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Admissible Representations and Characters of the Affine
Superalgebras $\widehat{osp}(1, 2)$ and $\widehat{sl}(2|1)$

by

Michael Robert Hayes

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of Durham

Department of Mathematical Sciences, 1998



13 JAN 1999

Abstract

In this thesis we compute characters and supercharacters of irreducible admissible representations for the two affine superalgebras $\widehat{osp}(1, 2; \mathbb{C})$ and $\widehat{sl}(2|1; \mathbb{C})$.

The work on $\widehat{osp}(1, 2; \mathbb{C})$ includes a derivation of the embedding diagram. We compute the modular transformations of the Neveu-Schwarz characters of $\widehat{osp}(1, 2; \mathbb{C})$ and show that they transform in a manner consistent with the different possible free fermion spin structures on a torus.

In chapter 3 we turn our attention to $\widehat{sl}(2|1; \mathbb{C})$. Characters and supercharacters are computed for three classes of admissible representation. We have to derive the embedding diagram for one of these classes. We show that the integrable characters in the classes we study are identical to characters of the $N = 4$ superconformal algebra and that some of the $\widehat{sl}(2|1; \mathbb{C})$ characters have a pole in a certain limit. The residue at this pole is computed and it is found to be proportional to an $N = 2$ minimal character. Specialising to fractional levels k of the form $k + 1 = 1/u, u \in \mathbb{N}$, we consider the $SL(2|1)/SL(2)$ coset theory and make a conjecture that it is a product of a parafermion theory and a rational torus model. The appearance of parafermion characters and rational torus model characters in the branching functions of some examples that we have worked out leads to a conjecture for the general form of the branching functions whenever the level k has the form $k + 1 = 1/u$.

The modular T transformation can be worked out easily for any character or supercharacter we have computed. We work out the S transformation of the Neveu-Schwarz characters in two examples and find that we get a unitary S -matrix in each case. The thesis finishes with some interesting identities between $\widehat{su}(2)$ string functions which are a corollary of the work on branching functions.

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Declaration

None of the material in this thesis has been submitted before for a degree in this or any other university. The results reported are those of the author unless otherwise stated.

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Chapter 1

Introduction and Background

1.1 Some Historical Physical Background.

The concept of symmetry in physics has had a long and fruitful history. The natural mathematical formalism for discussing symmetry is group theory and in contemporary theoretical physics one makes use of sophisticated structures like Lie groups and supergroups and their Lie (super)algebras. Early examples of the use of Lie symmetry were in the theory of isospin and in Gell Mann's "eight-fold way". The idea of isospin was introduced by Heisenberg to explain why protons and neutrons were affected equally and the three pions were affected equally by the strong nuclear force despite their different electric charges. Gell Mann's idea was an attempt to make some order of the plethora of hadrons that had been discovered at the time and it met with early success in the form of a prediction of a new particle which was duly found—the Ω^- .

The isospin symmetry can be described by the Lie algebra $su(2)$ of the group $SU(2)$. Gell Mann's theory relied on the Lie algebra $su(3)$. Any Lie group has a Lie algebra which can be thought of as being the "tangent space at the identity" of that group. The connection between the abstract mathematics of Lie algebras and observed particles is the representation theory of the algebra. The particles are thought of as being multiplets which carry a representation of the Lie algebra *i.e.*, the elements of the algebra (in the guise of their representation *e.g.*, a matrix) operate on the carrier space and can mix up the particles in the multiplet. These ideas have developed into modern gauge theory. We shall not be concerned with particulars of gauge theories at all in this work. We shall be more concerned with theories that exhibit *conformal symmetry*, for example string theory. However, as we shall see symmetry associated to Lie algebras will emerge again in the Wess-Zumino-Witten theories.

During the 1970's and early 1980's a new theory of elementary particles was being

formed—string theory. Here one considers a field theory defined on the world sheet of a string. Consider the theory of closed bosonic strings¹. Suppose we parametrize the string worldsheet by σ and τ with σ periodic with period 2π . Let $X^\mu(\sigma, \tau)$ be the spacetime coordinates of a point on the worldsheet *i.e.*, X^μ is an embedding of the worldsheet in spacetime. If we introduce the field $g_{\alpha\beta}(\sigma, \tau)$ as a metric on the worldsheet, we have the action (for free strings),

$$S(X, g) = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \quad (1.1)$$

where $g = \det g_{\alpha\beta}$, $\alpha, \beta = 0, 1$ and $(\sigma^0, \sigma^1) = (\tau, \sigma)$. T is a parameter called the string tension. If we now proceed to quantize our bosonic string theory we consider the path integral (we set $\hbar = 1$),

$$Z = \int D^d X Dg e^{-S(X, g)}. \quad (1.2)$$

We integrate over a d -dimensional spacetime and also over all possible metrics on the worldsheet. So that we only integrate over essentially different embeddings X and metrics g we need to know under which transformations S is invariant. It turns out that S is invariant under the following transformations: general coordinate transformations in two dimensions,

$$\begin{aligned} \sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha(\sigma) \\ g_{\alpha\beta} &\rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) \\ X^\mu(\sigma) &\rightarrow X^\mu(\tilde{\sigma}), \end{aligned} \quad (1.3)$$

local changes of scale (Weyl transformations),

$$\begin{aligned} g_{\alpha\beta} &\rightarrow e^{\lambda(\sigma)} g_{\alpha\beta} \\ X^\mu(\sigma) &\rightarrow X^\mu(\sigma), \end{aligned} \quad (1.4)$$

and finally, Poincaré invariance,

$$\begin{aligned} g_{\alpha\beta} &\rightarrow g_{\alpha\beta} \\ X^\mu(\sigma) &\rightarrow \Lambda^\mu_\nu X^\nu(\sigma) + a^\mu. \end{aligned} \quad (1.5)$$

There is a special case of the general coordinate transformations in (1.3) where the metric tensor is merely multiplied by a scalar function

$$g_{\alpha\beta}(\sigma) \rightarrow \Omega(\sigma) g_{\alpha\beta}(\sigma). \quad (1.6)$$

¹The following exposition is taken from the review by Dixon [Dix89]

Such transformations are called *conformal* transformations. Now we could undo the transformation (1.6) of the metric by a Weyl transformation (1.4) (which leaves the coordinates X^μ unchanged.) Thus there is a combined transformation (a conformal transformation then a Weyl transformation) of the worldsheet parameters that leaves the metric invariant and is a coordinate transformation acting on the X^μ . A field theory which is invariant under this type of transformation is called a conformal field theory. Thus we have a conformal field theory in two dimensions (the string worldsheet). We shall make a digression now to discuss some aspects of the conformal field theory ².

We shall begin this discussion in a d -dimensional spacetime with coordinates x^μ and then specialize to two dimensions. Recall that the conformal transformations are local changes of scale (1.6). A consequence of this is that the angle between vectors is conserved. The angle is given by $u \cdot v / |u||v| = g_{\alpha\beta} u^\alpha v^\beta (g_{\gamma\delta} u^\gamma u^\delta g_{\sigma\rho} v^\sigma v^\rho)^{-1/2}$ which is invariant under a rescaling of g . If we consider infinitesimal coordinate transformations $x^\mu \rightarrow x^\mu + \epsilon^\mu$ in flat Minkowski space where the metric is $\eta_{\mu\nu}$, then ϵ^μ must satisfy the conformal Killing equation:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial^\rho \epsilon_\rho \eta_{\mu\nu}, \quad (1.7)$$

For a d -dimensional Minkowski spacetime, $d > 2$, the solutions of the conformal Killing equation are translations, Lorentz transformations (which together comprise Poincaré transformations), global changes of scale (dilatations) and also special conformal transformations which are like inversions. The Poincaré transformations actually leave the metric invariant ($\Omega = 1$). In four dimensional Minkowski space in particular, the conformal group is fifteen dimensional. However, in a two dimensional spacetime *which is euclidean* the conformal Killing equation reduces to the Cauchy-Riemann equations and so the conformal group is infinite dimensional because any holomorphic function is then a conformal transformation. In two dimensions the finite conformal transformations are Möbius transformations. We return to a general d -dimensional spacetime momentarily. Every continuous symmetry has a conserved current associated to it. For conformal symmetry under which the coordinates transform as $x^\mu \rightarrow x^\mu + \epsilon^\mu$ we can take the conserved current j^μ to be $j^\mu = T^\mu_\nu \epsilon^\nu$ where $\partial_\mu j^\mu = 0$. T^μ_ν is the energy momentum tensor of the theory and ϵ satisfies the conformal Killing equation (1.7). Then, the invariance under translations, Lorentz transformations and global scaling lead to T being

²The following discussion on conformal field theory is largely taken from the lectures by Ginsparg [Gin89].

conserved, symmetric and traceless respectively.

Now return to a two-dimensional spacetime (perhaps a string worldsheet) and suppose that the space direction is periodic (a scenario which is useful to consider in the context of a theory of closed strings). If τ and σ are the coordinates on the string worldsheet, the periodicity is $X(\sigma + 2\pi) = X(\sigma)$ for any field X of the theory. Now do a Wick rotation so that the worldsheet is a euclidean space. We can conformally map this euclidean worldsheet onto the complex plane by,

$$z = e^{\tau+i\sigma} \quad \bar{z} = e^{\tau-i\sigma} \quad \sigma, \tau \in \mathbb{R} \quad (1.8)$$

and henceforth we consider z and \bar{z} to be independent variables. Then curves of equal time τ on the worldsheet get mapped to circles centered at the origin on the plane so translations in time correspond to moving radially on the plane. The infinite past ($\tau = -\infty$) gets mapped to $z = 0$ and the infinite future ($\tau = +\infty$) gets mapped to the point at infinity on the complex plane. Now that we are working on the complex plane, we have at our disposal all the machinery of complex analysis—contour integrals and the like.

In two dimensions a conserved, symmetric, traceless energy-momentum tensor has just two independent components: $T_{zz} \stackrel{d}{=} T(z)$ and $T_{\bar{z}\bar{z}} \stackrel{d}{=} \bar{T}(\bar{z})$. That these two components are functions of just z and \bar{z} respectively follows from the conservation of the energy-momentum tensor. We can expand these functions in Laurent series,

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}. \quad (1.9)$$

In a quantum theory where T and \bar{T} become operators, it is important to know the behaviour of products of (operator-valued) fields. To this end, the operator product expansion *e.g.*,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots, \quad (1.10)$$

is of fundamental importance. One can work out the OPE of any two fields. However the OPE of a field with the energy-momentum tensor is especially important because knowing this we can work out the small variation in a field under an infinitesimal conformal transformation in its arguments. More concretely if $z \rightarrow z + \epsilon(z)$ then the local field $X(z)$ changes by an amount,

$$\delta_\epsilon X(z) = \frac{1}{2\pi i} \int_{C_z} dw \epsilon(w) T(w) X(z), \quad (1.11)$$

where C_z is a contour which surrounds z and we integrate around it in the positive sense *i.e.*, anticlockwise. Thus the role of the energy-momentum tensor as the generator of conformal transformations is evident. It is not hard to show that the OPE (1.10) together with the inversion,

$$L_n = \frac{1}{2\pi i} \int_{C_0} dz T(z) z^{n+1}, \quad C_0 \text{ a contour around the origin} \quad (1.12)$$

of the Laurent series, imply the *Virasoro algebra*,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad [L_n, c] = 0. \quad (1.13)$$

Since the energy-momentum tensor is the generator of conformal transformations it is this algebra that characterises a conformal field theory in two dimensions. There is also a version of (1.13) involving \bar{L}_n 's derived from the other component $\bar{T}(\bar{z})$ but because the two algebras commute with one another we can confine our attention to one version.

Now we return to string theory in Minkowski spacetime. It is possible to obtain the Virasoro generators L_n from the bosonic string theory as quantities bilinear in the oscillator modes which are the Fourier modes of the solution to the equation of motion,

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0 \quad (1.14)$$

(*i.e.*, the wave equation) obtained from the action (1.1). The solution to this equation can be written,

$$X^\mu(\sigma, \tau) = x^\mu + l\alpha_0^\mu \tau + i\frac{1}{2}l \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\frac{\alpha_n^\mu}{n} e^{-i(\tau+\sigma)n} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-i(\tau-\sigma)n} \right). \quad (1.15)$$

That $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$ is evident. The parameter l is equal to $1/\sqrt{T}$ (where T is the string tension or energy per unit length) and is some fundamental length. In quantising the theory in the canonical manner, one makes the α_n^μ 's into operators and demand that they satisfy the following commutation relations,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\alpha_m^\mu, \tilde{\alpha}_n^\mu] = 0. \quad (1.16)$$

Then with,

$$L_n = \sum_{m \in \mathbb{Z}} \dot{\!:\!} \alpha_{n-m}^\mu \alpha_{m\mu} \dot{\!:\!}, \quad (1.17)$$

one can recover the Virasoro algebra from the commutation relations (1.16) by being very careful with the normal ordered products. In this particular case we obtain $c = d$, the dimension of spacetime.

In 1984 Witten [Wit84] showed how one could ensure that a σ -model would be conformally invariant (at least to low orders in perturbation theory.) His insight was to add another term, the Wess-Zumino term, to the usual σ -model action. We have then,

$$S_{WZW} = \frac{k}{16\pi} \int d\sigma d\tau \operatorname{Tr} ((\partial_m g)(\partial^m g^{-1})) + \frac{k}{24\pi} \int d^3y \epsilon^{mnl} \operatorname{Tr} ((g^{-1} \partial_m g)(g^{-1} \partial_n g)(g^{-1} \partial_l g)). \quad (1.18)$$

where the parameter k is to be quantised, in fact $k \in \mathbb{Z}$. The first term is the action of the principal σ -model associated to G and the second term is the Wess-Zumino action. In each term g is a map into G . In the first action the domain of g is the worldsheet, parametrised by σ and τ , and in the second term the domain of g has been extended into some 3-dimensional space which has the worldsheet as its boundary and such that on the boundary the value of g is the same as it was when defined just on the worldsheet. The theory defined by (1.18) is known as a Wess-Zumino-Witten (WZW) model. The action in (1.18) is invariant under another symmetry too,

$$g(\tau, \sigma) \rightarrow \Omega(\tau + \sigma)g(\tau, \sigma)\bar{\Omega}^{-1}(\tau - \sigma), \quad (1.19)$$

where Ω and $\bar{\Omega}$ are group-valued functions as g is. The symmetry (1.19) gives rise to the conserved currents $J(z) = J^a \tau_a$ and $\bar{J}(\bar{z}) = \bar{J}^a \tau_a$ ($z = e^{i(\tau+\sigma)}$ and $\bar{z} = e^{i(\tau-\sigma)}$) which are functions of independent complex variables z, \bar{z} . The τ_a are antihermitian generators of the Lie algebra \mathfrak{g} of the group G . That J and \bar{J} are functions of z and \bar{z} respectively follows from the conservation laws: $\partial_z J = \partial_{\bar{z}} \bar{J} = 0$. Expanding the currents in Laurent series,

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad (1.20)$$

and likewise for \bar{J} , we can recover the affine Kac-Moody algebra,

$$[J_m^a, J_n^b] = f_c^{ab} J_{m+n}^c + \frac{1}{2} k n \delta_{m+n,0}, \quad (1.21)$$

where f_c^{ab} are the structure constants of \mathfrak{g} . The quantised parameter k appears now as the level of the representation of the affine algebra. We get a copy of (1.21) from the \bar{J} currents and the two copies commute. Thus, the WZW model has a very large symmetry algebra—an affine Kac-Moody algebra and the Virasoro algebra. The two algebras are closely linked in this model though. The energy momentum tensor of the theory is bilinear in the current $J(z)$ and so its Laurent modes, the generators L_n of

the Virasoro algebra, are bilinear in the Kac-Moody generators J_m^a . That is to say the energy-momentum tensor has the Sugawara form. This is a characteristic feature of WZW models.

In 1986 Witten and Gepner [GW86] generalised the theory of closed bosonic strings moving in a flat background spacetime to strings moving in a spacetime, a component of which was a Lie group manifold *i.e.*, they took the WZW model (1.18) to be a string theory. So now we have a string theory with a greatly enhanced symmetry—the Virasoro algebra of the conformal symmetry and the Kac-Moody algebra arising from the symmetries of the WZW model.

The string theories we have considered so far are not realistic in as much as the strings cannot interact. We can imagine that interacting closed strings can split and join together. In such a theory the worldsheet is no longer just a plain cylinder but becomes a surface of higher genus *e.g.*, a torus (which has genus one.) For a theory of interacting closed strings, the complete path integral (1.2) takes the form,

$$Z = \sum_{g=0}^{\infty} g_s^{2g-2} \int D^d X Dg e^{-S(X,g)}. \quad (1.22)$$

according to Polyakov [Pol81a, Pol81b]. In this expression g is the genus of a surface and g_s is the string coupling constant. So we are to sum over the contributions from surfaces of different genus. This is analogous to Feynman diagrams with different numbers of loops in ordinary field theory. In the first term in the sum we have the contribution from the sphere which is like the tree level Feynman diagrams. The interacting theory has been formulated as a perturbative theory in the string coupling constant g_s . It is very important for the consistency of an interacting theory that the correlation functions (*i.e.*, Green's functions) are indifferent to the parametrisations of the particular two dimensional surface, sphere, torus etc. Now surfaces of genus > 1 , have a set of complex parameters or moduli to which one may make transformations which are not continuously connected to the identity. Such transformations change the value of the moduli but not the shape of the surface. These transformations are called *modular transformations*. The torus has just one modulus, conventionally denoted τ and modular invariance of the vacuum to vacuum amplitude on a torus (the “partition function”) is a very strong constraint on a conformal field theory. The set of modular transformations of a torus are the Möbius transformations on τ . That is to say,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z} \quad ad - cb = 1. \quad (1.23)$$

We can associate such a transformation to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group of such matrices is the group $SL(2, \mathbb{Z})$. In fact, since we obtain the same Möbius transformation by using $-a, -b, -c, -d$ instead, we actually use the group $PSL(2, \mathbb{Z}) \stackrel{d}{=} SL(2, \mathbb{Z})/\mathbb{Z}_2$. All Möbius transformations can be generated from just two, conventionally denoted S and T where

$$S : \tau \rightarrow -\frac{1}{\tau} \quad \text{and} \quad T : \tau \rightarrow \tau + 1 \quad (1.24)$$

so that,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (1.25)$$

We notice that, when acting on τ (by composition), $S^2 = 1$ and $(ST)^3 = 1$.

Now, the partition function for a WZW theory is given by [GW86] an expression that is a sum of products of a character and its complex conjugate of the affine Kac-Moody algebra associated to the WZW model. If it can be verified that the characters are modular invariant then the partition function is too and we have the result that we want. Gepner and Witten showed that for the theories they consider, the partition functions were modular invariant because the characters that they used were characters of integrable representations which carry a finite representation of the modular group as shown by Kac and Peterson [KP84]. More particularly, for a given value of the level of the representation, there are a finite number of characters. Upon performing a modular transformation on any character, it transforms into a linear combination of all of the characters at that level. Thus the set of characters as a whole is invariant.

Clearly, when studying any WZW theory it is of crucial importance to know the characters of the associated affine algebra. This is the problem that we tackle in this thesis. In particular our results will be useful in determining the exact nature of the correspondence between the theory of $N = 2$ non-critical strings and the $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ gauged WZW model although such theories are certainly not studied in this work. To paraphrase the introduction from [HT]: “The exact correspondence between the traditional approach to noncritical strings and G/G models is yet to be proven. However, a crucial ingredient in the description of the spectrum in the G/G picture is the representation theory of the corresponding affine Lie (super)algebra, $\hat{\mathfrak{g}}$, at fractional level $k = p/u - h^\vee$, $p \in \mathbb{Z} \setminus \{0\}$, $u \in \mathbb{N}$ and $\gcd(p, u) = 1$ with h^\vee the dual Coxeter

number of $\hat{\mathfrak{g}}$. For instance, the $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ topological quantum field theory obtained by gauging the anomaly free diagonal subgroup $SL(2|1; \mathbb{R})$ of the global $SL(2|1; \mathbb{R})_L \times SL(2|1; \mathbb{R})_R$ symmetry of the WZW model appears to be intimately related to the noncritical charged fermionic string, which is the prototype of $N = 2$ supergravity in two dimensions. A comparison of the ghost content of the two theories strongly suggests that the $N = 2$ noncritical string is equivalent to the tensor product of a *twisted* $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ WZW model with the topological theory of a spin $1/2$ system. It is however only when a one-to-one correspondence between the physical states and equivalence of the correlation functions of the two theories are established that one can view the twisted G/G model as the topological version of the corresponding noncritical string theory. For the bosonic string, the recent derivation of conformal blocks for admissible representations of $\widehat{sl}(2; \mathbb{R})$ is a major step in this direction [PRY95, FCP97]. As far as the $\hat{sl}(2|1; \mathbb{C})$ and $N = 2$ algebras are concerned, a deeper investigation of their representations is a prerequisite to obtaining similar results for the corresponding conformal models.

When the matter, which is coupled to supergravity in the $N = 2$ noncritical string, is minimal, *i.e.*, taken in an $N = 2$ super Coulomb gas representation with central charge,

$$c_{\text{matter}} = 3 \left(1 - \frac{2p}{u} \right), \quad p, u \in \mathbb{N}, \quad \gcd(p, u) = 1, \quad (1.26)$$

the level of the “matter” affine superalgebra $\hat{sl}(2|1; \mathbb{C})$ appearing in the $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ model is of the form,

$$k = \frac{p}{u} - 1. \quad (1.27)$$

That is, the level precisely takes the values for which admissible representations of $\hat{sl}(2|1; \mathbb{C})$ exist [KW88].” Thus, when in chapter 5, and afterwards, we concentrate on levels k of the form (1.27) but with $p = 1$, we see that it is motivated by a coupling of unitary minimal “matter” to supergravity. We shall also make some progress in showing that the characters we obtain comprise modular invariant sets. The equivalence between the $OSP(1, 2)/OSP(1, 2)$ gauged WZW model and $N = 1$ non-critical strings is discussed by Fan and Yu in [FYa].

The problem of computing the characters is also one of mathematical interest. Indeed Kac and Wakimoto [KW] themselves wished that more character formulae of affine superalgebras were known. They gave a partial answer for $\hat{sl}(2|1; \mathbb{C})$ (or $\hat{A}(1, 0)$) as they

call it.) Our results go a long way to completing the story for this algebra and indeed expand the exact results for affine superalgebra character formulae in general. To the best of our knowledge the only other affine superalgebra that has been studied in any detail is $\widehat{osp}(1, 2; \mathbb{C})$ ($\hat{B}(0, 1)$) and we include a review of the results for that algebra in this thesis too. See chapter 2.

1.2 Mathematical Background.

In superstring theory graded symmetry algebras are ubiquitous. For example the $N = 1, 2, 4$ superconformal algebras. These algebras have undergone a lot of study, [Dör95, GRR96, GRR, Dob, Dob87, Mat87, RY87, 63, ET88a, ET88b] to cite a fraction of the literature on this subject. Work on the $N = 2$ superconformal algebra in particular continues at present *e.g.*, [DGR]. However, graded Lie algebras *i.e.*, Lie superalgebras, have received less attention in the mainstream physical literature. We shall take the opportunity in the following subsection to review some of the important aspects of Lie superalgebras. In subsection 1.2.2 we shall say a little about the admissible representations of the affine superalgebras.

1.2.1 Lie Superalgebras

Much of the material to follow on superalgebras can be found in Cornwell's book [Cor89]. Lie superalgebras were first studied by Scheunert, Nahm and Rittenberg [SNR76a, SNR76b] and later by Kac in his paper [Kac78]. Lie superalgebras differ from Lie algebras in that one can find a basis, a *homogeneous* basis, in which some of the generators of the superalgebra are *even* and the remainder *odd*. To even elements we associate a *degree* of zero and to odd elements a degree of one. The superalgebra \mathfrak{g} is then a *graded vector space* and is written as a direct sum of the even part \mathfrak{g}_0 and the odd part \mathfrak{g}_1 , $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Elements in the even subspace are called even and those in the odd subspace, odd. An element which is wholly even or wholly odd is called homogeneous. This graded vector space may be real or complex so that we have a real or complex Lie superalgebra. As a vector space there is nothing mysterious about the odd subspace. Its special nature appears only when the extra structure of Lie brackets is brought in. A superalgebra is structured so that brackets between homogeneous types

obey the following rules,

$$[\text{even}, \text{even}] = \text{even}, \quad [\text{even}, \text{odd}] = \text{odd}, \quad \{\text{odd}, \text{odd}\} = \text{even}. \quad (1.28)$$

The different brackets in the last case are to indicate that we should compute the anticommutator of two odd quantities. (We have in mind superalgebras of matrices.) There is also a generalisation of the Jacobi identity to be satisfied. For homogeneous a , b and c it is,

$$[a, [b, c]](-1)^{(\deg a)(\deg c)} + [b, [c, a]](-1)^{(\deg b)(\deg a)} + [c, [a, b]](-1)^{(\deg c)(\deg b)} = 0, \quad (1.29)$$

and we choose the brackets to be a commutator or anticommutator as appropriate. It should be clear that the even subspace, \mathfrak{g}_0 is actually a Lie subalgebra but that the odd subspace \mathfrak{g}_1 is not. Superalgebras have an interesting structure in that the odd subspace carries a representation of the even subalgebra. An example of objects that obey the rules (1.28) and (1.29) are matrices which have been divided into block diagonal and block antidiagonal form. The former play the role of the even type and the latter are the odd type. That is,

$$\begin{pmatrix} A & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & D \end{pmatrix} \text{ is even and } \begin{pmatrix} 0 & \vdots & B \\ \dots & \dots & \dots \\ C & \vdots & 0 \end{pmatrix} \text{ is odd.} \quad (1.30)$$

A , B , C , D are themselves just ordinary matrices. *Graded representations* of Lie superalgebras involve the “graded” matrices of (1.30) rather than ordinary matrices. In general, of course, an element of a Lie superalgebra is not wholly even or wholly odd but we can always find a basis comprised of wholly even and wholly odd generators *i.e.*, the homogeneous basis mentioned above. The *dimension* of the superalgebra is the number of elements in a basis for it and, given a homogeneous basis, its *superdimension* is the number of even generators less the number of odd generators in that basis. When we meet the Killing form of a Lie superalgebra in the next chapter we shall need to know how to compute the *supertrace* of a graded matrix. It is defined as,

$$\text{str} \begin{pmatrix} A & \vdots & B \\ \dots & \vdots & \dots \\ C & \vdots & D \end{pmatrix} = \text{tr } A - \text{tr } D. \quad (1.31)$$

So the supertrace of an odd matrix vanishes by definition.

The theory of Lie algebras and Lie superalgebras is very similar except that whenever one talks about superalgebras, all vector spaces in the theory are to be graded. The Cartan subalgebra of a superalgebra however is comprised entirely of even elements. As we have already seen, the matrices which represent superalgebras are graded as are the carrier spaces allied to those representations and the space of weights and roots is graded too. There is one other difference that is quite important and it means that one has to tread a little carefully sometimes with superalgebras. The root space of a superalgebra need not be Euclidean. It is quite possible to have a root space with a metric of non-definite signature. The two algebras studied in this thesis, between them, exhibit the two possibilities for superalgebra root spaces. $osp(1, 2; \mathbb{C})$ has a Euclidean metric for its root space just like any ungraded Lie algebra but $sl(2|1; \mathbb{C})$ has a Minkowski type metric for its root space. In fact, $sl(2|1; \mathbb{C})$ has odd roots of zero length. This means that one can't always naively generalise formulae from ungraded to graded algebras because, for example, the coroot of a zero norm root is not defined. As an example consider the dual Coxeter number, h^\vee . The definition of this quantity for an ungraded algebra involves the so-called dual Dynkin labels which are the "coordinates" of the highest coroot when written in terms of the basis of simple coroots. But as we have just remarked, and as is realised in $sl(2|1; \mathbb{C})$, when there are zero length simple roots around, their coroots are not defined. So we have to find another formula for the dual Coxeter number. There is another way to compute h^\vee which involves the eigenvalue of the second Casimir operator C_2 in the adjoint representation. We have,

$$h^\vee = \frac{C_2^{\text{ad}}}{\theta \cdot \theta}, \quad (1.32)$$

where θ is the highest root. The " \cdot " denotes the scalar product on the root space which is taken with a metric proportional to the Cartan-Killing metric derived from the restriction of the Killing form to the Cartan subalgebra. We also have, for any highest weight representation of a superalgebra which is defined by a highest weight Λ ,

$$C_2 = (\Lambda + 2\rho) \cdot \Lambda, \quad (1.33)$$

where 2ρ is the sum of the positive even roots minus the sum of the positive odd roots (the Weyl vector). Substituting (1.33) into (1.32),

$$h^\vee = \frac{(\theta + 2\rho) \cdot \theta}{\theta \cdot \theta}. \quad (1.34)$$

because the adjoint representation has $\Lambda = \theta$. Notice that there is the same scalar product in the numerator and denominator. In particular, if we were to choose a different definition of scalar product, h^\vee remains invariant. So now choose a definition of scalar product that sets $\theta^2 = 2$. Then,

$$h^\vee = \frac{1}{2}(\theta + 2\rho) \circ \theta. \quad (1.35)$$

Now, (1.35) is just what we need for superalgebras for one never has to deal with dividing by zero. The claim is that (1.35) holds in any other basis of simple roots. An example will illustrate this point. Consider $sl(2|1; \mathbb{C})$. (See chapter 3 for more about the roots of $sl(2|1; \mathbb{C})$.) We have odd roots $\Delta_1 = \{\pm\alpha_1, \pm\alpha_2\}$ and even roots $\Delta_0 = \{\pm(\alpha_1 + \alpha_2)\}$. Choose simple roots $\Pi = \{\alpha_1, \alpha_2\}$. From the Cartan-Killing metric we have that $\alpha_1^2 = \alpha_2^2 = 0$. Then $(\alpha_1 + \alpha_2)$ is the highest root. If we normalize so that $(\alpha_1 + \alpha_2)^2 = 2$ then $\alpha_1 \circ \alpha_2 = 1$. In this basis $\rho = 0$. Clearly, (1.35) gives $h^\vee = 1$. Now choose $\Pi = \{(\alpha_1 + \alpha_2), -\alpha_2\}$. In this basis $\theta = \alpha_1$ and $\rho = \alpha_2$. Again (1.35) gives $h^\vee = 1$ since $\alpha_1 \circ \alpha_2 = 1$. Thus, as long as there is some basis in which $\theta^2 = 2$, (1.35) holds in any basis. We can also compute h^\vee for $osp(1, 2; \mathbb{C})$. Choose positive roots $\Delta^+ = \{\alpha, \alpha/2\}$. (See chapter 2 for more about the roots of $osp(1, 2; \mathbb{C})$.) Then $\theta = \alpha$ and $\rho = \alpha/4$. We obtain the dual Coxeter number $h^\vee = 3/2$. We can also compute $h^\vee = 2$ for $su(2)$.

The superalgebras $osp(1, 2; \mathbb{C})$ and $sl(2|1; \mathbb{C})$ are each *simple*, *classical* and *basic*. We shall define each of these terms now. The definitions are from Cornwell's book [Cor89]. A Lie superalgebra is simple if it is not abelian and it has no proper, invariant, graded subalgebra. A Lie superalgebra is said to be classical if the representation of its even part on its odd part is either irreducible or completely reducible. Non-classical superalgebras are known as Cartan Lie superalgebras. The even part of classical superalgebras is reductive. The classical superalgebras are classified into two types—basic and strange. These types are mutually exclusive. A classical, simple Lie superalgebra is basic if it has a bilinear, supersymmetric, consistent, invariant form. For the superalgebras studied in this work, the Killing form is such a form. It is defined in the next chapter.

The procedure of obtaining the affine version of a certain finite-dimensional superalgebra is just the same as for ungraded algebras. That is to say one generates the centrally extended loop algebra of the super Lie algebra. The root system is augmented and extended in the normal way.

1.2.2 Admissible Representations

This thesis is concerned with computing character functions for admissible representations of certain affine superalgebras. The admissible representations are of a more general nature than the more well-known integrable ones. The integrable representations of an affine algebra may be “exponentiated” to give a representation of the corresponding linear Lie group and the level of these representations is a non-negative integer. Furthermore, the characters span a space which is invariant under the modular group and the representations are derived from a *highest weight*. The last properties are very important for physics. One can build modular invariant partition functions from characters which carry a representation of the modular group and the notion of a highest weight corresponds to the notion of a ground state *i.e.*, a state of lowest energy. With the importance of modular invariance not in doubt, one can ask if there are any other representations of affine algebras whose characters carry a representation of the modular group. Kac and Wakimoto [KW88] discovered that there were and they called these *admissible*. They are derived from a highest weight like integrable representations but differ from the integrable ones in that the level can now be a rational number as well as an integer. Thus the integrable representations are a subset of the admissible ones. The details about the fractional level are as follows ([KW88]). Let k be the level of a representation of an affine algebra $\hat{\mathfrak{g}}$. For admissible representations we have $k = t/u$ where $t \in \mathbb{Z}$ and $u \in \mathbb{N}$ and $\gcd(t, u) = 1$. Furthermore, if h^\vee is the dual Coxeter number of the finite-dimensional algebra \mathfrak{g} associated to $\hat{\mathfrak{g}}$, we must have that $k + h^\vee \geq h^\vee/u$. Clearly, setting $u = 1$ and restricting t to be non-negative recovers the integrable type of level. The Verma modules of integrable representations can have an infinite number of singular vectors just as the integrable modules do. This is a necessary condition that the characters of irreducible, admissible representations can be written in terms of ϑ -functions. Another feature of admissible representations wherein they differ from the integrable ones is that the representations are inherently non-unitary.

1.3 Layout of the Thesis.

The remaining chapters of this thesis split naturally into two parts. The first part comprises just chapter 2 and chapters 3 to 6 comprise the second part. Chapter 7 brings together all that has gone before and there I draw some conclusions from the

work and speculate on further tasks that might be done.

In chapter 2 we give a detailed review of the calculation of irreducible, admissible characters of $\widehat{osp}(1, 2; \mathbb{C})$. Characters for $\widehat{sl}(2|1; \mathbb{C})$ are computed in chapter 4 and some of their properties are discovered there and in the following chapter. Chapter 6 presents the modular transformations of some $\widehat{sl}(2|1; \mathbb{C})$ characters and presents some new identities between string functions of $\widehat{su}(2)$.

The results of chapter 2 have appeared before. See the preprint by Fan and Yu [FYb]. I make no claim of originality for them. In fact, after I had finished my work on this algebra another preprint appeared which presented the same results again. This was the paper by Ennes, Ramallo and Sanchez de Santos [ERSdS]. The motivation for recomputing known characters was twofold. The work of Fan and Yu is a little disorganised. A more coherent exposition makes the calculation seem less magical. Also, the simplicity of the calculations for this algebra serve to highlight the difficulties of working with $\widehat{sl}(2|1; \mathbb{C})$. In particular, one can obtain the modular transformations of the $\widehat{osp}(1, 2; \mathbb{C})$ characters in full generality quite easily whereas I have been able to compute the modular S transformation for $\widehat{sl}(2|1; \mathbb{C})$ characters in only a few cases despite doing all the work of branching them into $\widehat{sl}(2; \mathbb{C})$ characters and knowing the full structure of the branching functions.

The results of chapter 4 are new. They do rely heavily though on the embedding diagrams computed in [BT97]. However, those results need to be augmented if we are to have the complete picture of admissible representations for this algebra. This is what the introduction of “class V” representations is all about. We’ll see more about this in chapter 3. I have computed also characters for the class I representations of $\widehat{sl}(2|1; \mathbb{C})$. These characters can be written in a neat way but they do not have the modular transformations which would allow them to be used in a modular invariant partition function. At this point we will move on to the much more interesting representations of classes IV and V which will be the focus of the remainder of the thesis. It should be said at this point that all the work on class IV and class V representations is done on the basis of the assumption that there are no sub-singular vectors in these modules. By “sub-singular vector” we mean a state in the Verma module which becomes singular after taking the quotient of the Verma module by a singular submodule ([DGR]).

Chapter 2

Characters of $\widehat{osp}(1, 2; \mathbb{C})$

2.1 Introduction.

In this chapter we compute the characters and the supercharacters of the admissible representations of the complex, affine superalgebra $\widehat{osp}(1, 2; \mathbb{C})$ ($= B(0, 1)^{(1)}$ in Kac's notation [Kac78]) for both the Ramond and Neveu-Schwarz sectors. In section 2.7 we will compute the effect of modular transformations on the Neveu-Schwarz characters to illustrate how such a calculation might be done. We shall see that they transform in a way consistent with the different possible free fermion spin structures on a torus.

We start by reviewing the finite-dimensional, complex superalgebra $osp(1, 2; \mathbb{C})$ and then move on to its untwisted, affine counterpart $\widehat{osp}(1, 2; \mathbb{C})$. The root system is introduced, a simple root is identified and the Cartan matrix is computed. We calculate the Sugawara tensor and its associated Virasoro algebra. Having dealt with these preliminaries we shall then see, in detail, the derivation of the structure of the submodules of a highest weight, admissible representation. This structure is encoded in the *embedding diagram*. Data provided by the Kac determinant is used to generate the embedding diagram. We use a formula for this determinant which is a generalization (due to Kac [Kac86] and, latterly, by Fan and Yu [FYb]) to the superalgebra case of the well known one for infinite dimensional, ungraded algebras obtained long ago by Kac and Kazhdan [KK79]. With the knowledge of the embedding diagram and the quantum numbers of the singular vectors, the characters are easily obtained for the irreducible, admissible representations. The deduction of the modular transformations in section 2.7 is given in some detail since these methods are not widely available in the literature.

Figure 2.1: The root diagram of $osp(1, 2; \mathbb{C})$.

2.2 Finite and Affine $osp(1, 2; \mathbb{C})$.

Before introducing the untwisted, complex, affine superalgebra $\widehat{osp}(1, 2; \mathbb{C})$, we give a very short review of $osp(1, 2; \mathbb{C})$ —the finite-dimensional superalgebra. It is a basic, classical, simple, complex Lie superalgebra. It has dimension 5 and superdimension 1. Recall that the superdimension is defined to be $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1$ where \mathfrak{g}_0 and \mathfrak{g}_1 are respectively the even and odd subspaces of the superalgebra. $osp(1, 2; \mathbb{C})$ is defined by the following brackets in the Cartan-Weyl basis.

$$\begin{aligned}
 \{f^+, f^-\} &= 2b^3 & \{f^\pm, f^\pm\} &= \pm 2b^\pm \\
 [b^3, f^\pm] &= \pm \frac{1}{2} f^\pm & [b^\pm, f^\mp] &= -f^\pm \\
 [b^+, b^-] &= 2b^3 & [b^3, b^\pm] &= \pm b^\pm.
 \end{aligned} \tag{2.1}$$

The even (bosonic) type generators are denoted by b and the odd (fermionic) type ones by f . Thus $[,]$ is a commutator and $\{, \}$ an anticommutator. We see that the even subalgebra is isomorphic to the complex Lie algebra A_1 . The Cartan subalgebra is 1-dimensional and is spanned by b^3 alone. Accordingly, $\text{rank}(osp(1, 2; \mathbb{C})) = 1$. From the brackets above we see that the roots are, $(\pm 1/2)$ and (± 1) . The first pair are called *odd* roots because their corresponding step operators (f^\pm) are odd. Similarly the second pair are called *even* roots. The root diagram is very simple and is displayed in figure 2.1. We can take the positive odd root $(1/2)$ to be the unique simple root spanning the root space. Cornwell [Cor89] calls a basis of simple roots with just one odd root in it a *distinguished* basis, Kac and Wakimoto [KW] call it a *standard* basis. The other positive root is $(+1)$ and it is the highest root. Henceforth let $\alpha \stackrel{\text{d}}{=} (+1)$. The Killing form g^{ab} for a superalgebra is defined to be,

$$g^{ab} \stackrel{\text{d}}{=} \text{str}((\text{ad } X^a)(\text{ad } X^b)), \tag{2.2}$$

where X^a are the generators of the algebra and the adjoint representation of a superalgebra is defined in just the same way as for an ungraded algebra—in terms of the structure constants. Notice the use of the supertrace (str). Matrix indices have been

suppressed on $(\text{ad } X)$. The Killing form is,

$$g^{ab} = \begin{pmatrix} \frac{3}{2} & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 3 & \vdots & 0 & 0 \\ 0 & 3 & 0 & \vdots & 0 & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 6 \\ 0 & 0 & 0 & \vdots & -6 & 0 \end{pmatrix} \quad (2.3)$$

Notice that when the indices run over the bosonic choices b^3, b^\pm in the top 3×3 block the matrix is symmetric and when the indices run over the fermionic choices f^\pm in the bottom right 2×2 block the matrix is antisymmetric. Such a bilinear form is said to be *supersymmetric*. When we want to compute the scalar (not inner¹) product between roots or weights we compute the inverse, g_{ab} , of g^{ab} and then use the the 1×1 block in the top left hand corner of g_{ab} as the Cartan-Killing metric. That is, the Cartan-Killing metric is $(2/3)$ and so,

$$\alpha^2 = \frac{2}{3}. \quad (2.4)$$

Having found the Killing form, we can compute the second (quadratic) Casimir operator, C_2 for $osp(1, 2; \mathbb{C})$. Using the matrix inverse to the one in (2.3),

$$C_2 = \frac{1}{3}(2J^3J^3 + J^+J^- + J^-J^+ - \frac{1}{2}j^+j^- + \frac{1}{2}j^-j^+). \quad (2.5)$$

C_2 commutes with all the generators of the algebra and is equal to the identity in the adjoint representation. For a general highest weight representation with highest weight state $|\Omega\rangle$ of spin j i.e., $b^3|\Omega\rangle = j|\Omega\rangle$, we have,

$$C_2|\Omega\rangle = \frac{1}{3}j(2j + 1)|\Omega\rangle. \quad (2.6)$$

Setting $j = 1$ we recover the adjoint representation. The Coxeter number h is given by $h = 3$ and the dual Coxeter number $h^\vee = 3/2$ upon using equation (1.35) as was explained in the first chapter. The Cartan matrix is just $A = (2)$ so the Kac-Dynkin diagram has just one node and, being so trivial, we omit it. With this short review we now move on to the untwisted, affine version of $osp(1, 2; \mathbb{C})$.

¹Products between roots are not inner products because the product need not be positive definite. The fermionic roots of $sl(2|1)$ are a case in point.

The (anti)commutation relations for $\widehat{osp}(1, 2; \mathbb{C})$ are as follows,

$$\begin{aligned}
\{f_r^+, f_s^-\} &= 2b_{r+s}^3 + r\tilde{k}\delta_{r+s,0} & \{f_r^\pm, f_s^\pm\} &= \pm 2b_{r+s}^\pm \\
[b_n^3, f_r^\pm] &= \pm \frac{1}{2}f_{n+r}^\pm & [b_n^\pm, f_r^\mp] &= -f_{n+r}^\pm \\
[b_n^+, b_m^-] &= 2b_{n+m}^3 + n\tilde{k}\delta_{n+m,0} & [b_n^3, b_m^\pm] &= \pm b_{n+m}^\pm \\
[b_m^3, b_n^3] &= \frac{1}{2}\tilde{k}m\delta_{n+m,0} & & \\
[d, b_n^i] &= nb_n^i & [d, f_n^j] &= nf_n^j
\end{aligned} \tag{2.7}$$

The derivative operator d is introduced to remove an infinite degeneracy in the non-zero roots. All the other possible brackets vanish. Let θ be the highest root of $osp(1, 2; \mathbb{C})$. If we denote the eigenvalue of the central generator \tilde{k} by \tilde{k} too, then the number,

$$k \stackrel{d}{=} 2\tilde{k}/\theta^2 \tag{2.8}$$

is called the *level* of that representation. In fact $\theta = \alpha$ when $\alpha/2$ is the simple root.

The subscripts on the even generators are always integral. The subscripts on the odd generators admit of two cases. They are integral for the Ramond sector and they are half-integral for the Neveu-Schwarz sector. One has to twist the Ramond algebra to obtain the Neveu-Schwarz one but the twist is trivial and the algebras for the two sectors are actually isomorphic. The following automorphism is what is used to move between the Ramond and Neveu-Schwarz sectors. Such a mapping is often known as “spectral flow”.

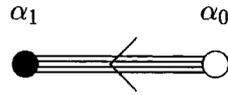
$$\begin{aligned}
\pm f_{n\pm\frac{1}{2}}^{\mp NS} &= f_n^{\pm R} & b_{n\pm 1}^{\mp NS} &= -b_n^{\pm R} \\
b_n^{3 NS} &= -b_n^{3 R} + \frac{1}{2}\tilde{k}\delta_{n,0} & d^{NS} &= d^R + b_0^{3 R} - \frac{1}{4}\tilde{k}
\end{aligned} \tag{2.9}$$

Now we describe the root system of $\widehat{osp}(1, 2; \mathbb{C})$. For a full account of the roots of affine superalgebras see the paper by Tsohantjis and Cornwell [TC90]. The Cartan subalgebra of $\widehat{osp}(1, 2; \mathbb{C})$ is spanned by the set $\{b_0^3, \tilde{k}, d\}$ which has the following commutation relations with the other generators.

$$\begin{aligned}
[b_0^3, b_n^\pm] &= \pm b_n^\pm & [b_0^3, f_n^\pm] &= \pm \frac{1}{2}f_n^\pm & [b_0^3, b_n^3] &= 0 \\
[\tilde{k}, b_n^\pm] &= 0 & [\tilde{k}, f_n^\pm] &= 0 & [\tilde{k}, b_n^3] &= 0 \\
[d, b_n^\pm] &= nb_n^\pm & [d, f_n^\pm] &= nf_n^\pm & [d, b_n^3] &= nb_n^3.
\end{aligned}$$

Reading down the columns we obtain the set, Δ , of roots,

$$\Delta \stackrel{d}{=} \{(\pm\alpha, 0, n), (\pm\frac{\alpha}{2}, 0, n), (0, 0, n) | n \in \mathbb{Z}\} \tag{2.10}$$

Figure 2.2: The Dynkin diagram of $\widehat{osp}(1, 2; \mathbb{C})$.

with step operators b_n^\pm, f_n^\pm, b_n^3 respectively. The roots,

$$\Delta_0 \stackrel{d}{=} \{(\pm\alpha, 0, n), (0, 0, n) | n \in \mathbb{Z}\} \quad (2.11)$$

are even and,

$$\Delta_1 \stackrel{d}{=} \{(\pm\frac{\alpha}{2}, 0, n) | n \in \mathbb{Z}\} \quad (2.12)$$

are the odd roots. Then $\Delta = \Delta_0 \cup \Delta_1$. The positive roots are,

$$\Delta^+ \stackrel{d}{=} \{(\alpha, 0, n-1), (\frac{\alpha}{2}, 0, n-1), (-\alpha, 0, n), (-\frac{\alpha}{2}, 0, n), (0, 0, n) | n \in \mathbb{N}\}. \quad (2.13)$$

The set of negative roots is $\Delta^- \stackrel{d}{=} \Delta \setminus \Delta^+$ and of course we have the zero root too. We have the following basis of simple roots, $\Pi = \{(\alpha/2, 0, 0), \alpha_0\}$ where, $\alpha_0 \stackrel{d}{=} (-\alpha, 0, 0) + (0, 0, 1)$. Note that $(\alpha/2, 0, 0)$ is odd and α_0 even. Henceforth let $(\alpha/2, 0, 0) \stackrel{d}{=} \alpha_1$. We can now compute the Cartan matrix of $\widehat{osp}(1, 2; \mathbb{C})$. The formula is

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}. \quad (2.14)$$

Note that there is no problem in this definition for the roots of $\widehat{osp}(1, 2; \mathbb{C})$ because they all have non-zero norm. This is not the case for $\hat{sl}(2|1; \mathbb{C})$ though as we shall see in the next chapter. This Cartan matrix has rank one (which is the dimension of the Cartan subalgebra of $osp(1, 2; \mathbb{C})$) and its determinant vanishes as it should for an affine algebra. The Dynkin diagram of the affine algebra is shown in figure 2.2. The fermionic simple root α_1 is denoted by a black node and the bosonic simple root α_0 by a white node.

We compute the two fundamental weights $\Lambda_{0,1}$ using the formula,

$$\frac{2\Lambda_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \delta_{ij}. \quad (2.15)$$

We easily find that

$$\Lambda_0 = (0, 1, 0) \quad \Lambda_1 = (\alpha/4, \frac{1}{2}, 0). \quad (2.16)$$

The affine Weyl vector ρ is,

$$\rho = \Lambda_0 + \Lambda_1 = \left(\frac{\alpha}{4}, \frac{3}{2}, 0\right) = \left(\frac{\alpha}{4}, h^\vee, 0\right). \quad (2.17)$$

Note that $\alpha/4$ is the Weyl vector of $osp(1, 2; \mathbb{C})$.

2.3 The Associated Virasoro Algebra.

In the 1960's Sugawara [Sug68] studied field theories in terms of their currents and wrote down an expression for the energy-momentum tensor which was bilinear in the currents. Sugawara's construction is used to construct a Virasoro algebra from any affine Kac-Moody (super)algebra. We recall from conformal field theory that the generators of the Virasoro algebra are the Laurent modes of the energy-momentum tensor in two dimensions. If we denote the generators of $\widehat{osp}(1, 2; \mathbb{C})$ by X_n^a where $X_n^1 = b_n^3$, $X_n^2 = b_n^+$, $X_n^3 = b_n^-$, $X_n^4 = f_n^+$ and $X_n^5 = f_n^-$ there are Virasoro generators defined by [GKO85, GO86],

$$L_n \stackrel{d}{=} \frac{1}{2\tilde{k} + C_2^{\text{ad}}} \sum_{m \in \mathbb{Z}} \sum_{a,b=1}^{\dim osp(1,2;\mathbb{C})} \bullet g_{ab} X_{m+n}^a X_{-m}^b \bullet. \quad (2.18)$$

In (2.18), \tilde{k} denotes the eigenvalue of the operator \tilde{k} , C_2^{ad} is the eigenvalue of the second Casimir operator in the adjoint representation *i.e.*, unity, and note that it is the inverse of the Killing form (2.3) that appears. The colons are to indicate that the operator product is to be normal ordered. We shall give our normal ordering prescription below. These operators L_n generate the *Virasoro algebra*,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \quad (2.19)$$

The generator c is a constant in any representation and commutes with all L 's. Originally, the coefficient $1/(2\tilde{k} + C_2^{\text{ad}})$ was thought to be just $1/2\tilde{k}$. The correct version arises when one is careful with the normal ordering. The correct version including the second Casimir was worked out for the case of affine algebras (rather than superalgebras) by Knizhnik and Zamolodchikov [KZ84], Goddard and Olive [GO85] and Todorov [Tod85] and earlier for particular cases by Bardakci and Halpern [BH71], Segal [Seg81] and Frenkel [Fre81]. The normal ordering prescription we will use in any definite calcu-

lations is,

$$\bullet b_m^i b_n^j \bullet = \begin{cases} b_m^i b_n^j & \text{if } n > m, \\ b_n^j b_m^i & \text{if } n < m, \\ \frac{1}{2}(b_n^i b_m^j + b_m^j b_n^i) & \text{if } n = m, \end{cases} \quad (2.20)$$

and,

$$\bullet f_r^i f_s^j \bullet = \begin{cases} f_r^i f_s^j & \text{if } s > r, \\ -f_s^j f_r^i & \text{if } s < r, \\ \frac{1}{2}(f_r^i f_s^j - f_s^j f_r^i) & \text{if } r = s. \end{cases} \quad (2.21)$$

The central charge c of the Virasoro algebra (2.19) when the generators are of the form (2.18) is,

$$c \stackrel{d}{=} \frac{2\tilde{k} \text{sdim } osp(1, 2; \mathbb{C})}{2\tilde{k} + C_2^{\text{ad}}}. \quad (2.22)$$

We see that it is the superdimension (sdim) that is used for superalgebras. This formula generalises to any superalgebra of course. In particular, since $C_2^{\text{ad}} = 1$, and $2\tilde{k} = \alpha^2 k = \frac{2}{3}k$ by (2.8) and (2.4) and $\text{sdim } osp(1, 2; \mathbb{C}) = 1$ we have,

$$c = \frac{2k}{2k + 3}. \quad (2.23)$$

It will be useful to have an explicit expression for the Virasoro generator L_0 when $\hat{\mathfrak{g}} = \widehat{osp}(1, 2; \mathbb{C})$ too. We get,

$$L_0 = \frac{3}{2k + 3} \sum_{m \in \mathbb{Z}} \bullet \frac{1}{3} (2b_m^3 b_{-m}^3 + b_m^+ b_{-m}^- + b_m^- b_{-m}^+ - \frac{1}{2}(f_m^+ f_{-m}^- - f_m^- f_{-m}^+)) \bullet. \quad (2.24)$$

Together the Virasoro algebra and its associated affine algebra have a semidirect product structure. Our total algebra comprises (2.7), (2.19) and

$$[L_m, b_n^i] = -n b_{n+m}^i \quad [L_m, f_r^\pm] = -r f_{r+m}^\pm. \quad (2.25)$$

Notice that $L_0 = -d$.

2.4 Singular Vectors and the Kac Determinant.

Since the generators of the Cartan subalgebra commute with one another they may be simultaneously diagonalized. As such they have simultaneous eigenvectors. All

vectors in the carrier space of the representation will be simultaneous eigenvectors of the isospin operator b_0^3 , the central element k and the derivative operator d . Because the derivative operator is the negative of the L_0 operator of the associated Virasoro algebra, its eigenvalue is the negative of the conformal weight of a vector. Since k commutes with everything else its eigenvalue on every vector is constant. The highest weight vector (and hence the whole representation) is specified by giving its eigenvalues of the three commuting generators b_0^3, k, d . Denote by h the conformal weight of a vector and by j its isospin. Then all vectors may be written in Dirac's notation as $|h, j, k\rangle$. However, since for any given representation k is the same for all vectors we shall not note it and shall write just $|h, j\rangle$ instead. We shall give now the definition of a highest weight state and then that of a singular vector.

Definition 2.1 (Highest Weight State.)

$|h, j\rangle$ is a highest weight state (hws) if,

$$b_n^+ |h, j\rangle = b_{n+1}^- |h, j\rangle = b_{n+1}^3 |h, j\rangle = f_r^+ |h, j\rangle = f_{r+1}^- |h, j\rangle = L_{n+1} |h, j\rangle = 0 \quad (2.26)$$

for $n, r \geq 0$ i.e., a hws is one annihilated by all the raising operators.

Remark 1

All of the above conditions on the raising operators are equivalent to just the following two:

$$b_1^- |h, j\rangle = f_0^+ |h, j\rangle = 0, \quad (2.27)$$

because these two together with the brackets (2.7) can generate (2.26).

Remark 2

The spectral flow mapping of (2.9) maps highest weight states to highest weight states.

Definition 2.2 (Singular Vector.)

$|h, j\rangle$ is a singular vector if it is a hws and if it has zero norm.

Definition 2.3 (Verma Module.)

The module generated by operating on a hws with lowering operators is called a Verma module.

Let F_η be the symmetric bilinear form on the Verma module. In [KK79], Kac and Kazhdan provided a formula for the determinant of F_η . This formula was subsequently generalized to the case of Kac-Moody superalgebras by Kac [Kac86]. Fan and Yu have

produced a formula applicable in the special case when there are no zero norm roots. See [FYb] for this. Kac's formula is,

$$\det F_\eta = \prod_{\substack{\alpha \in \Delta_0^+ \\ \frac{1}{2}\alpha \notin \Delta_1^+}} \prod_{n \in \mathbb{N}} \Phi(n, \alpha)^{P(\eta - n\alpha)} \prod_{\substack{\alpha \in \Delta_1^+ \\ \alpha^2 \neq 0}} \prod_{n \in 2\mathbb{N}-1} \Phi(n, \alpha)^{P(\eta - n\alpha)} \prod_{\substack{\alpha \in \Delta_1^+ \\ \alpha^2 = 0}} \Phi(\alpha)^{P_\alpha(\eta - \alpha)} \quad (2.28)$$

where

$$\Phi(n, \alpha) \stackrel{d}{=} (\Lambda_h + \rho) \cdot \alpha - \frac{1}{2}n\alpha \cdot \alpha \quad (2.29)$$

and

$$\Phi(\alpha) = \Phi(0, \alpha). \quad (2.30)$$

The affine Weyl vector is denoted ρ and may be written as the sum of the fundamental weights as in (2.17) above. Λ_h is the highest weight of the representation. P is called the Kostant partition function. Its value on a vector x is the number of sets of positive integers $\{n_1, n_2, \dots, n_r\}$ such that $x = \sum_{i=1}^r n_i \alpha_i$ where the α_i 's are some basis vectors. Obviously, for us, these basis vectors are the simple roots. Also $P(0) \stackrel{d}{=} 1$. $P_\alpha(x)$ is the number of partitions of x not involving α . We look for zeroes of the determinant *i.e.*, zeroes of Φ . We shall take the highest weight Λ_h to be the following combination of the fundamental weights,

$$\Lambda_h = (k - 2j)\Lambda_0 + 4j\Lambda_1 = (j\alpha, k, 0), \quad 2j \in \mathbb{Z}_+. \quad (2.31)$$

At this stage we ought to introduce two more parameters. In this chapter we are interested in admissible representations of $\widehat{osp}(1, 2; \mathbb{C})$. Admissible representations have a level k which is rational rather than just integral as it is for the integrable representations. In the following we have chosen to express k in terms of two other parameters *viz.* p and q . We set,

$$2k + 3 = \frac{p}{q}, \quad (2.32)$$

where $p, q \in \mathbb{N}$, $\gcd(q, (p+q)/2) = 1$ and $p > 1$. Setting $q = 1$ and restricting ourselves to odd p , we recover levels k of integrable type. The mysterious coprimality condition arises as follows. In their paper on admissible representations of $\widehat{osp}(1, 2; \mathbb{C})$, Fan and Yu [FYb] deduce expressions for characters by considering, like Kac and Wakimoto [KW88]

before them, the coset theory $OSP(1,2)/SL(2)$. When one looks at the coset in terms of the affine algebras, the $\hat{sl}(2; \mathbb{C})$ algebra inherits its level from that of $\widehat{osp}(1,2; \mathbb{C})$. We have,

$$k + \frac{3}{2} = \frac{p}{2q} \Leftrightarrow k + 2 = \frac{p+q}{2q}.$$

Recall that the dual Coxeter number of $\hat{sl}(2; \mathbb{C})$ is 2. If the fraction $(p+q)/2q$ is to be in its lowest terms then one can require that $\gcd(q, (p+q)/2) = 1$. That $(p+q)$ is even is ensured below in (2.44). Indeed, the coprimality condition allows both p and q to be even but only if one is divisible by 4 and the other not.

The Kac determinant (2.28) we want to analyse involves bosonic roots which are not twice some positive fermionic root. In the Ramond sector the third component of the fermionic roots are integers so we must only take the product over those bosonic roots that have their third component an *odd* integer. Thus the bosonic roots needed for the product are

$$\{(\pm\alpha, 0, 2m+1), (0, 0, 2m+1) | m \in \mathbb{Z}_+\}. \quad (2.33)$$

The product over the zero norm fermionic roots is redundant here but it will be needed in the next chapter when we study $\hat{sl}(2|1; \mathbb{C})$. We are to look for where $\Phi(n, \alpha) = 0$. Using $(\alpha, 0, 2m+1)$ gives $\Phi = 0$ if,

$$4j+1 = 2n - (2m+1)(2k+3), \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}_+. \quad (2.34)$$

And again, using $(-\alpha, 0, 2m+1)$ gives $\Phi = 0$ if,

$$4j+1 = -2n + (2m+1)(2k+3), \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}_+. \quad (2.35)$$

The other type of bosonic root to be taken into account in the product yields $\Phi = 0$ if $k = -\frac{3}{2}$ —a case which we specifically disallow since this would imply that $p = 0$ but we chose $p > 1$.

Now look at the fermionic part of the product formula. Putting $(\alpha/2, 0, m)$ into (2.29) gives $\Phi = 0$ if,

$$4j+1 = 2n - 1 - 2m(2k+3), \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}_+. \quad (2.36)$$

And again, using $(\alpha/2, 0, m)$ gives $\Phi = 0$ if,

$$4j+1 = -2n + 1 + 2m(2k+3), \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}_+. \quad (2.37)$$

From the four conditions above we see that the most general condition that the determinant vanishes is

$$4j + 1 = n - m(2k + 3), \quad (2.38)$$

where $(m + n)$ is odd, m and n have the same sign and $n \neq 0$. We can now use all the information gleaned from the vanishing of the determinant to obtain the following two theorems.

Theorem 2.1

If $|h, j\rangle$ is a hws such that its isospin j satisfies,

$$4j + 1 = n - m(2k + 3),$$

for $(m + n)$ odd and $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ then there exists a singular vector with conformal weight h' where,

$$h' = h + \frac{1}{2}nm, \quad (2.39)$$

and isospin j' such that

$$4j' + 1 = 4j + 1 - 2n = -n - m(2k + 3). \quad \blacksquare \quad (2.40)$$

If, on the other hand, the parameters m and n were negative we would have

Theorem 2.2

If $|h, j\rangle$ is a hws such that its isospin j satisfies,

$$4j + 1 = -n' + m'(2k + 3),$$

for $(m' + n')$ odd and $m' \in \mathbb{Z}_+$ and $n' \in \mathbb{N}$ then there exists a singular vector with conformal weight h' where,

$$h' = h + \frac{1}{2}n'm', \quad (2.41)$$

and isospin j' such that

$$4j' + 1 = 4j + 1 + 2n' = n' + m'(2k + 3). \quad \blacksquare \quad (2.42)$$

Notice that (2.38) is invariant under the simultaneous shifts in m and n ,

$$m \longrightarrow m + aq \quad n \longrightarrow n + ap, \quad a \in \mathbb{Z}. \quad (2.43)$$

Thus, given any solution to (2.38) we can obtain another by doing the shifts when the level is of the form (2.32). We will be interested in representations where the isospin j of the highest weight state satisfies the hypothesis of either theorem 2.1 or theorem 2.2. Now if the isospin of the highest weight state satisfies the hypothesis of theorem 2.2 we can, by doing simultaneous shifts on m' and n' turn it into the hypothesis of theorem 2.1 *provided* that $q \geq m' \geq 1$ and $p-1 \geq n' \geq 1$ (to keep their signs the same after shifting). This serves to ensure that $0 \leq m \leq q-1$ and $1 \leq n \leq p-1$. With $(m' + n')$ odd, we can only have $(m + n)$ odd after doing the shifts if we impose that

$$p + q \quad \text{is even.} \quad (2.44)$$

We see that when we are interested in representations with isospin j satisfying a condition like (2.38) we can, without loss of generality, take the parameters to be non-negative provided they are in the finite domains mentioned above. This is the same story as for admissible $\hat{sl}(2; \mathbb{C})$. However, these two cases are in stark contrast to admissible $\hat{sl}(2|1; \mathbb{C})$. In the next chapter we shall see that the condition (3.28) on the isospin of an admissible $\hat{sl}(2|1; \mathbb{C})$ representation is very like (2.38) except that now “ $n = 0$ ”. This means that we cannot consider the positive-parameter condition and negative-parameter condition as one by doing shifts, because shifting will throw up a non-zero “ n ” thereby changing the form of that condition. Thus the two cases are distinct and this is why we are led to introducing class V representations there as well as the class IV ones.

2.5 The Embedding Diagrams.

In this section we use theorems 2.1 and 2.2 to derive the embedding diagrams for the admissible representations of $\widehat{osp}(1, 2; \mathbb{C})$. Let the vector in the hypothesis of theorem 2.1 be the highest weight state that specifies the representation. Call it Z'_0 . Let that state have conformal weight h . Theorem 2.1 then tells us that there is a singular vector (which we shall call T'_0) with conformal weight and isospin,

$$h(T'_0) = h + \frac{1}{2}mn, \quad 4j(T'_0) + 1 = -n - m(2k + 3). \quad (2.45)$$

To reiterate again, henceforth we have $0 \leq m \leq q - 1$ and $1 \leq n \leq p - 1$. Now we also have,

$$\begin{aligned} 4j + 1 &= n - m(2k + 3) \\ &= -(p - n) + (q - m)(2k + 3). \end{aligned}$$

Theorem 2.2 provides us with a singular vector, which we will call Z_0 , with conformal weight and isospin,

$$h(Z_0) = h + \frac{1}{2}(n - p)(m - q), \quad 4j(Z_0) + 1 = -(n - 2p) - m(2k + 3). \quad (2.46)$$

Now sit at T'_0 .

$$4j(T'_0) + 1 = -n - m(2k + 3) = -(n + p) + (q - m)(2k + 3).$$

So, theorem 2.2 gives a singular vector Z'_1 with conformal weight and isospin,

$$h(Z'_1) = h + \frac{1}{2}(nq - mp + pq), \quad 4j(Z'_1) + 1 = n + 2p - m(2k + 3). \quad (2.47)$$

Again at T'_0 we have,

$$4j(T'_0) + 1 = -n - m(2k + 3) = (p - n) - (m + q)(2k + 3).$$

Theorem 2.1 gives a singular vector T_0 with conformal weight and isospin

$$h(T_0) = h + \frac{1}{2}(mp - nq + pq), \quad 4j(T_0) + 1 = n - 2p - m(2k + 3). \quad (2.48)$$

Now go to Z_0 . We have,

$$4j(Z_0) + 1 = (2p - n) - m(2k + 3),$$

and theorem 2.1 gives a singular vector with conformal weight and isospin identical to those of T_0 . Still at Z_0 ,

$$4j(Z_0) + 1 = (2p - n) - m(2k + 3) = -n + (2q - m)(2k + 3).$$

Theorem 2.2 gives a singular vector with exactly the same quantum numbers as those of Z'_0 .

Now sit at Z'_1 . We have,

$$4j(Z'_1) + 1 = n + 2p - m(2k + 3),$$

Series	conformal weight	$4 \times$ isospin
Z'_a	$h + \frac{1}{2}a(nq - mp + apq)$	$-1 + n + 2pa - m(2k + 3)$
Z_a	$h + \frac{1}{2}(n - (a + 1)p)(m - (a + 1)q)$	$-1 - n + 2(a + 1)p - m(2k + 3)$
T_a	$h + \frac{1}{2}(a + 1)(mp - nq + (a + 1)pq)$	$-1 + n - 2(a + 1)p - m(2k + 3)$
T'_a	$h + \frac{1}{2}(n + ap)(m + aq)$	$-1 - n - 2ap - m(2k + 3)$

Table 2.1: Quantum numbers of singular vectors of $\widehat{\mathfrak{osp}}(1, 2; \mathbb{C})$.

and we can use theorem 2.1 straightaway. It gives a new singular vector, T'_1 , with conformal weight and isospin,

$$h(T'_1) = h + \frac{1}{2}(n + p)(m + q), \quad 4j(T'_1) + 1 = -(n + 2p) - m(2k + 3). \quad (2.49)$$

Still at Z'_1 ,

$$4j(Z'_1) + 1 = n + 2p - m(2k + 3) = -(p - n) + (3q - m)(2k + 3).$$

Then theorem 2.2 gives another new singular vector, Z_0 , with,

$$h(Z_0) = h + \frac{1}{2}(n - 2p)(m - 2q), \quad 4j(Z_0) + 1 = 4p - n - m(2k + 3). \quad (2.50)$$

Now return to T_0 . Using the theorems we can hit again the singular vectors T'_1 and Z_0 . One continues in this way indefinitely, building up four sequences of singular vectors, Z'_a , T'_a , Z_a and T_a , $a \in \mathbb{Z}_+$. Their quantum numbers are collected in table 2.1. We have then the embedding diagram shown in figure 2.3 wherein the highest weight vector defining the representation is shown as a black node and singular vectors in its Verma module are shown as white nodes. In fact, if $p \mid n$ the diagram “collapses” and we obtain the diagram of figure 2.4. This, in effect, is what happens in the diagram for class IV and class V representations of $\hat{\mathfrak{sl}}(2|1; \mathbb{C})$.

Since the task of this thesis is not to study $\widehat{\mathfrak{osp}}(1, 2; \mathbb{C})$ in much depth, we shall not attempt to justify the fact that figure 2.3 is the complete embedding diagram. Other authors ([FYb]) have gone further than we shall in doing that.

2.6 The Characters and Supercharacters.

Now that we have the embedding diagrams we can write down the Ramond and Neveu-Schwarz sector characters and supercharacters. We shall construct the characters for

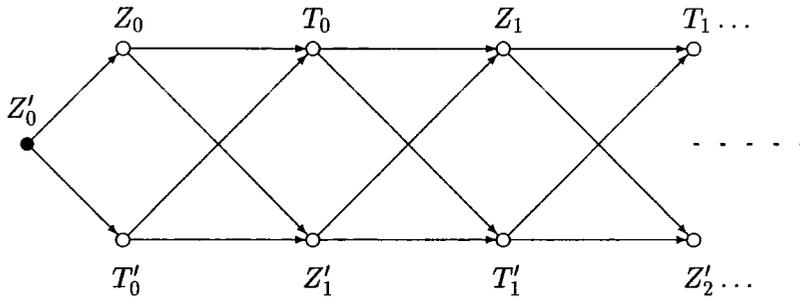


Figure 2.3: The embedding diagram of $\widehat{osp}(1, 2; \mathbb{C})$ for the case $n > 0$.



Figure 2.4: The embedding diagram of $\widehat{osp}(1, 2; \mathbb{C})$ for the case $n = 0$.

the $n > 0$ embedding diagram. Our definition of a (Ramond sector) character of an irreducible representation is,

$$\chi^R(\sigma, \tau) \stackrel{\text{d}}{=} \text{tr } q^{L_0^R - \frac{c}{24} z b_0^R}.$$

In the last equation and in all of the work following we define q and z in terms of two complex numbers τ and σ , where,

$$z \stackrel{\text{d}}{=} \exp(2\pi i\sigma), \quad q \stackrel{\text{d}}{=} \exp(2\pi i\tau),$$

with $\sigma, \tau \in \mathbb{C}$ and $\text{Im}(\tau) > 0$. We take the trace over an irreducible module. The definition can be realised by an alternating sum (deduced from the structure of the embedding diagram figure 2.3) of the characters of reducible modules built on hws's with the quantum numbers of the hws which specifies the entire representation and the singular vectors which are in the Verma module of that hws. That is to say that the Ramond sector character of the irreducible module built on the hws Z'_0 (as is the case in our embedding diagram) is,

$$\sum_{a \in \mathbb{Z}_+} \{ \chi_{V(Z'_a)} - \chi_{V(T'_a)} + \chi_{V(T_a)} - \chi_{V(Z_a)} \}, \tag{2.51}$$

where, by $\chi_{V(Z'_a)}$ for example, we mean,

$$\chi_{V(Z'_a)} = P^R(\sigma, \tau) q^{h(Z'_a) - \frac{c}{24}} z^{j(Z'_a)}. \quad (2.52)$$

We shall call $P^R(\sigma, \tau)$ the *prefactor* but it could also be known as the Kac-Weyl superdenominator. In concrete form,

$$P^R(\sigma, \tau) = \frac{\prod_{n=1}^{\infty} (1 + z^{\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} q^{n-1})}{\prod_{n=1}^{\infty} (1 - q^n)(1 - z q^n)(1 - z^{-1} q^{n-1})}. \quad (2.53)$$

This is merely the generating function for the states generated by the action of two fermionic generators which change isospin in units of $1/2$ (the numerator) and three bosonic generators two of which change isospin in units of 1 (the denominator). That the powers of q are all integers is a consequence of our being in the Ramond sector. Following Kac in his book [Kac90, p.252], we define the ϑ -function at integer level k

$$\vartheta_{m,k}(\sigma, \tau) \stackrel{\text{d}}{=} \sum_{a \in \mathbb{Z}} q^{k(a + \frac{m}{2k})^2} z^{k(a + \frac{m}{2k})} \quad m \in \mathbb{Z}. \quad (2.54)$$

For convenience we collect some properties of these ϑ -functions which will be used frequently hereafter.

$$\vartheta_{m,k}(\sigma, \tau) = \vartheta_{-m,k}(-\sigma, \tau), \quad \vartheta_{m+r,k}(\sigma, \tau) = \vartheta_{m,k}(\sigma, \tau) \quad \forall r \in 2\mathbb{Z}. \quad (2.55)$$

In terms of these ϑ -functions we can write Dedekind's η -function,

$$\eta(\tau) \stackrel{\text{d}}{=} q^{\frac{1}{24}} \prod_{n \in \mathbb{N}} (1 - q^n) \quad (2.56)$$

$$= \vartheta_{1,6}(0, \tau) - \vartheta_{5,6}(0, \tau). \quad (2.57)$$

By making use of Jacobi's triple product identity viz, (see for example the book by Hardy and Wright [HW54]),

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1} z^2)(1 + q^{2n-1} z^{-2}) = \sum_{n \in \mathbb{Z}} q^{n^2} z^{2n}, \quad (2.58)$$

we can rewrite $P^R(\sigma, \tau)$ as,

$$P^R(\sigma, \tau) = q^{\frac{1}{24}} z^{\frac{1}{4}} \frac{\vartheta_{1,2}(\frac{1}{2}\sigma, \tau) + \vartheta_{-1,2}(\frac{1}{2}\sigma, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))}. \quad (2.59)$$

Note that in (2.58), each side is convergent iff $|q| < 1$. The condition $\text{Im}(\tau) > 0$ ensures that $|q| < 1$. Then, filling in the data on conformal weights and isospin of singular vectors from table 2.1, we have that the character of an irreducible module built on Z'_0

which has conformal weight h and isospin j given by $4j + 1 = n - m(2k + 3)$, $m + n$ odd, $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, is, up to a factor of $P(\sigma, \tau)q^{-c/24}$,

$$\begin{aligned} & q^h z^{\frac{1}{4}(-1-m(2k+3))} \sum_{a=0}^{\infty} \left[q^{\frac{pq a^2}{2} + \frac{a(nq-mp)}{2}} z^{\frac{1}{4}n + \frac{ap}{2}} + q^{\frac{pq(a+1)^2}{2} - \frac{(a+1)(nq-mp)}{2}} z^{\frac{1}{4}n - \frac{(a+1)p}{2}} \right. \\ & \quad \left. - q^{\frac{pq a^2}{2} + \frac{a(nq+mp)}{2} + \frac{1}{2}mn} z^{-\frac{1}{4}n - \frac{ap}{2}} - q^{\frac{pq(a+1)^2}{2} - \frac{(a+1)(nq+mp)}{2} + \frac{1}{2}mn} z^{-\frac{1}{4}n + \frac{(a+1)p}{2}} \right] \\ &= q^h z^{\frac{1}{4}(-1-m(2k+3))} \left[\sum_{a \in \mathbb{Z}} q^{\frac{1}{2}pq(a^2 + a\frac{nq-mp}{pq})} z^{\frac{1}{4}n + \frac{ap}{2}} - \sum_{a \in \mathbb{Z}} q^{\frac{1}{2}pq(a^2 + a\frac{nq+mp}{pq} + \frac{mn}{pq})} z^{-\frac{1}{4}n - \frac{ap}{2}} \right] \end{aligned} \quad (2.60)$$

$$= q^{h - \frac{(nq-mp)^2}{8pq}} z^{-\frac{1}{4}} \sum_{a \in \mathbb{Z}} \left[q^{\frac{1}{2}pq(a + \frac{nq-mp}{2pq})^2} z^{\frac{pq}{2q}(a + \frac{nq-mp}{2pq})} - q^{\frac{1}{2}pq(a + \frac{nq+mp}{2pq})^2} z^{\frac{-pq}{2q}(a + \frac{nq+mp}{pq})} \right] \quad (2.61)$$

upon completing the square. With ϑ -functions as defined in (2.54), (2.61) may be written as,

$$q^{h - \frac{(nq-mp)^2}{8pq}} z^{-\frac{1}{4}} \left[\vartheta_{nq-mp, pq} \left(\frac{\sigma}{2q}, \frac{\tau}{2} \right) - \vartheta_{-nq-mp, pq} \left(\frac{\sigma}{2q}, \frac{\tau}{2} \right) \right] \quad (2.62)$$

using an elementary property from (2.55). To obtain the character from (2.59) and (2.62) we have to multiply by the factor $q^{\frac{c}{24}}$ where c is the central charge of the associated Virasoro algebra. From (2.23) we have that $c = \frac{2k}{2k+3}$. Also, from the expression (2.24) for L_0 it is not hard to show that if $|h, j\rangle$ is a hwv then,

$$L_0 |h, j\rangle = h|h, j\rangle = \frac{j(2j+1)}{2k+3} |h, j\rangle \quad (2.63)$$

i.e., the conformal weight of a hws can be written in terms of its isospin. If we also remember that $4j + 1 = n - m\frac{p}{q}$ then we have that the Ramond sector characters are,

$$\begin{aligned} \chi_{m,n}^R(\sigma, \tau) = & \frac{\vartheta_{1,2}(\frac{1}{2}\sigma, \tau) + \vartheta_{-1,2}(\frac{1}{2}\sigma, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))} \left(\vartheta_{nq-mp, pq} \left(\frac{\sigma}{2q}, \frac{\tau}{2} \right) - \vartheta_{-nq-mp, pq} \left(\frac{\sigma}{2q}, \frac{\tau}{2} \right) \right) \end{aligned} \quad (2.64)$$

with $(m+n)$ odd and $0 \leq m \leq q-1$ and $1 \leq n \leq p-1$. This expression is exactly the same as that exhibited by Ennes *et al* in [ERSdS]. Using this fact it is immediate that the character of the trivial representation ($k = j = 0$) is unity as it ought to be. I have chosen a different form for the prefactor because the version with level two ϑ -functions with an argument τ rather than ϑ -functions with an argument $\tau/2$ is better suited for the modular S transformation.

With the isomorphism (2.9) between the Ramond algebra and the Neveu-Schwarz algebra we can easily determine the NS sector characters in the following way. We have the definition of a NS character,

$$\begin{aligned}
\chi^{NS}(\sigma, \tau) &\stackrel{d}{=} \text{tr} \exp(2\pi i \tau (L_0^{NS} - \frac{c}{24})) \exp(2\pi i \sigma (b_0^{NS 3})) \\
&= \text{tr} \exp(2\pi i \tau (L_0^R - b_0^{R 3} + \frac{k}{4} - \frac{c}{24})) \exp(2\pi i \sigma (-b_0^{R 3} + \frac{k}{2})) \\
&= \text{tr} \exp(2\pi i \tau (L_0^R - \frac{c}{24} + \frac{k}{4})) \exp(2\pi i ((-\tau - \sigma) b_0^{R 3} + \sigma \frac{k}{2})) \\
&= q^{\frac{k}{4}} z^{\frac{k}{2}} \chi^R(-\sigma - \tau, \tau),
\end{aligned} \tag{2.65}$$

so that we need only perform a shift of variables on the Ramond characters. Then after some work we have the following for the Neveu-Schwarz sector characters,

$$\chi_{m,n}^{NS}(\sigma, \tau) = P^{NS}(\sigma, \tau) (\vartheta_{-nq+(m+1)p,pq}(\frac{\sigma}{2q}, \frac{\tau}{2}) - \vartheta_{nq+(m+1)p,pq}(\frac{\sigma}{2q}, \frac{\tau}{2})) \tag{2.66}$$

with $(m+n)$ odd and $0 \leq m \leq q-1$ and $1 \leq n \leq p-1$. We have also the Neveu-Schwarz prefactor,

$$P^{NS}(\sigma, \tau) \stackrel{d}{=} \frac{\vartheta_{0,2}(\frac{1}{2}\sigma, \tau) + \vartheta_{2,2}(\frac{1}{2}\sigma, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))}. \tag{2.67}$$

The supercharacters, $S\chi$, may be obtained from the characters by doing the following shift of variables and multiplying by $e^{-2\pi i j R, NS}$,

$$S\chi_{m,n}^{R,NS}(\sigma, \tau) = e^{-2\pi i j R, NS} \chi_{m,n}^{R,NS}(\sigma + 1, \tau). \tag{2.68}$$

This is identical to inserting the operator $(-1)^F$ ([Wit82]) into the definition of a character. The supercharacters “highlight” the fermionic states in the Verma module—fermionic states appear with a minus sign in a supercharacter. This fact could be used to project out the fermionic or bosonic states in a module by adding or subtracting the character and supercharacter for the same representation. This is just like the GSO [GSO76, GSO77, GSW88] projection of string theory where one projects out states which are even under the operator $(-1)^F$ (“ G -parity”) so as to render the string theory spacetime supersymmetric. We get the following for the Neveu-Schwarz sector supercharacters,

$$\begin{aligned}
S\chi_{m,n}^{NS}(\sigma, \tau) &= \frac{\vartheta_{0,2}(\frac{\sigma}{2}, \tau) - \vartheta_{2,2}(\frac{\sigma}{2}, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))} \\
&\quad \times (\vartheta_{-nq+(m+1)p,2pq}(\frac{\sigma}{2q}, \tau) + (-1)^p \vartheta_{2pq-nq+(m+1)p,2pq}(\frac{\sigma}{2q}, \tau) \\
&\quad - (-1)^n \vartheta_{nq+(m+1)p,2pq}(\frac{\sigma}{2q}, \tau) - (-1)^{n+p} \vartheta_{2pq+nq+(m+1)p,2pq}(\frac{\sigma}{2q}, \tau))
\end{aligned} \tag{2.69}$$

with $(m+n)$ odd and $0 \leq m \leq q-1$ and $1 \leq n \leq p-1$. The Ramond supercharacters can be written as,

$$\begin{aligned}
 S\chi_{m,n}^R(\sigma, \tau) &= \frac{\vartheta_{1,2}(\frac{\sigma}{2}, \tau) - \vartheta_{-1,2}(\frac{\sigma}{2}, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))} \\
 &\quad \times \left(\vartheta_{nq-mp, 2pq}(\frac{\sigma}{2q}, \tau) + (-1)^p \vartheta_{-2pq+nq-mp, 2pq}(\frac{\sigma}{2q}, \tau) \right. \\
 &\quad \left. - (-1)^n \vartheta_{-nq-mp, 2pq}(\frac{\sigma}{2q}, \tau) - (-1)^{n+p} \vartheta_{-2pq-nq-mp, 2pq}(\frac{\sigma}{2q}, \tau) \right) \quad (2.70)
 \end{aligned}$$

with $(m+n)$ odd and $0 \leq m \leq q-1$ and $1 \leq n \leq p-1$. If we try to write down characters for the case when $n = 0$ then we see that the correcting factor is such that it cannot be written in terms of ϑ -functions. This being the case these characters will not carry a representation of the modular group. We ignore this pathological case henceforth. Ennes *et al.* [ERSdS] have shown that certain of the characters above are singular in the limit as $\sigma \rightarrow 0$. They found that non-integrable Ramond sector characters with $m > 0$ were singular and that the residue was proportional to characters in the discrete series of $N = 1$ superconformal characters. These could be found in, for example [GKO86]. This result is not unexpected in view of the residue results of Mukhi and Panda [MP90] for non-integrable characters of $\hat{sl}(2; \mathbb{C})$ and our own results for $\hat{sl}(2|1; \mathbb{C})$ which are spelled out in chapter 4. See subsection 4.4.1 there.

At this stage we can derive a lemma which connects the occurrence of singular characters and infinite-dimensional representations of $osp(1, 2; \mathbb{C})$. Since $osp(1, 2; \mathbb{C})$ is a subalgebra of $\widehat{osp}(1, 2; \mathbb{C})$ the former “inherits” representations from the latter. Indeed at each grade of an affine representation there is a representation of the corresponding finite-dimensional algebra. Between them Kac, and Cornwell prove² that a representation of a finite-dimensional superalgebra is finite-dimensional itself if and only if the “numerical marks” corresponding to all roots with non-zero length in a distinguished basis are non-negative integers. The numerical mark n_i for a simple root α_i and highest weight Λ is defined to be,

$$n_i \stackrel{\text{d}}{=} \frac{2\Lambda \cdot \alpha_i}{\alpha_i \cdot \alpha_i}. \quad (2.71)$$

For $osp(1, 2; \mathbb{C})$ the unique simple root can be taken to be $\alpha/2$. Since this is odd we have a distinguished basis. Since we have just one simple root, let $n_1 = n$. Then $n = 6\Lambda \cdot \alpha$. Now the highest weight that $osp(1, 2; \mathbb{C})$ inherits from $\widehat{osp}(1, 2; \mathbb{C})$ is $\Lambda = j\alpha$ from (2.31). The quantity j satisfies (2.38). Thus $n = 4j$. Now we have the lemma,

²Kac proves sufficiency, Cornwell, necessity.

Lemma 2.1

When the highest weight Λ of a representation of $osp(1, 2; \mathbb{C})$ is of the form,

$$\Lambda = j\alpha,$$

with j satisfying (2.38), the representation is finite-dimensional if and only if $m = 0$ with n as in (2.38).

Proof. From the foregoing discussion, we require that $n = 4j \in \mathbb{Z}_+$. That is from (2.38),

$$n = 4j = (n - 1) - \frac{mp}{q} \in \mathbb{Z}_+, \quad (2.72)$$

where $0 \leq m \leq q - 1$ and $1 \leq n \leq p - 1$ and $\gcd(p, q) = 1$. Clearly $n \in \mathbb{Z}_+$ if and only if $m = 0$. ■

Corollary 2.1

Integrable representations of $\widehat{osp}(1, 2; \mathbb{C})$ provide finite-dimensional representations of $osp(1, 2; \mathbb{C})$.

Proof. Integrable representations have $q = 1 \Rightarrow m = 0$. ■

Lemma 2.2

The character of an admissible representation of $\widehat{osp}(1, 2; \mathbb{C})$ is regular in the limit as $\sigma \rightarrow 0$ ($z \rightarrow 1$), if and only if that representation provides finite-dimensional representations of $osp(1, 2; \mathbb{C})$.

Proof. From [ERSdS] the regular characters in the limit are those with $m = 0$. But from the previous lemma these are precisely the ones which provide finite-dimensional representations of $osp(1, 2; \mathbb{C})$. ■

2.7 The Modular Transformations.

Having obtained all the characters and supercharacters in the last section we shall now see an example of how to deduce their image under the modular transformations. First though, a reminder of some facts about modular transformations and the modular group.

When working with characters one always tries to write them in terms of ϑ -functions. The reason being that we know the image of the ϑ -functions under the modular group. For theories on a torus, the group of modular transformations is isomorphic to the group

$PSL(2, \mathbb{Z})$. The “ P ” is to indicate the fact that we have taken the quotient of $SL(2, \mathbb{Z})$ by the subgroup $\{I, -I\}$ where I is the identity of the group. By $SL(2, \mathbb{Z})$ we mean the group of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries a, b, c, d which are such that $ad - bc = 1$. One can generate the whole of $SL(2, \mathbb{Z})$ with just two of its elements. They are conventionally denoted S and T . In terms of matrices,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.73)$$

As far as the characters are concerned, the modular transformations act on their arguments σ and τ . The S and T transformations on the arguments (σ, τ) are,

$$S(\sigma, \tau) = \left(\frac{\sigma}{\tau}, \frac{-1}{\tau} \right) \quad T(\sigma, \tau) = (\sigma, \tau + 1). \quad (2.74)$$

We shall concentrate mainly on the S transformation since it is a lot more difficult to derive than the T transformation. It will be important for us to know how the η -function and the ϑ -functions transform under S . It is well-known that the η -function has the following behaviour (see the book by Knopp [Kno70] for one source and a derivation),

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \quad (2.75)$$

and from Kac’s book [Kac90] we have that the ϑ -functions as defined in (2.54) have the following transformation,

$$\vartheta_{m,k}\left(\frac{\sigma}{\tau}, \frac{-1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}} e^{-2\pi i k \sigma^2 / 2\tau} \sum_{r=0}^{2k-1} e^{\frac{-i\pi r m}{k}} \vartheta_{r,k}(\sigma, \tau). \quad (2.76)$$

The η -function and ϑ -function are said to have modular weight of $1/2$ because of the presence of the factor $\tau^{\frac{1}{2}}$ on the RHS of (2.75) and (2.76).

Now we are in a position to work out how the Neveu-Schwarz characters (2.66) transform under S . Consider the prefactor first. Using (2.75) and (2.76) it is easy to see that P^{NS} transforms as,

$$\frac{\vartheta_{0,2}(\frac{1}{2}\sigma, \tau) + \vartheta_{2,2}(\frac{1}{2}\sigma, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))} \xrightarrow{S} e^{3\pi i \sigma^2 / 2\tau} \frac{i}{\sqrt{-i\tau}} \frac{\vartheta_{0,2}(\frac{1}{2}\sigma, \tau) + \vartheta_{2,2}(\frac{1}{2}\sigma, \tau)}{\eta(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))}. \quad (2.77)$$

Now consider the ϑ -functions at level pq in the Neveu-Schwarz characters. Looking at them one sees that they each have an argument $\tau/2$. This is bad news because it means these ϑ -functions will not transform back to themselves under S . In fact one would obtain an argument 2τ . To get around this little problem one splits the

summation variable of the sum which is the ϑ -function into even values and odd values. This serves to generate two functions each with a second argument τ as we require and the level of those ϑ -functions doubles. The “ σ ” argument is not affected. Doing this procedure on the Neveu-Schwarz characters we get,

$$\begin{aligned} & \vartheta_{-nq+(m+1)p,pq}\left(\frac{\sigma}{2q}, \frac{\tau}{2}\right) - \vartheta_{nq+(m+1)p,pq}\left(\frac{\sigma}{2q}, \frac{\tau}{2}\right) \\ &= \vartheta_{-nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) + \vartheta_{2pq-nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) \\ & \quad - \vartheta_{nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) - \vartheta_{2pq+nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) \end{aligned} \quad (2.78)$$

From (2.76) we have that the S transform of the last expression is,

$$\begin{aligned} & \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1}^{4pq-1} \left(e^{\frac{-i\pi r(-nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) + e^{\frac{-i\pi r(2pq-nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) \right. \\ & \quad \left. - e^{\frac{-i\pi r(nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{-i\pi r(2pq+nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) \right). \end{aligned} \quad (2.79)$$

By splitting r into the ranges $r \in \{1, 2, \dots, 2pq - 1\}$ and $r \in \{2pq + 1, \dots, 4pq - 1\}$ we have that (2.79) is equal to,

$$\begin{aligned} & \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1, 2p|r}^{2pq-1} \left(e^{\frac{-i\pi r(-nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) + e^{\frac{-i\pi r(2pq-nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) \right. \\ & \quad \left. - e^{\frac{-i\pi r(nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{-i\pi r(2pq+nq+(m+1)p)}{2pq}} \vartheta_{r,2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \\ & + e^{\frac{-i\pi(r+2pq)(-nq+(m+1)p)}{2pq}} \vartheta_{r+2pq,2pq}\left(\frac{\sigma}{2q}, \tau\right) + e^{\frac{-i\pi(r+2pq)(2pq-nq+(m+1)p)}{2pq}} \vartheta_{r+2pq,2pq}\left(\frac{\sigma}{2q}, \tau\right) \\ & \quad \left. - e^{\frac{-i\pi(r+2pq)(nq+(m+1)p)}{2pq}} \vartheta_{r+2pq,2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{-i\pi(r+2pq)(2pq+nq+(m+1)p)}{2pq}} \vartheta_{r+2pq,2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \end{aligned} \quad (2.80)$$

We have neglected terms in the sum where $2p|r$ because the summand vanishes there anyway. One can count the number of ϑ -functions that the sum provides now and see that it is just the number we need *i.e.*, four times the number of characters for a representation with given level k . Now, in the last four terms of the summand change

summation variable to r' where $r' = 2pq - r$. We have then that (2.80) is equal to,

$$\begin{aligned}
& \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1, 2p \nmid r}^{2pq-1} \left(e^{\frac{-i\pi r(-nq+(m+1)p)}{2pq}} \vartheta_{r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) + e^{\frac{-i\pi r(2pq-nq+(m+1)p)}{2pq}} \vartheta_{r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right. \\
& \quad - e^{\frac{-i\pi r(nq+(m+1)p)}{2pq}} \vartheta_{r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{-i\pi r(2pq+nq+(m+1)p)}{2pq}} \vartheta_{r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \\
& \quad + e^{\frac{i\pi r(-nq+(m+1)p)}{2pq}} \vartheta_{-r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) + e^{\frac{i\pi r(2pq-nq+(m+1)p)}{2pq}} \vartheta_{-r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \\
& \quad \left. - e^{\frac{i\pi r(nq+(m+1)p)}{2pq}} \vartheta_{-r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{i\pi r(2pq+nq+(m+1)p)}{2pq}} \vartheta_{-r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \\
& = \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1, 2p \nmid r}^{pq-1} \left(e^{\frac{i\pi rn}{p}} - e^{-\frac{i\pi rn}{p}} \right) \left(e^{\frac{-i\pi r(m+1)}{q}} \vartheta_{2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{i\pi r(m+1)}{q}} \vartheta_{-2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right. \\
& \quad \left. + e^{\frac{-i\pi 2r((m+1)p+2pq)}{2pq}} \vartheta_{2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{i\pi 2r((m+1)p+2pq)}{2pq}} \vartheta_{-2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right). \quad (2.81)
\end{aligned}$$

after remembering that the first index on the ϑ -functions is to be taken mod $4pq$. Also, whenever r was odd the summand vanished so we wrote $2r$ for r and summed over half the range. One can easily see that for Neveu-Schwarz characters (2.66) at least, the first index, $\pm nq + (m+1)p$, on the ϑ -functions at level pq is always even. Now, in the terms involving the latter two ϑ -functions of the summand of (2.81) change summation variable to $r' = pq - r$. Then (2.81) is equal to,

$$\begin{aligned}
& \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1, p \nmid r}^{pq-1} \left(e^{\frac{i\pi rn}{p}} - e^{-\frac{i\pi rn}{p}} \right) \left(e^{\frac{-i\pi r(m+1)}{q}} \vartheta_{2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{i\pi r(m+1)}{q}} \vartheta_{-2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right. \\
& \quad \left. + e^{\frac{-i\pi 2r((m+1)p+2pq)}{2pq}} \vartheta_{2r+2pq, 2pq}\left(\frac{\sigma}{2q}, \tau\right) - e^{\frac{i\pi 2r((m+1)p+2pq)}{2pq}} \vartheta_{-2r-2pq, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \\
& = \sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{r=1, p \nmid r}^{pq-1} \left(e^{\frac{i\pi rn}{p}} - e^{-\frac{i\pi rn}{p}} \right) \left(e^{\frac{-i\pi r(m+1)}{q}} \left(\vartheta_{2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) + \vartheta_{2r+2pq, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \right. \\
& \quad \left. - e^{\frac{i\pi r(m+1)}{q}} \left(\vartheta_{-2r, 2pq}\left(\frac{\sigma}{2q}, \tau\right) + \vartheta_{-2r+2pq, 2pq}\left(\frac{\sigma}{2q}, \tau\right) \right) \right). \quad (2.82)
\end{aligned}$$

At this stage one can show that the value of the summand at r is the same as at $pq - r$ and thus one can halve the range again of r again. So now we sum over r from 1 to $\lfloor (pq-1)/2 \rfloor$ i.e., the integer part of $(pq-1)/2$.

Now we decompose r into p and q in the following way. In the terms involving the ϑ -functions with positive r , let $2r = \beta q - \alpha p$ and in the terms involving ϑ -functions with negative r , let $2r = -\beta q - \alpha p$. In both cases $1 \leq \alpha \leq p-1$, $1 \leq \beta \leq q$ and also we must have $\alpha + \beta$ even. We assert that in doing this one recovers all the terms of the sum

as they appear in (2.82) whatever p and q might be. We have no proof of this assertion but it holds in examples worked out. Substituting for r as described and simplifying one ends up with,

$$\begin{aligned} & \vartheta_{-nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) + \vartheta_{2pq-nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) \\ & \quad - \vartheta_{nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) - \vartheta_{2pq+nq+(m+1)p,2pq}\left(\frac{\sigma}{2q}, \tau\right) \xrightarrow{S} \\ & 4i\sqrt{\frac{-i\tau}{4pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{\substack{\alpha=1 \\ \alpha+\beta \text{ even}}}^{p-1} \sum_{\beta=1}^q \sin \frac{n\pi(\beta p - \alpha q)}{2p} e^{-\frac{i\pi(\beta p - \alpha q)(m+1)}{2q}} \chi_{\beta-1,\alpha}^{NS}(\sigma, \tau) / P^{NS}(\sigma, \tau) \end{aligned} \quad (2.83)$$

where P^{NS} is the prefactor of a Neveu-Schwarz character which was defined in equation (2.67). Then finally, combining (2.83) with (2.77) we have the result,

$$\chi_{m,n}^{NS}\left(\frac{\sigma}{\tau}, \frac{-1}{\tau}\right) = \frac{2}{\sqrt{pq}} e^{-\frac{\pi ip\sigma^2}{2q\tau}} \sum_{\substack{\alpha=1 \\ \alpha+\beta \text{ even}}}^{p-1} \sum_{\beta=1}^q \sin \frac{n\pi(-\beta p + \alpha q)}{2p} e^{-\frac{i\pi(\beta p - \alpha q)(m+1)}{2q}} \chi_{\beta-1,\alpha}^{NS}(\sigma, \tau). \quad (2.84)$$

In particular this transformation leaves the character of the trivial representation invariant as it ought to. (The character of the trivial rep. is unity). For integrable representations there is the slightly neater form,

$$\chi_n^{NS, \text{int}}\left(\frac{\sigma}{\tau}, \frac{-1}{\tau}\right) = \frac{2}{\sqrt{p}} e^{-\frac{\pi ip\sigma^2}{2\tau}} \sum_{\substack{n'=1 \\ n' \text{ odd}}}^{p-1} (-1)^{n+n'} \sin\left(\frac{\pi n n'}{p}\right) \chi_{n'}^{NS, \text{int}}(\sigma, \tau) \quad (2.85)$$

Now we turn to the problem of working out the result of doing the T transformation on the Neveu-Schwarz characters. This result is much easier to derive than the S transformation and we will not give the details. Suffice to say that the result is the following,

$$\chi_{m,n}^{NS}(\sigma, \tau + 1) = e^{2\pi i(h^{NS} - c/24)} S \chi_{m,n}^{NS}(\sigma, \tau). \quad (2.86)$$

The fact that m and n have different parities while p and q have the same parity is important in deriving this result. Also, (2.23), (2.63) and the spectral flow mapping from (2.9) are needed. A moment's reflection assures us that (2.86) is the result we should expect. In the Neveu-Schwarz sector fermionic states appear at half integer grades so the appearance of the supercharacter is to be expected because in it fermionic states appear with a negative sign. Remembering that a Neveu-Schwarz character has a factor

character	$S \longrightarrow$	$T \longrightarrow$
χ^R	$S\chi^{NS}$	χ^R
$S\chi^R$	$S\chi^R$	$S\chi^R$
χ^{NS}	χ^{NS}	$S\chi^{NS}$
$S\chi^{NS}$	χ^R	χ^{NS}

Table 2.2: Scheme of modular transformations of different characters and supercharacters.

of q (the power series parameter) at the front with a fractional power of $(h^{NS} - c/24)$, the phase in (2.86) is not alarming.

Whilst we have not explicitly checked the modular transformations of the other characters and supercharacters, the results above for the Neveu-Schwarz characters give one confidence that all would be well. In table 2.2 is recorded the complete scheme of transformations as it ought to be. Note that only the Ramond supercharacters are invariant under modular transformations.

Chapter 3

$\hat{sl}(2|1; \mathbb{C})$ and its Representations

3.1 Introduction.

The purpose of this chapter is to present the superalgebra $sl(2|1; \mathbb{C})$ and its affine counterpart $\hat{sl}(2|1; \mathbb{C})$ and to introduce the different classes of representations for which we will compute characters in the next chapter. The class V representations will receive special attention. We begin then, in section 3.2 with the definition of $sl(2|1; \mathbb{C})$ and $\hat{sl}(2|1; \mathbb{C})$ via their (anti)commutation relations. We discuss their roots and present Cartan matrices, Dynkin diagrams, the Killing form and second Casimir of $sl(2|1; \mathbb{C})$. In section 3.4 we turn to the task of classifying the possible admissible representations of $\hat{sl}(2|1; \mathbb{C})$. In their paper [BT97], Bowcock and Taormina made a detailed study of the irreducible, admissible representations of $\hat{sl}(2|1; \mathbb{C})$. By studying the zeroes of the Kac determinant they were able to identify four classes of admissible representations and, most important, obtain the embedding diagram for each class. The different classes of representations are defined according to the number of zeroes of the Kac determinant in the fermionic sector. That work is extended in this chapter. We shall see that a fifth class of representations exists. In the succeeding section we show how one might go about deriving the quantum numbers of singular vectors in class V modules. This prepares the ground for computing characters in the next chapter.

3.2 Finite and Affine $sl(2|1; \mathbb{C})$ and their Roots.

All of the results in this section can be found in [BT97]. We give here the (anti)commutation relations of the algebras, the root diagram of the finite-dimensional algebra and a set of simple roots, a Dynkin diagram and a Cartan matrix for each algebra. At the end of this section we mention the Sugawara construction for $\hat{sl}(2|1; \mathbb{C})$ and give the

Virasoro generator L_0 explicitly.

The superalgebra $sl(2|1; \mathbb{C})$ comprises four even generators and four odd generators. It is basic, classical, simple and complex. The even subalgebra is the direct sum of $sl(2; \mathbb{C})$ and a 1-dimensional abelian algebra. Thus the even subalgebra is not semisimple but reductive as it should be for a classical superalgebra. The non-zero (anti)commutation relations of $sl(2|1; \mathbb{C})$ in the homogeneous, Cartan-Weyl basis are,

$$\begin{aligned}
 [J^+, J^-] &= 2J^3 & [J^3, J^\pm] &= \pm J^\pm \\
 [J^\pm, j'^\mp] &= \pm j^\pm & [J^\pm, j^\mp] &= \mp j'^\pm \\
 [2J^3, j'^\pm] &= \pm j'^\pm & [2J^3, j^\pm] &= \pm j^\pm \\
 [2U, j'^\pm] &= \pm j'^\pm & [2U, j^\pm] &= \mp j^\pm \\
 \{j'^+, j'^-\} &= U - J^3 & \{j^+, j^-\} &= U + J^3 \\
 \{j'^\pm, j^\pm\} &= J^\pm & &
 \end{aligned} \tag{3.1}$$

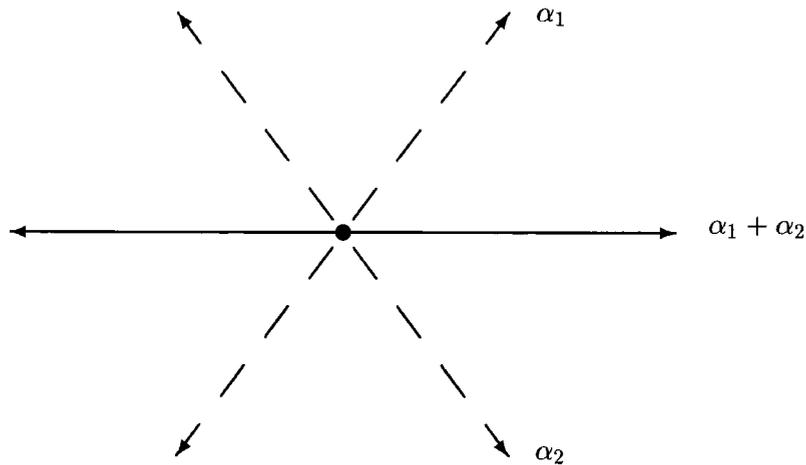
$J^{3,\pm}$ and U are the even generators of the algebra and j'^\pm and j^\pm are the odd generators. Thus $[,]$ and $\{, \}$ represent commutators and anticommutators respectively just as in the last chapter. The Cartan subalgebra is spanned by the set $\{J^3, U\}$. From (3.1) we may read off the set Δ of non-zero roots. We have $\Delta = \Delta_0 \cup \Delta_1$ where,

$$\begin{aligned}
 \Delta_0 &= \{\pm(\alpha_1 + \alpha_2)\} \\
 \Delta_1 &= \{\pm\alpha_1, \pm\alpha_2\}.
 \end{aligned} \tag{3.2}$$

with $\alpha_1 = (1/2, 1/2)$ and $\alpha_2 = (1/2, -1/2)$. The step operators corresponding to α_1 , α_2 and $(\alpha_1 + \alpha_2)$ are j'^+ , j^+ and J^+ respectively. Δ_0 and Δ_1 are the sets of even roots and odd roots respectively. There is also a two-fold degenerate zero root corresponding to the Cartan subalgebra itself. The root diagram is shown in figure 3.1. The Killing form g^{ab} is,

$$g^{ab} = \text{diag} \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \right) \tag{3.3}$$

The matrix is 8×8 and has block diagonal form. The order of the indices is $J^3, U, J^+, J^-, j'^+, j'^-, j^+, j^-$. When the indices run over “even” values in the top left hand corner

Figure 3.1: The root diagram of $sl(2|1; \mathbb{C})$.

the matrix is symmetric and when the indices run over the “odd” values in the bottom right hand corner the matrix is antisymmetric. So the form is supersymmetric. The Cartan-Killing metric on the root space is $\text{diag}(1, -1)$. We see its indefinite signature. This means that $\alpha_1^2 = \alpha_2^2 = 0$. Odd roots of zero norm are called *isotropic*. Also $\alpha_1 \cdot \alpha_2 = 1/2$. When we choose the positive roots to be

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.4)$$

$(\alpha_1 + \alpha_2)$ is the highest root and $(\alpha_1 + \alpha_2)^2 = 1$. With the Killing form to hand, the second Casimir, C_2 , is,

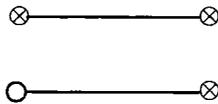
$$C_2 = J^3 J^3 - UU + \frac{1}{2}(J^+ J^- + J^- J^+ + j'^+ j'^- - j'^- j'^+ - j^+ j^- + j^- j^+). \quad (3.5)$$

This commutes with all the generators of the algebra and is equal to the identity for the adjoint representation. Furthermore, having chosen positive roots as above, for a general highest weight representation with highest weight state $|\Omega\rangle$ such that $J^3|\Omega\rangle = h_-/2|\Omega\rangle$ and $U|\Omega\rangle = h_+/2|\Omega\rangle$ (which defines the *isospin* $h_-/2$ and *charge* $h_+/2$ of a state¹), we have,

$$C_2|\Omega\rangle = \frac{1}{4}(h_-^2 - h_+^2)|\Omega\rangle. \quad (3.6)$$

Setting $h_- = 2$ and $h_+ = 0$ we recover the adjoint representation. Now we meet a feature which is not present in $osp(1, 2; \mathbb{C})$. Because some of the roots of $sl(2|1; \mathbb{C})$ have zero norm, the Weyl group of the algebra is smaller than it would otherwise be. In fact the

¹We retain the notation of [BT97]

Figure 3.2: The Dynkin diagrams of $sl(2|1; \mathbb{C})$.

Weyl group of $sl(2|1; \mathbb{C})$ is isomorphic to that of its even subalgebra $sl(2) \oplus u(1)$. That Weyl group is $W = \mathbb{Z}_2$ which is generated by the reflection in the plane perpendicular to the root $(\alpha_1 + \alpha_2)$. There are then several Weyl-inequivalent choices for the positive roots. For each choice there is a set of simple roots. If we choose the positive roots to be the set Δ^+ as in (3.4), then the simple roots are both odd, $\Pi = \{\alpha_1, \alpha_2\}$ and the Cartan matrix is,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We calculate this as follows. $A_{ij} = \alpha_i \cdot \alpha_j / \alpha_i \cdot \alpha_{i'}$ and $i' \neq i$ so that we avoid dividing by zero. See [Cor89, p.241]. The Dynkin diagram is very simple and is the first diagram in figure 3.2. The nodes are “grey” to indicate that the simple roots are isotropic. Compare figure 2.2 where the odd root is black because it has non-zero norm and the even root is white. With our choice, (3.4), of positive roots, the Weyl vector ρ vanishes, $\rho = 0$. All other choices of positive roots would give different Cartan matrices. In fact one gets,

$$A' = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \quad (3.7)$$

The Dynkin diagram in these cases is the second diagram in figure 3.2. The Coxeter number, h of $sl(2|1; \mathbb{C})$ is given by $h = 3$ (same as $osp(1, 2; \mathbb{C})$) and the dual Coxeter number h^\vee by, $h^\vee = 1$ which, as explained in chapter 1, is independent of the choice of simple roots. We shall choose now to remain with the choice (3.4) of positive roots. This means that our basis of simple roots is not distinguished.

We now move on to the untwisted, affine algebra $\hat{sl}(2|1; \mathbb{C})$. The non-zero (anti)com-

mutators of this affine algebra are as follows.

$$\begin{aligned}
 [J_m^+, J_n^-] &= 2J_{m+n}^3 + \tilde{k}m\delta_{m+n,0} & [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm \\
 [J_m^\pm, j_n^{\mp}] &= \pm j_{m+n}^\pm & [J_m^\pm, j_n^{\mp}] &= \mp j_{m+n}^\pm \\
 [2J_m^3, j_{m+n}^{\pm}] &= \pm j_{m+n}^{\pm} & [2J_m^3, j_n^\pm] &= \pm j_{m+n}^\pm \\
 [2U_m, j_n^{\pm}] &= \pm j_{m+n}^{\pm} & [2U_m, j_n^\pm] &= \mp j_{m+n}^\pm \\
 [J_m^3, J_n^3] &= \frac{\tilde{k}}{2}m\delta_{m+n,0} & [U_m, U_n] &= -\frac{\tilde{k}}{2}m\delta_{m+n,0} \\
 \{j_m^+, j_n^-\} &= U_{m+n} - J_{m+n}^3 - \tilde{k}m\delta_{m+n,0} & \{j_m^\pm, j_n^\pm\} &= J_{m+n}^\pm \\
 \{j_m^+, j_n^-\} &= U_{m+n} + J_{m+n}^3 + \tilde{k}m\delta_{m+n,0} & &
 \end{aligned}$$

The even generators have $m, n \in \mathbb{Z}$ and the odd generators have $m, n \in \mathbb{Z}$ in the Ramond sector and $m, n \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector. We shall work in the Ramond sector for the time being. There is also the even generator d —the derivative operator. Its brackets are of the form,

$$[d, X_n] = nX_n$$

where X is any of the other generators except \tilde{k} , the central generator, which commutes with all the other generators. The Cartan subalgebra is spanned by the set $\{J_0^3, U_0, \tilde{k}, d\}$. Weights are specified by their eigenvalues on these four operators—the *weight labels*. In the usual way we build the roots of the affine algebra from those of the corresponding finite dimensional algebra. With the choice above for the positive roots of $sl(2|1; \mathbb{C})$, the simple roots of $\hat{sl}(2|1; \mathbb{C})$ are,

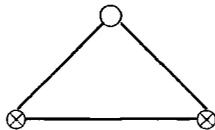
$$\Pi = \{\alpha_0, \alpha_1, \alpha_2\}$$

where $\alpha_0 \stackrel{d}{=} (-\alpha_1 - \alpha_2, 0, 1)$ and we denote an affine root which is an extension of a root of $sl(2|1; \mathbb{C})$ by the same symbol so that $\alpha_i = (\alpha_i, 0, 0)$ for $i = 1, 2$ now. The affine roots α_1 and α_2 are isotropic and α_0 is even with squared length of unity. The affine Weyl vector is,

$$\rho = (\rho, h^\vee, 0) = (0, 1, 0). \quad (3.8)$$

We can easily calculate a Cartan matrix for $\hat{sl}(2|1; \mathbb{C})$:

$$\begin{pmatrix}
 2 & -1 & -1 \\
 -1 & 0 & 1 \\
 -1 & 1 & 0
 \end{pmatrix}.$$

Figure 3.3: The Dynkin diagram of $\hat{sl}(2|1; \mathbb{C})$.

The Dynkin diagram is shown in figure 3.3.

Next we turn to the Sugawara construction as applied to $\hat{sl}(2|1; \mathbb{C})$. We can carry over the definition (2.18) from chapter 2. Note though that now the superdimension of the superalgebra vanishes. Thus (2.22) gives the central charge of the associated Virasoro algebra to be zero. The only non-trivial representations of a $c = 0$ conformal field theory are non-unitary [FQS84]. A useful quantity to know is the Virasoro generator L_0 . Equation (2.18) and the Killing form (3.3) (when inverted) give L_0 to be,

$$L_0 = \frac{1}{2(k+1)} \sum_{m \in \mathbb{Z}} \bullet 2J_m^3 J_{-m}^3 - 2U_m U_{-m} + J_m^+ J_{-m}^- + J_m^- J_{-m}^+ \\ + j_m^+ j_{-m}^- - j_m^- j_{-m}^+ - j_m^+ j_{-m}^- + j_m^- j_{-m}^+ \bullet \quad (3.9)$$

and we use the normal ordering prescription for even and odd generators that was given in (2.20) and (2.21) (with notation suitably changed.) Again, k denotes the level of the representation of $\hat{sl}(2|1; \mathbb{C})$. In the basis $\{\alpha_1, \alpha_2\}$ of simple roots of $sl(2|1; \mathbb{C})$, $(\alpha_1 + \alpha_2)$ is the highest root and $(\alpha_1 + \alpha_2)^2 = 1$ and so $k = 2\tilde{k}$ by (2.8).

For the admissible representations Bowcock and Taormina chose

$$k + h^\vee = p/u, \quad (3.10)$$

$p \in \mathbb{N}$, $u \in \mathbb{N}$ and $\gcd(p, u) = 1$. This choice for k allows us to consider integrable representations (where $k \in \mathbb{Z}_+$) by setting $u = 1$. We shall see more of integrable representations later.

The analysis is now similar to that in the last chapter. Next we give the definition of highest weight state which is appropriate for this algebra. The highest weight vector can be written in Dirac's notation as $|h, \frac{1}{2}h_-, \frac{1}{2}h_+\rangle$ where the L_0 eigenvalue (the conformal weight) is h , $\frac{1}{2}h_-$ is the isospin and $\frac{1}{2}h_+$ is the charge.

Definition 3.1 (Highest Weight State.)

$|h, \frac{1}{2}h_-, \frac{1}{2}h_+\rangle \stackrel{\text{d}}{=} |\Omega\rangle$ is a highest weight state if it is annihilated by all the raising opera-

tors, i.e.,

$$\begin{aligned}
 J_n^+|\Omega\rangle &= 0 & J_{n+1}^-|\Omega\rangle &= 0 \\
 J_{n+1}^3|\Omega\rangle &= 0 & U_{n+1}|\Omega\rangle &= 0 \\
 j_n^+|\Omega\rangle &= 0 & j_n'^+|\Omega\rangle &= 0 \\
 j_{n+1}^-|\Omega\rangle &= 0 & j_{n+1}'^-|\Omega\rangle &= 0 \\
 L_{n+1}|\Omega\rangle &= 0
 \end{aligned}$$

for $n \in \mathbb{Z}_+$.

We always choose the hws to be bosonic.

Remark 3

The conditions above for a highest weight vector can all be obtained from three conditions. We need only impose that,

$$J_1^-|\Omega\rangle = j_0^+|\Omega\rangle = j_0'^+|\Omega\rangle = 0.$$

These are the operators corresponding to simple roots of the (untwisted) affine algebra.

3.3 Twisting the algebra.

In this section we explain how to introduce a twist in the algebra to obtain the so-called Neveu-Schwarz version of the algebra. The Neveu-Schwarz version of $\hat{sl}(2|1; \mathbb{C})$ is obtained by twisting the Ramond algebra with an order 2 automorphism—an involution. Let this involution be denoted γ . Then $\gamma^2 = 1$. Let $|\Omega\rangle$ be a highest weight state as defined in definition 3.1 above. Then we also require of γ that $\gamma(X_n)|\Omega\rangle = 0$ iff $X_n|\Omega\rangle = 0$. This is to say that we want a Ramond sector highest weight state to be a Neveu-Schwarz sector highest weight state too so that upon transforming the character of a Ramond sector highest weight representation, we obtain the character of a Neveu-Schwarz sector highest weight representation. There are many automorphisms of the algebra $\hat{sl}(2|1; \mathbb{C})$ but not all of them satisfy these two criteria. Indeed in [BT97] there is a perfectly good automorphism but it does not preserve the hws as we require. However,

the following involution does have all the properties that we would like.

$$\begin{aligned}
\gamma(J_n^3) &= -J_n^3 + \frac{1}{2}k\delta_{n,0} & \gamma(U_n) &= U_n \\
\gamma(J_n^\pm) &= J_{n\pm 1}^\mp & \gamma(j_n^\pm) &= j_{n\pm\frac{1}{2}}^{\prime\mp} \\
\gamma(j_n^{\prime\pm}) &= j_{n\pm\frac{1}{2}}^\mp & \gamma(k) &= k \\
\gamma(L_n) &= L_n - J_n^3 + \frac{1}{4}k\delta_{n,0}. & &
\end{aligned} \tag{3.11}$$

The fact that the odd generators in the Neveu-Schwarz algebra have 1/2-integer indices is a consequence of the fact that γ is an order 2 automorphism. We can use this twist to easily obtain Neveu-Schwarz characters from Ramond ones in the next chapter. It is quite straightforward to check that γ leaves all the brackets invariant, that it is indeed an involution and that an operator of the Ramond algebra annihilates a hws iff the corresponding Neveu-Schwarz operator does. There are some differences between the two sectors though. The conformal weight and isospin and charge of any state change upon passing from one sector to the other. Indeed from (3.11) we have that,

$$\frac{1}{2}h_-^{NS} = -\frac{1}{2}h_-^R + \frac{1}{2}k \tag{3.12}$$

$$\frac{1}{2}h_+^{NS} = \frac{1}{2}h_+^R \tag{3.13}$$

$$h^{NS} = h^R - \frac{1}{2}h_-^R + \frac{1}{4}k. \tag{3.14}$$

The involution in (3.11) is not the canonical inner automorphism that might be found in [GO86] for example. We have not chosen the canonical automorphism because it does not map hws's to hws's.

3.4 Classification of Representations.

The formula for the Kac determinant for the case of Kac-Moody superalgebras is given in equation (2.28) in the last chapter but for the reader's convenience we give it again. Let F_η be the symmetric bilinear form on the Verma module. Then its determinant is given by

$$\det F_\eta = \prod_{\substack{\alpha \in \Delta_0^+ \\ \frac{1}{2}\alpha \notin \Delta_1^+}} \prod_{n \in \mathbb{N}} \Phi(n, \alpha)^{P(\eta - n\alpha)} \prod_{\substack{\alpha \in \Delta_1^+ \\ \alpha^2 \neq 0}} \prod_{n \in 2\mathbb{N}-1} \Phi(n, \alpha)^{P(\eta - n\alpha)} \prod_{\substack{\alpha \in \Delta_1^+ \\ \alpha^2 = 0}} \Phi(\alpha)^{P_\alpha(\eta - \alpha)} \tag{3.15}$$

where

$$\Phi(n, \alpha) \stackrel{d}{=} (\Lambda_h + \rho) \cdot \alpha - \frac{1}{2}n\alpha \cdot \alpha \quad (3.16)$$

and

$$\Phi(\alpha) = \Phi(0, \alpha). \quad (3.17)$$

The sets of even and odd roots are, again, Δ_0 and Δ_1 respectively. The affine Weyl vector is denoted ρ . Λ_h is the highest weight of the representation. We can write the highest weight Λ_h as,

$$\Lambda_h = \left(\frac{1}{2}h_-(\alpha_1 + \alpha_2) + \frac{1}{2}h_+(\alpha_1 - \alpha_2), k, 0\right), \quad (3.18)$$

where k is the level as it is in equation (3.10) and α_i are simple (fermionic) roots of $sl(2|1; \mathbb{C})$. The sets of affine roots we need are,

$$\Delta_0^+ = \left\{((\alpha_1 + \alpha_2), 0, m), (-(\alpha_1 + \alpha_2), 0, m + 1), (0, 0, m + 1) \mid m \in \mathbb{Z}_+\right\}, \quad (3.19)$$

and

$$\Delta_1^+ = \left\{(\alpha_i, 0, m), (-\alpha_i, 0, m + 1) \mid m \in \mathbb{Z}_+, i = 1, 2\right\}. \quad (3.20)$$

If we put in the root data above and use ρ given in (3.8) we obtain six² conditions for the determinant to vanish. These conditions are,

$$h_- - h_+ + 2(k + 1)m_1 = 0 \quad (3.21)$$

$$h_- + h_+ + 2(k + 1)m_2 = 0 \quad (3.22)$$

$$h_- - h_+ - 2(k + 1)(m_3 + 1) = 0 \quad (3.23)$$

$$h_- + h_+ - 2(k + 1)(m_4 + 1) = 0 \quad (3.24)$$

$$h_- + (k + 1)m_5 - n_1 = 0 \quad (3.25)$$

$$h_- - (k + 1)(m_6 + 1) + n_2 = 0 \quad (3.26)$$

where $m_i \in \mathbb{Z}_+$ for $i = 1, \dots, 6$ and $n_j \in \mathbb{N}$ for $j = 1, 2$. The first four conditions come from the factors involving the odd roots and these, as a whole, are what was referred to as the “fermionic sector” in the introduction of this chapter. Clearly some conditions are not compatible with others. Notice that by combining say, (3.21) and (3.22) it is

²Upon putting the even root $(0, 0, m + 1)$ into (3.16) gives the condition $(k + 1)(m + 1) = 0$ for the determinant to vanish. We specifically do not consider the case $k + 1 = 0$ in this work

possible to obtain a condition like (3.25) but with $n_1 = 0$. Note also that if we have an m_5 and n_1 such that (3.25) is satisfied then we can obtain another pair of integers that satisfy that condition by simply shifting $m_5 \rightarrow m_5 + au$ and $n_1 \rightarrow n_1 + ap$, $a \in \mathbb{N}$ since $k+1 = p/u$ where k is the level of a representation of $\hat{sl}(2|1; \mathbb{C})$. Likewise for m_6 and n_2 and (3.26). From these conditions one can deduce five lemmas, three of which we shall use subsequently to work out the quantum numbers of the singular vectors of class V. Copied directly from [BT97] the lemmas are,

Lemma 3.1

a) If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_- + (k+1)m - n = 0$ for some $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, there exists a singular vector corresponding to $\eta = n((\alpha_1 + \alpha_2), 0, m)$ with conformal weight $H + mn$, isospin $\frac{1}{2}H_- - n$ and charge $\frac{1}{2}H_+$.

b) If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_- - (k+1)(1+m) + n = 0$ for some $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, there exists a singular vector corresponding to $\eta = n(-(\alpha_1 + \alpha_2), 0, m+1)$ with conformal weight $H + (m+1)n$, isospin $\frac{1}{2}H_- + n$ and charge $\frac{1}{2}H_+$.

In fact, because of the invariance under simultaneous shifts of m and n we can restrict the coefficient of $(k+1)$ in the hypotheses of parts a and b of the lemma to be such that $0 \leq m \leq u-1$.

Lemma 3.2

If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_+ - H_- = 2(k+1)M$ for some $M \in \mathbb{Z}_+$, there exists a singular vector corresponding to $\eta = (\alpha_1, 0, M)$ with conformal weight $H + M$, isospin $\frac{1}{2}H_- - \frac{1}{2}$ and charge $\frac{1}{2}H_+ - \frac{1}{2}$.

Lemma 3.3

If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_+ - H_- = -2(k+1)(M+1)$ for some $M \in \mathbb{Z}_+$, there exists a singular vector corresponding to $\eta = (-\alpha_1, 0, M+1)$ with conformal weight $H + M + 1$, isospin $\frac{1}{2}H_- + \frac{1}{2}$ and charge $\frac{1}{2}H_+ + \frac{1}{2}$.

Lemma 3.4

If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_+ + H_- = 2(k+1)(M+1)$ for some $M \in \mathbb{Z}_+$, there exists a singular vector corresponding to $\eta = (-\alpha_2, 0, M+1)$ with conformal weight $H + M + 1$, isospin $\frac{1}{2}H_- + \frac{1}{2}$ and charge $\frac{1}{2}H_+ - \frac{1}{2}$.

Lemma 3.5

If χ is a singular vector such that $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$ and if $H_+ + H_- = -2(k+1)M$ for some $M \in \mathbb{Z}_+$, there exists a singular vector corresponding to $\eta = (\alpha_2, 0, M)$ with conformal weight $H + M$, isospin $\frac{1}{2}H_- - \frac{1}{2}$ and charge $\frac{1}{2}H_+ + \frac{1}{2}$.

We shall be interested in representations where the isospin $\frac{1}{2}h_-$ and charge $\frac{1}{2}h_+$ of the hws which defines the representation satisfy some of the conditions (3.21)–(3.26). Focus on the last two conditions, (3.25) and (3.26). If it ever happens that $p|n_1$ or $p|n_2$, we are reduced to the unpleasant case of a collapsed embedding diagram *e.g.*, figure 4.2. If none of the other conditions (3.21)–(3.24) are satisfied then the characters of such a representation cannot carry a representation of the modular group and such representations are of little interest to physics. We now restrict our attention to pairs (m_5, n_1) , such that $m_5 = au + \tilde{m}_5$ and $n_1 = au + \tilde{n}_1$ ($\tilde{n}_1 > 1$) *i.e.*, $m_5 \equiv \tilde{m}_5 \pmod{u}$, $n_1 \equiv \tilde{n}_1 \pmod{p}$ and to pairs (m_6, n_2) , such that $m_6 = au + \tilde{m}_6$ and $n_2 = au + \tilde{n}_2$ ($\tilde{n}_2 > 1$) *i.e.*, $m_6 \equiv \tilde{m}_6 \pmod{u}$, $n_2 \equiv \tilde{n}_2 \pmod{p}$. Because of their invariance under simultaneous shifts, (3.25) is satisfied iff (3.26) is and without loss of generality we can consider only the former.

Now we shall introduce some of the classes of representations as they were defined in [BT97]. The representations are classified according to how many of these conditions in the fermionic sector are satisfied simultaneously. Bowcock and Taormina defined representations of class I to be those for which the isospin $\frac{1}{2}h_-$ of the hws satisfied,

$$h_- + (k+1)m - n = 0, \quad (3.27)$$

where $0 \leq m \leq u-1$ and $1 \leq n \leq p-1$. The ranges for m and n are justified by the discussion at the end of the last paragraph. We shall not consider classes II and III in this work. It is believed that there are subsingular vectors in these representations. This being so it would not be easy to compute characters for these representations. Class IV was defined as being that for which the isospin $\frac{1}{2}h_-$ and the charge $\frac{1}{2}h_+$ of the hws satisfy,

$$h_- + (k+1)m = 0 \quad \text{and} \quad h_- - h_+ + 2(k+1)m' = 0 \quad (3.28)$$

where $0 \leq m \leq u-1$ and $m' - m \leq 0$. Conditions (3.28) are a combination of conditions (3.21) and (3.22). Now it is also quite possible that conditions (3.23) and (3.24) would be satisfied simultaneously. We take the representation defined by the conditions (3.23) and (3.24) holding simultaneously to be class V. Class V is not essentially different from

class IV in the way that classes I–IV are distinct. Nevertheless, it was not explicitly mentioned in [BT97] and it is crucial that it is considered along with class IV. Not surprisingly then, the structure of the class V embedding diagram is going to be very similar to that of class IV. Unfortunately the quantum numbers of the singular vectors will be different, so we have to derive them from scratch. This at least demonstrates how to produce the diagram too. The following derivation of quantum numbers will be very similar to that of the last chapter. For the rest of this work then, we define class V as that set of admissible representations with hws $|\Omega\rangle$ with conformal weight h , isospin $\frac{1}{2}h_-$ and charge $\frac{1}{2}h_+$ such that there exist two integers M and M' which satisfy,

$$\begin{aligned} h_- - h_+ - 2(k+1)(M'+1) &= 0 \\ h_- + h_+ - 2(k+1)(M+1) &= 0, \end{aligned} \quad (3.29)$$

where, $M, M' \in \mathbb{Z}_+$. Note that (3.29) means that,

$$h_- = (k+1)(M+M'+2). \quad (3.30)$$

At this stage we shall impose another condition which will be justified amply afterwards.

We restrict,

$$M+M'+2 \leq u. \quad (3.31)$$

From (3.30) we have,

$$\begin{aligned} h_- &= (k+1)(M+M'+2) \Leftrightarrow \\ h_- + (k+1)(u - (M+M'+2)) - p &= 0. \end{aligned} \quad (3.32)$$

According to lemma 3.1 part a, there is a singular vector with conformal weight and isospin,

$$H = h + (u - M - M' - 2)p, \quad \frac{1}{2}H_- = \frac{1}{2}h_- - p, \quad (3.33)$$

respectively. The charge is unchanged. Call this singular vector Z'_1 . Condition (3.30) cannot satisfy the hypothesis of lemma 3.1 part b in view of the comment after it.

Now we go to the top of the module with highest weight vector Z'_1 . We have,

$$h_-(Z'_1) - (k+1)(M+M'+2) + 2p = 0. \quad (3.34)$$

This satisfies the hypothesis for lemma 3.1 part b. Thus there exists a singular vector with conformal weight and isospin,

$$H = h + p(u + M + M' + 2), \quad \frac{1}{2}H_- = \frac{1}{2}h_- + p, \quad (3.35)$$

respectively. The charge is unchanged. Call this singular vector T'_1 . Because of the corollary, we cannot use part a of the lemma at Z'_1 . Now go to the submodule with T'_1 as hws.

$$\begin{aligned} h_-(T'_1) - (k+1)(M + M' + 2) - 2p &= 0 \Leftrightarrow \\ h_-(T'_1) + (k+1)(u - M - M' - 2) - 3p &= 0. \end{aligned} \quad (3.36)$$

But now $h_-(T'_1)$ satisfies the hypothesis of lemma 3.1 part a, so there exists a singular vector with conformal weight and isospin,

$$H = h + (2u - M - M' - 2)2p, \quad \frac{1}{2}H_- = \frac{1}{2}h_- - 2p. \quad (3.37)$$

Again, the charge is unchanged. Call this singular vector Z'_2 . We cannot use part b of the lemma when sitting at T'_1 . We can now verify that $H(Z'_1) \leq H(T'_1)$. This is immediate. We should also verify that $H(T'_1) \leq H(Z'_2)$. This so iff $M + M' + 2 \leq u$ and our assumption (3.31) is justified *a posteriori*. One just continues in this way *ad infinitum* to generate two sequences of singular vectors T'_{a+1} and Z'_{a+1} , $a \in \mathbb{Z}_+$ with quantum numbers,

$$\begin{aligned} H(T'_{a+1}) &= h + (a+1)p((a+1)u + M + M' + 2) \\ \frac{1}{2}H_-(T'_{a+1}) &= \frac{1}{2}h_- + (a+1)p \\ \frac{1}{2}H_+(T'_{a+1}) &= \frac{1}{2}h_+ \\ \\ H(Z'_{a+1}) &= h + (a+1)p((a+1)u - (M + M' + 2)) \\ \frac{1}{2}H_-(Z'_{a+1}) &= \frac{1}{2}h_- - (a+1)p \\ \frac{1}{2}H_+(Z'_{a+1}) &= \frac{1}{2}h_+, \end{aligned} \quad (3.38)$$

which are embedded as,

$$Z'_0 \rightarrow Z'_1 \rightarrow T'_1 \rightarrow Z'_2 \rightarrow T'_2 \cdots \quad (3.39)$$

if we denote by Z'_0 the hws on which the whole representation is built. Note that $A \rightarrow B$ means that B is a submodule of A and that there does not exist a module C such that

$B \rightarrow C \rightarrow A$. This completes the uncharged sector of the embedding diagram. Of course it remains to show the existence of the multiplicity two singular vectors which we expect in this sector because of class V's closeness to class IV. We shall defer that work until later in this section though.

Now return to (3.29). Consider the first condition there *i.e.*,

$$h_- - h_+ - 2(k+1)(M' + 1) = 0. \quad (3.40)$$

Let h_- and h_+ in this condition be quantum numbers (up to a factor of two) of Z'_0 as defined above. Suppose that Z'_0 has conformal weight h . Then we can apply lemma 3.3 and obtain a singular vector with conformal weight, isospin and charge as follows.

$$h + M' + 1, \quad \frac{1}{2}(h_- + 1), \quad \frac{1}{2}(h_+ + 1). \quad (3.41)$$

Call this state T_0^+ . We have now,

$$\begin{aligned} h_-(T_0^+) - (k+1)(M + M' + 2) - 1 &= 0 \\ \Leftrightarrow h_-(T_0^+) + (k+1)(u - M - M' - 2) - (1 + p) &= 0. \end{aligned}$$

And now we can use lemma 3.1 part a. We discover a singular vector with conformal weight and isospin,

$$\begin{aligned} H &= h - M - 1 + up + u - p(M + M' + 2), \\ \frac{1}{2}H_- &= \frac{1}{2}h_- - \frac{1}{2} - p, \end{aligned} \quad (3.42)$$

and the charge is the same as that of T_0^+ . Call this singular vector Z_1^+ . We cannot use part b of lemma 3.1. If we check now that $H(Z_1^+) \geq H(T_0^+)$, one finds that this is so iff $M + M' + 2 \leq u$ again. Now the isospin of Z_1^+ is such that,

$$\begin{aligned} h_-(Z_1^+) - h_- + 1 + 2p &= 0 \\ \Leftrightarrow h_-(Z_1^+) - (k+1)(M + M' + 2) + 1 + 2p &= 0, \end{aligned}$$

by (3.30). Now we can use lemma 3.1 part b to deduce the existence of a singular vector with conformal weight and isospin,

$$\begin{aligned} H &= h + M' + 1 + pu + u + p(M + M' + 2) \\ \frac{1}{2}H_- &= \frac{1}{2}h_- + \frac{1}{2} + p, \end{aligned} \quad (3.43)$$

and the charge is the same as Z_1^+ . Call this singular vector T_1^+ . Now, again, one continues *ad infinitum* in this manner and produces two sequences of singular vectors T_a^+ and Z_{a+1}^+ , ($a \in \mathbb{Z}_+$) with quantum numbers,

$$\begin{aligned} H(T_a^+) &= h + M' + 1 + a(apu + u + p(M + M' + 2)) \\ \frac{1}{2}H_-(T_a^+) &= \frac{1}{2}h_- + \frac{1}{2} + ap \\ \frac{1}{2}H_+(T_a^+) &= \frac{1}{2}h_+ + \frac{1}{2} \\ H(Z_{a+1}^+) &= h - M - 1 + (a + 1)((a + 1)p + u - p(M + M' + 2)) \\ \frac{1}{2}H_-(Z_{a+1}^+) &= \frac{1}{2}h_- - \frac{1}{2} - (a + 1)p \\ \frac{1}{2}H_+(Z_{a+1}^+) &= \frac{1}{2}h_+ + \frac{1}{2}, \end{aligned} \quad (3.44)$$

which are embedded as follows,

$$T_0^+ \rightarrow Z_1^+ \rightarrow T_1^+ \rightarrow Z_2^+ \rightarrow T_2^+ \dots \quad (3.45)$$

Again return to the defining conditions of class V in (3.29). Consider the second of these this time *i.e.*,

$$h_+ + h_- = 2(k + 1)(M + 1), \quad M \in \mathbb{Z}_+. \quad (3.46)$$

By lemma 3.4 there exists a singular vector with quantum numbers,

$$H = h + M + 1, \quad \frac{1}{2}H_- = \frac{1}{2}h_- + \frac{1}{2}, \quad \frac{1}{2}H_+ = \frac{1}{2}h_+ + \frac{1}{2}. \quad (3.47)$$

Call this singular vector T_0^- . Now go to the module with T_0^- as hws. We have,

$$\begin{aligned} h_-(T_0^-) - h_- - 1 &= 0 \\ \Leftrightarrow h_-(T_0^-) + (k + 1)(u - M - M' - 2) - (p + 1) &= 0, \end{aligned}$$

using (3.30). Lemma 3.1 part a then, provides a singular vector with quantum numbers,

$$\begin{aligned} H &= h - M' - 1 + pu + u - p(M + M' + 2) \\ \frac{1}{2}H_- &= \frac{1}{2}h_- - \frac{1}{2} - p \\ \frac{1}{2}H_+ &= \frac{1}{2}h_+ - \frac{1}{2}. \end{aligned} \quad (3.48)$$

Call this singular vector Z_1^- . We cannot use part b of the lemma when sitting at T_0^- . Again, $H(T_0^-) \leq H(Z_1^-)$ iff $M + M' + 2 \leq u$. Now go to the module with Z_1^- as hws.

$$\begin{aligned} h_-(Z_1^-) - h_- + 1 + 2p &= 0 \\ \Leftrightarrow h_-(Z_1^-) - (k + 1)(M + M' + 2) + 1 + 2p &= 0, \end{aligned}$$

using (3.30). We can use part b of lemma 3.1 and obtain a singular vector with quantum numbers,

$$\begin{aligned} H &= h + M + 1 + pu + u + p(M + M' + 2) \\ \frac{1}{2}H_- &= \frac{1}{2}h_- + \frac{1}{2} + p \\ \frac{1}{2}H_+ &= \frac{1}{2}h_+ - \frac{1}{2}. \end{aligned} \tag{3.49}$$

Call this singular vector T_1^- . We cannot use part a of the lemma. Again one continues this process indefinitely to produce two more sequences T_a^- and Z_{a+1}^- of singular vectors with quantum numbers,

$$\begin{aligned} H(T_a^-) &= h + M + 1 + a(apu + u + p(M + M' + 2)) \\ \frac{1}{2}H_-(T_a^-) &= \frac{1}{2}h_- + \frac{1}{2} + ap \\ \frac{1}{2}H_+(T_a^-) &= \frac{1}{2}h_+ - \frac{1}{2} \\ H(Z_{a+1}^-) &= h - M' - 1 + (a + 1)(up(a + 1) + u - p(M + M' + 2)) \\ \frac{1}{2}H_-(Z_{a+1}^-) &= \frac{1}{2}h_- - \frac{1}{2} - (a + 1)p \\ \frac{1}{2}H_+(Z_{a+1}^-) &= \frac{1}{2}h_+ - \frac{1}{2}, \end{aligned} \tag{3.50}$$

which are embedded as follows,

$$T_0^- \rightarrow Z_1^- \rightarrow T_1^- \rightarrow Z_2^- \rightarrow T_2^- \dots \tag{3.51}$$

We shall use these quantum numbers in the next chapter to write down characters and supercharacters for class V. Taking the three sequences of singular vectors which the Kac determinant has provided for us, the embedding diagram looks as in figure 3.4 at this stage. Clearly, as it stands now, the embedding diagram does not quite look like that shown in figure 4.3. In the rest of this chapter we shall give a similar level of justification for the true embedding diagram (figure 4.3) as was given in [BT97] for the class IV diagram. In particular we shall demonstrate the existence of two singular vectors with the quantum numbers of T_1' by making use of the notion of an “improved” state. We shall explain how the naive expression for T_0^- given by lemma 3.4 is actually zero because of the existence of nilpotent fermionic generators and how one can use again the idea of improving a state to obtain a non-zero T_0^- . Then we shall show how the two uncharged singular vectors at the T_1' position are actually descendents of T_0^+ and the improved T_0^- state. This serves to demonstrate the existence of arrows from T_0^+ and T_0^-

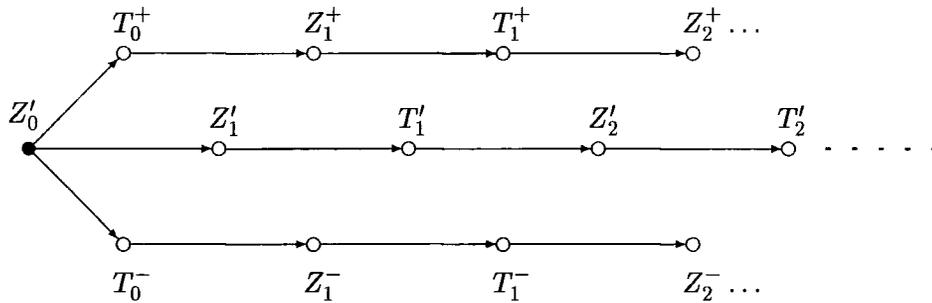


Figure 3.4: Preliminary form of the class IV/V embedding diagram

to T'_1 . The technique we shall use is a generalisation (due to Bowcock and Taormina) to $\hat{sl}(2|1; \mathbb{C})$ of the ideas of Malikov, Feigin and Fuks [MFF86]. They were able to obtain all the singular vectors in an integrable module of $\hat{sl}(2; \mathbb{C})$ by operating with operators w_0 and w_1 , say, alternately on a hws. That is to say they had two sequences *e.g.*, $w_1|hws\rangle, w_0w_1|hws\rangle, w_1w_0w_1|hws\rangle, \dots$ and $w_0|hws\rangle, w_1w_0|hws\rangle, w_0w_1w_0|hws\rangle, \dots$ where each $w_{0,1}$ contains information about the isospin of the state to its right. When the level k of the representation of the (super)algebra under study is not integral the construction is more complicated but the essential idea is the same. At this stage we ought to introduce the four operators $\tilde{w}_0^{(m)}, \tilde{w}_1^{(m)}, w_0^{(m)}$ and $w_1^{(m)}$ used in [BT97] (we retain their notation for easy comparisons). Let χ be a vector with $L_0\chi = H\chi$, $J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$. Then,

$$\tilde{w}_0^{(m)}\chi \stackrel{d}{=} \prod_{i=1}^m \left\{ (J_0^-)^{2i(k+1)-H_--1} \left((-2H_- + 4i(k+1))j_0^- j_0'^- + (H_+ + H_- - 2i(k+1))J_0^- \right) \times (J_{-1}^+)^{(2i-1)(k+1)-H_-} \right\} \chi, \quad (3.52)$$

which has quantum numbers,

$$\begin{aligned} H' &= H + m^2(k+1) - mH_- \\ H'_- &= H_- - 2m(k+1) \\ H'_+ &= H_+, \end{aligned} \quad (3.53)$$

also,

$$\tilde{w}_1^{(m)} \chi \stackrel{d}{=} \prod_{i=0}^{m-1} \left\{ (J_{-1}^+)^{(2i+1)(k+1)+H_-} (J_0^-)^{2i(k+1)+H_- - 1} \left((2H_- + 4i(k+1)) j_0^- j_0'^- + (H_+ - H_- - 2i(k+1)) J_0^- \right) \right\} \chi, \quad (3.54)$$

which has quantum numbers,

$$\begin{aligned} H' &= H + m^2(k+1) + mH_- \\ H'_- &= H_- + 2m(k+1) \\ H'_+ &= H_+. \end{aligned} \quad (3.55)$$

In terms of these, define two further operators,

$$w_0^{(m)} \chi \stackrel{d}{=} (J_{-1}^+)^{(2m+1)(k+1)-H_-} \tilde{w}_0^{(m)} \chi, \quad (3.56)$$

with quantum numbers,

$$\begin{aligned} H' &= H + (m+1)^2(k+1) - (m+1)H_- \\ H'_- &= -H_- + 2(m+1)(k+1) \\ H'_+ &= H_+, \end{aligned} \quad (3.57)$$

and,

$$\begin{aligned} w_1^{(m)} \chi &\stackrel{d}{=} (J_0^-)^{2m(k+1)+H_- - 1} \\ &\times \left((2H_- + 4m(k+1)) j_0^- j_0'^- + (H_+ - H_- - 2m(k+1)) J_0^- \right) \tilde{w}_1^{(m)} \chi, \end{aligned} \quad (3.58)$$

which has quantum numbers,

$$\begin{aligned} H' &= H + m^2(k+1) + mH_- \\ H'_- &= -H_- - 2m(k+1) \\ H'_+ &= H_+. \end{aligned} \quad (3.59)$$

The three generators J_1^- , j_0^+ and $j_0'^+$ commute with each of these four operators. This makes it easier to determine whether or not certain states are hws's. In all of the products above we take the convention that the factor for value "i" is to appear to the right of that for value "i + 1" i.e., the reverse of the normal convention for products. The operators $w_0^{(m)}$ and $w_1^{(m)}$ for any m have the property that the square of each of

them when acting on a state is just a c-number and the operators $\tilde{w}_0^{(m)}$ and $\tilde{w}_1^{(m)}$ have the property that each of the products $\tilde{w}_0^{(m)}\tilde{w}_1^{(m)}$ and $\tilde{w}_1^{(m)}\tilde{w}_0^{(m)}$ is just a c-number too again for any m when acting on some state. In fact we shall actually need just the latter result here which is precisely,

$$\tilde{w}_1^{(m)}\tilde{w}_0^{(m)}\chi = \prod_{i=1}^m (H_+ + H_- - 2i(k+1))(H_+ - H_- + 2i(k+1)), \quad (3.60)$$

where $J_0^3\chi = H_-/2\chi$ and $U_0\chi = H_+/2\chi$. The singular vectors obtained by using lemmas 3.1, 3.3 and 3.4 can be written in terms of the “ w ” operators as follows,

$$\begin{aligned} \text{lemma 3.1 part a : } & w_1^{(m)}\chi \\ \text{lemma 3.1 part b : } & w_0^{(m)}\chi \\ \text{lemma 3.3 : } & w_0^{(M)}j_0^-w_0^{(M)}\chi \\ \text{lemma 3.4 : } & w_0^{(M)}j_0'^-w_0^{(M)}\chi. \end{aligned} \quad (3.61)$$

With these we have that (recall the embedding diagram figure 3.4),

$$\begin{aligned} Z_1' &= w_1^{(u-M-M'-2)}Z_0' \\ T_1'^{(1)} &= w_0^{(M+M'+1)}Z_1' \\ T_0^+ &= w_0^{(M')}j_0^-w_0^{(M')}Z_0' \\ T_0^- &= w_0^{(M)}j_0'^-w_0^{(M)}Z_0', \end{aligned} \quad (3.62)$$

where $T_1'^{(1)}$ is just one of the singular vectors with the T_1' quantum numbers. We shall now construct the other. A natural starting point would be to consider the state $w_1^{(u-M-M'-2)}w_0^{(M+M'+1)}Z_0'$. However this leads to a problem. A possibility in superalgebras is the existence of nilpotent fermionic generators. There are such generators in $\hat{sl}(2|1; \mathbb{C})$. This leads to the fact that $w_0^{(M+M'+1)}Z_0'$ is identically zero as we shall now argue. We restrict ourselves to discussing the case $M - M'$ is odd and $M' < M - 1$. Using the explicit product form of $w_0^{(M+M'+1)}$ given above in equation (3.52) it is possible to find a factor in $w_0^{(M+M'+1)}$ in which the power of J_{-1}^+ vanishes and to find two further factors on each side of the the former in which the coefficient of J_0^- vanishes. Without a J_{-1}^+ between them, it is possible to bring together two similar fermionic operators (the product $\tilde{w}_1^{(\frac{1}{2}(M-M'-1))}\tilde{w}_0^{(\frac{1}{2}(M-M'-1))}$ appears between them but as we noted above this product is just a c-number when acting on some state.) But the square of any fermionic operator of $\hat{sl}(2|1; \mathbb{C})$ is zero and therefore the whole state $w_0^{(M+M'+1)}Z_0'$ is zero. However, the trick used in [BT97] is to build another state with the T_1' quantum numbers

not on Z'_0 but on an “improved” state instead. We use the expression encountered when showing that $w_0^{(M+M'+1)}Z'_0$ is zero but where the factor $(J_{-1}^+)^0$ appears, replace it with $\log J_{-1}^+$ instead. Then the previously annihilating fermionic generators are kept apart because of the “barrier” which is the logarithm. The new object $\log J_{-1}^+$ has no effect on the isospin of any state to its right, indeed,

$$[J_0^3, \log J_{-1}^+] = 1. \quad (3.63)$$

For reference, we include here another useful commutator involving $\log J_{-1}^+$.

$$[J_1^-, \log J_{-1}^+] = (J_{-1}^+)^{-1}(k+1-2J_0^3). \quad (3.64)$$

This is needed in verifying that certain states are hws's. The fermionic generators j_0^+ and $j_0'^+$ commute with $\log J_{-1}^+$. We take for the “improved” state $\hat{w}^{(M+M'+1)}Z'_0$,

$$\begin{aligned} \hat{w}^{(M+M'+1)}Z'_0 &\stackrel{d}{=} \\ &4h_+^2 w_0^{(M')} [\alpha (J_0^-)^{h_+-1} j_0^- j_0'^- \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} (\log J_{-1}^+) \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} (J_0^-)^{-h_+-1} \\ &\quad + \beta (J_0^-)^{-1} j_0'^- j_0^- w_0^{(M')} Z'_0. \end{aligned} \quad (3.65)$$

In this expression, α and β are c-number constants. The extra term proportional to β is introduced to ensure that the state is a hws. In checking this, by operating on the state with $j_0'^+$, we obtain the following constraint on α and β ,

$$\begin{aligned} -\beta h_+ \Delta &= \alpha \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} \Delta \\ &= \alpha \prod_{i=1}^{(M-M'-1)/2} (H_+ + H_- - 2i(k+1)) (H_+ - H_- + 2i(k+1)) \Delta, \end{aligned} \quad (3.66)$$

where H_+ and H_- are twice the charge and isospin respectively of the state $\Delta = j_0^- w_0^{(M')} Z'_0$ i.e., $H_- = H_+ = h_+$. Then we simply take,

$$T_1'^{(2)} = w_0^{(M+M'+1)} w_1^{(u-M-M'-2)} \hat{w}^{(M+M'+1)} Z'_0 \quad (3.67)$$

as a second state with the T_1' quantum numbers.

Using just the same sort of reasoning that was used to justify the vanishing of $w^{(M+M'+1)}$, we can show that nilpotency of fermionic generators implies that $w_0^{(M)} j_0'^- w_0^{(M)} Z'_0$ is identically zero too i.e., T_0^- is identically zero (equation (3.62)). To get around this difficulty we use the idea of an improved state again. We take instead,

$$\begin{aligned} \hat{T}_0^- &= 2h_+ w_0^{(M)} [\alpha' j_0'^- \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} (\log J_{-1}^+) \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} j_0'^- j_0^- (J_0^-)^{-h_+-1} \\ &\quad + \beta' (J_0^-)^{-h_+} j_0'^-] w_0^{(M')} Z'_0, \end{aligned} \quad (3.68)$$

with α' and β' constants. The term proportional to β' is there to ensure that we have a hws.

In checking that \hat{T}_0^- is a hws by operating with $j_0'^+$, we obtain the following constraint on α' and β' ,

$$\begin{aligned} \beta' h_+ \Psi &= \alpha' \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} \Psi \\ &= \alpha' \prod_{i=1}^{(M-M'-1)/2} (H_+ + H_- - 2i(k+1)) (H_+ - H_- + 2i(k+1)) \Psi \end{aligned} \quad (3.69)$$

where this time H_+ and H_- are twice the charge and isospin of Ψ and $H_-(\Psi) = H_+(\Psi) = h_+$. We note that this is reminiscent of (3.66) so that $\beta/\alpha = -\beta'/\alpha'$. We take $\beta' = -\beta$ and $\alpha' = \alpha$ in the following.

The next task is to show how $T_1'^{(1,2)}$ are descendents of T_0^\pm . This is not hard to discover. We notice that $T_1'^{(2)}$ and \hat{T}_0^- share some factors. Operating with $w_0^{(M)}$ on \hat{T}_0^- we have,

$$\begin{aligned} w_0^{(M)} \hat{T}_0^- &= 2h_+ N_0 [\alpha j_0'^- \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} (\log J_{-1}^+) \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} j_0^- j_0^- (J_0^-)^{-h_+-1} \\ &\quad - \beta (J_0^-)^{-h_+} j_0'^-] w_0^{(M')} Z'_0, \end{aligned} \quad (3.70)$$

where $N_0 \stackrel{d}{=} (w_0^{(M)})^2$ is a c-number as we mentioned above. Recalling also the form of $T_1'^{(1)}$ from (3.62) we can obtain,

$$\begin{aligned} N_0 (T_0'^{(2)} - 4\beta h_+^2 (w_0^{(M')})^2 T_0'^{(1)}) &= \\ &= 2h_+ w_0^{(M+M'+1)} w_1^{(u-M-M'-2)} w_0^{(M')} (J_0^-)^{h_+-1} j_0^- w_0^{(M)} \hat{T}_0^-, \end{aligned} \quad (3.71)$$

so that there is a descendent of \hat{T}_0^- which is a linear combination of the two singular vectors with T_0^- quantum numbers.

Consider now, $w_0^{(M')} T_0^+$. We have, from (3.62),

$$w_0^{M'} T_0^+ = (w_0^{(M')})^2 j_0^- w_0^{(M')} Z'_0 = N_1 j_0^- w_0^{(M')} Z'_0, \quad (3.72)$$

where N_1 is a constant c-number. We can use this expression for $j_0^- w_0^{(M')} Z'_0$ on the right hand end of the expression for $T_1'^{(2)}$ so as to write the latter state as a descendent of T_0^+ . One obtains,

$$\begin{aligned} N_1 T_1'^{(2)} &= 4h_+^2 w_0^{(M+M'+1)} w_1^{(u-M-M'-2)} w_0^{(M')} \times \\ &[\alpha (J_0^-)^{h_+-1} j_0^- j_0'^- \tilde{w}_1^{(\frac{1}{2}(M-M'-1))} (\log J_{-1}^+) \tilde{w}_0^{(\frac{1}{2}(M-M'-1))} (J_0^-)^{-h_+-1} j_0'^- \\ &\quad + \beta (J_0^-)^{-1} j_0'^-] w_0^{(M')} T_0^+, \end{aligned} \quad (3.73)$$

from (3.65) and (3.67). Clearly this discussion is completely parallel to that about class IV in [BT97].

The most important result of this chapter was to introduce what we call the class V admissible representations of $\hat{sl}(2|1; \mathbb{C})$. They are necessarily very similar to class IV but it is crucial that they be included in the work hereafter. For this reason, we have spelled out in some detail how the quantum numbers of the singular vectors in this class are derived and given some justification that classes IV and V share an embedding diagram. Having done this now we can compute the characters for class IV and V representations in the next chapter.

Chapter 4

Characters of $\hat{sl}(2|1; \mathbb{C})$

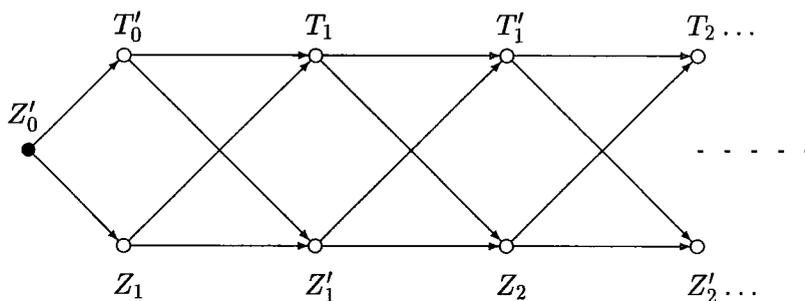
4.1 Introduction.

In this chapter we will compute the characters and supercharacters for irreducible, admissible representations of $\hat{sl}(2|1; \mathbb{C})$ in classes I, IV and V. We shall use the embedding diagrams as they appear in [BT97] for classes I and IV and, of course, the quantum numbers of singular vectors that go along with them. We also have to hand the data for class V which, as we saw in the last chapter, is very similar to class IV. Once we have an isomorphism between the Ramond and Neveu-Schwarz sectors, the Neveu-Schwarz characters can be obtained easily from the Ramond sector ones by a simple change of variables. That isomorphism *i.e.*, the spectral flow transformation, was given in the last chapter in equation (3.11). In section 4.4 we study the analytic properties of the characters of classes IV and V in the limit as $\sigma \rightarrow 0$. We discover that of the u^2 characters in classes IV and V together, u of these are regular in the limit in either sector (Ramond or Neveu-Schwarz). It turns out, as we shall see, that whenever a character is singular we can calculate the residue and that residue is proportional to a minimal $N = 2$ character which may be unitary or not. In section 4.5 there is a short discussion of characters of integrable representations of $\hat{sl}(2|1; \mathbb{C})$. The major results of this chapter have been published in [BHT98].

4.2 Class I.

In [BT97] we are given the quantum numbers of the singular vectors in the Verma module of class I representations. They are as in table 4.1 and the embedding diagrams are shown in figures 4.1 and 4.2. All singular vectors have the same charge as the hws Z'_0 . In the collapsed embedding diagram, figure 4.2, we have that $Z'_{a+1} = Z_{a+1}$ and

Series	Conformal Weight	$2 \times$ Isospin
Z'_a	$h + a(apu + un - pm)$	$n - m(k + 1) + 2ap$
T'_a	$h + mn + a(apu + un + pm)$	$-n - m(k + 1) - 2ap$
Z_{a+1}	$h + mn + (a + 1)((a + 1)pu - un - pm)$	$-n - m(k + 1) + 2(a + 1)p$
T_{a+1}	$h + (a + 1)((a + 1)pu - un + pm)$	$n - m(k + 1) - 2(a + 1)p$

Table 4.1: Quantum numbers of singular vectors in class I rep.'s of $\hat{sl}(2|1; \mathbb{C})$.Figure 4.1: The embedding diagram for class I reps. of $\hat{sl}(2|1; \mathbb{C})$ when $n > 0$.

$$T'_{a+1} = T_{a+1}.$$

We take as our definition of the Ramond sector character, whatever the class of representations,

$$\chi^R(\sigma, \tau, \nu) \stackrel{\text{d}}{=} \text{tr} \exp(2\pi i((L_0^R - c/24)\tau + J_0^{3,R}\sigma + U_0^R\nu)). \quad (4.1)$$

τ , σ and ν are arbitrary complex numbers except that we must have $\text{Im}(\tau) > 0$ if the character (as a power series in $q \stackrel{\text{d}}{=} e^{2\pi i\tau}$) is to converge. We shall be interested in taking the trace over irreducible modules. As we saw in chapter 3 though, c , the central charge of the Virasoro algebra associated to $\hat{sl}(2|1; \mathbb{C})$ through the Sugawara construction, actually vanishes because the superdimension of $\hat{sl}(2|1; \mathbb{C})$ vanishes. For all the Ramond sector characters that we shall compute in this chapter there is a common

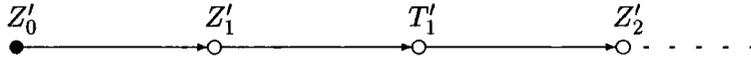


Figure 4.2: The embedding diagram for class I reps. of $\hat{sl}(2|1; \mathbb{C})$ when $n = 0$.

prefactor (the reciprocal of the Weyl-Kac superdenominator). It is,

$$F^R(\sigma, \nu, \tau) \stackrel{d}{=} \prod_{n=1}^{\infty} \frac{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{n-1})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{n-1})}{(1 - q^n)^2(1 - zq^n)(1 - z^{-1}q^{n-1})}. \tag{4.2}$$

Using Jacobi's triple product identity we can write the prefactor (4.2) as,

$$F^R(\sigma, \nu, \tau) = \frac{(\vartheta_{1,2}(\frac{\sigma+\nu}{2}, \tau) + \vartheta_{-1,2}(\frac{\sigma+\nu}{2}, \tau))(\vartheta_{1,2}(\frac{\sigma-\nu}{2}, \tau) + \vartheta_{-1,2}(\frac{\sigma-\nu}{2}, \tau))}{\eta^3(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))}. \tag{4.3}$$

See (2.54) and (2.56) for definitions of the ϑ -function and the η -function.

Just as in the last chapter we have to account for the vast reducibility of the Verma module by subtracting off characters for the submodules generated by the singular vectors. The correct form needed for the irreducible character is,

$$\sum_{a \in \mathbb{Z}_+} \left(\chi_{V(Z'_a)} + \chi_{V(T_{a+1})} - \chi_{V(T'_a)} - \chi_{V(Z_{a+1})} \right) \tag{4.4}$$

where, as in chapter 2, $\chi_{V(X)}$ denotes the character for the module generated by the hws X . Of course all of these modules except $V(Z'_0)$ are modules of singular vectors. With the data from the table above, (4.4) becomes (neglecting for the moment the necessary prefactor F^R),

$$\begin{aligned} & \zeta^{\frac{h_{\pm}}{2}} q^h z^{\frac{-m(k+1)}{2}} \sum_{a \in \mathbb{Z}_+} \left[q^{a^2 pu + a(un - pm)} z^{\frac{1}{2}(n+2ap)} + q^{(a+1)^2 pu - (a+1)(un - pm)} z^{\frac{1}{2}(n-2(a+1)p)} \right. \\ & \quad \left. - q^{a^2 pu + mn + a(un + pm)} z^{\frac{1}{2}(-n-2ap)} - q^{(a+1)^2 pu + mn - (a+1)(un + pm)} z^{\frac{1}{2}(-n+2(a+1)p)} \right] \\ &= \zeta^{\frac{h_{\pm}}{2}} q^h z^{\frac{-m(k+1)}{2}} \sum_{a \in \mathbb{Z}} \left[q^{a^2 pu + a(un - pm)} z^{\frac{1}{2}(n+2ap)} - q^{a^2 pu + mn + a(un + pm)} z^{\frac{1}{2}(-n-2ap)} \right] \tag{4.5} \\ &= \zeta^{\frac{h_{\pm}}{2}} q^h z^{\frac{-m(k+1)}{2}} \sum_{a \in \mathbb{Z}} \left[q^{pu(a + \frac{un - mp}{2pu})^2 - \frac{(un - mp)^2}{4pu}} z^{\frac{1}{u} pu(a + \frac{un - mp}{2pu}) + \frac{pm}{2u}} \right. \\ & \quad \left. - q^{pu(a + \frac{un + mp}{2pu})^2 - \frac{(un + mp)^2}{4up} + mn} z^{-\frac{1}{q} pu(a + \frac{un + mp}{2pu}) + \frac{pm}{2u}} \right] \end{aligned}$$

$$= \zeta^{\frac{h_+}{2}} q^{h - \frac{(un-pm)^2}{4pu}} \left(\vartheta_{un-pm,pu} \left(\frac{\sigma}{u}, \tau \right) - \vartheta_{un+pm,pu} \left(-\frac{\sigma}{u}, \tau \right) \right) \quad (4.6)$$

using the ϑ -functions defined in (2.54).

With the expression (3.9) for L_0 we deduce that

$$h = \frac{h_-^2 - h_+^2}{4(k+1)} \quad (4.7)$$

in the Ramond sector, by operating on the hws.

So with the expressions (4.3), (4.6) and (4.7) the character of the irreducible module with hws Z'_0 whose isospin $h_-^R/2$ is given by $h_-^R + (k+1)m - n = 0$, $0 \leq m \leq u-1$ and $1 \leq n \leq p-1$ (recall equation (3.27)), is,

$$\begin{aligned} \chi_{m,n}^{RI}(\sigma, \nu, \tau) &= \\ & q^{\frac{-h_+^R}{4(k+1)}} \zeta^{\frac{h_+^R}{2}} F^R(\sigma, \nu, \tau) \times \left(\vartheta_{un-pm,pu} \left(\frac{\sigma}{u}, \tau \right) - \vartheta_{un+pm,pu} \left(-\frac{\sigma}{u}, \tau \right) \right) \quad (4.8) \\ &= q^{\frac{-h_+^R}{4(k+1)}} \zeta^{\frac{h_+^R}{2}} \frac{\left(\vartheta_{1,2} \left(\frac{\sigma+\nu}{2}, \tau \right) + \vartheta_{-1,2} \left(\frac{\sigma+\nu}{2}, \tau \right) \right) \left(\vartheta_{1,2} \left(\frac{\sigma-\nu}{2}, \tau \right) + \vartheta_{-1,2} \left(\frac{\sigma-\nu}{2}, \tau \right) \right)}{\eta^3(\tau)} \\ & \quad \times \chi_{n-1,m;k-1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau), \quad (4.9) \end{aligned}$$

where, $0 \leq m \leq u-1$ and $1 \leq n \leq p-1$ and the $\hat{sl}(2;\mathbb{C})$ character at level $(k-1)$ is defined as,

$$\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) = \frac{\vartheta_{b_+,a} \left(\frac{\sigma}{u}, \tau \right) - \vartheta_{b_-,a} \left(\frac{\sigma}{u}, \tau \right)}{\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)} \quad (4.10)$$

where, in this case, the level k of the representation is, $k = t/u$, $(t, u) = 1$, $u \in \mathbb{N}$, $t \in \mathbb{Z}$, and, $0 \leq n \leq 2u + t - 2$ and $0 \leq n' \leq u - 1$. Also,

$$b_{\pm} \stackrel{d}{=} u(\pm(n+1) - n'(k+2)), \quad a \stackrel{d}{=} u^2(k+2). \quad (4.11)$$

This definition is from [MP90].

Because the class I characters are proportional to admissible $\hat{sl}(2;\mathbb{C})$ characters, they inherit properties of the latter. An interesting property of admissible $\hat{sl}(2;\mathbb{C})$ characters is that they can develop a singularity in the limit as $\sigma \rightarrow 0$. Mukhi and Panda [MP90] worked out which $\hat{sl}(2;\mathbb{C})$ characters actually are singular in this limit. We can transfer their result directly to our characters and say that a class I character has a singularity at $\sigma = 0$ whenever $m > 0$ because that is the behaviour of the $\hat{sl}(2;\mathbb{C})$ characters. The factor multiplying the $\hat{sl}(2;\mathbb{C})$ character in (4.9) does not vanish at $\sigma = 0$.

Now we see how to obtain the Neveu-Schwarz characters from the Ramond ones by using the twist of the last chapter (equation (3.11)). It is just the same trick that was used in chapter 2 but now adapted to the different automorphism. Again we take the trace over the irreducible module.

$$\begin{aligned}\chi^{NS}(\sigma, \tau, \nu) &\stackrel{d}{=} \text{tr} \exp(2\pi i(L_0^{NS}\tau + J_0^{3,NS}\sigma + U_0^{NS}\nu)) \quad (c/24 = 0) \\ &= \text{tr} \exp(2\pi i((L_0^R - J_0^{3,R} + \frac{1}{4}k)\tau + (-J_0^{3,R} + \frac{1}{2}k)\sigma + U_0^R\nu))\end{aligned}$$

by (3.11)

$$\begin{aligned}&= \text{tr} \exp(2\pi i(L_0^R\tau - J_0^{3,R}(\sigma + \tau) + U_0^R\nu))q^{\frac{1}{4}k}z^{\frac{1}{2}k} \\ &= \chi^R(-\sigma - \tau, \nu, \tau)q^{\frac{1}{4}k}z^{\frac{1}{2}k}.\end{aligned}\tag{4.12}$$

Then using (4.3), the Neveu-Schwarz prefactor is,

$$z^{\frac{1}{2}}q^{\frac{1}{4}} \frac{(\vartheta_{0,2}(\frac{\sigma+\nu}{2}, \tau) + \vartheta_{2,2}(\frac{\sigma+\nu}{2}, \tau))(\vartheta_{0,2}(\frac{\sigma-\nu}{2}, \tau) + \vartheta_{2,2}(\frac{\sigma-\nu}{2}, \tau))}{\eta^3(\tau)(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau))},\tag{4.13}$$

and, from (4.6), the transformed correcting factor is,

$$q^{-\frac{p}{4u}}z^{-\frac{p}{2u}}(\vartheta_{-un+(m+1)p, up}(\frac{\sigma}{u}, \tau) - \vartheta_{un+(m+1)p}(\frac{\sigma}{u}, \tau)).\tag{4.14}$$

Thus, combining the last two expressions and incorporating the factor of $q^{\frac{k}{4}}z^{\frac{k}{2}}$, the Neveu-Schwarz character is,

$$\begin{aligned}\chi_{m,n}^{NS I}(\sigma, \nu, \tau) &= \\ & q^{\frac{-h_+^{NS2}}{4(k+1)}} \zeta^{\frac{h_+^{NS}}{2}} \frac{(\vartheta_{0,2}(\frac{\sigma+\nu}{2}, \tau) + \vartheta_{2,2}(\frac{\sigma+\nu}{2}, \tau))(\vartheta_{0,2}(\frac{\sigma-\nu}{2}, \tau) + \vartheta_{2,2}(\frac{\sigma-\nu}{2}, \tau))}{\eta^3(\tau)} \\ & \quad \times \chi_{n-1, m+1; k-1}^{\hat{sl}(2; \mathbb{C})}(\sigma, \tau),\end{aligned}\tag{4.15}$$

where, $0 \leq m \leq u-1$ and $1 \leq n \leq p-1$. We can see that these Neveu-Schwarz characters cannot have a pole at $z = 1$ in contrast to the Ramond sector class I characters because $(m+1)$, the second index on the $\hat{sl}(2; \mathbb{C})$ character, cannot vanish.

Now we compute the supercharacters for class I. The transformation to turn a character into a supercharacter is,

$$S\chi_{m,n}^{R, NS}(\sigma, \nu, \tau) = e^{\pi i h_-^{R, NS}} \chi_{m,n}^{R, NS}(\sigma + 1, \nu, \tau).\tag{4.16}$$

The transformation is easily done and we obtain,

$$S\chi_{m,n}^{RI}(\sigma, \nu, \tau) = \zeta^{\frac{h_+^R}{2}} q^{\frac{-h_+^{R^2}}{4(k+1)}} \frac{(\vartheta_{1,2}(\frac{\sigma+\nu}{2}, \tau) - \vartheta_{-1,2}(\frac{\sigma+\nu}{2}, \tau))(\vartheta_{1,2}(\frac{\sigma-\nu}{2}, \tau) - \vartheta_{-1,2}(\frac{\sigma-\nu}{2}, \tau))}{\eta^3(\tau)} \times \chi_{n-1,m;k-1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau), \quad (4.17)$$

where, $0 \leq m \leq u - 1$ and $1 \leq n \leq p - 1$ and,

$$S\chi_{m,n}^{NSI}(\sigma, \nu, \tau) = \zeta^{\frac{h_+^{NS}}{2}} q^{\frac{-h_+^{NS^2}}{4(k+1)}} \frac{(\vartheta_{0,2}(\frac{\sigma+\nu}{2}, \tau) - \vartheta_{2,2}(\frac{\sigma+\nu}{2}, \tau))(\vartheta_{0,2}(\frac{\sigma-\nu}{2}, \tau) - \vartheta_{2,2}(\frac{\sigma-\nu}{2}, \tau))}{\eta^3(\tau)} \times \chi_{n-1,m+1;k-1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau), \quad (4.18)$$

where, $0 \leq m \leq u - 1$ and $1 \leq n \leq p - 1$. Looking ahead somewhat, we can count up the modular weight of all the functions in each of the characters and supercharacters above. The $\hat{sl}(2;\mathbb{C})$ characters have weight zero and the ϑ -functions have weight $\frac{1}{2}$ as does the η -function. Modular weights add when the functions are multiplied. We can see that the class I characters and supercharacters all have modular weight $-\frac{1}{2}$ so that when $h_+^R = h_+^{NS} = 0$ they stand a chance of spanning a space of modular forms of that weight. From the point of view of physical models though we would be more interested in representations which had characters with modular weight zero since with them we could build modular invariant partition functions. For this reason we shall not consider class I representations much any more except to note that the integrable characters of class I are already well known objects (section 4.5.)

4.3 Class IV and Class V.

In this section we turn to the computation of what will be the central objects of our study hereafter; the characters and supercharacters of representations of class IV and class V. In the last chapter we saw that the embedding diagrams of these two classes are identical and we derived the quantum numbers for class V singular vectors which were not given in [BT97]. For the reader's convenience we collect below all the data we will need to compute the characters. The embedding diagram is in figure 4.3 and the quantum numbers in tables 4.2, 4.3 and 4.4. The embedding diagram is exactly the same in structure as that for unitary, minimal representations of the $N = 2$ superconformal

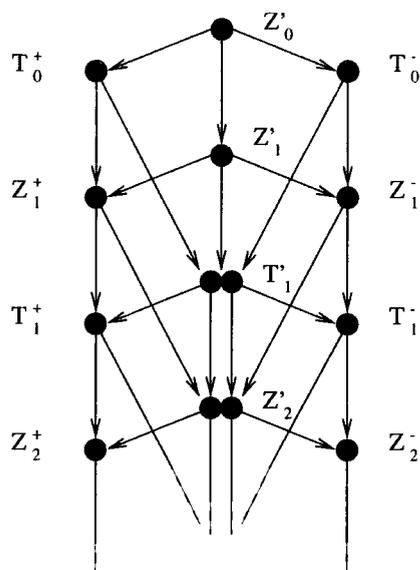


Figure 4.3: The embedding diagram for classes IV and V.

Series	Conformal Weight	$2 \times$ Isospin
Z'_a	$h + a(apu - pm)$	$2ap + h_-$
T'_{a+1}	$h + (a + 1)((a + 1)pu + pm)$	$-2(a + 1)p + h_-$
Z^-_{a+1}	$h + m' - m + (a + 1)((a + 1)pu + u - pm)$	$1 + 2(a + 1)p + h_-$
T^-_a	$h + m' + a(apu + u + pm)$	$-1 - 2ap + h_-$
Z^+_{a+1}	$h - m' + (a + 1)((a + 1)pu + u - pm)$	$1 + 2(a + 1)p + h_-$
T^+_a	$h + m - m' + a(apu + u + pm)$	$-1 - 2ap + h_-$

Table 4.2: Quantum numbers of the singular vectors of class IV rep.'s of $\hat{sl}(2|1; \mathbb{C})$.

Series	Conformal Weight
Z'_a	$h + a(apu - p(M + M' + 2))$
T'_{a+1}	$h + (a + 1)((a + 1)pu + p(M + M' + 2))$
Z^-_{a+1}	$h - M' - 1 + (a + 1)((a + 1)pu + u - p(M + M' + 2))$
T^-_a	$h + M + 1 + a(apu + u + p(M + M' + 2))$
Z^+_{a+1}	$h - M - 1 + (a + 1)((a + 1)pu + u - p(M + M' + 2))$
T^+_a	$h + M' + 1 + a(apu + u + p(M + M' + 2))$

Table 4.3: Conformal weights of the singular vectors of $\hat{sl}(2|1; \mathbb{C})$ class V rep.'s.

Series	$2 \times$ Isospin
Z'_a	$-2ap + h_-$
T'_{a+1}	$2(a + 1)p + h_-$
Z^-_{a+1}	$-1 - 2(a + 1)p + h_-$
T^-_a	$1 + 2ap + h_-$
Z^+_{a+1}	$-1 - 2(a + 1)p + h_-$
T^+_a	$1 + 2ap + h_-$

Table 4.4: Isospins of the singular vectors in class V rep.'s of $\hat{sl}(2|1; \mathbb{C})$.

algebra as worked out by Dörrzapf in his thesis [Dör95]. Even the singular vectors of multiplicity two in the centre of the $N = 2$ diagram appear in this diagram. There is a further link between representations of $\hat{sl}(2|1; \mathbb{C})$ and $N = 2$ which we shall see later. In the tables $a \in \mathbb{Z}_+$ and the singular vectors T_a^- and Z_{a+1}^- have charge $(\frac{1}{2}h_+ - \frac{1}{2})$ while the singular vectors T_a^+ and Z_{a+1}^+ have charge $(\frac{1}{2}h_+ + \frac{1}{2})$. It will also be useful to recall the definitions of classes IV and V which take the form of constraints on the isospin and charge of the highest weight state of a representation. Let the bosonic hws be $|h, \frac{1}{2}h_+, \frac{1}{2}h_-)$. For class IV, h_+ and h_- are to satisfy,

$$\begin{aligned} h_- + (k+1)m &= 0 \\ h_- - h_+ + 2m'(k+1) &= 0 \end{aligned} \quad (4.19)$$

where $0 \leq m \leq u-1$, $m' \in \mathbb{Z}_+$ and $m' - m \leq 0$. These are conditions as given in [BT97]. For class V, h_- and h_+ are to satisfy,

$$h_- + h_+ - 2(k+1)(M+1) = 0 \quad (4.20)$$

$$h_- - h_+ - 2(k+1)(M'+1) = 0 \quad (4.21)$$

where $M, M' \in \mathbb{Z}_+$. We saw in chapter 3 that we also need to restrict $M + M' + 2 \leq u$. Note that since in the Ramond sector the conformal weight h^R is given by (4.7), class IV and class V representations automatically have $h^R \geq 0$. To compute the Ramond characters we use the same prefactor as before *i.e.*, (4.2). Of course the correcting factor changes. For this diagram the appropriate correction to take account of the singular submodules is,

$$\sum_{a \in \mathbb{Z}_+} \left(\chi_{Z'_0} + \chi_{V(T'_{a+1})} + \chi_{V(Z'_{a+1})} + \chi_{V(Z'_{a+1})} - \chi_{V(Z'_{a+1})} - \chi_{V(T'_a)} - \chi_{V(T'_a)} \right). \quad (4.22)$$

Recall that by $\chi_{V(Z'_{a+1})}$ for example, we mean the character of the reducible module $V(Z'_{a+1})$ built on the hws Z'_{a+1} . In class I the hws was bosonic and all the singular vectors were bosonic too. In contrast, in class IV although the hws is bosonic many of the singular vectors are fermionic. In particular the four series Z_{a+1}^\pm, T_a^\pm mentioned above that have a charge different from the hws are all fermionic. Since the fermionic generators of $\hat{sl}(2|1; \mathbb{C})$ are nilpotent the fermionic singular vectors do not generate a complete Verma submodule. This being so the terms $\chi_{V(Z'_{a+1})}$ and $\chi_{V(T'_a)}$ in the correcting factor take a different form to that for a bosonic singular vector. The general

form for a fermionic singular vector X is,

$$\chi_{V(X)} = F^R \times \frac{q^{h(X)} z^{\frac{h_-(X)}{2}} \zeta^{\frac{h_+(X)}{2}}}{1 + q^{h(X)-h(Z)} z^{\frac{1}{2}(h_-(X)-h_-(Z))} \zeta^{\frac{1}{2}(h_+(X)-h_+(Z))}}, \quad (4.23)$$

where Z is the bosonic singular vector to which X is immediately connected in the diagram and $V(X) \subset V(Z)$. Then with the data in the table the correcting factor (4.22) is (and again we neglect F^R for the moment),

$$\begin{aligned} & \zeta^{\frac{h_+}{2}} q^h z^{\frac{h_-}{2}} \left\{ \sum_{a \in \mathbb{Z}_+} \left(q^{a^2 pu + apm} z^{-ap} \left(1 - \frac{q^{m-m'+au} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}}}{1 + q^{m-m'+au} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}}} - \frac{q^{m'+au} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}}}{1 + q^{m'+au} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}}} \right) \right) \right. \\ & \left. - \sum_{a \in \mathbb{N}} \left(q^{a^2 pu - apm} z^{ap} \left(1 - \frac{q^{-m'+au} z^{\frac{1}{2}} \zeta^{\frac{1}{2}}}{1 + q^{-m'+au} z^{\frac{1}{2}} \zeta^{\frac{1}{2}}} - \frac{q^{m'-m+au} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}}}{1 + q^{m'-m+au} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}}} \right) \right) \right\} \quad (4.24) \\ & = q^h z^{-\frac{h_-}{2}} \zeta^{\frac{h_+}{2}} \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} z^{-ap} \frac{1 - q^{m+2au} z^{-1}}{(1 + q^{m'+au} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{m-m'+au} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})} \end{aligned}$$

Thus with the Ramond sector prefactor, $F^R(\sigma, \nu, \tau)$ from the last section we have the class IV irreducible, admissible Ramond character,

$$\begin{aligned} \chi_{m,m'}^{R IV}(\sigma, \nu, \tau) &= q^h z^{\frac{h_-}{2}} \zeta^{\frac{h_+}{2}} F^R(\sigma, \nu, \tau) \times \\ & \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} z^{-ap} \frac{1 - q^{m+2au} z^{-1}}{(1 + q^{m'+au} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{m-m'+au} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.25) \end{aligned}$$

where $0 \leq m' \leq m \leq u - 1$. The isospin $h_-/2$ and charge $h_+/2$ of the hws Z'_0 are to be obtained from (4.19).

A similar analysis with the class V data yields,

$$\begin{aligned} \chi_{M,M'}^{R V}(\sigma, \nu, \tau) &= q^h z^{\frac{h_-}{2}} \zeta^{\frac{h_+}{2}} F^R(\sigma, \nu, \tau) \times \\ & \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(M+M'+2)} z^{ap} \frac{1 - q^{2au+M+M'+2} z}{(1 + q^{au+M+1} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{au+M'+1} z^{\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.26) \end{aligned}$$

with $0 \leq M + M' \leq u - 2$. The isospin $h_-/2$ and the charge $h_+/2$ of the hws Z'_0 are to be derived from (4.21).

There are some special cases of (4.25). One may consider integrable representations by setting $u = 1$. This forces m and m' to be zero. The character formula above simplifies and one recovers an expression that has the same q -expansion as a formula of Kac and Wakimoto in [KW]. In particular, when $p = 1$ so that $k = 0$, the integrable character reduces merely to unity in computer expansions. We shall see more of integrable characters in section 4.5.

Now we can use the same transformations ((4.12) and (4.16)) that we used in the previous section to obtain the Neveu-Schwarz characters and all the supercharacters. We obtain for class IV irreducible, admissible representations,

$$\chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h^{NS}} F^{NS}(\sigma, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(1+m)} z^{ap} \frac{1 - q^{2au+1+m} z}{(1 + q^{au+m'+\frac{1}{2}} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{au-m'+m+\frac{1}{2}} z^{\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.27)$$

where

$$F^{NS}(\sigma, \nu, \tau) \stackrel{d}{=} \prod_{n=1}^{\infty} \frac{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{n-\frac{1}{2}})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{n-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{n-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{n-\frac{1}{2}})}{(1 - q^n)^2 (1 - zq^n)(1 - z^{-1}q^{n-1})}, \quad (4.28)$$

and where $m, m' \in \mathbb{Z}_+$ and $0 \leq m' \leq m \leq u - 1$. Also, the characters for class V irreducible, admissible representations are,

$$\chi_{M,M'}^{NS,V}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h^{NS}} F^{NS}(\sigma, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(M+M'+1)} z^{-ap} \frac{1 - q^{2au+M+M'+1} z^{-1}}{(1 + q^{au+M+\frac{1}{2}} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{au+M'+\frac{1}{2}} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.29)$$

where $M, M' \in \mathbb{Z}_+$ and $0 \leq M + M' \leq u - 2$.

The supercharacters are, by (4.16), in the Ramond sector,

$$S\chi_{m,m'}^{R,IV}(\sigma, \nu, \tau) = q^{h^R} z^{\frac{1}{2}h^R} \zeta^{\frac{1}{2}h^R} F^R(\sigma + 1, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} z^{-ap} \frac{1 - q^{2au+m} z^{-1}}{(1 - q^{au+m'} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 - q^{au+m'-m} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.30)$$

where $m, m' \in \mathbb{Z}_+$ and $0 \leq m' \leq m \leq u - 1$, while the class V supercharacters are,

$$S\chi_{M,M'}^{R,V}(\sigma, \nu, \tau) = q^{h^R} z^{\frac{1}{2}h^R} \zeta^{\frac{1}{2}h^R} F^R(\sigma + 1, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(M+M'+2)} z^{ap} \frac{1 - q^{2au+M+M'+2} z}{(1 - q^{au+M+1} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 - q^{au+M'+1} z^{\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.31)$$

where $M, M' \in \mathbb{Z}_+$ and $0 \leq M + M' \leq u - 2$. In the Neveu-Schwarz sector, the class IV supercharacters are,

$$S\chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h^{NS}} F^{NS}(\sigma + 1, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(m+1)} z^{ap} \frac{1 - q^{2au+m+1} z}{(1 - q^{au+m'+\frac{1}{2}} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 - q^{au+m-m'+\frac{1}{2}} z^{\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.32)$$

where $m, m' \in \mathbb{Z}_+$ and $0 \leq m' \leq m \leq u - 1$ while for class V,

$$S\chi_{M,M'}^{NS,V}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h_-^{NS}} \zeta^{\frac{1}{2}h_+^{NS}} F^{NS}(\sigma + 1, \nu, \tau) \times \\ \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(M+M'+1)} z^{-ap} \frac{1 - q^{2au+M+M'+1} z^{-1}}{(1 - q^{au+M+\frac{1}{2}} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 - q^{au+M'+\frac{1}{2}} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})}, \quad (4.33)$$

where $M, M' \in \mathbb{Z}_+$ and $0 \leq M+M' \leq u-2$. This completes all the new (super)character formulae.

4.4 Poles in the Characters.

It is a phenomenon that some of the characters we have obtained above are not well defined in the limit as $\sigma \rightarrow 0$. We now proceed to ascertain which characters have a pole and then find the residue at that pole when a character does have a singularity. We present the result as a lemma. We prove the lemma and then state an easy corollary. The point of this lemma is that the prefactor, F^R has a potential pole at $z = 1$ i.e., $\sigma = 0$ arising from the factor $(1 - z^{-1})$ in its denominator. We look to find when this potential zero in the denominator is cancelled by a zero in the numerator. In view of results of Mukhi and Panda [MP90] and of Ennes, Ramallo and Sanchez de Santos [ERSdS] about singular admissible characters of $\widehat{sl}(2; \mathbb{C})$ and $\widehat{osp}(1, 2; \mathbb{C})$ respectively and their residues, our results are to be expected. Nevertheless, that we obtain the expected result gives us hope that the character formulae of the last section might be correct.

Lemma 4.1

The Ramond characters of class IV (resp. class V) are nonsingular at $\sigma = 0$ iff $m = 0$ (resp. $M + M' + 1 = u - 1$).

Proof. We prove the lemma only for class IV but the proof for class V is identical. The sum from (4.25) is,

$$\sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} z^{-ap} \frac{1 - q^{2ua+m} z^{-1}}{(1 + q^{u+m'} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{au+m-m'} z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})}$$

where $z \stackrel{d}{=} \exp(2\pi i \sigma)$. Then when σ is small the sum becomes,

$$\sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{2ua+m} - 2\pi i \sigma (ap + q^{2ua+m}(1 + ap)) + O(\sigma^2)}{f(\sigma; a)}$$

where the denominator $f(\sigma; a)$ is defined as,

$$f(\sigma; a) \stackrel{d}{=} (1 + q^{au+m-m'} \zeta^{\frac{1}{2}})(1 + q^{au+m'} \zeta^{-\frac{1}{2}}) \\ - i\pi\sigma(q^{au+m-m'} \zeta^{\frac{1}{2}} + q^{au+m'} \zeta^{-\frac{1}{2}} + 2q^{2au+m}) + O(\sigma^2).$$

We can pair up each term in the sum with the one that has the equal but opposite value of a . Then the sum becomes,

$$\frac{1}{2} \left\{ \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{2ua+m} - 2\pi i \sigma (ap + q^{2ua+m}(1+ap)) + O(\sigma^2)}{f(\sigma; a)} + (a \rightarrow -a) \right\} \\ = \frac{1}{2} \left\{ \sum_{a \in \mathbb{Z}} q^{a^2 pu} \frac{1 - q^{2ua} - 2\pi i \sigma (ap + q^{2ua}(1+ap)) + O(\sigma^2)}{f(\sigma; a)} + \right. \\ \left. \sum_{a \in \mathbb{Z}} q^{a^2 pu} \frac{1 - q^{-2ua} - 2\pi i \sigma (-ap + q^{-2ua}(1-ap)) + O(\sigma^2)}{f(\sigma; -a)} \right\}$$

iff $m = 0$. Then multiplying numerator and denominator of the second term by q^{2au} , we see that the last line vanishes at $\sigma = 0$. ■

Corollary 4.1

Integrable characters are nonsingular at $\sigma = 0$.

The proof of this is trivial. Similar arguments show that class IV Neveu-Schwarz characters are nonsingular only when $m = u - 1$. However, the type of argument above does not prove that the class V Neveu-Schwarz characters are ever nonsingular. In some examples we worked out and which are presented in the next section, the class V Neveu-Schwarz characters that appear are all singular. We conjecture that all of the class V Neveu-Schwarz characters are singular.

It is quite useful when studying affine algebras to draw a picture of all the states in a representation. One draws strata to represent the different grades that weights might have. The *grade* is the negative of the eigenvalue of the derivative operator d of the affine superalgebra. At each grade one draws a lattice which has the same dimension as the rank of the finite dimensional algebra from which we constructed our affine algebra. Thus at each stratum we can indicate all the quantum numbers of a state. For the algebra at hand, one should draw a 2-dimensional lattice at each grade for there are two quantum numbers (isospin and charge) to be shown for each state. If we concentrate on the Ramond version of $\hat{sl}(2|1; \mathbb{C})$ we see that it has a subalgebra—namely the algebra comprising all the generators with suffix zero. This algebra is isomorphic to the finite dimensional $sl(2|1; \mathbb{C})$. We can obtain a representation of $sl(2|1; \mathbb{C})$ from

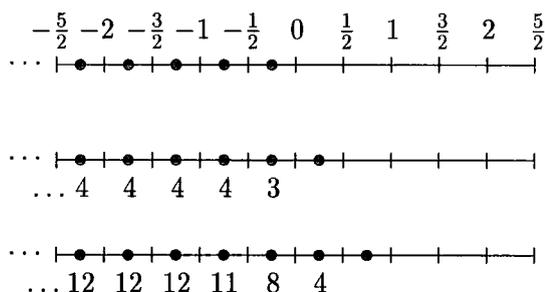


Figure 4.4: The states for the class IV rep. $k = -\frac{1}{2}$, $m = 1$, $m' = 0$.

one of $\hat{sl}(2|1; \mathbb{C})$ by looking for all the states at grade zero *i.e.*, on the first stratum on our picture. We can focus for a moment on the isospin of states in a representation of $sl(2|1; \mathbb{C})$. The effect of the $u(1)$ part of the even subalgebra of $sl(2|1; \mathbb{C})$ can be thought of as resolving extra detail about a state—another dimension on the lattice at each grade. The representations that we obtain for $sl(2|1; \mathbb{C})$ from $\hat{sl}(2|1; \mathbb{C})$ in this way can be infinite dimensional. That the representations are infinite dimensional is another way of saying that there is no lowest weight in those cases. Examples of this phenomenon are common. Choose the level of the representation to be $k = -\frac{1}{2}$ and $m = 1$ and $m' = 0$ in class IV (see (4.19)). Then the highest weight state has $\frac{1}{2}h_- = -1/4$ and we obtain an infinite sequence of states with everdecreasing isospin. The states are depicted in figure 4.4. The states continue indefinitely to the left with the multiplicity indicated below the state (unless the multiplicity is unity). The half integers across the top are the isospin values. Each diagram continues downward indefinitely. What is depicted are the states in the irreducible modules so that no singular vectors appear. Also, in each diagram we have suppressed the dimension which depicts the charge of states.

Compare this with the representation at the same level but with $m = m' = 0$ in class IV. This is a more conventional highest weight representation. Its states are shown in figure 4.5 In this case as far as $sl(2|1; \mathbb{C})$ is concerned, the highest weight state is the same as the lowest weight state—the representation is trivial for the finite subalgebra. Note that at the second grade in figure 4.5 there is an eight dimensional representation of $sl(2|1; \mathbb{C})$ *i.e.*, its adjoint representation. Another possibility is given by choosing the same level again and $M = M' = 0$ in class V (see (4.21)). We obtain an $sl(2|1; \mathbb{C})$ representation comprising an $sl(2)$ doublet with no charge and two charged states with no isospin. Its states are shown in figure 4.6.

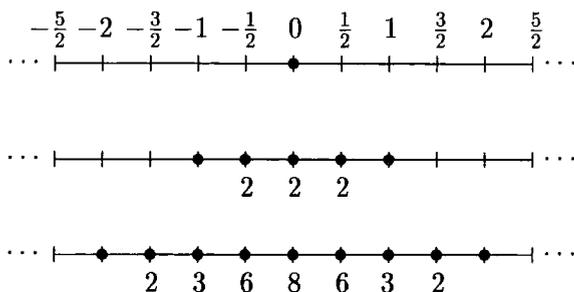


Figure 4.5: The states for the class IV rep. $k = -\frac{1}{2}$, $m = m' = 0$.

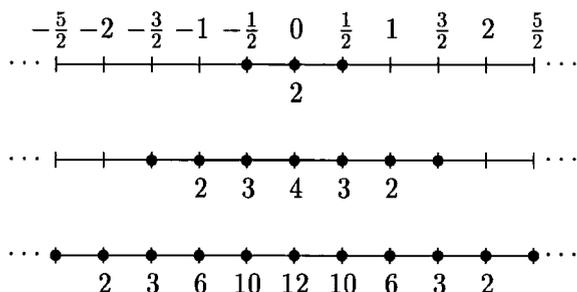


Figure 4.6: The states for the class V rep. $k = -\frac{1}{2}$, $M = M' = 0$.

Now we can obtain an alternative criterion for the existence of a pole in the admissible characters we found in the previous section in terms of the representation of the finite dimensional $sl(2|1; \mathbb{C}) \subset \hat{sl}(2|1; \mathbb{C})$.

Lemma 4.2

Let $\Lambda = (\frac{1}{2}h_-, \frac{1}{2}h_+)$ be the highest weight of $sl(2|1; \mathbb{C})$ where,

$$\frac{1}{2}h_- = -\frac{mp}{2u} \quad \text{and} \quad \frac{1}{2}h_+ = \frac{p}{2u}(2m' - m)$$

(from (4.19)). Then the highest weight representation with highest weight Λ is finite dimensional iff $m = 0$.

Proof. This follows the proofs of lemma 2.1 in the previous chapter. Let $\Pi = \{\alpha_1 + \alpha_2, -\alpha_1\}$ be a distinguished basis of simple roots for $sl(2|1; \mathbb{C})$. $\alpha_1 + \alpha_2$ is an even root and α_1 and α_2 are isotropic odd roots. $\alpha_1 \cdot \alpha_2 = 1/2$ so, $(\alpha_1 + \alpha_2)^2 = 1$. According to the Kac-Cornwell condition, the graded representation defined by Λ is finite-dimensional iff the numerical mark n_1 (corresponding to $\alpha_1 + \alpha_2$) is a non-negative integer, i.e., iff,

$$\Lambda \cdot (\alpha_1 + \alpha_2) \in \mathbb{Z}_+.$$

We can write Λ in terms of α_1 and α_2 as follows,

$$\Lambda = \frac{1}{2}h_-(\alpha_1 + \alpha_2) + \frac{1}{2}h_+(\alpha_1 - \alpha_2) = -\frac{mp}{u}\alpha_1 + \frac{m'p}{u}(\alpha_1 - \alpha_2).$$

Then,

$$\Lambda \cdot (\alpha_1 + \alpha_2) = -\frac{mp}{2u} \in \mathbb{Z}_+ \Leftrightarrow m = 0. \quad \blacksquare$$

Although Λ is different for class V, it is easy to see that the lemma holds there too for $M + M' + 1 = u - 1$.

Remark 4

The criterion of lemma 4.2 i.e., $m = 0$ gives also atypical representations of $sl(2|1; \mathbb{C})$. Thus, with the example above depicted in figure 4.5, we recover the known fact that the adjoint representation is commonly atypical ([Cor89, p.259]).

Now that we know which characters will have a pole we proceed to compute the residue of that pole.

4.4.1 The Residue.

We do the working for the Ramond characters of class IV. One simply does the same work for the class V Ramond characters and both classes of Neveu-Schwarz characters. From (4.2) we have that

$$q^h z^{\frac{h_-}{2}} \zeta^{\frac{h_+}{2}} F^R(\sigma, \nu, \tau) = q^{h+\frac{1}{8}} z^{\frac{h_-}{2} + \frac{1}{2}} \zeta^{\frac{h_+}{2}} \prod_{n=1}^{\infty} \frac{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{n-1})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{n-1})}{(1 - q^n)(\vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma))} \quad (4.34)$$

by Jacobi's triple product identity (2.58). Now, from [MP90],

$$\begin{aligned} \vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma) &= 2\pi i \sigma q^{\frac{1}{8}} \sum_{n \in \mathbb{Z}} q^{2n^2+n} (4n+1) + O(\sigma^2) \\ &= 2\pi i \sigma \eta^3(\tau) + O(\sigma^2), \end{aligned}$$

by an identity of Jacobi that could be found in Kac's book [Kac90] for example. So if we multiply (4.34) by $2\pi i \sigma$ and take $\lim_{\sigma \rightarrow 0}$ we obtain,

$$q^{h+\frac{1}{8}} \zeta^{\frac{h_+}{2}} \prod_{n=1}^{\infty} \frac{(1 + \zeta^{\frac{1}{2}} q^n)(1 + \zeta^{-\frac{1}{2}} q^{n-1})(1 + \zeta^{-\frac{1}{2}} q^n)(1 + \zeta^{\frac{1}{2}} q^{n-1})}{(1 - q^n) \eta^3(\tau)}$$

for (4.34). Bringing back in the other factor of the character from (4.25) we can write (using the proof of lemma 4.1),

$$\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{m,m'}^{R,IV}(\sigma, \nu, \tau) = \zeta^{\frac{h_+}{2}} q^{h+\frac{1}{12}} \left(\frac{\prod_{n=1}^{\infty} (1 + \zeta^{-\frac{1}{2}} q^{n-1})(1 + \zeta^{\frac{1}{2}} q^{n-1})}{\eta^2(\tau)} \right) \left(\frac{\prod_{n=1}^{\infty} (1 + \zeta^{\frac{1}{2}} q^n)(1 + \zeta^{-\frac{1}{2}} q^n)}{\prod_{n=1}^{\infty} (1 - q^n)^2} \right) \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{m+2au}}{(1 + q^{m'+au} \zeta^{-\frac{1}{2}})(1 + q^{m-m'+au} \zeta^{\frac{1}{2}})} \quad (4.35)$$

and of course $m \neq 0$ here. Now consider the products in (4.35). The products in the second bracket appear in an $N = 2$ Ramond character [RY87, Dob87] but we also need a factor of $(\zeta^{\frac{1}{4}} + \zeta^{-\frac{1}{4}})$ to make things just right. Then we must divide the product in the first bracket by the same factor. After a little work we can write (4.35) as,

$$\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{m,m'}^{R,IV}(\sigma, \nu, \tau) = \zeta^{\frac{h_+}{2}} q^h \times \frac{\vartheta_{1,2}(\tau, \frac{\nu}{2}) + \vartheta_{-1,2}(\tau, \frac{\nu}{2})}{\eta^3(\tau)} (\zeta^{\frac{1}{4}} + \zeta^{-\frac{1}{4}}) \frac{\prod_{n=1}^{\infty} (1 + \zeta^{\frac{1}{2}} q^n)(1 + \zeta^{-\frac{1}{2}} q^n)}{\prod_{n=1}^{\infty} (1 - q^n)^2} \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{m+2au}}{(1 + q^{m'+au} \zeta^{-\frac{1}{2}})(1 + q^{m-m'+au} \zeta^{\frac{1}{2}})}. \quad (4.36)$$

Recall that $h_+/2 = p(2m' - m)/2u$ and that $h = pm'(m - m')/u$. If we change notation as follows, $m = j + k$, $m' = j$ and $\zeta^{\frac{1}{2}} = y$ we have finally,

$$\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{m,m'}^{R,IV}(\sigma, \nu, \tau) = \frac{\vartheta_{1,2}(\tau, \frac{\nu}{2}) + \vartheta_{-1,2}(\tau, \frac{\nu}{2})}{\eta^3(\tau)} \chi_{j,k}^{R+,N=2}(y, q),$$

where,

$$\chi_{j,k}^{R+,N=2}(y, q) = y^{\frac{(j-k)p}{u}} q^{\frac{jkp}{u}} (y^{\frac{1}{2}} + y^{-\frac{1}{2}}) \frac{\prod_{n=1}^{\infty} (1 + yq^n)(1 + y^{-1}q^n)}{\prod_{n=1}^{\infty} (1 - q^n)^2} \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(j+k)} \frac{1 - q^{j+k+2au}}{(1 + q^{j+au} y^{-1})(1 + q^{k+au} y)}. \quad (4.37)$$

This is precisely the same as the Ramond(+) character in [RY87] when $p = 1$ and is the expression for a *non-unitary*, minimal $N = 2$ when $p > 1$ or when $m' = 0$ and $p = 1$. To be consistent the central charge of the $N = 2$ algebra should be

$$c = 3(1 - 2p/u). \quad (4.38)$$

If instead we had started with a class V Ramond character we could get the same residue by changing notation according to $j = M + 1$ and $k = M' + 1$ instead. This time the $N = 2$ characters are unitary or not according as $p = 1$ or $p > 1$.

Identical reasoning leads to,

$$\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) = \frac{\vartheta_{0,2}(\tau, \frac{\nu}{2}) + \vartheta_{2,2}(\tau, \frac{\nu}{2})}{\eta^3(\tau)} \chi_{j,k}^{NS,N=2}(y, q) \quad (4.39)$$

where,

$$\chi_{j,k}^{NS,N=2}(y, q) = y^{\frac{(j-k)p}{u}} q^{\frac{(jk-1/4)p}{u} - \frac{c}{24}} \frac{\prod_{n=1}^{\infty} (1 + yq^{n-\frac{1}{2}})(1 + y^{-1}q^{n-\frac{1}{2}})}{\prod_{n=1}^{\infty} (1 - q^n)^2} \times \sum_{a \in \mathbb{Z}} q^{a^2 pu + ap(j+k)} \frac{1 - q^{j+k+2au}}{(1 + q^{j+au}y^{-1})(1 + q^{k+au}y)}.$$

In this equation c is given by (4.38). Starting from a class IV Neveu-Schwarz character one defines, $m = j + k - 1$ and $m' = j - 1/2$. Starting from class V, $M = j - 1/2$ and $M' = k - 1/2$. In both cases $y = \zeta^{\frac{1}{2}}$. These $N = 2$ characters are unitary or not according as $p = 1$ or $p > 1$. Notice that in the Neveu-Schwarz case, j and k are half integers and in the Ramond case they are integers as they should be. Thus our new characters have exactly the residue one would expect of them. This information about the residue will prove crucial in determining the branching functions in the next chapter.

4.5 Characters of Integrable Representations.

In this section we specialise the admissible characters we worked out above to the case of integrable representations. This is easily done in the character formulas by simply setting $u = 1$. Recall that the level, k , was taken to be $k + 1 = p/u$ with p and u coprime positive integers. Kac and Wakimoto in [KW] give conditions on a highest weight module of $\hat{sl}(2|1; \mathbb{C})$ (or $A(1, 0)^{(1)}$ as they call it) to be integrable. Let α_1 and α_2 be simple roots of $sl(2|1; \mathbb{C})$. Let Λ be the highest weight of a module of $\hat{sl}(2|1; \mathbb{C})$. Then we can parametrize Λ as follows.

$$\Lambda = \left(\frac{1}{2}(h_- + h_+)\alpha_1 + \frac{1}{2}(h_- - h_+)\alpha_2, k, 0\right).$$

where $\frac{1}{2}h_-$ and $\frac{1}{2}h_+$ are the eigenvalues of the operators J_0^3 and U_0 respectively. The conditions that Kac and Wakimoto give are the following. We require $k \in \mathbb{Z}_+$ and $k \geq h_-$ and also that *either* $h_- \in \mathbb{N}$ and h_+ is unconstrained *or* $h_- = h_+ = 0$. It is quite plain to see that, of the classes that we study here, integrable representations can only appear in classes I and IV. Those provided by class I are the first type allowed by Kac-Wakimoto *i.e.*, $h_- \in \mathbb{N}$ and h_+ unconstrained. Note that for integrable class I

representations, $h_- = n \in \mathbb{N}$ and $n \leq p-1 = k$ so that $k \geq h_-$ is automatically satisfied. The integrable representation provided by class IV is the second type allowed by Kac-Wakimoto *i.e.*, $h_- = h_+ = 0$. Computer expansions of our formula for a Ramond sector integrable character from class IV and of the formula obtained by Kac and Wakimoto reveal that they are the same for many values of the level—at least to low order. Upon specialising the integrable character formula of Ramond sector class IV to $k = 0$ one obtains another form of the denominator identity that appears in §4 of [KW]. We have,

$$\sum_{a \in \mathbb{Z}} q^{a^2} z^{-a} \frac{1 - q^{2a} z^{-1}}{(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^a z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})} = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2 (1 - zq^n)(1 - z^{-1}q^{n-1})}{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{n-1})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^n)(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{n-1})}. \quad (4.40)$$

Now we come to the most intriguing aspect of the integrable characters. It is easy to show that they are the same as certain characters of the $N = 4$ superconformal algebra as computed by Eguchi and Taormina in [ET88a]. In particular, our class I characters are the same as what Eguchi and Taormina call *massive* characters of $N = 4$ and our class IV (vacuum) integrable characters are the same as their *massless* characters. These results hold in both the Ramond and Neveu-Schwarz sectors. Because Eguchi and Taormina were able to work out the modular transformations of some of their characters we have the modular transformations of some $\hat{sl}(2|1; \mathbb{C})$ integrable characters for free. If one looks at the structure of the Kac determinant formula for the $N = 4$ algebra as found by Kent and Riggs [KR87] one sees that it is quite similar to that of $\hat{sl}(2|1; \mathbb{C})$ and also the denominator of the characters is the same in each case. Thus it is plausible that there might be some similarity.

From [ET88a] we get the following $N = 4$ massless character (including the Casimir factor of $q^{-\frac{c}{24}} = q^{-\frac{k}{4}}$ on the LHS),

$$q^{-\frac{k}{4}} \text{ch}_0^R(k, \ell; \sigma, \nu) = F^R(\sigma, \nu, \tau) \times \sum_{m \in \mathbb{Z}} \left(\frac{z^{(k+1)m + \ell} q^{(k+1)m^2 + 2\ell m}}{(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{-m})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-m})} - \frac{z^{-(k+1)m - \ell} q^{(k+1)m^2 + 2\ell m}}{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{-m})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-m})} \right) \quad (4.41)$$

and we have changed their notation as follows, $2\pi\sigma = \theta$ and $2\pi\nu = \varphi$. Also $\ell \in \{0, \frac{1}{2}, \dots, \frac{k}{2}\}$ where $k \in \mathbb{Z}_+$ is the level of the $N = 4$ algebra. $F^R(\sigma, \nu, \tau)$ is given in (4.2). Now multiply the numerator and denominator of the second term of the sum by $q^{2m} z^{-1}$ while replacing m with $-m$ in the first term and setting $\ell = 0$. We have

now,

$$\sum_{m \in \mathbb{Z}} q^{(k+1)m^2} z^{-(k+1)m} \frac{1 - q^{2m} z^{-1}}{(1 + q^m z^{-\frac{1}{2}} \zeta^{\frac{1}{2}})(1 + q^m z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})},$$

for the sum in (4.41). With $k + 1 = p$ this is identical to the sum part of the integrable class IV character (see (4.25) with $u = 1, m = m' = 0$) and since the characters have F^R in common we get,

$$\chi_{\text{integrable}}^{R,IV}(\sigma, \nu, \tau) = q^{-\frac{1}{4}k} \text{ch}_0^R(k, \ell = 0; 2\pi\sigma, 2\pi\nu). \tag{4.42}$$

Which is to say that in the Ramond sector, the $h_- = h_+ = 0$ representation of $\hat{sl}(2|1; \mathbb{C})$ at level $k \in \mathbb{Z}_+$ is the same as the $\ell = 0$ representation of the $N = 4$ algebra at the same level. This relationship is preserved under spectral flow and one easily shows that,

$$\chi_{\text{integrable}}^{NS,IV}(\sigma, \nu, \tau) = q^{-\frac{1}{4}k} \text{ch}_0^{NS}(k, \ell = \frac{1}{2}k; 2\pi\sigma, 2\pi\nu). \tag{4.43}$$

One can look at [BHT98] or [ET88a] now and see the modular S transformation of the $N = 4$ i.e., $\hat{sl}(2|1; \mathbb{C})$ integrable characters at least when $k = 1$. We see that they do not carry a representation of the modular group since they mix with the class I characters under the modular S transformation. We saw in section 4.2 that the class I characters were not covariant under S either—they are modular forms¹ of weight $-1/2$.

Now we turn to class I and look at the integrable characters available to us from there. We shall see that they are the same as a certain subset of the massive $N = 4$ characters. Referring to [ET88a], the massive characters were defined to be those for which the conformal weight $h^{N=4}$ of the highest weight state satisfied $h^{N=4} \geq k/4$ where k is the level of the $N = 4$ algebra. This $h^{N=4}$ may take any real value above $k/4$. The class I integrable characters have $h^R \in \{n^2/4p | n = 1, \dots, p - 1\}$ —a finite, discrete set rather than a continuum. Thus we can reach just a subset of the massive $N = 4$ representations. In class I the condition on h_-^R and h_+^R that is satisfied is that h_+^R is unconstrained and $h_-^R \in \mathbb{N}$. Indeed recalling (3.27) we see that $h_-^R = n$ with $1 \leq n \leq p - 1 = k$. From (4.5) we have that (when $h_+^R = 0$),

$$\chi_{n; \text{integrable}}^{I,R}(\sigma, \nu, \tau) = q^{h^R} F^R(\sigma, \nu, \tau) \sum_{a \in \mathbb{Z}} q^{a^2(k+1)+ah_-} (z^{a(k+1)+h_-^R/2} - z^{-a(k+1)-h_-^R/2}). \tag{4.44}$$

¹A function $f(\tau)$ is a modular form of weight r if $f(-1/\tau) = \tau^r f(\tau)$. Modular forms of weight zero are called *modular functions*.

If we now compare with the expressions for massive characters in [ET88a] and multiply those by the Casimir factor $q^{-\frac{1}{4}k}$ and one also makes the identifications of σ and ν that appear after (4.41) and also set $h_- = 2\ell$ we see that the class I integrable character is the same as the $N = 4$ massive character. This class I integrable/massive $N = 4$ equality holds in the Neveu-Schwarz sector too. The reader might like to note that in equation (17) of [ET88a] the “ l ” is not the same as the “ l ” in equation (16). One can make (17) the spectral flow of (16) by replacing l in (17) by $k/2 - l$. There is no misprint as such, just a different definition of the quantity l in each equation. At this stage we can see that the $\hat{sl}(2|1; \mathbb{C})$ admissible characters are very closely connected to both $N = 2$ and $N = 4$ superconformal characters.

Having obtained the character formulae we wanted, in the next chapter we begin to study them more closely with a view towards working out their modular transformations. Wanting to do this leads us to try to expand our $\hat{sl}(2|1; \mathbb{C})$ characters in $\hat{sl}(2; \mathbb{C})$ characters. We can do this because $\hat{sl}(2; \mathbb{C})$ is a subalgebra of $\hat{sl}(2|1; \mathbb{C})$. Also, we already know the modular transformations of the $\hat{sl}(2; \mathbb{C})$ characters. The hard part of the analysis is in identifying the branching functions. However, we have managed to work out their structure completely (for a certain subset of the non-integrable representations) and it turns out that they are already-known functions with known modular transformations. The form of the branching functions is explained by a simple coset structure.

Chapter 5

Branching the Characters

5.1 Introduction.

The aim of this chapter is to put the class IV and class V Neveu-Schwarz characters into such a shape that their modular transformations could be computed. The modular T transformation is quite straightforward to deduce from the expressions we already have and we shall not discuss it further in this chapter but return to it in the next. The motivation for this chapter is the S transformation especially. Unfortunately, it is quite difficult to work this out in general unless one can write the characters in terms of functions that have known S transformations. This then is our task. Our method involves a certain coset construction [GKO85, GKO86]. In particular, we will use the $SL(2|1)/SL(2)$ coset and therefore branch the $\hat{sl}(2|1; \mathbb{C})$ characters into $\hat{sl}(2; \mathbb{C})$ characters. The branching functions can be identified as a product of characters [DVVV89, PT93] of the rational torus model $\mathcal{A}_{u(u-1)}$ and characters [GQ87, ADT93] of the \mathbb{Z}_{u-1} parafermion theory [ZF85] whenever the level k of the $\hat{sl}(2|1; \mathbb{C})$ algebra is of the form $k + 1 = 1/u$, $u \geq 2$. We shall show how this is realised in a series of examples in the last section of this chapter. In the next chapter the modular transformations will actually be worked out for some cases. The results of this chapter have been submitted to *Nuclear Physics* for publication as paper [HT].

5.2 The Coset.

One method of working out the modular transformation of the characters, in particular the S transformation, is to branch the characters into $\hat{sl}(2; \mathbb{C})$ characters. We can certainly try to do this because $sl(2; \mathbb{C})$ is a subalgebra of $sl(2|1; \mathbb{C})$. Following the seminal paper by Goddard, Kent and Olive [GKO85], we can do the Sugawara construction

for each of $\hat{sl}(2|1; \mathbb{C})$ and $\hat{sl}(2; \mathbb{C})$ producing two Virasoro algebras with central charges $c_{\hat{sl}(2|1; \mathbb{C})} (= 0)$ and $c_{\hat{sl}(2; \mathbb{C})}$ respectively. Having done this, we consider the Virasoro algebra defined by taking the “difference” of these two conformal algebras. This one commutes with $\hat{sl}(2; \mathbb{C})$ and has central charge $c_{\hat{sl}(2|1; \mathbb{C})} - c_{\hat{sl}(2; \mathbb{C})} = -c_{\hat{sl}(2; \mathbb{C})}$. An important observation in the context of the present work is the following. It was noticed that $-c_{\hat{sl}(2; \mathbb{C})}$ could be written as the sum of the central charges associated with two other known theories. In particular,

$$-c_{\hat{sl}(2; \mathbb{C})} = \frac{3k}{k+2} = \frac{3(1-u)}{1+u} = 1 + \frac{2(u-2)}{u+1} \quad (5.1)$$

when $k+1 = 1/u$. We can interpret the RHS as being the sum of central charges of a rational torus model ($c_{\text{torus}} = 1$) and a \mathbb{Z}_{u-1} parafermion theory. As we shall see below in section 5.4, the proposed coset can actually be realised as character sum rules in the three cases $u = 2, 3, 4$ *i.e.*, $\hat{sl}(2|1; \mathbb{C})$ characters can be written as a sum of products of $\hat{sl}(2; \mathbb{C})$, \mathbb{Z}_{u-1} and (as it turns out) $\mathcal{A}_{u(u-1)}$ characters. This mirrors the character sum rules proved by Goddard, Kent and Olive in [GKO86]. So far though the general sum rules (any u) have not been proven but we can make a conjecture based on the three cases ($u = 2, 3, 4$) we have worked out in detail.

In the next section we shall extract from the Neveu-Schwarz class IV and class V character formulae, factors which are precisely the admissible $\hat{sl}(2; \mathbb{C})$ characters. The $\hat{sl}(2; \mathbb{C})$ characters will be multiplied by rather unwieldy functions but we can, by making use of the fact that some class IV/V characters are singular at $\sigma = 0$ (see section 4.4), write the unwieldy coefficients in a very neat form which realises the proposed coset.

5.3 The ϑ -Function Form.

As they stand now the characters (4.27) and (4.29) are in the form of an infinite product times an infinite sum. The first step towards writing the characters in terms of ϑ -functions is to obtain a product form for the sum part of each character. Having obtained this we can use Jacobi’s triple product identity to write that new product in terms of ϑ -functions. We shall write down products for the class IV and class V Neveu-Schwarz sector characters only. One can use spectral flow to obtain the corresponding

expression for the Ramond sector. We propose that when the level $k = -1 + 1/u$,

$$\chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h_-^{NS}} \zeta^{\frac{1}{2}h_+^{NS}} F^{NS}(\sigma, \nu, \tau) \times \prod_{n \in \mathbb{N}} [(1 - q^{un})^2 (1 - zq^{u(n-1)+m+1})(1 - z^{-1}q^{un-m-1})/h_n(m, m'; \sigma, \nu, \tau)], \quad (5.2)$$

where,

$$h_n(m, m'; \sigma, \nu, \tau) = (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{u(n-1)+m-m'+\frac{1}{2}})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{u(n-1)+m'+\frac{1}{2}}) (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{un-m'-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{un+m'-m-\frac{1}{2}}). \quad (5.3)$$

For class V there is something very similar,

$$\chi_{M,M'}^{NS,V}(\sigma, \nu, \tau) = q^{h^{NS}} z^{\frac{1}{2}h_-^{NS}} \zeta^{\frac{1}{2}h_+^{NS}} F^{NS}(\sigma, \nu, \tau) \times \prod_{n \in \mathbb{N}} [(1 - q^{un})^2 (1 - zq^{un-M-M'-1})(1 - z^{-1}q^{u(n-1)+M+M'+1})/k_n(M, M'; \sigma, \nu, \tau)], \quad (5.4)$$

where,

$$k_n(M, M'; \sigma, \nu, \tau) = (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{un-M-\frac{1}{2}})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{un-M'-\frac{1}{2}}) (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{u(n-1)+M'+\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{u(n-1)+M+\frac{1}{2}}). \quad (5.5)$$

It is clear that these products have the same singularities as do the sums in the previous forms of these characters. Another step towards demonstrating the equivalence would be to prove that the new products here have the same residues at their poles as do the sums in the old form of the characters. We show how to do this in appendix A. It is found that the residues agree exactly. Two functions which have the same poles and residues need not be identical though. Their difference could be a function which is regular on the appropriate domain. However, by expanding each expression with a computer for many different representations we have found that the two are identical to a finite order at least. Also, whenever the $\hat{sl}(2|1; \mathbb{C})$ character is singular, upon working out the residue at the pole at $\sigma = 0$, we obtain again an $N = 2$ character of course, but now in the product form as Matsuo has it. See [Mat87] or [RY87]. All this evidence lends weight to the claim that the products exhibited above are the correct ones when $k + 1 = 1/u$.

The denominators of the characters (5.2) and (5.4) do not have 'nice' S transformations in as much as they do not transform back to themselves. To proceed we should remove the troublesome factors from the denominator. We will provide the details for

the class IV character, the details for the class V character being similar. The following reasoning mirrors that of Ravanini and Yang and their work on characters of the $N = 2$ superconformal algebra in [RY87].

It is clear that when we actually substitute values for m and m' and u into (5.2) all of the factors in h_n actually appear in the numerator too and thus many factors in the numerator may be cancelled off (though not all of them). We shall work out which factors remain in the numerator. Compare factors of $(1 + z^x \zeta^y (q^{\frac{1}{2}})^{2n-1})$ with $x = \pm \frac{1}{2}$ and $y = \pm \frac{1}{2}$ in the numerator and denominator. We find that *after cancelling common factors*, the only factors remaining in the numerator are ones where the powers of $q^{\frac{1}{2}}$ comprise the following sets:

$$x = \frac{1}{2}, y = \frac{1}{2}: \quad V \stackrel{d}{=} \{1, \dots, 2(m - m' + 1) - 3, 2(m - m' + 1) + 1, \dots, \\ 2(u + m - m' + 1) - 3, 2(u + m - m' + 1) + 1, \dots\} \quad (5.6)$$

$$x = \frac{1}{2}, y = -\frac{1}{2}: \quad W \stackrel{d}{=} \{1, \dots, 2(m' + 1) - 3, 2(m' + 1) + 1, \dots, \\ 2(u + m' + 1) - 3, 2(u + m' + 1) + 1, \dots\} \quad (5.7)$$

$$x = -\frac{1}{2}, y = \frac{1}{2}: \quad X \stackrel{d}{=} \{1, \dots, 2(u - m') - 3, 2(u - m') + 1, \dots, \\ 2(2u - m') - 3, 2(2u - m') + 1, \dots\} \quad (5.8)$$

$$x = -\frac{1}{2}, y = -\frac{1}{2}: \quad Y \stackrel{d}{=} \{1, \dots, 2(u + m' - m) - 3, 2(u + m' - m) + 1, \dots, \\ 2(2u + m' - m) - 3, 2(2u + m' - m) + 1, \dots\}. \quad (5.9)$$

Thus,

$$\chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) = \\ q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h_+^{NS}} \prod_{n \in \mathbb{N}} (1 - q^{un})^2 (1 - zq^{u(n-1)+m+1}) (1 - z^{-1}q^{un-m-1}) \times \\ \prod_{\alpha \in V} (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{\alpha}{2}}) \prod_{\beta \in W} (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{\beta}{2}}) \prod_{\gamma \in X} (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{\gamma}{2}}) \prod_{\delta \in Y} (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{\delta}{2}}) \times \\ \left[\prod_{n=1}^{\infty} (1 - q^n)^2 (1 - zq^n) (1 - z^{-1}q^{n-1}) \right]^{-1} \quad (5.10)$$

and the troublesome factors in the denominator have gone. Now each of V, W, X and Y is an infinite set. We wish to make each of them apparently finite by reducing their

elements mod $2u$. Each of these sets has $(u - 1)$ different residues mod $2u$. We can write down these sets of residues.

$$\begin{aligned}
\text{res}_V^{(2u)} &= \{1, \dots, 2(m - m') - 1, 2(m - m' + 2) - 1, \dots, 2(u) - 1\} = \emptyset \text{ if } m = m' \\
\text{res}_W^{(2u)} &= \{1, \dots, 2(m') - 1, 2(m' + 2) - 1, \dots, 2u - 1\} \\
&= \emptyset \quad \text{if } m' = 0 \\
\text{res}_X^{(2u)} &= \{1, \dots, 2(u - m' - 1) - 1, 2(u - m' + 1) - 1, \dots, 2u - 1\} \\
&= \emptyset \text{ if } m' = u - 1 \\
\text{res}_Y^{(2u)} &= \{1, \dots, 2(m' - m + u - 1) - 1, 2(m' - m + u + 1) - 1, \dots, 2u - 1\} \\
&= \emptyset \quad \text{if } m - m' = u - 1.
\end{aligned} \tag{5.11}$$

Then we can write each of the products over V, W, X and Y in (5.10) above so that the residues mod $2u$ in the power of $q^{\frac{1}{2}}$ are explicit. We obtain,

$$\begin{aligned}
\prod_{\alpha \in V} (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{\alpha}{2}}) &= \prod_{n \in \mathbb{N}} \overbrace{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2u-1)}{2}})}^{a_1} \cdots (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2(u-m+m')+1)}{2}}) \times \\
&\quad (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2(u-m+m'-2)+1)}{2}}) \cdots (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un-1}{2}}), \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
\prod_{\beta \in W} (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{\beta}{2}}) &= \prod_{n \in \mathbb{N}} \overbrace{(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2u-1)}{2}})}^{b_1} \cdots (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2(u-m')+1)}{2}}) \times \\
&\quad (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2(u-m'-2)+1)}{2}}) \cdots (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un-1}{2}}), \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
\prod_{\gamma \in X} (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{\gamma}{2}}) &= \prod_{n \in \mathbb{N}} (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2u-1)}{2}}) \cdots (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2(m'+1)+1)}{2}}) \times \\
&\quad (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un - (2(m'-1)+1)}{2}}) \cdots \overbrace{(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2un-1}{2}})}^{b_2}, \tag{5.14}
\end{aligned}$$

and finally,

$$\begin{aligned}
\prod_{\delta \in Y} (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{\delta}{2}}) &= \prod_{n \in \mathbb{N}} (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2u-1)}{2}}) \cdots (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2(m-m')+3)}{2}}) \\
&\quad \times (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un - (2(m-m'-1)+1)}{2}}) \cdots \overbrace{(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2un-1}{2}})}^{a_2}. \tag{5.15}
\end{aligned}$$

Now reorder the factors of (5.12) \times (5.13) \times (5.14) \times (5.15) and pair up a_1 with a_2 and then the factor after a_1 with the factor before a_2 and so on so that the residues mod $2u$

in the power of $q^{\frac{1}{2}}$ always add up to $2u$. There are obviously $(u - 1)$ such pairs. Also, pair up b_1 with b_2 and then the factor after b_1 with the factor before b_2 and so on again so that the residues mod $2u$ of the powers of $q^{\frac{1}{2}}$ add up to $2u$. And again there are $(u - 1)$ such pairs. Thus, eventually, the numerator of (5.10) can be written as,

$$\begin{aligned}
 & q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h^{NS}} \prod_{n \in \mathbb{N}} (1 - q^{un})^2 (1 - zq^{u(n-1)+m+1}) (1 - z^{-1}q^{un-m-1}) \times \\
 & \prod_{r=1}^u \prod_{n \in \mathbb{N}} \left(1 + (z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2r-u-1}{2}}) (q^{\frac{u}{2}})^{2n-1} \right) \left(1 + (z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2r-u-1}{2}})^{-1} (q^{\frac{u}{2}})^{2n-1} \right) \times \\
 & \prod_{s=1}^u \prod_{n \in \mathbb{N}} \left(1 + (z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2s-u-1}{2}}) (q^{\frac{u}{2}})^{2n-1} \right) \left(1 + (z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2s-u-1}{2}})^{-1} (q^{\frac{u}{2}})^{2n-1} \right)
 \end{aligned} \tag{5.16}$$

where, in the product \prod' we are to neglect the $r = m - m' + 1$ factor and in the product \prod'' we are to neglect the $s = m' + 1$ factor. When $u = 1$ these products do not appear at all since the sets (5.11) of residues are empty. Now that we have (5.16) we see that it is susceptible to Jacobi's triple product theorem if we put in some factors of $(1 - (q^{\frac{u}{2}})^{2n})$.

We need $(2u - 4)$ such factors. Thus, (5.16) can be written,

$$\begin{aligned}
 & q^{h^{NS}} z^{\frac{1}{2}h^{NS}} \zeta^{\frac{1}{2}h^{NS}} \prod_{n \in \mathbb{N}} (1 - (q^{\frac{u}{2}})^{2n})^{-(2u-4)} (1 - zq^{u(n-1)+m+1}) (1 - z^{-1}q^{un-m-1}) \times \\
 & \prod_{r=1}^u \prod_{n \in \mathbb{N}} (1 - (q^{\frac{u}{2}})^{2n}) \left(1 + (z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2r-u-1}{2}}) (q^{\frac{u}{2}})^{2n-1} \right) \left(1 + (z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{\frac{2r-u-1}{2}})^{-1} (q^{\frac{u}{2}})^{2n-1} \right) \times \\
 & \prod_{s=1}^u \prod_{n \in \mathbb{N}} (1 - (q^{\frac{u}{2}})^{2n}) \left(1 + (z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2s-u-1}{2}}) (q^{\frac{u}{2}})^{2n-1} \right) \left(1 + (z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{\frac{2s-u-1}{2}})^{-1} (q^{\frac{u}{2}})^{2n-1} \right),
 \end{aligned} \tag{5.17}$$

and we define $G(m, m', u; \sigma, \nu, \tau)$ to be the expression (5.17). Then a class IV Neveu-Schwarz character is written as,

$$\chi_{m,m'}^{IV,NS}(\sigma, \nu, \tau) = G(m, m', u; \sigma, \nu, \tau) / \prod_{n \in \mathbb{N}} (1 - q^n)^2 (1 - zq^n) (1 - z^{-1}q^{n-1}) \tag{5.18}$$

$$\begin{aligned}
 & = \prod_{r=1}^u \vartheta_{2r-1-u,u} \left(\frac{\sigma + \nu}{2u}, \frac{\tau}{2} \right) \prod_{s=1}^u \vartheta_{2s-1-u,u} \left(\frac{\sigma - \nu}{2u}, \frac{\tau}{2} \right) \times \\
 & \frac{(\vartheta_{-u+2(m+1),2u}(\frac{\sigma}{u}, \tau) - \vartheta_{u+2(m+1),2u}(\frac{\sigma}{u}, \tau))}{(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)) \eta(\tau) \eta^{2u-3}(u\tau)}
 \end{aligned} \tag{5.19}$$

$$\begin{aligned}
&= \prod_{t=1}^{u-1} \sum_{s=0}^1 \vartheta_{us+m-2m',u}\left(\frac{\nu}{u}, \tau\right) \vartheta_{u(s+1)+m+1+2t,u}\left(\frac{\sigma}{u}, \tau\right) \times \\
&\quad \frac{(\vartheta_{-u+2(m+1),2u}\left(\frac{\sigma}{u}, \tau\right) - \vartheta_{u+2(m+1),2u}\left(\frac{\sigma}{u}, \tau\right))}{(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)) \eta(\tau) \eta^{2u-3}(u\tau)} \quad (5.20)
\end{aligned}$$

after shifting the “gaps” in the products to the top of their ranges instead and shuffling the z and ζ factors within a product of two ϑ -functions. Continuing, (5.20) becomes,

$$\begin{aligned}
&\frac{\vartheta_{-u+2(m+1),2u}\left(\frac{\sigma}{u}, \tau\right) - \vartheta_{u+2(m+1),2u}\left(\frac{\sigma}{u}, \tau\right)}{(\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)) \eta(\tau) \eta^{2u-3}(u\tau)} \times \\
&\quad \sum_{n=1}^u \left\{ \vartheta_{m-2m',u}\left(\frac{\nu}{u}, \tau\right)^{u-n} \vartheta_{m-2m'+u,u}\left(\frac{\nu}{u}, \tau\right)^{n-1} \times \right. \\
&\quad \left. \sum_{\{p_i\}_{i=1}^{u-n} \subset S} \prod_{i=1}^{u-n} \vartheta_{m+1+u+2p_i,u}\left(\frac{\sigma}{u}, \tau\right) \prod_{j=1}^{n-1} \vartheta_{m+1+2p_{u-n+j},u}\left(\frac{\sigma}{u}, \tau\right) \right\}, \quad (5.21)
\end{aligned}$$

where the sum $\sum_{\{p_i\}_{i=1}^{u-n} \subset S}$ is over the $\frac{(u-1)!}{(u-n)!(n-1)!}$ possible subsets $S_{(n)}$ of $(u-n)$ distinct integers p_i included in the set $S = \{1, \dots, u-1\}$. For each choice of subset $S_{(n)}$, the variables $p_{u-n+1}, \dots, p_{u-1}$ take the distinct values in $S \setminus S_{(n)}$. We see that all the information is in ϑ -functions now—there are no loose factors of q or z or ζ anymore.

The final task is to extract $\hat{sl}(2; \mathbb{C})$ characters from (5.20). The details of this work are lengthy and they have been relegated to appendix B. At this point though we remind the reader of the form of admissible $\hat{sl}(2; \mathbb{C})$ characters that we will use. From [MP90] we have that admissible $\hat{sl}(2; \mathbb{C})$ characters are,

$$\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) = \frac{\vartheta_{b_+,a}\left(\frac{\sigma}{u}, \tau\right) - \vartheta_{b_-,a}\left(\frac{\sigma}{u}, \tau\right)}{\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)} \quad (5.22)$$

where, in this case, the level k of the representation is, $k = t/u$, $(t, u) = 1$, $u \in \mathbb{N}$, $t \in \mathbb{Z}$, and, $0 \leq n \leq 2u + t - 2$ and $0 \leq n' \leq u - 1$. Also,

$$b_{\pm} \stackrel{d}{=} u(\pm(n+1) - n'(k+2)), \quad a \stackrel{d}{=} u^2(k+2). \quad (5.23)$$

The ultimate formula with the $\hat{sl}(2; \mathbb{C})$ characters explicit is the following,

$$\begin{aligned} \chi_{m,m'}^{NS,IV}(\tau, \sigma, \nu) &= \eta(\tau)^{-1} \eta^{3-2u}(u\tau) \times \sum_{n=1}^u \mathcal{F}(n; \tau, \frac{\nu}{u}) \\ &\quad \sum_{\{p_i\}_{i=1}^{u-n} \subset S} \sum_{r=0}^{u-2} \sum_{D(\mu_1, \dots, \mu_{u-n-1}; \nu_1, \dots, \nu_{n-2}; \tau)} \mathcal{G}(p_1, \dots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) \\ &\quad \sum_{\ell=0}^{u-2} \vartheta_{u(u-n)(n-1)(1-2\ell)+2(n-1)(\bar{p}_{u-n}-u\tilde{\mu}_1)-2(u-n)(\bar{p}_{n-1}-u\tilde{\nu}_1), u(u-1)(u-n)(n-1)}(\tau) \\ &\quad \sum_{\lambda=0}^u \vartheta_{u[(u-1)(4\lambda+3)+2(u-n)(1-2\ell)-4r], 2u(u-1)(u+1)}(\tau) (-1)^\epsilon \chi_{[n_\epsilon], u-m-1}^{sl(2; \mathbb{C})}(\tau, \sigma) \end{aligned} \quad (5.24)$$

We shall now give a brief idea of how to go about obtaining the last equation from (5.21). The general strategy is quite straightforward. We multiply together all the ϑ -functions with a “ ν ” argument thereby pushing a ν into just one ϑ -function and do likewise for the ϑ -functions with a “ σ ” argument. We use the ϑ -function multiplication formula,

$$\vartheta_{m,k}(\tau, \sigma) \vartheta_{m',k'}(\tau, \sigma) = \sum_{\ell=1}^{k+k'} \vartheta_{mk'-m'k+2\ell kk', kk'(k+k')}(\tau) \vartheta_{m+m'+2\ell k, k+k'}(\tau, \sigma). \quad (5.25)$$

This results in terms with just one ϑ -function with ν argument and one ϑ -function with a σ argument and many factors of ϑ -functions with just a τ argument. These latter ϑ -functions often appear in combinations of say, N in number, which can be “reduced” into one ϑ -function which has a level reduced by a factor of N^2 . Now, it always turns out that the ϑ -functions with a σ argument can be grouped in combinations which are the numerators of $\hat{sl}(2; \mathbb{C})$ admissible characters. (In general one generates more such ϑ -functions than are desired but these extra ones cannot be put in the form of a $\hat{sl}(2; \mathbb{C})$ admissible character numerator and in fact they cancel themselves out.) Now, in the denominator of (5.21) we see a factor which is the denominator of an $\hat{sl}(2; \mathbb{C})$ character and thus the complete $\hat{sl}(2; \mathbb{C})$ can appear. That the denominator of (5.21) is so, is because $\hat{sl}(2; \mathbb{C})$ is a subalgebra of $\hat{sl}(2|1; \mathbb{C})$. The procedure just described will, unless u is just 2 or 3, produce many hundreds of terms and the number of terms grows very rapidly with u .

We need to know what are the quantities \mathcal{F} , \mathcal{G} , the sets S and D and the numbers p_i , μ_i , ν_i and n_ϵ .

Given a set of integers $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ introduce the sum,

$$\bar{\alpha}_i \stackrel{d}{=} \sum_{j=i}^n (n+1-j)\alpha_j \quad (5.26)$$

and the domain,

$$D(\alpha_1, \dots, \alpha_n; r) \stackrel{d}{=} \left\{ \alpha_j \mid 0 \leq \alpha_j \leq n+1-j, j=1, \dots, n \text{ and } \tilde{\alpha}_1 = k'(n+1) + r, k' \in \mathbb{N} \right. \\ \left. \text{and } 0 \leq \tilde{\alpha}_1 \leq \frac{1}{6}n(n+1)(2n+1) \right\} \quad (5.27)$$

In particular, one has,

$$D(\mu_1, \dots, \mu_{u-n-1}; \nu_1, \dots, \nu_{n-2}; r) = \left\{ (\mu_j, \nu_{j'}) \mid 0 \leq \mu_j \leq u-n-j, \right. \\ \left. j=1, \dots, u-n-1; 0 \leq \nu_{j'} \leq n-1-j', j'=1, \dots, n-2 \right. \\ \left. \text{and } \tilde{\mu}_1 + \tilde{\nu}_1 = k'(u-1) + r, k' \in \mathbb{N} \text{ and } 0 \leq \tilde{\mu}_1 \leq \frac{1}{6}(u-n-1)(u-n)(2(u-n-1)+1) \right. \\ \left. \text{and } 0 \leq \tilde{\nu}_1 \leq \frac{1}{6}(n-2)(n-1)(2(n-2)+1) \right\}. \quad (5.28)$$

The function \mathcal{G} is given by the following product,

$$\mathcal{G}(p_1, \dots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) = \prod_{i=1}^{u-n-1} \vartheta_{2P(0; u-n-i) - 2u\tilde{\mu}_i, (u-n-i)(u-n-i+1)u}(\tau) \\ \times \prod_{j=1}^{n-2} \vartheta_{2P(u-n; n-1-j) - 2u\tilde{\nu}_j, (n-1-j)(n-j)u}(\tau), \quad (5.29)$$

where one defines,

$$P(\alpha; \beta) \stackrel{d}{=} \bar{p}_{\alpha, \beta} - \beta p_{\alpha+\beta+1}, \quad \bar{p}_{j, n} \stackrel{d}{=} \sum_{k=1}^n p_{j+k}, \quad \bar{p}_{0, n} \equiv \bar{p}_n. \quad (5.30)$$

The function \mathcal{F} is defined in terms of \mathcal{G} ,

$$\mathcal{F}(n; \tau, \frac{\nu}{u}) = \sum_{s=0}^{u-n-1} \sum_{t=0}^{n-2} \sum_{D(\rho_1, \dots, \rho_{u-n-1}; s)} \sum_{D(\sigma_1, \dots, \sigma_{n-2}; t)} \mathcal{G}(0; \tilde{\rho}; \tilde{\sigma}) \times \\ \sum_{\lambda'=0}^{u-2} \vartheta_{u[-(u-n)(n-1)(2\lambda'+1) - 2(n-1)s + 2(u-n)t], u(u-1)(u-n)(n-1)}(\tau) \times \\ \vartheta_{u[-2(u-n)\lambda' - 2s - 2t + (n-1)] + (u-1)[m-2m'], u(u-1)}(\tau, \frac{\nu}{u}). \quad (5.31)$$

Finally, the label $[n_\epsilon]$ in the $sl(2; \mathbb{C})_k$ characters entering the formula (5.24) is the residue mod $2(u+1)$ of n_ϵ defined by,

$$n_\epsilon = -1 + (1 - 2\epsilon)((u-1)(2\lambda+1) + u - 2r + (u-n)(1 - 2\ell)), \quad \epsilon = 0, 1. \quad (5.32)$$

For each choice of variables λ, r, n and ℓ , either $[n_0]$ or $[n_1]$ is in the set $S = \{1, \dots, u-1\}$, or else, $[n_0] = [n_1]$. In the latter case, there is no contribution proportional to

$\chi_{[n_0], u-m-1}^{\hat{sl}(2; \mathbb{C})}$ in (5.24), while in the former case one gets a contribution $\chi_{[n_\epsilon], u-m-1}^{\hat{sl}(2; \mathbb{C})}$ with $\epsilon = 0$ (resp. 1) according to whether $[n_0]$ (resp. $[n_1]$) is in the set S . In appendix B we will give more details of the derivation of (5.24).

The class V version of this formula is obtained upon substituting $M + M'$ for m , M for m' , z^{-1} for z and multiplying the result by (-1) . One can always move from class IV Neveu-Schwarz expressions to class V Neveu-Schwarz expressions in this way. One should not try to transform the ranges of m, m' into a ranges for M and M' though. The virtue of (5.24) is the fact that the $\hat{sl}(2; \mathbb{C})$ characters are now explicit in it. Thus the coset is partially verified. 'Partially' because the structure of the branching functions is still not too clear. In the next section we shall explain how to derive a much more compact form of the branching functions by utilizing the fact that some of the characters are singular in the limit $\sigma \rightarrow 0$.

5.4 Some Examples.

Having produced the ultimate version of the general character formula, we now illustrate it with three examples. We shall work out the branching functions of the Neveu-Schwarz characters for the three cases $u = 2, 3, 4$. The first example has already appeared in [BHT98] and the others appeared in [HT]. These examples will make manifest the complete coset structure discussed above. We have also computed the modular S matrix explicitly for the first two examples. Those matrices will be presented in the next chapter.

Whenever the level of a representation is of the form $k + 1 = 1/u$, as we have it here, the set of characters for that representation can be thought of as being partitioned into three according to their analytic properties and how those behave under spectral flow. With classes IV and V taken together there are always u^2 characters in each sector. Because of lemma 4.1, between class IV and class V there are always u regular characters in the Ramond sector and owing to the remark about the Neveu-Schwarz characters afterwards, there are always u regular characters in that sector too, when both classes are taken into account. Now each character is mapped to another by spectral flow. But the point is that the u Ramond sector regular characters are mapped onto singular characters in the Neveu-Schwarz sector and there are u singular Ramond sector characters that are mapped onto the set of u regular Neveu-Schwarz characters. Thus there must be $(u^2 - 2u)$ characters which are singular in both sectors. It is a

fact that the admissible $\hat{sl}(2; \mathbb{C})$ characters exhibit exactly the same behaviour. See [MP90]. This is good news because we want to branch $\hat{sl}(2|1; \mathbb{C})$ characters into $\hat{sl}(2; \mathbb{C})$ ones. The scheme then is this. We can take one of the $(u^2 - u)$ singular Neveu-Schwarz characters and work out which $\hat{sl}(2; \mathbb{C})$ characters it branches into using the general formula from the last section. At this stage the branching functions are multinomials in ϑ -functions multiplied by $\eta^{3-2u}(u\tau)$. Then we compute the residue of (5.24) at the pole $\sigma = 0$. Mukhi and Panda [MP90] tell us that the residue of an admissible $\hat{sl}(2; \mathbb{C})$ character is proportional to a minimal Virasoro character. But from subsection 4.4.1 we also know that the residue of a $\hat{sl}(2|1; \mathbb{C})$ character is proportional to a minimal $N = 2$ character. Ravanini and Yang [RY87] have shown how to write these characters in terms of level $(u - 2)$ $\hat{su}(2)$ string functions. Their expression is,

$$\chi_{r,s}^{NS, N=2}\left(\tau, \frac{\nu}{2}\right) = \sum_{\tilde{m}=-u+3}^{u-2} c_{r+s-1, \tilde{m}}^{(u-2)}(\tau) \vartheta_{\tilde{m}u-(r-s)(u-2), u(u-2)}\left(\frac{\tau}{2}, \frac{\nu}{2u}\right) \quad (5.33)$$

when the central charge of this $N = 2$ theory is $c = 3(1 - 2/u)$. For this Neveu-Schwarz case, r and s belong to the domain, $\{r, s \in \mathbb{Z} + \frac{1}{2} | 0 < r, s, r + s \leq u - 1\}$. The objects $c_{a,b}^{(n)}(\tau)$ are called level n string functions. They have the following properties,

$$c_{a,b}^{(n)} = c_{a,-b}^{(n)} = c_{n-a, n-b}^{(n)} = c_{a, b+2jn}^{(n)}, \quad \forall j \in \mathbb{Z}. \quad (5.34)$$

We must also have $a \equiv b \pmod{2}$. The string functions are really the branching functions of the $SU(2)/U(1)$ coset theory and may be defined as [ADT93],

$$\eta^3(\tau) c_{a,b}^{(n)}(\tau) = \sum'_{x,y} \text{sign}(x) q^{x^2(n+2) - y^2n}, \quad (5.35)$$

where x and y are subject to three conditions: (i) $-|x| < y \leq |x|$, (ii) x or $(\frac{1}{2} - x)$ is congruent to $(a+1)/(2(n+2)) \pmod{1}$ and, (iii) y or $(y + \frac{1}{2})$ is congruent to $b/2n \pmod{1}$. In fact, as Kac and Peterson showed, the RHS of (5.35) is a Hecke indefinite modular form [Hec59]. Eholzer and Gaberdiel [EG97] were able to write minimal $N = 2$ characters in terms of the Hecke form too. To return to the $\hat{sl}(2|1; \mathbb{C})$ characters, we have two versions of the residue of one of them and equating the two versions of the residue one can eliminate the unpleasant multinomial expressions alluded to above and one obtains a form for the $\hat{sl}(2|1; \mathbb{C})$ character where each term involves an $\hat{sl}(2; \mathbb{C})$ character with the correct analytic properties, a ϑ -function and an $\hat{su}(2)$ string function. This realises the coset because, upon multiplying numerator and denominator of a branching function by an η -function, the character of an $\mathcal{A}_{u(u-1)}$ rational torus model [DVVV89, PT93]

and a \mathbb{Z}_{u-1} parafermion [GQ87, ADT93] theory appear. If an $\hat{sl}(2|1; \mathbb{C})$ character is regular then clearly this construction does not work. Instead one takes the image of that character under spectral flow in the other sector and applies the steps outlined above and then flows back to the original sector. Nothing is lost during spectral flow. In this way the branched form of any character in either sector can be achieved. We give below the explicit forms of the Neveu-Schwarz characters when $u = 2, 3, 4$. When $u = 2$ we have,

$$\begin{aligned}
\chi_{0,0}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{1,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{0,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)} + \chi_{0,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{2,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)}, \\
\chi_{1,0}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{1,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{1,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)} + \chi_{0,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{-1,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)}, \\
\chi_{1,1}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{1,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{-1,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)} + \chi_{0,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{1,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)}, \\
\chi_{0,0}^{NS,V}(\sigma, \nu, \tau) &= \chi_{0,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{0,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)} + \chi_{1,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \frac{\vartheta_{2,2}(\frac{1}{2}\nu, \tau)}{\eta(\tau)}. \tag{5.36}
\end{aligned}$$

In this case the parafermion characters are unity ($\sim \eta(\tau)/\eta(\tau)$) since the only string function at level 1 is $1/\eta(\tau)$. When $u = 3$ we have,

$$\begin{aligned}
\chi_{0,0}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{0,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) + \vartheta_{6,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) \right\} \\
&\quad + \chi_{1,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{3,6}(\frac{\nu}{3}, \tau) + \vartheta_{9,6}(\frac{\nu}{3}, \tau) \right) c_{1,1}^{(2)}(\tau) \right\} \\
&\quad + \chi_{2,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) + \vartheta_{6,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) \right\}, \tag{5.37}
\end{aligned}$$

$$\begin{aligned}
\chi_{1,0}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{0,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) + \vartheta_{8,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) \right\} \\
&\quad + \chi_{1,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{5,6}(\frac{\nu}{3}, \tau) + \vartheta_{11,6}(\frac{\nu}{3}, \tau) \right) c_{1,1}^{(2)}(\tau) \right\} \\
&\quad + \chi_{2,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) + \vartheta_{8,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) \right\}, \tag{5.38}
\end{aligned}$$

$$\begin{aligned}
\chi_{1,1}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{0,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) + \vartheta_{10,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) \right\} \\
&\quad + \chi_{1,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{1,6}(\frac{\nu}{3}, \tau) + \vartheta_{7,6}(\frac{\nu}{3}, \tau) \right) c_{1,1}^{(2)}(\tau) \right\} \\
&\quad + \chi_{2,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) + \vartheta_{10,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) \right\}, \tag{5.39}
\end{aligned}$$

$$\begin{aligned}
\chi_{2,0}^{NS,IV}(\sigma, \nu, \tau) &= \chi_{0,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) + \vartheta_{10,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) \right\} \\
&\quad + \chi_{1,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{1,6}(\frac{\nu}{3}, \tau) + \vartheta_{7,6}(\frac{\nu}{3}, \tau) \right) c_{1,1}^{(2)}(\tau) \right\} \\
&\quad + \chi_{2,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}(\frac{\nu}{3}, \tau) c_{2,2}^{(2)}(\tau) + \vartheta_{10,6}(\frac{\nu}{3}, \tau) c_{0,2}^{(2)}(\tau) \right\}, \tag{5.40}
\end{aligned}$$

$$\begin{aligned}
\chi_{2,1}^{NS,IV}(\sigma, \nu, \tau) = & \chi_{0,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) + \vartheta_{6,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) \right\} \\
& + \chi_{1,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{3,6}\left(\frac{\nu}{3}, \tau\right) + \vartheta_{9,6}\left(\frac{\nu}{3}, \tau\right) \right) c_{1,1}^{(2)}(\tau) \right\} \\
& + \chi_{2,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) + \vartheta_{6,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) \right\}, \quad (5.41)
\end{aligned}$$

$$\begin{aligned}
\chi_{2,2}^{NS,IV}(\sigma, \nu, \tau) = & \chi_{0,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) + \vartheta_{8,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) \right\} \\
& + \chi_{1,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{5,6}\left(\frac{\nu}{3}, \tau\right) + \vartheta_{11,6}\left(\frac{\nu}{3}, \tau\right) \right) c_{1,1}^{(2)}(\tau) \right\} \\
& + \chi_{2,0}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) + \vartheta_{8,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) \right\}, \quad (5.42)
\end{aligned}$$

$$\begin{aligned}
\chi_{0,0}^{NS,V}(\sigma, \nu, \tau) = & \chi_{0,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) + \vartheta_{6,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) \right\} \\
& + \chi_{1,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{3,6}\left(\frac{\nu}{3}, \tau\right) + \vartheta_{9,6}\left(\frac{\nu}{3}, \tau\right) \right) c_{1,1}^{(2)}(\tau) \right\} \\
& + \chi_{2,1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{0,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) + \vartheta_{6,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) \right\}, \quad (5.43)
\end{aligned}$$

$$\begin{aligned}
\chi_{0,1}^{NS,V}(\sigma, \nu, \tau) = & \chi_{0,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) + \vartheta_{8,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) \right\} \\
& + \chi_{1,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{5,6}\left(\frac{\nu}{3}, \tau\right) + \vartheta_{11,6}\left(\frac{\nu}{3}, \tau\right) \right) c_{1,1}^{(2)}(\tau) \right\} \\
& + \chi_{2,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{2,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) + \vartheta_{8,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) \right\}, \quad (5.44)
\end{aligned}$$

$$\begin{aligned}
\chi_{1,0}^{NS,V}(\sigma, \nu, \tau) = & \chi_{0,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) + \vartheta_{10,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) \right\} \\
& + \chi_{1,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{1,6}\left(\frac{\nu}{3}, \tau\right) + \vartheta_{7,6}\left(\frac{\nu}{3}, \tau\right) \right) c_{1,1}^{(2)}(\tau) \right\} \\
& + \chi_{2,2}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \vartheta_{4,6}\left(\frac{\nu}{3}, \tau\right) c_{2,2}^{(2)}(\tau) + \vartheta_{10,6}\left(\frac{\nu}{3}, \tau\right) c_{0,2}^{(2)}(\tau) \right\}. \quad (5.45)
\end{aligned}$$

For $u = 4$ we shall give just one character to give an indication of the shape to expect here. There are sixteen characters in total between class IV and class V when $u = 4$ and the branching functions of all of them have been worked out.

$$\begin{aligned}
\chi_{0,0}^{NS,IV}(\sigma, \nu, \tau) = & \chi_{0,3}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{4,12}\left(\frac{\nu}{4}, \tau\right) + \vartheta_{20,12}\left(\frac{\nu}{4}, \tau\right) \right) c_{3,1}^{(3)}(\tau) + \vartheta_{12,12}\left(\frac{\nu}{4}, \tau\right) c_{0,0}^{(3)}(\tau) \right\} \\
& + \chi_{1,3}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{8,12}\left(\frac{\nu}{4}, \tau\right) + \vartheta_{16,12}\left(\frac{\nu}{4}, \tau\right) \right) c_{1,1}^{(3)}(\tau) + \vartheta_{0,12}\left(\frac{\nu}{4}, \tau\right) c_{2,0}^{(3)}(\tau) \right\} \\
& + \chi_{2,3}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{4,12}\left(\frac{\nu}{4}, \tau\right) + \vartheta_{20,12}\left(\frac{\nu}{4}, \tau\right) \right) c_{1,1}^{(3)}(\tau) + \vartheta_{12,12}\left(\frac{\nu}{4}, \tau\right) c_{2,0}^{(3)}(\tau) \right\} \\
& + \chi_{3,3}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \left\{ \left(\vartheta_{8,12}\left(\frac{\nu}{4}, \tau\right) + \vartheta_{16,12}\left(\frac{\nu}{4}, \tau\right) \right) c_{3,1}^{(3)}(\tau) + \vartheta_{0,12}\left(\frac{\nu}{4}, \tau\right) c_{0,0}^{(3)}(\tau) \right\}. \quad (5.46)
\end{aligned}$$

Based on the branching functions computed for the three cases $u = 2, 3, 4$ we can make the following conjectures for the branching functions for general u .

$$\chi_{m,m'}^{IV,NS}(\sigma, \nu, \tau) = \sum_{i=0}^{u-1} \chi_{i,u-m-1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \times \left\{ \sum_{j=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(i+1)+2ui(\frac{u}{2}-\lfloor\frac{u}{2}\rfloor)-2uj,u(u-1)}\left(\frac{\nu}{u}, \tau\right) c_{i,i(u-1)+2i(\frac{u}{2}-\lfloor\frac{u}{2}\rfloor)-2j}^{(u-1)}(\tau) \right\} \quad (5.47)$$

and,

$$\chi_{M,M'}^{V,NS}(\sigma, \nu, \tau) = \sum_{i=0}^{u-1} \chi_{i,M+M'+1}^{\hat{sl}(2;\mathbb{C})}(\sigma, \tau) \times \left\{ \sum_{j=0}^{u-2} \vartheta_{(u-1)(M'-M)+u(u-1)i+2ui(\frac{u}{2}-\lfloor\frac{u}{2}\rfloor)-2uj,u(u-1)}\left(\frac{\nu}{u}, \tau\right) c_{i,i(u-1)+2i(\frac{u}{2}-\lfloor\frac{u}{2}\rfloor)-2j}^{(u-1)}(\tau) \right\}. \quad (5.48)$$

for character sum rules to realise the coset. The integer part of $u/2$ is denoted $\lfloor u/2 \rfloor$ here.

Our character formulae of the last chapter have now come a long way. We have seen how, by using the $SL(2|1)/SL(2)$ coset, one can factor our original expressions into known characters with known modular transformations. In the next chapter the general T transformation will be noted and the S transformation will be worked out for the cases $u = 2, 3$. In the process of working out the branching functions of all the characters displayed above, one comes across some interesting relations between the string functions. We shall explain how they arise and exhibit those that were found in the examples we have just worked out.

Chapter 6

Some Spin-off Results

6.1 Introduction.

In this chapter we present the modular transformations of some of the characters computed in the chapter 5 and another result which is a consequence of the work we did there on branching functions. The latter is a set of identities between the $\hat{su}(2)$ string functions. The identities express the $\hat{su}(2)$ string functions at level n in terms of those at level $n - 1$ where $n \geq 2$. We can also recover the well known expression for the unique level-1 string function (the reciprocal of the Dedekind η -function.) For small values of n we can recover known results but as n increases we obtain equations of which I am not aware.

6.2 The Modular Transformations.

In this section we will calculate the modular T transformation for all class IV and class V characters and supercharacters and the S for just two Neveu-Schwarz representations. Computing the general S is feasible given the results of the previous chapter but in practice is a difficult task.

6.2.1 The T Transformation.

To work this out we need only use the forms of the characters and supercharacters from chapter 4. It is quite plain to see that shifting $\tau \rightarrow \tau + 1$ gives the following results in

each class (classes do not mix.)

$$\begin{aligned}
 \chi^R(\sigma, \nu, \tau + 1) &\longrightarrow e^{2\pi i h^R} \chi^R(\sigma, \nu, \tau) \\
 S\chi^R(\sigma, \nu, \tau + 1) &\longrightarrow e^{2\pi i h^R} S\chi^R(\sigma, \nu, \tau) \\
 \chi^{NS}(\sigma, \nu, \tau + 1) &\longrightarrow e^{2\pi i h^{NS}} S\chi^{NS}(\sigma, \nu, \tau) \\
 S\chi^{NS}(\sigma, \nu, \tau + 1) &\longrightarrow e^{2\pi i h^{NS}} \chi^{NS}(\sigma, \nu, \tau).
 \end{aligned} \tag{6.1}$$

Thus we have just the same behaviour as with the $\widehat{osp}(1, 2; \mathbb{C})$ characters and supercharacters—the Ramond characters and supercharacters are invariant whereas the Neveu-Schwarz characters and supercharacters mix with one another. This result is true for *all* admissible levels k , $k + 1 = p/u$.

6.2.2 The S Transformation.

We shall restrict ourselves now to studying just the Neveu-Schwarz characters with level k given by $k + 1 = 1/u$ *i.e.*, just the objects that we studied in the last chapter. We confine ourselves to two particular cases of (5.47) and (5.48). Namely those where $u = 2$ and $u = 3$. In each case we find that the S-matrix is symmetric, unitary and its fourth power is the identity matrix. These are properties that such a matrix must have. The case of $u = 2$ was presented in [BHT98].

The easiest way to present the results is to simply give the S-matrix explicitly. Consider first the case $u = 2$. We should fix a labelling for the four characters. We shall label them as,

$$\begin{aligned}
 \chi_{0,0}^{NS,IV} &= \chi_1 & \chi_{1,0}^{NS,IV} &= \chi_2 \\
 \chi_{1,1}^{NS,IV} &= \chi_3 & \chi_{0,0}^{NS,V} &= \chi_4.
 \end{aligned} \tag{6.2}$$

We shall also need to know how the $\widehat{sl}(2; \mathbb{C})$ characters transform under S. We use,

$$\chi_{n,n'}^{\widehat{sl}(2;\mathbb{C})} \left(\frac{\sigma}{\tau}, -\frac{1}{\tau} \right) = e^{\pi i(u-p)\sigma^2/u\tau} \sum_{N,N'} S_{N,N'}^{n,n'} \chi_{N,N'}^{\widehat{sl}(2;\mathbb{C})}(\sigma, \tau), \tag{6.3}$$

where the matrix $S_{N,N'}^{n,n'}$ is given by [MP90],

$$S_{N,N'}^{n,n'} = \sqrt{\frac{2}{u(p+u)}} (-1)^{N'(n+1)+n'(N+1)} e^{-i\pi n' N'(p+u)/u} \sin \left(\frac{u\pi(n+1)(N+1)}{p+u} \right), \tag{6.4}$$

when the level k of the $\widehat{sl}(2; \mathbb{C})$ representation is $k + 1 = p/u$ (we will have $p = 1$).

When we look at the example with $u = 3$ below we shall need to know how the string

functions transform too. Gepner and Qiu [GQ87] give a neat formula for this (which is of course based on the original one by Kac and Peterson [KP84]). Gepner and Qiu say that for string functions at level K ,

$$c_{a,b}^{(K)}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{(-i\tau)K(K+2)}} \sum_{\substack{a'=0 \\ a' \equiv b' \pmod{2}}}^K \sum_{b'=-K+1}^K s(a,b,a',b') c_{a',b'}^{(K)}(\tau), \quad (6.5)$$

where the matrix $s(a,b,a',b')$ is,

$$s(a,b,a',b') = e^{\frac{i\pi bb'}{K}} \sin\left(\frac{\pi(a+1)(a'+1)}{K+2}\right). \quad (6.6)$$

Then we have,

$$\chi_{\alpha}\left(\frac{\sigma}{\tau}, \frac{\nu}{\tau}, \frac{-1}{\tau}\right) = e^{\pi i \sigma^2 / 2\tau} S_{\alpha\beta}^{(2)} \chi_{\beta}(\sigma, \nu, \tau) \quad (6.7)$$

where we sum over β and where,

$$S_{\alpha\beta}^{(2)} = \frac{1}{2} \begin{pmatrix} i & 1 & 1 & i \\ 1 & -i & i & -1 \\ 1 & i & -i & -1 \\ i & -1 & -1 & i \end{pmatrix}. \quad (6.8)$$

Now consider the case $u = 3$. When we label the characters as,

$$\begin{aligned} \chi_{0,0}^{NS,IV} &= \chi_1 & \chi_{1,0}^{NS,IV} &= \chi_2 & \chi_{1,1}^{NS,IV} &= \chi_3 \\ \chi_{2,0}^{NS,IV} &= \chi_4 & \chi_{2,1}^{NS,IV} &= \chi_5 & \chi_{2,2}^{NS,IV} &= \chi_6 \\ \chi_{0,0}^{NS,V} &= \chi_7 & \chi_{0,1}^{NS,V} &= \chi_8 & \chi_{1,0}^{NS,V} &= \chi_9, \end{aligned} \quad (6.9)$$

our result is that,

$$\chi_{\alpha}\left(\frac{\sigma}{\tau}, \frac{\nu}{\tau}, \frac{-1}{\tau}\right) = e^{2\pi i \sigma^2 / 3\tau} S_{\alpha\beta}^{(3)} \chi_{\beta}(\sigma, \nu, \tau), \quad (6.10)$$

where we sum over β and,

$$S_{\alpha\beta}^{(3)} = \frac{1}{3} \begin{pmatrix} e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & 1 & 1 & 1 & e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} \\ e^{\pi i/3} & 1 & e^{4\pi i/3} & e^{-\pi i/3} & -1 & e^{\pi i/3} & e^{2\pi i/3} & e^{-\pi i/3} & -1 \\ e^{\pi i/3} & e^{4\pi i/3} & 1 & e^{\pi i/3} & -1 & e^{-\pi i/3} & e^{2\pi i/3} & -1 & e^{-\pi i/3} \\ 1 & e^{-\pi i/3} & e^{\pi i/3} & e^{4\pi i/3} & 1 & e^{2\pi i/3} & -1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & e^{\pi i/3} & e^{-\pi i/3} & e^{2\pi i/3} & 1 & e^{4\pi i/3} & -1 & e^{4\pi i/3} & e^{2\pi i/3} \\ e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & -1 & -1 & -1 & e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} \\ e^{2\pi i/3} & e^{-\pi i/3} & -1 & e^{2\pi i/3} & 1 & e^{4\pi i/3} & e^{\pi i/3} & 1 & e^{4\pi i/3} \\ e^{2\pi i/3} & -1 & e^{-\pi i/3} & e^{4\pi i/3} & 1 & e^{2\pi i/3} & e^{\pi i/3} & e^{4\pi i/3} & 1 \end{pmatrix}. \quad (6.11)$$

6.3 Identities Between String Functions.

This section is based on the appendix B of [HT]. We shall present two sets of identities between $\hat{s}u(2)$ string functions. The identities are obtained when one calculates the residue of a singular $\hat{sl}(2|1; \mathbb{C})$ character in two different ways. The first way is to use the general branching formula (5.47) and the fact that $\hat{sl}(2; \mathbb{C})$ characters give rise to minimal Virasoro characters as residues in the limit as $\sigma \rightarrow 0$. The other is to use the $N = 2$ character form of the residue from (4.39). However, one must use the version of those characters which includes $\hat{s}u(2)$ string functions. This version was found by Ravanini and Yang [RY87]. They have,

$$\chi_{r,s}^{NS, N=2}(\tau, \frac{\nu}{2}) = \sum_{\tilde{m}=-u+3}^{u-2} c_{r+s-1, \tilde{m}}^{(u-2)}(\tau) \vartheta_{\tilde{m}u-(r-s)(u-2), u(u-2)}(\frac{\tau}{2}, \frac{\nu}{2u}) \quad (6.12)$$

when the central charge of this $N = 2$ theory is $c = 3(1 - 2/u)$. For the Neveu-Schwarz case, r and s belong to the domain, $\{r, s \in \mathbb{Z} + \frac{1}{2} | 0 < r, s, r + s \leq u - 1\}$. With this



form for the superconformal characters, we can rewrite (4.39) as follows.

$$\begin{aligned}
\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{m,m'}^{NS,IV}(\sigma, \nu, \tau) &= \frac{\vartheta_{0,2}(\tau, \frac{\nu}{2}) + \vartheta_{2,2}(\tau, \frac{\nu}{2})}{\eta^3(\tau)} \chi_{r,s}^{NS,N=2}(\tau, \frac{\nu}{2}) \\
&= \frac{\vartheta_{0,2}(\tau, \frac{\nu}{2}) + \vartheta_{2,2}(\tau, \frac{\nu}{2})}{\eta^3(\tau)} \sum_{\tilde{m}=-u+3}^{u-2} \tilde{c}_{r+s-1, \tilde{m}}^{(u-2)}(\tau) \vartheta_{\tilde{m}u-(r-s)(u-2), u(u-2)}(\frac{\tau}{2}, \frac{\nu}{2u}) \\
&= \frac{1}{\eta^3(\tau)} \sum_{\tilde{m}=-u+3}^{u-2} \tilde{c}_{r+s-1, \tilde{m}}^{(u-2)}(\tau) \left\{ \sum_{i=0}^{u-2} \sum_{j=0}^1 \vartheta_{\frac{1}{2}(\tilde{m}u-(2m'-m)(u-2))-(u-2)(2i-j), (u-1)(u-2)}(\tau) \right. \\
&\quad \left. \times \vartheta_{\frac{1}{2}(\tilde{m}u-(2m'-m)(u-2))+u(2i-j), u(u-1)}(\tau, \frac{\nu}{u}) \right\}, \tag{6.13}
\end{aligned}$$

where, $m = r + s - 1$ and $m' = r - 1/2$ and $m \neq u - 1$ since class IV Neveu-Schwarz characters are not singular when $m = u - 1$. We have used the ϑ -function multiplication formula which is given in appendix B. Taking the limit as $\sigma \rightarrow 0$ of (5.47) we can write,

$$\begin{aligned}
&\sum_{\tilde{m}=-u+3}^{u-2} \tilde{c}_{r+s-1, \tilde{m}}^{(u-2)}(\tau) \left\{ \sum_{i=0}^{u-2} \sum_{j=0}^1 \vartheta_{\frac{1}{2}(\tilde{m}u-(2m'-m)(u-2))-(u-2)(2i-j), (u-1)(u-2)}(\tau) \times \right. \\
&\quad \left. \vartheta_{\frac{1}{2}(\tilde{m}u-(2m'-m)(u-2))+u(2i-j), u(u-1)}(\tau, \frac{\nu}{u}) \right\} \tag{6.14} \\
&= \eta(\tau) \sum_{i=0}^{u-1} \chi_{u-m-1, i+1}^{\text{Vir}(u)}(\tau) \times \\
&\quad \sum_{j=0}^{u-2} \tilde{c}_{i, i(u-1)+2i(\frac{u}{2}-\lfloor \frac{u}{2} \rfloor)-2j}^{(u-1)}(\tau) \vartheta_{(u-1)(m-2m')+u(u-1)(i+1)+2iu(\frac{u}{2}-\lfloor \frac{u}{2} \rfloor)-2uj, u(u-1)}(\tau, \frac{\nu}{u}),
\end{aligned}$$

where $\lfloor \frac{u}{2} \rfloor$ denotes the integer part of $\frac{u}{2}$. For reference, we define the minimal Virasoro “level u ” ($u \geq 2$) characters to be,

$$\chi_{r,s}^{\text{Vir}(u)}(\tau) \stackrel{\text{d}}{=} \eta^{-1}(\tau) \left[\vartheta_{r(u+1)-su, u(u+1)}(\tau) - \vartheta_{r(u+1)+su, u(u+1)}(\tau) \right], \tag{6.15}$$

where the integers r and s are in the ranges, $r = 1, 2, \dots, u - 1$ and $s = 1, 2, \dots, r$ and the central charge of such a representation is in the FQS sequence ([FQS84]), $c = 1 - 6/u(u + 1)$. Note that $\chi_{r,s}^{\text{Vir}(u)} = \chi_{u-r, u+1-s}^{\text{Vir}(u)}$. Also, we have used the following equation for the residue of the pole at $\sigma = 0$ of the admissible $\hat{sl}(2; \mathbb{C})$ characters,

$$\lim_{\sigma \rightarrow 0} 2\pi i \sigma \chi_{n,n'}^{\hat{sl}(2; \mathbb{C})}(\sigma, \tau) = \eta^{-2}(\tau) \chi_{n', n+1}^{\text{Vir}(u)}(\tau), \tag{6.16}$$

where $n' > 0$. The integrable characters correspond to $n' = 0$ and they are all regular in the limit as Mukhi and Panda [MP90] discovered. The way to proceed now to obtain

identities is to equate coefficients of the ϑ -functions at level $u(u-1)$. We can do this since they are linearly independent functions. This gives an equation between string functions at level $(u-1)$ in terms of string functions at level $(u-2)$. We see this scheme potentially providing an iterative method of computing the string functions. For low values of u we recover already well known expressions for the string functions for, upon choosing $u=2$, $m=m'=0$ we get the unique level 1 string function $c_{0,0}^{(1)}(\tau) = 1/\eta(\tau)$. In fact we may restrict our attention to the $m=m'=0$ character all the time. For small values of u at least, all of the independent string functions of a given level appear in this character. A simple proof that this is the case for all u would put the potential iterative scheme on a more sound footing. Such a proof is lacking at this stage. However, we have worked out three cases. Namely, the level 1 string function as just mentioned, the three level 2 string functions in terms of the level 1 function and the four level 3 string functions in terms of those at level 2. These are easily obtained by setting $u=2, 3, 4$ in turn. We obtain then for $u=2$,

$$c_{0,0}^{(1)}(\tau) = \frac{1}{\eta(\tau)}, \quad (6.17)$$

for $u=3$,

$$c_{0,0}^{(2)}(\tau) = c_{0,0}^{(1)}(\tau) \frac{\vartheta_{2,2}(\tau) \chi_{2,1}^{\text{Vir}(3)}(\tau) - \vartheta_{0,2}(\tau) \chi_{1,1}^{\text{Vir}(3)}(\tau)}{\left(\chi_{2,1}^{\text{Vir}(3)^2}(\tau) - \chi_{1,1}^{\text{Vir}(3)^2}(\tau) \right) \eta(\tau)} = c_{0,0}^{(1)}(\tau) \chi_{1,1}^{\text{Vir}(3)}(\tau) \quad (6.18)$$

$$c_{0,2}^{(2)}(\tau) = c_{0,0}^{(1)}(\tau) \frac{\vartheta_{0,2}(\tau) \chi_{2,1}^{\text{Vir}(3)}(\tau) - \vartheta_{2,2}(\tau) \chi_{1,1}^{\text{Vir}(3)}(\tau)}{\left(\chi_{2,1}^{\text{Vir}(3)^2}(\tau) - \chi_{1,1}^{\text{Vir}(3)^2}(\tau) \right) \eta(\tau)} = c_{0,0}^{(1)}(\tau) \chi_{2,1}^{\text{Vir}(3)}(\tau) \quad (6.19)$$

$$c_{1,1}^{(2)}(\tau) = c_{0,0}^{(1)}(\tau) \frac{\vartheta_{1,2}(\tau)}{\chi_{2,2}^{\text{Vir}(3)}(\tau) \eta(\tau)} = c_{0,0}^{(1)}(\tau) \chi_{2,2}^{\text{Vir}(3)}(\tau). \quad (6.20)$$

The second equation on each line is a consequence of the fact that \mathbb{Z}_2 parafermion characters are identical to Ising model characters. For $u=4$,

$$f(\tau) c_{0,0}^{(3)}(\tau) = \left[\vartheta_{6,6}(\tau) c_{0,2}^{(2)}(\tau) + \vartheta_{0,6}(\tau) c_{0,0}^{(2)}(\tau) \right] \chi_{3,3}^{\text{Vir}(4)}(\tau) - \left[\vartheta_{6,6}(\tau) c_{0,0}^{(2)}(\tau) + \vartheta_{0,6}(\tau) c_{0,2}^{(2)}(\tau) \right] \chi_{3,2}^{\text{Vir}(4)}(\tau), \quad (6.21)$$

$$f(\tau) c_{0,2}^{(3)}(\tau) = \left[\vartheta_{2,6}(\tau) c_{0,2}^{(2)}(\tau) + \vartheta_{4,6}(\tau) c_{0,0}^{(2)}(\tau) \right] \chi_{3,3}^{\text{Vir}(4)}(\tau) - \left[\vartheta_{2,6}(\tau) c_{0,0}^{(2)}(\tau) + \vartheta_{4,6}(\tau) c_{0,2}^{(2)}(\tau) \right] \chi_{3,2}^{\text{Vir}(4)}(\tau), \quad (6.22)$$

$$f(\tau)c_{1,1}^{(3)}(\tau) = \left[\vartheta_{2,6}(\tau)c_{0,0}^{(2)}(\tau) + \vartheta_{4,6}(\tau)c_{0,2}^{(2)}(\tau) \right] \chi_{1,1}^{\text{Vir}(4)}(\tau) - \left[\vartheta_{2,6}(\tau)c_{0,2}^{(2)}(\tau) + \vartheta_{4,6}(\tau)c_{0,0}^{(2)}(\tau) \right] \chi_{3,1}^{\text{Vir}(4)}(\tau), \quad (6.23)$$

and finally,

$$f(\tau)c_{2,0}^{(3)}(\tau) = \left[\vartheta_{6,6}(\tau)c_{0,0}^{(2)}(\tau) + \vartheta_{0,6}(\tau)c_{0,2}^{(2)}(\tau) \right] \chi_{1,1}^{\text{Vir}(4)}(\tau) - \left[\vartheta_{6,6}(\tau)c_{0,2}^{(2)}(\tau) + \vartheta_{0,6}(\tau)c_{0,0}^{(2)}(\tau) \right] \chi_{3,1}^{\text{Vir}(4)}(\tau), \quad (6.24)$$

where $f(\tau) \stackrel{d}{=} \chi_{2,2}^{\text{Vir}(3)}(\tau)\eta(\tau)$. There are just four level-3 string functions and so we have new expressions for each of them. Since the level-3 string functions are proportional to level-5 Virasoro characters (\mathbb{Z}_3 parafermions \sim Tri-critical 3-states Potts model) we see that the last set of four equations are somewhat similar (though a little more complicated) to relations between the minimal Virasoro characters found a few years ago by Taormina [Tao94, TW].

Chapter 7

Conclusions

In this thesis character formulae for irreducible admissible representations of the affine superalgebras $\widehat{osp}(1, 2; \mathbb{C})$ and $\widehat{sl}(2|1; \mathbb{C})$ have been presented. We started by studying $\widehat{osp}(1, 2; \mathbb{C})$. By making use of data provided by the Kac determinant we explicitly derived the embedding diagram for the representations we were interested in. This diagram has just the same structure as that for admissible representations of $\widehat{sl}(2; \mathbb{C})$ for example or indeed, the Virasoro algebra. It was shown that the characters are easily written in terms of certain generalised ϑ -functions. The formula that we obtained here (equation (2.64)) was noted to be the same as that obtained by other authors [ERSdS]. The superficial difference in the “prefactor” amounts to nothing when one rewrites each as a product. Each form has its advantage. That of Ennes *et. al.* is advantageous to the extent that the character of the trivial representation (*i.e.*, highest weight zero) is immediately seen to be unity. On the other hand, the form of prefactor given in this work permits the modular S transformation to be more readily worked out. Having computed the Ramond sector characters, one can use the spectral flow transformation to easily get the Neveu-Schwarz sector characters. The Ramond and Neveu-Schwarz supercharacters were obtained too by doing another shift of variables on the corresponding characters. The last piece of work in chapter 2 is an explicit computation of the modular transformations (both S and T) of the Neveu-Schwarz characters. The qualitative results that were obtained *i.e.*, under S, Neveu-Schwarz characters transform to themselves and under T they transform to Neveu-Schwarz supercharacters, were entirely as expected. Although the transformations for the other types of “character” were not explicitly worked out, we felt sufficiently encouraged to conjecture table 2.2. This table is based on the different possible free fermion spin structures on a torus *i.e.*, the four different combinations of periodic and antiperiodic boundary conditions in each of two directions for a fermionic field defined on a torus. Periodic boundary conditions in both directions

corresponds to the super-Ramond sector, periodic in the “spatial” direction but antiperiodic in the “temporal” direction corresponds to the Ramond sector, antiperiodic and periodic in spatial and temporal directions respectively is the super-Neveu-Schwarz sector and antiperiodic in both directions is the Neveu-Schwarz sector.

In chapter 3 we turned to the other superalgebra we study, namely $\hat{sl}(2|1; \mathbb{C})$. The classification of admissible representations of the affine superalgebra, as first presented in [BT97] was explained. The fact that a certain possibility was not mentioned in the last cited paper leads to problems. These are fully remedied by including in the conditions that define class IV representations the possibility that the parameters m and m' may be negative (which is in fact provided by the Kac determinant for us). This possibility came to be known as class V. With negative parameters one has a virtually parallel theory to that for class IV. We derived the quantum numbers of all the singular vectors of class V and outlined how one justifies (section 3.4) that its embedding diagram (figure 4.3) is the same as that for class IV. Again, we see the multiplicity-two singular vectors in the uncharged sector, mirroring exactly the discovery of Dörrzapf [Dör95] for the $N = 2$ superconformal algebra. Indeed, when one comes to subtract off the submodules generated by the singular vectors so as to compute the irreducible characters, the fact that multiplicity-two singular vectors exist serves to make life easier. We ought to repeat again here that all along in classes I, IV and V, we have assumed that no subsingular vectors exist in these representations.

Chapter 4 contains the most important formulae in this thesis—the character formulae for admissible representations of class I, class IV and class V in both the Ramond and Neveu-Schwarz sectors. These were presented in equations (4.9), (4.15), (4.25), (4.26), (4.27) and (4.29). We also write down the supercharacter formulae in each sector. These were given in equations (4.30), (4.31), (4.32) and (4.33). The study of these characters fills the rest of the thesis. We concentrated exclusively on classes IV and V because these were the ones which stood a chance of yielding characters which carried a representation of the modular group and therefore would be useful in model building in physics. One can see immediately that the class I characters (equations (4.9) and (4.15)) will not span a space invariant under modular transformations for they do not have zero modular weight (*i.e.*, a factor of $(-i\tau)^{-1/2}$ remains after doing the S transform for example.) Therefore we left class I behind. We noted in section 4.3 that our character formula was identical to one produced by Kac and Wakimoto [KW] for the

vacuum integrable character when expanded by computer. After having seen the results of Mukhi and Panda [MP90], one might well ask whether or not any of the $\hat{sl}(2|1; \mathbb{C})$ characters are singular and if so which ones and what is the residue? In section 4.4 we answer all of these questions. In that section we have a corollary which states that characters of integrable representations of $\hat{sl}(2|1; \mathbb{C})$ are all nonsingular in the limit $\sigma \rightarrow 0$. This is completely in accord with Mukhi and Panda's results. Whenever a character is singular though, it turns out that the residue is proportional to characters of minimal $N = 2$ superconformal representations. This result fits in nicely with the known results for $\hat{sl}(2; \mathbb{C})$ and $\widehat{osp}(1, 2; \mathbb{C})$ characters; singular $\hat{sl}(2; \mathbb{C})$ characters have a residue proportional to minimal Virasoro characters and singular $\widehat{osp}(1, 2; \mathbb{C})$ characters have a residue proportional to minimal $N = 1$ superconformal characters. These results serve to further highlight the links between these (super)conformal algebras and (super) affine algebras¹ that is thought to exist through quantum Hamiltonian reduction. See for example Kimura's paper on this [Kim92]. Also, both Semikhatov [Sem96] and Wakimoto [Wak] have constructions of the $N = 2$ superconformal algebra from $\hat{sl}(2|1; \mathbb{C})$ thereby further underlining the connections between the two. There is another link between a superconformal algebra and $\hat{sl}(2|1; \mathbb{C})$. In section 4.5 we showed that characters of integrable representation of $\hat{sl}(2|1; \mathbb{C})$ were identical to $N = 4$ superconformal characters as proposed by Eguchi and Taormina in [ET88a]. This holds for both class I characters and class IV/V characters in both Ramond and Neveu-Schwarz sectors. It is interesting that these $N = 4$ characters were not deduced with the aid of an embedding diagram. However, as noted in [BHT98], there are similarities between the Kac determinant of $N = 4$ (as conjectured by Kent and Riggs [KR87]) and that for $\hat{sl}(2|1; \mathbb{C})$.

In the following chapter we began to work to write the class IV and class V character formulae in a form more amenable to modular transformations. In this chapter we restricted ourselves to representations of level k of the form $k + 1 = 1/u$. (We suspect that the analysis of the rest of this chapter does not follow through if we have $k + 1 = p/u, p > 1$.) This analysis requires quite a bit of work but in the end one obtains a complete understanding of the $SL(2|1)/SL(2)$ branching functions. The form of the branching functions as we obtain them lends weight to the interpretation of the coset

¹Note that a super affine algebra is not the same as an affine superalgebra. The former has an equal number of bosonic and fermionic generators. If this symmetry does not hold then one must add extra fields so as to make it hold. This is what Bershadsky and Ooguri did for the $osp(1, 2; \mathbb{C})$ superalgebra in [BO89]

structure which was suggested by a coincidence of central charges (section 5.2.) The examples we worked out in detail provide a foundation upon which to make a conjecture for the structure of the decomposition of $\hat{sl}(2|1; \mathbb{C})$ characters into $\hat{sl}(2; \mathbb{C})$, parafermion and rational torus model characters.

In principle the modular S transformation could be computed from the conjectured formulae. In fact we have worked out two examples ($u = 2, 3$) in chapter 6. We obtained modular matrices $S_{\alpha\beta}^{(2)}$ (equation (6.8)) and $S_{\alpha\beta}^{(3)}$ (equation (6.11)) which are symmetric and unitary and when raised to the fourth power give the identity matrix. It would be interesting to use these matrices to compute fusion rules using Verlinde's formula [Ver88].

Section 6.3 introduced some new identities between the $\hat{su}(2)$ string functions. If the conjectured general branching formulae (5.47) and (5.48) are correct then we have, potentially, an iterative procedure for computing the string functions.

The character formulae that I computed rely heavily on the embedding diagrams. It is not easy to derive these and the presence of nilpotent fermionic operators in $\hat{sl}(2|1; \mathbb{C})$ (in contrast to $\widehat{osp}(1, 2; \mathbb{C})$) makes life difficult. Despite these difficulties, the character formulae that one obtains have many nice properties and in many ways behave in an expected manner. A complete solution to the modular transformations of the class IV/V characters would have finished of this work nicely. Unfortunately, time did not permit this to be done.

The affine superalgebras are not an especially widely studied class of algebras. To the best of my knowledge, before the new results in this thesis were produced, only one other affine superalgebra had had its representations studied in any detail. That was $\widehat{osp}(1, 2; \mathbb{C})$. Therefore what we have done here significantly expands the body of work done on this class of superalgebra. Also, that we considered the admissible representations, rather than the less general integrable ones, makes our formulae all the more valuable.

One further result of the work done in this thesis has already been worked out—a new free field representation of $\hat{sl}(2|1; \mathbb{C})$ based on the discovery of the parafermionic component of the $SL(2|1)/SL(2)$ coset. That work has been submitted for publication to *Physics Letters B* in the form of paper [BHT].

The representations of classes II and III as presented in [BT97] have been completely neglected in this thesis. This is because it was feared that there might be sub-singular

vectors in these classes. Also, a computer-based check or a real proof that there are no sub-singular vectors in the classes we have studied here would close this chapter of the story. A thorough study of these issues is required so that we might have $\hat{sl}(2|1; \mathbb{C})$ under control.

Appendix A

Residues of Poles

In this appendix we show how to go about proving that the sum and product forms of the class IV Neveu-Schwarz character, exhibited in equations (4.27) and (5.2), have the same residues at their poles. Two functions which have the same poles and the same residues at those poles are equal up to a holomorphic function. In chapter 4 it was explained that the sum and product had been expanded on a computer for many different representations and in each case they were the same so that it looks as if the holomorphic function might be zero.

As we said above, we shall study the class IV representations here. It is straightforward to substitute different quantities to get the corresponding class V expressions. The sum and product under scrutiny are,

$$\sum_{a \in \mathbb{Z}} q^{a^2 p u + a p(m+1)} z^{ap} \frac{1 - q^{2au+m+1} z}{(1 + q^{au+m'+1/2} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{au+m-m'+1/2} z^{\frac{1}{2}} \zeta^{\frac{1}{2}})} \quad (\text{A.1})$$

and,

$$\prod_{n \in \mathbb{N}} [(1 - q^{un})^2 (1 - z q^{u(n-1)+m+1}) (1 - z^{-1} q^{un-m-1}) / h_n(m, m'; \sigma, \nu, \tau)] \quad (\text{A.2})$$

where,

$$h_n(m, m'; \sigma, \nu, \tau) = (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{u(n-1)+m-m'+\frac{1}{2}}) (1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{u(n-1)+m'+\frac{1}{2}}) \\ (1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{un-m'-\frac{1}{2}}) (1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{un+m'-m-\frac{1}{2}}). \quad (\text{A.3})$$

Consider the sum first. Suppose there exists an $a' \in \mathbb{Z}_+$ such that, say,

$$1 + q^{a'u+m'+1/2} z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} = 0. \quad (\text{A.4})$$

This might happen if for example, $z^{\frac{1}{2}} = -q^{-a'u-m'-1/2} \zeta^{\frac{1}{2}}$. Then one of the terms in the sum will be singular and the residue at that pole is,

$$q^{-a'^2 u + a'(m-2m')} \zeta^{a'}. \quad (\text{A.5})$$

Now suppose that (A.4) holds for the product too. That is to say that one of the factors in the denominator vanishes. Then substituting into the product this time for $z^{\frac{1}{2}}$ and z we obtain,

$$\prod_{n \in \mathbb{N}} \frac{(1 - q^{un})^2 (1 - \zeta q^{u(n-1-2a')-2m'+m}) (1 - \zeta^{-1} q^{u(2a'+n)+2m'-m})}{(1 - q^{u(n+a')}) (1 - \zeta^{-1} q^{u(n+a')+2m'-m}) (1 - \zeta q^{u(n-1-a')-2m'+m}) (1 - q^{u(n-1-a')})}$$

and $n \neq a' + 1$ in the last factor in the denominator. The numerator of this product may be rewritten,

$$(-1)^{2a'+1} \zeta^{2a'+1} q^{-(ua'+2m'-m)(2a'+1)} \times \prod_{n=1}^{\infty} (1 - q^{un})^2 (1 - \zeta^{-1} q^{u(n-1)+2m'-m}) (1 - \zeta q^{un-2m'+m}) \quad (\text{A.6})$$

and the denominator as,

$$-\zeta^{a'+1} q^{-ua'(a'+1)+(a'+1)(-2m'+m)} \times \prod_{n \in \mathbb{N}} (1 - q^{un})^2 (1 - \zeta^{-1} q^{u(n-1)+2m'-m}) (1 - \zeta q^{un-2m'+m}) \quad (\text{A.7})$$

so that upon dividing these two we obtain the same result for the residue as before in (A.5). One can proceed in this way for all the possible poles of the sum and show that the product has the same residue in each case.

Appendix B

The general branching formula

In this appendix we provide more details of the derivation of the branching formula (5.24) given in chapter 4. My supervisor assisted with the work in this appendix.

The use of the ϑ -function multiplication formula,

$$\vartheta_{m,k}(\tau, \sigma) \vartheta_{m',k'}(\tau, \sigma) = \sum_{\ell=1}^{k+k'} \vartheta_{mk' - m'k + 2\ell k k', k k' (k+k')}(\tau) \vartheta_{m+m'+2\ell k, k+k'}(\tau, \sigma), \quad (\text{B.1})$$

is crucial. In particular, it allows one to write,

$$\prod_{i=1}^N \vartheta_{p_i, u}(\tau, \frac{\sigma}{u}) = \sum_{r=0}^{N-1} \left\{ \sum_{D(n_1, \dots, n_{N-1}; r)} \prod_{i=1}^{N-1} \vartheta_{P(0; N-i) - 2u\tilde{n}_i, (N-i)(N-i+1)u}(\tau) \right\} \vartheta_{\bar{p}_N - 2ur, Nu}(\tau, \frac{\sigma}{u}), \quad (\text{B.2})$$

where $\tilde{n}_i = \sum_{j=i}^{N-1} (N-j)n_j$,

$$D(n_1, \dots, n_{N-1}; r) = \{n_j \mid 0 \leq n_j \leq N-j, j = 1, \dots, N-1 \text{ and } \tilde{n}_1 = k'N + r, k' \in \mathbb{N}\}, \quad (\text{B.3})$$

and

$$P(\alpha; \beta) = \bar{p}_{\alpha, \beta} - \beta p_{\alpha + \beta + 1}, \quad (\text{B.4})$$

with

$$\bar{p}_{j, N} = \sum_{k=1}^N p_{j+k}, \quad \bar{p}_{0, N} \equiv \bar{p}_N. \quad (\text{B.5})$$

Note that the invariance of the left hand side of (B.2) under permutations of $\{p_1, \dots, p_N\}$ provides identities between sums of products of ϑ -functions at fixed values of r . This type of identity is used in deriving (5.24), and will be mentioned at the appropriate time below.

We shall first discuss the occurrence of the function $\mathcal{F}(n; \tau, \frac{\nu}{u})$ in (5.24). It stems from the factors $\vartheta_{m-2m', u}(\tau, \frac{\nu}{u})^{u-n}$ and $\vartheta_{m-2m'+u, u}(\tau, \frac{\nu}{u})^{n-1}$ in (5.21). To see this, first note that in the term where $n = 1$ (resp. $n = u$), one is only left with the first (resp. second) factor. Applying formula (B.2) with $N = u - 1$ and $p_i = m - 2m'$ for $i = 1, \dots, u - 1$ (resp. $N = u - 1, p_i = m - 2m' + u$ for $i = 1, \dots, u - 1$) to this factor, one obtains $\mathcal{F}(1; \tau, \frac{\nu}{u})$ (resp. $\mathcal{F}(n; \tau, \frac{\nu}{u})$), where one sets ϑ -functions at level zero to 1, as well as sums and products ranging over negative values of their index. In the generic case where n is neither 1 nor u , apply formula (B.2) with $N = u - n, p_i = m - 2m'$ for $i = 1, \dots, u - n$ to the first factor, and with $N = n - 1, p_i = m - 2m' + u$ for $i = 1, \dots, n - 1$ to the second factor. This leads to,

$$\begin{aligned} & \vartheta_{m-2m', u}(\tau, \frac{\nu}{u})^{u-n} \vartheta_{m-2m'+u, u}(\tau, \frac{\nu}{u})^{n-1} = \\ & \left[\sum_{s=0}^{u-n-1} \left\{ \sum_{D(\rho_1, \dots, \rho_{u-n-1}; s)} \prod_{i=1}^{u-n-1} \vartheta_{-2u\tilde{\rho}_i, (u-n-i)(u-n-i+1)u}(\tau) \right\} \right. \\ & \quad \left. \vartheta_{(u-n)(m-2m')-2us, (u-n)u}(\tau, \frac{\nu}{u}) \right] \\ & \times \left[\sum_{t=0}^{n-2} \left\{ \sum_{D(\sigma_1, \dots, \sigma_{n-2}; t)} \prod_{j=1}^{n-2} \vartheta_{-2u\tilde{\sigma}_j, (n-1-j)(n-j)u}(\tau) \right\} \vartheta_{(n-1)(m-2m'+u)-2ut, (n-1)u}(\tau, \frac{\nu}{u}) \right]. \end{aligned} \quad (\text{B.6})$$

Now, use (B.1) to evaluate the product,

$$\begin{aligned} & \vartheta_{(u-n)(m-2m')-2us, (u-n)u}(\tau, \frac{\nu}{u}) \vartheta_{(n-1)(m-2m'+u)-2ut, (n-1)u}(\tau, \frac{\nu}{u}) = \\ & \sum_{\lambda'=0}^{u-2} \vartheta_{-u[(u-n)(n-1)(2\ell'+1)+2((n-1)s-(u-n)t)], u(u-1)(u-n)(n-1)}(\tau) \\ & \quad \times \vartheta_{(u-1)(m-2m')-u[2(s+t)-(n-1)+2(u-n)\ell'], u(u-1)}(\tau, \frac{\nu}{u}), \end{aligned} \quad (\text{B.7})$$

where the summation index ℓ' in (B.1) is rewritten as $\ell = (u-1)p - \lambda'$, with $p = 1, \dots, u$ and $\lambda' = 0, \dots, u - 2$ and the following relation has been used,

$$\vartheta_{m, u(u-1)(u-n)(n-1)}(\tau) = \sum_{p=1}^u \vartheta_{2pu^2(u-1)(u-n)(n-1)+um, u^3(u-1)(u-n)(n-1)}(\tau). \quad (\text{B.8})$$

From here, it is straightforward to obtain expression (5.31).

Let us now rewrite the σ -dependent factors in (5.21) in such a way that $sl(2; \mathbb{C})$ characters at level $k = \frac{1}{u} - 1$ appear. Note that the $sl(2; \mathbb{C})$ denominator ($\vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma)$) is already explicitly written in (5.21), but the numerator should be a difference of ϑ -functions in the variables τ and $\frac{\sigma}{u}$ at level $u(u+1)$ (see equation (5.22).) We

illustrate the derivation for the term $n = 1$ in (5.21), and first use (B.2) to rewrite,

$$\prod_{i=1}^{u-1} \vartheta_{m+1+u+2i,u}(\tau, \frac{\sigma}{u}) = \sum_{r=0}^{u-2} \sum_{D(\mu_1, \dots, \mu_{u-2}; r)} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau) \vartheta_{(m+1)(u-1)-2ur, u(u-1)}(\tau, \frac{\sigma}{u}). \tag{B.9}$$

In the above, we used that when $n = 1$, the possible values of p_i are all values in the set $S = \{1, 2, \dots, u - 1\}$. The next step is to evaluate the product

$$\prod_{i=1}^{u-1} \vartheta_{m+1+u+2i,u}(\tau, \frac{\sigma}{u}) \left[\vartheta_{2(m+1)-u, 2u}(\tau, \frac{\sigma}{u}) - \vartheta_{2(m+1)+u, 2u}(\tau, \frac{\sigma}{u}) \right]. \tag{B.10}$$

To do this use (B.1) with $\ell = (u + 1)p - \ell', p = 1, \dots, u$ and $\ell' = 0, \dots, u$ as well as the relation

$$\vartheta_{m, 2u(u-1)(u+1)}(\tau) = \sum_{p=1}^u \vartheta_{um+4u^2p(u-1)(u+1), 2u^3(u-1)(u+1)}(\tau), \tag{B.11}$$

and calculate,

$$\begin{aligned} &\vartheta_{(m+1)(u-1)-2ur, u(u-1)}(\frac{\sigma}{u}, \tau) \left[\vartheta_{2(m+1)-u, 2u}(\frac{\sigma}{u}, \tau) - \vartheta_{2(m+1)+u, 2u}(\frac{\sigma}{u}, \tau) \right] = \\ &\sum_{\ell=0}^u \{ \vartheta_{u(u-1)(4\ell+1)-4ur, 2u(u-1)(u+1)}(\tau) \vartheta_{u[(2\ell+1)(u-1)+1-2r]-(u-m-1)(u+1), u(u+1)}(\frac{\sigma}{u}, \tau) \\ &- \vartheta_{u(u-1)(4\ell+1)+4ur, 2u(u-1)(u+1)}(\tau) \vartheta_{-u[(2\ell+1)(u-1)+1+2r]-(u-m-1)(u+1), u(u+1)}(\frac{\sigma}{u}, \tau) \}. \end{aligned} \tag{B.12}$$

Also note that the above formula is obtained after defining $\ell'' = u + 1 - \ell'$ in the first sum, and using the fact that the term $\ell'' = u + 1$ is identical to the term $\ell'' = 0$. Now manipulate the above expression to make the $sl(2; \mathbb{C})$ character numerator,

$$\vartheta_{b_+, u(u+1)}(\tau, \frac{\sigma}{u}) - \vartheta_{b_-, u(u+1)}(\tau, \frac{\sigma}{u}) \tag{B.13}$$

appear for some b_{\pm} of the form,

$$b_{\pm} = \pm u(n' + 1) - n''(u + 1), \quad n', n'' \geq 0. \tag{B.14}$$

One identifies n'' with $u - m - 1$ i.e.,

$$n'' = u - m - 1 \geq 0, \quad n'' = 0, \dots, u - 1 \quad \text{since} \quad 0 \leq m \leq u - 1 \quad \text{in class IV.} \tag{B.15}$$

The identification of n' is more involved. When $r = 0$ in (B.12), the $sl(2; \mathbb{C})$ character appearing is $\chi_{n', n''}^{sl(2; \mathbb{C})}(\sigma, \tau)$ (resp. $-\chi_{n', n''}^{sl(2; \mathbb{C})}(\sigma, \tau)$) when $0 \leq n' \leq u - 1$ and n' is the residue modulo $2(u + 1)$ of $(u - 1)(2\ell + 1)$ (resp. $-(u - 1)(2\ell + 1) - 2$). If n' is the residue modulo $2(u + 1)$ of one of the two above expressions, but is greater than $u - 1$, then contributions cancel, and there is no $sl(2; \mathbb{C})$ character contribution. Whenever $r \neq 0$ in (B.12), the terms r and $u - 1 - r$ must be combined in order to produce the $sl(2; \mathbb{C})$ characters. The contribution to (B.10) from terms corresponding to $r \neq 0$ can be written,

$$\begin{aligned} & \sum_{r=1}^{u-2} \left[\sum_{D(\mu_1, \dots, \mu_{u-2}; r)} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau) \right] \times \\ & \sum_{\ell=0}^u \vartheta_{u(u-1)(4\ell+1)-4ur, 2u(u-1)(u+1)}(\tau) \vartheta_{u[(2\ell+1)(u-1)-2r+1]-(u-m-1)(u+1), u(u+1)}\left(\frac{\sigma}{u}, \tau\right) \\ & - \sum_{r'=1}^{u-2} \left[\sum_{D(\mu_1, \dots, \mu_{u-2}; r')} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau) \right] \times \\ & \sum_{\ell=0}^u \vartheta_{u(u-1)(4\ell+1)-4ur', 2u(u-1)(u+1)}(\tau) \vartheta_{-u[(2\ell+1)(u-1)-2r'+1]-(u-m-1)(u+1), u(u+1)}\left(\frac{\sigma}{u}, \tau\right), \end{aligned} \tag{B.16}$$

where we have changed variable from r to $r' = u - 1 - r$ in the second sum over r .

Finally, use the following identity,

$$\begin{aligned} & \sum_{D(\mu_1, \dots, \mu_{u-2}; r)} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau) = \\ & \sum_{D(\mu_1, \dots, \mu_{u-2}; u-1-r)} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau), \end{aligned} \tag{B.17}$$

which follows from the invariance of (B.2) under the permutations of p_1, \dots, p_{u-1} , and write, for all r -terms, including $r = 0$,

$$\begin{aligned} & \prod_{i=1}^{u-1} \vartheta_{m+1+u+2i, u}\left(\frac{\sigma}{u}, \tau\right) \left[\vartheta_{2(m+1)-u, 2u}\left(\frac{\sigma}{u}, \tau\right) - \vartheta_{2(m+1)+u, 2u}\left(\frac{\sigma}{u}, \tau\right) \right] = \\ & \sum_{r=0}^{u-2} \left[\sum_{D(\mu_1, \dots, \mu_{u-2}; r)} \prod_{i=1}^{u-2} \vartheta_{-(u-1-i)(u-i)-2u\tilde{\mu}_i, (u-1-i)(u-i)u}(\tau) \right] \\ & \sum_{\ell=0}^u \vartheta_{u(u-1)(4\ell+1)-4ur, 2u(u-1)(u+1)}(\tau) \\ & \left[\vartheta_{u[(2\ell+1)(u-1)-2r+1]-(u-m-1)(u+1), u(u+1)}\left(\frac{\sigma}{u}, \tau\right) - \right. \\ & \left. \vartheta_{-u[(2\ell+1)(u-1)-2r+1]-(u-m-1)(u+1), u(u+1)}\left(\frac{\sigma}{u}, \tau\right) \right]. \end{aligned} \tag{B.18}$$

The $sl(2; \mathbb{C})$ character appearing in the above is $\chi_{n', n''}^{sl(2; \mathbb{C})}(\sigma, \tau)$ (resp. $-\chi_{n', n''}^{sl(2; \mathbb{C})}(\sigma, \tau)$) when $0 \leq n' \leq u-1$ and n' is the residue modulo $2(u+1)$ of $(u-1)(2\ell+1) - 2r$ (resp. $-(u-1)(2\ell+1) - 2 + 2r$); n'' is identified with $u-m-1$. The discussion of the term $n = u$ is completely similar to the above case. The generic case where $n \neq 1$ and $n \neq u$ proceeds along the same lines, but is more involved since one has to apply (B.2) to the factors $\prod_{i=1}^{u-n} \vartheta_{m+1+u+2p_i, u}(\tau, \frac{\sigma}{u})$ and $\prod_{j=1}^{n-1} \vartheta_{m+1+2p_{u-n+j}, u}(\tau, \frac{\sigma}{u})$. This derivation is very close in spirit to the technique used at the beginning of this appendix when we were discussing the ν -dependence of (5.21). One obtains,

$$\prod_{i=1}^{u-n} \vartheta_{m+1+u+2p_i, u}(\frac{\sigma}{u}, \tau) \prod_{j=1}^{n-1} \vartheta_{m+1+2p_{u-n+j}, u}(\frac{\sigma}{u}, \tau) =$$

$$\sum_{\mu_1=0}^{u-n-1} \sum_{\mu_2=0}^{u-n-2} \dots \sum_{\mu_{u-n-1}=0}^1 \sum_{\nu_1=0}^{n-2} \sum_{\nu_2=0}^{n-3} \dots \sum_{\nu_{n-2}=0}^1 \mathcal{G}(p_1, \dots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) \times$$

$$\vartheta_{(u-n)(m+1+u)+2\bar{p}_{u-n}-2u\tilde{\mu}_1, (u-n)u}(\frac{\sigma}{u}, \tau) \vartheta_{(n-1)(m+1)+2\bar{p}_{u-n, n-1}-2u\tilde{\nu}_1, (n-1)u}(\frac{\sigma}{u}, \tau). \quad (\text{B.19})$$

Now, the product of the last two theta functions above can be written, using (B.1),

$$\sum_{\ell=0}^{u-2} \vartheta_{a, u(u-1)(u-n)(n-1)}(\tau) \vartheta_{(u-n)u(1-2\ell)+(u-1)(m+1+u)-2u(\tilde{\mu}_1+\tilde{\nu}_1), u(u-1)}(\frac{\sigma}{u}, \tau), \quad (\text{B.20})$$

where

$$a = u(u-n)(n-1)(1-2\ell) + 2(n-1)[\bar{p}_{u-n} - u\tilde{\mu}_1] - 2(u-n)[\bar{p}_{u-n, n-1} - u\tilde{\nu}_1]. \quad (\text{B.21})$$

Note that the partitioning of $\tilde{\mu}_1 + \tilde{\nu}_1 = k'(u-1) + r$, $k' \in \mathbb{N}$, $r = 0, \dots, u-2$ as described in (5.28) is devised to simplify the ϑ -function $\vartheta_{(u-n)u(1-2\ell)+(u-1)(m+1+u)-2u(\tilde{\mu}_1+\tilde{\nu}_1), u(u-1)}(\frac{\sigma}{u}, \tau)$ using the property

$$\vartheta_{2u(u-1), u(u-1)}(\frac{\sigma}{u}, \tau) = \vartheta_{0, u(u-1)}(\frac{\sigma}{u}, \tau). \quad (\text{B.22})$$

Now, using (B.1), we calculate,

$$\begin{aligned}
& \prod_{i=1}^{u-n} \vartheta_{m+1+2p_i+u, u} \left(\frac{\sigma}{u}, \tau \right) \prod_{j=1}^{n-1} \vartheta_{m+1+2p_{u-n+j}, u} \left(\frac{\sigma}{u}, \tau \right) \\
& \quad [\vartheta_{2(m+1)-u, 2u} \left(\frac{\sigma}{u}, \tau \right) - \vartheta_{2(m+1)+u, 2u} \left(\frac{\sigma}{u}, \tau \right)] = \\
& \sum_{\tau=0}^{u-2} \sum_{D(\mu_1, \dots, \mu_{u-n-1}; \nu_1, \dots, \nu_{n-2}; \tau)} \mathcal{G}(p_1, \dots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) \sum_{\ell=0}^{u-2} \vartheta_{a, u(u-1)(u-n)(n-1)}(\tau) \times \\
& \quad \sum_{\ell'=0}^u \left[\vartheta_{b+c, 2u(u+1)(u-1)}(\tau) \vartheta_{d+e+f, u(u+1)} \left(\frac{\sigma}{u}, \tau \right) - \right. \\
& \quad \left. \vartheta_{b-c, 2u(u+1)(u-1)}(\tau) \vartheta_{d-e+f, u(u+1)} \left(\frac{\sigma}{u}, \tau \right) \right], \quad (\text{B.23})
\end{aligned}$$

where

$$\begin{aligned}
b &= u(u-1)(4\ell' + 3), & c &= 2u(u-n)(1-2\ell) - 4ur, & d &= \frac{1}{2}c \\
e &= u[(u-1)(2\ell' + 1) + u], & f &= -(u-m-1)(u+1). \quad (\text{B.24})
\end{aligned}$$

One must now make the numerators of $sl(2; \mathbb{C})$ characters appear in the sum over ℓ' . Similar manipulations as in the case $n = 1$ described previously in this appendix allow one to rewrite the expression (B.23) as,

$$\begin{aligned}
& \sum_{\tau=0}^{u-2} \sum_{D(\mu_1, \dots, \mu_{u-n-1}; \nu_1, \dots, \nu_{n-2}; \tau)} \mathcal{G}(p_1, \dots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) \\
& \quad \times \sum_{\ell=0}^{u-2} \vartheta_{a, u(u-1)(u-n)(n-1)}(\tau) \sum_{\ell'=0}^u \vartheta_{b+c, 2u(u+1)(u-1)}(\tau) \\
& \quad \times \left[\vartheta_{d+e+f, u(u+1)} \left(\frac{\sigma}{u}, \tau \right) - \vartheta_{-d-e+f, u(u+1)} \left(\frac{\sigma}{u}, \tau \right) \right]. \quad (\text{B.25})
\end{aligned}$$

Call $n'' = u - m - 1$. The $sl(2; \mathbb{C})$ character is $\chi_{n', n''}^{sl(2; \mathbb{C})}(\tau, \sigma)$ (resp. $-\chi_{n', n''}^{sl(2; \mathbb{C})}(\tau, \sigma)$) when n' is the residue of $(u-1)(2\ell' + 2) - 2r + (u-n)(1-2\ell)$ (resp. $-(u-1)(2\ell' + 2) + 2r - (u-n)(1-2\ell) - 2$) in the range $0 \leq n' \leq u-1$.

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