\hat{H}_D(\xi,\xi')$ satisfies the boundary-value problem

$$\hat{\Delta}\hat{H}_D(\xi,\xi')=0,\quad \hat{H}_D(\xi,\xi')|_{\xi'\in B}=\hat{G}(\xi,\xi')|_{\xi'\in B}.$$

When $\xi=\xi'$ is on the bulk $\hat{H}_D(\xi,\xi)$ is Weyl invariant. Also on the whole plane there is no zero mode. Thus the Weyl transformation of $\hat{G}_D(\xi,\xi)$ is just given by the corresponding well known local change of $\hat{G}(\xi,\xi)$ [3, 24, 25, 26, 27, 33]

$$\delta_\rho \hat{G}_{D\epsilon}(\xi,\xi)=\frac{\rho(\xi)}{4\pi},\quad \xi\notin B. \tag{3.33}$$

Then applying Eqs. (3.13), (3.33) we conclude that the Weyl anomaly of Eq. (3.31) is given by

$$\delta_\rho\hat{\mathcal{F}}_D^1=\delta_\rho\hat{\mathcal{F}}_D^0-\alpha Q\oint d\hat{s}(\xi)\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi')+(\alpha^2-\alpha Q)\rho(\xi').$$

On the other hand, ignoring the non-local zero mode contributions which are all proportional to the net charge on the whole surface given in this order by $\oint d\hat{s}\hat{H}_D^{10}$, we use Eq. (3.26) and $\bar{Q}_B=2Q_B$ to find the Weyl anomaly of Eq. (3.32)

$$\delta_\rho\hat{\mathcal{F}}_D^{10}=\delta_\rho\hat{\mathcal{F}}_D^{00}+\bar{\alpha}_B Q_B\oint d\hat{s}(\xi)\rho(\xi)\partial_{\hat{n}}\hat{G}_D(\xi,\xi').$$

Thus we can easily see that to ensure Weyl invariance at the quantum level we must further set $Q_B=Q$, $\bar{\alpha}_B=\alpha$ and

$$1-\alpha Q+\alpha^2=0.$$

Here we took into account the contribution to the Weyl anomaly of the $\sqrt{\hat{g}}$ present in Eq. (3.30). Introducing the value of $Q$ we find

$$\alpha_\pm=\frac{1}{2\sqrt{6}}\left(\sqrt{25-d}\pm\sqrt{1-d}\right).$$

As we noted previously, these renormalised parameters only cancel the local contributions to the Weyl anomaly. As in the lowest order case we have to assume the charge selection rule associated with the zero mode integration to eliminate the non-local pieces. To find the renormalised parameters of the charge selection rule we need to glue the two discs to form a sphere enabling us to use the closed string result. We already know the value of $Q_0$ but now we also want the value of $\alpha_0$. The calculation goes exactly as before, all boundary contributions cancel out up to the length renormalisation counterterm and we find a zero mode integral which corresponds to a closed string selection rule with two bulk vertex operator charges $\alpha_0$ and a background gravity charge $Q_0 = Q$. This implies that $\alpha_0 = \alpha$ as expected.

With this calculation we are able to guarantee Weyl invariance at the quantum level for insertions of arbitrary numbers of gravitational Liouville vertex operators in the bulk. To see what happens when operators are inserted on the boundary let us consider the simplest case of just one such operator,

$$\oint d\hat{s} e^{\alpha_0 \Psi_0/2 + \alpha_B \bar{\Psi}/2}. \tag{3.34}$$

In this case only the boundary integration over $\psi$ gets changed. The renormalised boundary Liouville action is

$$S_c^{01}[\bar{\Psi}, \Psi_0, \hat{g}] = \frac{1}{2} \oint d\hat{s}(\xi) d\hat{s}(\xi') \bar{\Psi}(\xi) \hat{K}_D(\xi, \xi') \bar{\Psi}(\xi') + \oint d\hat{s} \hat{H}_D^{01} \bar{\Psi}$$
$$+ \left( \frac{Q_0 \chi_o}{2} - \frac{\alpha_0}{2} \right) \Psi_0$$

where we have introduced the current

$$\hat{H}_D^{01}(\xi) = \hat{H}_D^{00}(\xi) - \frac{\alpha_B}{2} \hat{\delta}_B(\xi - \xi').$$

The relevant functional is now

$$\hat{\mathcal{F}}_B^{01} = \hat{\mathcal{F}}_B^{00} - \frac{\alpha_B}{2} \oint d\hat{s}(\xi) \hat{H}_D^{00}(\xi) \hat{G}_K(\xi, \xi') + \frac{\alpha_B^2}{8} \hat{G}_K(\xi', \xi').$$

To find out our last renormalised parameter $\alpha_B$ we need the local Weyl transformation of $\hat{G}_K$ at coincident points given in Eq. (3.29). Thus the local anomaly vanish if all the other parameters keep their previous values and $1/2 - \alpha_B Q/2 + \alpha_B^2/2 = 0$, where the 1/2 term comes from the Weyl transformation of $d\hat{s}$ in Eq. (3.34). Thus $\alpha_B = \alpha$.

Since as before the non-local contributions cancel due to the charge selection rule this result shows that the full perturbative expansion is Weyl invariant at the quantum level for the values of the renormalised parameters found. Whenever we couple distinct Liouville vertex operators in higher orders there are no additional Weyl anomalous contributions.

### 3.2.3 Comments

Our results for the non-critical open string show that the gravitational sector can be interpreted as a conformally extended boundary Liouville field theory. In this picture $Q$ defines the central charge of the Liouville theory $c_\phi = 1 + 6Q^2$, which has its value fixed by demanding that it cancels the central charges of the matter and ghost systems $c_M + c_{gh} = d - 26$. Thus the central charge of the theory with boundary is equal to the central charge of the theory without boundary. This is to be expected since anomalies are local effects.

We have interpreted the Liouville field as an arbitrary Weyl scaling all over the open surfaces. Then we found that the value of $\alpha$ is exactly right to define a Liouville vertex operator $\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}$ of zero conformal weight. On the extended field theory it corresponds to a primary field $: e^{\alpha\phi} :$ of weight $(1,1)$. As expected $\alpha$ has the same value it takes when the surfaces are closed. We also found the right value for $\alpha_B$ in the sense that the boundary vertex operator $\oint d\hat{s}e^{\alpha_B\phi/2}$ has zero conformal weight corresponding to the boundary primary field $: e^{\alpha_B\phi/2} :$ of conformal weight $(1/2, 1/2)$. This means that the renormalisation of the Liouville field is the same all over the surface and is equal to the renormalisation on the closed surface as it should be.

As for the closed string we also find the need to restrict the validity of the approach to target space dimensions $d \leq 1$. Only in this way we have real renormalised parameters such that $e^{\alpha\phi}$ and $e^{\alpha_B\phi/2}$ can be interpreted as real Weyl scalings for a real scalar renormalised Liouville field $\phi$. From this we can see that our results extend very naturally those found for the closed string by David, Distler and Kawai. Since the analysis is fully local and we can choose the moduli integration measure to be independent of the conformal factor of the metric our results also generalise immediately to higher genus Riemann surfaces with just one boundary. Clearly more general boundary structures can also be considered. Here for simplicity we have just analysed the random surfaces one loop functional defined in Euclidean space [34, 41, 42]. Our results hold for an arbitrary number of loops. We also may consider non-smooth boundaries [33].

For the non-critical open string we also have the problem of extending to non-integer $s$ the perturbative Coulomb gas selection rule. Just as in the closed string we may also integrate the zero mode explicitly. For the partition function the renormalised zero mode integral will now be given by

$$I_0 = \int_{-\infty}^{+\infty} \frac{d\Psi_0}{\alpha_0} e^{-s\Psi_0 - \mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\Psi_o + \alpha\bar{\phi} + \bar{\alpha}_B \bar{\phi}_c} - \lambda_2 \oint d\hat{s} e^{\Psi_0/2 + \alpha_B \bar{\Psi}/2}},$$

where $s$ is determined by the equation

$$\frac{Q_0 \chi_o}{2} = \alpha_0 s\,.$$

The integration can also be performed exactly if we use the Mellin transform [43]. We then find

$$I_0 = \frac{2^{s+1}}{\alpha_0}\Big(\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}+\bar{\alpha}_B\bar{\phi}_c}\Big)^s \Gamma(-2s) \exp\left[\frac{\lambda_2^2\big(\oint d\hat{s} e^{\alpha_B \bar{\Psi}/2}\big)^2}{8\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}+\bar{\alpha}_B\bar{\phi}_c}}\right]$$

$$\times D_{2s}\left[\frac{\lambda_2 \oint d\hat{s} e^{\alpha_B\bar{\Psi}/2}}{\sqrt{2}\mu_2\big(\int d^2\xi\sqrt{\hat{g}} e^{\alpha\bar{\phi}+\bar{\alpha}_B\bar{\phi}_c}\big)^{1/2}}\right],$$

where $D_{2s}(z)$ are the parabolic cylinder functions [43]. For integer $2s$ we find

$$\frac{2}{\alpha_0}\Big(\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}+\bar{\alpha}_B\bar{\phi}_c}\Big)^s \Gamma(-2s) H_{2s}\left[\frac{\lambda_2 \oint d\hat{s} e^{\alpha_B\bar{\Psi}/2}}{2\mu_2\big(\int d^2\xi\sqrt{\hat{g}} e^{\alpha\bar{\phi}+\bar{\alpha}_B\bar{\phi}_c}\big)^{1/2}}\right],$$

where $H_{2s}(z)$ is the Hermite polynomial of degree $2s$. If $s$ is also taken as an integer we can apply our approach and find the same values for the renormalised parameters of the theory. Now the charge selection rule is automatic and so all non-local contributions to the Weyl anomaly are cancelled out. The partition function is then Weyl invariant at the quantum level and to proceed an analytic continuation on $s$ may now eventually be considered.

### 3.2.4 Tachyon gravitational dressings

Since this formalism is only valid for $d \leq 1$ the open string serves as a toy model for the more realistic $c \leq 1$ minimal series of boundary conformal field theories [17]. With this in mind let us see how bulk and boundary vertex operators get dressed by the gravitational sector.

We start by taking an $n$-point function of bulk tachyons with momentum $p_j$. This is given by Eq. (3.1) where the vertex operators are $\int d^2\xi_j \sqrt{\tilde{g}} e^{ip_j \cdot X}$. Now the matter action is

$$S_j[X, \tilde{g}] = \frac{1}{16\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_a X \cdot \partial_b X - i \sum_j p_j \cdot X(\xi_j) \,.$$

Shifting the field as we did for the partition function gives

$$S_j[X, \tilde{g}] = S_{cj}[X_c, \tilde{g}] + \bar{S}_j[\bar{X}, \tilde{g}] \,.$$

The action for $\bar{X}^\mu$ has the usual kinetic term with the covariant Laplacian plus the linear vertex operator current

$$\frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \tilde{J}_j \cdot \bar{X} \,,$$

where

$$\tilde{J}^\mu_j(\xi, \xi_j) = -8\pi i \sum_j p^\mu_j \frac{\delta^2(\xi - \xi_j)}{\sqrt{\tilde{g}(\xi)}} \,.$$

The boundary action $S_{cj}[X_c, \tilde{G}]$ is now an action for the orthogonal piece $\bar{Y}^\mu$ because of the momentum conservation law associated with the integral over the zero mode $Y_0$

$$\sum_j p_j^\mu = 0 .$$

We find

$$S_{cj}[X_c, \tilde{g}] = \frac{1}{2} \oint d\tilde{s}(\xi) d\tilde{s}(\xi') \bar{Y} \cdot \tilde{K}_D(\xi, \xi') \bar{Y}(\xi') + i \sum_j p_j \cdot \oint d\tilde{s}(\xi) \partial_{\tilde{n}} \tilde{G}_D(\xi, \xi_j) \bar{Y}(\xi) .$$

When we integrate $\bar{X}^\mu$ and the reparametrisation ghosts we get the usual Liouville action supplemented by the exponential of the following vertex operator functional

$$\begin{aligned} \tilde{\mathcal{F}}_{Djj'}[\tilde{g}] &= \frac{1}{16\pi} \int d^2\xi d^2\xi' \sqrt{\tilde{g}(\xi)} \tilde{J}_j(\xi) \tilde{G}_D(\xi, \xi') \sqrt{\tilde{g}(\xi')} \tilde{J}_{j'}(\xi') \\ &= -4\pi \sum_{jj'} p_j \cdot p_{j'} \tilde{G}_D(\xi_j, \xi_{j'}) . \end{aligned}$$

For $j \neq j'$ this is independent of the scale of the metric because of the conservation of momentum. For $j = j'$ the same condition plus the local Weyl anomaly of the Dirichlet Green's function allow us to write

$$\tilde{\mathcal{F}}_{Dj} = -\sum_j p_j^2 \varphi(\xi_j) .$$

On the other hand the integration over $\bar{Y}^\mu$ is to be absorbed in the length renormalisation counterterm because the corresponding current leads to a Weyl invariant contribution provided the conservation of momentum is operational.

So the bulk tachyon vertex operator gets dressed to

$$\int d^2\xi_j \sqrt{\hat{g}_j} e^{\gamma_{0j} \Psi_0 + \bar{\gamma}_j \bar{\phi}_c + \gamma_j \bar{\phi}} e^{i p_j \cdot X} ,$$

where quantum Weyl invariance demands that $\gamma_{0j} = \bar{\gamma}_j = \gamma_j$ and

$$\Delta_j^0 - \gamma_j (\gamma_j - Q) = 1 , \quad \Delta_j^0 = p_j^2 .$$

This equation shows that the dressed bulk tachyon vertex operator has zero conformal weight. The primary Liouville field $:e^{\gamma_j \phi}:$ dresses the tachyon field $:e^{ip_j \cdot X}:$ in such a way that $:e^{\gamma_j \phi} e^{ip_j \cdot X}:$ has conformal weight (1,1). Just as in the closed string, the solution for $\gamma_j$ in terms of $Q$ and $p_j^2$ is

$$\gamma_j = \frac{1}{2}\left[Q \pm \sqrt{Q^2 + 4(p_j^2 - 1)}\right] .$$

Just as $\alpha$ should be real for an arbitrary Weyl scaling so should be $\gamma_j$. This implies that $d \leq 1$ and $p_j^2 \geq (d-1)/24$.

Let us now consider the boundary tachyon vertex operator $\oint d\tilde{s}_j e^{ip_j \cdot X/2}$. Because the points $\xi_j$ are all in the boundary only the boundary action is going to be changed. Separating $\bar{X}^\mu$ and $X_c^\mu$ , and taking into account the momentum conservation we find

$$S_j[X_c, \tilde{g}] = \frac{1}{2}\oint d\tilde{s}(\xi) d\tilde{s}(\xi') \bar{Y} \cdot \tilde{K}_D(\xi, \xi') \bar{Y}(\xi') + \oint d\tilde{s} \tilde{H}_{Dj} \cdot \bar{Y} ,$$

where we have the current

$$\tilde{H}_{Dj}^\mu(\xi, \xi_j) = -\frac{i}{2}\sum_j p_j^\mu \tilde{\delta}_B(\xi - \xi_j) .$$

This current is to be shifted and then leads to the exponential of the following functional

$$\tilde{\mathcal{F}}_{Bjj'}[\tilde{g}] = -\frac{1}{8}\sum_{jj'} p_j \cdot p_{j'} \tilde{G}_K(\xi_j, \xi_{j'}) .$$

Due to the conservation of momentum this only contributes to the Liouville action when the points coincide

$$\tilde{\mathcal{F}}_{Bj} = -\frac{1}{2}\sum_j p_j^2 \varphi(\xi_j) .$$

So we have the dressed operator

$$\oint d\hat{s}_j e^{\gamma_{0j}\Psi_0/2 + \gamma_{Bj}\bar{\Psi}/2} e^{ip_j \cdot X/2} ,$$

where quantum Weyl invariance gives $\gamma_{Bj} = \gamma_j$. It means that the dressed field $:e^{\gamma_j \phi/2} e^{ip_j \cdot X/2}:$ considered in the boundary has conformal weight (1/2,1/2).

As expected these results parallel those of the closed string. If we extend the tachyon discussion to other vertex operators we also find that the conformal weight is equal to the symmetric of the critical invariant square mass of the open string states. For $d = 26$ the Liouville mode decouples and we find the critical string mass spectrum as given by the symmetric of the bare conformal weight. The result is the same as that contained in the no-ghost theorem.

## 3.3 Dirichlet boundary conditions

The analysis we have presented for the case of free boundary conditions puts us in a position where we can discuss Dirichlet boundary conditions on the Liouville field. We will find that if we do so the metric develops a discontinuity as the boundary is approached.

When we impose Dirichlet boundary conditions on the Liouville field the calculation follows the case of the free boundary conditions and stops at the renormalised lowest order boundary action $S_c^{00}$ given in Eq. (3.20) because we do not integrate over the boundary values of the Liouville field but leave them fixed. The Weyl anomaly of Eqs. (3.14) and (3.15) must now be cancelled by the Weyl transformation of $S_c^{00}$, together with a shift in the boundary value of the Liouville field, $\Psi$.

This simultaneous Weyl transformation on $\hat{g}_{ab}$ and shift in $\Psi$ is to be understood as follows

$$S_c^{00}[\Psi + \delta\Psi, \hat{g} + \delta_\rho \hat{g}] - S_c^{00}[\Psi, \hat{g}] = \delta_\Psi S_c^{00} + \delta_\rho S_c^{00} \,.$$

Now taking into account Eq. (3.26) we find

$$\delta_\Psi S_c^{00} = \oint d\hat{s}(\xi) d\hat{s}(\xi') \delta\Psi(\xi) \hat{K}_D(\xi, \xi') \Psi(\xi') + \oint d\hat{s} \hat{H}_D^{00} \delta\Psi \,,$$

$$\delta_\rho S_c^{00} = Q_B \oint d\hat{s}(\xi) d\hat{s}(\xi') \rho(\xi) \hat{K}_D(\xi, \xi') \Psi(\xi') \,.$$

So, with $\bar{Q}_B = 2Q_B$ and $Q_B = Q$ the anomaly is cancelled if we fix the shift in $\Psi$ to be $\delta\Psi = -Q\rho$.

Now the full metric is a reparametrisation of $e^{\alpha\Psi}\hat{g}_{ab}$, which should be invariant under this simultaneous Weyl transformation on $\hat{g}_{ab}$ and shift in $\Psi$, since the separation into reference metric and Liouville field is arbitrary. However, it is not because $Q \neq 1/\alpha$, as the correct relation, $1-\alpha Q+\alpha^2 = 0$, has an extra quantum piece. One way out of this would be to assume that the Liouville field is renormalised differently in the bulk and on the boundary, a phenomenon that occurs in $\phi^4$ theory in four dimensions [40]. However, this implies that the metric is discontinuous as the boundary is approached, and also that the functionals obtained by imposing Dirichlet boundary conditions cannot be sewn together to make closed surface functionals.

## 3.4 Neumann boundary conditions

The choice of boundary conditions always depends on the specific physical applications we have in mind. So far we have argued that in a proper coupling to 2D quantum gravity, the boundary conditions on the Liouville field have to be such that it can be interpreted as an arbitrary Weyl scaling on the whole surface and not just on its interior. As we said this rules out Dirichlet boundary conditions but we are free to choose Neumann boundary conditions for the conformal factor. To see what happens in this case let us for simplicity take also Neumann boundary conditions on the matter field $\partial_{\tilde{n}}X^\mu = 0$ and on the reparametrisation ghosts $\tilde{n}\cdot\delta\theta = 0$. Consider first the partition function. We can then follow the same reasoning as in the case of free boundary conditions with much more ease because the Neumann boundary condition simply eliminates the most part of the boundary contributions we had to worry about before. Then the standard calculation of the conformal anomaly on the disc based on the Neumann heat kernel leads to the Liouville action [38, 39, 33]

$$S_L^N[\varphi,\hat{g}] = -\frac{d-26}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\varphi\hat{\Delta}\varphi + \hat{R}\varphi\right) - \frac{d-26}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi$$

$$+\mu_1^2 \int d^2\xi \sqrt{\hat{g}} e^{\varphi} + \lambda_1 \oint d\hat{s} e^{\varphi/2}.$$

To integrate the Liouville mode we again apply the DDK renormalisation ansätse and introduce the canonical measure in the background $\hat{g}_{ab}$. The correspondent renormalised Liouville action is

$$S_L^N[\phi,\hat{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi\right) + \frac{\bar{Q}_B}{8\pi} \oint d\hat{s} k_{\hat{g}} \phi$$
$$+\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} + \lambda_2 \oint d\hat{s} e^{\alpha_B \phi/2}.$$

Here $Q$, $\bar{Q}_B$ refer to coupling renormalisation and $\alpha$, $\alpha_B$ are its field renormalisation counterparts.

Next we perturb on the exponentials of the renormalised cosmological constant area and length counterterms. First we factor out the contribution from the zero mode of the covariant Laplacian. This leads to our usual charge conservation selection rule

$$\int d^2\xi \sqrt{\hat{g}} \hat{J}_N^{MM'} = 0$$

for the non-zero contributions to the amplitude. Here the $(M + M' + 1)^{th}$ order Liouville gravity current is given by

$$\hat{J}_N^{MM'} = Q\hat{R} + \bar{Q}_B k_{\hat{g}} \hat{\delta}_B^2 - 8\pi\alpha \sum_{P=1}^{M} \delta^2(\xi - \xi_P)/\sqrt{\hat{g}(\xi)} - 4\pi\alpha_B \sum_{P=1}^{M'} \hat{\delta}_B(\xi - \xi_P),$$

where $\int d^2\xi \sqrt{\hat{g}} k_{\hat{g}} \hat{\delta}_B^2 = \oint d\hat{s} k_{\hat{g}}$. Once more this can be considered as an automatic charge selection rule when the zero mode is explicitly integrated without expanding the Liouville exponentials interactions.

Then in any order of the perturbative expansion we only need to shift the linear pieces associated with the Liouville gravity current. We introduce the Neumann Green's function in the reference metric to do the job

$$\hat{\Delta}\hat{G}_N(\xi,\xi') = \frac{\delta^2(\xi - \xi')}{\sqrt{\hat{g}(\xi)}} - \frac{1}{\int d^2\xi'' \sqrt{\hat{g}(\xi'')}}, \quad \partial_{\hat{n}} \hat{G}_N(\xi,\xi') = 0.$$

Here the Green's function is symmetric in its arguments and is orthogonal to the zero mode

$$\int d^2\xi \sqrt{\hat{g}(\xi)}\hat{G}_N(\xi,\xi') = 0\,.$$

Thus we find the non-local functional

$$\hat{\mathcal{F}}_N^{MM'} = \frac{1}{16\pi}\int d^2\xi' d^2\xi'' \sqrt{\hat{g}(\xi')}\hat{J}_N^{MM'}(\xi')\hat{G}_N(\xi',\xi'')\sqrt{\hat{g}(\xi'')}\hat{J}_N^{MM'}(\xi'')\,.$$

Considering the example of just one bulk Liouville vertex operator we get the infinitesimal Weyl transformation

$$\delta_\rho \hat{\mathcal{F}}_N^{10} = \frac{Q}{8\pi}\int d^2\xi \sqrt{\hat{g}}\hat{J}_N^{10}\rho + \alpha^2\rho(\xi_1) - \frac{Q}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{J}_N^{10}\delta_\rho \ln\int d^2\xi\sqrt{\hat{g}}$$
$$-\frac{1}{8\pi\int d^2\xi\sqrt{\hat{g}(\xi)}}\int d^2\xi\sqrt{\hat{g}}\hat{J}_N^{10}\int d^2\xi' d^2\xi''\sqrt{\hat{g}(\xi')}\rho(\xi')\hat{G}_N(\xi',\xi'')\sqrt{\hat{g}(\xi'')}\hat{J}_N^{10}(\xi'')\,.$$

Here we have already chosen $\bar{Q}_B = 2Q$. As in the case of free boundary conditions this eliminates the terms in $\partial_{\hat{n}}\rho$. The $\alpha^2$ term comes from the Weyl change of the Neumann Green's function at equal points on the bulk. We calculate it from the Green's function on the whole plane

$$\hat{G}_N(\xi,\xi') = \hat{G}(\xi,\xi') + \hat{H}_N(\xi,\xi')$$

where $\hat{H}_N(\xi,\xi')$ is defined by

$$\hat{\Delta}\hat{H}_N(\xi,\xi') = -\frac{1}{\int d^2\xi''\sqrt{\hat{g}(\xi'')}},\quad \partial_{\hat{n}}\hat{H}_N(\xi,\xi') = -\partial_{\hat{n}}\hat{G}(\xi,\xi').$$

In the bulk the variation of $\hat{H}_N(\xi,\xi)$ is due to the constant zero-mode. Then the local anomaly of the Neumann Green's function is $\delta_\rho\hat{G}_{N\varepsilon}(\xi,\xi) = \rho(\xi)/(4\pi)$ (plus a piece due to the zero-mode which ultimately decouples using the 'selection rule') thus leading to the above result. Hence the local contributions to the conformal anomaly are cancelled if we tune the local reference counterterms to zero and set

$$Q = \pm\sqrt{\frac{25-d}{6}},\quad 1-\alpha Q+\alpha^2 = 0.$$

Now let us consider just one boundary Liouville vertex operator. In this case we only need to put $\alpha_B = \alpha$ to ensure local Weyl invariance. This is a consequence of the local conformal change of the Neumann Green's function at equal points on the boundary. When $\xi = \xi'$ is on the boundary, $\hat{H}_N(\xi, \xi)$ is divergent because $\hat{G}(\xi, \xi)$ is singular. This short distance effect needs to be regularised and so introduces a local dependence on the scale of the reference metric. The result is an additional contribution to the Weyl anomaly of $\hat{G}_N(\xi, \xi)$ to which we must sum that coming from $\hat{G}(\xi, \xi)$. To find it we only need an explicit formula for $\hat{H}_N(\xi, \xi)$ that is valid in a neighbourhood of order $\sqrt{\varepsilon}$ around $\xi$. In this region the shape of the boundary is flat and we can take the problem in the upper-half plane. Then $\hat{G}_N(\xi, \xi)$ is defined by the method of images meaning that $\hat{H}_N(\xi, \xi) = \hat{G}(\xi, \xi)$. We then sum both closed string contributions to get $\delta_\rho \hat{G}_{N\varepsilon}(\xi, \xi) = \rho(\xi)/(2\pi)$ which leads to $\alpha_B = \alpha$.

Like before the non-local contributions to the Weyl anomaly which are present due to the Laplacian's zero mode are not eliminated by these assignments to the renormalised parameters. As in the case of the closed string they are all proportional to the net charge on the Riemann surface. Then Weyl invariance is ensured by the charge conservation condition which sets to zero any term which does not balance the charges on the surface. Also other orders in perturbation theory do not bring any new Weyl anomalous contributions.

Starting from a general open string bulk tachyon amplitude it is clear that we may follow the steps of the partition function calculation to find the equation for the gravitational dressing of the bulk tachyon vertex operator. A tachyon vertex operator with momentum $p_j$ gets dressed by the coupling to 2D quantum gravity

$$\int d^2\xi_j \sqrt{\hat{g}} e^{\gamma_j \phi} e^{ip_j \cdot X} ,$$

where $\Delta_j^0 - \gamma_j(\gamma_j - Q) = 1$, $\Delta_j^0 = p_j^2$. For the boundary tachyon vertex operator the coupling to gravity leads to the dressed operator

$$\oint d\hat{s}_j e^{\gamma_j \phi/2 + ip_j \cdot X/2}$$

of zero weight, where due to the effect of the Neumann Green's function on the boundary $\gamma_j$ satisfies the same equation as its bulk counterpart.

Thus so far we conclude that our results are exactly the same for Neumann and for free boundary conditions. This agrees with Jaskólski analysis of the Neumann case [44] and, apparently, supports the conjectured equivalence of free and Neumann boundary conditions put forward by Jaskólski and Meissner [45].

# Appendix. The Weyl anomaly of $\hat{G}_K$ and $\mathrm{Det}'\hat{K}_D$

Let us first calculate the non-local Weyl anomaly associated with the constant zero mode of $\hat{K}_D(\xi,\xi')$. We start by multiplying Eq. (3.22) by $d\hat{s}(\xi)$. Since $d\hat{s}(\xi)d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')$ and $d\hat{s}(\xi)\hat{\delta}_B(\xi-\xi')$ are Weyl invariant we then use $\delta_\rho d\hat{s}(\xi) = (1/2)\rho(\xi)d\hat{s}(\xi)$ to get

$$d\hat{s}(\xi)\oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\delta_\rho\hat{G}_K(\xi'',\xi') = -\frac{\rho(\xi)d\hat{s}(\xi)}{2\oint d\hat{s}(\eta)} + \frac{d\hat{s}(\xi)\oint d\hat{s}(\zeta)\rho(\zeta)}{2[\oint d\hat{s}(\eta)]^2}. \quad (3.35)$$

Next we multiply Eq. (3.35) by $\hat{G}_K(\xi,\xi''')$ and integrate on $\xi$. Using Eqs. (3.22) and (3.24) we find

$$\delta_\rho\hat{G}_K(\xi,\xi') = -\frac{\oint d\hat{s}(\xi'')\hat{G}_K(\xi'',\xi)\rho(\xi'')}{2\oint d\hat{s}(\xi''')} + \frac{\oint d\hat{s}(\xi'')\delta_\rho\hat{G}_K(\xi'',\xi')}{\oint d\hat{s}(\xi''')}$$

Finally the Weyl transformation of Eq. (3.24)

$$\oint d\hat{s}(\xi'')\delta_\rho\hat{G}_K(\xi'',\xi') = -\frac{1}{2}\oint d\hat{s}(\xi'')\rho(\xi'')\hat{G}_K(\xi'',\xi')$$

leads to Eq. (3.28).

At coincident points there is an extra local contribution coming from the regularisation of $\hat{G}_K$. To find it we use the reparametrisation invariant heat kernel associated with $\hat{K}_D(\xi,\xi')$. To construct it we follow Mansfield [27, 33]. In the process we also calculate the determinant of $\hat{K}_D$ and confirm that its anomaly is to be absorbed into the cosmological constant counterterm in the invariant world-sheet length of the boundary. First note that

$$\delta_\rho \ln \mathrm{Det}'\hat{K}_D = \sum_{N\neq 0} \frac{\delta_\rho \hat{\lambda}_N}{\hat{\lambda}_N}.$$

Then from Eq. (3.19) and using orthogonality of the eigenfunctions we find

$$\delta_\rho \hat{\lambda}_N = \oint d\hat{s}(\xi)\hat{v}_N(\xi) \oint \delta_\rho \left[ d\hat{s}(\xi')\hat{K}_D(\xi,\xi')\right] \hat{v}_N(\xi').$$

Now

$$\delta_\rho \left[ d\hat{s}(\xi')\hat{K}_D(\xi,\xi')\right] = -\frac{1}{2}\rho(\xi) d\hat{s}(\xi')\hat{K}_D(\xi,\xi').$$

If we then introduce the regularised integral representation of the eigenvalues

$$\hat{\lambda}_N = \int_\varepsilon^{+\infty} dt e^{-\hat{\lambda}_N t}$$

we get

$$\delta_\rho \ln \mathrm{Det}'\hat{K}_D = -\frac{1}{2}\int_\varepsilon^{+\infty} dt \sum_{N\neq 0} \oint d\hat{s}(\xi)\hat{v}_N(\xi)\rho(\xi) \oint d\hat{s}(\xi')\hat{K}_D(\xi,\xi')$$
$$\times \oint d\hat{s}(\xi'') e^{-t\hat{K}_D(\xi',\xi'')}\hat{v}_N(\xi'').$$

If we integrate in $t$ and then use the definition of the Green's function $\hat{G}_K(\xi,\xi')$ given in Eq. (3.22) we can find a formula which after using Eqs. (3.18), (3.23) and taking into account the zero mode of $\hat{K}_D(\xi,\xi')$ can be written as follows

$$\delta_\rho \ln \left( \frac{\mathrm{Det}'\hat{K}_D}{\oint d\hat{s}} \right) = -\frac{1}{2}\sum_N \oint d\hat{s}(\xi)\hat{v}_N(\xi)\rho(\xi) \oint d\hat{s}(\xi') e^{-\varepsilon \hat{K}_D(\xi,\xi')}\hat{v}_N(\xi').$$

If the heat kernel is defined by

$$\hat{\mathcal{G}}_K(t,\xi,\xi') = e^{-t\hat{K}_D(\xi,\xi')}$$

we can use Eq. (3.17) to find

$$\delta_\rho \ln\left(\frac{\mathrm{Det}'\hat{K}_D}{\oint d\hat{s}}\right) = -\frac{1}{2}\oint d\hat{s}(\xi)\rho(\xi)\hat{\mathcal{G}}_K(\varepsilon,\xi,\xi). \tag{3.36}$$

The heat kernel $\hat{\mathcal{G}}_K$ satisfies the generalised heat equation

$$-\frac{\partial}{\partial t}\hat{\mathcal{G}}_K(t,\xi,\xi') = \oint d\hat{s}(\xi'')\hat{K}_D(\xi,\xi'')\hat{\mathcal{G}}_K(t,\xi'',\xi'), \quad \hat{\mathcal{G}}_K(0,\xi,\xi') = \hat{\delta}_B(\xi-\xi')$$

and it admits the following expansion in terms of the eigenfunctions

$$\hat{\mathcal{G}}_K(t,\xi,\xi') = \sum_N \hat{v}_N(\xi)e^{-t\hat{\lambda}_N}\hat{v}_N(\xi'). \tag{3.37}$$

To calculate the Weyl anomaly in Eq. (3.36) we note that $\varepsilon$, now with the dimensions of length, can be made infinitesimally small. So the boundary heat kernel can only be sensitive to the invariant length of the boundary. Then reparametrisation invariance and dimensional analysis lead to the following expansion in powers of the proper time cutoff $\varepsilon$

$$\delta_\rho \ln\left(\frac{\mathrm{Det}'\hat{K}_D}{\oint d\hat{s}}\right) = \frac{C'}{\varepsilon}\oint d\hat{s}(\xi)\rho(\xi) + O(\varepsilon),$$

where $C'$ is a dimensionless constant. To find $C$ we just need the heat kernel for the upper half-plane $y \geq 0$. In this world-sheet the solutions of the wave equation with boundary values $\cos(kx)$ or $\sin(kx)$ which decay at infinity are $\cos(kx)\exp(-ky)$ or $\sin(kx)\exp(-ky)$ for $k > 0$. Since $\partial_{\hat{n}} = -\partial_y$ we have

$$\partial_{\hat{n}}\begin{pmatrix}\cos(kx)e^{-ky}\\ \sin(kx)e^{-ky}\end{pmatrix}\Big|_{y=0} = k\begin{pmatrix}\cos(kx)\\ \sin(kx)\end{pmatrix}.$$

Thus $\cos(kx)$ and $\sin(kx)$ are eigenfunctions of $\hat{K}_D$ with eigenvalue $k/(8\pi)$. They form a complete and orthonormal set of functions in the $x$-axis. So the boundary heat kernel in the upper half-plane is given by

$$\begin{aligned}\mathcal{G}_K^0(t,x,x') &= 8\int_0^{+\infty} dk\,[\cos(8\pi kx)\cos(8\pi kx') + \sin(8\pi kx)\sin(8\pi kx')]\,e^{-kt} \\ &= \frac{1}{8\pi^2}\frac{t}{[(t/(8\pi))]^2 + (x-x')^2}.\end{aligned}$$

Here we took into account the normalisation imposed on the heat kernel by the delta function condition as $t \to 0$. So we conclude that $C' = -4$.

To calculate the local Weyl anomaly of $\hat{G}_K$ we start by noting that the regularised Green's function can be written as

$$\hat{G}_{K\varepsilon}(\xi,\xi') = \sum_{N\neq 0} \hat{v}_N(\xi)\frac{e^{-\varepsilon\hat{\lambda}_N}}{\hat{\lambda}_N}\hat{v}_N(\xi'),$$

Using the heat kernel as given by Eq. (3.37) we write

$$\hat{G}_{K\varepsilon}(\xi,\xi') = \int_\varepsilon^\infty dt \left[\hat{\mathcal{G}}_K(t,\xi,\xi') - \frac{1}{\oint d\hat{s}(\xi'')}\right].$$

Since we work on the boundary consider the boundary parametrisation $\tau$ such that $\xi = \xi(\tau)$. We can now extend the analysis done for the covariant Laplacian Green's function [33]. Let $|\tau>$ be the eigenket of the boundary coordinate operator $T_j$

$$T_j|\tau> = \tau_j|\tau> .$$

The boundary delta function is then given by

$$<\tau|\tau'> = \delta(\tau - \tau').$$

Since $d\hat{s}(\xi) = d\tau\sqrt{\hat{g}^{ab}\dot{\xi}_a\dot{\xi}_b}$, where $\dot{\xi}_a = d\xi_a/(d\tau)$, we define the metric operator $\hat{\mu}$ satisfying

$$\hat{\mu}|\tau> = \sqrt{\hat{g}^{ab}\dot{\xi}_a\dot{\xi}_b}\,|\tau> .$$

We may now consider the eigenkets $|\hat{N}>$ of $\hat{K}_D\hat{\mu}$ and the eigenbras $<\hat{N}|$ of $\hat{\mu}\hat{K}_D$ correspondent to the eigenvalue $\hat{\lambda}_N$. Using the identity partition for the coordinate base

$$P_\tau = \int d\tau|\tau><\tau|$$

and the wave function $\hat{v}_N = <\tau|\hat{N}>$ it is clear that

$$< \tau|\hat{K}_D\hat{\mu}|\hat{N} >= \oint d\hat{s}(\xi')\hat{K}_D(\xi,\xi')\hat{v}_N(\xi').$$

So we get

$$\hat{K}_D\hat{\mu}|\hat{N} >= \hat{\lambda}_N|\hat{N} > .$$

The eigenket $|\hat{N} >$ is expressed in the coordinate basis as follows

$$|\hat{N} >= \int d\tau \hat{v}_N(\tau)|\tau > .$$

It also satisfies the orthogonality condition

$$< \hat{N}|\hat{\mu}|\hat{M} >= \delta_{NM}$$

and the projector for the eigenket basis is

$$\hat{P} = \sum_N |\hat{N} >< \hat{N}|\hat{\mu}.$$

From this we can write the heat kernel as follows

$$\hat{\mathcal{G}}_K(t,\tau,\tau') =< \tau|e^{-t\hat{K}_D\hat{\mu}}\hat{P}\hat{\mu}^{-1}|\tau' >$$

and so the regularised Green's function is

$$\hat{G}_{K\varepsilon}(\tau,\tau') = \int_\varepsilon^{+\infty} dt \left( < \tau|e^{-t\hat{K}_D\hat{\mu}}\hat{P}\hat{\mu}^{-1}|\tau' > - \frac{1}{\oint d\hat{s}} \right).$$

Because under an infinitesimal Weyl transformation the metric operator changes as $\delta_\rho\hat{\mu} = (1/2)\rho\hat{\mu}$ then we find

$$\delta_\rho(\hat{K}_D\hat{\mu}) = -\frac{1}{2}\rho\hat{K}_D\hat{\mu},$$

where $\rho|\tau >= \rho(\tau)|\tau >$. If we use the interaction picture then it follows that to first order we may write

$$\delta_\rho\left(e^{-t\hat{K}_D\hat{\mu}}\right) = \frac{1}{2}\int_0^t ds e^{-(t-s)\hat{K}_D\hat{\mu}}\rho\hat{K}_D\hat{\mu}e^{-s\hat{K}_D\hat{\mu}}.$$

Differentiating the integration limit and the integrand we find

$$-\frac{d}{dt}\left(\int_0^t ds e^{-(t-s)\hat{K}_D\hat{\mu}}\rho e^{-s\hat{K}_D\hat{\mu}}\hat{\mu}^{-1}\right) = -\rho e^{-t\hat{K}_D\hat{\mu}}\hat{\mu}^{-1}$$
$$+\int_0^t ds\hat{K}_D\hat{\mu}e^{-(t-s)\hat{K}_D\hat{\mu}}\rho e^{-s\hat{K}_D\hat{\mu}}\hat{\mu}^{-1}.$$

Integrating by parts in $s$ and using the Weyl change of $\hat{\mu}$ we get

$$\delta_\rho\left(e^{-t\hat{K}_D\hat{\mu}}\hat{\mu}^{-1}\right) = -\frac{1}{2}\frac{d}{dt}\left(\int_0^t ds e^{-(t-s)\hat{K}_D\hat{\mu}}\rho e^{-s\hat{K}_D\hat{\mu}}\hat{\mu}^{-1}\right).$$

Since the projector $\hat{P}$ is Weyl invariant we can introduce it in the above formula without any problem. Introducing this on the Weyl change of $\hat{G}_K$ and then integrating in $t$ lead us to the following local contribution

$$\delta_\rho\hat{G}_K(\tau,\tau') = \frac{1}{2} < \tau|\int_0^\varepsilon ds e^{-(\varepsilon-s)\hat{K}_D\hat{\mu}}\rho e^{-s\hat{K}_D\hat{\mu}}\hat{P}\hat{\mu}^{-1}|\tau' > .$$

So inserting $\hat{P}\hat{\mu}^{-1}\hat{\mu}$ just before $\rho$ and $P_{\tau''}$ after it we finally get

$$\delta_\rho\hat{G}_K(\xi,\xi') = \frac{1}{2}\int_0^\varepsilon ds \oint d\hat{s}(\xi'')\rho(\xi'')\hat{\mathcal{G}}_K(\varepsilon-s,\xi,\xi'')\hat{\mathcal{G}}_K(s,\xi'',\xi').$$

For $\xi = \xi'$ this anomaly is going to be given by an expansion in $\varepsilon$. That must be a dimensionless and covariant local function of $\xi$, $C''\rho(\xi) + O(\varepsilon)$. To find $C''$ we just need to use the flat heat kernel in the upper half-plane. Expanding around $\varepsilon$ we get

$$\mathcal{G}_K^0(\varepsilon-s,\xi,\xi') = \mathcal{G}_K^0(0,\xi,\xi') + O(\varepsilon), \quad \mathcal{G}_K^0(s,\xi,\xi') = \mathcal{G}_K^0(\varepsilon,\xi,\xi') + O(\varepsilon).$$

Thus

$$\delta_\rho\hat{G}_K(\xi,\xi) = 4\rho(\xi).$$

# Chapter 4

# Open string critical exponents and the saddle point limit

## 4.1 Introduction

In the theory of random surfaces we must be able to consider surfaces with boundaries [34, 41, 42]. It is by doing so that we may introduce fundamental loop functionals and thus analyse physical systems where the boundaries play an important role. Well known applications can be found in string theory and 2D quantum gravity, in the study of boundary effects in condensed matter and statistical physics and in the analysis of gauge field theories with particular emphasis on the attempts to describe the strongly coupled phase of QCD.

Polyakov's functional integral approach to string theory gives us the simplest operational definition of the loop functionals for continuous surfaces. For $n$-loops the amplitude is

$$G(l_1, \ldots, l_n) = \int_{M(l_i)} \mathcal{D}_{\tilde{g}}(X, \tilde{g}) e^{-S[X, \tilde{g}]} .$$

The closed string represents the partition function of the theory of random surfaces where no boundary loops are considered. When we have just one boundary the open string functional integral defines the one loop Green's functional also known as the Hartle-Hawking wave functional in the context of quantum gravity. For gauge theories this models the Wilson loop. In the case of two boundaries we have the string or universe propagator and for three boundaries the interaction vertex and so on for increasing number of boundaries. For each and all these functionals there is a topological expansion where higher perturbative orders correspond to an increasing number of closed string loops.

The loop functionals can also be precisely defined when we discretise the 2D Riemann surfaces by random triangulations and use matrix models to describe them. A continuum limit can then be properly defined and the correspondent critical properties calculated. In the end the results must be consistent with those obtained using theories of continuous fluctuating surfaces.

Let us start by considering Polyakov's sum over random surfaces with the topology of a disc. For definiteness we will take free boundary conditions. Generalising the closed string case the quantum partition function may now be written as an integral of the partition function for surfaces constrained to have fixed area, $A$, and perimeter, $L$, $\Gamma(A, L)$

$$Z = \int_0^{+\infty} \Gamma(A, L) e^{-\mu_0^2 A - \lambda_0 L} dA dL.$$

After integrating out the matter and reparametrisation ghost fields in the conformal gauge we find the following integral for $\Gamma(A, L)$

$$\Gamma(A, L) = \int \mathcal{D}_{\hat{g}}(\psi, \varphi) e^{-S_L[\varphi, \hat{g}]} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\varphi} - A\right) \delta\left(\oint d\hat{s} e^{\varphi/2} - L\right) \qquad (4.1)$$

where the Liouville action is given by

$$S_L^0[\varphi, \hat{g}] = \frac{26-d}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) + \frac{26-d}{24\pi} \oint d\hat{s} k_{\hat{g}} \varphi. \qquad (4.2)$$

In the DDK approach we renormalise the Liouville field and its couplings to 2D gravity to be able to work with a canonical measure and so write the partition function as

$$\Gamma(A,L) = \int \mathcal{D}_{\hat{g}}(\bar{\Psi},\bar{\phi})d\Psi_0\Big(\oint d\hat{s}\Big)^{1/2} e^{-S_c^{00}[\bar{\Psi},\Psi_0,\hat{g}]-\bar{S}^0[\bar{\phi},\hat{g}]}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}-A\right)$$
$$\times\delta\left(\oint d\tilde{s}e^{\alpha/2\phi}-L\right) . \tag{4.3}$$

Here we have factored out the cosmological constant counterterms left over from the renormalisation process. Note that in this process the initially infinite constants $\mu_0^2$ and $\lambda_0$ are changed into the finite constants $\mu_1^2$ and $\lambda_1$ before the DDK renormalisation and to the finite constants $\mu_2^2$ and $\lambda_2$ after it. As discussed before we have set $\nu_2 = 0$.

## 4.2 The open string susceptibility and Yang-Mills Feynman mass exponents

Consider Eq. (4.3) and shift the integration variable $\phi$ by a constant $\phi \to \phi + \rho/\alpha$. Since we keep $\hat{g}_{ab}$ fixed our functional integral must scale. Recall that the theory is invariant under arbitrary scalings of the reference metric once we have integrated $\phi$. So, it is only invariant under a shift of the integration variable provided this is compensated by a Weyl transformation of the reference metric. Because we consider a translational invariant quantum measure in Eq. (4.3), the scaling behavior is determined by the change in the action $S \equiv S_c^{00}[\bar{\Psi},\Psi_0,\hat{g}] + \bar{S}^0[\bar{\phi},\hat{g}]$, and in the delta functions which are used to fix the area $A$ and the perimeter $L$ of the surface. Being the shift constant only the zero mode $\Psi_0$ is actually changed. Thus the shift in the action is

$$S \to S + \frac{Q\chi_o}{2\alpha}\rho,$$

and the shifts in the delta functions are

$$\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}-A\right) \to e^{-\rho}\delta\left(\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi}-e^{-\rho}A\right) ,$$
$$\delta\left(\oint d\hat{s}e^{\alpha\phi/2}-L\right) \to e^{-\rho/2}\delta\left(\oint d\hat{s}e^{\alpha\phi/2}-e^{-\rho/2}L\right) .$$

Substituting back in the partition function we get the following scaling law

$$\Gamma(A,L) = e^{-\frac{\rho}{2}\left(\frac{\chi_o Q}{\alpha}+3\right)}\Gamma\left(e^{-\rho}A, e^{-\rho/2}L\right). \tag{4.4}$$

To be able to introduce critical exponents we have to define the partition function for fixed area $A$, $\Sigma(A)$, and the partition function for fixed perimeter $L$, $\Omega(L)$. Factoring out the appropriate counterterms we write

$$\Sigma(A,\lambda_2) = \int \Gamma(A,L)e^{-\lambda_2 L}dL, \quad \Omega(L,\mu_2^2) = \int \Gamma(A,L)e^{-\mu_2^2 A}dA.$$

Then introducing Eq. (4.4) leads us to two additional scaling laws

$$\Sigma(A,\lambda_2) = e^{-\rho\left(\frac{\chi_o Q}{2\alpha}+1\right)}\Sigma\left(e^{-\rho}A, \lambda_2 e^{\rho/2}\right), \tag{4.5}$$

$$\Omega(L,\mu_2^2) = e^{-\frac{\rho}{2}\left(\frac{\chi_o Q}{\alpha}+1\right)}\Omega\left(e^{-\rho/2}L, \mu_2^2 e^{\rho}\right). \tag{4.6}$$

The open string susceptibility exponent is defined just like in the closed string. In the case $\lambda_2 = 0$ we can continue to use the scaling argument. As $A \to +\infty$

$$\Sigma(A) \sim A^{\sigma(\chi_o)-3}.$$

and

$$\sigma(\chi_o) = 2 - \frac{\chi_o Q}{2\alpha}.$$

The last result is just the expected open string version of the closed string critical exponent. If we take the positive root for $Q$ and the corresponding negative one for $\alpha$ we find that in the semi-classical limit $d \to -\infty$

$$\sigma(\chi_o) = \frac{d-19}{12}\chi_o + 2.$$

For the open string we can also consider the asymptotic limit $L \to +\infty$ and introduce a mass critical exponent in close analogy with the the asymptotic limit $A \to +\infty$.

Here we take $\mu_2^2 = 0$. This case was considered by Durhuus, Olesen and Petersen [46] in connection with the calculation of the Wilson loop quark-antiquark potential. We define $\omega(\chi_o)$ by

$$\Omega(L) \sim L^{\omega(\chi_o)-3}.$$

Thus we find

$$\omega(\chi_o) = 2 - \frac{\chi_o Q}{\alpha},$$

to which we associate the semi-classical limit

$$\omega(\chi_o) = \frac{d-19}{6}\chi_o + 2.$$

We can interpret of $\omega(\chi_o)$ in the context of Yang-Mills gluon dynamics. To see this first note that the wave functional given in Eq. (3.3) models the Wilson loop ,W, for Yang-Mills theory [39, 46]. Consider the first quantised functional integral representing the propagator of a particle of mass $\lambda_2$ moving under the influence of a Yang-Mills field. At coincident points its trace is a gauge invariant expression

$$\mathrm{Tr}\, G_{\mathcal{A}}(x,x) = \int \mathcal{D}_{\tilde{g}} Y \,\mathrm{Tr}\, P\, e^{-\lambda_2 \oint d\tilde{s} - \oint d\tilde{Y}\cdot\mathcal{A}}.$$

If this is averaged over the Yang-Mills field we get

$$< \mathrm{Tr}\, G_{\mathcal{A}}(x,x) >_{\mathcal{A}} = \int \mathcal{D}_{\tilde{g}} Y\, e^{-\lambda_2 \oint d\tilde{s}}\, W = \int dL\, e^{-\lambda_2 L} \int \mathcal{D}_{\tilde{g}} Y\, \delta(L - \oint d\tilde{s})\, W$$

but this last functional integral is just what we mean by $\Omega$. Substituting the form that holds for $\mu_2^2 = 0$ we get

$$< \mathrm{Tr}\, G_{\mathcal{A}}(x,x) >_{\mathcal{A}} \propto {\lambda_2}^{\chi_o Q/\alpha} = {\lambda_2}^{2-\omega(\chi_o)},$$

valid for small $\lambda_2$, corresponding to large $L$. Thus $\omega(\chi_o)$ is the critical exponent associated with the Feynman propagator of a test particle which interacts with the Yang-Mills gauge fields.

So far we have expanded the cosmological terms so as to linearise the contribution of the exponential terms to the action. We will now discuss a different approach based on the semi-classical expansion.

## 4.3 The saddle point expansion

Consider the partition function for surfaces of area $A$ and perimeter $L$ as given by Eqs. (4.1) and (4.2) before DDK renormalisation. Representing the delta functions by integrals over (imaginary) Lagrange multipliers $p, q$, gives the Euclidean action

$$S_L[\varphi,\hat{g},p,q] = \frac{26-d}{48\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) + \frac{26-d}{24\pi}\oint d\hat{s}k_{\hat{g}}\varphi$$
$$-p\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi} - A\right) - q\left(\oint d\hat{s}e^{\varphi/2} - L\right). \qquad (4.7)$$

### 4.3.1 Möbius invariance

This action is invariant under Möbius transformations on the upper half-plane, i.e. $SL(2,\mathcal{R})$ invariant. These transformations preserve the conformal gauge, mapping the upper half-plane onto itself. Introducing the complex coordinates $\omega = x + iy$, $\bar{\omega} = x - iy$ we write the transformations as follows

$$\omega \to \omega' = \frac{a\omega + b}{c\omega + d}, \quad \varphi(\omega,\bar{\omega}) \to \varphi(\omega',\bar{\omega}') + 2\ln\left|\frac{d\omega'}{d\omega}\right|,$$

where $a, b, c, d \in \mathcal{R}$ and $ad - bc = 1$. The inverse mapping is then given by

$$\omega' \to \omega = \frac{d\omega' - b}{-c\omega' + a}, \quad \varphi(\omega',\bar{\omega}') \to \varphi(\omega,\bar{\omega}) + 2\ln\left|\frac{d\omega}{d\omega'}\right|.$$

To see the Möbius invariance consider the Euclidean action written in the upper-half plane. Taking into account the background charge placed at $\Lambda \to \infty$ in the boundary we use the Gauss-Bonnet theorem to write

$$S_L[\varphi,p,q] = \frac{26-d}{24\pi}\int d^2\omega\partial_\omega\varphi\partial_{\bar{\omega}}\varphi + \frac{26-d}{6}\varphi(\Lambda) - p\left(\int d^2\omega e^{\varphi} - A\right)$$
$$-q\left[\int \frac{1}{2}(d\omega + d\bar{\omega})\,e^{\varphi/2} - L\right],$$

where $d^2\omega = (1/2)d\omega d\bar{\omega}$ and on the boundary $\omega = \bar{\omega}$. First note that the area and boundary length integrals are clearly invariant under the $SL(2,\mathcal{R})$ transformation. In what concerns the kinetic term we start by writing

$$\int d^2\omega\partial_\omega\varphi\partial_{\bar{\omega}}\varphi = \int d^2\omega'\partial_{\omega'}\varphi\partial_{\bar{\omega}'}\varphi + \int d^2\omega'\partial_{\omega'}(\delta_M\varphi)\partial_{\bar{\omega}'}\varphi$$
$$+\int d^2\omega'\partial_{\omega'}\varphi\partial_{\bar{\omega}'}(\delta_M\varphi) + \int d^2\omega'\partial_{\omega'}(\delta_M\varphi)\partial_{\bar{\omega}'}(\delta_M\varphi), \tag{4.8}$$

where $\delta_M\varphi$ is the variation of the Liouville field under the $SL(2,\mathcal{R})$ transformation. Now if we apply the inverse Möbius transformation we find

$$\frac{d\omega}{d\omega'} = \frac{1}{(-c\omega'+a)^2}, \quad \frac{d\bar{\omega}}{d\bar{\omega}'} = \frac{1}{(-c\bar{\omega}'+a)^2}.$$

Then the variation of the Liouville mode is

$$\delta_M\varphi(\omega',\bar{\omega}') = 2\ln(-c\omega'+a)(-c\bar{\omega}'+a).$$

This leads to the following derivative of the variation

$$\partial_{\omega'}(\delta_M\varphi) = -\frac{2c}{-c\omega'+a}, \quad \partial_{\bar{\omega}'}(\delta_M\varphi) = -\frac{2c}{-c\bar{\omega}'+a}.$$

Now since $\partial_{\omega'}\partial_{\bar{\omega}'}(\delta_M\varphi) = \partial_{\bar{\omega}'}\partial_{\omega'}(\delta_M\varphi) = 2\pi\delta^2(\omega'-a/c)$ we can take a total derivative and note that on the boundary $\omega' = \bar{\omega}'$ to get

$$\int d^2\omega'\partial_{\omega'}(\delta_M\varphi)\partial_{\bar{\omega}'}\varphi + \int d^2\omega'\partial_{\omega'}\varphi\partial_{\bar{\omega}'}(\delta_M\varphi) = 4\pi\varphi(a/c)$$
$$-\int\frac{i}{2}\left(d\omega'+d\bar{\omega}'\right)\left[\left(\partial_{\omega'}-\partial_{\bar{\omega}'}\right)(\delta_M\varphi)\right]\varphi = 4\pi\varphi(a/c).$$

With the same calculation we also eliminate the boundary contribution from the last term in Eq. (4.8). From it we get $4\pi\delta_M\varphi(a/c)$. On the other hand

$$\varphi(\Lambda) \to \varphi(a/c) + \delta_M\varphi(a/c).$$

So if at infinity in the boundary we have

$$\varphi(\omega,\bar{\omega}) \to -4\ln(\omega\bar{\omega}) + \text{const}, \qquad |\omega| \to +\infty$$

then up to constants the Euclidean action in the upper half plane is in fact $SL(2,\mathcal{R})$ invariant. Note by the way that this does not happen for the $SL(2,\mathcal{C})$ group because the boundary terms do not cancel when complex group elements are involved.

In what follows it will be more convenient to work on the unit disc obtained from the upper half-plane by the complex Möbius transformation

$$\omega \to z = \frac{i-\omega}{i+\omega}, \quad \varphi(\omega,\bar{\omega}) \to \varphi(z,\bar{z}) + 2\ln\left|\frac{dz}{d\omega}\right| .$$

Note that this mapping does not preserve the conformal gauge. It changes from one gauge slice to another as allowed by the reparametrisation and Weyl invariances of the theory. For the inverse mapping we find

$$z \to \omega = i\frac{1-z}{1+z}, \quad \varphi(z,\bar{z}) \to \varphi(\omega,\bar{\omega}) + 2\ln\left|\frac{d\omega}{dz}\right| .$$

To map the Euclidean action onto the unit disc first note that the area and boundary length integrals transform to

$$\int d^2\omega e^{\varphi} = \int d^2 z e^{\varphi} ,$$

$$\int \frac{1}{2}\left(d\omega + d\bar{\omega}\right) e^{\varphi/2} = \int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right) e^{\varphi/2} ,$$

where on the boundary we now have $z = 1/\bar{z}$. For the kinetic term the boundary contribution is now non-zero and so we find

$$\frac{26-d}{24\pi}\int d^2\omega \partial_\omega \varphi \partial_{\bar{\omega}} \varphi = \frac{26-d}{24\pi}\int d^2 z \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{26-d}{24\pi}\int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi$$
$$+\varphi \text{ independent terms.}$$

Thus the Euclidean action on the unit disc is

$$S_L[\varphi,p,q] = \frac{26-d}{24\pi}\int d^2 z \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{26-d}{24\pi}\int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi$$
$$-p\left(\int d^2 z e^{\varphi} - A\right) - q\left[\int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right) e^{\varphi/2} - L\right] .$$

We can here note explicitly that the mapping does not preserve the conformal gauge leading to the extra piece in the geodesic curvature of the unit disc $k_{\hat{g}} = 1$.

To get the Möbius invariance on the unit disc we simply map the $SL(2,\mathcal{R})$ transformation. In complex coordinates we find

$$z' = \frac{\alpha z + \beta}{-\bar{\beta} z - \bar{\alpha}},$$

where $\alpha = c - b + i(a + d)$ and $\beta = -b - c + i(d - a)$. Calling $c_0 = \beta/\alpha$ we may write

$$z' = e^{i\theta_0} \frac{z + c_0}{1 + \bar{c}_0 z}, \tag{4.9}$$

where $\theta_0 \in \mathcal{R}$ is

$$\theta_0 = \pi + 2 \tan \left( \frac{a + d}{c - b} \right) \tag{4.10}$$

and

$$|c_0| = \sqrt{\frac{a^2 + b^2 + c^2 + d^2 - 2}{a^2 + b^2 + c^2 + d^2 + 2}} < 1. \tag{4.11}$$

This transformation is the conformal mapping of the unit disc onto itself. The inverse transformation is

$$z' \to z = e^{-i\theta_0} \frac{z' - c_0}{1 - \bar{c}_0 z'}.$$

With the usual transformation of the Liouville field,

$$\varphi(z, \bar{z}) \to \varphi(z', \bar{z}') + 2 \ln \left| \frac{dz'}{dz} \right|, \tag{4.12}$$

the definition of the Möbius symmetry on the unit disc is completed.

### 4.3.2 The classical Liouville field

In the saddle point approximation we expand around the solution of the following classical problem

$$\int d^2\zeta \frac{\delta S_L}{\delta \varphi}(\zeta)\Big|_c \chi(\zeta) = \frac{\delta S_L}{\delta p}\Big|_c m = \frac{\delta S_L}{\delta q}\Big|_c n = 0.$$

Here $|_c$ means evaluated at the classical point $\varphi_c$, $p_c$ and $q_c$. $\chi$, $m$ and $n$ are the quantum fluctuations around the classical solution. Differentiating $S_L[\varphi, p, q]$ as given by Eq. (4.7) we find

$$\frac{\delta S_L}{\delta\varphi(\zeta)} = \frac{1}{\gamma}\int d^2\xi\sqrt{\hat{g}}\left[\hat{g}^{ab}\partial_a\delta^2(\xi-\zeta)\partial_b\varphi + \hat{R}\delta^2(\xi-\zeta)\right] + \frac{2}{\gamma}\oint d\hat{s}(\xi)k_{\hat{g}}\delta^2(\xi-\zeta)$$
$$-p\int d^2\xi\sqrt{\hat{g}}e^{\varphi}\delta^2(\xi-\zeta) - \frac{q}{2}\oint d\hat{s}(\xi)e^{\varphi/2}\delta^2(\xi-\zeta), \tag{4.13}$$

where $\gamma = 48\pi/(26-d)$. After a total derivative we get

$$\int d^2\zeta\frac{\delta S_L}{\delta\varphi(\zeta)}\Big|_c\chi(\zeta) = \frac{1}{\gamma}\int d^2\xi\sqrt{\hat{g}}\left(\hat{\Delta}\varphi_c + \hat{R}\right)\chi + \frac{1}{\gamma}\oint d\hat{s}(\xi)\chi\partial_{\hat{n}}\varphi_c$$
$$+\frac{2}{\gamma}\oint d\hat{s}(\xi)k_{\hat{g}}\chi - p_c\int d^2\xi\sqrt{\hat{g}}e^{\varphi_c}\chi - \frac{q_c}{2}\oint d\hat{s}(\xi)e^{\varphi_c/2}\chi.$$

On the other hand we find

$$\frac{\delta S_L}{\delta p}\Big|_c m = -\left(\int d^2\xi\sqrt{\hat{g}}e^{\varphi_c} - A\right)m$$

and

$$\frac{\delta S_L}{\delta q}\Big|_c n = -\left(\oint d\hat{s}e^{\varphi_c/2} - L\right)n.$$

All this must be zero for all $\chi$, $m$ and $n$. So $\varphi_c$, $p_c$ and $q_c$ must satisfy the Liouville equation

$$\hat{R} + \hat{\Delta}\varphi_c = \eta e^{\varphi_c}, \qquad \int d^2\xi\sqrt{\hat{g}}e^{\varphi_c} = A$$

subjected to the boundary condition [5]

$$2k_{\hat{g}} + \partial_{\hat{n}}\varphi_c = ke^{\varphi_c/2}, \qquad \oint d\hat{s}e^{\varphi_c/2} = L,$$

where $\eta = p_c\gamma$ and $2k = q_c\gamma$.

We have written the classical problem in the conformal gauge $\tilde{g}_{ab} = e^{\varphi}\hat{g}_{ab}$. Using this and

$$\tilde{R} = e^{-\varphi}\left(\hat{R} + \hat{\Delta}\varphi\right), \quad k_{\tilde{g}} = e^{-\varphi/2}\left(k_{\hat{g}} + \frac{1}{2}\partial_{\hat{n}}\varphi\right)$$

we can write it in the following gauge invariant form

$$\tilde{R} = \eta, \quad 2k_{\tilde{g}} = k, \quad \int d^2\xi\sqrt{\tilde{g}} = A, \quad \oint d\tilde{s} = L.$$

Now $\eta$ and $k$ are not independent. To see it start by integrating the Liouville equation over the world-sheet. This is just $\eta A$ if we use the area condition. If we then apply the boundary conditions followed by the Gauss-Bonnet theorem on the disc we get

$$\eta A + kL = 4\pi.$$

On the upper half-plane the classical problem is

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_c = \eta e^{\varphi_c}, \quad \int dxdy e^{\varphi_c} = A,$$

with the boundary condition at $y = 0$

$$-\frac{\partial \varphi_c}{\partial y} = k e^{\varphi_c/2}, \quad \int dx e^{\varphi_c/2} = L.$$

We can write this with the complex coordinates $\omega$, $\bar{\omega}$

$$-4\partial_\omega\partial_{\bar{\omega}}\varphi_c = \eta e^{\varphi_c}, \quad \int d^2\omega e^{\varphi_c} = A,$$

$$-i\left(\partial_\omega - \partial_{\bar{\omega}}\right)\varphi_c = k e^{\varphi_c/2}, \quad \frac{1}{2}\int\left(d\omega + d\bar{\omega}\right)e^{\varphi_c/2} = L,$$

where at the boundary $\omega = \bar{\omega}$. To get the problem on the unit disc we perform the conformal mapping of the upper half-plane to the unit disc. Then we get

$$-4\partial_z\partial_{\bar{z}}\varphi_c = \eta e^{\varphi_c}, \quad \int d^2z e^{\varphi_c} = A,$$

$$\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right)\varphi_c + 2 = k e^{\varphi_c/2}, \quad \frac{i}{2}\int\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)e^{\varphi_c/2} = L,$$

where now the boundary is at $z = 1/\bar{z}$. We will work with the polar coordinates

$$z = \rho e^{i\theta}, \quad \bar{z} = \rho e^{-i\theta},$$

where $\rho \in [0,1]$ and $\theta \in [0, 2\pi]$. Then we find

$$-\left(\frac{\partial^2 \varphi_c}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \varphi_c}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \varphi_c}{\partial \theta^2}\right) = \eta e^{\varphi_c}, \quad \int d\theta \rho d\rho e^{\varphi_c} = A,$$

$$\frac{\partial \varphi_c}{\partial \rho} = k e^{\varphi_c/2} - 2, \quad \int d\theta e^{\varphi_c/2} = L,$$

where the boundary is at $\rho = 1$.

To solve the Liouville equation we will assume that $\varphi_c$ only depends on $\rho$. Then the equation becomes

$$-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d}{d\rho}\varphi_c\right) = \eta e^{\varphi_c}.$$

Take $\rho = e^r$, $\psi = \varphi_c + 2r$ and call $Y$ the derivative of $\psi$ with respect to $r$, $\dot{\psi}$. Then it is easy to find the following equation for $Y$ taken as a function of $\psi$

$$-\frac{1}{2}\frac{d}{d\psi}Y^2 = \eta e^{\psi}.$$

This can be integrated to yield $Y^2 = -2\eta e^{\psi} + 2C$, where $C$ is an integration constant. Choosing the + sign in front of the square root we now need to integrate the following equation

$$\frac{d\psi}{dr} = \sqrt{-2\eta e^{\psi} + 2C}.$$

This can be immediately done by separating the variables, thus leading to

$$r = \int^{\psi} d\bar{\psi} \left(-2\eta e^{\bar{\psi}} + 2C\right)^{-1/2} + C'',$$

where $C''$ is another integration constant. Let us change the variable on the integral to $\phi^2 = -2\eta e^{\bar{\psi}} + 2C$. Integrating this we obtain

$$r = C'' + C''' - \sqrt{\frac{2}{C}}\tanh^{-1}\sqrt{1 - \frac{\eta}{C}e^{\psi}},$$

where $C'''$ is still another integration constant. Calling $C' = C'' + C'''$ we can invert this function to find

$$\psi(r) = \ln\frac{C}{\eta}\left[\cosh\sqrt{\frac{C}{2}}(C' - r)\right]^{-2}.$$

This solution has to verify the boundary conditions. Introducing $\psi$ on them we find

$$\sinh C'\sqrt{\frac{C}{2}} = \frac{k}{\sqrt{2\eta}}, \quad \cosh C'\sqrt{\frac{C}{2}} = \frac{2\pi}{L}\sqrt{\frac{C}{\eta}}.$$

Using $\cosh^2 x - \sinh^2 x = 1$ we can determine an expression for $C$,

$$C = \frac{\eta L^2}{4\pi^2} + \frac{k^2 L^2}{8\pi^2}.$$

Finally we still have the area condition to satisfy. Using $\lim_{x\to+\infty}\tanh x = 1$ we get

$$-\frac{1}{\eta}\left(\sqrt{2C}\tanh C'\sqrt{\frac{C}{2}} - \sqrt{2C}\right) = \frac{A}{2\pi}.$$

Now

$$\tanh C'\sqrt{\frac{C}{2}} = \frac{kL}{2\pi\sqrt{2C}}.$$

Then $2\pi\sqrt{2C} = kL + \eta A$ which means that $C = 2$ because $kL + \eta A = 4\pi$.

We are now left with the following equations

$$\eta + \frac{k^2}{2} = \frac{8\pi^2}{L^2}, \quad \eta A + kL = 4\pi.$$

These can be solved for $\eta$ and $k$ as functions of $L$ and $A$. We get two solutions, $\eta = 0$, $k = 4\pi/L$ and

$$\eta = \frac{8\pi}{A}\left(1 - \frac{L^2}{4\pi A}\right), \quad k = \frac{2L}{A}\left(1 - \frac{2\pi A}{L^2}\right). \tag{4.14}$$

Since $e^{\psi} > 0$ we must have $\eta \geq 0$. Then $L^2 \leq 4\pi A$.

Now we can use the property $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$ and the value of $k$ in Eq. (4.14) to write

$$\psi(r) = 2\ln\frac{L}{2\pi}\left[\cosh r + \left(1 - \frac{L^2}{2\pi A}\right)\sinh r\right]^{-1}.$$

Using the expressions for $\cosh r$ and $\sinh r$ in terms of $e^r$ and substituting $e^r = \rho$ we finally get

$$\varphi_c(\rho) = 2\ln\frac{2A}{L}\left[1 + \left(\frac{4\pi A}{L^2} - 1\right)\rho^2\right]^{-1}.$$

This is regular all over the disc. At the center $\rho = 0$ and at the boundary $\rho = 1$ we find

$$\varphi_c(0) = 2\ln\frac{2A}{L}, \quad \varphi_c(1) = 2\ln\frac{L}{2\pi}.$$

The case $\eta = 0$ is obtained when $L^2 = 4\pi A$. Then $\varphi_c = 2\ln L/(2\pi)$.

Due to the $\theta$ independence this is the metric of a spherical cap of length $L$ and area $A$. When $\eta = 0$ we have a flat disc with perimeter $L$ and area $L^2/(4\pi)$.

### 4.3.3 The tree level partition function

The saddle point tree level approximation is given by the classical functional

$$\Gamma(A, L) = e^{-S_L[\varphi_c, \hat{g}, p_c, q_c]}.$$

To calculate the classical Euclidean action let us start by writing it on the unit disc

$$S_L[\varphi, p, q] = \frac{2}{\gamma}\int d^2z\partial_z\varphi\partial_{\bar{z}}\varphi + \frac{2}{\gamma}\int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi - p\left(\int d^2ze^{\varphi} - A\right)$$
$$-q\left[\int \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)e^{\varphi/2} - L\right].$$

After a total derivative we get

$$S_L[\varphi, p, q] = -\frac{2}{\gamma}\int d^2z\varphi\partial_z\partial_{\bar{z}}\varphi + \frac{1}{2\gamma}\int\frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi\left(z\partial_z\varphi + \bar{z}\partial_{\bar{z}}\varphi\right)$$

$$+\frac{2}{\gamma}\int\frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi - p\left(\int d^2ze^{\varphi} - A\right) - q\left[\int\frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)e^{\varphi/2} - L\right].$$

Applying the classical equations of motion we then can write the classical Euclidean action on the unit disc as follows

$$S_L[\varphi_c, p_c, q_c] = \frac{\eta}{2\gamma}\int d^2z\varphi_c e^{\varphi_c} + \frac{k}{2\gamma}\int\frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi_c e^{\varphi_c/2} + \frac{1}{\gamma}\int\frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)\varphi_c.$$

In polar coordinates this is

$$S_L[\varphi_c, p_c, q_c] = \frac{\eta}{2\gamma}\int d\theta\rho d\rho\varphi_c e^{\varphi_c} + \frac{k}{2\gamma}\int d\theta\varphi_c e^{\varphi_c/2} + \frac{1}{\gamma}\int d\theta\varphi_c.$$

Let us introduce the new coordinate $\varrho$ such that $\rho$ is given by

$$\rho = \frac{1}{\sqrt{4\pi A/L^2 - 1}}\tan(\varrho/2).$$

For $\rho \in [0,1]$ we find $\varrho \in [0, 2\arctan\sqrt{4\pi A/L^2 - 1}]$. Then at the boundary $\rho = 1$ we find $\cos^2(\varrho/2) = L^2/(4\pi A)$ and so $\cos\varrho \in [-1 + L^2/(2\pi A), 1]$.

Now if we integrate introducing the values of $\varphi_c$, $\eta$ and $k$ we find

$$\frac{\eta}{2\gamma}\int d\theta\rho d\rho\varphi_c e^{\varphi_c} = \frac{1}{\gamma}\left[\frac{2L^2}{A}\left(\frac{4\pi A}{L^2} - 1\right)\ln\frac{2A}{L} - \frac{2L^2}{A}\ln\frac{L^2}{4\pi A} - \frac{2L^2}{A}\left(\frac{4\pi A}{L^2} - 1\right)\right],$$

$$\frac{k}{2\gamma}\int d\theta\varphi_c e^{\varphi_c/2} = \frac{2L^2}{A\gamma}\left(1 - \frac{2\pi A}{L^2}\right)\ln\frac{L}{2\pi},$$

$$\frac{1}{\gamma}\int d\theta\varphi_c = \frac{4\pi}{\gamma}\ln\frac{L}{2\pi}.$$

Summing this up leads to

$$S_L[\varphi_c, p_c, q_c] = \frac{26-d}{6}\ln\frac{2A}{L} - \frac{26-d}{6} + \frac{2L^2}{A\gamma}.$$

Thus

$$\Gamma(A,L) = \left(\frac{A}{L}\right)^{(d-26)/6} e^{-2L^2/(A\gamma)}.$$

In the semi-classical limit $d \to -\infty$ we get

$$\Gamma(A,L) = e^{d/12\rho}\Gamma\left(e^{-\rho}A, e^{-\rho/2}L\right).$$

If we take the branch $\alpha_-$, $\chi_o = 1$ and the limit $d \to -\infty$ we reproduce this scaling law from Eq. (4.4) so that in the case of the disc topology both methods match in the asymptotic limit $A \to +\infty$, $L \to +\infty$ such that $A/L^2 \to$ const.

### 4.3.4 The one loop partition function

If we go to one loop we must consider

$$S_L[\varphi,p,q] = S_L[\varphi_c,p_c,q_c] + \frac{1}{2}\int d^2\zeta d^2\zeta' \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta\varphi(\zeta')}\Big|_c \chi(\zeta)\chi(\zeta')$$
$$+\frac{1}{2}\int d^2\zeta \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta p}\Big|_c \chi(\zeta)m + \frac{1}{2}\int d^2\zeta \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta q}\Big|_c \chi(\zeta)n.$$

Here we have already taken into account that

$$\frac{\delta^2 S_L}{\delta^2 p} = \frac{\delta^2 S_L}{\delta^2 q} = \frac{\delta^2 S_L}{\delta p \delta q} = 0.$$

Now using Eq. (4.13) it is easy to see that

$$\frac{1}{2}\int d^2\zeta d^2\zeta' \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta\varphi(\zeta')}\Big|_c \chi(\zeta)\chi(\zeta') = \frac{1}{2\gamma}\int d^2\xi \sqrt{\hat{g}}\hat{g}^{ab}\partial_a\chi\partial_b\chi - \frac{1}{2\gamma}\eta\int d^2\xi\sqrt{\hat{g}}e^{\varphi_c}\chi^2$$
$$-\frac{1}{4\gamma}k\oint d\hat{s}e^{\varphi_c/2}\chi^2.$$

We also find

$$\frac{1}{2}\int d^2\zeta \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta p}\Big|_c \chi(\zeta)m = -\frac{m}{2}\int d^2\xi\sqrt{\hat{g}}e^{\varphi_c}\chi$$

and

$$\frac{1}{2}\int d^2\zeta \frac{\delta^2 S_L}{\delta\varphi(\zeta)\delta q}\Big|_c \chi(\zeta) n = -\frac{n}{4}\oint d\hat{s}\sqrt{\hat{g}}e^{\varphi_c/2}\chi.$$

This last two contributions can be used to recover the delta functions. So we get

$$\Gamma(A,L) = e^{-S_L[\varphi_c,\hat{g},p_c,q_c]}\int \mathcal{D}_{g_c}(\phi,\chi)\delta\left(\int d^2\xi\sqrt{g_c}\chi\right)\delta\left(\oint ds_c\phi\right)e^{-S_1[\chi,\phi,g_c]},$$

where we have written $\chi|_B = \phi$. The metric ${g_c}^{ab}$ is given by $e^{\varphi_c}\hat{g}^{ab}$ and the one loop action is

$$S_1[\chi,\phi,g_c] = \frac{1}{2\gamma}\int d^2\xi\sqrt{g_c}{g_c}^{ab}\partial_a\chi\partial_b\chi - \frac{1}{2\gamma}\eta_c\int d^2\xi\sqrt{g_c}\chi^2 - \frac{1}{4\gamma}k_c\oint ds_c\phi^2.$$

To integrate this we need to factor out the Möbius invariance. Consider $\chi = \bar{\chi}+\chi_b$, where $\bar{\chi}$ is an homogeneous Dirichlet field, $\bar{\chi}|_B = 0$, and $\chi_b$ is a background field fixed by the boundary value of $\chi$, $\chi_b|_B = \phi$. Since the Möbius symmetry refers to the whole Liouville field, $\bar{\chi}$ is fixed at the boundary and $\phi$ can take any value we will use the boundary integration measure to deal with the Möbius invariance. In the one loop semi-classical approximation we write

$$||\delta\phi||^2 = \oint d\hat{s}e^{\varphi/2}(\delta\phi)^2 \approx \oint ds_c(\delta\phi)^2.$$

On the unit disc we consider the Möbius invariance as given by Eqs. (4.9)-(4.12). Then we separate it out from $\phi$ as follows

$$\phi = \phi_\perp + 2\ln\left|\frac{dz'}{dz}\right|.$$

Here $\phi_\perp$ is defined to be the component of $\phi$ that is orthogonal to the Möbius zero mode piece. Note that it includes a constant contribution corresponding to the constant zero mode of the covariant Laplacian $\Delta_c$. For the infinitesimal case we consider $z' = z+\varepsilon(z)$, where $\varepsilon(z) = \delta c_0 + i\delta\theta_0 z - \delta\bar{c}_0 z^2$. Then it is clear that

$$\delta\phi_M \equiv 2\delta\ln\left|\frac{dz'}{dz}\right| \approx -2\delta\bar{c}_0 z - 2\delta c_0\bar{z}.$$

Since by definition $\delta\phi_M$ is orthogonal to $\delta\phi_\perp$ we can now write

$$\|\delta\phi\|^2 = \oint ds_c(\delta\phi_\perp)^2 + \oint ds_c(\delta\phi_M)^2.$$

Now in polar coordinates at the boundary $\rho = 1$ we have

$$(\delta\phi_M)^2 = 8\delta c_0\delta\bar{c}_0 + 8(\delta\rho_0)^2\cos 2(\theta - \delta\varphi_0),$$

where we have introduced $\delta c_0 = \delta\rho_0 e^{i\delta\varphi_0}$, $\delta\rho_0 = \sqrt{(\delta a)^2 + (\delta b)^2}$ if $c_0 = a + ib$. Then the integration leads to

$$\|\delta\phi\|^2 \approx \|\delta\phi_\perp\|^2 + 8L\delta c_0\delta\bar{c}_0,$$

where

$$\|\delta\phi_\perp\|^2 \approx \frac{L}{2\pi}\int d\theta(\delta\phi_\perp)^2.$$

So we write for the integration measure in the one loop saddle point approximation the following formula

$$\mathcal{D}_{g_c}(\phi, \bar{\chi}) \approx L dc_0 d\bar{c}_0 \mathcal{D}_{g_c}(\phi_\perp, \bar{\chi}).$$

The above result means that the operator $\mathcal{O}_c = \Delta_c - \eta$ has two zero modes $\delta_M\varphi$ which also must satisfy the boundary condition $\mathcal{O}_c^B\delta_M\varphi = 0$, where $\mathcal{O}_c^B = \partial_{n_c} - k/2$. This can be seen as follows. Since the Euclidean action $S_L[\varphi]$ is invariant under the Möbius transformations $\varphi \to \varphi + \delta_M\varphi$, $\delta_M S_L = 0$, we must have

$$\int d^2\zeta' \frac{\delta}{\delta\varphi(\zeta')}(\delta_M S_L) = \int d^2\zeta' \frac{\delta}{\delta\varphi(\zeta')}\left[\int d^2\zeta \delta_M\varphi(\zeta)\frac{\delta S_L}{\delta\varphi(\zeta)}\right] = 0.$$

Then we get

$$\int d^2\zeta d^2\zeta' \left\{\frac{\delta}{\delta\varphi(\zeta')}\left[\delta_M\varphi(\zeta)\right]\frac{\delta S_L}{\delta\varphi(\zeta)} + \delta_M\varphi(\zeta)\frac{\delta^2 S_L}{\delta\varphi(\zeta')\delta\varphi(\zeta)}\right\} = 0.$$

Taking this at the classical point $\varphi_c$, $p_c$ and $q_c$ we find

$$\int d^2\zeta d^2\zeta' \delta_M\varphi(\zeta)\frac{\delta^2 S_L}{\delta\varphi(\zeta')\delta\varphi(\zeta)}\Big|_c = 0.$$

Integrating in $\zeta$ and $\zeta'$ after the appropriate total derivatives lead us to

$$\int d^2\xi\sqrt{g_c}\mathcal{O}_c\delta_M\varphi + \oint ds_c\mathcal{O}_c^B\delta_M\varphi = 0.$$

So we conclude

$$\mathcal{O}_c\delta_M\varphi = 0, \quad \mathcal{O}_c^B\delta_M\varphi = 0.$$

Let us now separate the constant zero mode of the covariant Laplacian $\Delta_c$. To do so we write $\phi_\perp = \phi_0 + \bar{\phi}_\perp$, where $\oint ds_c\bar{\phi}_\perp = 0$. Note that this means that $\chi_b = \phi_0 + \bar{\chi}_b$, where $\bar{\chi}_b|_B = \bar{\phi}_\perp$. Then the functional integration measure is

$$\int \mathcal{D}_{g_c}(\phi,\chi) \approx L^{3/2}\int d\phi_0 dc_0 d\bar{c}_0\mathcal{D}_{g_c}(\bar{\phi}_\perp,\bar{\chi}).$$

Since $\oint ds_c\bar{\phi}_\perp = 0$ we can use the delta function for the integral along the boundary of $\phi_\perp$ to eliminate the zero mode $\phi_0$

$$\delta\left(\oint ds_c\phi_\perp\right) = \frac{1}{L}\delta(\phi_0).$$

Now integrating the zero mode and defining $\bar{\chi}_b$ as the solution of the boundary-value problem $\mathcal{O}_c\bar{\chi}_b = 0$, $\bar{\chi}_b|_B = \bar{\phi}_\perp$ lead us to

$$\Gamma(A,L) = e^{-S_L[\varphi_c,\hat{g},p_c,q_c]}\int L^{1/2}dc_0 d\bar{c}_0\mathcal{D}_{g_c}\bar{\phi}_\perp \exp\left(-\frac{1}{2\gamma}\oint ds_c\bar{\chi}_b\mathcal{O}_c^B\bar{\chi}_b\right)$$
$$\times\int \mathcal{D}_{g_c}\bar{\chi}\,\delta\left[\int\sqrt{g_c}\,(\bar{\chi}+\bar{\chi}_b)\right]\exp\left(-\frac{1}{2\gamma}\int d^2\xi\sqrt{g_c}\bar{\chi}\mathcal{O}_c\bar{\chi}\right).$$

However unlike the closed string case we still have another delta function which involves the other orthogonal modes of $\chi$. Unfortunately this means we are left with a functional integral too difficult to be solved here.

All these calculations can be attempted taking homogeneous Neumann boundary conditions on the Liouville field $\partial_{\hat{n}}\varphi = 0$. The results for the critical exponents using the scaling argument are the same. However we run into difficulties in performing the semi-classical expansion because the classical solution $\varphi_c$ does not satisfy homogeneous Neumann boundary conditions so if the full Liouville field does, then the classical field and the quantum fluctuation are not independent, but rather are related with each other on the boundary $\partial_{\hat{n}}\varphi_c + \partial_{\hat{n}}\chi = 0$. So we conclude that the free boundary conditions are much better suited for the semi-classical expansion.

## 4.4 The tachyon gravitational scaling dimensions

Let us now calculate the gravitational scaling dimensions of the tachyon vertex operators for free boundary conditions. For the anomalous gravitational scaling dimension of the bulk tachyon vertex operator we consider the expectation value of the 1-point function at fixed area $A$

$$< W_j > (A) = \frac{1}{\Gamma(A)} \int \mathcal{D}_{\hat{g}}(\bar{\phi}, \bar{\Psi}) d\Psi_0 \Big( \oint d\hat{s} \Big)^{1/2} e^{-S_c^{00}[\bar{\Psi},\Psi_0,\hat{g}] - \bar{S}^0[\bar{\phi},\hat{g}]}$$
$$\times \delta \left( \int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A \right) \int d^2\xi_j \sqrt{\hat{g}} e^{\gamma_j \phi}.$$

By definition the bulk gravitational scaling dimension is as in the closed string $< W_j > (A) \sim A^{1-\Delta_j}$. Applying the scaling argument we find $\Delta_j = 1 - \gamma_j/\alpha$ and this leads to the KPZ equation for the anomalous gravitational dimension in the open string

$$\Delta_j - \Delta_j^0 = -\alpha^2 \Delta_j (\Delta_j - 1).$$

Similarly we define the anomalous gravitational scaling dimension of the boundary tachyon vertex operator by $< W_j^B > (A) \sim A^{1/2 - \Delta_j^B}$. Then the scaling argument gives $\Delta_j^B = \Delta_j/2$.

We can also define critical exponents associated with the expectation values at fixed length $L$. These should also be interpreted as anomalous gravitational scaling dimensions. In this case the asymptotic limits are $< W_j > (L) \sim L^{1-\Delta_j}$ and $< W_j^B > (L) \sim L^{1/2-\Delta_j^B}$, where $\Delta_j$ and $\Delta_j^B$ are given as in the case of fixed area $A$.

## 4.5 A connection with matrix models

These results generalise to other models and physical systems. As we observed before the open string is a toy model for the $c \leq 1$ boundary conformal field theories [17] coupled to 2D quantum gravity. In the next chapter we show that similar results can be written down for this more realistic class of models. Here we finish by considering a comparison with exact results of matrix models at genus zero [47]. According to ref. [47] we may deduce from matrix models calculations the following exact expression for $\Gamma(A, L)$ when the surface has the topology of a disc

$$\Gamma(A, L) = A^x L^y e^{-L^2/A},$$

where $x = -Q/\alpha$ and $y = -3 + Q/\alpha$. This formula is consistent with our scaling laws given in Eqs. (4.4)-(4.6). Introducing it in the definitions of $\Sigma(A)$ and $\Omega(L)$ we find

$$\sigma(1) = x + y/2 + 7/2, \quad \omega(1) = 2x + y + 5.$$

When we substitute back the values of $x$ and $y$ we get the same results for $\sigma(1)$ and $\omega(1)$ as we did using the David, Distler and Kawai's scaling argument.

This is an indication that our results should be in agreement with those obtained in models of dynamically triangulated open random surfaces. However it should be emphasised that a full comparison is beyond the scope of the present work.

# Chapter 5

# Minimal models on open random surfaces

The open string analysis can now be easily extended to $c \leq 1$ minimal conformal field theories on open random surfaces if we represent the matter sector by a conformally extended Liouville theory. The curious affinity between the matter and gravitational sector Liouville theories that emerges for closed surfaces generalises to the case with boundaries. We simply take the matter action of Eq. (2.14) with additional boundary terms

$$S_M[\Phi, \tilde{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\tilde{g}} \left[ \frac{1}{2} \tilde{g}^{ab} \partial_a \Phi \partial_b \Phi + i \left( \beta - 1/\beta \right) \tilde{R} \Phi \right]$$
$$+ \frac{i}{4\pi} \left( \beta - 1/\beta \right) \oint d\tilde{s} k_{\tilde{g}} \Phi \; + \mu^2 \int d^2\xi \sqrt{\tilde{g}} \left( e^{i\beta\Phi} + e^{-i/\beta\Phi} \right)$$
$$+ \lambda \oint d\tilde{s} \left[ e^{i\beta\Phi/2} + e^{-i/(2\beta)\Phi} \right]. \tag{5.1}$$

This is the conformally extended Toda field theory defined on an open surface for the Lie algebra $A_1$. It has recently been considered as a Coulomb gas description of the $c \leq 1$ minimal conformal matter in the case of Neumann boundary conditions imposed on the matter field [18].

## 5.1 Boundary minimal models and the Coulomb gas

That $c \leq 1$ minimal boundary conformal field theories may have a Coulomb gas description in terms of the extended Liouville action given in Eq. (5.1) can be seen as follows. Consider on the upper half-plane the basic bulk conformal operator

$$U(jj') =: e^{-i(j\beta - j'/\beta)\Phi(\zeta)} :,$$

where the normal order regularisation for the product of fields at the same point has been used [16]. Let us start with Neumann boundary conditions on $\Phi$ [18]. The action we need to consider is that of a free field

$$S[\Phi] = \frac{1}{16\pi} \int d^2\xi \Phi \Delta \Phi,$$

where as usual $\Delta$ is the covariant Laplacian. To describe the $c \leq 1$ theories we still have to introduce the background charge corresponding to the terms in the action with the scalar and geodesic curvatures. According to the Gauss-Bonnet theorem this will involve a boundary operator placed at some point $\Lambda \to +\infty$

$$U_B(\Lambda) =: e^{-\frac{i}{2}(\beta - 1/\beta)\Phi(\Lambda)} : .$$

Expanding the matter exponential interactions in powers of $\mu^2$ and $\lambda$ we naturally implement the Coulomb gas insertion of screening operators. In the presence of boundaries we should consider both bulk and boundary screening operators. Then the 1-point function of $U$ is defined by

$$< U >= \lim_{\Lambda \to +\infty} \Lambda^{f(\beta)} < e^{-\frac{i}{2}(\beta - 1/\beta)\Phi(\Lambda)} e^{-i(j\beta - j'/\beta)\Phi(\zeta)} \prod_{m=1}^{M} \oint ds(\xi_m) e^{i\frac{\beta}{2}\Phi(\xi_m)}$$

$$\times \prod_{m'=1}^{M'} \oint ds(\xi_{m'}) e^{-i\frac{1}{2\beta}\Phi(\xi_{m'})} > .$$

Here we consider the notation $<\ \ >$ for the functional integral over $\Phi$ weighted by $e^{-S[\phi]}$. We also consider the correlator divided by the partition function $\int \mathcal{D}\Phi e^{-S[\Phi]}$.

Moreover we have omitted the normal ordering sign which should be considered on all of the operators. To satisfy the zero mode charge conservation selection rule we must put $M = 2j+2$ and $M' = 2j'+2$. Also note that no bulk screening operators are needed.

Using the Neumann Green's function $G_N$ to shift out the linear terms in the action leads us to

$$< U >= \lim_{\Lambda \to +\infty} \Lambda^{f(\beta)} \prod_{m=1}^{2j+2} \oint ds(\xi_m) \prod_{m'=1}^{2j'+2} \oint ds(\xi_{m'}) e^{\mathcal{F}},$$

where

$$\mathcal{F} = \frac{1}{16\pi} \int d^2\xi d^2\xi' J(\xi) G_N(\xi, \xi') J(\xi'),$$

with the current $J$ given by

$$J(\xi) = 4\pi i \left(\beta - 1/\beta\right) \delta^2(\xi - \Lambda) + 8\pi i \left(j\beta - j'/\beta\right) \delta^2(\xi - \zeta) - 4\pi i\beta \sum_{m=1}^{2j+2} \delta^2(\xi - \xi_m)$$
$$+ \frac{4\pi i}{\beta} \sum_{m'=1}^{2j'+2} \delta^2(\xi - \xi_{m'}).$$

The Neumann Green's function on the upper half-plane is given by the method of images in terms of the Green's function on the whole plane

$$G_N(\xi, \xi') = G(\xi, \xi') + G(\bar{\xi}, \xi'),$$

where $\bar{\xi} = (x, -y)$ is the image point of $\xi = (x, y)$ and

$$G(\xi, \xi') = -\frac{1}{4\pi} \ln \frac{(x - x')^2 + (y - y')^2}{R}.$$

Above $R \to +\infty$ is the size of the upper half-plane. Because of the charge selection rule this infinite constant term does not contribute to the amplitude.

Now with $f(\beta) = 2(\beta - 1/\beta)^2$, $\zeta = (x, y)$ we get

$$< U >= \int \prod_{m=1}^{2j+2} dx_m \int \prod_{m'=1}^{2j'+2} dx_{m'} \prod_{m \neq k}^{2j+2} (x_m - x_k)^{\beta^2} \prod_{m=1}^{2j+2} \prod_{m'=1}^{2j'+2} (x_m - x_{m'})^{-2}$$

$$\times \prod_{m' \neq k'}^{2j'+2} (x_{m'} - x_{k'})^{1/\beta^2} \left\{ \prod_{m=1}^{2j+2} \left[ (x - x_m)^2 + y^2 \right] \right\}^{-2\beta(j\beta - j'/\beta)}$$

$$\times \left\{ \prod_{m'=1}^{2j'+2} \left[ (x - x_{m'})^2 + y^2 \right] \right\}^{-2(j\beta - j'/\beta)/\beta} (4y^2)^{(j\beta - j'/\beta)^2}.$$

Since $< W > \equiv F(y)$ we may scale $y \to \rho y$, $x_m \to \rho x_m$ and $x_{m'} \to \rho x_{m'}$ to get $F(\rho y) = \rho^p F(y)$, where $p = -2\Delta^0_{jj'}$ with $\Delta^0_{jj'}$ the bare conformal weight of $U$ given by Kač formula

$$\Delta^0_{jj'} = [(j + 1/2)\beta - (j' + 1/2)/\beta]^2 - (\beta - 1/\beta)^2/4.$$

Thus in agreement with the Möbius invariance we find

$$< U > \propto y^{-2\Delta^0_{jj'}}.$$

Up to a constant factor defined by the integral over the screening variables this is Cardy's result for the 1-point function of a bulk primary field [17, 18]. In principle the explicit calculation of the integral would select those operators $U(jj')$ which can be identified as conformal primary fields when the check with Cardy's 1-point functions is made.

If we consider the boundary operator

$$U_B(jj') =: e^{-\frac{i}{2}(j\beta - j'/\beta)\Phi(\zeta)} :$$

it is easy to see that $< U_B >= 0$ for all boundary fields $U_B$. This is also what is found by Cardy for boundary primary fields [17]. So all boundary operators $U_B$ might be primary fields.

Let us now consider the 2-point functions of these bulk and boundary operators. We start with

$$< UU' >= \lim_{\Lambda \to +\infty} \Lambda^{2(\beta - 1/\beta)^2} < e^{-\frac{i}{2}(\beta - 1/\beta)\Phi(\Lambda)} e^{-i(j\beta - j'/\beta)\Phi(\zeta)} e^{-i(j\beta - j'/\beta)\Phi(\zeta')}$$

$$\times \prod_{m=1}^{M} \oint ds(\xi_m) e^{i\frac{\beta}{2}\Phi(\xi_m)} \prod_{m'=1}^{M'} \oint ds(\xi_{m'}) e^{-i\frac{1}{2\beta}\Phi(\xi_{m'})} > .$$

Again no bulk screening operators are needed. Note that although we could use them to cancel the charges of the operators $U$ and $U'$, the same cannot be done with the background gravity charge placed at infinity. Of course with the boundary screening operators the selection rule is satisfied provided $M = 4j+2$ and $M' = 4j'+2$. Then integrating and using $\zeta = (x,y)$ and $\zeta' = (x',y')$ we find

$$
\begin{aligned}
< UU' >= &\int \prod_{m=1}^{4j+2} dx_m \int \prod_{m'=1}^{4j'+2} dx_{m'} \left\{ \prod_{m=1}^{4j+2} \left[ (x-x_m)^2 + y^2 \right] \right\}^{-2\beta(j\beta - j'/\beta)} (4y^2)^{(j\beta-j'/\beta)^2} \\
&\times \left\{ \prod_{m'=1}^{4j'+2} \left[ (x-x_{m'})^2 + y^2 \right] \right\}^{-2(j\beta-j'/\beta)/\beta} \left\{ \prod_{m=1}^{4j+2} \left[ (x'-x_m)^2 + y'^2 \right] \right\}^{-2\beta(j\beta-j'/\beta)} \\
&\times \left[ \prod_{m\neq k}^{4j+2} (x_m - x_k)^2 \right]^{\beta^2/2} \left[ \prod_{m=1}^{4j+2} \prod_{m'=1}^{4j'+2} (x_m - x_{m'})^2 \right]^{-1} \left[ \prod_{m'\neq k'}^{4j'+2} (x_{m'} - x_{k'})^2 \right]^{2/\beta^2} \\
&\times \left[ (x-x')^2 + (y-y')^2 \right]^{2(j\beta-j'/\beta)^2} \left[ (x-x')^2 + (y+y')^2 \right]^{2(j\beta-j'/\beta)^2} \\
&\times \left\{ \prod_{m'=1}^{4j'+2} \left[ (x'-x_{m'})^2 + y'^2 \right] \right\}^{-2(j\beta-j'/\beta)/\beta} (4y'^2)^{(j\beta-j'/\beta)^2} .
\end{aligned}
$$

When we scale $< UU' >\equiv F(\zeta,\zeta')$ we get $F(\rho\zeta,\rho\zeta') = \rho^q F(\zeta,\zeta')$, with $q = -4\Delta^0_{jj'}$. Now $F$ looks like a 4-point function on the whole plane. It depends on three distances $|\zeta - \bar{\zeta}|$, $|\zeta - \zeta'|$ and $|\zeta - \bar{\zeta}'|$. Global conformal invariance then implies

$$
< UU' >= F(r) \left[ \frac{(z-\bar{z})(z'-\bar{z}')}{(z-z')(\bar{z}-\bar{z}')(z-\bar{z}')(\bar{z}-z')} \right]^{2\Delta^0_{jj'}},
$$

where we have used the complex coordinates $z = x+iy$, $\bar{z} = x - iy$ and $r = (z - z')(\bar{z}-\bar{z}')/(z-\bar{z})(z'-\bar{z}')$. Once more this is Cardy's result up to the hypergeometric function $F(r)$ defined by the integral over the screening variables we wrote above [17, 18].

For the case of the 2-point function of boundary operators we follow the same procedure to conclude that

$$
< U_B U'_B >\propto (x-x')^{-2\Delta^0_{jj'}} .
$$

As in the previous cases this would still match Cardy's result if the right value for the constant factor is found after the calculation of the integral over the $2j + 2j' + 4$ screening variables [17, 18].

In a similar fashion we may attempt the calculation of higher point functions. The explicit calculation of the screening integrals could in principle define the set of bulk and boundary primary fields leading to a Coulomb gas formulation of the minimal boundary conformal field theories [18]. Here we will not attempt this explicit calculation. Nevertheless, we would like to point out that the analysis we have considered here should at least be generalisable to the case of free boundary conditions on the matter field. This is to be hoped for because we must be able to sew boundary conformal field theories and obtain conformal field theories on the whole plane. This can only be done using the free boundary conditions. In what follows we will assume that the coupling of the minimal boundary conformal field theories to 2D quantum gravity can be described by the conformally extended Liouville theory. We will allow for free, Neumann and Dirichlet boundary conditions on the matter and gravitational sectors. In all cases we will find a full Weyl invariant non-critical theory at the quantum level to all orders in the hypothetical Coulomb gas perturbation theory. However the Dirichlet boundary condition on the gravitational Liouville mode will lead to a discontinuity on the metric as it approaches the boundary.

## 5.2 Anomaly cancellation and critical exponents

For definiteness we take here the free boundary conditions on all fields. The central charge of the matter theory is $c_M = 1 - 6(\beta - 1/\beta)^2$. Requiring that the sum of this and the central charges of the gravitational sector Liouville field and the reparametrisation ghosts vanish gives $\gamma = \pm i\beta$, where $\gamma$ relates to our previous string $Q$, $Q = i(\gamma - 1/\gamma)$. The Liouville field renormalisation parameter must satisfy the equation $1 - \alpha(\beta + 1/\beta) + \alpha^2 = 0$ which, as before, gives us two branches $\alpha_+ = \beta$ and $\alpha_- = 1/\beta$. All the boundary renormalisation parameters relate to $\alpha$

and $\gamma$ as happened for the string case. We find dressed vertex operators of vanishing conformal weight on the bulk

$$U^D(jj') = \int d^2\xi\sqrt{\hat{g}}\exp\left[\left(l\beta + \frac{l'}{\beta}\right)\phi\right]\exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\Phi\right]$$

where $l = -j$, $l' = j' + 1$ or $l = j + 1$, $l' = -j'$. On the boundary we also define dressed primary vertex operators of vanishing conformal weight consistent with the need to consider the Liouville field as an arbitrary Weyl scaling on the whole surface

$$U_B^D(jj') = \oint d\hat{s}\exp\left[\left(l\beta + \frac{l'}{\beta}\right)\frac{\phi}{2}\right]\exp\left[-i\left(j\beta - \frac{j'}{\beta}\right)\frac{\Phi}{2}\right].$$

As occurred for the string, Dirichlet boundary conditions on the Liouville field imply that we have no dynamical quantum degrees of freedom on the boundary, and hence no boundary vertex operators. Although they still allow the cancellation of the Weyl anomaly provided the metric has a discontinuity as the boundary is approached.

The open string formulas for the critical exponents generalise to these models. Thus the susceptibility exponent is $\sigma(\chi_o) = 2 - \chi_o Q/(2\alpha)$, the Feynman mass exponent is $\omega(\chi_o) = 2 - \chi_o Q/\alpha$. The semi-classical limit is obtained for $\beta \to +\infty$ and, just like for closed surfaces, selects the classical branch $\alpha_+ = \beta$. As in the open string the saddle point expansion singles out the free boundary conditions on the Liouville field. Similarly we find the same expressions for the anomalous gravitational scaling dimensions of the primary vertex operators. In the end the gravitational scaling dimension of a boundary operator is half that of a bulk operator, the latter being related to its bare conformal dimension by the KPZ equation.

# Chapter 6

# Non-critical dual membranes

## 6.1 T-duality and D-branes

Over the years string theory has been establishing itself as the leading candidate to a unified description of particle physics and gravity. The latest and exciting development has been the rediscovery of string duality [19, 20, 48, 49]. As a symmetry of the exact theory it has shed new light into its strongly coupled phase. Particularly fascinating is the idea that different string theories at weak and strong coupling are in fact equivalent among each other and to new and mysterious higher dimensional M and F theories. New extended objects such as the D-branes have been revealed by string duality and interesting new links with particles, solitons and black holes have now emerged.

In this chapter we are specially interested in investigating the possible role that non-critical dual membranes may play in the understanding of the strongly coupled phase of the non-critical string theory. Since this will involve showing that they actually exist let us start with a review of the connection between T-duality and D-branes in the critical bosonic string [19].

Consider the closed oriented bosonic string theory. Take this to be compactified on a torus $X^i = X^i + 2\pi m_i R_i$, where $R_i$ is the compactification radius and $m_i \in \mathcal{Z}$ the correspondent winding number. Here $i, j, \ldots$ range from $26-k+1$ to 26 representing the $k$ compactified dimensions. For the non-compact dimensions we will use the notation $m, n, \ldots$ and for all we use the Greek letters $\mu, \nu, \ldots$. Respectively these range from 1 to $26 - k$ and from 1 to 26.

That this theory is self-dual when $R_i \to 0$ can be seen as follows. The solution of the wave equation satisfying the boundary condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$ is well known to be

$$X_R^\mu(z) = X_0^\mu + i\frac{\alpha'}{2} p_R^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{z^{-n}}{n} \alpha_n^\mu$$

$$X_L^\mu(\bar{z}) = \tilde{X}_0^\mu + i\frac{\alpha'}{2} p_L^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\bar{z}^{-n}}{n} \tilde{\alpha}_n^\mu,$$

where $z = e^{\tau + i\sigma}$ and as usual $\tau$, $\sigma$ are respectively the world-sheet time and space coordinates. Since

$$p_R^\mu \ln z + p_L^\mu \ln \bar{z} = (p_R^\mu + p_L^\mu)\tau + i(p_R^\mu - p_L^\mu)\sigma$$

we conclude that for non-compact dimensions $p_R^m = p_L^m$ and for compact dimensions

$$p_{R,L}^i = \frac{n_i}{R_i} \pm \frac{m_i R_i}{\alpha'}, \quad m_i, n_i \in \mathcal{Z}.$$

The mass squared is given by

$$M^2 = p_R^i p_R^i + N = p_L^i p_L^i + \tilde{N},$$

where $N$ and $\tilde{N}$ are the sums of the right and left moving oscillator levels. So we see that the state $(n_i, m_i)$ at $R_i$ has the same mass squared as the state $(m_i, n_i)$ at $R'_i = \alpha'/R_i$. As $R_i \to 0$ all states with non-zero momentum in the compact dimension become infinitely heavy, while the states with $n_i = 0$ which wind around the compact dimension become light going over to a continuum. On top of this the interactions are equal if $R_i \to 0$ and the dual string coordinates are defined by

$$Y_R^\mu(z) = X_R^\mu(z), \quad Y_L^\mu(\bar{z}) = (-1)^{S^\mu} X_L^\mu(\bar{z})$$

where $S^\mu = 0$ for a non-compact and 1 for a compact dimension.

In the case of the open oriented bosonic string we know that it cannot be self-dual because there is no winding number and so there are no states that become light as $R_i \to 0$. In this case the open strings propagate in $26 - k$ dimensions although their oscillations still span the full 26 dimensions. To see the nature of the dual theory first note that the standard Neumann boundary condition is translated into a Dirichlet boundary condition in the dual compact coordinates

$$\partial_\sigma X^i = 0 \to \partial_\tau Y^i = 0.$$

Now $Y^i = X_R^i - X_L^i$ where

$$X_R^\mu(z) = \frac{1}{2}X_0^\mu + i\frac{\alpha'}{2}p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{z^{-n}}{n}\alpha_n^\mu$$

$$X_L^\mu(\bar{z}) = \frac{1}{2}X_0^\mu + i\frac{\alpha'}{2}p^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{z}^{-n}}{n}\alpha_n^\mu.$$

So $Y^i(\sigma = 2\pi) - Y^i(\sigma = 0) = -2\pi\alpha' p^i = -2\pi\alpha' n_i/R_i = -2\pi n_i {R'}_i$. Thus we get an open string with its end points confined to a $26 - k$ hyperplane $Y^i = 0$, the D-brane.

Because open strings contain closed strings at the loop level we expect a dynamical D-brane. In fact the dual closed strings still propagate and vibrate in 26 dimensions and of course they contain gravity. At low energy only the massless states survive to generate the background fields representing the quantum fluctuations of the geometry of the target spacetime. In the case of the open string with its ends attached to the D-brane the only massless state is that of the dual of the perpendicular $U(1)$ gauge boson, $\oint d\tilde{s} A^i(Y^m)\partial_{\tilde{n}} Y^i$. To the polarisation $A^i$ we associate the background $U(1)$ gauge field $\mathcal{A}^i$ which should then reflect the collective motions of the D-brane.

To verify this consider the low energy effective action for the membrane embedded in a target space with metric $G_{\mu\nu}(Y)$

$$S_D = \mathrm{T}' \int d^{26-k}\sigma \sqrt{g} g^{a'b'} \partial_{a'} Y^\mu \partial_{b'} Y^\nu G_{\mu\nu}(Y),$$

where T′ is the membrane tension. Then expand around flat spacetime and flat membrane using the membrane coordinates

$$\sigma^a = Y^a, \quad G_{\mu\nu}(Y) = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(Y), \quad Y^i(\sigma) = \frac{1}{\sqrt{\mathrm{T}'}}\phi^i(\sigma).$$

The first order $h\phi$ term is given by

$$S_D \approx 2\kappa\sqrt{\mathrm{T}'}\int d^{26-k}\sigma\partial^a\phi^i h_{ai}.$$

Changing to momentum space

$$\phi_j(\sigma) = \int d^{26-k}q\phi_j(q)e^{iq\cdot\sigma}, \quad h_{ai}(\sigma) = \int d^{26-k}q' h_{ai}(q')e^{iq'\cdot\sigma}.$$

leads us to

$$S_D \approx 2i\kappa\sqrt{\mathrm{T}'}\delta^{ji}\int d^{26-k}qq^a\phi_j(q)h_{ai}(q).$$

Thus we find the $\phi^j h^{ai}$ vertex,

$$-2\kappa\sqrt{\mathrm{T}'}q^a\delta^{ji}.$$

This corresponds to a graviton hitting the D-brane which then vibrates

$$\int \mathcal{D}Y\mathcal{D}ge^{-S} = \int \mathcal{D}Y\mathcal{D}ge^{-S_D^0}\Big[1 - 2\kappa\mathrm{T}'$$
$$\times\int d^{26-k}\sigma\sqrt{g}g^{a'b'}\partial_{a'}Y^\mu\partial_{b'}Y^\nu h_{\mu\nu}(Y) + \cdots\Big],$$

where the flat spacetime membrane action is

$$S_D^0 = \mathrm{T}'\int d^{26-k}\sigma\sqrt{g}g^{a'b'}\partial_{a'}Y^\mu\partial_{b'}Y^\nu\eta_{\mu\nu}.$$

To calculate T′ we consider the open string disc amplitude with two vertex operators, one for the dual photon

$$W_\gamma = g_\gamma\oint d\tilde{s}\varepsilon(q)\cdot\partial_{\tilde{n}}Ye^{iq\cdot Y}$$

and the other for the graviton

$$W_G = g_G \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_a Y^\mu \partial_b Y^\nu \varepsilon_{\mu\nu}(q') e^{iq' \cdot Y}.$$

The amplitude is then given by the following functional integral

$$A = \beta_{st}^{-1} \int \mathcal{D}_{\tilde{g}}(Y, \tilde{g}) W_\gamma W_G e^{-S},$$

where $\beta_{st}$ is the dimensionless string coupling and the string action includes the usual set of renormalisation counterterms and Polyakov's matter action

$$S = \frac{\mathrm{T}}{2} \int d^2\xi \sqrt{\tilde{g}} \tilde{g}^{ab} \partial_a Y^\mu \partial_b Y^\nu \eta_{\mu\nu} + \mu_0^2 \int d^2\xi \sqrt{\tilde{g}} + \lambda_0 \oint d\tilde{s} + \nu_0 \oint d\tilde{s} k_{\tilde{g}}.$$

The string tension is as usual $\mathrm{T} = 1/(2\pi\alpha')$. This amplitude can be written as follows

$$A = \beta_{st}^{-1} g_\gamma g_G \oint d\tilde{s}_1 \tilde{n}_1^c \varepsilon^\lambda(q) \int d^2\xi_2 \sqrt{\tilde{g}_2} \tilde{g}_2^{ab} \varepsilon^{\mu\nu}(q') \frac{\partial}{\partial q_1^{\lambda c}} \frac{\partial}{\partial q_2^{\mu a}} \frac{\partial}{\partial q_3^{\nu b}} \int \mathcal{D}_{\tilde{g}}(Y, \tilde{g}) e^{-S'} \Big|_{q_j = 0},$$

where $j = 1, 2, 3$ and omitting the counterterms

$$S' = S - iq \cdot Y(\xi_1) - iq' \cdot Y(\xi_2) - q_1^{\alpha d} \partial_d Y_\alpha(\xi_1) - q_2^{\beta e} \partial_e Y_\beta(\xi_2) - q_3^{\delta f} \partial_f Y_\delta(\xi_2).$$

To calculate this amplitude we of course work in the conformal gauge $\tilde{g}^{ab} = e^\varphi \hat{g}^{ab}$ and impose the T-duality mixed boundary conditions on $Y^\mu$. On the non-compact dimensions we have Neumann boundary conditions $\partial_{\tilde{n}} Y^m = 0$, $m = 1, \ldots, 26 - k$, and on the compact dimensions we have homogeneous Dirichlet boundary conditions $Y^i|_B = 0$, $i = 26 - k + 1, \ldots, 26$.

After a total derivative we find

$$S' = \frac{\mathrm{T}}{2} \int d^2\xi \sqrt{\tilde{g}} \left( Y \cdot \tilde{\Delta} Y + \tilde{J} \cdot Y \right),$$

where

$$\tilde{J}^\alpha = -\frac{2i}{\mathrm{T}} \frac{\delta^2(\xi - \xi_1)}{\sqrt{\tilde{g}}} q^\alpha - \frac{2i}{\mathrm{T}} \frac{\delta^2(\xi - \xi_2)}{\sqrt{\tilde{g}}} q'^\alpha + 2 \frac{q_1^{\alpha d}}{\mathrm{T}\sqrt{\tilde{g}}} \partial_d \delta^2(\xi - \xi_1) + 2 \frac{q_2^{\alpha e} + q_3^{\alpha e}}{\mathrm{T}\sqrt{\tilde{g}}} \partial_e \delta^2(\xi - \xi_2)$$

$$-2\frac{q^{\alpha}_{1\tilde{n}}}{\mathrm{T}}\tilde{\delta}^2_B\delta^2(\xi-\xi_1)-2\frac{q^{\alpha}_{2\tilde{n}}+q^{\alpha}_{3\tilde{n}}}{\mathrm{T}}\tilde{\delta}^2_B\delta^2(\xi-\xi_2).$$

Now we can separate the compact and non-compact coordinates and shift the corresponding currents using respectively the Dirichlet and Neumann Green's functions. We then find

$$S'=\frac{1}{2}\int d^2\xi\sqrt{\tilde{g}}Y\cdot\tilde{\Delta}Y-\tilde{\mathcal{F}}_N-\tilde{\mathcal{F}}_D,$$

where

$$\tilde{\mathcal{F}}_{N,D}=\frac{\mathrm{T}}{8}\int d^2\xi' d^2\xi''\sqrt{\tilde{g}(\xi')}\tilde{J}^{m,i}(\xi')\tilde{G}_{N,D}(\xi',\xi'')\sqrt{\tilde{g}(\xi'')}\tilde{J}_{m,i}(\xi'')$$

The zero mode associated with the Neumann Green's function demands the momentum conservation on the non-compact dimensions $q'^m=-q^m$ because

$$\int d^2\xi\sqrt{\tilde{g}}\tilde{J}^m=\frac{-2i}{\mathrm{T}}\left(q^m+q'^m\right)=0.$$

For the compact dimensions we do not need this because the Dirichlet boundary condition sets the zero mode to zero. Thus we can write the amplitude as follows

$$A=\beta_{st}^{-1}g_\gamma g_G\oint d\tilde{s}_1\tilde{n}^c_1\varepsilon^\lambda(q)\int d^2\xi_2\sqrt{\tilde{g}_2}\tilde{g}_2^{ab}\varepsilon^{\mu\nu}(q')\frac{\partial}{\partial q_1^{\lambda c}}\frac{\partial}{\partial q_2^{\mu a}}\frac{\partial}{\partial q_3^{\nu b}}e^{\tilde{\mathcal{F}}_N+\tilde{\mathcal{F}}_D}\Big|_{q_j=0}Z,$$

where the partition function is

$$Z=\int\mathcal{D}_{\tilde{g}}(Y,\tilde{g})e^{-\frac{1}{2}\int d^2\xi\sqrt{\tilde{g}}Y\cdot\tilde{\Delta}Y}.$$

Let us now expand $\tilde{\mathcal{F}}_N$ and $\tilde{\mathcal{F}}_D$. After some straightforward calculations involving taking total derivatives we find

$$\begin{aligned}\tilde{\mathcal{F}}_N=\frac{1}{2\mathrm{T}}\Big[&-q^m q_m\tilde{G}_N(\xi_1,\xi_1)-2q^m q'_m\tilde{G}_N(\xi_1,\xi_2)+2iq^m q^d_{1m}\partial_d\tilde{G}_N(\xi_1,\xi')\Big|_{\xi'=\xi_1}\\&+2iq^m\left(q^e_{2m}+q^e_{3m}\right)\partial_e\tilde{G}_N(\xi_1,\xi')\Big|_{\xi'=\xi_2}-q'^m q'_m\tilde{G}_N(\xi_2,\xi_2)\\&+2iq'^m q^d_{1m}\partial_d\tilde{G}_N(\xi_2,\xi')\Big|_{\xi'=\xi_1}+2iq'^m\left(q^e_{2m}+q^e_{3m}\right)\partial_e\tilde{G}_N(\xi_2,\xi')\Big|_{\xi'=\xi_2}\\&+q_1^{md}q^{d'}_{1m}\partial_d\partial_{d'}\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi'=\xi_1}+2q_1^{md}q^e_{2m}\partial_d\partial_e\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi_1,\xi'=\xi_2}\end{aligned}$$

$$+2q_1^{md}q_{3m}^{f}\partial_d\partial_f\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi_1,\xi'=\xi_2} + q_2^{me}q_{2m}^{e'}\partial_e\partial_{e'}\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi'=\xi_2}$$
$$+2q_2^{me}q_{3m}^{f}\partial_e\partial_f\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi'=\xi_2} + q_3^{mf}q_{3m}^{f'}\partial_f\partial_{f'}\tilde{G}_N(\xi,\xi')\Big|_{\xi=\xi'=\xi_2}\Bigg]$$

and a similar formula for $\tilde{\mathcal{F}}_D$ obtained changing $N \to D$ and $m \to i$.

Taking into account the boundary conditions we know that for the Weyl anomaly cancellation we need $q^m q_m = 0$, $\varepsilon \cdot q = 0$, $q' \cdot q' = 0$, ${q'}_\mu \varepsilon^{\mu\nu} = {q'}_\nu \varepsilon^{\mu\nu} = 0$ and $\varepsilon^\mu_\mu = 0$. If we then use the boundary conditions, the conservation of momentum in the non-compact dimensions, eliminate the quadratic terms in the $q_j$, $j = 1, 2, 3$ and select only those terms which contribute to the $\varepsilon^i(q)\varepsilon_{mi}(-q)q^m$ vertex we may take the derivatives on the $q_j$ to find

$$A = \frac{2ig_\gamma g_G}{\beta_{st}\mathrm{T}^2}\varepsilon^i(q)\varepsilon_{mi}(-q)q^m \oint d\hat{s}_1 \int d^2\xi_2\sqrt{\hat{g}(\xi_2)}\hat{g}^{ab}(\xi_2)\partial_{\hat{n}_1}\partial_b\hat{G}_D(\xi,\xi')\Big|_{\xi=\xi_1,\xi'=\xi_2}$$
$$\times\left[\partial_a\hat{G}_N(\xi_1,\xi')\Big|_{\xi'=\xi_2} - \partial_a\hat{G}_N^R(\xi_2,\xi')\Big|_{\xi'=\xi_2}\right] Z,$$

where $\hat{G}_N^R$ is the regular part of the Neumann Green's function. It is important to note that this result is divergent. That can easily be seen by doing a total derivative in the above formula. It is a consequence of the $SL(2,\mathcal{R})$ invariance of the theory and so we need to factor out the infinite volume of the group. To do so let us write the amplitude on the upper half-plane with the complex coordinates $z = x + iy$, $\bar{z} = x - iy$. The Neumann and Dirichlet Green's functions are given by the method of images in terms of the Green's function on the whole plane

$$G_{N,D}(z,z') = G(z,z') \pm G(\bar{z},z'), \quad G_N^R(z,z') = G(\bar{z},z'),$$

where up to the now irrelevant constant zero mode

$$G(z,z') = -\frac{1}{4\pi}\ln|z-z'|^2.$$

Then after the simple calculation of the derivatives of the Green's functions we find

$$A = -\frac{ig_\gamma g_G}{4\pi^2\beta_{st}\mathrm{T}^2}\varepsilon^i(q)\varepsilon_{mi}(-q)q^m \oint (dz_1 + d\bar{z}_1)\int dz_2 d\bar{z}_2\Bigg[\frac{1}{(z_1-z_2)^2}\Bigg(\frac{1}{z_1-\bar{z}_2}$$

$$-\frac{1}{z_2-\bar{z}_2}\Big)+\frac{1}{(z_1-\bar{z}_2)^2}\left(\frac{1}{z_1-z_2}-\frac{1}{\bar{z}_2-z_2}\right)\Big]Z$$

Now on the upper half-plane we have three conformal Killing vectors $U^z = A_0 + A_1 z + A_2 z^2$, where $A_j \in \mathcal{R}$, $j = 1, 2, 3$. We can use them to fix three points in the integrals [27]

$$z_1 = u_1 + iU^z(z_1), \quad z_2 = u_2 + iU^z(z_2), \quad \bar{z}_2 = \bar{u}_2 - iU^{\bar{z}}(\bar{z}_2).$$

Then

$$dz_1 dz_2 d\bar{z}_2 = (\bar{u}_2 - u_2)(u_1 - u_2)(u_1 - \bar{u}_2) i \prod_j dA_j.$$

Substituting back in the amplitude we get

$$A = -\frac{g_\gamma g_G}{\pi^2 \beta_{st} \mathrm{T}^2} \varepsilon^i(q) \varepsilon_{mi}(-q) q^m Z \int \prod_j dA_j.$$

As it is the volume of the Möbius group is not properly normalised [50]. To do it let us work on the unit disc. As we have seen the conformal transformations are given by

$$z \to z' = e^{i\theta_0} \frac{z + c_0}{1 + \bar{c}_0 z},$$

where $|c_0| < 1$ and $\theta_0 \in \mathcal{R}_+$ for an oriented disc or $\theta_0 \in \mathcal{R}$ for an unoriented disc. In the case of infinitesimal transformations we find $\delta z = c_0 + i\theta_0 z - \bar{c}_0 z^2$. Putting $c_0 = c_1 + ic_2$ we may write $\delta z = c_1(1 - z^2) + ic_2(1 + z^2) + i\theta_0 z$. To normalise the conformal Killing vectors we set $(U, U) = \int d^2\xi \sqrt{\hat{g}} \hat{g}_{ab} U^a U^b = 1$ and use the metric on the unit sphere $d^2\hat{s} = 4(1 + \rho^2)^{-2} \rho d\theta d\rho$. Then we find the normalised vectors [50]

$$\sqrt{\frac{3}{16\pi}}\begin{pmatrix}1 - z^2 \\ 1 - \bar{z}^2\end{pmatrix}, \quad i\sqrt{\frac{3}{16\pi}}\begin{pmatrix}1 + z^2 \\ -1 - \bar{z}^2\end{pmatrix}, \quad i\sqrt{\frac{3}{4\pi}}\begin{pmatrix}z \\ -\bar{z}\end{pmatrix},$$

which lead us to the normalised group volume

$$\mathrm{Vol(CKV)} = \left(\frac{16\pi}{3}\right)^{3/2} \int d^2 c_0 d\theta_0 = \left(\frac{16\pi}{3}\right)^{3/2} \int \prod_j A_j.$$

On the other hand

$$Z\mathrm{Vol(CKV)} = (2\pi)^{-(26-k)/2}\left(\frac{\mathrm{Det}_N{}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(26-k)/2}\left(\mathrm{Det}_D\hat{\Delta}\right)^{-k/2}\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}.$$

Since the theory is Weyl invariant and all the divergencies have been renormalised this is just a finite constant which can be defined using the Selberg trace formulas for the determinants [50, 51]. We find

$$\mathrm{Det}_D\hat{\Delta}\mathrm{Det}_N{}'\hat{\Delta} = \mathrm{Det}_S{}'\hat{\Delta}, \quad \mathrm{Det}_N{}'\hat{\Delta} = \sqrt{2\pi}\sqrt{\mathrm{Det}_S{}'\hat{\Delta}}, \quad \mathrm{Det}'\hat{P}^\dagger\hat{P} = \sqrt{\mathrm{Det}_S{}'\hat{P}^\dagger\hat{P}},$$

where $S$ indicates that the determinants refer to the Riemann sphere. Calculating the area integral using the metric of the unit sphere we then get

$$Z\mathrm{Vol(CKV)} = 2^{13/2}\pi^{-(13-k)/2}\sqrt{Q_S^{26}},$$

where

$$Q_S^{26} = \left(\mathrm{Det}_S{}'\hat{\Delta}\right)^{-13}\sqrt{\mathrm{Det}_S{}'\hat{P}^\dagger\hat{P}}.$$

We may introduce the open string coupling in the effective $26-k$ dimensional theory on the world-sheet $g_{26-k}$ using dimensional analysis on the photon vertex operator $\varepsilon^i = \mathrm{T}^{3/2}\phi^i/g_{26-k}$. In this way $g_\gamma$ is a dimensionless constant. Similarly $g_G$ is dimensionless if $\varepsilon_{mi} = \mathrm{T}\kappa h_{mi}$. This leads to

$$A = -\frac{g_\gamma g_G \kappa\sqrt{\mathrm{T}}}{\pi^2\beta_{st}g_{26-k}}\pi^{-(13-k)/2}\sqrt{Q_S^{26}}\phi^i(q)h_{mi}(-q)q^m.$$

So if we normalise the coupling constant factor [50] as

$$\frac{g_\gamma g_G}{\beta_{st}} = \pi^{(16-k)/2}Q_S^{26^{-1/2}}$$

we get

$$A = -\frac{\kappa\sqrt{\mathrm{T}}}{\sqrt{\pi}g_{26-k}}\phi^i(q)h_{mi}(-q)q^m,$$

which when compared with the low energy vertex leads to the membrane tension [19]

$$\mathrm{T}' = \frac{\mathrm{T}}{\pi g_{26-k}^2}.$$

Thus we conclude that the T-dual to a theory of open and closed oriented bosonic strings is a theory of closed oriented bosonic strings plus a $26 - k$ dimensional D-brane. The D-brane is defined as the membrane where the dual open strings have their end points attached. It is a dynamical object interacting with the closed strings and its collective motions are represented by dual open string perpendicular $U(1)$ gauge bosons.

## 6.2 Non-critical D-instantons

When we consider the case of a string propagating in a non-critical target space we have to take into the picture the deeply non-linear dynamics of the conformally invariant path integral Liouville theory. In a weakly coupled phase the effect of the functional measure can be taken into account by the DDK renormalisation ansätse. In this theory the boundary conditions we may impose on the Liouville mode are constrained by quantum Weyl invariance. So while the Dirichlet boundary conditions lead to a discontinuity in the metric as the boundary is approached, the Neumann and free boundary conditions both allow a fully smooth and Weyl invariant theory. For simplicity let us consider the case of Neumann boundary conditions on the Liouville mode.

Since the DDK approach is only certainly valid for $d \leq 1$ we will start with the disc non-critical partition function with the case of the D-instanton in mind [20]. We are thus looking for stringy non-perturbative effects of the order $e^{-O(1/\beta_{st})}$ [20, 21]. Consider the dual of an open bosonic string theory with $k$ dimensions compactified on a torus. The dual string field satisfies homogeneous Dirichlet boundary conditions on the compact dimensions $Y^i|_B = 0$, $i = d - k + 1, \ldots, d$ and Neumann boundary conditions on the non-compact dimensions $\partial_{\tilde{n}} Y^m = 0$, $m = 1, \ldots, d-k$. Omitting the counterterms the partition function is given by

$$Z = \beta_{st}^{-\chi_o} \int \mathcal{D}_{\tilde{g}}(Y, \tilde{g}) \exp\left(-\frac{\mathrm{T}}{2}\int d^2\xi \sqrt{\tilde{g}} Y \cdot \tilde{\Delta} Y\right),$$

where for the disc $\chi_o = 1$. Integrating the matter fields and the reparametrisation ghosts we find after the DDK renormalisation

$$Z = \beta_{st}^{-\chi_o}\left(\frac{\mathrm{T}}{2\pi}\right)^{(d-k)/2}\left(\frac{\mathrm{Det}_N{}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(d-k)/2}\left(\mathrm{Det}_D\hat{\Delta}\right)^{-k/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}$$
$$\times \int \mathcal{D}_{\hat{g}}\phi e^{-S_L[\phi,\hat{g}]},$$

where the renormalised Liouville action is

$$S_L[\phi,\hat{g}] = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\phi\hat{\Delta}\phi + Q\hat{R}\phi\right) + \frac{Q}{4\pi}\oint d\hat{s}k_{\hat{g}}\phi$$
$$+2\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi} + \lambda_2\oint d\hat{s}e^{\alpha\phi/2}.$$

Here $\mu_2^2$ and $\lambda_2$ are arbitrary finite constants left over from the renormalisation process. For Weyl invariance we have $Q = \pm\sqrt{(25-d)/6}$ and for the Liouville field renormalisation $\alpha_\pm = (1/2)(Q \pm \sqrt{Q^2-4})$.

To integrate the Liouville mode let us separate out its zero mode. To do so we take the semi-classical branch of $\alpha$, $\alpha_-$, which in what follows will be denoted as $\alpha$. We then write $\phi = \phi_0 + \bar{\phi}$, where $\int d^2\xi\sqrt{\hat{g}}\bar{\phi} = 0$. So we get

$$S_L[\phi,\hat{g}] = S_L^0[\bar{\phi},\hat{g}] + \frac{Q\chi_o}{2}\phi_0 + 2e^{\alpha\phi_0}\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\bar{\phi}} + e^{\alpha\phi_0/2}\lambda_2\oint d\hat{s}e^{\alpha\bar{\phi}/2},$$

where

$$S_L^0[\bar{\phi},\hat{g}] = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\bar{\phi}\hat{\Delta}\bar{\phi} + Q\hat{R}\bar{\phi}\right) + \frac{Q}{4\pi}\oint d\hat{s}k_{\hat{g}}\bar{\phi}.$$

The measure is $\mathcal{D}_{\hat{g}}\phi = 1/(4\pi)\left(\int d^2\xi\sqrt{\hat{g}}\right)^{1/2} d\phi_0\mathcal{D}_{\hat{g}}\bar{\phi}$ so we need to integrate

$$I_0 = \frac{1}{\alpha}\int_{-\infty}^{+\infty} d\phi_0 \exp\left(-s\phi_0 - 2e^{\phi_0}\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\alpha\bar{\phi}} - e^{\phi_0/2}\lambda_2\oint d\hat{s}e^{\alpha\bar{\phi}/2}\right),$$

where $\alpha s = Q\chi_o/2$. Changing the integration variable from

$$\phi_0 \to \phi_0{}' = e^{\phi_0/2}\Big(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\Big)^{1/2}$$

the integral becomes

$$I_0 = \frac{2}{\alpha}\int_0^{+\infty} d\phi_0 \Big(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\Big)^s \phi_0^{-2s-1} \exp\left[-\phi_0^2 - \phi_0 \frac{\lambda_2 \oint d\hat{s} e^{\alpha\bar{\phi}/2}}{(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}})^{1/2}}\right].$$

Using the Mellin transform [43] we get

$$I_0 = \frac{2^{s+1}}{\alpha}\Big(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\Big)^s \Gamma(-2s) \exp\left[\frac{\lambda_2^2 \big(\oint d\hat{s} e^{\alpha\bar{\phi}/2}\big)^2}{16\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}}\right]$$

$$\times D_{2s}\left[\frac{\lambda_2 \oint d\hat{s} e^{\alpha\bar{\phi}/2}}{2\mu_2 \big(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\big)^{1/2}}\right],$$

where $D_{2s}(z)$ are the parabolic cylinder functions [43]. For integer $s$ this is

$$I_0 = \frac{2}{\alpha}\Big(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\Big)^s \Gamma(-2s) H_{2s}\left[\frac{\lambda_2 \oint d\hat{s} e^{\alpha\bar{\phi}/2}}{2\sqrt{2}\mu_2 \big(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\big)^{1/2}}\right],$$

where $H_{2s}(z)$ is the Hermite polynomial of degree $2s$.

Thus our partition function is now written as

$$Z = \beta_{st}^{-\chi_o} \left(\frac{\mathrm{T}}{2\pi}\right)^{(d-k)/2} \left(\frac{\mathrm{Det}_N{}'\hat{\Delta}}{\int d^2\xi \sqrt{\hat{g}}}\right)^{-(d-k)/2} \left(\mathrm{Det}_D \hat{\Delta}\right)^{-k/2} \frac{\sqrt{\mathrm{Det}' \hat{P}^\dagger \hat{P}}}{\mathrm{Vol(CKV)}}$$

$$\times \int \mathcal{D}_{\hat{g}}\bar{\phi} \Big(\int d^2\xi \sqrt{\hat{g}}\Big)^{1/2} \frac{2^{s-1}}{\pi\alpha}\Big(2\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\Big)^s \Gamma(-2s) \exp\left[\frac{\lambda_2^2 \big(\oint d\hat{s} e^{\alpha\bar{\phi}/2}\big)^2}{16\mu_2^2 \int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}}\right]$$

$$\times D_{2s}\left[\frac{\lambda_2 \oint d\hat{s} e^{\alpha\bar{\phi}/2}}{2\mu_2 \big(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\bar{\phi}}\big)^{1/2}}\right] e^{-S_L^0[\bar{\phi},\hat{g}]}.$$

Taking into account eventual analytical continuations in $s$ [30, 31, 32] this is a formula which in principle can be used for any $d$ and $\chi_o$. Let us consider the case where $k = d$. Following Polchinski [20] the partition function on the disc is to be interpreted as minus the D-instanton action, which appears in the one D-instanton amplitude $A_1 \sim e^Z$. This only gives the right weight $e^{-O(1/\beta_{st})}$ if the partition function is negative. In the critical dimension this is true because of the negative value of the renormalised Möbius volume [52]. Here we have the additional problem associated with the multiple scattering of the Liouville vertex operators. Let us try to calculate the partition function in the limiting case $d = 1$. In this case $Q = 2$, $\alpha = 1$ lead to $s = 1$. Then we find $H_2(z) = 4z^2 - 2$ which leads to

$$Z = \frac{\Gamma(-2)}{2\pi\beta_{st}}\left(\mathrm{Det}_D\hat{\Delta}\right)^{-1/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}\int\mathcal{D}_{\hat{g}}\bar{\phi}\left(\int d^2\xi\sqrt{\hat{g}}\right)^{1/2}\left[\lambda_2^2\left(\oint d\hat{s}e^{\bar{\phi}/2}\right)^2 -4\mu_2^2\int d^2\xi\sqrt{\hat{g}}e^{\bar{\phi}}\right]e^{-S_L^0[\bar{\phi},\hat{g}]}.$$

Integrating over $\bar{\phi}$ on the upper half-plane we find

$$Z = \frac{\Gamma(-2)}{2\pi\beta_{st}}\left(\mathrm{Det}_D\hat{\Delta}\right)^{-1/2}\left(\frac{\mathrm{Det}_N\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-1/2}\frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}\varepsilon^{-6}\Lambda^8 \left[\lambda_2^2\int_{-\infty}^{+\infty}dxdy(x-y)^{-2} - \frac{\mu_2^2}{2}\int_{-\infty}^{+\infty}dxdyy^{-2}\right].$$

Here $\varepsilon \to 0$ is a short distance cutoff which comes from the Green's functions calculated at equal points. We have also introduced the background gravity charge in the boundary at $\Lambda \to \infty$. These terms and $\Gamma(-2) > 0$ are to be absorbed in the coupling constants $\beta$, $\lambda_2$ and $\mu_2$. As is clear the integrals over the whole plane are divergent. Once more this is a consequence of the Möbius invariance of the theory. To factor out the divergence we fix the bulk vertex operator on the center of the unit disc and the two boundary operators on 0 and $\pi/2$. Then the integrals are

$$\int_{-\infty}^{+\infty}dxdy(x-y)^{-2} = \frac{1}{2}\int_{-\infty}^{+\infty}dxdy(y)^{-2} = 4\int d^2c_0.$$

The renormalisation of the Möbius volume leads to [52]

$$\mathrm{Vol(CKV)} = -\left(\frac{16\pi}{3}\right)^{3/2}\frac{\pi^2}{2},$$

for the oriented theory. For the unoriented case we just have to multiply by 2. Since $\int d^2c_0 = \pi$ gives the area of the unit disc we get

$$Z = -\left(\frac{3}{16\pi}\right)^{3/2} \frac{4}{\pi\sqrt{\pi}\beta_{st}} \left(\lambda_2^2 - \mu_2^2\right) \sqrt{Q_S^2},$$

where

$$Q_S^2 = \left(\mathrm{Det}_S{}'\hat{\Delta}\right)^{-1} \sqrt{\mathrm{Det}_S{}'\hat{P}^{\dagger}\hat{P}}$$

and now $\lambda_2$ and $\mu_2^2$ are taken as dimensionless constants.

Since we may absorb $\lambda_2^2$ into $\beta_{st}$ this result depends only on $\beta_{st}$ and on the ratio of the two Liouville couplings $\mu_2/\lambda_2$. Note that this is a general result. If we consider an arbitrary positive integer $s$ given by $2s = Q\chi_o/\alpha$ it is easy to see by looking at the Hermite polynomials that we may always absorb $\lambda_2^{Q/\alpha}$ into $\beta_{st}$ and so obtain amplitudes which only depend on $\beta_{st}$ and on the ratio $\mu_2/\lambda_2$. By analytical continuation this is valid for any $s$, that is for any $d$ and $\chi_o$. Within the DDK approach the one dimensional non-critical string theory can be viewed as a two dimensional critical theory defined in a consistent background given by the matter and Liouville systems [32, 53]. So the ratio of the Liouville couplings defines different possible backgrounds for the critical string theory and so different non-critical theories. From perturbative quantum Weyl invariance alone all positive values of the couplings are allowed. They are simply the finite left-overs of the renormalisation process which are positive so that the right damping positive sign in the Liouville action is obtained. What we are going to see next is that not all of these values and so not all perturbatively consistent backgrounds lead to a consistent theory. Absorbing $\lambda_2^2$ in $\beta_{st}$ and setting $\omega_2 = \mu_2/\lambda_2$ the partition function can be written as

$$Z = -\left(\frac{3}{16\pi}\right)^{3/2} \frac{4}{\pi\sqrt{\pi}\beta_{st}} \left(1 - \omega_2^2\right) \sqrt{Q_S^2}. \tag{6.1}$$

Now, the problem we face here is that we have a relative sign between the bulk and boundary contributions which means that the ratio $\omega_2$ controls the sign of the partition function. In fact the zeta function regularisation of the determinants always leads to positive values. On the other hand $\omega_2$ can be any positive number. This means that only for $\omega_2 < 1$ we can find

$$Z \propto -\frac{1}{\beta_{st}}.$$

So we conclude that the strength of the stringy non-perturbative effects off the critical dimension is of the order $e^{-O(1/\beta_{st})}$ in agreement with the result found in matrix models [21] and in the critical string theory [20]. This is to be expected since the weight of boundaries only depends on the topology of the string world-sheet. However we have found that the correct sign can only be obtained if the boundary Liouville coupling constant is actually bigger than its bulk counterpart $\lambda_2 > \mu_2$. If they are equal we find no non-perturbative effects. If $\mu_2 > \lambda_2$ the theory is inconsistent.

Note that Eq. (6.1) is only valid if $\lambda_2 > 0, \mu_2 \geq 0$. If we take $\lambda_2 = 0, \mu_2 \geq 0$ we just have to go back to the integral of the Liouville zero mode $\phi_0$ to see that in this case it does not bring an extra minus sign to $Z$. So, for $\lambda_2 = 0, \mu_2 \geq 0$ we also find the right weight $e^{-O(1/\beta_{st})}$ due to the negative value of the renormalised Möbius volume.

We may also analyse the range $d < 1$ within the DDK approach. This corresponds to the coupling to 2D quantum gravity of the $c < 1$ minimal boundary conformal field theories. In this case $d = c = 1 - 6(\beta - 1/\beta)^2$. Since the background gravity charge only depends on the world-sheet topology we still find $\alpha s = Q\chi_o/2$, where $Q = \beta + 1/\beta$ and $\alpha = \alpha_+ = \beta$. Then on the disc we find $s = (1+\beta^2)/(2\beta^2)$ which is a rational number because $\beta^2 = (2+k')/(2+k)$, where $k, k'$ are positive integers. To see if the analysis of the limiting case $d = 1$ still holds for general $k, k'$ or indeed for $d > 1$ we would have to get involved with the intricate dynamics of the multiple Liouville scattering where $s$ does not have to be an integer [32]. However if as in the case $d = 1$ the boundary dominates the bulk, $\lambda_2 > \mu_2$, the stringy non-perturbative effects should still be of the order $e^{-O(1/\beta_{st})}$. If just one of the Liouville couplings is zero the same result is to be expected. These remarks are based on a possible Coulomb gas representation of the minimal boundary conformal field theories. They suggest that the strength of the non-perturbative effects is of the order $e^{-O(1/\beta_{st})}$. A precise definition of the effects needs a careful analysis of the partition function on this type of models.

## 6.3 The size of non-critical D-instantons

To look for D-branes we may also consider the case where there is an exchange of closed string states while they are separated by a distance $Y^iY_i$ [49]. This corresponds to a dual open string with Chan-Paton charges on its end points which lie on the D-branes. It also corresponds to the case where the string does not wind around the compact dimensions so that all fields are single valued. The diagram for such an exchange is an open string annulus. Let us consider $p+1$ uncompactified dimensions $m = 0, \ldots, p$ and $d-p-1$ compact ones. According to T-duality we use Neumann boundary conditions for the $p+1$ D-brane coordinates and Dirichlet boundary conditions on the other compact coordinates. We consider that $Y^i|_{B_1} = -\mathcal{Y}^i/2$ and $Y^i|_{B_2} = \mathcal{Y}^i/2$ where $\mathcal{Y}^i \geq 0$, for all $i = p+1, \ldots, d$.

As usual the partition function is $Z = Z_{nc}Z_c$, where $Z_{nc}$, $Z_c$ are respectively the partition function on the non-compact and compact space. To integrate on the compact coordinates we have to factorise the classical solution $Y^i = Y_c^i + \bar{Y}^i$, where $\bar{Y}^i$ satisfies homogeneous Dirichlet conditions on both boundaries and $\tilde{\Delta}Y_c^i = 0$, $Y_c^i|_{B_1} = -\mathcal{Y}^i/2$, $Y_c^i|_{B_2} = \mathcal{Y}^i/2$. Then integrating the matter fields and the reparametrisation ghosts in the standard way we find

$$Z = V_{p+1}\left(\frac{\mathrm{T}}{2\pi}\right)^{(p+1)/2} \int d\tau \frac{\det(\psi, B)}{\sqrt{\det(B,\hat{}B)}} \frac{\sqrt{\mathrm{Det}'\hat{P}^{\dagger}\hat{P}}}{\mathrm{Vol(CKV)}} \left(\frac{\mathrm{Det}_N{}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(p+1)/2}$$

$$\times \left(\mathrm{Det}_D\hat{\Delta}\right)^{-(d-p-1)/2} \exp\left(-\frac{\mathrm{T}}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a Y_c^i \partial_b Y_{ci}\right) \int \mathcal{D}_{\hat{g}}\varphi e^{-S_L[\varphi,\hat{g}]},$$

where $V_{p+1}$ is the D-brane world volume and $\tau$ is the modulus of the annulus. In the case of the annulus $\chi_o = 0$. So we can choose $\hat{R} = k_{\hat{g}} = 0$ without the introduction of a background gravity charge. So we will not have to deal with the Liouville amplitudes because $s = 0$. After the DDK renormalisation we can integrate $\phi$ with Neumann boundary conditions to obtain

$$Z = \frac{V_{p+1}\Gamma(0)}{2\pi\alpha}\left(\frac{\mathrm{T}}{2\pi}\right)^{(p+1)/2}\int d\tau \frac{\det(\psi, B)}{\sqrt{\det(\hat{B}, B)}} \frac{\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}}}{\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}_N{}'\hat{\Delta}}{\int d^2\xi\sqrt{\hat{g}}}\right)^{-(p+2)/2}$$

$$\times\left(\mathrm{Det}_D\hat{\Delta}\right)^{-(d-p-1)/2}\exp\left(-\frac{\mathrm{T}}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a Y^i_c\partial_b Y_{ci}\right).$$

In this formula it is clear that the only effect of the Liouville mode within the DDK approach is to introduce just another dimension into the problem and so the calculation can be immediately carried out by following the steps already taken in the critical dimension [49]. To describe the annulus we will choose the parameter domain to be a square $\{0 \leq \xi^a \leq 1\}$ of area $\tau$. In the complex coordinates $z = \xi^1 + i\tau\xi^2$, $\bar{z} = \xi^1 - i\tau\xi^2$ the area element is $d^2s = dzd\bar{z} = d\xi^1 d\xi^1 + \tau^2 d\xi^2 d\xi^2$. So the reference metric is

$$\hat{g}_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^2 \end{bmatrix}, \quad \hat{g}^{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\tau^2 \end{bmatrix}.$$

The Beltrami differential is then given by

$$\psi_{ab} = \frac{d}{d\tau}\hat{g}_{ab} = \begin{bmatrix} 0 & 0 \\ 0 & 2\tau \end{bmatrix}.$$

Since $\tau$ can reach infinity the conformal Killing vectors must be regular there. If they are periodic in $\xi^1$, $U^z(0) = U^z(1)$ and $U^z(i\tau) = U^z(1+i\tau)$. Then if they satisfy Neumann boundary conditions we get $U^z = U^{\bar{z}} = A_0 \in \mathcal{R}$. In the $\xi^a$ coordinates we then have $U^z = U^1 + iU^2$, where $U^1 = A_0$ and $U^2 = 0$. Because $\chi_o = 0$ the Riemann-Roch theorem tells us that we also have just one holomorphic quadratic differential $B_{ab}$. It is a traceless and symmetric tensor which satisfies $\hat{P}^\dagger(B) = 0$. Then under regularity at infinity, Neumann boundary conditions and periodicity like in the case of $U$ we conclude that $B_{z\bar{z}} = B_{\bar{z}z} = 0$, $B_{zz} = B_{\bar{z}\bar{z}} = B_0 \in \mathcal{R}$, which in the $\xi^a$ coordinates is

$$B = B_0 \begin{bmatrix} 2 & 0 \\ o & -2\tau^2 \end{bmatrix}.$$

Then we find $(U, U) = \tau A_0^2$. For the case of the holomorphic quadratic differential we use the same inner product we have used for the world-sheet metric. Since $B$ is traceless we get $(B, B) = 8\tau B_0^2$ and $(\psi, B) = -4B_0$. With the normalisation [49, 54] $B_0 < 0$ and $|A_0| = 1$ we may finally write

$$\frac{\det(\psi, B)}{\sqrt{\det(B, B)\mathrm{Vol(CKV)}}} = \frac{\sqrt{2}}{\tau}.$$

We also have to deal with the infinite determinants associated with the covariant Laplacian. In our metric this operator is written as

$$\hat{\Delta} = -\frac{1}{\tau}\left(\tau^2\partial_1^2 + \partial_2^2\right).$$

The eigenfunctions of the Laplacian with Neumann and Dirichlet boundary conditions are

$$\begin{pmatrix} N(\xi^1, \xi^2) \\ D(\xi^1, \xi^2) \end{pmatrix} = e^{2\pi i n \xi^1} \begin{pmatrix} \cos \pi m \xi^2 \\ \sin \pi m \xi^2 \end{pmatrix}, \quad \begin{pmatrix} m \geq 0 \\ m > 0 \end{pmatrix}, n \in \mathcal{Z}.$$

The eigenvalues have a common expression

$$\lambda_{mn} = \frac{\pi^2}{\tau^2}|2i\tau n - m|^2.$$

Using the generalised Riemann zeta function [27, 26, 43]

$$\tilde{\zeta}(s) = \sum_{mn}{}' |2i\tau n - m|^{-2s},$$

where the prime means the omission of the single zero mode $m = n = 0$, the determinant of the Laplacian is given by

$$\ln \mathrm{Det}'\hat{\Delta} = \frac{1}{2}\ln\frac{\pi^2\tilde{\zeta}(0)}{\tau^2} - \frac{1}{2}\tilde{\zeta}'(0).$$

Then after the Sommerfeld-Watson trick [26, 27] we apply the residue theorem at $s = 0$ and manipulate the resulting formula to find

$$\mathrm{Det}_N{}'\hat{\Delta} = \mathrm{Det}_D\hat{\Delta} = 2\tau|\eta(2i\tau)|^2,$$

where the Dedekind $\eta$ function is

$$\eta(2i\tau) = e^{-\pi\tau/6}\prod_{n=1}^{+\infty}\left(1 - e^{-4\pi n\tau}\right).$$

Since the tensor operator $(\hat{P}^\dagger\hat{P})^a_c$ is just the vector Laplacian $2\delta^a_c\hat{\Delta}$ we also get

$$\sqrt{\mathrm{Det}'\hat{P}^\dagger\hat{P}} = \sqrt{2}\tau|\eta(2i\tau)|^2.$$

To be able to write out the partition function we still need to define the classical action. First note that $Y^i_c = \mathcal{Y}^i\xi^2 - \mathcal{Y}^i/2$. So $\int d^2\xi\sqrt{\hat{g}}\hat{g}^{ab}\partial_a Y^i_c\partial_b Y^i_c = \mathcal{Y}^2/\tau$. If we put all this together and change the variable from $\tau$ to $1/(2\tau)$ taking into account that $|\eta(i/\tau)| = \sqrt{\tau}|\eta(i\tau)|$, we get the following result

$$Z = V_{p+1}\Gamma(0)(2\pi\alpha^2\mathrm{T})^{-1/2}\int\frac{d\tau}{\tau}\left(\frac{\mathrm{T}}{4\pi\tau}\right)^{(p+2)/2}\exp\left(-\mathrm{T}\mathcal{Y}^2\tau\right)|\eta(i\tau)|^{1-d}.$$

Note that here the modular group acts like an ordinary change of variables and so it does not lead to a cutoff in the integration range. Although the integrand is not modular invariant the integral is and so there is no anomaly. The transformation $\tau$ to $1/(2\tau)$ corresponds to the exchange of $\xi^1$ and $\xi^2$ and so it takes us to Polchinski's cylinder [49]. If we then rescale it by $1/\tau$ as we are allowed by Weyl invariance we get our cylinder with fixed circunference but now with length $1/\tau$ instead of $\tau$.

Introducing the $\eta$ function we now need to consider the asymptotics $\tau \to +\infty$, $q \to 0$ where $q = e^{-\pi\tau}$ [49]. To lowest order

$$|\eta(i\tau)|^{1-d} = q^{(1-d)/12}\left[\prod_{n=1}^{+\infty}\left(1 - q^{2n}\right)\right]^{1-d} \approx q^{(1-d)/12} + (d-1)q^{(25-d)/12} + \cdots.$$

Using again the modular transformation $|\eta(i/\tau)| = \sqrt{\tau}|\eta(i\tau)|$ we get the asymptotics for $\tau \to 0$. Like in the critical theory [49] this corresponds to the ultra-violet limit for the one loop open string channel as can be seen by shrinking the circunference of Polchinski's cylinder to zero. By duality, this is an infrared limit in the closed string channel. This is clear in the rescaled Polchinski's cylinder which has an infinitely long length. It is thus dominated by the lightest closed string states. We obtain

$$Z = V_{p+1}\Gamma(0)(2\pi\alpha^2\mathrm{T})^{-1/2}\int\frac{d\tau}{\tau}\left(\frac{\mathrm{T}}{4\pi\tau}\right)^{(p+2)/2}\exp\left(-\mathrm{T}\mathcal{Y}^2\tau\right)\tau^{(d-1)/2}\Big[e^{(d-1)\pi/(12\tau)} + (d-1)e^{(d-25)\pi/(12\tau)} + \cdots\Big].$$

So we find a tower of massive poles for $d < 1$ and one massless tachyon pole for $d = 1$. This massless tachyon pole corresponds to the massless tachyon in the two dimensional critical theory where the Liouville mode is interpreted as Euclidean time [32] (with only Neumann or free boundary conditions since the Dirichlet boundary conditions lead to a discontinuity as the boundary is approached). In fact from the dressing condition $p_j^2 - \gamma_j(\gamma_j - Q) = 1$ we may define the tachyon momentum as $(E_j, p_j)$ where $E_j = \gamma_j + Q/2$. Then the tachyon mass is $m_j^2 = E_j^2 - p_j^2 = (1-d)/24$ which is zero for $d = 1$. If we consider a Coulomb gas representation of the $c \leq 1$ minimal boundary conformal field theories it is clear that we get exactly the same result as for the string because $s = 0$ and the topological background charge can be set to zero. However this only confirms the suggestion that the order of magnitude of the non-perturbative effects should be the same as that of the string. For $p = -1$ the massless pole contribution is

$$Z_p = \frac{\Gamma(0)}{2\pi\sqrt{2}}\int d\tau\tau^{-3/2}\exp\left(-\mathrm{T}\mathcal{Y}^2\tau\right).$$

This can be integrated using the Gamma function giving

$$Z_p = -\frac{\Gamma(0)}{\sqrt{2\pi}}\mathrm{T}^{1/2}\mathcal{Y}.$$

Let us now regularise the divergence associated with the above Gamma function [32]. If $\lambda_2 > 0$, $\mu_2 \geq 0$ we use the KPZ scaling on the disc for fixed boundary length $L$

$$\lambda_2^{2s}\Gamma(-2s) = \int_0^{+\infty} dL L^{-2s-1} e^{-\lambda_2 L}.$$

For $s = 0$ we may cutoff the integral from $[0, +\infty]$ to $[\varepsilon, 1/\lambda_2]$, where $\varepsilon \to 0$ and $\lambda_2 \to 0$. So we get $-\ln(\varepsilon\lambda_2)$ which is a positive number for sufficiently small $\varepsilon$ and $\lambda_2$. Now the Liouville zero mode is $\phi_0 = \ln L$. So $\ln\varepsilon < \phi_0 < \ln(1/\lambda_2)$ and $\ln(1/\lambda_2)$ defines the Liouville wall [32]. We then write

$$Z_p = -\ln(1/\lambda_2)\Big(\frac{\mathrm{T}}{2\pi}\Big)^{1/2}\mathcal{Y}.$$

which diverges like $\ln(1/\lambda_2)$ as the now dimensionless $\lambda_2 \to 0$. If $\lambda_2 = 0$ we use the KPZ scaling with fixed area $A$ to get a similar divergence with a dimensionless $\mu_2^2$. These are the divergences which correspond to that already known to appear in the matrix models [22, 23].

Now recall that according to Polchinski's string picture [20] the one D-instanton amplitude is given by $A_1 \sim e^Z$, where $Z$ is minus the D-instanton action. From the field theory of instantons in one dimension [55] the one instanton amplitude in the semi-classical approximation to one loop is

$$A_1 = A\Big(\frac{\omega}{\pi\hbar}\Big)^{1/2} e^{-S_0/\hbar - \omega T/2}\left[1 + O(\hbar)\right],$$

where $S_0 = \int_{-a}^{a} dx (2V)^{1/2}$ is the instanton action, $\omega = V''(\pm a)$ and $A$ is a normalisation constant. Here $V$ is a symmetric double well potential with two minima at $x = \pm a$, where $V(\pm a) = 0$. In this theory $A_1$ is the one instanton contribution to the transition amplitude, $< a|e^{-HT/\hbar}| - a >$, for a first quantised particle of unit mass to go from $x = -a$ to $x = a$ as the Euclidean time $t$ goes from $t = -T/2$ to $t = T/2$ for large $T$. We consider the instanton with its center fixed at $t = 0$. In this picture the instanton is a well localised extended object with a size of the order $1/\omega$. To connect with the string description we have to interpret our dual string coordinate field $Y$ as the Euclidean time $t$. Then the string has its end points fixed in the center of the instanton. To the disc level the string can only see a point-like object being sensitive just to the instanton action. So to tree level both descriptions agree if $Z = -S_0/\hbar$.

In the case of two D-instantons we have seen that at low energy corresponding to a large distance $\mathcal{Y}$, they interact by exchanging the massless Klein-Gordon fields. Following Polchinski [20] the two D-instanton amplitude for point-like events is then

$$A_2 \sim e^{2Z+Z_p}.$$

Once more from the theory of instantons [55] the two instanton amplitude in the semi-classical approximation to one loop is

$$A_2 = A^2 \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-S_0/\hbar-\omega T/2} \left[1 + O(\hbar)\right].$$

In the above formula $A_2$ is the two instanton contribution to the transition amplitude, $< -a|e^{-HT/\hbar}| - a >$, for the particle to go from $x = -a$ to $x = a$ and back to $x = -a$ as time goes from $t = -T/2$ to $t = T/2$, $T \to +\infty$. In this case the two instantons are in fact an instanton and an anti-instanton with widely separated centers respectively fixed close to $t = -T/2$ and to $t = T/2$. In the string picture the instantons are seen as point-like objects located at $t = -\mathcal{Y}/2$ and at $t = \mathcal{Y}/2$. They interact with a force $Z_p = -\omega\mathcal{Y}/2$. So to one loop both descriptions agree provided

$$\omega = \frac{\ln(1/\lambda_2)}{\pi\sqrt{\alpha'}}, \quad \lambda_2 > 0,\ \mu_2 \geq 0$$

or

$$\omega = \frac{\ln(1/\mu_2^2)}{\pi\sqrt{\alpha'}}, \quad \lambda_2 = 0,\ \mu_2 > 0.$$

From the field theory semi-classical calculation of $A_1$ to one loop $1/\omega$ is interpreted as the average size of the instanton. Thus we conclude that the size of the D-instantons is of the order of $\sqrt{\alpha'}/\ln(1/\lambda_2)$ for $\lambda_2 > 0$, $\mu_2 \geq 0$ or of the order of $\sqrt{\alpha'}/\ln(1/\mu_2^2)$ for $\lambda_2 = 0$, $\mu_2 > 0$.

Let us now consider the case $d = 1$, $p = 0$ where the scalar field is compactified on a circle of radius $R$. Here we have to take into account that the field is not single valued when the string is winding around the compact dimension [56]

$$Y(\xi^1, \xi^2) = Y(\xi^1, \xi^2) + 2\pi n R, \quad n = 1, 2, \ldots .$$

Thus the periodicity in $\xi^1$ can be written as

$$Y(\xi^1 + 1, \xi^2) = Y(\xi^1, \xi^2) + 2\pi n R,$$

which leads us to

$$Y(\xi^1, \xi^2) = 2\pi n R \xi^1 + \bar{Y}(\xi^1, \xi^2),$$

where $\bar{Y}$ is now a single valued field which in this case satisfies Neumann boundary conditions. In the case $\lambda_2 > 0$, $\mu_2 \geq 0$ we thus find the partition function

$$Z = \frac{R}{4\pi\sqrt{\alpha'}} \ln(1/\lambda_2) \int_0^{+\infty} \frac{d\tau}{\tau^2} \sum_{n=1}^{+\infty} \exp\left(-\frac{\pi n^2 R^2}{2\alpha' \tau}\right).$$

Integrating this using the zeta function and the Bernoulli numbers [43]

$$\sum_{n=1}^{+\infty} n^{-2} = \zeta(2) = \pi^2 B_2 = \frac{\pi^2}{6}$$

we get

$$Z = \ln(1/\lambda_2) \frac{\sqrt{\alpha'}}{12R}.$$

This result is to be compared with the partition function on closed surfaces which agrees with the corresponding matrix model [22, 23]. Note that we also have used the standard normalisation of the sum over continuum surfaces. The extra factor of 2 comes from the open string Liouville zero mode integral. Also we have naturally lost the self-dual nature of the closed string. As for the closed surfaces we expect an agreement with the boundary matrix model. A proof of this however needs the explicit calculation which is beyond the scope of this work.

To finish let us consider the partition functions with fixed area $A$ and fixed boundary length $L$. It is then easy to see that the only change we need to make is in the integration of the Liouville zero mode. For the case $Z(A)$ we set $\lambda_2 = 0$ which is the case where we can continue to use the DDK scaling argument. Then for $s = 0$ we get $I_0(A, \lambda_2 = 0) = 1/(\alpha A)$. In the case $Z(L)$ we set $\mu_2 = 0$ and for $s = 0$ we find $I_0(L, \mu_2 = 0) = 2/(\alpha L)$. So the partition functions are

$$Z(A) = \frac{1}{6\alpha AR}, \quad Z(L) = \frac{1}{3\alpha LR}.$$

If as in the case of closed surfaces [23, 57] it could be proved that the partition functions of the boundary conformal models can be written as a linear combination of the partition functions of a scalar field compactified on some particular radii, it would be an easy matter to extend these results to those models. Although this is something we would like to expect the required proof is still not available.

If we consider a string which winds around the compact dimension but still has its end points located on two D-branes a distance $Y$ apart we might be able to discuss non-perturbative phenomena in the boundary conformal models. We then have to take

$$Y(\xi^1, \xi^2) = 2\pi n R \xi^1 + \mathcal{Y}\xi^2 - \mathcal{Y}/2 + \bar{Y}(\xi^1, \xi^2).$$

Thus in the case $\lambda_2 > 0$, $\mu_2 \geq 0$ the partition function is

$$Z = \frac{1}{2\pi\sqrt{2}} \ln(1/\lambda_2) \int_0^{+\infty} d\tau \tau^{-3/2} \sum_{n=1}^{+\infty} \exp\left(-\mathrm{T}\mathcal{Y}^2\tau - \frac{\mathrm{T}}{\tau}\pi^2 n^2 R^2\right).$$

Integrating using the Mellin transform [43] we find

$$Z = \frac{1}{\pi} \ln(1/\lambda_2) \sum_{n=1}^{+\infty} \left(\frac{\mathcal{Y}}{2\pi n R}\right)^{1/2} K_{-1/2}\left(\frac{nR\mathcal{Y}}{\alpha'}\right),$$

where we have introduced the modified Bessel function [43]

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \quad I_\nu(z) = \sum_{m=0}^{+\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu + m + 1)}.$$

Since $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$ we may write

$$Z = \ln(1/\lambda_2)\frac{\sqrt{\alpha'}}{2\pi}\sum_{n=1}^{+\infty}\frac{e^{-nR\mathcal{Y}/\alpha'}}{nR}.$$

To sum the series let us first note that we may write it as an integral of a geometric series with argument $z = e^{-x}$

$$\sum_{n=1}^{+\infty}\frac{e^{-nx}}{n} = \int_x^{+\infty} dt\left(\sum_{n=0}^{+\infty} e^{-nt} - 1\right).$$

Since the geometric series is $1/(1-e^{-t})$ we simply substitute above, change the variable to $y = e^{-t}$ and integrate to find

$$Z = -\ln(1/\lambda_2)\frac{\sqrt{\alpha'}}{2\pi R}\ln\left(1 - e^{-R\mathcal{Y}/\alpha'}\right).$$

For large distances $\mathcal{Y} \to +\infty$ we may expand the logarithm to first order and get

$$Z = \ln(1/\lambda_2)\frac{\sqrt{\alpha'}}{2\pi R}e^{-R\mathcal{Y}/\alpha'}.$$

For small distances $\mathcal{Y} \to 0$ we expand the exponential to find

$$Z = -\ln(1/\lambda_2)\frac{\sqrt{\alpha'}}{2\pi R}\ln\frac{R\mathcal{Y}}{\alpha'},$$

which means that for $\mathcal{Y} = 0$ we have an extra ultraviolet divergence due to the winding of the string around the compact dimension. This may be expressed using the Riemann zeta function

$$Z(\mathcal{Y} = 0) = \ln(1/\lambda_2)\frac{\sqrt{\alpha'}}{2\pi R}\zeta(1).$$

If we consider the partition functions for fixed area $A$ and fixed boundary length $L$ we just have to put $\alpha' = 4$ for the different normalisation associated with the conformal models and change the Liouville zero mode integral to be able to write

$$Z(A) = -\frac{1}{\pi\alpha AR}\ln\left(1 - e^{-R\mathcal{Y}/\alpha'}\right), \quad Z(L) = -\frac{2}{\pi\alpha LR}\ln\left(1 - e^{-R\mathcal{Y}/\alpha'}\right).$$

# Chapter 7

# Conclusions

In this thesis we have shown how to extend the approach of David, Distler and Kawai to the coupling of boundary conformal field theories to 2D quantum gravity. The organising principle behind their approach is Weyl invariance at the quantum level applied to a perturbative expansion analogous to the Coulomb gas. We used this to determine the renormalised parameters, gravitational dressings and surface critical exponents such as the susceptibility of random surfaces, the anomalous gravitational scaling dimensions of primary vertex operators and the Feynman mass exponent. The crucial problem is the choice of boundary conditions on the Liouville field. We have discussed free, Neumann and Dirichlet boundary conditions on the Liouville field. The first two lead to similar results within this perturbative approach, but Dirichlet conditions imply that the metric is discontinuous as the boundary is approached. We have also considered the semi-classical expansion and advocated the free boundary conditions for the Liouville field, since homogeneous Neumann boundary conditions do not allow a clean split between the classical and quantum pieces of the field, but rather couple them together. As would be expected the bulk properties are equal for open and closed surfaces. This approach may also be naturally extended to higher genus and more complex boundary structures. Unfortunately as for closed surfaces the results only apply to the weak coupling of $c \leq 1$ boundary conformal field theories to gravity.

At the end an application to the theory of dual membranes was considered. We have identified a non-critical D-instanton as a dynamical object which consistently interacts with closed strings defined in a one dimensional target space. Its action gives the leading stringy non-perturbative effects which are associated with the presence of boundaries. We have shown that these are of the order of $e^{-O(1/\beta_{st})}$, a result which not surprisingly is in agreement with those found in the 26 dimensional critical theory and in the analysis of matrix models. We have seen that this does not hold for all the non-critical theories allowed by perturbative Weyl invariance. Only those which satisfy $\lambda_2 \geq \mu_2$ with $\lambda_2 > 0$, $\mu_2 \geq 0$ or those where $\lambda_2 = 0$, $\mu_2 \geq 0$ lead to consistent non-perturbative effects. Using the one loop open string partition function, we have calculated the force between two D-instantons due to the exchange of massless Klein-Gordon fields and shown that the size of the D-instanton is of the order of $\sqrt{\alpha'}/\ln(1/\lambda_2)$ for small $\lambda_2 > 0$, $\mu_2 \geq 0$ or of the order of $\sqrt{\alpha'}/\ln(1/\mu_2^2)$ for $\lambda_2 = 0$ and small $\mu_2 > 0$. We also discussed the partition functions when the string winds around the compact dimension. The possible but still unfinished extension to the minimal $c \leq 1$ boundary conformal models was also considered.

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