Brane probes and gauge theory/gravity dualities

Page, David C.

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David C. Page

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Centre for Particle Theory
Department of Mathematical Sciences
University of Durham
England

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Dedicated to

My parents

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Abstract

We examine the use of branes as probes of supergravity geometries which arise in the study of gauge theory/gravity dualities. We investigate the moduli spaces of supersymmetric gauge theories through moduli spaces of brane probes in the dual gravity theories. Preferred coordinate systems emerge in which the supergravity geometries can readily be compared to the gauge theory and various gauge theory quantities such as anomalous scaling dimensions can be read off.

We also consider the physics of certain expanded brane configurations, called giant gravitons. We identify supergravity solutions which represent coherent states of these objects. We find a degeneracy between giant graviton probes and massless particles in a broad class of supergravity backgrounds and uncover a close relationship with charged particle states in lower dimensions.

Declaration

The work in this thesis is based on research carried out between October 1999 and July 2002 at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham , England.

No part of this thesis has been submitted elsewhere for any other degree or qualification and is all my own work unless referenced to the contrary in the declaration or in the text. Chapter 1 consists of a review of preliminary material. The introductory sections of chapters 2,3 and 4 also contain review material which is clearly marked as such in the text . The remainder of the thesis is based on my own original research performed in collaboration with the coauthors of the papers [1–4] where many of these results appear.

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Contents

Chapter 1

Introduction

The underlying theme of this thesis is to investigate various aspects of the rôle of brane physics in string theory. A complete understanding of the framework for incorporating a quantum theory of branes into string theory is still lacking and provides a serious challenge for the future. However, much has been learnt about the important rôle which branes have to play in understanding many diverse areas of quantum gravity and in this thesis we shall take the point of view of asking 'What can branes teach us about string theory?'

A particular area of current interest in string theory in which branes are proving to play a crucial rôle is the field of gauge theory/gravity dualities. The original motivation for these remarkable dualities between non-gravitational field theories and higher dimensional theories of quantum gravity came from considering a duality in the description of branes in string theory. Much about these gauge theory/gravity dualities remains mysterious and in the following chapters we investigate ways in which brane physics can help us to probe the correspondence.

In this introductory chapter, we review some basic facts about branes and gauge theory/gravity correspondences. In chapter 2 we focus on a particular gauge theory and its dual description in supergravity and study the physics of a brane probe in this background. This allows us to learn certain facts about the moduli space of vacua of the supersymmetric gauge theory.

In chapter 3 we investigate further the link between the dynamics of brane probes in gravitational duals of gauge theories and the low energy physics on the moduli space of the gauge theories. We find that there are certain natural choices for coordinate systems which arise in this context and allow for easy comparison between the two sides. In particular, we are able to rederive certain non-perturbative results about the anomalous dimensions of gauge theory operators from the gravity dual.

In chapters 4 and 5 we study the so-called giant gravitons which are certain compact brane states which exist in string theory backgrounds arising in the study of gauge theory/gravity dualities. In chapter 4 we investigate the gravitational description of these states and identify certain geometries which represent coherent states of giant gravitons. In chapter 5 we consider the physics of giant graviton probes in a class of backgrounds which arise as lifts of a five-dimensional gauged supergravity theory and show that in these backgrounds, the ground state description of giant gravitons is degenerate with that of massless particles.

1.1 Branes

We discuss the complementary descriptions of branes as classical solutions of supergravity and as D-branes in perturbative string theory. Some of the statements we make will be true for more general string theories but when we need to be specific we shall mostly refer to the type IIB string theory in ten dimensions. A review of the necessary string theory background is contained in [5], whilst D-branes are well covered in [6–8].

1.1.1 Supergravity p-branes

The low energy limit of closed string theory is described by supergravity. The simplest way to understand this is to start from string perturbation theory on flat space. The massless modes of the string form a supergravity multiplet. There should be an effective Lagrangian for these modes to describe string theory at low energies. Since α' is the only dimensionful parameter in string theory, this effective Lagrangian will be an expansion in powers of α'. The leading order term is found to be a supergravity action. This was discovered in [9] by considering the conditions for conformal invariance of the worldsheet action for strings in curved backgrounds.

This result can be easily understood since the supergravity action is the unique lowest order action with the relevant spectrum and supersymmetries.

For this reason, supergravity provides a powerful tool for studying low energy string theory. In particular it can be used to learn information about regimes of strong coupling where string perturbation theory is not valid. This has been crucial in the development of many non-perturbative string dualities.[1].

Branes provide a particularly interesting class of supergravity solutions. As an example we shall review the 3-brane solution of type IIB supergravity. The idea behind the brane solutions is to find objects which are charged under the various $(p+2)$-form field strengths which appear in the spectrum of supergravity theories. For example, type IIB supergravity contains $(p+2)$-form field strengths $F^{(p+2)}$ for $p+2=1,3$ and 5. [2] Such a field strength can be expressed locally in terms of a potential $F^{(p+2)}=dC^{(p+1)}$. We can integrate a $(p+1)$-form over a $(p+1)$-dimensional surface and thus we expect to find $(p+1)$-dimensional brane solutions which are charged under these potentials.

The simplest such supergravity solution corresponding to a stack of N flat, coincident p-branes has $p+1$ dimensional Poincaré invariance along the brane and $SO(9-p)$ rotational symmetry in the transverse space. It also preserves half the maximal number of supersymmetries - a fact which is very useful in practice in finding the solutions. We present the brane solution and subsequent brane action in Einstein frame. For N 3-branes the solution takes the form [11]:

$$\begin{aligned} ds^2 &= H(y)^{-1/2}\sum_{\mu,\nu=0}^{3}\eta_{\mu\nu}dx^\mu dx^\nu + H(y)^{1/2}\sum_{i=1}^{6} dy^i dy^i \\ e^\phi &= g_s = \text{constant} \\ C^{(0)} &= \text{constant} \\ F^{(5)} &= G^{(5)} + *G^{(5)}, \end{aligned} \tag{1.1}$$

[1]For a review with references see [10]

[2]The 5-form field strength is self-dual. The other p-forms have magnetic duals which are also relevant to our discussion.

where

$$G^{(5)} = d(H^{-1}) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \ , \quad y^2 = \sum_i y^i y^i \quad \text{and} \quad H = 1 + \frac{L^4}{y^4}.$$

As we have mentioned, the supergravity action is only the leading order term in the α' expansion of the string theory low energy effective action. Thus a supergravity solution is to be trusted only in regions where the typical length scales of the solution are much larger than the string length $l_s \sim \sqrt{\alpha'}$. The 3-brane solution has uniformly small curvature so long as the length scale L is much larger than the string length. L is given in terms of the number of branes N and the string coupling by the relation:

$$L^4 = 4\pi N g_s \alpha'^2. \tag{1.2}$$

Since the brane solutions contain sources for $(p+1)$-form fields, they are not solutions of the pure supergravity theory. A diffeomorphism invariant action which provides such a source is given by:

$$S = T_p \int_{M_{(p+1)}} C^{(p+1)}, \tag{1.3}$$

where T_p is the charge of the brane. In fact the brane also has a mass and so the source action should contain couplings to the metric also. An action which provides a suitable source can be found which couples supersymmetrically to the supergravity [12]. The bosonic part of this action is given by a sum of Dirac-Born-Infeld and Wess-Zumino-Witten terms and for the 3-brane it takes the form [13]:

$$S = S_{DBI} + S_{WZW}, \tag{1.4}$$

where

$$S_{DBI} = -T_3 \int_{\mathcal{M}_4} d^4\sigma \sqrt{-\det[\mathcal{P}(G)_{ab} + e^{-\phi/2}\mathcal{F}_{ab}]} \tag{1.5}$$

and

$$S_{WZW} = T_3 \int_{\mathcal{M}_4} \left(\mathcal{P}(C^{(4)}) + \mathcal{P}(C^{(2)}) \wedge \mathcal{F} + \frac{1}{2}\mathcal{P}(C^{(0)})\, \mathcal{F} \wedge \mathcal{F} \right). \tag{1.6}$$

In these expressions, $\mathcal{F}_{ab} = \mathcal{P}(B)_{ab} + 2\pi\alpha' F_{ab}$, where F_{ab} is a $U(1)$ gauge field strength on the brane. T_3 is the tension of a D3-brane and $\mathcal{P}$ denotes the pullback of a ten-dimensional field to the brane worldvolume, $\mathcal{M}_4$, with coordinates $\sigma^0, \ldots, \sigma^3$. So, for example:

$$\mathcal{P}(G)_{ab} = G_{\mu\nu}\frac{\partial x^{\mu}}{\partial \sigma^{a}}\frac{\partial x^{\nu}}{\partial \sigma^{b}} . \quad (1.7)$$

The appearance of a $U(1)$ gauge field in this action can be understood from various points of view. In particular, the spacetime gauge invariance of B gets incorporated into the brane action by combining B with the gauge field to form the gauge invariant quantity $\mathcal{F}$. The simplest understanding of the appearance of a worldvolume gauge field will come in the next section when we discuss the description of branes in perturbative string theory.

1.1.2 D-Branes in String Theory

We have discussed the existence of various supergravity solutions corresponding to p-branes carrying charges under $(p+1)$-form potentials. The mass of these objects is proportional to g_s^{-1} and in the limit relevant to perturbative string theory $g_s \to 0$ the solutions reduce to flat space everywhere except at the brane where the curvature diverges. Clearly, supergravity does not give an accurate description of the physics here and another picture is needed.

Dp-branes are $(p+1)$-dimensional surfaces where open strings may end. This means that instead of considering a theory of only closed strings we admit also open strings which have endpoints on a given hypersurface. Open string loop diagrams can be reinterpreted as the emission or absorption of closed strings and so D-branes must be a source of the closed string fields. A classic calculation due to Polchinski [14] shows that Dp-branes are charged under the $(p+1)$-form potential with a charge equal to their mass.

Once we introduce open strings the spectrum of the low energy theory is modified. In addition to the massless closed string states which form the normal supergravity multiplet there is a gauge multiplet of massless open string states with end points on the D-brane. In particular, for a single D3-brane of type IIB string theory, the massless open string states form a $U(1)$ $\mathcal{N}=4$ vector multiplet.

We may also consider D-branes in curved backgrounds. Conformal invariance of the open string worldsheet once again provides constraints and the D-brane has to solve equations of motion coming from a DBI action [15]. Combining this with

Polchinski's result for the coupling to the RR fields and arguments involving e.g. T-duality [6, 7, 16] it is possible to reproduce, through string theory arguments, the full form of the brane action given above. Historically, the string theory derivation of the brane action preceded the supergravity result although we have chosen to present things in this manner to emphasise the fact that this action may be used even in situations in which perturbative string theory is not understood.

1.1.3 M-branes and 11 dimensional supergravity

This approach of using low energy supersymmetric actions to constrain the form of the dynamics in regions in which perturbative string theory is not valid, becomes even more useful when we move up to eleven dimensions. It has been conjectured [17] that the strong coupling limit of type IIA string theory is described by an eleven dimensional theory known as M-theory. The low energy dynamics of this theory are described by eleven dimensional supergravity. The relation to the type IIA theory is that eleven dimensional supergravity compactified on a circle of small radius is described at low energies by type IIA supergravity. The type IIA string coupling constant is proportional to the radius of the circle.

Eleven dimensional supergravity contains a four form field strength which couples electrically to an M2-brane and magnetically to an M5-brane. These objects can be described as supergravity solutions [18, 19] and can also be described through superbrane actions [20, 21] which couple to eleven dimensional supergravity. They have certain properties which are similar to the D3-branes of type IIB string theory although they lack the simple perturbative description which we have for D-branes which makes them harder to work with. We will have more to say about these M-theory branes during the course of the thesis.

1.2 Brane probes

Since D-branes are dynamical objects we may consider the dynamics of D-brane probes in supergravity backgrounds. The idea of such a probe experiment is to neglect the backreaction of the probe brane on the geometry and just study the brane dynamics in a fixed background. This may be seen as a first approximation to studying the full dynamics of the supergravity coupled to brane worldvolume actions. In fact, in many cases we will study situations in which a brane is probing a background sourced by branes of the same type in which case it may be possible to fully solve the coupled brane/supergravity equations by placing the probe branes at the source of the geometry. In other situations, there may be some supersymmetry preserved by the supergravity/probe configuration and we might be able to argue that our first order analysis should be a good approximation to an exact solution.

Such probe 'experiments' will be a running theme of this thesis. We start with a simple illustrative example of the technique. Consider a D3-brane probing the type IIB supergravity background sourced by a stack of N flat, parallel D3-branes in flat space. This background is given in equation (1.1). We shall choose our probe to lie parallel to the stack of branes and moving slowly in the transverse directions. Also for simplicity we will set the $U(1)$ gauge field on the brane worldvolume to zero.

We can fix a convenient choice of worldvolume coordinates for the brane as follows. We set $\sigma^0 = x^0 \equiv t$, $\sigma^i = x^i$, $i = 1, 2, 3$ and $y^m = y^m(t)$, where the σ^μ are worldvolume coordinates and the x^μ, y^m are the spacetime coordinates of eqn. (1.1). In this gauge (commonly called static or physical gauge) it is easy to write down the resulting probe action:

$$S = -T_3 \int d^4x \, \sqrt{-\det(\mathcal{P}(G))} + T_3 \int \mathcal{P}(C^{(4)}), \tag{1.8}$$

where,

$$\mathcal{P}(G) = \begin{pmatrix} G_{00} + G_{mn}\dot{y}^m\dot{y}^n & 0 & 0 & 0 \\ 0 & G_{11} & 0 & 0 \\ 0 & 0 & G_{22} & 0 \\ 0 & 0 & 0 & G_{33} \end{pmatrix} \tag{1.9}$$

and

$$\mathcal{P}(C^{(4)}) = \left(C^{(4)}_{0123} + \dot{y}^m C^{(4)}_{m123}\right) d^4x. \tag{1.10}$$

Substituting the values for the metric and R–R four-form given in equation (1.1) we find:

$$S = -T_3 \int d^4x \, \left(H^{-1}\sqrt{1 - H\sum_m (\dot{y}^m)^2} - H^{-1}\right). \tag{1.11}$$

For a slowly moving probe we can expand out the square root keeping terms with at most two time derivatives. The final result for the effective action of the brane is:

$$S = T_3 \int d^4x \, \frac{1}{2} \sum_m (\dot{y}^m)^2. \tag{1.12}$$

We can see that the potential terms in this action have vanished and furthermore all dependence on the harmonic function H has dropped out of the kinetic terms. Thus the brane responds to the background of the other branes precisely as it would to flat space, in this low energy limit. Further interpretation of this result will be provided after the next section in which we discuss the gauge theory which lives on a D-brane worldvolume.

1.3 D-branes and gauge theory

If we had not set the gauge field on the brane to zero in the calculation of the previous section, but rather maintained terms in the action with at most two derivatives acting on A also, we would have found a low energy Lagrangian corresponding to a $U(1)$ gauge theory. The scalars corresponding to the transverse positions of the brane $\phi^m \sim y^m/\alpha'$ and the fermions, which we have suppressed, combine with the gauge field A to make up a vector multiplet of $\mathcal{N} = 4$ supersymmetry in four-dimensions. The low energy Lagrangian which we would have found if we had kept track of all the terms is that of the $\mathcal{N} = 4$ supersymmetric $U(1)$ Yang-Mills theory in four-dimensions.

If we had a system comprising N parallel branes, we might expect to find a $U(1)^N$ field theory. In fact a surprising enhancement of the gauge symmetry takes place. If we place the N branes close together it is possible for light open strings to start on one brane and end on another. When these extra light states are properly accounted for the massless states form a $U(N)$ adjoint vector multiplet. The low energy theory on N coincident D3-branes is given by the $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory in four dimensions [22].

Myers wrote down various terms in a non-abelian generalisation of the brane action [23]. As it should, this reproduces the U(N) SYM action at low energies in flat space. An interesting point about this action is that it includes terms which describe the coupling of branes to higher dimensional R–R forms via matrix commutator couplings. This suggests that collections of lower dimensional branes can behave like a single higher dimensional brane in backgrounds with non-trivial R–R flux. The description of the lower dimensional brane worldvolumes is in terms of non-commutative geometry in which the positions of the branes at minima of the potential energy are given by genuinely matrix valued coordinates.

Giant gravitons are an analogous phenomenon in which Kaluza-Klein modes expand into spherical branes. This occurs, for example, in backgrounds of the form $AdS \times S$ with RR flux through the sphere. This will be described in chapter 4 and will be followed by investigations into the supergravity description of these objects and their appearance in more general backgrounds.

1.4 The AdS/CFT Correspondence

In this section we review some arguments of Maldacena [24] which point to a duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory. We start by examining more closely the physics of a stack of N coincident D3-branes of type IIB string theory. As we have explained, the perturbative dynamics of such a system in flat space is described by open strings ending on the branes and closed strings propagating in the bulk. At energies below the string scale, we expect to be able to describe the physics in terms of an effective action for the massless modes :

$$S_{eff} = S_{bulk} + S_{brane}, \tag{1.13}$$

where S_{brane} contains the dynamics of the massless modes on the brane and interactions between the brane and the bulk. As we have discussed, certain terms of the brane action are captured by a (non-abelian) DBI + WZW action. The bulk action, meanwhile, contains a supergravity action plus higher derivative corrections.

We can study the low energy limit of this physics in two complementary ways. The first way is to linearise the action about flat space and flat branes and then take a low energy limit. After taking this low energy limit the flat space/ flat brane ansatz is a solution to the equations of motion and we can consistently study the low energy physics in this background. The second way is to find a solution of the coupled equations, which corresponds to a set of flat branes sitting in their own background fields, and then take a low energy limit. We will see that these two approaches lead to very different pictures of the same physics.

Let us first consider the expansion of the effective action about flat space. The leading order terms in the bulk supergravity action are found by applying a linearised ansatz for the supergravity fields about flat space. The resulting action describes a free theory of the supergravity modes. Interaction terms in the supergravity action and higher curvature terms in the full bulk action are suppressed in the low energy limit.

The leading order terms in the brane action are found by substituting the linearised ansatz for the supergravity fields into the non-abelian DBI-WZW action and then linearising this around an ansatz for flat coincident branes. For our purposes it will not be important that we do not know the full non-abelian action, since we shall only keep the known leading order terms. The terms which survive the low energy limit are those in the action of $\mathcal{N} = 4$ $U(N)$ SYM theory. All interaction terms between the brane and the bulk are suppressed in this limit. The fact that these interaction terms vanish is crucial in ensuring the consistency of our ansatz.

Thus, the low energy physics of a set of N coincident branes in flat space is described by a system consisting of two decoupled sectors: free supergravity in ten-dimensions and $\mathcal{N} = 4$ $U(N)$ super Yang-Mills in four dimensions. In fact, at low energies the $U(N)$ theory is essentially a product of an $SU(N)$ theory with a decoupled $U(1)$ sector which represents the centre of mass dynamics of the branes.

We now consider the problem from a different viewpoint. We know a solution to the coupled equations arising from the action (1.13). This was discussed in the previous sections and involves N flat coincident branes in the supergravity 3-brane background. We wish to investigate the low energy physics of this system. The brane solution contains a horizon at $r = 0$ behind which the branes are positioned. We should, however, be able to describe the whole physics in terms of the region outside the horizon. Thus we are studying the low energy physics of string theory in the brane solution.

At first sight it might seem that this physics can be described simply by free supergravity describing the long wavelength modes. This certainly describes one sector of the low energy theory since the cross section for absorption by the brane of supergravity modes goes to zero at low energies [25, 26]. However, the presence of a horizon means that there are other states in the low energy theory. This is because we measure energies at infinity and states near the horizon undergo a significant red-shift when measured at infinity. The correct way to isolate the degrees of freedom which survive the low energy limit was found in [24] and involves taking the near-horizon limit of the D3-brane geometry. In this limit we 'zoom-in' on the region near $y = 0$ where the geometry becomes that of $AdS_5 \times S^5$. More precisely, to take a low energy limit, we can send α' to zero whilst keeping the mass y/α' of a stretched string fixed in this limit. We can do this by dropping the constant term in the definition of $H(y)$ after equation (1.1). In this region, all string theory excitations survive the low energy limit and we find another sector of the low energy theory corresponding to string theory on $AdS_5 \times S^5$.

The Maldacena conjecture proposes a correspondence between the two descriptions of the brane physics which we have discussed. It is proposed that string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$.

We now discuss the range of validity of the two sides of the correspondence. The field theory has a good perturbative description in the limit of small 't Hooft coupling $\lambda = g_{YM}^2 N$ [27]. The factor of N in this coupling occurs because of combinatorial factors in loop diagrams in which traces are taken over the gauge group indices. On the other hand, the supergravity description is valid when the string coupling $g_s \ll 1$, in order to suppress loop diagrams, and when the radius of curvature of the geometry L is much larger than the string length $L^4/l_s^4 \sim g_s N \sim \lambda \gg 1$. We see that the two weakly coupled descriptions are valid in complementary regimes.

In its weakest form, the conjectured duality is expected to hold in the limit of large 't Hooft coupling, λ. The strongest form of the duality postulates that the gauge theory is dual to the full string theory for all values of $g_s \sim g_{YM}^2$ and N.

Finally we comment that there are versions of the AdS/CFT correspondence which hold for the theories on N coincident M2- or M5-branes [24]. The near horizon geometry of a stack of N M2-branes is $AdS_4 \times S^7$ and M-theory on this space is expected to be dual to the low energy limit of the theory living on the stack of branes. This is an $\mathcal{N} = 8$ supersymmetric theory which is the IR fixed point of the reduction to three dimensions of the $\mathcal{N} = 4$ $SU(N)$ gauge theory in four dimensions.

For M5-branes, the near horizon geometry is $AdS_7 \times S^4$ whilst the low energy theory on the worldvolume is a six dimensional $\mathcal{N} = (2,0)$ superconformal field theory. Some details of this theory are given in [28].

1.5 Aspects of AdS/CFT

In this section we discuss various checks on the AdS/CFT correspondence and briefly outline the procedure for matching gauge theory quantities with results in string theory.

A basic requirement is that the global symmetries of the two theories should match. The $\mathcal{N} = 4$ gauge theory is in fact conformally invariant and so its global symmetry group contains the corresponding superconformal group $SU(2,2|4)$. This contains the conformal group $SO(2,4)$ and an $SU(4) \sim SO(6)$ R-symmetry group as bosonic subgroups. There is also a doubling in the number of supersymmetries because of the failure of the ordinary supersymmetries to commute with generators of the conformal group.

The $AdS_5 \times S^5$ solution has the same $SU(2,2|4)$ superisometry group. In particular, the $SO(6)$ subgroup corresponds to the isometry group of the S^5 and the $SO(2,4)$ is the group of isometries of AdS_5. The solution is maximally supersymmetric and thus has 32 preserved supercharges in accord with the field theory. We will see other examples of gauge theory/ gravity dualities in later chapters and a first check will always be to match symmetries.

A more detailed test of the correspondence involves matching the spectra of the two theories. In order to do this we need to be more specific about the way in which quantities from gauge theory match with objects in string theory.

A first step [29, 30] is to identify the boundary of the AdS_5 space with the 3+1 dimensional space on which the field theory is defined. This is a reasonable thing to do since the boundary of AdS_5 (in Poincaré coordinates) is conformally equivalent to four dimensional Minkowski space. Also, the $SO(2,4)$ isometry group of the AdS_5 space acts as the conformal group on the boundary.

Next we need some method for comparing correlation functions of gauge invariant operators, which are the objects of interest in the field theory, to quantities defined in string theory. The relevant object to compute in the string theory turns out to be the string partition function (or, in the limit of classical supergravity, the exponential of the supergravity action on a solution of the equations of motion) subject to certain asymptotic boundary conditions for the fields.

We can treat supergravity on $AdS_5 \times S^5$ as a five dimensional theory by expanding fields in terms of spherical harmonics. This will lead to an infinite number of coupled fields on AdS_5. Near the boundary of AdS_5 these fields will interact weakly with each other and we can perform an approximate analysis by considering fields which only couple to the background metric. Consider for example a scalar field of mass m, minimally coupled to the AdS_5 metric. The action for such a field is:

$$S = \frac{1}{2} \int d^5x \sqrt{g}(|d\phi|^2 + m^2\phi^2) \tag{1.14}$$

For large y,[3] the two linearly independent solutions of the equations of motion derived from this action behave as y^γ, where γ is a root of the quadratic [29]

$$\gamma(\gamma + 4) = m^2. \tag{1.15}$$

We can thus specify boundary conditions for ϕ of the form:

$$\phi \sim y^\gamma \phi_0, \tag{1.16}$$

where ϕ_0 is an arbitrary function on the boundary. Under scaling transformations $y \to \alpha y$, the boundary metric scales by a factor of α^2 whilst ϕ_0 scales by α^γ. Thus, ϕ_0 is some object of mass dimension $-\gamma$ in the boundary theory. The idea is to interpret a boundary value with dimension given by one root of (1.15) as a source

[3] y is the coordinate introduced in eqn. (1.1)

for an operator of dimension $4+\gamma$ whilst the other root corresponds to a vev for such an operator [31, 32].

A natural proposal for matching quantities between the boundary field theory and the bulk string theory is then[4] [29, 30]:

$$\left\langle e^{\int dx O(x)\phi_0(x)} \right\rangle_{\text{CFT}} = Z_{\text{string}}(\phi_0), \tag{1.17}$$

where the left hand side is the CFT partition function with an external source $\phi_0(x)$ coupled to a gauge invariant scalar operator and the right hand side is the string theory partition function with boundary values for a scalar field controlled, as in the discussion above, by ϕ_0. Similar relations hold for operators of other spins.

Note that to make a complete specification of the correspondence it is necessary to determine a precise dictionary between operators of the boundary theory and fields in the bulk. Also note that this relation is a formal identity for computing correlation functions in the $\mathcal{N}=4$ theory and comparing with string theory in the $AdS_5 \times S^5$ background. However, it immediately suggests an extension to non-perturbative regimes where the sources are set to finite values. This leads to a whole family of gauge theory/ gravity dualities, some of which will be considered in the following chapters.

Finally, note that in the limit in which classical supergravity is a valid approximation to the physics, the string partition function becomes:

$$Z_{\text{string}}(\phi_0) = e^{iS_{\text{sugra}}(\phi(x,y))}, \tag{1.18}$$

where $\phi(x,y)$ is a solution of the supergravity equations of motion whose asymptotic behaviour is controlled by the boundary field $\phi_0(x)$.

With this statement of the conjecture it is possible to make various more detailed checks of the correspondence. First of all it is possible to show that the relation (1.17) produces boundary correlation functions of the general form required for a conformal field theory. This beautiful result confirms that it is reasonable to conjecture a duality between a bulk gravity theory on an AdS space and a boundary conformal

[4]The following discussion is schematic. A more careful approach involves regulating both quantities by the addition of suitable counterterms - see refs. [29, 33].

field theory. Furthermore it ensures that the forms of two and three point functions agree between the two theories since these are determined up to normalisation by conformal symmetry.

In the limit in which classical supergravity is applicable, the spectrum of the bulk theory on AdS_5 is known [34, 35]. In this limit, the string modes become infinitely massive and so they consistently decouple from the supergravity modes.

In order to compare with the boundary theory we need to know the gauge theory spectrum in the limit of large t' Hooft coupling, which seems like a tall order. However, the states in the supergravity Kaluza-Klein spectrum are all in short representations of the superconformal algebra and have masses which are determined by their SO(6) representations. Consequently, we can reliably look for such states in the field theory at strong coupling.

This programme has been carried out in some detail [29, 36] and a complete matching is found between Kaluza-Klein states of supergravity and operators of protected dimension in field theory. This provides a highly non-trivial test of the Maldacena correspondence. As an example we list one family of protected operators which will concern us later on.

The family of field theory operators which we consider are gauge invariant scalars of the form:

$$O_k = \mathrm{tr}\left(\mathrm{X}^{(\mathrm{i}_1}\mathrm{X}^{\mathrm{i}_2}\dots\mathrm{X}^{\mathrm{i}_k)}\right) - \mathrm{traces}, \tag{1.19}$$

where the trace in the first term is taken over the gauge indices, to ensure $SU(N)$ gauge invariance, whilst the second term subtracts $SO(6)$ traces. It can be shown [29, 36] that these operators are superconformal primary operators [5] since they do not arise by acting with the supercharges Q on other operators. Furthermore, these operators are chiral since they are annihilated by some of the Q's. We conclude that the superconformal representations which are built by acting on these operators with the Q's are short.

The scaling dimensions of these operators are related to their $SO(6)$ representation [38]. In particular, the operator O_k falls into the k'th symmetric traceless

[5] See the review [37] for more details

representation of $SO(6)$ and has scaling dimension k.

According to the conjectured duality there should therefore be a family of scalar fields, in the Kaluza-Klein spectrum of type IIB supergravity on $AdS_5 \times S^5$, in the k'th symmetric traceless representation of $SO(6)$ and with mass $m^2 = k(k-4)$. This is in agreement with the supergravity analysis [34, 35].

Having seen how to relate gauge theory operators to bulk KK fields, it is possible to make a further check on the correspondence. This involves the explicit computation of various n-point functions of gauge invariant operators from a field theory and a supergravity perspective. For four-point functions and higher, the functional form of n-point functions is no longer fully determined by conformal invariance. Furthermore, the relative normalisation of two- and three-point functions can be tested. Various comparisons have been made and in some cases, bulk calculations have led to new predictions for gauge theory correlators which have subsequently proved to be correct. These predictions are generally non-renormalisation theorems which were conjectured on the basis that the strong coupling result from gravity agrees with a free-field computation in gauge theory. This provides further strong evidence in support of the AdS/CFT correspondence.

1.5.1 Branes in the bulk

It is interesting at this point to review the brane probe computation of section 1.2 in the context of the AdS/CFT correspondence. We found that a D3-brane probing the background sourced by a collection of N parallel D3-branes feels no potential and has a scalar kinetic term given by a flat metric. Since we expect that in the full version of the Maldacena correspondence, the theory dual to the boundary gauge theory will be the full type IIB string theory on $AdS_5 \times S_5$, it is interesting to consider the rôle of states corresponding to branes in the bulk. Furthermore, given the motivation for the correspondence in terms of the physics of a stack of branes, it is clear that a single brane in the bulk which is parallel to the background branes should correspond to a picture in which there is a stack of N coincident branes and one other brane at some small distance. On the gauge theory side, this corresponds to breaking the low energy gauge group from $SU(N+1)$ to $SU(N) \times U(1)$ by turning on scalar vevs. With this interpretation in mind we see that the vanishing of the scalar potential in the D3-brane probe action indicates that there is a six-dimensional moduli space of vacua which involve moving one brane away from the stack. This is mirrored in the gauge theory. Furthermore, the flat metric in front of the scalar kinetic term corresponds to a flat metric on moduli space in agreement with the requirements of $\mathcal{N} = 4$ supersymmetry in the gauge theory.

In the next chapter we discuss extensions of the Maldacena conjecture to other more complicated gauge theory/gravity dualities and develop techniques involving brane probes which allow us to further investigate the correspondence.

Chapter 2

Probing An $\mathcal{N} = 1$ Renormalisation Group Flow

In the previous chapter we have introduced the prototype example of a gauge theory/gravity duality, between $\mathcal{N} = 4$ $SU(N)$ SYM in the conformal phase and type IIB string theory on $AdS_5 \times S^5$. It is of great interest to extend this correspondence to theories with less supersymmetry and without conformal invariance, since such theories display a richer set of dynamics and are closer to realistic gauge theories. Furthermore, such theories provide powerful checks on the validity of the AdS/CFT correspondence and allow us to gain new insights into the duality. In this chapter we shall consider deformations of the original correspondence which lead to renormalisation group flows away from the $\mathcal{N} = 4$ theory in the UV to new theories in the IR.

First we review the subject of holographic renormalisation group flows. We discuss a particular $\mathcal{N} = 1$ supersymmetric field theory which may be obtained as a relevant deformation of the $\mathcal{N} = 4$ theory. Next we review a supergravity solution which has been conjectured to be the dual of this renormalisation group flow. We give details of the calculations of [1, 2] in which we consider a brane probe in this supergravity background and reproduce various results from the field theory. We also make strong coupling predictions about the metric on moduli space.

2.1 Holographic renormalisation group flows

A simple way to introduce scale dependence into a conformal field theory is to perturb by a relevant operator in the UV. The effect of such a perturbation grows with decreasing energy scale and induces a renormalisation group flow. The IR physics of the theory may then be described by a trivial theory, a free theory or a new interacting conformal field theory. (Alternatively the perturbation may leave a theory which has no stable vacuum in the IR.) It is natural to ask whether we can model such a renormalisation group flow in the dual string theory description.

The starting point for our discussion of holographic renormalisation group flows is the equation relating Green's functions of the boundary field theory to quantities in the dual gravity theory, which we discussed in the previous chapter. We can write this relation formally as:

$$\left\langle e^{i(S_{CFT}+\int dxO(x)\phi_0(x))}\right\rangle = e^{iS_{sugra}(\phi(r,x))}, \tag{2.1}$$

where the equality is between series expansions about $\phi_0(x) = 0$ and we are working in the regime in which classical supergravity is a valid approximation to the bulk theory. (Here we have chosen to use a new radial coordinate, r, for AdS space which is related to the coordinate, y, of the last chapter by $y = e^{r/L}$). If we take this formula seriously we might expect that the two sides should agree at non-zero $\phi_0(x)$. If we expand around $\phi_0(x) = \lambda$ we are computing Green's functions in a new field theory in which we have deformed the Lagrangian by a term

$$\int dxO(x)\lambda. \tag{2.2}$$

Thus we should expect that a new field theory which is related to the original CFT by a relevant or marginal perturbation will be described in gravity by a background with perturbed boundary asymptotics. In particular if $O(x)$ is an operator of conformal dimension Δ in the CFT, then to first order in the perturbation we would expect to have to study a geometry with

$$\phi(r,x) \to e^{(\Delta-d)r/L}\lambda, \tag{2.3}$$

as $r \to \infty$. Here, d is the dimension of the boundary, which was taken to be four in the corresponding expressions of the previous chapter.

2.2 The LS flow

In this section we identify a particular relevant deformation of the $\mathcal{N} = 4$ SYM theory which preserves $\mathcal{N} = 1$ supersymmetry and flows to a conformal field theory in the IR. The $\mathcal{N} = 1$ supersymmetry gives us some control over the non-perturbative physics of the field theory. The fact that the theory is conformal in the UV and IR means that the dual geometry flows between two smooth AdS regions and so the supergravity description should be uniformly valid (at large N, λ.)

We start by writing the $\mathcal{N} = 4$ SYM Lagrangian in terms of $\mathcal{N} = 1$ superfields. The field content of the $\mathcal{N} = 4$ theory is a single $\mathcal{N} = 4$ gauge multiplet which consists of six real scalars, X_i, four fermions, λ_a and a gauge field, A_μ, all in the adjoint representation of $SU(N)$. We can combine the gauge field with one of the fermions, λ_4, into an $\mathcal{N} = 1$ vector superfield. Meanwhile the other three fermions can be paired with complex scalars, $\phi_j = X_{2j-1} + iX_{2j}$, to form three $\mathcal{N} = 1$ chiral superfields, Φ_j.

The superpotential for the chiral superfields is:

$$W = h\mathrm{Tr}(\Phi_3[\Phi_1, \Phi_2]) \ , \tag{2.4}$$

where $\mathcal{N} = 4$ supersymmetry requires $h = g_{YM}$. (We label h by a different letter from the gauge coupling g_{YM} since these are independent couplings once we break the $\mathcal{N} = 4$ supersymmetry and undergo renormalisation group flow.)

In this form, the theory has a manifest $\mathcal{N} = 1$ supersymmetry with a $U(1)$ R-symmetry and an $SU(3)$ flavour symmetry which rotates the three Φ_i's. In order to introduce some scale dependence we perturb the theory by giving a mass to Φ_3,

$$\delta W = \frac{1}{2} m\mathrm{Tr}(\Phi_3^2) \ . \tag{2.5}$$

To see that this is a relevant deformation we note that in the pure $\mathcal{N} = 4$ theory, the chiral superfields Φ_i have dimension 1 and thus the chiral operator Φ_3^2

has dimension 2. This leads to a perturbation of dimension 3 in the Lagrangian after performing the integral $\int d^2\theta$. Thus we expect that the perturbation will become less important at higher energies where the theory will return to the $\mathcal{N}=4$ theory. (The reason that we spell this out in detail is that the regime where supergravity is valid is the strong coupling regime of large λ and so we cannot merely count engineering dimensions of operators to see if they are relevant or not. In particular an operator like $\sum_i X_i X_i$, which is not a chiral operator, is relevant at small λ, but believed to have an anomalous dimension which grows like $\lambda^{1/4}$ as $\lambda \to \infty$ and is thus irrelevant at strong coupling [30].)

In order to discuss the low energy physics of this flow we need to make use of various non-renormalisation theorems for $\mathcal{N}=1$ supersymmetric theories. In particular we shall use the results for the exact beta functions for the physical couplings in the effective Lagrangian:

$$\begin{aligned} \beta_g &\sim \frac{3}{2}T(adj) - \frac{1}{2}\sum_i T(R_i)(1-\gamma_i) \\ \beta_\alpha &\sim 3 - d_O - \frac{1}{2}\sum_i n_i\gamma_i. \end{aligned} \tag{2.6}$$

The first of these equations gives the NSVZ exact beta function [39] for the gauge coupling in a supersymmetric gauge theory and the second equation gives the exact beta function for the coupling α of an operator in the superpotential:

$$\alpha O = \alpha \prod_i \Phi_i^{n_i}. \tag{2.7}$$

The terms γ_i are the anomalous dimensions of the operators Φ_i coming from the kinetic terms in the action, which are not protected by a non-renormalisation theorem. Finally, R_i is the representation of the gauge group under which Φ_i transforms and T denotes the index of the representation.

For the example at hand we find beta functions for g and the two couplings h and m :

$$\begin{aligned} \beta_g &\sim N(\gamma_1+\gamma_2+\gamma_3) \\ \beta_h &\sim \gamma_1+\gamma_2+\gamma_3 \\ \beta_m &\sim 1-2\gamma_3. \end{aligned} \tag{2.8}$$

The important point to notice here is that the beta function for the gauge coupling β_g is proportional to β_h. In order for a theory to be conformally invariant, it is necessary that the beta functions vanish. In this case we would need:

$$\gamma_1 = \gamma_2 = -\frac{1}{4}, \qquad \gamma_3 = \frac{1}{2}. \tag{2.9}$$

Note that $\gamma_1 = \gamma_2 \equiv \gamma$ because of an unbroken $SU(2)$ symmetry rotating Φ_1 into Φ_2. The linear dependence of the beta functions suggests that there is a one-dimensional space of fixed point theories in the IR. In particular this tells us that the IR theory should have an exactly marginal operator and is thus an interacting conformal field theory.

We finish this section by making a few further comments about the field theory. The theory preserves an $SU(2)$ subgroup of the $SU(3)$ flavour symmetry which was manifest in the $\mathcal{N} = 1$ superfield formulation of the pure $\mathcal{N} = 4$ theory. This $SU(2)$ rotates the two remaining massless chiral superfields, Φ_1 and Φ_2. There is also a U(1) R-symmetry under which the fields Φ_1, Φ_2 and Φ_3 have charges $\frac{1}{2}$, $\frac{1}{2}$ and 1 respectively so that each term in the superpotential has total R-charge 2.

To determine the moduli space of supersymmetric vacua of the theory we need to find solutions to the vacuum equations for the complex adjoint scalars and then divide out by the gauge group. The vacuum equations which follow from the superpotential (2.4), (2.5) are:

$$\begin{aligned} h\,[\phi_1, \phi_2] + m\phi_3 &= 0 \\ [\phi_2, \phi_3] &= 0 \\ [\phi_3, \phi_1] &= 0. \end{aligned} \tag{2.10}$$

Solving these requires:

$$\phi_3 = 0, \qquad [\phi_1, \phi_2] = 0. \tag{2.11}$$

In this case the moduli space is parametrized by the space of complex diagonal matrices ϕ_1 and ϕ_2. Later when we study the physics of a D-brane probe in the dual geometry, we expect to find four flat directions in the potential. This corresponds to giving a vev to a single eigenvalue of each of the four real scalars X_1, X_2, X_3, X_4. Furthermore, we expect to find that the metric on moduli space should share the

$SU(2) \times U(1)$ symmetry of the theory. We can say rather more than this by realising the restrictions placed on the low energy Lagrangian for motions on moduli space, by $\mathcal{N} = 1$ supersymmetry. We will return to this later.

2.3 The dual geometry

In this section we discuss the supergravity description of the LS flow introduced in the previous section. In general finding exact solutions to the non-linear type IIB supergravity equations, with given boundary conditions, is not an easy task. Fortunately, the particular perturbation which we are studying only involves bilinear operators and such perturbations may be studied in the context of a much simpler five dimensional supergravity.

2.3.1 $\mathcal{N} = 8$ $SO(6)$ gauged supergravity

The $\mathcal{N} = 8$ $SO(6)$ gauged supergravity theory in five-dimensions [40, 41] is believed to be a consistent truncation of type IIB supergravity compactified on $AdS_5 \times S^5$. The gauged supergravity represents the dynamics of a finite subset of the Kaluza-Klein modes which is closed in the following sense. If we set the remaining Kaluza-Klein fields to zero and study the dynamics of the fields of the consistent truncation, these do not act as sources for the fields we have set to zero. Thus any solution of the gauged supergravity theory corresponds to a solution of the full type IIB theory.

The reasons why one should be able to make such a truncation (in the absence of any symmetry arguments) remain obscure. However there is strong evidence that the truncation is consistent and in many cases it is known how to 'lift' solutions of the five dimensional theory to solutions of ten dimensional type IIB supergravity - i.e. it is known how to relate the fields of the five dimensional solution to ten dimensional fields via a highly non-trivial lift ansatz.

Another way to look at this is as follows. Consistent truncation is equivalent to a particular kind of ansatz for solutions of type IIB supergravity. The most trivial kind of ansatz we can have is a single solution to the ten-dimensional theory. A less trivial ansatz will specify a family of solutions in terms of certain functions which themselves obey some field equations. We can start from a single solution - $AdS_5 \times S^5$ - and ask if it is possible to embed this in a family of ansätze. The $AdS_5 \times S^5$ solution itself has an $SO(6)$ isometry on the compact space and has 32 supercharges in ten-dimensions. It turns out that we can gauge these symmetries in the five non-compact directions in the following sense. We can construct an ansatz which involves fields in the non-compact directions such that local SO(6) rotations of the S^5 may be compensated by SO(6) gauge transformations of the fields and likewise for local supersymmetry transformations. The type IIB field equations automatically produce equations for the five-dimensional fields with these symmetries which are thus determined to be the equations of the $\mathcal{N} = 8$ gauged supergravity.

2.3.2 The bulk field/ boundary operator dictionary

The fields of the five-dimensional $\mathcal{N} = 8$ $SO(6)$ gauged supergravity are in one-to-one correspondence with operators in the energy momentum tensor supermultiplet of the $\mathcal{N} = 4$ SYM theory. Here we shall focus on scalars in the two theories.

The five-dimensional gauged supergravity has 42 scalar fields. 20 of the scalars are in the symmetric traceless component of a product of two 6's of $SO(6)$. They have the correct masses to be dual to field theory scalar bilinears of the form

$$O_2 = \mathrm{Tr}\left(X^{(i}X^{j)}\right) - \text{traces}. \tag{2.12}$$

A further 20 supergravity scalars are dual to fermion bilinears

$$[Q^2, O_2] = \mathrm{Tr}\left(\lambda_a\lambda_b\right) + \mathrm{Tr}\left([\phi^I, \phi^J]\phi^K\right). \tag{2.13}$$

and their complex conjugates. The final two scalars represent the complexified gauge coupling of the boundary field theory.

The perturbation which we discussed in the last section involved adding a mass term for one of the chiral superfields Φ_3. In terms of component fields this corresponds to adding masses for the fermion λ_3 and the scalars X_5, X_6. When we take into account operator mixing which occurs when we perturb the $\mathcal{N} = 4$ theory [1] we

[1] This is discussed in some detail in the review of ref. [37]

find that the fields we need to excite are dual to the operators:

$$\mathrm{Tr}(\lambda_3\lambda_3) + \mathrm{Tr}(\phi_1[\phi_2, \phi_3]) + \mathrm{h.c.} \tag{2.14}$$

and

$$\sum_{i=1}^{4} \mathrm{Tr}(X_i X_i) - 2\sum_{i=5}^{6} \mathrm{Tr}(X_i X_i) \tag{2.15}$$

2.3.3 The five dimensional solution

We have seen how the problem of finding a solution of type IIB supergravity with certain boundary conditions may be reduced to that of solving the five-dimensional $\mathcal{N} = 8$ $SO(6)$ gauged supergravity. However, this remains an extremely complicated theory so that further simplifications are necessary. The strategy is to use the known symmetries of the field theory to make further simplifying ansätze.

The most obvious simplification comes from the requirement of Poincaré symmetry in the boundary theory. This means that the five dimensional solution should also have four dimensional Poincaré invariance and thus coordinates can be chosen such that the metric has the form:

$$ds^2_{1,4} = e^{2A(r)}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2 \ , \tag{2.16}$$

whilst the scalars depend only on r. The other fields of the theory are taken to vanish.

As we have discussed the field theory has an $SU(2) \times U(1)$ invariance all along the flow and this should be reflected in the dual geometry. It is possible to find such solutions by truncating the gauged supergravity to the space of scalars which are singlets of this group. (In fact the route followed in [42] was to construct the truncation to singlets of the $SU(2)$ and then find $U(1)$ invariant solutions of this theory. Details can be found in that paper.)

The final symmetry requirement is that the flow should preserve $\mathcal{N} = 1$ supersymmetry. Analysis of the Killing spinor equations or BPS type conditions can be used to show that the scalars obey first order gradient flow equations coming from a superpotential. This superpotential is expressed in terms of two supergravity scalars

ρ and χ as:

$$W = \frac{1}{4}\rho^4(\cosh 2\chi - 3) - \frac{1}{2\rho^2}(\cosh 2\chi + 1)\ , \tag{2.17}$$

The final result is that the five dimensional geometry is described by a metric of the form (2.16) along with the two scalars ρ and χ. The equations of motion for ρ, χ and the scale factor A are :

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6L}\rho^2\frac{\partial W}{\partial \rho} = \frac{1}{6L}\left(\frac{\rho^6(\cosh(2\chi)-3)+2\cosh^2\chi}{\rho}\right)\\
\frac{d\chi}{dr} &= \frac{1}{L}\frac{\partial W}{\partial \chi} = \frac{1}{2L}\left(\frac{(\rho^6-2)\sinh(2\chi)}{\rho^2}\right)\\
\frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L}\left(\rho^4(\cosh 2\chi - 3) - \frac{2}{\rho^2}(\cosh 2\chi + 1)\right)\ .
\end{aligned} \tag{2.18}$$

There are two solutions of these equations with constant scalars, corresponding to the UV and IR fixed points of the flow. These fixed point solutions are given by:

$$UV: \qquad \rho = 1, \qquad \chi = 0, \qquad A = \frac{r}{L} \tag{2.19}$$

$$IR: \qquad \rho = 2^{\frac{1}{6}}, \qquad \chi = \frac{1}{2}\log 3, \qquad A = \frac{2^{5/3}}{3}\frac{r}{L}. \tag{2.20}$$

An analytic solution to the flow equations with non-constant scalars is not known. However, it may be checked numerically that there does exist a flow solution between the UV and IR fixed points. Furthermore it is possible to analyse the asymptotic solutions in the scaling regions close to the UV and IR fixed points and compare to field theory expectations [42]. The UV asymptotics ($r \to \infty$) are:

$$\chi(r) \to a_0 e^{-r/L} + \ldots\ ; \qquad \alpha(r) \equiv \log\rho(r) \to \frac{2}{3}a_0^2\frac{r}{L}e^{-2r/L} + \frac{a_1}{\sqrt{6}}e^{-2r/L} + \ldots \tag{2.21}$$

Following the discussion of section 2.1 we see that χ is dual to an operator of dimension 3 and α is dual to an operator of dimension 2 in the boundary theory. The constant a_0 is related to the perturbing mass m in the field theory Lagrangian. Meanwhile, a_1 is related to a choice of vacuum state in the field theory. In order to flow to the conformal fixed point in the IR we need to choose

$$\hat{a} = \frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}}\log a_0 = -1.4694\ldots \tag{2.22}$$

where the constant $\hat{a}$ is invariant under additive shifts in r and this value has been determined numerically. Other values of $\hat{a}$ correspond to unphysical flows or flows to the Coulomb branch of the theory.

Rather more can be said about this five dimensional solution and its relation to the boundary field theory. In particular strong evidence was gathered for the duality by computing the Weyl anomaly in the field theory and finding that it agrees with that calculated in the bulk. Furthermore, the spectrum of linear perturbations of the gauged supergravity around the IR fixed point was calculated and found to match with the spectrum of bilinear operators of the IR field theory.

Our focus will be on some new results on the moduli space physics which uses brane probe techniques and thus requires knowledge of the full ten-dimensional solution. This is the subject we turn to next.

2.4 The lift to ten dimensions

In this section we present the fields of the ten-dimensional lift of the flow solution. The ten-dimensional solution was found in ref. [43].The authors of that paper were able to make a lift ansatz suitable for supersymmetric flows which only involve the scalars ρ and χ. This ansatz gives a solution to the type IIB supergravity equations of motion subject to ρ, χ and A satisfying the flow equations (2.18).

The Einstein frame metric is of the form [43, 44]:

$$ds_{10}^2 = \Delta^{1/2} ds_{1,4}^2 + \Delta^{-1/2} ds_5^2 \,, \tag{2.23}$$

where,

$$ds_5^2 = L^2 \left[\rho^2 (dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2) + \rho^{-4} (dz^3 d\bar{z}^3) + \frac{\sinh^2 \chi}{X} | \sum_i \bar{z}^i dz^i |^2 \right] . \tag{2.24}$$

with

$$\begin{aligned} \Delta &= X \cosh^2 \chi \\ X &= \rho^{-2} \cos^2 \theta + \rho^4 \sin^2 \theta . \end{aligned} \tag{2.25}$$

The z^i's are complex coordinates on C^3 subject to the restriction $\sum_i |z^i|^2 = 1$ which defines a five–sphere. We have introduced the angle θ satisfying $z^1 \bar{z}^1 + z^2 \bar{z}^2 = \cos^2 \theta$,

$z^3\bar{z}^3 = \sin^2\theta$. The metric has a $U(2)$ invariance under which (z^1, z^2) transform as a doublet and also a $U(1)$ invariance under phase rotations in the z^3 plane. This symmetry is broken to the expected $SU(2) \times U(1)$ by the other fields in the solution.

Expressions for the R–R two–form potential, $C_{(2)}$, and the NS–NS two–form potential $B_{(2)}$ are presented in ref. [43]. If we were to consider fermionic terms in the brane action we would need to consider the coupling to these fields. [2] However, if we restrict to bosonic terms in the worldvolume action then, for the probes of interest to us, the pullback of these fields vanishes.

We do need the result for the RR four form potential $C_{(4)}$ to which the D3–brane couples. In ref. [43], an expression was given for the field strength $F_{(5)} = dC_{(4)}$, but in fact it is possible to integrate this up using the flow equations for ρ and χ to give:

$$\begin{aligned} C_{(4)} &= -4w(r,\theta)\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \; + \ldots, \\ \text{where} \quad w(r,\theta) &= \frac{e^{4A}}{8\rho^2}[\rho^6 \sin^2\theta(\cosh(2\chi) - 3) - \cos^2\theta(1 + \cosh(2\chi))] \end{aligned} \tag{2.26}$$

and the remaining terms in $C_{(4)}$ do not couple to the D3–brane probe which we consider.

2.5 A D3-brane probe

We now move on to the brane probe calculation. As in the example of chapter 1, the idea is to pull a single D3-brane off the stack which provides the supergravity background and thus probe a part of the Coulomb branch of the gauge theory where the $SU(N+1)$ gauge group is broken to $SU(N) \times U(1)$. The relevant part of the D3-brane action in Einstein frame is:

$$S = -T_3 \int_{\mathcal{M}_4} d^4\xi \, \det{}^{1/2}[G_{ab} + e^{-\Phi/2}\mathcal{F}_{ab}] + T_3 \int_{\mathcal{M}_4} \left(C_{(4)} + C_{(2)} \wedge \mathcal{F} + \frac{1}{2} C_{(0)}\, \mathcal{F} \wedge \mathcal{F} \right) \tag{2.27}$$

The meaning of the various terms in this action was described in chapter 1 after equation (1.6).

[2] See ref. [45] for a related example

The probe of interest to us is aligned along the $\{t, x^1, x^2, x^3\}$ directions and moving slowly in the transverse directions. Accordingly we partition the spacetime coordinates into $x^i = \{t, x^1, x^2, x^3\}$ and $y^m = \{r, z^i, \bar{z}^i\}$. Our choice of static gauge is:

$$x^i = \xi^i , \quad y^m = y^m(t) . \tag{2.28}$$

Substituting the values for the supergravity fields into the brane action and performing a low energy expansion (i.e. keeping only terms with up to two derivatives), we find an effective Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{T_3}{2} e^{2A} \Delta^{\frac{1}{2}} G_{mn} \dot{y}^m \dot{y}^n - T_3 \sin^2\theta e^{4A} \rho^4 (\cosh(2\chi) - 1) \\ &= \frac{T_3}{2} e^{2A} \left[\Delta \dot{r}^2 + \rho^2 \left(|\dot{z}^1|^2 + |\dot{z}^2|^2 \right) + \rho^{-4} |\dot{z}^3|^2 + \frac{\sinh^2\chi}{X} |\sum_i \bar{z}^i \dot{z}^i|^2 \right] \\ &\quad - T_3 \sin^2\theta e^{4A} \rho^4 (\cosh(2\chi) - 1) \end{aligned} \tag{2.29}$$

The second term in the Lagrangian is a potential energy. In the flow to the conformal phase of the IR theory, the values of ρ^4 and $\cosh(2\chi) - 1$ are non-zero for $r \leq \infty$ and so the potential is minimized only at $\theta = 0$. Setting $\theta = 0$ leaves a 4 (real) dimensional space spanned by $\{r, z^1, \bar{z}^1, z^2, \bar{z}^2\}$. This agrees with the field theory prediction for the dimension of this part of moduli space. Furthermore, $\theta = 0$ is a fixed subspace under the $SU(2)$ symmetry. This agrees with the field theory result in which the moduli space is given by $\phi_3 = 0$ and is also fixed under $SU(2)$.

In order to further interpret our low energy effective Lagrangian in the context of the boundary field theory, we need to fix coordinates in the bulk such that the coordinates transverse to the brane can be matched to gauge theory scalars. This will be the subject of the next chapter.

2.6 Discussion

In this chapter we have reviewed the gravitational description of a particular $\mathcal{N} = 1$ supersymmetric field theory renormalisation group flow and presented the results of a D3-brane probe computation in this background. The gravitational background with a single brane in the bulk is expected to describe the field theory on the Coulomb branch where the gauge group $SU(N+1)$ has been broken to $SU(N) \times U(1)$. The low energy Lagrangian for this brane probe should encode the low energy physics of the $U(1)$ sector of this theory. Thus our probe computation provides predictions at large N and large 't Hooft coupling for this sector of the low energy physics in the field theory.

One immediate result which we can check is that the probe sees the correct number of dimensions for the gauge theory moduli space. As we have described, the four flat directions in the probe potential agree with field theory predictions. Also the $SU(2) \times U(1)$ symmetry of the field theory is inherited by the supergravity geometry and thus is present in the brane probe action. Further checks and predictions will be made in the next chapter.

We close with a comment on a remarkable feature of holography which is nicely illustrated by this example. Normally in discussions of holography the radial coordinate r is thought of as an energy scale in the boundary theory. This is borne out in the flow solution of this chapter since the geometry asymptotes to $AdS_5 \times S^5$ for large r and to a warped product geometry of the (schematic) form $AdS_5 \times M^5$ for small r. Here M^5 is the compact manifold with $SU(2) \times U(1)$ symmetry which emerges at the $r \to -\infty$ end of the flow. This geometry reflects the fact that the field theory at high energies approaches the $\mathcal{N} = 4$ supersymmetric fixed point whilst at low energies it is described by a new fixed point theory. This relationship can be made more precise by studying field theory Green's functions at different energy scales via computations in the bulk. These bulk computations are only sensitive to the details of the geometry over a range of values of r related to the energy scale of the operators in the Green's function.

However a remarkable feature becomes evident when branes are used to probe the bulk. As we have seen a D3-brane in the bulk represents a state of the Coulomb branch of the theory. In particular, the position of the brane represents the vevs for certain gauge theory scalars. Thus the bulk geometry encodes a picture of the gauge theory moduli space as well as the renormalisation group flow of the theory. This is brought out particularly clearly in the example of this chapter since the geometry describes a field theory with non-trivial RG flow and non-trivial moduli space. Of course the two pictures of the transverse geometry are interrelated since the scale of the scalar vevs sets the energy scale at which the $U(1)$ theory on the probe decouples from the $SU(N)$ theory.

Chapter 3

The Kähler Structure of Supersymmetric Holographic RG Flows

3.1 Introduction

In order to extract useful predictions from gauge theory/gravity dualities we need to have a dictionary for comparing quantities between the two sides of the correspondence. It is not always obvious how this should work. As an example, supergravity has an invariance under local coordinate transformations which may nonetheless drastically alter the appearance of a solution. Similarly, field theories have an invariance under field redefinitions. In order to compare predictions from the two sides of the correspondence it is useful to fix simple coordinate systems on both sides which make the process of matching straightforward. This is the main subject of the present chapter in which we use the special properties of D-brane probe moduli spaces to fix preferred coordinates in supergravity.

The basic reason why D-branes are useful for fixing coordinates in this way is that the world volume theory on the brane is expected to match directly with a sector of the gauge theory dynamics. The world volume action for the brane encodes couplings between the bulk fields and the fields on the brane (i.e. the transverse scalars, gauge fields and fermionic superpartners) which correspond to fields of the boundary theory.

We pick up where we left off in the previous chapter with the effective Lagrangian for a D3-brane probe in the type IIB supergravity background dual to the Leigh-Strassler flow. The expectation is that this Lagrangian should represent the low energy dynamics of the decoupled $U(1)$ sector of the $SU(N+1)$ field theory broken to $SU(N) \times U(1)$ by scalar vevs. This field theory has $\mathcal{N} = 1$ supersymmetry as well as a $U(1)$ R-symmetry and an $SU(2)$ flavour symmetry. By considering the restrictions which such symmetries place on the low energy effective Lagrangian for the $U(1)$ theory on the brane we can find strong consistency conditions on our result. In particular, the metric on moduli space (part of the kinetic term in eqn. (2.29)) should be a Kähler metric. Once suitable coordinates have been fixed, our probe Lagrangian can be seen to have this property.

Having thus used the symmetries of the field theory/ probe Lagrangian to fix coordinates on both sides of the correspondence, we can interpret our results directly in the field theory. In particular, the probe result predicts the anomalous dimensions of the fields on moduli space, agreeing with a non-perturbative result from field theory described in the previous chapter. This pleasing result provides some highly non-trivial evidence for the duality.

Another result which we find is an exact formula for the Kähler potential all along the flow in terms of the scalars of the five-dimensional supergravity. We find a simple differential equation satisfied by the Kähler potential. It is not immediately apparent why the Kähler potential should have such a simple form in terms of the five-dimensional supergravity scalars. A direct comparison with the gauge theory is difficult because of the lack of reliable strong coupling results for the Kähler potential.

For the remainder of the chapter, we consider various other supersymmetric flows in the literature and once again use $\mathcal{N} = 1$ supersymmetry along with the relevant global symmetries to constrain the form of the low energy effective action for a brane probe. We compare this to supergravity results for the brane probes and again find preferred coordinate systems in which to study the duality. In all cases, the Kähler potential obeys the same simple differential equation (or a related one in the case of eleven dimensional flows) which allows us to solve exactly for the Kähler potential along the flow.

3.2 $\mathcal{N}$ = 1 supersymmetric Lagrangians in four dimensions

The idea of our approach is to use the strong constraints which $\mathcal{N} = 1$ supersymmetry and various global symmetries place on the form of the low energy effective Lagrangian in order to fix simple coordinates in field theory and supergravity. Supersymmetric Lagrangians are most simply expressed in terms of superfields. We may represent the matter content of a four dimensional $\mathcal{N} = 1$ theory in terms of chiral superfields. The most general terms of a low energy Lagrangian which involve at most two derivatives of these fields are:

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \Phi^{\dagger\bar{i}}) - \int d^2\theta W(\Phi^i) + c.c. \tag{3.1}$$

In this equation K is a real function of the chiral superfields Φ^i and $\Phi^{\dagger\bar{i}}$ but not their derivatives whilst W is a superpotential. There may also be couplings to the gauge fields but these shall not concern us. We shall focus on the low energy physics of probes moving on moduli space and so the superpotential does not contribute to the action here. Expanding out the kinetic terms for the scalar components, ϕ^i, of the chiral superfields we find:

$$\mathcal{L} = g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\dagger\bar{j}} + \dots \tag{3.2}$$

where $g_{i\bar{j}}$ is the Kähler metric defined by the Kähler potential K, i.e.

$$g_{i\bar{j}} = \frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\phi^{\dagger\bar{j}}}K(\phi^k, \phi^{\dagger\bar{k}}). \tag{3.3}$$

Now suppose that we have some theory with $\mathcal{N} = 1$ supersymmetry and global symmetry group G, which furthermore has a moduli space of supersymmetric vacua. We may pick complex coordinates on (a part of) the moduli space such that the symmetry group G is realised linearly and the kinetic term for motions on moduli space is given by a G-invariant Kähler metric. Such a metric may in general be constructed from a G-invariant Kähler potential and so the task of determining the metric on moduli space is reduced to specifying a G-invariant real function of the complex fields parametrizing moduli space.

In the following section we shall illustrate this procedure with the example of finding the form of an $SU(2) \times U(1)$ invariant Kähler metric on the relevant part of the moduli space of the LS flow.

3.3 The Leigh-Strassler flow revisited

3.3.1 A Kähler metric on moduli space

We recall the main results of the previous chapter. The Leigh-Strassler flow is described in supergravity by an $SU(2) \times U(1)$ invariant solution with $\mathcal{N} = 1$ supersymmetry. A single brane probe of the geometry has a low energy effective action given by equation (2.29). The moduli space has four (real) dimensions and the reduction of the low energy Lagrangian to this space is given by:

$$\mathcal{L} = \frac{T_3}{2} e^{2A} \left[\Delta \dot{r}^2 + \rho^2 \left(|\dot{z}^1|^2 + |\dot{z}^2|^2 \right) + \frac{\sinh^2 \chi}{X} |\sum_{i=1}^{2} \bar{z}^i \dot{z}^i|^2 \right], \tag{3.4}$$

corresponding to the metric on moduli space:

$$ds^2 = \frac{T_3}{2} e^{2A} \left[\Delta dr^2 + \rho^2 \left(|dz^1|^2 + |dz^2|^2 \right) + \frac{\sinh^2 \chi}{X} |\sum_{i=1}^{2} \bar{z}^i dz^i|^2 \right]. \tag{3.5}$$

Recall that as in the previous chapter, the z^i coordinates are constrained by the equation $\sum_i |z^i|^2 = 1$ and thus (on the moduli space $z^3 = 0$) parametrize a three-sphere.

This metric inherits an $SU(2) \times U(1)$ invariance from the supergravity solution under which the fields transform linearly as:

$$\begin{aligned} SU(2) \quad &: \quad \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to U \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \quad , \; U \in SU(2), \\ U(1) \quad &: \quad \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Note that $SU(2) \times U(1)$ acts transitively on the z^i coordinates whilst r is inert under this symmetry.

We wish to find out if the metric on moduli space (3.5) is indeed Kähler. We shall use the method outlined in the previous section to construct the most general Kähler metric in two complex dimensions with an $SU(2) \times U(1)$ invariance realised as in eqn. (3.6) on the complex coordinates.

In order to proceed, we choose complex coordinates $(u^1, \bar{u}^1, u^2, \bar{u}^2)$ which transform in the same way as the z coordinates under $SU(2) \times U(1)$. We demand that the Kähler potential, K, is invariant under this $SU(2) \times U(1)$. Thus K is only a function of the quantity q defined as:

$$q = u^1\bar{u}^1 + u^2\bar{u}^2. \tag{3.7}$$

The Kähler metric can now be easily determined in terms of derivatives of K with respect to q. For example:

$$g_{1\bar{1}} = \frac{\partial}{\partial u^1}\frac{\partial}{\partial \bar{u}^1}K(q) = \frac{d}{dq}K + u^1\bar{u}^1\frac{d^2}{dq^2}K. \tag{3.8}$$

The complete expression for the metric in terms of the u coordinates is given by:

$$ds^2 = (du^1d\bar{u}^1 + du^2d\bar{u}^2)\frac{dK}{dq} + |u^1d\bar{u}^1 + u^2d\bar{u}^2|^2\frac{d^2K}{dq^2}. \tag{3.9}$$

We may now ask if our probe result for the metric on moduli space (3.5) may be brought into this form by a change of coordinates. Since the $SU(2) \times U(1)$ transformation of the u's was chosen to match that of the z's our only freedom is in choosing q as a function of r. Thus the coordinates should be related according to:

$$u^i = \sqrt{q(r)}z^i \tag{3.10}$$

It is a simple exercise to transform eqn.(3.9) into r, z coordinates to find:

$$\begin{aligned} ds^2 = \frac{1}{4q^2}\left(\frac{dq}{dr}\right)^2\left(q\frac{d}{dq}\right)^2 K\, dr^2 + \left(q\frac{d}{dq}\right)K\,(dz^1d\bar{z}^1 + dz^2d\bar{z}^2) \\ + \left(q^2\frac{d^2}{dq^2}\right)K\,|z^1d\bar{z}^1 + z^2d\bar{z}^2|^2\,, \end{aligned} \tag{3.11}$$

Comparing this metric with the probe result (3.5) and demanding that the two are equal, we find three equations for the two unknown functions q(r) and K(r):

$$\frac{1}{4q^2}\left(\frac{dq}{dr}\right)^2\left(q\frac{d}{dq}\right)^2 K = \frac{T_3}{2}\frac{\cosh^2\chi}{\rho^2}e^{2A}\,, \tag{3.12}$$

$$\left(q\frac{d}{dq}\right) K = \frac{T_3}{2}L^2\rho^2e^{2A}\,, \tag{3.13}$$

$$\left(q^2\frac{d^2}{dq^2}\right) K = \frac{T_3}{2}L^2\rho^2e^{2A}\sinh^2\chi\,. \tag{3.14}$$

In fact it is more convenient to add the last two equations to get:

$$\left(q\frac{d}{dq}\right)^2 K = \frac{T_3}{2}L^2\rho^2e^{2A}\cosh^2\chi \tag{3.14'}$$

Fixing q as a function of r corresponds to a preferred choice of complex coordinates (in particular the complex structure is determined) whilst fixing K corresponds to a choice of Kähler metric. Since there are 3 equations to satisfy and only 2 unknown functions, we expect that there will be one non-trivial consistency condition which needs to be satisfied if everything is to work out.

Substituting (3.14′) into (3.12) we get an equation for q:

$$\frac{dq}{dr} = \frac{2}{L\rho^2}q \tag{3.15}$$

Substituting this result for q into (3.13) we find a strikingly simple equation for K:

$$\frac{dK}{dr} = T_3Le^{2A}\,, \tag{3.16}$$

Finally we need to check that (3.14′) is satisfied. Substituting (3.15) into (3.14′) we find that

$$\frac{d}{dr}(\rho^2e^{2A}) = \frac{2}{L}e^{2A}\cosh^2\chi\,, \tag{3.17}$$

is needed for consistency. Remarkably, this equation is a consequence of the supergravity flow equations (2.18) which encode the $\mathcal{N}=1$ supersymmetry of the geometry. We have thus performed a strong check on the results of [43] and found that the metric on moduli space is indeed Kähler.

The differential equations (3.15,3.16) defining q and K are tantalisingly simple. It is possible to show in particular that K can be expressed in terms of the supergravity fields as:

$$K = \frac{T_3L^2e^{2A}}{4}\left(\rho^2+\frac{1}{\rho^4}\right)\,. \tag{3.18}$$

This is a pleasing and unexpectedly simple result.

3.3.2 Scaling dimensions

One result of the last section was that we were able to fix unique coordinates on moduli space by demanding that the supersymmetry and flavour symmetries be realised linearly in the brane probe action. To recap, the $SU(2)$ flavour symmetry uniquely fixed (up to rigid rotations) the angular part of the complex coordinates, $(u^1, \bar{u}^1, u^2, \bar{u}^2)$, whilst the requirement of a Kähler structure fixed the radial part. The coordinates u^i are thus expected to match directly with (the non-zero eigenvalues of) the complex scalar fields ϕ^i , $i = 1, 2$.

In the UV and IR limits, the field theory approaches conformal fixed points and as we have discussed in the previous chapter, there are definite predictions for the scaling dimensions of the chiral superfields in these limits. We can now ask if these scaling dimensions are correctly reproduced by the u^i coordinates in the supergravity solution.

We start at the UV end of the flow which is just the standard $AdS_5 \times S^5$ geometry. The AdS_5 part of the metric is

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \qquad \text{where } A = \frac{r}{L} , \tag{3.19}$$

which has a symmetry under which a rescaling of the brane coordinates x^μ is compensated by a shift in the radial coordinate r:

$$x \to \frac{1}{\alpha}x \qquad e^A \to \alpha e^A . \tag{3.20}$$

In fact this is a symmetry of all the fields in the supergravity solution (since the scalars ρ and χ are constant) and is thus a symmetry of the action of a brane probing this background. In terms of the coordinates on moduli space derived in the previous section, we have $\sqrt{q} \sim e^A$ for large r and so the scaling symmetry becomes

$$x \to \frac{1}{\alpha}x \qquad \sqrt{q} \to \alpha\sqrt{q} . \tag{3.21}$$

In other words the fields on moduli space have scaling dimension 1 which matches with the field theory prediction for the scalar components of these chiral superfields in the $\mathcal{N} = 4$ theory.

Next we consider the IR end of the flow solution. Here the solution again has the scaling symmetry (3.20) except that $A = \frac{2^{5/3}r}{L}$ in this case. The coordinate $\sqrt{q}$

goes like $\sqrt{q} \sim \exp\left(\frac{r}{2^{1/3}L}\right) \sim (e^A)^{3/4}$ and thus the scaling symmetry becomes

$$x \to \frac{1}{\alpha} x \qquad \sqrt{q} \to \alpha^{3/4}\sqrt{q}\,. \tag{3.22}$$

Therefore, we see that the massless fields have scaling dimension 3/4 here. This agrees with the field theory result (2.9) which followed from a nonperturbative analysis of the vanishing of the exact beta functions. It is very pleasing that this result is encoded in the supergravity solution in this way!

3.3.3 Comments on the Kähler potential

The scaling arguments of the previous section only require knowledge of the complex coordinates on moduli space and the asymptotic form of the supergravity solution. In section 3.3.1 we also found a simple expression for the Kähler potential all along the flow. It is difficult to make a direct comparison between this result and the gauge theory owing to the lack of non-renormalisation theorems to protect the field theory result at strong coupling. However, a few observations are possible.

The UV and IR limits of the Kähler potential are particularly simple as the Kähler metric becomes conical in these limits. [1] Consequently, the Kähler potential is a simple power of q in these limits. This power is determined by dimensional analysis. The Kähler potential has dimension 2 (since it appears in the Lagrangian with an $\int d^4\theta$). In the UV, q has dimension 2 and so $K \sim q$ whilst in the IR q has dimension 3/2 and so $K \sim q^{4/3}$. It is straightforward to check that our result for K (3.18) has these asymptotics.

A further check would be to investigate the next to leading order corrections to K in the limit of large vevs. It is straightforward to expand the result for K as a series expansion at large r and it should be possible to compare this to a perturbative calculation in field theory. This would provide an interesting test of the result.

Perhaps the most intriguing result which we have found is the simple differential equation satisfied by K (3.16). As we shall see this result also holds for a variety

[1] This is because we have a situation in which conformal invariance is being broken by a single scale - the position of the brane in the bulk - and thus the physics should appear the same at all radii, leading to a conical metric.

of other holographic renormalisation group flows which preserve at least $\mathcal{N} = 1$ supersymmetry in four dimensions. We now turn to the investigation of these other flows using the techniques we have developed in this section. The next solution we consider represents a flow to the Coulomb branch of the Leigh-Strassler theory.

3.4 A More General $\mathcal{N} = 1$ Flow

3.4.1 The Ten Dimensional Solution

In this section we consider a related ten dimensional supergravity flow which was presented in ref [46] and is another lift from five-dimensional gauged supergravity. The five-dimensional solution involves the same scalars ρ and χ as in the previous flow, but allows a new scalar, $\beta = \log \nu$, to vary. Switching on β breaks the $SU(2) \times U(1)$ symmetry to $U(1)^2$, and corresponds to turning on a vev on the Coulomb branch of the $\mathcal{N} = 1$ theory which we explored previously.

The new ten-dimensional metric is of the same general structure as given in equation (2.23), with equation (2.16) for the first component. The new warp factor Ω is given by:

$$\Omega^2 = \cosh\chi \left((\nu^2 \cos^2\phi + \nu^{-2} \sin^2\phi) \frac{\cos^2\theta}{\rho^2} + \rho^4 \sin^2\theta \right)^{\frac{1}{2}} , \tag{3.23}$$

whilst the metric on the compact space is:

$$\begin{aligned} ds_5^2 = \quad & \frac{L^2}{\Omega^2} \left[\rho^{-4} \left(\cos^2\theta + \rho^6 \sin^2\theta \, (\nu^{-2} \cos^2\phi + \nu^2 \sin^2\phi) \right) d\theta^2 \right. \\ & + \rho^2 \cos^2\theta (\nu^2 \cos^2\phi + \nu^{-2} \sin^2\phi) d\phi^2 \\ & - 2\rho^2 (\nu^2 - \nu^{-2}) \sin\theta \cos\theta \sin\phi \cos\phi d\theta d\phi \\ & \left. + \rho^2 \cos^2\theta (\nu^{-2} \cos^2\phi d\varphi_1^2 + \nu^2 \sin^2\phi d\varphi_2^2) + \rho^{-4} \sin^2\theta d\varphi_3^2 \right] \\ & + \frac{L^2}{\Omega^6} \sinh^2\chi \cosh^2\chi (\cos^2\theta (\cos^2\phi d\varphi_1 - \sin^2\phi d\varphi_2) - \sin^2\theta d\varphi_3)^2 \end{aligned} \tag{3.24}$$

The $U(1)^2$ symmetry is generated by the Killing vectors $\partial/\partial\varphi_1$ and $\partial/\partial\varphi_2$. The superpotential for this flow is given by [46]:

$$W = \frac{1}{4}\rho^4 (\cosh 2\chi - 3) - \frac{1}{4\rho^2} (\nu^2 + \nu^{-2})(\cosh 2\chi + 1) , \tag{3.25}$$

which generalises the superpotential in equation (2.17). The equations of motion for the supergravity fields are:

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6L}\rho^2\frac{\partial W}{\partial \rho} = \frac{1}{12L}\left(\frac{2\rho^6(\cosh 2\chi - 3) + (\nu^2+\nu^{-2})(\cosh 2\chi + 1)}{\rho}\right) , \\
\frac{d\nu}{dr} &= \frac{1}{2L}\nu^2\frac{\partial W}{\partial \nu} = -\frac{1}{4L}\left(\frac{(\cosh 2\chi + 1)\nu(\nu^2-\nu^{-2})}{\rho^2}\right) , \\
\frac{d\chi}{dr} &= \frac{1}{L}\frac{\partial W}{\partial \chi} = \frac{\sinh 2\chi}{2L}\left(\frac{\rho^6 - (\nu^2+\nu^{-2})}{\rho^2}\right) , \\
\frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L}\left(\frac{\rho^6(\cosh 2\chi - 3) - (\nu^2+\nu^{-2})(\cosh 2\chi + 1)}{\rho^2}\right) . \qquad (3.26)
\end{aligned}$$

The authors of ref. [46] probed the solution with a D3–brane. Once again the moduli space has four real dimensions , which agrees with the interpretation of the solution as the LS theory on Coulomb branch. The result for the metric on moduli space is:

$$\begin{aligned}
ds^2 &= \frac{1}{2}T_3 e^{2A}\left[\zeta(\rho^{-2}\cosh^2\chi dr^2 + L^2\rho^2 d\phi^2)\right. \\
&\qquad + L^2\rho^2(\nu^{-2}\cos^2\phi d\varphi_1^2 + \nu^2\sin^2\phi d\varphi_2^2) \\
&\qquad\qquad \left. + L^2\rho^2\sinh^2\chi\zeta^{-1}(\cos^2\phi\, d\varphi_1 - \sin^2\phi\, d\varphi_2)^2\right] , \\
\text{where } \zeta &\equiv (\nu^2\cos^2\phi + \nu^{-2}\sin^2\phi) . \qquad (3.27)
\end{aligned}$$

3.4.2 A Kähler Potential

Again we wish to use $\mathcal{N} = 1$ supersymmetry and the flavour and R–symmetries of the theory to choose a special set of coordinates in which the action for the brane probe can be compared to field theory expectations. In this case the $SU(2)\times U(1)$ symmetry has been broken to $U(1)^2$ by giving a vev to one of the massless fields.

The $U(1)^2$ symmetries are given by constant shifts in φ_1 and φ_2. We wish to find a complex structure in which this metric is Kähler and the $U(1)^2$ symmetries are realised linearly. We can choose:

$$z_1 = \sqrt{u(r,\phi)}e^{i\varphi_1} , \qquad z_2 = \sqrt{v(r,\phi)}e^{-i\varphi_2} , \qquad (3.28)$$

and a $U(1)^2$ invariant Kähler potential

$$K = K(z_1\bar{z}_1, z_2\bar{z}_2) = K(u,v) . \qquad (3.29)$$

Proceeding as before we write down the form of the metric which results from this Kähler potential. A short calculation gives

$$
\begin{aligned}
ds^2 &= \frac{1}{4u^2}\left(u\frac{\partial}{\partial u}\right)^2 K du^2 + \frac{1}{2uv}\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K du dv + \frac{1}{4v^2}\left(v\frac{\partial}{\partial v}\right)^2 K dv^2 \\
&+ \left(u\frac{\partial}{\partial u}\right)^2 K d\varphi_1^2 - 2\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K d\varphi_1 d\varphi_2 + \left(v\frac{\partial}{\partial v}\right)^2 K d\varphi_2^2 \,. \qquad (3.30)
\end{aligned}
$$

Comparison with equation (3.27) gives the following set of equations for u, v and K.

$$
\begin{aligned}
\left(u\frac{\partial}{\partial u}\right)^2 K &= \frac{T_3}{2} e^{2A} L^2 \rho^2 (\nu^{-2}\cos^2\phi + \sinh^2\chi \zeta^{-1}\cos^4\phi) \\
&= 4u^2 e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial u}\right)^2 + L^2\rho^2\left(\frac{\partial\phi}{\partial u}\right)^2\right) , \\
\left(v\frac{\partial}{\partial v}\right)^2 K &= \frac{T_3}{2} e^{2A} L^2 \rho^2 (\nu^{2}\sin^2\phi + \sinh^2\chi \zeta^{-1}\sin^4\phi) \\
&= 4v^2 e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial v}\right)^2 + L^2\rho^2\left(\frac{\partial\phi}{\partial v}\right)^2\right) , \\
\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K &= \frac{T_3}{2} e^{2A} L^2 \rho^2 \sinh^2\chi \zeta^{-1}\cos^2\phi\sin^2\phi \qquad (3.31) \\
&= 4uv e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) + L^2\rho^2\left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial\phi}{\partial v}\right)\right) .
\end{aligned}
$$

The solutions for u and v are

$$u = f(r)\cos^2\phi \,, \qquad v = g(r)\sin^2\phi \,,$$

where

$$\frac{df}{dr} = \frac{2\nu^2}{L\rho^2} f \,, \qquad \frac{dg}{dr} = \frac{2}{L\rho^2\nu^2} g \,. \qquad (3.32)$$

We find an exact solution for the Kähler potential:

$$K = \frac{T_3}{2} L^2 e^{2A}\left(\rho^2(\nu^2 - \nu^{-2})\sin^2\phi + \frac{1}{2}(\rho^2\nu^{-2} + \rho^{-4})\right) . \qquad (3.33)$$

As before the equations of motion (3.26) were needed in order to find a solution. The specific combinations of the equations used are rather simple and we reproduce them below:

$$
\begin{aligned}
\frac{d(e^{2A}\rho^2\nu^{-2})}{dr} &= \frac{d(e^{2A}\rho^2\nu^2)}{dr} = \frac{2}{L} e^{2A}\cosh^2\chi \,, \\
\frac{d(e^{2A}\rho^{-4})}{dr} &= \frac{e^{2A}}{L}(3 - \cosh(2\chi)) \,. \qquad (3.34)
\end{aligned}
$$

In fact, this allows us to write down a remarkably simple solution for A as a function of ρ and ν for the case $\nu \neq 1$:

$$e^{2A} = \frac{k}{\rho^2(\nu^2 - \nu^{-2})} , \tag{3.35}$$

where k is a constant. Using this expression we can simplify the Kähler potential:

$$K = \frac{T_3}{2} L^2 k \sin^2\phi + \frac{T_3 L^2 e^{2A}}{4} \left(\frac{\rho^2}{\nu^2} + \frac{1}{\rho^4} \right) . \tag{3.36}$$

Once again K satisfies the simple differential equation (3.16).

3.5 Pure Coulomb Branch Flows

3.5.1 Switching off the mass

It is quite interesting to study a special case of the above. Let us switch off the mass deformation by setting $\chi = 0$. We can consistently do this in the flow equations (3.26) whilst allowing ρ and ν to vary. In this case we are studying a purely $\mathcal{N} = 4$ Coulomb branch deformation. With $\chi = 0$, the equations (3.34) simplify to yield the result

$$e^{2A}\rho^2\nu^2 = e^{2A}\rho^2\nu^{-2} + k = e^{2A}\rho^{-4} + l , \tag{3.37}$$

where k is the constant appearing in equation (3.35) and l is another integration constant. We can also write down a solution for equation (3.32) in this case:

$$f = \frac{L^2}{\alpha'^2} e^{2A}\rho^2\nu^{-2}, \qquad g = \frac{L^2}{\alpha'^2} e^{2A}\rho^2\nu^2 . \tag{3.38}$$

Now if we substitute into the expression (3.36) for the Kähler potential, we find

$$\begin{aligned} K &= \frac{T_3}{2} L^2 e^{2A} (\rho^2\nu^2 \sin^2\phi + \rho^2\nu^{-2}\cos^2\phi) \\ &= \frac{1}{8\pi^2 g_{YM}^2}(u + v) \\ &= \frac{1}{8\pi^2 g_{YM}^2}(z_1\bar{z}_1 + z_2\bar{z}_2) . \end{aligned} \tag{3.39}$$

This is the expected probe result [7] for a two complex dimensional subspace of the flat three complex dimensional moduli space which exists for the $\mathcal{N} = 4$ theory on Coulomb branch.

3.5.2 The General Case of the Coulomb Branch

There is in fact a broader class of $\mathcal{N} = 4$ supersymmetric Coulomb branch flows which are accessible via five dimensional gauged supergravity. The five dimensional supergravity equations describing these flows were given in ref. [47] and a way to decouple these equations in order to find exact solutions was presented in ref. [48]. The ten–dimensional lift corresponds to a continuous distribution of parallel D3–branes [49] and so the metric must take the form

$$ds^2 = H^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}(dy_1^2 + dy_2^2 + \ldots + dy_6^2) , \qquad (3.40)$$

for some harmonic function $H(y_i)$. A D3–brane probing this background has a flat metric on moduli space [7]:

$$ds^2 = \frac{T_3}{2}(dy_1^2 + dy_2^2 + \ldots + dy_6^2) = \frac{T_3}{2}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3) , \qquad (3.41)$$

coming from a Kähler potential

$$K = \frac{T_3}{2}(y_1^2 + y_2^2 + \ldots + y_6^2) . \qquad (3.42)$$

In ref. [48] it was shown that it is natural to write the lift solution in terms of a radial coordinate F which satisfies, in the coordinates of our discussion:

$$\frac{dF}{dr} = 2Le^{2A} , \qquad (3.43)$$

and is related to the y_i by

$$y_i = (F - b_i)^{\frac{1}{2}}\hat{x}_i , \qquad (3.44)$$

where b_i are constants and $\hat{x}_i$ are coordinates on a unit S^5. Changing to F coordinates we find that the Kähler potential is given by

$$K = \frac{T_3}{2}\sum_i (F - b_i){\hat{x}_i}^2 = \frac{T_3}{2}(F - \sum_i b_i {\hat{x}_i}^2) , \qquad (3.45)$$

from which we can read

$$\frac{\partial K}{\partial F} = \frac{T_3}{2} , \qquad (3.46)$$

which when combined with equation (3.43) results in our equation (3.16) once again. In fact it is straightforward to generalise these results to the case of M2– and M5–branes on their analogues of the Coulomb branch by adapting the results of ref. [50].

In these cases we find

$$\frac{\partial K}{\partial r} = T_{M2} L e^{A} , \tag{3.47}$$

for M2–branes and

$$\frac{\partial K}{\partial r} = T_{M5} L e^{4A} , \tag{3.48}$$

for M5–branes. It seems that the Kähler potentials governing the metric on moduli space for supersymmetric holographic RG flows arising from gauged supergravities, always satisfy these equations. We have shown this for purely Coulomb branch flows, for the Leigh–Strassler flow and a generalisation, and in the next two sections demonstrate that it is also true for another three families of examples.

3.6 An $\mathcal{N} = 2$ Flow in $D = 4$

In this section we consider the ten–dimensional supergravity dual of an $\mathcal{N} = 2$ supersymmetric field theory RG flow. This field theory flow results from perturbing the $\mathcal{N} = 4$ theory in the UV by an equal mass for two of the chiral superfields. The ten-dimensional flow solution was constructed in ref. [51], and has been studied via brane probing in refs. [52–54]. Again we wish to find an explicit form of the Kähler potential and check that equation (3.16) is satisfied. In this case the moduli space for a D3–brane probe is a one complex dimensional space and so it is simple to find coordinates in which the metric is Kähler, since these are just the coordinates in which the metric is conformally flat:

$$ds^2 = \partial\bar{\partial}K \, dz \, d\bar{z} . \tag{3.49}$$

In fact such coordinates have already been found in refs. [52, 53], and we reproduce the result here

$$ds^2 = \frac{T_3}{2} k^2 L^2 \frac{c}{(c+1)^2} \, dz \, d\bar{z} , \tag{3.50}$$

where

$$c = \cosh(2\chi) \quad \text{and} \quad z = e^{-i\phi} \sqrt{\frac{(c+1)}{(c-1)}} . \tag{3.51}$$

Note that this fixes the complex structure on moduli space but that in this case the flavour symmetries are insufficient to pin down a unique choice of complex *coordinates.* $\mathcal{N}=2$ supersymmetry is used in ref. [52] to match the scalar kinetic term with the kinetic term of the $U(1)$ gauge field on the brane and so fix a unique set of coordinates[2]. However, for our purposes we only need the correct complex structure and so $\mathcal{N}=1$ arguments are enough. To proceed we set $u=z\bar{z}$ and find

$$ds^2 = \frac{d}{du}\left(u\frac{d}{du}\right) K\, dz\, d\bar{z} = \frac{T_3}{2}k^2L^2\frac{(u^2-1)}{4u^2}\, dz\, d\bar{z} \ , \tag{3.52}$$

which has solution

$$K = \frac{T_3}{2}k^2L^2\left(\frac{1}{4}(u-u^{-1}) + a\log(u) + b\right) \ , \tag{3.53}$$

where a and b are constants. To get this expression for the Kähler potential into the form we want, we need to use the following solutions [51] for the supergravity fields A and ρ,

$$\begin{aligned} e^A &= k\frac{\rho^2}{\sinh 2\chi} \\ \rho^6 &= \cosh 2\chi + \sinh^2 2\chi\left(\gamma + \log\left[\frac{\sinh\chi}{\cosh\chi}\right]\right) \ . \end{aligned} \tag{3.54}$$

If we choose $a=-\frac{1}{2}$ and $b=\gamma$ then the Kähler potential simplifies to

$$K = \frac{T_3}{2}L^2\rho^2e^{2A} \ , \tag{3.55}$$

which on applying the relevant supergravity equations of motion

$$\begin{aligned} \rho\frac{d\rho}{dr} &= \frac{1}{3L}\left(\frac{1}{\rho^2} - \rho^4\cosh 2\chi\right) \ , \quad \frac{d\chi}{dr} = -\frac{1}{2L}\rho^4\sinh 4\chi \ , \\ \frac{dA}{dr} &= \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh 2\chi\right) \ , \end{aligned} \tag{3.56}$$

can be shown to satisfy equation (3.16).

3.7 Two $\mathcal{N}=2$ Flows in $D=3$

Finally, we examine two examples of flows in eleven dimensions constructed in ref. [55]. The three-dimensional field theories on the M2-branes which source these eleven dimensional geometries are close cousins of the Leigh-Strassler field theory in

[2]See also ref. [54] for further work and generalisations.

four-dimensions. Consequently, the eleven-dimensional flows are closely related to the ten-dimensional geometry dual to the Leigh-Strassler flow, which we studied in some detail at the start of this chapter.

The first eleven dimensional solution flows between geometries of the form $AdS_4 \times S^7$ in the UV and $AdS_4 \times M^7$ in the IR, where M^7 is a compact seven-dimensional manifold. The flow preserves 4 supercharges (corresponding to $\mathcal{N}=2$ supersymmetry on the three-dimensional brane worldvolume) and has an $SU(3) \times U(1)$ invariance. The similarity to the LS flow lies in the fact that this field theory also comes from a mass perturbation for a single chiral multiplet in the UV and induces a flow between conformal field theories. Some aspects of the field theory are discussed in [56], where the flow was first studied in four dimensional gauged supergravity.

The second eleven dimensional flow which we consider arises from a second lift of the same four dimensional gauged supergravity solution which gives rise to the first flow. The geometry at the UV end of this flow is not $AdS_4 \times S^7$, but rather the near horizon limit of a collection of M2-branes at a conical singularity in flat five-dimensional Minkowski space $\times$ the conifold. We shall discuss this example in more detail later.

3.7.1 An $SU(3) \times U(1)$ invariant flow in gauged supergravity

In this section we present details of the analogue of the LS flow in four-dimensional $\mathcal{N}=8$ gauged supergravity. This solution was constructed in [56] (see also [55]). The flow preserves $2+1$ dimensional Poincaré invariance and so the metric can be written as:

$$e^{2A(r)}dx^\mu dx_\mu + dr^2. \tag{3.57}$$

Two scalars are turned on and we call these $\rho(r)$ and $\chi(r)$ in analogy with the corresponding five-dimensional flow. The superpotential is

$$W(\rho,\chi) = \frac{1}{8}\rho^6(\cosh(2\chi)-3) - \frac{3}{8\rho^2}(\cosh(2\chi)+1) \tag{3.58}$$

leading to flow equations:

$$\frac{dA}{dr} = -\frac{2}{L}W\,, \qquad \frac{d\rho}{dr} = \frac{\rho^2}{6L}\frac{\partial W}{\partial\rho}\,, \qquad \frac{d\chi}{dr} = \frac{2}{L}\frac{\partial W}{\partial\chi}\,. \tag{3.59}$$

3.7.2 The $SU(3)\times U(1)$ invariant lift

We now consider the first of the two lifts of this four-dimensional flow solution to eleven dimensions . The lift ansatz for the eleven dimensional metric in ref. [55] is:

$$ds_{11}^2 = \Delta^{-1}(dr^2 + e^{2A(r)}(\eta_{\mu\nu}dx^\mu dx^\nu)) + \Delta^{\frac{1}{2}}L^2 ds^2(\rho,\chi) \ , \tag{3.60}$$

where

$$ds^2(\rho,\chi) = \rho^2(dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2 + dz^3 d\bar{z}^3) + \rho^{-6}dz^4 d\bar{z}^4 + \frac{\sinh^2(\chi)}{X}|z^i d\bar{z}^i|^2 \ , \tag{3.61}$$

with

$$\begin{aligned} X &= \rho^{-2}\cos^2\theta + \rho^6\sin^2\theta \ , \\ \Delta &= (X\cosh\chi)^{-4/3}. \end{aligned} \tag{3.62}$$

The complex coordinates z^i are constrained by the equation $|z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1$ and thus span an S^7. The angle theta is defined via the equation $|z^4|^2 = \sin^2\theta$.

The three-form potential, which couples to an M2 brane probe, is also given in ref. [55] and a brane probe calculation is performed which reveals that the moduli space has 3 complex dimensions (as expected for a deformation of the UV theory by a mass for a single chiral multiplet) and is situated at $\sin^2\theta = 0$. Using these results it is straightforward to read off the metric on moduli space for an M2-brane probe:

$$ds^2 = \frac{T_{M2}}{2}e^A\left(\frac{\cosh^2\chi}{\rho^4}dr^2 + L^2 ds^2(\rho,\chi)|_{\text{moduli}}\right) \ , \tag{3.63}$$

where

$$ds^2(\rho,\chi)|_{\text{moduli}} = \rho^2\sum_{i=1}^{3}dz^i d\bar{z}^i + \rho^2\sinh^2\chi|\sum_{i=1}^{3}z^i d\bar{z}^i|^2 \ . \tag{3.64}$$

The next stage is to calculate the form of the most general $SU(3)\times U(1)$ invariant metric in three complex dimensions with which to compare this result. As in the similar example of section 3.3.1, $SU(3)$ invariance will automatically imply invariance under the larger group $SU(3)\times U(1)$.

In fact it will be useful to have the general form for an $SU(n)$ invariant Kähler metric on $\mathbb{C}^n$. Let $w^1, w^2, \ldots, w^n$ be coordinates on $\mathbb{C}^n$. We assume that the coordinates $(\bar{w}^1,\ldots,\bar{w}^n)$ transform in the fundamental representation of $SU(n)$. Invariance under this group implies that the Kähler potential K depends only on the combination

$$q = w^1\bar{w}^1 + w^2\bar{w}^2 + \ldots + w^n\bar{w}^n \ . \tag{3.65}$$

Let us reparametrise $\mathbb{C}^n \sim \mathbb{R}^+ \times S^{2n-1}$ with coordinates $z^1,\ldots,z^n,q$. The z^i's are complex coordinates on an S^{2n-1} of unit radius and are related to the w's by

$$w^i = \sqrt{q}z^i \ . \tag{3.66}$$

A short calculation gives the Kähler metric in these coordinates as:

$$ds^2 = \frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K\,dq^2 + \left(q\frac{d}{dq}\right) K \sum_{i=1}^{n} dz^i d\bar{z}^i + \left(q^2\frac{d^2}{dq^2}\right) K\,|\sum_{i=1}^{n} z^i d\bar{z}^i|^2 \ , \tag{3.67}$$

Now we wish to compare this result (for $n=3$) to the metric on moduli space of the flow solution. Substituting equation (3.64) into (3.63) and comparing with equation (3.67), we get three equations for the two unknown functions $q(r)$ and $K(r)$:

$$\begin{aligned}
\frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K dq^2 &= \frac{T_{M2}}{2}e^A\Delta^{-\frac{3}{2}}dr^2 \ , && (3.68)\\
\left(q\frac{d}{dq}\right) K &= \frac{T_{M2}}{2}e^A L^2\rho^2 \ , && (3.69)\\
\left(q^2\frac{d^2}{dq^2}\right) K &= \frac{T_{M2}}{2}e^A L^2\rho^2\sinh^2\chi \ . && (3.70)
\end{aligned}$$

Consistency of these equations requires that

$$\frac{d}{dr}(e^A\rho^2) = \frac{2}{L}e^A\cosh^2\chi \ , \tag{3.71}$$

which is a consequence of the supergravity equations of motion. Then the solutions for K and q are:

$$K = \frac{T_{M2}L^2}{4}e^A\left(\rho^2 + \frac{1}{\rho^6}\right) \ , \qquad \frac{dq}{dr} = \frac{2}{L\rho^2}q \ . \tag{3.72}$$

We observe again that this Kähler potential satisfies the equation (3.47).

3.7.3 The $T^{1,1}$ flow

Interestingly, there is a second lift of the same four dimensional gauged supergravity solution which led to the eleven dimensional example of the previous section. This was also constructed in ref. [55]. The second eleven dimensional geometry is similar to the first, but with the stretched five spheres (spanned by $z^i, \bar{z}^i$, $i=1,2,3$) replaced by stretched $T^{1,1}$'s. The details are in ref. [55] and since we shall only be interested in the moduli space of an M2–brane probe, we shall not go into detail here.[3]

The metric on moduli space for an M2–brane probing this geometry is again given by equation (3.63) except that equation (3.64) is replaced by

$$ds^2(\rho,\chi)|_{\rm moduli} = \rho^2 ds^2_{T^{1,1}} + \rho^2 \sinh^2\chi \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \ , \qquad (3.73)$$

and the metric on $T^{1,1}$ is [57]:

$$ds^2_{T^{1,1}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \ . \qquad (3.74)$$

In particular, the S^5's in equation (3.64) have been replaced by $T^{1,1}$'s so that for large r ($\rho \to 1$, $\chi \to 0$) the two moduli spaces approach flat $\mathbb{R}^6$, and the Ricci flat Kähler conifold, respectively.

To proceed we first note that the $SU(2)\times SU(2)$ symmetry of the conifold metric is preserved for all values of r. Therefore, we need to find the general form of an $SU(2)\times SU(2)$ invariant Kähler metric on the conifold and compare to equations (3.63), (3.73). Such metrics have been studied in ref. [57] and we shall rederive a result of that paper below.

The conifold is a surface in $\mathbb{C}^4$ parametrised by four complex coordinates z_1, z_2, z_3

[3]It may seem rather surprising that there should be two different lifts of a single four dimensional solution. A possible explanation is that the four dimensional solution may be embeddable in two ways in a larger four dimensional gauged supergravity with other fields representing e.g. twisted sector modes of M-theory on an orbifold of $AdS_4 \times S^7$. In particular, if there were some decoupling between twisted and untwisted sector modes, the same values for the untwisted fields ρ, χ could perhaps occur with different constant values of the twisted fields. There would then be two different solutions to lift to eleven dimensions which might be expected to differ in topology in roughly the way which we find here.

and z_4, which satisfy an equation

$$z_1^2+z_2^2+z_3^2+z_4^2=0. \tag{3.75}$$

An $SU(2)\times SU(2)=SO(4)$ invariant metric depends only on the combination

$$p=z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3+z_4\bar{z}_4. \tag{3.76}$$

Thus, to construct an $SU(2)\times SU(2)$ invariant metric on the conifold we can start with our equation (3.67) for an $SU(4)$ invariant metric on $\mathbb{C}^4$ and then restrict to the conifold using equation (3.75). For convenience and in order to adjust a couple of notations we reproduce the equation for an $SU(4)$ invariant metric on $\mathbb{C}^4$ here:

$$ds^2=\frac{1}{4p^2}\left(p\frac{d}{dp}\right)^2K\,dp^2+\left(p\frac{d}{dp}\right)K\,d\hat{z}_i d\hat{\bar{z}}_i+\left(p^2\frac{d^2}{dp^2}\right)K\,|\hat{z}_i d\hat{\bar{z}}_i|^2, \tag{3.77}$$

where the $\hat{z}_i$'s parametrise an S^7, *i.e.* $\Sigma\hat{z}_i\hat{\bar{z}}_i=1$. In fact it will be more convenient to work in terms of a radial coordinate $q=\frac{3}{2}p^{2/3}$ for reasons which will become clear. In these coordinates the $SU(4)$ invariant metric becomes:

$$ds^2=\frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2K\,dq^2+\left(q\frac{d}{dq}\right)K\,\left[\frac{2}{3}d\hat{z}_i d\hat{\bar{z}}_i-\frac{2}{9}|\hat{z}_i d\hat{\bar{z}}_i|^2\right]+\frac{4}{9}\left(q^2\frac{d^2}{dq^2}\right)K\,|\hat{z}_i d\hat{\bar{z}}_i|^2. \tag{3.78}$$

Finally, we need to restrict to the conifold by applying equation (3.75) to the $\hat{z}_i$'s. If we reparametrise in terms of the coordinates[4] on $T^{1,1}$ introduced in equation (3.74), then we find the following form for the general $SU(2)\times SU(2)$ invariant metric on the conifold:

$$\begin{aligned} ds^2 \;=\; & \frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2K\,dq^2+\left(q\frac{d}{dq}\right)K\,[ds^2_{T^{1,1}}] \\ & +\frac{1}{9}\left(q^2\frac{d^2}{dq^2}\right)K\,(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2. \end{aligned} \tag{3.79}$$

It is now straightforward to compare this metric with the metric on moduli space (3.63), (3.73), to extract equations for K and q. The equations which K and q satisfy are precisely (3.68), (3.69) and (3.70) with solution (3.72). It should be noted, however, that q has a different definition than it did in the flow of the previous section.

[4]The reader should refer to ref. [57] for the explicit form of these coordinates in terms of the $\hat{z}_i$'s.

3.7.4 Scaling dimensions

Since the eleven dimensional geometries which we are considering flow between asymptotic geometries containing an AdS_4 factor, the corresponding field theories flow between conformal fixed points in the UV and IR. Consequently we may once again discuss scaling dimensions of the field theory operators. Our field theory arguments in this section are rather heuristic owing to the lack of a Lagrangian description of the IR fixed point theory on a stack of N M2-branes and should be considered as providing a rough way to understand the supergravity results, which we believe to be correct. We could perhaps make these arguments more rigorous by staring in the UV with the dimensional reduction of the four dimensional $\mathcal{N} = 4$ gauge theory but we shall not attempt this here.

First we consider the flow from the $AdS_4 \times S^7$ geometry in the UV. Let us consider the field theory which in the UV contains four chiral superfields ϕ_i, with $i = 1 \ldots 4$. Suppose that all four chiral superfields are coupled together in the Lagrangian. Then the scaling dimensions (d_i) of the fields satisfy

$$d_1 + d_2 + d_3 + d_4 = 2 , \tag{3.80}$$

and $d_i = \frac{1}{2}$ by symmetry. The flow solution corresponds to perturbing the superpotential by a mass term for Φ_4

$$\delta W = \frac{m}{2} \mathrm{Tr} \Phi_4^2 . \tag{3.81}$$

This leads to a β–function

$$\beta_m = m(2d_4 - 2) . \tag{3.82}$$

Thus the IR values of the scaling dimensions are $d_4 = 1$, $d_1 = d_2 = d_3 = \frac{1}{3}$.

Let us compare with the limits of the flow solution written in the coordinates of section 3.7.2. The UV end of the flow is just $AdS_4 \times S^7$ and has a symmetry under the scaling (3.20), with $A = \frac{2r}{L}$. The radial coordinate on moduli space from section 6 is $\sqrt{q} \sim \exp(\frac{r}{L}) \sim (e^A)^{1/2}$ and so the scaling symmetry becomes

$$x \to \frac{1}{\alpha} x \qquad \sqrt{q} \to \alpha^{1/2} \sqrt{q}. \tag{3.83}$$

At the IR end of the flow $A \sim \frac{3^{3/4}r}{L}$ and $\sqrt{q} \sim \exp(\frac{r}{3^{1/4}L}) \sim (e^A)^{1/3}$. The scaling symmetry is

$$x \to \frac{1}{\alpha}x \qquad \sqrt{q} \to \alpha^{1/3}\sqrt{q}. \tag{3.84}$$

These are all in agreement with the scaling dimensions of the massless scalar fields derived above.

We can now consider the Kähler potential at either end of the flow, as we did in section 3.3.3. In three dimensions K should have scaling dimension 1. At the UV end of the flow, since $\sqrt{q}$ has scaling dimension 1/2, $K \sim q$ as before. For the IR end, $\sqrt{q}$ has scaling dimension 1/3, and so K should obey $K \sim q^{3/2}$. Let us compare this with the results we found in section 3.7, equation (3.72). We see that for $r \to -\infty$, $K \sim e^A$ and $q \sim (e^A)^{2/3}$. This implies $K \sim q^{3/2}$, matching the result from the classical scaling argument above. Note that the scaling argument does not give the Kähler potential for arbitrary r (where the scaling symmetry no longer holds).

For completeness, we note that it is possible to extract scaling dimensions for the UV and IR ends of the conifold flow of section 3.7.3. Since the large field limit of the flow gives the moduli space as the conifold, it is natural to assume that the dual field theory will be an orbifold, with the M2–branes at the origin. The conifold arises as the moduli space of vacua of a set of fields restricted by a D–term equation which is the defining equation (3.75) of the conifold. Then, a good description of the field theory on moduli space is in terms of fields A_i and B_j, as in ref. [58] (called X_i and Y_j in ref. [59]). There is an additional complex field, say Φ, which has a superpotential giving it a mass, which drives the flow. Carrying out the supergravity analysis as above, we find that the scaling dimensions of the A's and B's are 3/8 in the UV and 1/4 in the IR. These numbers are simply 3/4 times the dimensions of the fields in the other flow. A guess for the superpotential which couples all the fields is $W \sim \epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_iB_kA_jB_l\Phi)$, for which the β–function vanishes if Φ has dimension 1/2 in the UV, and 1 in the IR, the latter also fitting nicely with a mass term $m\mathrm{Tr}\Phi^2$.

3.8 Conclusions

In this chapter we have continued to investigate the physics of the moduli space of holographic gauge theories by probing with branes. We have succeeded in finding good coordinates in which to study the supergravity duals by putting the metric on moduli space for a brane probe into a manifestly Kähler form. One nice point in these calculations has been the way in which the first order supergravity equations which ensure the $\mathcal{N} = 1$ supersymmetry of the full supergravity geometry are precisely what is needed to ensure that the metric on moduli space is Kähler.

Furthermore, we have shown that by working in these natural coordinates we can give a simple derivation of the scaling dimensions of the massless chiral superfields. The usual method for calculating dimensions of fields from the dual supergravity involves linearising fluctuations about a given background and is computationally rather tedious. Also these analyses of the linearised fluctuations lead to predictions for the spectrum of gauge invariant composite operators rather than the basic chiral superfields.

We have studied a wide variety of flows preserving different numbers of supersymmetries and in different dimensions. Remarkably, we were able to find an exact expression for the Kähler potential in terms of the supergravity fields in every case. This is exciting since gauge theory results for this quantity at strong coupling were not previously available for theories with a low amount of supersymmetry.

It certainly seems worthwhile to study these results further and to try to extract the gauge theory predictions in their clearest form. In particular, the expression for the Kähler potential in terms of the supergravity fields is not directly related to an expression in terms of the basic fields in the gauge theory but rather relates the renormalisation group flow equations of the gauge theory coupling constants to that of the Kähler potential. This is because the supergravity scalars represent renormalisation group profiles of gauge theory couplings.

As an example of the way in which this works we recall the maximally supersymmetric Coulomb branch flows. In these examples, supersymmetry constrains the Kähler potential to have a very simple form in terms of the chiral superfields since the metric on moduli space is flat. However, it looks more complicated in terms of the supergravity scalars. We can of course change coordinates in this case to see that the supergravity reflects the simplicity of the gauge theory.

In another example, the two eleven dimensional lifts of the same four dimensional gauged supergravity of section 3.7 have identical Kähler potentials in terms of the supergravity fields, but represent quite different field theories.

In each of the examples we have considered, the Kähler potential satisfies a remarkably simple differential equation (3.16) (or (3.47), (3.48) in eleven dimensions). It would be very interesting to try to understand better the origins of this equation in the gauged supergravity and also its interpretation in the dual gauge theory.

Chapter 4

Superstars and Giant Gravitons

4.1 Introduction

In the previous chapters, we have seen that certain states of the boundary theory in gauge theory/ gravity dualities are well described by branes in the bulk. In particular we have studied the coulomb branch of gauge theories. The picture in the bulk is of branes breaking off from the stack which provides the background geometry. This is a very natural picture given the way in which the brane description was used to motivate the AdS/CFT correspondence. Below we review some arguments which led to the discovery of another class of bulk states which are described by branes - the giant gravitons.

In the simplest case of string theory on $AdS_5 \times S^5$, giant gravitons are compact spherical D3-branes stabilised against collapse by a magnetic coupling to the background RR field. Alternatively, they should have a microscopic description as non-commutative bound states of Kaluza-Klein states of the supergravity blowing-up in the presence of RR flux according to a version of the Myers mechanism. The details of the microscopic description in this case are not yet fully understood (although see [60, 61]) and here we will focus on the macroscopic description in terms of D3-branes.

The chapter is structured as follows. In the next section we review some field theory arguments which hint at the rôle of brane physics in describing certain states of string theory on $AdS_5 \times S^5$. Following that we review the original probe calculation of [62] which brought to light the existence of giant graviton states. We also discuss various extensions of this construction [63], [64] and [65].

Next we discuss the description of giant gravitons as supergravity solutions. This is based on work of [66] and our paper [3]. We identify supergravity solutions with the relevant masses and angular momenta to represent coherent states of giant gravitons and find that dipole moments in the form fields are present as expected for collections of spherical branes. These 'superstar' solutions were originally found when considering a consistent truncation of the type IIB supergravity theory dimensionally reduced to AdS_5 where they appear as the supersymmetric limits of charged black hole solutions. Once lifted back to the full ten–dimensional theory, the solutions carry internal momentum along the three commuting Killing angles on the five–sphere. In both five and ten dimensions, the supersymmetric solutions display naked singularities. However, within the context of ten–dimensional type IIB superstring theory, there is a physical interpretation in which the singularities are generated by distributions of giant gravitons.

We also study the analogous superstar solutions in M–theory compactified on $AdS_7 \times S^4$ and $AdS_4 \times S^7$. We show that these eleven–dimensional supergravity solutions can be interpreted as being sourced by distributions of giant gravitons which in this case are M2 or M5 branes.

4.2 The stringy exclusion principle

It remains an outstanding problem to provide a full string theory description of quantum gravity on $AdS_5 \times S^5$. In particular, it is not known even how to perform perturbative string theory in this background.

The SYM description is expected to be useful for small 't Hooft coupling, λ, and the small g_s, large N, large λ limit should be well-described by supergravity. As we have reviewed in chapter one, certain protected operators in the spectrum can be compared and found to match between these two descriptions.

It is straightforward to extend the analysis of the SYM spectrum to finite N. We can then use this as a guide to what we should expect to find in the spectrum of the bulk theory away from the supergravity limit. For simplicity we consider the same operators as in chapter one, i.e. operators of the form :

$$O_k = \mathrm{tr}\left(\mathrm{X}^{(i_1}\mathrm{X}^{i_2}\ldots\mathrm{X}^{i_k)}\right) - \mathrm{traces}. \tag{4.1}$$

For finite N these operators are not all algebraically independent. In particular, for $k > N$, O_k can be written as a sum of products of O_j with $j \leq k$. According to the interpretation of single trace operators as single particle states in the bulk[1], this suggests that the Kaluza-Klein towers of particles should truncate at $k = N$. This idea, first investigated in the context of AdS_3 black holes [68], is known as the stringy exclusion principle.

Attempts to explain this phenomenon in the string theory picture led to the discovery of giant graviton states. The problem is that a cutoff on Kaluza-Klein states with momentum of order N is difficult to understand from a supergravity perspective since N is encoded in the large-scale structure of the geometry (e.g. via the length scale L). The resolution is that particle states expand into branes which are extended objects and thus sensitive to the large scale geometry. As more and more momentum is added the branes expand further until they reach a maximum size given by the radius of the compact sphere, L. This mechanism can account for the kind of cutoff expected from the stringy exclusion principle. In the next section we review a brane probe calculation which gives the relation between Kaluza-Klein momentum and size for the expanded branes and produces a family of BPS states with the expected form of cutoff.

4.3 Giant gravitons

In this section we review the calculations by various author's [62–64] which show that there are stable expanded brane states in backgrounds of the form $AdS \times S$.

We consider the dynamics of a D3-brane in $AdS_5 \times S^5$. We shall work in global coordinates on the AdS space. The metric is :

[1]See [67] for a nice explanation of this interpretation and its limitations.

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2+\left(1+\frac{r^2}{L^2}\right)^{-1}dr^2+r^2d\tilde{\Omega}_3^2+L^2\left(d\theta^2+\cos^2\theta d\phi^2+\sin^2\theta d\Omega_3^2\right), \tag{4.2}$$

where $d\tilde{\Omega}_3^2$ is the metric on the three sphere $\tilde{S}^3$ at constant r and t in the AdS space and $d\Omega_3^2$ is the metric on the three sphere S^3 at constant θ and ϕ on the five-sphere. The utility of writing the metric in this way will become clear shortly. The self-dual RR five-form field strength is:

$$F^{(5)} = \frac{4}{L}\left(\epsilon_{AdS_5} + \epsilon_{S^5}\right), \tag{4.3}$$

where ϵ_{AdS_5} is the volume form on the AdS space and ϵ_{S^5} is the volume form on the five-sphere. We can use the coordinates introduced above to integrate this locally as :

$$F^{(5)} = dC^{(4)}_{electric} + dC^{(4)}_{magnetic} \tag{4.4}$$

where

$$C^{(4)}_{electric} = -\frac{r^4}{L}dt\wedge\epsilon_{\tilde{S}^3}, \qquad C^{(4)}_{magnetic} = L^4\sin^4\theta d\phi\wedge\epsilon_{S^3}. \tag{4.5}$$

The labels 'electric' and 'magnetic' which we have given to the two pieces of the four-form potential refer to the fact that the electric part couples to a static brane (owing to the presence of the dt term) whereas the magnetic part does not. In order for a brane to couple to the magnetic potential it will have to have some angular momentum on the five-sphere. In fact it is easy to see that a D3-brane wrapped on the S^3 and moving in the ϕ direction will couple to this potential.

Thus we consider an ansatz in which the brane is a spherical D3-brane at constant θ and moving in the ϕ direction. For consistency, it should also follow a geodesic in the AdS space and we choose the geodesic at $r = 0$. Finally, we can consistently set the gauge field on the brane to zero. The action for this brane probe is :

$$\begin{aligned} S &= -T_3\int d^4\sigma\sqrt{-det\left(\mathcal{P}(G)\right)} + T_3\int\mathcal{P}(C^{(4)}) \\ &= \frac{N}{L}\int dt\left(-\sin^3\theta\sqrt{1-\cos^2\theta L^2\dot{\phi}^2} + sin^4\theta L\dot{\phi}\right), \quad . \end{aligned} \tag{4.6}$$

where in the second line we have substituted in our ansatz for the brane motion and integrated over S^3. Also, we have substituted $N = T_3 V_3 L^4$ where V_3 is the volume of a unit S^3.

Since ϕ is a cyclic variable we replace it by the conjugate angular momentum P_ϕ and find a Hamiltonian:

$$\mathcal{H} = \frac{N}{L}\sqrt{p^2 + \tan^2\theta(p - \sin^2\theta)^2}, \tag{4.7}$$

where $p = P_\phi/N$. This Hamiltonian is a sum of squares and so it is straightforward to minimise over θ for fixed p. The minima are at $\theta = 0$ and $\sin^2\theta = p$. The first minimum corresponds to a pointlike configuration as can be seen by inspection of the metric at $\theta = 0$. The second minimum corresponds to an expanded giant graviton. Both minima have the same energy :

$$\mathcal{H} = P_\phi/L. \tag{4.8}$$

This probe calculation indicates that there are expanded brane states which are degenerate with massless particle states with the same angular momentum and energy. Since the expanded branes are compact and do not wrap a topologically non-trivial cycle, they carry no net D3-brane charge. Thus, all their quantum numbers match those of the Kaluza-Klein states and we expect to find mixing between the two. As expected, the expanded states form a finite family of BPS states with $P_\phi \leq N$ since $\sin^2\theta \leq 1$.

This calculation generalises to the case of spherical M2 brane and M5 brane probes in $AdS_7 \times S^4$ and $AdS_4 \times S^7$ respectively. (Note that the probe brane is the magnetic dual of the branes which form the background since the probe couples to the magnetic potential of the background.) These calculations are carried out explicitly in the references [62,63].

For M2 brane probes of $AdS_7 \times S^4$ there are degenerate minima at $\theta = 0$ and $\sin\theta = P_\phi/N$, where θ, ϕ are the corresponding angles on the four-sphere[2]. Mean-

[2]We pick coordinates on a D-dimensional sphere such that the metric is $d\Omega_D^2 = d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\Omega_{D-2}^2$

while for M5 brane probes of $AdS_4 \times S^7$, the degenerate minima are at $\theta = 0$ and $\sin^4\theta = P_\phi/N$. In all cases the BPS relation is given by eqn. (4.8).

4.3.1 Giant gravitons from holomorphic surfaces

In fact, there is a beautiful geometrical construction of a more general class of giant gravitons, due to Mikhailov [65]. The idea is to think of the S^5 of $AdS_5 \times S^5$ as being embedded in $R^6 \simeq C^3$. We think of C^3 as a cone over S^5 and then Killing spinors on S^5 are given by parallel spinors on C^3 [69]. There are a large class of supersymmetric branes in C^3 given by holomorphic surfaces since such a surface is calibrated[3] by $\omega \wedge \omega$ where ω is the Kähler form on C^3. The idea is then that these holomorphic surfaces should lead to supersymmetric brane configurations on the S^5 base of the cone.

The details of the construction are contained in [65] and we just present the result here. Supersymmetric giant gravitons on $AdS_5 \times S^5$ are given as follows. They are the intersection of holomorphic surfaces in C^3 with the S^5 base, rotating at the speed of light in the direction $\sum_i \frac{\partial}{\partial\phi_i}$, given by acting with the complex structure on the radial direction of the cone.

It is easy to see that the giant gravitons we have considered are a special case in which the holomorphic surface is a flat complex two plane. In this case the intersection with the S^5 is a three-sphere. Furthermore, since the ϕ_2 and ϕ_3 directions lie completely within the spatial world volume of the brane, rotation at the speed of light in the $\sum_i \frac{\partial}{\partial\phi_i}$ direction reduces to rotation at less than the speed of light in the transverse ϕ_1 direction.

We note that if we chose a null time coordinate in the $\frac{\partial}{\partial t} + \sum_i \frac{\partial}{\partial\phi_i}$ direction the giant gravitons would be static branes. It would then presumably be possible to modify the set up of [72] to directly describe these as generalized calibrations.

[3]See refs. [70, 71] for a discussion of the connection between calibrations and supersymmetry for branes.

4.3.2 Giant gravitons as baryons

The construction of the previous section shows that there are rather a wide variety of supersymmetric giant graviton states in the bulk theory. This calls into question the idea that these states should be dual to the single trace operators O_k described previously. In fact a more detailed investigation suggests that these are not precisely the correct operators to consider.

The first thing to note is that the construction of the previous section generalises immediately to other supersymmetric geometries which arise as the near-horizon limits of branes at conical singularities. In order for such a near-horizon geometry to be supersymmetric, the cone must be Calabi-Yau [58, 69, 73]. In this case holomorphic surfaces are once again calibrating and the whole construction of giant graviton states can be precisely repeated.

Examples of such supersymmetric near-horizon geometries are $AdS_5 \times T^{11}$ and various orbifold examples. In these cases, some of the giant gravitons, corresponding to branes wrapping topologically non-trivial cycles, had already been studied independently and had been proposed to correspond to baryon operators in the boundary field theory [74–76]. This was the clue for the proposals of [67, 77, 78] to identify giant gravitons in general with baryon-type operators in the field theory.

This proposal has now passed a number of tests. In particular these operators carry the correct R-charges and fall into finite families with cutoffs to match the momentum bound on giant gravitons. Indeed a detailed matching of baryon states and supersymmetric giant graviton states has been performed [78]. Also these operators have been shown to be orthogonal at large N, even for large values of the R-charge and are thus reasonable candidates for the duals of single brane states [67].

It may be a little confusing that we have introduced two sets of operators - the trace operators O_k and now some baryon operators - in the context of giant gravitons. The point is that these operators are different choices of bases for the chiral operators in the gauge theory. It is to be expected that the stringy exclusion principle should hold whatever choice of basis we make for the chiral operators. The O_k states are a good description of supergravity modes of low angular momentum. Meanwhile, the baryon states are a good description of the expanded giant gravitons. There is no contradiction here, since the expanded brane description and the supergravity mode description are valid in complementary regimes - the branes are a good description so long as they are large and so carry large momentum whilst the supergravity perturbations are a good description for small angular momentum. Since there is no good semi-classical description which interpolates between these two regimes it is not suprising that we find ourselves working in two different bases of operators in the dual gauge theory.

4.4 Giant gravitons in supergravity

For the remainder of the chapter we discuss a complementary way of studying giant gravitons in the bulk. If Kaluza-Klein particles are expanding into brane states this should be reflected in the supergravity solutions. In particular, we would expect to see dipole moments of the RR five-form field strength excited by the presence of expanded D3-branes. A first signal of this can be seen in the analysis of the linearised perturbations of supergravity around $AdS_5 \times S^5$. In contrast with perturbations about flat space, the mass eigenstates in this case are mixtures of graviton and five-form modes. This suggests that branes rather than particles are being produced. (A similar effect was discussed by Polchinski and Strassler [79] in investigating the role of dielectric branes in the description of $\mathcal{N} = 1*$ vacua.)

In the following sections we discuss solutions to the full non-linear supergravity equations which display the expected properties of giant gravitons.

4.5 The Superstar solutions

Here we review some charged black hole solutions of gauged supergravity theories in four, five and seven dimensions and their lifts to ten dimensional type IIB and eleven dimensional supergravity. These solutions have supersymmetric limits which generally display naked singularities from which they get the name superstars. In the following sections we will describe the interpretation of these singularities as corresponding to coherent states of giant gravitons in IIB string theory and M-theory.

4.5.1 Superstars in $AdS_5 \times S^5$

As discussed previously, type IIB string theory has a spontaneous compactification on $AdS_5 \times S^5$ and there is believed to be a consistent truncation of the theory down to a five dimensional gauged supergravity with $\mathcal{N} = 8$ supersymmetry and $SO(6)$ gauge group corresponding to the isometries of the five-sphere. Here we consider a further truncation of this theory to a five-dimensional $\mathcal{N} = 2$ supergravity theory with $U(1)^3$ gauge group. This gauge group is the maximal abelian subgroup of $SO(6)$. An advantage of using this theory is that the lift ansatz to ten dimensions is explicitly known [80]. Moreover, the $U(1)^3$ gauge symmetry corresponds to an unbroken $U(1)^3$ isometry of the ten-dimensional theory which allows us to consider objects with conserved angular momenta along three commuting killing angles on the five-sphere. These objects will be giant gravitons.

The bosonic fields of the five-dimensional $\mathcal{N} = 2$ supergravity with gauge group $U(1)^3$ are the metric, three $U(1)$ gauge fields A^i $(i = 1,2,3)$ and two scalar fields which are usefully parametrized as X_i obeying $X_1X_2X_3 = 1$.

The Lagrangian for the theory is

$$e^{-1}\mathcal{L} = R - \frac{1}{2}\sum_i (X_i^{-1}\partial X_i)^2 + \frac{4}{L^2}\sum_i X_i^{-1} - \frac{1}{4}\sum_i X_i^{-2}(F^i_{(2)})^2 - \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F^1_{\mu\nu}F^2_{\rho\sigma}A^3_\lambda \tag{4.9}$$

which leads to the equations of motion:

$$d(X_1^{-2} *_{(1,4)} F^1) = -F^2 \wedge F^3 \quad \text{and cyclic permutations,} \tag{4.10}$$

$$d(X_i^{-1} *_{(1,4)} dX_i) = \sum_j M_{ij}[\frac{4}{L^2}\epsilon_{(1,4)}X_j^{-1} - X_j^{-2}F^j \wedge *_{(1,4)}F^j], \tag{4.11}$$

where $M_{ij} = \delta_{ij} - 1/3$.

The five-dimensional $\mathcal{N} = 2$ supergravity admits charged AdS black hole solutions [81, 82] which in the extremal limit can be written in the form

$$ds^2_{1,4} = -(H_1H_2H_3)^{-2/3}f dt^2 + (H_1H_2H_3)^{1/3}(f^{-1}r^2dr^2 + d\Omega_3^2), \tag{4.12}$$

$$A^i = -q_iH_i^{-1}dt \qquad i = 1,2,3, \tag{4.13}$$

$$X_i = H_i^{-1}(H_1 H_2 H_3)^{1/3}, \tag{4.14}$$

where we have introduced

$$f = r^4 + \frac{1}{L^2} H_1 H_2 H_3, \tag{4.15}$$

$$H_i = r^2 + q_i - q_3, \tag{4.16}$$

The q_i's are the three $U(1)$ charges and without loss of generality we have chosen $q_1 \geq q_2 \geq q_3 \geq 0$. In this extremal limit there is a naked singularity at $r = 0$.

The mass of this black hole is [81]

$$M = \frac{\pi}{4G_5} \sum q_i. \tag{4.17}$$

The ten dimensional lift ansatz is [80] [4]:

$$ds_{10}^2 = \Delta^{1/2} ds_{1,4}^2 + \Delta^{-1/2} \sum_i X_i^{-1} (L^2 d\mu_i^2 + \mu_i^2 [L d\phi_i + A^i]^2) \tag{4.18}$$

and $F_{(5)} = G_{(5)} + *G_{(5)}$ where

$$\begin{aligned} G_{(5)} &= \frac{2}{L} \sum_i (X_i^2 \mu_i^2 - \Delta X_i) \epsilon_{(1,4)} + \frac{L}{2} \sum_i X_i^{-1} *_{(1,4)} dX_i \wedge d(\mu_i^2) \\ &+ \frac{L}{2} \sum_i X_i^{-2} d(\mu_i^2) \wedge [L d\phi_i + A^i] \wedge *_{(1,4)} F_{(2)}^i \end{aligned} \tag{4.19}$$

Here, $ds_{1,4}^2$ and $\epsilon_{(1,4)}$ are the metric and volume form of the five dimensional geometry and $*_{(1,4)}$ denotes the Hodge dual with respect to the five dimensional metric. Δ is given by

$$\Delta = \sum_i X_i \mu_i^2. \tag{4.20}$$

The three quantities μ_i are non-negative real variables subject to the constraint $\sum_i \mu_i^2 = 1$ and we parametrize them by

$$\mu_1 = \cos\theta_1, \;\; \mu_2 = \sin\theta_1 \cos\theta_2, \;\; \mu_3 = \sin\theta_1 \sin\theta_2. \tag{4.21}$$

where $0 \leq \theta_1 \leq \pi/2$ and $0 \leq \theta_2 \leq \pi/2$. The angles ϕ_i are unconstrained with $0 \leq \phi_i < 2\pi$. In other words each pair $\{\mu_i, \phi_i\}$ are planar polar coordinates and the constraint on the μ_i describes the embedding of a unit S^5 in flat $\mathbf{R}^6$.

[4]The lift in the case without the scalar fields (i.e. $X_i = 1$) was first found in [83].

Finally the mass of the superstar solution can be written in terms of ten dimensional quantities as:

$$M = \frac{N^2}{2L^3} \sum_i q_i. \tag{4.22}$$

This follows from eqn. (4.17) and the standard relations between the five-dimensional Newton's constant and ten-dimensional quantities (see [66]).

4.5.2 Superstars in $AdS_7 \times S^4$

There is a similar story for eleven-dimensional supergravity compactified on $AdS_4 \times S^7$ and $AdS_7 \times S^4$. The S^4 compactification results in a consistent truncation to an $\mathcal{N} = 4$ supersymmetric gauged $SO(5)$ supergravity in seven dimensions [84, 85]. Here we consider a truncation of this theory to an $\mathcal{N} = 2$ supergravity theory with gauge group $U(1)^2$, the Cartan subgroup of $SO(5)$. The bosonic fields of this theory are the metric, two $U(1)$ gauge fields A^i $(i = 1, 2)$ and two scalar fields X_i (see refs. [86, 87] for more details).

The seven-dimensional $\mathcal{N} = 2$ supergravity admits charged AdS black hole solutions which in the extremal limit can be written in the form [88],

$$ds^2_{1,6} = -(H_1H_2)^{-4/5} f dt^2 + (H_1H_2)^{1/5} \left(f^{-1}dr^2 + r^2 d\Omega_5^2\right), \tag{4.23}$$

$$A^i = \left(H_i^{-1} - 1\right) dt \qquad i = 1, 2\,, \tag{4.24}$$

$$X_i = (H_1H_2)^{2/5} H_i^{-1}, \tag{4.25}$$

where we have introduced

$$f = 1 + \frac{r^2}{4L^2} H_1 H_2, \tag{4.26}$$

$$H_i = 1 + \frac{q_i}{r^4}, \tag{4.27}$$

and the q_i's are the charges of the superstar.

The mass of this asymptotically AdS_7 solution is [89]

$$M = \frac{\pi^2}{4G_7} \sum_i q_i. \tag{4.28}$$

The eleven dimensional lift ansatz is [86, 87]

$$ds^2_{11} = \Delta^{1/3} ds^2_{1,6} + \Delta^{-2/3} \left(\frac{1}{X_0} L^2 d\mu_0^2 + \sum_{i=1}^{2} \frac{1}{X_i} \left[L^2 d\mu_i^2 + \mu_i^2 \left(L d\phi_i + A^i\right)^2\right]\right) \tag{4.29}$$

for the metric, and

$$*F_{(4)} = -\frac{2}{L}\sum_{\alpha=0}^{2}\left(X_\alpha^2\mu_\alpha^2 - \Delta X_\alpha\right)\epsilon_{(1,6)} - \frac{1}{L}\Delta X_0\epsilon_{(1,6)} - \frac{L}{2}\sum_{\alpha=0}^{2}\frac{1}{X_\alpha} *_{(1,6)} dX_\alpha \wedge d\mu_\alpha^2$$
$$-\frac{L^2}{2}\sum_{i=1}^{2}\frac{1}{X_i^2}d\mu_i^2 \wedge \left(d\phi_i + A^i/L\right) \wedge *_{(1,6)}F_{(2)}^i, \quad (4.30)$$

for the field strength of the supergravity three–form. The four–sphere is parametrized by embedding it into R^5. We split R^5 into two orthogonal two–planes parametrized by $\{\mu_i, \phi_i\}$ $(i = 1, 2)$ and a real line parametrized by μ_0. The equation defining the sphere is $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The ϕ_i's are the two Killing angles on the sphere and we reparametrize the μ_i's by

$$\mu_0 = \sin\theta_1 \sin\theta_2 \quad \mu_1 = \cos\theta_1 \quad \mu_2 = \sin\theta_1\cos\theta_2 \ , \quad (4.31)$$

The $*_{(1,6)}$ denotes the Hodge dual with respect to the seven–dimensional metric $ds_{1,6}^2$. Also $X_0 \equiv (X_1X_2)^{-1}$ and $\Delta = \sum_{\alpha=0}^2 X_\alpha\mu_\alpha^2$.

Finally, the mass of the superstar (4.28) can be re-expressed in terms of eleven dimensional quantities as:

$$M = \frac{N^3}{24L^5}\sum_i q_i, \quad (4.32)$$

where N is the number of four form flux quanta through the four-sphere.

4.5.3 Superstars in $AdS_4 \times S^7$

Finally we consider charged black hole solutions of $\mathcal{N} = 2$ $U(1)^4$ gauge theory in four dimensions which is a consistent truncation of the $\mathcal{N} = 8$ $SO(8)$ theory arising from eleven-dimensional supergravity compactified on $AdS_4 \times S^7$.

The bosonic fields of the four dimensional $\mathcal{N} = 2$ supergravity are the metric, four scalars labelled by X_i $(i = 1, 2, 3, 4)$ $(X_1X_2X_3X_4 = 1)$ and four one–form gauge fields A^i. (see, refs. [86, 87] for more details).

This theory admits quadruply charged AdS black hole solutions which in the extremal limit take the form [87, 90],

$$ds_{1,3}^2 = -\left(H_1H_2H_3H_4\right)^{-1/2} f dt^2 + \left(H_1H_2H_3H_4\right)^{1/2}\left(f^{-1}dr^2 + r^2 d\Omega_2^2\right), \quad (4.33)$$

$$A^i = \left(H_i^{-1} - 1\right)dt \qquad i = 1, 2, 3, 4 \ , \quad (4.34)$$

$$X_i = \left(H_1H_2H_3H_4\right)^{1/4} H_i^{-1}, \tag{4.35}$$

where we have introduced

$$f = 1 + \frac{4r^2}{L^2} H_1H_2H_3H_4, \tag{4.36}$$

$$H_i = 1 + \frac{q_i}{r}. \tag{4.37}$$

The mass of this AdS_4 black hole is [89]

$$M = \frac{1}{4G_4} \sum_i q_i. \tag{4.38}$$

The eleven dimensional lift ansatz which may be used to lift four–dimensional solutions to solutions of the eleven–dimensional supergravity equations of motion, is [86, 87]

$$ds_{11}^2 = \Delta^{2/3} ds_{1,3}^2 + \Delta^{-1/3} \sum_{i=1}^{4} \frac{1}{X_i} \left(L^2 d\mu_i^2 + \mu_i^2 \left(L d\phi_i + A^i \right)^2 \right) \tag{4.39}$$

for the metric, and

$$F_{(4)} = \frac{2}{L} \sum_{i=1}^{4} \left(X_i^2 \mu_i^2 - \Delta X_i \right) \epsilon_{(1,3)} + \frac{L}{2} \sum_{i=1}^{4} \frac{1}{X_i} *_{(1,3)} dX_i \wedge d\mu_i^2$$
$$- \frac{L^2}{2} \sum_{i=1}^{4} \frac{1}{X_i^2} d\mu_i^2 \wedge \left(d\phi_i + A^i/L \right) \wedge *_{(1,3)} F_{(2)}^i, \tag{4.40}$$

for the field strength of the supergravity three–form. The seven sphere is parametrized by seven angular variables. We start by embedding the sphere into R^8 with coordinates μ_i, ϕ_i in four orthogonal two–planes. The ϕ_i's are Killing angles and the other angular variables are defined through

$$\mu_1 = \cos\theta_1 \quad \mu_2 = \sin\theta_1 \cos\theta_2 \quad \mu_3 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \quad \mu_4 = \sin\theta_1 \cos\theta_2 \cos\theta_3, \tag{4.41}$$

such that $\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1$. Also $\Delta = \sum_{i=1}^4 X_i \mu_i^2$.

Finally the mass of the superstar (4.38) may be re-expressed in terms of eleven dimensional quantities as:

$$M = \frac{4\sqrt{2} N^{3/2}}{3L^2} \sum_i q_i, \tag{4.42}$$

where N is the number of seven-form flux quanta through the seven-sphere.

4.6 Interpreting the superstar singularities

In this section we would like to understand the physics of the lifts to ten and eleven dimensions of the superstar geometries. We shall see that the naked singularities have various properties which suggest that they can be interpreted as particular distributions of giant gravitons.

4.6.1 D3 brane superstars

First we shall study the lift to ten dimensions of the five-dimensional superstar of equations (4.12)–(4.14). For simplicity we restrict to the case where only a single charge q_1 is non-zero. Inspecting the form of the lift ansatz for the metric (4.18) we see that the solution has angular momentum in the ϕ_1 direction coming from the term:

$$ds_{10}^2 = \Delta^{-1/2} L^2 \frac{\mu_1^2}{X_1} [L d\phi_1 + A^1]^2 + \dots \tag{4.43}$$

with $A^1 \to -(q_1/r^2)dt$ asymptotically. This is the first piece of evidence that the geometry contains giant gravitons. We can see more generally that the presence of electric charge for the gauge field A^i in a five-dimensional solution will lead to angular momentum in the ϕ_i direction in ten-dimensions.

Next we look at the lift ansatz for the RR five-form field strength (4.19). This time the relevant piece to pick out is the term:

$$F_{(5)} = \frac{L}{2} \sum_i X_i^{-2} *_{(1,4)} F_{(2)}^i \wedge d(\mu_i^2) \wedge [L d\phi_i + A^i] + \dots \tag{4.44}$$

In the absence of electric sources in five dimensions, the equations of motion (4.10), (4.11) imply that $dF_{(5)} = 0$. However, if we have a source term on the right hand side of the five-dimensional equation of motion (4.10) then this will produce a source in ten-dimensions. Explicitly, for the singly charged superstar:

$$d(X_1^{-2} *_{(1,4)} F^1) = -2q_1 \delta(r) dr \wedge d^3\Omega, \tag{4.45}$$

corresponding to an electric charge of strength $\sim q_1$ at $r = 0$.

Inspecting the form of $F_{(5)}$ from eqn. (4.44) we see that in ten dimensions we get a non-zero integral from:

$$\int_{\mathcal{M}_5} F_{(5)} \sim q_1, \tag{4.46}$$

for a surface $\mathcal{M}_5$ which is extended in the μ_1, ϕ_1 directions on the five-sphere and on a three-sphere enclosing $r = 0$ in the AdS space. These are the directions transverse to a spherical D3-brane wrapping the θ_2, ϕ_2 and ϕ_3 directions on the five-sphere at $r = 0$ in the AdS space. We conclude that charged sources in five-dimensions are lifted to collections of spherical D3-branes in ten dimensions.

In fact we can do rather better and find the precise distribution of giant gravitons on the S^5. In order to detect the presence of D3-brane charge we need to choose a compact six-dimensional ball, B_6, which intersects with the D3-brane source in a single point. The number of D3-branes captured in this way is given by the charge integral:

$$n_1 = \frac{1}{16\pi G_{10} T_3} \int_{\mathcal{M}_5} F_{(5)}, \tag{4.47}$$

where $\mathcal{M}_5 \equiv \partial B_6$.

We expect that the source is a collection of spherical D3-branes wrapping the θ_2, ϕ_2, ϕ_3 directions on the five–sphere at $r = 0$ and so the transverse spatial directions are $r, \theta_1, \phi_1, \alpha_1, \alpha_2$ and α_3. We can choose a six-dimensional compact surface which intersects the source as follows. First of all the surface can span the S^3 parametrized by the α_i's. We can also choose the surface to span the compact ϕ_1 direction. In fact, since ϕ_1 is a Killing direction, we expect the source branes to be distributed uniformly along this circle and we will be counting the total number of branes around the circle.

Finally we need to choose a two-dimensional compact surface in the r, θ_1 directions. We could choose to integrate over the θ_1 direction and bound the region at finite r. This would count the total number of source branes. However, the branes are distributed non-uniformly in the θ_1 direction. In order to find the precise form of this distribution we choose the following integration surface: $\theta_1^0 \leq \theta_1 \leq \theta_1^0 + \delta\theta_1$, $r \leq r_0$, where $\delta\theta_1$ is arbitrarily small and r_0, θ_1^0 are constants. This surface only intersects branes near $\theta_1 = \theta_1^0$.

In order to find the number of D3-branes we now need to perform the flux integral (4.47) over the boundary of the region which we have specified. This boundary lies in the $\phi_1, \alpha_i, i = 1, 2, 3$ directions and splits into three pieces in the r, θ_1 space. Two of these pieces are at $\theta_1 = \theta_1^0$ and $\theta_1 = \theta_1^0 + \delta\theta_1$ and run in the r direction. The flux integral through these directions vanishes as the appropriate term in $F_{(5)}$ is zero. The non-zero contribution to the flux integral comes from the boundary at $r = r_0$ and gives:

$$\frac{dn_1}{d\theta_1} = \frac{N}{4\pi^3 L^4} \int_{S^3\{\alpha_i\}\times S^1\{\phi_1\}} \iota_{\frac{\partial}{\partial\theta_1}} F_{(5)} = N\frac{q_1}{L^2} \sin 2\theta_1. \tag{4.48}$$

where ι denotes the interior product of a vector with a form. We can also integrate over θ_1 to find the total number of giant gravitons:

$$n_1 = N\frac{q_1}{L^2}. \tag{4.49}$$

To find the total angular momentum carried by giant gravitons we can use the result $P_1 = N\sin^2\theta_1$ for probe giant gravitons in $AdS_5 \times S^5$. We find that the total angular momentum of the geometry carried by giant gravitons is:

$$P_1^{total} = \int d\theta_1 P_1(\theta_1)\frac{dn_1}{d\theta_1} = \frac{N^2}{2}\frac{q_1}{L^2}, \tag{4.50}$$

This is equal [66] to the total angular momentum carried by the five-dimensional geometry [81]. The simplest way to see this is to use the BPS relation (4.8) to relate the angular momentum of the giant gravitons to their energy. We find that the total energy of the giant gravitons is:

$$E^{total} = \frac{N^2}{2L^3}q_1. \tag{4.51}$$

which agrees with the formula (4.22) for the mass of the superstar geometries. This lends support to the proposal that the superstar singularities are sourced by giant gravitons.

In the next chapter we will provide some probe calculations in the superstar background to give further support for this. In particular we will derive the result $P_1 = N\sin^2\theta_1$ for probes in the *superstar* background as opposed to $AdS_5 \times S^5$. This amounts to taking into account the backreaction of the branes and is an important step in justifying the approach we have taken.

We can generalize the calculations of this section to the case of the multi-charged superstar. The proposal is that the source is composed of three types of spherical giant gravitons with momenta around the ϕ_1, ϕ_2 and ϕ_3 directions. We find the distribution of each type of giant graviton individually as for the single-charge background. The result for the total angular momentum carried by each set of giant gravitons is simply:

$$P_i^{total} = \frac{N^2}{2}\frac{q_i}{L^3} \tag{4.52}$$

and the total mass of giant gravitons is given by adding up the contributions from the three sets:

$$E^{total} = \frac{N^2}{2L^3}\sum_i q_i. \tag{4.53}$$

Once again this agrees with the mass formula (4.22) for the superstar background.

4.6.2 M2 brane superstars

Here we repeat the analysis of the previous section for the $AdS_7 \times S^4$ superstars. We begin for simplicity by considering superstars that have a single non-zero charge, q_1. The lift ansatz for the metric (4.29) assures that this solution has angular momentum along ϕ_1.

$$ds_{11}^2 = \Delta^{-2/3}L^2\frac{\mu_1^2}{X_1}[Ld\phi_1 + A^1]^2 + \dots \tag{4.54}$$

where $A^1 \to -(q_1/r^4)dt$ asymptotically.

The dipole moment excited by the M2-brane giant gravitons is given by the term:

$$*F_{(4)} = \frac{L}{2}\sum_i X_i^{-2} *_{(1,6)} F_{(2)}^i \wedge d(\mu_i^2) \wedge [Ld\phi_i + A^i] + \dots \tag{4.55}$$

For the singly charged superstar:

$$d(X_1^{-2} *_{(1,6)} F^1) = -2q_1\delta(r)dr \wedge d^5\Omega, \tag{4.56}$$

corresponding to an electric charge of strength q_1 at $r = 0$.

Inspecting the form of $*F_{(4)}$ from eqn. (4.55) we see that in eleven dimensions we get a non-zero integral

$$\int_{\mathcal{M}_7} *F_{(4)} \tag{4.57}$$

for a surface $\mathcal{M}_7$ which is extended in the μ_1, ϕ_1 directions on the four-sphere and on a five-sphere enclosing $r = 0$ in the AdS space. We conclude that charged sources in four-dimensions are lifted to collections of spherical M2-branes carrying angular momentum in ten dimensions.

Once again, we can find the total number and distribution of giant gravitons on the four-sphere. We need to use the relation:

$$n_1 = \frac{1}{16\pi G_{11} T_{M2}} \int_{\mathcal{M}_7} *F_4, \tag{4.58}$$

where G_{11} is the eleven-dimensional gravitational constant, T_{M2} the tension of an M2-brane and n_1 the total number of giant gravitons enclosed.

Proceeding as before, we choose a compact eight-dimensional surface which intersects with the M2-brane sources. The surface is extended over the five-sphere spanned by the α_i's and also along the ϕ_1 direction over which the branes are uniformly distributed. It is chosen to include a finite range of r, $0 \leq r \leq r_0$ and an arbitrarily small range of θ_1, $\theta_1^0 \leq \theta_1 \leq \theta_1^0 + \delta\theta_1$. Performing the flux integral through the boundary of this region we find:

$$\frac{dn_1}{d\theta_1} = \frac{q_1 N^2}{8L^4} \cos\theta_1 \sin\theta_1. \tag{4.59}$$

According to the test-brane analysis in an $AdS_7 \times S^4$ background [62], the size and angular momentum of a giant graviton are related by $P_{\phi_1} = N \sin\theta_1$. Combining this with eq. (4.59) and integrating over θ_1, we find the total angular momentum of the distribution:

$$P_1^{total} = N \int_0^{\pi/2} d\theta_1 \sin\theta_1 \frac{dn_1}{d\theta_1} = \frac{1}{24} \frac{q_1 N^3}{L^4}. \tag{4.60}$$

Using the BPS relation between momentum and energy (4.8) we find the following result for the total energy of the giant gravitons:

$$E^{total} = \frac{1}{24} \frac{q_1 N^3}{L^5}. \tag{4.61}$$

This agrees with the mass of the superstar geometry given in equation (4.32).

It is straightforward to generalize the calculation to the case of superstars with two non-vanishing charges: $q_1 \neq 0$ and $q_2 \neq 0$. The total angular momentum carried by each set of giant gravitons is:

$$P_i^{total} = \frac{1}{24} \frac{q_i N^3}{L^4} \tag{4.62}$$

and the total energy of the giant gravitons is:

$$E^{total} = \frac{1N^3}{24L^5}\sum_i q_i. \tag{4.63}$$

Once again these results agree with the supergravity calculation of the superstar mass (4.32) and provide further evidence that the superstars are built out of giant gravitons.

4.6.3 M5 brane superstars

Finally we turn to the analysis of the superstars in $AdS_4 \times S^7$. Following the same route as in the previous two sections, we start by considering superstars with only one non-zero charge, q_1. Inspection of the asymptotic geometry:

$$ds_{11}^2 = \Delta^{-1/3}L^2\frac{\mu_1^2}{X_1}[Ld\phi_1 + A^1]^2 + \dots, \tag{4.64}$$

with $A^1 \to -(q_1/r)dt$ asymptotically, tells us that there is angular momentum in the ϕ_1 direction.

Furthermore, the lift ansatz for $F_{(4)}$ contains a piece:

$$F_{(4)} = \frac{L}{2}\sum_{i=1}^{4} X_i^{-2} *_{(1,3)} F_{(2)}^i \wedge d\mu_i^2 \wedge \left[Ld\phi_i + A^i\right].+\dots \tag{4.65}$$

For the single-charged superstar:

$$d(X_1^{-2} *_{(1,3)} F^1) = -2q_1\delta(r)dr\wedge d^2\Omega, \tag{4.66}$$

corresponding to an electric charge of strength q_1 at r=0. According to eqn. (4.65) this will lift to a collection of M5-brane giant gravitons in eleven dimensions, whose spatial world-volume is the five-sphere parametrized by θ_2, θ_3, ϕ_2, ϕ_3, ϕ_4. To see the details of this we compute the number and distribution of giant gravitons on the seven-sphere.

An M5–brane is magnetically charged with respect to the supergravity three–form potential. We consider a compact five-dimensional region which intersects the source M5-branes and is spanned by the two-sphere with coordinates α_i, the circle spanned by ϕ_1 and the wedge $0 \le r \le r_0$, $\theta_1^0 \le \theta_1 \le \theta_1^0 + \delta\theta_1$. The flux through the surface is proportional to the number of M5–branes enclosed,

$$n_1 = \frac{1}{16\pi G_{11} T_{M5}} \int_{M_4} F_4, \tag{4.67}$$

where G_{11} is the eleven–dimensional gravitational constant, T_{M5} is the tension of an M5–brane and n_1 is the total number of giant gravitons making the superstar.

Performing this flux integral for the region we have specified, with $F_{(4)}$ given by eqn. (4.65), we find the following distribution of branes in the θ_1 direction:

$$\frac{dn_1}{d\theta_1} = \frac{8q_1 N^{1/2}}{\sqrt{2}L} \cos\theta_1 \sin\theta_1. \tag{4.68}$$

Next we calculate the energy of the above distribution. The test–brane analysis of [62] gives the relation between the angular momentum of a giant graviton and its position on the sphere as:

$$P_{\phi_1} = N \sin^4\theta_1. \tag{4.69}$$

Combining this result with eq. (4.68), we find the total angular momentum of the distribution,

$$P_1^{total} = N \int_0^{\pi/2} d\theta_1 \sin\theta_1 \frac{dn_1}{d\theta_1} = \frac{4\sqrt{2}}{3} \frac{q_1 N^{3/2}}{L}. \tag{4.70}$$

On applying the BPS relation between angular momentum and mass (4.8) we find a total energy :

$$E^{total} = \frac{4\sqrt{2}}{3} \frac{q_1 N^{3/2}}{L^2}. \tag{4.71}$$

This agrees with the energy eq. (4.42) of the corresponding superstar. This is evidence that the asymptotically $AdS_4 \times S^7$ superstars are indeed composed of M5-brane giant gravitons.

We can easily generalize to the case of multi-charge superstars. The total angular momentum carried by each set of giant gravitons is

$$P_i^{total} = \frac{4\sqrt{2}}{3} \frac{q_i N^{3/2}}{L}, \tag{4.72}$$

corresponding to a total energy

$$E^{total} = \frac{4\sqrt{2} N^{3/2}}{3L^2} \sum_i q_i. \tag{4.73}$$

Once again this agrees with the supergravity result.

4.7 Discussion

In this chapter we have described various aspects of the physics of giant gravitons. We reviewed their description in terms of probe branes and as states of the boundary field theory. We then identified certain supersymmetric geometries as corresponding to the fields external to a collection of giant gravitons. These superstar geometries contain naked singularities and we have argued that giant gravitons have the correct properties to act as sources for these.

An important omission from our analysis was a proper derivation of the momentum size relations for giant gravitons in the superstar backgrounds. Furthermore, to complete the analysis and identify the superstars as solutions of supergravity coupled to the worldvolume actions of the giant gravitons, we should investigate the physics of probe branes in the superstar backgrounds. This is the subject to which we turn in the next chapter.

Chapter 5

Probing with giant gravitons

A well known example of a consistent truncation in supergravity is the reduction of the $\mathcal{N} = 1$ theory in eleven dimensions on a circle. This produces type IIA supergravity in ten dimensions. The ten dimensional theory has a solution corresponding to a fundamental string which when lifted to eleven dimensions becomes a membrane wrapped on the circle. A complementary result is that a membrane *probe* of the eleven dimensional lift ansatz, which is wrapped on the circle, behaves like a fundamental string probe of the type IIA theory [91].

In the previous chapter, we found that the extremal limits of charged black hole solutions in five dimensional $\mathcal{N} = 2$ $U(1)^3$ gauged supergravity lift to collections of giant gravitons in ten dimensions. The (extremal) charged black hole solutions represent the external fields sourced by a massive charged particle. In this chapter we investigate giant graviton probes in ten dimensions and we ask whether such a probe of the ten dimensional lift ansatz might have an expression as a charged particle probe of the five-dimensional theory.

On the one hand this seems a reasonable expectation. We start with a D3-brane action which couples in a supersymmetric way to the type IIB supergravity. The lift ansatz preserves a $U(1)^3$ symmetry (corresponding to the three commuting Killing angles ϕ_i) and so we can consider probes which carry a non-zero, conserved angular momentum under one of these $U(1)$'s. A probe carrying such an angular momentum on the sphere [1] will carry a $U(1)$ charge from a five-dimensional perspective. We might expect that we can minimize the energy of a D3-brane probe with a fixed angular momentum by partially solving the equations of motion in the compact directions whilst leaving the motion in the remaining five non-compact dimensions undetermined. If so this should produce a five dimensional action for a massive charged particle which couples supersymmetrically to the five dimensional supergravity.

On the other hand, such a result will require a good deal of luck. In order for the ten-dimensional problem to admit a general solution for minimal energy configurations on the five-sphere whilst leaving the *AdS* dynamics completely unspecified requires a number of remarkable 'conspiracies' in the structure of the truncation ansatz. In fact this is what happens and for minimal energy giant gravitons with a given angular momentum, the ten dimensional dynamics simplifies to that of a charged five-dimensional particle probe.

This then is the main result of the present chapter - a giant graviton probe of the ten-dimensional lift ansatz behaves as a charged particle probe of the five-dimensional gauged supergravity. We do not show explicitly that this particle probe action is the bosonic part of a superparticle action. However, we observe a couple of facts which suggest that this is the case. In particular, the charge of the particle is equal to its mass in agreement with the BPS relation (4.17) for the extremal superstar solutions. Furthermore, the scalar field couplings to the probe are of the correct form to source the superstar geometry.

We see that the lift ansatz encodes more than just the embedding of one supergravity theory into another. It also encodes the relation between the branes of the two theories. It secretly contains an ansatz for lifting solutions of the lower dimensional supergravity, dynamically coupled to a particle, to solutions of the higher dimensional system of supergravity coupled to a brane.

As a warm up to the main result, we also consider massless particle probes of the ten-dimensional lift ansatz. We take massless particles with angular momentum

[1] Note that for brevity we sometimes refer loosely to the non-compact part of these geometries as the '*AdS*' directions and the compact part as the 'sphere' directions.

on the five-sphere which are expected to be degenerate with giant gravitons at least in the pure $AdS_5 \times S^5$ background. We find that these particles also behave like charged particles in five-dimensions with the same action as the minimal energy giant gravitons. Thus we confirm that the degeneracy between massless particles and giant gravitons persists in more complicated backgrounds.

As a corollary of our giant graviton probe calculations, we can confirm that the relation between momentum and size for the giant gravitons - namely $P = N \sin^2 \theta$ - which we used in the previous chapter, persists in any ten-dimensional solution which arises as a lift of the five-dimensional gauged supergravity. This fills in a crucial step in the logic of the previous chapter in which we argued that giant gravitons could correctly account for all of the angular momentum/energy of the superstar geometries.

We can go further and probe the superstar backgrounds with giant gravitons with a view to finding out whether the branes which are supposed to source the singularities can in fact reach that region of the geometry. These probe calculations are very straightforward since they can be performed entirely within the context of the five-dimensional solutions. We simply have to probe the five-dimensional superstars with charged particle probes[2].

Our results are somewhat surprising. In the case of the singly charged superstars, the probes can be brought all the way in to the singularity where they sit at a minimum of their potential energy. In this way we see that the coupled equations of motion of the gravity/brane system are solved by a collection of giant gravitons in the single charge background. However, for the generic multi-charge geometries, the singularities appear to be repulsive - at least at the level of our analysis. This casts some doubt over their interpretation as physical solutions sourced by branes, although it is possible that curvature corrections could alter this conclusion.

As a possible resolution of this puzzle, we consider so-called dual giant gravitons as probes of the superstar geometries. Since these probes carry the same angular momenta and masses as the ordinary giants it might be possible that they play a rôle

[2]This case is in contrast to accepted lore that singularities of lower dimensional supergravities can only be explored properly within the context of their ten/eleven dimensional lifts.

in resolving the superstar singularities. We find that there always exist dual giants at BPS minima in the superstar backgrounds and speculate that the singularities expand into configurations of dual giants.

5.1 A massless particle probe

We start by considering the physics of a massless particle probe in a ten-dimensional geometry of the form given in equation (4.18), i.e. in a geometry which arises as a lift of a solution of the $\mathcal{N}=2$ $U(1)^3$ gauged supergravity in five-dimensions. Such a ten-dimensional geometry has a $U(1)^3$ isometry and so we may consider a particle with conserved angular momenta P_i, $i=1,2,3$ along the three killing angles ϕ_i. [3]

Our strategy is to partially solve the equations of motion for such a particle probe by making an ansatz that the particle should have no kinetic energy in the S^5 directions beyond that which is related to the motion in the ϕ_i directions. In other words, we set $\dot{\theta}_1 = \dot{\theta}_2 = 0$ and then solve for the values of θ_1, θ_2 such that the equations of motion are satisfied.

It is a non-trivial statement that we should be able to do this without specifying the form of the five-dimensional fields or the motion of the probe in the non-compact space. In fact, these details decouple form the θ_1, θ_2 equations of motion under our ansatz and we find a simple five-dimensional action describing the effective dynamics in the non-compact directions.

For convenience we start with the action for a massive particle in ten-dimensions and later take the mass m to zero.

$$S = -m \int dt \sqrt{-\det(\mathcal{P}(g))}, \tag{5.1}$$

where $\mathcal{P}(g)$ is the pullback of the spacetime metric onto the particle's world line and is given by

$$\mathcal{P}(g) = g_{MN} \dot{X}^M \dot{X}^N. \tag{5.2}$$

Here X^M are coordinates on ten-dimensional space whilst we shall refer to coordinates on the five-dimensional non-compact space as x^μ. $x^0 \equiv t$ and $\dot{X}^M$ is the

[3]See section 4.5 for definitions of the S^5 coordinates.

derivative of X^M wrt. t. The metric g_{MN} is given by equation (4.18).

We consider a particle with angular momenta in the ϕ_i directions but stationary in the θ_1, θ_2 directions. The Lagrangian becomes:

$$\mathcal{L} = -m\sqrt{-\Delta^{1/2}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) - \Delta^{-1/2}\sum_i X_i^{-1}\mu_i^2(L\dot{\phi}_i + A^i_\mu\dot{x}^\mu)^2} \qquad (5.3)$$

Recall that Δ defined in equation (4.20) depends on the five-dimensional scalar fields X_i as well as the sphere coordinates θ_1, θ_2.

Since all the fields in the lift ansatz are independent of ϕ_i, the action we have written down has no explicit ϕ_i dependence and we can replace $\dot{\phi}_i(t)$ with conjugate momenta P_i which are conserved in time. Since we have put no restrictions on the motion in the five non-compact directions and we have set $\dot{\theta}_i = 0$ it is convenient to remain in a Lagrangian formulation for these variables. The resulting Routhian is found to be[4]:

$$\mathcal{R} = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\left(\sum_i \frac{P_i^2 X_i \Delta}{L^2\mu_i^2}\right)^{1/2} - \frac{1}{L}\sum_i \dot{x}^\mu A^i_\mu P_i. \qquad (5.6)$$

where we have now taken the limit $m \to 0$.

We need to find the minimum of the energy with respect to the μ_i. If we define

[4]We will be required to perform several Legendre transforms of this type and it will be useful to have to hand a result which covers all the cases at once. Suppose we have a Lagrangian of the form:

$$\begin{aligned} \mathcal{L} &= \frac{N}{L}\left[-YX^{1/2} + Z + \sum LW_i\dot{\phi}_i\right] \\ X &= A - \sum L^2 B_i\left(\dot{\phi}_i - \frac{1}{L}C_i\right)^2 \end{aligned} \qquad (5.4)$$

where A, B_i, C_i, W_i, X, Y and Z are functions of r and the other angles, but not of $\dot{\phi}_i$. The Hamiltonian (or Routhian) reached by performing a Legendre transform on $\dot{\phi}_i$ is:

$$\mathcal{H} = \frac{N}{L}\left[A^{1/2}\left(\sum\frac{(p_i - W_i)^2}{B_i} + Y^2\right)^{1/2} + \sum C_i(p_i - W_i) - Z\right] \qquad (5.5)$$

where $p_i = P_{\phi_i}/N$.

vectors **U** and **V** by:

$$\begin{aligned} U_i &= \sqrt{\frac{X_i}{\mu_i^2}\frac{P_i}{L}} \text{ for i = 1,2,3} \\ V_i &= \sqrt{X_i\mu_i^2} \text{ for i = 1,2,3} \end{aligned}$$

Then

$$\begin{aligned} \left(\sum_i \frac{P_i^2 X_i \Delta}{L^2\mu_i^2}\right)^{1/2} &= |\mathbf{U}||\mathbf{V}| \\ &\geq \mathbf{U}.\mathbf{V} \qquad \text{(Schwarz inequality)} \end{aligned}$$

with equality iff. **U** and **V** are parallel. Thus the minimum of the energy occurs at $\mu_i^2 = (P_i/\sum_j P_j)$, taking into account the constraint $\sum_i \mu_i^2 = 1$.

If we now set the μ_i's to these values we find a remarkable simplification in the Lagrangian for motion in the remaining directions. The resulting charged particle Lagrangian in five dimensions is:

$$\mathcal{L} = \frac{1}{L}\sum_i \left(-P_i X_i \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \dot{x}^\nu A_\nu^i P_i\right). \tag{5.7}$$

We emphasize once more that in the calculation above, the choice of five dimensional geometry (and background fields) and the motion of the particle probe in the five non-compact directions remains arbitrary throughout. It is perhaps rather surprising that it is possible to minimize the energy of a massless probe in the compact space independently of these details. Presumably this is illustrating some very special properties of the lift ansatz. It would be interesting to understand this in more detail.

In the following sections we turn our attention to probe calculations involving giant gravitons. We have already seen that in $AdS_5 \times S^5$, the physics of giant gravitons mirrors that of massless particles. If this is to happen for giant graviton probes of the more general class of solutions, which arise as lifts from five-dimensions, it will require even more special structure in order to bring about the necessary cancellations and decouple the physics of the branes on the sphere from the details of the dynamics in the remaining directions.

5.2 The R–R four-form potential

If we wish to probe the lift ansatz of eqns. (4.18) (4.19), with a giant graviton, we are presented with a new technical difficulty which was not present in the particle probe computation. We need to know the expression for the R-R four-form potential to which a D3-brane couples. The lift ansatz is presented in terms of $G_{(5)}$, a part of the self-dual five-form field strength $F_{(5)} = G_{(5)} + *G_{(5)}$ with indices mostly in the non-compact space. Since the giant graviton spans a three-surface inside the S^5, we will need the explicit expression for $*G_{(5)}$ also. Furthermore we need to integrate $F_{(5)}$ to produce an expression for the four-form potential $C_{(4)}$ satisfying $dC_{(4)} = F_{(5)}$. Details of these rather lengthy computations are presented in this section. The most important results are found at the end of the section starting with equation (5.24).

5.2.1 From $G_{(5)}$ to $*G_{(5)}$

In order to dualise $G_{(5)}$ we will need to dualise several forms in ten-dimensions which split into a p-form, $\alpha_{(p)}$, in the AdS directions and a q-form, $\beta_{(q)}$, in the sphere directions. The following result will be useful:

$$*_{(10)}(\alpha \wedge \beta) = (-)^{q(5-p)} \Delta^{(q-p)/2} (*_{(1,4)}\alpha \wedge *_{(5)}\beta), \tag{5.8}$$

where $*_{(1,4)}$ and $*_{(5)}$ refer to Hodge duals taken in the AdS directions and the sphere directions respectively.

The remaining difficulty in dualising $G_{(5)}$ resides in the fact that the sphere metric is given in terms of constrained variables (the μ_i's.) Consider R^3 spanned by the μ_i's (without the constraint $\sum_i \mu_i^2 = 1$) and with metric:

$$ds_3^2 = \sum_i X_i^{-1} d\mu_i^2 \tag{5.9}$$

Let S be the surface given by $\sum_i \mu_i^2 = 1$ and denote the restriction of this metric to S by ds_2^2 . Suppose that e_1, e_2, e_3 are a vielbein for R^3 with the metric ds_3^2, such that $e_3 = \Delta^{-1/2} \sum_i \mu_i d\mu_i$, i.e. e_3 is a unit normal covector to S. Then the following identity allows us to dualise forms within the surface S in a straightforward fashion:

$$*_{(2)}\alpha = *_{(3)}(e_3 \wedge \alpha). \tag{5.10}$$

Thus, for example

$$Vol_S = *_{(2)}1 = *_{(3)}\Delta^{-1/2}\sum_i \mu_i d\mu_i = \Delta^{1/2}W, \quad (5.11)$$

$$*_{(2)}d(\mu_i^2) = *_{(3)}\Delta^{-1/2}\sum_j \mu_j d\mu_j \wedge d(\mu_i^2) = -2\Delta^{-1/2}\sum_j X_i X_j \mu_i \mu_j Z_{ij} \quad (5.12)$$

where

$$W = \frac{1}{2}\sum_{ijk} \epsilon_{ijk}\mu_i d\mu_j \wedge d\mu_k = \sin\theta_1 d\theta_1 \wedge d\theta_2 \quad (5.13)$$

is the volume form on the sphere $\sum_i \mu_i^2 = 1$ embedded in flat R^3 and $Z_{ij} = \sum_k \epsilon_{ijk} d\mu_k$.

It is now relatively simple to dualise $G_{(5)}$ as given by equation (4.19). We find:

$$\begin{aligned} *G_{(5)} = & - L\frac{2U}{\Delta^2}W \bigwedge_i \mu_i[Ld\phi_i + A^i] \\ & - \frac{L}{\Delta^2}\sum_{ij} X_j dX_i \wedge Z_{ij}\mu_i\mu_j \bigwedge_l \mu_l[Ld\phi_l + A^l] \\ & - \frac{L}{\Delta}\sum_{ij} F^i_{(2)} \wedge Z_{ij}\mu_j X_j \bigwedge_{l\neq i} \mu_l[Ld\phi_l + A^l] \end{aligned}$$

where $U = \sum_i (X_i^2\mu_i^2 - \Delta X_i)$. After a little more algebra we find:

$$\begin{aligned} *G_{(5)} = & -\frac{2}{L\Delta^2}\sum_i (X_i^2\mu_i^2 - \Delta X_i)L^2 W \bigwedge_k \mu_k[Ld\phi_k + A^k] \\ & -\sum_i \partial_\nu\left(\frac{X_i\mu_i}{\Delta}\right) dx^\nu \wedge LZ_i \bigwedge_k \mu_k[Ld\phi_k + A^k] \\ & +\frac{L}{\Delta}\sum_{ij} X_i\mu_i Z_{ij} \bigwedge_{k\neq j} \mu_k[Ld\phi_k + A^k] \wedge F^j_{(2)} \end{aligned} \quad (5.14)$$

where $Z_i = \epsilon_{ijk}\mu^j d\mu^k$. For one-forms $d\phi_i$ our notation is:

$$\bigwedge_{k\neq i} d\phi_k = \frac{1}{2}\sum_{jk} \epsilon_{ijk} d\phi_j \wedge d\phi_k$$

5.2.2 Integrating $F_{(5)}$

We now have the complete expression for $F_{(5)}$ and we can proceed to integrate to find $C_{(4)}$. Our tactic will be to guess pieces of the expression for $C_{(4)}$, differentiate and subtract from $F_{(5)}$. When differentiating expressions involving the fields of the five-dimensional theory, we may use the five-dimensional equations of motion. This reflects the fact that the Bianchi identity for the self-dual five-form, $dF_{(5)} = 0$, is only satisfied modulo these equations of motion. We encountered a similar phenomenon when integrating the five-form potential in chapter 2.

We start by evaluating

$$\begin{aligned}
& d\left(\frac{L}{2}\sum_i \mu_i^2 \left[X_i^{-1} *_{(1,4)} dX_i + X_i^{-2}[Ld\phi_i + A^i] \wedge *_{(1,4)}F^i_{(2)}\right]\right) \\
= \; & \frac{L}{2}\sum_i d(\mu_i^2) \wedge X_i^{-1} *_{(1,4)} dX_i + \frac{L}{2}\sum_i \mu_i^2 d(X_i^{-1} *_{(1,4)} dX_i) \\
& + \frac{L}{2}\sum_i d(\mu_i^2) \wedge X_i^{-2}[Ld\phi_i + A^i] \wedge *_{(1,4)}F^i_{(2)} + \frac{L}{2}\sum_i \mu_i^2 X_i^{-2} F^i_{(2)} \wedge *_{(1,4)}F^i_{(2)} \\
& - \frac{L}{2}\sum_i \mu_i^2 [Ld\phi_i + A^i] \wedge d(X_i^{-2} *_{(1,4)} F^i_{(2)})
\end{aligned} \tag{5.15}$$

We can use the five-dimensional equations of motion (4.10) and (4.11) to replace the second and fifth terms. We find:

$$\begin{aligned}
& d\left(\frac{L}{2}\sum_i \mu_i^2 \left[X_i^{-1} *_{(1,4)} dX_i + X_i^{-2}[Ld\phi_i + A^i] \wedge *_{(1,4)}F^i_{(2)}\right]\right) \\
= \; & \frac{L}{2}\sum_i d(\mu_i^2) \wedge X_i^{-1} *_{(1,4)} dX_i + \frac{2}{L}\sum_{ij} \mu_i^2 M_{ij} \epsilon_{(1,4)} X_j^{-1} \\
& + \frac{L}{6}\sum_i X_i^{-2} F^i_{(2)} \wedge *_{(1,4)}F^i_{(2)} + \frac{L}{2}\sum_i d(\mu_i^2) \wedge X_i^{-2}[Ld\phi_i + A^i] \wedge *_{(1,4)}F^i_{(2)} \\
& + \frac{L}{2}\sum_i \mu_i^2 [Ld\phi_i + A^i] \bigwedge_{k\neq i} F^k_{(2)}
\end{aligned} \tag{5.16}$$

It is now easy to check, using the identity

$$U = \sum_i (X_i^2 \mu_i^2 - \Delta X_i) = \sum_i X_i^{-1}(\mu_i^2 - 1)$$

that this differs from the expression for $G_{(5)}$ given in equation (4.19) by

$$-\frac{4}{3L}\sum_i X_i^{-1}\epsilon_{(1,4)} - \frac{L}{2}\sum_i \mu_i^2 [Ld\phi_i + A^i] \bigwedge_{k\neq i} F^k_{(2)} - \frac{L}{6}\sum_i X_i^{-2} F^i_{(2)} \wedge *_{(1,4)} F^i_{(2)}$$

We next evaluate

$$
\begin{aligned}
& d\left(-\sum_i \left(\frac{X_i\mu_i}{\Delta}\right) LZ_i \bigwedge_k \mu_k[Ld\phi_k + A^k]\right) \\
= & -\sum_i \partial_\nu \left(\frac{X_i\mu_i}{\Delta}\right) dx_\nu LZ_i \bigwedge_k \mu_k[Ld\phi_k + A^k] \\
& -L\sum_i \left(\frac{X_i}{\Delta}(\mu_i^2 - 1) - \frac{X_i\mu_i}{\Delta^2}\sum_j 2\mu_j X_j(\mu_i\mu_j - \delta_{ij})\right) W \bigwedge_k \mu_k[Ld\phi_k + A^k] \\
& -2LW \bigwedge_k \mu_k[Ld\phi_k + A^k] + \left(\frac{1}{\Delta}\sum_i X_i - 3\right) LW \bigwedge_k \mu_k[Ld\phi_k + A^k] \\
& +\sum_{ij} \left(\frac{X_i\mu_i}{\Delta}\right) LZ_i \bigwedge_{k\neq j} \mu_k[Ld\phi_k + A^k] \wedge \mu_j F^j_{(2)} \\
= & -\sum_i \partial_\nu \left(\frac{X_i\mu_i}{\Delta}\right) dx_\nu LZ_i \bigwedge_k \mu_k[Ld\phi_k + A^k] - \left(4 + \frac{2U}{\Delta^2}\right) LW \bigwedge_k \mu_k[Ld\phi_k + A^k] \\
& +\sum_{ij} \left(\frac{X_i\mu_i}{\Delta}\right) LZ_i\mu_j \bigwedge_{k\neq j} \mu_k[Ld\phi_k + A^k] \wedge F^j_{(2)} \qquad (5.17)
\end{aligned}
$$

This expression can now be seen to differ from equation (5.14) for $*G_{(5)}$ by

$$4LW \bigwedge_k \mu_k[Ld\phi_k + A^k] - L\sum_i Z_i \bigwedge_{k\neq i} \mu_k[Ld\phi_k + A^k] \wedge F^i_{(2)}$$

As well as the equations of motion (4.11), we have used various useful identities which are listed below for convenience:

$$
\begin{aligned}
dZ_i &= 2\mu_i W && (5.18)\\
Z_i \wedge d\mu_j &= (\delta_{ij} - \mu_i\mu_j)W && (5.19)\\
Z_{ij}\mu_k + Z_{ki}\mu_j + Z_{jk}\mu_i &= 0 && (5.20)\\
Z_j - \frac{1}{\Delta}\sum_i X_i\mu_i Z_i\mu_j &= -\frac{1}{\Delta}\sum_i X_i\mu_i Z_{ij} && (5.21)
\end{aligned}
$$

So, adding together $G_{(5)}$ and $*G_{(5)}$, we see that

$$\begin{aligned} F_{(5)} = \quad & d\left(\frac{L}{2}\sum_i \mu_i^2\left[X_i^{-1} *_{(1,4)} dX_i + X_i^{-2}[Ld\phi_i + A^i]\wedge *_{(1,4)}F^i_{(2)}\right]\right) \\ + \quad & d\left(-\sum_i\left(\frac{X_i\mu_i}{\Delta}\right) LZ_i \bigwedge_k \mu_k[Ld\phi_k + A^k]\right) \\ - \quad & \frac{4}{3L}\sum_i X_i^{-1}\epsilon_{(1,4)} - \frac{L}{6}\sum_i X_i^{-2}F^i_{(2)}\wedge *_{(1,4)}F^i_{(2)} \\ - \quad & \frac{L}{2}\sum_i \mu_i^2[Ld\phi_i + A^i]\bigwedge_{k\neq i} F^k_{(2)} + 4LW\bigwedge_k \mu_k[Ld\phi_k + A^k] \\ - \quad & L\sum_i Z_i \bigwedge_{k\neq i}\mu_k[Ld\phi_k + A^k]\wedge F^i_{(2)} \end{aligned} \tag{5.22}$$

The first two terms of this expression have been written as the derivative of a globally well-defined potential (assuming that the A^i's are well-defined, i.e. there are no magnetic strings in five dimensions.) The next two terms are five forms on AdS_5 and so are manifestly closed, but their integrals will depend on the precise solution and are not needed here. The sum of the final three terms is closed but not exact. We can (partly) integrate these terms locally for $\mu_1 \neq 0$ to obtain the expression

$$d\left(-\frac{L}{2}\mu_2^2\bigwedge_{k\neq 3}[Ld\phi_k + A^k]\wedge F^3_{(2)} + \frac{L}{2}\mu_3^2\bigwedge_{k\neq 2}[Ld\phi_k + A^k]\wedge F^2_{(2)}\right)$$

$$+d\left(L\frac{Z_1}{\mu_1}\bigwedge_k \mu_k[Ld\phi_k + A^k]\right) - \frac{L}{2}[Ld\phi_1 + A^1]\wedge F^2_{(2)}\wedge F^3_{(2)}$$

Putting this together with equation (5.21) (for $j = 1$) we can write the five-form field strength in terms of a four-form potential, well-defined for $\mu_1 \neq 0$, plus terms which do not couple to the D3-branes we consider. Our final expression is

$$\begin{aligned} F_{(5)} = \quad & d\left(\frac{L}{2}\sum_i \mu_i^2\left[X_i^{-1} *_{(1,4)} dX_i + X_i^{-2}[Ld\phi_i + A^i]\wedge *_{(1,4)}F^i_{(2)}\right]\right) \\ + \quad & d\left(-\frac{L}{2}\mu_2^2\bigwedge_{k\neq 3}[Ld\phi_k + A^k]\wedge F^3_{(2)} + \frac{L}{2}\mu_3^2\bigwedge_{k\neq 2}[Ld\phi_k + A^k]\wedge F^2_{(2)}\right) \\ + \quad & d\left(\frac{L}{\Delta\mu_1}\sum_i X_i\mu_i Z_{1i}\bigwedge_k \mu_k[Ld\phi_k + A^k]\right) - \frac{L}{2}[Ld\phi_1 + A^1]\wedge F^2_{(2)}\wedge F^3_{(2)} \\ - \quad & \frac{4}{3L}\sum_i X_i^{-1}\epsilon_{(1,4)} - \frac{L}{6}\sum_i X_i^{-2}F^i_{(2)}\wedge *_{(1,4)}F^i_{(2)} \end{aligned} \tag{5.23}$$

From this expression it is easy to read off the pieces of $C_{(4)}$:

$$\begin{aligned} C_{(4)} &= \frac{L}{2}\sum_i \mu_i^2 \left[X_i^{-1} *_{(1,4)} dX_i + X_i^{-2}[Ld\phi_i + A^i] \wedge *_{(1,4)}F^i_{(2)}\right] \\ &\quad -\frac{L}{2}\mu_2^2 \bigwedge_{k\neq 3}[Ld\phi_k + A^k] \wedge F^3_{(2)} + \frac{L}{2}\mu_3^2 \bigwedge_{k\neq 2}[Ld\phi_k + A^k] \wedge F^2_{(2)} \\ &\quad +\frac{L}{\Delta\mu_1}\sum_i X_i\mu_i Z_{1i} \bigwedge_k \mu_k[Ld\phi_k + A^k] + \tilde{C}_{(4)} \end{aligned} \tag{5.24}$$

where $\tilde{C}_{(4)}$ is a four-form satisfying

$$d\tilde{C}_{(4)} = -\frac{4}{3L}\sum_i X_i^{-1}\epsilon_{(1,4)} - \frac{L}{6}\sum_i X_i^{-2}F^i_{(2)} \wedge *_{(1,4)}F^i_{(2)} - \frac{L}{2}[Ld\phi_1 + A^1] \wedge F^2_{(2)} \wedge F^3_{(2)} \tag{5.25}$$

In section 5.3 we will consider a D3-brane at constant θ_1 so the above expression is well-defined (for $\mu_1 = \cos\theta_1 \neq 0$) and after some simple manipulations, we find:

$$C^{(4)}_{\phi_1\theta_2\phi_2\phi_3} = -\frac{L^4}{\Delta}\sin^4\theta_1 \cos\theta_2 \sin\theta_2 \left(X_2\cos^2\theta_2 + X_3\sin^2\theta_2\right) \tag{5.26}$$

Finally in this section, we comment that our expression for $C_{(4)}$ makes manifest the fact that $F_{(5)}$ satisfies the Bianchi identity $dF_{(5)} = 0$. This is certainly a requirement for consistent truncation and was not previously demonstrated to our knowledge. In the reference [80] the Bianchi identity for $G_{(5)}$ is partly checked, but $*G_{(5)}$ is not found.

5.3 Giant graviton probes

In this section we consider probing the ten-dimensional lift ansatz with giant gravitons. We find that, as with the massless particle probes of the previous section, it is possible to find minimal energy configurations in the compact directions without specifying a particular five-dimensional solution or any particular motion of the probe in the five non-compact dimensions. As with the massless particle probes, the giant gravitons behave simply as massive charged particles in five dimensions.

Our giant graviton probe will be a D3-brane with the topology of an S^3 lying inside the S^5. More precisely, the brane will 'wrap' the θ_2, ϕ_2 and ϕ_3 directions whilst moving rigidly in the ϕ_1 direction at fixed θ_1. The motion of the probe in the non-compact directions remains arbitrary with the assumption that it is independent of θ_2, ϕ_2 and ϕ_3, i.e. we only consider rigid motion of the brane. While it is not initially obvious that this is a consistent way of embedding the brane, we will see that it does in fact give a minimal energy configuration. Specifically we find that the brane action reduces to a particle action in five dimensions – independent of θ_2, ϕ_2 and ϕ_3 and with θ_1 and $\dot{\phi}_1$ constant.

The action for the D3-brane probe is:

$$S_3 = -T_3 \int dt d\theta_2 d\phi_2 d\phi_3 \left[\sqrt{-\det(\mathcal{P}(g))} + \dot{x^\nu} C^{(4)}_{\nu\theta_2\phi_2\phi_3} + \dot{\phi}_1 C^{(4)}_{\phi_1\theta_2\phi_2\phi_3}\right] \tag{5.27}$$

Our first task in evaluating this action will be to find the pieces of the RR four-form potential $C^{(4)}$ which couple to the probe. It is straightforward to read off the relevant pieces of $C^{(4)}$ from equation (5.24). We find

$$\dot{x^\nu} C^{(4)}_{\nu\theta_2\phi_2\phi_3} + \dot{\phi}_1 C^{(4)}_{\phi_1\theta_2\phi_2\phi_3} = -\frac{L^3}{\Delta}\sin^4\theta_1\cos\theta_2\sin\theta_2\alpha\dot{\Phi} \tag{5.28}$$

where $\alpha = X_2\cos^2\theta_2 + X_3\sin^2\theta_2$ and $\dot{\Phi} \equiv L\dot{\phi}_1 + \dot{x^\nu} A^1_\nu$.

Combining this with the terms coming from the pullback of the metric the action becomes:

$$\begin{aligned} S_3 &= -T_3 L^3 \int dt d\theta_2 d\phi_2 d\phi_3 \left[\frac{\sin^3\theta_1}{\sqrt{\Delta}} X_1 \cos\theta_2 \sin\theta_2 \alpha^{1/2}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{\cos^2\theta_1}{X_1\Delta}\dot{\Phi}^2}\right. \\ &\left. -\frac{1}{\Delta}\sin^4\theta_1\cos\theta_2\sin\theta_2\alpha\dot{\Phi}\right] \end{aligned} \tag{5.29}$$

We stress that the action at this point appears to couple together the fields representing the position of the probe in the sphere directions with the position in the *AdS* directions in a very complicated way. In particular note the coupling between the angles θ_1, θ_2 and the five-dimensional scalar fields X_i in the terms involving Δ and α.

Since all the fields in the lift ansatz are independent of ϕ_1, the action we have written down has no explicit ϕ_1 dependence and we can replace $\dot{\phi}_1$ with a conjugate momentum $P_{\phi_1}(\theta_2, \phi_2, \phi_3)$ which is independent of time. The resulting Routhian is found to be:

$$\begin{aligned}
\mathcal{R} &= \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\left[\frac{X_1\Delta}{\cos^2\theta_1}\left(P_{\phi_1} - N\frac{\alpha\sin^4\theta_1\cos\theta_2\sin\theta_2}{\Delta V_3}\right)^2\right. \\
&\qquad \left. + N^2\frac{\sin^6\theta_1 X_1^2\alpha\cos^2\theta_2\sin^2\theta_2}{\Delta V_3^2}\right]^{1/2} - \frac{1}{L}\dot{x}^\nu A^1_\nu P_{\phi_1} \\
&= \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}X_1\left[P^2_{\phi_1} + \frac{\alpha}{X_1}\tan^2\theta_1(P_{\phi_1} - \frac{1}{V_3}N\sin^2\theta_1\cos\theta_2\sin\theta_2)^2\right]^{1/2} \\
&\qquad -\frac{1}{L}\dot{x}^\nu A^1_\nu P_{\phi_1}, \qquad (5.30)
\end{aligned}$$

where $N = V_3T_3L^4$ and $V_3 = 2\pi^2$ is the volume of a unit S^3 in $\mathbf{R}^4$.

Minimizing over θ_1 is now straightforward. The expanded minimum[5] occurs when $P_{\phi_1}(\theta_2, \phi_2, \phi_3) = \frac{1}{V_3}P_1\cos\theta_2\sin\theta_2$ for some constant P_1 and $N\sin^2\theta_1 = P_1$. This is the promised derivation of the momentum-size relation for expanded giant gravitons in these more general backgrounds, which was needed to fill in a missing step in the argument of the previous chapter. In section 5.6, we provide a derivation of the corresponding result for the eleven dimensional, single-charge superstars only, by considering these geometries directly rather than by probing the general lift ansatz. Probe calculations of the general lift ansatz in eleven dimensions along the lines of the calculations presented here for the ten-dimensional case are currently being considered in [92].

It is easy to check that our result for P_{ϕ_1} indeed implies that $\dot{\phi}_1$ is constant and thus that our ansatz is consistent. If we substitute these values back into the Routhian and integrate over θ_2, ϕ_2 and ϕ_3 the resulting particle Lagrangian is:

$$\mathcal{L} = \frac{1}{L}\left(-P_1X_1\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \dot{x}^\nu A^1_\nu P_1\right) \qquad (5.31)$$

It is remarkable that the complicated action of equation (5.29) should reduce to such a simple form when we consider minimal energy configurations in the compact directions. The crucial step is that the Routhian (5.30) rearranges into a sum of squares to make minimization over θ_1 simple. That this should occur independently

[5]There is also a zero-size minimum at $\theta_1 = 0$ corresponding to the massless particle probe of the previous section.

of the details of the five-dimensional solution which we are lifting and of the motion in the AdS directions suggests that our ansatz for the motion of the brane in the compact directions manages to preserve some supersymmetry in the five non-compact dimensions. Naturally, when we introduce a particular five-dimensional solution, supersymmetry may be completely broken.

Equation (5.31) is the same Lagrangian for a charged particle in five dimensions[6] which we saw in the previous section. The conclusion is that probing with the giant graviton is equivalent to probing with a massless particle in ten-dimensions and both are equivalent to probing with a charged massive particle in five-dimensions.

5.4 Superstars and giant gravitons

In this section we apply the giant graviton probe results, which we have found for general lifts of the five-dimensional supergravity, to study the ten-dimensional superstar geometries of [66, 80]. We discussed these geometries in some detail in the last chapter where we argued that they represent the supergravity backgrounds sourced by a collection of giant gravitons in $AdS_5 \times S^5$.

In order to identify the naked singularities in the superstar geometries as corresponding to a collection of giant gravitons we should check whether giant graviton probes minimize their energy at the singularity. We consider giant graviton probes carrying angular momentum in the ϕ_i direction. The results of the previous section show that it is equivalent to probe the five-dimensional charged black holes with the charged particle probe of equation (5.31). We insert the fields of equations (4.12)–(4.14) into the probe action (5.31) and look for a stationary solution ($\dot{x}^\mu = 0$ for $\mu \neq 0$.) The resulting energy of the probe is:

$$E_i = \frac{P_i}{L}\frac{f^{1/2} + q_i}{H_i} \tag{5.32}$$

[6]Our giant graviton probes only cover the case of a singly-charged particle, whereas the massless particles could carry charges under all three $U(1)$'s simultaneously. It would be interesting to understand whether the more general giant gravitons related to holomorphic surfaces might play a rôle here as multiply charged states.

The question we are interested in answering is whether the singularity $r = 0$ corresponds to a BPS minimum ($E_i = P_i/L$) for each type of probe. It turns out that there are several distinct cases to consider.

- $q_1 = q_2 = q_3 = 0$ – All three types of probe have a BPS minimum at $r = 0$ as expected since this is the case of pure $AdS_5 \times S^5$.

- $q_2 = q_3 = 0$ with q_1 non-zero – The probe coupling to A^1 has a BPS minimum while the probes coupling to A^2 or A^3 appear to have non-BPS minima ($E_{2,3} = \sqrt{1 + q_{2,3}/L^2}P_{2,3}/L$) at $r = 0$.

- $q_3 = 0$ with q_1 and q_2 non-zero – For the probes coupling to A^1 or A^2, the energy saturates the BPS bound at $r = 0$ but the gradient of the potential is non-zero indicating an attractive force at the singularity. For the probe coupling to A^3 the energy diverges as $r \to 0$ and there is an infinite repulsive force.

- All q_i non-zero – The singularity is repulsive to all three types of probe. Furthermore there is an infinite repulsive force on the probe coupling to A^3 (and on the probes coupling to A^i in the special cases $q_i = q_3$.)

The meaning of these results is not entirely clear. For a singly charged superstar, the fact that a probe with the same type of charge as the background has a BPS minimal energy configuration at the singularity agrees with the interpretation of the singularity as a collection of giant gravitons. However, in the other cases agreement is not reached. A possible resolution is that curvature corrections to the supergravity background and to the D3-brane action could modify the results in these less supersymmetric cases.

Another possibility would be that other states that carry the same angular momenta and energy as giant gravitons could appear in the bulk and change the interior geometry, removing the need for the superstar singularities. In the next section we take a first step towards understanding this possibility by probing [3,66] the superstars with so-called dual giants [63,64]. A next step would be to look for supergravity geometries with dual giants in the bulk. We haven't yet considered this in detail.

5.5 Dual giants in the superstar background

We search for solutions corresponding to expanded BPS states of dual giant graviton probes in the superstar geometries. These results first appeared in [66] although our method for finding the BPS solutions is original.

The idea of the dual giants is to couple a probe brane to the electric RR potential rather than the magnetic potential to which the ordinary giants couple. It turns out that there are once again expanded states with angular momentum and energy degenerate with the giant gravitons [63, 64]. A discussion of these states in a field theory context appears in [64].

For the asymptotically $AdS_5 \times S^5$ superstars, the relevant probe is a D3-brane wrapped on the S^3 at constant r in the AdS space and orbiting in the ϕ_i directions at fixed θ_1, θ_2. As with the massless particle probes we can consider a dual giant with three non-zero angular momenta.

The action for such a probe is:

$$S_3 = -T_3 \int dt d^3\alpha_i \left[\sqrt{-\det(\mathcal{P}(g))} + C^{(4)}_{t\alpha_1\alpha_2\alpha_3} + \dot{\phi}_1 C^{(4)}_{\phi_1\alpha_1\alpha_2\alpha_3}\right] \tag{5.33}$$

Substituting in the fields of the multi-charge superstar and integrating over the three-sphere over which the brane is wrapped, we find the following Lagrangian:

$$\mathcal{L} = \frac{\tilde{N}}{L}\left[-\Delta\sqrt{\frac{f}{\Pi} - \frac{1}{\Delta}\sum_i H_i\mu_i^2[L\dot{\phi}_i - \frac{q_i}{L^2 H_i}]^2} + \Delta + \sum_i \frac{\mu_i^2 q_i}{L^2}(L\dot{\phi}_i - 1)\right].$$

The corresponding Hamiltonian is:

$$\begin{aligned}\mathcal{H} &= \frac{N}{L}\left[\sqrt{\sum_i \frac{\mu_i^2}{H_i}\left(\frac{(r^2-q_3)^2}{L^4} + \Pi\right)}\sqrt{\frac{\sum_i (p_i - \frac{\mu_i^2 q_i}{L^2})^2}{H_i\mu_i^2} + \Pi\sum_i \frac{\mu_i^2}{H_i}}\right. \\ &\quad \left. -\sum_i \frac{(r^2-q_3)}{L^2 H_i}\left(p_i - \frac{\mu_i^2 q_i}{L^2}\right) - \Delta\right] + \frac{N}{L}\sum_i p_i \\ &= \frac{N}{L}\mathcal{H}_0 + \frac{N}{L}\sum_i p_i \end{aligned} \tag{5.34}$$

We will find BPS minima satisfying $\mathcal{H} = \frac{N}{L}\sum_i p_i$ whenever $\mathcal{H}_0$ vanishes. If we define vectors X and Y by:

$$\begin{aligned}
X_i &= \sqrt{\frac{\mu_i^2}{H_i}\frac{(r^2-q_3)}{L^2}} \quad \text{for i = 1,2,3} \\
X_4 &= \sqrt{\sum_j \frac{\mu_j^2}{H_j}\Pi} = \sqrt{\Delta} \\
Y_i &= \sqrt{\frac{1}{H_i\mu_i^2}}(p_i - \frac{\mu_i^2 q_i}{L^2}) \quad \text{for i = 1,2,3} \\
Y_4 &= \sqrt{\sum_j \frac{\mu_j^2}{H_j}\Pi} = \sqrt{\Delta}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{H}_0 &= |X||Y| - X.Y \\
&\geq 0 \qquad \text{(Schwarz inequality)}
\end{aligned}$$

with equality iff X and Y are parallel ($X = \lambda Y$.)

Inspecting the metric ansatz (4.18) we see that Δ controls the size of the sphere which the brane wraps and in particular expanded branes are given by $\Delta \neq 0$. In this case, the solutions of $\mathcal{H}_0 = 0$ are at r, μ_i given by:

$$\mu_i^2\frac{(r^2-q_3)}{L^2} = p_i - \frac{\mu_i^2 q_i}{L^2} \tag{5.35}$$

There may also be pointlike solutions with $\Delta = 0$. This can only happen at $r^2 = 0$ and thus these states cannot play a rôle in resolving the singularity.

We can find the radius at which the dual giants can appear in the bulk by summing the three equations in (5.35) over i. This gives:

$$\frac{r^2}{L^2} = \frac{q_3}{L^2} + \sum_i \left(p_i - \frac{\mu_i^2 q_i}{L^2}\right). \tag{5.36}$$

To understand whether these dual giants have a rôle in resolving the superstar singularities we would need to look for supergravity solutions which asymptotically approach the superstar solutions, with branes in the bulk to carry the angular momentum. It might then be possible to use the momentum-radius relation, derived here for the BPS dual giants, to perform an analysis similar to that involved in interpreting the superstar singularities.

5.6 Some eleven dimensional results

In this section we briefly report on some giant graviton probe calculations in eleven dimensions. We have not considered giant gravitons probes of the general eleven dimensional lift ansätze although we would expect similar results to carry across to these cases also (see [92]). However, we have considered giant graviton probes in the single charge superstar backgrounds and found that (a) the required size-momentum relations used in chapter 4 do indeed hold and (b) the branes can sit at the singularity and thus provide a physical source for the solution. This mimics the result for the single-charge solution in ten-dimensions.

We have also considered dual giant probes in the multi-charge eleven dimensional superstar backgrounds. Once again it is always possible to find BPS minima corresponding to expanded dual giants.

5.6.1 Probing superstars in $AdS_7 \times S^4$

In the following, we present a brief summary of the M2-brane probe calculations for the superstars presented in section 4.5. We will restrict our attention to the simplest case of a singly charged superstar (i.e. only q_1 is nonvanishing).

Our giant graviton probe will be a spherical M2-brane inside the S^4 at constant θ_1 and moving on a circle in the ϕ_1 direction. We will first consider this configuration at a finite radius in the anti–de Sitter space (i.e. away from the singularity) and then consider taking the test–brane to $r = 0$. The M2-brane couples to the three form potential $A^{(3)}$ satisfying $dA^{(3)} = F^{(4)}$. Hence to evaluate the worldvolume action, we need to dualize $*F^{(4)}$ given in eq. (4.30) and integrate to find $A^{(3)}$. The result is

$$\begin{aligned} F^{(4)} &= -\frac{1}{L}H_1\frac{(\Delta+2)}{\Delta^2}\sin^2\theta_1\cos\theta_1 Ld\theta_1\wedge\sin\theta_2\cos\theta_2 Ld\theta_2\bigwedge_j[Ld\phi_j+A^j_{(1)}] \\ &\quad +\dots \\ &= d\left(-H_1\frac{\sin^3\theta_1}{\Delta}\wedge\sin\theta_2\cos\theta_2 Ld\theta_2\bigwedge_j[Ld\phi_j+A^j_{(1)}]\right)+\dots \\ &= dA^{(3)}. \end{aligned} \tag{5.37}$$

The probe action is then

$$\begin{aligned} S_2 &= -T_{M2}\int dt d\theta_2 d\phi_2 \left[\sqrt{-g} - A^{(3)}_{t\theta_2\phi_2} - \dot{\phi}_1 A^{(3)}_{\phi_1\theta_2\phi_2}\right] \\ &= \frac{N}{L}\int dt \left[-\Delta^{-1/2}\sin^2\theta_1\left(\frac{f}{H_1} - \Delta^{-1}H_1\mu_1^2[L\dot{\phi}_1 + (H_1^{-1}-1)]^2\right)^{1/2} \right. \\ &\quad \left. + \frac{H_1}{\Delta}\sin^3\theta_1(L\dot{\phi}_1 + H_1^{-1} - 1)\right], \end{aligned} \tag{5.38}$$

Now fixing the momentum $p \equiv P_{\phi_1}/N$, we find the Hamiltonian

$$\mathcal{H} = \frac{N}{L}\left[\sqrt{\frac{f}{H_1}\left(\frac{p^2}{H_1} + \tan^2\theta_1(p - \sin\theta_1)^2\right)} + (1 - H_1^{-1})p\right]. \tag{5.39}$$

Minimizing $\mathcal{H}(r,\theta_1)$ with respect to θ_1 yields $\theta_1 = 0$ or $\sin\theta_1 = p$ for arbitrary values of r. However, to find a true solution of the equations of motion, we must also minimize with respect to the radius. Setting $\sin\theta_1 = p$, a short calculation shows that the minimum energy configuration is at r=0, and that the energy of this configuration satisfies the BPS relation:

$$\mathcal{H}(r=0, \sin\theta_1 = p) = \frac{N}{L}p = \frac{P_{\phi_1}}{L}. \tag{5.40}$$

5.6.2 Probing superstars in $AdS_4 \times S^7$

The four–form and metric corresponding to the superstars in $AdS_4 \times S^7$ were given in section 3. The probe which we shall be studying in this case is a spherical M5-brane inside the S^7 at constant θ_1 and moving on a circle in the ϕ_1 direction. Again, we begin by placing the probe away from the singularity and consider the limit as $r \to 0$. The M5-brane couples to the six–form potential $A^{(6)}$ satisfying $dA^{(6)} = *F^{(4)}$ and so our first task is to dualize $F^{(4)}$ given in eq. (4.40) and integrate to find $A^{(6)}$. We find

$$\begin{aligned} *F^{(4)} &= -\frac{2}{L}H_1\frac{(2\Delta+1)}{\Delta^2}\sin^5\theta_1\cos\theta_1 Ld\theta_1 \wedge \sin^3\theta_2\cos\theta_2 Ld\theta_2 \\ &\quad \wedge \sin\theta_3\cos\theta_3 Ld\theta_3 \bigwedge_j [Ld\phi_j + A^j_{(1)}] + \ldots \\ &= d\left(-H_1\frac{\sin^6\theta_1}{\Delta} \wedge \sin^3\theta_2\cos\theta_2 Ld\theta_2 \wedge \sin\theta_3\cos\theta_3 Ld\theta_3 \bigwedge_j [Ld\phi_j + A^j_{(1)}]\right) \\ &\quad + \ldots \\ &= dA^{(6)}. \end{aligned} \tag{5.41}$$

Evaluating the probe action for the configuration described above yields

$$\begin{aligned} S_5 &= -T_{M5}\int dtd\theta_2 d\theta_3 d\phi_2 d\phi_3 d\phi_4 \left[\sqrt{-g} + A^{(6)}_{t\theta_2\theta_3\phi_2\phi_3\phi_4} + \dot{\phi}_1 A^{(6)}_{\phi_1\theta_2\theta_3\phi_2\phi_3\phi_4}\right] \\ &= \frac{N}{L}\int dt\left[-\Delta^{-1/2}\sin^5\theta_1\left(\frac{f}{H_1} - \Delta^{-1}H_1\mu_1^2[L\dot{\phi}_1 + (H_1^{-1}-1)]^2\right)^{1/2}\right. \\ &\quad \left. + \frac{H_1}{\Delta}\sin^6\theta_1(L\dot{\phi}_1 + H_1^{-1} - 1)\right]. \end{aligned} \tag{5.42}$$

Again fixing $p \equiv P_{\phi_1}/N$, we find the Hamiltonian

$$\mathcal{H} = \frac{N}{L}\left[\sqrt{\frac{f}{H_1}\left(\frac{p^2}{H_1} + \tan^2\theta_1(p - \sin^4\theta_1)^2\right)} + (1 - H_1^{-1})p\right]. \tag{5.43}$$

Extremizing $\mathcal{H}$ with respect to θ_1 yields minima at $\theta_1 = 0$ and $\sin^4\theta_1 = p$ independent of the radius. The r-dependence of the Hamiltonian is essentially as before and we once again find a solution to the equations of motion at $r = 0$ satisfying the BPS relation :

$$\mathcal{H}(r = 0, \sin^4\theta_1 = p) = \frac{P_{\phi_1}}{L}. \tag{5.44}$$

5.6.3 Dual giants in $AdS_7 \times S^4$

We have also considered probing the M–theory superstars with dual giant graviton probes. In $AdS_7 \times S^4$, the dual giants are spherical M5-branes, spanning the S^5 of the AdS$_7$ space at constant r. They orbit on the S^4 at constant θ_1 and θ_2 with fixed angular momentum P_{ϕ_i} conjugate to the angles ϕ_i.

The probe Lagrangian takes the following form (after integrating over the 5-sphere):

$$\begin{aligned} \mathcal{L} &= \frac{\tilde{N}}{L}\left[-\Delta^{1/2}\left(\frac{r}{\tilde{L}}\right)^5\left(f\left(\mu_0^2 + \sum_i\frac{\mu_i^2}{H_i}\right) - \sum_i H_i\mu_i^2[L\dot{\phi}_i + (H_i^{-1}-1)]^2\right)^{1/2}\right. \\ &\quad \left. + \frac{r^6}{\tilde{L}^6}\Delta + \sum_i\frac{\mu_i^2 q_i}{\tilde{L}^4}(L\dot{\phi}_i - 1)\right] \end{aligned} \tag{5.45}$$

Here $\tilde{L} = 2L$ is the radius of the AdS space and $\tilde{N} = L\tilde{L}^2 A_2 T_{M2}$, where A_2 is the area of a unit S^2 and T_{M2} is the tension of an M2 brane. Fixing the value of

$\tilde{p}_i \equiv P_{\phi_i}/\tilde{N}$ we find the Hamiltonian:

$$\mathcal{H} = \frac{\tilde{N}}{L}\left[\sqrt{f\left(\mu_0^2+\sum_i\frac{\mu_i^2}{H_i}\right)\left(\sum_i\frac{(\tilde{p}_i-\frac{\mu_i^2 q_i}{\tilde{L}^4})^2}{H_i\mu_i^2}+\Delta\frac{r^{10}}{\tilde{L}^{10}}\right)}\right.$$
$$\left.+\sum_i(1-H_i^{-1})\left(\tilde{p}_i-\frac{\mu_i^2 q_i}{\tilde{L}^4}\right)-\Delta\frac{r^6}{\tilde{L}^6}+\sum_i\frac{\mu_i^2 q_i}{\tilde{L}^4}\right] \tag{5.46}$$

Expanded minima of this action saturating the BPS bound $\mathcal{H} = (N/L)\sum p_i$ occur whenever the following equations are satisfied:

$$\frac{r^4}{\tilde{L}^4}\mu_i^2 = \tilde{p}_i - \frac{q_i\mu_i^2}{\tilde{L}^4}. \tag{5.47}$$

5.6.4 Dual giants in $AdS_4 \times S^7$

In $AdS_4 \times S^7$, the giant gravitons are M5-branes whereas the dual giants are spherical M2-branes, spanning the S^2 of the AdS$_4$ space at constant r. They orbit on the S^7 at constant θ_1, θ_2 and θ_3 with fixed angular momentum P_{ϕ_i} conjugate to the angles ϕ_i.

The probe Lagrangian takes the following form (after integrating over the 2-sphere):

$$\mathcal{L} = \frac{\tilde{N}}{L}\left[-\Delta^{1/2}\frac{r^2}{\tilde{L}^2}\left(\sum_i\frac{\mu_i^2}{H_i}f-\sum_i H_i\mu_i^2[L\dot{\phi}_i+(H_i^{-1}-1)]^2\right)^{1/2}\right.$$
$$\left.+\frac{r^3}{\tilde{L}^3}\Delta+\sum_i\frac{\mu_i^2 q_i}{\tilde{L}}(L\dot{\phi}_i-1)\right] \tag{5.48}$$

Here we have introduced $\tilde{L} = L/2$ as the radius of the AdS space and $\tilde{N} = L\tilde{L}^5 A_5 T_5$ where A_5 is the area of a unit S^5 and T_{M5} is the tension of an M5-brane. Fixing the value of $\tilde{p}_i \equiv P_{\phi_i}/\tilde{N}$ we find the Hamiltonian:

$$\mathcal{H} = \frac{\tilde{N}}{L}\left[\sqrt{\left(\sum_i\frac{\mu_i^2}{H_i}f\right)\left(\sum_i\frac{(\tilde{p}_i-\frac{\mu_i^2 q_i}{\tilde{L}})^2}{H_i\mu_i^2}+\Delta\frac{r^4}{\tilde{L}^4}\right)}\right.$$
$$\left.+\sum_i(1-H_i^{-1})\left(\tilde{p}_i-\frac{\mu_i^2 q_i}{\tilde{L}}\right)-\Delta\frac{r^3}{\tilde{L}^3}+\sum_i\frac{\mu_i^2 q_i}{\tilde{L}}\right] \tag{5.49}$$

Expanded minima of this action saturating the BPS bound $\mathcal{H} = (N/L)\sum p_i$ occur whenever the following equations are satisfied:

$$\frac{r}{\tilde{L}}\mu_i^2 = \tilde{p}_i - \frac{q_i \mu_i^2}{\tilde{L}}. \tag{5.50}$$

5.7 Conclusions

We have considered giant gravitons probing solutions of type IIB supergravity which are lifts of solutions of five-dimensional gauged supergravity. In particular we have shown that the structure of the lift ansatz ensures that the action for a minimal energy giant graviton reduces to that of a massive charged particle in five dimensions. The mass and charge of this particle are equal, suggesting that this is the bosonic part of a superparticle action. So it seems that the consistent truncation ansatz applies not only to the pure supergravity fields but also to allowed sources in the form of brane actions which can be coupled to the supergravity action. The derivation of the particle action in this way is similar to the derivation of type IIA string and D-brane actions from M-brane actions using the truncation of eleven-dimensional supergravity to ten-dimensional type IIA supergravity [91, 93].

In section 5 we looked at a specific example of a solution of five-dimensional gauged supergravity which lifts to the superstar geometry in ten dimensions discussed in [3, 66]. We found, in the multi-charge case, that charged particle probes are repelled by the naked singularity and hence from the results of section 4, giant graviton probes are repelled by the superstar naked singularity. A possible conclusion might be that these singular geometries could be resolved by the presence of configurations of dual giant gravitons in the bulk. Alternatively it is possible that higher derivative corrections to the probe action or the supergravity solution might change the situation, since we are investigating regions of strong curvature.

It is certainly interesting that it is possible to find BPS states corresponding to expanded dual giant gravitons in the general superstar backgrounds. This is consistent with expectations from supersymmetry since these probes break the same supersymmetries as the background geometry. However, in order to determine whether these states do play a rôle in resolving the singularities it would be necessary to find supergravity backgrounds which match the superstars at large r but contain dual giants in the bulk modifying the small r behaviour.

Chapter 6

Conclusions

In this thesis we have investigated the physics of branes in the light of new insights provided by gauge theory/ gravity dualities. The study of branes probing holographic gauge theories, undertaken in chapters 2 and 3, has provided information about the strong coupling physics on the moduli spaces of these theories. Where gauge theory results were available, such as for non-perturbative scaling dimensions of operators, our techniques have provided a new derivation of these results from an alternative perspective. On the other hand, specific results for the Kähler metrics on moduli space, at strong coupling, are new and it would be interesting to compare to computations in gauge theory. Particularly intriguing is the simple equation (3.16) and its eleven dimensional counterparts, which suggest a simple behaviour of the Kähler potential under renormalisation group flow at strong coupling.

An interesting extension to the investigations of moduli space physics would be to try to reproduce the full low energy Lagrangian for the gauge theory on a single brane probe. It should be possible in particular to incorporate superpotential terms and kinetic terms for motion off moduli space into a full supersymmetric Lagrangian, at least in the regime of large vevs in which the Higgs bosons are very massive and can be consistently integrated out. We have made some progress in understanding the potential terms in this regime [2] and it would be interesting to see how far this can be pushed. Likewise, it is possible to make some progress with the kinetic terms for motions of the probe off moduli space. We can even reproduce the scaling dimension of the massive superfields in the LS flow [2] by putting the metric on a subspace orthogonal to moduli space into a Kähler form. However, it has proved difficult to find a metric on the full space of fields which is of the expected form and this remains a worthwhile problem for future study.

Whereas in the early chapters our brane probe results led to new information about field theory, in chapter 4 and 5 the flow of ideas was mainly in the other direction. Motivated by puzzles arising from the known spectrum of BPS operators in field theory, the authors of [62] were inspired to find a new class of light states in string theory on backgrounds with RR flux, which are represented by compact branes carrying no topological charge. The existence of such states leads to fascinating possibilities regarding the description of quantum gravity on spaces with RR flux. In particular, detailed investigations of the spectrum instigated in [77] and continued in [78] confirm that we need to treat the compact branes quantum mechanically. Furthermore, the correct microscopic description of these states is likely to be in terms of non-commutative bound states of graviton modes expanding according to the Myers effect. The implications that extended objects and/or matrix valued fields should play a rôle in quantizing Kaluza-Klein gravity in these backgrounds seem sure to lead to developments in our understanding.

Our contribution to this subject has been to show that these branes are evident even at the level of classical supergravity solutions. The description in supergravity adds extra weight to the arguments that Kaluza-Klein states do expand and also admits the possibility of a different way to investigate these states. One example of this is a conjectured duality between configurations with a small number of background branes and a large number of giant gravitons to configurations with a large number of background branes and a small number of giant gravitons which are based on a study of the relevant superstar geometries [94].

We have also found results about giant graviton probes in some more general backgrounds and seen that the degeneracy with Kaluza-Klein modes persists. We have found that these probes are naturally related to charged particles in lower dimensional gauged supergravities. We believe that this observation, which generalises similar remarks about simpler circle reductions (see e.g. [91]), could lead to new geometrical insights into the old problem of consistent truncations.

In summary, whilst there is much still to learn about branes we can feel sure that they will play a central rôle in our understanding of quantum gravity. We have enjoyed finding surprising new ways in which branes fit into the beautiful structure of string theory and look forward to future developments with great interest.

Bibliography

[1] C. V. Johnson, K. J. Lovis, and D. C. Page, "Probing some N = 1 AdS/CFT RG flows," JHEP **05** (2001) 036, http://arXiv.org/abs/hep-th/0011166.

[2] C. V. Johnson, K. J. Lovis, and D. C. Page, "The Kaehler structure of supersymmetric holographic RG flows," JHEP **10** (2001) 014, http://arXiv.org/abs/hep-th/0107261.

[3] F. Leblond, R. C. Myers, and D. C. Page, "Superstars and giant gravitons in M-theory," JHEP **01** (2002) 026, http://arXiv.org/abs/hep-th/0111178.

[4] D. C. Page and D. J. Smith, "Giant gravitons in non-supersymmetric backgrounds," JHEP **07** (2002) 028, http://arXiv.org/abs/hep-th/0204209.

[5] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring theory. Vols. 1 and 2,". Cambridge, UK: Univ. Pr. (1987) (Cambridge Monographs On Mathematical Physics).

[6] J. Polchinski, "String theory. Vols. 1 and 2,". Cambridge, UK: Univ. Pr. (1998).

[7] C. V. Johnson, "D-brane primer," http://arXiv.org/abs/hep-th/0007170.

[8] C. V. Johnson, "D-branes,". Cambridge, UK: Univ. Pr. (2002) (Cambridge Monographs On Mathematical Physics).

[9] C. G. Callan, E. J. Martinec, M. J. Perry, and D. Friedan, "Strings in background fields," Nucl. Phys. **B262** (1985) 593.

[10] A. Sen, "An introduction to non-perturbative string theory," http://arXiv.org/abs/hep-th/9802051.

[11] G. T. Horowitz and A. Strominger, "Black strings and P-branes," Nucl. Phys. **B360** (1991) 197–209.

[12] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell, and A. Westerberg, "The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity," Nucl. Phys. **B490** (1997) 179–201, http://arXiv.org/abs/hep-th/9611159.

[13] M. Cederwall, A. von Gussich, B. E. W. Nilsson, and A. Westerberg, "The Dirichlet super-three-brane in ten-dimensional type IIB supergravity," Nucl. Phys. **B490** (1997) 163–178, http://arXiv.org/abs/hep-th/9610148.

[14] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," Phys. Rev. Lett. **75** (1995) 4724–4727, http://arXiv.org/abs/hep-th/9510017.

[15] R. G. Leigh, "Dirac-Born-Infeld action from dirichilet sigma model," Mod. Phys. Lett. **A4** (1989) 2767.

[16] M. R. Douglas, "Branes within branes," http://arXiv.org/abs/hep-th/9512077.

[17] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. **B443** (1995) 85–126, http://arXiv.org/abs/hep-th/9503124.

[18] M. J. Duff and K. S. Stelle, "Multi-membrane solutions of D = 11 supergravity," Phys. Lett. **B253** (1991) 113–118.

[19] R. Gueven, "Black p-brane solutions of D = 11 supergravity theory," Phys. Lett. **B276** (1992) 49–55.

[20] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Supermembranes and eleven-dimensional supergravity," Phys. Lett. **B189** (1987) 75–78.

[21] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, "World-volume action of the M-theory five-brane," Nucl. Phys. **B496** (1997) 191–214, http://arXiv.org/abs/hep-th/9701166.

[22] E. Witten, "Bound States Of Strings And p-Branes," Nucl. Phys. **B460** (1996) 335–350, http://arXiv.org/abs/hep-th/9510135.

[23] R. C. Myers, "Dielectric-branes," JHEP **12** (1999) 022, http://arXiv.org/abs/hep-th/9910053.

[24] J. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231–252, http://arXiv.org/abs/hep-th/9711200.

[25] I. R. Klebanov, "World-volume approach to absorption by non-dilatonic branes," Nucl. Phys. **B496** (1997) 231–242, http://arXiv.org/abs/hep-th/9702076.

[26] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, "String theory and classical absorption by three-branes," Nucl. Phys. **B499** (1997) 217–240, http://arXiv.org/abs/hep-th/9703040.

[27] G. 't Hooft, "A planar diagram theory for the strong interactions," Nucl. Phys. **B72** (1974) 461.

[28] N. Seiberg, "Notes on theories with 16 supercharges," Nucl. Phys. Proc. Suppl. **67** (1998) 158–171, http://arXiv.org/abs/hep-th/9705117.

[29] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253–291, http://arXiv.org/abs/hep-th/9802150.

[30] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. **B428** (1998) 105–114, http://arXiv.org/abs/hep-th/9802109.

[31] V. Balasubramanian, P. Kraus, and A. E. Lawrence, "Bulk vs. boundary dynamics in anti-de Sitter spacetime," Phys. Rev. **D59** (1999) 046003, `http://arXiv.org/abs/hep-th/9805171`.

[32] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, "Holographic probes of anti-de Sitter space-times," Phys. Rev. **D59** (1999) 104021, `http://arXiv.org/abs/hep-th/9808017`.

[33] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **07** (1998) 023, `http://arXiv.org/abs/hep-th/9806087`.

[34] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "The mass spectrum of chiral N=2 d = 10 supergravity on S**5," Phys. Rev. **D32** (1985) 389.

[35] M. Gunaydin and N. Marcus, "The spectrum of the S**5 compactification of the chiral N=2, D = 10 supergravity and the unitary supermultiplets of U(2, 2/4)," Class. Quant. Grav. **2** (1985) L11.

[36] S. Ferrara, C. Fronsdal, and A. Zaffaroni, "On N = 8 supergravity on AdS(5) and N = 4 superconformal Yang-Mills theory," Nucl. Phys. **B532** (1998) 153–162, `http://arXiv.org/abs/hep-th/9802203`.

[37] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183–386, `http://arXiv.org/abs/hep-th/9905111`.

[38] S. Minwalla, "Restrictions imposed by superconformal invariance on quantum field theories," Adv. Theor. Math. Phys. **2** (1998) 781–846, `http://arXiv.org/abs/hep-th/9712074`.

[39] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus," Nucl. Phys. **B229** (1983) 381.

[40] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, "Gauged N=8 D = 5 supergravity," Nucl. Phys. **B259** (1985) 460.

[41] M. Gunaydin, L. J. Romans, and N. P. Warner, "Gauged N=8 supergravity in five dimensions," Phys. Lett. **B154** (1985) 268.

[42] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c-theorem," Adv. Theor. Math. Phys. **3** (1999) 363–417, `http://arXiv.org/abs/hep-th/9904017`.

[43] K. Pilch and N. P. Warner, "N = 1 supersymmetric renormalization group flows from IIB supergravity," `http://arXiv.org/abs/hep-th/0006066`.

[44] M. Cvetic, H. Lu, and C. N. Pope, "Geometry of the embedding of supergravity scalar manifolds in D = 11 and D = 10," Nucl. Phys. **B584** (2000) 149–170, `http://arXiv.org/abs/hep-th/0002099`.

[45] M. Grana and J. Polchinski, "Supersymmetric three-form flux perturbations on AdS(5)," Phys. Rev. **D63** (2001) 026001, `http://arXiv.org/abs/hep-th/0009211`.

[46] A. Khavaev and N. P. Warner, "An N = 1 supersymmetric Coulomb flow in IIB supergravity," Phys. Lett. **B522** (2001) 181–188, `http://arXiv.org/abs/hep-th/0106032`.

[47] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, "Continuous distributions of D3-branes and gauged supergravity," JHEP **07** (2000) 038, `http://arXiv.org/abs/hep-th/9906194`.

[48] I. Bakas and K. Sfetsos, "States and curves of five-dimensional gauged supergravity," Nucl. Phys. **B573** (2000) 768–810, `http://arXiv.org/abs/hep-th/9909041`.

[49] P. Kraus, F. Larsen, and S. P. Trivedi, "The Coulomb branch of gauge theory from rotating branes," JHEP **03** (1999) 003, `http://arXiv.org/abs/hep-th/9811120`.

[50] I. Bakas, A. Brandhuber, and K. Sfetsos, "Domain walls of gauged supergravity, M-branes, and algebraic curves," Adv. Theor. Math. Phys. **3** (1999) 1657–1719, `http://arXiv.org/abs/hep-th/9912132`.

[51] K. Pilch and N. P. Warner, "N = 2 supersymmetric RG flows and the IIB dilaton," Nucl. Phys. **B594** (2001) 209–228, http://arXiv.org/abs/hep-th/0004063.

[52] A. Buchel, A. W. Peet, and J. Polchinski, "Gauge dual and noncommutative extension of an N = 2 supergravity solution," Phys. Rev. **D63** (2001) 044009, http://arXiv.org/abs/hep-th/0008076.

[53] N. Evans, C. V. Johnson, and M. Petrini, "The enhancon and N = 2 gauge theory/gravity RG flows," JHEP **10** (2000) 022, http://arXiv.org/abs/hep-th/0008081.

[54] J. Babington, N. Evans, and J. Hockings, "Secrets of the metric in N = 4 and N = 2* geometries," JHEP **07** (2001) 034, http://arXiv.org/abs/hep-th/0105235.

[55] R. Corrado, K. Pilch, and N. P. Warner, "An N = 2 supersymmetric membrane flow," http://arXiv.org/abs/hep-th/0107220.

[56] C.-h. Ahn and J. Paeng, "Three-dimensional SCFTs, supersymmetric domain wall and renormalization group flow," Nucl. Phys. **B595** (2001) 119–137, http://arXiv.org/abs/hep-th/0008065.

[57] P. Candelas and X. C. de la Ossa, "Comments on conifolds," Nucl. Phys. **B342** (1990) 246–268.

[58] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," Nucl. Phys. **B536** (1998) 199–218, http://arXiv.org/abs/hep-th/9807080.

[59] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," Adv. Theor. Math. Phys. **3** (1999) 1–81, http://arXiv.org/abs/hep-th/9810201.

[60] S. R. Das, S. P. Trivedi, and S. Vaidya, "Magnetic moments of branes and giant gravitons," JHEP **10** (2000) 037, http://arXiv.org/abs/hep-th/0008203.

[61] B. Janssen and Y. Lozano, "A microscopical description of giant gravitons," http://arXiv.org/abs/hep-th/0207199.

[62] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," JHEP **06** (2000) 008, http://arXiv.org/abs/hep-th/0003075.

[63] M. T. Grisaru, R. C. Myers, and O. Tafjord, "SUSY and Goliath," JHEP **08** (2000) 040, http://arXiv.org/abs/hep-th/0008015.

[64] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," JHEP **08** (2000) 051, http://arXiv.org/abs/hep-th/0008016.

[65] A. Mikhailov, "Giant gravitons from holomorphic surfaces," JHEP **11** (2000) 027, http://arXiv.org/abs/hep-th/0010206.

[66] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," JHEP **11** (2001) 009, http://arXiv.org/abs/hep-th/0109127.

[67] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, "Giant gravitons in conformal field theory," JHEP **04** (2002) 034, http://arXiv.org/abs/hep-th/0107119.

[68] J. M. Maldacena and A. Strominger, "AdS(3) black holes and a stringy exclusion principle," JHEP **12** (1998) 005, http://arXiv.org/abs/hep-th/9804085.

[69] C. Bar, "Real killing spinors and holonomy," Comm. Math. Phys. **154** (1993) 509–521.

[70] J. P. Gauntlett, N. D. Lambert, and P. C. West, "Branes and calibrated geometries," Commun. Math. Phys. **202** (1999) 571–592, http://arXiv.org/abs/hep-th/9803216.

[71] G. W. Gibbons and G. Papadopoulos, "Calibrations and intersecting branes," Commun. Math. Phys. **202** (1999) 593–619, http://arXiv.org/abs/hep-th/9803163.

[72] J. Gutowski, G. Papadopoulos, and P. K. Townsend, "Supersymmetry and generalized calibrations," Phys. Rev. **D60** (1999) 106006, http://arXiv.org/abs/hep-th/9905156.

[73] A. Kehagias, "New type IIB vacua and their F-theory interpretation," Phys. Lett. **B435** (1998) 337–342, http://arXiv.org/abs/hep-th/9805131.

[74] E. Witten, "Baryons and branes in anti de Sitter space," JHEP **07** (1998) 006, http://arXiv.org/abs/hep-th/9805112.

[75] S. S. Gubser and I. R. Klebanov, "Baryons and domain walls in an N = 1 superconformal gauge theory," Phys. Rev. **D58** (1998) 125025, http://arXiv.org/abs/hep-th/9808075.

[76] S. Gukov, M. Rangamani, and E. Witten, "Dibaryons, strings, and branes in AdS orbifold models," JHEP **12** (1998) 025, http://arXiv.org/abs/hep-th/9811048.

[77] D. Berenstein, C. P. Herzog, and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," JHEP **06** (2002) 047, http://arXiv.org/abs/hep-th/0202150.

[78] C. E. Beasley, "BPS branes from baryons," http://arXiv.org/abs/hep-th/0207125.

[79] J. Polchinski and M. J. Strassler, "The string dual of a confining four-dimensional gauge theory," http://arXiv.org/abs/hep-th/0003136.

[80] M. Cvetic *et. al.*, "Embedding AdS black holes in ten and eleven dimensions," Nucl. Phys. **B558** (1999) 96–126, http://arXiv.org/abs/hep-th/9903214.

[81] K. Behrndt, M. Cvetic, and W. A. Sabra, "Non-extreme black holes of five dimensional N = 2 AdS supergravity," Nucl. Phys. **B553** (1999) 317–332, http://arXiv.org/abs/hep-th/9810227.

[82] K. Behrndt, A. H. Chamseddine, and W. A. Sabra, "BPS black holes in N = 2 five dimensional AdS supergravity," Phys. Lett. **B442** (1998) 97–101, http://arXiv.org/abs/hep-th/9807187.

[83] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Charged AdS black holes and catastrophic holography," Phys. Rev. **D60** (1999) 064018, http://arXiv.org/abs/hep-th/9902170.

[84] H. Nastase, D. Vaman, and P. van Nieuwenhuizen, "Consistent nonlinear K K reduction of 11d supergravity on AdS(7) x S(4) and self-duality in odd dimensions," Phys. Lett. **B469** (1999) 96–102, http://arXiv.org/abs/hep-th/9905075.

[85] H. Nastase, D. Vaman, and P. van Nieuwenhuizen, "Consistency of the AdS(7) x S(4) reduction and the origin of self-duality in odd dimensions," Nucl. Phys. **B581** (2000) 179–239, http://arXiv.org/abs/hep-th/9911238.

[86] M. J. Duff, "Lectures on branes, black holes and anti-de Sitter space," http://arXiv.org/abs/hep-th/9912164.

[87] M. J. Duff and J. T. Liu, "Anti-de Sitter black holes in gauged N = 8 supergravity," Nucl. Phys. **B554** (1999) 237–253, http://arXiv.org/abs/hep-th/9901149.

[88] M. Cvetic and S. S. Gubser, "Phases of R-charged black holes, spinning branes and strongly coupled gauge theories," JHEP **04** (1999) 024, http://arXiv.org/abs/hep-th/9902195.

[89] R.-G. Cai, "The Cardy-Verlinde formula and AdS black holes," Phys. Rev. **D63** (2001) 124018, http://arXiv.org/abs/hep-th/0102113.

[90] W. A. Sabra, "Anti-de Sitter BPS black holes in N = 2 gauged supergravity," Phys. Lett. **B458** (1999) 36–42, http://arXiv.org/abs/hep-th/9903143.

[91] M. J. Duff, P. S. Howe, T. Inami, and K. S. Stelle, "Superstrings in D = 10 from supermembranes in D = 11," Phys. Lett. **B191** (1987) 70.

[92] E. Hackett-Jones and D. J. Smith, "Giant gravitons as probes of gauged supergravity,".

[93] P. K. Townsend, "D-branes from M-branes," Phys. Lett. **B373** (1996) 68–75, http://arXiv.org/abs/hep-th/9512062.

[94] V. Balasubramanian and A. Naqvi, "Giant gravitons and a correspondence principle," Phys. Lett. **B528** (2002) 111–120, http://arXiv.org/abs/hep-th/0111163.