Black hole entropy and models

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A thesis submitted for the degree of Doctor of Philosophy

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August 1997

Preface

This thesis is based on the study and research carried out independently by the author between October 1994 and August 1997 at the University of Durham. No part of it has been submitted previously for any degree at any university.

The content of this thesis is a blend of review of known work and research of my own. When others' work are referred to, proper citations have been given. It is believed that the arguments in section 2.2.3, the idea in section 3.3, and the models in chapter 4 are original to the author.

Acknowledgement

I am grateful to Professor E. J. Squires, my former supervisor who sadly and untimely died last summer, for giving me an opportunity to pursue a PhD degree in Durham. I would like to thank Professor E. Corrigan for taking the responsibility as my supervisor afterwards. In particular, I want to thank him for giving me great encouragement when I was at the most difficult moment. Thanks are also given to the staffs in Department and University for general support in various aspects.

Abstract

No universally accepted statistical explanation of black hole entropy exists up to now, therefore, *it is worth another try. Admittedly,* black hole entropy does not have to have a statistical origin. If the "black hole entropy" is called "black hole index" instead, someone might be lured to give it an economic explanation.

Nonetheless, the only way to justify one's claim about the statistical origin of black hole entropy is to compute it statistically. This is the motivation for the construction of black hole models.

In chapter 1, I first review the four laws of classical black hole mechanics which form the basis for the introduction of black hole thermodynamics. After observing the formal analogy between the four laws of the black hole mechanics and that of the ordinary thermodynamics, I further explore the thermodynamic properties of black holes in chapter 2 by reviewing the phenomenon of Hawking radiation and introducing the idea of black hole entropy.

Three statistical explanations of black hole entropy are introduced in chapter 3. I will start with 't Hooft's brick wall model. Then, à la Brown and York, I review the approach based on the gravitational degrees of freedom via path integral. In the final part of this chapter, I present my own version of a quantum statistical explanation of black hole entropy by regarding a black hole as a cavity *with thermal states inside.*

The final chapter will be devoted to the construction of black hole models to materialise the idea that a black hole, in some sense, can be regarded as a cavity where thermalised quantum states reside with quantised spectrum. These quantum states and the corresponding spectrum will then justify the statistical explanation of black hole entropy presented in the final section of chapter 3.

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Introduction

Although astronomers can still only talk about the evidence of the existence of black holes [44], I suppose few theorists doubt their existence. The process of a real (astronomical) black hole formation from the gravitational collapse of a sufficiently massive star inevitably involves very complicated dynamics due to the enormous degrees of freedom contained. However, theoretically, our understanding of black hole physics, at least classically, has enhanced a lot since the Schwarzschild solution of the Einstein field equation was discovered eighty years ago.

I suppose it is uncontroversial to say that the interests in black hole physics received a boost thirty years ago after the combined work of Hawking and Penrose about the singularity theorem [33]. Especially, after the introduction of black hole entropy [4], the formulation of four laws of black hole mechanics [2], the discoveries of Hawking radiation [30] and Unruh effect [64], quantum theory around/of black holes also experienced a boom in the past two decades, as can be seen clearly from the abundant references in the SLAC pre-print archive [37].

Although intensive efforts have been put on this area, some aspects are still as enigmatic as they were twenty years ago. Quantum gravity aside, one of them is the statistical explanation of black hole entropy [7]. In recent years, programs for understanding black hole entropy became ever more sophisticated.[1] Complete references can be found in the pre-print archive [37].

The subject I am going to address in this thesis is the statistical origin of black hole entropy. Admittedly, the faith that the black hole entropy indeed has a statistical

[1]See review articles [7], [26], and [59] for examples.

origin can be challenged because, as far as I can see, no universally accepted proposal of the statistical explanation of black hole entropy exists up to now.

Therefore, it will be interesting to ask why people take the conjecture of Bekenstein's about black hole entropy so seriously. There must be some hints indicating that it could be true. As reflected by Bekenstein himself in an article [7], the idea of black hole entropy was not embraced without doubt after it was proposed. I suppose one of the reasons is that at that time, most of the understanding of black hole physics was classical.

It was the discovery of the Hawking radiation that changed our understanding about black hole physics. Perhaps it is more appropriate to say that Hawking radiation raised our understanding of black hole physics from the classical level to a quantum-mechanical one. Only then the idea is justified that the black hole entropy is genuinely quantum-mechanical [4], and that black hole entropy might have a statistical origin.

My opinion about the statistical origin of black hole entropy is that, phenomenologically, it is just like the entropy of thermal radiation: There are quantum states inside a cavity and their distribution is governed by certain statistical distribution law regarding to the statistics and the quantised spectrum.

Though the dynamics involved in black hole physics and the thermal radiation in a cavity are totally different, the analogy between the four laws of black hole mechanics and thermodynamics, and the phenomenon of black hole radiation seem to indicate that the phenomenological similarities are worth exploring further. In order to understand what those states are and what the corresponding spectrum is, I will construct a model of a black hole to realise the idea that in some sense, a black hole is just like a cavity with thermal radiation inside.

Outline

The outline of this thesis is as follows:[2]

Chapter 1 I will review the four laws of classical black hole mechanics which form the basis for the introduction of black hole thermodynamics. However, before we dive

[2]For more detailed abstract and references, see the beginning of each chapter.

into the various properties of black holes, it makes sense to convince oneself that black holes exist at least theoretically. I therefore start with reviewing black hole formation in the 2-D CGHS dilatonic gravity.

Chapter 2 After observing the formal analogy between the black hole mechanics and ordinary thermodynamics in chapter 1, I further explore the thermodynamic properties of black holes in this chapter. The Hawking radiation is reviewed within the 2-D CGHS dilatonic gravity model, which realises the thermodynamic properties of a black hole by endowing it with a temperature. Afterwards, Bekenstein's idea of black hole entropy is introduced.

Chapter 3 Three different statistical explanations of black hole entropy are introduced to contrast their approaches and show the way which leads me to the idea that the quantum states responsible for black hole entropy should contain both the quantum field and gravitational degrees of freedom. I will start with 't Hooft's brick wall model which applies the conventional quantum field theory in a straightforward manner. Then, à la Brown and York, I review the approach based on the gravitational degrees of freedom via path integral. In the final section, I present my own version of a quantum statistical explanation of black hole entropy by regarding a black hole as a cavity with thermal states inside.

Chapter 4 This chapter will be devoted to the construction of black hole models to materialise the idea that a black hole, in some sense, can be regarded as a cavity where thermalised quantum states reside with quantised spectrum. These states and the corresponding spectrum will then justify the statistical explanation of black hole entropy presented in section 3.3, chapter 3.

Constraints In this thesis, I will confined myself to spherically symmetric cases only. Most of the results in the first three chapters can be extended to rotating black holes, too. However, solving PDE's numerically is inevitable in chapter 4 if one is interested in rotating black holes.

For charged black holes, I also exclude the extreme cases. The thermal properties of extreme black holes are still under debate.[3] In fact, extreme charged black holes are ruled out as unstable within my model as will be explained in section 4.3, chapter 4.

In recent years, there has been great interest in understanding black hole physics from the point of view of superstring theory [60]. Since the philosophy involved in superstring theory is totally different from mine which is based on the Einstein field equation and the quantum field theory, I am not able to address that line of approach.

However, since my models are based on the Kleinian signature, $(--++)$, I think it is necessary to point out that it seems the superstring theory is a proper context to address the problem of two time co-ordinates [3].

In fact, there are more questions raised than solved in my approach. Basically, all of them are related to the extra time co-ordinate. However, the problem of *two-time* is a big issue by itself. I have to leave it for future investigations.

[3]See article [21] and references therein.

Chapter 1

Classical Black Hole Physics

In this chapter, I review some of the classical aspects of black hole physics in order to establish the classical black hole mechanics. Though, chronologically, the idea of black hole entropy was introduced before the proof of the four laws of black hole mechanics[1] and the discovery of black hole radiation, I think it is more appropriate to review them in the other order because it is the latter two which support the idea of black hole entropy.

Abstract of chapter 1

Section 1.1 I at first briefly describe the general development about the description of black hole formation in subsection 1.1.1. Then I review the black hole formation in the 2-D CGHS dilatonic gravity model [11] in subsection 1.1.2, which will provide the background for the discussion of Hawking radiation in section 2.1, chapter 2.

Section 1.2 The formal analogy between the four laws of black hole mechanics and thermodynamics relies on the similarity of the statements and the appearance of mathematical formulae of these two sets of laws. I therefore at first compare the statements of these two sets of laws [2, 36] in subsection 1.2.1. I then review the derivations of the

[1]However, note that the second law—the area theorem—was proved before the proposal of the black hole entropy [33]. In fact, as noted by Bekenstein in reference [7], the area theorem constituted one of the initiatives for the idea of black hole entropy.

first two of the four laws of black hole mechanics. The zeroth law is foremost derived in subsection 1.2.2, which, basically, involves straightforward algebra only [2, 67]. In subsection 1.2.3, I explicitly derive the first law for non-rotating charged black holes within the Einstein-Maxwell theory following the Noether charge method developed by Wald [70, 71]. This method is perhaps the most general one. It can be applied to any metric theory that can be given a covariant Lagrangian formalism.[2]

Unfortunately,[3] the proof of the second law [33] cannot be reviewed within one section. On the other hand, I know of no rigorous proof of the third law, though there are *failed* attempts to disprove it reported in reference [66]. I therefore only describe the statements of the second and third laws.

Appendix A I introduce the notation and convention employed. Some basic elements of the CGHS model are explained.

Appendix B The convention and notation regarding the differential forms are explained.

[2]The readers are referred to references [71, 72] for a flavour of the generality of this method.
[3]Perhaps "Fortunately" is the word. Anyway, it is not a simple topic.

1.1 Black hole formation from classical matter

If it doesn't quantum-mechanically, so be it.

1.1.1 General developments

OS model When the Schwarzschild solution was derived more then eighty years ago, it was noted by him that the region of $r < r_s$ (where r_s is the Schwarzschild radius) is unphysical [74]. Later, some models concerning the interior of a star were developed.[4] Amongst them, those related to a black hole are the dynamically gravitational stellar collapse. The simplest one is perhaps the OS model developed by Oppenheimer and Snyder in 1932 [74]. In this model, a pressureless dust of uniform density is allowed to contract due to its own gravity. The exterior region can be chosen to be the Schwarzschild solution as dictated by Birkhoff's theorem [74]. It is attached to an interior solution of contracting spatially homogeneous, isotropic cosmological model with proper boundary condition across the boundary of the dust. Since there is no pressure among those particles constituting the dust, they are in the motion of free falling. Two of the most important observations from this model are [74]:

1. A fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite proper energy density in a finite proper time (co-moving time).

2. The collapse to the Schwarzschild radius appears to an outside observer to take an infinite time, and therefore the collapse to $R = 0$ is utterly unobservable from outside (where R is proportional to the Schwarzschild co-ordinate r by a finite factor).

Numerical studies In recent years, due to renewed interest in black hole physics and the advance in computer technology, black hole formation is re-addressed from both analytical and numerical points of view.

[4] See textbook [74] and references therein.

The difficulties in solving the four dimensional, dynamic Einstein field equations should never be overestimated. Therefore, numerical study has become another important alternative to understand the classical dynamics of the Einstein field equation. In fact, some observations have prompted new understanding about the dynamic aspects of the Einstein field equation. One of the most interesting observations is probably the critical phenomenon related to a phase transition in a gravitational collapse [14]. If one prepares a family (characterised by one parameter, p) of spherically symmetric initial states of matter field in which the parameter p characterises the energy contained in a particular initial state, then one finds that the final states depend on the initial states in a certain manner: If $p < p_c$ in which p_c is a critical value, then all matter will eventually be scattered off to infinity. However, as $p > p_c$, a black hole forms and its mass, M, can be put in a simple formula,

$$M \propto (p - p_c)^\gamma ,$$

where γ is the critical exponent. The universality of this exponent is still under debate. From the present evidence, it seems that the exponent depends on the matter content [29]. However, such a phenomenon is hardly to be expected without the evidence from numerical simulation.

Analytical studies Regarding this observation, some analytical analyses have been put forward to understand this aspect [29, 42]. This development opened a new channel of studies of dynamic Einstein field equation.

Another development is based on the analytical approach in low dimensional theories. One of the reasons for considering low dimensional theories is, obviously, that more can be done analytically. On the other hand, its advantage is also its disadvantage: Since the dimension is reduced, some physics is also erased from the spectrum. Though Einstein's theory of general relativity in 2-D is trivial because the Einstein-Hilbert action is a topological constant and all space-times are conformally flat, some new features appear if a dilaton field is included. Especially, the 4-D spherically symmetric system can be recast in the form of 2-D gravity coupled to a dilaton field.

1.1.2 Black hole formation in the CGHS model

The equations In the rest of this section, I will review the black hole formation from classical matter in the CGHS dilatonic gravity model by considering the following action [11],[5]

$$S = \frac{1}{4G}\int d^2x\sqrt{|g|}\left[e^{-2\phi}(-\mathcal{R} + 4\nabla_a\phi\nabla^a\phi + 4\lambda^2) - 8\pi G\nabla_a\varphi\nabla^a\varphi\right] .$$

In the conformal gauge with the metric parametrised as

$$ds^2 = -e^{2\rho}dx^+dx^- = -e^{2\rho}(dt^2 - dx^2) ,$$

we can always find a gauge such that $\rho = \phi$. Furthermore, if the scalar field φ is restricted to be x^+-dependent only, the equations (A.8)-(A.12) can be simplified to

$$\partial_+\partial_-(e^{-2\phi}) + \lambda^2 = 0 , \tag{1.1}$$

$$\partial_-\partial_-(e^{-2\phi}) = 0 , \tag{1.2}$$

$$\partial_+\partial_+(e^{-2\phi}) + 8\pi G T_{++} = 0 , \tag{1.3}$$

$$\nabla_a\varphi\nabla^a\varphi = 0 . \tag{1.4}$$

The black hole solutions The general solution of equations (1.1)-(1.4) for ϕ $(= \rho)$ is

$$e^{-2\phi} = -\lambda^2 x^+ x^- - 8\pi G\int_0^{x^+} dy^+\int_0^{x^+} dy^+ T_{++}(y^+) \tag{1.5}$$

$$= \frac{8\pi G M(x^+)}{\lambda} - \lambda^2 x^+\left[x^- + 8\pi G P(x^+)\right] , \tag{1.6}$$

where

$$M(x^+) = \lambda\int_0^{x^+} dy^+ y^+ T_{++}(y^+) , \tag{1.7}$$

$$P(x^+) = \frac{1}{\lambda^2}\int_0^{x^+} dy^+ T_{++}(y^+) , \tag{1.8}$$

[5] See appendix A for an introduction of some basic elements of 2-D dilatonic gravity and the CGHS model.

and I have assumed that the matter field φ is restricted within a period of advanced time, say from x_i^+ (> 0) to x_f^+ $(> x_i^+)$. Define the following quantities for convenience,

$$M_\infty = M(x_f^+) \ , \qquad P_\infty = P(x_f^+) \ . \tag{1.9}$$

In the asymptotic flat form To see that a black hole always forms classically if the energy-momentum is non-zero, we consider the late advanced time behaviour (i.e., $x^+ > x_f^+$) of the metric in the new co-ordinate system $\sigma^\pm$ $(= \sigma^0 \pm \sigma^1)$ which is related to $x^\pm$ by the following relations

$$\lambda x^+ = e^{\lambda\sigma^+} \ , \qquad \lambda(x^- + 8\pi G P_\infty) = -e^{-\lambda\sigma^-} \ . \tag{1.10}$$

Then the metric is

$$ds^2 = \frac{-d\sigma^+ d\sigma^-}{1 + \frac{8\pi G M_\infty}{\lambda} e^{-2\lambda\sigma^1}} \ . \tag{1.11}$$

It is clearly seen that the metric is asymptotically flat and the mass of the black hole is M_∞.[6] Its Penrose diagram can be derived by attaching the Penrose diagram of an eternal black hole solution to that of a vacuum solution (see figure 1.1).

Though the exact classical black hole solution can be found in the CGHS model, the full quantum theory is still beyond control. However, there are some discussions at the semi-classical level [26].

[6] See appendix A for explanation.

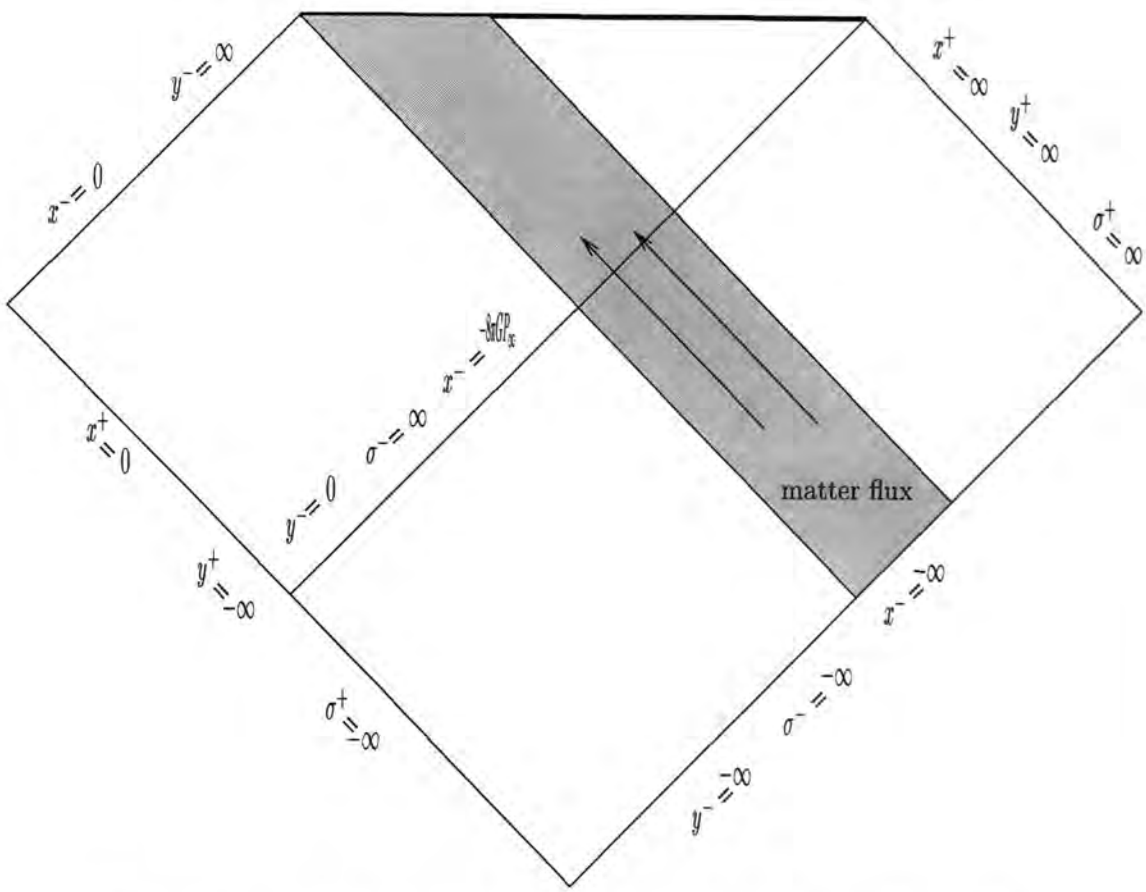

Figure 1.1: Penrose diagram for a space-time of black hole formation.

1.2 Four laws of black hole mechanics

1.2.1 Two *four laws*

Four laws of thermodynamics Let us at first review the four laws of thermodynamics [12, 36]:

0. **The zeroth law**: For a thermal equilibrium system, the temperature, T, is a constant over the system.

1. **The first law**: For two neighbouring equilibrium states, the following relation holds,

$$dU = TdS + \mu dN \ , \tag{1.12}$$

where U is the internal energy, S the entropy, μ the chemical potential, and N the particle number.

2. **The second law**: [7] For a thermally isolated system, the state of equilibrium is the state of maximum entropy consistent with the external constraints. Or, equivalently, the total entropy of a thermally isolated system does not decrease with time.[8]

3. **The third law**: It is impossible to reduce the temperature of a system to absolute zero by a finite sequence of operations.[9]

Four laws of black hole mechanics The four laws of black hole mechanics have very similar statements [2]:

0. **The zeroth law**: For a static black hole, the surface gravity, κ, is a constant over the event horizon.

[7] Though the second law is often stated in terms of entropy, this indeed can be derived as a consequence of either the Kelvin's or the Clausius' statement of the second law [36].

[8] Unlike the second law of black hole mechanics, the second law of thermodynamics cannot be exactly true at microscopic level [36].

[9] There are at least two different versions of the third law [36], I adopt the one which is used in the third law of black hole mechanics [2].

1. **The first law**: For two neighbouring static black hole solutions, the following relation holds,[10]

$$d\mathcal{E} = \frac{\kappa}{2\pi} dA + \Phi dq \ , \tag{1.13}$$

where $\mathcal{E}$ is the ADM mass of the black hole, A the area of the cross-section of the event horizon, Φ the Coulomb potential at the event horizon, and q the total charge of the black hole.

2. **The second law**: The area of the cross-section of the event horizon of a black hole does not decrease with time.

3. **The third law**: It is impossible to reduce the surface gravity, κ, to zero by a finite sequence of operations.

The difference The analogy between these two sets of laws are too obvious to ignore though I cheat a bit by stating them in almost the same manner. Nonetheless, there is one great difference between them that one has to keep in mind: The thermodynamic laws are true at macroscopic level, they can only be derived from the underlying microscopic statistical models through a kind of coarse-graining, say, by assuming that the scattering process between those constituent particles are Markov processes [1]. On the other hand, the laws of black hole mechanics are exact *classical* laws which can be derived from the equations of motion governing the system concerned.[11]

I thereby review the derivations of the zeroth and the first law in the rest of this section [2, 67, 69, 70, 71, 73].[12]

[10] The explicit form of this relation depends on the content of the theory. The above one is applicable to the Einstein-Maxwell theory.

[11] It is because of this difference that when these laws were first derived by Barteen, et al., in article [2], they opposed the idea that the black hole area should be regarded as a kind of thermodynamic entropy though they had observed the similarity between them.

[12] The reasons for not reviewing the derivation of the other two laws have been given in the abstract in the beginning of this chapter.

1.2.2 Derivation of the zeroth law

The statement of the first law The more precise statement of the zeroth law of black hole mechanics is: the surface gravity, κ, is a constant on any Killing horizon[13] of a solution of the Einstein field equation if the energy-momentum satisfies the dominant energy condition.[14]

The definition of surface gravity The surface gravity, κ, of a Killing horizon is defined by the relation

$$\xi^{\mu}\nabla_{\nu}\xi_{\mu} = -\kappa\xi_{\nu} \ . \tag{1.14}$$

Note that the surface gravity is defined on the Killing horizon only. In order to prove its constancy over a Killing horizon, we at first extend κ outside the Killing horizon analytically so that the differentiation $\nabla_{\mu}\kappa$ is well-defined. The following result is independent of how the extension is performed.

Derivation If we write $\nabla_{\nu}\kappa = c_{\xi}\xi_{\nu} + c_n n_{\nu} + c_1 m^1_{\nu} + c_2 m^2_{\nu}$ where (ξ, n, m^1, m^2) forms a local orthogonal basis with ξ and n null, which are normalised as $\xi^{\nu}n_{\nu} = -1$, and with m^i's space-like tangent vectors of the Killing horizon, then $c_n = 0$ because $\xi^{\nu}\nabla_{\nu}\kappa = 0$.[15] The constancy of κ over the Killing horizon is equivalent to the statements that $c_1 = c_2 = 0$.[16] And this is a consequence of the following identity,

$$(\xi_{\nu}\nabla_{\mu} - \xi_{\mu}\nabla_{\nu})\kappa = 0 \ . \tag{1.15}$$

[13] A Killing horizon is a null hypersurface which is invariant under the action of the group of isometry generated by a Killing vector field, ξ, and on which $\xi^{\nu}\xi_{\nu} = 0$ [67]. A Killing vector field, ξ, of a space-time is a vector field which generates the isometry of it, i.e., $\delta g_{\mu\nu} = \nabla_{\nu}\xi_{\mu} + \nabla_{\mu}\xi_{\nu} = 0$ under the co-ordinate transformation $x_{\mu} \longrightarrow x_{\mu} - \xi_{\mu}$ [67].

[14] The dominant energy condition states that $-T^{\alpha}{}_{\beta}\chi^{\beta}$ is future time-like or null-like for any time-like or null-like vector, χ [33, 67].

[15] This can be proved by applying $\xi^{\nu}\nabla_{\nu}$ to equation (1.14) and using equation (1.17) below.

[16] The value of c_{ξ} is determined by the way it is extended outside the Killing horizon. Its value is irrelevant.

To prove equation (1.15), we at first apply $\xi_{[\gamma}\nabla_{\delta]}$[17] to both side of equation (1.14). With the help of equation (1.14) itself and the identities true for a Killing vector field on the Killing horizon[18]

$$\begin{aligned} \xi_\alpha \nabla_\beta \xi_\gamma &= -2\xi_{[\beta}\nabla_{\gamma]}\xi_\alpha \ , & (1.16) \\ \nabla_\beta \nabla_\gamma \xi_\alpha &= -\xi_\sigma \mathcal{R}^\sigma{}_{\beta\gamma\alpha} \ , & (1.17) \end{aligned}$$

we have

$$\xi_\beta \xi_{[\gamma}\nabla_{\delta]}\kappa = \xi^\sigma \xi^\alpha \mathcal{R}_{\alpha\beta\sigma[\delta}\xi_{\gamma]} \ . \qquad (1.18)$$

Then, we apply $\xi_{[\delta}\nabla_{\lambda]}$ to equation (1.16) and, with the help of equations (1.16) and (1.17), we have the following relation

$$\xi_\alpha \xi_{[\delta}\mathcal{R}^\sigma{}_{\lambda]\beta\gamma}\xi_\sigma = 2\xi_{[\delta}\mathcal{R}^\sigma{}_{\lambda]\alpha[\gamma}\xi_{\beta]}\xi_\sigma \ . \qquad (1.19)$$

After contracting α with λ, the LHS is zero and from the RHS we arrive at

$$\xi_\beta \mathcal{R}^\sigma{}_{[\gamma}\xi_{\delta]}\xi_\sigma = \xi^\sigma \xi^\alpha \mathcal{R}_{\alpha\beta\sigma[\delta}\xi_{\gamma]} \ . \qquad (1.20)$$

Equation (1.18) minus equation (1.20) gives

$$\xi_{[\gamma}\nabla_{\delta]}\kappa = \mathcal{R}^\sigma{}_{[\gamma}\xi_{\delta]}\xi_\sigma \ . \qquad (1.21)$$

Now, if the Einstein field equation holds, $\mathcal{R}_{\gamma\sigma}\xi^\gamma\xi^\sigma = 0$[19] implies $T_{\gamma\sigma}\xi^\gamma\xi^\sigma = 0$ since ξ is null. Furthermore, if the dominant energy condition holds, namely, $-T^\gamma{}_\sigma\xi^\sigma$ is future time-like or null-like, then we can conclude that $-T_{\gamma\sigma}\xi^\sigma \propto \xi_\gamma$. This thus implies $\xi_{[\delta}T_{\gamma]\sigma}\xi^\sigma \propto \xi_{[\delta}\xi_{\gamma]} = 0$. The Einstein field equation then implies the RHS of equation (1.21) is zero. We thus arrive at the desideratum that $\xi_{[\gamma}\nabla_{\delta]}\kappa = 0$.

Another version of the zeroth law which does not employ the Einstein equation, but with other constraints required, can be found in reference [53].

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[17]The total antisymmetrisation is defined as

$$\xi^1_{[\nu_1}\xi^2_{\nu_2}\cdots\xi^n_{\nu_n]} = 1/(n!)\sum sign(\sigma)\xi^1_{\nu_{\sigma(1)}}\xi^2_{\nu_{\sigma(2)}}\cdots\xi^n_{\nu_{\sigma(n)}} \ ,$$

in which the summation is over all possible permutations, σ, of $(1, 2, \ldots, n)$.

[18]The first one follows from the Frobenius's theorem that if the normal, ξ, is surface forming, then $\xi_{[\alpha}\nabla_\beta\xi_{\gamma]} = 0$. Using the relation, $2\nabla_{[\alpha}\nabla_{\beta]}\xi_\gamma = -\xi_\sigma\mathcal{R}^\sigma{}_{\gamma\beta\alpha}$, the second one follows from the summation of three such relations with the index orders, (α, β, γ), (β, α, γ), (α, γ, β), and the identity $\mathcal{R}^\sigma{}_{[\alpha\beta\gamma]} = 0$.

[19]This relation can be derived from the equation (1.19) by contracting λ with γ and β with δ.

1.2.3 Derivation of the first law

The whole derivation is more transparent and elegant if it is done with the help of differential forms. However, when explicit calculations are involved, the tensorial indices are inevitable, I therefore use the Latin and Greek letters to denote the indices of differential forms and tensors, respectively.

Symplectic potential 3-form $\mathbf{\Theta}$

Consider a system described by the action[20]

$$S = \int dx^4 \sqrt{|g|}(L_g + L_a + L_m) = \int (\mathbf{L}_g + \mathbf{L}_a + \mathbf{L}_m) \ ,$$

where

$$\begin{aligned} \mathbf{L}_g &= \boldsymbol{\epsilon} L_g = \frac{-\boldsymbol{\epsilon}}{16\pi G}\mathcal{R} \ , \\ \mathbf{L}_a &= \boldsymbol{\epsilon} L_a = \frac{-\epsilon_A \boldsymbol{\epsilon}}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \ , \\ \mathbf{L}_m &= \boldsymbol{\epsilon} L_m = \frac{-\boldsymbol{\epsilon}}{2}(\nabla_\nu\varphi\nabla^\nu\varphi + m^2\varphi^2) \ , \end{aligned} \tag{1.22}$$

where $\boldsymbol{\epsilon}$ is the volume 4-form and $\epsilon_A = 1$.

Then, the variation of $\mathbf{L} = \mathbf{L}_g + \mathbf{L}_a + \mathbf{L}_m$ can be written as

$$\begin{aligned} \delta\mathbf{L} &= (\mathbf{T}_g^{\alpha\beta} + \mathbf{E}_g^{\alpha\mu\nu\kappa}\mathcal{R}^\beta{}_{\mu\nu\kappa})\delta g_{\alpha\beta} + \mathbf{d}\mathbf{\Theta}_g \\ &\quad + \mathbf{E}_a^\nu \delta A_\nu + \frac{1}{2}\mathbf{T}_a^{\alpha\beta}\delta g_{\alpha\beta} + \mathbf{d}\mathbf{\Theta}_a \\ &\quad + \mathbf{E}_m\delta\varphi + \frac{1}{2}\mathbf{T}_m^{\alpha\beta}\delta g_{\alpha\beta} + \mathbf{d}\mathbf{\Theta}_m \\ &= \sum_{i=g,a,m} \mathbf{E}_i\delta\phi_i + \mathbf{d}\mathbf{\Theta} \ , \end{aligned} \tag{1.23}$$

where I have used the symbol ϕ_i for either $g_{\mu\nu}$, or A_ν, or φ, and $\mathbf{\Theta} = \sum_{i=g,a,m}\mathbf{\Theta}_i$. Explicitly, we have

$$\mathbf{T}_g^{\alpha\beta} = \boldsymbol{\epsilon} T_g^{\alpha\beta} = \frac{\boldsymbol{\epsilon}}{8\pi G}(\mathcal{R}^{\alpha\beta} - \frac{1}{4}g^{\alpha\beta}\mathcal{R}) \ , \tag{1.24}$$

[20] See appendix B for convention and notation related to differential forms.

$$
\begin{aligned}
\mathbf{T}_m^{\alpha\beta} &= \boldsymbol{\epsilon} T_m^{\alpha\beta} = \boldsymbol{\epsilon}[\nabla^\alpha\varphi\nabla^\beta\varphi - \frac{g^{\alpha\beta}}{2}(\nabla^\nu\varphi\nabla_\nu\varphi + m^2\varphi^2)] \ , & (1.25)\\
\mathbf{E}_g^{\alpha\mu\nu\kappa} &= \boldsymbol{\epsilon} E_g^{\alpha\mu\nu\kappa} = \frac{-\boldsymbol{\epsilon}}{32\pi G}(g^{\mu\kappa}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\kappa}) \ , & (1.26)\\
\Theta_i(\phi_i, \delta\phi_i) &= \theta_i \cdot \boldsymbol{\epsilon} \ , \qquad i = g, a, m \ , & (1.27)
\end{aligned}
$$

with

$$
\begin{aligned}
\theta_g^\kappa &= \frac{-1}{16\pi G} g^{\mu\kappa}g^{\alpha\nu}(\nabla_\mu\delta g_{\alpha\nu} - \nabla_\alpha\delta g_{\mu\nu}) \ ,\\
\theta_a^\kappa &= -\epsilon_A\mathcal{F}^{\kappa\nu}\delta A_\nu \ , \qquad \theta_m^\kappa = -\nabla^\kappa\varphi\delta\varphi \ .
\end{aligned}
$$

The $\mathbf{\Theta}$ is called the *symplectic potential 3-form* [71]. There are ambiguities in the choices of $\mathbf{\Theta}_i$ in general [71]. However, the above explicit calculation fixes the choice.

Noether current 3-form J

If the variation in the $\mathbf{\Theta}_i$'s is generated by a smooth vector field, χ, i.e., $\delta\phi_i = \mathcal{L}_\chi\phi_i$, where $\mathcal{L}_\chi$ is the Lie derivative with respect to χ, a *Noether current 3-form*, $\mathbf{J}(\chi)$, associated with χ and any field configuration can be defined as follows [71]

$$
\begin{aligned}
J(\chi)_{bcd} &= (\mathbf{J}_g + \mathbf{J}_a + \mathbf{J}_m)_{bcd}\\
&= \sum_{i=g,a,m} (\mathbf{\Theta}_i(\phi_i, \mathcal{L}_\chi\phi_i) - \chi\cdot\mathbf{L}_i)_{bcd}\\
&= -\chi_\alpha(\frac{1}{8\pi G}\mathcal{G}^{\alpha\beta} + T_a^{\alpha\beta} + T_m^{\alpha\beta})\epsilon_{\beta bcd}\\
&\quad +\frac{1}{8\pi G}\nabla_\alpha(\nabla^{[\alpha}\chi^{\beta]})\epsilon_{\beta bcd} + \epsilon_A\nabla_\alpha(A^\nu\chi_\nu\mathcal{F}^{[\alpha\beta]})\epsilon_{\beta bcd} \ , \qquad (1.28)
\end{aligned}
$$

where the identity $\mathcal{G}^{\mu\nu} = 16\pi G(T_g^{\mu\nu} + E^{\mu\alpha\beta\gamma}\mathcal{R}^\nu{}_{\alpha\beta\gamma})$ and the definitions of $\mathbf{\Theta}$, $T_{g,m}^{\alpha\beta}$ in equations (1.27), (1.24), (1.25) have been used in the third line.

With the help of the Bianchi identity and the conservation of the energy-momentum tensor, we find

$$
\begin{aligned}
\mathbf{dJ} &= \mathbf{d\Theta} - \mathbf{d}(\chi\cdot\mathbf{L})\\
&= \mathbf{d\Theta} - \mathcal{L}_\chi\mathbf{L} + \chi\cdot\mathbf{dL}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{d\Theta} - (\sum_{i=g,a,m} \mathbf{E}_i \mathcal{L}_\chi \phi_i + \mathbf{d\Theta}) \\
&= -\sum_{i=g,a,m} \mathbf{E}_i \mathcal{L}_\chi \phi_i \ , \qquad (1.29)
\end{aligned}
$$

where I have used equation (B.1), the fact that $\mathbf{dL} = 0$, and the following relation from the variation of the Lagrangian (1.23),

$$
\mathcal{L}_\chi \mathbf{L} = \sum_{i=g,a,m} \mathbf{E}_i \mathcal{L}_\chi \phi_i + \mathbf{d\Theta}(\phi, \mathcal{L}_\chi \phi) \ .
$$

Noether charge 2-form Q

If the field configuration ϕ_i satisfies the equations of motion, i.e., $\mathbf{E}_i = 0, i = g, a, m$, then it is seen from equation (1.29) that $\mathbf{J}$ is closed, i.e., $\mathbf{dJ} = 0$. Furthermore, it is easily seen from equation (1.28) that as the equations of motion are satisfied, the Noether current 3-form, $\mathbf{J}(\chi)$, is exact,[21] namely,

$$
\mathbf{J}(\chi) = \mathbf{dQ}(\chi), \qquad (1.30)
$$

where $\mathbf{Q}(\chi)$ is the *Noether charge 2-form* defined as [71]

$$
\mathbf{Q}(\chi) = \mathbf{X}^{\mu\nu} \nabla_{[\mu} \chi_{\nu]} + \chi^\nu \mathbf{W}_\nu \ , \qquad (1.31)
$$

with

$$
\begin{aligned}
X^{\mu\nu}{}_{ab} &= \frac{-\epsilon^{\mu\nu}{}_{ab}}{16\pi G} \ , \qquad (1.32) \\
W_{\nu ab} &= -\epsilon_A \mathcal{F}^{\alpha\beta} A_\nu \epsilon_{\alpha\beta ab} \ . \qquad (1.33)
\end{aligned}
$$

Symplectic current 3-form ω and the Hamiltonian H

Symplectic current 3-form ω Before we are able to prove the first law, we need another concept, the Hamiltonian. Define the *symplectic current 3-form*, $\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$, as [43]

$$
\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \mathbf{\Theta}(\phi, \delta_2\phi) - \delta_2 \mathbf{\Theta}(\phi, \delta_1\phi) \ . \qquad (1.34)
$$

[21] This is a general result following from a theorem in reference [68].

A *symplectic form*, Ω, with respect to a Cauchy surface,[22] C, can be defined from $\boldsymbol{\omega}$,

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \int_C \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) \ . \tag{1.35}$$

Hamiltonian H If the variation, $\delta_2\phi$, in Ω is generated by a symmetry of the system with respect to a vector field,[23] ξ, i.e., $\delta_2\phi = \mathcal{L}_\xi\phi$, then we can define the variation of the *Hamiltonian*,[24] H, with respect to ξ as

$$\delta H = \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) \ .$$

Note that $\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi)$ is conserved, i.e., $\mathbf{d}\boldsymbol{\omega} = 0$, if both ϕ and $\phi + \delta\phi$ satisfy the equations of motion because, with the help of equation (1.22) and the definition of $\mathbf{J}$ in equation (1.28), we have

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \delta\mathbf{J}(\xi) - \mathbf{d}(\xi \cdot \Theta) \ . \tag{1.36}$$

The result $\mathbf{d}\boldsymbol{\omega} = 0$ follows from the exactness of $\mathbf{J}(\xi)$ (see equation (1.30)).

Using equations (1.36) and (1.30), we can write

$$\delta H = \int_C \delta\left(\mathbf{J}(\xi) - \mathbf{d}[\xi \cdot \Theta(\phi, \delta\phi)]\right) = \int_{\partial C} [\delta\mathbf{Q}(\xi) - \xi \cdot \Theta(\phi, \delta\phi)] \ .$$

Now, assuming that the solution, ϕ, in the symplectic current 3-form $\boldsymbol{\omega}$ is invariant under the symmetry generated by ξ, i.e., $\mathcal{L}_\xi\phi = 0$ and the boundary, ∂C, is comprised of two disconnected 2-surfaces such that one is at spatial infinity, denoted by ∞, and another one is denoted by Σ,[25] we then arrive at

$$\begin{aligned} \delta\mathcal{E} &\stackrel{def}{=} \int_\infty [\delta\mathbf{Q}(\xi) - \xi \cdot \Theta] \\ &= \int_\Sigma [\delta\mathbf{Q}(\xi) - \xi \cdot \Theta] \ , \end{aligned} \tag{1.37}$$

[22]Since in general the Cauchy surface is non-compact, some asymptotic boundary condition should be imposed on the field configurations [43].

[23]The vector field, ξ^ν, is defined with respect to the *background field metric*, $g_{\mu\nu}$, it is thus fixed irrespective of any variation, $\delta g_{\mu\nu}$.

[24]Recall that if the Lagrangian is not explicitly time-dependent, the Hamilton's equation is: $dH = (\partial_t q)dp - (\partial_t p)dq = d(\partial_t qp) - \partial_t(pdq)$.

[25]We should use the freedom of co-ordinate transformation to enforce the boundary, Σ, defined on both solutions of $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$ coincident, and, consequently, the variation, δ, and integration on Σ can be interchanged.

where we have use the fact that $\delta H = 0$ and the orientation of the surface integral has been chosen such that $\int_{\partial C} = \int_\infty - \int_\Sigma$.

Equation (1.37) is the basic result which will be used in the following to prove the first law of black hole mechanics.

The first law

The ADM mass In a static black hole background which is asymptotically flat,[26] we consider the case in which the Killing vector field, ξ, is the time translation, ∂_t, asymptotically at ∞. Then a 3-form $\mathbf{B}(\phi)$ exists such that[27]

$$\delta \int_\infty \xi \cdot \mathbf{B}(\phi) = \int_\infty \xi \cdot \boldsymbol{\Theta}(\phi, \delta\phi) \ .$$

It is a straightforward exercise to show that the $\mathcal{E}$, defined as $\mathcal{E} = \int_\infty (\mathbf{Q}(\xi) - \xi \cdot \mathbf{B})$, is in fact the ADM mass of the black hole [67].[28]

Variational formula on any cross-section of the Killing horizon We have from equation (1.31)

$$\delta \mathbf{Q}(\xi) = \delta(\mathbf{X}^{\mu\nu} \nabla_{[\mu} \xi_{\nu]}) + \delta(\xi^\nu \mathbf{W}_\nu) \ . \tag{1.38}$$

To calculate $\delta \nabla_{[\mu}\xi_{\nu]}$, recall at first that $\delta \xi^\nu = 0$.[29] And note that $\nabla_{[\mu}\xi_{\nu]} = \kappa \epsilon_{\mu\nu} + a_i \xi_{[\mu} m^i_{\nu]}$ on a Killing horizon where $\epsilon_{\mu\nu} = \xi_\mu n_\nu - \xi_\nu n_\mu$ is the binormal of the Killing horizon with

[26] This can be generalised to stationary cases. See reference [71] for a full discussion.

[27] It is unclear if this is always true in any metric theory of gravity, but, for our case,

$$\xi \cdot \mathbf{B} = \frac{-1}{16\pi G}[\partial_1 g_{00} - \partial_0 g_{10} + \eta^{ij}(\partial_i g_{1j} - \partial_1 g_{ij})]\bar{\epsilon} \ ,$$

where $\bar{\epsilon} = r^2 \sin(\theta) d\theta \wedge d\phi$ is the volume element on the 2-surface at ∞, the g_{ij} is the spatial part of the metric $g_{\mu\nu}$, and we require that $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$, $\partial_\alpha g_{\mu\nu} = O(1/r^2)$, $\varphi = O(1/r)$, $\partial_\alpha \varphi = O(1/r^2)$, $\mathcal{A}_\nu = O(1/r)$, and $\partial_\alpha \mathcal{A}_\nu = O(1/r^2)$ as $r \longrightarrow \infty$ asymptotically.

[28] This is because

$$\int_\infty \mathbf{Q}(\xi) = \frac{1}{16\pi G}\int_\infty (\partial_0 g_{10} - \partial_1 g_{00})\bar{\epsilon} - \epsilon_A \int_\infty \mathcal{A}_0 \mathcal{F}^{01} \bar{\epsilon} \ ,$$

where the electrostatic term should be zero because we have chosen the gauge so that $\mathcal{A}_\nu = O(1/r)$ at spatial infinity.

[29] See footnote 23 on page 19.

$n^\nu n_\nu = 0$, $\xi^\nu n_\nu = -1$.[30]

Define the tensor $\Delta_{\mu\nu}$ as

$$\Delta_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]} - \kappa\epsilon_{\mu\nu} \; .$$

Then, for its variation, we have

$$\begin{aligned}
\delta\Delta_{\mu\nu} &= \delta\left(\nabla_{[\mu}\xi_{\nu]} - \kappa\epsilon_{\mu\nu}\right) \\
&= \delta\left[g_{\alpha[\nu}(\nabla_{\mu]}\xi^\alpha - \kappa\epsilon_{\mu]}{}^\alpha)\right] \\
&= h_{\alpha[\nu}\Delta_{\mu]}{}^\alpha + (\delta\nabla_{[\mu})\xi_{\nu]} - (\delta\kappa)\epsilon_{\mu\nu} + \delta w_{\mu\nu} \; ,
\end{aligned} \tag{1.39}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$, and

$$\delta w_{\mu\nu} = -\kappa g_{\alpha[\nu}\delta\epsilon_{\mu]}{}^\alpha \; .$$

For the first term on the LHS of equation (1.38), we can write

$$\begin{aligned}
\delta(\mathbf{X}^{\mu\nu}\nabla_{[\mu}\xi_{\nu]}) &= \delta\left[\mathbf{X}^{\mu\nu}(\Delta_{\mu\nu} + \kappa\epsilon_{\mu\nu})\right] \\
&= \kappa\delta(\mathbf{X}^{\mu\nu}\epsilon_{\mu\nu}) + \mathbf{X}^{\mu\nu}\delta w_{\mu\nu} \\
&\quad + \delta\mathbf{X}^{\mu\nu}\Delta_{\mu\nu} + \mathbf{X}^{\mu\nu}(h_{\alpha[\nu}\Delta_{\mu]}{}^\alpha + (\delta\nabla_{[\mu})\xi_{\nu]}) \; .
\end{aligned} \tag{1.40}$$

Combining equations (1.37), (1.38), and (1.40), we arrive at

$$\begin{aligned}
\delta\mathcal{E} &= \frac{\kappa}{2\pi}\delta S + \int_\Sigma \left[\delta(\xi^\nu \mathbf{W}_\nu) - \xi\cdot\mathbf{\Theta}\right] \\
&\quad + \int_\Sigma \left[\mathbf{X}^{\mu\nu}\delta w_{\mu\nu} + \delta\mathbf{X}^{\mu\nu}\Delta_{\mu\nu} + \mathbf{X}^{\mu\nu}(h_{\alpha[\nu}\Delta_{\mu]}{}^\alpha + (\delta\nabla_{[\mu})\xi_{\nu]})\right] \; ,
\end{aligned} \tag{1.41}$$

where Σ is any cross-section of the Killing horizon and we have defined the black hole entropy, S, as

$$S = 2\pi\int_\Sigma \mathbf{X}^{\mu\nu}\epsilon_{\mu\nu} \; . \tag{1.42}$$

The integration involving $\mathbf{X}^{\mu\nu}\delta w_{\mu\nu}$ is zero because $\mathbf{X}^{\mu\nu}$ and $\delta w_{\mu\nu}$ have only normal-normal and normal-tangential components[31] with respect to Σ, respectively.

[30] The m^i, $i = 1, 2$, are two linear independent tangent vectors of the Killing horizon and a^i's are constants. (See the paragraph **Derivation** in subsection 1.2.2 on page 14.) This can be proved by expanding $\nabla_{[\mu}\xi_{\nu]} = \xi_{[\mu}v_{\nu]}$. Using the definition and the surface-forming property of a Killing vector field, it is seen that we can do such an expansion.

[31] A tensor, $w_{\mu\nu}$, is said to have only normal-tangent components with respect to a 2-surface, Σ, if

Variational formula on the bifurcation surface of the Killing horizon If we choose the Σ as the bifurcation 2-surface of the black hole on which the Killing vector field, ξ, vanishes,[32] then the second line of equation (1.41) is zero because $\Delta_{\mu\nu} = 0$.[33] From equations (1.41), (1.33), and (1.27), we have

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\delta S - \epsilon_A \int_\Sigma \left[\delta(\xi^\nu \mathcal{A}_\nu \mathcal{F}^{\alpha\beta}\epsilon_{\alpha\beta\gamma\mu}) - \xi^\lambda \mathcal{F}^{\nu\tau}\delta\mathcal{A}_\nu \epsilon_{\lambda\tau\gamma\mu}\right] dx^\gamma \wedge dx^\mu \ . \tag{1.43}$$

We have kept terms involving $\mathcal{A}_\nu$ because they have to be dealt with greater care. Recall that for a Reissner-Nordström black hole[34] the Killing vector is $\xi = \partial_t \propto V\partial_V - U\partial_U$, and the gauge field is $\mathcal{A} = \mathcal{A}_t dt \propto \frac{\partial_U}{U} - \frac{\partial_V}{V}$. Therefore, $\xi^\nu \mathcal{A}_\nu = constant$[35] on the Killing horizon even though the Killing vector, ξ^ν, is zero and the gauge field is infinite on the bifurcation 2-surface on which $U = V = 0$. However, noting that, formally, $\sum_{\gamma<\mu} \xi^\nu \delta\mathcal{A}_\nu \mathcal{F}^{\alpha\beta}\epsilon_{\alpha\beta\gamma\mu} dx^\gamma \wedge dx^\mu = \sum_{\gamma<\mu} \xi^\lambda \delta\mathcal{A}_\nu \mathcal{F}^{\nu\tau}\epsilon_{\lambda\tau\gamma\mu} dx^\gamma \wedge dx^\mu$ when evaluated on any cross-section of the Killing horizon of a static black hole background, we then arrive at the desired expression,

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\delta S + \Phi\delta q \ , \tag{1.44}$$

where

$$\Phi = -\xi^\nu \mathcal{A}_\nu \ , \tag{1.45}$$

$$q = \epsilon_A \int_\Sigma \mathcal{F}^{\alpha\beta}\epsilon_{\alpha\beta\gamma\mu} dx^\gamma \wedge dx^\mu \ . \tag{1.46}$$

$w_{\mu\nu} l_1^\mu l_2^\nu = 0$ except when one of the l_i's is normal, and another one is tangent to Σ. Similar definition extends to normal-normal case. For $\mathbf{X}^{\mu\nu}$, this is obvious because, from equation (1.32), $\mathbf{X}^{\mu\nu}$ is in fact the volume 4-form. For $\delta w_{\mu\nu}$, at first expand $\delta\epsilon_\mu{}^\nu = \delta\xi_\mu n^\nu + \xi_\mu \delta n^\nu - \xi^\nu \delta n_\mu$. Then it can be checked that $\delta w_{\mu\nu}$ has only normal-tangential components by multiplying $\delta w_{\mu\nu}$ with various combinations of ξ and n and with the help of the identity, $\xi^\nu \delta n_\nu = 0$. (This is because of the normalisation condition $\xi^\nu n_\nu = -1$.)

[32] It has been proved in reference [33] that the event horizon of a stationary black hole is always a Killing horizon. The generality of extending this Killing horizon to a bifurcation Killing horizon is discussed in reference [52, 53].

[33] Note that, by explicit calculation for specific cases, $\epsilon_{\mu\nu}$ is non-zero even though it is linear in ξ. Therefore, explicit calculation is necessary to identify the specific form of the first law.

[34] For the solution of Reissner-Nordström black holes, see section D.2 in appendix D at the end of chapter 4.

[35] The fact that $\xi^\nu \mathcal{A}_\nu = constant$ on a Killing horizon can be proved as follows: The Einstein-Maxwell equation and the identity, $\mathcal{R}_{\mu\nu}\xi^\mu\xi^\nu = 0$, on a Killing horizon imply that $\xi^\nu \mathcal{F}_{\nu\mu} \propto \xi_\mu$. Using the fact that $\mathcal{L}_\xi \mathcal{A}_\mu = 0$, we then arrive at $\nabla_\mu(\xi^\nu \mathcal{A}_\nu) \propto \xi_\mu$. We then conclude that $\xi^\nu \mathcal{A}_\nu$ is a constant on the Killing horizon.

Note that the differential formula of the first law (1.44) is obtained by employing the vanishing property of ξ^ν on the bifurcation 2-surface. It is unclear to me if the various terms involving $\delta g_{\mu\nu}$ in equation (1.41) can be exactly cancelled on any cross-section of the Killing horizon, though it is so for $\delta \mathcal{A}_\nu$. However, the black hole entropy, S, can be calculated on any cross-section of the Killing horizon due to the static nature of the background metric [40, 71].

Example On the space-time of a Schwarzschild black hole in the Kruskal co-ordinate (T, X, θ, ϕ) [67], we have

$$\begin{aligned} \xi &= \kappa(X\partial_T + T\partial_X) \ , \\ \kappa &= \frac{1}{4\pi GM} \ , \end{aligned} \tag{1.47}$$

$$\epsilon^{TX}{}_{\phi\theta} = -\left[\frac{32(MG)^3 e^{-r/(2MG)}}{r}\right]^{-1} r^2 \sin(\theta) \ , \tag{1.48}$$

$$\epsilon_{TX} = \frac{32(MG)^3 e^{-r/(2MG)}}{r} \ . \tag{1.49}$$

Note that due to the abnormal choice of orientation of boundary, Σ, in equation (1.37), the volume element on Σ is $\epsilon^{TX}{}_{\phi\theta} d\theta d\phi$, instead of $\epsilon^{TX}{}_{\theta\phi} d\theta d\phi$. It is then easily seen, from equations (1.42) and (1.32), that $S = 4\pi GM^2 = A/4$ in which A is the area of the cross-section of the event horizon.

* * *

Appendix A

Some Basic Elements of 2-D Dilatonic Gravity

Some basic elements of the 2-D dilatonic gravity is introduced. I employ units such that $\hbar = c = k_B = 1$ in which k_B is the Boltzmann's constant. The gravitational constant, G, will be shown explicitly, though $G = 1$ as its value is used in numerical calculations. The signature of Lorentzian space-time is $(- + ++)$. The definitions of various geometric quantities follow Weinberg's [74] such that

$$\begin{aligned} \mathcal{R}^{\lambda}{}_{\mu\nu\kappa} &= \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} , \\ \mathcal{R}_{\mu\nu} &= \mathcal{R}^{\alpha}{}_{\mu\alpha\nu} , \end{aligned}$$

where $\mathcal{R}^{\lambda}{}_{\mu\nu\kappa}$ satisfy, for an arbitrary vector, χ_μ,

$$\nabla_\kappa \nabla_\nu \chi_\mu - \nabla_\nu \nabla_\kappa \chi_\mu = -\chi_\lambda \mathcal{R}^{\lambda}{}_{\mu\nu\kappa} .$$

When only 4-D physics is involved, I will use the Greek letters to denote the tesorial indices which range from 0 to 3. Frequently, we will consider a spherically symmetric system. Then we have the following correspondence of co-ordinate indices: $(0, 1, 2, 3) = (t, r, \theta, \phi)$.

A.1 Dilaton field via dimensional reduction

In order to motive the introduction of dilaton field from the point of view of 4-D gravity,[1] we start with spherically symmetric 4-D system with the action

$$S = \frac{1}{2}\int d^4x\sqrt{|\tilde{g}|}(-\frac{\tilde{\mathcal{R}}}{8\pi G} - \tilde{\nabla}_\mu\varphi\tilde{\nabla}^\mu\varphi) \ .$$

The most general metric with respect to spherical symmetry is [74]

$$ds^2 = g_{ab}dx^a dx^b + e^{-2\phi}d\Omega^2 \ ,$$

where g_{ab} and ϕ are functions of x^a only, $d\Omega^2$ is the angular part of the metric, and the first and second half of Latin indices run over $0, 1$ and $2, 3$, respectively. In order to distinguish the intrinsic 2-D geometric quantities defined by g_{ab} from the 4-D ones, I will use a *tilde* to indicate explicitly the 4-D character of that quantity when 2-D quantities are involved.

Straightforward algebra gives us Einstein field equation as follows

$$\begin{aligned} \tilde{\mathcal{R}}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{\mathcal{R}} &= \mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} \\ &\quad -2\nabla_a\nabla_b\phi + 2\nabla_a\phi\nabla_b\phi - g_{ab}(3\nabla_a\phi\nabla^a\phi - 2\Box\phi - e^{2\phi}) \\ &= -8\pi G T_{ab} \ , \end{aligned} \tag{A.1}$$

$$\tilde{\mathcal{R}}_{mm} - \frac{1}{2}\tilde{g}_{mm}\tilde{\mathcal{R}} = -g_{mm}(\frac{1}{2}\mathcal{R} - \Box\phi + \nabla_a\phi\nabla^a\phi) = 0 \ , \tag{A.2}$$

where

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}(\nabla_a\varphi\nabla^a\varphi) \ ,$$

and note that in 2-D

$$\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} = 0 \ .$$

Because the curvature tensor, $\mathcal{R}_{abcd}$, has only one algebraically independent component, $\mathcal{R}_{1212}$, the curvature scalar can be written as $\mathcal{R} = 2\mathcal{R}_{1212}/g$ [74].

Effectively, equations (A.1) and (A.2) can be derived from the following action

$$S = \frac{1}{4G}\int d^2x\sqrt{|g|}e^{-2\phi}(-\mathcal{R} + 2\nabla_a\phi\nabla^a\phi + 2e^{2\phi} - 8\pi G\nabla_a\varphi\nabla^a\varphi) \ .$$

[1] For the origin of the dilaton field in higher dimensional theories, the reader is referred to article [11] and references therein.

A.2 The CGHS model

Unfortunately, the above system is still too complicated to be solved analytically. The so called CGHS[2] model considers the following action [11],

$$S = \frac{1}{4G}\int d^2x\sqrt{|g|}\left[e^{-2\phi}(-\mathcal{R} + 4\nabla_a\phi\nabla^a\phi + 4\lambda^2) - 8\pi G\nabla_a\varphi\nabla^a\varphi\right] . \tag{A.3}$$

The equations of motion are

$$\mathcal{R} + 4(\nabla_a\phi\nabla^a\phi - \Box\phi - \lambda^2) = 0 , \tag{A.4}$$

$$2\nabla_a\nabla_b\phi + 2g_{ab}(\nabla_a\phi\nabla^a\phi - \Box\phi - \lambda^2) - 8\pi G e^{2\phi}T_{ab} = 0 , \tag{A.5}$$

$$\Box\varphi = 0 , \tag{A.6}$$

where

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\nabla^c\varphi\nabla_c\varphi .$$

In 2-D, we can always choose a conformal gauge locally such that the metric is written as

$$ds^2 = -e^{2\rho}dx^+dx^- ,$$

where $x^\pm = x^0 \pm x^1$.[3] Note that there is still a residual gauge invariance within the conformal gauge because a co-ordinate transformation like $x^\pm = x^\pm(z^\pm)$ changes the metric to

$$ds^2 = -e^{2\rho'}dz^+dz^- , \tag{A.7}$$

[2]The CGHS refers to the initials of *C*allan, *G*iddings, *H*arvey, and *S*trominger.

[3]This can be seen by considering the identity [16]

$$ds^2 = g_{ab}dy^a dy^b = (\sqrt{g_{00}}dy^0 + \frac{g_{01}+\sqrt{-g}}{\sqrt{g_{00}}}dy^1)(\sqrt{g_{00}}dy^0 + \frac{g_{01}-\sqrt{-g}}{\sqrt{g_{00}}}dy^1) = -e^{2\rho}dx^+dx^- .$$

With the help of the continuity condition, $\partial_{y^0}\partial_{y^1}x^a = \partial_{y^1}\partial_{y^0}x^a$, one arrives at the condition, $\mathcal{L}_B x^a = 0$, where the Beltrami's operator, $\mathcal{L}_B$, is defined as

$$\mathcal{L}_B = \partial_{y^0}(\frac{g_{01}\partial_{y^1} - g_{11}\partial_{y^0}}{\sqrt{-g}}) + \partial_{y^1}(\frac{g_{01}\partial_{y^0} - g_{00}\partial_{y^1}}{\sqrt{-g}}) .$$

The theory of partial differential equation teaches us that if g_{ab} is analytic, then there are solutions which are one-to-one between (y^0, y^1) and (x^0, x^1).

where $\rho' = \rho + \frac{1}{2}\ln(\partial_{z^+}x^+\partial_{z^-}x^-)$.

In conformal gauge we then have $\mathcal{R} = -8e^{-2\rho}\partial_+\partial_-\rho$, and equations (A.4)-(A.6) can be simplified to

$$2\partial_+\partial_-\rho + 4\partial_+\phi\partial_-\phi - 4\partial_+\partial_-\phi + \lambda^2 e^{2\rho} = 0\ , \tag{A.8}$$

$$4\partial_+\phi\partial_-\phi - 2\partial_+\partial_-\phi + \lambda^2 e^{2\rho} - 8\pi G e^{2\phi}T_{+-} = 0\ , \tag{A.9}$$

$$\partial_+\partial_+\phi - 2\partial_+\rho\partial_+\phi - 8\pi G e^{2\phi}T_{++} = 0\ , \tag{A.10}$$

$$\partial_-\partial_-\phi - 2\partial_-\rho\partial_-\phi - 8\pi G e^{2\phi}T_{--} = 0\ , \tag{A.11}$$

$$\nabla_a\nabla^a\varphi = 0\ . \tag{A.12}$$

Equations (A.8) and (A.9) give us $2\partial_+\partial_-(\rho - \phi) = 8\pi G e^{2\phi}T_{+-}$. The general solution is then

$$\rho - \phi = f_+(x^+) + f_-(x^-) + 4\pi G\int^{x^+} dy^+ \int^{x^-} dy^- e^{2\phi}T_{+-}\ . \tag{A.13}$$

However, using the residual co-ordinate invariance described in equation (A.7) we can chose a ρ' such that $\rho' = \rho + f_+(x^+) + f_-(-)$. Therefore, we can set $f_\pm = 0$ in equation (A.13).

A.3 Eternal black hole solutions

Let us consider the eternal black hole solution as the scalar field $\varphi = constant$.

Solutions Then equations (A.8)-(A.11) can be further simplified to

$$\partial_+\partial_-(e^{-2\phi}) + \lambda^2 = 0\ , \tag{A.14}$$

$$\partial_\pm\partial_\pm(e^{-2\phi}) = 0\ , \tag{A.15}$$

with $\rho = \phi$ because of equation (A.13). The general solution[4] for the metric of the above equations is

$$ds^2 = -e^{2\phi}dx^+dx^- = \frac{-dx^+dx^-}{\frac{8\pi GM}{\lambda} - \lambda^2 x^+x^-} ,$$

where M is a constant. We then have

$$\mathcal{R} = \frac{-32\pi GM\lambda^2}{8\pi GM - \lambda^3 x^+x^-} .$$

The metric has a space-like singularity at $M = \lambda^3 x^+x^-/8\pi G$. On the other hand, it is seen to be asymptotically flat by performing the co-ordinate transformation, $\lambda x^\pm = \pm e^{\pm\lambda\sigma^\pm}$ $(\sigma^\pm = \sigma^0 \pm \sigma^1)$. Then we arrive at

$$ds^2 = \frac{-d\sigma^+d\sigma^-}{1 + \frac{8\pi GM}{\lambda}e^{\sigma^- - \sigma^+}} . \tag{A.16}$$

Mass of the black hole The constant M is the mass of the eternal black hole. This can be understood from the Noether charge method developed by Wald [71].[5]

For the Lagrangian (A.3) with $\varphi = constant$, the symplectic potential 1-form, $\Theta(g_{ab}, \delta g_{ab}, \phi, \delta\phi) = \Theta_e dx^e$, is

$$\begin{aligned} \Theta_e &= \left[\frac{e^{-2\phi}}{2\pi G}(\nabla^d\phi)\delta\phi + 2E_g^{abcd}\nabla_b\delta g_{ac} - \nabla_b(2E_g^{adcb})\delta g_{ac}\right]\epsilon_{de} \\ &= \frac{e^{-2\phi}\epsilon_{ae}}{16\pi G}\Big[8(\nabla^a\phi)\delta\phi - g^{bd}\nabla^a\delta g_{bd} + \nabla^b\delta g_b{}^a \\ &\quad -2(\nabla^a\phi)g^{bd}\delta g_{bd} + 2(\nabla^b\phi)g^{ad}\delta g_{bd}\Big] , \end{aligned} \tag{A.17}$$

where $\boldsymbol{\epsilon} = \sum_{a<b}\epsilon_{ab}dx^a \wedge dx^b$ is the volume 2-form and

$$E_g^{abcd} = \frac{-e^{-2\phi}}{32\pi G}(g^{ac}g^{bd} - g^{ad}g^{bc}) .$$

[4]The most general solution should be

$$e^{-2\phi} = \frac{8\pi GM}{\lambda} - \lambda^2(x^+ + B_+)(x^- + B_-) ,$$

where $B_\pm$ are constants. They can be removed by shifting the origin of the co-ordinates $x^\pm$.

[5]See also section 1.2.3 for an introduction.

The Noether current 1-form, $\mathbf{J} = J_a dx^a$, with respect to a Killing vector field, ξ, is

$$J_a = \frac{-\epsilon_{ba}}{8\pi G}\nabla_c\left(e^{-2\phi}\nabla^{[b}\xi^{c]} + 2\xi^{[b}\nabla^{c]}e^{-2\phi}\right) = (\mathbf{d}Q)_a \ ,$$

where Q is the Noether charge 0-form, defined as

$$Q = \frac{-\epsilon_{ab}}{16\pi G}(e^{-2\phi}\nabla^a\xi^b + 2\xi^a\nabla^b e^{-2\phi}) \ .$$

The variation of Hamiltonian, δH, is

$$\delta H = \delta Q(\infty) - \xi\cdot\mathbf{\Theta}(\infty) \ ,$$

where the relevant quantities are calculated at spatial infinity, denoted by ∞. Consider the black hole solution of equation (A.16) with the Killing vector field, $\xi = \partial_{\sigma^0}$, and the variation, δM, we found that $\xi\cdot\mathbf{\Theta}(\infty) = 0$. Therefore, the variation of the mass of the black hole, $\delta\mathcal{E}$, is

$$\delta\mathcal{E}{=}\delta H = \delta Q(\infty) = \frac{\epsilon^{01}}{16\pi G}(e^{-2\phi}\partial_{\sigma^1}\xi_0 - 2\xi_0\partial_{\sigma^1}e^{-2\phi}) = \delta M \ . \tag{A.18}$$

The mass of the black hole is thus $Q(\infty) = M$.

Penrose diagram Since the metric (A.16) is similar to the t–r_* part of a Schwarzschild black hole in Kruskal co-ordinates, its Penrose diagram can be derived following reference [33]. We consider the co-ordinate transformation $u^{\pm} = \arctan(x^{\pm})$ such that $-\pi/2 < u^{\pm} < \pi/2$. We then have

$$ds^2 = \frac{1}{\frac{8\pi GM}{\lambda} - \lambda x^+x^-}\frac{-du^+du^-}{\cos^2(u^+)\cos^2(u^-)} \ .$$

With the help of the relation, $\tan(u^+ + u^-) = \frac{2t}{1-\tan(u^+)\tan(u^-)}$, we find the constraint that $-\pi/2 < u^+ + u^- < \pi/2$. The Penrose diagram is shown in figure A.1.

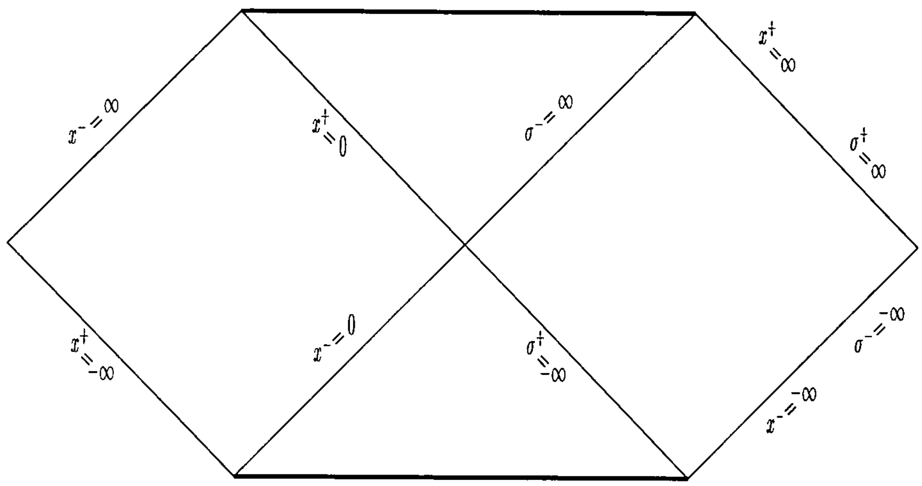


Figure A.1: Penrose diagram for an eternal black hole.

Appendix B

Differential Forms

The adopted convention and notation for differential forms are introduced. Two useful formulae are listed. For more detailed discussion, see textbook [67].

I will use the Latin letters to denote the indices of differential forms which range from 0 to 3.

B.1 Convention and notation

Symbols of differential forms I will use the **boldface** of a letter to denote the differential form of the corresponding totally anti-symmetric tensor, e.g.,

$$\mathbf{f} = \sum_{a<b<c<d} f_{abcd}\, dx^a \wedge dx^b \wedge dx^c \wedge dx^d \ ,$$

where f_{abcd} is totally anti-symmetric.

Volume 4-form The volume 4-form for a Lorentzian space-time is

$$\boldsymbol{\epsilon} = \sum_{a<b<c<d} \epsilon_{abcd}\, dx^a \wedge dx^b \wedge dx^c \wedge dx^d \ ,$$

where ϵ_{abcd} is normalised according to $\epsilon^{abcd}\epsilon_{abcd} = -4!$ in which the indices are raised by g^{ab}.

Contraction The notation $\xi \cdot \mathbf{f}$ indicates the contraction of the first index of $\mathbf{f}$ with the vector ξ, e.g.,

$$\xi \cdot \mathbf{f} = \sum_{\substack{\alpha \\ b<c}} \xi^\alpha f_{\alpha bc}\, dx^b \wedge dx^c \ .$$

Exterior derivative The symbol, $\mathbf{d}$, stands for the exterior derivative, e.g.,

$$\begin{aligned} \mathbf{d}(\sum_{b<c} f_{bc}\, dx^b \wedge dx^c) &= \sum_{\substack{a \\ b<c}} \nabla_a f_{bc}\, dx^a \wedge dx^b \wedge dx^c \ , \\ &= \sum_{a<b<c} (\nabla_a f_{bc} - \nabla_b f_{ac} + \nabla_c f_{ba}) dx^a \wedge dx^b \wedge dx^c \ . \end{aligned}$$

Lie derivative The Lie derivative, $\mathcal{L}_\chi$ of a tensor, $T^\alpha{}_\beta$, with respect to a vector χ^ν is defined as

$$\mathcal{L}_\chi T^\alpha{}_\beta = \chi^\nu \nabla_\nu T^\alpha{}_\beta + T^\alpha{}_\nu \nabla_\beta \chi^\nu - T^\nu{}_\beta \nabla^\alpha \chi_\nu \ .$$

Its generalisation to higher rank tensors is straightforward. The Lie derivative of a differential form is defined accordingly.

B.2 Two nice formulae

The First The following identity for components of the volume 4-form with respect to a Lorentzian metric holds:

$$\sum_{c_{j+1},\cdots,c_4} \epsilon^{a_1 \dots a_j c_{j+1} \dots c_4} \epsilon_{b_1 \dots b_j c_{j+1} \dots c_4} = -(4-j)!\, j!\, \delta^{[a_1}{}_{b_1} \cdots \delta^{a_j]}{}_{b_j} \ . \tag{B.1}$$

The Second The following identity for an arbitrary vector field χ and a differential form $\mathbf{f}$ will be useful:

$$\mathbf{d}(\chi \cdot \mathbf{f}) = \mathcal{L}_\chi \mathbf{f} - \chi \cdot \mathbf{d}\mathbf{f} \ . \tag{B.2}$$

Chapter 2

Semi-classical Black Hole Physics

The black hole thermodynamics was firmly established after Hawking discovered that a black hole is not black at all. It radiates, too; and one can therefore attribute a temperature to a black hole [30]. Only then, the concept of black hole entropy had a more solid foundation because in the textbook thermodynamics, the entropy and temperature form a conjugate pair of variables. In this chapter, I review two of the semi-classical aspects of black hole physics—black hole temperature and entropy—in order to establish black hole thermodynamics, which, in turn, will motivate the search for a statistical explanation of black hole entropy.

Abstract of chapter 2

Section 2.1 The derivation of Hawking radiation is reviewed within the context of the 2-D CGHS dilationic gravity model [11] which provides a more transparent and tractable description than the 4-D case originally developed by Hawking [30].

The underlying principle which governs the black hole radiation is the Bogoluibov transformation, I thus foremost review the basic idea of the Bogoluibov transformation in subsection 2.1.1. In subsection 2.1.2, the black hole radiation is derived by applying the Bogoluibov transformation to the specific case of quantum field theory on a black hole forming background. No discussion about black hole radiation could be complete these days without mentioning the issue of information puzzle [32]. However, this issue by itself is so big and difficult that it deserves another PhD thesis. I thus only briefly, in subsection 2.1.3, describe the source of this problem and record two proposals which, personally, I think most likely to be able to offer the resolution.

Section 2.2 After having determined the temperature of a black hole, the concept of black hole entropy proposed by Bekenstein [4] and the generalised laws of thermodynamics [5] are introduced in subsection 2.2.1. Before I present my own arguments in support of the generalised second law in subsection 2.2.3, I comment on previous attempts to prove it in subsection 2.2.2 in order to contrast their approach with my attitude.

2.1 Black hole radiation

Deep in the hole, dormant in the black; where sunshine dares not stay. But ardour will never wane.

2.1.1 Bogoliubov transformation

Phenomenologically, the Hawking radiation from a black hole resulting from the gravitational collapse can be regarded as a reminiscence of the electron-positron pair creation in strong electric field background [8].

Following the arguments in reference [25], we employ the Bogoliubov transformation method which has become a standard approach since it was first used by Hawking to derive black hole radiation [30]. Though, in 2-D there is another more elegant and quicker method based on the relation between trace anomaly of conformally coupled matter field and the curvature scalar [8], I still apply the more mundane, but general method based on the Bogoliubov transformation.

Bogoliubov transformation—basic idea

Mode decomposition The basic idea behind the Bogoliubov transformation is quite straightforward [8]. In quantum field theory on a classical, stationary space-time background, the real, massless scalar field operator, $\hat{\varphi}$, can be expanded in terms of the eigen-modes, u_w, as follows,

$$\hat{\varphi} = \int_0^\infty dw(\hat{a}_w u_w + \hat{a}_w^\dagger u_w^*) \ ,$$

with the normalisation condition

$$(u_w, u_{w'})_\Sigma = -(u_w^*, u_{w'}^*)_\Sigma = 2\pi\delta_{ww'} \ ,$$

where

$$(f, g)_\Sigma = -i \int_\Sigma (f \nabla_\mu g^* - (\nabla_\mu f) g^*) \, d\Sigma^\mu \ .$$

The integration region Σ is a suitably chosen Cauchy surface.[1]

[1] In the massless case we are considering, the convenient choice of the Cauchy surface, Σ, is a null hypersurface. Though the same symbol, Σ, will be used throughout the text, the precise choice of Σ

Unique in a flat space-time If the space-time background is stationary as it is in the case of quantum field theory in Minkowski background, the eigen-modes for initial states (*in* modes) and final states (*out* modes) can be chosen the same. They thus define the same vacuum states and Fock spaces. The isomorphism between the initial Fock space and the final one in an interacting theory is indeed a built-in principle of quantum field theory in flat space-time [38]. However, when the background space-time is non-stationary, situation is much more complicated and interesting.

Need for the time-like Killing vector field However, as far as I am aware, there has not existed a satisfactory quantum field theory in a dynamic, classical background up to now, not to mention addressing the problem of quantum field theory of gravitational field. However, as the energy scale involved in the quantum field is much less than the Planck energy, the so called quantum field theory on curved space-times is reliable at leading order [8].[2] Even so, there are still some complication when the space-time is dynamic. The one which is related to the Bogoliubov transformation is the choice of eigen-modes. This is then related to the well-known issue of time because an eigen-mode is necessarily an eigen-mode of Hamiltonian. The definition of Hamiltonian is in fact a definition of time-variable. If the space-time background is stationary, i.e., there exist a time-like Killing vector field, then the Hamiltonian can be chosen as the generator of the Killing vector field.

Non-unique in a dynamic space-times In a dynamic space-time background, there is no privileged choice of time-variable. Though, if the change of space-time background is not so rapid with respect to a prejudicially chosen time-variable, then the adiabatic

depends on the context.

[2]This statement might be a backfire to our approach because, as shown later, the Hawking radiation necessarily involves trans-planck frequency modes. Therefore, the back-reaction of the energy-momentum resulting from the radiation cannot be ignored. The semi-classical approach is much more involved because, in general, the Einstein field equation cannot be solved exactly anymore. Fortunately, in certain models, e.g., the RST model, this can be achieved [55]. Nonetheless, the back-reaction only changes the background metric, the correlation between various radiating modes, perhaps also the final fate of an evaporating black hole [59], but not the fact of radiating and the late time thermal behaviours. I thus confine myself to the classical background case.

expansion is still applicable [8]. Black hole formation is definitely not the case. Nonetheless, if the initial and final stages of the dynamic process are stationary, there are still well defined eigen-modes there. But, generally, the *in* and *out* modes will not be the same. However, if we still assume that the Hilbert spaces, $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ generated by the *in* and *out* modes are the same, we can still expand the field operator in terms of *in* and *out* modes, respectively. However, since there are two different sets of eigen-modes, the expansion can be done in two ways. Explicitly, we can write

$$\begin{aligned} \hat{\varphi} &= \int_0^\infty dw(\hat{a}_w u_w + \hat{a}_w^\dagger u_w^*) \\ &= \int_0^\infty dw(\hat{b}_w v_w + \hat{b}_w^\dagger v_w^* + \hat{\bar{b}}_w \bar{v}_w + \hat{\bar{b}}_w^\dagger \bar{v}_w^*) \ , \end{aligned} \tag{2.1}$$

with the normalisation condition (with unlisted being zero)

$$(u_w, u_{w'})_\Sigma = (v_w, v_{w'})_\Sigma = (\bar{v}_w, \bar{v}_{w'})_\Sigma = 2\pi\delta_{ww'} \ , \tag{2.2}$$

$$(u_w^*, u_{w'}^*)_\Sigma = (v_w^*, v_{w'}^*)_\Sigma = (\bar{v}_w^*, \bar{v}_{w'}^*)_\Sigma = -2\pi\delta_{ww'} \ , \tag{2.3}$$

where u_w stands for *in* modes and v_w and $\bar{v}_w$ for *out* modes.[3] We have assumed that the Hilbert space, $\mathcal{H}_{out}$, is a direct product of two Hilbert spaces spanned by v_w and $\bar{v}_w$ whose defining regions are *exclusive*, i.e., $\mathcal{H}_{out} = \mathcal{H}_{v_w} \otimes \mathcal{H}_{\bar{v}_w}$. We thus label the normalisation conditions with Σ to indicate that the normalisation is performed on the respective defining region.

As we will see later, it is indeed because of this *exclusiveness* that the exciting phenomenon of Hawking radiation occurs.

Bogoliubov coefficients α and β

Mode transformations For equation (2.1) to hold, we should be able to expand v_w and $\bar{v}_w$ in terms of u_w and u_w^*, i.e., we can write

$$v_w = \int_0^\infty dw'(\alpha_{ww'} u_{w'} + \beta_{ww'} u_{w'}^*) \ , \tag{2.4}$$

$$\bar{v}_w = \int_0^\infty dw'(\bar{\alpha}_{ww'} u_{w'} + \bar{\beta}_{ww'} u_{w'}^*) \ . \tag{2.5}$$

[3] We will call v_w and $\bar{v}_w$ the *external* and *internal* out modes, respectively.

The *Bogoliubov coefficients*, $\alpha_{ww'}$ and $\beta_{ww'}$, are determined from the normalisation,

$$(v_w, u_{w'})_\Sigma = 2\pi\alpha_{ww'} \ , \qquad (v_w, u^*_{w'})_\Sigma = -2\pi\beta_{ww'} \ , \tag{2.6}$$

where the inner products are evaluated on a proper Cauchy surface, Σ. There are similar relations for $\bar{\alpha}_{ww'}$ and $\bar{\beta}_{ww'}$. With the help of these relations, we can expand u_w in terms of v_w, $\bar{v}_w$, and their complex conjugates as

$$u_w = \int_0^\infty dw'(\alpha^*_{w'w}v_{w'} - \beta_{w'w}v^*_{w'} + \bar{\alpha}^*_{w'w}\bar{v}_{w'} - \bar{\beta}_{w'w}\bar{v}^*_{w'}) \ .$$

Operator transformations Consequently, we arrived at the relations between various creation and annihilation operators,

$$\hat{a}_{w'} = \int_0^\infty dw(\hat{b}_w\alpha^*_{ww'} + \hat{b}^\dagger_w\beta^*_{ww'} + \hat{\bar{b}}_w\bar{\alpha}^*_{ww'} + \hat{\bar{b}}^\dagger_w\bar{\beta}^*_{ww'}) \ , \tag{2.7}$$

$$\hat{b}_{w'} = \int_0^\infty dw(\hat{a}_w\alpha^*_{w'w} - \hat{a}^\dagger_w\beta^*_{w'w}) \ , \tag{2.8}$$

$$\hat{\bar{b}}_{w'} = \int_0^\infty dw(\hat{a}_w\bar{\alpha}^*_{w'w} - \hat{a}^\dagger_w\bar{\beta}^*_{w'w}) \ . \tag{2.9}$$

Meaning of Bogoliubov coefficients The meaning of coefficient β[4] can be understood by observing that

$${}_{in}\langle 0|\hat{N}^v_w|0\rangle_{in} = \int_0^\infty dw'\beta_{ww'}\beta^*_{ww'} \ , \tag{2.10}$$

where $\hat{N}^v_w = \hat{b}^\dagger_w\hat{b}_w$ is the number operator for the *external out* modes and $|0\rangle_{in}$ is the *in* vacuum such that $\hat{a}_w|0\rangle_{in} = 0$. In words, it means that the with respect to the *out* eigen-modes, the *in* vacuum is not the vacuum. The Bogoliubov coefficients then contain the information of the particle content of the *in* vacuum with respect to *out* eigen-modes.

From the above formal discussion, we have found that in order to observe that particles are created from the vacuum, the *in* and *out* vacua have to be different. In other words, the natural choice of *in* and *out* eigen-modes are different. Hawking radiation is in fact such a phenomenon due to gravitational collapse as we will illustrate below in the 2-D CGHS model following the approach in references [25, 26].

[4]Similar expression can also be found for α.

2.1.2 Hawking radiation

Mode decomposition

Before the black hole forms In section 1.1.1 the classical solution of black hole formation from matter has been given. As $x^+ < x_i^+$, the solution is the dilatonic vacuum (*in* vacuum) because there is no matter anywhere. The metric is given by equation (1.6) with M and P set to zero, i.e.,

$$ds^2 = \frac{dx^+ dx^-}{\lambda^2 x^+ x^-} = -dy^+ dy^- \ , \tag{2.11}$$

where

$$\lambda x^+ = e^{\lambda y^+} \ , -\lambda x^- = \lambda 8\pi G P_\infty e^{-\lambda y^-} \ , \tag{2.12}$$

and P_∞ is defined in equation (1.9). From equation (2.11) it is seen that the natural choice for a right-moving *in* eigen-mode, u_w is

$$u_w = \frac{1}{\sqrt{2w}} e^{-iwy^-} \ .$$

Recall that when compared with the 4-D case, the x^1 co-ordinate is the analogue of the r co-ordinate, a right-moving modes is thus the analogue of a particle escaping from the neighbourhood of the black hole. Those left-moving ones, at late time, are thus swallowed by the black hole formed. To recover the observations performed at future infinity, we need considering right-moving modes only.

After the black hole formed At late time after the black formed, i.e., $x^+ > x_f^+$, the metric is (see equations (1.6) and (1.11))

$$ds^2 = \frac{-dx^+ dx^-}{\frac{8\pi G M_\infty}{\lambda} - \lambda^2 x^+ (x^- + 8\pi G P_\infty)} = \frac{-d\sigma^+ d\sigma^-}{1 + \frac{8\pi G M_\infty}{\lambda} e^{\lambda(\sigma^- - \sigma^+)}} \ .$$

The natural choice for a right-moving, *external out* eigen-mode, v_w, is

$$v_w = \frac{\theta(-y^-)}{\sqrt{2w}} e^{-iw\sigma^-} \ ,$$

where $y^{\pm}$ are related to $x^{\pm}$ through the relation (2.12) and θ is a step function such that $\theta(z>0)=1$, otherwise zero. The reason for the appearance of θ function is that, in fact, there are two *exclusive regions* as $x^{+}\longrightarrow\infty$. The boundary of these two regions is the event horizon $x^{-}=-8\pi GP_{\infty}$ which corresponds to $y^{-}=0$ from equation (2.12) and $\sigma^{-}=\infty$ from equation (1.10) (see figure 1.1). A physical measurement made at exterior region should be independent of the choice of the *internal out* eigen-modes, $\bar{v}_{w}$, because they are unobservable, and therefore should be traced out . However, a good choice makes the explicit calculation easier. The following is a convenient choice [25]

$$\bar{v}_{w}=\frac{\theta(y^{-})}{\sqrt{2w}}e^{-iw\bar{\sigma}^{-}} ,$$

where $e^{\lambda\bar{\sigma}^{-}}=8\pi G\lambda P_{\infty}(e^{\lambda y^{-}}-1)$.

Resulting Bogoliubov coefficients Using the definition of Bogoliubov coefficients, equation (2.6), we have

$$\begin{aligned}
\alpha_{ww'} &= \frac{-i}{2\pi}\int_{-\infty}^{\infty}dy^{-}(v_{w}\partial_{-}u^{*}_{w'}-(\partial_{-}v_{w})u^{*}_{w'}) && (2.13)\\
&= \frac{-i}{\pi}\int_{-\infty}^{0}dy^{-}v_{w}\partial_{-}u^{*}_{w'} , && (2.14)
\end{aligned}$$

where in the second line we have used the fact that the right-moving, *external out* modes are defined only in the lower half of real y^{-} axis. We also performed integration by parts once. The resulting surface terms are discarded by replacing w and w' with $w-i\epsilon$ and $w'-i2\epsilon$ ($\epsilon>0$), respectively. As shown later, such pole insertion is needed to define Bogoliubov coefficients properly. With the help of the transformation that $\lambda(x^{-}+8\pi GP_{\infty})=-e^{-\lambda\sigma^{-}}$ (cf. equation (1.10)), we can write

$$\begin{aligned}
\alpha_{ww'} &= \frac{1}{2\pi}\frac{w'}{w}\int_{-\infty}^{0}dy^{-}e^{i\frac{w}{\lambda}\ln(\lambda 8\pi GP_{\infty}(e^{-\lambda y^{-}}-1))+iw'y^{-}}\\
&= \frac{1}{2\pi}\frac{w'}{w}(\lambda 8\pi GP_{\infty})\int_{0}^{1}dz(1-z)^{i\frac{w}{\lambda}}z^{-1+i\frac{w'-w}{\lambda}} , && (2.15)
\end{aligned}$$

where we have changed the variable to $z=e^{\lambda y^{-}}$ in the second line. It is easily seen that the pole prescription given above makes the integral finite.

Late time behaviour of Bogoliubov coefficients

For α Now, if we are interested in the late time ($x^+ \longrightarrow \infty$) behaviour of the Bogoliubov coefficients, we can approximate the integrand in the first line of equation (2.15) by its value near event horizon [25], $y^- \sim 0$, so we have

$$\alpha_{ww'} \simeq \frac{1}{2\pi}\frac{w'}{w}\int_{-\infty}^{0} dy^- e^{i\frac{w}{\lambda}\ln(-\lambda^2 8\pi G P_\infty y^-)+iw'y^-} . \tag{2.16}$$

This is equivalent to approximating the exponent of the *exp* in the integrand by its saddle point if $\frac{w'}{w} \gg 1$ since that is where the dominated contribution comes from and we expect those contributions from large phases cancel off. Physically, the condition $\frac{w'}{w} \gg 1$ is understandable due to the extremely large red-shift a wave suffers while it is emerging from the neighbourhood of a black hole [74]. Recall the relation between the *in* mode, $u_{w'}$, and *out* mode, v_w, is given by equation (2.4). Therefore, the observed *out* mode of low frequency w started off with an extremely high frequency w'. The involvement of trans-planck frequency modes in the derivation of the Hawking radiation has been causing concerns. Though the phenomenology of Hawking radiation seldom raises doubt, there are indeed considerably efforts trying to avoid the involvement of the ultrahigh frequencies. [5] Since certain modification in the approach used above is inevitable due to the effect of quantum gravity induced by those trans-planck modes, I have to leave this question aside.

For β Similarly, we can write down the late time behaviour of β as[6]

$$\beta_{ww'} = \frac{1}{2\pi}\frac{w'}{w}\int_{-\infty}^{0} dy^- e^{i\frac{w}{\lambda}\ln(\lambda 8\pi G P_\infty (e^{-\lambda y^-}-1))-iw'y^-} \tag{2.17}$$

$$\simeq \frac{1}{2\pi}\frac{w'}{w}\int_{-\infty}^{0} dy^- e^{i\frac{w}{\lambda}\ln(-\lambda^2 8\pi G P_\infty y^-)-iw'y^-} . \tag{2.18}$$

Relating the late time behaviour of α and β The difference between the $\alpha_{ww'}$ and $\beta_{ww'}$ is so small that it seems to suggest that there is a relation between them. Note that for the negative frequency *in* modes, $u^*_{w'}$, it is analytic on the lower half w' and y^-

[5] See reference [39] for an example.

[6] Note that the pole prescription in this case is $w \longrightarrow w - i\epsilon$ and $w' \longrightarrow w' + i2\epsilon$

complex planes, respectively, if the branch cut is chosen at upper half complex plane, i.e., $\ln(-1) = -i\pi$. We can then analytically continue the integration range in $\alpha_{ww'}$ from the negative real y^- axis to positive axis (cf. equation (2.14)). From equation (2.16), we therefore arrive at

$$
\begin{aligned}
\alpha_{ww'} &\simeq \frac{-1}{2\pi}\frac{w'}{w}\int_0^\infty dy^- e^{i\frac{w}{\lambda}\ln(-\lambda^2 8\pi G P_\infty y^-)+iw'y^-} \\
&= \frac{-1}{2\pi}\frac{w'}{w}\int_\infty^0 dy^- e^{i\frac{w}{\lambda}\ln(-\lambda^2 8\pi G P_\infty y^-)-iw'y^-+\frac{\pi w}{\lambda}} \\
&= -e^{\frac{\pi w}{\lambda}}\beta_{ww'} \; .
\end{aligned}
\tag{2.19}
$$

Planck distribution With the help of the relation[7]

$$
\begin{aligned}
\delta_{ww'} &= \int_0^\infty dw''(\alpha_{ww''}\alpha^*_{w'w''} - \beta_{ww''}\beta^*_{w'w''}) \\
&= (e^{\frac{\pi}{\lambda}(w+w')} - 1)\int_0^\infty dw''\beta_{ww''}\beta^*_{w'w''} \; ,
\end{aligned}
\tag{2.20}
$$

we can now calculate the particle number of *out* mode between frequency w and $w+dw$ (cf. equation (2.10)),

$$
\begin{aligned}
{}_{in}\langle 0|\hat{N}^v_w|0\rangle_{in} dw &= dw\int_0^\infty dw'\int_0^\infty dw''\beta_{ww'}\beta^*_{w''w'}\delta_{ww''} \\
&= \frac{dw\,\delta(0)}{e^{\frac{2\pi w}{\lambda}} - 1} \; ,
\end{aligned}
\tag{2.21}
$$

where $dw\delta(0)$ is normalised to a constant. To understand the above expression properly, we can introduce a periodic boundary condition so that the spectrum is discrete. Then the continuous integrating, $\int dw$, on the left hand side of equation (2.21) corresponding to a discrete summation, $\sum_{w_j}$, on the right hand side. The above equation shows that the observed spectrum distributes according to the Planck rule with temperature $\frac{2\pi}{\lambda}$.[8] This is one of the basic conclusion of Hawking's seminal work of article [30].

[7] The first line follows from the normalisation condition that $(v_w, v_{w'}) = 2\pi\delta_{ww'}$. Use has been made of the relation (2.19) in the second line.

[8] Surely, the temperature depends on the context. In the 4-D cases, the temperature, T, is given by $T = \frac{\hbar c^3}{k_B}\frac{\kappa}{2\pi}$ where κ is the surface gravity of the background black hole. It is proportional the temperature appeared in the first law of black hole mechanics (cf. equation (1.13)).

Black hole radiation is thermal

Constructing new *out* modes Another even more far-reaching conclusion is that, at late time ($x^+ > x_f^+$), the observed particle states are thermal radiation. This can be seen by expanding the *in* vacuum, $|0\rangle_{in}$, in terms of the *out* modes, v_w and $\bar{v}_w$. Usually, this is achieved through a trick by noting that the positive frequency mode e^{-iwy^-} is analytic and bounded on the lower half complex y^- plane [64]. Using the same approximation at $y^- \sim 0$ which was used to derive equation (2.16), we have

$$v_w = e^{-iw\sigma^-}\theta(-y^-) = (e^{-\lambda\sigma^-})^{i\frac{w}{\lambda}}\theta(-y^-) \sim (-\lambda^2 8\pi G P_\infty y^-)^{i\frac{w}{\lambda}}\theta(-y^-) \ , \quad (2.22)$$

$$\bar{v}_w = e^{iw\bar{\sigma}^-}\theta(y^-) = (e^{\lambda\bar{\sigma}^-})^{i\frac{w}{\lambda}}\theta(y^-) \sim (\lambda^2 8\pi G P_\infty y^-)^{i\frac{w}{\lambda}}\theta(y^-) \ . \quad (2.23)$$

Next, consider the following combination,

$$u^1_{w/\lambda} = (1 - e^{-2\pi\frac{w}{\lambda}})^{-1/2}(v_w + e^{-\pi\frac{w}{\lambda}}\bar{v}^*_w) \sim (\lambda^2 8\pi G P_\infty y^-)^{w/\lambda - \pi\frac{w}{\lambda}} \ , \quad (2.24)$$

$$u^2_{\frac{w}{\lambda}} = (1 - e^{-2\pi\frac{w}{\lambda}})^{-1/2}(\bar{v}_w + e^{-\pi\frac{w}{\lambda}}v^*_w) \sim (\lambda^2 8\pi G P_\infty y^-)^{-i\frac{w}{\lambda}} \ , \quad (2.25)$$

where the branch cut, as described in the paragraph above equation (2.19), is chosen at upper half complex plane of y^- such that $\ln(-1) = -i\pi$. They have the same analytic properties as $u_w \propto e^{-iwy^-}$. Therefore the new *out* modes, $u^1_{w/\lambda}$ and $u^2_{w/\lambda}$, are positive frequency modes with respect to y^-. The annihilation operator, $\hat{a}^1_{w/\lambda}$ ($\hat{a}^2_{w/\lambda}$), corresponding to the mode, $u^1_{w/\lambda}$ ($u^2_{w/\lambda}$), then annihilates the *in* vacuum because $u^1_{w/\lambda}$ ($u^2_{w/\lambda}$) is a superposition of u_w only.

From equations (2.24) and (2.25), we have

$$\hat{a}^1_{w/\lambda} = (1 - e^{-2\pi\frac{w}{\lambda}})(\hat{b}_w - e^{-\pi\frac{w}{\lambda}}\hat{\bar{b}}^\dagger_w) \ , \quad (2.26)$$

$$\hat{a}^2_{w/\lambda} = (1 - e^{-2\pi\frac{w}{\lambda}})(\hat{\bar{b}}_w - e^{-\pi\frac{w}{\lambda}}\hat{b}^\dagger_w) \ . \quad (2.27)$$

The content of the *in* vacuum Regarding the fact that

$$0 = (\hat{a}^{1\dagger}_{w/\lambda}\hat{a}^1_{w/\lambda} - \hat{a}^{2\dagger}_{w/\lambda}\hat{a}^2_{w/\lambda})|0\rangle_{in} \propto (\hat{b}^\dagger_w\hat{b}_w - \hat{\bar{b}}^\dagger_w\hat{\bar{b}}_w)|0\rangle_{in} \ ,$$

we see that the *in* vacuum contains the same number of particle and anti-particle for each *out* mode. Therefore, we can write down the following general expression

$$|0\rangle_{in} = \sum_{\{n_w\}} c(\{n_w\})|\{n_w\}\rangle|\{\bar{n}_w\}\rangle \ ,$$

where n_w $(= \bar{n}_w)$ is the particle (anti-particle) number of *out* mode with frequency w. With the help of equation (2.26) and the fact that $\hat{a}^1_{w/\lambda}|0\rangle_{in} = 0$, we arrived at for a single mode

$$c(n_w) = e^{-\pi\frac{w}{\lambda}} c(n_w - 1) \ .$$

By iteration, we then have $c(\{n_w\}) = c(0)e^{-\frac{\pi}{\lambda}\sum_w n_w w}$ in which $c(0)$ is a normalisation constant.

2.1.3 Information puzzle

The decline of the pure According to the physical picture given in reference [30], after a particle pair is created, the one with negative energy falls into the black hole, another one with positive energy escapes to infinity.[9] We therefore cannot observe the in-falling particles. From an outsider's point of view, s/he does not observe the pure state of the *in* vacuum, instead, s/he can only measure the density matrix arrived from integrating over the in-falling particle states, i.e., the physical quantities are coded in the following density matrix $\rho_{out}(\{n_w\}\{n'_w\})$

$$\begin{aligned} \rho_{out}(\{n_w\}\{n'_w\}) &= \sum_{\{\bar{n}_w\}} \langle n_w|\langle \bar{n}_w|0\rangle_{in\,in}\langle 0|\bar{n}_w\rangle|n'_w\rangle \\ &= e^{-\frac{2\pi}{\lambda}\sum_w n_w w}\delta_{\{n_w\}\{n'_w\}} \ . \end{aligned} \tag{2.28}$$

Assume that the black hole evaporates completely,[10] what is left is a mixed state of thermal radiation. If we accept this conclusion and we also assume that the in-falling matter which collapses to form the black hole is in a pure state initially, we then arrived at the so called information puzzle [32]: A pure state can involve into a mixed state.

The possible salvation Whether the above conclusion can be accepted or not is still under much debate.[11] There are several opinions around. I only record the two

[9]The Hawking radiation!

[10]It seems inevitably that as the black hole mass is reduced to that of several Planck mass, the quantum field theory on curved space-times used to derive above picture breaks down. However, there is no clear picture about what will/should happen at this stage. For competitive candidates, see the review article [50] and references therein.

[11]See reference [50] for a comprehensive review.

possibilities which I *feel*[12] most likely to provide the resolution. The first was proposed a long time ago by Page [49]: The subtle correlation between various modes of the thermal radiation will return us a pure state. The second is proposed by Myers recently [47]: A pure state cannot form a black hole. The latter one is of particular interest because it tries to solve the problem by illegitimating the question. The argument given in reference [47] is based on the approach of superstring theory. However, I wonder this idea can be tested in the context of quantum theory plus the Einstein field equation.

[12]Regarding to the fact that neither are there compelling arguments endorsing any one, thus also ruling out others, nor can I provide my own arguments, I *feel* it is better to resort to one's intuition.

2.2 Black hole entropy

A simple, but beautiful idea. That is the extreme of physics.

2.2.1 Black hole entropy and the generalised laws of thermodynamics

Black hole entropy—a short history

The conjecture that the area of a black hole is its measure of entropy was given by Bekenstein in reference [4]. He wrote,

> We take the area of a black hole as a measure of its entropy—entropy in the sense of inaccessibility of information about its internal configuration.

Note that this happened before Hawking's discovery that a black hole radiates [30]. The motivation behind this conjecture was recounted by Bekenstein in reference [7]. They included the Christoloulou's irreducible mass, Wheeler's suggestion of a demon who violates the second law with the help of a black hole, Penrose and Floyd's observation that the event horizon area tends to grow and Hawking's area theorem. As noted by Bekenstein [7],

> Carter and Bardeen, Carter, and Hawking were aware of the analogy between horizon area and entropy as reflected in their first and second laws of black hole mechanics, but did not take the analogy seriously.

It should not surprise us very much if some opposed this idea for the reason that the laws of black hole mechanics are exact classical laws following the Einstein field equation and several assumed energy conditions [2]. On the other hand, the laws of thermodynamics are postulates true only at macroscopic level [36].[13] Furthermore, it is

[13]One may wonder if the thermodynamic laws can be *derived* from the statistical mechanics. As far as I can see, the zeroth law cannot be derived from statistical mechanics because one cannot derive the concept of equilibrium (in terms of constant temperature) from statistical mechanics though one definitely can construct models such that the temperature can be identified with certain measurable quantities with the *assumption* that the system is in equilibrium. Then, one needs introduce certain

unclear, in the context of black hole physics, what is the quantity that corresponds to the temperature of a thermal system which is needed in order to define entropy.

The generalised second law of thermodynamics

Nonetheless, in the same article, Bekenstein also proposed a generalised second law of thermodynamics (GSL) [4]:

> When some common entropy goes down a black hole, *the black hole entropy plus the common entropy in the black hole exterior never decreases.*

According to Bekenstein's personal historical experiences recounted in reference [7], the idea of black hole entropy

> was embraced widely after Hawking's demonstration that black holes radiate thermally. By the end of the 1970's it was generally accepted that a black hole, at least quasi-statically, and semi-classically evolving one, is endowed with an entropy $S_{bh} = A/(4G\hbar)$.

Since it is not my intention to give a historical account of how this transition of the attitude towards the black hole entropy occurred, I think it is appropriate to stop such historical retracing here.

The generalised laws of thermodynamics

Specifying the systems When a thermal system consisted of black holes and ordinary material, the generalised second law combines to a single one the statements from that of thermodynamics and black hole mechanics. It is reasonable to expect that such a generalisation can be extended to the other three laws.[14] Before I specify the various generalised laws of thermodynamics, it is necessary to be more specific about

coarse-graining to derive the second law. However, the choice of the coarse-graining is, in a loose sense, arbitrary, and the validity of the second law is used as a criterion for a *good* one. As to the first law, its un-derivability can be traced back to the explicit involvement of temperature in its statement. For the third law, the reader is referred to discussions in textbook [36].

[14] It is not my intention to give a systematic development of thermodynamics of a gravitational system. They are simply straightforward generalisation.

the system concerned. In the rest of this section, i.e., section 2.2, I will be considering a spherically symmetric[15] thermal system enclosed in a cavity of radius R_c. This cavity is not included as part of this system; it only serves to confine those material to form a well-defined thermal system. It is made of energy restrictive wall (ERwall) [12].

Furthermore, it is well-known that within a gravitational field the temperature, like frequency, also suffers from the red-shift or blue-shift. The Tolman relation gives the transformation rules [63].

The statements Then for such systems, we can state the following generalised laws of thermodynamics:

0. **The zeroth law**: For a thermal equilibrium system, the temperature, T, is a constant over the wall of the cavity and the surface gravity, κ, is also a constant over the event horizon of the black hole (if there is one black hole). The temperature for the rest of the system,[16] is determined by the Tolman relation.[17]

1. **The first law**: For the material part of the system, we have

$$dS_m = \beta dU - \tilde{\mu} dN \ , \tag{2.29}$$

where $\beta = 1/T$ and $\tilde{\mu} = \beta\mu$. For the black hole, as before we have

$$d\mathcal{E} = \frac{\kappa}{2\pi} dS_b + \Phi dq \ , \tag{2.30}$$

such that $d\mathcal{E} + dU =$ total energy input from outside the cavity.[18]

[15]The reason for restriction to spherically symmetric system is because a stationary black hole, only which can be regarded as in thermal equilibrium, is either spherically symmetric or axi-symmetric [33]. Since I have always been considering a spherically symmetric black hole, I therefore also restrict the other part of the system having the same symmetry.

[16]Namely, outside the black hole but inside the cavity.

[17]Note that, such a simple-minded extrapolation in fact give us an arbitrarily high temperature just outside a black hole. However, if the temperature is always combined with other quantities in use so that the combined quantities are finite, then such simple-mindedness can be justified within our present understanding of the thermodynamics around the black hole. Obviously, this problem is related to the trans-planck frequency encountered in the derivation of Hawking temperature (see section 2.1.2).

[18]For more detail about the prescription of various quantities, see subsection 2.2.3 below.

2. **The second law**: For a thermally isolated system, the state of equilibrium is the state of maximum entropy consistent with external constraints. Or, equivalently, the total entropy of a thermally isolated system does not decrease with time.[19]

3. **The third law**: It is impossible to reduce the temperature (as measured on the wall of the cavity) of a system to absolute zero by a finite sequence of operations, though, the surface gravity could be reduced to zero due to black hole radiating.

Since the justification of thermodynamic laws relies on experiments very much, it is not something that can be done by a theorist. However, some heuristic arguments in support of the generalised second law can be found in the literature. Furthermore, from purely thermodynamic point of view, some arguments can be devised to support it.

2.2.2 Comments to proofs of the GSL from thermodynamic point of view

How I see thermodynamics

Thermodynamics is simple and general Thermodynamics (see [12, 36] for an introduction) is one of the physical disciplines in which physical laws are governed by simplicity and generality (S&G). Due to the largeness of physical degrees of freedom, most macroscopic systems are untraceable microscopically. Therefore, a systematic way, based on macroscopic S&G regardless of the microscopic details, is needed in order to extract the information we are interested in, amongst which one of the most important is perhaps the equilibrium states. Thermodynamics suffices such task by employing extremising (maximising or minimising) principles [12]. Even though a microscopic model is introduced later to give thermodynamic quantities a statistical-mechanical interpretation, the major roles of thermodynamic quantities, and hence the extremising principles, are unquestionable.

[19]This generalised second law suffers from the same problem as the second law of thermodynamics.

Thermodynamic laws are postulates Like all other micro/macro-scopic physical laws, the status of the second law of thermodynamics (SLT) is more a postulate than a theorem [12, 36]: It is taken as one of the starting points for a long journey of searching statistical descriptions of a physical system. Therefore, it has to be checked up again and again throughout the journey. In other words, it can only be verified in a *self-consistent* way, or in a *circular* way by Callen's word [12]. And because SLT is an experimental law applicable only at macroscopic scale, it cannot survive under the closest scrutiny from the viewpoint of microscopic unitary evolution. Even so, up to now, we have all of the reasons to believe that its S&G is unquestionable if we do not go beyond the border.

Including gravity into thermodynamics On the other side of physics, we are used to thinking of space-time in geometric language after Einstein formulated general relativity, which is always regarded as a dynamic theory. Presumably, it would have surprised him very much, as we are, that, along with the development of black hole physics [33, 67], gravitational degrees of freedom can also be cast in the language of thermodynamics [41], as can be seen most transparently from the identification of the area of an event horizon with entropy [4, 30] and the formulation of four laws of black hole mechanics [2].

However, as being pointed out by Callen [12], thermodynamics by itself is not a theory; it is a way of thinking: thinking about the laws of nature which are universal and revealed in macroscopic scale whatever the microscopic compositions and dynamics the system has. From this point of view, it should not surprise us anymore that gravity, which is usually neglected in thermodynamics because of its weakness, can/should also be incorporated into thermodynamics.

An early high tide amongst these developments was Bekenstein's conjecture [4] about the generalised second law of thermodynamics.

After Bekenstein offered his conjecture of the GSL, strong evidences for its truism have been given in references [4, 5, 20, 57, 58, 65, 76]. It is thus natural to regard the GSL as a special case of the SLT which involves a black hole.

They proved it Nonetheless, with above attitude towards the SLT in mind, I feel that the *proofs* of the GSL available to me are unsatisfactory in two aspects: The first, the status of the GSL in thermodynamics is not revealed explicitly. As being stressed above, the GSL of thermodynamics is not a consequence of any other physical laws within thermodynamics (or statistical mechanics); it is the starting point of the following story: By maximising entropy we can determine the equilibrium states of the system. It is virtually hopeless to do this following the microscopic dynamic evolution. Conversely, as far as I know, only an equilibrium state has well-defined thermodynamic functions of state; entropy is one of them. The second, the flavour of thermodynamics—simplicity and generality—is veiled by the detailed microscopic dynamics.

In next subsection, I would like to offer examples to see how the GSL works and evidences for its truism from the point of view described above. Because we have accepted its truism as the *first law*, the arguments are *not* a proof, but self-consistent statements that serves as its foundation.

Before I present my approach, I will give a few comments on those proofs. Although Frolov and Page have given several comments to those prior to theirs [20], I would like to add some to contrast those approaches with my attitude.

Comments to proofs of the GSL

The first Since Bekenstein offered his conjecture before Hawking radiation was discovered, his proof [4, 5] suffered from the unavoidable incompleteness in which the entropy of radiation was missed. He proposed a lower bound of spatial expansion (with respect to fixed S and U) for ordinary thermodynamic system (m, for matter) as a remedy. We think this is neither necessary (as will be shown later) nor sufficient: Consider the case in which the initial and final masses of the black hole are the same, then the entropy difference comes purely from those matter outside the black hole. Due to the universality of Bekenstein's bound [6], it cannot inform us how to calculate this difference.

The second In Unruh and Wald's version [65], the importance of the entropy contribution from radiation was stressed, and it was used to remedy the incompleteness of Bekenstein's proof by considering the buoyancy force originated from radiation. This buoyancy force will be felt by m in a stationary Schwarzschild frame (SSF), but not in a locally inertial frame (LIF). As a consequence, after the rope is cut, which ties m to someone standing outside the system concerned, such buoyancy force can be forgotten. On the other hand, because the GSL concerns the never-decreasing property of entropy of a thermally closed system, what we are interested in is the entropy change during the period of approaching equilibrium *after* the rope is cut when the system can be regarded as thermally closed. Nevertheless, the non-negligibility of the entropy of Hawking radiation will never be over-stressed. We will see later that the GSL is rescued not only by the existence of Hawking radiation, but also by its massless and thermal properties.

The third Zurek's proof [76](later generalised by Schumacher [57]) gave a strong support to Bekenstein's statistical interpretation of black hole entropy as the lack of information about the internal configurations of a black hole. Nonetheless, his proof only achieved half of the goal: if the surrounding thermal radiation has a higher temperature than that of the black hole, the GSL will be violated. However, as we will show later, if the equilibrium states can be handled properly, a final equilibrium state with higher entropy can always be found.

The fourth In Frolov and Page's proof [20], quantum modes in the interior of a black hole were assumed to be the CPT reversal of those outside a black hole. We suppose that this is an assumption. And, since the black hole entropy can be determined at spatial infinity, it seems unnecessary to worry about what is happening inside a black hole. On the other hand, in the last step of their proof, they compared two Massieu functions[20] defined on different spaces—exterior and interior of the black hole. However,

[20] A Massieu function is a Legendre transform of the entropy [12]. For example, consider the entropy, $S(U)$, as a function of the internal energy, U, so we have the differential relation, $dS(U) = dU/T$. One can perform a Legendre transformation so that the entropy is a function of the inverse temperature,

entropy is an extensive quantity (this is still *roughly* true for a composite system which is not separated by an impermeable wall), its value depends on the volume where the system is confined. In the terminology of quantum field theory, it depends on the normalisation. It is thus unclear, since the practical computation scheme was not given in such a general approach, whether their difference can be explicitly calculated to be semi-positive definite, as claimed in their proof.

I like this one, the fifth The proof provided by Sorkin [58] is perhaps the most general one. The author explicitly chooses a coarse-graining by assuming what happens at the exterior region of the black hole can be approximated by a Markov process.[21] The statement of the GSL is effectively reduced to a statement of SL of conventional thermodynamics for the matter outside the black hole, i.e., $\delta(S_{out} - \beta\langle\hat{E}_{out}\rangle) > 0$ where $\langle\hat{E}_{out}\rangle$ is the quantum mechanical expectation value of the energy of matter. By requiring $\langle\hat{E}_{out}\rangle + M_{bh} = constant$, the GSL follow immediately. This approach is akin to mine most in flavour. However, it is unclear to me how general the assumption of Markov process is in the present context. On the other hand, due to the process of matter-absorption and radiation-emission, it is unclear how to justify the assumption that the transition probability, T_{kl} between different states *outside* the black hole is unitary,[22] which is used in Sorkin's proof.

Is it possible? Furthermore, I am sceptical about the possibility of proving the GSL from a *dynamic* point of view within current understanding about the dynamics underlying the generalised thermodynamics. Before one starts to prove it, s/he has to answer the following question: Is the underlying dynamics unitary or non-unitary?

If the answer is non-unitary as claimed by Hawking [32], then the GSL should be

$1/T$, i.e., define $S(1/T) = S(U) - U/T$. Then we have $dS(1/T) = -Ud(1/T)$. The $S(1/T)$ can be seen relating to the Helmholz free energy, F, by the relation, $S(1/T) = -F/T$.

[21] A Markov process is the next-to-leading order approximation of stochastic process [1]. Roughly speaking, by discretising time with spacing dt, the event at arbitrary time $t + dt$ depends on events happened at time t only.

[22] There are two problems: The first, the Hilbert space is changing. The second, the unitary rule is hard to justify.

a straightforward consequence of the non-unitary dynamics.[23] If the answer is unitary, then one has to specify the coarse-graining which results in the entropy increasing.

It is unclear to me that if the unitarity/non-unitarity concerned about the black hole entropy can be derived from the ultimate theory (U-theory—whatever it is) or it should be regarded as one of the building blocks of the U-theory, i.e., one has to make up one's mind right from the very beginning of constructing a theory if s/he wants to make a unitary or non-unitary theory. This question is related to the intriguing problem about the relation between phenomena and the theory (or model): If something is missed, one can always insist that the underlying phenomenon should be unitary so that all we have to do is work harder to sort it out. Whatever the outcome of this debate is, to me, it seems that no physics can totally exclude the subjectivities of the physicists involved.

On the other hand, the origin of the thermal radiation has never been addressed in above mentioned proofs. They are assumed to be there from the very beginning. As explained in section 2.1, the thermal radiation arises from the emerging of the event horizon during a collapsing process. A dynamic proof, from my point of view, should be able to give detailed explanation, say using Feynman diagrams, how the transformation between different matter forms happens, because if whatever the black hole swallows up, it always returns us "TV sets", it is quite unlikely that the generalised second law could be true.

2.2.3 My arguments

The scenario

To help clarify the scenario of my approach, let me review the basic statement of the SLT at first. From the point of view of entropy, it says that the entropy (S) for a thermally closed system ($\delta Q = 0$) will never decrease. For a thermally closed system in equilibrium, *by definition* (since this is how we determine the equilibrium state), S is maximum, and the temperature $T = 1/\beta$, defined as $\beta = (\partial S/\partial U)_V$, is a constant throughout the system.

[23]I cannot rule out the possibility that a non-unitary dynamics will *decrease* the entropy. But, I think it is quite unlikely.

How can we benefit from the SLT? Consider a composite system which consists of two boxes of matter attached to each other along a wall which is restrictive with respect to energy (ERWall)[12]; and each one is in equilibrium on its own. Then, we change the wall to a heat-permeable (or particle-permeable) one with negligible effects on the system so that they can approach final equilibrium state. The final state is determined by maximising the entropy with internal energy and volume kept fixed. Even though this is not the only way to get the information about the final state, this is perhaps the simplest one. In this case, the SLT is verified directly from daily experiences.

I will apply a similar picture to the GSL. As mentioned previously, the GSL will be violated without Hawking radiation. Consequently, we need to include Hawking radiation to form an equilibrium state involving a black hole. We confine a black hole (H, we consider Schwarzschild black holes only) and thermal radiation (R) of total internal energy (ADM mass) $U_i = M_{Hi} + U_{Ri}$ at temperature $T_i = 1/\beta_i$ in a ball (B) of volume V (black hole has no volume by prescription). This picture was pioneered by Hawking [31].

Prescription for various thermodynamic quantities

Some remarks are in order since V and U_{Ri} are divergent in the SSF (stationary Schwarzschild frame) without a proper prescription. Consider, at spatial infinity, a thin spherical shell of box of volume $\delta V = a\delta r$ (where a is the area orthogonal to radial co-ordinate and δr is radial expansion in the co-ordinates of the SSF) containing thermal radiation of temperature T. The entropy is then $S_\delta = \beta(U_\delta - F_\delta)$, where F_δ is the corresponding Helmholtz free energy. If we put this shell at co-ordinate r, to an observer in a LIF (locally inertial frame), the energy and temperature then scale in the same manner by a factor $\chi = (1-2M/r)^{-1/2}$ [63]. Therefore, the entropy is independent of frames (SSF or LIF) even though χ diverges at the event horizon. Furthermore, if we use a box with the same δr, then the energy density is also frame-independent since the volume also scales by a factor of χ (but entropy density is thus zero, and Stefan's constant is *not* a fundamental constant). We therefore build up the ball at spatial infinity at first, then we pull it to the nearby of the black hole while at the same time keeping the radial co-ordinate expansion fixed. Since in our equations only entropy, hence the combinations of βU and βF, will appear, we can dismiss the factor χ totally. We thus can use these quantities as if we are in spatial infinity. This perhaps is one of the reasons that entropy is more important and interesting than other quantities.

The criteria for the existence of various configurations of the system described above have been analysed in [31, 23]. In our approach, these criteria are not used explicitly because it is not necessary to require that there is a black hole in either the initial or final state.

To see how the GSL works, we attach to the outer surface of B a spherical shell of box (b) of volume v which can contain any kind of ordinary matter (m) of temperature $T_{\bar{\imath}}$ with internal energy $U_{\bar{\imath}}$ and Helmholtz free energy $F_{\bar{\imath}}$.

Two routes

From here, we can have two different approaches. The first, B and b is always separated by a heat-permeable wall while they are approaching equilibrium, so there is only heat exchange between them. The final matter form is still m. The second, the wall between B and b will be removed so that m will fall into the hole (if there is one), thus the final matter form is thermal radiation (r). The second case shows a new feature of thermodynamics involving a black hole in which the black hole acts as a matter-to-radiation transformer (if m is not thermal radiation in the first place). We will explain its significance later. We consider these two cases separately.

The first From the first case we will learn how the GSL works and this will provide us with a basis for the consideration of the second case. The whole entropy change can be separated into three parts,

$$dS = dS_H + dS_R + dS_m \ , \tag{2.31}$$

where

$$dS_H \quad = \quad 4\pi M_{Hf}^2 - 4\pi M_{Hi}^2 \ , \tag{2.32}$$

$$
\begin{aligned}
dS_R &= \beta_f(U_{Rf} - F_{Rf}) - \beta_i(U_{Ri} - F_{Ri}) \;, \\
dS_m &= \beta_f(U_{mf} - F_{mf}) - \beta_{\bar{i}}(U_{m\bar{i}} - F_{m\bar{i}}) \;.
\end{aligned} \tag{2.33}
$$

U_{ab} and F_{ab} are internal energy and Helmholtz free energy for matter form a at temperature T_b, respectively.

To understand why the final state of the triple-phase system (if $M_{Hf} \neq 0$) has the highest entropy without doing a calculation, let us consider the following slow-motion thought experiment in which the whole system approaches equilibrium through an infinite-step procedure: We cover the black hole by an ERWall at first, then let R and m approach equilibrium with $U_R + U_m$ being fixed. According to the SLT, the entropy change is semi-positive definite. Afterwards, we remove the ERWall around the black hole, but cover B with an ERWall. Then, let H and R approach equilibrium with $M_H + U_R$ fixed. According to the GSL, the entropy change is also semi-positive definite. We then carry on the above procedure again and again until the whole system, H+R+m, arrives at equilibrium. Because entropy is a function of state, the entropy change is unique for a thermally closed system if the final state determined by maximising entropy is so (we assume it is). Therefore, the total entropy change is semi-positive definite. (The reader may suspect that with delicate arrangement, the above procedure could have no definite final state, just like the sequence $1, -1, 1, -1, \ldots$ has no limit. However, it is not difficult to convince oneself that if $T_{\bar{i}} > T_i$ ($T_{\bar{i}} < T_i$), then T of m will decrease (increase) only. And if a phase transition happens, i.e., $M_H \neq 0 \to M_H = 0$ or $M_H = 0 \to M_H \neq 0$, then the other direction will not happen. Therefore, we can safely expect that the final equilibrium state will be the limit state of the above procedure.) Alternatively, one can write down those entropy terms explicitly and maximise the total entropy.

The second The second case is generic to the GSL because it involves transformations between different matter forms. The total entropy change can be separated into four parts,

$$
dS = dS_H + dS_R + dS_r + dS_m \;,
$$

where dS_H and dS_R are respectively as those in (2.32) and (2.33), and

$$\begin{aligned} dS_r &= \beta_f(U_{rf} - F_{rf}) - \beta_{\bar{\bar{\imath}}}(U_{r\bar{\bar{\imath}}} - F_{r\bar{\bar{\imath}}}) , \\ dS_m &= \beta_{\bar{\bar{\imath}}}(U_{r\bar{\bar{\imath}}} - F_{r\bar{\bar{\imath}}}) - \beta_{\bar{\imath}}(U_{m\bar{\imath}} - F_{m\bar{\imath}}) . \end{aligned}$$

$U_{r\bar{\bar{\imath}}}$ and $\beta_{\bar{\bar{\imath}}} = 1/T_{\bar{\bar{\imath}}}$ are determined by replacing initial matter m in b with thermal radiation r of the same internal energy, namely, $U_{r\bar{\bar{\imath}}} = U_{m\bar{\imath}}$. Now, if we consider a system with initial state $H_i + R_i + r_{\bar{\bar{\imath}}}$, then we can borrow the conclusion of the first case that the entropy change, $dS_H + dS_R + dS_r$, is semi-positive definite. However, if the final state of our original system $H_i + R_i + m_{\bar{\imath}}$ contains a black hole (i.e., m will be swallowed by H), then this final state is just the same final state arrived at from the initial state $H_i + R_i + r_{\bar{\bar{\imath}}}$. Therefore, if dS_m is semi-positive definite, we then arrive at the desideratum. Though it is quite unlikely to give a proof to this statement, it seems intuitively true. On the other hand, if the final state of $H_i + R_i + m_{\bar{\imath}}$ does not contain a black hole, then we come back to the first case.

From the other point of view, by accepting the truism of GSL in the first place (as I did), we can indeed turn the logic around to make the conjecture: Given fixed volume V and fixed internal energy U as constraints, massless thermal radiation (if no black hole forms) has the largest entropy amongst all possible kinds of matter. In this way, we find that a black hole is a nature-born entropy generator by way of transforming matter into thermal radiation. How cute Nature is to realise the GSL in such a delicate way! In Jacobson's approach of space-time thermodynamics [41], the status of the first law of thermodynamics is *lowered* to a more fundamental one than that of Einstein field equation. We wonder if this can also be done to the GSL?

Obviously, it is of vital importance that Hawking radiation is massless thermal radiation. From Page's estimation [48], this is indeed the case for large mass black holes for which thermodynamics can be ensured to make sense.

* * *

Chapter 3

Three Statistical Explanations of Black Hole Entropy

In previous chapters, I have introduced those results which could serve as the initiatives for the pursuing of the statistical origin of black hole entropy. It seems that everyone in this field has her/his own version of explanation. Since the event horizon separates the space-time to two regions plus a boundary—the event horizon, all attempts can be catalogued, by force, according to where the entropy calculated resides.

However, most of the them are done at the outside, or on the boundary, of the black hole. There are also attempts which use both of the inside and outside regions of a black hole.[1]

All attempts can also be classified according to the signature of space-times used. The most common ones are Lorentzian and Euclidean signatures. As far as I know, the only example using Kleinian signature is reported by the author.[2]

The methods employed vary according to a researcher's intuition or taste because, it seems that no one has yet had a clue why a black hole, a classically almost bald[3] animal, should/could have so much quantum hair so hard to comb. Some apply quantum field theory, some adopt path integral. I borrow some tricks from history.

[1]More accurately, they use both wedges of the maximal extension of a Schwarzschild black hole.

[2]In a higher dimension theory, presumably, there are even more possibilities.

[3]In 4-D, according no-hair theorem, a stationary black hole has only three classical hairs: the mass, charge, and angular momentum. See the article [70] for a short review and references therein.

Abstract of chapter 3

Section 3.1 I start with 't Hooft's brick wall model [62] in subsection 3.1.1 because, it seems that this is the first reported attempt to calculate the black hole entropy statistically. The entropy derived is formally divergent. 't Hooft introduced a straightforward, but simple-minded, cut-off to regularise the divergent. After I give two short comments to the model in subsection 3.1.2, two attempts trying to justify such a cut-off are reviewed afterward [15, 55].

Section 3.2 I review the explanation proposed by Brown and York [9, 10], via path integral, by attributing the origin of black hole entropy to the gravitational degrees of freedom. In order to understand their prescription of the path integral representation of partition functions, I at first consider an ordinary statistical system as an example following references [9, 18] in subsection 3.2.1. The prescription of micro-canonical action used to evaluate the black hole entropy in subsection 3.2.3 is reviewed in subsection 3.2.2.

Section 3.3 I present my own version of a statistical explanation of black hole entropy from the point of view of quantum states inside a black hole. Each quantum state contains degrees of freedom of both the quantum and gravitatinal fields. [4] This idea is, basically, derived from the unsatisfactoriness to the previous two attempts in which these two degrees of freedom are dealt with separately. I foremost in subsection 3.3.1 give several general assumptions regarding a static black hole as a thermal equilibrium system. I then calculate the quantum statistical entropy of neutral and $U(1)$ charged black holes in subsections 3.3.2 and 3.3.3, respectively.

Appendix C Various standard kinematic formulae are presented in C.1. The simplifications of several equations used in section 3.2 are shown in section C.2.

[4]See section 4.1.2, chapter 4 for detail.

3.1 Brick wall model

't Hooft considers a thermal bath propagating outside the horizon of a black hole [62]. The black hole entropy is identified as the entropy of the thermal bath. Though the entropy is proportional to the area of the cross-section of the horizon, its coefficient is divergent. The divergence can be traced back to the infinite large time-dilation[5] factor near the horizon. In order to regularise this divergent coefficient, 't Hooft introduces a naïve cut-off—brick wall—near the horizon and imposes the boundary condition that the quantum field vanishes on the wall. By choosing a suitable proper distance between the brick wall and the horizon, it is possible to adjust the thermal entropy to the value of the black hole entropy.

Not everyone is happy with the naïve building-up of a brick wall. There are two attempts trying to regularise the divergence in a more systematic manner. The first is given in reference [15] by employing the Pauli-Villars regularisation. Another one is reported in reference [54] by replacing the brick wall with an apparent horizon.

3.1.1 The model

Thermal bath outside a black hole

The equations We consider a massive, neutral scalar field propagating outside a Schwarzschild black hole with the boundary condition in Schwarzschild co-ordinates $x^\mu = (t, r, \theta, \phi)$,

$$\varphi(x) = 0 \ , \qquad \text{as} \ \ r \le r_s + h \ \ \text{and} \ \ r \ge L \ ,$$

where $r_h = r_s + h$ is the radial co-ordinate of the brick wall. Another large distance regulator, L, is also introduced. If the scalar field is expanded as

$$\varphi = e^{-iEt} Y_{lm}(\theta, \phi) R(r) \ ,$$

[5]Namely, the red- or blue-shift factors depending on one's viewpoint.

where $Y_{lm}(\theta,\phi)$ is the spherical harmonics of order (l,m), then the radial part, $R(r)$, satisfies the equation,

$$\hbar^2\left[\frac{1}{r^2}\partial_r(r(r-r_s)\partial_r R)\right]-(\frac{l(l+1)}{r^2}+m^2)R+\frac{eE^2}{r-r_s}R=0\ , \tag{3.1}$$

where $\hbar$ have been written down explicitly. To proceed computing the leading contribution to the free energy, F, 't Hooft employs the WKB approximation [56] by writing $R = R_0 e^{iS/\hbar}$ in which R_0 is a *constant* and S is a slow varying phase. Expanding equation (3.1) in terms of S and keeping the lowest order terms of $\hbar$, we arrive at

$$(S')^2=(1-\frac{r_s}{r})^{-2}\left[E^2-(1-\frac{r_s}{r})(\frac{l(l+1)}{r^2}+m^2)\right]\ .$$

Energy quantisation rule The WKB quantisation condition [56] gives

$$(n+\frac{1}{2})\pi=\int_{r_h}^{L}dr\partial_r S\ , \tag{3.2}$$

where the E is always chosen such that the integrand is real. To estimate the degeneracy, $\nu(E)$, of eigen-modes with eigen-energy E, we sum over those l which ensure the reality of the integrand in equation (3.2), i.e.,

$$\nu(E)=\sum(2l+1)\int_{r_h}^{L}\frac{dr}{\pi}\partial_r S\ . \tag{3.3}$$

Thermodynamic quantities

The free energy F The entropy of a thermal bath at temperature of $1/\beta$ can be calculated from the free energy, F, defined as

$$\begin{aligned} F &= \frac{1}{\beta}\sum_E \ln(1-e^{-\beta E})\nu(E) \\ &\simeq \frac{1}{\beta}\int_{E_0}^{\infty}dE\frac{d\nu(E)}{dE}\ln(1-e^{-\beta E}) \\ &= -\int_{E_0}^{\infty}\frac{\nu(E)dE}{e^{\beta E}-1}\ , \end{aligned} \tag{3.4}$$

where in the last line an integration by parts has been used and the surface terms do not contribute to F. Note that the lower bound E_0 in equation (3.4) depends on the mass, m, and the regulator, L. By approximating the summation over l by integration in equation (3.3),[6] we have

$$F \simeq \frac{-2}{3\pi}\int_{E_0}^{\infty}\frac{dE}{e^{\beta E}-1}\int_{r_h}^{L}\frac{dr\, r^2}{(1-\frac{r_s}{r})^2}(E^2-(1-\frac{r_s}{r})m^2)^{3/2}$$
$$= \frac{-2}{3\pi}\int_{E_0}^{\infty}\frac{dE}{e^{\beta E}-1}\int_{h}^{\rho_L}d\rho\frac{(\rho+r_s)^4}{\rho^2}\left[1+\frac{r_s}{\rho+r_s}\frac{m^2}{E^2-m^2}\right]^{3/2}(E^2-m^2)^{3/2}\ , \tag{3.5}$$

where $\rho = r - r_s$ has been used in the last line. At this moment, we are interested in those terms in equation (3.5) diverging as $h = r_h - r_s \longrightarrow 0$, i.e., as the brick wall approaches the horizon, and the bulk term which behaves as L^3. They correspond to terms of r_s^4/ρ^2, $4r_s^3/\rho$, and ρ^2 in the expansion of $(\rho + r_s)^4/\rho^2$. For terms of r_s^4/ρ^2 and $4r_s^3/\rho$, we approximate ρ as 0 in the rest of the integrands in equation (3.5). For the term of ρ^2, we set $r_s = 0$ since $L \gg r_s$. We then arrive at the approximated F as

$$\begin{aligned} F \ \simeq\ & \frac{-2}{3\pi}\int_0^{\infty}\frac{dE E^3}{e^{\beta E}-1}\int_h^L d\rho\left[\frac{r_s^4}{\rho^2}+\frac{4r_s^3}{\rho}\right] \\ & +\frac{-2}{3\pi}\int_m^{\infty}\frac{dE(E^2-m^2)^{3/2}}{e^{\beta E}-1}\int_h^L d\rho\rho^2\ . \end{aligned} \tag{3.6}$$

Doing the integration,[7] we arrive at

$$F \simeq \frac{-2\pi^3}{45}\left[(\frac{r_s}{\beta})^4\frac{1}{h}-\frac{4r_s^3}{\beta^4}\ln(\frac{h}{r_s})\right]-\frac{2}{9\pi}L^3\int_m^{\infty}dE\frac{(E^2-m^2)^{3/2}}{e^{E\beta-1}}\ , \tag{3.7}$$

where a constant term has been added in to constructed $\ln(h/r_s)$. It is seen there is a *surface* contribution apart from the expected volume contribution. In the following, we will ignore the volume term, i.e., term proportional to L^3, and concentrate on the surface terms.

---

[6] Note that the suitable upper bound of dr depends on E and m.

[7] The following integral is used [28]:

$$\int_0^{\infty}\frac{x^3dx}{e^x-1}=\frac{6}{90}\pi^4\ .$$

The entropy $\mathbf{S_{th}}$ The entropy of the thermal bath, $S_{th} = \beta^2 \partial_\beta F$, is

$$S_{th} \simeq \frac{8\pi^3}{45} (\frac{r_s}{\beta})^3 \left[\frac{r_s}{h} - 4 \ln(\frac{h}{r_s}) \right] . \tag{3.8}$$

It is then obvious that the entropy, in the leading order, diverges as h^{-1}.

Building up the wall

In order to prescribe a finite entropy that is proportional to the area, 't Hooft introduces a proper distance cut-off, ϵ_h, which is related to h by

$$\epsilon_h = \int_{r_s}^{r_s+h} \frac{dr}{(1-\frac{r_s}{r})^{1/2}} \simeq 2(r_s h)^{1/2} . \tag{3.9}$$

The value of ϵ_h is determined by equating the thermal entropy, S_{th}, with the black hole entropy, $S_{bh} = 4\pi G M^2$. Then $\epsilon_h^2 \simeq 1/(90\pi)$. It is of the order of Planck length.

The introduction of a cut-off is inevitable in continuous quantum field theory. However, it is unclear why the ϵ_h should be chosen in such a manner. Before we review two attempts trying to overcome this seemly arbitrary choice, two comments are in order.

3.1.2 Comments

Why black hole entropy? It is unclear to me in what sense the thermal entropy, S_{th}, *is* the black hole entropy. If one accepted the original interpretation of black hole entropy proposed by Bekenstein[8] that the black hole entropy has something do to with lost information about the interior of the black hole, then, it seems obvious that the S_{th} has no such character.

Does a neutron star have gravitational entropy? Furthermore, if S_{th} can be attributed to the entropy of the object enclosed by the brick wall, then, as far as I can see, such a calculation can also be done around, say, a neutron star. The radius of the neutron star can serve as a natural co-ordinate cut-off r_h, though the proper distance

[8]See section 2.2.1.

cut-off, ϵ_h, will be of macroscopic order. However, no conceptual obstacle to rule out such a calculation. Then, 't Hooft's interpretation seems to suggest that a neutron star, and generally any star, also has gravitational entropy in addition to the usual thermal entropy associated with the matter. This is contradictory to the conventional wisdom that a star has no gravitational entropy because of the lack of an event horizon [22]. Admittedly, perhaps a normal star can indeed be attributed with certain gravitational entropy, but, the question raised in previous paragraph remains.

3.1.3 Removing the divergence

There are two attempts trying to evade the naïve brick wall cut-off. One is given by Russo by replacing the brick wall with an apparent horizon [54]. Another one is reported in reference [15] with the help of Pauli-Villars regularisation.

Pauli-Villars regularisation

The difficulties it faces In this approach, several auxiliary fields of large masses are introduced. To achieve the goal of regularisation, their masses, statistics (hence the signs of free energies) are chosen so that the divergent terms in equation (3.7) are cancelled. Though the conclusion from this approach is quite remarkable, I think this approach has its limit due to the great approximation involved in the calculation of free energy, F, in equation (3.7). Recall that two basic approximations are used in order to arrive at the expression (3.7): WKB approximation in equation (3.2) and the boundary condition in the integral in equation (3.6). Unlike in quantum field theory on a flat space-time, few of the computations involved on a curved background can be done exactly. It is then unclear if the power of Pauli-Villars regularisation can be fully generalised to the present case. But, let us have a quick look how it is done.

A quick look at this method As in reference [15], we introduce 5 auxiliary fields with masses[9] $m_i = (m^2 + a_i\mu^2)^{1/2}$, $i = 1, \ldots, 5$, where a_i's are *constants* and μ the

[9]The mass of the original field will be denoted as m_0.

cut-off scale. Repeat the calculation leading to equation (3.7) for each field, we have the total free energy,[10]

$$F_{tot} = \sum_{i=0}^{5} \triangle_i \frac{-2}{3\pi} \int_0^\infty \frac{dE E^3}{e^{\beta E} - 1} \int_h^L d\rho \left[\frac{r_s^4}{\rho^2} + \frac{4r_s^3}{\rho} \right] , \tag{3.10}$$

where $\triangle_i = 1$ for commuting fields and $\triangle_i = -1$ for anti-commuting fields. Note that the integrand and the boundary of integral are independent of i, so the divergent terms from various fields will cancel if we arrange $\sum_{i=0}^{5} \triangle_i = 0$. Back to equation (3.5), for the F_{tot}, we have

$$F_{tot} \simeq \frac{-2r_s^3}{3\pi} \int_0^\infty \frac{dE}{e^{\beta E} - 1} \int_0^{L'} \frac{ds}{s^2(1-s)^4} \sum_{i=0}^{5} \triangle_i (E^2 - s m_i^2)^{3/2} ,$$

where $s = 1 - r_s/r$, $L' = 1 - r_s/L$, and the lower bound of dE is approximated by zero. One can then follow reference [15] to derive the entropy and make it finite by considering the one-loop renormalisation of the gravitational constant. However, as explained in previous paragraph, the whole derivation is done with great approximations; though above result might suggest that Pauli-Villars regularisation could still work in an exact computation.

Apparent horizon as the brick wall

Because I think the scope of the above method is quite vague, I now review another method given by Russo based on the idea of the apparent horizon [54]. The basic idea is: The global apparent horizon is a natural place to impose the cut-off if the back-reaction of the black hole radiation is taken into account.

Assumption The underlying assumption behind the replacement of the brick wall with the apparent horizon is: *The quantum mechanical information of a black hole is encoded in the Hawking radiation* [54]. If one adopts this attitude, then one should recover a finite entropy from the black hole radiation. Furthermore, its value should be comparable to the black hole entropy. However, as shown in section 3.1.1, this can be

[10]Note that the cut-off near the horizon is still imposed at this moment.

achieved only when a cut-off, i.e., the brick wall, is imposed at a proper position. One is then motivated to justify this particular choice.

The importance of being dynamic At first, recall that the divergences are originated from the arbitrarily large red-shift a mode suffers near the event horizon of a *static* black hole. However, if this black hole is radiating and it totally disappears, due to the evaporation, at a finite moment of time with respect to an observer at spatial infinity, then the overall red-shift effect should be finite.[11] Therefore, the question has been reduced to the identification of a particular boundary, in a dynamic black hole background, where the radiation originates.

Building the wall at the apparent horizon Now, with the above assumption in mind, the black hole radiation should be causally connected with the future infinity. The last frontier where this is possible is obviously the event horizon. However, the event horizon is a global concept that its identification needs the global knowledge of the space-time [33]. Locally, one can only determine the apparent horizon[12] which is the concept most similar to the event horizon from a local point of view. In order for a mortal to *find* the place to build up the wall, the last frontier should be chosen as the apparent horizon.[13]

The background We now calculate the position of the apparent horizon following Russo's method [54]. For a large black hole, the metric is given by the Schwarzschild

[11]Otherwise, to this observer, the black hole disappears in an infinite amount of time due to the infinite time dilation, which then corresponds to the infinite red-shift.

[12]An apparent horizon is a boundary where a bundle of outgoing null geodesics begin to converge. In the spherically symmetric case, its position is determined by the condition $\frac{\partial r}{\partial v} = 0$ where r is the radial co-ordinate and $v = t + r_*$ in which r_* is the tortoise co-ordinate corresponding to r [54]. Intuitively, the condition $\frac{\partial r}{\partial v} < 0$ just means that the outgoing null geodesics are bent inward, so they are converging.

[13]In fact, the identification of this particular boundary with the *global apparent horizon* is given as an ansatz in Russo's article [54]. I only describe things in a slightly different manner. And note that there is no difference between the *apparent horizon* and the *global apparent horizon* after the in-falling matter flux has stopped. Though, I have to admit that I do not understand the prescription given by Russo that one can construct the global apparent horizon from the late time apparent horizon by *analytic continuation*.

solution,

$$\begin{aligned} ds^2 &= -(1-\frac{2GM}{r})dudv + r^2 d\Omega^2 \\ &= -\frac{32G^3M^3}{r}e^{-\frac{r}{2GM}}dUdV + r^2 d\Omega^2 , \end{aligned} \tag{3.11}$$

where

$$u = t - r_*, \qquad v = t + r_*, \tag{3.12}$$

$$U + P = -e^{\frac{-u}{4GM}} , \qquad V = e^{\frac{v}{4GM}} , \tag{3.13}$$

and $r_* = r + 2GM\ln(\frac{r}{2GM} - 1)$. Since the black hole is radiating, the M is a function of time, t. The factor p in equation (3.13) is an arbitrary factor to shift the origin of the U (cf. equation (1.10)) so that the end point of evaporation corresponds to $(U, V) = (-P, V_e)$ (see figure 3.1) and the final flat space-time is[14]

$$\begin{aligned} ds^2 &= -dt^2 + dr_*^2 + r_*^2 d\Omega^2 \\ &= -dudv + r_*^2 d\Omega^2 \\ &= -16G^2M^2 e^{-\frac{r_*}{2GM}} dUdV + r_*^2 d\Omega^2 , \end{aligned} \tag{3.14}$$

where u, v and U, V are defined as in equations (3.12) and (3.13), respectively.

Locating the apparent horizon From the definition of the apparent horizon $\frac{\partial r}{\partial v} = \frac{\partial r}{\partial V} = 0$ [54], we have

$$\frac{dr}{dU} - \partial_U r = \partial_U V \partial_V r = 0 . \tag{3.15}$$

To determine $\partial_U r$, we at first solve r in terms of U and V from the definitions in equation (3.13). Write $r = 2GM + \delta$, we can solve δ as $\delta \ll 1$ and $\frac{dM}{dt} \ll 1$,

$$\delta \simeq -2GMV(U+P) . \tag{3.16}$$

We then arrive at

$$\partial_U r \simeq -2GMV . \tag{3.17}$$

[14] Ignore the curvature resulting from the Hawking radiation.

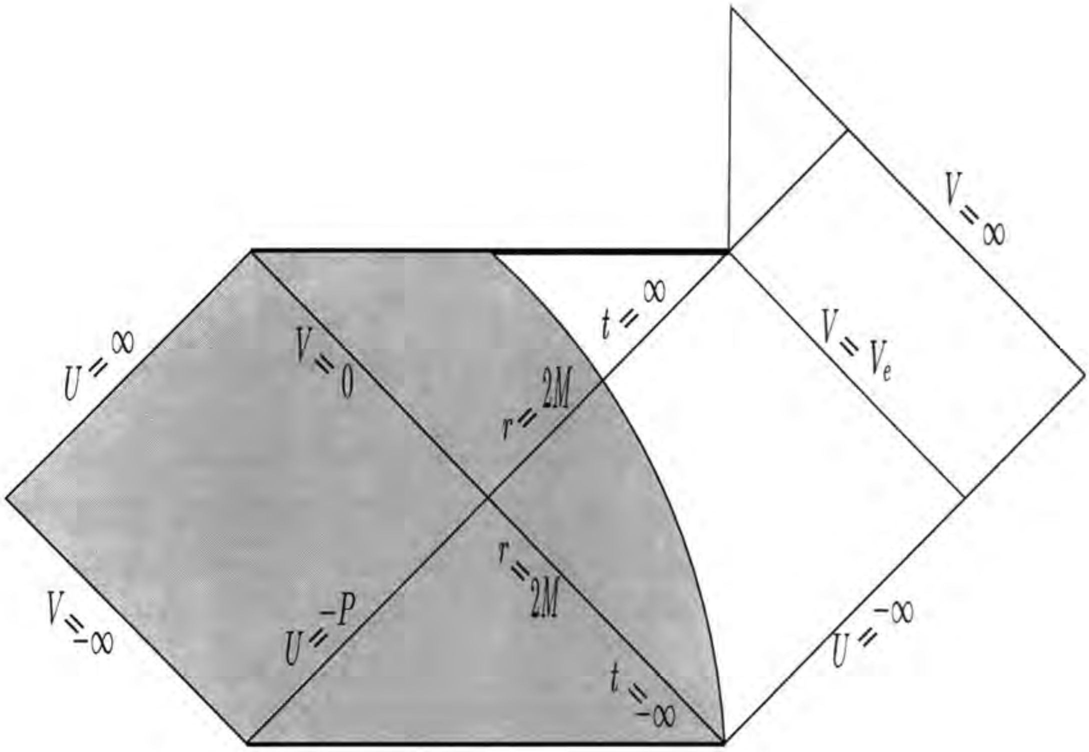


Figure 3.1: Penrose diagram for an evaporating black hole. The shaded region is the analytic continuation of the exterior Schwarzschild sulotion (white region) of a collapsing star.

To calculate $\frac{dr}{dU}$, recall the black hole evaporation rate is $\frac{dM}{du} = \frac{dM}{dt} = -\alpha\pi^2 T^4 A = \frac{-\gamma}{G^2M^2}$ where $\gamma = \frac{\alpha}{256\pi}$ and α is a constant of order 1 [8]. Then with the help of equations (3.16) and (3.13), we have

$$\frac{dr}{dU} = 2G\frac{du}{dU}\frac{dM}{du} \simeq \frac{8\gamma}{(U+P)M} \,. \tag{3.18}$$

Combining equations (3.18) and (3.17), we arrive at $\delta \simeq \frac{8\gamma}{M}$ following the condition (3.15).

From equation (3.9) with $h = \delta$, we arrive at the proper distance between the event horizon and the apparent horizon, $\epsilon_\delta \simeq \sqrt{16G\gamma}$, which is also of the order of Planck length. The entropy can be calculated from equation (3.8), which is $S_{th} = \frac{2}{45\alpha}4\pi GM^2$.

3.2 Black hole entropy in terms of gravitational degrees of freedom

Brown and York advocate that the statistical origin of black hole entropy can be traced back to the gravitational degrees of freedom [9, 10]. In standard statistical mechanics, the entropy can be calculated accordingly if one can write down the partition function of the system concerned [36]. Following the observation by Gibbons and Hawking [22], they proposed a partition function of the gravitational degrees of freedom via path integral [9]. In order to motivate the prescription they adopted, I at first review how they write down the partition function via path integral for an ordinary non-relativistic statistical system[9, 19].

3.2.1 Partition function and entropy via path integral

Partition function for an ordinary statistical system

The canonical partition function, $Z(\beta)$, for a system with positive energy spectrum, E, at inverse temperature, β, can be formally expressed as

$$Z(\beta) = Tre^{-\beta \hat{H}} = \int_0^\infty dE\, Tr\left(\delta(E-\hat{H})\right) e^{-\beta E} = \int_0^\infty dE \nu(E) e^{-\beta E} \ , \qquad (3.19)$$

where $\hat{H}$ is the Hamiltonian, $\nu(E)$ the density of states of energy E, and Tr the trace over all states. If we are able to rewrite the $\nu(E)$ via path integral, then we almost arrive at our aim.

Density of states via path integral We see that $\nu(E) = Tr(\delta(E-\hat{H}))$ from equation (3.19). Taking the trace in configuration space with co-ordinate x, we have

$$\nu(E) = \int dx \langle x|\delta(E-\hat{H})|x\rangle \ . \qquad (3.20)$$

In turn, the matrix element of the delta function can be re-expressed as

$$\langle x'|\delta(E-\hat{H})|x\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dT e^{iET/\hbar} \langle x'|e^{-i\hat{H}T/\hbar}|x\rangle \ . \qquad (3.21)$$

Using the particle path integration ansatz given by Feynman [19], the matrix element in the above equation can be represented as

$$\langle x'|e^{-i\hat{H}T/\hbar}|x\rangle = \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x e^{iI_T/\hbar} , \tag{3.22}$$

where $\mathcal{D}x$ is the functional integration over particle paths, I_T is the Hamilton's action of the system with the time interval fixed to value T, and note that, the time interval, T, is allowed to be negative in the present case.

Combining equations (3.20)-(3.22), we arrive at the path integral representation for $\nu(E)$,

$$\nu(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dT \int_{x(0)=x}^{x(T)=x} \mathcal{D}x e^{i(I_T+ET)/\hbar} . \tag{3.23}$$

The above expression can be further recast in terms of the Jacobi's action[15] as follows [9],

$$\nu(E) = \int \mathcal{D}H_p e^{iI_E/\hbar} , \tag{3.24}$$

where I_E is the Jacobi's action with energy E and $\mathcal{D}H_p$ indicates that the path integration is done over all periodic paths.

Canonical partition function via path integral Putting equation (3.23) into (3.19) and performing the change of variables, $T \longrightarrow -i\tau$,we arrive at the following representation for the partition function,

$$Z(\beta) = \int_0^{\infty} \frac{dE}{2\pi i} e^{-\beta E} \int_{-i\infty}^{i\infty} \frac{d\tau}{\hbar} e^{E\tau/\hbar} \left[\int_{x(0)=x}^{x(T)=x} \mathcal{D}x e^{iI_T/\hbar} \right]_{T=-i\tau} . \tag{3.25}$$

Note that the above equation can be formally read as the Laplace transform of the inverse Laplace transform of the quantity inside the square brackets. We then finally arrive at the path integral representation of the canonical partition function as

$$Z(\beta) = \int_{x(0)=x}^{x(T)=x} \mathcal{D}x e^{iI_T/\hbar} |_{T=-\hbar\beta} . \tag{3.26}$$

[15]Hamilton's action is just the usually so-called *action*, which is used to derived the Euler-Lagrange equation via the *variational principle* in which the time-interval of the path history is fixed. The Jacobi's action is the action used in the *least action principle* in which, instead of the time-interval, the energy is fixed. For a closed system in which the energy is conserved, the Jacobi's action can be derived from the Hamilton's through a certain algebra [27].

This expression is foremost derived by Feynman [19].

Partition function for a self-gravitating system

Brown and York propose that the path integral representations of the density of states, $\nu(E)$, in equation (3.24);[16] and the canonical partition function, $Z(\beta)$, in equation (3.26), can be generalised to the gravitational degrees of freedom of a self-gravitating system. Formally,

$$\nu(E) = \int \mathcal{D}H_p e^{iI_E/\hbar} , \qquad (3.27)$$

$$Z(\beta) = \int \mathcal{D}H_p e^{iI_T/\hbar} |_{T=-i\hbar\beta} , \qquad (3.28)$$

where I_T is the Hamilton's action and I_E the corresponding Jacobi's action. The explicit prescription of I_E of Brown and York's is shown in next subsection. Note that we have gloss over the difficult problem about the choice of the measure, $\mathcal{D}H_p$.[17] Also note that the time interval in equation (3.27) is real and unfixed; contrarily, it is imaginary and fixed in equation (3.28).

Saddle point approximation of entropy

Given the canonical partition function in equation (3.19), the formal expression of the entropy is

$$S(\beta) = -\int dE\nu(E)P(E)\ln P(E) = \ln Z(\beta) + \beta\langle E\rangle . \qquad (3.29)$$

where $P(E) = e^{-\beta E}/Z(\beta)$. It is a formidable task to do the computation exactly. However, if we are content with the leading order contribution, then we can approximate the partition function by the saddle point of the exponent of the *exp*;[18] i.e.,

$$Z(\beta) = \int dE\nu(E)e^{-\beta E} \sim e^{\ln\nu(E^*)-\beta E^*} , \qquad (3.30)$$

[16] An alternative name for $\nu(E)$ is the *micro-canonical* partition function because no energy exchange is allowed between members of an ensemble.

[17] Although I use the same symbol, $\mathcal{D}H_p$, for the measure in both expressions, it by no means implies that they need to be the same.

[18] Assuming that the saddle point is a maximum point, otherwise, this system is unstable.

where E^* is the energy of the solution of $\partial_E \ln \nu(E) = \beta$. Within this approximation, the average energy, $\langle E \rangle$, is

$$\langle E \rangle = \int dE \nu(E) P(E) \sim E^* \frac{e^{\ln \nu(E^*) - \beta E^*}}{e^{\ln \nu(E^*) - \beta E^*}} = E^* \ . \tag{3.31}$$

Combining equations (3.31) and (3.29), we arrive at

$$S(\beta) \sim \ln \nu(E^*) \ . \tag{3.32}$$

If we also approximate the path integral representation of $\nu(E)$ in equation (3.24) in terms of its saddle point, we can write $\nu(E^*) \sim e^{iI_{E^*}}$. We then arrive at the simple relation between the entropy and the Jacobi's action that[19]

$$S(\beta) \sim iI_{E^*} \ . \tag{3.33}$$

Such an approximation will be generalised to the gravitational degrees of freedom later to evaluate the black hole entropy.

3.2.2 Micro-canonical action

As described in previous section, at leading order, the evaluation of the entropy is equivalent to the evaluation of the action[20] at a suitable solution. It is a laboured job to determine the form of the micro-canonical action,[21] and this is the content of Brown and York's prescription [9, 10]. Since the essential point of their prescription concerns the boundary terms in the Hamilton's action, and their occurrence can be traced back to the variational process. We start with the variational principle of the Hamilton's action.

[19]In order to make sense of this expression, we optimistically hope that the action calculated from the saddle point is purely imaginary so that the entropy is real and positive.

[20]Since this action appears in the micro-canonical partition function, i.e., the density of states, $\nu(E)$, it is therefore called the *micro-canonical action* [9]. It is the counterpart of the Jacobi's action for an ordinary statistical system.

[21]I will leave some length algebra in appendix C at the end of this chapter.

The Hamilton's action for a gravitational system

Need for boundary terms in the variational principle One usually omits the boundary terms resulting from the Stoke's theorem while one is considering the variational principle because the spatial boundaries are always put at spatial infinity. However, it has been argued by York in reference [10] that a black hole in an infinitely large space is unstable. In order to stabilise a black hole, it is then necessary to confine the black hole within a cavity by imposing certain boundary condition at the surface of the cavity. On the other hand, in order to arrive at the Hamiltonian formulation, one needs to foliate a space-time into a family of space-like hypersurfaces, we therefore encounter space-like boundaries, too. Since certain boundary conditions will be imposed on them, both space-like and time-like boundaries, one has to keep track of those boundaries terms.

The appearance of the boundary terms We start with the Einstein-Hilbert action in equation (1.22) on a space-time, $(M, g_{\mu\nu})$, of topology $I \times \Sigma$ and with boundaries ∂M.[22] Under the variation of the metric, we arrive at (cf. equation (1.23))

$$\delta \mathbf{L} = \mathbf{G}^{\mu\nu} \delta g_{\mu\nu} + \mathbf{d}\Theta_g \ ,$$

where $\mathbf{G}^{\mu\nu} = \frac{\boldsymbol{\epsilon}}{16\pi G}(\mathcal{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\mathcal{R})$. Using the Stoke's theorem, we rewrite $\mathbf{d}\Theta_g$ as an integral on the boundary, ∂M. On a particular piece of boundary, ∂M_n, we have the integrand (cf. equation (1.27))

$$\Theta_g = \theta_g \cdot \boldsymbol{\epsilon} = -\tau n_\nu \theta^\nu n \cdot \boldsymbol{\epsilon} \ .$$

From equation (C.52), we arrive at, on the boundary ∂M_n with unit outward-pointing normal n^μ,

$$\Theta_g = -\tau \mathbf{P}_{\mu\nu} \delta h^{\mu\nu} + \frac{\tau}{8\pi G} \delta (K n \cdot \boldsymbol{\epsilon}) - \frac{\tau}{16\pi G} \mathbf{d} \left[\delta n \cdot (n \cdot \boldsymbol{\epsilon}) \right] \ . \qquad (3.34)$$

[22] See appendix C at the end of this chapter for further explanation of some basic properties of M we endowed.

Since K involves first order derivatives of the metric, it is seen that it is not enough to ensure a solution of the Einstein equation being a stationary point of the Einstein-Hilbert action by requiring the metric variation, $\delta g_{\mu\nu}$, vanishing on the boundary. Therefore, first proposed by Gibbons and Hawking [22], a surface action term, $-\tau \int Kn\cdot\epsilon/8\pi G$, is added to cancel the second term in equation (3.34) resulted from performing variation.

Need also for corner terms This not yet enough. Extra "corner terms" are needed to cancel the contribution from $\mathbf{d}[\delta n\cdot(n\cdot\epsilon)]$ [35]. At a particular junction corner between a space-like boundary, ∂M_n and a time-like one, ∂M_u, the total corner contribution, $\delta\angle$, is

$$\delta\angle = \frac{-1}{16\pi G}\int\left[\delta n\cdot(n\cdot\epsilon) - \delta u\cdot(u\cdot\epsilon)\right] \ . \tag{3.35}$$

From equation (C.66), we see that

$$\delta\angle = -\frac{1}{8\pi G}\int r\cdot(n\cdot\epsilon)\delta\alpha \ , \tag{3.36}$$

where $\alpha = \sinh^{-1}(\eta)$. Therefore, if a corner action term, $\int \alpha\ r\cdot(n\cdot\epsilon)/8\pi G$, is added into the Einstein-Hilbert action, the variation of the action will depend on $\delta g_{\mu\nu}$ only, but not its derivatives.

The resulting Hamilton's action I'_H: The general case Writing them explicitly, for a manifold, M, with two space-like boundaries, the upper one is $\partial M_{n''} = \Sigma_{t''}$ and the lower one is $\partial M_{-n'} = \Sigma_{t'}$, one time-like boundary, $\partial M_u = B$, and two corners, $C'' = \Sigma_{t''}\cap B$ and $C' = \Sigma_{t'}\cap B$ (cf. figure 3.2), we have the following variation,

$$\begin{aligned}\delta I'_H &= \delta\left[\int_M \mathbf{L}_g - \frac{1}{8\pi G}\int_{\Sigma_{t'}}^{\Sigma_{t''}} Kn\cdot\epsilon + \frac{1}{8\pi G}\int_B \Gamma u\cdot\epsilon + \frac{1}{8\pi G}\int_{C'}^{C''}\alpha r\cdot(n\cdot\epsilon)\right] \\ &= \int_M \mathbf{G}^{\mu\nu}\delta g_{\mu\nu} - \int_{\Sigma_{t'}}^{\Sigma_{t''}} \mathbf{P}_{\mu\nu}\delta h^{\mu\nu} + \int_B \mathbf{\Pi}_{\mu\nu}\delta\gamma^{\mu\nu} \\ &\quad -\frac{1}{16\pi G}\int_{C'}^{C''}\alpha r\cdot(n\cdot\epsilon)\sigma_{\mu\nu}\delta\sigma^{\mu\nu} \ . \end{aligned} \tag{3.37}$$

The action, I'_H, is the Hamilton's action for a gravitational system with boundaries.

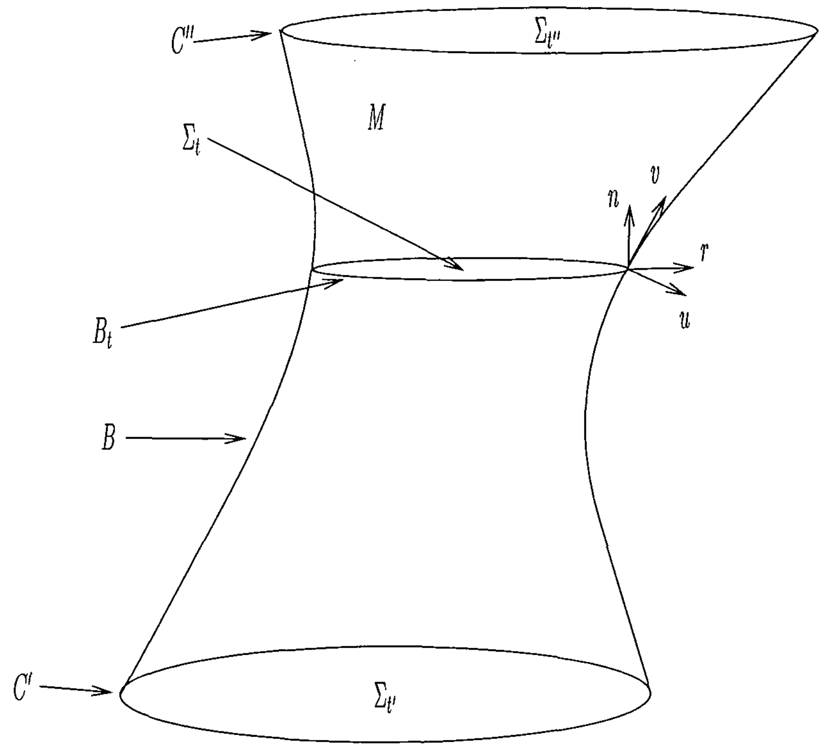


Figure 3.2: Manifold M.

The Hamilton's action I_H: Special case of $n^\mu u_\mu = \eta = 0 = N^\mu u_\mu$ In the rest of this section, following Brown and York [9, 10], we consider the special case in which the family of space-like hypersurfaces, Σ_t, intersect the time-like boundary, B, orthogonally, i.e., $n^\mu u_\mu = \eta = 0$, and the Hamiltonian does not generate the spatial diffeomorphisms which map the field variables across the time-like boundary, i.e., $N^\mu u_\mu = 0$. Therefore, all corner terms discussed in previous paragraph disappeared. We will call I_H the action resulted from I'_H in equation (3.37) by subtracting corner terms (cf. equation (C.26)). From the results in section C.2.4 (cf. equation (C.76)), we can rewrite equation (3.37) as

$$\delta I_H \;=\; \int_M \mathbf{G}^{\mu\nu}\delta g_{\mu\nu} + \int_{\Sigma_{t'}}^{\Sigma_{t''}} \mathbf{P}^{\mu\nu}\delta h_{\mu\nu}$$

$$- \int_B \frac{u \cdot \epsilon}{N} (-\frac{N}{2} s^{\mu\nu} \delta\sigma_{\mu\nu} + \varepsilon \delta N - j_\mu \delta N^\mu) \ . \qquad (3.38)$$

Note that we have use the relations $-\mathbf{P}_{\mu\nu}\delta h^{\mu\nu} = \mathbf{P}^{\mu\nu}\delta h_{\mu\nu}$ and $-s_{\mu\nu}\delta\sigma^{\mu\nu} = s^{\mu\nu}\delta\sigma_{\mu\nu}$. In the Hamiltonian formulation of general relativity [67], the $\mathcal{H}$ and $\mathcal{H}_\mu$ (see equations (C.13) and (C.14)) are interpreted as the energy-momentum conjugate to N and N^μ. It is then nature to interpret ε and j_μ as the conjugate energy-momentum of N and N^μ with respect to the boundary B. These variables are the counterparts of the energy and chemical potentials in an ordinary thermodynamic system. The variable $\sigma_{\mu\nu}$ determine the size and shape of the system, it is therefore the counterpart of volume.

The preliminary micro-canonical action

The action, I_H, suitable for variation with fixed metric at boundary is derived from I'_H in equation (3.37) with corner terms ignored. If we add the following term, I_L, to I_H, (cf. equations (C.31) and (C.32))

$$I_L = -I_k - I_p = \int_B \varepsilon \ u \cdot \epsilon - \int_B \frac{u \cdot \epsilon}{N} j_\mu N^\mu \ , \qquad (3.39)$$

the variation of $I_E = I_H + I_L$ is

$$\begin{aligned} \delta I_E &= \delta(I_H + I_L) \\ &= \int_M \mathbf{G}^{\mu\nu}\delta g_{\mu\nu} + \int_{\Sigma_{t'}}^{\Sigma_{t''}} \mathbf{P}^{\mu\nu}\delta h_{\mu\nu} \\ &\quad + \int_B \left[\frac{u \cdot \epsilon}{2} s^{\mu\nu}\delta\sigma_{\mu\nu} + N\delta(\varepsilon \frac{u \cdot \epsilon}{N}) - N^\mu \delta(j_\mu \frac{u \cdot \epsilon}{N}) \right] \ . \end{aligned} \qquad (3.40)$$

It is obvious that, at the boundary B, now the suitable boundary condition are the fixed energy-momentum density, $\varepsilon \frac{u \cdot \epsilon}{N}$, $j_\mu \frac{u \cdot \epsilon}{N}$, and the induced metric, $\sigma_{\mu\nu}$.

If we regard ε, j_μ, and $\sigma_{\mu\nu}$ as the thermodynamic quantities which characterises a system enclosed by the boundary B, then what we are doing is performing a Legendre transformation which transforms the functional dependence of a state function amongst the conjugate pair [9]. If the path integral prescription of micro-canonical partition function (cf. equation (3.27)) is generalised to a gravitational system so that the integrated paths are constrained to satisfy the imposed boundary condition of fixed ε,

j_μ, and $\sigma_{\mu\nu}$ at B, the action, I_E, will be a proper candidate as the micro-canonical action for evaluating the micro-canonical partition function. We therefore write $\nu(\varepsilon, j, \sigma)$ henceforth.[23]

Micro-canonical action without space-like boundary

Hamilton's action without space-like boundary In order to compute the black hole entropy via path integral, as prescribed in section 3.2.1 (cf. equation (3.27)), the integrated paths are constrained being periodic in time co-ordinate. Therefore, the topology of the manifold, M, is $S^1 \times \Sigma$. This causes some changes in the form of the Hamilton's action, I_H, because we do not add I_K[24] into the Einstein-Hilbert action, I_R.[25] Furthermore, we are interested in the case that the topology of Σ is $I \times S^2$ in which the two end points of I corresponding to two time-like boundaries, B_o and B_i;[26] which have topology $S^1 \times S^2$. Though there are two of them, the boundary condition is prescribed only on the outer one, B_o [9].[27] Therefore, only the I_Γ term (cf. equation (C.29)) of the outer boundary, denoted as I_{Γ_o}, is add to I_R. The resulting Hamilton's action, I_h, is

$$\begin{aligned}
I_h = I_R + I_{\Gamma_o} &= \frac{-1}{16\pi G}\int_M \mathcal{R}\epsilon + \frac{1}{8\pi G}\int_{B_o} \Gamma u \cdot \epsilon \\
&= \int_M (\mathbf{P}^{\mu\nu}\dot{h}_{\mu\nu} - N\mathbf{H} - N^\mu \mathbf{H}_\mu) + \frac{1}{8\pi G}\int_{B_o} k\ u \cdot \epsilon - 2\int_{B_o} \frac{1}{N} N^\mu \sigma_{\mu\nu}\mathcal{P}^{\nu\beta} u_\beta\ u \cdot \epsilon \\
&\quad - \frac{1}{8\pi G}\int_{B_i} l_\mu a^\mu\ l \cdot \epsilon - 2\int_{B_i} \frac{1}{N} N^\mu \sigma_{\mu\nu}\mathcal{P}^{\nu\beta} l_\beta\ l \cdot \epsilon \\
&= I_E + I_k + I_p - \frac{1}{8\pi G}\int_{B_i} l_\mu a^\mu\ l \cdot \epsilon - 2\int_{B_i} \frac{1}{N} N^\mu \sigma_{\mu\nu}\mathcal{P}^{\nu\beta} l_\beta\ l \cdot \epsilon\ , \qquad (3.41)
\end{aligned}$$

where l^μ is the unit outward-pointing normal of the inner boundary, B_i.

[23]Recall that the integrated paths in the path integral representation of the density of states is periodic with respect to time. Therefore, there is no dependence on $h_{\mu\nu}$ at space-like hypersurfaces.

[24]ee equations (C.27)-(C.29) for the definitions of various action terms.

[25]Alternatively, notice that the periodicity in time co-ordinate means the two I_K terms cancelled.

[26]Therefore, the topology of M is (*annulus*) $\times S^2$.

[27]This could mean that we can only measure the relevant quantities on the outer boundary.

Micro-canonical action without space-like boundary Performing the Legendre transformation by adding in I_L (cf. equation (3.39)), we arrive at the micro-canonical action, I_m,

$$I_m = I_h + I_L = I_E - \frac{1}{8\pi G}\int_{B_i} l_\mu a^\mu \, l\cdot\epsilon - 2\int_{B_i}\frac{1}{N}N^\mu\sigma_{\mu\nu}\mathcal{P}^{\nu\beta}l_\beta \, l\cdot\epsilon \,. \tag{3.42}$$

The micro-canonical action, I_m, will be used in next section to evaluate the entropy of a spherically symmetric black hole.

3.2.3 Entropy for spherically symmetric black holes

I will content myself with the spherically symmetric cases only.[28]

The saddle point

Boundary condition In order to calculate the entropy of a black hole, the boundary condition, ε, j_μ, and $\sigma_{\mu\nu}$, at B is necessarily derived from the black hole solution—a Lorentzian one. This can be done by, at first, choosing a 2 sphere, B_2, on a hypersurface, Σ_3, of $t = constant$ of the black hole solution, one can then obtain the boundary condition by embedding B_2 as a hypersurface in Σ_3, as described in section C.1.3, appendix C. To implement the saddle point approximation described in section 4.2.1 (cf. equation (3.32)), we need to find the stationary point of the micro-canonical action which satisfies the imposed boundary condition at $B = S^1 \times B_2$.

Need for complex solution Though the boundary condition at B is derived from a $t = constant$ hypersurface of a Lorentzian black hole, this black hole solution cannot be the stationary point of the micro-canonical action because it contains two wedges, which is separated by the bifurcation 2 surface of the Killing horizon, therefore it cannot be placed on a manifold with one single boundary, B, of topology $S^1 \times S^2$ [9]. However, if we analytically continue this solution to a complex one, this complex black hole solution could have a single boundary with topology $S^1 \times S^2$.[29]

[28] For more general cases, see reference [9, 10].

[29] Therefore, either the measure of the path integral should include complex metrics or analytic continuation should be used to rotate the integral path on the complex metric plane.

Too see this, we at first write the metric in the ADM form, (cf. equation (C.1))

$$g_{\mu\nu} = -\bar{N}^2(r)dt^2 + g_{rr}(r)dr^2 + \sigma_{ab}dx^a dx^b \ .$$

where the shift vector is zero because of spherical symmetry and $\bar{N}$, g_{rr}, σ_{ab} are t-independent. The horizon 2-surface is located at the "bolt" where $\bar{N} = 0$ [24]. Analytically continuing this solution to a complex one, in fact, an Euclidean one, by $t \longrightarrow -i\tau$, we arrive at the solution,

$$g_{\mu\nu} = -N^2 d\tau^2 + g_{rr}(r)dr^2 + \sigma_{ab}dx^a dx^b \ , \tag{3.43}$$

where τ is real and $N = -i\bar{N}$. Away from the horizon 2-surface of $\bar{N} = 0$, the above metric satisfies the vacuum Einstein field equation because the equation is also analytic with respect to t. By requiring the t co-ordinate being periodic with period, $P = constant$, we could put this solution in a manifold with a single boundary of topology $S^1 \times S^2$ if the boundary corresponds to the hypersurface of fixed r co-ordinate. However, as $\bar{N} = 0$, a conical singularity could arise [24]. To avoid the appearance of a conical singularity, a certain constraint should be imposed on $\bar{N}$ regarding to the period, P [9].

Regularity condition Rewrite the metric in equation (3.43) in the Gaussain normal form, we have,

$$g_{\mu\nu} = \bar{N}^2(r_*)dt^2 + dr_*^2 + \sigma_{ab}dx^a dx^b \ . \tag{3.44}$$

Near $\bar{N} = 0$, the $t - r_*$ section of the metric describes an Euclidean 2-surface. Therefore, the conical singularity is avoided if, as $\bar{N} \longrightarrow 0$, the proper circumference, $P\bar{N}$, approaches $2\pi r_*$ asymptotically, namely, the Euclidean 2-surface approaches an Euclidean 2-plane. In terms of the original co-ordinates, this is equivalent to the condition that

$$\partial_{r_*}(P\bar{N}) = P\sqrt{g^{rr}}\partial_r \bar{N} = -Pl^\mu \partial_\mu \bar{N} = 2\pi, \tag{3.45}$$

where l^μ is the unit inward-pointing normal of the bolt, i.e., the outward-pointing normal of the boundary, B_i.

The entropy

In order to employing the saddle point approximation as described in section 3.2.1 (cf. equation (3.33)) for the solution of equation (3.43), we consider a manifold of topology $(annulus) \times S^2$ with two boundaries, B_o and B_i, corresponding to the two edge circles of the annulus [9]. The outer one, B_o, is therefore corresponding to B where the boundary condition is specified.[30] No boundary condition is specified at the inner boundary, B_i, because we have to take the limit such that the radius of the hole inside the annulus shrinks to zero. The limit of B_i is therefore the bolt corresponding to $\bar{N} = 0$. We then evaluate the micro-canonical action, I_m, in equation (3.42) for the Euclidean solution of equation (3.43).

Explicitly, the action is (cf. equations (3.42) and (C.30))

$$I_m = \int_M (\mathbf{P}^{\mu\nu} \dot{h}_{\mu\nu} - N\mathbf{H} - N^\mu \mathbf{H}_\mu) - \frac{1}{8\pi G} \int_{B_i} l_\mu a^\mu \; l \cdot \epsilon - 2 \int_{B_i} \frac{1}{N} N^\mu \sigma_{\mu\nu} \mathcal{P}^{\nu\beta} l_\beta \; l \cdot \epsilon \; . \tag{3.46}$$

The first integral on the RHS of the above equation is zero because the stationary solution satisfies the Einstein field equation. The third integral is also zero because the shift vector, N^μ is zero. For the remaining term, note that from equation (C.38), we have $l \cdot \epsilon = \sqrt{-\det \gamma} = N\sqrt{\det \sigma}$. From equation (C.36), we have $a_r = \partial_r N / N$. The remaining term can therefore be rewritten as

$$I_m = \frac{-1}{8\pi G} \int_{B_i} d^3x \sqrt{\det \sigma} \; l^\mu \partial_\mu N = \frac{-1}{8\pi G} \int_0^P dt \int d^2x \sqrt{\det \sigma} \; l^\mu \partial_\mu N \; . \tag{3.47}$$

With the help of the constraint of equation (3.45), and note that $N = -i\bar{N}$, we arrive at

$$I_m = \frac{-i}{4G} A_H \; , \tag{3.48}$$

where $A_H = \int d^2x \sqrt{\det \sigma}$ is the area of the horizon 2-surface. Therefore, from equation (3.33), the entropy is $S = A_H / 4G$.

[30] See the paragraph **Boundary condition** on page 79

3.3 Statistical origin of black hole entropy from inside

In this section, a version of quantum statistical explanation of Black hole entropy is present from the point of view of quantised thermal states inside a black hole. The whole approach is in fact modelled from the case for blackbody radiation [19]. This is the simplest method I know of. I suppose this is also its strength. Nonetheless, I also have to admit that the underlying physics is perhaps not as easy to justify as the computation itself could be. However, it points me a route to investigate.

3.3.1 General assumptions

Standard statistical mechanics is applicable If one assumes, as I do, that the statistical aspect of a statistical explanation of black hole entropy can be borrowed from the textbook statistical mechanics, then the two basic ingredients for computing the statistical entropy of a thermal equilibrium system are the spectrum of the states in the system and the statistical distribution law governing them.

A static black hole is a thermal equilibrium system of non-interacting states Within the standard thermodynamics, only equilibrium systems have well-defined state functions; entropy is one of them. Or, we call those systems which have well-defined state functions equilibrium systems. I therefore assume that a static Schwarzschild black hole of mass M is in fact a thermal equilibrium system which consists of *non-interacting* [31] neutral states whose distribution is governed by the Bose-Einstein statistics. As for a static, non-rotating, charged black hole, I will assume the black hole is composed of non-interacting charged bosonic states.[32]

The temperature is $\mathbf{T} = \kappa/2\pi$ A thermal equilibrium system is characterised by a constant temperature. I thus assume that the temperature, T, of a static black hole is

[31] This is a very strong assumption. Nonetheless, an equilibrium system, at leading order, can be described by free states like the *free* electrons inside a piece of metal [36].

[32] More generally, a mixture of neutral and charged states.

$T = \kappa/2\pi$ where κ is the surface gravity of the black hole. I am not able to justify why the temperature of the black hole should be the temperature of the thermal radiation observed at future infinity if the black hole is radiating, though, in the cases I am considering it is not. However, since it is the temperature that appears in the first law of black hole mechanics (cf. equation (1.13)), I assume the temperature is so.

The energy spectrum is $\mathbf{E_j = \kappa j}$ Furthermore, I assume that the spectrum of these states is given by $E_j = \kappa j$, $j = 1, 2, 3, \ldots$. This seems to be an arbitrary assumption. Its justification will be left to next chapter[33] where I will explain how the Kleinian signature forces me to make such a choice. At this moment, I only remind the readers to notice the similarity between this choice and that used by Planck for a thermal radiation.

3.3.2 For Schwarzschild black holes

Thermodynamic quantities

Partition function and Helmholtz free energy Having given the spectrum and the distribution law, the thermal properties of a Bose system can be calculated from the logarithm of the partition function Z [19],

$$\ln Z = -\beta F_b = n_b \ln \left[\sum_{\{\tilde{n}_j\}} e^{-\beta \sum_j \tilde{n}_j E_j} \right] = -n_b \sum_{j=1}^{\infty} \ln(1 - e^{-\beta E_j}) \;, \qquad (3.49)$$

where n_b is a degeneracy factor, F_b the Helmholtz free energy, and $\tilde{n}_j$ the state-number for the j-th eigen-state. The summation is over all possible sets of numbers, $\{\tilde{n}_j\} = (\tilde{n}_1, \tilde{n}_2, \ldots)$, $\tilde{n}_j = 0, 1, 2, \ldots$.

Definitions I define the following quantities for convenience:

$$n_j = \frac{1}{e^{2\pi j} - 1} \;,$$

$$b_0 = -\sum_{j=1}^{\infty} \ln\left(\frac{e^{-2\pi j}}{n_j}\right) \;, \qquad b_1 = \sum_{j=1}^{\infty} n_j \;, \qquad b_2 = \sum_{j=1}^{\infty} 2\pi j n_j \;.$$

[33] See equation (4.14).

$U = TS + F$ Then the relation between the Helmholtz free energy, F_b, internal energy, U_b, and entropy, S_b is

$$U_b = n_b \sum_{\{\tilde{n}_j\}} \tilde{n}_j E_j P(\{\tilde{n}_j\}) = -(\partial_\beta \ln Z)_{E_j,n_b} = TS_b + F_b = Tn_b b_2 \ , \tag{3.50}$$

where

$$S_b = -n_b \sum_{\{\tilde{n}_j\}} P(\{\tilde{n}_j\}) \ln P(\{\tilde{n}_j\}) = \beta^2 (\partial_\beta F_b)_{E_j,n_b} = n_b(b_2 + b_0) \ , \tag{3.51}$$

and $P(\{\tilde{n}_j\})$ is the probability of finding the configuration with the state-number set $\{\tilde{n}_i\}$. Explicitly, we have

$$P(\{\tilde{n}_j\}) = \frac{e^{-\beta \sum_j \tilde{n}_j E_j}}{\prod_j (1 - e^{-\beta E_j})^{-1}} \ .$$

Total number of state $\mathbf{N_b}$ The total number of states, N_b, is

$$N_b = n_b \sum_{\{\tilde{n}_j\}} \tilde{n}_j P(\{\tilde{n}_j\}) = n_b \sum_{j=1}^{\infty} n_j = n_b b_1 \ . \tag{3.52}$$

Entropy per state is a constant Then the entropy per state $\bar{s}_b$ is,

$$\bar{s}_b = \frac{S_b}{N_b} = \frac{b_0 + b_2}{b_1} \ ,$$

which is independent of M due to the specific dependence of T and E_j on κ.

Quantisation rule of black hole masses

Determination of $\mathbf{n_b}$ In order to determine the normalisation constant, n_b, we equate the state function for entropy from equation (3.50) and that from black hole mechanics,[34]

$$S_{bh} = \beta \frac{M}{2} \ . \tag{3.53}$$

The normalisation constant then is determined to be

$$n_b = \frac{4\pi G M^2}{b_0 + b_2} \ . \tag{3.54}$$

Mass quantisation rule The above relation then give us a quantisation rule for the masses of Schwarzschild black holes: Since the total number of state, N_b, should be a positive integer,[35] consequently, from equations (3.52) and (3.54), we obtain

$$M = \sqrt{\frac{b_0 + b_2}{4\pi G b_1} N_b} \ .$$

This conforms (up to a pre-factor) with various derivations.[36] Note that there is only one independent variable for neutral black holes. Classically, it is the mass, M, of the black hole. Within the present quantum statistical explanation, it can be replaced by either n_b or N_b.

Reminders

From above approach we see that, phenomenological, there are great similarities between the calculation of black hole entropy and the entropy of blackbody radiation in a cavity if the assumptions in section 3.2.1 are accepted. This is one of the lessons which strengthen

[34]Effectively, I use the value of entropy to determine the normalisation constant, n_b. Therefore, I am *not* deriving the value of black hole entropy. Instead, I am explaining its statistical origin. Alternatively, one can interpret the n_b as the area (corresponding the to volume, V, of the cavity which contains the black-body radiation) and a quantity like U_b/n_b is then the internal energy density (per area). However one performs the normalisation, generally, one arrives at $S_b \propto M^2$, $U \propto M$, and $n_b \propto M^2$, as required. The latter fact conforms with the general opinions that the proper extensive quantity regarding the size of a black hole is the area, instead of the volume. See reference [58] for an example.

[35]In the expression of the partition function, equation (3.49), I have assumed that the bosonic state number is not conserved due to state-antistate pair creation. Therefore, it is not obvious why the average number of state, N_b, should be an integer. However, if N_b is interpreted as a kind of additive quantum (say, baryon) number such that the quantum number of an antistate is the negative of a state, then it is nature to require that the total quantum number being an integer.

[36]See article [45] and references therein.

my faith that a black hole indeed has statistical entropy and its origin should be resorted to the quantum states inside a black hole.

On the other hand, I think my explanation, compared with those two reviewed in previous sections of this chapter, has the merit of simplicity. Although I have ignored the possible dynamic complexity due to the non-linearity of gravitational field, I think, if entropy indeed has a statistical explanation which can be extrapolated from the textbook statistical mechanics as I have assumed in the beginning of this section, then the simplicity and generality of statistical description of a equilibrium thermal system should also be applied to a black hole. I will explore this point further in next chapter.

3.3.3 For Reissner-Nordström black holes

There are two independent variables

A charged black hole is a mixture of charged and neutral states The calculation of quantum statistical entropy of charged black holes is, for the most parts, the same to the case of Schwarzschild black holes. However, recall that there are *two* independent variables in the first law of (charged) black hole mechanics: the black hole mass, M $(= \mathcal{E})$, and the total charge, q. In a statistical explanation of black hole entropy based on a microscopic material model, there exists necessarily a certain relation between the mass and the charge if all of the material constituents of the charged black hole are charged states. Then, effectively, there is only one independent variable. Fortunately, there are chargeless states as explained in previous section. Therefore, I will assume that, in general, a charged black hole is a mixture of charged and neutral states.

Chemical potential Since an extra energy, the Coulomb potential, enters the first law, one can expect the extra energy behaves as a chemical potential [4]. In order to find the relation between the chemical potential and the Coulomb potential of the charged black hole, I will further assume that the total charge, q, observed from the outside of the black hole is determined by a charge unit, e, through the relation, $q = N_c e$ where N_c

is the total number of charged states.[37] The first law of black hole mechanics, equation (1.13), says

$$dM = TdS + \Phi dq \ .$$

With the assumption that $q = N_c e$, we can then rewrite the first law as

$$dM = TdS + \mu dN_c \ ,$$

where $\mu = e\Phi = eq/r_+$ is therefore the chemical potential.[38] According to the assumption about the spectrum that $E_j = \kappa j$, I will require that $E_j > \mu$ for all allowed j. This constraint follows from the requirement that those quantum states inside a black hole should be bounded states, as will be explained in section 4.3.2, chapter 4 (cf. equation (4.40)). At this moment, this condition can also be understood from a purely statistical point of view: If the chemical potential, μ, is larger than the eigen-energy, E, of a particular state, then the sum $\sum_{n=0}^{\infty} e^{-\beta n(E-\mu)}$ is divergent. In other word, such a system cannot be dealt with by conventional statistical mechanics. Admittedly, it could hint that the conventional statistical mechanics needs being generalised to include such systems. But, based on my strategy of approach and current understanding of statistical mechanics, I rule out such possibility.

Thermodynamic quantities

Partition functions and Helmholtz free energies We can therefore write down the logarithm of the partition function,

$$\begin{aligned} \ln Z &= -n_c \sum_{j=1}^{\infty} \ln \left[1 - e^{-\beta(E_j-\mu)}\right] - n_b \sum_{j=1}^{\infty} \ln(1 - e^{-\beta E_j}) \\ &= -\beta(F_c + F_b) \ , \end{aligned} \tag{3.55}$$

[37] I will explain in section 4.3, chapter 4 while I am constructing a model for charged black holes that the situation is, in fact, much more complicated than this simple-minded assumption can grasp.

[38] See section D.2 in appendix D at the end of chapter 4 for the explanation of Reissner-Nordström solution and the definitions of various quantities.

where n_c and n_b are normalisation constants;[39] F_c and F_b the Helmholtz free energy for the charged and neutral states, respectively.[40] The β is the inverse of the temperature as determined by the first law of black hole mechanics and μ is the chemical potential.

More definitions It is convenient to further define the following quantities:

$$s_j = j - \frac{\mu}{\kappa} , \qquad m_j = \frac{1}{e^{2\pi s_j} - 1} , \tag{3.56}$$

$$c_0 = -\sum_{j=1}^{\infty} \ln(\frac{e^{-2\pi s_j}}{m_j}) , \qquad c_1 = \sum_{j=1}^{\infty} m_j , \qquad c_2 = \sum_{j=1}^{\infty} 2\pi s_j m_j , \tag{3.57}$$

where κ is the surface gravity of a Reissner-Nordström black hole (cf. equation (D.12)).

The entropy Then following the derivation in section 3.3.2 and by equating the state functions for entropy, we have

$$\begin{aligned} S_{bh} &= \frac{\pi r_+^2}{G} \\ &= S_c + S_b = n_c(c_0 + c_2) + n_b(b_0 + b_2) . \end{aligned} \tag{3.58}$$

The total number of states, N, is

$$N = N_c + N_b = n_c c_1 + n_b b_1 , \tag{3.59}$$

where N_c and N_b are the total numbers of the charged and neutral states, respectively. Then equation (3.58) turns to

$$\frac{\pi r_+^2}{G} = N_c \frac{c_0 + c_2}{c_1} + N_b \frac{b_0 + b_2}{b_1} . \tag{3.60}$$

Mass quantisation rule for charged black holes By requiring N_b and N_c being positive integers, equation (3.60) gives us a quantisation rule of the masses of charged black hole. Because the three c-factors in equation (3.57) depend on M and q, hence

[39] Effectively, they account for the two independent variables.

[40] I will use subscript c and b to denote the quantities corresponding to charged and neutral states, respectively.

N_b and N_c, in a very complicated manner, it is easier to *read* the quantised masses by looking for the intersections of the two curves respectively from the LHS and RHS of equation (3.60) as functions of N_b and N_c. Since $(b_0 + b_2)/b_1$ is independent of N_b and N_c, the curve of the RHS with $N_b \neq 0$ can be arrived easily from the special case that $N_b = 0$ by shifting the curve vertically. Two examples with $N_b = 0$ are shown in figures 3.3 and 3.5 for two different choices of charge unit, e. Note that although only one intersection point is shown, say in figure 3.3.1, there are in fact two of them, as in figure 3.3.7. They are not shown up because the initial points of M in those figures are not close enough to $M_{min}(N_c)$ which is determined by equation (4.42) following from the requirement that $E_j > \mu$, or, equivalentlt, $s_j > 0$ for all possible j. In figures 3.4 and 3.6, the local magnifications are presented to provide a better view around the intersection points close to $M_{min}(N_c)$.[41]

Therefore, for fixed e and small N_b, equation (3.60) has two solutions of M as $N_c < N_{cd}(N_b, e)$ in which $N_{cd}(N_b, e)$ is a degenerate point, obviously depending on N_b and e, where the two solutions coincide. For $N_c > N_{cd}$, solution of M does not exist. However, as N_b is large, there is only one solution of M as $N_c < N_{cd}(N_b)$ in general. The physical significance of such transitions is unclear to me at this moment.

*　　　*　　　*

[41]Analytically, from equations (3.56) and (3.57), we see that as s_1 approaches 0 (equivalently, as M approaches $M_{min}(N_c)$), m_1 diverges as $1/s_1$. Consequently, $(c_0 + c_2)/c_1 \longrightarrow 0$ because $c_0 \propto \ln m_1$, $c_1 \propto m_1$, and c_2 is finite as $s_1 \longrightarrow 0$. However, the LHS of equation (3.60) is strictly positive.

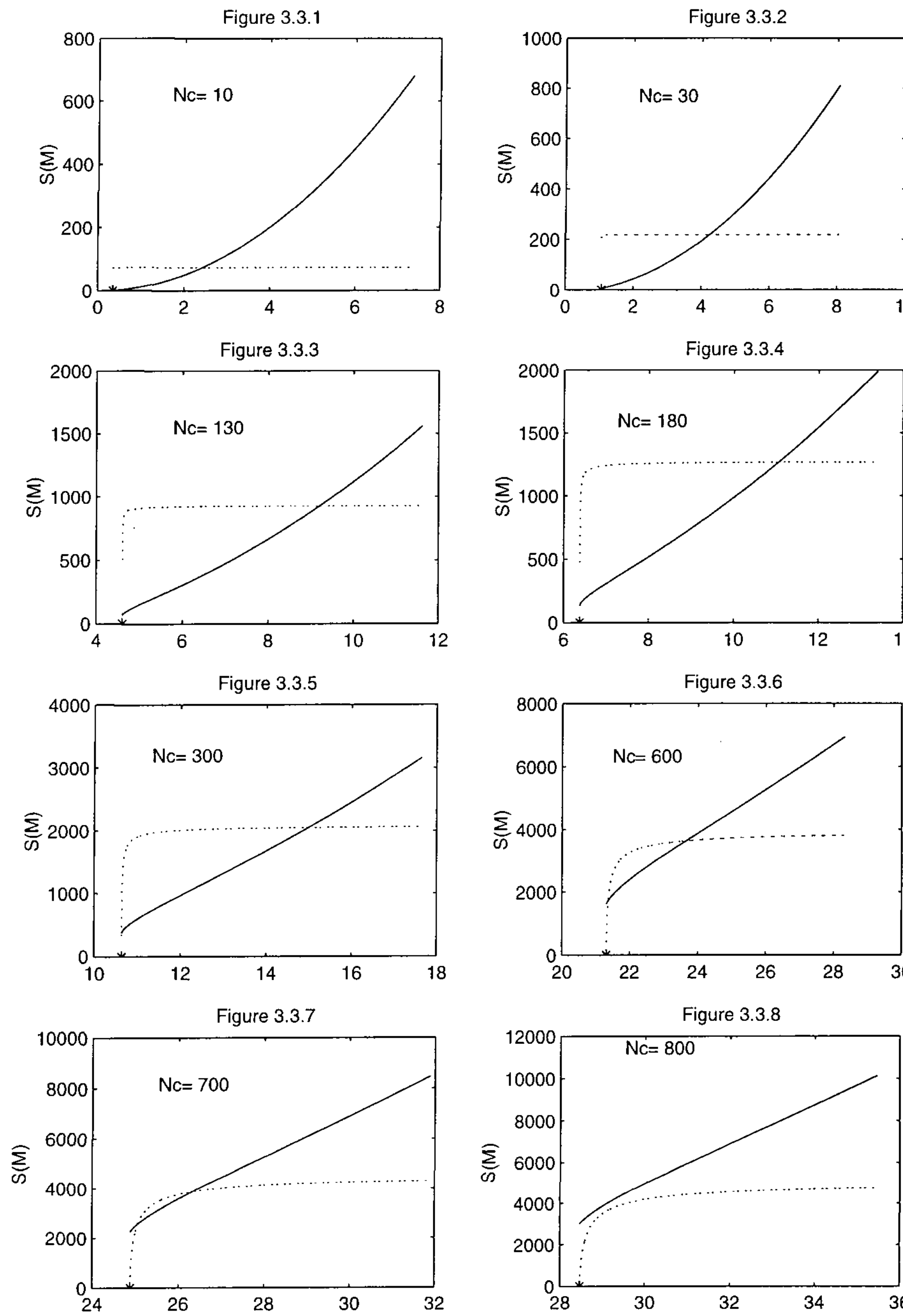


Figure 3.3: Examples of quantised mass of charged black hole with respect to N_c as $N_b = 0$ and $e = 0.01$. The — and $\cdots$ lines corresponds to the LHS and RHS of equation (3.60), respectively. The value of $M_{min}(N_c)$ is indicated by a $*$.

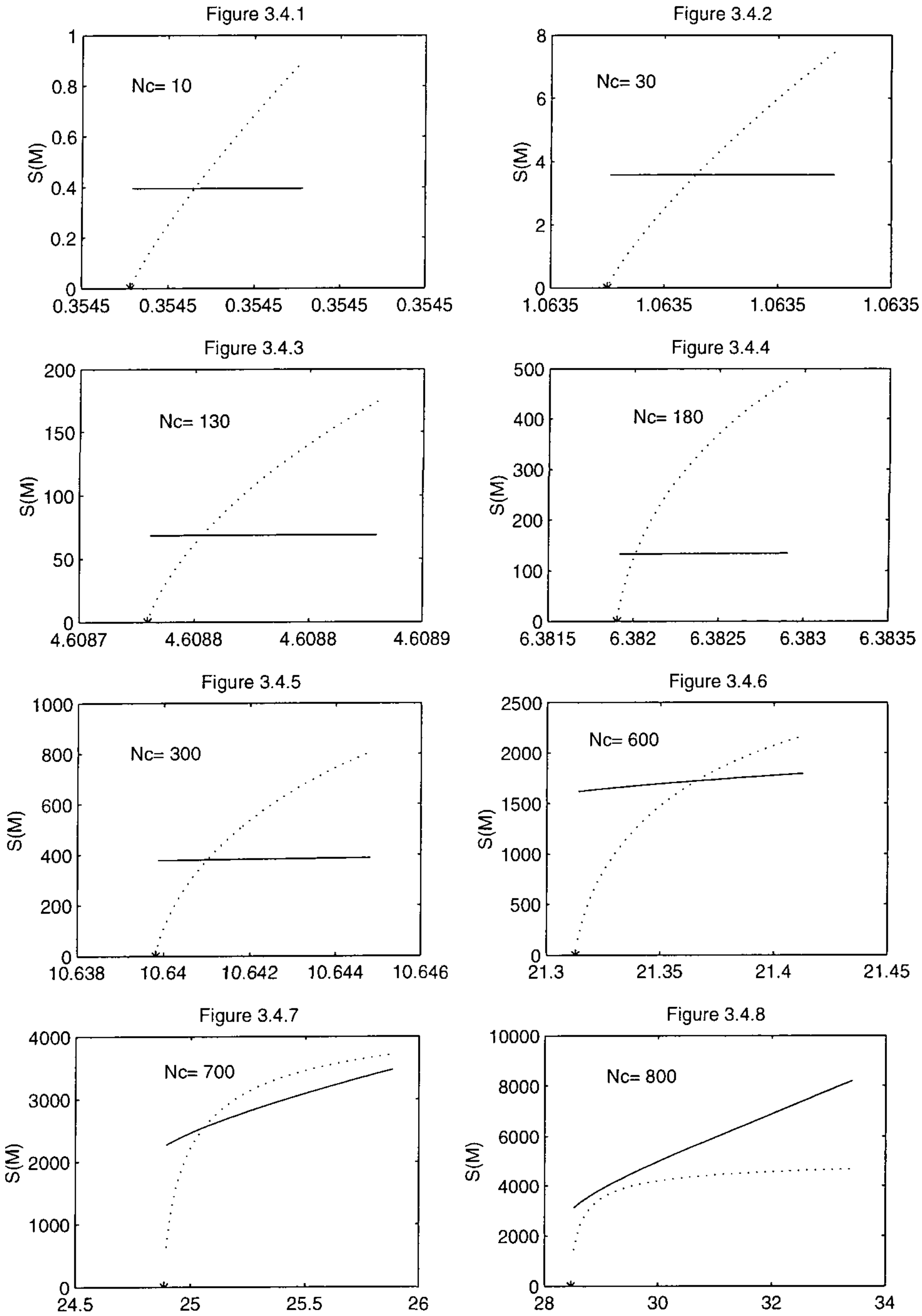


Figure 3.4: Local magnification around $M = M_{min}$ of figure 3.3.

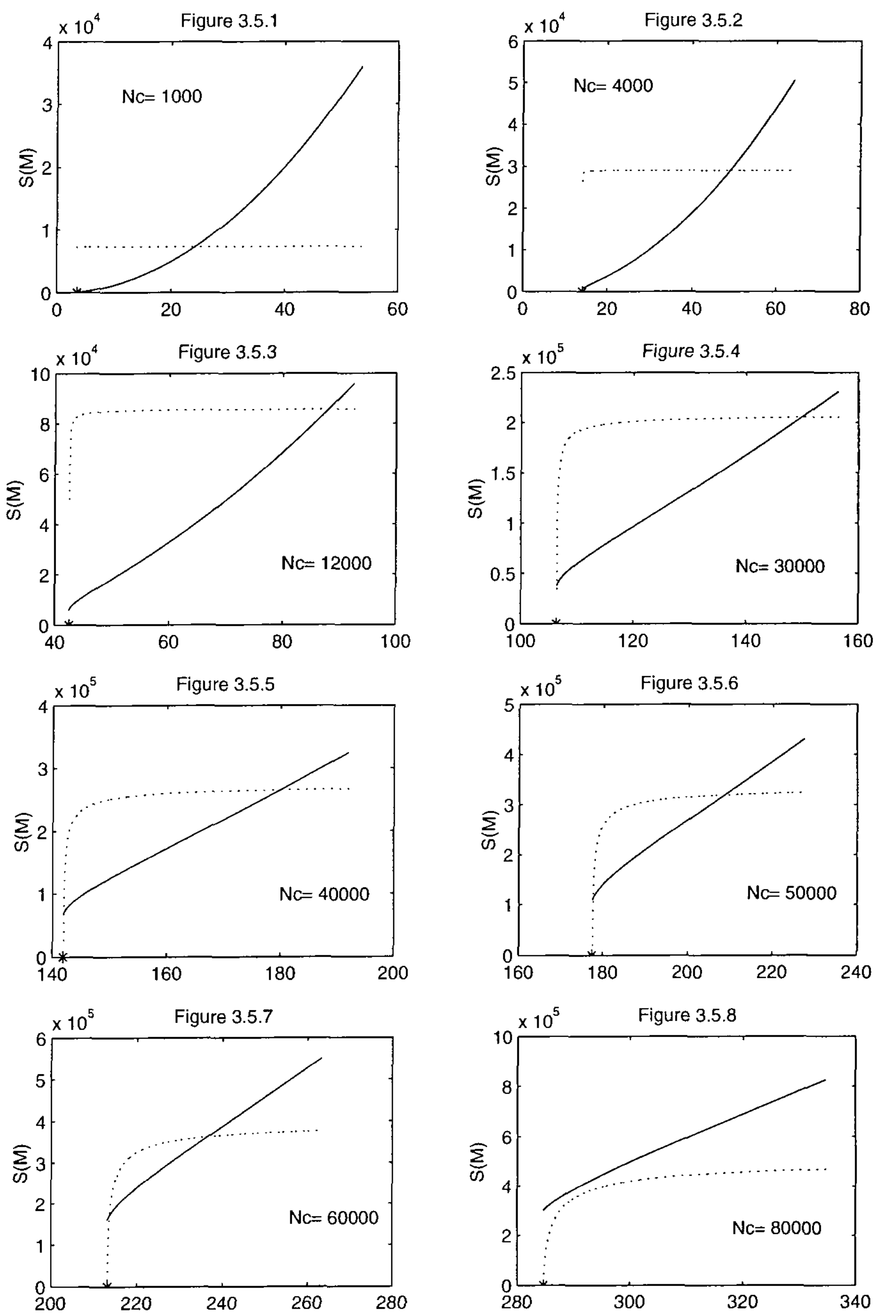


Figure 3.5: Examples of quantised mass of charged black hole with respect to N_c as $N_b = 0$ and $e = 0.001$. The two solutions of M for $N_c = 50000$ are $M_s \simeq 177.66$ and $M_l \simeq 208.65$, which will be used as example masses for constructing numerically the eigen-states in section 4.3.

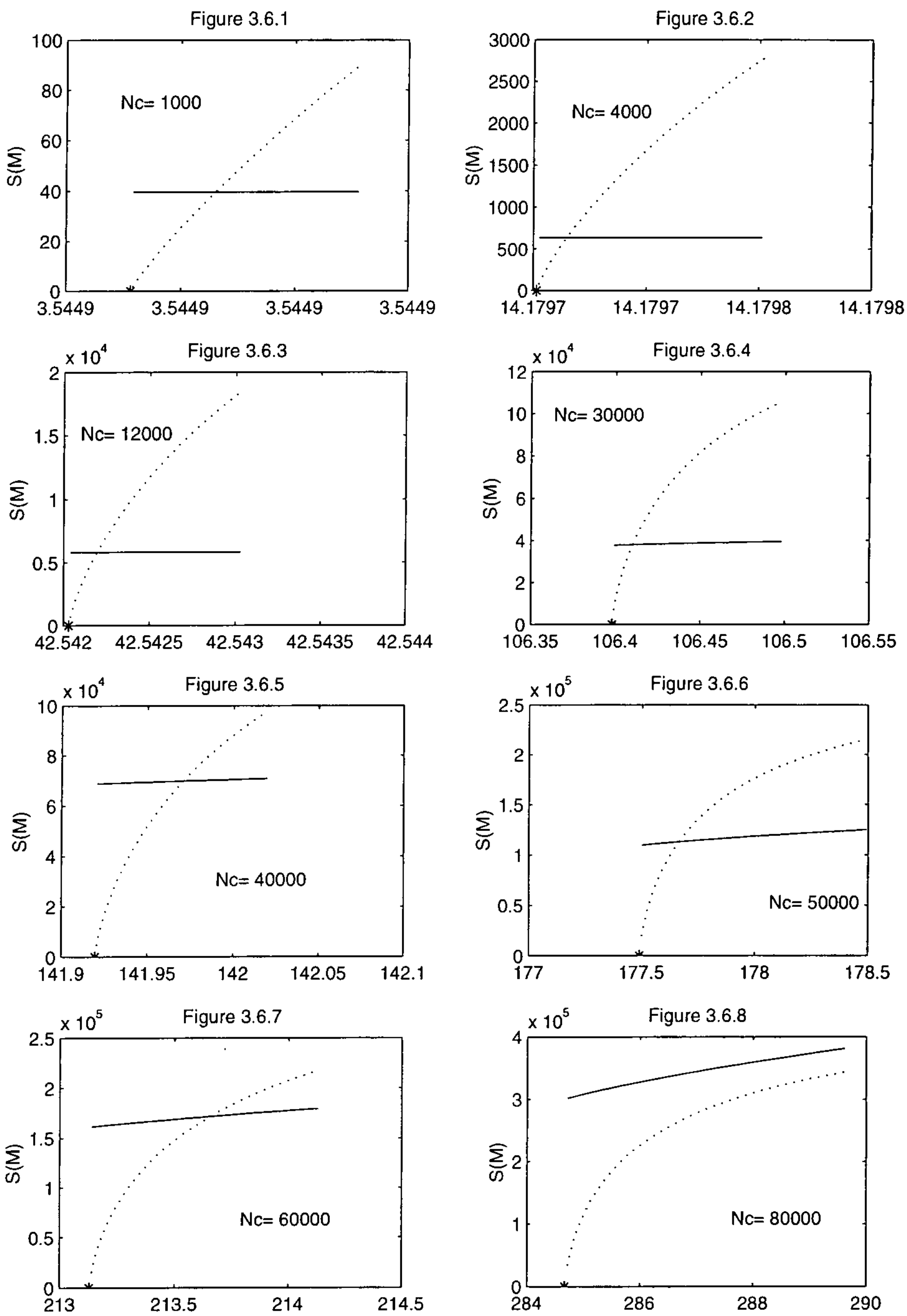


Figure 3.6: Local magnification around $M = M_{min}$ of figure 3.5.

Appendix C

Kinematics

In this appendix, I present some basic equations used in section 3.2, chapter 3, following references [9, 34, 35, 67]. At first, I specify some concepts employed throughout the current appendix and section 3.2, chapter 3. We consider a 4 dimensional space-time $(M, g_{\mu\nu})$, admitting a scalar time function, t, which foliates M into a family of space-like hypersurfaces, Σ_t. The boundary, ∂M, of M could consist of several disconnected pieces of boundary, ∂M_n's. Each one of them is characterised by its unit outward-pointing normal vector, n^ν, which is normalised to be $n^\nu n_\nu = -\tau$. They could be either time-like ($\tau = 1$) or space-like ($\tau = -1$). Furthermore, at the junction of two pieces of boundary, e.g., ∂M_n and ∂M_u, the normal is allowed to be discontinuous, i.e., the boundary ∂M needs not being smooth at the "corners". However, we will assume that the normal vector is continuos on either a single space-like or time-like boundary. In other words, the corners only appear at the junction between a space- and time-like boundaries. (See figure 3.2 for an example.)

For the notation concerning differential forms, see appendix B at the end of chapter 1. The induced orientation of a particular piece of boundary, ∂M_n, is derived from the orientation of M by the following method: Permute the normal, n^μ, to the first slot in an orthogonal basis, $(n^\mu, t_1^\mu, t_2^\mu, t_3^\mu)$, which has the correct orientation of M and where $(t_1^\mu, t_2^\mu, t_3^\mu)$ forms a basis of ∂M_n. Then, the induced orientation of ∂M_n is determined by $(t_1^\mu, t_2^\mu, t_3^\mu)$.

Since there exists a time function, à la Arnowitt-Deser-Misner [67], we can decompose the metric $g_{\mu\nu}$ in a particular basis, $(t^\nu, s_i^\nu), i = 1, 2, 3$, so that

$$g_{\mu\nu} = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ . \tag{C.1}$$

The N and N^i in the above equation are defined through the relation

$$t^\nu = Nn^\nu + N^i s_i^\nu = Nn^\nu + N^\nu \ , \tag{C.2}$$

where $n_\mu = -N\partial_\mu t$ is the future-pointing unit normal of a space-like hypersurface, and s_i^ν's are three linear independent tangent vectors of a hypersurface. Similar decomposition can be performed with respect to time-like hypersurfaces, B_r,

$$g_{\mu\nu} = \tilde{M}^2 dr^2 + \gamma_{ij}(dx^i + \tilde{M}^i dr)(dx^j + \tilde{M}^j dr) \ , \tag{C.3}$$

where $i, j = 0, 2, 3$. The h_{ij} and γ_{ij} can further be decomposed as

$$h_{ij} = M^2 dr^2 + \sigma_{ab}(dx^a + M^a dr)(dx^b + M^b dr) \ , \tag{C.4}$$

$$\gamma_{ij} = -\tilde{N}^2 dt^2 + \sigma_{ab}(dx^a + \tilde{N}^a dt)(dx^b + \tilde{N}^b dt) \ , \tag{C.5}$$

where $a, b = 2, 3$.[1]

C.1 Induced quantities on hypersurfaces

C.1.1 Space-like hypersurfaces Σ_t

The induced metric, $h_{\mu\nu}$, and the induced covariant derivative, D_μ, for a tensor, e.g., $T^\alpha{}_\beta$, on a hypersurface, Σ_t, with time-like, future-pointing unit normal, n^ν, are

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \ , \tag{C.6}$$

$$D_\mu T^\alpha{}_\beta = h_\mu{}^\nu h^\alpha{}_\lambda h_\beta{}^\gamma \nabla_\nu T^\lambda{}_\gamma \ . \tag{C.7}$$

Note that D_α is compatible with $h_{\mu\nu}$, i.e., $D_\alpha h_{\mu\nu} = 0$. The extrinsic curvature, $K_{\mu\nu}$, is

$$K_{\mu\nu} = h_\mu{}^\alpha \nabla_\alpha n_\nu = \frac{1}{16\pi G}(\dot{h}_{\mu\nu} - 2D_{(\mu}N_{\nu)}) \ , \tag{C.8}$$

[1] There are certain relations between N, M, $\tilde{N}$, and $\tilde{M}$ resulting from the equivalence between these two decomposition. See reference [35] for detail.

where $\dot{h}_{\mu\nu}$ is the Lie derivative of $h_{\mu\nu}$ along t_ν. From the Gauss-Codazzi's relations

$$R_{\alpha\beta\gamma}{}^{\rho} = h_\alpha{}^\lambda h_\beta{}^\tau h_\gamma{}^\sigma h^\rho{}_\kappa \mathcal{R}_{\lambda\tau\sigma}{}^\kappa + K_{\alpha\gamma}K_\beta{}^\rho - K_{\beta\gamma}K_\alpha{}^\rho \,, \tag{C.9}$$

$$\mathcal{R}_{\gamma\lambda}n^\gamma h^\lambda{}_\mu = -D_\nu K^\nu \mu + D_\mu K \,, \tag{C.10}$$

where $\mathcal{R}^\mu{}_{\nu\lambda\gamma}$ and $R^\mu{}_{\nu\lambda\gamma}$, respectively, satisfy $2\nabla_{[\gamma}\nabla_{\lambda]}\chi_\nu = -\chi_\mu \mathcal{R}^\mu{}_{\nu\lambda\gamma}$ and $2D_{[\gamma}D_{\lambda]}\chi_\nu = -\chi_\mu R^\mu{}_{\nu\lambda\gamma}$ for an arbitrary vector χ, one can derive

$$\mathcal{R} = R - K_{\mu\nu}K^{\mu\nu} + K^2 - 2\nabla_\mu(n^\mu K - a^\mu) \,, \tag{C.11}$$

where $a^\mu = n^\nu \nabla_\nu n^\mu$. Define the following quantities;[2]

$$\mathcal{P}^{\mu\nu} = \frac{1}{16\pi G}(K^{\mu\nu} - h^{\mu\nu}K) \,, \qquad \mathbf{P}^{\mu\nu} = \mathcal{P}^{\mu\nu} n\cdot\epsilon \,, \tag{C.12}$$

$$\mathcal{H} = 8\pi G(2\mathcal{P}_{\mu\nu}\mathcal{P}^{\mu\nu} - \mathcal{P}^2) - \frac{1}{16\pi G}R \,, \qquad \mathbf{H} = \mathcal{H} n\cdot\epsilon \,, \tag{C.13}$$

$$\mathcal{H}_\mu = -2D_\nu \mathcal{P}^\nu{}_\mu \,, \qquad \mathbf{H}_\mu = \mathcal{H}_\mu n\cdot\epsilon \,. \tag{C.14}$$

From equations (C.9)-(C.14), one can derive

$$\frac{N}{16\pi G}(-R + K_{\mu\nu}K^{\mu\nu} - K^2) = \mathcal{P}^{\mu\nu}\dot{h}_{\mu\nu} - N\mathcal{H} - N^\nu \mathcal{H}_\nu - 2D_\nu(\mathcal{P}^{\nu\mu}N_\mu) \,. \tag{C.15}$$

C.1.2 Time-like boundary ∂M_u

The induced metric, $\gamma_{\mu\nu}$, and the extrinsic curvature, $\Gamma_{\mu\nu}$, of ∂M_u with space-like unit outward-pointing normal u^ν are

$$\gamma_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \,, \tag{C.16}$$

$$\Gamma_{\mu\nu} = \gamma_\mu{}^\alpha \nabla_\alpha u_\nu \,. \tag{C.17}$$

Define three quantities, η, λ, α, and a vector, v_μ, as

$$\eta = n_\mu u^\mu \,, \qquad \lambda = \frac{1}{\sqrt{1+\eta^2}}, \qquad \alpha = \sinh^{-1}(\eta) \,, \tag{C.18}$$

$$v_\mu = \lambda\gamma_{\mu\nu}n^\nu = \lambda(n_\mu - \eta u_\mu) \,. \tag{C.19}$$

[2]They are in fact the conjugate energy-momenta with respect to, $h_{\mu\nu}$, N, and N^μ.

It can be easily seen that $v_\nu v^\nu = -1$, $v_\nu u^\nu = 0$, and $v_\nu n^\nu = -1/\lambda$. In words, if n^ν and u^ν are not orthogonal, then v^ν is the vector which lies on the boundary, ∂M_u, and is orthogonal to u^ν.

The momentum conjugated to $\gamma_{\mu\nu}$ is

$$\pi^{\mu\nu} = \frac{1}{16\pi G}(\Gamma^{\mu\nu} - \Gamma\gamma^{\mu\nu}) \; , \Pi^{\mu\nu} = \pi^{\mu\nu} u \cdot \epsilon \; . \tag{C.20}$$

C.1.3 Foliation of ∂M_u in terms of 2 surfaces B_t

Since the manifold, M, is foliated into a family of hypersurfaces, Σ_t, the time-like 3 dimensional boundary, ∂M_u, can also be foliated into a family of 2 dimensional surfaces, B_t, accordingly. The 2-surface B_t is indeed the intersection of Σ_t and ∂M_u. In order to regard B_t as a hypersurface in Σ_t, and define its induced metric and extrinsic curvature accordingly, we need its unit normal, r_ν, which lies in Σ_t (cf. figure 3.2). This can be done by projecting the normal, u_ν, of ∂M_u to Σ_t. Define, r_ν, as

$$r_\nu = \lambda h_\nu{}^\alpha u_\alpha = \lambda(u_\nu + \eta n_\nu) \; , \tag{C.21}$$

it can be verified that $r_\nu n^\nu = 0$, $r^\nu r_\nu = 1$, $r^\nu v_\nu = -\eta$, and $r^\nu u_\nu = 1/\lambda$.

The induced metric, $\sigma_{\mu\nu}$, and the extrinsic curvature, $k_{\mu\nu}$, of B_t as a hypersurface of Σ_t are

$$\begin{aligned} \sigma_{\mu\nu} &= h_{\mu\nu} - r_\mu r_\nu = \gamma_{\mu\nu} + n_\mu n_\nu \\ &= g_{\mu\nu} - \lambda^2(n_\mu n_\nu - u_\mu u_\nu - 2\eta n_{(\mu} u_{\nu)}) \; , \end{aligned} \tag{C.22}$$

$$k_{\mu\nu} = \sigma_\mu{}^\alpha D_\alpha r^\nu = \sigma_\mu{}^\alpha h_\alpha{}^\beta h_\nu{}^\tau \nabla_\beta r_\tau \; . \tag{C.23}$$

Taking the trace of $k_{\mu\nu}$ and with the help of the definitions of K, Γ, one can derive

$$k = \lambda(\Gamma + \eta K - u_\nu a_\nu + \lambda v^\nu \nabla \eta) \; . \tag{C.24}$$

For the special case that $\eta = u^\nu n_\nu = 0$, i.e., the family of Σ_t intersect ∂M_u orthogonally, the following relation will be used;[3]

$$k_{\mu\nu} = \Gamma_{\mu\nu} + n_\mu n_\nu u_\beta a^\beta - 2n_{(\mu}\sigma_{\nu)}{}^\alpha K_{\alpha\beta}u^\beta \ . \tag{C.25}$$

C.1.4 Special case of $n^\mu u_\mu = 0$ and $N^\mu u_\mu = 0$

Consider the special case that M is foliated by a family of space-like hypersurfaces, Σ_t, with the upper and lower space-like boundaries, $\partial M_{n''} = \Sigma_{t''}$ and $\partial M_{-n'} = \Sigma_{t'}$,[4] respectively, and one time-like boundary, $\partial M_u = B$, which connects these two space-like boundaries. Furthermore, we require that B intersects the family of Σ_t orthogonally, i.e., $n^\mu u_\mu = 0$ on B, and $N^\mu u_\mu = 0$. Then, the following identity will be useful,

$$I_H \stackrel{def}{=} I_R + I_K + I_\Gamma = I_E + I_p + I_k \ , \tag{C.26}$$

where[5]

$$\begin{aligned}
I_R &= \frac{-1}{16\pi G}\int_M \mathcal{R}\epsilon \ , && \text{(C.27)}\\
I_K &= -\frac{1}{8\pi G}\int_{\Sigma_{t'}}^{\Sigma_{t''}} K n\cdot\epsilon \ , && \text{(C.28)}\\
I_\Gamma &= \frac{1}{8\pi G}\int_B \Gamma u\cdot\epsilon \ , && \text{(C.29)}\\
I_E &= \int_M (\mathbf{P}^{\mu\nu}\dot{h}_{\mu\nu} - N\mathbf{H} - N^\mu \mathbf{H}_\mu) \ , && \text{(C.30)}\\
I_p &= -2\int_B \frac{1}{N} N^\mu \sigma_{\mu\nu}\mathcal{P}^{\nu\beta}u_\beta \ u\cdot\epsilon \stackrel{def}{=} \int_B \frac{u\cdot\epsilon}{N} j_\mu N^\mu \ , && \text{(C.31)}\\
I_k &= \frac{1}{8\pi G}\int_B k \ u\cdot\epsilon \stackrel{def}{=} -\int_B \varepsilon \ u\cdot\epsilon \ . && \text{(C.32)}
\end{aligned}$$

[3]This can be derived by expanding the RHS of the identity

$$\Gamma_{\mu\nu} = \delta_\mu{}^\alpha \delta_\nu{}^\beta \Gamma_{\alpha\beta} = (h_\mu{}^\alpha - n_\mu n^\alpha)(h_\nu{}^\beta - n_\nu n^\beta)\Gamma_{\alpha\beta} \ .$$

And note that for $\eta = 0$, $\gamma_\mu{}^\alpha h_\alpha{}^\lambda = h_\mu{}^\alpha \gamma_\alpha{}^\lambda = \sigma_\mu{}^\alpha h_\alpha{}^\lambda$.

[4]Note that both n'' and n' are future-pointing.

[5]In the second line, we will use the shorthand,

$$\int_{\Sigma_{t'}}^{\Sigma_{t''}} = \int_{\Sigma_{t''}} - \int_{\Sigma_{t'}} \ .$$

C.1.5 Co-ordinate conditions

Within the ADM decomposition, the following specific results will be useful.

At first, we have the matrix identity,

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det A} \det A(i,j) \ , \tag{C.33}$$

where $A(i,j)$ is the matrix arrived by removing the i-th row and j-th column from A.

Using the above identity, for the space-like decomposition in subsection C.1.1 with the metric form (C.1), we have

$$n_\mu = (-N, 0, 0, 0) \ , \tag{C.34}$$

$$n^\mu \epsilon_{\mu 123} = \sqrt{\det h_{ij}} = \frac{\sqrt{-\det g_{\mu\nu}}}{N} \ , \tag{C.35}$$

When the shift vector $N^\mu = 0$ we have

$$\Gamma^0_{00} = \frac{\dot{N}}{N} \ , \qquad \Gamma^0_{0i} \frac{\partial_i N}{N} \ .$$

With the definition $a_\mu = n^\nu \nabla_\nu n_\mu$, we then obtain

$$a_0 = 0 \ , \qquad a_i = \frac{\partial_i N}{N} \ . \tag{C.36}$$

For the time-like decomposition in subsection C.1.2 with the metric form (C.3), we have

$$u_\mu = (0, \tilde{M}, 0, 0) \ , \tag{C.37}$$

$$u^\mu \epsilon_{\mu 032} = \sqrt{-\det \gamma_{ij}} = \frac{\sqrt{-\det g_{\mu\nu}}}{\tilde{M}} \ , \tag{C.38}$$

Furthermore, for the decomposition in equation (C.5) and the foliation in section C.1.3, we have

$$\sqrt{-\det \gamma_{ij}} = N\lambda \sqrt{\det \sigma_{ab}} \ . \tag{C.39}$$

C.2 Collection of equations

C.2.1 Simplification of Θ on a 3-hypersurface

Expanding Θ Considering the term, $\mathbf{d}\Theta$, in equation (1.23), it can be reduced to surface integral via Stoke's theorem when it is evaluated on M with boundary ∂M. On a particular piece of boundary, ∂M_n, the integrand is (cf. equation (1.27))

$$\begin{aligned}\Theta_g &= \theta_g \cdot \epsilon = -\tau n_\alpha \theta^\alpha n \cdot \epsilon \\ &= \frac{\tau}{16\pi G}\left[n^\lambda \nabla_\lambda (g^{\alpha\beta}\delta g_{\alpha\beta}) - n^\alpha \nabla^\beta \delta g_{\alpha\beta}\right] n \cdot \epsilon \,.\end{aligned} \tag{C.40}$$

The first term in the square brackets on the second line can be rewritten as

$$n^\lambda \nabla_\lambda (g^{\alpha\beta}\delta g_{\alpha\beta}) = -n^\alpha \nabla_\alpha (h_{\mu\nu}\delta h^{\mu\nu}) + 2\tau n^\alpha \nabla_\alpha (n_\nu \delta n^\nu) \,. \tag{C.41}$$

Similarly, the second term is

$$n^\alpha \nabla^\beta \delta g_{\alpha\beta} = K_{\alpha\beta}\delta h^{\alpha\beta} + \tau n^\alpha \nabla_\alpha (n_\beta \delta n^\beta) + \tau K n_\alpha \delta n^\alpha - \tau a_\alpha \delta n^\alpha - \nabla_\beta \delta n^\beta \,. \tag{C.42}$$

In above equations, the symbols $h_{\mu\nu}$, $K_{\mu\nu}$, and a_ν have been defined in previous section.

Combining equations (C.40)-(C.42) and after some algebra, we have

$$\begin{aligned}\Theta_g &= -\tau \mathbf{P}_{\alpha\beta}\delta h^{\alpha\beta} \\ &+ \frac{\tau n \cdot \epsilon}{16\pi G}\left[\nabla_\beta (h^\beta{}_\alpha \delta n^\alpha) - \nabla_\beta (n^\beta h_{\alpha\nu}\delta h^{\alpha\nu}) + \tau a_\alpha \delta n^\alpha - 2\tau K n_\alpha \delta n^\alpha\right] \,,\end{aligned} \tag{C.43}$$

where $\mathbf{P}_{\alpha\beta}$ is defined in equation (C.12).

Expanding $\delta(Kn \cdot \epsilon)$ The $\delta(Kn \cdot \epsilon)$ can be written as the sum of two terms,

$$\delta(K n^\mu \epsilon) = \delta K n \cdot \epsilon + K \delta(n \cdot \epsilon) \,. \tag{C.44}$$

Using the definition of $K_{\mu\nu}$ in equation (C.8), one can derive the following expression for δK,

$$\delta(K_{\mu\nu}h^{\mu\nu}) = \nabla_\nu (h^\nu{}_\beta \delta n^\beta - \frac{1}{2} n^\nu h_{\beta\mu}\delta h^{\beta\mu}) - \tau K n_\alpha \delta n^\alpha + \frac{1}{2} K h_{\alpha\beta}\delta h^{\alpha\beta} \,. \tag{C.45}$$

Note that in deriving above equation, the following gauging fixing condition has been used,

$$n_\alpha \delta h^{\alpha\beta} = 0 \ . \tag{C.46}$$

This constraint is necessary in order for the perturbed metric preserving the form of ADM decomposition (cf. equation (C.1)). Consequently, we have $h^{\alpha\beta}\delta n_\beta = 0$ because of the normalisation condition that $h^{\alpha\beta} n_\beta = 0$. The second term give us

$$K\delta(n\cdot\boldsymbol{\epsilon}) = \left[-\tau K n_\beta \delta n^\beta - \frac{1}{2}K(h_{\mu\beta}\delta h^{\mu\beta} - 2\tau n_\beta \delta n^\beta)\right] n\cdot\boldsymbol{\epsilon} \ . \tag{C.47}$$

Overall, we arrive at

$$\delta(Kn\cdot\boldsymbol{\epsilon}) = \left[\nabla_\nu(h^\nu{}_\beta \delta n^\beta - \frac{1}{2}n^\nu h_{\beta\mu}\delta h^{\beta\mu}) - \tau K n_\mu \delta n^\mu\right] n\cdot\boldsymbol{\epsilon} \ . \tag{C.48}$$

Expanding $\mathbf{d}[\delta n\cdot(n\cdot\boldsymbol{\epsilon})]$ At first, expand $\mathbf{d}[\delta n\cdot(n\cdot\boldsymbol{\epsilon})]$ according to the definition of exterior derivative,

$$\mathbf{d}\left[\delta n\cdot(n\cdot\boldsymbol{\epsilon})\right] = \sum_{b<c<d} 3\left[\nabla_{[b}(\delta^\beta n^\alpha)\epsilon_{cd]\alpha\beta}\right] dx^b \wedge dx^c \wedge dx^d \ . \tag{C.49}$$

On the other hand, because there is only one algebraically independent 3-form on a 3-hypersurface with outward-pointing unit normal n^μ, we can write $\mathbf{d}[\delta n\cdot(n\cdot\boldsymbol{\epsilon})] = f n\cdot\boldsymbol{\epsilon}$. With the help of equation (B.1), f can be determined. So we have

$$\mathbf{d}\left[\delta n\cdot(n\cdot\boldsymbol{\epsilon})\right] = \left[\nabla_\beta \delta n^\beta + \tau n^\beta \nabla_\beta(n_\mu \delta n^\mu) - \tau a_\mu \delta n^\mu + \tau K n_\mu \delta n^\mu)\right] n\cdot\boldsymbol{\epsilon} \ . \tag{C.50}$$

The result From equation (C.48) and (C.50), we found that

$$\begin{aligned} &2\delta(Kn\cdot\boldsymbol{\epsilon}) - \mathbf{d}\left[\delta n\cdot(n\cdot\boldsymbol{\epsilon})\right] \\ &\quad = \left[\nabla_\beta(h^\beta{}_\alpha \delta n^\alpha) - \nabla_\beta(n^\beta h_{\alpha\nu}\delta h^{\alpha\nu}) + \tau a_\alpha \delta n^\alpha - 2\tau K n_\alpha \delta n^\alpha\right] n\cdot\boldsymbol{\epsilon} \ . \end{aligned} \tag{C.51}$$

From equation (C.51) and (C.43), we arrive at

$$\boldsymbol{\Theta}_g = -\tau \mathbf{P}_{\alpha\beta}\delta h^{\alpha\beta} + \frac{\tau}{8\pi G}\delta(Kn\cdot\boldsymbol{\epsilon}) - \frac{\tau}{16\pi G}\mathbf{d}\left[\delta n\cdot(n\cdot\boldsymbol{\epsilon})\right] \ . \tag{C.52}$$

C.2.2 Simplification of $\delta(\alpha r \cdot (n \cdot \epsilon))$ on a 2-surface

Consider a 2 dimensional surface which is the intersection of two 3-hypersurfaces with outward-pointing unit normals, time-like n^μ and space-like u^μ, respectively. Define r^μ and α according to equations (C.21) and (C.18). Expanding $\delta(\alpha r \cdot (n \cdot \epsilon))$, we arrive at four terms,

$$\begin{aligned} \delta\left[\alpha r \cdot (n \cdot \epsilon)\right] = \delta\alpha r \cdot (n \cdot \epsilon) + \alpha r \cdot (n \cdot \delta\epsilon) \\ +\alpha\delta r \cdot (n \cdot \epsilon) + \alpha r \cdot (\delta n \cdot \epsilon) \ . \end{aligned} \tag{C.53}$$

It is easily see that $\delta\alpha = \lambda\delta\eta$ and $\delta\epsilon = \frac{-1}{2} g_{\mu\nu}\delta g_{\mu\nu}\epsilon$. With the help of the equation (B.1), one can check that

$$\delta r \cdot (n \cdot \epsilon) = r_\nu \delta r^\nu r \cdot (n \cdot \epsilon) \ , \tag{C.54}$$

$$r \cdot (\delta n \cdot \epsilon) = -n_\nu \delta n^\nu r \cdot (n \cdot \epsilon) \ . \tag{C.55}$$

Note that r^ν and n^ν are space- and time-like, respectively. Using the definition of r^ν (cf. equation (C.21)), one can check

$$r_\nu \delta r^\nu - n_\nu \delta n^\nu = \lambda^2 (u_\nu \delta u^\nu - n_\nu \delta n^\nu) \ . \tag{C.56}$$

Combine equations (C.53)-(C.56), we arrive at

$$\delta\left[\alpha r \cdot (n \cdot \epsilon)\right] = \left[\delta\alpha + \alpha\lambda^2 (u_\nu \delta u^\nu - n_\nu \delta n^\nu) - \frac{\alpha}{2} g_{\mu\nu}\delta g^{\mu\nu}\right] r \cdot (n \cdot \epsilon) \ . \tag{C.57}$$

Using the definition of σ (cf. equation (C.22)), one can derive

$$g_{\mu\nu}\delta g^{\mu\nu} = \frac{2\delta\lambda}{\lambda} + 2\eta\lambda^2\delta\eta + 2\lambda^2 u_\nu \delta u^\nu + 2(\eta^2\lambda^2 - 1) n_\nu \delta n^\nu + \sigma_{\mu\nu}\delta\sigma^{\mu\nu} \ . \tag{C.58}$$

Note that we have used the identity $r_\mu r_\nu \delta\sigma^{\mu\nu} = 0$ in deriving the above equation. This is a consequence of $n_\mu n_\nu \delta\sigma^{\mu\nu} = u_\mu u_\nu \delta\sigma^{\mu\nu} = n_\mu u_\nu \delta\sigma^{\mu\nu} = 0$. They can be further traced back to the gauging fixing conditions[6]

$$n_\nu \delta h^{\nu\mu} = 0 = u_\nu \delta\gamma^{\nu\mu} \ . \tag{C.59}$$

With the help of equation (C.58), as expected, equation (C.57) turns to

$$\delta(\alpha r \cdot (n \cdot \epsilon)) = (\delta\alpha - \frac{\alpha}{2}\sigma_{\mu\nu}\delta\sigma^{\mu\nu}) r \cdot (n \cdot \epsilon) \ . \tag{C.60}$$

[6]The origin of the condition, $u_\nu \delta\gamma^{\nu\mu} = 0$, is the same to $n_\nu \delta h^{\nu\mu} = 0$. See equation (C.46).

C.2.3 Simplification of $\delta n \cdot (n \cdot \epsilon) - \delta u \cdot (u \cdot \epsilon)$ on a 2-surface

Again, we consider a 2 dimensional surface as described in the beginning of previous subsection. At first, rewrite $\delta n \cdot (n \cdot \epsilon)$ and $\delta u \cdot (u \cdot \epsilon)$ in terms of $r \cdot (n \cdot \epsilon)$, we have

$$\delta n \cdot (n \cdot \epsilon) = r_\nu \delta n^\nu r \cdot (n \cdot \epsilon) \ , \tag{C.61}$$

$$\delta u \cdot (u \cdot \epsilon) = -v_\nu \delta u^\nu r \cdot (n \cdot \epsilon) \ . \tag{C.62}$$

With the help of the identities,[7]

$$n_\nu \delta u^\nu = \delta\eta - \eta n_\nu \delta n^\nu \ , \tag{C.63}$$

$$u_\nu \delta n^\nu = \delta\eta + \eta u_\nu \delta u^\nu \ , \tag{C.64}$$

we have

$$r_\nu \delta n^\nu + v_\nu \delta u^\nu = 2\delta\alpha \ . \tag{C.65}$$

Finally, from equation (C.65), it is seen that equation (C.61) minus equation (C.62) gives us

$$\delta n \cdot (n \cdot \epsilon) - \delta u \cdot (u \cdot \epsilon) = 2\delta\alpha r \cdot (n \cdot \epsilon) \ . \tag{C.66}$$

C.2.4 Simplification of $\pi_{\mu\nu}\delta\gamma^{\mu\nu}$ as $n_\mu u^\mu = \eta = 0$

On a time-like hypersurface, B, using the definition, $\gamma^{\mu\nu} = \sigma^{\mu\nu} - n^\mu n^\nu$ (cf. equation (C.22)),[8] and with the help of the identities, $n_\nu \sigma^{\nu\mu} = u_\nu \sigma^{\nu\mu} = 0$, we have

$$\pi_{\mu\nu}\delta\gamma^{\mu\nu} = \pi_{\mu\nu}\delta\sigma^{\mu\nu} - 2\pi_{\mu\nu} n^\mu \delta n^\nu \ , \tag{C.67}$$

where $\pi_{\mu\nu} = \frac{1}{16\pi G}(\Gamma_{\mu\nu} - \Gamma\gamma_{\mu\nu})$. Recalling the definition of $\Gamma_{\mu\nu}$ in equations (C.17), and with the help of equations (C.24), (C.25), and the gauge conditions, equation (C.59), one can rewrite the first term in the above equation as follows,[9]

$$\pi_{\mu\nu}\delta\sigma^{\mu\nu} = \frac{1}{16\pi G}\left[k_{\mu\nu} - (k + u_\alpha a^\alpha)\sigma_{\mu\nu}\right]\delta\sigma^{\mu\nu} \ . \tag{C.68}$$

---

[7] From the relations, $\delta u^\nu = u_\mu \delta g^{\mu\nu} + g^{\mu\nu}\delta u_\mu$, $\delta g^{\mu\nu} = \delta h^{\mu\nu} - \delta n^\mu n^\nu - n^\mu \delta n^\nu$, and $\delta\eta = \delta n^\nu u_\nu + n^\nu \delta u_\nu$, the first one follows. The second one can be derived similarly.

[8] Note that because $\eta = 0$, $v^\mu = n^\mu$ and $r^\mu = u^\mu$.

[9] The identity, $n_\nu \delta\sigma^{\nu\mu} = -u^\mu n_\nu \delta u^\nu = 0$, as $\eta = 0$, will be used. (cf. equation (C.63))

The second term can be transformed to

$$-2\mathcal{T}_{\mu\nu}n^{\mu}\delta n^{\nu} = \frac{-1}{8\pi G}(-kn_{\mu} - \sigma_{\mu\alpha}K^{\alpha\beta}u_{\beta})\delta n^{\mu} \ . \tag{C.69}$$

Combining equations (C.69) and (C.68), we can rewrite equation (C.67) as

$$\mathcal{T}_{\mu\nu}\delta\gamma^{\mu\nu} = -\frac{1}{2}s_{\mu\nu}\delta\sigma^{\mu\nu} - \varepsilon n_{\mu}\delta n^{\mu} - j_{\mu}\delta n^{\mu} \ , \tag{C.70}$$

where

$$\varepsilon = \frac{-k}{8\pi G} \ , \tag{C.71}$$

$$j_{\mu} = -2\sigma_{\mu\alpha}\mathcal{P}^{\alpha\beta}u_{\beta} \ , \tag{C.72}$$

$$s_{\mu\nu} = \frac{-1}{8\pi G}\left[k_{\mu\nu} - (k + u_{\alpha}a^{\alpha})\sigma_{\mu\nu}\right] \ . \tag{C.73}$$

When the metric is ADM-decomposed (cf. equation (C.1)), we have[10]

$$n_{\mu}\delta n^{\mu} = \frac{\delta N}{N} \ , \tag{C.74}$$

$$\sigma_{\mu\alpha}\delta n^{\mu} = \frac{1}{N}\sigma_{\mu\alpha}\delta(Nn^{\mu}) = -\frac{1}{N}\sigma_{\mu\alpha}\delta N^{\mu} \ . \tag{C.75}$$

We can then rewrite equation (C.70) as follows,

$$\mathcal{T}_{\mu\nu}\delta\gamma^{\mu\nu} = \frac{1}{N}(-\frac{N}{2}s_{\mu\nu}\delta\sigma^{\mu\nu} - \varepsilon\delta N + j_{\mu}\delta N^{\mu}) \ . \tag{C.76}$$

[10]The first equation is a straightforward result of equations (C.1) and (C.34). To derive the second one, use definition (C.2) and note that $\sigma_{\mu\alpha}n^{\mu} = 0$ and $\delta t^{\mu} = 0$.

Chapter 4

Quantum Material Models of Spherically Symmetric Black Holes

It is unclear to me why most of the attempts to explain the statistical origin of black hole entropy are done at the outside or surface of a black hole. Presumably, it is because of a singularity sitting inside a black hole and the ambiguous status of time there. However, intuitively, I feel it is more natural to attribute the origin of the black hole entropy to the matter residing inside a black hole. I have assumed that *there exists matter inside a black hole.* I believe it is incontrovertible to assume that no one on earth has ever gone into a black hole, this is thus an open question. Personally, I do not feel like to descend into a black hole because, though life has not always been easy on the earth, I still have a lot of the unforgettable. Nonetheless, imagination has no border. A painter realises her/his visions with colours. A poet realises her/his dreams with pens. A physicist realises her/his imagination with models. I therefore try to explore the interior of a black hole, staying outside.

Because my attempt is to give an *outsider's* statistical explanation of black hole entropy from *inside*, I then, out of ignorance at this moment as explained in section 3.3, chapter 3, extend to the interior of a black hole the applicability of the concepts of statistical mechanics in Lorentzian space-time.[1] There, a statistical explanation of

[1]Newtonian space-time is perhaps more accurate because, in my knowledge, there is no fully developed relativistic statistical mechanics.

black hole entropy has been given and the spectrum of eigen-states is chosen as $E_j = \kappa j$. Besides the spectrum, another important ingredient of the model is the corresponding eigen-states, which should be obtained from some Schrödinger-like equation. The main theme of this chapter is therefore to justify the chosen spectrum and to identify the corresponding eigen-states.

Abstract of chapter 4

Section 4.1 In subsection 4.1.1, I will foremost explain in more detail the motivations behind the construction of black hole models: the lesson from atomic physics and the analogy from blackbody radiation. Subsection 4.1.2 is then devoted to the prescription of the Schrödinger-like equation—the quantum-mechanical, static Einstein field equation—which will be imposed on the matter wave function and the associated metric in order to construct the matter-metric energy eigen-states inside a black hole.

Section 4.2 The models for Schwarzschild black hole are presented. In subsection 4.2.1, I shall explain at first why the model is constructed on space-times with Kleinian signature $(- - ++)$. I will then be able to justify the spectrum $E_j = \kappa j$. Afterwards, two models based on different parametrisations of the metric are given in the next two subsections, 4.2.2 and 4.2.3, respectively. We will therefore see that, within my approach, all Schwarzschild black holes can be classified by a positive integer, N, the analogue of atomic number.

Section 4.3 A model for charged, non-rotating black holes is then explained. Due to the appearance of the extra energy scale, the coupling constant, e, of $U(1)$ gauge field, the whole situation is complicated. However, following the naïve assumption given in section 3.3.3, I can normalise the wave function accordingly.

Section 4.4 In the final section, I will review the whole approach from various points of view. Several important questions are raised which should be addressed in order to provide more complete pictures of black hole models and the statistical origin of black hole entropy.

4.1 Motivation and prescription

The innermost motivation is to complete my PhD. The ultimate prescription is, don't do it at all.

4.1.1 Motivation for constructing black hole models

Analogy has always been an important feature in the progress of physics [51]. This can be seen clearly from the development of atomic, nuclear, and later, particle physics. Another example is the simulation of QCD from QED. It seems that such analogies have had no great impact in the field of gravity up to now. One of the reasons is perhaps that the underlying principles governing the dynamics of gravity differ from other physical fields. This is indeed also the reason why gravity is still excluded from the unification scheme which works so well for other three forces.

However, this does not rule out the possibility of borrowing some ideas from other fields to tackle problems involving gravity. The spontaneous symmetry breaking, which is employed in standard model for the generation of Higgs particle, is at first developed in condensed matter physics [38]. On the other hand, the renormalisation group approach, which has had very successful applications in condensed matter physics, is foremost formulated in particle physics [75]. One of the essentials of making progress is to let the ideas flow. My approach to the models of black holes has in fact evolved under such a strategy and spirit.

Lesson from atomic physics

Let us start with the singularity theorem of classical general relativity. The singularity theorem states that once a trapped surface forms, a singularity is inevitable [33]. If classical general relativity is still applicable beyond the event horizon, matter will simply fall into the singularity without any sigh. Since a self-consistent quantum gravity is still beyond our scope, it is unclear how the fortune of matter falling into a black hole will be changed.

Nevertheless, it is not too naïve to expect that the quantum effect could turn such a catastrophe into a new chance of surviving—much like what happens inside an atom: If one treats an electron around a proton classically, then the catastrophe of the electron falling into the proton is inevitable. The quantum effect rescues the electrons, and us, from the death penalty of being annihilated. Encouraged by the lesson from atomic physics, we think it makes sense to ask this question: How could this also happen inside a black hole so that the quantum effect can prevent the matter from falling into the singularity?

Analogy from thermodynamics

On the other hand, the idea of black hole entropy and the phenomenon of Hawking radiation seem to suggest that there are great similarities, at least phenomenologically, between a black hole and a blackbody. Let us look at the statistical explanation of black hole entropy formulated in section 3.3, chapter 3 from another point of view. At a deeper level, i.e., if one see things from the point of view of QED, the interaction between matter which forms the cavity and the radiation is much more complicated than the simple picture of Planck's quantised radiation can provide. However, as far as the statistical, hence the thermodynamic, properties of the blackbody radiation are concerned, the simple quantisation rule suffices to account for them. This is one of the most important lessons I will elaborate in this chapter.

Historically, Planck proposed quantised energy to explain the observed spectra of blackbody radiation far before people understood QED. It seems that this is the first instance that the idea of quantised radiation field appeared in physical history. And, in fact, it is the phenomenological understanding of blackbody radiation which in part induced the understanding of quantum mechanics. We therefore ask: How could a similar quantised energy spectrum arise in the interior of a black hole which will then provide a basis for the explanation of thermal properties of a black hole?

Entropy as a low energy quantum phenomenon

There are various claims that the black hole entropy could only be properly explained once we have a quantum gravity[2]. Since no one has yet come up with a properly understood and universally accepted quantum gravity, it is not clear to me what a quantum gravity should look like. However, I suppose that it is appropriate to consider the following analogy:

$$\begin{array}{rcl} \text{Classical Electrodynamics} & — & \text{Classical General Relativity} \\ \\ \text{Quantum Electrodynamics} & — & \text{Quantum Gravity}\ . \end{array}$$

If the so-called quantum gravity corresponds to QED, then I think that we need quantum gravity to explain black hole entropy no more than we need QED to explain the entropy of black-body radiation. Admittedly, it also depends on what one would like to regard as an explanation.

With the lesson from atomic physics and the analogy from thermodynamics in mind, my opinion regarding to the relation between black hole entropy and the quantum gravity is thus more conservative, and perhaps also more simple-minded: By accepting the *phenomenon* of black hole entropy, a kind of quantum theory of gravity is needed in order to calculate the quantised spectrum and the corresponding states. However, quantum gravity as a kind of QED in above analogy is too high-brow for this purpose. Though, quantum gravity is inevitable to explain how these states can exist at first instance, just like we need QED to explain why the matter constituting the cavity radiates.

Nonetheless, if history has another say about the development of a physical theory, then the lesson of the development from classical mechanics to quantum mechanics, finally arriving at quantum field theory seems to indicate that we need a kind of quantum mechanics which incorporates the effect of gravity in an explicit manner, which will eventually form the basis for the understanding of quantum field theory of gravity. And, to me, the difference between quantum mechanics and quantum field theory is

[2] I only provide one reference [61] as a representative.

the energy scale involved. Though the concept of energy is ambiguous when gravity is involved. I think it is appropriate to regard the energy involved in the collision of two black holes, or the big bang, is larger than that of the gravitational collapse of a star. If such an energy hierarchy can be established, I think it is reasonable to assume that there is also a hierarchy in the corresponding theories.

Material goal

Therefore, my goal is to construct the eigen-states inside a black hole and the corresponding eigen-energies. As explained in section 1.1, chapter 1, our present understanding of black hole formation are strongly related to the gravitational collapse. Though, due to both conceptual and technical difficulties, I will work on an eternal, i.e., strictly static black hole, I believe that a black hole is not just a kind of gravitational field devoid of matter content. I thus feel, intuitively,[3] that those eigen-states should be associated with matter which forms the black hole, however they get there at the first instance. Moreover, for static black hole, all matter is confined within it, those eigen-states should therefore be bound states within the black hole. I am thus motivated to construct a material model based on an appropriate quantum theory.

4.1.2 Quantum-mechanical prescription of static Einstein field equation

Semi-classical prescription

In quantum field theory in curved space-times [8], the semi-classical Einstein field equation is give by $\mathcal{R}_{\mu\nu} - 1/2 g_{\mu\nu}\mathcal{R} = -8\pi G\langle\widehat{\mathcal{T}}_{\mu\nu}\rangle$ in which $\mathcal{R}_{\mu\nu}$ is the Ricci tensor, $\widehat{\mathcal{T}}_{\mu\nu}$ the energy-momentum tensor, and G the gravitational constant. In words, the geometry is determined by the expectation value of energy-momentum tensor with respect to a certain state. In conventional approach the expectation value is taken with respect to the vacuum state, one then constructs an effective action and renormalises the energy-momentum tensor. In such an approach, the background space-time has to be

[3]Since it is a kind of intuition, I can only feel about it, instead of thinking.

prescribed beforehand in order to construct the eigen-states of Hamiltonian. Then the field variable can be expanded in terms of these eigen-states. If one wishes to consider the influences of the energy-momentum on the background space-time, one has to resort to considering back-reaction.

However, I think one of the most important features of the Einstein field equation, which should be respected, is that it has to be solved self-consistently. In the classical level, it means that the Einstein field equation and the Euler-Lagrange equations for the matter which produce the relevant energy-momenta should be used together to determine the geometry and the matter distribution once and for all. This is well-known to be hard to implement.

Quantum-mechanical prescription

Furthermore, at quantum-mechanical level, my opinion is that such an attitude should also be preserved so that a matter eigen-state and the associated space-time geometry should be solved together. Different matter eigen-states will induce different space-time geometry according to the Einstein field equation. In other words, a matter eigen-state and the associated space-time geometry together should be regarded as a matter-metric eigen-state. More precisely, the quantum-mechanical, static Einstein field equation can be written as $\mathcal{R}_{\mu\nu} - 1/2 g_{\mu\nu}\mathcal{R} = -8\pi G\langle j|{:}\widehat{\mathcal{T}}_{\mu\nu}{:}|j\rangle$ in which $|j\rangle$ is the j-th eigen-states and $:(\ ):$ denotes the normal ordering as one used in quantum field theory. Note that the difference between the prescription in the conventional quantum field theory in curved space-time and ours rests principally in the interpretation. However, it is indeed such an interpretative jump that leads a *classical* Schrödinger equation to a *quantum-mechanical* one.

The above quantum-mechanical prescription is understandably irrelevant if the system of interest is an astronomical planet system, such as the sun and the earth. Nevertheless, when the system concerned is a black hole, we think it is important to proceed quantum-mechanically as one is required to treat an electron in an atom quantum-mechanically.

Application to spherically symmetric system

Equations Our prescription is not yet precise enough for us to do anything. In the rest of this section, we will confine ourselves to a specific system to see how to implement this prescription practically.

We will consider a spherically symmetric, self-gravitating system of real scalar field. The Euler-Lagrange equation and the (classical) Einstein field equation can be written as

$$\phi'' + \phi'(\frac{1}{r} + \frac{q'}{q} + \frac{h'}{h}) + \frac{r^2}{q^2h^2}\partial_t^2\phi = 0 \ , \tag{4.1}$$

$$\frac{h'}{h} + 8\pi G\left(\frac{1}{2}\frac{r^3}{q^2h^2}(\partial_t\phi)^2 - \frac{r}{2}\phi'^2\right) = 0 \ , \tag{4.2}$$

$$\frac{h'}{h} - \frac{1}{q}(1-q') = 0 \ , \tag{4.3}$$

$$\frac{h''}{h} + \frac{h'}{h}(\frac{3}{2}\frac{q'}{q} - \frac{1}{2r}) + \frac{1}{2}\frac{q''}{q} + \frac{1-q'}{rq} + 8\pi G\frac{r^2}{q^2h^2}(\partial_t\phi)^2 = 0 \ , \tag{4.4}$$

where a prime denotes the differentiation with respect to r. Note that we do not list the $t-r$ compenent of the Einstein equation, which is exactly zero in the present static case according our prescription (see later). These equations are derived from the action

$$I = \frac{1}{2}\int d^4x\sqrt{|g|}(-\frac{\mathcal{R}}{8\pi G} - \nabla_\mu\phi\cdot\nabla^\mu\phi) \ ,$$

and we have written the metric in the standard form [74],

$$ds^2 = h^2\frac{q}{r}dt^2 + \frac{r}{q}dr^2 + r^2d\Omega^2 \ . \tag{4.5}$$

We have a spherically symmetric system in mind, so there is no angular dependence in our equations. Since we are giving a quantum-mechanical prescription of *static* Einstein field equation, the metric is t-independent. There are in fact only three independent equations due to the Bianchi identity. Note that up to now, everything is classical. To implement the quantum-mechanical prescription, we will dress a $\widehat{\text{hat}}$ to the field variable ϕ which will be realised as operator-valued henceforth.

Probability density function Before we implement the quantum-mechanical prescription, we have to introduce another important concept in a quantum theory: the probability density function (square of the moduli of wave functions). We expand $\hat{\phi}$ as (with $E_j > 0$)

$$
\begin{aligned}
\hat{\phi} &= \hat{\varphi} + \hat{\varphi}^\dagger \ , \\
\hat{\varphi} &= \sum_j \hat{\varphi}_j = \sum_j \hat{a}_j e^{-iE_j t} R_j(r) \ , \\
\hat{\varphi}^\dagger &= \sum_j \hat{\varphi}_j^\dagger = \sum_j \hat{a}_j^\dagger e^{iE_j t} R_j(r) \ ,
\end{aligned}
\tag{4.6}
$$

where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators for j-th eigen-state $|j\rangle$ such that $[\hat{a}_j, \hat{a}_i^\dagger] = \delta_{ji}$.[4] We can then regard $\hat{J}^t = -ig^{tt}(\hat{\varphi}^\dagger \cdot \partial_t \hat{\varphi} - \partial_t \hat{\varphi}^\dagger \cdot \hat{\varphi})$ as the probability density operator so that the function, $\langle j|\hat{J}^t|j\rangle$, will be identified as the probability density function of matter in a matter-metric eigen-state. The normalisation condition is

$$
\int dr d\Omega \sqrt{|g|} \langle j|\hat{J}^t|j\rangle = -\int dr d\Omega \sqrt{|g_s g^{tt}|} \langle j|\hat{J}_t|j\rangle = N_j \ , \tag{4.7}
$$

where N_j is the normalisation of the matter wave function and g_s is the spatial part of the metric in the j-th matter-metric eigen-state. This definition is a generalisation of number density operator in the quantum field theory in flat space-time [8].

Operational interpretation We would like to remind the reader again that in the present static case our approach is a quantum-mechanical one. The role of $\hat{\phi}$ is thus more like a superposition of wave functions in quantum mechanics, rather than a field operator in quantum field theory. The terms $exp(-iE_j t)$ and $exp(iE_j t)$ are thus like

[4]The main purpose of introducing operators is to project the field operator to a single eigen-state. It is possible to do without operators if one writes down the Einstein field equation with the energy-momentum tensor resulting from a complex scalar field. One then regards the complex scalar field as the wave function of an eigen-state. One may wonder how to go beyond the static case. I intend to understand the dynamics from the viewpoint of path integral. The most significant quantities in the dynamic case are probably the transition amplitudes between initial and final states. The dynamics responsible for a transition is decreed by the propagator. The initial and final states are therefore the energy eigen-states. The path integral approach to gravity has been explored in the paradigm of *Quantum Cosmology*. See anthology [17] and references therein.

the t-dependent phase terms of an energy eigen-state in quantum mechanics. The wave function in quantum mechanics is realised at operational level, i.e., only the probability density can be associated with experimental outcomes. We also give the probability density function, $\langle j|\hat{J}^t|j\rangle$, such an operational meaning.

It is therefore inappropriate to interpret the $\hat{\varphi}_j$ as propagating on the corresponding space-time geometry, which is solved self-consistently using the quantum-mechanical static Einstein field equation, because the underlying principle of our approach is: There is no prescribed space-time background. The Einstein field equation has to be dealt with in its full value.

Consequently, a question like initial-value problem for the field ϕ has to be asked carefully if one would like to adopt the above attitude; one should not try to formulate such a problem in a background space-time, even though that space-time geometry is the solution in a particular matter-metric eigen-state.

A challenging question is: How to interpret the static space-time geometry in a particular matter-metric eigen-state? I interpret it in this manner: Theoretically, it is calculated according the quantum-mechanical, static Einstein field equation. Experimentally, it is the space-time geometry a particle (i.e., the matter in a matter-metric eigen-state) is experiencing while it is being measured. We need sufficient amount of measurement outcomes to draw a fairly good picture of the probability density function. Similarly, we also need the same amount of measurements to build up the structure of the metric. A single measurement will not tell us anything about the probability density function and the metric.

Implementing We can now implement the quantum-mechanical prescription by replacing ϕ in equations (4.1)–(4.4) with $\hat{\phi}$ expanded as in equation (4.6). With the new variables $W = 8\pi G R_j^2 \langle j|: \hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger :|j\rangle$, $x = E_j r$ and $\bar{f}$, they can be rewritten as follows after taking the expectation value of $\widehat{T}_{\mu\nu}$ with respect to $|j\rangle$,

$$\frac{\ddot{W}}{2} + \frac{\dot{W}}{2}\left(\frac{\dot{\bar{f}}}{\bar{f}} + \frac{1}{x}\right) - \frac{x^2}{\bar{f}^2}W - \frac{1}{4}\frac{\dot{W}^2}{W} = 0 \ , \tag{4.8}$$

$$\frac{\dot{h}}{h} + \frac{x^3}{2\bar{f}^2}W - \frac{x}{8}\frac{\dot{W}^2}{W} = 0 \,, \tag{4.9}$$

$$\frac{\dot{\bar{f}}}{\bar{f}} - \frac{1}{\bar{q}} = 0 \,, \tag{4.10}$$

$$\ddot{\bar{f}} + \dot{\bar{f}}\left(\frac{x^3}{2\bar{f}^2}W - \frac{x}{8}\frac{\dot{W}^2}{W}\right) = 0 \,, \tag{4.11}$$

where

$$\bar{f} = \bar{q}h = E_j q h = E_j f \,,$$

and a dot denotes differentiation with respect to x.

Normal ordering Before we turn to next section, where we will solve the matter-metric eigen-states inside a black hole using the prescription just given, a remark is in order for the operation of normal ordering. We will interpret this operation in a line similar to that in quantum field theory. Therefore, we are only interested in the difference of energies, though the definition of energy has always been an intriguing issue when the theory of general relativity is involved. It is unclear to us if it is possible to formulate a quantum theory without ever mentioning things like Hamiltonian or energy. (Even though this can be achieved theoretically, it is unclear if experimenters will be happy with it.) Our present understanding, experimentally and theoretically, about the universe depends on the concept of energy so much that we will try to conform ourselves with this fact at this moment.

4.2 Models of static Schwarzschild black holes

Nothing could be darker than a human's mind; nor could anything brighter than a human's intelligence: It is sad to learn that Schwarzschild died in a battle field.

4.2.1 The Kleinian signature $(--++)$

Bound states

In order to motivate our approach, we at first consider the conventional quantum field theory on the background of the interior of a Schwarzschild black hole. We write the background metric in the standard form as in equation (4.5) and decompose an eigenstate of the massless real scalar field as (with $E_j > 0$)

$$\phi_j = e^{-iE_j t} R_j(r) + c.c \,, \tag{4.12}$$

like that in equation (4.6). The Euler-Lagrange equation (4.1) is then reduced to

$$R'' + R'(\frac{1}{r} + \frac{q'}{q} + \frac{h'}{h}) = \frac{E^2 r^2}{q^2 h^2} R \,. \tag{4.13}$$

It can be easily checked that $q = r - r_s$ ($r_s = 2GM$ in which M is the mass of the black hole) and $h = constant$ is a solution of the vacuum Einstein field equation. The choice of $h = i$ then corresponds to the well-known Schwarzschild solution. From equation (4.13) we clearly see that if we choose $h = i$, then the wave function oscillates as $(r_s - r)^{\pm i r_s E} + c.c.$ as r tends to r_s from inside. However, if we choose $h = 1$ (the signature of space-time is therefore $(--++)$), then we could have bound states within the region $0 < r < r_s$ which asymptotically behave as $(r_s - r)^{+r_s E}$ as r tends to r_s. Though these wave functions then diverge logarithmically near the origin $r = 0$, they are normalisable. But, how to identify the spectrum?

The spectrum

Similar to the Euclidean Schwarzschild solution [22], the Kleinian solution (i.e., with signature $(--++)$) can also be derived from the Lorentzian one through analytic continuation. By introducing Kruskal co-ordinates, we can write the metric of the Lorentzian Schwarzschild solution as (see equation (D.4))

$$ds^2 = \frac{1}{\kappa^2}\frac{r_s}{r}e^{-2\kappa r}(-dT^2 + dX^2) + r^2 d\Omega \ ,$$

where t and r are related to T and X by relations (see equations (D.8) and (D.9))

$$\begin{aligned} \tanh(\kappa t) &= \frac{T}{X} \ , \\ (\frac{r}{r_s} - 1)e^{2\kappa r} &= X^2 - T^2 . \end{aligned}$$

If we analytically continue T to $-iT$ and t to $-it$, we then arrive at the Euclidean Schwarzschild solution with the constraint $r \geq r_s$. However, instead of T, we can analytically continue X to iX. Combined with continuing t to $-it$, we arrive at the Kleinian Schwarzschild solution which is confined within the interior of the black hole with the restriction that $-1 < -(T^2 + X^2) < 0$ in which the end points -1 and 0 corresponding to $r = 0$ and $r = r_s$, respectively. As in the Euclidean solution, t is also required to be periodic with period $\beta = 2\pi/\kappa$ in the Kleinian case to avoid conical singularity. Back to equation (4.12), we are thus constrained to choose the spectrum as

$$E_j = \frac{j}{2r_s} = \kappa j \ , \qquad j = 1, 2, 3, \ldots \ , \tag{4.14}$$

where κ $(= 1/2r_s)$ is the surface gravity so that $E_j\beta = 2\pi j$. Note that this spectrum is the one we used in section 3.3, chapter 3 to calculate the statistical entropy of a black hole.

The problem of two times

Based on the above observations, I thus propose that we should understand the statistical (thermodynamic) properties of a black hole from inside using Kleinian signature.

This definitely raises alarm questioning how we cope with the two time-like co-ordinates in a Kleinian space-time; even more, we compactified one of them so that it is periodic. A classical particle whose trajectory is required only to be time-like could travel around by moving along the non-t time-like direction with t co-ordinate frozen. If it does travel along the compactified time-like direction, a closed time-like curve could form. More basically, what is the concept of a particle of which our understandings have always been associated with the Lorentzian space-time? We therefore refer to a state on a Kleinian space-time as a generalised *state*; moreover, as one is destined to arrive at incorrect conclusions if one treats an electron around a proton classically, we think only a genuine quantum theory (in the sense given in section 4.1) makes sense in a Kleinian space-time.

On the other hand, according to current understanding, we live in a Lorentzian space-time, not a Kleinian one.[5] However, just like one uses Euclidean space-times as a mathematical tool for formulating thermal field theories and quantum gravity, we should/can adopt the same attitude toward Kleinian space-times.[6] As discussed above, the Euclidean and Kleinian Schwarzschild solutions cover the exterior and the interior of the black hole, respectively. They therefore can be regarded as a complimentary pair so that the thermal properties of the whole Schwarzschild black hole, both inside and ouside, can be accounted for.

4.2.2 The first model: $h - q$ parametrisation

Though we have recovered the desired spectrum (4.14), there is one grave unsatisfactoriness: Since we intend to interpret the quantum field as the constituent components of the black hole, the metric should not be the vacuum solution of the Einstein field equation. We thus have to take the energy-momentum tensor of the quantum field into account, i.e., we need to implement the quantum mechanical, static Einstein field equation (4.8)–(4.11).

We then regard the metric variable h, q, and the W as unknown variables, with proper boundary conditions at $x_i \sim x_s = E_j r_s$, we can solve them numerically.

[5] Nonetheless, see reference [3] for a proposal for the *possible* fundamental signature (Kleinian) and dimensions (13 or 14) of space-times.

[6] I am grateful to I. Moss for stressing the importance of this point and pointing out to me that one cannot match a Kleinian Schwarzschild solution to a Lorentzian one.

Boundary conditions

We chose the following boundary condition,

$$\bar{q} \sim -(x_s - x) + \bar{q}_{j+1}(x_s - x)^{j+1} ,$$
$$h \sim 1 + h_j(x_s - x)^j , \quad W \sim w_j(x_s - x)^j .$$

We then find the following self-consistent conditions

$$x_s = \frac{j}{2} , \qquad j = 1, 2, 3, \dots ,$$
$$\bar{q}_{j+1} = h_j = \frac{-j}{j+1}\frac{w_j}{2} ,$$

where w_j is a free parameter and will be determined by the normalisation condition of W. Note that we have chosen j as positive integers which is the consequence of the choice $E_j = \kappa j$. Though we have no compulsive reason to make such a choice, this is a natural one if we parametrise the metric as

$$ds^2 = f'\frac{f}{r}(dt^2 + \frac{\rho^2}{f_0^2}d\rho^2) + r^2 d\Omega^2 , \tag{4.15}$$

where $f_0 = \rho - \rho_s$, $\rho_s = r_s$, and ρ is implicitly defined by the equation

$$\frac{d\rho}{dr} = \frac{f_0}{\rho}\frac{r}{f} . \tag{4.16}$$

By comparing the co-ordinates t and ρ in equation (4.15) with the co-ordinates t and r in a Kleinian Schwarzschild solution, it is thus natural to endow t with the character of an angular co-ordinate. The above choice of spectrum is thus demanded for a topological reason. We will call the solution of $\bar{f}$ and W corresponding to j the j-th eigen-state.

Asymptotic behaviour

From equations (4.8)–(4.11) we can discover the asymptotic behaviour at $x \sim 0$ of the j-th eigen-state: $h \sim x^{a_j}$, $\bar{q} \sim x^{-a_j}$, and $W \sim 2a_j \ln^2(x)$ in which a_j is positive. Then W is always normalisable (cf. equation (4.7)).

Divergence of the energy-momentum tensor The integral of the expectation value of ${}_t$-^{t} component of the energy-momentum tensor is

$$\begin{aligned}
\mathcal{E} &\stackrel{def}{=} \int dr d\Omega \sqrt{|g|} \langle j| : \hat{T}_t^t : |j\rangle \\
&= \frac{-1}{2GE_j} \int_0^{j/2} dx |\bar{f}| \left(\frac{1}{2} \frac{x^3}{\bar{f}^2} W - \frac{x}{8} \frac{\dot{W}^2}{W} \right) \\
&= \frac{1}{2GE_j} \int_0^{j/2} dx |\bar{f}| \frac{\dot{h}}{h} \, .
\end{aligned} \tag{4.17}$$

It diverges logarithmically.

Using h-q parametrisation in equation (4.5), we found that it is possible to obtain sensible solutions for the fully implemented quantum-mechanical, static Einstein field equation. However, There are two drawbacks in the model based on the h-q parametrisation. The first, since the analytical solution is unavailable, it is a bit tricky to integrate the equation (4.16). Particularly, we are interested in knowing the range of co-ordinate ρ. The second, the $\mathcal{E}$ in equation (4.17) is divergent. It should be interesting to construct another model to bypass these two problems.

4.2.3 The second model: $\eta - r$ parametrisation

We consider a model based on a different parametrisation of the metric. We parametrise the metric as

$$ds^2 = \frac{\eta}{r}(dt^2 + \frac{\rho^2}{f_0^2} d\rho^2) + r^2 d\Omega^2 \, , \tag{4.18}$$

with η and r being regarded as unknown variables.

The equations

Then it is a straightforward exercise to write down the Euler-Lagrange and Einstein field equations. With proper initial condition at $y_i \sim y_s = E_j \rho_s$, we will show numerically that the boundary condition at y_s and the asymptotic behaviour at $y \sim 0$ given below can be linked together (see figures 4.1–4.3 for examples) [46]. We will give the relevant equations directly,

$$\frac{\ddot{W}}{2}+\frac{\dot{W}}{2}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0}-\frac{1}{y}+2\frac{\dot{x}}{x}\right)-\frac{y^2}{\bar{f}_0^2}W-\frac{1}{4}\frac{\dot{W}^2}{W}=0\ , \tag{4.19}$$

$$\frac{1}{2}\frac{(x^2)^{\cdot\cdot}}{x^2}+\frac{1}{2}\frac{(x^2)^{\cdot}}{x^2}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0}-\frac{1}{y}\right)-\frac{\bar{\eta}y^2}{x^3\bar{f}_0^2}=0\ , \tag{4.20}$$

$$\ddot{\bar{\sigma}}+\dot{\bar{\sigma}}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0}-\frac{1}{y}\right)+\frac{\bar{\eta}y^2}{x^3\bar{f}_0^2}+\frac{y^2}{\bar{f}_0^2}W+\frac{1}{4}\frac{\dot{W}^2}{W}=0\ , \tag{4.21}$$

where $y = E_j\rho$, $\bar{\sigma} = ln(-\bar{\eta})$, $\bar{f}_0 = y - y_s$, $x = E_jr$, and a dot denotes differentiation with respect to y.

Boundary conditions

We consider the following boundary conditions at y_s,[7]

$$\begin{aligned}\bar{\eta} &\sim -(y_s-y)+\bar{\eta}_{j+1}(y_s-y)^{j+1}\ ,\\ W &\sim w_j(y_s-y)^j\ ,\quad x\sim y+x_{j+1}(y_s-y)^{j+1}\ ,\end{aligned} \tag{4.22}$$

where $y_s = j/2, j = 1, 2, 3...$, then x_{j+1} and $\bar{\eta}_{j+1}$ are determined by w_j from the following relations,

$$\bar{\eta}_{j+1}=\frac{w_j}{2}\ ,\quad x_{j+1}=\frac{\bar{\eta}_{j+1}}{(j+1)^2}\ . \tag{4.23}$$

In deriving the above relations, the spectrum $E_j = \kappa j$ $(j = 1, 2, 3, ...,)$ has been chosen to conform with the interpretation that the t co-ordinate in the metric (4.18) has the character of an angular co-ordinate so that t is periodic with a period of $2\pi/\kappa$. The choice of w_j will be determined by the normalisation condition of W.

Asymptotic behaviour

Asymptotically near $y = 0$, W, x, and $\bar{\eta}$ behave as

$$W\sim w_0+w_2y^2\ ,\quad x\sim x_0+x_2y^2\ ,\quad \bar{\eta}\sim\bar{\eta}_0+\bar{\eta}_2y^2\ , \tag{4.24}$$

[7]Note that, apart from the junction conditions mentioned above, the boundary condition also satisfies the regularity condition of equation (3.45).

where w_i, x_i, and $\bar{\eta}_i$ $(i = 1, 2)$ are constants. The numerical results suggest that $x_0 \neq 0$, in contrast to the model based on the h-q parametrisation in which the range of x is $0 < x < x_s = E_j r_s$. The origin of the difference lies on the boundary conditions. For the h-q parametrisation (4.5), it can be derived, using equation (4.16) by requiring self-consistence, that $x = y + o\left((y_s - y)^{j+2}\right)$, in contrast to equation (4.22).

Finiteness of the energy-momentum tensor With the help of the boundary condition (4.22) and the asymptotic behaviour (4.24), it is seen that the integral of the expectation value of ${}_t$-^{t} component of the energy-momentum tensor for any eigen-state is finite because

$$\begin{aligned} \mathcal{E} &= 4\pi \int_0^{\rho_s} d\rho \sqrt{|g|} \langle j | \colon \hat{\mathcal{T}}_t^t \colon | j \rangle \\ &= \frac{-1}{2GE_j} \int_0^{j/2} dy \frac{yx^2}{|\bar{f}_0|} \left(\frac{1}{2} W - \frac{1}{8} \frac{\bar{f}_0^2}{y^2} \frac{\dot{W}^2}{W} \right) \\ &\overset{def}{=} \frac{1}{2GE_j} \int_0^{j/2} dy \mathcal{E}_j(y) \ . \end{aligned} \tag{4.25}$$

Normalisation condition

The numerical value of w_j are determined by the normalisation condition (4.7) with N_j yet to be specified. Recall that in our statistical explanation of black hole entropy in section 3.3, chapter 3, the probability of finding matter in the j-th eigen-state is $n_b n_j$. Therefore, we should set $N_j = n_b n_j$. With the help of equations (3.52), (3.54), and the definition of κ, equation (4.7) can be reduced to

$$\int_0^{j/2} dy \frac{yx^2}{|\bar{f}_0|} W \overset{def}{=} \int_0^{j/2} dy \mathcal{W}_j(y) = \frac{\pi}{2} \frac{j^2 n_j}{b_0 + b_2} \overset{def}{=} \mathcal{N}_j \ . \tag{4.26}$$

Note that the above expression is independent of the mass of a black hole. Consequently, within our model, all static Schwarzschild black holes can be classified according to an integer—the black-hole number, N. The first three eigen-states are shown in figures 4.1-4.3.

4.3 Model of charged, non-rotating black holes

When I gave a statistical explanation of the (charged) black hole entropy in section 3.3.3, chapter 3, I have assumed that, in general, a charged black hole is a mixture of charged and neutral states. Especially, there is no interaction between those states. Therefore, those neutral and charged eigen-states can be solved independently. The examples of neutral states have been given in previous section. In this section, I show examples of charged states. Due to the appearance of a new length scale, the charge unit, e, the situation is much more complicated. In particular, we should not expect the equations and the normalisation condition can be written in a scale-independent manner. In order to perform the numerical computation, values of charge unit, e, and gravitational constant, G, have to be chosen. I set $e = 10^{-3}$ and $G = 1$. The choice of G is for convenience and it sets up the length scale. The choice of e is arbitrary. It is one of the values used in figures 3.3-3.6 to *read* the quantised masses.

4.3.1 The model

The system

The action The action of the charged sector is

$$I = \int d^4x\sqrt{|g|}(-\frac{\mathcal{R}}{16\pi G} - \varepsilon_\theta(\mathcal{D}^\mu\vartheta)^\dagger \cdot \mathcal{D}_\mu\vartheta - \varepsilon_A\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \ ,$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu \ , \quad \mathcal{D}_\mu = \nabla_\mu - ie\mathcal{A}_\mu \ , \quad \varepsilon_\theta, \varepsilon_A = \pm 1 \ ,$$

$\mathcal{R}$ is the curvature scalar, $\mathcal{A}_\mu$ the $U(1)$ gauge field, ϑ the complex scalar field.

The parametrisation I adopt the η-r parametrisation used in section 4.2.3 so that

$$ds^2 = \frac{\eta}{r}(dt^2 + \frac{\rho^2}{f_0^2}d\rho^2) + r^2d\Omega^2 \ , \tag{4.27}$$

where $f_0 = f_1(\rho - \rho_+)$. To recover the Reissner-Nordström solution, we set

$$\rho = r \ , f_1 = i\frac{\rho - \rho_-}{\rho} \ , \eta = -\frac{\rho - \rho_-}{\rho}(\rho - \rho_+) \ , \tag{4.28}$$

where $r_\pm = \rho_\pm = GM \pm \sqrt{G^2M^2 - 4\pi G q^2}$.

The choice of ϵ_A and ϵ_θ In section 4.2.1, I rotated the signature of the interior space-time of a Schwarzschild black hole to Kleinian type by analytic continuation through the Kruskal co-ordinates. A Kruskal co-ordinates for a Reissner-Nordström black hole can be also be constructed in a similar line.[8] However, if one wishes to recover an Euclidean or a Kleinian Reissner-Nordström solution, one has also to change the sign of ϵ_A from the conventional $+1$ to -1. Although it is unclear if there exists any mechanism to induce such a change associated with the change of signature, I will consider $\epsilon_A = -1$.[9] On the other hand, I choose the conventional $\epsilon_\theta = +1$.

To solve the charged eigen-states, the variables, η and r, in metric (4.27) will be regarded as unknown variables. We set $f_1 = (r_+ - r_-)/r_+$ in order to endow t co-ordinate the character of an angular co-ordinate with period $\beta = 2\pi/\kappa$ where $\kappa = (r_+ - r_-)/2r_+^2$.

The Equations

The Euler-Lagrange equations of $\theta = \sqrt{8\pi G}\vartheta$, $A = \sqrt{8\pi G}A_t$ and the relevant components of the classical Einstein field equation are (with $\tilde{e} = e/\sqrt{8\pi G}$)

$$\theta'' + \theta'(\frac{f_0'}{f_0} - \frac{1}{\rho} + 2\frac{r'}{r}) + \frac{\rho^2}{f_0^2} D_t^2\theta = 0 \; , \tag{4.29}$$

$$A'' + A'\left(\frac{(r^3 f_0)'}{r^3 f_0} - \frac{(\eta\rho)'}{\eta\rho}\right) - i\varepsilon_\theta \varepsilon_A \tilde{e}\frac{\eta\rho^2}{r f_0^2}(\theta^\dagger \cdot D_t\theta - (D_t\theta)^\dagger \cdot \theta) = 0 \; , \tag{4.30}$$

$$\frac{1}{2}\frac{(r^2)''}{r^2} + \frac{1}{2}\frac{(r^2)'}{r^2}\left(\frac{f_0'}{f_0} - \frac{1}{\rho}\right) - \frac{\eta\rho^2}{r^3 f_0^2} - \varepsilon_A \frac{1}{2}\frac{r}{\eta}A'^2 = 0 \; , \tag{4.31}$$

$$\sigma'' + \sigma'(\frac{f_0'}{f_0} - \frac{1}{\rho}) + \frac{\eta\rho^2}{r^3 f_0^2} + 2\varepsilon_\theta \frac{\rho^2}{f_0^2}(D_t\theta)^\dagger \cdot D_t\theta + 2\varepsilon_\theta \theta'^\dagger\theta' + \varepsilon_A \frac{3}{2}\frac{r}{\eta}A'^2 = 0 \; , \tag{4.32}$$

where $\sigma = \ln(-\eta)$, $D_\mu = \partial_\mu - i\tilde{e}A$, and a prime denotes differentiation with respect to ρ.

Mode decomposition Following the quantum-mechanical prescription of static Einstein equation in section 4.1.2, $\hat{\vartheta}$ is at first expanded as (realised as operator-valued

[8]See appendix D.2.

[9]For the possibility of using $\epsilon_A = 1$, see the discussion in the paragraph, $\boldsymbol{\epsilon_A = \pm 1}$, in section 4.4.

henceforth)

$$\hat{\vartheta} = \sum_j \hat{\vartheta}_j = \sum_j \hat{a}_j e^{-iE_j t} R_j(r) + \hat{b}_j^\dagger e^{iE_j t} R_j(r)$$

in which $\hat{a}_j$ is the annihilation operator of the j-th eigen-state $|j\rangle$, and $\hat{b}_j^\dagger$ is the creation operator of the j-th eigen-anti-state $|j^\dagger\rangle$ such that $[\hat{a}_j, \hat{a}_i^\dagger] = [\hat{b}_j, \hat{b}_i^\dagger] = \delta_{ji}$.

Rationalised equations After employing the quantum-mechanical prescription that $\mathcal{R}_{\mu\nu} - 1/2 g_{\mu\nu}\mathcal{R} = -8\pi G\langle j|{:}\hat{\mathcal{T}}_{\mu\nu}{:}|j\rangle$, equations (4.29)–(4.32) can be rewritten as follows (with new variables $W = 16\pi G R_j^2 \langle j|\hat{a}_j^\dagger \hat{a}_j|j\rangle$, $x = E_j r$, $\bar{\eta} = E_j \eta$, $y = E_j \rho$, and $\bar{e} = \tilde{e}/E_j = e/(8\pi G E_j)$),

$$\frac{\ddot{W}}{2} + \frac{\dot{W}}{2}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0} - \frac{1}{y} + 2\frac{\dot{x}}{x}\right) - \frac{1}{4}\frac{\dot{W}^2}{W} - \frac{y^2}{\bar{f}_0^2} W(1 + \bar{e}A)^2 = 0 \ , \tag{4.33}$$

$$\ddot{A} + \dot{A}\left(\frac{(x^3 \bar{f}_0)^{\cdot}}{x^3 \bar{f}_0} - \frac{(\bar{\eta} y)^{\cdot}}{\bar{\eta} y}\right) - \varepsilon_\theta \varepsilon_A \bar{e} \frac{\bar{\eta} y^2}{x \bar{f}_0^2} W(1 + \bar{e}A) = 0 \ , \tag{4.34}$$

$$\frac{1}{2}\frac{(x^2)^{\cdot\cdot}}{x^2} + \frac{1}{2}\frac{(x^2)^{\cdot}}{x^2}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0} - \frac{1}{y}\right) - \frac{\bar{\eta} y^2}{x^3 \bar{f}_0^2} - \varepsilon_A \frac{1}{2}\frac{x}{\bar{\eta}}\dot{A}^2 = 0 \ , \tag{4.35}$$

$$\ddot{\bar{\sigma}} + \dot{\bar{\sigma}}\left(\frac{\dot{\bar{f}}_0}{\bar{f}_0} - \frac{1}{y}\right) + \frac{\bar{\eta} y^2}{x^3 \bar{f}_0^2} + \varepsilon_\theta \frac{y^2}{\bar{f}_0^2} W(1 + \bar{e}A)^2 + \varepsilon_\theta \frac{1}{4}\frac{\dot{W}^2}{W} + \varepsilon_A \frac{3}{2}\frac{x}{\bar{\eta}}\dot{A}^2 = 0 \ , \tag{4.36}$$

where $\bar{\sigma} = \ln(-\bar{\eta})$, $\bar{f}_0 = f_1(y - y_+)$, and a dot denotes differentiation with respect to y.

The normalisation

General case The normalisation condition for states is

$$\int d\rho d\Omega \sqrt{|g|}\langle j|\hat{J}^t|j\rangle = -\int d\rho d\Omega \sqrt{|g_s g^{tt}|}\langle j|\hat{J}_t|j\rangle = N_j \ ,$$

where

$$\hat{J}_t = -i(\hat{\vartheta}^\dagger \cdot \partial_t \hat{\vartheta} - \partial_t \hat{\vartheta}^\dagger \cdot \hat{\vartheta}) \ ,$$

N_j is the normalisation, g_s is the spatial part of the metric, and the integration region is the interior of the black hole. It can be reduced to

$$\int_0^{y_+} dy \frac{y x^2}{|\bar{f}_0|} W = \int_0^{y_+} dy \mathcal{W}_j = 2 G E_j^2 N_j \ . \tag{4.37}$$

Note that although the above probability density operator is not the one appeared in the Maxwell equation, it is still conserved. Using above normalisation condition amounts to normalise the number of states, instead of the total charges.

Special case In order to provide numerical examples of the solutions of charged eigen-state, we need explicit value of N_j. Recalling the assumption and calculation made in section 3.3.3, chapter 3, we have $N_j = N_c m_j / c_1$ (cf. equation (3.59)). However, the N_c is determined by equation (3.60) which depends on the value of N_b, the total number of neutral states inside the charged black hole. To simplify the situation, I assume that $N_b = 0$. Then, for the RHS of equation (4.37), we have

$$\mathrm{RHS} = 2GE_j^2 N_j = \frac{\pi}{2} \frac{j^2 m_j}{c_0 + c_2} (1 - \frac{r_-}{r_+})^2 \stackrel{def}{=} \mathcal{N}_j \ , \tag{4.38}$$

where $r_\pm$ are defined in equations (D.10) and (D.11). The numerical solutions of the first three eigen-states with $N_c = 5 \times 10^4$ and $e = 10^{-3}$ are shown in figures 4.4-4.9.

Ambiguity in the meaning of the charge

The relation between e and q is indeed unclear in our system since the self-interaction has been involved in the Maxwell equation explicitly. There are three charge-related terms in this system: total charge q, coupling constant e, and the charge density that is defined from the charge density operator, $\hat{\mathcal{J}}_t$, appeared in the Maxwell equation,

$$\hat{\mathcal{J}}_t = -i \left[\hat{\vartheta}^\dagger \cdot D_t \hat{\vartheta} - (D_t \hat{\vartheta})^\dagger \cdot \hat{\vartheta} \right] \ .$$

If the lesson from the conventional quantum field theory is valuable, we expect some renormalisation schemes should be brought in to related these three terms. The situation is in fact more intricate because, as shown later, we can confine the effect of charges inside the black hoe totally so that even though the boundary condition is from that of a neutral black hole (i.e., $q = 0$), the complex scalar field and the $U(1)$ gauge field could be non-zero inside.

Naïve assumption I will follow the naïve assumption in section 3.3.3, chapter 3 that $q = N_c e$ in order to implement the normalisation condition.

4.3.2 The solutions

General solutions

Boundary conditions I consider the following boundary conditions near $y_+ = E_j\rho_+$ (with $z = y_+ - y$ and $s_j > 0$),

$$W \sim w_j z^{s_j} \ , \quad \bar{\eta} \sim \bar{\eta}_c(z) + \bar{\eta}_{j+1} z^{s_j+1} \ ,$$
$$x \sim x_c(z) + x_{j+1} z^{s_j+1} \ , \quad A \sim A_c(z) + a_{j+1} z^{s_j+1} \ , \tag{4.39}$$

where $\bar{\eta}_c(z)$, $x_c(z)$, and $A_c(z)$ are the *classical* solution of equations (4.34)–(4.36) with $W = 0$. By expanding $\bar{\eta}_c$, x_c, and A_c in terms of polynomials of z, it can be checked that the classical solution[10] is uniquely determined by the following boundary condition at $z = 0$

$$\bar{\eta}_c(0) = \bar{\eta}_c = 0 \ , \quad \dot{\bar{\eta}}_c(0) = \dot{\bar{\eta}}_c = \bar{f}_1 \ ,$$
$$x_c(0) = x_c = y_s \ , \quad \dot{x}_c(0) = \dot{x}_c = 1 \ ,$$
$$A_c(0) = A_c = -\frac{\bar{q}}{y_+} \ , \quad \dot{A}_c(0) = \dot{A}_c = \frac{\bar{q}}{y_+^2} \ ,$$

where $\bar{f}_1 = (y_+ - y_-)/y_+$ and $\bar{q} = \sqrt{8\pi G}E_j q$.

Using equations (4.33)–(4.36), we obtain

$$s_j = \frac{2x_c(1 + \bar{e}A_c)}{\bar{f}_1} = \frac{E_j}{\kappa}(1 - \frac{eq}{y_+}) = \frac{E_j}{\kappa} - \frac{eq}{\kappa r_+} \ , \tag{4.40}$$

where $s_j > 0$ in order to form bounded states and the rest of coefficients are determine by w_j from the following relations

$$\bar{\eta}_{j+1} = \varepsilon_\theta \dot{\bar{\eta}}_c \frac{w_j}{2} \ ,$$
$$a_{j+1} = \frac{\dot{A}_c}{\dot{\bar{\eta}}_c}\frac{\bar{\eta}_{j+1}}{1+s} - \varepsilon_\theta \varepsilon_A \bar{e} \frac{\dot{\bar{\eta}}_c x_c}{s(1+s)\bar{f}_1^{\,2}}(1 + \bar{e}A_c) w_j \ ,$$
$$x_{j+1} = (\frac{1}{\bar{f}_1^2} - \frac{\varepsilon_A}{2}\frac{x_c^2}{\dot{\bar{\eta}}_c}\dot{A}_c^{\,2})\frac{\bar{\eta}_{j+1}}{(1+s)^2} + \varepsilon_A \dot{A}_c \frac{x_c^2}{\dot{\bar{\eta}}_c}\frac{a_{j+1}}{1+s} \ .$$

[10] This is just the Kleinian Reissner-Nordström black hole solution as $\epsilon_A = -1$.

The spectrum To conform with our interpretation that the t co-ordinate has the character of an angular co-ordinate with period $2\pi/\kappa$, we should set $E_j = \kappa j$. The choice of w_j will be determined by the normalisation condition of W.

Asymptotic behaviour Asymptotically, near $x = 0$, W, $\bar{\eta}$, x, and A behave as

$$
\begin{aligned}
W \sim w_0 + w_2 y^2 \ , \quad \bar{\eta} \sim \bar{\eta}_0 + \bar{\eta}_2 y^2 \ , \\
x \sim x_0 + x_2 y^2 \ , \quad A \sim a_0 + a_2 y^2 \ ,
\end{aligned}
$$

where w_i, a_i, x_i, and $\bar{\eta}_i$ $(i = 1, 2)$ are constants.

Finiteness of the energy-momentum tensor With the help of the boundary condition and the asymptotic behaviour, it is seen that the integral of the expectation value of ${}_t\text{-}^t$ component of the energy-momentum tensor for any eigen-state is finite because

$$
\begin{aligned}
\mathcal{E} &= 4\pi \int_0^{\rho_s} d\rho \sqrt{|g|} \langle j |\!: \widehat{\mathcal{T}}_t^t :\!| j \rangle \\
&= \frac{-1}{2GE_j} \int_0^{y_s} dy \frac{yx^2}{|f_0|} \left(\varepsilon_\theta \frac{1}{2} W (1 + \bar{e}A)^2 - \varepsilon_\theta \frac{1}{8} \frac{\bar{f}_0^2}{y^2} \frac{\dot{W}^2}{W} + \varepsilon_a \frac{1}{2} \frac{x \bar{f}_0^2}{\bar{\eta} y^2} \dot{A}^2 \right) \\
&= \frac{1}{2GE_j} \int_0^{y_s} dy \mathcal{E}_j(y) \ .
\end{aligned} \tag{4.41}
$$

Large and small charge limits

There are two special situations worth being pointed out. One is the large q limit. Another one is the limit of $q = 0$.

Large q limit Within my model, the extreme black holes satisfying $GM^2 = 4\pi q^2$ is ruled out by requiring those states inside a charged black hole being bound states, i.e., $s_j > 0, j = 1, 2, 3, \ldots$. Because it implies

$$
M^2 > \frac{(1 - eq)^2}{1 - 2eq} \frac{4\pi}{G} q^2 \ , \tag{4.42}
$$

with the constraint that $eq = e^2 N_c < \frac{1}{2}$.[11] On the other hand, for the non-extreme Reissner-Nordström black, we need $M^2 > \frac{4\pi}{G} q^2$, it is seen that the boundedness implies the non-extremeness.

Limit of q = 0 For the limit that $q = 0$, there exists an interesting phenomenon of *charge confinement* in the sense that even though the black hole appears as a Schwarzschild black hole to an observer outside of it, the gauge field at the inside is non-zero. By taking the limit of $q = 0$, the boundary condition (4.39) can be reduced to[12]

$$\begin{aligned} W &\sim w_j (y_s - y)^j \,, & \bar{\eta} &\sim -(y_s - y) + \bar{\eta}_{j+1} (y_s - y)^{j+1} \,, \\ x &\sim y + x_{j+1} (y_s - y)^{j+1} \,, & A &\sim a_0 + a_{j+1} (y_s - y)^{j+1} \,, \end{aligned} \tag{4.43}$$

where $y_s = j/2, j = 1, 2, 3 \ldots$, and x_{j+1}, $\bar{\eta}_{j+1}$, and a_{j+1} are determined by w_j from the following relations,

$$\bar{\eta}_{j+1} = \varepsilon_\theta \frac{w_j}{2} \,, \quad a_{j+1} = \frac{-\varepsilon_\theta \epsilon_A \bar{e}}{j+1} \frac{w_j}{2} \,, \quad x_{j+1} = \frac{\bar{\eta}_{j+1}}{(j+1)^2} .$$

The coefficient a_0 is a gauge degree of freedom. It is set to zero so that the gauge field is continuous across the event horizon. The asymptotic behaviours near the origin $y = 0$ look just like that for $q \neq 0$ cases. Examples of numerical solutions are shown in figures 4.10-4.12.

---

[11] With the help of the defintion of κ in equation (D.12), we see that $s_1 = 1 - eq/(\kappa r_+) > 0$ implies $1 - 2eq > r_-/r_+$. This constraint follows from the requirement that $r_-/r_+ > 0$. In general, one can require that $s_j > 0$ only for $j \geq J$, where J is a positive constant, i.e., those eigen-states with $j < J$ do not exist. Then equation (4.42) becomes

$$M^2 > \frac{(1 - \frac{eq}{J})^2}{1 - \frac{2eq}{J}} \frac{4\pi}{G} q^2 \,,$$

with the constraint that $eq = e^2 N_c < \frac{J}{2}$, and the summation of j in partition functions in equation (3.55) should be modified accordingly. I have implicitly assumed that $J = 1$. The physical significance of this constraint is unclear at this moment. Also note that in order for the solution of M of equation (3.60) existing, N_c is constrained by a maximum value, $N_{cd}(N_b, e)$. See the last paragraph of section 3.3.3 on page 89 for further explanation.

[12] It is easier to check this by working on the equations directly with the following ansatz.

Regarding to the phenomenon of charge confinement, we can thus ask how much of the total charge will be revealed if such a black hole is formed from the collapse of charged matter. If not all of them, then to an observer at the outside of the black hole, the charge is not conserved. Admittedly, this claim is hard to justify at this moment since I am considering an eternal black hole. However, I think this question is still interesting in a general setting.

Since the q observed from the outside cannot represent the total charge at the inside, this further worsens the problem concerning the ambiguity of the meaning of charge mentioned in section 4.3.1. It seems that a more fundamental approach is needed in order to solve this problem. However, the naïve choice made in section 3.3.3, chapter 3 such that $q = N_c e$ removes, by force, the ambiguity temporarily.

4.4 Prospects and conclusions

As a theory, my approach is only a starter. Objections can be easily raised and a lot of questions are awaiting to be addressed. Nonetheless, I am trying to see things from a point of view that is different from the one adopted in conventional quantum field theory which, I believe, has its limitation.

In order to understand the statistical origin (in the sense of the textbook statistical mechanics) of black hole entropy, I felt obliged to put matter into a black hole. In order to construct matter bound states, I rotated the signature of the interior space-time of a black hole to Kleinian type. I also gave a quantum-mechanical prescription of static Einstein field equation as the building foundation of the matter-metric eigen-states. None of these is easy to justify by itself standing along. However, things began to make sense as they were combined to form a logical argument.

The urgent questions The most urgent question concerning my approach is the existence of two-time co-ordinates. It seems that a higher dimensional theory is a more nature context to deal with it, where the two-time co-ordinates originates from other fundamental principles [3].

How to choose a good time As mentioned in the text, the concept of energy, hence time, in the theory of general relativity is not so clear. I have chosen, a priori, a time variable by hand. In fact, I have sacrificed the covariance of general relativity in favour of the prejudicial time variable in quantum mechanics. I do not think this is a drawback of my approach; nonetheless, it does reflect one of the most basic questions we have to face as dynamics is brought in. The so-called *problem of time* has been re-appearing again and again in different contexts. I am not able to review the various opinions at this moment. However, a relevant question that can be asked immediately about our approach is: We have got a finite energy $\mathcal{E}$ (cf. equation (4.25)), what are we going to do with it? Can we associate the black hole mass M to any quantities calculated locally? We also encounter another (thermodynamic) energy in the calculation of statistical entropy. What are the relations between all these energy terms? A complete *physical* picture of black holes can emerge only after these physical questions have been answered satisfactory.

$\epsilon_{\mathbf{A}} = \pm\mathbf{1}$ I changed not only the signature, but also the relative sign of Lagrangians. As far as a physical theory is concerned, it seems that the relative sign of various Lagrangian terms can only be determined experimentally. It is unclear to me if there is any theoretical principle or self-consistency constraint which can help us determining this. In an unified theory, perhaps all children Lagrangians can be derived from a single parent.

One may wonder if solutions exist for the case that $\epsilon_A = 1$. For the explicitly charged cases, namely, $q \neq 0$, I did not find satisfactory solutions.[13] However, for the limit of $q = 0$ discussed in the end of section 4.2.2, the solution between $\epsilon_A = 1$ and $\epsilon_A = -1$, superficially, do not differ much except that the sign of gauge field, A is reversed due to the ϵ_A-dependence of the boundary condition (4.3.2). To compare, I show in figures 4.13-4.15 the solution of $\epsilon_A = 1$ with all other parameter the same to those used to integrate figures 4.10-4.12.

This consideration therefore raises another important question: How trustful are those numerical solutions? I am no expert of numerical analysis. However, I think the existence of the eigen-states for Schwarzschild black holes is unquestionable because they can be regarded as perturbations of the classical vacuum solution. As j increases, the solutions approach the Schwarzschild solution because the matter content decreases. For the charged cases, again, the problem is complicated by the extra length scale, the charge unit, e. Though I only show one set of solutions corresponding to $e = 10^{-3}$, my results suggest the inner horizon cannot survive in a material black hole.[14] I am not able to scan a very wide range of possible choice of e,[15] because I do not believe that ten, or a hundred sets of data could convince anyone who does not appeal to numerical solutions.

[13]In fact, Matlab rejected to solve them at all due to the divergent behaviours of the solutions.

[14]In a loose sense, this agrees with the well-known result that the inner horizon of a Reissner-Nordström black hole is unstable against perturbation [13].

[15]On the other hand, I do not see the necessity.

As a physical theory, the second triumph of Schrödinger atomic theory is that the eigen-states can be solved exactly for the simplest, perhaps also the most significant, case of an hydrogen atom.[16] It thus makes sense to try to construct a simpler model in which the eigen-states can be solved exactly, however one prescribes the relevant equations.

What is the right statistical mechanics inside a black hole? I have assumed that the concepts from textbook statistical mechanics can be generalised to the region inside a black hole, based on the faith of the generality of the underlying principle of statistical mechanics. This should be a reasonable assumption. However, I also assumed that those states behaves as free states. Due to the non-linearity of the Einstein field equation, it is hard to imagine why it must be so, though the self-interaction of an eigen-state has been included. However, based on the assumption that entropy is a kind of low energy quantum phenomenon (see section 4.1.1), I suppose that the mutual-interaction belongs to the high energy regime, which needs a quantum gravity to account for.

What is the correct interpretation? In the models, the matter-metric eigen-states are confined within the black hole totally. Since all eigen-states in a model derive their boundary conditions from the same Schwarzschild solution, they can be regarded as residing in the same black hole. However, if the transition between different matter-metric eigen-sates is allowed to happen, then it seems necessary to put different eigen-states inside different black holes because as the contents of a black hole changes, its radius should change accordingly. Then the meaning of a model probably needs being re-addressed: Should it be regarded as a model of a black hole or, an ensemble of black holes such that each eigen-state lives inside a different black hole? In the later case, we can then evade the difficult problem of the interactions between different eigen-states since the members of an statistical asemble are not allowed to interact with each other.

[16] The first triumph is that, it works!

How to parametrise physically? I have given models based on different parametrisations and boundary conditions. Obviously, there is no reason to rule out the possibility of constructing further models based on other parametrisations and boundary conditions, or totally different approaches. As far as my approach is concerned, it will be important for us, as a guidance, to choose a proper parametrisation and boundary conditions if further physical criteria for choosing a co-ordinate gauge can be given. In the $\eta - r$ parametrisation where the energy-momentum is finite, the two variable, η and r, are indeed conformal factors at the sub-topology sectors, R^2 and S^2, respectively. It is not yet clear to me what this suggests, but it seems obvious that the conformal factors do play special role in a gravity theory.

Introducing coupling If we introduce the simplest coupling between the gravity and the scalar field, then for the neutral black hole case, the action is

$$I = \frac{1}{2}\int d^4x\sqrt{|g|}\left(-\frac{\mathcal{R}}{8\pi G} - \partial_\mu\phi\partial^\mu\phi - \xi\mathcal{R}\phi^2\right) ,$$

where ξ is a coupling constant such that $\xi = 0$ and $\xi = -\frac{1}{6}$ correspond to the minimally and conformally coupled cases, respectively. The Euler-Lagrange equation and one component of the Einstein field equation are (with η-r parametrisation)

$$\frac{W''}{2} + \frac{W'}{2}\left(\frac{f_0'}{f_0} - \frac{1}{\rho} + 2\frac{r'}{r}\right) - \frac{1}{4}\frac{W'^2}{W} - \frac{\rho^2}{f_0^2}E_j^2 W - \xi\frac{\eta\rho^2}{r f_0^2}W\mathcal{R} = 0 ,$$

$$(1+\xi W)\left(\frac{\eta'}{\eta}\frac{r'}{r} - \frac{\eta\rho^2}{r^3 f_0^2}\right) + \frac{1}{2}\left(\frac{\rho^2}{f_0^2}E_j^2 W - \frac{1}{4}\frac{W'^2}{W}\right)$$
$$-\frac{\xi}{1+6\xi}\frac{\eta\rho^2}{r f_0^2}\mathcal{R} - \frac{\xi}{2}W'' - 2\xi\left(\frac{r^2 f_0}{\eta\rho}\right)'\frac{\eta\rho}{r^2 f_0}W = 0 ,$$

where a prime denotes the differentiation with respect to ρ, and $\xi = -1/6$, $\mathcal{R} = 0$ for the conformally coupled case, $\xi \neq 0, -1/6$,

$$\mathcal{R} = \frac{-(1+6\xi)}{1+\xi(1+6\xi)W}\left(\frac{r}{\eta}E^2 W + \frac{1}{4}\frac{r f_0^2}{\eta\rho^2}\frac{W'^2}{W}\right)$$

for the general case. Given the ansatz (4.22) at $\rho \sim \rho_s$, it is found that no self-consistent solution exists due to the appearance of W'' in the Einstein field equation. Similar conclusion applies to the charged case. Admittedly, our statement does not serve as a proof. Nonetheless, it could hint that the minimally coupled case is privileged.

Back to classical limit It seems widely accepted that black hole entropy is a genuine quantum phenomenon. However, recent progress concerning classical black hole mechanics shows that the zeroth and first law are true in a variety of gravity theories [73]. One may thus wonder if the so-called black hole entropy has something more to do with the classical physics. Even though the black hole entropy does need a quantum interpretation, one may wonder if we should require it having such a generality that it can be applied to various gravity theories in which the classical laws of black hole mechanics hold.

On the other hand, the four laws of black hole mechanics are classical laws. Any quantum mechanical explanation of black hole entropy seems inevitably face the challenge to recover them as a limit in some sense, which I have to leave out.

*   *   *

Appendix D

Black Hole Solutions

Some basic properties of Schwarzschild and Reissner-Nordström black holes, which are used, are reviewed [33, 67].

D.1 Schwarzschild solution

The metric in Schwarzschild co-ordinate is

$$ds^2 = -(1-\frac{r_s}{r})dt^2 + (1-\frac{r_s}{r})^{-1}dr^2 + r^2 d\Omega^2 . \tag{D.1}$$

Define the tortoise co-ordinate, $r_* = r + r_s \ln(\frac{r}{r_s} - 1)$, $r > r_s$ and $u = t - r_*$, $v = t + r_*$, then (D.1) can be written as

$$ds^2 = -\frac{r_s}{r}e^{-2\kappa r + \kappa(v-u)}dudv + r^2 d\Omega^2 , \tag{D.2}$$

where $\kappa = 1/2r_s$, is the surface gravity. Introducing Kruskal co-ordinates U and V such that $U = -e^{-\kappa u}$, $V = e^{\kappa v}$, we obtain the maximal extension of Schwarzschild solution

$$ds^2 = -\frac{1}{\kappa^2}\frac{r_s}{r}e^{-2\kappa r}dUdV + r^2 d\Omega^2 . \tag{D.3}$$

Finally, define co-ordinates T and X, such that $2T = U + V$ and $2X = V - U$, we can write

$$ds^2 = \frac{1}{\kappa^2}\frac{r_s}{r}e^{-2\kappa r}(-dT^2 + dX^2) + r^2 d\Omega^2 , \tag{D.4}$$

where t and r are related to T and X by relations

$$e^{2\kappa t} = \frac{T+X}{X-T}, \tag{D.5}$$

$$(\frac{r}{r_s}-1)e^{2\kappa r} = X^2-T^2. \tag{D.6}$$

From the Lorentzian Schwarzschild solution, the Euclidean Schwarzschild solution can be derived that is useful for investigating the thermal properties of the Schwarzschild black hole [22, 23]. At first, rewrite equation (D.5) as

$$\tanh(\kappa t) = \tanh\left[\ln(\frac{T+X}{X-T})^{1/2}\right] = \frac{T}{X}\,. \tag{D.7}$$

If we analytically continue T to $-iT$ and t to $-it$, we then arrive at the Euclidean Schwarzschild solution such that

$$\tan(\kappa t) = \frac{T}{X}, \tag{D.8}$$

$$(\frac{r}{r_s}-1)e^{2\kappa r} = X^2+T^2, \tag{D.9}$$

with the constraint $r \geq r_s$. It is seen that the t co-ordinate has a nature interpretation as the angular co-ordinate at the $t-r$ plan if the period of t is $\beta = \frac{2\pi}{\kappa}$. This is indeed necessary in order for the Euclidean Schwarzschild solution being devoid of the conical singularity and satisfying the vacuum Einstein field equation everywhere for $r \geq r_s$ [22].

D.2 Reissner-Nordström solution

The metric of a Reissner-Nordström black hole is

$$ds^2 = -(1-\frac{2GM}{r}+\frac{4\pi GQ^2}{r^2})dt^2 + (1-\frac{2GM}{r}+\frac{4\pi GQ^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2\,.$$

Define the tortoise co-ordinate r_*, $(r > r_+, r_-)$

$$\begin{aligned} r_* &= r + \frac{r_+^2}{r_\pm}\ln(\frac{r}{r_+}-1) - \frac{r_-^2}{r_\pm}\ln(\frac{r}{r_-}-1) \\ &= r + \frac{1}{2\kappa}\ln\left[(\frac{r}{r_+}-1)(\frac{r}{r_-}-1)^\alpha\right], \end{aligned}$$

in which $\alpha = -(\frac{r_-}{r_+})^2$, $r_\pm = r_+ - r_-$, and

$$r_+ = GM + \sqrt{(GM)^2 - 4\pi GQ^2}\ , \tag{D.10}$$

$$r_- = GM - \sqrt{(GM)^2 - 4\pi GQ^2}\ . \tag{D.11}$$

Introduce co-ordinates u and v such that $u = t - r_*$ and $v = t + r_*$, the metric can be written as

$$ds^2 = -\frac{r_+ r_-^\alpha}{r^{1+\alpha}}(1 - \frac{r_-}{r})^{1-\alpha} e^{-2\kappa r + \kappa(v-u)} du dv + r^2 d\Omega^2\ ,$$

in which the surface gravity, κ, is

$$\kappa = r_\pm / 2r_+^2\ . \tag{D.12}$$

Introducing Kruskal-like co-ordinates U and V such that $U = -e^{-\kappa u}$ and $V = e^{\kappa v}$, we obtain

$$ds^2 = -\frac{1}{\kappa^2}\frac{r_+ r_-^\alpha}{r^{1+\alpha}}(1 - \frac{r_-}{r})^{1-\alpha} e^{-2\kappa r} dU dV + r^2 \Omega^2\ .$$

Finally, define $2T = (U + V)$ and $2X = (V - U)$, we obtain the desideratum

$$ds^2 = \frac{1}{\kappa^2}\frac{r_+ r_-^\alpha}{r^{1+\alpha}}(1 - \frac{r_-}{r})^{1-\alpha} e^{-2\kappa r}(-dT^2 + dX^2) + r^2 \Omega^2\ .$$

The relations between co-ordinates t, r and T, X are

$$X^2 - T^2 = e^{2\kappa r}(\frac{r}{r_+} - 1)(\frac{r}{r_-} - 1)^\alpha\ , \tag{D.13}$$

$$\frac{T + X}{X - T} = e^{2\kappa t}\ . \tag{D.14}$$

Similar to the Schwarzschild black hole case, by continuing analytically T to $-iT$ and t to $-it$, we arrived at the Euclidean Reissner-Nordström solution with the constraint $r \geq r_+$.

Note that, the maximal extension copied from the Schwarzschild black hole case are good only for the region of $r > r_-$.

The gauge potential, $\mathcal{A}$, in a Reissner-Nordström solution is

$$\mathcal{A} = \mathcal{A}_t dt = -\frac{q}{r} dt\ .$$

The associated Coulomb potential, Φ, is defined as

$$\Phi = -\xi^\mu \mathcal{A}_\mu = \frac{q}{r} ,$$

where $\xi = \partial_t$ is the Killing vector field.

Figures

Figures for section 4.2

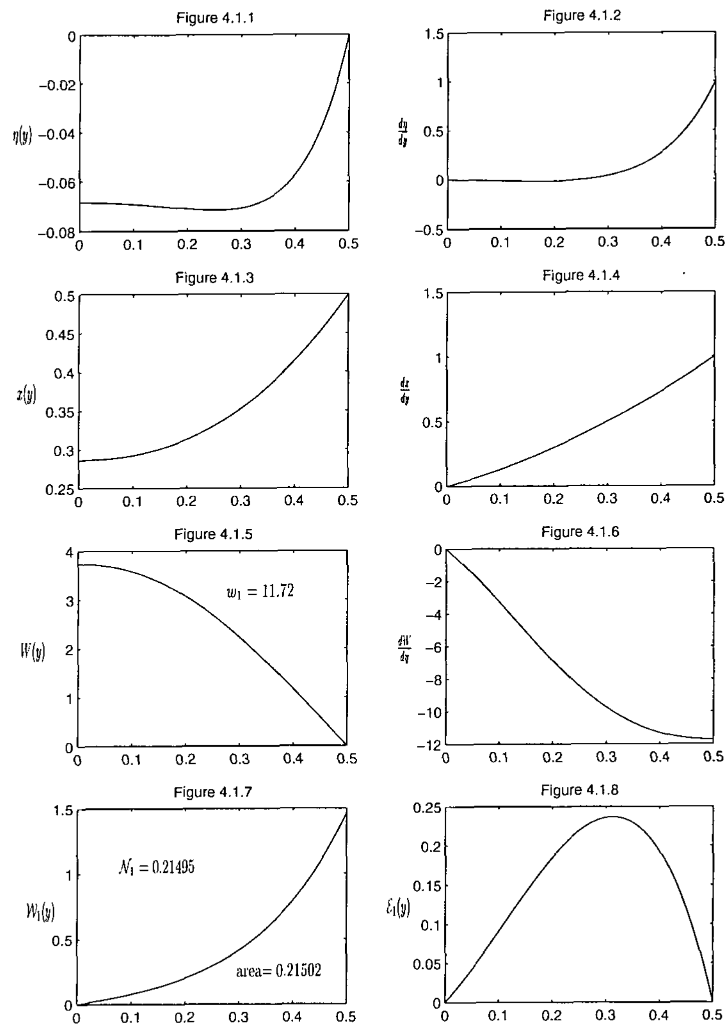


Figure 4.1: The eigen-state of $j = 1$. The functions, $\mathcal{E}_j$, $\mathcal{W}_j$, and $\mathcal{N}_j$ are defined in equations (4.25) and (4.26), respectively. The w_j is the coefficient in equation (4.22). It is expected that the value of w_j depends on the initial position from which the equations are integrated by the Runge-Kutta-Fehlberg method. Nonetheless, I beleive the curves presented can represent the typical qualitative features of the solutions.

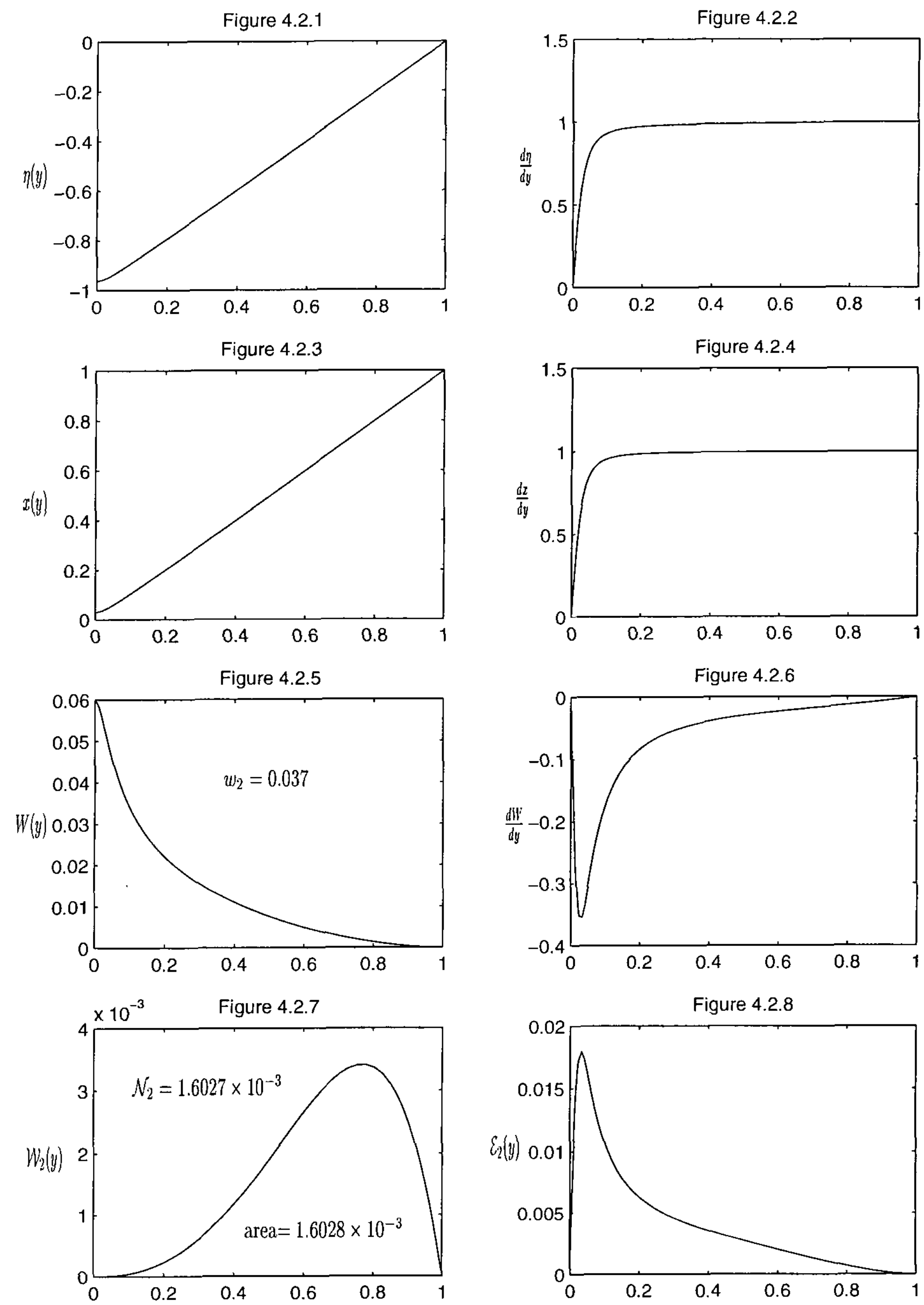


Figure 4.2: The eigen-state of $j = 2$.

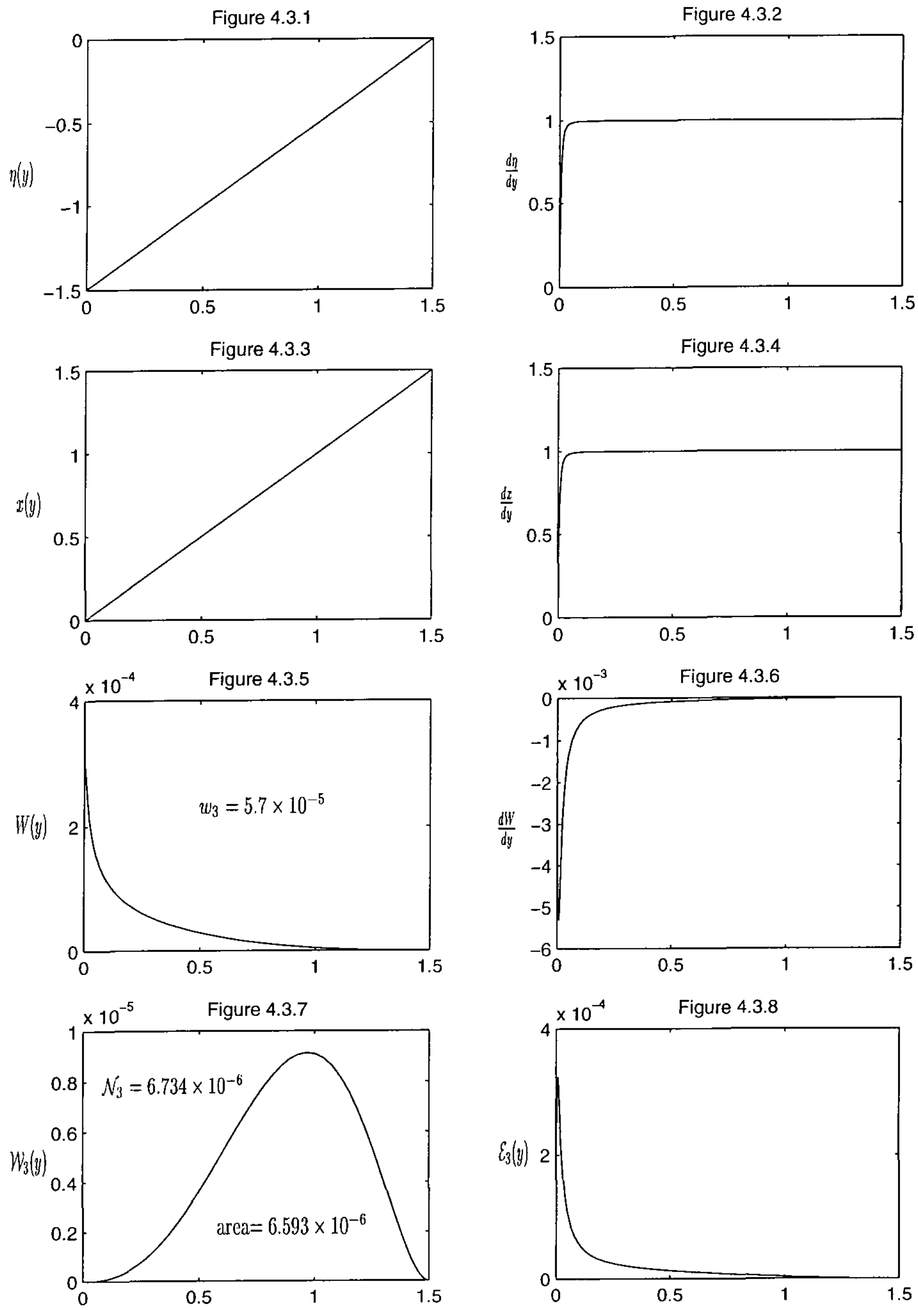


Figure 4.3: The eigen-state of $j = 3$.

Figures for section 4.3

$M = M_l$

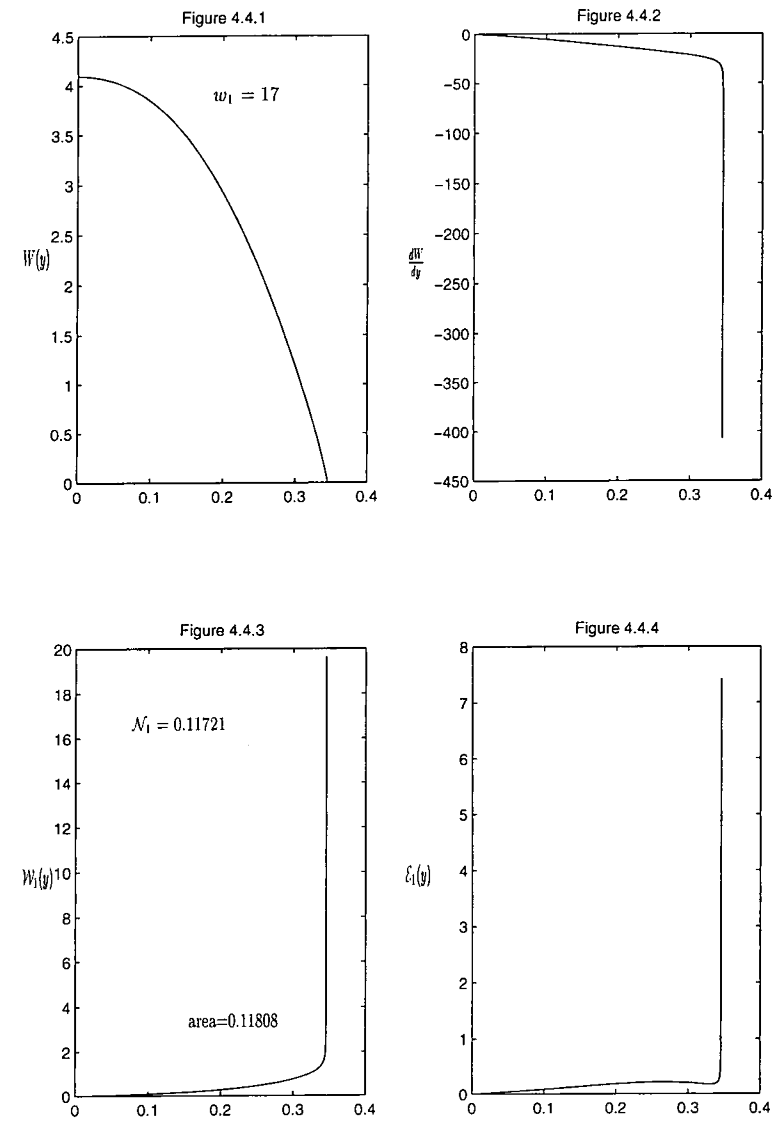


Figure 4.4: The eigen-state of $j = 1$ for the interior of a Reissner-Nordström black hole with parameters, $\epsilon_A = -1$, $\epsilon_\theta = 1$, $M = M_l$, $N_c = 5 \times 10^4$, $e = 10^{-3}$. The $\mathcal{E}_j$, $\mathcal{N}_j$, and $\mathcal{W}_j$ are defined in equations (4.41), (4.38), and (4.37), respectively.

Figure 4.4.5

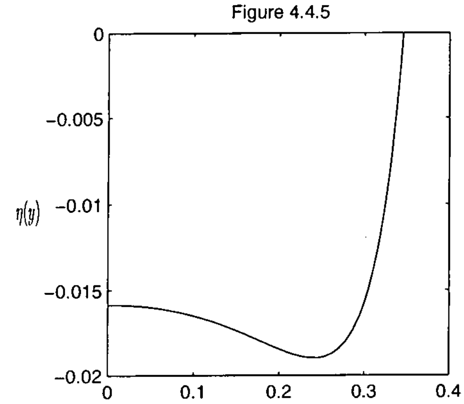


Figure 4.4.6

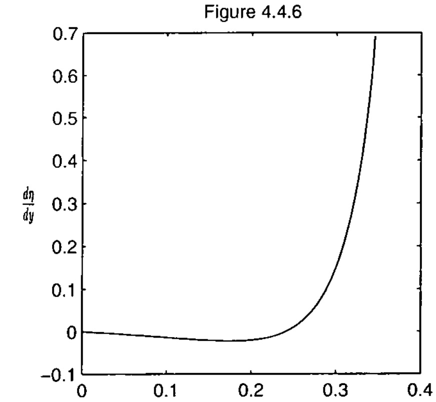


Figure 4.4.7

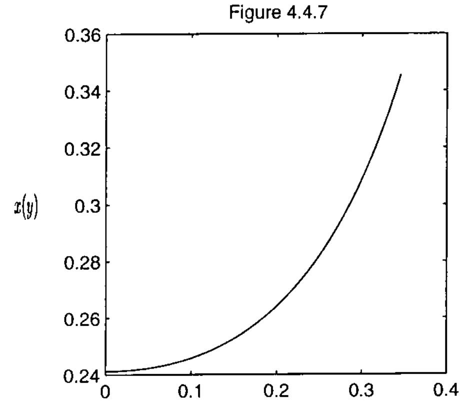
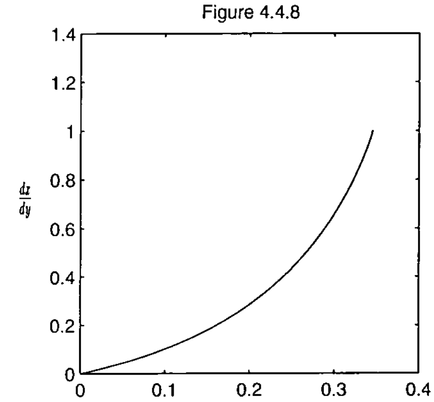


Figure 4.4.8



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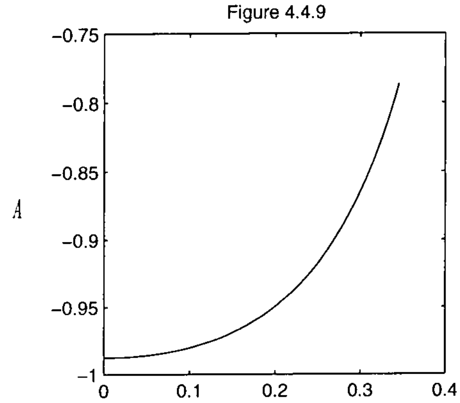
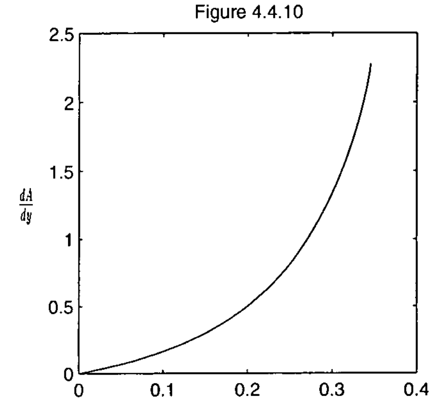


Figure 4.4.10



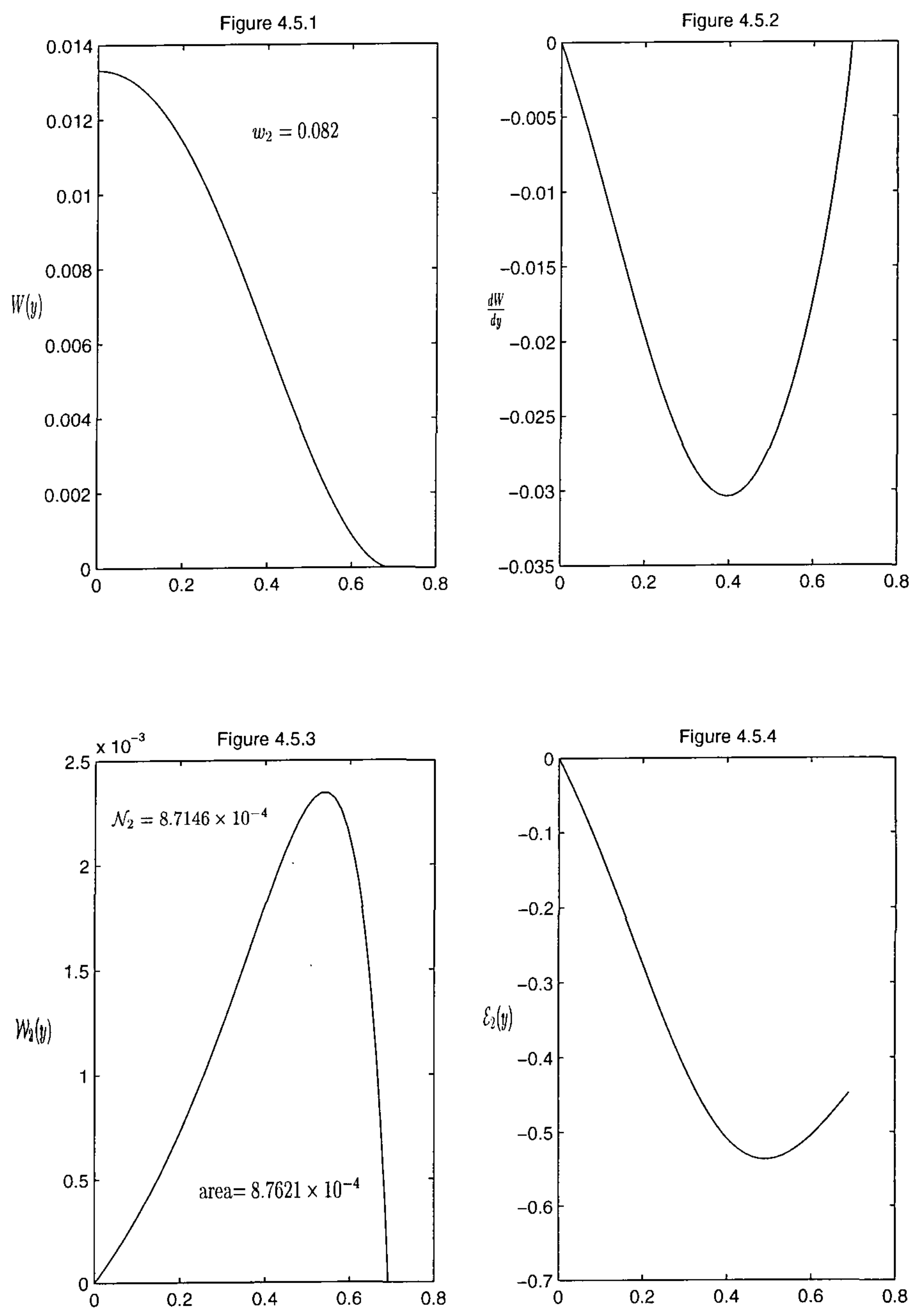
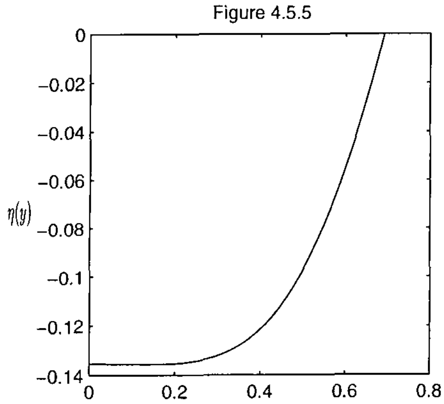


Figure 4.5: The eigen-state of $j = 2$.

Figure 4.5.5



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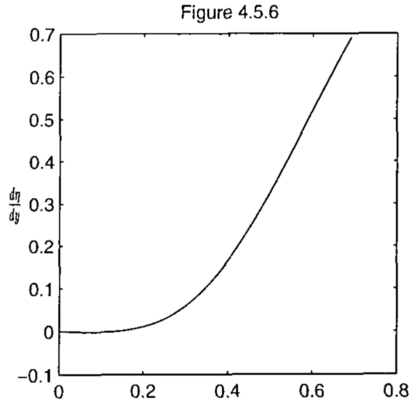

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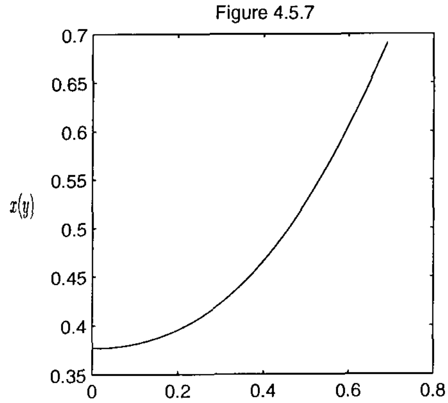

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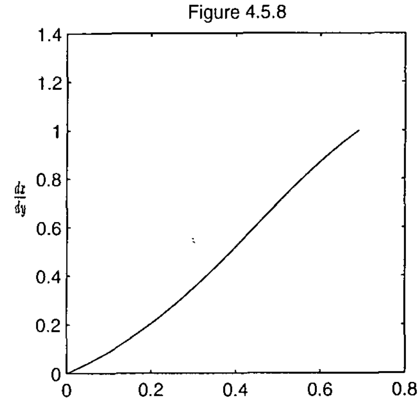

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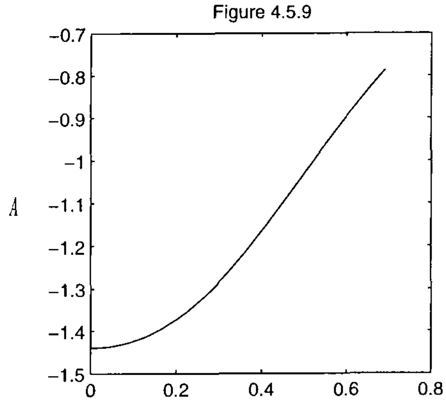

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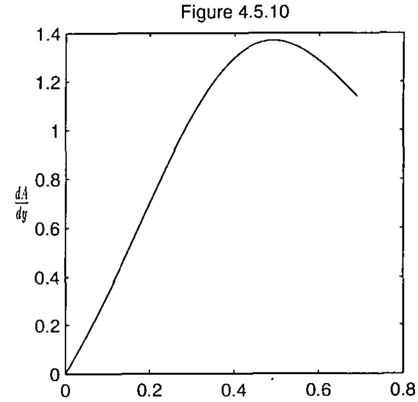


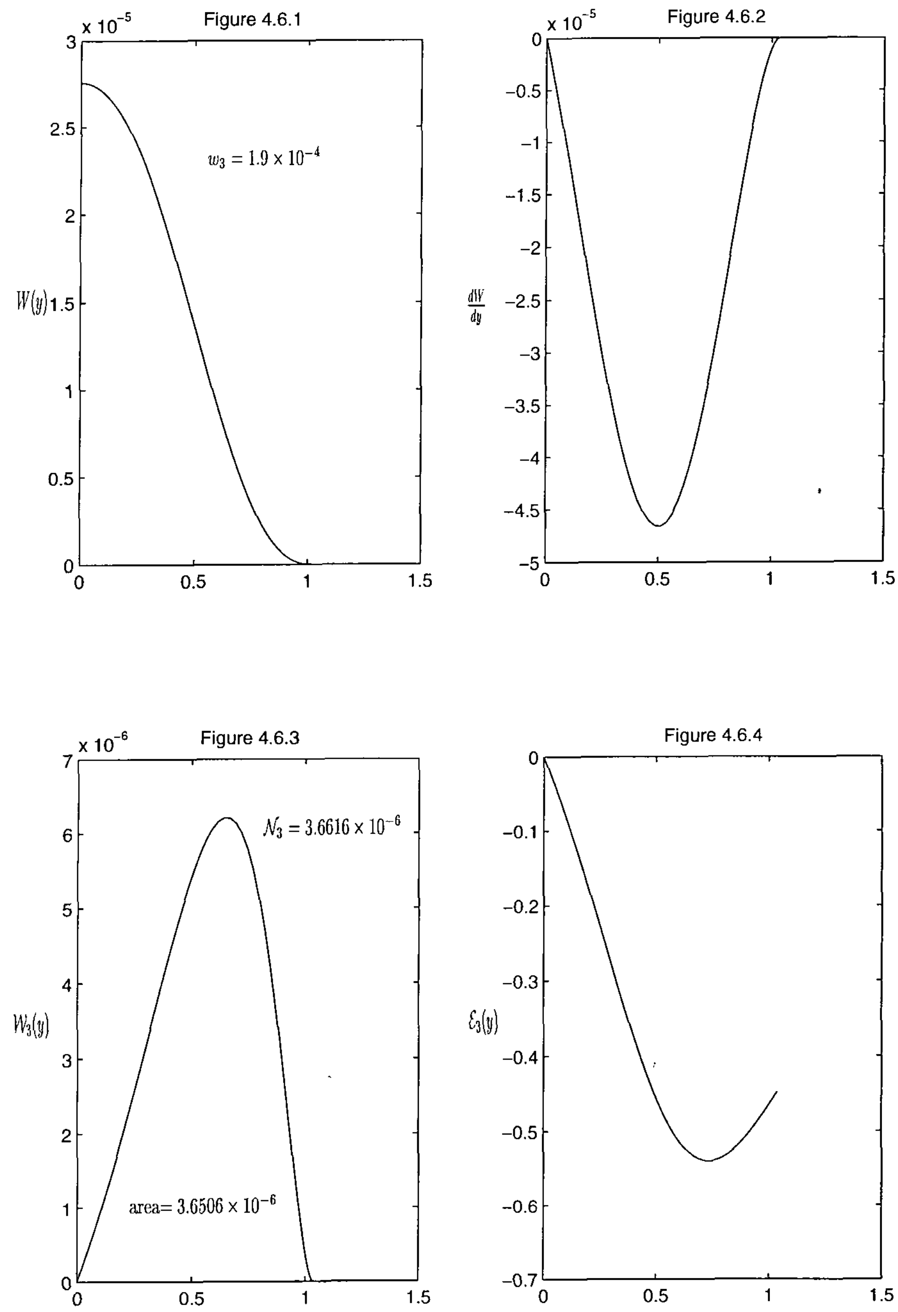
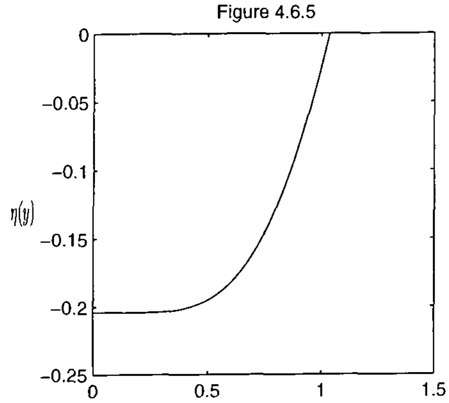


Figure 4.6: The eigen-state of $j = 3$.



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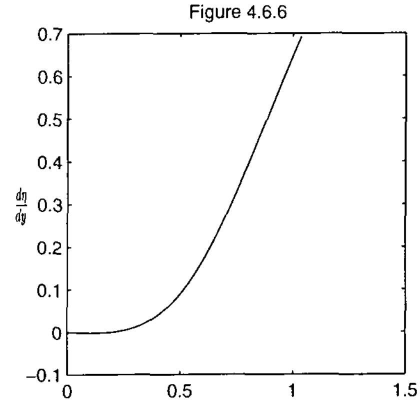

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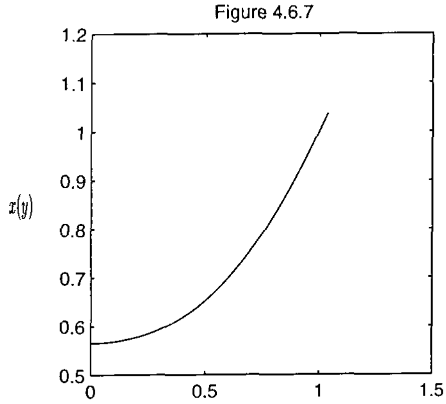

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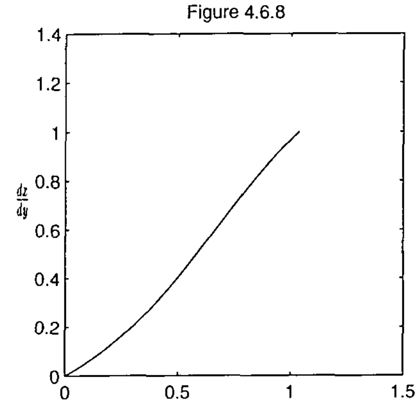

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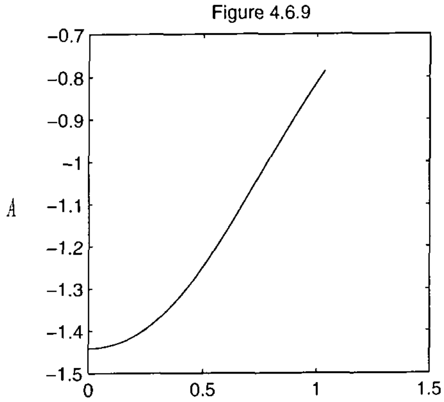

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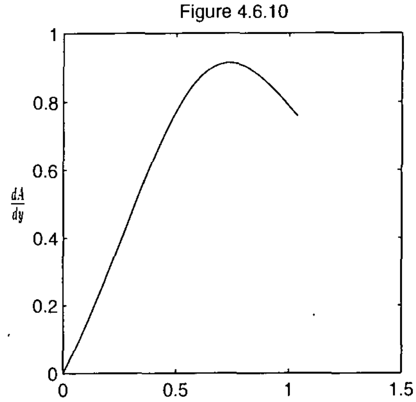


Figure 4.6.10

$M = M_s$

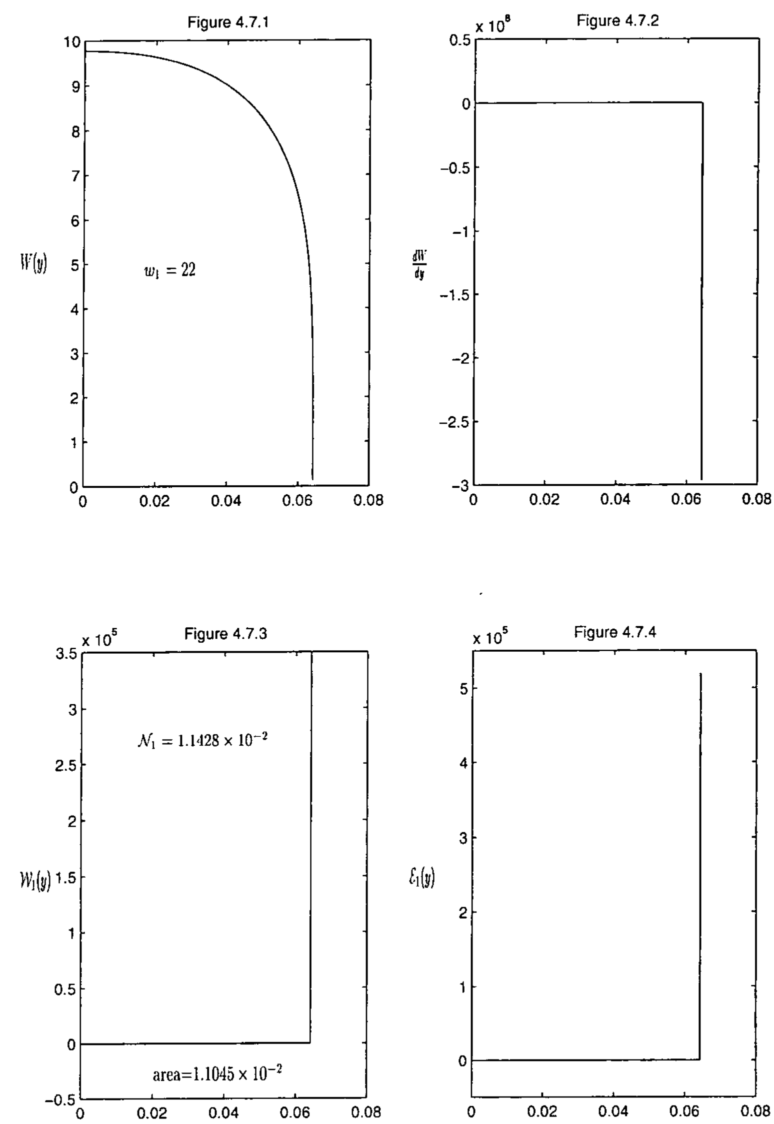


Figure 4.7: The eigen-state of $j = 1$ for the interior of a Reissner-Nordström black hole with parameters, $\epsilon_A = -1$, $\epsilon_\theta = 1$, $M = M_s$, $N_c = 5 \times 10^4$, $e = 10^{-3}$.

Figure 4.7.5

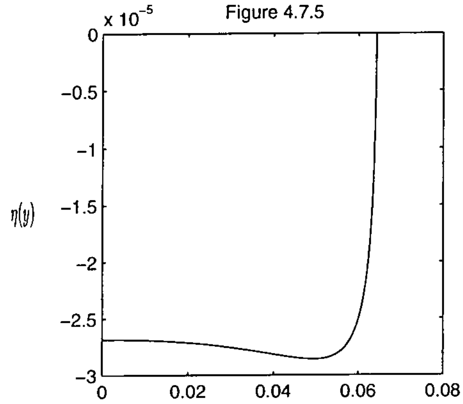
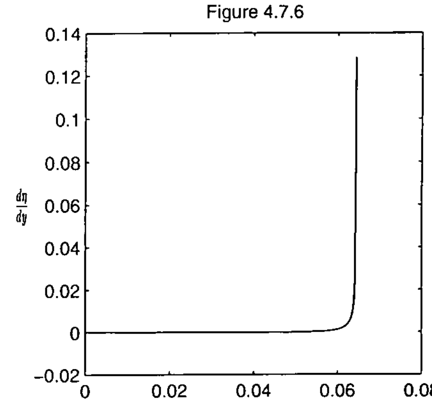


Figure 4.7.6



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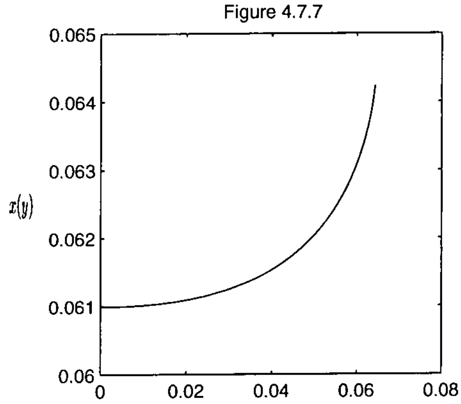

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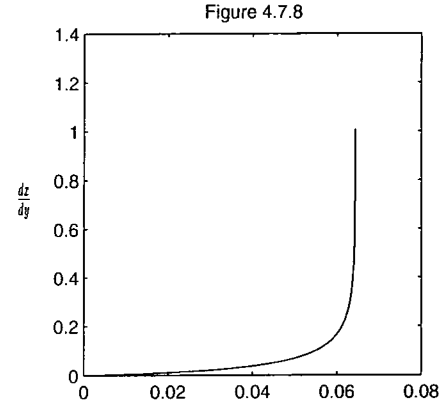

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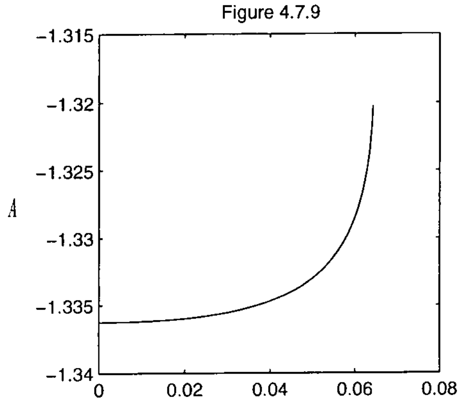

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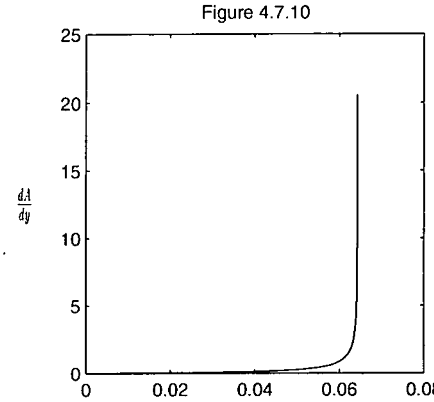


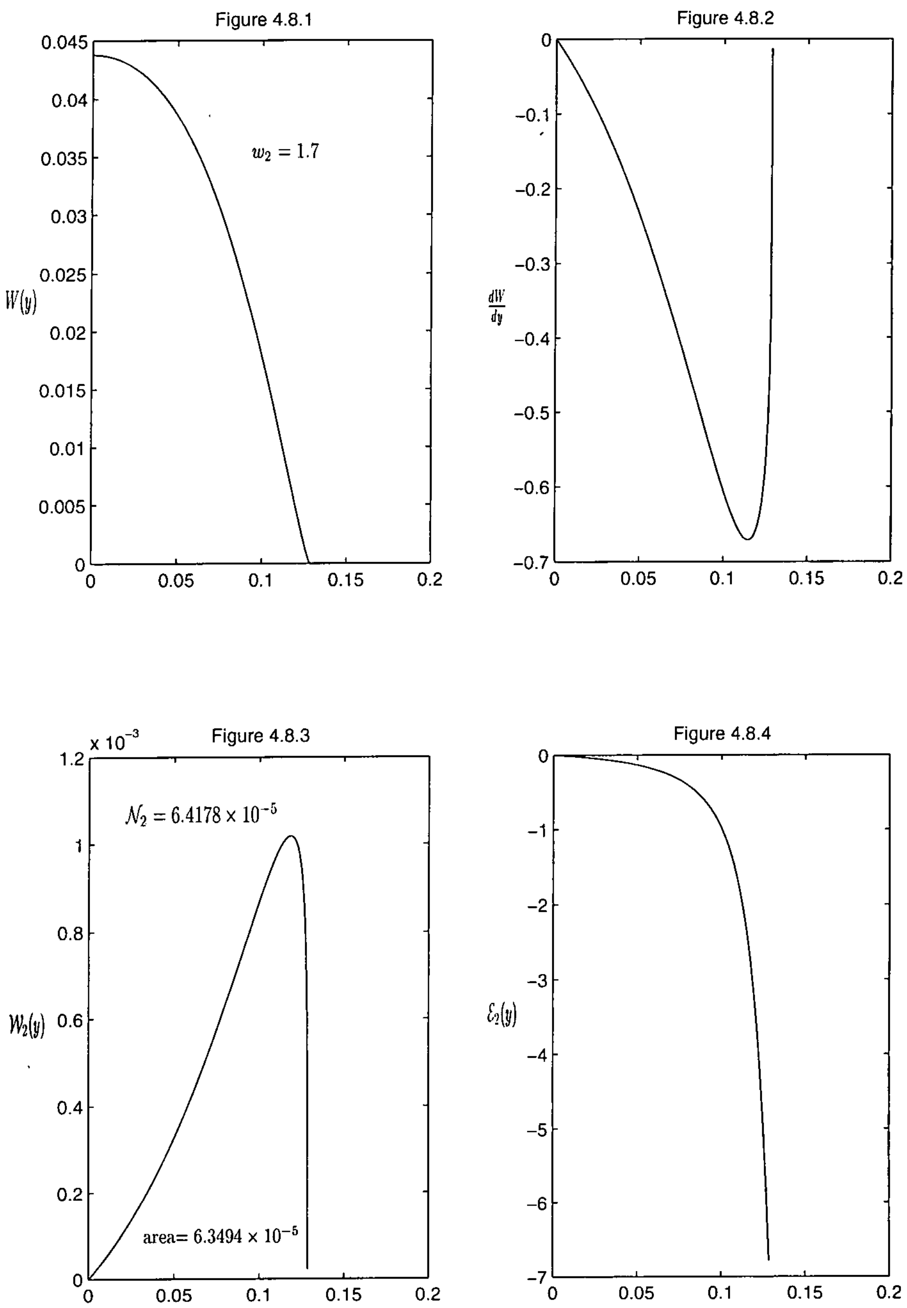
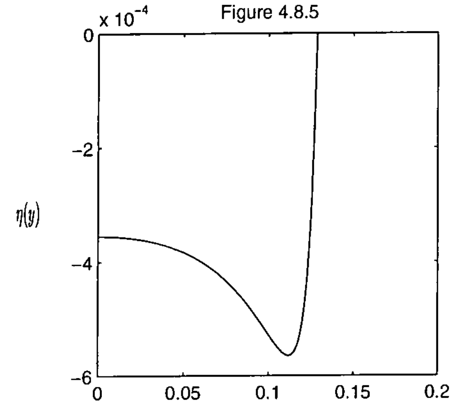


Figure 4.8: The eigen-state of $j = 2$.

Figure 4.8.5



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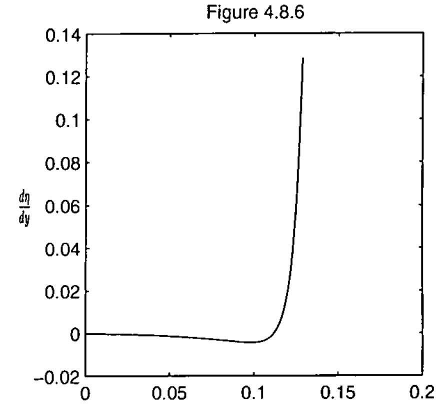

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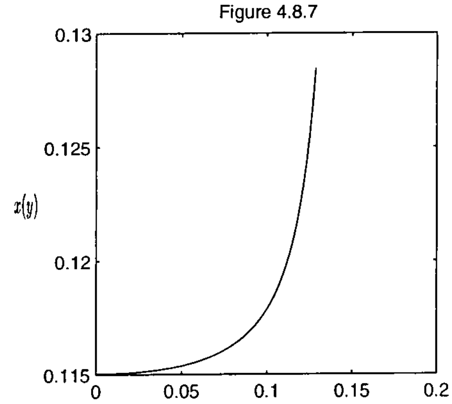

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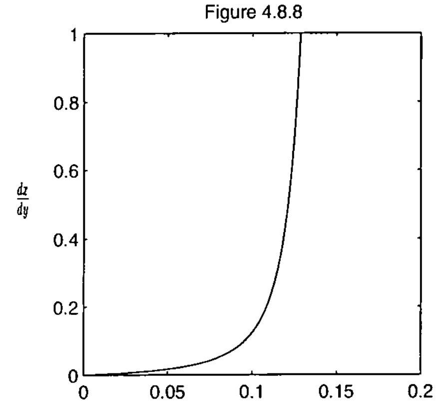

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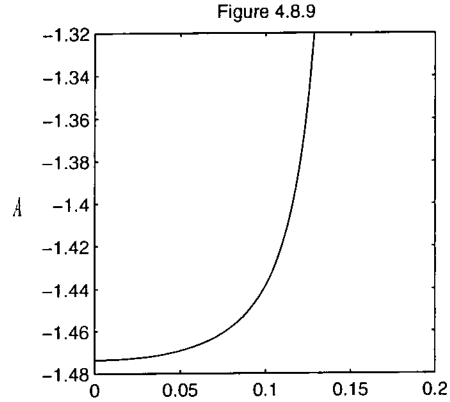

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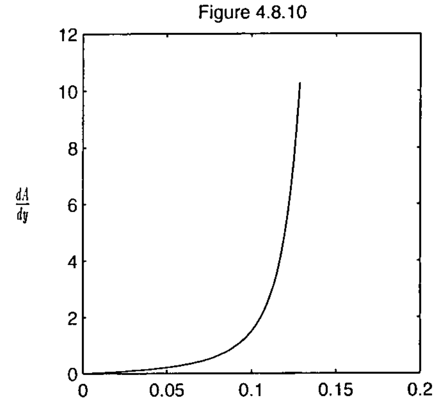


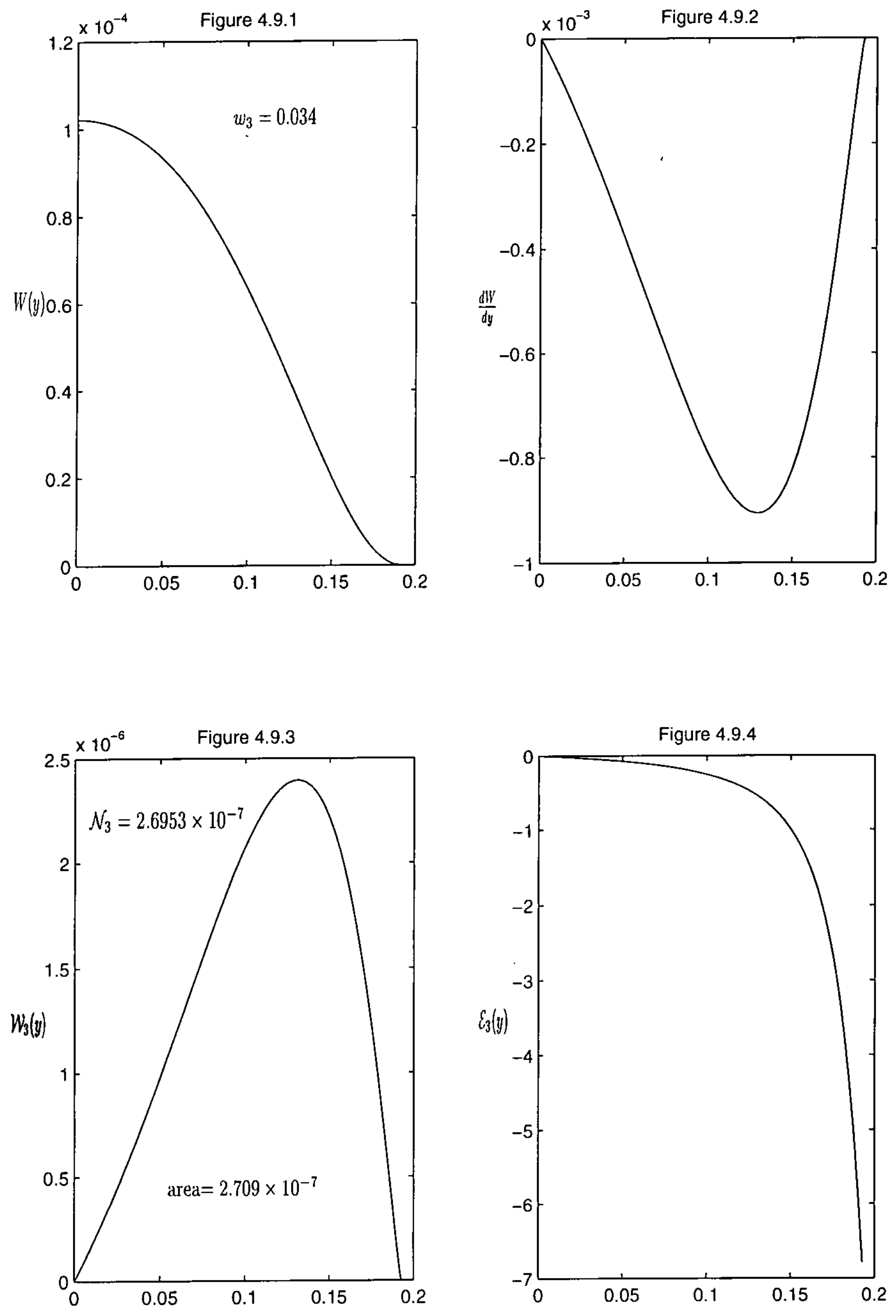
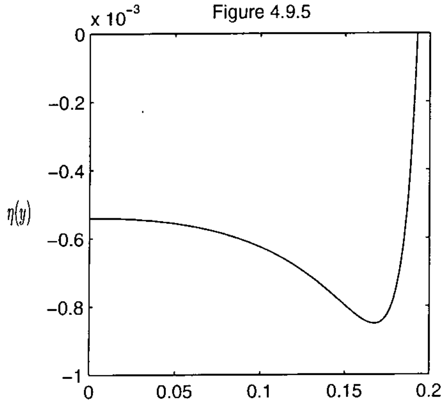


Figure 4.9: The eigen-state of $j = 3$.

Figure 4.9.5



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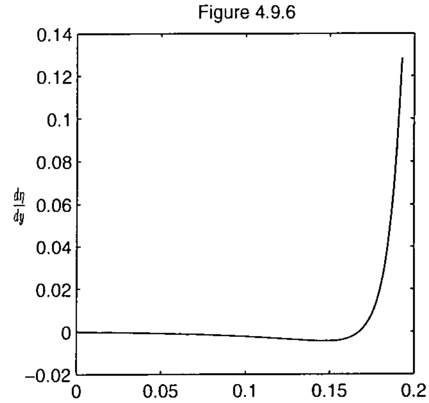

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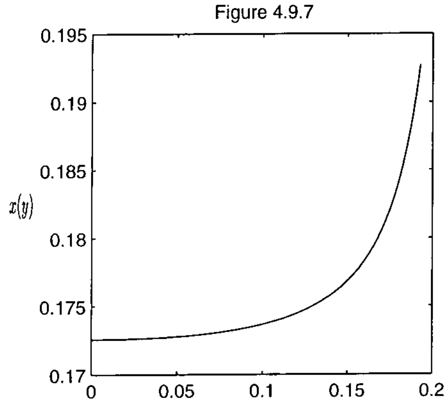

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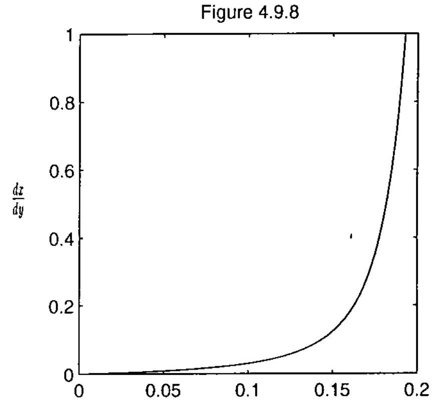

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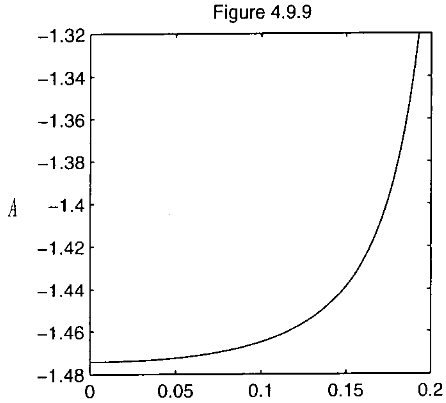

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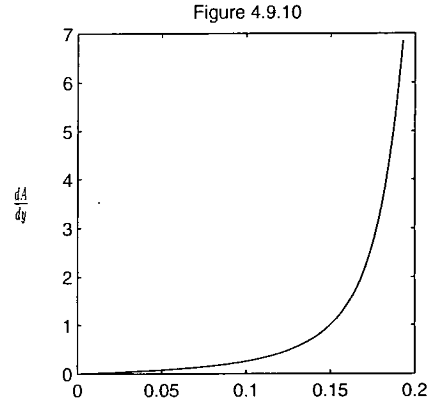


Limit of $q = 0$

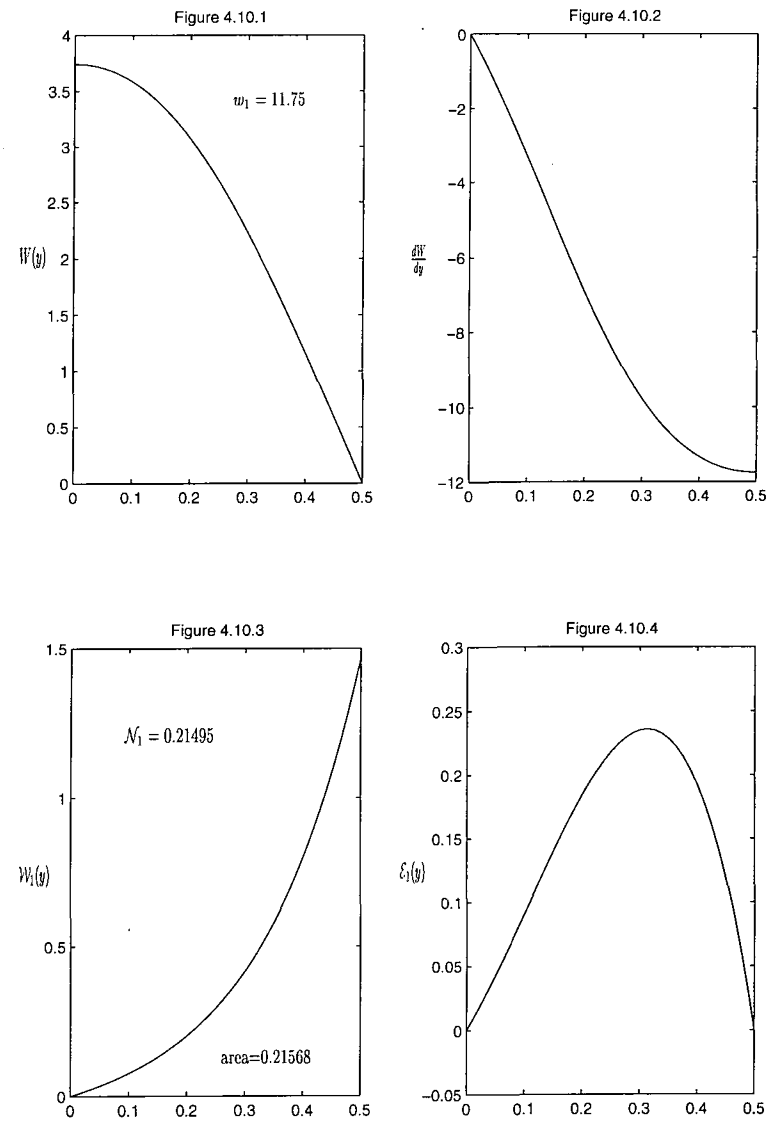


Figure 4.10: The eigen-state of $j = 1$ for the *charged* interior of a Schwarzschild black hole with parameters, $q = 0$, $\epsilon_A = -1$, $\epsilon_\theta = 1$, $M = M_l$, $e = 10^{-3}$. The $\mathcal{E}_j$, $\mathcal{N}_j$, and $\mathcal{W}_j$ are defined in equations (4.41), (4.26), and (4.37), respectively.

Figure 4.10.5

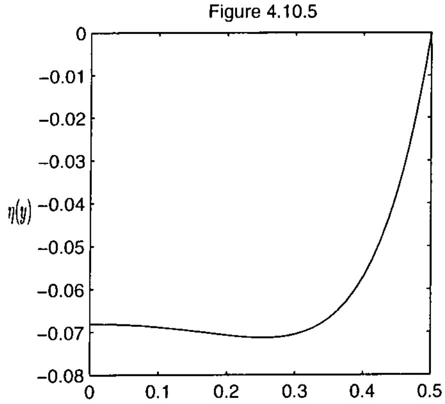


Figure 4.10.6

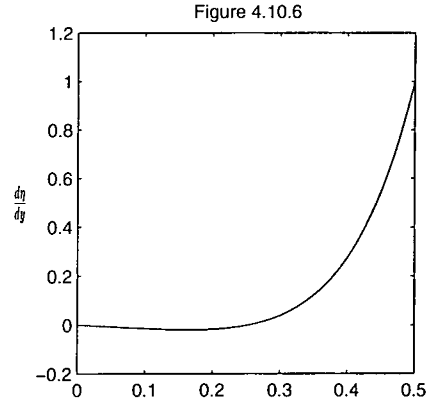


Figure 4.10.7

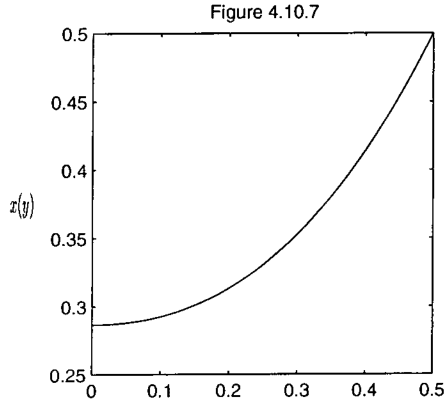
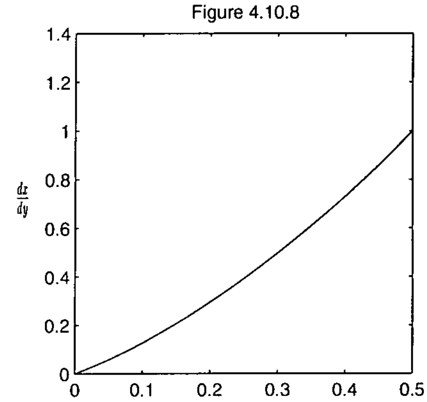


Figure 4.10.8



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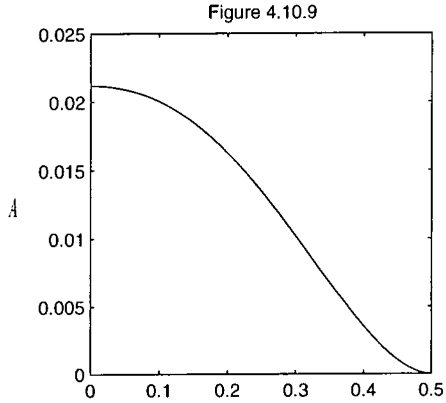

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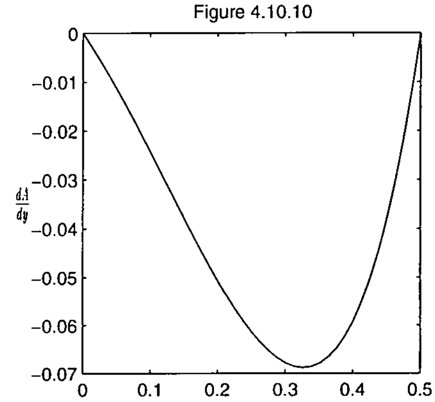


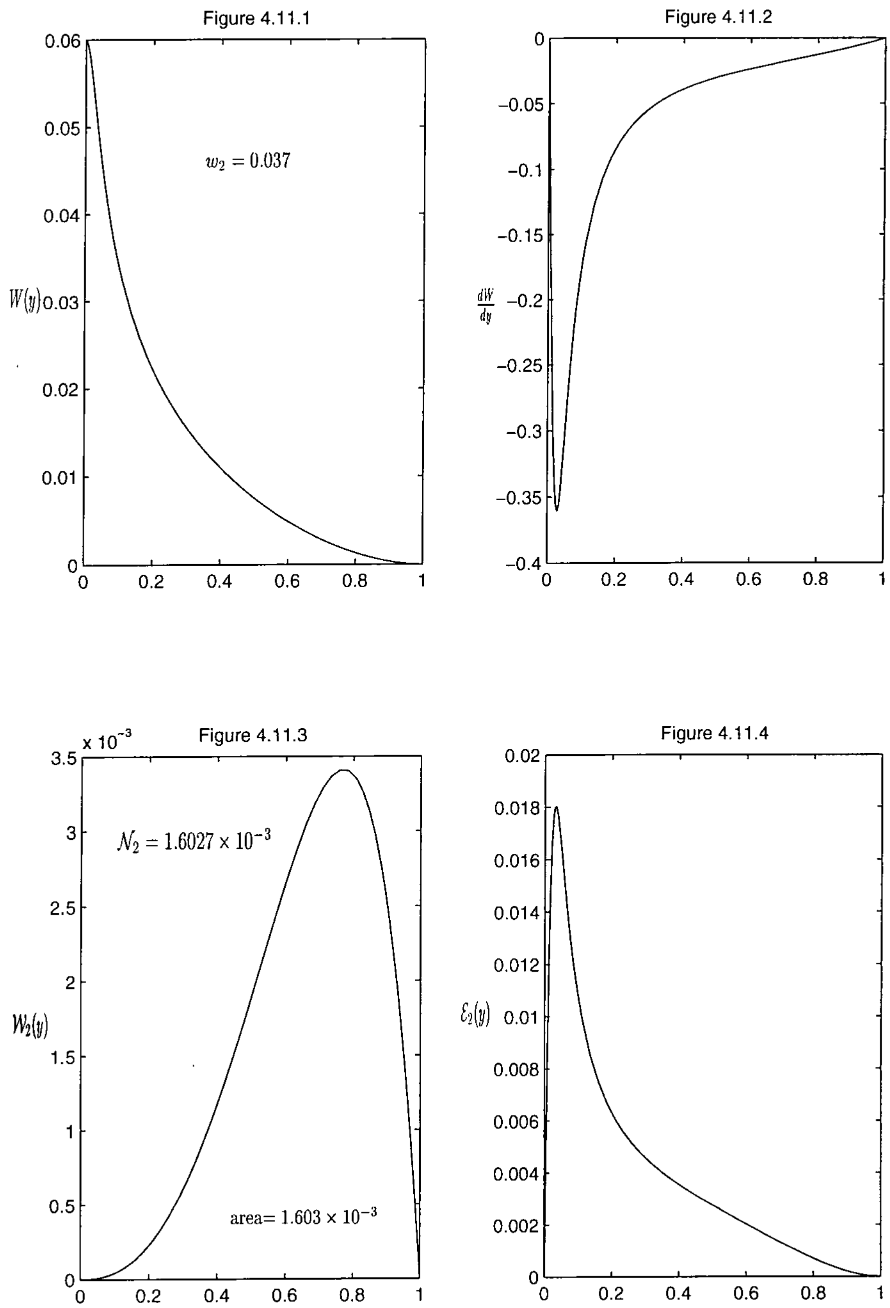


Figure 4.11: The eigen-state of $j = 2$.

Figure 4.11.5

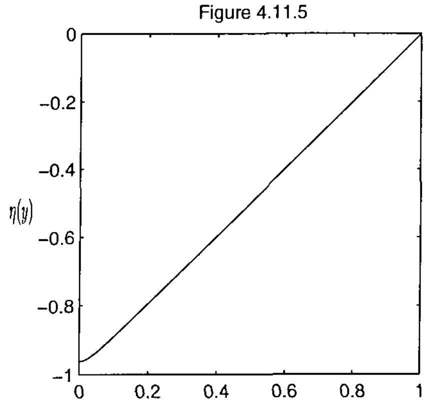


Figure 4.11.6

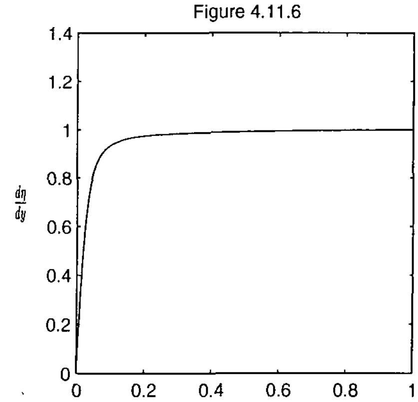
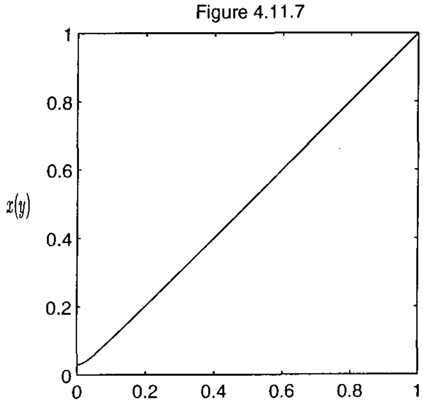


Figure 4.11.7



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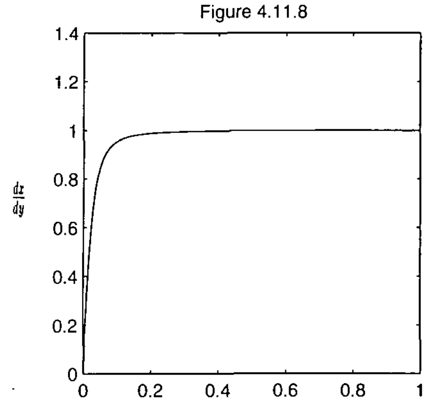
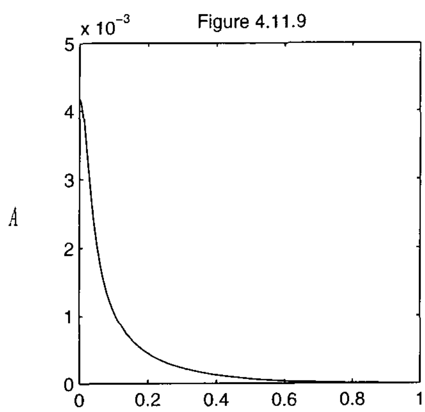


Figure 4.11.9



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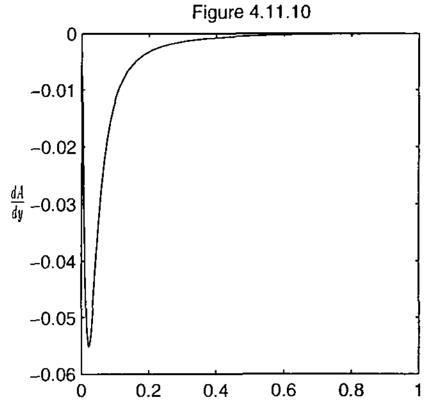


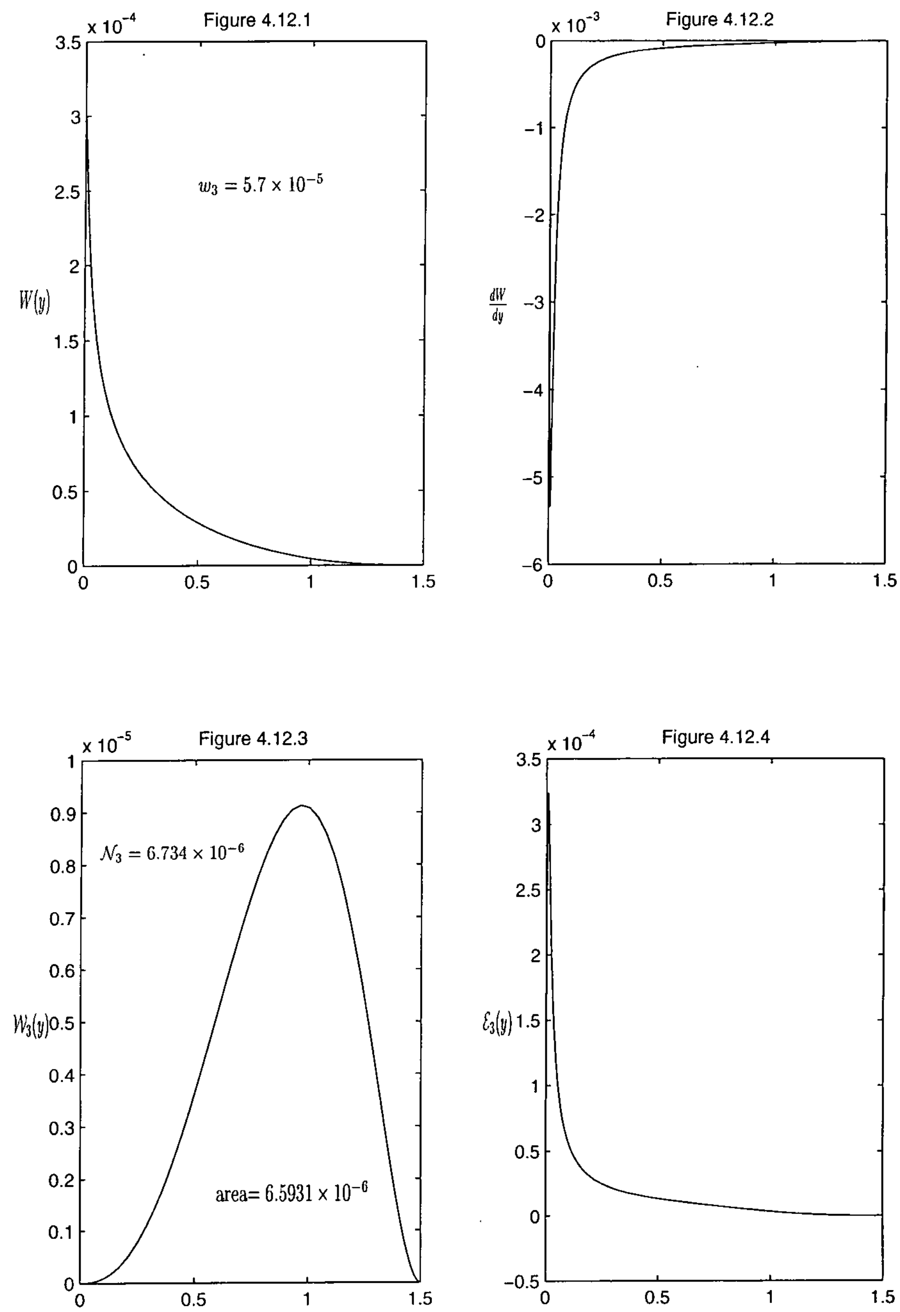


Figure 4.12: The eigen-state of $j = 3$.

Figure 4.12.5

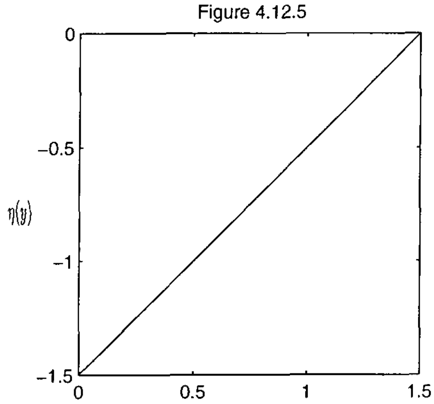
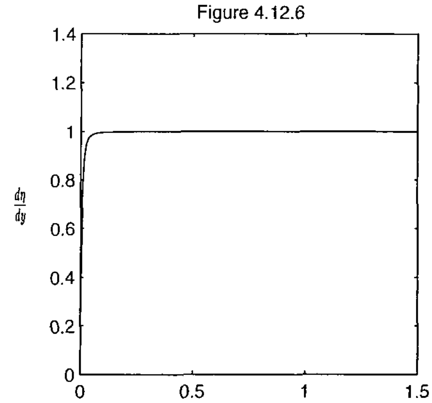


Figure 4.12.6



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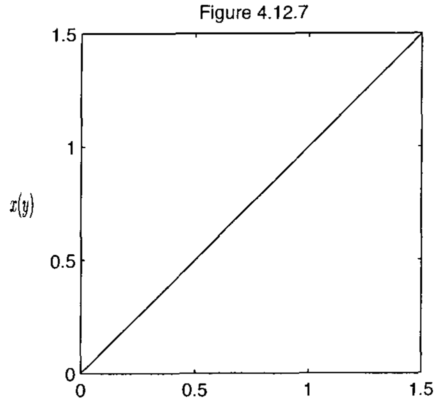

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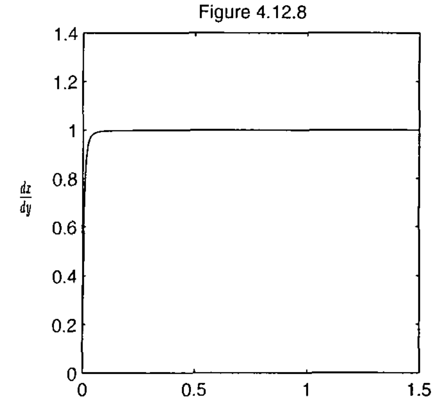

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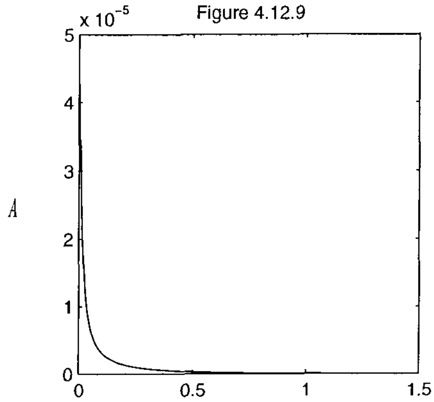

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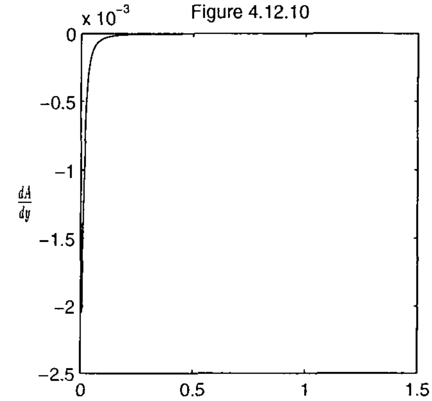


Figures for section 4.4

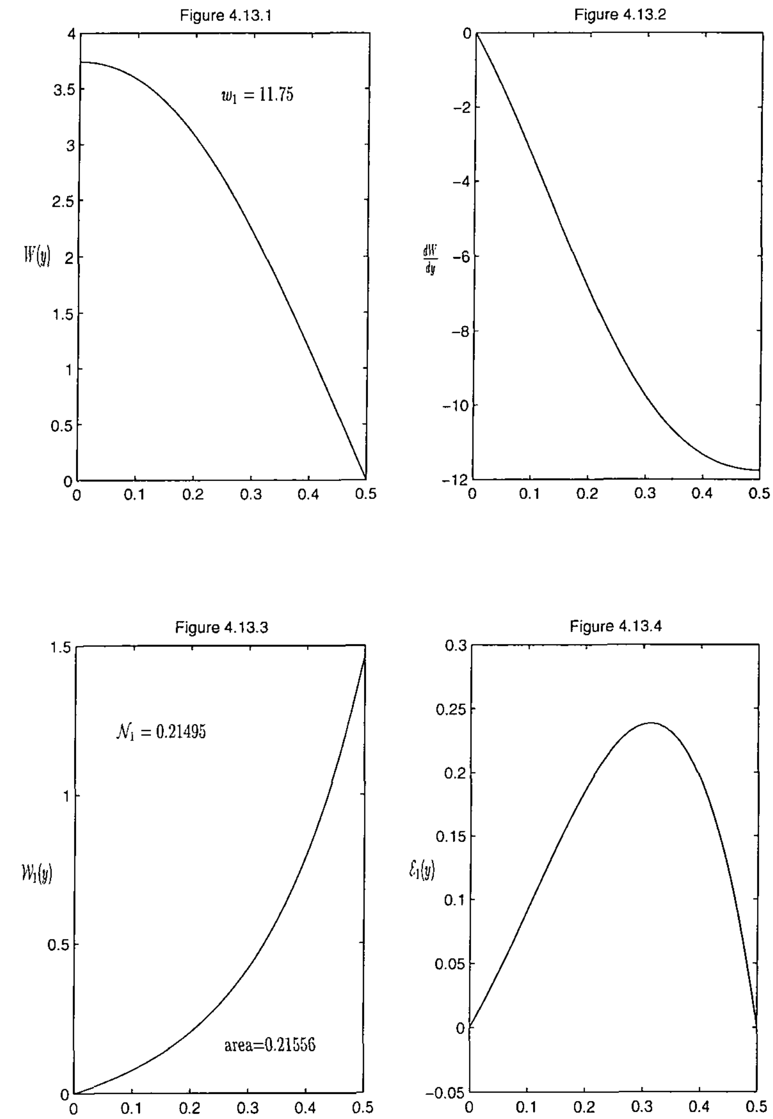


Figure 4.13: The eigen-state of $j = 1$ for the *charged* interior of a Schwarzschild black hole with parameters, $q = 0$, $\epsilon_A = \epsilon_\theta = 1$, $M = M_l$, $e = 10^{-3}$.

Figure 4.13.5

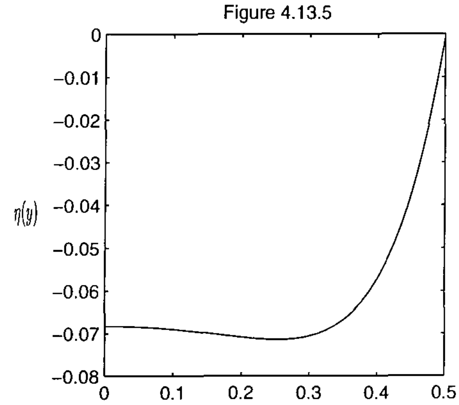
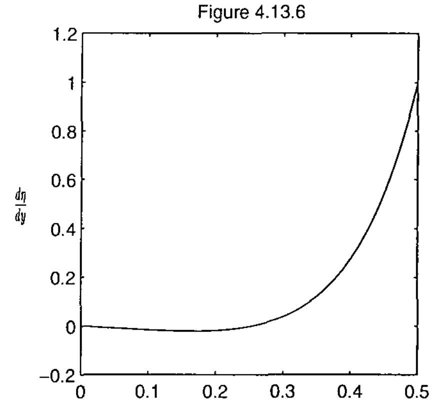


Figure 4.13.6



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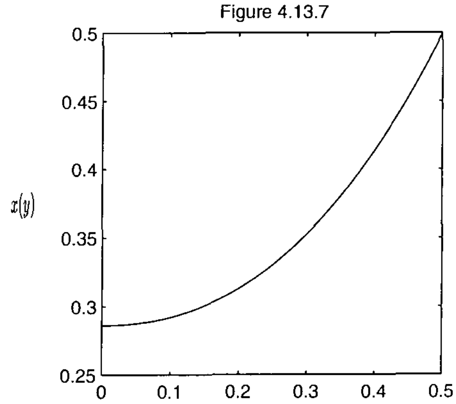

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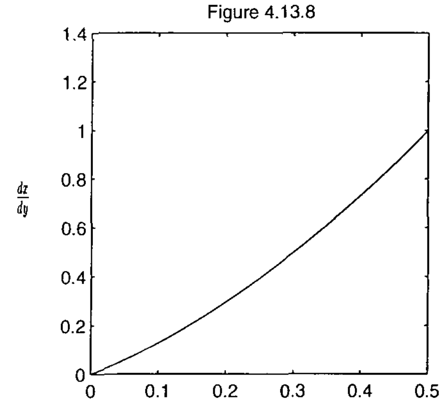

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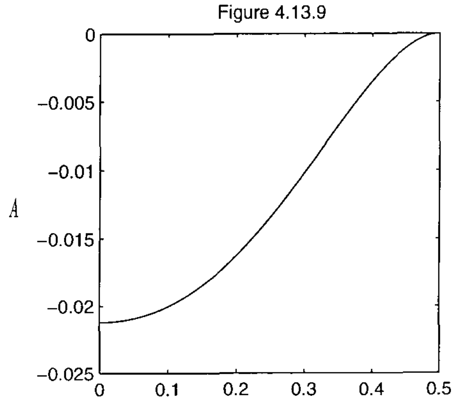

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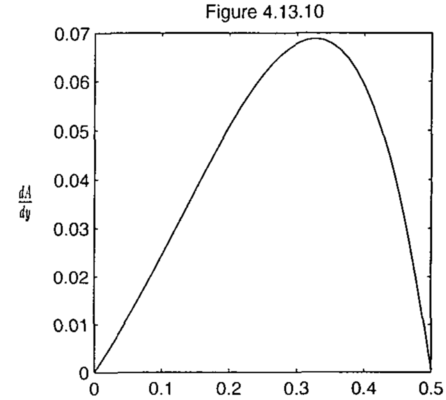


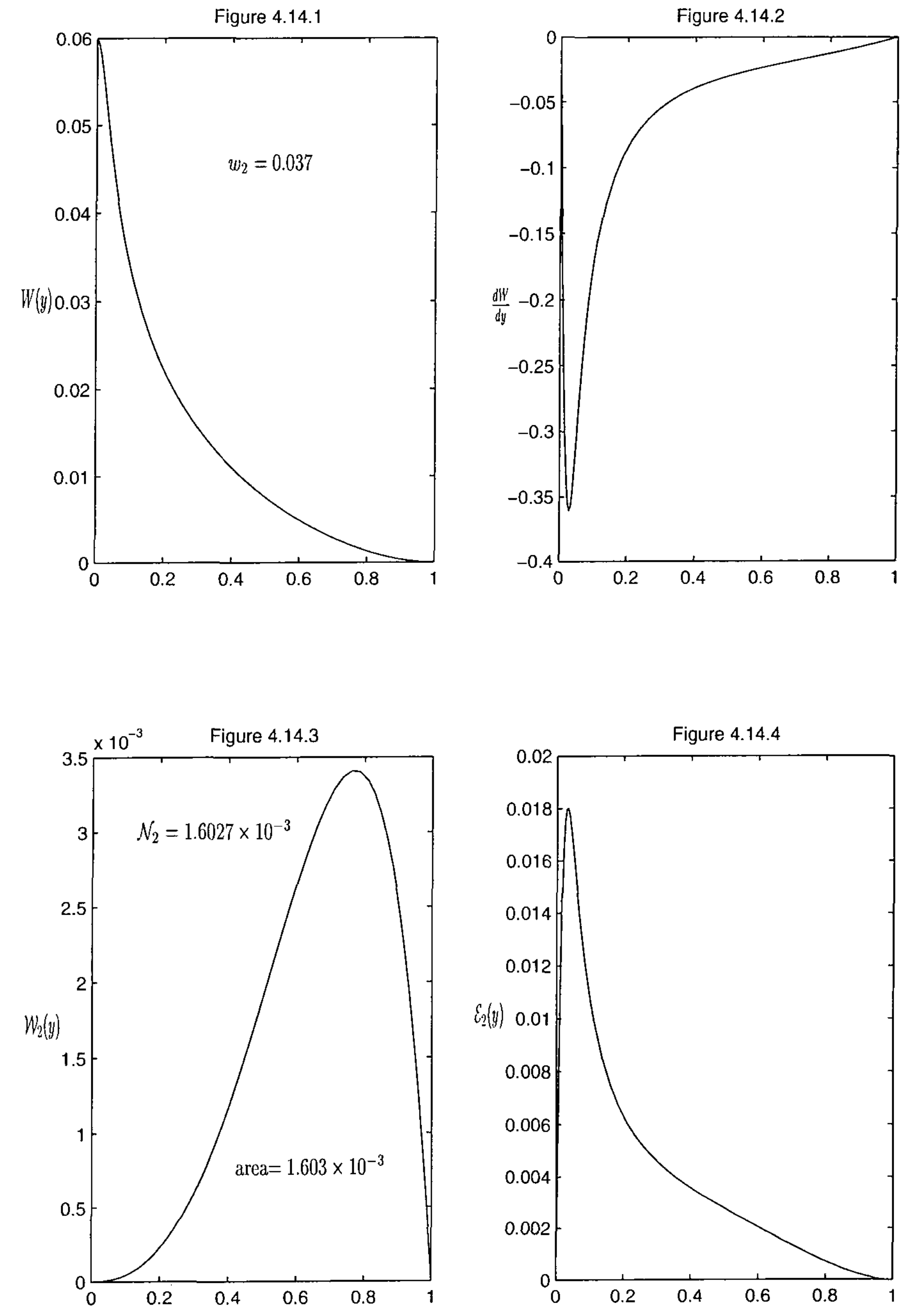


Figure 4.14: The eigen-state of $j = 2$.

Figure 4.14.5

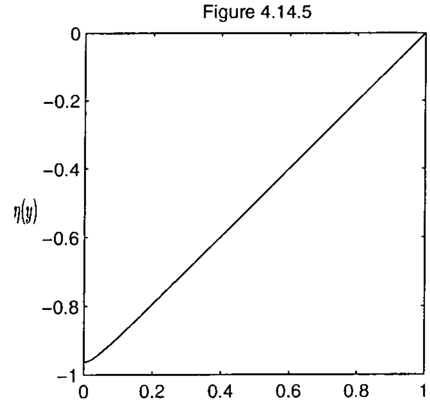


Figure 4.14.6

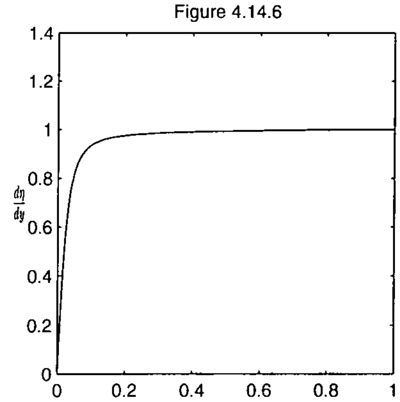


Figure 4.14.7

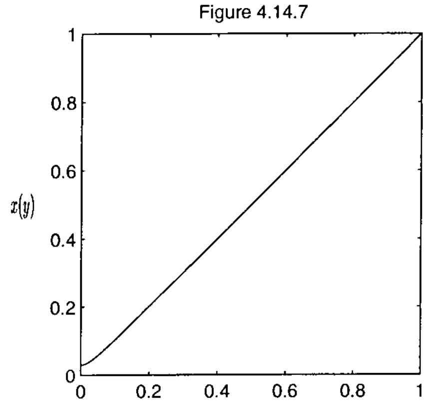


Figure 4.14.8

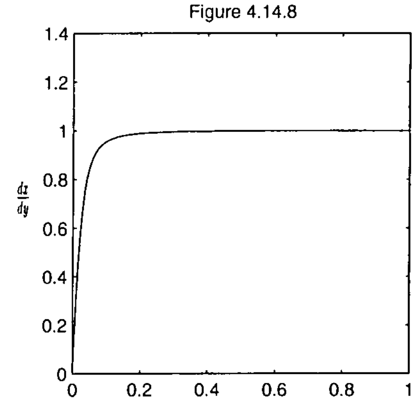


Figure 4.14.9

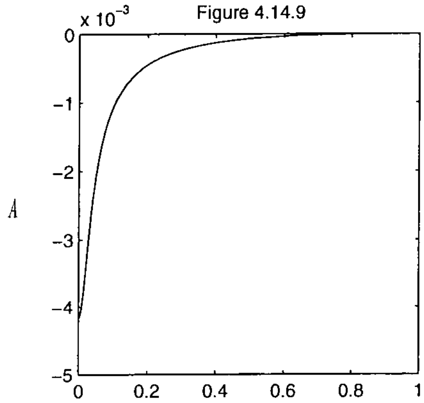
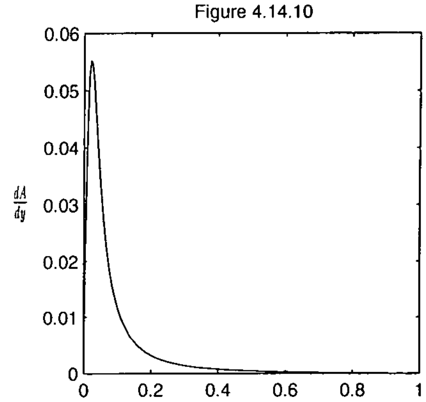


Figure 4.14.10



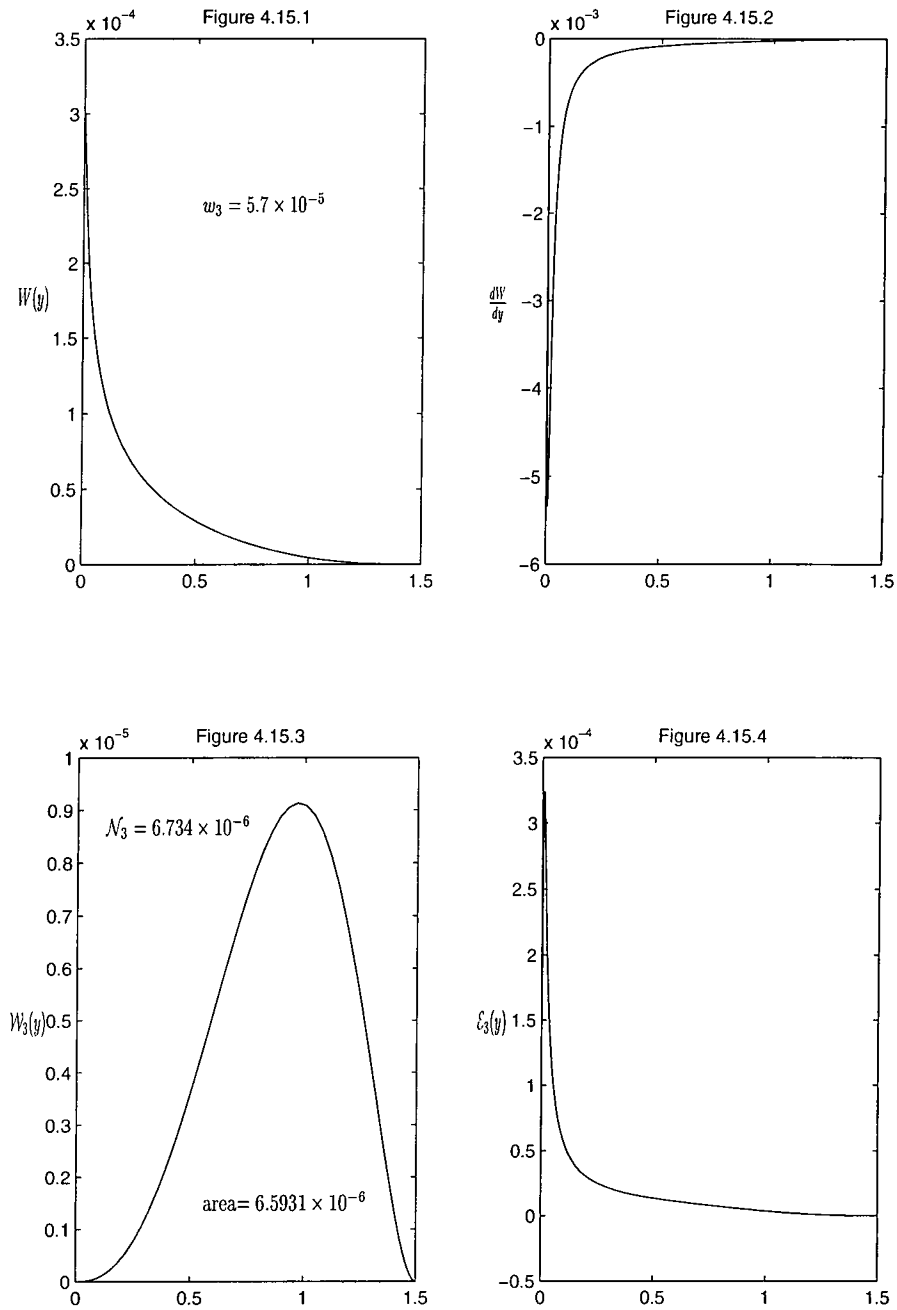


Figure 4.15: The eigen-state of $j = 3$.

Figure 4.15.5

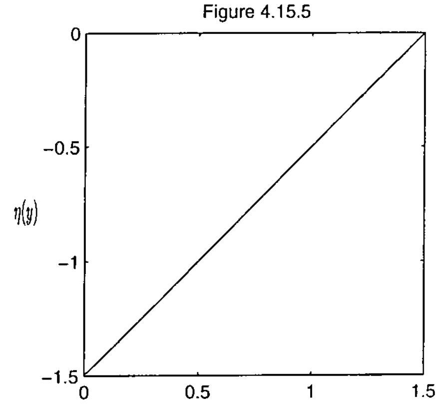


Figure 4.15.6

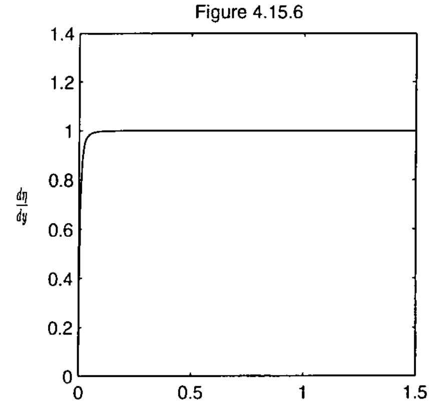


Figure 4.15.7

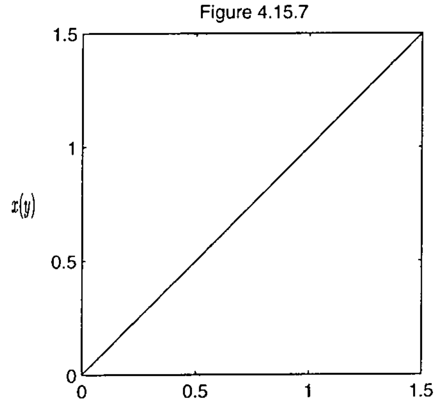


Figure 4.15.8

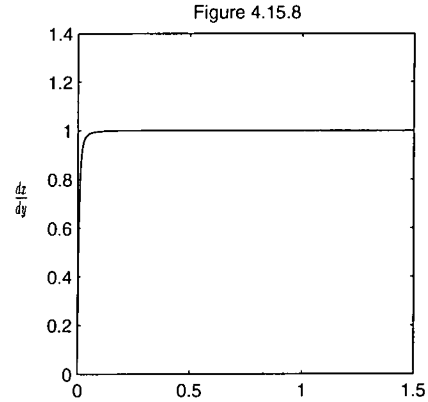


Figure 4.15.9

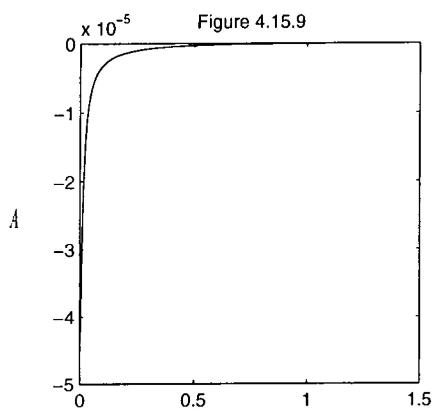
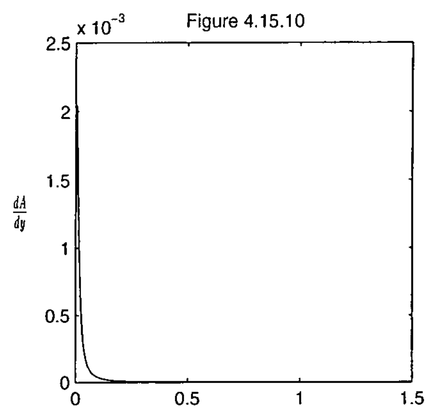


Figure 4.15.10



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