braneworld cosmology and holography

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Braneworld Cosmology and Holography

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For my Mum y para mi Papaito

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Abstract

This thesis is devoted to studying two important aspects of braneworld physics: their cosmology and their holography. We examine the Einstein equations induced on a general $(n-2)$-brane of arbitrary tension, embedded in some n-dimensional bulk. The brane energy-momentum tensor enters these equations both linearly and quadratically. From the point of view of a homogeneous and isotropic brane we see quadratic deviations from the FRW equations of the standard cosmology. There is also a contribution from a bulk Weyl tensor. We study this in detail when the bulk is AdS-Schwarzschild or Reissner-Nordström AdS. This contribution can be understood holographically. For the AdS-Schwarzschild case, we show that the geometry on a brane near the AdS boundary is just that of a radiation dominated FRW universe. The radiation comes from a field theory that is dual to the AdS bulk. We also develop a new approach which allows us to consider branes that are not near the AdS boundary. This time the dual field theory contributes quadratic energy density/pressure terms to the FRW equations. Remarkably, these take exactly the same form as for additional matter placed on the brane by hand, with no bulk Weyl tensor.

We also derive the general equations of motion for a braneworld containing a domain wall. For the critical brane, the induced geometry is identical to that of a vacuum domain wall in $(n-1)$-dimensional Einstein gravity. We develop the tools to construct a nested Randall-Sundrum scenario whereby we have a "critical" domain wall living on an anti-de Sitter brane. We also show how to construct instantons on the brane, and calculate the probability of false vacuum decay.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and is all my own work unless referenced to the contrary in the text.

Chapter 2 of this thesis is a review of necessary background material. Chapter 3 is also a review, although many of the results have been generalised to arbitrary dimension for later convenience. Chapter 4 is original work done in collaboration with my supervisor, Dr. Ruth Gregory [1,2]. Chapter 5 contains some initial review, but section 5.5 onwards is all my own work [3]. Chapter 6 contains original work done in collaboration with James Gregory [4].

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Chapter 1

Introduction

1.1 From three to four dimensions

For centuries, physicists and philosophers have puzzled over the dimension of our universe. Why is it we only experience three spatial dimensions? Kepler [5] reasoned that the threefold nature of the Holy Trinity [6] was responsible. The advent of Special Relativity [7] and Maxwell's theory of electromagnetism led to Minkowski's suggestion [8] that we should understand physics geometrically in four-dimensional *spacetime* rather three-dimensional *space*. As observers, we only notice the "mixing" of space and time at very high speeds, through phenomena such as length contraction and time dilation. Ever since Minkowski's breakthrough, physicists have been tempted to play with the dimensionality of our universe, either to find new explanations to old problems, or to "tidy up" existing theories. A particularly important example of this was *Kaluza-Klein theory* [9–11]. For a nice introduction to higher dimensions, see [12].

1.2 Kaluza-Klein theory

Kaluza's [9] aim was to unify gravity and electrodynamics. Gravity is well described at a classical level by the General Theory of Relativity [13]. This states that matter causes the universe to curve, with particles moving along geodesics in this curved geometry. If matter is described by the four-dimensional energy-momentum tensor,

$T_{\mu\nu}$, and G is Newton's constant, then

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

where $g_{\mu\nu}$, R and $R_{\mu\nu}$ are the metric, Ricci scalar and Ricci tensor of our universe. The Einstein equations (1.1) can be derived from the Einstein-Hilbert action

$$S_G = S_m + \frac{1}{16\pi G}\int d^4x\ \sqrt{g}R \tag{1.2}$$

where $g = \det g_{\mu\nu}$ and

$$T_{\mu\nu} = \left(\frac{2}{\sqrt{g}}\right)\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{1.3}$$

Meanwhile, the Maxwell equations for a gauge potential, A_μ, coupled to a source of electromagnetic current, j_μ, are given by

$$\nabla_\mu F^{\mu\nu} = -\mu_0 j^\nu, \tag{1.4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Equation (1.4) can be derived from the following action

$$S_{EM} = \tilde{S}_m - \frac{1}{4\mu_0}\int d^4x\ \sqrt{g}F^2 \tag{1.5}$$

where

$$j^\mu = \left(\frac{1}{\sqrt{g}}\right)\frac{\delta \tilde{S}_m}{\delta A_\mu} \tag{1.6}$$

If we add together the actions (1.2) and (1.5) we get Einstein-Maxwell theory for gravity coupled to an electromagnetic field. Kaluza's idea was to consider pure gravity in five dimensions. Ignoring matter terms, the five-dimensional action is simply

$$S = \int d^4x dz \sqrt{\tilde{g}}\tilde{R} \tag{1.7}$$

where $\tilde{g}_{AB}$ is the five dimensional metric, and $\tilde{R}$ is the corresponding Ricci scalar. Note that we have the original four dimensions labelled with coordinates x^μ where $\mu = 0, 1, 2, 3$. The fifth dimension is compactified on a circle and is labelled by the coordinate $0 \leq z \leq L$.

Now we can expand the metric as a Fourier series of the form

$$\tilde{g}_{AB}(x, z) = \sum_n \tilde{g}^{(n)}_{AB}(x)e^{inz/L}. \tag{1.8}$$

We find that we get an infinite number of fields in four dimensions. Modes with $n \neq 0$ correspond to massive fields with mass $|n|/L$. The zero mode corresponds to a massless field. As we take L to be smaller and smaller we see that the mass of the first massive field becomes very large. This means that if we compactify on a small enough circle we can truncate to massless modes in the four-dimensional theory. We can only see the extra dimension by exciting massive modes which are at energies beyond our reach.

Let us now focus on the zero mode, $\tilde{g}_{AB}(x)$. We could define $\tilde{g}_{\mu\nu}$, $\tilde{g}_{\mu z}$ and $\tilde{g}_{zz}$ to be the four-dimensional fields $g_{\mu\nu}$, A_μ and ϕ. In order that our results are more transparent we will actually define the components of the metric in the following way:

$$\tilde{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu, \qquad \tilde{g}_{\mu z} = e^{2\beta\phi} A_\mu, \qquad \tilde{g}_{zz} = e^{2\beta\phi}. \tag{1.9}$$

where $\alpha = 1/2\sqrt{3}$ and $\beta = -1/\sqrt{3}$. Since we have truncated to the massless fields, we can integrate out the z part of the action (1.7). We find that the four-dimensional effective action is given by

$$S_{\text{eff}} = L \int d^4x \sqrt{g} \left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4} e^{-\sqrt{3}\phi} F^2 \right) \tag{1.10}$$

Although we had set out to obtain Einstein-Maxwell theory, we have ended up with an additional coupling to the scalar field ϕ. It turns out we cannot consistently set this field to zero. This was a worry to the original authors but today we are more comfortable with the idea that scalar fields might exist, such as the Higgs. Here, ϕ is known as the dilaton.

Kaluza-Klein type compactifications can be more complicated than simply compactifying on a circle. The important thing is that the extra dimension is small so that we do not excite massive modes. We can truncate to massless modes and read off the effective theory in four dimensions.

We need not restrict ourselves to just one extra dimension either. In fact, higher dimensions have become very fashionable in the last twenty years, mainly due to the success of *string theory* as a possible quantum theory of gravity. At the quantum level, bosonic string theory is only consistent[1] in twenty-six (!) dimensions, although

[1] Actually, bosonic string theory contains a tachyon, but we will ignore that here.

this figure is reduced to ten when we introduce supersymmetry. Furthermore, there are five distinct string theories which can be viewed as different elements of an embracing new theory, M-theory [14–16]. M-theory lives in eleven dimensions and has eleven-dimensional supergravity as its low energy limit.

Traditionally we achieve the reduction down to four dimensions using Kaluza-Klein techniques. If we start with a $(4+n)$-dimensional theory, we compactify on a small n-dimensional manifold. Different manifolds generally give different effective theories in four dimensions. The one thing all of these manifolds have in common is that they are very small, and compact.

There is, however, an alternative to Kaluza-Klein compactification. This is the idea that we live on something called a *braneworld*, where the extra dimension can be infinite.

1.3 Introduction to braneworlds

The idea is that our four-dimensional world is nothing more than an infinitesimally thin 3-brane, embedded in a $(4+n)$-dimensional spacetime [17, 18]. All Standard Model fields are bound to the brane, although gravity may propagate into the extra dimensions.

Of particular interest to us here are the Randall-Sundrum braneworlds [19, 20]. There are in fact two models. The Randall-Sundrum I model [19] is introduced in detail in section 2.1. Here we have two 3-branes of equal and opposite tension separated by some five-dimensional anti-de Sitter bulk. In order to preserve Poincaré invariance on the branes, we fine tune the brane tensions against the bulk cosmological constant.

The most important quality of the Randall-Sundrum I model is that it provides an ingenious approach to the *hierarchy problem*. We will describe what this is in more detail at the beginning of section 2.1. For now, we note that it is the problem of the Planck scale being so much larger than the weak scale. Braneworld models avoid this by stating that the fundamental Planck scale is of similar size to the fundamental weak scale. It is only when we examine the *effective* theory

on the brane that we see the hierarchy between scales emerge. Unfortunately, the simplest braneworld models simply transfer the problem by requiring that the extra dimensions be very large. The Randall-Sundrum I model, however, is more subtle than this. By having anti-de Sitter space between the branes we get an exponential warp factor in the metric. This ensures that the effective four-dimensional Planck scale is much larger than the weak scale, even when there is no hierarchy in the fundamental five-dimensional theory. Crucially, this is achieved without the need for the extra dimension to be very large.

Despite this success of RS1, there are still some physical problems with the model, such as how one should stabilise the extra dimension. For this reason, we will focus on its successor, the Randall-Sundrum II model [20], which we discuss in detail in section 2.2. This time there is only one brane and an infinitely large anti-de Sitter bulk. The brane tension is positive and is once again fine tuned against the bulk cosmological constant to ensure Poincaré invariance on the brane. The warp factor in the bulk metric does not play the role of solving the hierarchy problem like in RS1. Here it ensures that gravity is localised on the brane.

Recall that standard Kaluza-Klein compactifications ensure that gravity looks four-dimensional by stating that the extra dimensions should be small. In Randall-Sundrum II, the extra dimension is infinite! Gravity is allowed to propagate into the extra dimension so we would expect it to look five-dimensional even to an observer on the brane. However, the warp factor causes metric perturbations to be damped as they move away from the brane. This has the effect that gravity looks four-dimensional, at least perturbatively, to a braneworld observer. Randall-Sundrum II offers an interesting "alternative to compactification".

RS2 branes are often referred to as *critical* because the brane tension is fine tuned to a critical value. This ensures that the metric induced on the brane is Minkowski. If we relax this fine tuning we obtain *non-critical* branes, which are discussed in section 2.2.3. Branes whose tension exceed the critical value have a de Sitter induced metric. Those with a tension smaller than the critical value have an anti-de Sitter induced metric. The de Sitter brane in particular is important because our universe may have a small positive cosmological constant [21, 22].

1.4 Braneworld cosmology

The initial success of RS2, from a gravitational point of view, sparked off a lot of interest, especially amongst cosmologists. In particular, Shiromizu *et al* [23] calculated the Einstein equations induced on the brane. In chapter 3, we generalise their work to arbitrary dimensions. By this we mean considering the geometry induced on an $(n-2)$-brane in an n-dimensional bulk. We start by writing the energy-momentum tensor for the brane in the following way:

$$S_{ab} = -\sigma h_{ab} + \mathcal{T}_{ab} \tag{1.11}$$

where σ is the brane tension, h_{ab} the brane metric and $\mathcal{T}_{ab}$ the energy-momentum of additional matter on the brane. In the linearised analysis of chapter 2, we take $\mathcal{T}_{ab}$ to be small and ignore quadratic contributions. However, from a cosmological point of view, it is important to consider situations where $\mathcal{T}_{ab}$ is not small. In this instance, we use the Gauss-Codazzi formalism to derive the Einstein tensor on the brane. Leaving the details until chapter 3, we will give a rough version of the result. If R_{ab} and R are the Ricci tensor and scalar on the $(n-2)$-brane, then

$$R_{ab} - \tfrac{1}{2}Rh_{ab} = -\Lambda_{n-1}h_{ab} + 8\pi G_{n-1}\mathcal{T}_{ab} + \mathcal{T}^{(2)}_{ab} - E_{ab}. \tag{1.12}$$

The first two terms on the right hand side are what we would have expected from Einstein gravity in $(n-1)$ dimensions: a cosmological constant term and a linear matter term. The brane cosmological constant depends on σ and the bulk cosmological constant. As we stated at the end of the last section, it vanishes for critical branes, but not for non-critical branes. The Newton's constant on the brane, G_{n-1}, turns out to be proportional to the bulk Newton's constant, G_n, and the brane tension. This dependence on the brane tension is often ignored although it turns out to be very important when we study braneworld holography on non-critical branes in chapter 5.

The last two terms on the right hand side of equation (1.12) are the most interesting. The E_{ab} term is often referred to as the electric part of the bulk Weyl tensor. It vanishes for a pure anti-de Sitter bulk, but can be non-zero if (say) we have a bulk black hole. This term is best understood from a holographic point of view so we will postpone its discussion until the next section.

The $\mathcal{T}^{(2)}_{ab}$ term is actually quite complicated. The important thing is that it is quadratic in $\mathcal{T}_{ab}$. In section 3.2.1, we consider a Friedmann-Robertson-Walker brane. The $\mathcal{T}^{(2)}_{ab}$ terms show up in the FRW equations as quadratic terms in energy density and pressure. If these quantities are small, we can neglect the quadratic contribution. However, this might not be the case in the early universe so the $\mathcal{T}^{(2)}_{ab}$ terms could be important.

Braneworld cosmology deviates slightly from pure Einstein gravity in $(n-1)$ dimensions. In chapter 4, we consider non-perturbative gravity on the brane in a different way. We investigate what happens when we have a strongly gravitating object such as a domain wall on the brane [1, 2]. We can think of this as a domain wall within a domain wall. It turns out that the equations of motion for this kind of configuration are completely integrable.

The most interesting solutions are the following: the domain wall living on a critical RS brane, the nested Randall-Sundrum scenario, and the Coleman-De Luccia instantons. The first of these yields a remarkable result. It turns out that the geometry induced on the $(n-2)$-brane agrees exactly with what we would have expected from $(n-1)$-dimensional Einstein gravity. Let us make this a little clearer: suppose we have a domain wall of tension, T, sitting in $(n-1)$-dimensional flat space. If we do Einstein gravity in $(n-1)$-dimensions we find that our flat spacetime has a certain geometry. This geometry is *exactly* the same as the geometry on an $(n-2)$-brane containing a nested domain wall, also of tension, T. We see that we have exact Einstein gravity on the brane, even at a non-perturbative level.

Although the original motivation was to look at strong gravity on the brane, we have developed tools that enable us to construct other interesting configurations. The nested Randall-Sundrum scenario has a "critical" nested domain wall living on an anti-de Sitter brane. The geometry induced on the brane is the traditional RS2 geometry, in $(n-1)$ dimensions.

Staying with the cosmological theme, in section 4.4 we show how to construct gravitational instantons on the brane. These are the braneworld analogue of the Coleman-De Luccia instantons [24]. In this paper, the authors calculate the probability of (say) a flat bubble spacetime nucleating in a de Sitter false vacuum. This

kind of instanton describes a first order phase transition in the early universe. We show how to patch together our solutions so as to create these instantons on a brane. We do the same probability calculations and find that they agree with [24], at least in certain limits.

1.5 Braneworld holography

Having examined brane cosmology and strong brane gravity, we change direction in chapter 5, and discuss *braneworld holography*. We begin by reviewing the holographic principle. For now, all we need to say is that this involves projecting all the degrees of freedom in some volume on to its boundary surface. The AdS/CFT correspondence [25–27] is the first concrete example of this principle in action. We find that a gravity theory on $AdS_5 \times S^5$ is dual to a conformal field theory on the boundary. Braneworld holography is slightly different to AdS/CFT. The bulk gravity theory is conjectured to be dual to a field theory on the brane. This field theory is cut-off in the ultra-violet, and unlike in the AdS/CFT correspondence, it is coupled to gravity on the brane.

The difficulty with braneworld holography is that we do not know the precise nature of the dual field theory. We can, however, make use of the coupling to gravity. If we place a black hole in the bulk, the Hawking radiation causes the brane to heat up. Any dual field theory that lives on the brane should absorb energy which we can try to calculate.

This procedure was first carried out for critical branes [28], and is reviewed in detail in section 5.4. To summarise, we place a black hole of mass, M, in an n-dimensional bulk, and consider a critical FRW brane near the boundary of AdS. M is measured by an observer using the bulk time coordinate, t. This should translate into the energy of the dual field theory [29]. However, the field theory lives on the brane, so we should use the brane time coordinate, τ. To find its energy, we need to scale the black hole mass with some red-shift factor, $\dot{t}$, where dot denotes differentiation with respect to τ. By using conservation of energy, we can also calculate the pressure on the brane.

Given that we have a FRW brane, we can write down FRW equations for its cosmological evolution. If $Z(\tau)$ is the scale factor, and $H = \dot{Z}/Z$ is the Hubble parameter, then

$$H^2 = -\frac{1}{Z^2} + \frac{c}{Z^{n-1}} \tag{1.13a}$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} \tag{1.13b}$$

where c is proportional to M. This black hole mass term comes from the non-trivial bulk Weyl tensor, E_{ab}. Using the ideas just described, we can calculate the energy density, ρ, and the pressure, p, of the dual field theory, in terms of M, or equivalently, c. We find that we can rewrite the FRW equations entirely in terms of field theory quantities:

$$H^2 = -\frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho \tag{1.14a}$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) \tag{1.14b}$$

These are the FRW equations of the standard cosmology in $(n-1)$ dimensions. We see that we do indeed have a holographic description. On the one hand the brane cosmology is driven by the bulk black hole. On the other hand it is driven by the energy-momentum of a dual field theory. It turns out that for an uncharged black hole in the bulk, this field theory behaves like radiation.

In section 5.5, we attempt to extend these ideas to de Sitter and anti-de Sitter branes [3]. This is not as straightforward as we might have thought. We have to be more careful than to say that the bulk energy is given by the black hole mass. Our calculation of the bulk energy is affected by cutting the spacetime off at the brane. We use Euclidean quantum gravity techniques to calculate the bulk energy from first principles, and then multiply by a red-shift factor to get the energy of the field theory. It turns out that various factors combine to give us a similar holographic description to before. The only difference is that the FRW equations now contain a cosmological constant term corresponding to the de Sitter or anti-de Sitter brane, as appropriate.

The main problem with all the analysis of chapter 5 is that its relies on a number of approximations. In particular, we assume that the brane is near the AdS bound-

ary. This has two implications. The first is that it enables us to get a reasonable approximation for the bulk energy. The second is that it means the cut-off in the field theory is fairly insignificant. The dual field theory is nearly conformal, which is consistent with it behaving like radiation. However, a general brane trajectory does not need to go near the AdS boundary. In chapter 6, we take a completely different approach to braneworld holography [4]. We modify the Hamiltonian technique of Hawking and Horowitz [30] to calculate the energy of the dual field theory *exactly*, with no assumptions made about the position of the brane. As a result, we can also get an exact expression for the pressure. We end up with a highly non-trivial equation of state that simplifies to radiation only as the brane gets nearer to the AdS boundary. The really interesting result, however, lies in the effect on the FRW equations. When we express these equations using the exact braneworld quantities, we find that they take the following form:

$$H^2 = a - \frac{1}{Z^2} + \frac{8\pi G_n \sigma_n}{n-2}\rho + \left(\frac{4\pi G_n}{n-2}\right)^2 \rho^2 \tag{1.15a}$$

$$\dot{H} = \frac{1}{Z^2} - 4\pi G_n \sigma_n (\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \tag{1.15b}$$

where where we have included the possibility of a brane cosmological constant in the a term, and $\sigma_n = 4\pi G_n/(n-2)$. Although these equations do not correspond to the FRW equations for the standard cosmology, they have exactly the same form as the unconventional braneworld cosmology we discussed in the last section, complete with quadratic energy-momentum terms. When these equations are encountered in chapter 3, they correspond to a brane moving in a pure anti-de Sitter bulk, with additional matter placed on the brane by hand. In chapter 6, they have a very different origin. There is no additional matter on the brane although we now have a black hole in the bulk. When we derive properties for the dual field theory from the black hole, we find that the field theory behaves exactly as if it had been placed on the brane by hand. This means that the dual descriptions of chapter 5 are merely an approximation of this larger relationship.

We conclude this thesis in chapter 7 with some general thoughts and discussion. The main results are stated and interpreted as we go along.

Chapter 2

Randall-Sundrum Braneworlds

2.1 Randall-Sundrum I (RS1)

In a four-dimensional world there are at least two fundamental energy scales: the weak scale, $m_{EW} \sim 10^3$ GeV and the Planck scale, $m_{pl} \sim 10^{19}$ GeV. Physics is well described by the Standard Model at least up to 100 GeV or so. At the Planck scale, gravity becomes as strong as the SM interactions and a quantum theory of gravity is required. Why is there such a vast difference between the two scales? This question is the essence of the *hierarchy problem.* Consider the Higgs boson whose physical mass, $m_H \sim m_{EW}$. Now suppose our theory is cut-off at some large scale Λ, where $m_H \ll \Lambda$. When we calculate the one loop correction for the Higgs mass we find that $\delta m_H^2 \sim \Lambda^2$. The bare mass must then be of order $-\Lambda^2$ to give a renormalised mass near the weak scale. If we believe that our fundamental theory contains scales as high as the Planck scale, then the cancellation just described is disturbingly precise, given the huge numbers involved. What is more, this bizarre precision is required again at all subsequent orders of perturbation theory.

Traditionally, it is thought that this vast desert between the weak and the Planck scales must be populated with new theories, such as supersymmetry. Above the scale of supersymmetry breaking, the problems with radiative corrections to the Higgs mass are solved, although we may still ask why the desert exists at all. There is, however, another solution to the hierarchy problem that is radically different to supersymmetry. We assume that there is only one fundamental energy scale, the

weak scale. The large (effective) Planck scale comes from extra dimensions, beyond the traditional four. As observers, we are bound to a braneworld embedded in a $(4+n)$-dimensional spacetime. The $(4+n)$-dimensional Planck scale, M, is now the fundamental scale of gravity, and is taken to be of order the weak scale. The extra dimensions are given by an n-dimensional compact space of volume V_n. In the simplest cases [31–33], our effective four-dimensional Planck scale is given by

$$m_{pl}^2 = M^{n+2} V_n. \tag{2.1}$$

By taking V_n to be sufficiently large we can recover $m_{pl} \sim 10^{19}$ GeV. However, in some sense the hierarchy problem has not gone away. There is now a new hierarchy between the weak scale and the compactification scale, $1/V_n^{1/n} \ll m_{EW}$. Fortunately, the Randall-Sundrum I (RS1) model [19] is an extension of these ideas that does not appear to transfer the problem in this way[1].

2.1.1 The model

In RS1, we have two 3-branes embedded in a five dimensional anti-de Sitter bulk spacetime. We define x^μ to be the familiar four-dimensional coordinates while $0 \leq z \leq z_c$ is the coordinate for the extra dimension. Since our spacetime clearly fails to fill out all of the five dimensions we need to specify boundary conditions: identify (x^μ, z) with $(x^\mu, -z)$ and take z to be periodic with period $2z_c$. The orbifold fixed points at $z = 0,\ z_c$ are the positions of the two branes, which we will take to have tension $\sigma_0,\ \sigma_c$ respectively. These fixed points may also be thought of as the boundaries of the five-dimensional spacetime so that the action describing this model is given by

$$S = M^3 \int d^4x \int_{-z_c}^{z_c} dz \sqrt{g}\,(R - 2\Lambda) \; - \sigma_0 \int_{z=0} d^4x \sqrt{h_0} \; - \sigma_c \int_{z=z_c} d^4x \sqrt{h_c}\,. \tag{2.2}$$

where g is the bulk metric and $h_0,\ h_c$ are the metrics on the branes at $z = 0,\ z_c$ respectively. M is of course the five-dimensional Planck scale. We now require the

[1] Actually, the hierarchy problem remains if we consider fluctuations in the "radion" field. We will comment on this later.

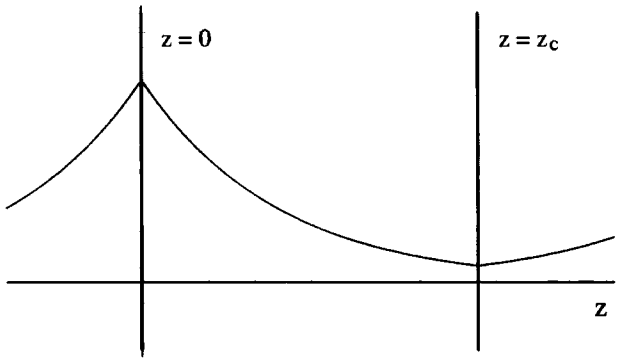


Figure 2.1: The behaviour of the warp factor in the RS1 model

3-branes to exhibit four-dimensional Poincaré invariance and choose the metric to take the following form

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \tag{2.3}$$

The bulk equations of motion with orbifold boundary conditions impose a fine tuning of the brane tensions against the bulk cosmological constant

$$\sigma_0 = -\sigma_c = 12M^3k, \ \Lambda = -6k^2 \tag{2.4}$$

We are also free to set $a(0) = 1$ so that we arrive at the following solution for the metric

$$ds^2 = e^{-2k|z|}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \quad \text{for} \ -z_c \leq z \leq z_c. \tag{2.5}$$

The $\mathbb{Z}_2$ symmetry about $z = 0$ is explicit whereas the other boundary conditions should be understood. We also note that the constant z slicings exhibit Poincaré invariance as required. The metric (2.5) contains an exponential warp factor which is seen graphically in figure 2.1. Notice the peak in the warp factor at the positive tension brane and the trough at the negative tension brane. At this point we should emphasize that RS1 is really only a toy model. It is, however, possible to construct string theory/supergravity models that have similar properties [34–37].

2.1.2 Tackling the hierarchy problem

In order to tackle the hierarchy problem, we will need to derive the (effective) four-dimensional Planck scale, m_{pl} in terms of the five-dimensional scales M, k, z_c. We

do this by identifying the four-dimensional low energy effective theory. This comes from massless graviton fluctuations. In principle, we should also include massless fluctuations in the brane separation [38], often referred to as the radion field. This does not affect the calculation of m_{pl} directly [39] so we will ignore the radion in this section and assume the brane separation is stabilised at z_c. The gravitational zero modes now take the form

$$ds^2 = e^{-2k|z|}\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + dz^2 \quad \text{where } \bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{2.6}$$

and we interpret $h_{\mu\nu}$ as the physical graviton in the four-dimensional effective theory. We now substitute equation (2.6) into the action (2.2) to derive the effective action. Focusing on the curvature term we find that

$$S_{eff} = M^3 \int d^4x \, \sqrt{\bar{g}}\bar{R} \int_{-z_c}^{z_c} dz \, e^{-2k|z|} + \ldots \tag{2.7}$$

where $\bar{R}$ is the Ricci scalar built out of $\bar{g}_{\mu\nu}(x)$. We now perform the z-integral to obtain

$$m_{pl}^2 = \frac{M^3}{k}\left[1 - e^{-2kz_c}\right]. \tag{2.8}$$

This tells us that m_{pl} depends weakly on z_c in the limit of large kz_c. We will see that this is not the case for the physical masses in the SM.

Suppose we live on the negative tension brane at $z = z_c$. Consider a fundamental Higgs field bound to this brane. If it has a five-dimensional mass parameter, m_0, then the matter part of the action near the brane is given by

$$S_c = \int_{z=z_c} d^4x\sqrt{g_c}\left[g_c^{\mu\nu}\nabla_\mu H^\dagger \nabla_\nu H - \lambda\left(|H|^2 - m_0^2\right)^2\right] \tag{2.9}$$

where ∇_μ is the covariant derivative corresponding to g_c. The metric at $z = z_c$ is $\bar{g}_{c\mu\nu} = e^{-2kz_c}\bar{g}_{\mu\nu}$ so that

$$S_c = \int_{z=z_c} d^4x\sqrt{\bar{g}}e^{-4kz_c}\left[e^{2kz_c}\bar{g}^{\mu\nu}\nabla_\mu H^\dagger \nabla_\nu H - \lambda\left(|H|^2 - m_0^2\right)^2\right] \tag{2.10}$$

We now renormalise the Higgs wavefunction, $H \to e^{kz_c}H$, to derive the following part of the effective action

$$S_{eff} = \int_{z=z_c} d^4x\sqrt{\bar{g}}\left[\bar{g}^{\mu\nu}\nabla_\mu H^\dagger \nabla_\nu H - \lambda\left(|H|^2 - e^{-2kz_c}m_0^2\right)^2\right] + \ldots \tag{2.11}$$

An observer on the brane will therefore measure the physical mass of the Higgs to be

$$m_H = e^{-kz_c} m_0. \tag{2.12}$$

This result generalises to any mass parameter on the negative tension brane.

We shall now address the hierarchy problem directly. Assume that the bare Higgs mass, m_0, and the fundamental Planck mass, M, are both around 10^{19} GeV, thereby eliminating any hierarchy between the two scales in the five-dimensional theory. The physical masses in the effective theory are given by equations (2.8) and (2.12). To ensure that $m_H \sim 10^3$ GeV and $m_{pl} \sim 10^{19}$ GeV we require that $e^{kz_c} \sim 10^{15}$. The presence of the exponential here is crucial because all we really need is $kz_c \sim 50$. We see that we have solved the hierarchy problem without introducing a second hierarchy involving the compactification scale, $1/z_c$ or the AdS length, $1/k$. We should emphasize here that this is only true if the radion is stabilised. If not, its fluctuations appear in the exponential, spoiling the solution to the problem.

At this point we should note that we have set the fundamental mass scale to be around 10^{19} GeV. We could easily have chosen the fundamental scale to be as low as a few TeV because what really matters is the ratio between the physical masses, as this is a dimensionless quantity. We can see this explicitly if we change coordinates $x^\mu \to e^{kz_c} x^\mu$. The warp factor at $z = z_c$ is unity, whereas at $z = 0$ it is exponentially large, e^{2kz_c}. This time, the Higgs mass does not get rescaled, $m_H \sim m_0$, unlike the Planck mass which behaves like $m_{pl}^2 \sim e^{2kz_c} \frac{M^3}{k}$. If both M and m_0 are around a few TeV, we again only need $kz_c \sim 50$ to recover the correct physical masses in the effective theory.

To summarise, even though all scales in the fundamental theory are near the weak scale, the extra dimension ensures that m_{pl} is close to the large value we observe in Nature. What is more, this is achieved without the need for the extra dimension to be disturbingly large. From a phenomenological point of view this is particularly exciting. If the fundamental scale of gravity is indeed as low as a few TeV then we would expect quantum gravity effects to start showing up in forthcoming collider experiments. The path to a "theory of everything" could be dictated by experiment rather than the imagination.

2.2 Randall-Sundrum II (RS2)

When we introduced braneworlds at the start of this chapter we stated that the Standard Model fields are localised on the brane [17,18] in contrast to gravity which can propagate into the fifth dimension. This should worry a braneworld observer because Newton's $1/r^2$ law for gravitational force is a property of four-dimensional gravity and is experimentally verified as low as $r \sim 0.2$ mm. The problem is solved if the extra dimension is small and compact owing to the large mass gap between the graviton zero mode and the first heavy Kaluza-Klein mode. This ensures that gravity behaves four dimensionally, except at very high energies near the heavy mode masses. In braneworld models we have seen how the extra dimension can be of order one or larger so we would naively expect gravity to look five dimensional even at fairly low energies. This would violate Newton's law and be unacceptable. The RS2 model is more subtle than this. Even though it has an infinite extra dimension it still manages to reproduce Newton's law on the brane. This is because we have a negative cosmological constant in the bulk. RS2 does not solve the hierarchy problem in the way that RS1 does, and is of interest from a purely gravitational point of view.

2.2.1 The model

To arrive at the RS2 model we start with RS1, and extend the brane separation to infinity so that we are left with a single brane of positive tension. The old negative tension brane will act as a regulator in the subsequent analysis. The geometry of this new set-up is again described by the metric (2.5) with $z_c \to \infty$. We can see the behaviour of the warp factor in figure 2.2. It has a peak at $z = 0$ indicating that the brane there has positive tension. Note also the $\mathbb{Z}_2$ symmetry about $z = 0$ which is, of course, explicit in the metric.

2.2.2 Localisation of gravity

In the absence of any additional matter, we have a single brane with tension $\sigma = 12M^3k$ embedded in five-dimensional anti-de Sitter space with cosmological constant

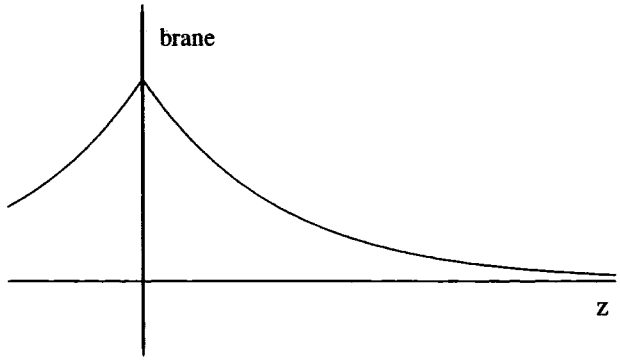


Figure 2.2: The behaviour of the warp factor in the RS2 model

$\Lambda = -6k^2$. In order to investigate whether gravity is localised on the brane, we will consider small gravitational perturbations about the background metric

$$ds^2 = \tilde{g}_{ab}dx^a dx^b = e^{-2k|z|}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \tag{2.13}$$

This may be achieved by placing a point mass on the brane, and solving the relevant perturbation equations. In the event of gravity localisation we would hope to see the graviton zero mode dominating at large enough distances. This would reproduce observed phenomena such as Newton's inverse square law and gravitational light bending. In the remainder of this section we will adopt Garriga and Tanaka's delightful approach to gravity in the Randall-Sundrum model [40].

2.2.2.1 The Newtonian potential on the brane

We begin by deriving the Newtonian potential due to a point mass, m_0, bound to the brane. If we denote the perturbed metric by $g_{ab} = \tilde{g}_{ab} + h_{ab}$, the Randall-Sundrum gauge [20] is given by

$$h_{zz} = h_{\mu z} = 0, \quad h_\mu{}^\nu{}_{,\nu} = 0, \quad h^\mu{}_\mu = 0. \tag{2.14}$$

Since we have no additional matter in the bulk, the bulk equations of motion for h_{ab} are given by

$$0 = \delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab} \tag{2.15}$$

where Δ_L is the Lichnerowicz operator[2]. We are free to take the RS gauge (2.14) everywhere in the bulk [20] so that equation (2.15) is reduced to

$$\left[e^{2k|z|}\Box^{(4)} + \partial_z^2 - 4k^2\right] h_{\mu\nu} = 0. \tag{2.16}$$

Boundary conditions for this equation are given by the jump conditions at the brane. However, if we take the RS gauge in the bulk then additional matter causes the brane to bend and we can no longer say that it lies at $z = 0$. For this reason, we will temporarily relax our choice of gauge and work in Gaussian normal (GN) coordinates, denoted by $(\hat{x}^\mu, \hat{z})$. By definition, we now have $\hat{h}_{zz} = \hat{h}_{\mu z} = 0$ and can set the brane to be located at $\hat{z} = 0$. By using the Israel junction conditions [41] we can relate the jump in extrinsic curvature[3], ΔK_{ab}, across the brane to the energy-momentum tensor, S_{ab} on the brane.

$$\Delta K_{ab} = -8\pi G_5 \left(S_{ab} - \frac{1}{3} S \hat{g}_{0ab}\right). \tag{2.17}$$

Here, $\hat{g}_{0ab} = \hat{g}_{ab}(\hat{z} = 0)$ is the induced metric on the brane and $G_5 = 1/16\pi M^3$ is the five-dimensional Newton's constant. Note that the energy momentum tensor is dominated by the brane tension, σ with a small additional contribution coming from the point mass, $\mathcal{T}_{ab}$. Explicitly

$$S_{ab} = -\sigma \hat{g}_{0ab} + \mathcal{T}_{ab}. \tag{2.18}$$

By imposing $\mathbb{Z}_2$ symmetry across the brane we arrive at

$$(\partial_z + 2k)\Big|_{z=0^+} \hat{h}_{\mu\nu} = -8\pi G_5 \left(\mathcal{T}_{\mu\nu} - \frac{1}{3}\mathcal{T}\eta_{\mu\nu}\right) \tag{2.19}$$

where we have used the fine-tuning conditions (2.4) and have ignored all terms non-linear in $\hat{h}_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$. Note that there are no μz or zz components of equation (2.17) because we chose a GN coordinate system. We will now attempt to construct

[2]The Lichnerowicz operator is defined by $\Delta_L h_{ab} = \Box h_{ab} - 2\tilde{\nabla}_{(a}\tilde{\nabla}_{|c|}\bar{h}^c_{b)} - 2\tilde{R}_{c(a}h^c_{b)} + 2\tilde{R}_{acbd}h^{cd}$ where $\bar{h}_{ab} = h_{ab} - \frac{1}{2}h\tilde{g}_{ab}$ and the covariant derivative and Riemann tensor are constructed out of the unperturbed metric $\tilde{g}_{ab}$.

[3]$\Delta K_{ab} = K^+_{ab} - K^-_{ab}$ where $K^-_{ab} = \hat{g}_0{}^c{}_a \hat{g}_0{}^d{}_b \nabla_{(c} n_{d)}$ and n^a is the unit normal to the brane pointing in the direction of increasing z, and $\hat{g}_{0ab}$ is the induced metric on the brane.

the junction condition (2.19) in the RS gauge. The most general transformation between GN and RS gauge is given by

$$\xi^z = f(x^\rho), \qquad \xi^\mu = -\frac{1}{2k} e^{2k|z|} \eta^{\mu\nu} \partial_\nu f + F^\mu(x^\rho) \tag{2.20}$$

where f and F^μ are independent of z. The perturbation in the RS gauge, $h_{\mu\nu}$, is related to its GN counterpart by

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{k} f_{,\mu\nu} - 2k e^{-2k|z|} \eta_{\mu\nu} f + e^{-2k|z|} \eta_{\rho(\nu} F^\rho{}_{,\nu)}, \tag{2.21}$$

Inserting this back into (2.19) we derive the junction condition in the RS gauge

$$(\partial_z + 2k) \Big|_{z=0^+} h_{\mu\nu} = -\Sigma_{\mu\nu} \tag{2.22}$$

where

$$\Sigma_{\mu\nu} = 8\pi G_5 \left(\mathcal{T}_{\mu\nu} - \frac{1}{3} \mathcal{T} \eta_{\mu\nu} \right) + 2 f_{,\mu\nu}. \tag{2.23}$$

Equations (2.16) and (2.22) fully define the bulk equations of motion with boundary conditions at the brane. Given that a solution must be $\mathbb{Z}_2$ symmetric about $z = 0$, we see that $\partial_z h_{\mu\nu}$ must be discontinuous there. Both (2.16) and (2.22) can be contained in a single equation if we include delta functions at the discontinuity.

$$\left[e^{2k|z|} \Box^{(4)} + \partial_z^2 - 4k^2 + 4k\delta(z) \right] h_{\mu\nu} = -2\delta(z) \Sigma_{\mu\nu} \tag{2.24}$$

Before we can solve equation (2.24) we need to identify $f(x)$. Nevertheless, we shall proceed blindly and define $G_R(x, z; x', z')$ to be the five-dimensional retarded Green's function satisfying

$$\left[e^{2k|z|} \Box^{(4)} + \partial_z^2 - 4k^2 + 4k\delta(z) \right] G_R(x, z; x', z') = \delta^{(4)}(x - x') \delta(z - z'). \tag{2.25}$$

The solution to the perturbation equation (2.24) is then given by

$$h_{\mu\nu}(x, z) = -2 \int d^4x' G_R(x, z; x', 0) \Sigma_{\mu\nu}(x') \tag{2.26}$$

where we have integrated across the surface $z' = 0$. Since we are in the RS gauge, $h^\mu_\mu = 0$ and so

$$\Sigma^\mu_\mu = 0 \qquad \Rightarrow \qquad \Box^{(4)} f = \frac{4\pi G_5}{3} \mathcal{T}. \tag{2.27}$$

$f(x)$ represents the brane position in RS gauge and in principle we can calculate it by solving equation (2.27). Here we see explicitly that the brane is bent by the presence of additional matter because $\mathcal{T}$ acts as a source for $f(x)$.

In order to evaluate the full Green's function we will use techniques from Sturm Liouville theory. We will simply state the result here although a detailed derivation can be found in appendix A.1.

$$G_R(x,z;x',z') = -\int \frac{d^4p}{(2\pi)^4} e^{ip_\mu(x^\mu - x'^\mu)} \left[\frac{e^{-2k(|z|+|z'|)}k}{\mathbf{p}^2 - (\omega+i\epsilon)^2} + \int_0^\infty dm \, \frac{v_m(z)v_m(z')}{m^2+\mathbf{p}^2-(\omega+i\epsilon)^2} \right], \tag{2.28}$$

where

$$v_m(z) = \frac{\sqrt{m/2k}\left[J_1(m/k)Y_2(me^{k|z|}/k) - Y_1(m/k)J_2(me^{k|z|}/k)\right]}{\sqrt{J_1(m/k)^2 + Y_1(m/k)^2}}. \tag{2.29}$$

and J_n, Y_n are Bessel's functions of integer order n.

If we return to GN coordinates, we can define the stationary point mass m_0 to be located at $(t,\mathbf{x},z) = (t,\mathbf{0},0)$ so that its energy momentum tensor on the brane is given by

$$\mathcal{T}_{ab} = m_0\delta^{(3)}(\mathbf{x})\mathrm{diag}(1,0,0,0,0) \tag{2.30}$$

Combining equation (2.21) with equation (2.26) we obtain an expression for the gravitational perturbation in this gauge.

$$\hat{h}_{\mu\nu}(x,z) = h^{(m)}_{\mu\nu} + h^{(f)}_{\mu\nu} + \frac{1}{k}f_{,\mu\nu} + 2ke^{-2k|z|}\eta_{\mu\nu}f - e^{-2k|z|}\eta_{\rho(\nu}F^{\rho}{}_{,\nu)}, \tag{2.31}$$

where the matter part and the brane bending part are given by

$$h^{(m)}_{\mu\nu} = -16\pi G_5 \int d^4x' \; G_R(x,z;x',0)\left(\mathcal{T}_{\mu\nu} - \frac{1}{3}\mathcal{T}\eta_{\mu\nu}\right) \tag{2.32}$$

$$h^{(f)}_{\mu\nu} = -4\int d^4x' \; G_R(x,z;x',0)f_{,\mu\nu} \tag{2.33}$$

Since we are only interested in the perturbation on the brane, we set $z=0$, and can choose F^μ appropriately so that

$$\hat{h}_{\mu\nu}(x,0) = 2k\eta_{\mu\nu}f - 16\pi G_5 \int d^4x' \; G_R(x,0;x',0)\left(\mathcal{T}_{\mu\nu} - \frac{1}{3}\mathcal{T}\eta_{\mu\nu}\right) \tag{2.34}$$

To evaluate $f(x)$, we solve equation (2.27) with $\mathcal{T} = m_0\delta^{(3)}(\mathbf{x})$. Note that our source is stationary so we look for time independent solutions. With this ansatz,

the differential operator in equation (2.27) is reduced to the Laplacian so that

$$f(x) = \frac{G_5 m_0}{3r} \tag{2.35}$$

where $r = |\mathbf{x}|$. We now evaluate the matter part of the perturbation $h^{(m)}_{\mu\nu}(x,0)$ when we insert the energy momentum tensor (2.30).

$$h^{(m)}_{\mu\nu}(x,0) = -\frac{16\pi G_5 m_0}{3}\mathrm{diag}(2,1,1,1)\int dt'\, G_R(t,\mathbf{x},0;t',\mathbf{0},0) \tag{2.36}$$

where

$$\int dt'\, G_R(t,\mathbf{x},0;t',\mathbf{0},0) = -\frac{k}{4\pi r} - \int_0^\infty dm\, \frac{e^{-mr}}{4\pi r}[v_m(0)]^2 \tag{2.37}$$

The integration over m is exponentially suppressed for $m > 1/r$. For small m,

$$[v_m(0)]^2 = \frac{m}{2k} + \mathcal{O}(m/k)^2 \tag{2.38}$$

where we have used the fact that

$$J_n(m/k) \sim \frac{1}{n!}(m/2k)^n\,, \qquad Y_n(m/k) \sim \frac{(n-1)!}{\pi}(m/2k)^{-n} \tag{2.39}$$

in this limit. The matter part of the perturbation is therefore given by

$$h^{(m)}_{\mu\nu}(x,0) = \frac{2G_5 k m_0}{3r}\mathrm{diag}(2,1,1,1)\left[2 + \frac{1}{k^2r^2} + \mathcal{O}(1/r^3)\right] \tag{2.40}$$

Inserting the solution (2.35) for f into equation (2.34) yields the full metric perturbation

$$\hat{h}_{\mu\nu}(x,0) = \frac{2G_5 k m_0}{r}\left[\mathrm{diag}(1,1,1,1) + \frac{1}{3k^2r^2}\mathrm{diag}(2,1,1,1) + \mathcal{O}(1/r^3)\right] \tag{2.41}$$

We are ready to read off the Newtonian potential, $\phi(r)$, measured by a braneworld observer distance r away from the source. This is given by

$$\phi(r) = \frac{1}{2}\hat{h}_{00} = \frac{G_5 k m_0}{r}\left[1 + \frac{2}{3k^2r^2} + \mathcal{O}(1/r^3)\right] \tag{2.42}$$

This is the Newtonian potential of four-dimensional gravity, with Yukawa type corrections at short distances ($r < 1/k$). Note that the four-dimensional Newton's constant on the brane, $G_4 = G_5 k$. We conclude that this model does not contradict experimental tests of Newton's inverse square law for the force of gravitational attraction.

2.2.2.2 The graviton propagator

In the previous section we were careful to include the scalar field f corresponding to brane bending. This appeared because additional matter on the brane acted as a source for the field. However, consider what would have happened had we naively ignored it and worked in the RS gauge throughout, with the brane at a fixed position. The Newtonian potential would still have behaved like $1/r$ to leading order. We would have been conned into thinking we had derived four-dimensional gravity.

However, the Newtonian potential is not the only property of four-dimensional gravity that we can consider. There is also the form of the massless graviton propagator. In a five-dimensional theory, there is an extra polarization state that alters the tensor structure of the propagator. This extra degree of freedom must be removed from the effective theory so that the massless propagator on the brane looks four-dimensional. If this didn't happen, the bending of light, for example, would be $\frac{3}{4}$ of the value accurately predicted by General Relativity [42].

In RS2 we also have massive KK gravitons. Even in the small mass limit the tensor structure of their propagator is five dimensional [42–45]. Since these are only important at high energies we will ignore them in our effective theory and focus on the massless graviton bound state.

From equation (2.32), the matter part of the metric perturbation on the brane is given by

$$h_{\mu\nu}^{(m)} = -16\pi G_5 \int d^4x' \; G_R(x,0;x',0) \left(\mathcal{T}_{\mu\nu} - \frac{1}{3}\,\mathcal{T}\eta_{\mu\nu} \right) \tag{2.43}$$

If we ignore the massive modes then the Green's function takes the following truncated form

$$G_R(x,0;x',0) = \frac{k}{\Box^{(4)}} \tag{2.44}$$

where

$$\frac{1}{\Box^{(4)}} = -\int \frac{d^4p}{(2\pi)^4} \, \frac{e^{ip_\mu(x^\mu - x'^\mu)}}{\mathbf{p}^2 - (\omega + i\epsilon)^2} \tag{2.45}$$

is the massless scalar Green's function for four-dimensional Minkowski space [46,47]. If we insert the truncated Green's function (2.44) into equation (2.43) we see that

we do not have the usual propagator for a massless four-dimensional graviton. We need the factor of $\frac{1}{3}$ to be replaced by $\frac{1}{2}$. This task is carried out by the brane bending term as we shall now demonstrate.

The *full* metric perturbation (2.34) contains a term proportional to f. We can express f in terms of the four-dimensional Green's function using equation (2.27)

$$f(x) = \frac{4\pi G_5}{3} \int d^4x' \, \frac{1}{\Box^{(4)}} \mathcal{T}. \tag{2.46}$$

When this is introduced into equation (2.34) we find that the (massless) metric perturbation is given by

$$h_{\mu\nu} = -16\pi G_5 k \int d^4x' \, \frac{1}{\Box^{(4)}} \left(\mathcal{T}_{\mu\nu} - \frac{1}{2} \mathcal{T} \eta_{\mu\nu} \right) \tag{2.47}$$

This has the correct tensor structure for a four-dimensional massless graviton. The extra degree of freedom in the five-dimensional propagator has been compensated for by the brane bending scalar field f.

The two results derived in this section are good evidence that braneworld gravity agrees with General Relativity, at least for small perturbations about the background metric. The warped geometry of the bulk causes these perturbations to be damped away from the brane, so that gravity is localised. The fact that the brane has positive tension is crucial as the warp factor is a maximum there. In RS1, we chose to live on the negative tension brane which is at a minimum of the warp factor. We would not therefore expect gravity to be localised on this type of braneworld, which makes its solution to the hierarchy problem a little pointless. However, the ideas of both models can be combined such they solve the hierarchy problem and exhibit localisation of gravity [48]. In this case there are two positive tension branes, the Planck brane and the TeV brane. The Planck brane has a much larger tension than the TeV brane, which in some sense is regarded as a probe. The hierarchy problem is solved in exactly the same way as in RS1 provided we live on the TeV brane. In a similar way to RS2, we find that gravity looks four-dimensional at least up to a few TeV on both branes.

2.2.3 Non-critical braneworlds

Although the RS2 model agrees with Newton's Law and other properties of four-dimensional gravity, it certainly contradicts one recent experimental observation. The study of supernovae suggest that the universe contains a small positive cosmological constant [21, 22]. In RS2, we have Minkowski space on the brane which has a vanishing cosmological constant. In this section we shall show how to extend the model to allow for de Sitter or anti-de Sitter braneworlds.

Recall that we have so far demanded that our braneworlds should exhibit four-dimensional Poincaré invariance. This led to the ansatz (2.3) which has Minkowski spacetime induced on the brane. We found that we then had to fine tune the brane tension, σ against the bulk cosmological constant, Λ, in the following way

$$\frac{4\pi G_5 \sigma}{3} = k, \qquad \Lambda = -6k^2 \tag{2.48}$$

This is the *criticality condition* and as such the flat braneworlds that satisfy it are known as *critical*. We now generalise the ansatz (2.3) to allow for dS and AdS branes.

$$ds^2 = a^2(z) g_{\mu\nu} dx^\mu dx^\nu + dz^2 \tag{2.49}$$

where $g_{\mu\nu}$ can be Minkowski, de Sitter or anti-de Sitter. The solutions to the bulk equations of motion with appropriate boundary conditions are derived in [49–51] although a review may be found in appendix A.2. In this section we will proceed as in [52] and simply quote the results.

$$\text{de Sitter}: \quad a(z) = \frac{1}{k}\sqrt{\frac{\lambda}{3}}\sinh(c - k|z|) \qquad k = \sqrt{\frac{\lambda}{3}}\sinh c, \tag{2.50}$$

$$\text{Minkowski}: \quad a(z) = e^{-k|z|}, \tag{2.51}$$

$$\text{anti-de Sitter}: \quad a(z) = \frac{1}{k}\sqrt{-\frac{\lambda}{3}}\cosh(c - k|z|), \qquad k = \sqrt{-\frac{\lambda}{3}}\cosh c, \tag{2.52}$$

where the cosmological constant on the brane is given by

$$\lambda = 3(\tilde{\sigma}^2 - k^2), \qquad \tilde{\sigma} = \frac{4\pi G_5 \sigma}{3}. \tag{2.53}$$

When σ takes its critical value we have $\tilde{\sigma} = k$, and the cosmological constant on the brane vanishes. For de Sitter branes, σ exceeds its critical value ($\tilde{\sigma} > k$) where

as the opposite is true for anti-de Sitter branes. For this reason we refer to dS and AdS branes as *supercritical* and *subcritical* branes respectively.

In section 2.2.2 we saw how gravity was localised on critical braneworlds. This was due to the behaviour of the warp factor, which damped gravitational perturbations as they went further into the bulk. We can ask whether the same is true for supercritical and subcritical braneworlds. Without performing a detailed analysis we can see the behaviour of the warp factors in figures 2.3 and 2.4. In each case,

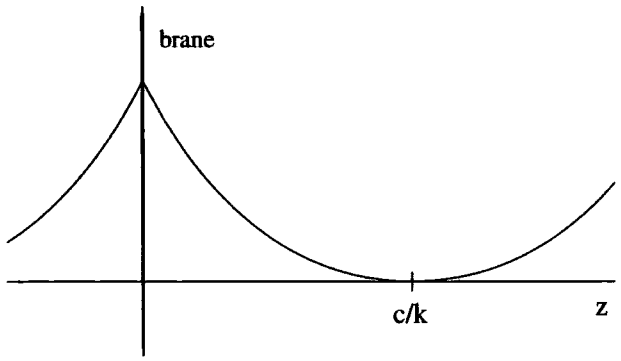


Figure 2.3: The behaviour of the warp factor around a supercritical (ie de Sitter) brane.

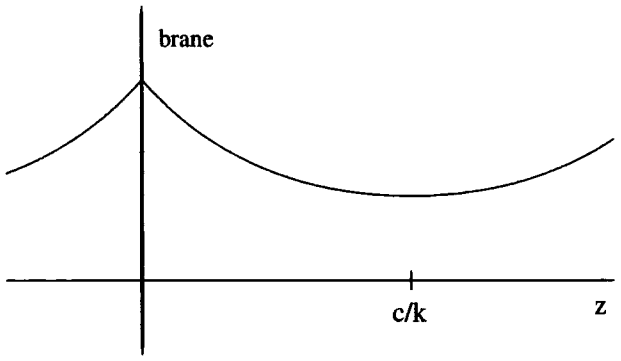


Figure 2.4: The behaviour of the warp factor around a subcritical (ie anti-de Sitter) brane.

there is a turnaround in the warp factor. For the de Sitter brane this corresponds to the de Sitter horizon where the warp factor vanishes altogether, and the spacetime

ends. It is clear that de Sitter branes are even more likely to exhibit four-dimensional gravity than flat branes, because the damping is greater. This is argued in [52] and proven in [53, 54]. Unlike in RS2, there is a mass gap between the zero mode and the heavy modes in the metric perturbations. We further note that the Newton's constant on the brane is found to be proportional to the brane tension, σ, as opposed to the bulk quantity k.

The situation for the anti-de Sitter brane is less clear. Near the brane the fluctuations in the metric behave in the same way as for de Sitter and flat branes. However, the warp factor does not vanish at the turnaround point, and beyond this the metric perturbations start to grow. If we assume that this point lies far from the brane we might yet believe that gravity is localised at low enough energies. At finite temperature we could even hide the point behind a black hole horizon. Despite the absence of a normalisable zero mode the case for localisation is presented in [52].

Finally, in this section we have seen how braneworld models can exhibit four-dimensional gravity in line with experimental observations. They also provide an unusual resolution of the hierarchy problem, without the need for an unacceptably large (but finite) extra dimension. Given our extension to non-critical branes, we could also rephrase the cosmological constant problem. This is now a question of balancing the tension and other matter fields on the brane against the bulk cosmological constant [55, 56].

Chapter 3

Brane Cosmology

3.1 Introduction

We have seen how Randall-Sundrum braneworlds provide a radical new way of thinking about our universe and the extra dimensions that might exist. If this extra dimension is warped anti-de Sitter space then it can be infinitely large and still exhibit localisation of gravity on the brane. We have also seen how to generalise the RS2 model to include super/subcritical braneworlds which have a positive/negative cosmological constant in four dimensions.

To better understand these models we can and should generalise further. We note that in the last section we always assumed a five-dimensional bulk which was $\mathbb{Z}_2$ symmetric about a brane of codimension one. In this section we will consider bulk spacetimes which are n-dimensional and in some cases relax the $\mathbb{Z}_2$ symmetry. We will not generalise to branes of higher codimension although they have been studied (see for example [57–59]).

Another very important assumption of the last section was the fact that perturbations about the background spacetime were small: the energy-momentum due to additional matter on the brane was far less than the brane tension.

$$\mathcal{T}_{00} \ll \sigma \tag{3.1}$$

Unfortunately, life is not so easy as to be fully described by perturbative physics. We will begin a study of non-perturbative physics on the brane by examining their

cosmology. There are two main approaches: the *brane based* approach and the *bulk based* approach, although we will show that these are in fact equivalent. Each approach has its advantages and disadvantages. For example, if we wished to examine non-$\mathbb{Z}_2$ symmetric theories it would be much easier to use the latter. However, we begin with a review of the brane based approach of Shiromizu *et al* [23], and although we will retain $\mathbb{Z}_2$ symmetry we will generalise their work to n-dimensions.

3.2 Brane based braneworld cosmology

Consider a timelike $(n-2)$-brane, (M, h_{ab}), in an n-dimensional bulk spacetime (V, g_{ab}). The induced metric on M is given by

$$h_{ab} = g_{ab} - n_a n_b \tag{3.2}$$

where n^a is the unit normal to M (see figure 3.1). By using the Gauss-Codazzi

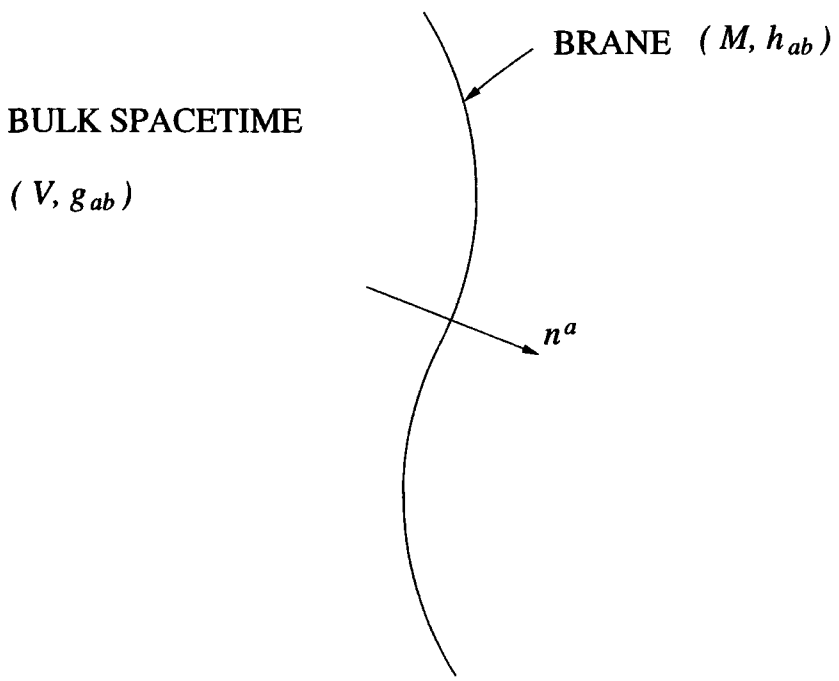


Figure 3.1: $(n-2)$-brane embedded in an n-dimensional bulk.

equations [60] we can relate the $(n-1)$-dimensional geometry on M to its extrinsic

curvature $K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)}$ in V and the bulk geometry. If we label curvature tensors with an n or $(n-1)$ depending on whether they correspond to the bulk or the brane respectively, we have

$$
\begin{aligned}
{}^{(n-1)}R_{abcd} &= {}^{(n)}R_{pqrs} h_a^p h_b^q h_c^r h_d^s + K_{ac}K_{bd} - K_{ad}K_{bc} && \text{(3.3a)} \\
D_b(K_a^b - K h_a^b) &= {}^{(n)}R_{cd} n^c h_a^d && \text{(3.3b)} \\
-2{}^{(n)}G_{ab} n^a n^b &= {}^{(n-1)}R - K^2 + K_{ab}K^{ab} && \text{(3.3c)}
\end{aligned}
$$

where D_a is the covariant derivative made out of h_{ab}. When there is no $\mathbb{Z}_2$ symmetry, we label the "left hand" bulk with a "$-$" and the "right hand" bulk with a "$+$". There is a version of equations (3.3a) to (3.3c) for both "$+$" and "$-$", so in principle we should label each of the bulk quantities (${}^{(n)}R_{abcd}$ and K_{ab}) with the appropriate sign. However, for now we shall assume $\mathbb{Z}_2$ symmetry so we drop the labels.

From equation (3.3a) we are able to construct the Einstein tensor on the brane

$$
\begin{aligned}
{}^{(n-1)}G_{ab} &= {}^{(n)}G_{cd} h_a^c h_b^d - {}^{(n)}R_{pqrs} n^p n^r h_a^q h_b^s + {}^{(n)}R_{cd} n^c n^d h_{ab} \\
&\quad + K K_{ab} - K_a^c K_{bc} - \frac{1}{2} h_{ab} \left(K^2 - K^{cd} K_{cd} \right) && \text{(3.4)}
\end{aligned}
$$

We now use the bulk equations of motion

$$
{}^{(n)}G_{ab} = {}^{(n)}R_{ab} - \frac{1}{2}{}^{(n)}R g_{ab} = -\Lambda_n g_{ab} + 8\pi G_n T_{ab} \qquad \text{(3.5)}
$$

where Λ_n is the bulk cosmological constant, G_n is the Newton's constant in n-dimensions, and T_{ab} is the energy-momentum tensor due to any additional bulk fields. We can also express the bulk Riemann tensor in terms of the Weyl and Ricci tensors.

$$
\begin{aligned}
{}^{(n)}R_{abcd} &= {}^{(n)}C_{abcd} + \frac{1}{n-2} \left({}^{(n)}R_{ac} g_{bd} - {}^{(n)}R_{ad} g_{bc} + {}^{(n)}R_{bd} g_{ac} - {}^{(n)}R_{bc} g_{ad} \right) \\
&\quad - \frac{1}{(n-1)(n-2)} {}^{(n)}R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) && \text{(3.6)}
\end{aligned}
$$

Inserting equations (3.5) and (3.6) into equation (3.4) we find

$$
{}^{(n-1)}G_{ab} = -\Lambda_n \left(\frac{n-3}{n-1} \right) h_{ab} - E_{ab} + K K_{ab} - K_a^c K_{bc} - \frac{1}{2} h_{ab} \left(K^2 - K^{cd} K_{cd} \right) \qquad \text{(3.7)}
$$

where

$$
E_{ab} = C_{pqrs} n^p n^r h_a^q h_b^s - \left(\frac{n-3}{n-2} \right) \left[h_a^c h_b^d + n^c n^d h_{ab} - \frac{1}{n-1} g^{cd} h_{ab} \right] 8\pi G_n T_{cd} \qquad \text{(3.8)}
$$

This term is often described as the "electric" part of the Weyl tensor although this is only the case when there are no extra bulk fields and $T_{ab} \equiv 0$. We can make sense of the extrinsic curvature terms by using the Israel equations [41] at the brane

$$\Delta K_{ab} = -8\pi G_n \left(S_{ab} - \frac{1}{n-2} S h_{ab} \right) \tag{3.9}$$

where the energy-momentum tensor for the brane is given by

$$S_{ab} = -\sigma h_{ab} + \mathcal{T}_{ab} \tag{3.10}$$

with $\mathcal{T}_{ab} n^b = 0$. Here we understand σ to correspond to brane tension and $\mathcal{T}_{ab}$ to additional matter, although it not obvious that we should do this. In section 2.2.2.1 we assumed the additional matter $\mathcal{T}_{ab}$ was much smaller than the brane tension. This meant that the split between tension and extra matter in equation (3.10) was natural. However, we are now allowing for larger values of $\mathcal{T}_{ab}$ which makes the split an arbitrary one. It is not clear why we should have tension σ rather than (say) $\sigma/2$ because we could always redefine $\mathcal{T}_{ab}$ to absorb the left over terms. However, we shall see in chapter 6 some evidence that we are in fact interpreting equation (3.10) in the right way.

At this stage we are assuming $\mathbb{Z}_2$ symmetry across the brane so we have $\Delta K_{ab} = 2K_{ab}$. Using the Israel equation (3.9) we can replace the extrinsic curvature terms in equation (3.7) with terms involving σ and $\mathcal{T}_{ab}$.

$$^{(n-1)}G_{ab} = -\Lambda_{n-1} h_{ab} + 8\pi G_{n-1} \mathcal{T}_{ab} + (4\pi G_n)^2 \Pi_{ab} - E_{ab} \tag{3.11}$$

where

$$\Lambda_{n-1} = \frac{1}{2}(n-2)(n-3)\left[\sigma_n^2 + \frac{2}{(n-1)(n-2)}\Lambda_n\right] \tag{3.12}$$

$$G_{n-1} = \frac{G_n \sigma_n (n-3)}{2} \tag{3.13}$$

$$\Pi_{ab} = -\mathcal{T}_a^c \mathcal{T}_{bc} + \frac{1}{n-2}\mathcal{T}\,\mathcal{T}_{ab} + \frac{1}{2}\mathcal{T}^{cd}\mathcal{T}_{cd} h_{ab} - \frac{1}{2n-4}\mathcal{T}^2 h_{ab} \tag{3.14}$$

and

$$\sigma_n = \frac{4\pi G_n \sigma}{n-2} \tag{3.15}$$

The most striking feature of equation (3.11) is the presence of the quadratic matter terms contained in Π_{ab}. We will discuss these in more detail later on. Meanwhile, we

see that we should interpret Λ_{n-1} and G_{n-1} as the braneworld cosmological constant and Newton's constant respectively. As we hinted at the end of section 2.2.3, G_{n-1} is proportional to the brane tension, rather than $\sqrt{|\Lambda_n|}$. This is highly relevant to non-critical branes, although it is often ignored.

The other term in equation (3.11) is of course the "Weyl tensor" term, E_{ab}. It contains information about the bulk but is constrained by the matter on the brane. We might hope to fully determine E_{ab} from knowledge of this matter, but this turns out not to be the case. In general we need to solve the bulk equations of motion to derive E_{ab} and then insert it into the braneworld Einstein equation. We will discuss this mysterious term from a holographic point of view in chapters 5 and 6.

3.2.1 A Friedmann-Robertson-Walker brane

We will now simplify the discussion further by assuming that the bulk spacetime has negative cosmological constant with no additional fields, that is

$$\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2, \qquad T_{ab} \equiv 0 \tag{3.16}$$

where k_n is the inverse AdS length in n-dimensions. The cosmological constant on the brane is now given by

$$\Lambda_{n-1} = \frac{1}{2}(n-2)(n-3)\left[\sigma_n^2 - k_n^2\right] \tag{3.17}$$

Note that equations (3.15) and (3.17) are the n-dimensional analogue of equation (2.53). Critical branes are now defined as those satisfying the n-dimensional criticality condition $\sigma_n = k_n$. Super/subcritical branes now have $\sigma_n > k_n / \sigma_n < k_n$ respectively. For a study of cosmology it is important to examine the behaviour of a homogeneous and isotropic braneworld described by a Friedmann-Robertson-Walker (FRW) metric.

$$ds_{n-1}^2 = h_{ab}dx^a dx^b = -d\tau^2 + Z^2(\tau)d\mathbf{x}_\kappa^2 \tag{3.18}$$

where $d\mathbf{x}_\kappa^2$ is the metric on an $(n-2)$-dimensional Euclidean space, X of constant curvature, $\kappa = 0, \pm 1$.

$$X = \begin{cases} S^{n-2} & \text{for } \kappa = 1 \\ \mathbb{R}^{n-2} & \text{for } \kappa = 0 \\ H^{n-2} & \text{for } \kappa = -1 \end{cases} \tag{3.19}$$

where $S^{n-2}, \mathbb{R}^{n-2}, H^{n-2}$ are the unit sphere, plane, and hyperboloid respectively. $Z(\tau)$ represents the scale factor for our braneworld. We will assume the matter on the brane is given by a homogeneous perfect fluid of density $\rho(\tau)$ and pressure $p(\tau)$ so that

$$\mathcal{T}_{ab} = \rho \tau_a \tau_b + p(h_{ab} + \tau_a \tau_b) \tag{3.20}$$

where τ^a are the components of $\frac{\partial}{\partial \tau}$. Finally, we avoid difficulties with E_{ab} by setting it to zero, which corresponds to *pure* anti-de Sitter space in the bulk. We now use the braneworld Einstein equation (3.11) to derive the FRW equations for the cosmological evolution of the brane. Defining the Hubble parameter, $H = \dot{Z}/Z$, where dot denotes differentiation with respect to τ, we find

$$H^2 = a - \frac{\kappa}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho + \left(\frac{4\pi G_n}{n-2}\right)^2 \rho^2 \tag{3.21a}$$

$$\dot{H} = \frac{\kappa}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \tag{3.21b}$$

where $a = \sigma_n^2 - k_n^2$. These are not the standard FRW equations because they contain terms quadratic in ρ and p. Braneworld cosmology is therefore different to the standard cosmology. This unconventional behaviour was first discovered in five dimensions by Binetruy *et al* [61]. Notice that we recover the standard cosmology for large values of the scale factor, because we can ignore the non-linear density terms.

3.3 Bulk based braneworld cosmology

In the last section we saw a number of the limitations of the brane based approach to braneworld cosmology. We chose to impose $\mathbb{Z}_2$ symmetry across the brane and ignored the possibility of non-zero Weyl terms. These were difficult to get a handle

on because we were working with a static brane in a dynamic bulk. The bulk based approach turns everything around by having a dynamic brane in a static bulk. This allows us to include non-$\mathbb{Z}_2$ symmetric branes and non-vanishing Weyl terms. The disadvantage now is that we will only be considering FRW branes, and will not have the generalisation provided by equation (3.11).

3.3.1 Generalised Birkhoff's Theorem

Since the bulk based approach works on the premise of there being a static bulk spacetime, we immediately think of Birkhoff's Theorem [62,63]. This states that *if the geometry of a given region of spacetime is spherically symmetric and a solution to the vacuum Einstein equations, then it is necessarily a piece of the Schwarzschild geometry.* In order to bridge the gap between the brane based approach to braneworld cosmology and the bulk based approach, we will prove a generalised version of this theorem. This was first shown by Bowcock *et al* [64] in five dimensions, but once again we will extend the ideas to n-dimensions.

We start by assuming that our spacetime contains a codimension two Euclidean surface of constant curvature. This will ultimately provide us with spatial homogeneity on our braneworld. The most general metric admitting this symmetry is given by [1,2]

$$ds^2 = A^{\frac{2}{n-2}} d\mathbf{x}_\kappa^2 + e^{2\nu} A^{-\left(\frac{n-3}{n-2}\right)} (-dt^2 + dz^2) \tag{3.22}$$

where A and ν are functions of t and z to be determined by the bulk Einstein equations, as well as the jump conditions across the brane. Again, $d\mathbf{x}_\kappa^2$ represents the metric on the Euclidean surface of constant curvature, $\kappa = 0, \pm 1$. Here we have used the fact that the rest of the metric is two dimensional and therefore conformally flat. Without loss of generality, we can say that the brane sits at $z = 0$[1].

We will assume that the bulk spacetime contains no additional matter ($T_{ab} \equiv 0$). When we insert our metric ansatz into the bulk Einstein equations (3.5) we arrive

[1] If the brane sits at $z' = \zeta(t')$ we use the conformal transformation $t' \pm z' = t \pm z \pm \zeta(t \pm z)$ to shift the wall back to $z = 0$ without spoiling the form of the metric (3.22) [64].

at the following set of differential equations

$$A_{,tt} - A_{,zz} = \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)\kappa A^{-\frac{1}{n-2}}\right] e^{2\nu} \tag{3.23a}$$

$$\nu_{,tt} - \nu_{,zz} = \left[\frac{\Lambda_n}{n-2} A^{-\left(\frac{n-3}{n-2}\right)} + \frac{n-3}{2}\kappa A^{-\left(\frac{n-1}{n-2}\right)}\right] e^{2\nu} \tag{3.23b}$$

$$A_{,tt} + A_{,zz} = 2\nu_{,z} A_{,z} + 2\nu_{,t} A_{,t} \tag{3.23c}$$

$$A_{,tz} = \nu_{,z} A_{,t} + \nu_{,t} A_{,z} \tag{3.23d}$$

It is convenient to change to lightcone coordinates

$$u = \frac{t-z}{2}, \qquad v = \frac{t+z}{2} \tag{3.24}$$

so that we now have

$$A_{,uv} = \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)\kappa A^{-\frac{1}{n-2}}\right] e^{2\nu} \tag{3.25a}$$

$$\nu_{,uv} = \left[\frac{\Lambda_n}{n-2} A^{-\left(\frac{n-3}{n-2}\right)} + \frac{n-3}{2}\kappa A^{-\left(\frac{n-1}{n-2}\right)}\right] e^{2\nu} \tag{3.25b}$$

$$2\nu_{,u} A_{,u} = A_{,u}\left[\ln(A_{,u})\right]_{,u} \tag{3.25c}$$

$$2\nu_{,v} A_{,v} = A_{,v}\left[\ln(A_{,v})\right]_{,v} \tag{3.25d}$$

We can easily integrate equations (3.25c) and (3.25d) to give

$$\text{Case I}: \ A \text{ is constant}$$

$$\text{Case II}: \ A = A(u), \qquad e^{2\nu} = A'(u)V'(v)$$

$$\text{Case III}: \ A = A(v), \qquad e^{2\nu} = A'(v)U'(u)$$

$$\text{Case IV}: \ A = A(u,v), \qquad e^{2\nu} = V'(v)A_{,u} = U'(u)A_{,v}$$

where $U'(u)$ and $V'(v)$ are arbitrary non-zero functions of u and v respectively. Note that *prime* denotes differentiation with respect to the unique argument of the function. Cases I to III imply that $\Lambda_n = \kappa = 0$, which is not relevant here (see [64,65] for some discussion). We will focus on case IV, for which it is easy to see that

$$A = A(U(u) + V(v)), \qquad e^{2\nu} = A'U'V' \tag{3.26}$$

so that equation (3.25a) is reduced to an ODE

$$A'' - \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)\kappa A^{-\frac{1}{n-2}}\right] A' = 0 \tag{3.27}$$

$$\Rightarrow A' - 2\left(\frac{n-2}{n-1}\right)\Lambda_n A^{\left(\frac{n-1}{n-2}\right)} + (n-2)^2\kappa A^{\left(\frac{n-3}{n-2}\right)} = (n-2)^2 c \tag{3.28}$$

where c is a constant of integration. Notice that equation (3.25b) just gives the derivative of the ODE, and is satisfied automatically. We are now ready to impose the jump conditions on the brane. Once again we will assume that the matter on the brane is homogeneous and isotropic so that

$$S_{ab} = -\sigma h_{ab} + \mathcal{T}_{ab}, \qquad \mathcal{T}_{ab} = \rho\tau_a\tau_b + p(h_{ab} + \tau_a\tau_b)$$

where τ^a is the unit timelike vector parallel to $\frac{\partial}{\partial t}$. When there is $\mathbb{Z}_2$ symmetry across the brane at $z = 0$, the Israel equations (3.9) give

$$4\pi G_n(\sigma + \rho) = -e^{-\nu}A^{-\frac{1}{2}\left(\frac{n-1}{n-2}\right)}A_{,z} = \frac{1}{2}e^{-\nu}A^{-\frac{1}{2}\left(\frac{n-1}{n-2}\right)}[U' - V']A' \tag{3.30}$$

$$\begin{aligned} &4\pi G_n\left[\frac{n-3}{n-2}(\sigma + \rho) - \sigma + p\right] = -\partial_z\left[e^{-\nu}A^{\frac{1}{2}\left(\frac{n-3}{n-2}\right)}\right] \\ &= \frac{1}{4}e^{-\nu}A^{\frac{1}{2}\left(\frac{n-3}{n-2}\right)}\left[(V' - U')\left(\frac{A''}{A} - \left(\frac{n-3}{n-2}\right)\frac{A'}{A}\right) + \frac{V''}{V} - \frac{U''}{U}\right] \end{aligned} \tag{3.31}$$

Note that we could use equation (3.28) to eliminate A' and A''. If we make the following coordinate transformation

$$u \to f(u), \qquad v \to f(v) \tag{3.32}$$

then the boundary conditions at the brane are unchanged[2]. This symmetry is related to the conformal symmetry on the $t - z$ plane. To eliminate this unphysical gauge freedom we choose $f = V$, thereby setting $V = v$. We are now left with only one physical degree of freedom, $U(u)$. Setting

$$Z = A^{\frac{1}{n-2}}, \qquad T = (n-2)(v - U) \tag{3.33}$$

we see that the bulk metric can locally be written in the explicitly static form

$$ds_n^2 = -h(Z)dT^2 + \frac{dZ^2}{h(Z)} + Z^2 d\mathbf{x}_\kappa^2 \tag{3.34}$$

where

$$h(Z) = -\frac{Z'}{n-2} = -\frac{A'A^{-\left(\frac{n-3}{n-2}\right)}}{(n-2)^2} \tag{3.35}$$

[2]This is seen if we note that the brane is given by $u = v$, where the coordinate change gives $U' \to f'(u)U'$, $V' \to f'(u)V'$ and $e^{-\nu} = 1/\sqrt{A'U'V'} \to e^{-\nu}/f'(u)$. A' and A'' are unchanged

From equation (3.28)

$$h(Z) = -\frac{2\Lambda_n}{(n-1)(n-2)}Z^2 + \kappa - \frac{c}{Z^{n-3}} \tag{3.36}$$

For $c > 0$, the metric (3.34) clearly takes the form of the Schwarzschild black hole in de Sitter, flat or anti-de Sitter space, depending on the value of Λ_n. Given that our starting point was that our braneworld contained spatial geometry of constant curvature, we conclude that we have indeed proved a generalised version of Birkhoff's theorem. In this work we assumed our bulk physics was described by pure Einstein gravity with a cosmological constant. Similar proofs have been carried out for Einstein-Maxwell gravity [66] and Gauss-Bonnet gravity [67].

Although this generalisation of Birkhoff's Theorem is of interest from a mathematical point of view, our focus is on braneworld physics. We have shown that we can express the bulk geometry in the static form given by equation (3.34), although in doing so we can no longer say that we have a static brane sitting quietly at $z = 0$. On the contrary, we now have a dynamic brane, whose trajectory in the new coordinates is far more complicated. Braneworld cosmology from this perspective was first studied by Ida [68], although moving branes in a static anti-de Sitter bulk were considered earlier by Kraus [69].

3.3.2 A dynamic brane in a static bulk

Having bridged the gap from the brane based approach to braneworld cosmology we are ready to give a generalisation of Ida's bulk based approach. We will see that by transferring the dynamics of the system from the bulk to the brane we allow ourselves more flexibility regarding the structure of the bulk spacetime. We will no longer assume $\mathbb{Z}_2$ symmetry across the brane and will even allow the cosmological constant on either side to differ.

We start by taking the general static solution (3.34) to the Einstein equations with cosmological constant, Λ_n. To construct the brane solution, we treat the brane as the boundary

$$X^a = (\mathbf{x}^\mu, t(\tau), Z(\tau)) \tag{3.37}$$

of the bulk (3.34). We now patch this bulk spacetime (labelled with a"$-$") onto

another appropriate bulk (labelled with a "+") with the same boundary value $Z(\tau)$. Note that we have reintroduced the "$\pm$" notation to indicate which side of the brane a given quantity resides[3]. We set the parameter τ to correspond to the proper time with respect to an observer comoving with the brane. This imposes the conditions

$$-h^{\pm}\dot{t}_{\pm}^2 + \frac{\dot{Z}^2}{h^{\pm}} = -1 \tag{3.38}$$

so that whichever side of the brane you look from, the induced metric on the brane takes the standard FRW form

$$ds_{n-1}^2 = h_{ab}dx^a dx^b = -d\tau^2 + Z^2(\tau)d\mathbf{x}_\kappa^2 \tag{3.39}$$

and $Z(\tau)$ is understood to be the scale factor of the brane universe. It is clear that the bulk metric is continuous across the brane because both τ and $Z(\tau)$ agree there. Note that t can be discontinuous at the brane, because neither g_{ab} nor h_{ab} depend on it explicitly.

In order to produce the type of brane required, it is important we patch together the two bulk spacetimes in such a way that the Israel equations (3.9) are satisfied. We take the energy momentum tensor of the brane to be given by a tension σ and a perfect fluid of energy density ρ and pressure p (that is, equation (3.10) with $\mathcal{T}_{ab}$ given by (3.20)). In defining the extrinsic curvature of the brane on either side, we need some knowledge of the outward normal.

$$n_a^{\pm} = \epsilon_{\pm}(\mathbf{0}, -\dot{Z}(\tau), \dot{t}_{\pm}(\tau)) \tag{3.40}$$

where $\epsilon^{\pm} = \pm 1$ depending on which part of the spacetime is kept[4]. With reference to appendix A.3, the Israel equations now yield the following

$$\frac{1}{Z}\overline{[\epsilon h\dot{t}]} = \frac{4\pi G_n}{n-2}(\sigma + \rho) \tag{3.41}$$

$$\overline{\left[\frac{\ddot{Z} + \frac{1}{2}h'}{\epsilon h\dot{t}}\right]} = \frac{4\pi G_n}{n-2}\left[\sigma - (n-3)\rho - (n-2)p\right] \tag{3.42}$$

[3]For example, g_{ab}^+ and Λ_n^+ are the bulk metric and cosmological constant on the "+" side of the brane.

[4]If we wished to keep (say) $Z < Z(\tau)$ on the "$-$" side we would choose $\epsilon_- = 1$, assuming of course that $\dot{t}_- > 0$.

where $\overline{Q} = \frac{Q_+ + Q_-}{2}$ for a given quantity Q. Note that while $K^-_{ab} = h^c_a h^d_b \nabla_{(c} n^-_{d)}$, the process of gluing together spacetimes causes the "+" side to flip orientation so that we must define $K^+_{ab} = -h^c_a h^d_b \nabla_{(c} n^+_{d)}$. We now refer back to the third Gauss-Codazzi equation (3.3c), with the understanding that it is valid on both sides of the brane, and $G^\pm_{ab} = -\Lambda^\pm_n g^\pm_{ab}$. If we now take the difference between the "+" equation and the "−" equation we find that

$$-\Delta\Lambda_n = \overline{K}\Delta K - \overline{K}_{ab}\Delta K^{ab}. \tag{3.43}$$

Inserting the values of the extrinsic curvature found in appendix A.3 we obtain

$$-\Delta\Lambda_n = 4\pi G_n(n-2)(\sigma - p)\frac{\Delta[\epsilon h\dot{t}]}{Z} + 4\pi G_n(\sigma+\rho)\Delta\left[\frac{\ddot{Z} + \frac{1}{2}h'}{\epsilon h\dot{t}}\right]. \tag{3.44}$$

After careful and tedious manipulations of equations (3.41), (3.42) and (3.44) we arrive at the following expressions for derivatives of the scale factor

$$\dot{Z}^2 = -\overline{h} + \left[\frac{4\pi G_n}{n-2}(\sigma+\rho)Z\right]^2 + \left[\frac{(n-2)\Delta h}{16\pi G_n(\sigma+\rho)Z}\right]^2 \tag{3.45}$$

and

$$\begin{aligned}\ddot{Z} &= -\frac{1}{2}\overline{h'} - \left(\frac{4\pi G_n}{n-2}\right)^2 (\sigma+\rho)\left[(n-3)\rho + (n-2)p - \sigma\right]Z \\ &\quad + \left(\frac{(n-2)\Delta h}{16\pi G_n(\sigma+\rho)Z}\right)^2 \left[\frac{(n-3)\rho + (n-2)p - \sigma}{(\sigma+\rho)Z}\right] \\ &\quad + \left(\frac{n-2}{16\pi G_n(\sigma+\rho)Z}\right)^2 \Delta h \Delta h'.\end{aligned} \tag{3.46}$$

Note that for equations (3.45) and (3.46) to be consistent, we require that the conservation of energy equation holds on the brane

$$\dot{\rho} = -(n-2)\frac{\dot{Z}}{Z}(\rho + p). \tag{3.47}$$

Here we have seen the beauty of the bulk based approach to braneworld cosmology. We have found the cosmological evolutions equations (3.45) and (3.46) for the brane without assuming $\mathbb{Z}_2$ symmetry. This is particularly important when studying braneworld models that have differing cosmological constants on either side of the brane (eg. [48,70]). Furthermore, by considering general values of h, we have allowed the bulk Weyl tensor on either side to be non-zero. Recall that in the brane based approach the Weyl tensor contribution was just hidden away behind the mysterious E_{ab} term, without any real understanding of its effects. That is not the case here.

3.3.2.1 A $\mathbb{Z}_2$ symmetric brane in AdS-Schwarzschild

As a consistency check, we will now examine the evolution equations when we do indeed have $\mathbb{Z}_2$ symmetry across the brane. This has the effect that for a given quantity Q, $\overline{Q} \to Q$ and $\Delta Q \to 0$. We will also assume that the bulk cosmological constant is negative, and set

$$\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2. \tag{3.48}$$

Our bulk solution is therefore given by equation (3.34) with

$$h(Z) = k_n^2 Z^2 + \kappa - \frac{c}{Z^{n-3}} \tag{3.49}$$

Note that the integration constant c gives the Weyl tensor contribution. For $c = 0$, (3.34) represents pure AdS space with the appropriate slicing (depending on κ). For $c > 0$ we have the AdS-Schwarzschild metric, with its horizon at the point where h vanishes. In the spirit of Randall-Sundrum, we will construct the brane by cutting away the AdS boundary in each bulk, and then gluing together. This imposes the choice $\epsilon = 1$. Again, defining $H = \dot{Z}/Z$, we find that the cosmological evolution equations now simplify somewhat

$$\frac{h\dot{t}}{Z} = \frac{4\pi G_n}{n-2}(\sigma + \rho) \tag{3.50a}$$

$$H^2 = a - \frac{\kappa}{Z^2} + \frac{c}{Z^{n-1}} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho + \left(\frac{4\pi G_{n-1}}{n-2}\right)^2 \rho^2 \tag{3.50b}$$

$$\dot{H} = \frac{\kappa}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \tag{3.50c}$$

where we recall that $a = \sigma_n^2 - k_n^2$ represents the cosmological constant on the brane, and σ_n is defined by equation (3.15). We have also used the relationship (3.13) to include the $(n-1)$-dimensional Newton's constant. Notice that equations (3.50b) and (3.50c) agree with equations (3.21a) and (3.21b) derived using the brane based approach. However we have now been able to explicitly include the the bulk Weyl term, which we were not able to do previously.

Although we have come a long way using the bulk based approach, this is as far as we can go. The main limitation is that we can only consider FRW branes, but

that is fine if we wish to examine cosmological branes. The brane based approach had the advantage that we can generalise to more complicated brane geometries.

To conclude this section, we reiterate two interesting features to arise in brane cosmology. The first is the quadratic energy-momentum terms. One can generally ignore these if the densities are small (for example, when the scale factor is very large), although not otherwise. The second feature is the effect of the bulk Weyl tensor on these cosmologies. We will see in chapters 5 and 6 how this can be understood from the point of view of AdS/CFT.

Chapter 4

Bubbles and ribbons on the brane

4.1 Introduction

In chapter 2, we saw why the RS2 model was so compelling, and why it has been taken as a viable toy model for our universe. The key feature is that gravity on the brane is precisely Einstein gravity at low energies, *i.e.*,

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab} \tag{4.1}$$

This result is of course perturbative [20, 40], and does not include the effect of the short-range KK corrections. Strictly speaking it is only valid for a single brane universe – the presence of a second wall, as in RS1 [19], introduces a radion, representing the distance between the branes and modifying the Einstein gravity to Brans-Dicke gravity [38, 40, 46, 71]. Non-perturbative results however, particularly understanding the effect of the KK modes, are somewhat sparse. In chapter 3, we began a study of non-perturbative braneworld gravity by examining their cosmology. The most notable effect was the deviation from the standard four-dimensional cosmology via quadratic energy density and pressure terms in the FRW equations. The most obvious example of strong brane gravity would be a black hole bound to the brane. Although this has been well understood for a 2-brane in four dimensions [72], we know very little about the higher dimensional analogue.

In this chapter we will investigate non-perturbative gravity by considering the effect of a domain wall living entirely on the brane [1, 2]. Recall that braneworld

universes are really only domain walls themselves [17], so the codimension 2 objects (or vortices) we are considering can be regarded as *nested* domain walls (see figure 4.1) These kind of objects can arise naturally from domain wall configurations.

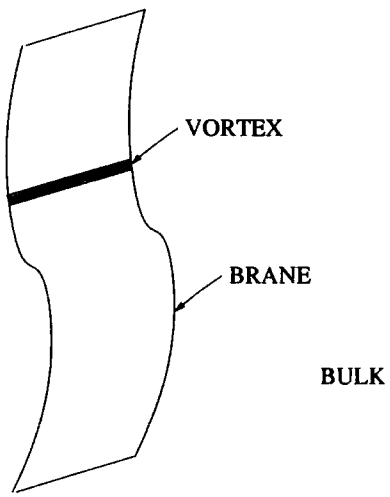


Figure 4.1: A nested domain wall, or vortex on an $(n-1)$-brane.

For example, suppose we have a $\lambda\phi^4$ kink interacting with an additional scalar, σ, via a potential of the form

$$V(\phi,\sigma) = \frac{\lambda}{4}(\phi^2-\eta^2)^2 + \frac{\tilde{\lambda}}{4}\sigma^4 + (\phi^2-m^2)\sigma^2. \tag{4.2}$$

In the true vacuum, $\langle\phi\rangle = \pm\eta$, the state $\langle\sigma\rangle = 0$ is energetically favoured. However, this is not the case in the core of the wall. For example, when $\langle\phi\rangle = 0$, the potential is minimised when $\langle\sigma\rangle = \pm m\sqrt{2/\tilde{\lambda}}$. We see, then, that we can generate a kink in the σ field within the core of the domain wall. Such a configuration is quite well studied in the context of nested topological defects in field theory [73–77], although gravity is absent. This particular configuration is known as a *domain ribbon* [73, 74]. In this chapter we will show that we can fully derive the gravitational field associated with these nested defects. This will not only give us an insight into strong gravity

on the brane, it will also enable us to construct a whole class of new configurations, including *nested braneworlds* and the braneworld analogue of the *Coleman-De Luccia instantons* [24].

4.2 Equations of motion for the domain ribbon

Consider the gravitational field generated by a domain ribbon source. In general, it will depend on only two spacetime coordinates, r and z say, with z roughly representing the direction orthogonal to the brane and r, the direction orthogonal to the domain ribbon (or vortex) within our brane universe. Schematically, the energy-momentum tensor of this source will have the following form:

$$T_{ab} = -\sigma h_{ab}\frac{\delta(z)}{\sqrt{g_{zz}}} - \mu\gamma_{ab}\frac{\delta(z)\delta(r)}{\sqrt{g_{zz}g_{rr}}} \tag{4.3}$$

where h_{ab} is the induced metric on the brane universe, and γ_{ab} the induced metric on the vortex. The symmetries of this energy-momentum tensor mean that we can treat the vortex as a constant curvature spacetime. The most general metric consistent with these symmetries can, in n dimensions, be reduced to the form

$$ds_n^2 = A^{\frac{2}{n-2}}d\mathbf{x}_\kappa^2 + e^{2\nu}A^{-\left(\frac{n-3}{n-2}\right)}(dr^2 + dz^2) \tag{4.4}$$

where $d\mathbf{x}_\kappa^2$ represents the 'unit' metric on an $(n-2)$-dimensional spacetime of constant curvature ($\kappa = 0$ corresponds to a Minkowski spacetime, $\kappa = \pm 1$ to de-Sitter and anti-de Sitter spacetimes). A and ν are functions of r and z to be determined by the equations of motion. Here, the brane universe sits at $z = 0$ with the vortex at $r = z = 0$. This is basically an analytic continuation of the cosmological metric (3.22) in section 3.3, where it is the time translation symmetry ∂_t which is broken, rather than ∂_r. The key result of that section of relevance here was to show that the conformal symmetry of the t, z plane meant that the gravity equations were completely integrable in the bulk, and the brane universe was simply a boundary $(\mathrm{x}^\mu, \mathrm{t}(\tau), \mathrm{Z}(\tau))$ of that bulk (identified with another boundary of another general bulk). The dynamical equations of the boundary reduced to pseudo-cosmological equations for $Z(\tau)$. We now briefly review this argument in the context of the current problem.

First of all, transform the (r, z) coordinates to complex coordinates $(\omega, \bar{\omega})$ where $\omega = z + ir$, $\bar{\omega} = z - ir$, in which the bulk equations of motion reduce to:

$$\partial\bar{\partial}A = -\frac{1}{2}\Lambda_n A^{\frac{1}{n-2}} e^{2\nu} + \frac{(n-2)(n-3)}{4}\kappa A^{-\frac{1}{n-2}} e^{2\nu} \tag{4.5a}$$

$$\partial\bar{\partial}\nu = -\frac{1}{4(n-2)}\Lambda_n A^{-\left(\frac{n-3}{n-2}\right)} e^{2\nu} - \frac{n-3}{8}\kappa A^{-\left(\frac{n-1}{n-2}\right)} e^{2\nu} \tag{4.5b}$$

$$\partial A \partial[\ln \partial A] = 2\partial\nu\partial A \tag{4.5c}$$

$$\bar{\partial} A \bar{\partial}[\ln \bar{\partial} A] = 2\bar{\partial}\nu\bar{\partial}A \tag{4.5d}$$

where ∂ and $\bar{\partial}$ denote partial differentiation with respect to ω and $\bar{\omega}$ respectively. For non-zero Λ_n or κ, equations (4.5c) and (4.5d) can be integrated to give $e^{2\nu} = A' f' g'$, where $A = A\big(f(\omega) + g(\bar{\omega})\big)$ with f and g being arbitrary functions of the complex variables. The remaining equation (4.5a) for A becomes an ODE.

Were the brane not present, we could use the fact that the metric depends only on the combination $f + g$ to make a coordinate transformation in the bulk which would give the metric in the familiar simple canonical form

$$ds_n^2 = Z^2 d\mathbf{x}_\kappa^2 + h(Z) dR^2 + \frac{dZ^2}{h(Z)} \tag{4.6}$$

where $d\mathbf{x}_\kappa^2$ is now a constant curvature Lorentzian spacetime, and in general the function h is

$$h(Z) = -\frac{2\Lambda_n}{(n-1)(n-2)} Z^2 + \kappa - \frac{c}{Z^{(n-3)}} \tag{4.7}$$

The addition of the brane, however, requires that the Israel conditions be satisfied at $z = 0$ in the original coordinates. These turn out to have a scaling symmetry $\omega \to W(\omega)$, $\bar{\omega} \to W(\bar{\omega})$, so we are free to choose f or g (but not both) as we wish. The net result is that our brane becomes some boundary of the bulk (4.6) identified with the boundary of some other general bulk. The vortex (or ribbon), in these coordinates, becomes a kink on this boundary as we shall see. Introducing the affine parameter ζ which parametrizes geodesics on the brane normal to the vortex, the brane is now given by the section $(\mathbf{x}^\mu, R(\zeta), Z(\zeta))$ of the general bulk metric. Note that we now have the condition

$$hR'^2 + \frac{Z'^2}{h} = 1 \tag{4.8}$$

In an exactly analogous procedure to section 3.3, we consider the Israel equations for the jump in extrinsic curvature across the brane, as well as the normal component of the Einstein equations, and thus obtain the equations of motion for the source:

$$Z'^2 = \bar{h} - \sigma_n^2 Z^2 - \left(\frac{\Delta h}{4\sigma_n Z}\right)^2 \tag{4.9a}$$

$$\begin{aligned} Z'' &= \frac{\bar{h}'}{2} - \sigma_n^2 Z - \frac{\mu_n}{2}\sigma_n Z\delta(\zeta) + \frac{1}{Z}\left(\frac{\Delta h}{4\sigma_n Z}\right)^2 \\ &\quad - \frac{\Delta h' \Delta h}{(4\sigma_n Z)^2} + \frac{\mu_n \delta(\zeta)}{2\sigma_n Z}\left(\frac{\Delta h}{4\sigma_n Z}\right)^2 \end{aligned} \tag{4.9b}$$

$$\overline{\epsilon h R'} = \sigma_n Z \tag{4.9c}$$

The brane and vortex tensions now appear in σ_n and μ_n respectively. These are defined as follows

$$\sigma_n = \frac{4\pi G_n \sigma}{n-2}, \qquad \mu_n = 8\pi G_n \mu. \tag{4.10}$$

As in section 3.3, the quantity ϵ in (4.9c) is related to the sign of the the outward normal to the boundary of the bulk spacetime, the boundary of course being the brane. In particular, it is given by

$$n_a^{\pm} = \epsilon_{\pm}(0, -Z', R'_{\pm}) \tag{4.11}$$

where $\epsilon_{\pm} = \pm 1$, depending on which part of the spacetime is kept. Recall that we should define the extrinsic curvature on the "+" with an extra minus sign. This is to account for reversing its orientation when we glue it onto the "−" spacetime.

For simplicity, we will now assume our brane universe is $\mathbb{Z}_2$ symmetric. This has the effect that for any intrinsic bulk quantity Q, $\overline{Q} \to Q$ and $\Delta Q \to 0$. We also assume that the integration constant, c, vanishes and that the bulk cosmological constant is given by

$$\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2 \tag{4.12}$$

This definition is aimed at studying an anti-de Sitter bulk, which is of course what we find in RS models. However, we can easily extend to a Minkowski/de Sitter bulk by allowing k_n to vanish/take imaginary values, as required. Rewriting equation (4.9)

for the trajectory of the source we obtain the $\mathbb{Z}_2$ symmetric equations of motion:

$$Z'^2(\zeta) = \left(k_n^2 - \sigma_n^2\right) Z^2 + \kappa \tag{4.13a}$$

$$Z''(\zeta) = \left(k_n^2 - \sigma_n^2\right) Z - \frac{\mu_n}{2}\sigma_n Z\delta(\zeta) \tag{4.13b}$$

$$R'(\zeta) = \frac{\sigma_n Z}{(k_n^2 Z^2 + \kappa)} \tag{4.13c}$$

Note that we have chosen $\epsilon = 1$. This ensures that the brane has positive tension and that in the spirit of Randall-Sundrum, we retain the $Z < Z(\zeta)$ part of the bulk. In fact, the Randall-Sundrum brane (in n dimensions) is given by setting $\kappa = \mu = 0$ (flat, no vortex) and $\sigma_n = k_n$. The bulk metric is then

$$ds_n^2 = Z^2(-dt^2 + dx_i^2 + k_n^2 dR^2) + \frac{dZ^2}{k_n^2 Z^2} \tag{4.14}$$

with the brane given by $Z = Z_0$ a constant, and $kR = \zeta/Z_0$. Letting $Z_0 = 1$, and $Z = e^{-k_n z}$ gives the usual RS coordinates.

Before turning to the instanton solutions, we will remark on a few straightforward domain ribbon solutions in order to gain an understanding of the geometrical effect of the ribbon. In particular, we will discuss the gravity of nested domain walls from the point of view of observers on the brane.

4.3 Domain ribbon solutions

In this section we examine the solutions to (4.13), exploring their qualitative features as well as some useful illustrative special cases. We begin by integrating the Z-equation (4.13a) away from the vortex:

$$Z = \begin{cases} \frac{1}{\sqrt{a}} \cos\left[\pm\sqrt{a}(\zeta - \zeta_0)\right] & a > 0,\ \kappa = 1 \text{ only} \\ Z_0 \pm \kappa\zeta & a = 0,\ \kappa = 0, 1 \\ \frac{1}{2\sqrt{|a|}}\left[e^{\pm\sqrt{|a|}(\zeta-\zeta_0)} - \kappa e^{\mp\sqrt{|a|}(\zeta-\zeta_0)}\right] & a < 0,\ \kappa = 0, \pm 1 \end{cases} \tag{4.15}$$

where $a = \sigma_n^2 - k_n^2$. Recall that $a = 0$ for a critical brane, whereas $a > 0$ ($a < 0$) for a super (sub) critical brane respectively. In the absence of the vortex, a critical brane has a Minkowski induced metric and corresponds to the original RS scenario [19,20]. A supercritical brane has a de-Sitter induced metric, and can be regarded as an

inflating cosmology [49,69,78,79], whereas the subcritical brane has an AdS induced metric (see [52,80,81] for discussion of its phenomenology). Staying away from the vortex, we can use (4.13c) and the square root of (4.13a) to obtain an ODE for $R(Z)$:

$$R'(Z) = \pm \frac{\sigma_n Z}{\kappa + k_n^2 Z^2} \left(\kappa - aZ^2\right)^{-\frac{1}{2}} \tag{4.16}$$

This is easily integrated to give

$$2k_n(R - R_0) = \pm \begin{cases} \ln\left(1 + k_n^2 Z^2\right) & \kappa = 1, \quad a = 0 \\ \ln\left|\frac{k_n\sqrt{1-aZ^2}-\sigma_n}{k_n\sqrt{1-aZ^2}+\sigma_n}\right| & \kappa = 1, \quad a \neq 0 \\ -\frac{2\sigma_n}{k_n\sqrt{|a|}Z} & \kappa = 0, \quad a < 0 \\ 2\tan^{-1}\left(\frac{k_n\sqrt{1-aZ^2}}{\sigma_n}\right) & \kappa = -1, \quad a < 0. \end{cases} \tag{4.17}$$

where the choice of signs refers to the sign of $Z'(\zeta)$. Note that these trajectories are invariant under Euclideanization of the metric, therefore instanton trajectories will also have this form.

In order to see how these trajectories embed into the bulk AdS spacetime, it is useful to transform into conformal coordinates, $(\tilde{t}, \tilde{\mathbf{x}}, u)$ in which the metric is conformally flat:

$$ds_n^2 = \frac{1}{k_n^2 u^2}\left[-d\tilde{t}^2 + d\tilde{\mathbf{x}}^2 + du^2\right] \tag{4.18}$$

For the $\kappa = 1$ spacetimes needed to construct the braneworld instantons, this requires the bulk coordinate transformation

$$k_n u = e^{k_n R}/\sqrt{1 + k_n^2 Z^2} \tag{4.19a}$$

$$\left(\tilde{t}, \tilde{\mathbf{x}}\right) = k_n u Z \left(\sinh t, \cosh t \mathbf{n}_{n-2}\right) \tag{4.19b}$$

where $\mathbf{n}_{n-2}$ is the unit vector in $(n-2)$ dimensions. Under such a transformation the trajectories $R(Z)$ in (4.17) generally take the form

$$(u \mp u_0)^2 + \tilde{\mathbf{x}}^2 - t^2 = u_1^2 \tag{4.20}$$

for $a \neq 0$, where

$$u_0 = \frac{\sigma_n}{k_n} u_1 = \frac{\sigma_n}{k_n} \frac{e^{k_n R_0}}{|a|^{1/2}} \tag{4.21}$$

This means that braneworlds (4.20) have the form of hyperboloids (or spheres in the Euclidean section) in the conformal metric (4.18). In particular, for *subcritical branes* ($a < 0$), we have $u_0 < u_1$, and both branches of the hyperboloid (4.20) are allowed, each intersecting the AdS boundary (see figures 4.2(a) and 4.2(b)). An analysis of the normals to the braneworld shows that for a positive tension $\mathbb{Z}_2$-symmetric braneworld, the upper root $Z' > 0$ corresponds to keeping the interior of the hyperboloid, whereas for $Z' < 0$ the exterior is kept. *Supercritical branes* ($a > 0$) on the other hand have only the upper root for u_0, and as $u_0 > u_1$ in a Euclidean signature they represent spheres which are entirely contained within the AdS spacetime. For a supercritical brane of positive tension the interior of the hyperboloid (or sphere) is kept (see figure 4.2(d)). For a *critical brane*, ($a = 0$) there are once again two possible trajectories, one having the form of (4.20) but with $u_0 = u_1 = e^{k_n R_0}/2k_n$ (figure 4.2(c)), and the other having $u = \text{const.}$ – the RS braneworld.

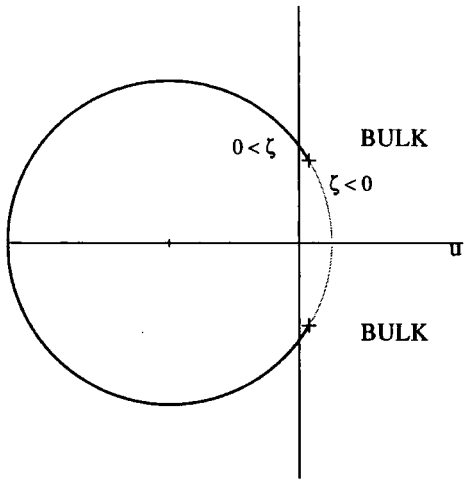
To put a vortex on the braneworld, we require solutions with non-zero μ_n, and hence a discontinuity in Z'. To achieve this, we simply patch together different branches of the solutions (4.15) for $\zeta > 0$ and $\zeta < 0$. We immediately see that critical and supercritical branes can only support a vortex if $\kappa = 1$, that is, if the induced metric on the vortex itself is a de-Sitter universe. A subcritical brane on the other hand can support all induced geometries on the vortex. Defining $k_{n-1}^2 = |a|$, these trajectories are

$$Z = \begin{cases} \frac{1}{2k_{n-1}}\left[e^{\alpha - k_{n-1}|\zeta|} - \kappa e^{k_{n-1}|\zeta| - \alpha}\right] & \text{subcritical brane} \\ Z_0 - |\zeta| & \text{critical brane } (\kappa = 1 \text{ only}) \\ \frac{1}{k_{n-1}}\cos\left(k_{n-1}|\zeta| + \beta\right) & \text{supercritical brane } (\kappa = 1 \text{ only}) \end{cases} \tag{4.22}$$

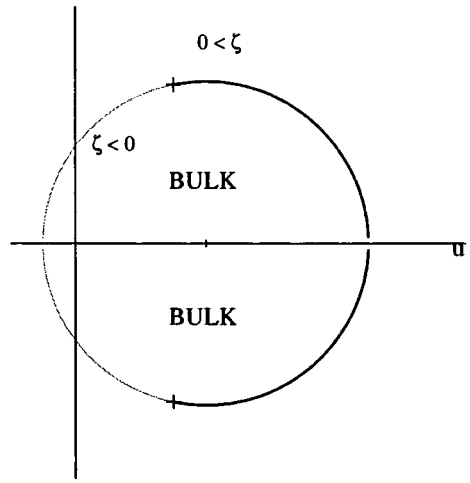
where

$$\mu_n = \begin{cases} \frac{4k_{n-1}}{\sigma_n}\left[\frac{e^{2\alpha}+\kappa}{e^{2\alpha}-\kappa}\right] & \text{subcritical brane} \\ \frac{4}{k_n Z_0} & \text{critical brane} \\ \frac{4k_{n-1}}{\sigma_n}\tan\beta & \text{supercritical brane} \end{cases} \tag{4.23}$$

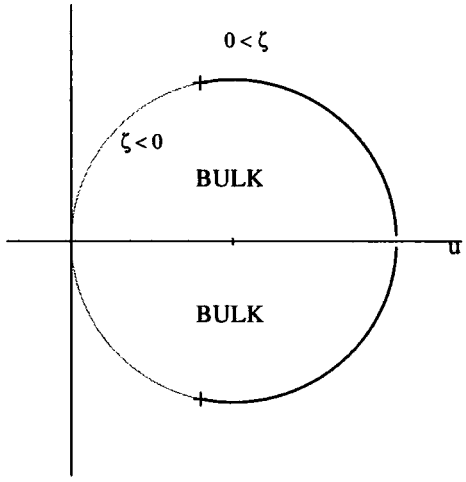
respectively.



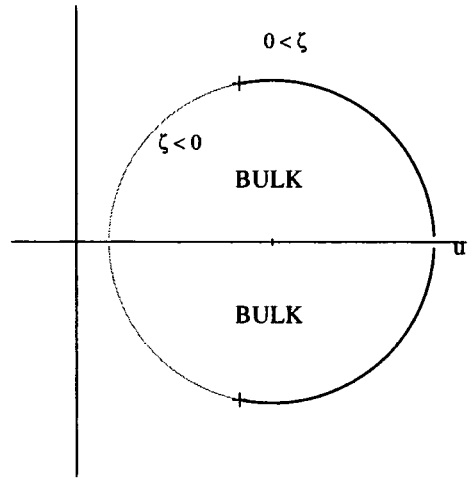
(a) Subcritical brane centred on $-u_o$



(b) Subcritical brane centred on $+u_o$



(c) Critical brane



(d) Supercritical brane

Figure 4.2: Braneworld trajectories given by equation 4.20, in Euclidean signature. In each case the location of the bulk spacetime is indicated. In addition we have the simple critical brane trajectory given by $u = \text{const}$. Here the bulk lies to the right of the brane.

4.3.1 The domain ribbon in a vacuum bulk

In order to examine the geometry of the ribbon it is useful to consider a vacuum bulk spacetime. This will obviously represent a vortex living on a supercritical braneworld. There is no warping of the bulk due to the cosmological constant so we can clearly compare the ribbon spacetime to that of an isolated vortex ($\sigma_n = 0$) or a pure de-Sitter domain wall ($\mu_n = 0$). Since the bulk cosmological constant vanishes, we have

$$k_n = 0 \;\Rightarrow\; k_{n-1} = \sigma_n \tag{4.24}$$

Note that the pure domain wall universe is a hyperboloid in Minkowski spacetime [82, 83]. Specifically it is an accelerating bubble of proper radius σ_n^{-1}, with $\kappa = 1$. Meanwhile, we also note that the pure δ-function isolated vortex solution has a conical deficit metric

$$ds_n^2 = -dt^2 + d\mathbf{x}^2 + d\rho^2 + \left(1 - \frac{\Delta\theta}{2\pi}\right)^2 \rho^2 d\theta^2 \tag{4.25}$$

where $\Delta\theta \simeq \mu_n$ for small μ_n [84].

We can read off the domain ribbon trajectory from equations (4.22) and (4.23). In (R, Z) space, this gives

$$Z = \frac{1}{\sigma_n\sqrt{16+\mu_n^2}}\left[4\cos(\sigma_n\zeta) - \mu_n \sin(\sigma_n|\zeta|)\right] \tag{4.26a}$$

$$R = \frac{1}{\sigma_n\sqrt{16+\mu_n^2}}\left[4\sin(\sigma_n\zeta) \pm \mu_n[\cos(\sigma_n\zeta) - 1]\right] \tag{4.26b}$$

where we preserve the region $Z < Z(\zeta)$ of the bulk:

$$ds_n^2 = Z^2\left[-dt^2 + \cosh^2 t\; d\Omega_{n-3}^2\right] + dZ^2 + dR^2 \tag{4.27}$$

where $d\Omega_{n-3}^2$ is the metric on a unit $(n-3)$-sphere. This is of course simply a coordinate transformation of Minkowski spacetime, with the appropriate limit of (4.19) being $(\tilde{t}, \tilde{\mathbf{x}}) = (Z\sinh t, Z\mathbf{n}\cosh t)$. Transforming into Minkowski coordinates, we find that the vacuum braneworld domain ribbon is given by two copies of the interior of the sliced hyperboloid

$$\tilde{\mathbf{x}}^2 - \tilde{t}^2 + \left(|R| + \frac{\mu_n}{\sigma_n\sqrt{16+\mu_n^2}}\right)^2 = \frac{1}{\sigma_n^2} \tag{4.28}$$

If $\mu_n = 0$ this is clearly the standard domain wall hyperboloid. However, when $\mu_n > 0$, this represents a hyperboloid which has had a slice of width $2\mu_n/\sigma_n\sqrt{16+\mu_n^2}$ removed from it (see figure 4.3). This corresponds rather well with the intuitive notion that walls are obtained by slicing and gluing spacetimes.

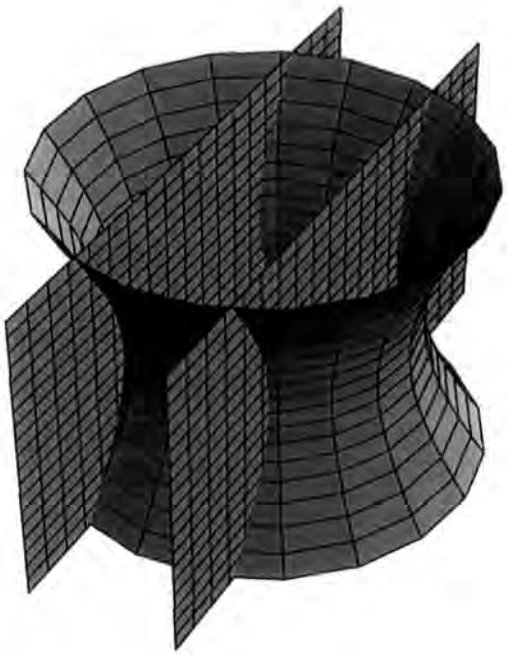


Figure 4.3: Constructing a domain ribbon on a vacuum domain wall. The hyperboloid interior has a slice of thickness $2\mu_n/\sigma_n\sqrt{16+\mu_n^2}$ removed from it, and is re-identified. The full spacetime consists of a second copy identified across the hyperboloid.

Looking at a constant time slice (figure 4.4) we also see how the domain ribbon looks like a vortex, with the identifications giving rise to a conical deficit angle in terms of the overall n-dimensional spacetime. We find that

$$\Delta\theta = 4\tan^{-1}\mu_n/4 \tag{4.29}$$

Note that for small μ we have $\Delta\theta \simeq \mu_n$, which agrees with the case of the isolated vortex. A crucial difference however, appears to be that for the ribbon spacetime,

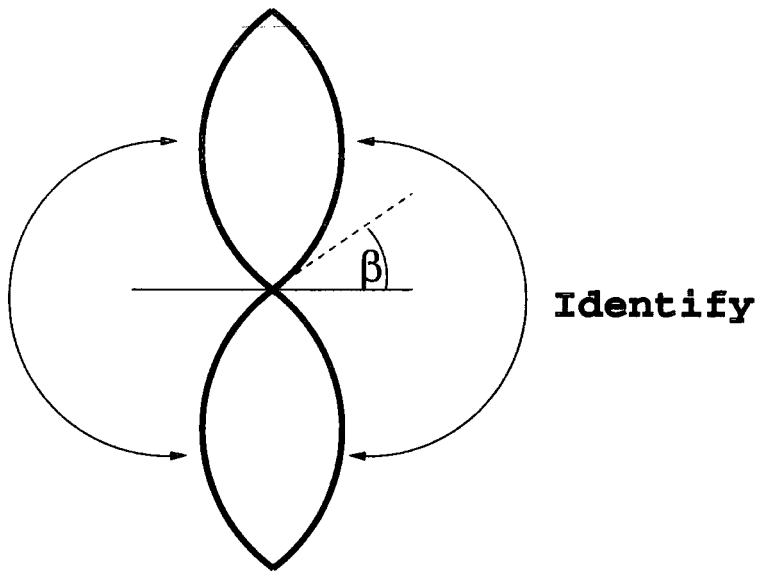


Figure 4.4: Taking a constant time slice through the vacuum domain wall plus vortex spacetime shows how the deficit angle is built up.

the vortex can have an arbitrarily large energy per unit length, as we simply cut out more and more of the hyperboloid. Indeed, the deficit angle approaches 2π only as μ_n approaches infinity! Contrast this with the spacetime of a pure vortex, (4.25), in which the deficit angle approaches 2π as $\mu_n \simeq 1$ [85]. The ribbon is clearly not behaving as a vortex for large μ. On the other hand, a domain wall has the effect of compactifying its spatial sections (the interior of the hyperboloid) and the transverse dimension only shrinks to zero size as the tension of the wall becomes infinite. Therefore in this sense, the ribbon spacetime really does behave as a domain wall.

Finally, we note that the induced metric on the brane is given by

$$ds^2_{n-1} = \frac{[4\cos\sigma_n\zeta - \mu_n \sin\sigma_n|\zeta|]^2}{\sigma_n^2(16+\mu_n^2)}\left[-dt^2 + \cosh^2 t d\Omega^2_{n-3}\right] + d\zeta^2 \tag{4.30}$$

This is the metric of an $(n-2)$-dimensional domain wall in an $(n-1)$-dimensional de-Sitter universe of tension, μ. We see this by examining the Israel equations in $(n-1)$ dimensions, at $\zeta = 0$. For a wall of tension T,

$$\Delta K_{ab} = -\frac{8\pi G_{n-1} T}{n-3}\gamma_{ab} \tag{4.31}$$

Meanwhile, from (4.30), the jump in extrinsic curvature at $\zeta = 0$ is given by

$$\Delta K_{ab} = -\frac{\sigma_n \mu_n}{2} \gamma_{ab} \tag{4.32}$$

With the identification (3.13) we conclude that $T \equiv \mu$. In this sense the geometry of the braneworld seems to know nothing about the extra dimension. Gravity on the brane appears $(n-1)$-dimensional even in this non-perturbative regime.

4.3.2 The domain ribbon on a critical RS brane

Having constructed this symmetric vacuum domain ribbon spacetime, we now see the general principle involved in having a domain ribbon. Whereas a braneworld without a vortex consists of two segments of AdS (or vacuum/dS) spacetime glued across a boundary, the domain ribbon consists of two copies of an AdS spacetime with a kinked boundary identified together. The kink itself could be viewed as two copies of an AdS bulk glued together across a tensionless boundary. Recall that our original motivation was to investigate the behaviour of domain walls on branes, and in particular the critical RS brane. With our current insight, we would expect a domain ribbon on a critical RS brane, in conformal coordinates, to be the critical hyperboloid sliced by a critical flat RS wall (see figure 4.5). We will investigate this presently.

The tension of the critical RS brane satisfies the relation $\sigma_n = k_n$. Here we have pure AdS space in the bulk so $k_n > 0$. Since $a = 0$, a domain ribbon on this brane *must* have $\kappa = 1$, that is 'spherical' spatial geometry. In (R, Z) space, the brane trajectory is given by

$$Z = \frac{4}{\mu_n k_n} - |\zeta| \tag{4.33a}$$

$$R = \mp \frac{1}{2k_n} \ln \left[\frac{\mu_n^2 + (4 - k_n \mu_n |\zeta|)^2}{\mu_n^2 + 16} \right] \tag{4.33b}$$

For the full spacetime we keep the region $Z < Z(\zeta)$ of the bulk:

$$ds_n^2 = Z^2 \left[-dt^2 + \cosh^2 t \, d\Omega_{n-3}^2 \right] + \frac{dZ^2}{k_n^2 Z^2 + 1} + (k_n^2 Z^2 + 1) dR^2 \tag{4.34}$$

At first sight neither the trajectory nor bulk looks like the original RS scenario,

Figure 4.5: A representation of the domain ribbon on a critical RS brane.

however, the coordinate transformation

$$k_n u = e^{k_n R}/\sqrt{1+k_n^2 Z^2} \tag{4.35a}$$

$$(\tilde{t}, \tilde{\mathbf{x}}) = k_n u Z(\sinh t, \cosh t\ \mathbf{n}_{n-2}) \tag{4.35b}$$

(where $\mathbf{n}_{n-2}$ is the unit vector in $(n-2)$ dimensions) gives

$$ds_n^2 = \frac{1}{k_n^2 u^2}\left[-\tilde{t}^2 + d\tilde{\mathbf{x}}^2 + du^2\right] \tag{4.36}$$

This is the familiar planar AdS metric in conformal coordinates. The trajectory (4.33) now becomes

$$\zeta < 0: \quad u = u_0 \tag{4.37a}$$

$$\zeta > 0: \quad \left(u - \frac{1}{2k_n^2 u_0}\right)^2 + \tilde{\mathbf{x}}^2 - \tilde{t}^2 = \frac{1}{4k_n^4 u_0^2} \tag{4.37b}$$

where

$$u_0 = \frac{\mu_n}{k_n\sqrt{16+\mu_n^2}} \tag{4.38}$$

The change of coordinates means that the trajectory is no longer manifestly $\mathbb{Z}_2$ symmetric. However, the $\zeta < 0$ branch now becomes a subset of the RS planar domain wall, specifically, the interior of the hyperboloid

$$\frac{\tilde{\mathbf{x}}^2 - \tilde{t}^2}{k_n^2 u_0^2} = \frac{16}{k_n^2\mu_n^2} = [2\pi G_{n-1}\mu]^{-2} \tag{4.39}$$

where we have used equation (3.13) with $\sigma_n = k_n$. Recall that the global spacetime structure of a vacuum domain wall is that of two identified copies of the interior of a hyperboloid in Minkowski spacetime, with proper radius $1/2\pi G_{n-1}\mu$, [82,83,86]. We conclude that (4.39) corresponds identically with what we would expect from $(n-1)$-dimensional Einstein gravity.

Meanwhile, the $\zeta > 0$ branch is a hyperboloid in the bulk centered on $u = 1/2k_n^2u_0$ with comoving radius $1/2k_n^2u_0$. As μ increases, more and more of the hyperboloid is removed, with the spacetime 'disappearing' only as $\mu \to \infty$. This is the same behaviour as we found in section 4.3.1 for the domain ribbon in a vacuum bulk. As before, this is normal behaviour for a domain wall, but very different to what one would expect from a vortex.

In order to examine the geometry on the brane more carefully, we note that the induced metric on the brane is given by

$$ds_{n-1}^2 = \left(1 - \frac{\mu_n k_n|\zeta|}{4}\right)^2 \left[-d\hat{t}^2 + \left(\frac{4}{\mu_n k_n}\right)^2 \cosh^2\left(\frac{\mu_n k_n \hat{t}}{4}\right) d\Omega_{n-3}^2\right] + d\zeta^2 \tag{4.40}$$

where $\hat{t} = 4t/\mu_n k_n$. This is *precisely* the metric of a self-gravitating domain wall of tension μ in $(n-1)$-dimensional Einstein gravity [84,86]. Again, this is best seen by examining the Israel equations (at $\zeta = 0$) in $(n-1)$ dimensions. The jump in extrinsic curvature across a wall of tension T is given by equation (4.31). However, from the metric (4.40), the jump in extrinsic curvature at $\zeta = 0$ is given by

$$\Delta K_{ab} = -\frac{k_n\mu_n}{2}\gamma_{ab} \tag{4.41}$$

If we once again use equation (3.13) with $\sigma_n = k_n$, we can conclude $T \equiv \mu$. This proves that the geometry on the brane is indeed behaving in an $(n-1)$-dimensional

way, just as it did for the vacuum bulk in section 4.3.1. We have shown that even in this non-perturbative case, the RS model exhibits exact $(n-1)$-dimensional Einstein gravity on the brane, even though the model is manifestly n-dimensional.

4.3.3 Nested braneworlds

We now have the tools to construct nested Randall-Sundrum type configurations, that is, a flat ($\kappa = 0$) ribbon on an AdS brane with an AdS bulk. Fortunately, we see from (4.22) that a subcritical (AdS) brane can sustain a flat ribbon. From equations (4.17), (4.22) and (4.23), the brane trajectory is given by

$$Z = Z_0 e^{-k_{n-1}|\zeta|}, \qquad k_n R = \pm\frac{4}{k_n \mu_n}\left(Z^{-1} - Z_0^{-1}\right) \tag{4.42}$$

with $\mu = 4k_{n-1}/\sigma_n$. Transforming to conformal coordinates $(u, v) = (1/k_n Z, k_n R)$, the brane trajectories become

$$v = \pm\frac{4}{\mu_n}(u - u_0) \tag{4.43}$$

Each branch of this trajectory is a subcritical brane, which, if it were not for the vortex at $(u_0, 0)$ would reach the AdS boundary at $v = \mp\sigma_n u_0/k_{n-1}$.

Notice that the induced metric on the braneworld

$$ds_{n-1} = Z_0^2 e^{-2k_{n-1}|\zeta|}[-dt^2 + dx_i^2] + d\zeta^2 \tag{4.44}$$

is indeed that of a RS universe in $(n-1)$ dimensions. We would expect there to be an analogue of the criticality condition for flat branes. Again this arises from the Israel equations at $\zeta = 0$. As expected, we find that

$$k_{n-1} = \frac{4\pi G_{n-1}\mu}{n-3} \tag{4.45}$$

where we have used the condition (3.13). This is precisely the RS criticality condition $\sigma_n = 4\pi G_n \sigma/(n-2) = k_n$ adjusted for one dimension less.

We conclude this section by emphasizing its main result. In each of the examples we have looked at, the geometry on the brane has been in exact agreement with the geometry predicted by $(n-1)$-dimensional Einstein gravity, without any knowledge of the bulk. This is a remarkable result because it means that, at least in this highly

symmetric set up, RS braneworld models exhibit localisation of gravity on the brane, even in the non-perturbative regime.

We have had the added bonus that we have seen how to construct nested braneworld configurations. In the next section we will use the same tools to construct braneworld instantons.

4.4 Instantons and tunneling on the brane

Traditionally, instantons correspond to classical Euclidean solutions to the equations of motion. In many cases, they represent a quantum tunneling from a metastable false vacuum to a true vacuum. In [24], Coleman and de Luccia discussed the effect of gravity on these decays. Such processes, of course have direct relevance for cosmology, as they correspond to a first order phase transition, and hence a dramatic change in the structure of our universe.

In [24], the authors evaluated the probability of nucleation of a true vacuum bubble in a false vacuum background. They focussed on two particular configurations: a flat bubble spacetime in a de Sitter false vacuum; and an AdS bubble spacetime in a flat false vacuum. This was before the idea of large extra dimensions was fashionable, so the analysis was done in just the usual four dimensions.

We now have the tools to develop these ideas in a braneworld set up. To replicate the configurations of [24], we just have to patch together our brane trajectories in the right way. Recall that these trajectories are given by equation (4.20), along with the critical brane solution, $u = \text{const}$. In Euclidean signature, the former are shaped like spheres and were illustrated in figures 4.2(a) to 4.2(d). However, when patching these solutions together, we should be aware of a slight subtlety. In equation (4.13b), the $\mu_n \sigma_n \delta(\zeta)$ term does not make sense if we have branes of different type either side of the vortex. Suppose we have a brane of tension σ_n^+ in $\zeta > 0$, and σ_n^- in $\zeta < 0$, we must then modify equation (4.13b) by replacing σ_n with $\bar{\sigma}_n$, where

$$\bar{\sigma}_n = \frac{\sigma_n^+ + \sigma_n^-}{2} \tag{4.46}$$

It is easy to see that this is the right thing to do. Regard the vortex as a thin wall limit of some even energy distribution. Mathematically, this corresponds to $\mu_n \delta(\zeta)$

being the limit of some even function $\mu_n f(\zeta)$. The weight of the distribution is the same on either side of $\zeta = 0$, so we pick up the average of the brane tensions.

We are now in a position to reproduce the work of [24] in our higher dimensional environment. Let us consider first the decay of a de Sitter false vacuum, and the nucleation of a flat bubble spacetime.

4.4.1 Nucleation of a flat bubble spacetime in a de Sitter false vacuum

We now describe the braneworld analogue of the nucleation of a flat bubble spacetime in a de Sitter false vacuum. The de Sitter false vacuum is given by a supercritical brane of tension $\sigma_n^{dS} > k_n$ with no vortex (see figure 4.2(d)). This metastable state decays into a "bounce" configuration given by a critical brane (tension $\sigma_n^{flat} = k_n$) patched on to a supercritical brane (tension $\sigma_n^{dS} > k_n$). If we are to avoid generating an unphysical negative tension vortex we must patch together trajectories in the following way:

$$Z = \begin{cases} \frac{1}{k_{n-1}^{dS}} \cos(k_{n-1}^{dS}\zeta - \zeta_0) & \zeta > 0 \\ \zeta + \frac{1}{k_{n-1}^{dS}} \cos\zeta_0 & \zeta < 0 \end{cases} \tag{4.47}$$

where $(k_{n-1}^{dS})^2 = (\sigma_n^{dS})^2 - k_n^2$. The vortex tension μ, is related to the constant ζ_0 in the following way:

$$\frac{\mu_n \bar{\sigma}_n}{2k_{n-1}^{dS}} = \sec\zeta_0 - \tan\zeta_0 \tag{4.48}$$

It is useful to have a geometrical picture of this bounce solution. We just patch together the $u =$ const. critical brane trajectory and the supercritical brane trajectory given by figure 4.2(d) to get figure 4.6. Note that we have two copies of the bulk spacetime because we imposed $\mathbb{Z}_2$-symmetry across the branes.

It is natural to calculate the probability, $\mathcal{P}$, that this flat bubble spacetime does indeed nucleate on the de Sitter brane.

$$\mathcal{P} \propto e^{-B} \tag{4.49}$$

where B is the difference between the Euclidean actions of the bounce solution and

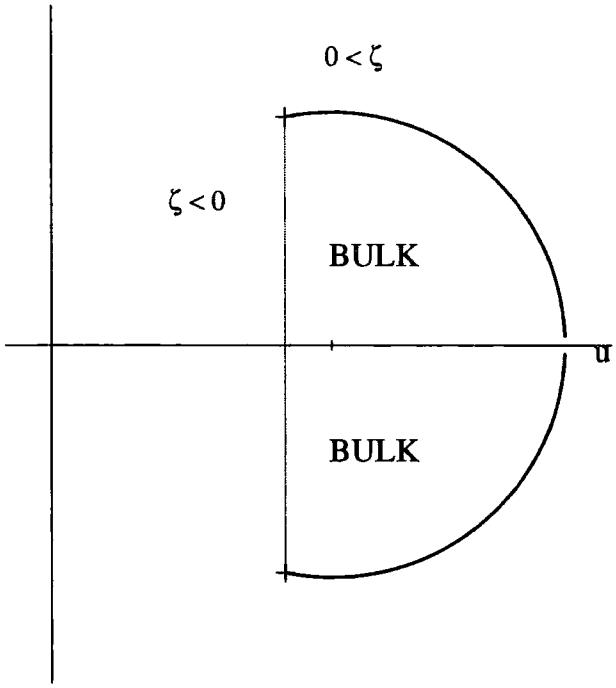


Figure 4.6: An example of a critical-supercritical brane "bounce" solution. This looks like a flat bubble spacetime has nucleated on a de Sitter brane.

the false vacuum solution, that is:

$$B = S_{bounce} - S_{false} \tag{4.50}$$

Given our geometrical picture it is straightforward to write down an expression for the bounce action:

$$S_{bounce} = S_{bulk} + S_{flat} + S_{dS} + S_{vortex} \tag{4.51}$$

where the contribution from the bulk, critical brane (flat), supercritical brane (de Sitter), and vortex are as follows

$$S_{bulk} = -\frac{1}{16\pi G_n}\int_{bulk} d^n x\sqrt{g}(R - 2\Lambda_n) \tag{4.52a}$$

$$S_{flat} = -\frac{1}{16\pi G_n}\int_{flat} d^{n-1} x\sqrt{h}(-2\Delta K - 4(n-2)\sigma_n^{flat}) \tag{4.52b}$$

$$S_{dS} = -\frac{1}{16\pi G_n}\int_{dS} d^{n-1} x\sqrt{h}(-2\Delta K - 4(n-2)\sigma_n^{dS}) \tag{4.52c}$$

$$S_{vortex} = \mu\int_{vortex} d^{n-2} x\sqrt{\gamma} = -\frac{1}{16\pi G_n}\int_{vortex} d^{n-2} x\sqrt{\gamma}(-2\mu_n) \tag{4.52d}$$

Note that ΔK, the jump in the trace of the extrinsic curvature across the brane, contains the Gibbons-Hawking boundary term [87] for each side of the brane, and h_{ab}, γ_{ab} are the induced metrics on the brane and vortex respectively. We should point out that due to the presence of the vortex, there is a delta function in the extrinsic curvature that exactly cancels off the contribution of S_{vortex}. The expression for S_{false} is similar except that there is no flat brane or vortex contribution, and no delta function in the extrinsic curvature. After some calculation (see appendix A.4), we find that our probability term, B is given by:

$$B = \frac{4k_n^{2-n}\Omega_{n-2}}{16\pi G_n}\left(I_n - \left(\frac{1}{n-1}\right)\left(\frac{k_n\cos\zeta_0}{k_{n-1}^{dS}}\right)^{n-1}\right) \tag{4.53}$$

where Ω_{n-2} is the volume of an $(n-2)$ sphere and the integral I_n is given by:

$$I_n = \int_{u_0-u_1}^{u_c} du \quad \left(u_0\frac{[\rho(u)]^{n-3}}{u^{n-1}} - \frac{[\rho(u)]^{n-1}}{u^n}\right) \tag{4.54}$$

where

$$u_0 = \frac{\sigma_n^{dS}}{(k_{n-1}^{dS})^2}(\sigma_n^{dS} + k_n\sin\zeta_0)u_c \tag{4.55a}$$

$$u_1 = \frac{k_n}{\sigma_n^{dS}}u_0 \tag{4.55b}$$

$$\rho(u) = \sqrt{u_1^2 - (u-u_0)^2} \tag{4.55c}$$

Note that u_c is an arbitrary constant so we are free to choose it as we please (think of the flat brane as being at $u = u_c$). This integral is non-trivial and although we can in principle solve it for any integer n it would not be instructive to do so. Instead, we will restrict our attention to the case where $n = 5$. This means that our braneworld is four dimensional, so comparisons with [24] are more natural. Given that $\Omega_3 = 2\pi^2$, we find that:

$$B = \frac{8\pi^2 k_5^{-3}}{16\pi G_5}\left[\log\left[\frac{\sigma_5^{dS} + k_5\sin\zeta_0}{\sigma_5^{dS} + k_5}\right] - \frac{1}{2}\left(\frac{k_5\cos\zeta_0}{k_4^{dS}}\right)^2 + \frac{k_5\sigma_5^{dS}}{(k_4^{dS})^2}(1-\sin\zeta_0)\right] \tag{4.56}$$

Equation (4.48) in five dimensions enables us to replace the trigonometric functions using:

$$\cos\zeta_0 = \frac{2\lambda}{1+\lambda^2} \tag{4.57a}$$

$$\sin\zeta_0 = \frac{1-\lambda^2}{1+\lambda^2} \tag{4.57b}$$

where

$$\lambda = \frac{\mu_5 \bar{\sigma}_5}{2k_4^{dS}} \tag{4.58}$$

This leads to a complicated expression. It is perhaps more instructive to examine the behaviour for small μ *i.e.,* in the regime where we have a vortex with a low energy density. In this regime we find that:

$$B = \frac{256\pi^5}{(k_4^{dS})^6}(G_5\bar{\sigma}_5)^3\mu^4 + \mathcal{O}(\mu^5) = \frac{256\pi^5}{(\mathrm{k}_4^{\mathrm{dS}})^6}(\bar{\mathrm{G}}_4)^3\mu^4 + \mathcal{O}(\mu^5) \tag{4.59}$$

where $\bar{G}_4$ is the average of the four dimensional Newton's constants on the flat brane and the de Sitter brane. The presence of this average as opposed to a single four dimensional Newton's constant is due to the difference in brane tension on either side of the vortex. From equation (3.13) we see that this induces a difference in the Newton's constants on each brane.

We now compare this to the result we would have got had we assumed no extra dimensions. The analogous probability term, B', is calculated in [24]. When the energy density of the bubble wall, μ, is small, we now find that[1]:

$$B' = \frac{256\pi^5}{(k_4)^6}(G_4)^3\mu^4 + \mathcal{O}(\mu^5) \tag{4.60}$$

If we associate G_4 in equation (4.60) with $\bar{G}_4$ in equation (4.59) we see that the approach of [24], where no extra dimensions are present, yields exactly the same result to the braneworld setup, at least for small μ.

Before we move on to discuss alternative instanton solutions we should note that in the above analysis we have assumed $\frac{\pi}{2} > \zeta_0 > 0$. The bounce solution presented is therefore really only valid if we have $\lambda < 1$. However, the extension to regions where $\lambda > 1$ corresponds to allowing ζ_0 to take negative values and everything holds.

[1] In order to reproduce equation (4.60) using separating a flat bubble spacetime and a de Sitter spacetime whose radius of curvature is $\frac{1}{k_4}$. Then substitute into the relevant equations and take μ to be small.

4.4.2 Nucleation of an AdS bubble spacetime in a flat false vacuum

We now turn our attention to the decay of a flat false vacuum, and the nucleation of an AdS bubble spacetime. The braneworld analogue of the flat false vacuum is given by a critical brane of tension $\sigma_n^{flat} = k_n$ with no vortex (u = const). This decays into a new "bounce" configuration given by a subcritical brane (tension $\sigma_n^{AdS} < k_n$) patched onto a critical brane (tension $\sigma_n^{flat} = k_n$). Again, in order to avoid generating an unphysical vortex, we must patch together trajectories in the following way:

$$Z = \begin{cases} \zeta + \frac{1}{k_{n-1}^{AdS}} \sinh \zeta_0 & \zeta > 0 \\ \frac{1}{k_{n-1}^{AdS}} \sinh(k_{n-1}^{AdS}\zeta + \zeta_0) & \zeta < 0 \end{cases} \tag{4.61}$$

where $(k_{n-1}^{AdS})^2 = k_n^2 - (\sigma_n^{AdS})^2$. The vortex tension μ is related to the constant ζ_0 in the following way:

$$\frac{\mu_n \bar{\sigma}_n}{2k_{n-1}^{AdS}} = \coth\zeta_0 - \operatorname{cosech}\zeta_0 \tag{4.62}$$

By patching together u = const. and figure 4.2(a) we again obtain a geometrical picture of our bounce solution (see figure 4.7).

As before we now consider the probability term, B, given by the difference between the Euclidean actions of the bounce and the false vacuum. We shall not go into great detail here as the calculation is very similar to that in the previous section. We should emphasize, however, that the bounce action will include as before an Einstein-Hilbert action with negative cosmological constant in the bulk, a Gibbons Hawking surface term on each brane, and tension contributions from each brane and the vortex. Again we find that the delta function in the extrinsic curvature of the brane exactly cancels off the contribution from the vortex tension. The false vacuum action omits the AdS brane and vortex contributions, and contains no delta functions from extrinsic curvature. Recall that in each case we have two copies of the bulk spacetime because of $\mathbb{Z}_2$-symmetry across the brane.

This time, we find that our probability term, B, is given by:

$$B = \frac{4k_n^{2-n}\Omega_{n-2}}{16\pi G_n}\left(I_n + \left(\frac{1}{n-1}\right)\left(\frac{k_n \sinh\zeta_0}{k_{n-1}^{AdS}}\right)^{n-1}\right) \tag{4.63}$$

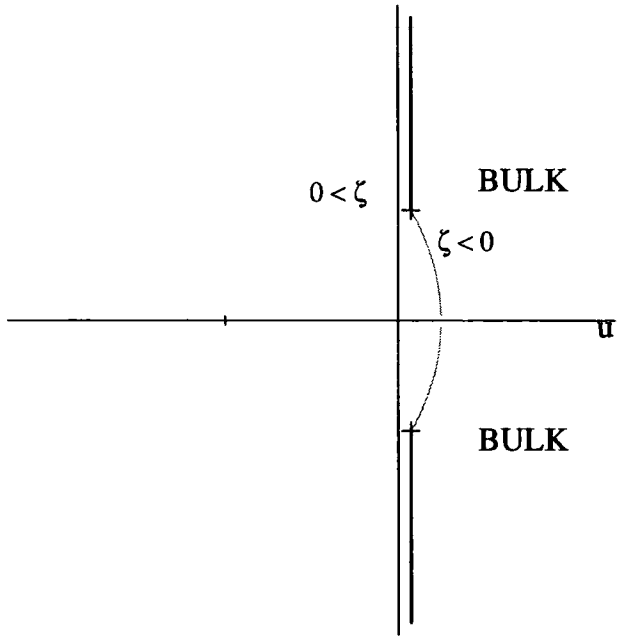


Figure 4.7: An example of a subcritical-critical brane "bounce" solution. This looks like an AdS bubble spacetime has nucleated on a flat brane.

where the integral I_n is now given by:

$$I_n = \int_{-u_0+u_1}^{u_c} du \quad \left(u_0 \frac{[\rho(u)]^{n-3}}{u^{n-1}} + \frac{[\rho(u)]^{n-1}}{u^n} \right) \tag{4.64}$$

where u_c is an arbitrary constant corresponding to the position of the flat brane, and

$$u_0 = \frac{\sigma_n^{AdS}}{(k_{n-1}^{AdS})^2}(\sigma_n^{dS} + k_n \cosh\zeta_0)u_c \tag{4.65}$$

$$u_1 = \frac{k_n}{\sigma_n^{AdS}} u_0 \tag{4.66}$$

$$\rho(u) = \sqrt{u_1^2 - (u+u_0)^2} \tag{4.67}$$

Again, although we could in principle solve this integral for any positive integer n, we shall restrict our attention to $n = 5$. In this case we now find that:

$$B = \frac{8\pi^2 k_5^{-3}}{16\pi G_5}\left[-\log\left[\frac{\sigma_5^{AdS} + k_5\cosh\zeta_0}{\sigma_5^{AdS} + k_5}\right] + \frac{1}{2}\left(\frac{k_5 \sinh\zeta_0}{k_4^{AdS}}\right)^2 + \frac{k_5\sigma_5^{AdS}}{(k_4^{AdS})^2}(1-\cosh\zeta_0)\right] \tag{4.68}$$

We can now replace the hyperbolic functions using equation (4.62):

$$\sinh\zeta_0 = \frac{2\lambda}{1-\lambda^2} \tag{4.69a}$$

$$\cosh\zeta_0 = \frac{1+\lambda^2}{1-\lambda^2} \tag{4.69b}$$

where

$$\lambda = \frac{\mu_5\bar{\sigma}_5}{2k_4^{AdS}} \tag{4.70}$$

This is again an ugly expression. It is more interesting to examine the behaviour at small μ:

$$B = \frac{256\pi^5}{(k_4^{AdS})^6}(G_5\bar{\sigma}_5)^3\mu^4 + \mathcal{O}(\mu^5) = \frac{256\pi^5}{(\mathrm{k}_4^{\mathrm{AdS}})^6}(\bar{\mathrm{G}}_4)^3\mu^4 + \mathcal{O}(\mu^5) \tag{4.71}$$

This is very similar to what we had for the nucleation of a flat bubble spacetime in a de Sitter false vacuum with $\bar{G}_4$ now representing the average of the Newton's constants on the AdS brane and the flat brane. Again we compare this to the result from [24] where we have no extra dimensions. When the energy density of the bubble wall is small, the expression for the probability term is again given by equation (4.60), where $\frac{1}{k_4}$ now corresponds to the radius of curvature of the AdS spacetime. Once again we see that the braneworld result agrees exactly with [24] in the small μ limit, provided we associate G_4 with $\bar{G}_4$.

Note that again we have assumed $\zeta_0 > 0$ and therefore, the bounce solution given here is only valid for $\lambda < 1$. The extension to $\lambda > 1$ is more complicated than for the nucleation of the flat bubble in the previous section. We now have to patch together figure 4.2(b) and figure 4.2(c). However, in [24], the analogue of $\lambda > 1$ violates conservation of energy as one tunnels from the false vacuum to the new configuration. In the braneworld set up we should examine what happens as λ approaches unity from below. In this limit, ζ_0 becomes infinite, and the AdS bubble encompasses the entire brane. The probability, $\mathcal{P}$, of this happening is zero and so there is no vacuum decay. Beyond this, in analogy with [24], we would suspect that the energy of the false vacuum is insufficient to allow the nucleation of a bubble with a large wall energy density. This is indeed the case. When we calculate the probability term, B, for the AdS bubble in a flat, spherical false vacuum, we find that it is divergent and the probability of bubble nucleation vanishes. This divergence

comes from the fact that the false vacuum touches the AdS boundary whereas the bounce solution does not.

Finally, we could also have created an AdS bubble in a flat spacetime using a $\kappa = 0$ vortex. However, it is of no interest since the probability of bubble nucleation is exponentially suppressed by the vortex volume.

4.4.3 Ekpyrotic Instantons

The notion of the Ekpyrotic universe [88] proposes that the Hot Big Bang came about as the result of a collision between two braneworlds. The model claims to solve many of the problems facing cosmology without the need for inflation. Although the authors work mainly in the context of heterotic M theory, they acknowledge that an intuitive understanding can be gained by considering Randall-Sundrum type braneworlds. In this context, we regard the pre-Big Bang era in the following way. We start off with two branes of equal and opposite tension: the hidden brane of positive tension, σ, and the visible brane of negative tension, $-\sigma$. A bulk brane with a small positive tension, $\frac{\epsilon}{2}$, then "peels off" the hidden brane causing its tension to fall to $\sigma - \epsilon$. The bulk brane is then drawn towards our universe, the visible brane, until it collides with us, giving rise to the Hot Big Bang.

The process of "peeling off" is not really considered in great detail in [88]. They suggest that the hidden brane undergoes a small instanton transition with the nucleation of a bubble containing a new hidden brane with decreased tension, and the bulk brane. The walls of this bubble then expand at the speed of light until it envelopes all of the old hidden brane. Given that all the branes in this model are critical we can illustrate the instanton solution in the simplified RS set-up by using a suitable combination of critical brane solutions. In conformal coordinates, critical branes look like planes ($u = \text{const}$) or spheres (see figure 4.2(c)). In describing the Ekpyrotic instanton we present the visible and hidden branes (old and new) as planes. The bulk brane is given by a sphere that intersects the hidden brane, separating the old and new branches (see figure 4.8).

Given this geometrical picture we can calculate the probability of tunneling to this configuration from the initial two brane state. We proceed much as we did in

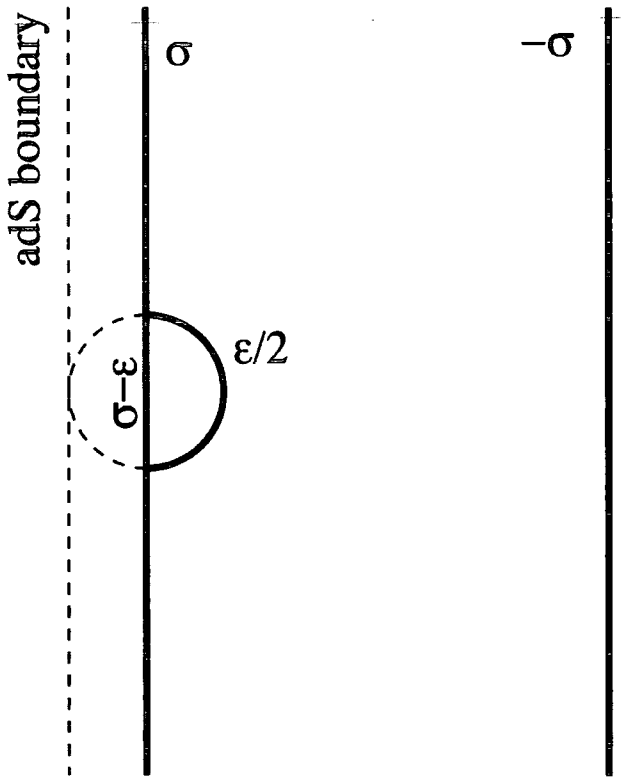


Figure 4.8: The Ekpyrotic Instanton

the previous section, and obtain the following expression for the probability term:

$$B = \frac{\pi^2 \epsilon}{4}\left(\frac{3}{k^3}\ln(1+k^2 Z_0^2) - \frac{2k^{-2}Z_0^2}{1+k^2Z_0^2} - \frac{Z_0^2}{k^2} - \frac{Z_0^4}{4}\right) + \mathcal{O}(\epsilon^2) \tag{4.72}$$

where k is related to the cosmological constant in the bulk of the initial state ($\Lambda = -6k^2$), and Z_0 is a free parameter related to the "size" of the bubble: the larger the value of Z_0, the larger the bubble. We should not be worried by this freedom in Z_0, as we are working with Randall-Sundrum braneworlds which are much simpler than the M5 branes of heterotic M theory. When we return to the M theory context, we lose a number of degrees of freedom and one might expect the value of Z_0 to be fixed. However, since we are dealing with a "small" instanton, we might expect Z_0 itself to be small, and the probability term approximates to the following:

$$B = \frac{\pi^2}{16}\epsilon Z_0^4 + \mathcal{O}(\epsilon^2, Z_0^6) \tag{4.73}$$

We should once again stress however, that this is an extremely simple and naive calculation that ignores any dynamics of the additional scalars, or other fields, that

result from a five-dimensional heterotic M-theory compactification [89]. Another point to note is that while we can have a small brane peel off from the positive tension braneworld, we cannot have one peel off from the negative tension braneworld, as a quick glance at figure 4.8 shows. Such a brane, being critical, must have the form of a sphere grazing the AdS boundary, which therefore necessarily would intersect the positive tension brane as well.

4.5 The AdS soliton

Recall that at the end of section 4.2 we set the integration constant, c, to zero. Now consider what happens when we allow for non-zero values. We will assume that we have a negative cosmological constant in the bulk given by equation (4.12). The bulk spacetime is now described by the metric (4.6) with

$$h(Z) = k_n^2 Z^2 + \kappa - \frac{c}{Z^{n-3}} \tag{4.74}$$

For $c < 0$, the metric becomes singular at the AdS horizon, $Z = 0$. Of greater interest is the case $c > 0$, when the metric takes the form of the AdS soliton [90]. This is the double analytic continuation of the AdS-Schwarzschild metric (3.34). For this reason the (R, Z) plane behaves like a Euclidean AdS black hole, with a horizon at Z_H where $h(Z_H) = 0$. In order to avoid a conical singularity at Z_H, we cut the spacetime off there, and identify R as a angular coordinate with periodicity $\Delta R = 4\pi/h'(Z_H)$. The geometry (up to an AdS warp factor) is therefore the familiar cigar shape with a smooth tip at $Z = Z_H$.

We can clearly try to play the same game as before and investigate branes and vortices in the AdS soliton background. The equations of motion (4.9) for a $\mathbb{Z}_2$-symmetric brane become:

$$Z'^2 = -aZ^2 + \kappa - \frac{c}{Z^{n-3}} \tag{4.75a}$$

$$Z'' = -aZ + \left(\frac{n-3}{2}\right)\frac{c}{Z^{n-2}} - 4\pi G_n \sum_i \mu_i \sigma_n Z \delta(\zeta - \zeta_i) \tag{4.75b}$$

$$R' = \frac{\sigma_n Z}{k_n^2 Z^2 + \kappa - \frac{c}{Z^{n-3}}} \tag{4.75c}$$

where $a = \sigma_n^2 - k_n^2$ as before, and (4.75b) now allows for a multitude of vortices of tension μ_i located at ζ_i.

Although we will explicitly solve these equations for $n = 5$, we will simply describe the qualitative behaviour of solutions for arbitrary values of n. The generic trajectory (which must be periodic in R) will consist of two segments of $Z(\zeta)$ of opposite gradient. These patch together at a positive tension vortex at $R = 0$, say, and a negative tension vortex at $R = \Delta R/2$. This is exactly analogous to the usual situation with a domain wall spacetime when we need both positive and negative tension walls to form a compact extra dimension. However, we see that with the bulk "mass" term, c, there are now also other possibilities. This is because (4.75a) now has at least one root for $Z > Z_H$, and for supercritical branes ($a > 0$) there are two roots. These roots correspond to zeros of Z', and enable a smooth transition from the positive branch of Z' to the negative branch. We can therefore form a trajectory which loops symmetrically around the cigar, and has only one kink – which we can fix to be a positive tension vortex. Of course, the tension of this vortex will be determined by the other parameters of the set-up: the bulk mass, cosmological constant and the braneworld tension, but this is no worse a fine tuning than is already present in conventional RS models. Note that this is now distinct from a domain wall on a compact extra dimension, as we can construct a domain ribbon spacetime with only a single positive tension vortex on asymptotically de Sitter, flat and anti-de Sitter branes. In addition, for a supercritical brane (asymptotically de Sitter) we have the possibility of dispensing with the ribbon altogether. In this case we have a smooth trajectory with two roots of Z', where the brane smoothly wraps the cigar, although a fine tuned mass term is required

In all cases the induced geometry on the brane has the form

$$ds_{n-1}^2 = Z^2(\zeta) d\mathbf{x}_\kappa^2 + d\zeta^2 \tag{4.76}$$

(where of course ζ has a finite range). The energy momentum tensor of this space-

time is

$$\begin{aligned} 8\pi G_{n-1} T^{\mu}_{\nu} &= \left[(n-3)\frac{Z''}{Z} + \frac{(n-3)(n-4)}{2}\frac{(Z'^2-\kappa)}{Z^2}\right]\delta^{\mu}_{\nu} \\ &= -\frac{(n-3)}{2}\left[(n-2)a - \frac{c}{Z^{n-1}} + \mu_n \sigma_n \delta(\zeta)\right]\delta^{\mu}_{\nu} \qquad (4.77a) \\ 8\pi G_{n-1} T^{\zeta}_{\zeta} &= \frac{(n-2)(n-3)}{2}\frac{(Z'^2-\kappa)}{Z^2} \\ &= -\frac{(n-2)(n-3)}{2}\left(a + \frac{c}{Z^{n-1}}\right) \qquad (4.77b) \end{aligned}$$

which has three distinct components. There is a cosmological constant (the a-term) which reflects the lack of criticality of the braneworld when it is non-vanishing. The domain ribbon terms (μ_i) when present indicate the presence of a nested $(n-3)$-brane – note the normalisation is precisely correct for the induced $(n-1)$-dimensional Newton's constant. Finally, the c-term corresponds to a negative stress-energy tensor and can be directly associated to the Casimir energy of field theory living in the braneworld. We will discuss holographic interpretations like this in more detail in chapters 5 and 6.

4.5.1 The AdS soliton in five dimensions

We will now present explicit solutions to equations (4.75a) and (4.75c) when we specialise to $n = 5$. We also restrict attention to the case where $\kappa = 1$, which in any case is the only possibility for supercritical and critical branes. So as not to be littered with confusing suffices let us adopt the notation of chapter 2 and replace k_5 and σ_5 with k and $\tilde{\sigma}$ respectively. The set of equations we wish to solve are therefore just:

$$Z'^2 = -aZ^2 + 1 - \frac{c}{Z^2} \qquad (4.78)$$

$$R' = \frac{\tilde{\sigma} Z}{k^2 Z^2 + 1 - \frac{c}{Z^2}} \qquad (4.79)$$

where $a = \tilde{\sigma}^2 - k^2$. We can easily solve (4.78) to obtain $Z(\zeta)$. The solutions are:

$$Z(\zeta)^2 = \begin{cases} \frac{1}{2|a|}\left[-1 + \sqrt{1-4ac}\cosh(2\sqrt{|a|}(\zeta-\zeta_0))\right] & a < 0 \\ c + (\zeta-\zeta_0)^2 & a = 0 \\ \frac{1}{2a}\left[1 + \sqrt{1-4ac}\cos(2\sqrt{a}(\zeta-\zeta_0))\right] & a > 0 \end{cases} \tag{4.80}$$

where ζ_0 is just a constant of integration. Notice that the solution for the supercritical wall is only valid when $c \leq \frac{1}{4a}$. As a consistency check we observe that (4.80) gives (4.15) when $c = 0$. In order to construct branes containing domain ribbons we patch together solutions with the opposite sign in Z'. This corresponds to taking the opposite sign in the square root of (4.80).

We now tackle the more interesting problem of expressing R in terms of Z. The governing equation is given by:

$$\frac{dR}{dZ} = \pm \frac{\tilde{\sigma} Z}{k^2 Z^2 + 1 - \frac{c}{Z^2}} \frac{1}{\sqrt{-aZ^2 + 1 - \frac{c}{Z^2}}} \tag{4.81}$$

Consider first critical branes with $a = 0$. Define:

$$x_\pm = \frac{-1 \pm \sqrt{1+4k^2c}}{2k^2} \tag{4.82}$$

$$\mu_\pm = 1 - \frac{2x_\pm}{c} \tag{4.83}$$

$$\nu_\pm = \frac{2}{c}\sqrt{\pm x_\pm (c - x_\pm)} \tag{4.84}$$

Given that for critical branes, $\tilde{\sigma} = k$, the solution is:

$$\begin{aligned} R(Z) = R_0 \pm \frac{1}{k}\Bigg[& \cosh^{-1}\left(\frac{Z}{\sqrt{c}}\right) \\ & + \frac{c - x_+}{\sqrt{1+4k^2c}}\left(\frac{2}{\nu_+ c}\right)\tan^{-1}\left(\frac{u(Z)+\mu_+}{\nu_+}\right) \\ & - \frac{c - x_-}{\sqrt{1+4k^2c}}\left(\frac{1}{\nu_- c}\right)\log\left|\frac{u(Z)+\mu_- - \nu_-}{u(Z)+\mu_- + \nu_-}\right|\Bigg] \end{aligned} \tag{4.85}$$

where

$$u(Z) = \exp\left[2\cosh^{-1}\left(\frac{Z}{\sqrt{c}}\right)\right] \tag{4.86}$$

and R_0 is an integration constant. When we consider the non critical branes we find that equation (4.81) gives an elliptic integral. The best we can do is express the solution in terms of canonical forms for elliptic integrals. We will require the

incomplete elliptic integrals of the first and the third kind. They are defined below for $0 \leq x \leq 1$ [91]:

$$F(x|t) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-tz^2)}} \tag{4.87}$$

$$\Pi(n;x|t) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-tz^2)}} \left(\frac{1}{1-nz^2}\right) \tag{4.88}$$

where $0 < t < 1$. We will also need to define the following:

$$\lambda_\pm = \frac{1 \pm \sqrt{1-4ac}}{2a} \tag{4.89}$$

$$n_\pm = \frac{\lambda_+ - \lambda_-}{\lambda_+ - x_\pm} \tag{4.90}$$

$$m_\pm = \frac{x_\pm}{x_\pm - \lambda_-} \tag{4.91}$$

$$q = \frac{\lambda_+ - \lambda_-}{\lambda_+} \tag{4.92}$$

Consider now the supercritical branes with $a > 0$. The solution is:

$$\begin{aligned} R(Z) = R_0 \mp \frac{\tilde{\sigma}}{k^2\sqrt{a\lambda_+}} \Bigg[& F(v(Z)|q) \\ & + \frac{1}{\sqrt{1+4k^2c}} \left(\frac{c-x_+}{\lambda_+ - x_+}\right) \Pi(n_+; v(Z)|q) \\ & - \frac{1}{\sqrt{1+4k^2c}} \left(\frac{c-x_-}{\lambda_+ - x_-}\right) \Pi(n_-; v(Z)|q) \Bigg] \end{aligned} \tag{4.93}$$

where

$$v(Z) = \sqrt{\frac{\lambda_+ - Z^2}{\lambda_+ - \lambda_-}} \tag{4.94}$$

For subcritical branes, with $a < 0$, the solution is:

$$\begin{aligned} R(Z) = R_0 \pm \frac{\tilde{\sigma}\lambda_-}{k^2\sqrt{|a|(\lambda_- - \lambda_+)(1+4k^2c)}} \Bigg[& \frac{1}{x_+} \left(\frac{c-x_+}{\lambda_- - x_+}\right) \Pi(m_+; w(Z)|q^{-1}) \\ & - \frac{1}{x_-} \left(\frac{c-x_-}{\lambda_- - x_-}\right) \Pi(m_-; w(Z)|q^{-1}) \Bigg] \end{aligned} \tag{4.95}$$

where

$$w(Z) = \sqrt{\frac{Z^2 - \lambda_-}{Z^2}}. \tag{4.96}$$

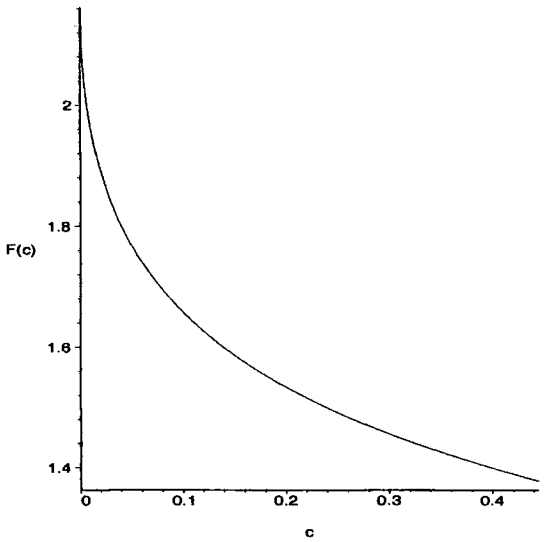


Figure 4.9: A plot of $\mathcal{F}(c)$ for $0 \leq c \leq 1/4a$ when $\tilde{\sigma} = 1.25$ and $k = 1$. Note that there is no solution to $\mathcal{F}(c) = 0$.

As a mathematical exercise, the derivation of these solutions has been of some use. However, do we gain any further understanding of braneworld physics? We suggested earlier that for the supercritical brane, we might be able to place a brane on this compact soliton background, without any need for a vortex. This is because the supercritical brane solution is periodic, so we can have a smooth brane trajectory wrapping the soliton cigar. We are now in a position to investigate this more closely.

If such a configuration does exist, then the following would be true:

$$2N[R(Z_{max}) - R(Z_{min})] = \Delta R \tag{4.97}$$

where $N = 1, 2, 3...$, and $Z_{max} = \sqrt{\lambda_+}$, $Z_{min} = \sqrt{\lambda_-}$ are the maximum and minimum values of Z respectively. This amounts to the fine tuning conditon on the mass term. In particular, the value of c that satisfies

$$\mathcal{F}(c) = 2[R(Z_{max}) - R(Z_{min})] - \Delta R = 0 \tag{4.98}$$

gives us the fine tuned valued for $N = 1$. It is not obvious that (4.98) has a solution in the allowed range, $0 \leq c \leq 1/4a$. However, we can choose $\tilde{\sigma}$ and k and then hope to solve $\mathcal{F}(c) = 0$ numerically. For example, in figure 4.9, we see that there is

no solution for $\tilde{\sigma} = 1.25$ and $k = 1$. Whether or not this behaviour is true for all choices of $\tilde{\sigma}$ and k is unclear and would require a detailed analytic investigation.

Chapter 5

Braneworld holography

We will now change direction in our study of braneworlds, and focus on how they fit into the realms of *holography*. We will discover that the brane cosmology described in chapter 3 can be understood from a "holographic" point of view, by adjusting the properties of the bulk geometry. Before describing this in detail, it is important we review some of the fundamental ideas behind *the holographic principle* [92,93] as well as its most celebrated example, the AdS/CFT correspondence [25–27].

5.1 The holographic principle

The holographic principle is a radical idea that rose from attempts to understand gravity and quantum field theory simultaneously. The natural tool with which to do this is the black hole. As we move close enough to the black hole singularity, the curvature of spacetime becomes of order the Planck scale. At this point the gravitational interactions become as strong as the weak interactions, and the classical description of gravity is inadequate. The time has come to apply quantum physics.

There are two very important results that arise from a quantum description of black holes. The first concerns the black hole entropy, which rather surprisingly turns out to be [94–97]

$$S = \frac{A}{4G_n} \tag{5.1}$$

where A is the area of the black hole horizon. The second result is due to Hawking [98] who noticed that black holes are not as black as they seem. They emit

thermal radiation (Hawking radiation!) and can eventually evaporate.

Now consider a spherical region, Γ, of volume V, in an asymptotically flat spacetime. We will place no restrictions on the matter contained within, and will only state that the boundary, $\delta\Gamma$ has area A. We begin by using local quantum field theory to calculate the maximum entropy, S_{max}, of the quantum mechanical system contained in Γ. By definition,

$$S_{max} = \ln[N_{states}] \tag{5.2}$$

where N_{states} is the total number of possible states of Γ. We can think of the maximum entropy as counting the total number of degrees of freedom. Locality tells us that there is at least one degree of freedom at each spatial point, so we conclude that $S_{max} = \infty$. Even if we say that Γ is not continuous but discrete we still find that $S_{max} \propto V$, as we will now explain. Suppose that Γ is really a lattice with lattice spacing α. The number of cells is approximately V/α^{n-1}, where n is the spacetime dimension. We assume that each cell has m possible states, and deduce that

$$N_{states} = m^{V/\alpha^{n-1}} \tag{5.3}$$

The maximum entropy is then

$$S_{max} = \ln[N_{states}] = \frac{V \ln m}{\alpha^{n-1}} \propto V \tag{5.4}$$

We now use our knowledge of gravity and black holes to calculate the maximal entropy. First consider how much mass can be contained in Γ. We cannot continue to add mass to Γ indefinitely because eventually we will start to form a black hole. As we wish to avoid gravitational collapse we have an upper bound on the mass. It corresponds to the mass, M of the black hole that just fits inside Γ, with its horizon coinciding exactly with the boundary $\delta\Gamma$. Such a black hole has entropy given by equation (5.1). If the mass is smaller than M we can avoid gravitational collapse. If it is M or larger, gravitational collapse is inevitable.

Now suppose that Γ starts of with mass, E and entropy S. We must have $E < M$ to ensure that gravitational collapse has not already taken place. Now consider a spherically symmetric shell of matter with entropy S'. The combined system has

initial entropy

$$S_{initial} = S + S' \tag{5.5}$$

We now assume that the shell is collapsing to form a black hole inside Γ. If the shell has mass $M - E$, then we might expect that the final state will be the black hole described in the last paragraph. The final entropy of the combined system is therefore given by

$$S_{final} = \frac{A}{4G_n} \tag{5.6}$$

However, the second law of thermodynamics tells us that the entropy of a thermodynamical system cannot decrease. This means that $S_{initial} \leq S_{final}$, and since $S' \geq 0$ we conclude that

$$S \leq \frac{A}{4G_n} \qquad \Rightarrow \qquad S_{max} = \frac{A}{4G_n}. \tag{5.7}$$

The gravitational approach and the QFT approach are clearly at odds with one another. Gravity predicts $S_{max} \propto A$ whereas QFT predicts $S_{max} \propto V$. It turns out that its the QFT approach that is wrong because it badly over-estimates the number of degrees of freedom. Each cell of the lattice described earlier has volume $V_{cell} = \alpha^{n-1}$. How much mass, E_{cell}, can be contained in a particular cell without the threat of gravitational collapse? Again, we can have no more than the mass of the largest black hole that can fit into the cell. The mass of a black hole is given by its radius, so we see that $E_{cell} \lesssim \alpha$. This implies that the total mass contained in Γ, is no greater than

$$E_{max} = \alpha \frac{V}{\alpha^{n-1}} = \frac{V}{\alpha^{n-2}} \tag{5.8}$$

However, the mass, M of the largest black hole that can fit inside Γ is given by the radius of Γ, so that

$$M \sim V^{1/n-1} \tag{5.9}$$

We require $E_{max} \leq M$ which gives $V \lesssim \alpha^{n-1}$. So the upper mass limit E_{max} is only valid if Γ is the size of a single cell. The lattice approach permits total energies that exceed the mass, M of the largest black hole. This means that although black holes will not form in each individual cell, they will form on larger scales.

We could of course reject the gravitational approach if we accept the possibility of gravitational collapse. Let us suppose that the number of possible states of Γ is

indeed given by equation (5.3). If Γ contains total mass M, the matter within will collapse to form a black hole. After collapse, the number of possible states is given by $e^{A/4G_n}$. This violates unitarity because the number of initial states is greater than the number of final states. Hawking argued that unitarity broke down in black holes [99]. If we accept that the maximum entropy of a spatial region is proportional to the area of its boundary, rather than its volume, then we can retain unitarity in black holes. This is how the holographic principle was first formulated.

Note that we have made various assumptions so far, such as spherical volumes and asymptotic flatness. We might think that the maximum entropy of a spatial region is still given by $A/4G_n$, even when we drop these assumptions. This is known as the spacelike entropy bound, but it is clearly not valid. Suppose we have a contracting universe. Entropy does not decrease but the boundary area of a given region does. As we shrink smaller and smaller the entropy will eventually exceed the boundary area.

We can however form the covariant entropy bound [93, 100] using light sheets and the focusing theorem of General Relativity (for a nice review see [101] or [102]). Briefly this states that given a (codimension 2) boundary surface, $\delta\Gamma$, of area A, the entropy on any light sheet of $\delta\Gamma$ cannot exceed $A/4G_n$. A light sheet is made up of the light rays passing through $\delta\Gamma$, as long as they are not expanding. Note that a light sheet is a null surface whereas Γ is a spacelike surface.

Depending on the structure of the spacetime, we can use the covariant entropy bound to place bounds on spacelike surfaces. Consider anti-de Sitter space (see figure 5.1). Since light sheets are not allowed to expand, the warped geometry of AdS means that light sheets point away from the AdS boundary towards the AdS horizon. Now take a static (codimension 2) surface, $\delta\Gamma$, near the AdS boundary and consider the region Γ bounded by $\delta\Gamma$, including the AdS horizon. Since the future light sheet of $\delta\Gamma$ points towards the horizon, matter contained in Γ will eventually pass through it. Suppose the entropy contained in Γ is S. When the matter in Γ passes through the light sheet its entropy is $S' \geq S$, in accordance with the second law of thermodynamics. By the covariant entropy bound we have $S' \leq A/4G_n$, and

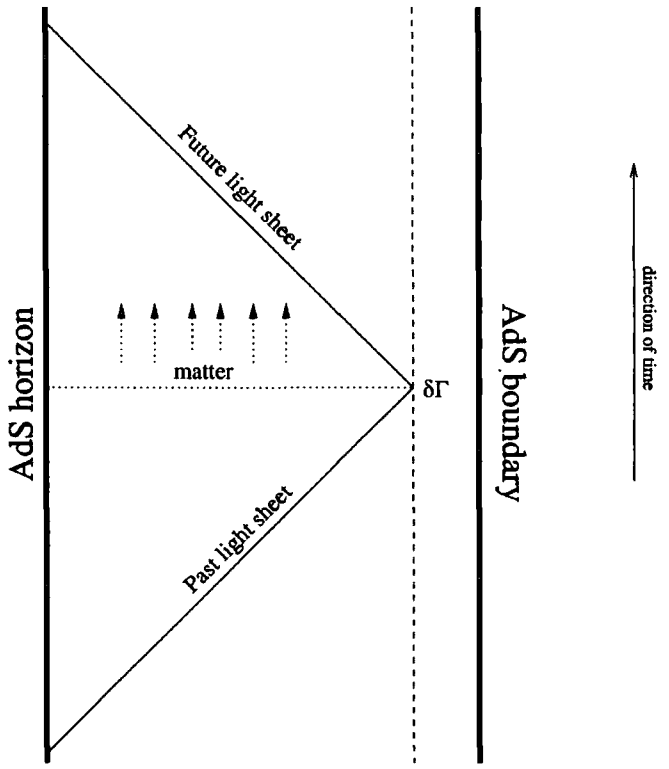


Figure 5.1: In AdS space, matter contained within Γ crosses the future light sheet of the boundary $\delta\Gamma$.

therefore

$$S \leq \frac{A}{4G_n} \tag{5.10}$$

This is an important property of AdS space. There is a timelike Killing vector (so we can define static surfaces like $\delta\Gamma$) and the entire geometry can be foliated by spacelike surfaces satisfying the holographic bound. This means that we have the counterintuitive result: the total number of physical degrees of freedom in a region, Γ of AdS is proportional to the area of the boundary, $\delta\Gamma$, rather than the volume. Note that we can take $\delta\Gamma$ to be as close to the AdS boundary as we wish, so in this sense the holographic principle applies to the whole of AdS. With all this in mind,

it is no surprise that the first concrete example of holography involves anti-de Sitter space. This is the AdS/CFT correspondence [25–27], which we will now describe, albeit very briefly.

5.2 The AdS/CFT correspondence

Consider type IIB string theory on 10 dimensional Minkowski space. The fundamental dimensionful parameter of the theory is the string tension, $T \propto l_s^{-2}$, where l_s is a stringy length scale. We define g_s to be the string coupling, which we will hold fixed. IIB string theory contains objects known as Dp-branes [103]. A Dp-brane is a timelike brane of p dimensions where the ends of open strings can terminate. Its world volume is therefore $(p+1)$-dimensional. The low energy physics of a single brane is described by a $U(1)$ gauge theory. For N distinct branes we naturally have a $U(1)^N$, although N coincident branes are described by an $SU(N)$, where we neglect an overall centre of mass degree of freedom. Furthermore, we can think of a D-brane as a source of energy momentum in the bulk spacetime as well as a source of other supergravity fields. It couples to the bulk by absorbing and emitting closed strings.

Now consider N coincident D3-branes and take the following low energy supergravity limit,

$$l_s \to 0, \qquad u \equiv r/l_s^2 = \text{fixed}. \tag{5.11}$$

Here u represents a typical energy scale corresponding to an open string stretched by an amount r. In this limit, the closed string physics in the bulk can be shown to decouple from the open string physics on the brane [25]. The open string physics is described by $\mathcal{N} = 4$ $SU(N)$ super Yang Mills on Minkowski space in 3+1 dimensions.

As we stated earlier we can think of D-branes as sources of 10-dimensional supergravity fields, in this case the metric, dilaton and 4-form potential, $C_{(4)}$, with field strength $F_{(5)} = dC_{(4)}$. The *extremal* black D3-brane solution is given by [104]

$$ds_{10}^2 = H^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(r)\left[dr^2 + r^2 d\Omega_5^2\right] \tag{5.12}$$

where the dilaton is constant and

$$H(r) = 1 + 4\pi g_s N \left(\frac{l_s}{r}\right)^4. \tag{5.13}$$

$d\Omega_5^2$ is the metric on S^5 and x^μ, for $\mu = 0, 1, 2, 3$, are the D-brane coordinates. The D-brane stack is located at $r = 0$, which also corresponds to the horizon in the extremal case. Finally, we note that the 5-form flux through the 5-sphere surrounding the D-brane source is integer valued,

$$\int_{S^5} F_{(5)} = N. \tag{5.14}$$

As before we define $u = r/l_s^2$. In the low energy supergravity limit we have $l_s \to 0$, so holding u fixed corresponds to taking the *near horizon limit.* The limiting form of the metric is just $AdS_5 \times S^5$,

$$ds_{10}^2 = l_s^2 \left[\sqrt{4\pi g_s N}\frac{du^2}{u^2} + \frac{u^2}{\sqrt{4\pi g_s N}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{4\pi g_s N}d\Omega_5^2\right] \tag{5.15}$$

Note that the AdS_5 and the S^5 both have the same radius given by

$$l^2 = l_s^2\sqrt{4\pi g_s N}. \tag{5.16}$$

and that we still have integer 5-form flux across the S^5. Can we really trust this supergravity description? We can if the curvature is small compared to the string scale. This means

$$l \gg l_s \qquad \Rightarrow \qquad g_s N \gg 1 \tag{5.17}$$

For classical supergravity we also wish to suppress stringy loop corrections so we assume $g_s < 1$. This means we really need $N \gg 1$.

The preliminary conjecture is that IIB supergravity on $AdS_5 \times S^5$ describes the same physics as a large N Yang Mills theory. However, when N is not large, we can no longer trust the supergravity description and need to go to the full string theory. We now formally state the Maldacena conjecture:

The following theories are equivalent

- *Type IIB string theory on $AdS_5 \times S^5$ where both the AdS_5 and the S_5 have the same radius, and the 5-form has integer flux, N, across the S^5.*

- *$\mathcal{N} = 4$ super Yang Mills on 3+1 dimensional Minkowski space, with gauge group $SU(N)$.*

Naturally, if we are to make sense of the correspondence we ought to provide a dictionary that translates the gauge theory language into the gravity language, and vice-versa. Two important entries are

$$g_{YM}^2 = g_s \tag{5.18}$$

$$\left(\frac{l}{l_s}\right)^4 = 4\pi g_{YM}^2 N \tag{5.19}$$

where g_{YM} is the Yang Mills coupling constant. The quantity $g_{YM}^2 N$ is known as the 't Hooft coupling. This is the natural loop counting parameter and we note from (5.17) that it should be large. It was 't Hooft who initiated the study of large N gauge theories in an attempt to understand their behaviour at strong coupling [105].

The boundary of $AdS_5 \times S^5$ is given by Minkowski space in 3+1 dimensions, and is invariant under conformal transformations of the metric. $\mathcal{N} = 4$ super Yang Mills is also conformally invariant and we think of it as living on this boundary. It is decoupled from gravity in the bulk. This means that the correspondence is indeed holographic, as all the degrees of freedom of the bulk gravity theory are projected on to the boundary.

As we will see later on, from the point of view of braneworlds, the most important feature of the AdS/CFT correspondence is *the UV/IR connection* [106]. This states that the ultra-violet degrees of freedom in the CFT correspond to the infra-red in the bulk theory. How can we see this? Consider a string stretched from a D-brane probe in the AdS bulk all the way to the boundary. From the CFT perspective the string looks like a point charge. The mass of the string is proportional to its proper length, which is divergent near the boundary. On the CFT side this corresponds to the divergent self-energy of the point charge. In order to regularize the divergence in the bulk the string is only allowed to approach to within some finite distance of the boundary. This is a long distance, or infra-red cut-off in the length of the string. In the CFT, this turns out to be equivalent to cutting out a shell of small radius around the point charge. This time we have a short distance, or ultra-violet cut-off.

5.2.1 AdS-Schwarzschild/Finite temperature CFT

We will now change the picture slightly by relaxing the condition that the D branes should be extremal. Instead we will consider *near extremal* D branes. The supergravity solution for a non-extremal black brane is given by

$$ds_{10}^2 = H^{-1/2}(r)\left[-f(r)dt^2 + \delta_{ij}dx^i dx^j\right] + H^{1/2}(r)\left[f^{-1}(r)dr^2 + r^2 d\Omega_5^2\right] \tag{5.20}$$

where

$$H(r) = 1 + \left(\frac{l}{r}\right)^4, \qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^4 \tag{5.21}$$

and the constants r_0 and l are related to the overall brane tension and Ramond-Ramond charge. Again the brane is located at $r = 0$, although this time it is hidden behind a horizon at $r = r_0$. For a near extremal brane we take $r_0 \ll l$ and the near horizon limit corresponds to taking $r_0 < r \ll l$. This gives

$$ds_{10}^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + \left(\frac{r}{l}\right)^2 \delta_{ij}dx^i dx^j + l^2 d\Omega_5^2 \tag{5.22}$$

where

$$h(r) = \left(\frac{r}{l}\right)^2\left[1 - \left(\frac{r_0}{r}\right)^4\right] \tag{5.23}$$

This is Schwarzschild-$AdS_5 \times S^5$. When we rotate to Euclidean signature we get a conical singularity at the horizon, r_0. In order to avoid this we cut the spacetime off at the horizon, and identify time t with time $t + \beta$, where

$$\beta = \frac{4\pi}{h'(r_0)} = \frac{\pi l^2}{r_0}. \tag{5.24}$$

This black hole is at temperature $T = 1/\beta$, and its entropy is given by the area of the horizon

$$S_{BH} = \frac{A}{4G} \tag{5.25}$$

where

$$A = \left(\frac{r_0}{l}\right)^3 V_3 \,.\, \Omega_5 l^5 \sim V_3 T^3 \tag{5.26}$$

and $V_3 = \int_{\mathbb{R}^3} dx^1 dx^2 dx^3$.

Staying in Euclidean signature, the boundary of this black hole spacetime has topology $\mathbb{R}^3 \times S^1$, where the S^1 has radius $\beta/2\pi$. A gauge theory living on this boundary would be heated to a finite temperature, T, due to Hawking radiation

from the bulk black hole. A large T we would expect the entropy of the gauge theory to scale like the spatial volume, ie $S \sim V_3$. In the case of $\mathcal{N} = 4$ super Yang Mills we started out with a *conformal* field theory, and although conformal invariance is broken at finite temperature, the only scale we have introduced in T. On dimensional grounds we conclude then that

$$S_{YM} \sim V_3 T^3 \tag{5.27}$$

which is in agreement with the black hole entropy (5.25).

Here we have only given an intuitive argument but more precise calculations of the CFT entropy have been carried out [107, 108]. To sum up, we find that when we switch on a finite temperature, we can associate the temperature, entropy and indeed the energy of the CFT with the corresponding black hole quantities in the bulk [26, 29].

So vast is the subject, we have not been able to present the AdS/CFT correspondence in all its glory[1]. The hope is that we now have a feeling for AdS/CFT and can embark on a study of holography in the context of braneworlds.

5.3 Braneworld holography

We can think of the RS2 braneworld model as two identical copies of AdS space patched together in such a way as to form a brane of given tension. Consider one of these copies of AdS. Notice that we have cut the spacetime off before reaching the AdS boundary. From the point of view of AdS/CFT this corresponds to a long distance, or infra-red cut-off in the bulk. We learnt from the UV/IR connection that an infra-red cut-off in the bulk corresponds to an ultra-violet cut-off in the CFT. Therefore when studying braneworlds we might expect some version of AdS/CFT whereby the gravity theory in the bulk is dual to a CFT with a UV cut-off [111].

At this point we note that our language is rather misleading. The notion of a *conformal* field theory with a momentum cut-off is paradoxical. What we really have is a *broken* conformal field theory. By chopping off part of AdS near the boundary,

[1] See [109] or [110] for a more extensive review.

we broke translational invariance in the "radial" direction. Since we have introduced a scale, this corresponds to breaking conformal invariance in the dual field theory.

There is, however, a twist in the tale. Recall that in the traditional picture of AdS/CFT, the CFT on the boundary is decoupled from gravity in the bulk. We can understand this in the following way. Consider a bulk graviton propagating towards the boundary. It cannot reach the boundary because the background AdS metric blows up there. Gravity is therefore decoupled from the boundary theory.

The situation for braneworlds, meanwhile, is slightly different. The metric at the brane does not blow up. This time, the bulk graviton *can* reach the brane, and gravity is coupled to the field theory there.

Braneworld holography can be summed up in the following statement:

> *Randall-Sundrum braneworld gravity is dual to a CFT with a UV cut-off, coupled to gravity on the brane.*

This is nothing more than a conjecture, and is far from proven. One of the difficulties in studying this type of holography is our lack of knowledge regarding the dual field theory. It is some abstract field theory which we know very little about.

However, consider what happens when we switch on a finite temperature. We have seen how this corresponds to creating a black hole in the bulk, where we now have a non-zero Weyl tensor. Casting our mind back to the Einstein equations on the brane (3.11), we see that the presence of the bulk Weyl tensor affects the geometry on the brane. The hope is that we can understand this effect from a holographic perspective. Intuitively we might think that Hawking radiation from the bulk black hole heats up the brane, giving energy to the dual field theory. We can then examine how this energy enters (say) the cosmological equations on the brane, and compare this to what happens when there is no bulk black hole and we put mass on the brane by hand. If we find the same behaviour we have evidence for braneworld holography. The remainder of this thesis will be devoted to this problem.

5.4 CFTs on critical branes

Consider two n-dimensional spacetimes with negative cosmological constant

$$\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2 \tag{5.28}$$

and glue them together across an $(n-1)$-dimensional brane of tension σ. We saw in section 3.3 that a generalised Birkhoff's theorem admits the following solution for the bulk metric

$$ds_n^2 = -h(Z)dt^2 + \frac{dZ^2}{h(Z)} + Z^2 d\Omega_{n-2}^2 \tag{5.29}$$

where

$$h(Z) = k_n^2 Z^2 + 1 - \frac{c}{Z^{n-3}}. \tag{5.30}$$

Here we have taken the $\kappa = 1$ slicing, with $d\Omega_{n-2}^2$ giving the metric on a unit $(n-2)$-sphere. Recall that $c = 0$ corresponds to pure AdS in the bulk, whereas $c > 0$ corresponds to AdS-Schwarzschild. We wish to see the effect when there is a non-vanishing Weyl tensor, so we will consider the latter.

As in section 3.3, we parametrise the brane using the affine parameter τ. The brane is then given by the section $(\mathbf{x}^\mu, t(\tau), Z(\tau))$ of the bulk metric. Since τ corresponds to the proper time of an observer comoving with the brane, we have the condition

$$-h\dot{t}^2 + \frac{\dot{Z}^2}{h} = -1 \tag{5.31}$$

where dot denotes $\partial/\partial\tau$. This condition ensures that the induced metric on the brane takes the standard FRW form (3.39). Again, we treat $Z(\tau)$ as the scale factor of our brane universe, and construct the Hubble parameter $H = \dot{Z}/Z$.

Now suppose that we have a *critical* brane, ie

$$\sigma_n = \frac{4\pi G_n \sigma}{n-2} = k_n \tag{5.32}$$

so that the induced cosmological constant, $\Lambda_{n-1} = 0$. We further assume that there is no additional matter on the brane so that the brane energy-momentum consists only of brane tension. We can read off the cosmological evolution equations from equations (3.50a) to (3.50c) by setting $\kappa = 1$, $a = \sigma_n^2 - k_n^2 = 0$ and $\rho = p = 0$. The

brane evolution is therefore given by

$$\dot{t} = \frac{k_n Z}{h} \tag{5.33}$$

$$H^2 = -\frac{1}{Z^2} + \frac{c}{Z^{n-1}} \tag{5.34}$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} \tag{5.35}$$

This cosmology is very similar to the standard $\kappa = 1$ cosmology of closed FRW universes. We start off with a Big Bang at $Z = 0$ and experience a period of cosmological expansion, crossing the black hole horizon[2]. Eventually, the rate of expansion slows down and we reach a maximum value of Z. After this, the brane starts to contract until Armageddon, when we disappear with a Big Crunch. The shape of the brane trajectory is shown in figure 5.2.

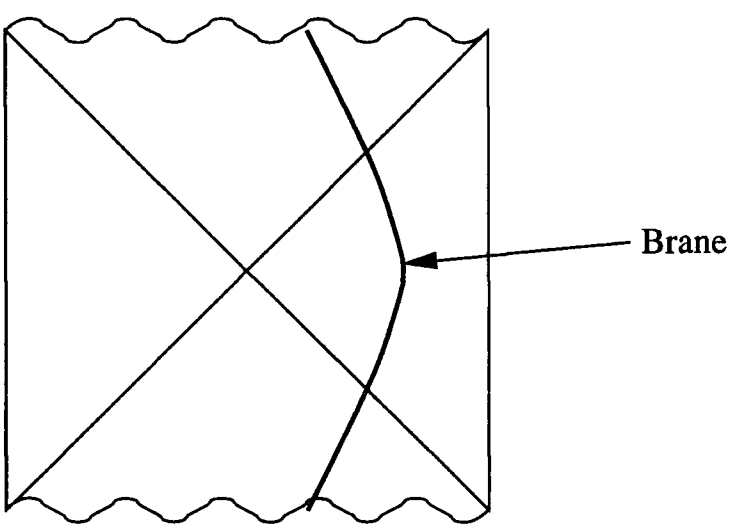


Figure 5.2: The Penrose diagram showing the trajectory of a critical brane in an AdS-Schwarzschild bulk. We have two copies of the bulk glued together at the brane. We have only shown one of those copies here.

[2]Recall that the horizon of the bulk black holes is given by $Z = Z_H$, where $h(Z_H) = 0$.

It is clear from equations (5.34) and (5.35) that the brane cosmology is driven by the terms like c/Z^{n-1}. These come from the mass of the bulk black holes. How should we understand them from a braneworld perspective? Motivated by braneworld holography, we might expect them to correspond to the energy density and pressure of a dual field theory. Given that they go like $1/Z^{n-1}$, this field theory will probably look like radiation. However, the conformal nature of radiation suggests that this will only be the case when there is only a small UV cut-off, and the brane is near the AdS boundary.

5.4.1 Energy density and pressure of the dual CFT

We will now attempt to calculate the energy density/pressure of the dual field theory, at least when the brane is near the boundary of AdS [28]. We can think of these as being the energy density/pressure measured by a braneworld observer.

If we use the bulk time, t as our time coordinate, we would measure the total energy to be given by the sum of the black hole masses, that is

$$E = 2M \tag{5.36}$$

where the mass of an AdS black hole is given by the standard formula [112]:

$$M = \frac{(n-2)\Omega_{n-2}c}{16\pi G_n} \tag{5.37}$$

and Ω_{n-2} is the volume of the unit $(n-2)$-sphere. However, an observer on the brane uses the CFT time, τ as his time coordinate, and will therefore measure the energy differently. To arrive at the CFT energy, E_{CFT}, we need to scale E by $\dot{t}$. We have assumed we are near the AdS boundary. This means that Z is large and we can say that $\dot{t} \approx 1/k_n Z$. The CFT energy is therefore given by

$$E_{CFT} = E\dot{t} \approx \frac{(n-2)\Omega_{n-2}c}{8\pi G_n}\left(\frac{1}{k_n Z}\right) \tag{5.38}$$

In order to calculate the energy *density* we must first evaluate the spatial volume of the CFT, which is just the spatial volume of the brane,

$$V_{CFT} = \Omega_{n-2}Z^{n-2} \tag{5.39}$$

The CFT energy density is given by the ratio of energy to volume[3],

$$\rho_{CFT} = \frac{E_{CFT}}{V_{CFT}} \approx \frac{(n-2)}{8\pi G_n k_n}\left(\frac{c}{Z^{n-1}}\right) \tag{5.40}$$

To calculate the pressure of the CFT, we use the standard formula from thermodynamics[4]

$$p_{CFT} = -\left(\frac{Z}{n-2}\right)\frac{\partial \rho_{CFT}}{\partial Z} - \rho_{CFT} \tag{5.41}$$

Using the expression (5.40) in (5.41) we see that the equation of state for the CFT is indeed that of radiation.

$$p_{CFT} \approx \frac{1}{8\pi G_n k_n}\left(\frac{c}{Z^{n-1}}\right) \approx \frac{\rho_{CFT}}{n-2} \tag{5.42}$$

5.4.2 The cosmological evolution equations

Now that we know the CFT energy density and pressure in terms of c, we can substitute back into (5.34) and (5.35) and examine the brane cosmology in terms of CFT quantities. Before we do this, we recall that for a critical brane, the induced Newton's constant is given by

$$G_{n-1} = \frac{G_n k_n (n-3)}{2} \tag{5.43}$$

We now obtain a more useful expression for ρ_{CFT},

$$\rho_{CFT} \approx \frac{(n-2)(n-3)}{16\pi G_{n-1}}\left(\frac{c}{Z^{n-1}}\right). \tag{5.44}$$

Substituting this and equation (5.42) into (5.34) and (5.35) gives the cosmological evolution equations for the brane.

$$H^2 = -\frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho_{CFT} \tag{5.45}$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho_{CFT} + p_{CFT}) \tag{5.46}$$

These are the standard FRW equations in $(n-1)$ dimensions. They correspond to a spatially spherical universe, with no cosmological constant. The braneworld

[3]Since we are concerned with this ratio, the extension of these ideas to $\kappa \neq 1$ is presumably trivial.

[4]This expression is easily derived from $p = -\partial E/\partial V$, using $E = \rho V$ and $V \sim Z^{n-2}$.

observer sees the normal cosmological expansion driven by the dual CFT. The CFT behaves like radiation in this instance.

We should emphasise that we now have two very different ways of interpreting this cosmology. On the "gravity" side, we think of the cosmological expansion/contraction as being driven by the bulk black holes. On the "field theory" side we think of it being driven by the dual CFT, in the standard way.

5.5 CFTs on non-critical branes

We will now attempt to generalise the above analysis to de Sitter and anti-de Sitter branes. This corresponds to relaxing the criticality condition so that

$$\sigma_n \neq k_n. \tag{5.47}$$

We proceed exactly as before, except this time we allow for $a = \sigma_n^2 - k_n^2 \neq 0$. Equations (5.33) to (5.35) generalise to

$$\dot{t} = \frac{\sigma_n Z}{h} \tag{5.48}$$

$$H^2 = a - \frac{1}{Z^2} + \frac{c}{Z^{n-1}} \tag{5.49}$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}}. \tag{5.50}$$

For $a < 0$, we have subcritical branes, which are asymptotically anti-de Sitter. In this case the brane evolves in much the same way as for critical branes. We start off with a Big Bang and expand to some maximum value of Z, and then contract back to the Big Crunch. As before, the brane crosses black hole horizon. The Penrose diagram for this trajectory is more or less the same as the critical brane trajectory given in figure 5.2.

For $a > 0$, we have supercritical branes, which are asymptotically de Sitter. This time there are four different possible trajectories for the brane depending on the various parameters. This is summarised in the following table,

where

$$a_{crit} = \left(\frac{n-3}{n-1}\right)\left(\frac{2}{(n-1)c}\right)^{\frac{2}{n-3}} \tag{5.51}$$

Case	Trajectory	Conditions
a	Z runs from 0 to ∞.	$a \geq a_{crit}$, Z starts out small.
b	Z runs from ∞ to 0.	$a \geq a_{crit}$, Z starts out large.
c	Z runs from 0 up to a maximum, and then down to 0.	$a \leq a_{crit}$, Z starts out small.
d	Z runs from ∞ down to a minimum, and then up to ∞.	$a \leq a_{crit}$, Z starts out large.

For cases (a) to (c) the brane crosses the black hole horizon. Case (d) is sometimes known as the "bounce" solution, and in this case the brane does not cross the horizon. Each of these possible trajectories are shown in figures 5.3(a) to 5.3(d).

Notice that if $a = a_{crit}$, we can have either cases (a) and (c), or (b) and (d), depending on how Z starts out. We can also have $Z = \text{const}$, although this solution is presumably very unstable.

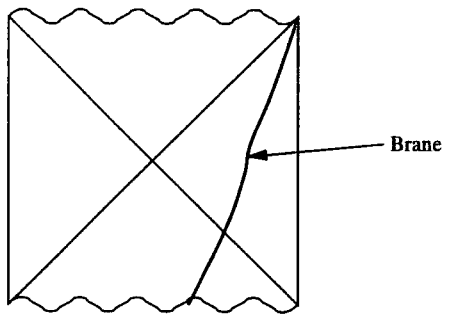
Once again, our goal is to understand the terms like c/Z^{n-1} in the evolution equations (5.49) and (5.50), from the point of view of AdS/CFT. Can we think of this cosmology as being driven by a dual field theory? We will start by blindly adopting the approach of [28], as described in the last section. We will run into problems, but it is nevertheless illustrative to see how things go wrong. We will then give a correct approach which agrees with [28] for critical branes, but not for non-critical branes.

5.5.1 CFT energy density/pressure: naive approach

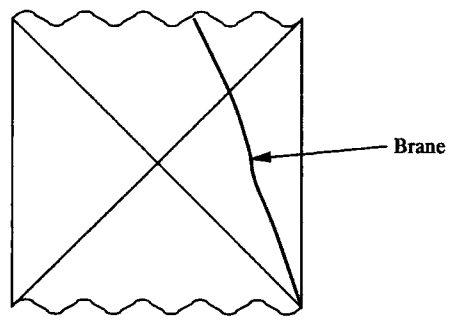
As in section 5.4, we assume that the energy of bulk spacetime is given by

$$E = 2M \tag{5.52}$$

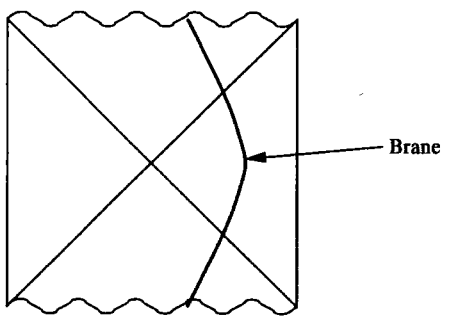
In order to calculate the energy of the CFT, we should once again scale E by $\dot{t}$ so that it is measured with respect to the CFT time, τ, rather than the bulk time, t.



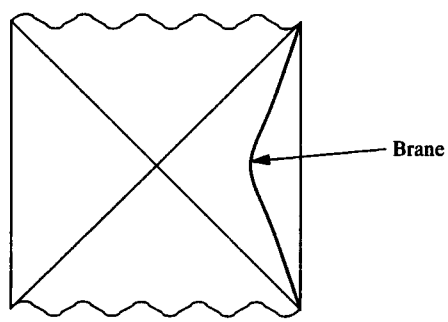
(a) Z starts out small, $a \geq a_{crit}$.



(b) Z starts out large, $a \geq a_{crit}$.



(c) Z starts out small, $a \leq a_{crit}$.



(d) Z starts out large, $a \leq a_{crit}$.

Figure 5.3: Penrose diagrams showing possible trajectories for supercritical (de Sitter) branes in an AdS-Schwarzschild bulk.

However, for large Z, we have from equation (5.48),

$$\dot{t} = \frac{\sigma_n Z}{k_n^2 Z^2 + 1 - \frac{c}{Z^{n-3}}} \approx \frac{\sigma_n}{k_n^2 Z} \tag{5.53}$$

The energy of the CFT is then

$$E_{CFT} = E\dot{t} \approx 2M\left(\frac{\sigma_n}{k_n^2 Z}\right) \tag{5.54}$$

Since the spatial volume of the CFT is just $V_{CFT} = \Omega_{n-2} Z^{n-2}$, we have the following expression for the energy density of the CFT.

$$\rho_{CFT} \approx \frac{2M}{\Omega_{n-2} Z^{n-2}}\left(\frac{\sigma_n}{k_n^2 Z}\right) = \frac{(n-2)}{8\pi G_n \sigma_n}\left(\frac{c}{Z^{n-1}}\right)\left(\frac{\sigma_n^2}{k_n^2}\right) \tag{5.55}$$

where we have used equation (5.37).

At this stage we note an important feature of non-critical branes: the induced Newton's constant on the brane is proportional to the brane tension. More precisely, from (3.13),

$$G_{n-1} = \frac{G_n \sigma_n (n-3)}{2} \tag{5.56}$$

We can insert this back into (5.55) to give

$$\rho_{CFT} \approx \frac{(n-2)(n-3)}{16\pi G_{n-1}}\left(\frac{c}{Z^{n-1}}\right)\left(\frac{\sigma_n^2}{k_n^2}\right) \tag{5.57}$$

Now consider what happens when we express the evolution equations in terms of ρ_{CFT}. In particular, equation (5.49) now reads

$$H^2 = a - \frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho_{CFT}\left(\frac{k_n^2}{\sigma_n^2}\right) \tag{5.58}$$

Note that for critical branes the factor of k_n^2/σ_n^2 disappears and we recover the standard Friedmann equation. However, for non-critical branes, $k_n^2/\sigma_n^2 \neq 1$. This means that equation (5.58) does not resemble the standard FRW cosmology. Either holography has failed, or we have tackled the problem in the wrong way. We shall now see that it is the latter.

5.5.2 CFT energy density/pressure: better approach

Unlike in flat space, when one derives the mass of an AdS black hole (5.37), the leading order contribution comes from the bulk [112]. Furthermore, this derivation

includes contributions from the AdS-Schwarzschild spacetime all the way up to the AdS boundary. In our case, we have a brane that has cut off our bulk spacetime before it was able to reach the boundary. We should not therefore include contributions from "beyond" the brane and must go back to first principles in order to calculate the energy of the bulk [3].

We will begin by Wick rotating to Euclidean signature.

$$t \to t_E = it, \qquad \tau \to \tau_E = i\tau$$

This analytic continuation is well defined for the subcritical brane, critical brane and for the supercritical brane, cases (c) and (d). For cases (a) and (b) we find that $Z(\tau_E)$ is not a real function, so they are excluded from this analysis.

Our bulk metric is now given by

$$ds_n^2 = h(Z)dt_E^2 + \frac{dZ^2}{h(Z)} + Z^2 d\Omega_{n-2}^2 \tag{5.60}$$

As discussed in section 5.2.1, we wish to avoid a conical singularity at the horizon, $Z = Z_H$. In order to do this we cut the spacetime off at the horizon and associate t_E with $t_E + \beta$ where $\beta = 4\pi/h'(Z_H)$. The brane is now given by the section $(\mathbf{x}^\mu, \mathrm{t_E}(\tau_\mathrm{E}), \mathrm{Z}(\tau_\mathrm{E}))$ of the Euclidean bulk. The new equations of motion of the brane are the following:

$$\frac{dt_E}{d\tau_E} = \frac{\sigma_n Z}{h} \tag{5.61}$$

$$\left(\frac{dZ}{d\tau_E}\right)^2 = -aZ^2 + 1 - \frac{c}{Z^{n-3}} \tag{5.62}$$

$$\frac{d^2Z}{d\tau_E^2} = -aZ + \left(\frac{n-3}{2}\right)\frac{c}{Z^{n-2}} \tag{5.63}$$

It is not difficult to see that for both critical and non-critical branes, $Z(\tau_E)$ has a minimum value. In contrast to Lorentzian signature, in Euclidean signature none of these branes cross the black hole horizon. The supercritical branes have a maximum value of Z, whilst the critical and subcritical branes may stretch to the AdS boundary. This will not be a problem because the integrand in our overall action will remain finite, as we shall see.

In calculating the energy we could go ahead and evaluate the Euclidean action of this solution and then differentiate with respect to β. We must however, remember

to take off the contribution from a reference spacetime [30]. In this context, the most natural choice of the reference spacetime would be pure AdS cut off at a surface, Σ whose geometry is the same as our braneworld.

The bulk metric of pure AdS is given by the following:

$$ds_n^2 = h_0(Z)dT^2 + \frac{dZ^2}{h_0(Z)} + Z^2 d\Omega_{n-2}^2 \tag{5.64}$$

where

$$h_0(Z) = k_n^2 Z^2 + 1 \tag{5.65}$$

As we said earlier, the cut-off surface, Σ, should have the same geometry as our braneworld. The induced metric on this surface is therefore

$$ds_{n-1}^2 = d\tau_E^2 + Z(\tau_E)^2 d\Omega_{n-2}^2 \tag{5.66}$$

To achieve this, we must regard our cut-off surface as a section $(\mathbf{x}^\mu, \mathrm{T}(\tau_\mathrm{E}), \mathrm{Z}(\tau_\mathrm{E}))$, where

$$h_0 \left(\frac{dT}{d\tau_E}\right)^2 + \frac{1}{h_0}\left(\frac{dZ}{d\tau_E}\right)^2 = 1 \tag{5.67}$$

Let us now evaluate the difference ΔI between the Euclidean action of our AdS-Schwarzschild bulk, I_{BH} and that of our reference background, I_{AdS}.

$$I_{BH} = -\frac{1}{16\pi G_n}\int_{bulk} d^n x\sqrt{g}(R - 2\Lambda_n) - \frac{1}{8\pi G_n}\int_{brane} d^{n-1}x\sqrt{h}\, 2K \tag{5.68}$$

$$I_{AdS} = -\frac{1}{16\pi G_n}\int_{ref.\ bulk} d^n x\sqrt{g}(R - 2\Lambda_n) - \frac{1}{8\pi G_n}\int_{\Sigma} d^{n-1}x\sqrt{h}\, 2K_0 \tag{5.69}$$

where K and K_0 are the trace of the extrinsic curvature of the brane and Σ respectively. Now recall that we have the Einstein equations in the bulk

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -\Lambda_n g_{ab} \tag{5.70}$$

and the ($\mathbb{Z}_2$-symmetric) Israel equations across the brane

$$K_{ab} = \sigma_n h_{ab}. \tag{5.71}$$

Given that $\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2$, we can immediately read off the following:

$$R - 2\Lambda_n = -2(n-1)k_n^2 \tag{5.72}$$

$$2K = 2(n-1)\sigma_n \tag{5.73}$$

The unit normal to the cut-off surface, Σ is given by $n_a = (\mathbf{0}, -\frac{dZ}{d\tau_E}, \frac{dT}{d\tau_E})$. We use this to find

$$2K_0 = (n-1)\frac{2\sigma_n^2 Z(\tau_E) + cZ(\tau_E)^{2-n}}{h_0 \frac{dT}{d\tau_E}}. \tag{5.74}$$

We will also need the correct form of the measures and the limits in each case. If we say that $-\frac{\beta}{2} \leq t_E \leq \frac{\beta}{2}$, then we obtain the following (see appendix A.5 for a detailed derivation):

$$\int_{bulk} d^n x\sqrt{g}\,(R-2\Lambda_n) = 2\Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \frac{Z(\tau_E)^{n-1} - Z_H^{n-1}}{n-1}\,(R-2\Lambda_n) \tag{5.75}$$

$$\int_{ref.\ bulk} d^n x\sqrt{g}\,(R-2\Lambda_n) = 2\Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \left(\frac{\frac{dT}{d\tau_E}}{\frac{dt_E}{d\tau_E}}\right)\frac{Z(\tau_E)^{n-1}}{n-1}\,(R-2\Lambda_n) \tag{5.76}$$

$$\int_{brane} d^{n-1}\sqrt{h}\,2K = \Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \left(\frac{1}{\frac{dt_E}{d\tau_E}}\right) Z(\tau_E)^{n-2}\,2K \tag{5.77}$$

$$\int_{\Sigma} d^{n-1}\sqrt{h}\,2K_0 = \Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \left(\frac{1}{\frac{dt_E}{d\tau_E}}\right) Z(\tau_E)^{n-2}\,2K_0 \tag{5.78}$$

The factor of two in equations (5.75) and (5.76) just comes from the fact that we have two copies of the bulk spacetime in each case. Notice that the expressions for the integrals over the brane and the cut-off surface Σ are the same. This is a consequence of the two surfaces having the same geometry. Also using equations (5.61) and (5.67), we put everything together and arrive at the following expression for the difference in the Euclidean action:

$$\begin{aligned}\Delta I \;=\; & \frac{\Omega_{n-2}k_n^2}{4\pi G_n}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E Z^{n-1}\left[1-\left(1+\frac{cZ^{1-n}}{\sigma_n^2}\right)^{\frac{1}{2}}\left(1-\frac{cZ^{1-n}}{k_n^2}\left(1+\frac{1}{k_n^2Z^2}\right)^{-1}\right)\right] \\ & -\frac{\Omega_{n-2}}{4\pi G_n}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E\,(n-1)h(Z)Z^{n-3}\left[1-\tfrac{1}{2}\left(1+\tfrac{cZ^{1-n}}{\sigma_n^2}\right)^{\frac{1}{2}}-\tfrac{1}{2}\left(1+\tfrac{cZ^{1-n}}{\sigma_n^2}\right)^{-\frac{1}{2}}\right] \\ & -\frac{\Omega_{n-2}k_n^2}{4\pi G_n}\beta Z_H^{n-1}\end{aligned} \tag{5.79}$$

To proceed further, we are going to have to make things a little bit simpler. In the spirit of AdS/CFT, we want the brane to be close to the AdS boundary. This corresponds to taking c to be large, so our bulk is at a very high temperature. By considering this regime we guarantee that we focus on the "holographic" energy density, and can ignore contributions from matter on the brane. We have not included

any such contributions in our analysis so it is appropriate for us to assume that we are indeed working at large c. To leading order:

$$Z_H \approx \left(\frac{c}{k_n^2}\right)^{\frac{1}{n-1}} \tag{5.80}$$

$$\beta \approx \frac{4\pi}{(n-1)k_n^2}\left(\frac{k_n^2}{c}\right)^{\frac{1}{n-1}} \tag{5.81}$$

For supercritical and critical branes we can assume $Z(\tau_E) \gg c^{\frac{1}{n-1}}$. For subcritical branes this is true provided $|a| \ll 1$ (see appendix A.6). Given this scenario, we now evaluate ΔI to leading order in c.

$$\Delta I = -\frac{\Omega_{n-2}c\beta}{4\pi G_n} + \frac{\Omega_{n-2}k_n^2 c}{4\pi G_n}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E\left(\frac{1}{k_n^2} - \frac{1}{2\sigma_n^2}\right) + \ldots = -\frac{\Omega_{n-2}c\beta}{8\pi G_n}\left(\frac{k_n^2}{\sigma_n^2}\right) + \ldots \tag{5.82}$$

The entire leading order contribution comes from the bulk rather than the brane, which is consistent with [112]. We can now determine the energy of our bulk space-time.

$$E = \frac{d\Delta I}{d\beta} \approx \frac{(n-2)\Omega_{n-2}c}{8\pi G_n}\left(\frac{k_n^2}{\sigma_n^2}\right) \tag{5.83}$$

Notice that in this large c limit, $E \approx 2M\left(\frac{k_n^2}{\sigma_n^2}\right)$, so for critical branes the choice $E = 2M$ would indeed have worked. Our aim was to calculate the energy of the dual CFT, rather than the bulk AdS-Schwarzschild. We must therefore scale E, by $\dot{t}$, so that it is measured with respect to the CFT time τ. Recall that when Z is large, $\dot{t} \approx \sigma_n/k_n^2 Z$ and the energy of the CFT is given by:

$$E_{CFT} = E\dot{t} \approx \frac{(n-2)\Omega_{n-2}c}{8\pi G_n}\left(\frac{k_n^2}{\sigma_n^2}\right)\left(\frac{\sigma_n}{k_n^2 Z}\right) = \frac{(n-2)\Omega_{n-2}c}{8\pi G_n}\left(\frac{1}{\sigma_n Z}\right) \tag{5.84}$$

We divide this by the spatial volume of the CFT, $V_{CFT} = \Omega_{n-2}Z^{n-2}$ to find the CFT energy density.

$$\rho_{CFT} = \frac{E_{CFT}}{V_{CFT}} \approx \frac{(n-2)}{8\pi G_n \sigma_n}\left(\frac{c}{Z^{n-1}}\right) \tag{5.85}$$

To calculate the pressure of the CFT, we just use equation (5.41). This yields an expression that is consistent with the CFT corresponding to radiation:

$$p_{CFT} \approx \frac{1}{8\pi G_n \sigma_n}\left(\frac{c}{Z^{n-1}}\right) \approx \frac{\rho_{CFT}}{n-2} \tag{5.86}$$

5.5.3 The cosmological evolution equations

As before, we want to understand the cosmological equations (5.49) and (5.50) for the brane in terms of braneworld quantities only. This means making use of the correct expression for the induced Newton's constant (3.13). The CFT energy density is now given by

$$\rho_{CFT} \approx \frac{(n-2)(n-3)}{16\pi G_{n-1}} \left(\frac{c}{Z^{n-1}}\right). \tag{5.87}$$

We are now ready to insert this and equation (5.86) into equations (5.49) and (5.50) to derive the cosmological evolution equations for our braneworld.

$$H^2 = a - \frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho_{CFT} \tag{5.88}$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho_{CFT} + p_{CFT}) \tag{5.89}$$

As in section 5.4.2, these are the standard FRW equations in $(n-1)$ dimensions, although this time we have a cosmological constant term a. As was the case for flat branes, we can think of the cosmology as being driven by a dual CFT corresponding to radiation. Alternatively, from a "gravity" perspective, the brane cosmology is driven by the bulk black holes.

The important thing about this analysis was that went beyond the work of [28], which concentrated only on flat braneworlds. Recent observations that we may live in a universe with a small positive cosmological constant [21, 22] suggest that it is important that we extend the discussion at least to de Sitter braneworlds. We have considered de Sitter branes satisfying $a \leq a_{crit}$. In the large c limit, $a_{crit} \ll 1$, so we actually have $a \ll 1$. Our analysis also applies to anti-de Sitter branes satisfying $|a| \ll 1$.

Given the mounting evidence for holography in the literature, we are not really surprised by our result. What is interesting is the way in which we were forced to prove it. The proof offered by [113] is unacceptable because it relies on the assumption that:

$$G_{n-1} = \frac{G_n k_n (n-3)}{2} \tag{5.90}$$

This is true for critical branes, but one should replace k_n in the above expression with σ_n when one considers non-critical branes. We also see in section 5.5.1 that if

we had applied the approach of [28] to non-critical branes, a factor of k_n^2/σ_n^2 would have appeared in front of the CFT terms in equations (5.88) and (5.89). This comes from assuming that the bulk energy is just given by the sum of the black hole masses. As we stated in section 3, this involves an over-counting because it includes energy contributions from "beyond" the brane. The correct calculation of the bulk energy given in this paper ensures that the undesirable factor of k_n^2/σ_n^2 does not appear.

Finally, we end with a note of caution. In the spirit of AdS/CFT we have consistently assumed large Z, and for various reasons, large c. This means that our results are only approximate. We suspect that one could find corrections to higher orders in $1/Z$ and $1/c$. Clearly we should be more careful, and seek an alternative approach that gives us *exact* results, even at finite values of Z and c. Furthermore, because of the limitations imposed by Wick rotation, we were not able to say anything about cases (a) and (b) for supercritical branes. In the next chapter we will adopt a new approach to braneworld holography that does not suffer from any of these limitations or approximations.

Chapter 6

Exact braneworld holography

6.1 Introduction

In the last section, we tried to interpret the Weyl tensor contribution to the Einstein equation induced on a brane. Specifically, we embedded the brane in a AdS-Schwarzschild spacetime so that the non-trivial Weyl tensor manifested itself as a "radiation" term in the FRW equations for the brane universe. Using the ideas of AdS/CFT, we could interpret this term in two ways: (i) it came from the mass of the bulk black holes or (ii) it came from the energy-momentum tensor of some dual conformal field theory.

However, our analysis was based on the assumption that the brane probed deep into AdS, near to the boundary. This allowed us to assume that the energy density of the braneworld was small, and the true holographic description of an $(n-1)$ dimensional braneworld in an n dimensional bulk was understood. Unfortunately, these results were all approximations in the sense that for a general brane evolution it is not necessary for the brane to remain close to the boundary. In this chapter, we will undertake a new study in which we calculate the energy of the field theory on the brane exactly, regardless of the brane's position in the bulk [4].

In order to emphasize the full generality of these results, we will allow the bulk black holes to couple to an electromagnetic field. We are therefore generalising from AdS-Schwarzschild in the bulk, to Reissner-Nordström AdS. We will also allow the brane tension to be arbitrary, thereby including both critical and non-critical branes.

6.2 Branes in a charge black hole background

Consider an $(n-1)$ dimensional brane of tension σ sandwiched in between two n dimensional black holes. Although our brane will be uncharged, we will allow the black holes to be charged. Since this means that lines of flux must not converge to or diverge from the brane, we must have black holes of equal but opposite charge. In this case, the flux lines will pass through the brane since one black hole will act as a source for the charge whilst the other acts as a sink. It should be noted that although we do not have $\mathbb{Z}_2$ symmetry across the brane for the electromagnetic field, the geometry *is* $\mathbb{Z}_2$ symmetric.

We denote our two spacetimes by $\mathcal{M}^+$ and $\mathcal{M}^-$ for the positively and negatively charged black holes respectively. Their boundaries, $\partial\mathcal{M}^+$ and $\partial\mathcal{M}^-$, both coincide with the brane. This scenario is described by the following action:

$$\begin{aligned} S &= \frac{1}{16\pi G_n}\int_{\mathcal{M}^+ + \mathcal{M}^-} d^n x\sqrt{g}\left(R - 2\Lambda_n - F^2\right) + \frac{1}{8\pi G_n}\int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1}x\sqrt{h}K \\ &\quad + \frac{1}{4\pi G_n}\int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1}x\sqrt{h}F^{ab}n_a A_b + \sigma\int_{brane} d^{n-1}x\sqrt{h}, \end{aligned} \tag{6.1}$$

where g_{ab} is the bulk metric and h_{ab} is the induced metric on the brane. K is the trace of the extrinsic curvature of the brane, and n_a is the unit normal to the brane pointing from $\mathcal{M}^+$ to $\mathcal{M}^-$. Notice the presence of the Hawking-Ross term in the action (6.1) which is necessary for black holes with a fixed charge [114].

The bulk equations of motion which result from this action are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda_n g_{ab} + 2F_{ac}F_b{}^c - \frac{1}{2}g_{ab}F^2 \tag{6.2}$$

$$\partial_a\left(\sqrt{g}F^{ab}\right) = 0 \tag{6.3}$$

These admit the following 2 parameter family of electrically charged black hole solutions for the bulk metric

$$ds_n{}^2 = -h(Z)dt^2 + \frac{dZ^2}{h(Z)} + Z^2 d\Omega_{n-2}^2, \tag{6.4}$$

in which

$$h(Z) = k_n{}^2 Z^2 + 1 - \frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}, \tag{6.5}$$

and the electromagnetic field strength

$$F = dA \quad \text{where} \quad A = \left(-\frac{1}{\kappa}\frac{q}{Z^{n-3}} + \Phi\right)dt \quad \text{and} \quad \kappa = \sqrt{\frac{2(n-3)}{n-2}}. \tag{6.6}$$

Recall that $d\Omega_{n-2}^2$ is the metric on a unit $(n-2)$ sphere. k_n is related to the bulk cosmological constant by $\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2$, whereas c and q are constants of integration. If q is set to zero in this solution, we regain the AdS-Schwarzschild solution discussed in the last chapter, where c introduces a black hole mass. The presence of q introduces black hole charge for which Φ is an electrostatic potential difference. In this general metric, $h(Z)$ has two zeros, the larger of which, Z_+, represents the event horizon of the black hole.

Here, charge is a localised quantity. It can be evaluated from a surface integral on any closed shell wrapping the black hole (Gauss' Law). In $\mathcal{M}^{\pm}$ the total charge is

$$Q = \pm\frac{(n-2)\kappa\Omega_{n-2}}{8\pi G_n}q, \tag{6.7}$$

The mass of each black hole, meanwhile, is same as for the uncharged case [112].

$$M = \frac{(n-2)\Omega_{n-2}c}{16\pi G_n}. \tag{6.8}$$

Let us now consider the dynamics of our brane embedded in this background of charged black holes. Once again, we use the affine parameter, τ to parametrise the brane so that it is given by the section $(\mathbf{x}^\mu, t(\tau), Z(\tau))$ of the bulk metric. The Israel equations for the jump in extrinsic curvature across the brane give the brane's equations of motion. One might suspect that the presence of the Hawking-Ross term in the action will affect the form of these equations. However, since the charge on the black holes is fixed, the flux across the brane does not vary and the Israel equations take their usual form

$$K_{ab} = \sigma_n h_{ab}, \tag{6.9}$$

where

$$K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)} \quad \text{and} \quad n_a = (\mathbf{0}, -\dot{Z}, \dot{t}). \tag{6.10}$$

As in the uncharged case, we also have the condition

$$-h(Z)\dot{t}^2 + \frac{\dot{Z}^2}{h(Z)} = -1 \tag{6.11}$$

This ensures that the induced metric on the brane once again takes the standard FRW form (3.39). Again we think of $Z(\tau)$ as the scale factor on the brane, and $H =$

$\dot{Z}/Z$, is the Hubble parameter. We find that the cosmological evolution equations are given by

$$\dot{t} = \frac{\sigma_n Z}{h(Z)} \tag{6.12a}$$

$$H^2 = a - \frac{1}{Z^2} + \frac{c}{Z^{n-1}} - \frac{q^2}{Z^{2n-4}} \tag{6.12b}$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} + (n-2)\frac{q^2}{Z^{2n-4}}. \tag{6.12c}$$

Let us examine these equations in more detail. Equation (6.12b) contains the cosmological constant term $a = \sigma_n^2 - k_n^2$. For subcritical and critical branes, Z has a maximum and minimum value. For supercritical branes, we have two possibilities: either Z is bounded above and below or it is only bounded below and may stretch out to infinity. All trajectories cross the horizon, except the unbounded supercritical one.

Our real interest in equations (6.12b) and (6.12c), lies in understanding the c and q^2 terms. If we take the brane to be close to the AdS boundary, we have already seen how the c term behaves like radiation from a dual CFT. If we make the same approximations, we find that the q^2 term behaves like stiff matter[1] [115]. In the next section we will not make any of these approximations. We will modify the Hamiltonian approach of [30] to calculate the energy density and pressure of the field theory on the brane exactly.

6.3 Energy density on the brane

Consider an observer living on the brane. He measures time using the braneworld coordinate, τ, rather than the bulk time coordinate, t. We saw in the last chapter how this can affect his measurement of the energy density. Since we are trying to understand physics on the brane, we will calculate the energy with respect to τ.

We begin by focusing on the contribution from the positively charged black hole spacetime, $\mathcal{M}^+$ and its boundary, $\partial\mathcal{M}^+$. This boundary of course coincides with the brane. Consider the timelike vector field defined on $\partial\mathcal{M}^+$

[1] Stiff matter has the equation of state $\rho_{CFT} \approx p_{CFT}$.

$$\tau^a = (\mathbf{0}, \dot{t}, \dot{Z}). \tag{6.13}$$

This maps the boundary/brane onto itself, and satisfies $\tau^a \nabla_a \tau = 1$. In principle we can extend the definition of τ^a into the bulk, stating only that it approaches the form given by equation (6.13) as it nears the brane. We now introduce a family of spacelike surfaces, Σ_τ, labelled by τ that are always normal to τ^a. This family provide a slicing of the spacetime, $\mathcal{M}^+$ and each slice meets the brane orthogonally. As usual we decompose τ^a into the lapse function and shift vector, $\tau^a = N r^a + N^a$, where r^a is the unit normal to Σ_τ. However, when we lie on the brane, τ^a *is* the unit normal to Σ_τ, because there we have the condition (6.11). Therefore, on $\partial\mathcal{M}^+$, the lapse function, $N = 1$ and the shift vector, $N^a = 0$. Before we consider whether or not we need to subtract off a background energy, let us first state that the relevant part of the action at this stage of our analysis is the following:

$$I^+ = \frac{1}{16\pi G_n} \int_{\mathcal{M}^+} R - 2\Lambda_n - F^2 + \frac{1}{8\pi G_n} \int_{\partial\mathcal{M}^+} K + \frac{1}{4\pi G_n} \int_{\partial\mathcal{M}^+} F_{ab} n^a A^b. \tag{6.14}$$

As stated earlier, we do not include any contribution from $\mathcal{M}^-$ or $\partial\mathcal{M}^-$, nor do we include the term involving the brane tension. This is because we want to calculate the gravitational Hamiltonian, without the extra contribution of a source. The brane tension has already been included in the analysis as a cosmological constant term, and it would be wrong to double count.

Given the slicing Σ_τ, the Hamiltonian that we derive from I^+ is given by

$$\begin{aligned} H^+ \;=\; & \frac{1}{8\pi G_n} \int_{\Sigma_\tau} N\mathcal{H} + N^a \mathcal{H}_a - 2NA_\tau \nabla_a E^a \\ & - \frac{1}{8\pi G_n} \int_{S_\tau} N\Theta + N^a p_{ab} n^b - 2NA_\tau n_a E^a + 2NF^{ab} n_a A_b \end{aligned} \tag{6.15}$$

where $\mathcal{H}$ and $\mathcal{H}_a$ are the Hamiltonian and momentum constraints respectively [30]. p^{ab} is the canonical momentum conjugate to the induced metric on Σ_τ and E^a is the momentum conjugate to A_a. The surface S_τ is the intersection of Σ_τ and the brane, while Θ is the trace of the extrinsic curvature of S_τ in Σ_τ (see figure 6.1).

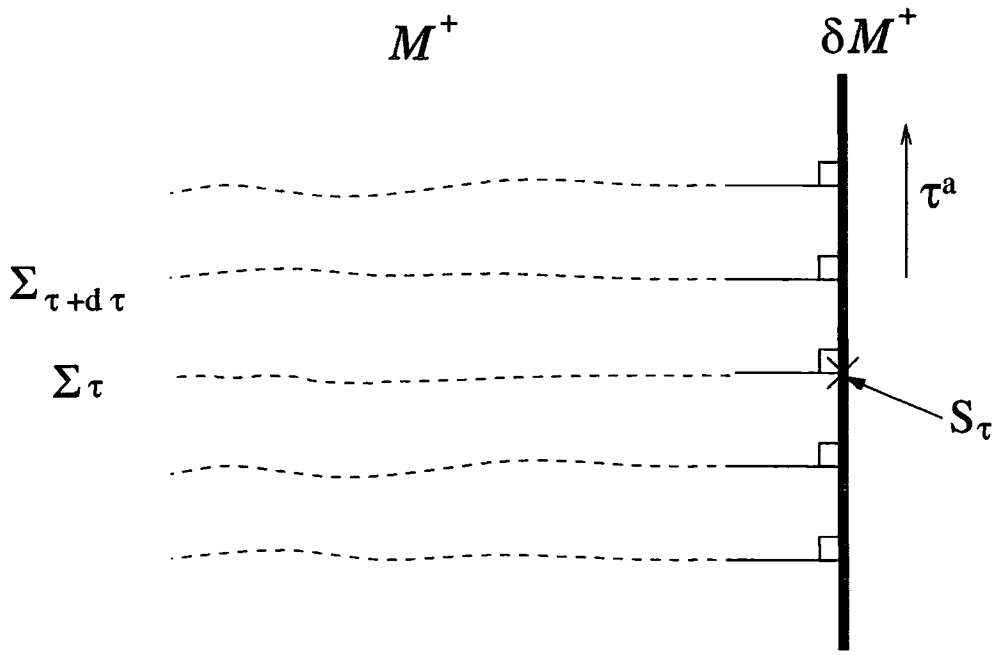


Figure 6.1: Foliation of $\mathcal{M}^+$ into spacelike surfaces Σ_τ. These surfaces meet the brane orthogonally as shown.

Note that the momentum $E^a = F^{a\tau}$. In particular, $E^\tau = 0$ and we regard A_τ as an ignorable coordinate. We will now evaluate this Hamiltonian for the RNAdS spacetime described by equations (6.4), (6.5) and (6.6). Each of the constraints vanish because this is a solution to the equations of motion.

$$\mathcal{H} = \mathcal{H}_a = \nabla_a E^a = 0. \tag{6.16}$$

The last constraint is of course Gauss' Law. When evaluated on the surface S_τ, the potential, $A = \left(-\frac{1}{\kappa}\frac{q}{Z(\tau)^{n-3}} + \Phi\right) \dot{t}\, d\tau$. The important thing here is that it only has components in the τ direction. This ensures that the last two terms in the Hamiltonian cancel one another. Since $N = 1$ and $N^a = 0$ on S_τ, it only remains to evaluate the extrinsic curvature Θ. If γ_{ab} is the induced metric on S_τ, it is easy to

show that

$$\Theta = \Theta_{ab}\gamma^{ab} = K_{ab}\gamma^{ab} = (n-2)\frac{h(Z)\dot{t}}{Z}. \tag{6.17}$$

The energy is then evaluated as

$$\mathcal{E} = -\frac{1}{8\pi G_n}\int_{S_\tau}(n-2)\frac{h(Z)\dot{t}}{Z}. \tag{6.18}$$

We will now address the issue of background energy. This is usually necessary to cancel divergences in the Hamiltonian. In our case, the brane cuts off the spacetime. If the brane does not stretch to the AdS boundary there will not be any divergences that need to be cancelled. However it is important to define a zero energy solution. In this work we will choose pure AdS space. This is because the FRW equations for a brane embedded in pure AdS space would include all but the holographic terms that appear in equations (6.12b) and (6.12c). These are the terms we are trying to interpret with this analysis.

We will denote the background spacetime by $\mathcal{M}_0$. We have chosen this to be pure AdS space cut off at a surface $\partial\mathcal{M}_0$ whose geometry is the same as our brane. As is described in section 5.5.2, this means we have the bulk metric given by

$$ds_n{}^2 = -h_{AdS}(Z)dT^2 + \frac{dZ^2}{h_{AdS}(Z)} + Z^2 d\Omega_{n-2}, \tag{6.19}$$

in which

$$h_{AdS}(Z) = k_n{}^2 Z^2 + 1. \tag{6.20}$$

There is of course no electromagnetic field. The surface $\partial\mathcal{M}_0$ is described by the section $(\mathbf{x}^\mu, T(\tau), Z(\tau))$ of the bulk spacetime. In order that this surface has the same geometry as our brane we impose the condition

$$-h_{AdS}(Z)\dot{T}^2 + \frac{\dot{Z}^2}{h_{AdS}(Z)} = -1 \tag{6.21}$$

which is analogous to the condition given in equation (6.11).

We now repeat the above evaluation of the Hamiltonian for the background spacetime. This gives the following value for the background energy

$$\mathcal{E}_0 = -\frac{1}{8\pi G_n}\int_{S_\tau}(n-2)\frac{h_{AdS}(Z)\dot{T}}{Z}. \tag{6.22}$$

Making use of equations (6.12a), (6.12b) and (6.21) we find that the energy of $\mathcal{M}^+$ above the background $\mathcal{M}_0$ is given by

$$E_+ = \mathcal{E} - \mathcal{E}_0 = \frac{(n-2)}{8\pi G_n}\int_{S_\tau}\sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n \tag{6.23}$$

where

$$\Delta h = h(Z) - h_{AdS}(Z) = -\frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}. \tag{6.24}$$

In this relation Δh is negative everywhere outside of the black hole horizon and so it is clear that E_+ is positive. We now turn our attention to the contribution to the energy from $\mathcal{M}^-$. Since the derivation of E_+ saw the cancellation of the last two terms in the Hamiltonian (6.15) we note that the result is purely geometrical. Even though $\mathcal{M}^+$ and $\mathcal{M}^-$ have opposite charge, they have the same geometry and so $E_+ = E_-$. We deduce then that the total energy

$$E = E_+ + E_- = \frac{(n-2)}{4\pi G_n}\int_{S_\tau}\sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n \tag{6.25}$$

Since the spatial volume of the braneworld $V = \int_{S_\tau} = \Omega_{n-2}Z^{n-2}$, we arrive at the exact expression for the energy density measured by an observer living on the brane

$$\rho = \frac{(n-2)\sigma_n}{4\pi G_n}\left(\sqrt{1 - \frac{\Delta h}{\sigma_n^2 Z^2}} - 1\right) \tag{6.26}$$

where we have pulled out a factor of σ_n.

6.4 Pressure on the brane and equation of state

Using equation (5.41), we can derive the pressure, p, measured on the brane:

$$p = -\rho + \frac{1}{8\pi G_n \sigma_n}\left(1 - \frac{\Delta h}{\sigma_n^2 Z^2}\right)^{-\frac{1}{2}}\left[\frac{(n-1)c}{Z^{n-1}} - \frac{2(n-2)q^2}{Z^{2n-4}}\right] \tag{6.27}$$

This is not very illuminating as it stands. If we take $\Delta h/\sigma_n^2 Z^2 \ll 1$, we recover the approximate results for when the brane is near the AdS boundary:

$$\rho \approx \frac{(n-2)}{8\pi G_n \sigma_n}\left(\frac{c}{Z^{n-1}} - \frac{q^2}{Z^{2n-4}}\right) \tag{6.28a}$$

$$p \approx \frac{(n-2)}{8\pi G_n \sigma_n}\left(\frac{c}{(n-2)Z^{n-1}} - \frac{q^2}{Z^{2n-4}}\right) \tag{6.28b}$$

Here we can clearly see how the pressure is made up of a "radiation" and a "stiff matter" contribution:

$$p \approx \frac{\rho_{\text{rad}}}{n-2} + \rho_{\text{stiff}} \tag{6.29}$$

However, the equation of state in our exact analysis is far more complicated. In the simpler case when $q^2 = 0$, we can express the equation of state in the following way:

$$p = -\rho + \frac{(n-1)\sigma_n}{8\pi G_n}\left[\left(1 + \frac{4\pi G_n}{(n-2)\sigma_n}\rho\right) - \left(1 + \frac{4\pi G_n}{(n-2)\sigma_n}\rho\right)^{-1}\right] \tag{6.30}$$

This simplifies to the radiation state $p = \frac{\rho}{n-2}$ when $\rho \ll 1$. The c/Z^{n-1} term that appears in the FRW equations is often referred to as the radiation term. We have shown that this is only true when ρ is small, and the brane is near to the AdS boundary. More generally, the equation of state is not as simple as that of radiation, and nor should we expect it to be. By introducing a significant UV cut-off in our field theory on the brane, we have completely lost the conformal properties of the theory, and therefore its resemblance to radiation.

When we consider non-zero values of q^2 it is even harder to write down an expression like (6.30). In the limit of small ρ, we have shown that the equation of state simplifies to (6.29), but we cannot say much more.

6.5 The cosmological evolution equations

We shall now insert our expressions for the braneworld energy density (6.26) and pressure (6.27) into the cosmological evolution equations (6.12b) and (6.12c). We find

$$H^2 = a - \frac{1}{Z^2} + \frac{8\pi G_n \sigma_n}{n-2}\rho + \left(\frac{4\pi G_n}{n-2}\right)^2 \rho^2 \tag{6.31a}$$

$$\dot{H} = \frac{1}{Z^2} - 4\pi G_n \sigma_n(\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \tag{6.31b}$$

These are clearly not the standard Friedmann equations for an $(n-1)$ dimensional universe with energy density ρ and pressure p. This should come as no surprise. We have not made any approximations in arriving at these results so it is possible that we would see non-linear terms. What is exciting is that the quadratic terms we see here have exactly the same form as the unconventional terms that we discussed in section 3.3.2.1. In that case, one places extra matter on the brane to discover this unconventional cosmology. We have no extra matter on the brane but by including a bulk black hole, we get exactly the same type of cosmology. Clearly there is an alternative description.

We also note that in section 3.3.2.1, the energy momentum tensor on the brane is split between tension and additional matter in an arbitrary way. In the analysis we have just carried out, the tension is the *only* explicit source of energy momentum on the brane so there is no split required. With this in mind we are able to interpret each term in the FRW equations more confidently, in particular, the cosmological constant term. Furthermore, we have not yet made any assumptions on the form of the braneworld Newton's constant.

Finally, we see that for small ρ and p, we can neglect the ρ^2 and ρp terms and recover the standard Friedmann equations for an $(n-1)$ dimensional universe:

$$H^2 = a - \frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho \tag{6.32}$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) \tag{6.33}$$

where we have taken the induced Newton's constant on the brane to be given by (3.13). We see, then, how the relationships noticed in sections 5.4 and 5.5 are just an approximation of the relationship described here.

Chapter 7

Discussion

Having been on a long, and sometimes difficult journey through the braneworld, we might wonder whether or not such objects really exist in Nature. Moreover, do we actually live on a brane? It is highly unlikely that the Randall-Sundrum models [19,20] accurately describe the structure of our universe. As we emphasized in chapter 2, these are merely toy models. Nature, meanwhile, is far more complicated than this. In particular, neither RS1 nor RS2 includes any supersymmetry, which, although yet to be discovered, is commonly thought to exist. Furthermore, if we believe that something like M-theory represents a "theory of everything" we have to accept that we might have more than just five dimensions. However, despite their simplicity, the RS models have contributed in at least two very important ways:

- they provide a viable "alternative to compactification".
- they give us new tools with which to study holography and its applications.

We will now discuss each point in turn, with emphasis on the relevance to this thesis.

7.1 An alternative to compactification

In chapter 1, we noted that to be consistent at a quantum level, superstring theory and M-theory need to live in 10 and 11 dimensions respectively. We have generally believed that the reason we do not see more than four dimensions is that the extra dimensions are very small, and we require very high energies to probe them. In

RS2, we have seen that generically this need not be the case. In RS2, the extra dimension is infinite, and yet preliminary results suggest that an observer on the brane would see four-dimensional physics up to at least a few TeV. This is achieved in the following way: standard model fields are bound to a domain wall, or brane, although gravity can propagate into the bulk. The bulk geometry is warped, and this warp factor ensures that gravitational perturbations are damped as they move away from the brane. This is known as localisation of gravity.

In this thesis, we began a study of gravity localisation at a non-perturbative level. In chapter 3 we discussed cosmology on the brane. The most interesting feature of this was the quadratic energy-momentum terms that appeared in the FRW equations [23,61,116]. We can neglect the effect of these terms at low density. However, if the universe was very small at some time, these terms become important. This does not disagree with the idea that extra dimensions might show up in the very early universe.

We should mention at this stage that some braneworld cosmologies do not possess a Big Bang singularity. In chapters 5 and 6 we saw that there exist brane trajectories that do not pass through $Z = 0$, where Z is the scale factor of the brane universe. These "bounce" solutions are made possible by modifying the structure of the bulk space-time. By introducing a non-trivial Weyl tensor in the bulk we obtain "dark matter" terms in the FRW equations that prevent the brane from shrinking to zero size. We will discuss "dark matter" terms more in the section on holography.

In chapter 4 we attacked the issue of non-perturbative gravity in a very different way. Our approach was to place a strongly gravitating object on the brane and examine how that affected the geometry there. When we think of a strongly gravitating object, we immediately think of a black hole. However, finding a solution for a black hole bound to the brane is an outstanding problem. Instead, we chose to study a domain wall on the brane. This is a codimension two object living entirely on the brane. For this reason, we refer to it as a vortex. Because there are only two dimensions transverse to the vortex, the transverse part of the bulk metric is conformally flat. This conformal flatness ensures that our equations of motion are completely integrable and we can find an exact solution for the bulk

geometry. Remarkably, when we examine the geometry induced on the brane we find that it behaves as if there were no extra dimensions. This is one of the main results of this thesis: the geometry on a brane containing a vortex of tension T is the same as the geometry that arises from a domain wall of the same tension in $(n-1)$-dimensional Einstein gravity. In this non-perturbative example, gravity is localised on the brane *exactly*. This exactness is probably due to the high degree of symmetry in the problem. Nevertheless, the result has added to the claims that $(n-1)$-dimensional gravity can be reproduced even when a large n-th dimension is present.

Finally, the techniques used in this analysis opened up a number of possibilities. Firstly, we were able to construct *nested* Randall-Sundrum scenarios where the geometry on the *brane* is the traditional RS geometry (in $(n-1)$-dimensions) and we live on the vortex. We could try and play the whole Randall-Sundrum game again and see if "an alternative to compactification" can work with two large extra dimensions. We might also consider the implications this has for holography although more on that later. We also have the tools to construct *braneworld instantons*. This means we can discuss first order phase transitions whereby a true vacuum bubble nucleates in a false vacuum, and then grows. In particular we have shown how one could start off with a de Sitter false vacuum which corresponds to an inflationary era. We have calculated the probability that a flat bubble universe nucleates in this background. The result seems to agree with [24], where we have no extra dimensions.

7.2 A tool for holography

We have seen how the presence of the AdS warp factor in the bulk ensures that gravity is localised on a braneworld. In chapter 5, we came across another important property of AdS space: it can be foliated by a family of spacelike surfaces, each of which satisfy the holographic entropy bound. This makes AdS space a prime candidate for a holographic description. The first concrete example of this is the AdS/CFT correspondence, where we have a duality relating gravity in the bulk to a conformal field theory on the boundary. Specifically, type IIB superstring theory

on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang Mills on the boundary.

Braneworld holography is not so precise. We have Einstein gravity with a negative cosmological constant in the bulk. This is thought to be dual to a field theory on the brane that is cut-off in the ultra-violet. We do not know what the field theory is. However, whereas in the original Maldacena conjecture, gravity decouples from the CFT, this is not the case for the braneworld theory. Although we know very little about this field theory, we can use its coupling to gravity to derive some of its properties. To study braneworld holography we usually require two things: a FRW brane and a black hole in the bulk.

The intuition is as follows: the bulk black hole emits Hawking radiation that heats the brane to a finite temperature. If the braneworld theory exists, it should be hot, and have a non-zero energy density and pressure. In the original work of Verlinde and Savonije [28], they found that we could interpret the brane cosmology in two different ways. Either it is driven by the bulk black hole or it is driven by a dual field theory. In the latter case, the FRW equations are those of the standard cosmology. If the bulk black hole is uncharged, the field theory behaves like radiation.

In chapter 5, we saw that the extension of these ideas to de Sitter and anti-de Sitter branes was non-trivial. We need to be careful when using our AdS/CFT dictionary. The method of Verlinde and Savonije was to take the black hole mass and calculate the energy of the dual CFT by scaling with some appropriate redshift. Although this method works for flat braneworlds, it does not quite work for dS and AdS branes. The AdS/CFT dictionary should really state that bulk energy, rather than black hole mass, translates into the energy of the field theory. Since we have a brane present, the bulk space-time is cut-off before it reaches the AdS boundary. The presence of the bulk cosmological constant ensures that this can affect the calculation of the bulk energy. In chapter 5, we use Euclidean quantum gravity techniques to calculate the bulk energy properly. We find that the bulk energy differs from the black hole mass in just the right way. The dual description described at the end of the last paragraph for flat branes carries over to de Sitter and anti-de Sitter branes.

From a phenomenological point of view, a study of the de Sitter brane is important as recent observations suggest our universe has a small positive cosmological constant [21, 22]. However, from a holographic point of view, we might be more interested in the anti-de Sitter brane. We have already discussed the nested Randall-Sundrum scenario described in chapter 4. Perhaps in this case we could do holography twice and project all degrees of freedom on to the vortex.

We could criticise this kind of braneworld holography for being too imprecise. However, in chapter 6 we saw that we can actually do much more exact calculations. In the approximate braneworld holography of chapter 5, we assumed that the brane was close to the AdS boundary. We can relax this assumption if we use a hamiltonian approach to calculate the energy on the brane. By allowing the brane trajectory to move far away from the boundary, we can see the effect of the UV cut-off in the dual field theory. Although the field theory is nowhere near being conformal, braneworld holography survives. This is another very important result of this thesis. It enables us to make the following exact statement:

The cosmological evolution equations on the brane have the same form whether we have

(i) a black hole in the bulk with no additional matter on the brane.

or (ii) no bulk black hole with additional matter placed on the brane by hand.

For case (ii), we saw in chapter 3 how the evolution equations contain quadratic energy density/pressure terms. When we calculate the energy density/pressure of the dual field theory in (i) we find that they contribute to the evolution equations in exactly the same way. A braneworld observer cannot tell whether the energy that drives his cosmology comes from additional brane matter or a bulk black hole. In this way, the bulk black hole behaves like "dark matter" on the brane: you cannot see it, but you can tell it is there.

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Appendix A

Detailed Calculations

A.1 Green's function in RS2

In order to construct the full Green's function in the RS2 model, we will use Sturm Liouville theory techniques. We begin by reintroducing the negative tension brane at $z = z_c$ so that it acts as a regulator, and an additional boundary condition is imposed

$$(\partial_z + 2k)\Big|_{z=z_c^-} h_{\mu\nu} = 0 \tag{A.1.1}$$

This places new constraints on the (regulated) Green's function so we modify equation (2.25) appropriately

$$\left[e^{2k|z|}\Box^{(4)} + \partial_z^2 - 4k^2 + 4k\delta(z) - 4k\delta(z - z_c)\right] G_R(x, z; x', z') = \delta^{(4)}(x - x')\delta(z - z'). \tag{A.1.2}$$

We now take Fourier transforms with respect to x^μ,

$$\left[-e^{2k|z|}p^2 + \partial_z^2 - 4k^2 + 4k\delta(z) - 4k\delta(z - z_c)\right] \tilde{G}_R(p; z, z') = \delta(z - z') \tag{A.1.3}$$

where

$$\tilde{G}_R(p; z, z') = \int d^4x e^{-ip_\mu(x^\mu - x'^\mu)} G_R(x, z; x', z'). \tag{A.1.4}$$

For $z \neq z'$, the Green's function satisfies the following Sturm Liouville equation

$$\left(\partial_z^2 - 4k^2\right) \tilde{G}_R = p^2 e^{2k|z|} \tilde{G}_R \tag{A.1.5}$$

with boundary conditions

$$(\partial_z + 2k)\Big|_{z=0^+} \tilde{G}_R = 0, \qquad (\partial_z + 2k)\Big|_{z=z_c^-} \tilde{G}_R = 0 \tag{A.1.6}$$

We wish to find eigenstates, $u_m(z)$, for this problem, with eigenvalues $p^2 = -m^2$. The zero mode eigenstate is trivially given by

$$u_0(z) = N_0 e^{-2k|z|} \tag{A.1.7}$$

where N_0 is some normalisation constant. Note that we have inserted the $\mathbb{Z}_2$ symmetry about $z = 0$ explicitly. In order to determine the massive eigenstates we will change variables to $y = me^{k|z|}/k$, so that equation (A.1.5) is transformed into Bessel's equation with $n = 2$ [117]

$$\left[y^2\partial_y^2 + y\partial_y + (y^2 - 4)\right]\tilde{G}_R = 0 \tag{A.1.8}$$

with boundary conditions

$$(y\partial_y + 2)\Big|_{y=m/k}\tilde{G}_R = 0, \qquad (y\partial_y + 2)\Big|_{y=y_c}\tilde{G}_R = 0 \tag{A.1.9}$$

where $y_c = me^{kz_c}/k$. Equation (A.1.8) has solutions $J_2(y)$ and $Y_2(y)$ which satisfy the following recurrence relations [117]

$$(y\partial_y + 2)\,J_2(y) = yJ_1(y), \qquad (y\partial_y + 2)\,Y_2(y) = yY_1(y) \tag{A.1.10}$$

We deduce then that the massive eigenstates are given by

$$u_m(z) = N_m\left[J_1(m/k)Y_2(y) - Y_1(m/k)J_2(y)\right] \tag{A.1.11}$$

where N_m is the normalisation constant. Note that the boundary condition at $y = y_c$ ($z = z_c$) is only satisfied for quantised values of m satisfying the following condition

$$J_1(m/k)Y_1(me^{kz_c}/k) - Y_1(m/k)J_1(me^{kz_c}/k) = 0 \tag{A.1.12}$$

For large z, the asymptotic behaviour of Bessel's functions is given by

$$\begin{aligned} J_n(me^{kz}/k) &\sim \sqrt{\frac{2ke^{-kz}}{\pi m}}\cos\left(\frac{me^{kz}}{k} - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ Y_n(me^{kz}/k) &\sim \sqrt{\frac{2ke^{-kz}}{\pi m}}\sin\left(\frac{me^{kz}}{k} - \frac{n\pi}{2} - \frac{\pi}{4}\right). \end{aligned} \tag{A.1.13}$$

As we send the regulator brane towards infinity ($z_c \to \infty$), equations (A.1.12) and (A.1.13) imply that m is quantised in units of $\pi k e^{-kz_c}$. The normalisation constants, meanwhile, are determined by the following normalisation condition

$$\int_{-z_c}^{z_c} dz\; e^{2k|z|}u_m(z)u_n(z) = \delta_{mn}. \tag{A.1.14}$$

For the zero mode, it is easy to see that this gives

$$N_0^2 = k\left(1 - e^{-2kz_c}\right)^{-1} \tag{A.1.15}$$

The normalisation for the heavy modes is less obvious. However, we note that for large z_c, the dominant contribution to the integral (A.1.14) lies near $|z| = z_c$. Using the asymptotic behaviour (A.1.13) we find that

$$N_m^2 = \frac{\pi m}{2} e^{-kz_c}\left[J_1(m/k)^2 + Y_1(m/k)^2\right]^{-1} + \mathcal{O}(e^{-2kz_c}) \tag{A.1.16}$$

The (Fourier transformed) Green's function satisfies

$$\left(\partial_z^2 - 4k^2 - p^2 e^{2k|z|}\right)\tilde{G}_R = \delta(z - z') \tag{A.1.17}$$

and can be expressed in terms of the complete set of eigenstates, $\{u_m(z)\}$.

$$\tilde{G}_R(p; z, z') = -\frac{u_0(z)u_0(z')}{p^2} - \sum_m \frac{u_m(z)u_m(z')}{m^2 + p^2} \tag{A.1.18}$$

where we ensure $p^2 \neq -m^2$ by adding a small imaginary part in the "time" direction, ie. $p^\mu = (\omega + i\epsilon, \mathbf{p})$. We now remove the regulator brane completely by sending $z_c \to \infty$. The quantisation in m disappears and we go to a continuum limit, replacing the sum in equation (A.1.18) with the following integral

$$\sum_m \frac{u_m(z)u_m(z')}{m^2 + p^2} \longrightarrow \int_0^\infty dm \lim_{z_c \to \infty} \frac{1}{\pi k e^{-kz_c}} \left(\frac{u_m(z)u_m(z')}{m^2 + p^2}\right) \tag{A.1.19}$$

The extra term appearing in the integral is just a "density of states" factor that will cancel the vanishing part of the normalisation constant. Inverting the Fourier transform (A.1.4), we find that the full Green's function is given by

$$G_R(x, z; x', z') = -\int \frac{d^4p}{(2\pi)^4} e^{ip_\mu(x^\mu - x'^\mu)} \left[\frac{e^{-2k(|z|+|z'|)}k}{\mathbf{p}^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \, \frac{v_m(z)v_m(z')}{m^2 + \mathbf{p}^2 - (\omega + i\epsilon)^2}\right], \tag{A.1.20}$$

where

$$v_m(z) = \frac{\sqrt{m/2k}\left[J_1(m/k)Y_2(me^{k|z|}/k) - Y_1(m/k)J_2(me^{k|z|}/k)\right]}{\sqrt{J_1(m/k)^2 + Y_1(m/k)^2}}. \tag{A.1.21}$$

Finally we should note that we did not include eigenstates satisfying $p^2 = m^2 > 0$. These would be linear combinations of "modified" Bessel's functions, but would not be normalisable and are therefore omitted.

A.2 Warp factor around non-critical branes

Given the ansatz (2.49) we need to solve the bulk equations of motion with cosmological constant, $\Lambda = -6k^2$. Our solution must then satisfy the boundary conditions imposed at the brane of (positive) tension σ, sitting at $z = 0$.

The bulk equations of motion are just given by the Einstein equations with the appropriate cosmological constant.

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda g_{ab} \tag{A.2.22}$$

If we define λ to be the cosmological constant *on the brane*, this gives

$$\mu\nu \text{ equation :} \qquad \frac{\lambda}{a^2} - 3\left(\frac{a'}{a}\right)^2 - \frac{a''}{a} = -4k^2, \tag{A.2.23}$$

$$zz \text{ equation :} \qquad -4\frac{a''}{a} = -4k^2. \tag{A.2.24}$$

where 'prime' denotes differentiation with respect to z. These equations have three classes of solutions, depending on whether λ is positive, negative or zero.

$$\lambda > 0 : \qquad a(z) = \frac{1}{k}\sqrt{\frac{\lambda}{3}}\sinh(\pm kz + c) \tag{A.2.25}$$

$$\lambda = 0 : \qquad a(z) = e^{\pm kz + c} \tag{A.2.26}$$

$$\lambda < 0 : \qquad a(z) = \frac{1}{k}\sqrt{-\frac{\lambda}{3}}\cosh(\pm kz + c) \tag{A.2.27}$$

where c is a constant of integration.

The boundary conditions are given by the Israel junction conditions [41] at the brane.

$$\Delta K_{ab} = -\frac{8\pi G_5}{3}\sigma g_{0ab} \tag{A.2.28}$$

where g_{0ab} is the induced metric on the brane. Given our ansatz (2.49) and the fact that we have $\mathbb{Z}_2$ symmetry across the brane, we find that

$$\frac{a'}{a}\Big|_{z=0^+} = -\frac{4\pi G_5}{3}\sigma \tag{A.2.29}$$

Since we are assuming $\sigma > 0$ we find that we are left with

$$\lambda > 0 : \qquad a(z) = \frac{1}{k}\sqrt{\frac{\lambda}{3}}\sinh(-k|z| + c) \tag{A.2.30}$$

$$\lambda = 0 : \qquad a(z) = e^{-k|z| + c} \tag{A.2.31}$$

$$\lambda < 0 : \qquad a(z) = \frac{1}{k}\sqrt{-\frac{\lambda}{3}}\cosh(-k|z| + c) \tag{A.2.32}$$

with the following conditions

$$\lambda > 0 \ : \qquad \tilde{\sigma} = k \coth c > k \tag{A.2.33}$$

$$\lambda = 0 \ : \qquad \tilde{\sigma} = k \tag{A.2.34}$$

$$\lambda < 0 \ : \qquad \tilde{\sigma} = k \tanh c < k \tag{A.2.35}$$

where $\tilde{\sigma} = 4\pi G_5 \sigma/3$. We are also free to set $a(0) = 1$ in each case giving

$$\lambda > 0 \ : \qquad k = \sqrt{\frac{\lambda}{3}} \sinh c \tag{A.2.36}$$

$$\lambda = 0 \ : \qquad c = 0 \tag{A.2.37}$$

$$\lambda < 0 \ : \qquad k = \sqrt{-\frac{\lambda}{3}} \cosh c \tag{A.2.38}$$

Equations (A.2.33) to (A.2.38) fix the cosmological constant on the brane to be

$$\lambda = 3(\tilde{\sigma}^2 - k^2) \tag{A.2.39}$$

with the final solutions given by equations (2.50),(2.51) and (2.52).

A.3 Extrinsic curvature of a dynamic brane

Suppose we have a bulk spacetime whose metric is given by

$$ds_n^2 = -h(Z)^2 dt^2 + \frac{dZ^2}{h(Z)} + Z^2 d\mathbf{x}_\kappa^2 \tag{A.3.40}$$

cut off at a brane given by the section

$$X^a = (\mathbf{x}^\mu, \mathrm{t}(\tau), \mathrm{Z}(\tau)) \tag{A.3.41}$$

where τ is the proper time for an observer comoving with the brane. This gives the condition

$$-h\dot{t}^2 + \frac{\dot{Z}^2}{h} = -1 \tag{A.3.42}$$

so that the induced metric on the brane is given by equation (3.39). Now suppose the normal to the brane is defined as

$$n_a == \epsilon(\mathbf{0}, -\dot{Z}(\tau), \dot{t}(\tau)) \tag{A.3.43}$$

and define the extrinsic curvature of the brane to be $K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)}$. We first find that

$$K_{\mu\nu} = \nabla_{(\mu} n_{\nu)} = -\Gamma^a_{\mu\nu} n_a = \frac{\epsilon h \dot{t}}{Z} h_{\mu\nu} \tag{A.3.44}$$

The components of $\partial/\partial\tau$ are given by

$$\tau^a = (\mathbf{0}, \dot{t}(\tau), \dot{Z}(\tau)) \tag{A.3.45}$$

which is normal to n_a. The last non-zero component of the extrinsic curvature is then

$$\begin{aligned} K_{\tau\tau} &= \tau^a \tau^b \nabla_a n_b = -\tau^a n_b \nabla_a \tau^b = -n_c(\dot{\tau}^c + \Gamma^c_{ab}\tau^a\tau^b) \\ &= \epsilon \dot{Z}\left[\ddot{t} + \frac{h'}{h}\dot{t}\dot{Z}\right] - \epsilon\dot{t}\left[\ddot{Z} + \left(\frac{h'}{2}\right) h\dot{t}^2 - \left(\frac{h'}{2}\right)\frac{\dot{Z}^2}{h}\right] \\ &= \frac{\ddot{Z} + \frac{1}{2}h'}{\epsilon h \dot{t}} \end{aligned} \tag{A.3.46}$$

where we have used equation (A.3.42).

A.4 Probability of bubble nucleation on the brane

In section 4.4 we calculated the probability of bubble nucleation in a number of braneworld situations. The details of these calculations are remarkably similar for both the flat bubble and the AdS bubble. In this section we shall present the calculation for the flat bubble spacetime forming in a de Sitter false vacuum.

Consider now equations (4.52a-d). Our solution satisfies the equations of motion both in the bulk and on the brane. The bulk equations of motion are just the Einstein equations (in Euclidean signature) with a negative cosmological constant:

$$R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda_n g_{ab} \tag{A.4.47}$$

from which we can quickly obtain

$$R - 2\Lambda_n = \frac{4\Lambda_n}{n-2} = -2(n-1)k_n^2 \tag{A.4.48}$$

where we have used the relation (4.12). The brane equations of motion are just the Israel equations given that we have a brane tension and a nested domain wall:

$$\Delta K_{ab} - \Delta K h_{ab} = 8\pi G_n \sigma h_{ab} + 8\pi G_n \mu \delta(\zeta) \gamma_{ab} \tag{A.4.49}$$

where σ is σ^{flat} and σ^{dS} on the flat and de Sitter branes respectively. We can therefore read off the following expression:

$$\Delta K = -2(n-1)\sigma_n - \mu_n \delta(\zeta) \tag{A.4.50}$$

where we have also used (3.15) and $\mu_n = 8\pi G_n \mu$. We are now ready to calculate the action. Inserting (A.4.48) and (A.4.50) in (4.52), we immediately see that the contribution from the vortex is cancelled off by the delta function in the extrinsic curvature and we are left with

$$S_{bounce} = \frac{2(n-1)k_n^2}{16\pi G_n}\int_{bulk} d^n x\sqrt{g} - \frac{4\sigma_n^{flat}}{16\pi G_n}\int_{flat} d^{n-1}x\sqrt{h} - \frac{4\sigma_n^{dS}}{16\pi G_n}\int_{dS} d^{n-1}x\sqrt{h} \tag{A.4.51}$$

The expression for S_{false} is similar except that there is of course no flat brane contribution and the limits for the bulk and de Sitter brane integrals run over the whole of the de Sitter sphere interior and surface respectively.

Working in Euclidean conformal coordinates (*i.e.*, the metric (4.18) rotated to Euclidean signature) the bulk measure is simply

$$\sqrt{g}d^n x = \frac{\rho^{n-2}}{(k_n u)^n} du\, d\rho\, d\Omega_{n-2} \tag{A.4.52}$$

where $d\Omega_{n-2}$ is the measure on a unit $n-2$ sphere. From (4.20) and (4.55), the de Sitter brane is given by

$$(u-u_0)^2 + \rho^2 = u_1^2 \tag{A.4.53}$$

so the induced metric on this brane is given by:

$$ds_{n-1}^2 = \frac{1}{k_n^2 u^2}\left[\left(\frac{k_n u_0}{\sigma_n^{dS}\rho(u)}\right)^2 du^2 + \rho(u)^2 d\Omega_{n-2}^2\right] \tag{A.4.54}$$

where $\rho(u)$ is given in (4.55c). As we did for the bulk, we can now read off the de Sitter brane measure:

$$\sqrt{h}d^{n-1}x = u_1 \frac{\rho(u)^{n-3}}{(k_n u)^{n-1}}\, du\, d\Omega_{n-2}\,. \tag{A.4.55}$$

Now consider the flat brane. This is given by $u = u_c$ where u_c is given by (4.55a), and the measure can be easily seen to be

$$\sqrt{h}d^{n-1}x = \frac{\rho^{n-2}}{(k_n u_c)^{n-1}}\, d\rho\, d\Omega_{n-2}. \tag{A.4.56}$$

Now we are ready to evaluate the probability term $B = S_{bounce} - S_{false}$. Given each of the measures we have just calculated and taking care to get the limits of integration right for both the bounce action and the false vacuum action, we arrive at the following expression:

$$\begin{aligned} B &= -\frac{4(n-1)k_n^2}{16\pi G_n}\Omega_{n-2}\int_{u_0-u_1}^{u_c} du \int_0^{\rho(u)} d\rho \frac{\rho^{n-2}}{(k_n u)^n} \\ &\quad -\frac{4\sigma_n^{flat}}{16\pi G_n}\Omega_{n-2}\int_0^{\rho(u_c)} d\rho \frac{\rho^{n-2}}{(k_n u_c)^{n-1}} + \frac{4\sigma_n^{dS}}{16\pi G_n}\Omega_{n-2}\int_{u_0-u_1}^{u_c} du u_1 \frac{\rho(u)^{n-3}}{(k_n u)^{n-1}} \end{aligned} \tag{A.4.57}$$

We should note that we have a factor of two in the bulk part of the above equation arising from the fact that we have two copies of the bulk spacetime. If we use the fact that:

$$\rho(u_c) = \frac{k_n u_c}{k_{n-1}^{dS}} \cos\zeta_0 \tag{A.4.58}$$

along with $\sigma_n^{flat} = k_n$ and equation (4.55b), we can simplify (A.4.57) to arrive at equation (4.53).

A.5 Limits and measures for action integrals

Let us consider in more detail each contribution to the action integrals given in equations (5.68) and (5.69). We will start by looking at the bulk integral for the black hole action:

$$\int_{bulk} = \int_{bulk} d^n x\sqrt{g}(R - 2\Lambda_n) \tag{A.5.59}$$

From equation (5.72), we see that $R - 2\Lambda_n$ is constant and so does not cause us any problems. Given that the AdS-Schwarzschild bulk is cut off at the brane, $Z(\tau_E)$, and the horizon, Z_H, we find that:

$$\int_{bulk} = 2\Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \int_{Z_H}^{Z(\tau_E)} dZ\, Z^{n-2}(R-2\Lambda_n) = 2\Omega_{n-2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \frac{Z(\tau_E)^{n-1} - Z_H^{n-1}}{n-1}(R-2\Lambda_n) \tag{A.5.60}$$

which is just equation (5.75). The factor of two comes in because we have two copies of AdS-Schwarzschild. The factor of Ω_{n-2} just comes from integrating out $\int d\Omega_{n-2}$. We now turn our attention to the bulk integral for the reference action:

$$\int_{ref.\ bulk} = \int_{ref.\ bulk} d^n x \sqrt{g}(R - 2\Lambda_n) \tag{A.5.61}$$

Again, $R - 2\Lambda_n$ is constant and does not worry us. This time the AdS bulk is cut off at Σ (given by $Z = Z(\tau_E)$), and at $Z = 0$. The periodicity of the T coordinate is β' rather than β. The bulk integral for the reference action is then:

$$\int_{ref.\ bulk} = 2\Omega_{n-2} \int_{-\frac{\beta'}{2}}^{\frac{\beta'}{2}} dT \int_0^{Z(\tau_E)} dZ\, Z^{n-2}(R-2\Lambda_n) = 2\Omega_{n-2} \int_{-\frac{\beta'}{2}}^{\frac{\beta'}{2}} \frac{Z(\tau_E)^{n-1}}{n-1}(R-2\Lambda_n) \tag{A.5.62}$$

β' is fixed by the condition that the geometry of Σ and the brane should be the same. This just amounts to saying that $T^{-1}(\pm\frac{\beta'}{2}) = \pm\tau_{max} = t_E^{-1}(\pm\frac{\beta}{2})$ where $-\tau_{max} \leq \tau_E \leq \tau_{max}$ on both Σ and the brane. As illustrated below by changing coordinates to τ_E and then t_E, we arrive at equation (5.76):

$$\begin{aligned} \int_{ref.\ bulk} &= 2\Omega_{n-2} \int_{-\tau_{max}}^{\tau_{max}} d\tau_E \frac{dT}{d\tau_E} \frac{Z(\tau_E)^{n-1}}{n-1}(R - 2\Lambda_n) \\ &= 2\Omega_{n-2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \frac{d\tau_E}{dt_E} \frac{dT}{d\tau_E} \frac{Z(\tau_E)^{n-1}}{n-1}(R - 2\Lambda_n) \end{aligned} \tag{A.5.63}$$

Consider now the brane integral:

$$\int_{brane} = \int_{brane} d^{n-1} x \sqrt{h}\, 2K \tag{A.5.64}$$

We will use the coordinate τ_E to begin with and then change to t_E, thus arriving at equation (5.77):

$$\int_{brane} = \Omega_{n-2} \int_{-\tau_{max}}^{\tau_{max}} d\tau_E\, Z(\tau_E)^{n-2}\, 2K = \Omega_{n-2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt_E \frac{d\tau_E}{dt_E} Z(\tau_E)^{n-2}\, 2K \tag{A.5.65}$$

The procedure for arriving at equation (5.78) is exactly the same, owing to the fact that Σ and the brane have the same geometry.

A.6 Justifying $Z(\tau_E) \gg c^{\frac{1}{n-1}}$ in large c limit

Let us consider the claim made in section 5.5.2 that for most brane solutions, $Z(\tau_E) \gg c^{\frac{1}{n-1}}$ in the large c limit. The governing equation for the branes in Euclidean AdS-Schwarzschild is given by equation (5.62):

$$\left(\frac{dZ}{d\tau_E}\right)^2 = -aZ^2 + 1 - \frac{c}{Z^{n-3}} \tag{A.6.66}$$

Now in each case, $Z \geq Z_{min}$ where Z_{min} is the minimum value of Z on the brane. It is sufficient to show that $Z_{min} \gg c^{\frac{1}{n-1}}$. At $Z = Z_{min}$, $\frac{dZ}{d\tau_E} = 0$. For $a = 0$, we have:

$$Z_{min} = c^{\frac{1}{n-3}} \gg c^{\frac{1}{n-1}} \tag{A.6.67}$$

For $a > 0$, we have:

$$Z_{min} \geq c^{\frac{1}{n-3}} \gg c^{\frac{1}{n-1}} \tag{A.6.68}$$

We see that our claim holds for supercritical and critical branes. For subcritical branes with $a < 0$ we need to be more careful. Z_{min} satisfies:

$$Z_{min}^{n-3}(1 + |a|Z_{min}^2) = c \tag{A.6.69}$$

If $Z_{min}^2 \ll |a|^{-1}$ then $Z_{min} \approx c^{\frac{1}{n-3}}$. If $Z_{min}^2 \sim |a|^{-1}$ then $(1 + |a|Z_{min}^2) \sim c^0$ and therefore $Z_{min} \sim c^{\frac{1}{n-3}}$. In each case we have $Z_{min} \gg c^{\frac{1}{n-1}}$. Finally, when $Z_{min}^2 \gg |a|^{-1}$:

$$Z_{min} \approx \left(\frac{c}{|a|}\right)^{\frac{1}{n-1}} \tag{A.6.70}$$

Provided $|a| \ll 1$ we can say:

$$Z_{min} \gg c^{\frac{1}{n-1}} \tag{A.6.71}$$

We see, therefore that the claim made in section 5.5.2 was indeed valid: $Z(\tau_E) \gg c^{\frac{1}{n-1}}$ for subcritical branes with $|a| \ll 1$ and all supercritical and critical branes.