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Abstract

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This thesis is concerned with the Schrödinger representation of quantum field theory. We describe techniques for solving the Schrödinger equation which supplement the standard techniques of field theory. Our aim is to develop these to the point where they can readily be used to address problems of current interest. To this end, we study realistic models such as gauge theories coupled to dynamical fermions. For maximal generality we consider particles of all physical spins, in various dimensions, and eventually, curved spacetimes.

We begin by considering Gaussian fields, and proceed to a detailed study of the Schwinger model, which is, amongst other things, a useful model for (3+1) dimensional gauge theory.

One of the most important developments of recent years is a conjecture by Maldacena which relates supergravity and string/M-theory on anti-de-Sitter spacetimes to conformal field theories on their boundaries. This correspondence has a natural interpretation in the Schrödinger representation, so we solve the Schrödinger equation for fields of arbitrary spin in anti-de-Sitter spacetimes, and use this to investigate the conjectured correspondence. Our main result is to calculate the Weyl anomalies arising from supergravity fields, which, summed over the supermultiplets of type IIB supergravity compactified on $AdS_5 \times S^5$ correctly matches the anomaly calculated in the conjecturally dual $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory. This is one of the few existing pieces of evidence for Maldacena's conjecture beyond leading order in N.

Declaration

This thesis is the result of research carried out between October 1996 and August 2000. The material presented in this thesis has never previously been submitted for any degree at any university. Most of Chapters 2 and 4 have been published in [51], and Chapters 5 and 6 in [52, 53, 54].

Chapter 2 is original apart from some standard results; for a review see for example [17]. Chapter 3 makes significant use of [3] and [4]-[7]. Chapters 5 and 6 are the result of an ongoing collaboration with Prof. Paul Mansfield. All other work is original to the author, except as explicitly referenced.

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Dedicated to my family.

Contents

Chapter 1

Introduction

Since its inception, quantum theory has been fraught with difficulties of interpretation. Unlike relativity, which is built up from fundamental physical concepts, quantum mechanics is difficult to motivate at a physical level except by reference to its phenomenal experimental success.

In quantum field theory (QFT) the situation is aggravated by the technical demands of the subject. In addition, perturbation theory, which underlies the standard approach to QFT, is unable to provide reliable information in the infra-red sector, which governs many important phenomena such as confinement and chiral symmetry breaking, as well as probably being crucial to the interpretation of realistic theories as the low-energy limit of some more complete theory such as string theory. Recently, many techniques have been developed to probe non-perturbative behaviour, but the underlying physics is not always transparent. It is all too easy to get lost in a conceptual maze. It is important to retain as much conceptual unity as possible between diverse approaches.

The Schrödinger representation (SR) is a familiar concept in quantum mechanics, where it underlies many standard techniques, and forms the basis for our conceptual understanding of quantum physics. In this approach, physical states are thought of as wave-functions which satisfy the Schrödinger equation. These wave-functions are ordinary functions of space-time whose modulus squared gives the probability density of finding a particle at some space-time point. Their physical status is on a par with the classical particle concept, giving rise to the celebrated wave/particle duality.

Quantum field theory generalizes quantum mechanics to an infinite number of degrees of freedom. Particles become fields, and these fields are similarly subject to a probability distribution. Hence physical states may be thought of as wave-functionals, whose arguments are physical field configurations, and whose modulus squared gives the probability density of finding the fields in those configurations. By analogy with quantum mechanics, one might expect the wave-functionals, as physical concepts, to be of as much interest as the fields themselves. But this expectation is not supported by the historical development of field theory, where the existence of the Schrödinger equation was demonstrated less than 20 years ago by Symanzik [3].

The reasons for this relative lack of interest are not difficult to perceive. Symmetries such as Lorentz invariance were of paramount importance in the early days of QFT, and it is doubtful if renormalization theory, for example, could have been developed without Lorentz invariance—which is not manifest in the Schrödinger representation—as a guide. Experiments also played a more significant leading rôle at that time, and thus the standard approach to QFT was based on calculating cross-sections within a perturbation expansion, a framework ideally suited to comparison with experiment. Finally, in the absence of a developed formalism for the Schrödinger representation in QFT, many of the standard techniques of quantum mechanics were no longer available, and new ones were invented.

These factors no longer have the same relevance. Breaking the manifest symmetries of the theory is less of a disadvantage than it at first appears. Symmetries such as Lorentz invariance and gauge invariance are encoded in the wave-functionals in a perfectly satisfactory way. For example, in what follows we will invariably integrate out all gauge degrees of freedom, but gauge invariance will remain an important guide. Indeed, provided we identify suitable gauge invariant variables right from the start, thereafter everything is guaranteed to be gauge invariant.

Many recent developments in QFT have forsaken experiment completely for a more theoretical frame of reference. Whether one looks to theoretical or experimental technology for paving the way to new experimental success, it is clear that there is an enormous gap between theory and experiment. But the SR has some important advantages in this situation.

Because its wave-functional solutions have a well-defined physical interpretation, the functional Schrödinger equation has long been known for generating some useful physical insights. But to perceive the extent of its utility, one must appreciate the following:

1. The standard techniques of quantum mechanics have analogues which provide a whole new set of calculational tools for QFT.

2. Many existing results in QFT can be derived in novel and extremely elegant ways in the SR.

3. The formalism is flexible enough to describe both perturbative and non-perturbative results in the same framework, and crucially, allows both numerical and analytical tools to be implemented.

For these reasons, we believe that the further development of the SR for QFT holds great promise of narrowing the gap between theory and experiment, and of furthering the cause of conceptual unity in physics.

This thesis will be an attempt to develop the SR to the point where it can be used to address problems of particular current interest, such as testing the Maldacena conjecture. This will involve incorporating various new ideas; fields of arbitrary spin, gauge symmetries, supersymmetry and curved spacetime. The claims made above will be well-illustrated by our results; we will describe new techniques for QFT which solve problems which have proved intractable using conventional methods, we will rederive some well-known results, and go on to indicate how our methods may be used to provide new information about QFT.

To begin with, in Chapter 2 we set up the SR for fields of arbitrary spin. As will eventually become clear, the representations of scalar and spin 1/2 fields form a basis for representing all others, even superfields. Thus the most crucial step is a better understanding of the representations of fermionic fields.

In Chapter 3 we describe how perturbation theory and renormalization may be performed in the Schrödinger representation, and discuss a large-distance derivative expansion which provides a general non-perturbative approach to the solution of the Schrödinger equation. Here is a brief description of this expansion:

In quantum mechanics wavefunctions are often studied by expanding them in a basis of functions. A similar expansion exists for QFT. Consider the vacuum functional. Provided the masses of all particles are bounded away from zero, and we restrict our attention to the infra-red, its logarithm has an expansion in local derivatives. By this we mean that it may be expressed as an integral over an infinite sum of terms, each of which is a finite product of fields and their derivatives, evaluated at the same space-time point.

If the spectrum includes massless particles, we can re-express the vacuum as the large time limit of the Schrödinger functional, which describes the propagation of fields over a finite time interval. This interval acts as a natural infra-red regulator, and at distances which are large on this scale, the Schrödinger functional again admits a derivative expansion. Since excited states can be constructed either by applying creation operators to the vacuum wave-functional, or by studying the large time asymptotics of the Schrödinger functional, these two wave-functionals are the central objects of study in a Schrödinger representation approach to QFT.

This approach has been shown to be useful in studying the analytical properties of QFT at large distances; for example in [4] it was shown that the leading order term in a strong coupling expansion of the Yang-Mills vacuum functional leads to confinement, via a kind of dimensional reduction. This has been generalized to the study of the Wheeler-de-Witt equation in quantum gravity [15].

But the use of the local expansion as an analytical tool is not limited to the study of large distance phenomena. By exploiting analyticity in a complex scale parameter [5], small distance behaviour can be reconstructed from the large distance expansion. Thus a knowledge of this expansion is sufficient to understand physics at all scales.

From the point of view of the Schrödinger equation, obtaining wave functionals in the form of a local expansion is equivalent to solving an infinite set of coupled algebraic equations. If we truncate the local expansion at some large but finite order, the resulting equations are well-suited to numerical treatment. Thus this approach provides a useful source of numerical information, even when exact results are unavailable [5, 6, 7]. Most of this thesis is concerned with analytical results, but we will indicate along the way how numerical techniques can be used to solve more general problems.

Chapter 4 is a detailed study of the Schwinger model, which is the simplest example of a theory involving fermions coupled to gauge fields. This model can be solved exactly, but it illustrates many issues which are relevant to the study of more complicated gauge theories. In particular, although the physical spectrum exhibits a (dynamically generated) mass gap, the local expansion property of the wave-functionals breaks down as a result of gauge invariance. However, it is still possible to reconstruct the wave-functionals from a local expansion of the Schrödinger functional, which we obtain explicitly. This illustrates a technique which may be applied in a similar fashion to QCD in (3+1) dimensions. Also, the Schrödinger functional may be interpreted as the density matrix of the finite temperature model.

Many well-known features of the Schwinger model (vacuum angle, bosonization, confinement, etc.) are exhibited by the wave-functional solutions. Most of this applies to other (1+1) dimensional gauge theories as well. We discuss a straightforward method for extracting VEV's. Using similar methods, we show how the massive Schwinger model can be solved up to n-point interactions by expanding the kernels of the wave-functional in a suitable basis of functions.

One of the most important recent developments in theoretical high energy physics is a conjecture by Maldacena that string theory propagating in an anti-de-Sitter spacetime (AdS) is equivalent to supersymmetric Yang-Mills propagating in the Minkowski spacetime at the anti-de-Sitter boundary. This conjecture is extremely useful since it allows the strong coupling behaviour of either theory to be calculated from the weak coupling limit of the other. But it is extremely difficult to test, especially beyond tree level in AdS—indeed there are virtually no existing calculations of quantum corrections in this context. Yet testing the conjecture at this level is essential to its application to the real world.

In Chapters 5 and 6 we develop the technology for understanding quantum fields in AdS. The boundary partition functions which are the main objects of study in the AdS/CFT correspondence have a natural interpretation in terms of the wave-functionals of the AdS theory. We show how these wave-functionals may be obtained from the Schrödinger equation, and use them to calculate the two-point functions and scaling dimensions of the corresponding boundary fields. We also discuss the numerical evaluation of higher n-point functions in this formalism.

Now because the conjectured correpondence relates strong and weak coupling, the perturbation expansions in the two theories are valid in different domains. So in order to test the conjecture we must find quantities whose exact coupling constant dependence can be calculated. In practise this restricts us to quantities such as global anomalies. We will consider the Weyl anomaly, which measures the effect of a rescaling of the boundary metric. This involves generalizing the correpondence to incorporate a curved boundary.

We calculate the Weyl anomalies arising from particles of arbitrary spin in AdS space-time, finding that the result is discontinuous in the particle mass, and non-zero for precisely the mass values appearing in the Kaluza-Klein compactifications of supergravity.

Maldacena's conjecture relates type IIB string theory compactified on S^5 to $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory. From the point of view of the string theory, the Weyl anomaly should arise from the sum of the contributions from the supergravity fields. On the other hand, the anomaly can be calculated for free fields in super-Yang-Mills, and the result is protected by supersymmetry from loop corrections. We find that the contribution to the anomaly from tree level gravity coincides with the SYM result for large N, whereas (for a Ricci-flat boundary metric) the one-loop anomalies cancel in each supergravity multiplet to give a vanishing result. This is also in accordance with the result obtained from SYM. In the same way, the contributions to the renormalization of the cosmological and Newton's constants cancel within multiplets, as is required for the finiteness of the boundary theory. We outline how these results can be extended to a general Einstein metric on the boundary.

In conclusion, our results provide substantial new evidence for Maldacena's conjecture at finite N. In the future we hope to apply our methods to other instances of the AdS/CFT correspondence, for instance M-theory on $AdS_7 \times S^4$ and the many compactifications of string theory on AdS_3.

Chapter 2

The Schrödinger representation for fields of arbitrary spin

2.1 Scalar fields

For scalar fields, the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi \partial_\nu \phi - m^2\phi^2 - V(\phi)). \tag{2.1}$$

In the usual way, we define a field canonically conjugate to ϕ

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \tag{2.2}$$

and impose equal-time commutation relations

$$\begin{aligned} [\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] &= i\hbar\delta^d(\mathbf{x} - \mathbf{x}') \\ [\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] &= [\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = 0. \end{aligned}$$

Now in analogy with quantum mechanics, we define a Hilbert space on which the operators $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ act. In order to satisfy the commutation relations, we choose to diagonalize ϕ

$$\langle\varphi|\hat{\phi}(\mathbf{x},t) = \varphi(\mathbf{x})\langle\varphi|, \tag{2.3}$$

and π is represented by functional differentiation

$$\langle\varphi|\hat{\pi}(\mathbf{x},t) = -i\hbar\frac{\delta}{\delta\varphi(\mathbf{x})}\langle\varphi|. \tag{2.4}$$

The eigenstates $\langle\varphi|$ form a complete orthonormal set:

$$\begin{aligned}\langle\varphi|\tilde{\varphi}\rangle &= \delta[\varphi-\tilde{\varphi}]\\ \int\mathcal{D}\varphi|\varphi\rangle\langle\varphi| &= 1,\end{aligned}$$

and by virtue of the commutation relations we can extract their explicit φ dependence

$$\langle\varphi| = \langle D|\exp(i\int d\mathbf{x}\hat{\pi}\varphi), \tag{2.5}$$

where $\langle D|$ is annihilated by $\hat{\phi}$.

Wave-functionals (WF's) are inner-products of these states with the physical states of the theory

$$\Psi[\varphi] = \langle\Psi|\varphi\rangle, \tag{2.6}$$

and themselves have an inner-product defined by

$$\langle\Psi_1|\Psi_2\rangle = \int\mathcal{D}\varphi\Psi_1^*[\varphi]\Psi_2[\varphi]. \tag{2.7}$$

They satisfy the Schrödinger equation

$$H\Psi[\varphi] = i\hbar\frac{\partial}{\partial t}\Psi[\varphi], \tag{2.8}$$

where the Hamiltonian is given by

$$H = \int d^dx(\pi\dot{\phi} - \mathcal{L}) = \tfrac{1}{2}\int d^dx(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 + V(\phi)). \tag{2.9}$$

WF's also have a path-integral representation. For example, according to the standard correspondence between expectation values and path-integrals, the Schrödinger functional $\langle\varphi|e^{-Ht}|\tilde{\varphi}\rangle$ may be written as

$$\langle\varphi|e^{-Ht}|\tilde{\varphi}\rangle = \int \mathcal{D}\phi e^{-S_E[\phi]}, \tag{2.10}$$

where S_E is the Euclidean action on a space-time volume bounded by surfaces time t apart, and the functional integration is subject to the boundary conditions $\phi(\mathbf{x},0) = \tilde{\varphi}(\mathbf{x})$, $\phi(\mathbf{x},t) = \varphi(\mathbf{x})$.

We can enforce the boundary conditions by adding boundary terms to S_E in (2.10). On a space-time with boundaries, it is necessary to choose a definite direction of propagation for all physical fields. Changing this direction is equivalent to performing an integration by parts in the bulk action and leads to boundary terms. Appropriate boundary conditions are ones which do not allow currents to leak across the boundary. To begin with we will propagate everything forward in time, which correponds to selecting the advanced Green's function for the propagator. Then we can express S_E as

$$S_E = i\int d^{d+1}x \left\{(\tfrac{1}{2}\mathcal{T}_{-\epsilon}\phi)(\partial^2 + m^2)\phi\right\}, \tag{2.11}$$

where $\mathcal{T}_{-\epsilon}$ implements a small negative time shift. We can get back to the form (2.1) which is appropriate for passing to the Hamiltonian formulation by integrating by parts:

$$S_E = i\int d^{d+1}x \left\{(\tfrac{1}{2}\mathcal{T}_{-\epsilon}\phi)(\partial^2 + m^2)\phi - (\delta(x_0 - t) - \delta(x_0))\,(\mathcal{T}_{-\epsilon}\phi)\dot{\phi}\right\}. \tag{2.12}$$

Now on the initial boundary there is nowhere for fluctuations to propagate from, so we have a Neumann boundary condition $\dot{\phi}|_{x_0=0} = 0$. On the final boundary fluctuations can propagate from the bulk, but they cannot cross the boundary, so we have a Dirichlet boundary condition $\phi|_{x_0=t} = 0$. Of course there is nothing to stop us from making other choices of boundary condition. From a physical point of view we might like to take a Feynman prescription for the propagator, so that negative energy states are propagated backwards in time in accordance with our usual notions of causality. This complicates the quantization procedure considerably when things like gauge invariance are taken into account. But it doesn't really matter what choice we make, because we can always change it later by integrating over the boundary values of the fields (which is analogous to changing from the position to the momentum representation in quantum mechanics). So we will do whatever seems simplest. In (2.10) we chose Dirichlet conditions on both boundaries, which is perfectly consistent.

So introducing the source terms arising from (2.5) and suppressing the ϵ dependence, we have

$$S_E = i \int d^{d+1}x \left\{ \tfrac{1}{2}\phi(\partial^2 + m^2)\phi - \delta(x_0 - t)(\phi - \varphi)\dot{\phi} + \delta(x_0)(\phi - \tilde{\varphi})\dot{\phi} \right\}, \qquad (2.13)$$

and the boundary conditions in (2.10) are now automatically satisfied.

From the Schrödinger functional we can construct eigenstates of the Hamiltonian. Inserting a complete set of such eigenstates we find

$$\langle \varphi | e^{-Ht} | \tilde{\varphi} \rangle = \sum_n \langle \varphi | E_n \rangle e^{-tE_n} \langle E_n | \tilde{\varphi} \rangle. \qquad (2.14)$$

Normalizing the vacuum energy to zero, we have, as $t \to \infty$

$$\langle \varphi | e^{-Ht} | \tilde{\varphi} \rangle \sim \Psi_0[\tilde{\varphi}] \Psi_0^*[\tilde{\varphi}], \qquad (2.15)$$

where Ψ_0 is the vacuum WF. Excited states can be extracted in a similar way.

To illustrate the solution of the Schrödinger equation consider free scalar fields, ie. $V(\phi) = 0$ in 2.1. The Schrödinger functional satisfies

$$\dot{\Psi}[\varphi, \tilde{\varphi}] = \frac{1}{2}\left(-\frac{\delta^2}{\delta\varphi^2} + \varphi\omega^2\varphi\right)\Psi[\varphi, \tilde{\varphi}], \tag{2.16}$$

with the initial condition $\Psi[\varphi, \tilde{\varphi}]|_{t=0} = \delta[\varphi - \tilde{\varphi}]$, and $\omega^2 = -\nabla^2 + m^2$. It also satisfies a similar equation with φ and $\tilde{\varphi}$ interchanged. On general grounds we expect Ψ to be Gaussian, so we make the ansatz

$$\Psi[\varphi, \tilde{\varphi}] = f(t) \exp \int d^d x \left(\varphi\Gamma\varphi + \varphi\Xi\tilde{\varphi} + \tilde{\varphi}\Upsilon\tilde{\varphi}\right), \tag{2.17}$$

which leads to the following differential equations

$$\begin{aligned}
2\dot{f} &= -\mathrm{tr}\Gamma f = -\mathrm{tr}\Upsilon f \\
2\dot{\Gamma} &= \omega^2 - \Gamma^2 = -\Xi^2 \\
2\dot{\Xi} &= -\Gamma\Xi = -\Upsilon\Xi \\
2\dot{\Upsilon} &= -\Xi^2 = \omega^2 - \Upsilon^2.
\end{aligned} \tag{2.18}$$

These have the unique solution (with the overall sign chosen so that the WF is normalizable)

$$\begin{aligned}
f &= \tfrac{1}{2}\mathrm{tr}\omega\mathrm{cosech}(\tfrac{1}{2}\omega t) \\
\Gamma &= -\omega\coth(\tfrac{1}{2}\omega t) \\
\Xi &= -\omega\mathrm{cosech}(\tfrac{1}{2}\omega t) \\
\Upsilon &= -\omega\coth(\tfrac{1}{2}\omega t).
\end{aligned} \tag{2.19}$$

As $t \to 0$, $\Psi \sim \frac{1}{t}e^{-\int d^d x(\varphi-\tilde{\varphi})^2/2t} \to \delta[\varphi - \tilde{\varphi}]$ so the initial condition is satisfied. As

$t \to \infty$ we have

$$\Psi \sim e^{-\frac{1}{2}\int d^dx(\varphi\omega\varphi + \tilde{\varphi}\omega\tilde{\varphi})}, \tag{2.20}$$

from which we identify the vacuum functional as $\Psi_0 = e^{-\int d^dx \frac{1}{2}\varphi\omega\varphi}$.

We can construct excited states by the action of creation operators on the vacuum WF; for free fields the annihilation and creation operators are respectively ($\phi(\mathbf{p})$ is the Fourier transform of $\phi(\mathbf{x})$, etc.)

$$\begin{aligned} a(\mathbf{p}) &= \frac{1}{\sqrt{2}}(\omega^{1/2}(\mathbf{p})\hat{\phi}(\mathbf{p}) + i\omega^{-1/2}(\mathbf{p})\hat{\pi}(\mathbf{p})) \\ a^\dagger(\mathbf{p}) &= \frac{1}{\sqrt{2}}(\omega^{1/2}(\mathbf{p})\hat{\phi}(\mathbf{p}) - i\omega^{-1/2}(\mathbf{p})\hat{\pi}(\mathbf{p})). \end{aligned} \tag{2.21}$$

Using (2.3) and (2.4) we can easily verify that $a(\mathbf{p})\Psi_0 = 0$, while an n-particle excited state is given (up to normalization) by $\varphi(\mathbf{p}_1)\ldots\varphi(\mathbf{p}_n)\Psi_0[\varphi]$, and has energy $E_0 + \sum_i \omega(\mathbf{p}_i)$.

For interacting fields, it is sometimes easier to construct the n-particle states by solving the time-independent Schrödinger equation $\hat{H}\Psi_n = E_n\Psi_n$ using the ansatz

$$\Psi_n[\varphi] = \Phi[\varphi]\Psi_0[\varphi], \tag{2.22}$$

and taking $\Phi[\varphi]$ to be of order n in φ.

Now for fields which vary slowly on the scale of the mass m, we can expand ω as

$$\omega = \sqrt{-\nabla^2 + m^2} = m - \frac{\nabla^2}{2m} - \frac{\nabla^4}{8m^3} + \cdots \tag{2.23}$$

This allows us to express the logarithm of a physical WF as an integral over a sum of local terms, each involving φ and its derivatives evaluated at a single point. Similarly, for small t, we can expand (2.19) in powers of t, which leads to a local expansion for the logarithm of the Schrödinger functional, even when $m = 0$. The terms of this expansion can be obtained from the Schrödinger equation.

In later chapters we will see that similar expansions exist for all theories, including interacting ones.

2.2 Yang-Mills fields

The Lagrangian density for $SU(N)$ Yang-Mills theory is

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}, \tag{2.24}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $A_\mu = gA_\mu^A T_A$, with T_A the N^2-1 generators of $SU(N)$. In the Weyl gauge $A_0 = 0$ we can write

$$H = -\frac{1}{g^2}\int d^3x \mathrm{tr}(\mathbf{E}^2 + \mathbf{B}^2), \tag{2.25}$$

$$\mathbf{E} = -\dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \wedge \mathbf{A} + \mathbf{A} \wedge \mathbf{A}. \tag{2.26}$$

The fields conjugate to $\mathbf{A}$ are $-g^2\mathbf{E}$, and are represented as

$$\mathbf{E} = ig^2\hbar\frac{\delta}{\delta\mathbf{A}}. \tag{2.27}$$

The Euler-Lagrange equations of motion are obtained by varying A_μ in (2.24)

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0, \tag{2.28}$$

and the $\nu = 0$ component of this equation is a constraint (Gauss' law). Substituting (2.27) into it gives an operator which annihilates physical WF's

$$G\Psi[\mathbf{A}] = 0, \qquad G = (\delta^{AB}\partial_\alpha + f^{ACB}A_\alpha^C)\frac{\delta}{\delta A_\alpha^B}. \tag{2.29}$$

This is simply the statement that WF's are invariant under the time-independent gauge transformations generated by G.

On the other hand, gauge transformations in other homotopy classes do not leave WF's invariant in general, but produce a phase; for a transformation $\mathcal{G}_n$ in the nth homotopy class

$$\mathcal{G}_n \Psi[\mathbf{A}] = e^{-in\theta}\Psi[\mathbf{A}]. \tag{2.30}$$

This is the origin of the so-called θ-angle. We can implement this property by writing

$$\Psi[\mathbf{A}] = e^{-i\theta w[\mathbf{A}]}\Phi[\mathbf{A}], \tag{2.31}$$

where Φ is invariant under all gauge transformations and in four dimensions[1] w is a Chern-Simons term

$$w[\mathbf{A}] = -\frac{1}{16\pi^2}\int d^3x \epsilon^{ijk} \mathrm{tr}(F_{ij}A_k - \frac{2}{3}A_iA_jA_k), \tag{2.32}$$

which takes integer values for monopole-like configurations of the fields. So "instanton number" can effectively be interpreted as monopole number in this context. The Chern-Simons term may be cancelled by adding a total derivative to the Lagrangian

$$\mathcal{L} \to \mathcal{L} + \theta\frac{dw}{dt}, \tag{2.33}$$

and it is easily verified that

$$\frac{dw}{dt} = -\frac{1}{16\pi^2}\mathrm{tr}\,{}^*F^{\mu\nu}F_{\mu\nu}, \qquad {}^*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}. \tag{2.34}$$

For the Abelian theory we can easily solve the Schrödinger equation for the vac-

[1] the corresponding expression in two dimensions is given in Chapter 4.

uum functional. It takes the following form

$$\frac{1}{2}\int d^3x\left(-\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\cdot\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}+\mathbf{B}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})\right)\Psi[\mathbf{A}]=E_0\Psi[\mathbf{A}]. \tag{2.35}$$

Since the Hamiltonian is quadratic, we expect a Gaussian ground state, so we write

$$\Psi_0[\mathbf{A}]=e^{-W[\mathbf{A}]},\qquad W=\frac{1}{2}\int d^3xd^3yA^i(\mathbf{x})\Gamma_{ij}(\mathbf{x},\mathbf{y})A^j(\mathbf{y}). \tag{2.36}$$

Substituting into the Schrödinger equation (2.35) we find

$$\int d^3\mathbf{y}\Gamma_{ij}(\mathbf{x},\mathbf{y})\Gamma_{jk}(\mathbf{y},\mathbf{z})=(-\nabla^2\delta_{ik}+\partial_i\partial_k)\delta^3(\mathbf{x}-\mathbf{z}), \tag{2.37}$$

which in momentum space has the solution

$$\Gamma_{ij}=-|\mathbf{p}|\delta_{ij}+p_ip_j/|\mathbf{p}|. \tag{2.38}$$

Transforming back to position space, we have

$$\Psi_0[\mathbf{A}]=\exp\left(-\frac{1}{8\pi^2}\int d^3xd^3y\frac{F_{ij}(\mathbf{x})F_{ij}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\right). \tag{2.39}$$

For non-Abelian fields, which are self-interacting, the Schrödinger equation is of course much harder to solve. Nevertheless, Feynman conjectured [39] that the vacuum WF in (2+1) dimensions has the following form

$$\Psi_0[\mathbf{A}]=\exp\left\{-\beta\mathrm{tr}\int F_{ij}(\mathbf{x})S(\mathbf{x},\mathbf{y})F_{ij}(\mathbf{x})S(\mathbf{y},\mathbf{x})f(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\right\}, \tag{2.40}$$

$$S(\mathbf{x},\mathbf{y})=\mathcal{P}\exp\left\{i\int_{\mathbf{y}}^{\mathbf{x}}\mathbf{A}(\mathbf{x}')\cdot d\mathbf{x}'\right\} \tag{2.41}$$

and this has been conjectured to hold in (3+1) dimensions as well [40, 60]. It is difficult to verify this directly because of the intractability of calculations involving functional derivatives of the Wilson line $S(\mathbf{x}, \mathbf{y})$. Nevertheless, at short distances (2.40) reduces to (2.39) in accordance with asymptotic freedom.

At large distances one might hope to expand the bilocal integrand in (2.40) about a single point

$$\Psi_0[\mathbf{A}] = \exp\left(-\lambda_0 \int d\mathbf{x} \mathrm{tr}(F_{ij})^2 - \lambda_2 \int d\mathbf{x}(D_i F_{ij})^2 + \ldots\right). \tag{2.42}$$

The leading order term in this expansion dominates in the large distance limit, as suggested by Greensite [41], and leads to confinement via a kind of dimensional reduction [4]. However, Schrödinger representation calculations [42] in (2+1) dimensional Yang-Mills suggest that (2.42) is not quite right; there are non-local terms arising from the non-local nature of the weight function $f(\mathbf{x}, \mathbf{y})$. This is very similar to the situation in (1+1) dimensional QED, which we will study in detail later. There the non-local terms are associated with the existence of massless non-physical modes which are necessary to preserve gauge-invariance. Thus in spite of the existence of a dynamically generated mass gap, the anticipated existence of a derivative expansion of the WF's fails to materialize. This appears to be a general feature of gauge theories.

It is still possible to obtain the vacuum functional in an expansion of the form (2.42) by expressing it as a large-time limit of the Schrödinger functional, as we did for scalar fields. Alternatively, the form of the non-local terms may be investigated in more detail to generalize the large-distance expansion (2.42) in an appropriate way. In either case, using the techniques described in Chapter 3 the small distance behaviour of (2.40) can be reconstructed from the large-distance expansion, whose coefficients may be determined from the Schrödinger equation.

2.3 Fermions

The functional representation of fermions is complicated by the self-conjugate nature of fermion fields; for a free Majorana fermion with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} i\bar{\psi}\gamma \cdot \partial\psi, \tag{2.43}$$

we can reproduce the canonical anti-commutation relation

$$\{\psi_a(\mathbf{x}), \psi_b(\mathbf{x}')\} = \delta_{ab}\delta^3(\mathbf{x} - \mathbf{x}') \tag{2.44}$$

by using the Floreanini-Jackiw representation [15]

$$\langle u|\hat{\psi} = \frac{1}{\sqrt{2}}\left(u + \frac{\delta}{\delta u}\right)\langle u|. \tag{2.45}$$

The difficulty is that (2.45) is reducible. But this reducibility may be removed by interpreting the representation in a different way. Consider the arbitrary projection operators $Q_\pm = \frac{1}{2}(1 \pm Q)$. Suppose we diagonalize $Q_+\psi$ and represent $Q_-\psi$ by functional differentiation:

$$\langle u|Q_+\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{2}}Q_+u(\mathbf{x})\langle u|, \qquad \langle u|Q_-\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{2}}Q_-\frac{\delta}{\delta u(\mathbf{x})}\langle u|. \tag{2.46}$$

As before, we can solve for the u-dependence explicitly[2]

$$\langle u| = \langle Q|\exp(\sqrt{2}uQ_-\hat{\psi}), \tag{2.47}$$

where $\langle Q|$ is defined by the property $\langle Q|Q_+\hat{\psi} = 0$. This shows that WF's depend only on Q_-u and we can take $Q_+u = 0$. The expression corresponding to (2.47) for

[2] We adopt the convention that when bilinears such as uQu or $u\hat{\psi}$ are written down without arguments, the spatial argument is integrated over: $uQu = \int d\mathbf{x} u_a(\mathbf{x})Q_{ab}u_b(\mathbf{x})$, etc.

the representation (2.45) is

$$\langle u| = \langle \text{any}| \exp(\sqrt{2}u\hat{\psi}), \tag{2.48}$$

and if we choose $\langle \text{any}| = \langle Q|$ then this coincides with (2.47) up to a trivial factor e^{uQu} which vanishes when the constraint $Q_+ u = 0$ is imposed.

As pointed out in [15], the choice of Q is a choice of Dirac sea. But as in the bosonic case, it is neither necessary nor desirable to make it a physical Dirac sea, which would make the reference state $\langle u|$ an excitation of the physical vacuum. We wish to make a choice which is independent of the dynamics or the specific theory under consideration. It is convenient to take Q to be a local operator. One particularly useful choice is $Q = \gamma_0$, which is the unique choice preserving gauge invariance in odd dimensional spacetimes, and corresponds to a vanishing vacuum angle in even dimensional ones.

To define an inner-product we construct dual states $|v\rangle$ defined by

$$\hat{\psi}|v\rangle = \frac{1}{\sqrt{2}}\left(v - \frac{\delta}{\delta v}\right)|v\rangle, \tag{2.49}$$

or equivalently

$$Q_+\hat{\psi}|v\rangle = \frac{1}{\sqrt{2}}Q_+ v|v\rangle, \qquad Q_-\hat{\psi}|v\rangle = \frac{1}{\sqrt{2}}Q_-\frac{\delta}{\delta v}|v\rangle, \tag{2.50}$$

with (2.47) becoming

$$|v\rangle = \exp(\sqrt{2}vQ_+\hat{\psi})|Q\rangle, \tag{2.51}$$

where $Q_-\psi|Q\rangle = 0$. We have $\langle u|v\rangle = \langle Q|Q\rangle = 1$ and the completeness relation is

$$1 = \int DuDv|v\rangle\langle u|. \tag{2.52}$$

As before, WF's are overlaps $\Psi[u] = \langle u|\Psi\rangle$ and have the inner-product

$$\langle\Psi_1|\Psi_2\rangle = \int DuDv\Psi_1^*[v]\Psi_2[u]. \tag{2.53}$$

It is easy to see that with repect to this inner-product the Hermitian conjugate of u is $\frac{\delta}{\delta u}$, so that $\hat{\psi}$ is Hermitian, as it should be.

Now charged fermions can be represented in a similar way[3]

$$\begin{aligned}
\langle\langle u, u^\dagger|\hat{\psi} &= \frac{1}{\sqrt{2}}(u + \frac{\delta}{\delta u^\dagger})\langle\langle u, u^\dagger| \\
\langle\langle u, u^\dagger|\hat{\psi}^\dagger &= \frac{1}{\sqrt{2}}(u^\dagger + \frac{\delta}{\delta u})\langle\langle u, u^\dagger|.
\end{aligned} \tag{2.54}$$

The notation of (2.54) indicates that u and $u^\dagger$ are treated as unconstrained fields, eg. for the purpose of taking functional derivatives. However, as before, the reducibility of (2.54) may be removed by imposing the constraints $Q_-u = u^\dagger Q_+ = 0$. In the presence of these constraints (2.54) corresponds to diagonalizing $Q_+\psi$ and $\psi^\dagger Q_-$:

$$\begin{aligned}
\langle u, u^\dagger|Q_+\hat{\psi} &= \sqrt{2}Q_+u\langle u, u^\dagger| \\
\langle u, u^\dagger|\hat{\psi}^\dagger Q_- &= \sqrt{2}u^\dagger Q_-\langle u, u^\dagger|,
\end{aligned} \tag{2.55}$$

and representing the other projections by functional differentiation

$$\begin{aligned}
\langle u, u^\dagger|Q_-\hat{\psi} &= \frac{1}{\sqrt{2}}Q_-\frac{\delta}{\delta u^\dagger}\langle u, u^\dagger| \\
\langle u, u^\dagger|\hat{\psi}^\dagger Q_+ &= \frac{1}{\sqrt{2}}\frac{\delta}{\delta u}Q_+\langle u, u^\dagger|.
\end{aligned} \tag{2.56}$$

[3]The notation is a little misleading, since $\frac{\delta}{\delta u}$ (and *not* $u^\dagger$) is the Hermitian conjugate of u.

Again, physical WF's will always depend only on the constrained fields. The two representations are exactly equivalent, but $\langle\langle u, u^\dagger|$ and $\langle u, u^\dagger|$ differ by a relative factor of $e^{-u^\dagger Q u}$, which guarantees that the functional derivatives acting in (2.54) do not take the state out of the subspace defined by Q.

The explicit dependence on u, $u^\dagger$ is given by

$$
\begin{aligned}
\langle u, u^\dagger| &= \langle Q|\exp[\sqrt{2}(u^\dagger\hat{\psi} - \hat{\psi}^\dagger u)] \\
&= \langle Q|\exp[\sqrt{2}(u^\dagger Q_-\hat{\psi} - \hat{\psi}^\dagger Q_+ u)],
\end{aligned} \tag{2.57}
$$

where $\langle Q|$ satisfies

$$
\langle Q|Q_+\hat{\psi} = \langle Q|\hat{\psi}^\dagger Q_- = 0. \tag{2.58}
$$

Similarly, defining $|u, u^\dagger\rangle = \exp[\sqrt{2}(u^\dagger\hat{\psi} - \hat{\psi}^\dagger u)]|Q\rangle$ with $Q_-\hat{\psi}|Q\rangle = \hat{\psi}^\dagger Q_+|Q\rangle = 0$, we have

$$
\begin{aligned}
\hat{\psi}|u, u^\dagger\rangle &= \frac{1}{\sqrt{2}}(u - \frac{\delta}{\delta u^\dagger})|u, u^\dagger\rangle \\
\hat{\psi}^\dagger|u, u^\dagger\rangle &= \frac{1}{\sqrt{2}}(u^\dagger - \frac{\delta}{\delta u})|u, u^\dagger\rangle.
\end{aligned} \tag{2.59}
$$

The inner-product is a little more difficult to define in this case. The above definitions give rise to the following equations:

$$
\begin{aligned}
0 &= (\hat{\psi}^\dagger - \sqrt{2}u^\dagger)Q_+\langle\langle u, u^\dagger|v, v^\dagger\rangle\rangle \\
&= Q_+(\hat{\psi} + \sqrt{2}v)\langle\langle u, u^\dagger|v, v^\dagger\rangle\rangle \\
&= (\hat{\psi}^\dagger + \sqrt{2}v^\dagger)Q_-\langle\langle u, u^\dagger|v, v^\dagger\rangle\rangle \\
&= Q_-(\hat{\psi} - \sqrt{2}u)\langle\langle u, u^\dagger|v, v^\dagger\rangle\rangle,
\end{aligned} \tag{2.60}
$$

where we have used the canonical commutation relations. The field operators in (2.60) may be represented by either pair of fields, and the equations solved to give

$$\langle\langle u, u^\dagger | v, v^\dagger \rangle\rangle = \exp(u^\dagger Q u - u^\dagger 2Q_- v + v^\dagger 2Q_+ u + v^\dagger Q v). \tag{2.61}$$

Hence

$$\langle u, u^\dagger | v, v^\dagger \rangle = \exp(2v^\dagger u - 2u^\dagger v), \tag{2.62}$$

which differs slightly from the version given by other authors [37].

To calculate inner-products of WF's we use the partition of unity

$$1 = \int DuDu^\dagger DvDv^\dagger e^{(2v^\dagger u - 2u^\dagger v)} |v, v^\dagger\rangle\langle u, u^\dagger|. \tag{2.63}$$

2.4 Fermions coupled to gauge fields

We can combine the representations described in the previous two sections to give a representation of QED or QCD. We start with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} i\bar{\psi}(\gamma \cdot D - m)\psi + \tfrac{1}{2g^2}\mathrm{tr} F^{\mu\nu}F_{\mu\nu}, \tag{2.64}$$

with D the covariant derivative $D_\mu = \partial_\mu - eA_\mu$. This is invariant under the gauge transformations

$$\psi \to g^{-1}\psi, \qquad A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g. \tag{2.65}$$

For many purposes it is convenient to require physical WF's to be gauge invariant also. We wish to choose Q so that this is possible. In general, we could consider Q to have some gauge field dependence. However, any field dependence or non-locality of Q will cause the representation to transform non-trivially under (2.65) (this can be seen from (2.57) and (2.58)). The only gauge invariant representations are when we take Q to be a local, field independent operator. In particular, if we take $Q_\pm$ to be the (non-local) projections $P_\pm$ onto +ve/-ve energy eigenstates, the resulting WF's are invariant under time-independent gauge transformations of the fields, but in general they do not satisfy Gauss' law [15, 16]. However, provided we take Q to satisfy the conditions above, Gauss' law is automatically satisfied, and the WF is a genuinely gauge invariant object.

Since $|u, u^\dagger, \mathcal{A}\rangle$ is not a physical state we need to check that its overlap with physical states is well-defined and non-vanishing . Consider the vacuum of the theory defined by (2.64). With $P_\pm$ defined as above, it satisfies

$$\hat{\psi}^\dagger P_+|0\rangle = P_-\hat{\psi}|0\rangle = 0. \tag{2.66}$$

The conditions (2.58) and (2.66) and the representation (2.54) lead to the following equations:

$$\begin{aligned}
&P_-\Big(u + \frac{\delta}{\delta u^\dagger}\Big)\langle\!\langle u, u^\dagger, \mathcal{A}|0\rangle = 0 \\
&\Big(u^\dagger + \frac{\delta}{\delta u}\Big)P_+\langle\!\langle u, u^\dagger, \mathcal{A}|0\rangle = 0 \\
&Q_+\Big(u - \frac{\delta}{\delta u^\dagger}\Big)\langle\!\langle u, u^\dagger, \mathcal{A}|0\rangle = 0 \\
&\Big(u^\dagger - \frac{\delta}{\delta u}\Big)Q_-\langle\!\langle u, u^\dagger, \mathcal{A}|0\rangle = 0.
\end{aligned} \tag{2.67}$$

These have the solution

$$\langle\!\langle u, u^\dagger, \mathcal{A}|0\rangle = \langle Q|Be^{u^\dagger M u}\exp\Big(i\mathcal{A}\cdot\hat{E}\Big)|0\rangle, \tag{2.68}$$

where $M = A^{-1}C = C^{-1}A$, and we define $A = \{P_-, Q_+\}$, $C = [P_-, Q_+]$. The constant of integration B corresponds to the determinant of the Dirac operator, and solving (2.67) is equivalent to performing the fermion integration, since (2.68) no longer involves fermion field operators. For the moment we assume that A^{-1} exists; the non-invertibility of A would signal the presence of zero modes of the Dirac operator, which would have to be treated separately. But we can always choose $Q_\pm$ so that A is invertible.

We can reproduce the result in a path integral representation. The vacuum is represented as $\lim_{t\to\infty} e^{-\hat{H}t}|S\rangle$, where S is any physical state not orthogonal to the vacuum. This leads, via the usual correspondence of matrix elements with path integrals, to the expression

$$\begin{aligned}\Psi[u, u^\dagger, \mathcal{A}] = \quad & \int DA_i D\psi^\dagger D\psi \exp\Big\{-S_B - S_F \\ & +\Big[\sqrt{2}(u^\dagger Q_-\psi - \psi^\dagger Q_+ u) + u^\dagger Q u + i\mathcal{A}\cdot \dot{A}\Big]\Big\}. \end{aligned} \tag{2.69}$$

Here S_B and S_F are the bosonic and fermionic parts of the Euclidean action, and the integral is evaluated with the following boundary conditions implied by (2.58):

$$A_i|_{t=0} = Q_+\psi|_{t=0} = \psi^\dagger Q_-|_{t=0} = 0. \tag{2.70}$$

These boundary conditions may be implemented by boundary terms as before. For fermions we choose $\psi^\dagger Q_+$ and $Q_-\psi$ to be propagated forward in time, and $\psi^\dagger Q_-$, $Q_+\psi$ backwards; this is effected by adding the boundary term

$$(\mathcal{T}_\epsilon\psi^\dagger)Q_+\psi \tag{2.71}$$

to the action, while the gauge field boundary condition is implemented by the boundary term

$$i(\mathcal{T}_\epsilon A)\cdot \dot{A}. \tag{2.72}$$

The fermion integration is now easily done by standard methods; we obtain the path integral version of (2.68)

$$\Psi[u, u^\dagger, \mathcal{A}] = \int DA_i \det D\, \exp\Big\{-S_B + \mathrm{Tr}\Big[u^\dagger A^{-1} C u + i(\mathcal{A} - A)\cdot \dot{A}\Big]\Big\}. \tag{2.73}$$

D is the Dirac operator. Similarly, the vacuum functional for free fermions or fermions in a classical background field is given by

$$\Psi[u, u^\dagger] = \exp \operatorname{Tr}\Big[u^\dagger A^{-1} C u\Big]. \tag{2.74}$$

Now consider the possible choices of Q. The solution we found above depended on the existence of the operator $A^{-1}C$, which is equivalent to the invertibility of $Q + P$ for all momenta. Q satisfying this condition are given in Appendix E of [15]; in even spacetime dimensions they take the form

$$Q = a\gamma^0 + ib\gamma^0\gamma^5 + c\gamma^5 \tag{2.75}$$

where $a^2 + b^2 + c^2 = 1$, with the additional conditions

$$(1 - c^2)^{-1}(-a \pm ibc) \notin [1, \infty), \tag{2.76}$$

for $m > 0$, and $c^2 \neq 1$ for $m = 0$. In odd spacetime dimensions $\gamma^5 = \gamma^0$, so the only possibility is $Q = \gamma_0$.

So what happens if we choose a Q which does not satisfy these conditions? The invertibility of $Q + P$ implies the absence of zero modes of the Dirac operator. This completely pins down the topological properties of the wave-functional, which will be useful when we consider subtleties like the vacuum angle. But other choices lead to equally well-defined wave-functionals provided we take account of zero modes. And it is always possible to alter Q by means of a functional Fourier transform.

So assume for the moment that we impose (2.75), which we can rewrite as

$$Q_+ = (a\gamma^0 + ib\gamma^0\gamma^5)\frac{1}{2}(1 + a'\gamma^0 + ib'\gamma^0\gamma^5), \tag{2.77}$$

$$Q_- = \frac{1}{2}(1 - a'\gamma^0 - ib'\gamma^0\gamma^5)(a\gamma^0 + ib\gamma^0\gamma^5), \tag{2.78}$$

where $a' = \frac{a+ibc}{a^2+b^2}$ and $b' = \frac{b-iac}{a^2+b^2}$; from (2.55) and (2.56) we see that this is equivalent to choosing $Q_{\pm} = \frac{1}{2}(1 \pm a'\gamma^0 \pm ib'\gamma^0\gamma^5)$. Since $a'^2 + b'^2 = 1$, this means that we can set $c = 0$ in (2.75) without loss of generality. Thus the possible choices for Q are parametrized by the single, arbitrary complex number $z = a - ib$ (which could in principle be taken to have local variations in space).

Since the Dirac operator has no zero modes when these boundary conditions are imposed, what happens to topological objects like instantons?

We will see that the phase of z corresponds to the vacuum angle. In two dimensions our choice of boundary conditions mean the instanton number is not quantized, but corresponds to the non-integrable phase. In four dimensions we observed a similar phenomenon, and argued that the corresponding phase was related to monopole number. Our representation seems to furnish a useful new perspective on these non-perturbative structures.

2.5 Superfields

In order to work with supersymmetric theories it is useful to set up the Schrödinger representation in the superfield formalism. The representation we give here extends to all superfields. It would be extremely interesting to study Super-Yang-Mills in this way, both because issues of regularization are much easier to deal with than in ordinary QCD, and because, as we saw in the last section non-perturbative and topological properties have very nice interpretations in the Schrödinger formalism. In particular, z must parametrize the $SL(2, Z)$ duality of $\mathcal{N} = 4$ SYM.

Just to illustrate the use of the superfield formalism in the Schrödinger representation, we will describe the vacuum solution of the Wess-Zumino model. The action is written in terms of anti-commuting coordinates θ as

$$S = \int d^4x d^2\theta d^2\bar{\theta}\, \tfrac{1}{2}\bar{\phi}(x, \theta, \bar{\theta})\phi(x, \theta, \bar{\theta}), \tag{2.79}$$

where $\phi(x,\theta,\bar\theta)$ and $\bar\phi(x,\theta,\bar\theta)$ are chiral superfields

$$\begin{aligned}
\phi(x,\theta,\bar\theta) &= e^{-i\bar\theta\not\partial\theta}\varphi(x,\theta) \\
\bar\phi(x,\theta,\bar\theta) &= e^{i\bar\theta\not\partial\theta}\varphi(x,\bar\theta).
\end{aligned} \tag{2.80}$$

We can expand $\varphi(x,\theta)$ and $\bar\varphi(x,\bar\theta)$ in powers of θ

$$\begin{aligned}
\varphi(x,\theta) &= A + 2\theta\psi - \theta^2 F \\
\bar\varphi(x,\bar\theta) &= B + 2\bar\theta\bar\psi - \bar\theta^2 G
\end{aligned} \tag{2.81}$$

and choosing projection operators $Q_\pm$ as in the fermionic case, we can take as a complete set of conjugate variables $\varphi(x,Q_+\theta)$, $\bar\varphi(x,Q_-\bar\theta)$, and their conjugates

$$\begin{aligned}
\pi(x,Q_+\theta) = \frac{\delta S}{\delta\varphi(x,Q_+\theta)} &= \tfrac{1}{2}\int d^2\bar\theta\, i\bar\theta\theta\, e^{2i\bar\theta\not\partial\theta}\bar\varphi(x,Q_-\bar\theta) \\
\bar\pi(x,Q_-\bar\theta) = \frac{\delta S}{\delta\dot{\bar\varphi}(x,Q_-\bar\theta)} &= \tfrac{1}{2}\int d^2\theta\, i\bar\theta\theta\, e^{-2i\bar\theta\not\partial\theta}\varphi(x,Q_+\theta).
\end{aligned} \tag{2.82}$$

Defining $\phi_c = e^{-i\bar\theta\not\nabla\theta}\varphi(x,Q_+\theta)$, $\bar\phi_c = e^{i\bar\theta\not\nabla\theta}\bar\varphi(x,Q_-\bar\theta)$, and similarly for π_c, $\bar\pi_c$, we find that the Hamiltonian is given by

$$\begin{aligned}
H &= \int d^3x d^2\theta\, \pi\dot\varphi + \int d^3x d^2\bar\theta\, \bar\pi\dot{\bar\varphi} - \mathcal{L} \\
&= \int d^3x d^4\theta \left(\tfrac{1}{2}\bar\pi_c\pi_c + \tfrac{1}{2}\bar\phi_c\phi_c\right).
\end{aligned} \tag{2.83}$$

As before we represent π and $\bar\pi$ by functional differentiation:

$$\pi(x,Q_+\theta) = -i\frac{\delta}{\delta\varphi(x,Q_+\theta)}, \qquad \bar\pi(x,Q_-\bar\theta) = -i\frac{\delta}{\delta\bar\varphi(x,Q_-\bar\theta)}, \tag{2.84}$$

and the Schrödinger equation $H\Psi = E_0\Psi$ is easily solved for the vacuum WF:

$$\Psi_0[\varphi,\bar\varphi] = \exp\left(\int d^3x d^4\theta\, \bar\varphi\bar\theta Q_- \frac{i\not\nabla}{\sqrt{\nabla^2}} Q_+\theta\, \tfrac{1}{2}\varphi\right). \tag{2.85}$$

Note that the vacuum energy vanishes

$$E_0 = \mathrm{tr}\frac{1}{2}\int d^4\theta\, e^{2i\bar{\theta}\not{\nabla}\theta}\,\bar{\theta}Q_-\frac{i\not{\nabla}}{\sqrt{\nabla^2}}Q_+\theta = 0, \tag{2.86}$$

and therefore H is a regular operator, unlike previous cases where the divergent vacuum energy had to be subtracted.

2.6 Gravity

Quantum gravity is of course non-renormalizable, but it can be studied in the semi-classical expansion using Schrödinger representation methods. The Lagrangian is

$$\int d^3x\sqrt{g}(R(g)+\Lambda), \tag{2.87}$$

where Λ is the cosmological constant.

It is useful to expand the metric about a classical solution of the Einstein equations

$$g_{ij} = g_{ij}^{\text{classical}} + h_{ij}. \tag{2.88}$$

To quadratic order in the perturbation h_{ij}, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{4}\left(-h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} + h_{\lambda}^{\lambda,\nu}h_{\mu,\nu}^{\mu} - 2h_{\mu,\nu}^{\mu}h_{,\lambda}^{\lambda\nu} + 2h_{\lambda\mu,\nu}h_{,\mu}^{\lambda\nu}\right). \tag{2.89}$$

The theory is invariant under the gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, and a convenient choice of gauge is given by

$$h_{ij,j} = 0, \quad h_{ii} = 0, \quad h_{0\mu} = 0, \tag{2.90}$$

so that (2.89) reduces to

$$\mathcal{L} = \frac{1}{4} \int d^3x \, (h_{ij,0}h_{ij,0} - h_{ij,k}h_{ij,k}) \, . \tag{2.91}$$

The conjugate momenta are

$$\pi^{ij} = \frac{\delta \mathcal{L}}{\delta h_{ij,0}} = \tfrac{1}{2} h_{ij,0}, \tag{2.92}$$

and the Hamiltonian is

$$\int d^3x \, (\pi^{ij}\pi^{ij} + \tfrac{1}{4} h_{ij,k}h_{ij,k}). \tag{2.93}$$

This has precisely the same form as the Hamiltonian of free electrodynamics (2.35), allowing us to obtain the ground state of linearized gravity by direct analogy:

$$\Psi_0[h_{ij}] = \exp\left(\frac{1}{8\pi^2} \int d^3x d^3y \frac{h_{ij,k}(\mathbf{x})h_{ij,k}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \right) . \tag{2.94}$$

The analogy between particles whose spin differs by one is extremely useful; even in the case of loop effects and interacting theories it allows us to generalize many results to arbitrary spin without repeating the calculations explicitly. We will make extensive use of this in Chapter 6.

Chapter 3

Perturbative and non-perturbative approaches to the Schrödinger equation

3.1 Loop expansion and renormalization

Our main incentive for studying the Schrödinger representation is as a non-perturbative approach to field theory, but there are some issues, such as renormalization, which are best dealt with in perturbation theory. Also, it is useful to be able to make contact with standard perturbative results.

For a generic quantum field theory, the vacuum WF can be written as $\Psi_0 = \exp(W/\hbar)$ where W generates *connected* Feynman diagrams. So, for example, the Schrödinger equation for the scalar field vacuum becomes

$$E_0 = \frac{1}{2}\int d^dx \left(\varphi(\mathbf{x})(m^2 - \nabla^2)\varphi(\mathbf{x}) - \left(\frac{\delta W}{\delta\varphi(\mathbf{x})}\right)^2 - \hbar\frac{\delta^2 W}{\delta\varphi(\mathbf{x})^2}\right) + V[\varphi]. \tag{3.1}$$

We can expand W in powers of $\hbar$

$$W = W^{(0)} + \hbar W^{(1)} + \hbar^2 W^{(2)} + \ldots \tag{3.2}$$

and $W^{(i)}$ in powers of φ

$$W^{(i)} = \sum_{n=2}^{\infty} \int d^d x_1 \dots d^d x_n f_n^{(i)}(\mathbf{x}_1, \dots, \mathbf{x}_n)\varphi(\mathbf{x}_1)\dots\varphi(\mathbf{x}_n). \tag{3.3}$$

For the vacuum WF we can assume that the functions $f_n^{(i)}$ are translation invariant; this ensures that the vacuum is a zero eigenstate of the momentum operator. We can also assume without loss of generality that they are symmetric. Substituting the expansions into (3.1) gives us simple equations which we can solve order by order in $\hbar$ and φ. The order $\hbar^0$ or tree-level contribution gives the Hamilton-Jacobi equation for W_0, which corresponds to the classical action. At $O(\varphi^2)$ this is

$$\int d^d y f_2^{(0)}(\mathbf{x}, \mathbf{y}) f_2^{(0)}(\mathbf{y}, \mathbf{z}) = (m^2 - \nabla^2)\delta(\mathbf{x} - \mathbf{z}), \tag{3.4}$$

so that $f_2^{(0)}(\mathbf{x}, \mathbf{y}) = -\frac{1}{2}\sqrt{m^2 - \nabla^2}\delta(\mathbf{x} - \mathbf{y})$, ie. the tree-level contribution to the two-point function reproduces the result we found earlier for free fields. Now suppose we include a φ^4 interaction term

$$V[\varphi] = \frac{g}{4}\int d^d x \varphi(\mathbf{x})^4 = \frac{g}{4}\int \frac{d^d p_1}{(2\pi)^d}\cdots\frac{d^d p_1}{(2\pi)^d}\delta(\mathbf{p}_1 + \dots + \mathbf{p}_4)\varphi(\mathbf{p}_1)\dots\varphi(\mathbf{p}_4). \tag{3.5}$$

Then according to (3.1) the tree-level four-point function (in momentum space) satisfies

$$2\sum_{i=1}^{4} f_2^{(0)}(\mathbf{p}_i) f_4^{(0)}(\mathbf{p}_1, \dots, \mathbf{p}_4) = \frac{g}{4}\delta(\mathbf{p}_1, \dots, \mathbf{p}_4), \tag{3.6}$$

with the solution

$$f_4^{(0)}(\mathbf{p}_1, \dots, \mathbf{p}_4) = -\frac{g}{4}\frac{1}{\sum \omega(\mathbf{p}_i)}\delta(\mathbf{p}_1, \dots, \mathbf{p}_4). \tag{3.7}$$

Similarly the tree-level six-point function is obtained as

$$f_6^{(0)}(\mathbf{p}_1, \dots, \mathbf{p}_6) = \frac{1}{2\sum \omega(\mathbf{p}_i)}\int \frac{d^d p}{(2\pi)^d} f_4^{(0)}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) f_4^{(0)}(-\mathbf{p}, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6). \tag{3.8}$$

The one-loop or $O(\hbar)$ n-point functions are obtained in a similar fashion from (3.1). At $O(\varphi^2)$ we have

$$\frac{6g}{4}\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{2(\omega(p)+\omega(q))} = 2\int \frac{d^dq}{(2\pi)^d}\omega(q)f_2^{(1)}(q), \tag{3.9}$$

so that

$$f_2^{(1)}(q) = \frac{3g}{8}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(\omega(p)+\omega(q))}\frac{1}{\omega(q)}. \tag{3.10}$$

If we regulate the p-integral by introducing a momentum cutoff $p^2 < \Lambda^2$, then for $d = 3$ we find

$$f_2^{(1)}(q) \sim \frac{A\Lambda^2}{\omega(q)} + B\omega(q)\ln\Lambda + C\Lambda + f_{\text{reg}}(q), \tag{3.11}$$

where f_{reg} is regular as $\Lambda \to \infty$. The first two divergences can be absorbed by performing the usual mass and wavefunction renormalizations, and the third may be removed by subtracting a counterterm $C\Lambda\int d^dx\varphi(\mathbf{x})^2$ from the Lagrangian. In [3] it was shown that this is a general feature of QFT in the Schrödinger representation; wave-functionals are finite as the cutoff is removed, provided that in addition to the usual renormalization we subtract suitable local counterterms from the action. These additional counterterms are necessitated by the boundary terms which define the quantization surface. Their form is constrained on dimensional grounds and must respect all the symmetries of the theory; for this reason it is easy to show that they are finite in number.

3.1.1 Renormalization of Yang-Mills

In gauge theory, gauge invariance is thought to preclude such additional counterterms [48, 50], so that physical WF's should be finite provided we renormalize masses, coupling constants, etc. in the usual way. For example, in Yang-Mills theory, we can make the coupling constant g an appropriate function of a momentum cutoff, and subtract the vacuum energy from the Hamiltonian. Thus the Schrödinger equation for the vacuum becomes

$$\lim_{s\searrow 0}\left(-\frac{\hbar^2 g^2(s)}{2}\Delta_s+\frac{1}{2g^2(s)}\mathcal{B}-E(s)\right)\Psi_0=0 \tag{3.12}$$

where

$$\Delta_s=\int d^3x\,d^3y\frac{\delta}{\delta A^R_\rho(\mathbf{y})}\Lambda(\mathbf{x},\mathbf{y})\frac{\delta}{\delta A^R_\rho(\mathbf{x})},\quad \mathcal{B}=\int d^3x\,\mathrm{tr}\mathbf{B}^2. \tag{3.13}$$

Λ is a kernel regularizing the coincident functional derivatives in the functional Laplacian Δ (which represents $\mathbf{E}^2$ in the Hamiltonian). The regularization must be done in a gauge invariant fashion, so we choose

$$\Lambda(\mathbf{x},\mathbf{y})=S(\mathbf{x},\mathbf{y})\int_{\mathbf{p}^2<1/s}\frac{d^3p}{(2\pi)^3}\,e^{i\,\mathbf{p}.(\mathbf{x}-\mathbf{y})} \tag{3.14}$$

where as before S is a Wilson line. Again, we can expand everything in powers of $\hbar$

$$W[\mathbf{A}]=\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n W_n[\mathbf{A}],\quad g^2=\sum_{n=0}^{\infty}\hbar^n g_n^2,\quad E(s)=\sum_{n=1}^{\infty}\hbar^n E_n(s). \tag{3.15}$$

Substituting into the Schrödinger equation, we have at tree-level the Hamilton-Jacobi equation

$$g_0^4\int\frac{d^3p}{2\pi^3}\frac{\delta W_0}{\delta A^R_\alpha(\mathbf{p})}\frac{\delta W_0}{\delta A^R_\alpha(-\mathbf{p})}=\mathcal{B}, \tag{3.16}$$

and at one-loop

$$\lim_{s\searrow 0}\left\{\left(g_0^2\Delta_s W_0+2g_0^2\int d^3x\frac{\delta W_0}{\delta A^R}\cdot\frac{\delta W_1}{\delta A^R}\right)+\frac{2g_1^2}{g_0^4}\mathcal{B}+2E_1(s)\right\}=0. \tag{3.17}$$

Now $\Delta_s W_0$ diverges as $s\searrow 0$, but we can choose g_1^2 in such a way as to cancel this divergence. Consider the $O(\mathbf{A}^2)$ part of $\Delta_s W_0$. Because of gauge invariance, this contains a term of the form

$$c\ln s \int \frac{d^3p}{(2\pi)^3} A_\mu^A(\mathbf{p}) A_\nu^A(-\mathbf{p}) \big(p^\mu p^\nu - p^2 \nabla^{\mu\nu}\big) \tag{3.18}$$

for some constant c. We can compute $c = 11N/(24\pi^2)$ by evaluating W_0 up to $O(\mathbf{A}^4)$ from the Hamilton-Jacobi equation [9]. Similarly, to $O(\mathbf{A}^2)$ $\mathcal{B}$ has the form

$$\int \frac{d^3p}{(2\pi)^3} A_\mu^A(\mathbf{p}) A_\nu^A(-\mathbf{p}) \big(p^\mu p^\nu - p^2 \nabla^{\mu\nu}\big) \tag{3.19}$$

and we conclude that to cancel the divergent term (3.18) we must take $2g_1^2/g_0^4 = c\ln(s\mu^2)$ for some mass scale μ. So to one loop, the coupling is

$$g^2 = g_0^2 + \hbar g_1^2 = g_0^2 + \hbar \frac{11 g_0^4 N \log(s\mu^2)}{48\pi^2}, \tag{3.20}$$

allowing us to calculate the one-loop beta-function

$$\beta = \mu \frac{\partial g^2}{\partial \mu} = -\hbar \frac{11 g_0^4 N}{24\pi^2}. \tag{3.21}$$

In perturbation theory we can use similar methods to determine higher loop corrections to $g(s)$ and $E(s)$. In principle these could be computed non-perturbatively as well, but in four dimensions an analytical treatment is probably beyond current technology. When fermions are included there are of course additional renormalizations of the wavefunction and the fermion mass to be dealt with.

3.2 Large-distance expansion

If, at least with current knowledge, renormalization is most easily dealt with in perturbation theory, there are nevertheless many results which may be obtained non-perturbatively in the Schrödinger representation. Also, the formalism is flexible enough to allow the simultaneous use of perturbative and non-perturbative strategies in any concrete calculation. The non-perturbative techniques which we are about to describe will be of considerable use even within perturbation theory.

In the last chapter we saw various examples of WF's which can be expanded in local terms for slowly varying fields. For example, we expect that the Schrödinger functional of any QFT can be expanded in this way for fields which vary slowly on the scale of the time parameter. For theories with a mass gap, we expect the vacuum WF to have such an expansion for fields which vary slowly on the scale of the lightest mass. Of course, as we discussed earlier and will see explicitly in the next chapter, gauge invariance can spoil the local nature of this expansion by introducing massless modes which connect gauge equivalent configurations.

Although expansions of this type are valid only for slowly varying fields, we can use the analyticity properties of the WF's under complex rescalings to reconstruct their behaviour for large momentum. Thus from a local expansion of the WF's, we can in principle calculate all physically interesting information, including that which is sensitive to UV modes, such as the computation of the particle spectrum, or the beta function.

From the point of view of the Schrödinger equation, solving for the local expansion of a WF is equivalent to solving an infinite set of coupled algebraic equations for the coefficients of the local terms. However, in solving the Schrödinger equation, care must be taken with the regularization procedure, since it will not in general commute with the expansion in local terms.

If a WF admits a local expansion, it can also be expanded locally order by order in the loop expansion. Thus this approach to the Schrödinger equation may be implemented within a perturbation theory approach as well.

To illustrate the general strategy we are outlining, we will first show that the analyticity properties of WF's allow them to be reconstructed from their local expansions, and then show how the coefficients of these expansions may be obtained from the Schrödinger equation.

3.2.1 Analyticity of Wave-Functionals

To begin with we will show how the small distance cutoff dependence of the free scalar vacuum energy can be reconstructed from its large distance behaviour. Consider the Hamiltonian (2.9). The vacuum energy is given by the action of $\hat{\pi}^2$, which is represented by a functional Laplacian Δ, on the vacuum functional $\Psi_0 = e^{-\frac{1}{2}\varphi\omega\varphi}$. This clearly diverges, so as before we introduce a momentum cutoff which regulates the Laplacian

$$\Delta_s = \int_{p^2<1/s} \frac{d^d p}{(2\pi)^d} \frac{\delta^2}{\delta\varphi(\mathbf{p})\delta\varphi(-\mathbf{p})}, \tag{3.22}$$

and leads to the well-defined vacuum energy density

$$\mathcal{E} = \frac{\Delta_s \omega}{2V} = \frac{1}{2}\int_{p^2<1/s} \frac{d^d p}{(2\pi)^d}\sqrt{p^2+m^2} \sim \frac{\text{const.}}{s^{(d+1)/2}} \quad \text{as} \quad s \to 0. \tag{3.23}$$

Now suppose we insert the local expansion (2.23) of ω into (3.23). This does *not* give the correct behaviour as $s \to 0$, because the expansion is only valid for $p^2 < m^2$. But we can remedy this by resumming the series expansion to obtain the correct small distance behaviour. If we rewrite (3.23) as

$$\mathcal{E}(s) = \frac{1}{2s^{d/2}} \int_{p^2<1} \frac{d^d p}{(2\pi)^d}\sqrt{p^2/s+m^2}, \tag{3.24}$$

then it is easy to see that it extends to an analytic function of s on the complex plane with the negative real axis removed. We can expand the square root provided that $|s|m^2 > 1$. Consider the integral

$$I(\lambda) = \frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\lambda s}\mathcal{E}(s), \tag{3.25}$$

where C is a keyhole contour which runs under the negative real axis up to $s = -R$, where $R > 1/m^2$, around the circle of radius R about the origin, and back to $s = -\infty$

above the negative real axis. We can evaluate this integral using the local expansion

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{\alpha_n}{(n+d/2)!} \left(\frac{\lambda}{m^2}\right)^{n+d/2} , \tag{3.26}$$

where $\alpha_n = \frac{V_d m^{d+1} \frac{1}{2}!}{2(2\pi)^d (d+2n) n! (1/2-n)!}$ with V_d the volume of the unit sphere in d dimensions. Alternatively, we can evaluate (3.25) by collapsing the contour until it just surrounds the negative real axis. As $\lambda \to \infty$ the contribution from the negative real axis is exponentially suppressed, and we are left with $\mathcal{E}(0)$.

In practise we will want to truncate the series to a finite number of terms. Since (3.26) is an alternating series, truncating it at $n = N$, say, gives an error less than

$$\left| \frac{\alpha_{N+1}}{(N+d/2+1)!} \left(\frac{\lambda}{m^2}\right)^{N+d/2+1} \right| \sim \frac{1}{N} \left(\frac{e\lambda}{Nm^2}\right)^{N+d/2+1} . \tag{3.27}$$

If we take $\lambda = N\mu^2$ with $\mu^2 < m^2$ then the truncation error goes to zero for large N. We can choose μ so that the error due to approximating $\mathcal{E}(0)$ by (3.25) is comparable in magnitude to (3.27).

Now we will describe how a similar technique allows WF's to be reconstructed from their large distance expansions. If we write WF's as functionals of a scaled field $\varphi^s(\mathbf{x}) = \varphi(\mathbf{x}/\sqrt{s})$ we wish to show that they form analytic functions of s in the cut plane. This will allow us to express their behaviour for small s in terms of their behaviour for large s, via Cauchy's theorem, just as we did for the vacuum energy.

Consider the path integral representation (2.10) for the Schrödinger functional, written in terms of the scaled fields

$$\int \mathcal{D}\phi \, \exp -i \int d^{d+1}x \left\{ \phi(\partial^2 + m^2)\phi - 2\delta(x_0 - \tau)(\phi - \varphi^s)\dot{\phi} + 2\delta(x_0)(\phi - \tilde{\varphi}^s)\dot{\phi} \right\} . \tag{3.28}$$

Suppose we have only one spatial dimension. Since we have chosen the space-time to be Euclidean, we can interchange the rôle of the time coordinate with that of the spatial coordinate to obtain an integral over the space-time $-\infty < t < \infty$, $0 < x < \tau$. The time derivatives in (3.28) become space derivatives, and we can reinterpret the path integral as a time-ordered vacuum expectation value in the rotated space-time

$$\Psi_\tau[\varphi^s, \tilde{\varphi}^s] = T\langle 0_r| \exp \int dt \left(2\varphi^s \phi'|_{x=\tau} - 2\tilde{\varphi}^s \phi'|_{x=0}\right)|0_r\rangle. \tag{3.29}$$

Here $|0_r\rangle$ is the vacuum of the theory with Euclidean action

$$S_E = i \int d^2x \left\{\phi(\partial^2 + m^2)\phi + 2(\delta(x_0) - \delta(x - \tau))\phi\dot{\phi}\right\} \tag{3.30}$$

defined on the rotated space-time. We can expand the exponential in (3.29) and use the time-evolution operator $e^{-\hat{H}_r t}$ of the rotated theory to generate the time-dependence. We obtain

$$\begin{aligned}\Psi_\tau[\varphi^s, \tilde{\varphi}^s] &= \sum_{n,m} \int_{-\infty}^{\infty} dt_n \int_{-\infty}^{t_n} dt_{n-1} \ldots \int_{-\infty}^{t_2} dt_1 \frac{1}{\pi^n} \int dp_1 \ldots dp_n\, e^{ip_i t_i} \\ &\quad \times \varphi^s(p_1) \ldots \varphi^s(p_m)\tilde{\varphi}^s(p_{m+1}) \ldots \tilde{\varphi}^s(p_n) \langle 0_r|\hat{\phi}'(0) e^{-(t_n - t_{n-1})\hat{H}^r} \hat{\phi}'(0) \ldots \\ &\quad \ldots e^{-(t_2 - t_1)\hat{H}^r} \hat{\phi}'(0)|0_r\rangle,\end{aligned} \tag{3.31}$$

where the first m occurrences of $\hat{\phi}'(0)$ are evaluated at $x = \tau$, and the remaining ones at $x = 0$. The time integrals may be performed to give

$$\begin{aligned}\Psi_\tau[\varphi^s, \tilde{\varphi}^s] &= \sum_{n,m} \int dp_1 \ldots dp_n \delta(\textstyle\sum p_i) \varphi^s(p_1) \ldots \varphi^s(p_m) \tilde{\varphi}^s(p_{m+1}) \ldots \tilde{\varphi}^s(p_n) \\ &\quad \times s^{n/2} \langle 0_r|\hat{\phi}'(0) \frac{1}{\sqrt{s}\hat{H}^r + i(\sum p_i)} \hat{\phi}'(0) \ldots \hat{\phi}'(0) \frac{1}{\sqrt{s}\hat{H}^r + ip_1} \hat{\phi}'(0)|0_r\rangle\end{aligned} \tag{3.32}$$

Inserting a complete set of energy eigenstates between occurrences of the Hamiltonian in (3.32) allows us to write this as a sum of terms in which the s-dependence is contained in fractions of the form $1/(E - i\sum_{i=1}^{a} p_i)$; since the eigenvalues of the

Hamiltonian are real, this shows that Ψ_τ is analytic in s apart from singularities occuring on the negative real axis.

To extend this result to fields in more than one spatial dimension, we can consider scaling each dimension seperately, and applying the same argument as above. Similarly, this result extends straightforwardly to fields of higher spin, including fermions and gauge fields.

Now consider W_τ, the logarithm of Ψ_τ. To any finite order in the sources φ and $\tilde{\varphi}$ this may be written as a sum of terms which are products of the terms appearing in (3.32), and it is thus also analytic in the complex s-plane with the negative real axis removed. Because particles are restricted to the range $0 < x < \tau$ the theory has a mass-gap of value $1/\tau$, even if $m = 0$. Thus all the fractions $1/(E - i\sum_{i=1}^{a} p_i)$ have either $E = 0$ or $E > 1/\tau$. In the latter case, provided the sources have compact support in momentum space, and we take s to be sufficiently large, we can expand the fractions in positive powers of the momenta, leading to a local expansion of the wave-functional. The fractions with $E = 0$ cannot be expanded, but these denominators must cancel against powers of momentum in the numerator to avoid violating the cluster decomposition property of W_τ. This cancellation is highly non-trivial, but it can be verified explicitly at each order by laborious calculations.

So in conclusion, for sufficiently large s, $W_\tau[\varphi^s, \tilde{\varphi}^s]$ can be expanded in terms which are local functionals of the sources. Using the fact that $W_\tau[\varphi^s, \tilde{\varphi}^s]$ is analytic in s allows us to reconstruct the behaviour for small s from from this expansion, writing

$$W_\tau[\varphi, \tilde{\varphi}] = \lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C \frac{ds}{s-1} e^{\lambda(s-1)}\sqrt{s}\,W_\tau[\varphi^s, \tilde{\varphi}^s], \tag{3.33}$$

with C the same contour as before. We can improve the convergence of the series by applying an additional resummation:

$$\lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C' \frac{dr}{r} e^{\lambda^2 r} I(1/\sqrt{r}). \tag{3.34}$$

3.2.2 The Schrödinger equation for the local expansion

In principle, by substituting the local expansion of a WF into the Schrödinger equation, we can reduce it to a set of algebraic equations for the coefficients of the local terms. But as mentioned earlier, the need for regularization complicates this procedure. Consider the simple case of φ^4 theory in (1+1) dimensions. This theory is super-renormalizable, and the Hamiltonian can be regularized by normal-ordering with respect to an arbitrary mass scale. Alternatively, the regularization may be performed by subtracting mass and vacuum energy counterterms. It is a relatively straightforward procedure to obtain the mass and energy counterterms as explicit functions of a momentum cutoff; we require functions $m(s)$ and $\mathcal{E}(s)$ with the property that

$$:\hat{H} := \lim_{s\searrow 0} \hat{H}_s = \lim_{s\searrow 0}\int dx\left(\hat{\pi}_s^2 + m^2(s)\phi_s^2 + \frac{g}{4}\phi_s^4 - \mathcal{E}(s)\right) \tag{3.35}$$

for the cut-off fields

$$\phi_s = \int dy \int_{p^2<1/s} \frac{dp}{2\pi} e^{ip(x-y)}\phi(y), \quad \hat{\pi}_s = \int dy \int_{p^2<1/s} \frac{dp}{2\pi} e^{ip(x-y)}\hat{\pi}(y). \tag{3.36}$$

These are given by

$$m^2(s) = m^2 - \hbar\frac{6g}{4}\int_{p^2<1/s}\frac{dp}{2\pi}\frac{1}{\sqrt{p^2+m^2}}, \tag{3.37}$$

and

$$\mathcal{E}(s) = \frac{\hbar}{2}\int_{p^2<1/s}\frac{dp}{2\pi}\left(\sqrt{p^2+m^2} + \frac{m^2(s)-m^2}{2\sqrt{p^2+m^2}}\right) + \frac{g\hbar^2}{32}\left(\int_{p^2<1/s}\frac{dp}{2\pi}\frac{1}{\sqrt{p^2+m^2}}\right)^2. \tag{3.38}$$

We chose to normal order with respect to the classical mass m, but we could have normal-ordered with respect to any other mass, reflecting the freedom in our choice of subtraction point which lends the usual arbitrariness to the counterterms.

Now the Schrödinger equation may be written as

$$\lim_{s\searrow 0}\left(\frac{\partial}{\partial\tau}\Psi_\tau[\varphi^s,\tilde{\varphi}^s]+\hat{H}_s\Psi_\tau[\varphi^s,\tilde{\varphi}^s]\right)=0. \tag{3.39}$$

Using the argument already described, we can show that the LHS of this equation extends to an analytic function on the s plane with the negative real axis removed. This allows us to express (3.39) as

$$\lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_C\frac{ds}{s}e^{\lambda s}\left(\frac{\partial}{\partial\tau}\Psi_\tau[\varphi^s,\tilde{\varphi}^s]+\hat{H}_s\Psi_\tau[\varphi^s,\tilde{\varphi}^s]\right)=0. \tag{3.40}$$

where in this expression we can assume that $\Psi_\tau=e^{W_\tau}$, where W_τ is a sum of local functionals

$$W_\tau=\sum_{n,m,i}\int\frac{dp_1}{2\pi}\dots\frac{dp_{2n}}{2\pi}\varphi(p_1)\dots\varphi(p_{2n})\delta(p_1+\dots+p_{2n})a_{n,m}^{i_1,\dots,i_{2n}}p_1^{i_1}\dots p_{2n}^{i_{2n}}\tau^m. \tag{3.41}$$

Substituting this into (3.40) gives an infinite set of algebraic equations for the coefficients $a_{n,m}^{i_1,\dots,i_{2n}}$. In the following chapters we will encounter many more examples of equations like this; in many of the cases we will consider they can be solved analytically, but in general they are well suited to numerical treatment.

It is worth noting that the renormalization counterterms can also be obtained non-perturbatively from these equations, provided suitable renormalization conditions are imposed.

Chapter 4

The Schwinger model

We now wish to illustrate the non-perturbative solution of the Schrödinger equation by studying a simple interacting theory, QED in (1+1) dimensions with massless fermions, known as the Schwinger model since it was shown by Schwinger to be exactly solvable [36]. This model exhibits many of the features of QCD in higher dimensions, such as the vacuum angle, chiral symmetry breaking, and confinement, but since it can be solved exactly, it is a useful model for studying these phenomena analytically.

Another motivation for studying this model is to shed some light on the apparent breakdown of the local expansion property which seems to result from gauge invariance. In the Schwinger model we will see that this is related to charge screening. This supports the idea that gauge invariant WF's do not in general admit local expansions, even in the presence of a mass gap. However the Schrödinger functional *does* always have an expansion, as we will verify for the Schwinger model by constructing the Schrödinger functional explicitly.

Many of the phenomena which we describe for the Schwinger model, such as the vacuum angle, bosonization, and much of the detailed structure of the Schrödinger equation, carries over directly to more general gauge theories in (1+1) dimensions. Though exact WF solutions have not yet been found for non-abelian gauge groups, our approach may be applied to this problem without further development. We will also investigate what happens when we give the fermions a non-zero mass; we will show how to solve the Schrödinger equation for the exact n-point functions.

In two dimensions, the general form of a physical WF is

$$\Psi = \langle u, u^\dagger, \mathcal{A}|\Psi_f\rangle = \sum_{a=0}^{\infty} \frac{1}{a!} \prod_{n=1}^{a} \int dx_n dy_n u^\dagger(x_n)\gamma^5 u(y_n) e^{ie\int_{y_n}^{x_n} \mathcal{A}} f^a(x_1, y_1, \ldots, x_a, y_a), \tag{4.1}$$

where the Wilson lines $e^{ie\int_{y_n}^{x_n} \mathcal{A}}$ guarantee gauge invariance and f^a are arbitrary functions. It is useful to note that in the presence of the constraints on u and $u^\dagger$, two dimensional gamma matrices have only one effective degree of freedom; ie. in (4.1) we could have written γ^0 or γ^1 in place of γ^5.

Using this and (2.63) we can calculate (dropping a divergent constant)

$$\langle\Psi_g|\Psi_f\rangle = g^0 f^0 + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \epsilon_{i_1 \ldots i_a} g^a(x_1, y_{i_1}, \ldots, x_a, y_{i_a})^* f^a(x_1, y_1, \ldots, x_a, y_a). \tag{4.2}$$

A specific representation of the two dimensional Euclidean gamma matrices is given $\gamma^0 = \sigma^1$, $\gamma^1 = \sigma^2$, $\gamma^5 = \sigma^3$. We will use the fermion representation described in chapter 2, with the gauge invariant representations having

$$Q_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm z \\ \pm z^{-1} & 1 \end{pmatrix}. \tag{4.3}$$

4.1 The Schwinger model vacuum

4.1.1 Path integral solution

To begin with, we will solve the path integral expression (2.73) for the vacuum WF. Since the effective theory is gaussian, the calculation is particularly simple. Later we will show how the same result may be obtained from the Schrödinger equation.

The Schwinger model has the Euclidean action

$$\int d^2x(\bar{\psi}\gamma\cdot(\partial+ieA)\psi+\frac{i}{4}F^{\mu\nu}F_{\mu\nu}). \tag{4.4}$$

For massless fermions we can obtain the fermion determinant by integrating the anomaly [32]; with a U(1) gauge group the result is

$$\det D=\exp\left\{-\frac{e^2}{2\pi}(\partial_\mu\phi)^2\right\}, \tag{4.5}$$

where $A_\mu=\partial_\mu\eta+\epsilon_{\mu\nu}\partial_\nu\phi$. $P_\pm$ are obtained as equal time limits of the Dirac propagator

$$P_\pm=\lim_{\delta t\to 0^\pm}D^{-1}, \tag{4.6}$$

and

$$D=e^{ie(\eta+\gamma^5\phi)}\gamma^0\gamma.\partial e^{-ie(\eta+\gamma^5\phi)} \tag{4.7}$$

At $t=0$ we want $A_1=\mathcal{A}$, which implies that $\eta=\int^x(\mathcal{A}+\dot{\phi})$. For ϕ we can without loss of generality choose the boundary condition $\phi|_{t=0}=0$. Thus we have

$$P_\pm=\frac{1}{2}\left(\delta(x-y)\pm\frac{\gamma^5}{\pi}\mathcal{P}\frac{1}{x-y}\right)\exp\left\{ie\int_y^x(\mathcal{A}+\dot{\phi})\right\}. \tag{4.8}$$

Here $\mathcal{P}$ denotes the principal part. The operator $A^{-1}C$ is easily found; (2.73) becomes

$$\begin{aligned}\Psi[u,u^\dagger,\mathcal{A}]= &\int D\phi\exp\left\{-\frac{1}{2}\phi(\partial^4+M^2\partial^2)\phi\right.\\ &\left.+\frac{2}{\pi}\int dxdy\left[u^\dagger(x)\gamma^5u(y)\mathcal{P}\frac{1}{x-y}\exp\left\{ie\int_y^x(\mathcal{A}+\dot{\phi})\right\}\right]\right\}\end{aligned} \tag{4.9}$$

where $M^2=\frac{e^2}{\pi}$. We can integrate this to give

$$
\begin{aligned}
\Psi[u,u^\dagger,\mathcal{A}] = \quad & \sum_{a=0}^{\infty}\frac{1}{a!}\prod_{n=0}^{a}\left[\frac{2}{\pi}\int dx_n dy_n u^\dagger(x_n)\gamma^5 u(y_n)\mathcal{P}\frac{1}{x_n-y_n}e^{ie\int_y^x \mathcal{A}}\right.\\
& \left.\times\exp\left\{\sum_{i,j=1}^{a}\Phi(x_i-y_i)-\sum_{j>i=1}^{a}\left[\Phi(x_i-x_j)+\Phi(y_i-y_j)\right]\right\}\right] \quad (4.10)
\end{aligned}
$$

where

$$
\Phi(x)=\int\frac{dp}{2\pi}\left(\frac{1}{|p|}-\frac{\sqrt{p^2+M^2}}{p^2}\right)(1-\cos px). \quad (4.11)
$$

It may be explicitly verified that (4.10) satisfies both Gauss' law and the Schrödinger equation. Note that this expression is both UV and IR convergent.

Since the Dirac operator is flavour neutral, all of this is perfectly valid for the Schwinger model with N_f flavours, provided we put a flavour index on the fermions and set $M^2 = N_f\frac{e^2}{\pi}$.

4.1.2 VEV's, chiral condensate and vacuum angle

Define

$$
\tilde{C}(x)=\int\frac{dp}{2\pi}\left(\frac{1}{|p|}-\frac{\sqrt{p^2+M^2}}{p^2}\right)e^{ipx}, \quad (4.12)
$$

so that $\Phi(x)=\tilde{C}(0)-\tilde{C}(x)$.

From the analysis of [43] it is clear that the two-point function $\langle\bar{\psi}(x)_{L/R}\psi(y)_{L/R}\rangle$ is given by

$$
\begin{aligned}
\langle\bar{\psi}(x)_L\psi(y)_L\rangle &= \frac{e^{i\theta}}{2\pi}e^{C(x-y)}\\
\langle\bar{\psi}(x)_R\psi(y)_R\rangle &= \frac{e^{-i\theta}}{2\pi}e^{C(x-y)}, \quad (4.13)
\end{aligned}
$$

where $C(x-y)=\int\frac{dp}{2\pi}(\frac{1}{|p|}-\frac{1}{\sqrt{p^2+M^2}})e^{ipx}$ is the propagator of $\phi(x)$ (whereas $\Phi(x-y)$ is the propagator of $\int^x\dot{\phi}$).

In particular, since $\lim_{x\to 0} C(x) = \ln \frac{M}{2} + \gamma$, we have for the chiral condensate

$$\langle \bar{\psi}\psi \rangle_\theta = \frac{Me^\gamma}{2\pi} \cos\theta, \tag{4.14}$$

which corresponds with the well-known result [38].

Since the effective theory has only two-point interactions, the other n-point functions are obtained from $C(x)$ by a trivial extension of this analysis. When we consider the massive model, we will have to include n-point interactions in this analysis.

We have identified the theta angle as a parameter in our choice of boundary conditions. We now demonstrate that this is the same parameter appearing in the instanton physics of the pure gauge theory. From this point of view the vacuum angle is obtained by inserting a term

$$\frac{ie\theta}{2\pi} \int dx \epsilon^{\mu\nu} F_{\mu\nu} \tag{4.15}$$

into the path-integral expression (2.69). This term is proportional to the "instanton number", which is *not* quantized, as a result of the boundary conditions. We have $\epsilon^{\mu\nu}F_{\mu\nu} = -2\partial^2\phi$, but on the other hand $\mathcal{A} = -\int_{-\infty}^{0} dt \partial^2\phi$, so that

$$\int d^2x \epsilon^{\mu\nu} F_{\mu\nu} = 2 \int dx \mathcal{A}. \tag{4.16}$$

Hence

$$\Psi_\theta[u, u^\dagger, \mathcal{A}] = \Psi[u, u^\dagger, \mathcal{A}] e^{\frac{ie\theta}{\pi} \int dx \mathcal{A}} \tag{4.17}$$

so that under a large gauge transformation $\int dx\mathcal{A} \to \int dx\mathcal{A} + \frac{2\pi n}{e}$, we have $\Psi_\theta \to \Psi_\theta e^{2in\theta}$. This phase is twice what we might expect; because of the relation (4.16) large gauge transformations change the instanton number by multiples of two. To change the instanton number by *one* unit, we must perform a large gauge transformation of winding-number one half $\int dx\mathcal{A} \to \int dx\mathcal{A} + \frac{\pi}{e}$. These "half-integer" transformations were first discussed in [33], but should not be thought of as contributing to the vacuum degeneracy [34]. The periodicity of θ is explained by the fact that it is energetically favourable to create an electron-positron pair when $\theta > 2\pi$ [35].

Because of the chiral anomaly, by performing a chiral rotation

$$\psi \to e^{i\theta\gamma^5/2}\psi, \qquad \psi^\dagger \to \psi^\dagger e^{-i\theta\gamma^5/2}, \tag{4.18}$$

in (2.69) the term (4.15) is cancelled. This returns us to the original expression, but with Q replaced by $Q' = Qe^{i\theta\gamma^5}$ and u, $u^\dagger$ chirally rotated so that they satisfy the constraints with Q' instead of Q. In other words, the term (4.15) just modifies the boundary conditions in the expected way. Incidentally, this implies that

$$\langle\Psi_\theta^1|\hat{O}(\hat{\psi},\hat{\psi}^\dagger)|\Psi_\theta^2\rangle = \langle\Psi_0^1|\hat{O}(e^{i\theta\gamma^5/2}\hat{\psi},\hat{\psi}^\dagger e^{-i\theta\gamma^5/2})|\Psi_0^2\rangle, \tag{4.19}$$

which is useful for calculating expectation values.

This discussion is unaffected if the fermions have a non-zero mass, except that the mass term is also altered by the chiral rotation (4.18); $m \to me^{i\theta\gamma^5}$. This indicates that the vacuum angle has a true physical significance in this case, whereas for massless fermions it can be altered by a suitable redefinition of the fields.

4.1.3 Bosonization

The confining nature of the theory is easily seen from (4.10); since the potential (4.36) goes like $|x|$ at large distances, configurations in which a field source goes off to infinity without an accompanying anti-source, are exponentially damped. Infrared slavery and asymptotic freedom are similarly seen by scaling the momentum in (4.36); the effect of such scaling is to make the coupling constant large for small momenta and vice-versa.

Bosonization may be understood by transferring the sources to currents instead of fields. By a gauge transformation, we can always set $\mathcal{A}=0$. Now consider the following object:

$$\Psi[\xi]=\int Du^\dagger Du e^{\int d^2x u^\dagger\gamma^5 u}\Psi[u,\frac{\sqrt{\pi}}{2}\xi u^\dagger,0]. \tag{4.20}$$

Inserting the path-integral expression (4.9) and performing the $u, u^\dagger$ integrations gives (with boundary conditions as before)

$$\Psi[\xi]=\int D\phi D\psi^\dagger D\psi e^{-S_B-S_F+\sqrt{\pi}\mathrm{Tr}(\xi\psi^\dagger\gamma^5\psi)}. \tag{4.21}$$

This is a functional whose argument couples to a local fermion current. Using (4.5) the fermion integration yields

$$\begin{aligned}\Psi[\xi] &= \int D\phi e^{-S_B-\frac{1}{2\pi}\int d^2x(\partial\varphi)^2}\\ &= \int D\phi e^{-\frac{1}{2}\int d^2x(\partial^2\phi)^2-\frac{1}{2\pi}\int d^2x\left(e^2\phi\partial^2\phi-2e\tilde{\phi}\phi+\tilde{\phi}\frac{1}{\partial^2}\tilde{\phi}\right)},\end{aligned} \tag{4.22}$$

where $\varphi=e\phi-\frac{1}{\partial^2}\tilde{\phi}$ and $\tilde{\phi}=\epsilon_{\lambda\mu}\partial_\lambda J_\mu$ with $J_0=0,\ J_1=i\sqrt{\pi}\xi\delta(t)$. Hence $\tilde{\phi}=i\sqrt{\pi}\frac{\partial}{\partial t}(\xi\delta(t))$ and performing the remaining integration, we find

$$\Psi[\xi]=e^{-\frac{1}{2}\int dx\xi\sqrt{-\partial_x^2+M^2}\xi}, \tag{4.23}$$

which is the vacuum WF of a free boson field of mass M.

From this we can easily reproduce the standard correspondence between bosonic and fermionic operators. (4.21) implies that

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\frac{\delta}{\delta\xi}\Psi[\xi] &= \int D\phi D\psi^\dagger D\psi\psi^\dagger\gamma^5\psi e^{-S_B-S_F+\sqrt{\pi}\mathrm{Tr}(\xi\psi^\dagger\gamma^5\psi)}\\ &= \int Du^\dagger Du e^{\int d^2x u^\dagger\gamma^5 u}\hat{\psi}^\dagger\gamma^5\hat{\psi}\Psi[u,\frac{\sqrt{\pi}}{2}\xi u^\dagger,0].\end{aligned} \tag{4.24}$$

The fermion operators are represented as in (2.55) and (2.56). Now define

$$\Psi[\xi,\chi]=\int D\phi D\psi^{\dagger}D\psi e^{-S_B-S_F+\sqrt{\pi}\mathrm{Tr}(\xi\psi^{\dagger}\gamma^5\psi+\chi\psi^{\dagger}\psi)}. \tag{4.25}$$

Performing the fermion integration we find $J_1 = i\sqrt{\pi}\xi\delta(t)$, as before, and $J_0 = \sqrt{\pi}\chi\delta(t)$, so that $\tilde{\phi}=i\sqrt{\pi}\frac{\partial}{\partial t}(\xi\delta(t))+\sqrt{\pi}\frac{\partial}{\partial t}(\chi\delta(t))$. Thus

$$\Psi[\xi,\chi]=e^{\int dx\left(-\frac{1}{2}\xi\sqrt{-\partial^2+M^2}\xi-\chi\xi'-\frac{1}{2}\chi\partial_x^2(-\partial^2+M^2)^{-1/2}\chi\right)}. \tag{4.26}$$

Hence

$$\begin{aligned}-\frac{1}{\sqrt{\pi}}\xi'\Psi[\xi] &= \frac{1}{\sqrt{\pi}}\frac{\delta}{\delta\chi}\Psi[\xi,\chi]|_{\chi=0}\\ &= \int Du^{\dagger}Du e^{\int d^2x u^{\dagger}\gamma^5 u}\hat{\psi}^{\dagger}\hat{\psi}\Psi[u,\frac{\sqrt{\pi}}{2}\xi u^{\dagger},0].\end{aligned} \tag{4.27}$$

If we represent $\dot{\xi}$ as $-i\frac{\partial}{\partial\xi}$ then (4.24) and (4.27) imply that

$$\begin{aligned}&\int Du^{\dagger}Du e^{\int d^2x u^{\dagger}\gamma^5 u}\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi}\Psi[u,\frac{\sqrt{\pi}}{2}\xi u^{\dagger},0]\\ &\quad=\int Du^{\dagger}Du e^{\int d^2x u^{\dagger}\gamma^5 u}\frac{-1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\xi\Psi[u,\frac{\sqrt{\pi}}{2}\xi u^{\dagger},0].\end{aligned} \tag{4.28}$$

It follows that $\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi}\sim\frac{-1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\xi$ on all physical states. This equivalence may be exploited in a number of ways; for example it immediately allows us to identify the creation and annihilation operators of the theory

$$a_{\pm}(p)=\frac{1}{\sqrt{2}}\left(\omega^{1/2}j_0(p)/p\mp\omega^{-1/2}j_5(p)\right), \tag{4.29}$$

where $j_0(p)$ and $j_5(p)$ are the Fourier transforms of $j_0(x)=\hat{\psi}^{\dagger}(x)\hat{\psi}(x)$ and $j_5(x)=\hat{\psi}^{\dagger}(x)\gamma^5\hat{\psi}(x)$ respectively. Also, in calculating physical expectation values

$$\langle\Psi_1|O(\hat{\psi},\hat{\psi}^{\dagger})|\Psi_2\rangle \tag{4.30}$$

the operator O can be represented in terms of equivalent bosonic operators; the resulting calculations are generally much easier. To illustrate this, consider the chiral condensate

$$\begin{aligned}
\langle\hat{\bar{\psi}}\hat{\psi}\rangle_\theta &= \frac{\langle\theta|\hat{\bar{\psi}}\hat{\psi}|\theta\rangle}{\langle\theta|\theta\rangle} \\
&= \frac{\langle 0|\hat{\bar{\psi}}e^{i\theta\gamma^5}\hat{\psi}|0\rangle}{\langle 0|0\rangle},
\end{aligned} \tag{4.31}$$

where we have used (4.19). Now the bosonic operators corresponding to the chiral densities $\hat{\bar{\psi}}(1\pm\gamma^5)\hat{\psi}$ are

$$\frac{Me^\gamma}{2\pi} : e^{\pm i\sqrt{4\pi}\xi} : \tag{4.32}$$

(this can be verified in a similar way to the other bosonization formulae). The normal-ordering is perfomed with respect to the scale M, and γ is the Euler constant. Thus $\hat{\bar{\psi}}e^{i\theta\gamma^5}\hat{\psi}$ corresponds to $\frac{Me^\gamma}{2\pi} : \cos(\sqrt{4\pi}\xi+\theta) :$ whose vacuum expectation value is easily ascertained from (4.23). We have as before

$$\langle\hat{\bar{\psi}}\hat{\psi}\rangle_\theta = \frac{Me^\gamma}{2\pi}\cos\theta. \tag{4.33}$$

4.1.4 Solution from the Schrödinger equation

A solution for the vacuum WF of the Schwinger model (on a circle) was given in [37], where it was found by solving the Schrödinger equation. We now wish to show how a similar calculation can reproduce our result. Define

$$\hat{J}_M(x,y) = \hat{\psi}^\dagger(x)M\hat{\psi}(y)e^{ie\int_y^x \hat{A}}. \tag{4.34}$$

Gauss' law implies that $\partial_x \hat{E}(x) \sim e\hat{\jmath}_0(x)$ on physical states, where $\hat{\jmath}_0 = \hat{\psi}^\dagger\hat{\psi}$. By Fourier transforming, this in turn gives us

$$\int dx \hat{E}^2(x) \sim \int dxdy \hat{\jmath}_0(x)\hat{\jmath}_0(y)\bar{V}(x-y) + \hat{F}^2, \tag{4.35}$$

where $\hat{F} = \int dx \hat{E}$ and

$$\bar{V}(x) = \frac{e^2}{2\pi}\int \frac{dp}{p^2} e^{ipx}. \tag{4.36}$$

Thus the Hamiltonian acting on physical states can be written as

$$\hat{H} = \int dxdy \hat{\jmath}_0(x)\hat{\jmath}_0(y)\bar{V}(x-y) - i\int dx \lim_{y\to x} \partial_y \hat{J}_{\gamma^5}(x,y) + \hat{F}^2. \tag{4.37}$$

Note that the fermion interaction is completely determined by Gauss' law. The $\hat{F}$ term is represented by $\frac{\partial}{\partial(\int dx \mathcal{A})}$, and may be diagonalized by inserting a factor of $e^{\frac{ie\theta}{\pi}\int dx\mathcal{A}}$ into the WF's, ie. it gives the vacuum angle. For the moment we will take this to vanish, so that this term may be ignored.

Substituting in (4.37) using (2.55) and (2.56) gives

$$\begin{aligned}\hat{H} &= \int dxdy \bar{V}(x-y)\Big(u^\dagger(x)\frac{\delta}{\delta u^\dagger(x)} + \frac{\delta}{\delta u(x)}u(x)\Big)\Big(u^\dagger(y)\frac{\delta}{\delta u^\dagger(y)} + \frac{\delta}{\delta u(y)}u(y)\Big) \\ &\quad + \int dx \lim_{y\to x}\partial_y\Big[\Big(\frac{1}{2}\frac{\delta}{\delta u(x)}\gamma^5\frac{\delta}{\delta u^\dagger(y)} + 2u^\dagger(x)\gamma^5 u(y)\Big)e^{ie\int_y^x \mathcal{A}}\Big].\end{aligned} \tag{4.38}$$

Gauge invariant WF's have the form (4.1). If the distributions f^a are taken to be translation invariant, then (4.1) is simultaneously a zero eigenstate of momentum. Inspired by the path integral calculation of the last section we make the Ansatz

$$\begin{aligned}&f^a(x_1,y_1,\ldots,x_a,y_a) \\ &= 2\prod_{n=1}^{a}\Delta(x_n-y_n)\exp\Big\{\sum_{i,j=1}^{a}\Phi(x_i-y_i) - \sum_{j>i=1}^{a}[\Phi(x_i-x_j)+\Phi(y_i-y_j)]\Big\}\end{aligned} \tag{4.39}$$

where $\Delta(x) = \frac{i}{\pi}\mathcal{P}\frac{1}{x-y}$. The equal-time limits of the free-fermion propagator are

$\frac{1}{2}(1 \pm \gamma^5\Delta)$, and Φ is the two-point function of the effective theory which results from integrating out the fermions . In a more general theory we would include higher n-point functions in (4.39), but there are no higher n-point functions in this case, as the effective theory is non-interacting. We take Φ symmetric and $\Phi(0) = 0$, so we can write

$$\Phi(x) = \int dpC(p)(1 - \cos(px)), \tag{4.40}$$

and we have only to determine $C(p)$. Let us write

$$\hat{H}\Psi = \sum_{a=0}^{\infty} \frac{1}{a!} \prod_{n=1}^{a} \int dx_n dy_n u^\dagger(x_n)\gamma^5 u(y_n) e^{ie\int_{y_n}^{x_n} A} \tilde{f}^a(x_1, y_1, \ldots, x_a, y_a). \tag{4.41}$$

We will interpret sums and products from 1 to 0 as 0 and 1 respectively. An explicit computation, as in [37], gives

$$\begin{aligned}
\tilde{f}^a(x_1, y_1, \ldots, x_a, y_a) &= V(x_1, y_1, \ldots, x_a, y_a) f^a(x_1, y_1, \ldots, x_a, y_a) \\
&\quad - i \int dx \lim_{y \to x} \partial_y f^{a+1}(x_1, y_1, \ldots, x_a, y_a, y, x) \\
&\quad + i \int dx \lim_{y \to x} \partial_y \sum_{b=1}^{a} f^{a+1}(x_1, y_1, \ldots, x_b, x, y, y_b, \ldots, x_a, y_a) \\
&\quad - i \sum_{b=1}^{a} f^{a-1}(x_1, y_1, \ldots, \not{x}_b, \not{y}_b, \ldots, x_a, y_a)\delta'(x_b - y_b),
\end{aligned} \tag{4.42}$$

where $\not{x}$ denotes the omission of that argument, and we define

$$V(x_1, y_1, \ldots, x_a, y_a) = \sum_{i,j=1}^{a} V(x_i - y_i) - \sum_{j>i=1}^{a} \left[V(x_i - x_j) + V(y_i - y_j)\right], \tag{4.43}$$

and $V(x) = 2(\bar{V}(0) - \bar{V}(x))$.

We wish to verify that Ψ solves the Schrödinger equation

$$\hat{H}\Psi = E_0\Psi. \tag{4.44}$$

For $a = 0$ (4.44) reduces to

$$E_0 = -i \int dx \lim_{y \to x} \left\{ \partial_y (\Delta(y-x) e^{\Phi(y-x)}) \right\}, \tag{4.45}$$

and gives us the divergent vacuum energy. For $a = 1$ we find

$$\begin{aligned} E_0 \Delta(x_1 - y_1) &= V(x_1 - y_1)\Delta(x_1 - y_1) \\ &+ i \int dx \Delta(x_1 - x)\Delta(x - y_1)(\Phi'(x - y_1) - \Phi'(x - x_1)) \\ &- i\Delta(x_1 - y_1) \int dx \lim_{y \to x} \partial_y \left(\Delta(y - x) e^{g(x,y,x_1,y_1)} \right), \end{aligned} \tag{4.46}$$

where $g(x, y, x_1, y_1) = \Phi(y - x) + \Phi(x_1 - x) + \Phi(y - y_1) - \Phi(y - x_1) - \Phi(x - y_1)$, and we have used $\int dx \Delta(x_1 - x)\Delta(x - x_2) = \delta(x_1 - x_2)$. Now by Taylor expanding a test function $f(x)$ it is clear that

$$\begin{aligned} \lim_{x \to 0} \Delta(x) f(x) &= \frac{i}{\pi} f'(0) + \dots \\ \lim_{x \to 0} \Delta'(x) f(x) &= \frac{-i}{2\pi} f''(0) + \dots \end{aligned} \tag{4.47}$$

The ellipses represent potentially divergent terms which contribute to the vacuum energy but cancel in (4.46). Hence we have

$$-i \int dx \lim_{y \to x} \partial_y \left(\Delta(y - x) e^{g(x,y,x_1,y_1)} \right) = E_0 + \frac{1}{2\pi} \int dx (\Phi'(x - y_1) - \Phi'(x - x_1))^2, \tag{4.48}$$

and

$$\frac{1}{2\pi} \int dx (\Phi'(x - y_1) - \Phi'(x - x_1))^2 = \int dp p^2 C(p)^2 (1 - \cos(p(x_1 - y_1))). \tag{4.49}$$

Now the identity

$$\int dx \Delta(x_1 - x)\Delta(x - y_1) f(x) = \delta(x_1 - y_1) f(x_1) + \Delta(x_1 - y_1) \times$$

$$\int \frac{dp}{2\pi}\epsilon(p)\tilde{f}(p)(e^{-ipx_1} - e^{-ipy_1}) \tag{4.50}$$

corresponds in momentum space to the identity

$$\epsilon(p)\epsilon(p+q) = 1 + \epsilon(p)\epsilon(q) - \epsilon(q)\epsilon(p+q) \tag{4.51}$$

(ϵ is the step function). This implies that

$$\begin{aligned} & i\int dx\Delta(x_1 - x)\Delta(x - y_1)(\Phi'(x - y_1) - \Phi'(x - x_1)) \\ & = 2\Delta(x_1 - y_1)\int dp|p|C(p)(1 - \cos(p(x_1 - y_1))). \end{aligned} \tag{4.52}$$

Thus (4.46) reduces to the quadratic equation

$$\frac{e^2}{\pi p^2} + 2|p|C(p) - p^2C(p)^2 = 0, \tag{4.53}$$

and to get a normalizable WF we must take the appropriate root

$$C(p) = \frac{|p| - \sqrt{p^2 + M^2}}{p^2}. \tag{4.54}$$

This reproduces the result (4.10). It is straightforward to show that (4.44) is satisfied by this solution for all $a > 1$; using the identities (4.49) and (4.50) the $a = n$ equation may be reduced to multiple copies of the $a = 1$ equation.

4.2 The local expansion

4.2.1 The Schrödinger functional

We will now describe how the result (4.10) can be reconstructed from a derivative expansion, whose terms are local expressions in the fields. This allows the techniques of [4]-[7], which are generalizable to higher dimensions, to be used to provide an alternative solution of the theory. Naively, the presence of a mass gap in the theory should mean that all propagators are exponentially damped at large distances, so that the logarithm of (4.10) should reduce for slowly varying fields to a sum of local terms, ie. integrals over finite powers of the fields and their derivatives, evaluated at a single spatial point. Unfortunately, because of gauge-invariance (4.10) contains massless modes not appearing in the physical spectrum, and this simplification of the vacuum WF does not occur. This may be seen by noting the existence of screened large-distance configurations.

We will get around this problem by considering instead the Schrödinger functional

$$\Psi_\tau[\tilde{u}, \tilde{u}^\dagger, u, u^\dagger] = \langle \tilde{u}, \tilde{u}^\dagger | e^{-H\tau} | u, u^\dagger \rangle. \tag{4.55}$$

The vacuum functional is just the $\tau \to \infty$ limit of this object. We can also extract excited states by inserting a basis of energy eigenstates

$$\Psi_\tau[0, 0, u, u^\dagger] \sim \sum_E \Psi_E[u, u^\dagger] e^{-E\tau}. \tag{4.56}$$

For simplicity we will only consider the case $\tilde{u} = \tilde{u}^\dagger = 0$. The parameter τ, which corresponds to Euclidean time, acts as an inverse mass for all states of the theory—physical or otherwise. Provided we work with fields that vary slowly on the scale of τ, the logarithm of Ψ_τ has an expansion in positive powers of τ, each term of which is itself a finite sum of local terms. As we will see, the large τ behaviour can be reconstructed from this expansion.

We will begin by finding the free fermion solution. The Schrödinger equation is

$$-\frac{\partial}{\partial \tau} \Psi_\tau = \hat{H} \Psi_\tau, \tag{4.57}$$

where

$$\hat{H} = \frac{1}{2}\int d^d x(u^\dagger + \frac{\delta}{\delta u})h(u + \frac{\delta}{\delta u^\dagger}), \qquad h = i\gamma^5\partial_x. \tag{4.58}$$

We also have the initial condition

$$\Psi_0 = \langle Q|u, u^\dagger\rangle\rangle = e^{\int dx(u^\dagger Q u)}, \tag{4.59}$$

If we put

$$\Psi_\tau = e^{\int dx(u^\dagger \Gamma(\tau) u)}, \tag{4.60}$$

then (4.42) becomes

$$\dot{\Gamma} = -\frac{1}{2}(1-\Gamma)h(1+\Gamma), \tag{4.61}$$

with $\Gamma(0) = Q$. This equation is solved by the ansatz

$$\Gamma = (\Sigma + Q_-)(\Sigma - Q_-)^{-1}, \tag{4.62}$$

and substitution into (4.61) gives $\dot{\Sigma} = h\Sigma$, $\Sigma(0) = Q_+$, with the solution

$$\Sigma = e^{-i\gamma^5\tau\partial_x}Q_+. \tag{4.63}$$

Hence we obtain

$$\Gamma = Q + 2Q_-\gamma^5\tanh(-i\tau\partial_x)Q_+, \tag{4.64}$$

which has a derivative expansion for small τ, as promised. It is also easy to find the (non-local) large-time limit:

$$\lim_{\tau\to\infty}\Gamma(x-y) = Q + 2Q_-\frac{i\gamma^5}{\pi}\mathcal{P}\frac{1}{x-y}Q_+, \tag{4.65}$$

which coincides with the solution $\Gamma = A^{-1}C$ that we found before.

Now consider the interacting theory. Substituting the result (4.42) into the timedependent Schrödinger equation (4.57) gives

$$\begin{aligned}
-\dot{f}^a(x_1,y_1,\ldots,x_a,y_a) &= V(x_1,y_1,\ldots,x_a,y_a)f^a(x_1,y_1,\ldots,x_a,y_a) \\
&\quad -i\int dx \lim_{y\to x}\partial_y f^{a+1}(x_1,y_1,\ldots,x_a,y_a,y,x) \\
&\quad +i\int dx \lim_{y\to x}\partial_y \sum_{b=1}^{a} f^{a+1}(x_1,y_1,\ldots,x_b,x,y,y_b,\ldots,x_a,y_a) \\
&\quad -i\sum_{b=1}^{a} f^{a-1}(x_1,y_1,\ldots,\not{x}_b,\not{y}_b,\ldots,x_a,y_a)\delta'(x_b-y_b) \\
&\quad -E_0 f^a(x_1,y_1,\ldots,x_a,y_a) \qquad (4.66)
\end{aligned}$$

Here we have made $\hat{H}$ regular by subtracting the zero-point energy (4.45), ie. $\hat{H}\to\hat{H}-E_0$; this may be achieved by normal-ordering. In order that we recover the solution (4.64) in the free-fermion limit $e\to 0$, we use the Ansatz (4.39) with $\Delta(x-y)=\tanh(i\tau\partial_x)\delta(x-y)$. The $a=0$ part of (4.66) is of course trivial; for $a=1$ it becomes

$$\begin{aligned}
0 &= \Delta(x_1-y_1)\partial_\tau\Phi(x_1-y_1)+V(x_1-y_1)\Delta(x_1-y_1) \\
&\quad +i\int dx\Delta(x_1-x)\Delta(x-y_1)(\Phi'(x-y_1)-\Phi'(x-x_1)) \\
&\quad -i\Delta(x_1-y_1)\int dx\lim_{y\to x}\partial_y\left(\Delta(y-x)e^{g(x,y,x_1,y_1)}\right)-E_0\Delta(x_1-y_1) \quad (4.67)
\end{aligned}$$

In obtaining this we used the identity

$$\begin{aligned}
\dot{\Delta}(x_1-y_1) &= i(1-\tanh^2(i\tau\partial))\delta'(x_1-y_1) \\
&= i\delta'(x_1-y_1)-i\int dx\Delta(x_1-x)\Delta'(x-y_1). \qquad (4.68)
\end{aligned}$$

Now for small x

$$\Delta(x)=\frac{i\pi}{\tau}\mathrm{cosech}\left(\frac{\pi^2 x}{\tau}\right)\sim\frac{i}{\pi x}+O(x), \qquad (4.69)$$

so that

$$\lim_{x\to 0}\Delta(x)f(x) = \frac{i}{\pi}f'(0)+\ldots$$
$$\lim_{x\to 0}\Delta'(x)f(x) = \frac{-i}{2\pi}f''(0)+\ldots \tag{4.70}$$

The ellipses have the same meaning as in (4.47). Hence as before

$$-i\int dx \lim_{y\to x}\partial_y\left(\Delta(y-x)e^{g(x,y,x_1,y_1)}\right) = E_0 - \int dpp^2C(p,\tau)^2(1-\cos(p(x_1-y_1))). \tag{4.71}$$

On the other hand, we find that

$$i\int dx\Delta(x_1-x)\Delta(x-y_1)(\Phi'(x-y_1)-\Phi'(x-x_1)) =$$
$$2\Delta(x_1-y_1)\int dp\coth(\tau p)pC(p,\tau)(1-\cos(p(x_1-y_1))) \tag{4.72}$$

Thus (4.67) reduces to

$$\dot{C}(p,\tau)+2p\coth(p\tau)C(p,\tau)-p^2C^2(p,\tau)+\frac{M^2}{p^2}=0. \tag{4.73}$$

The initial condition (4.59) is satisfied if $C(p,0)=0$. For small τ we can expand C as

$$C=\sum_{n=1}^{\infty}c_n\tau^n, \tag{4.74}$$

and substituting into (4.73) leads to a recursion relation which is easily solved for the coefficients c_n, which are polynomials in positive powers of p, divided by $1/p^2$. We note that (4.73) reduces to (4.53) for large τ, so that we recover the vacuum functional, as expected.

Just as for the vacuum functional, it is easily shown that the solution given by (4.73) satisfies (4.66) for all a.

4.2.2 Solution from a local Ansatz

The Ansatz (4.39) is very useful in two dimensions, but it is not clear that it generalizes in any way to higher dimensions. Thus we will now describe an Ansatz which *does* generalize.

Provided we work with fields that vary slowly on the scale of τ, $\Delta(x) = \tanh(i\partial_x\tau)\delta(x)$ can be expanded in derivatives of the delta function. It follows from (4.39) that the logarithm of Ψ_τ has a local expansion (this should also be true in higher dimensions). This allows us to write $\log\Psi_\tau$ in the form (4.1), where we take the f^a to be local:

$$\begin{aligned} f^a(p_1,\ldots,p_{2a}) &= \sum_{n_1,\ldots,n_{2a}=0}^{\infty} b_{n_1}p_1^{n_1}\ldots b_{n_{2a}}p_{2a}^{n_{2a}} \\ f^a(x_1,y_1,\ldots,x_a,y_a) &= \int\frac{dp_1}{2\pi}\ldots\frac{dp_{2a}}{2\pi}\delta(p_1+\ldots+p_{2a})f^a(p_1,\ldots,p_{2a}) \end{aligned} \tag{4.75}$$

Since we can gauge away the electromagnetic field, this leads to the following Ansatz

$$\begin{aligned} \Psi_\tau &= 1+\int dx\sum_{n,q=0}^{\infty}a_n^{(q)}\tau^q u^\dagger(x)\gamma^5\partial_x^n u(x) \\ &\quad +\int dxdy\sum_{n,m,q=0}^{\infty}a_{nm}^{(q)}\tau^q u^\dagger(x)\gamma^5(\partial_x^n u(x))u^\dagger(y)\gamma^5(\partial_y^m u(y))+\ldots \end{aligned} \tag{4.76}$$

where the coefficients $a_{ij\ldots}$ are obtained from the b_n in the obvious way. This Ansatz can be inserted directly into the Schrödinger equation (4.66), which thus reduces to an infinite set of algebraic equations (a finite set for each order in τ). We have explicitly verified for the first few terms the that the local expansion which results from solving the Schrödinger equation in this way is the same as that which is obtained by expanding the small-τ solution found in the last section.

Now the local expansion depended on expanding Ψ_τ in positive powers of τ for small τ, but as described in the last chapter we can reconstruct the large τ behaviour from a knowledge of the local expansion alone. If we evaluate Ψ_τ for scaled fields $u(x/\sqrt{\rho})$ and $u^\dagger(x/\sqrt{\rho})$ then it can be analytically continued to the complex ρ-plane with the negative real axis removed. The proof of this is a straightforward generalization of the arguments of [4]-[7]. Then by using Cauchy's theorem, we can relate the value of the functional at $\rho = 1$ to the value at large ρ, where the fields are slowly varying, so that the τ-expansion converges for large τ. Specifically, we study the following integral:

$$\frac{1}{2\pi i}\int_C \frac{d\rho}{\rho - 1} e^{\lambda\rho}\Psi_\tau[u(x/\sqrt{\rho}, u^\dagger(x/\sqrt{\rho})]. \tag{4.77}$$

If we take C to be a circle of sufficiently large radius we can evaluate this integral by inserting the local expansion of Ψ_τ for any value of τ. Alternatively, we can collapse the contour around the point $\rho = 1$, and the negative real axis, the latter contribution being exponentially suppressed for large λ. Thus we have successfully expressed Ψ_τ for large τ in terms of the local expansion. In practise we will wish to truncate the expansion at some order, in which case the truncation error is minimized by taking λ to be large but finite [5, 7].

The Ansatz (4.75) generalizes straightforwardly to higher dimensions, and could form the starting point for a numerical approach to (3+1) dimensional QCD. The main steps involved in implementing a numerical approach to higher dimensional QCD based on solving the Schrödinger equation for the local expansion are as follows. First, to obtain the generalization of (4.35) to non-Abelian fields; this is straightforward, and allows the Hamiltonian to be written in terms of gauge-invariant currents. Second, to write down the local ansatz which generalizes (4.76). The local gauge-invariant terms which make up the local expansion are easily identified. Renormalization may be dealt with by adding counterterms familiar from perturbation theory. Or, we can consider supersymmetric theories in which they are absent. Finally, after finding the action of the Hamiltonian on the terms of the local expansion, the solution of the Schrödinger equation proceeds as described above.

4.3 The massive Schwinger model

The gaussian nature of the massless Schwinger model makes it possible to find a path-integral solution, but when we give the electrons a non-zero mass, the model is no longer gaussian, and is not completely soluble by any known method. However, if we limit our attention to n-point interactions for any finite n, we can still solve the Schrödinger equation in much the same way as before.

Throughout this section we take $Q = \gamma^1$. It is obvious from the preceding sections how to generalize to other values of Q. The Hamiltonian becomes

$$\hat{H} = \int dxdy \hat{j}_0(x)\hat{j}_0(y)\bar{V}(x-y) - i\int dx \lim_{y\to x} \partial_y \hat{J}_{\gamma^5}(x,y) + m\int dx \lim_{y\to x} \hat{J}_{\gamma^0}(x,y), \tag{4.78}$$

and we have

$$P_{\pm} = \frac{1}{2}\exp\left\{ie\int_y^x (\mathcal{A}+\dot{\phi})\right\}\left(1 \pm \frac{i\gamma^5\partial + \gamma^0 m}{\sqrt{m^2-\partial^2}}\right)\delta(x-y). \tag{4.79}$$

We do not have a closed expression for the fermion determinant D_F, but we do have the following path-integral expression for the vacuum WF:

$$\begin{aligned}\Psi[u, u^\dagger, \mathcal{A}] = \quad & \int D\phi \exp\Big\{-\frac{1}{2}\phi\partial^4\phi + \ln D_F \\ & +2\int dxdy \Big[u^\dagger(x)\gamma^5 u(y)\Delta(x-y)\exp\Big\{ie\int_y^x(\mathcal{A}+\dot{\phi})\Big\}\Big]\Big\}, \end{aligned} \tag{4.80}$$

where

$$\Delta(x) = i\frac{\partial_x - m}{\omega}\delta(x), \tag{4.81}$$

and $\omega^2 = m^2 - \partial^2$. This can be formally solved to give the Ansatz

$$\Psi[u, u^\dagger, \mathcal{A}] = \sum_{a=0}^{\infty} \frac{1}{a!} \prod_{n=1}^{a} \int dx_n dy_n u^\dagger(x_n)\gamma^5 u(y_n) e^{ie\int_y^x \mathcal{A}} f^a(x_1, y_1, \ldots, x_a, y_a), \tag{4.82}$$

with

$$\begin{aligned} f^a(x_1, y_1, \ldots, x_a, y_a) &= \prod_{n=1}^{a} \Delta(x_n - y_n) \exp\Bigg\{ \sum_{i,j=1}^{a} \Phi(x_i - y_i) \\ &\quad - \sum_{j>i=1}^{a} [\Phi(x_i - x_j) + \Phi(y_i - y_j)] + \ldots \Bigg\} \end{aligned} \tag{4.83}$$

The ellipsis represents non-factorizing contributions from higher n-point functions. Because of the combinatorics of the formal path integral solution, we can take these to be functions of $x_1, y_1, \ldots, x_n, y_n$ which are symmetric in all their arguments and vanish when any two arguments coincide. We will concentrate on the 2-point function. As before, we can write

$$\Phi(x) = \int dp C(p)(1 - \cos(ipx)). \tag{4.84}$$

The Schrödinger equation becomes

$$\begin{aligned} E_0 f^a(x_1, y_1, \ldots, x_a, y_a) &= V(x_1, y_1, \ldots, x_a, y_a) f^a(x_1, y_1, \ldots, x_a, y_a) \\ &\quad - i \int dx \lim_{y\to x} (\partial_y + m) f^{a+1}(x_1, y_1, \ldots, x_a, y_a, y, x) \\ &\quad + i \int dx \lim_{y\to x} (\partial_y + m) \sum_{b=1}^{a} f^{a+1}(x_1, y_1, \ldots, x_b, x, y, y_b, \ldots, x_a, y_a) \\ &\quad - i \sum_{b=1}^{a} f^{a-1}(x_1, y_1, \ldots, \not{x}_b, \not{y}_b, \ldots, x_a, y_a)(\partial - m)\delta(x_b - y_b) \end{aligned} \tag{4.85}$$

and for $a = 0$ reduces to

$$E_0 = -i \int dx \lim_{y \to x} \left\{ (\partial_y + m) \left(\Delta(y - x) e^{\Phi(y-x)} \right) \right\} \tag{4.86}$$

For $a = 1$ we find

$$\begin{aligned} E_0 \Delta(x_1 - y_1) &= V(x_1 - y_1)\Delta(x_1 - y_1) \\ &\quad + i \int dx \Delta(x_1 - x)\Delta(x - y_1)(\Phi'(x - y_1) - \Phi'(x - x_1)) \\ &\quad - i\Delta(x_1 - y_1) \int dx \lim_{y \to x} (\partial_y + m) \left(\Delta(y - x) e^{g(x,y,x_1,y_1)} \right) \end{aligned} \tag{4.87}$$

where we have used the identity $\int dx \Delta(x_1 - x)\Delta(x - y_1) = \frac{\partial - m}{\partial + m}\delta(x_1 - y_1)$. Now for small x

$$\Delta(x) \sim \frac{i}{\pi x} - \frac{im}{\pi} \ln x + O(1), \tag{4.88}$$

so that

$$\begin{aligned} \lim_{x \to 0} \Delta(x) f(x) &= \frac{i}{\pi} f'(0) + \dots \\ \lim_{x \to 0} \Delta'(x) f(x) &= \frac{-i}{2\pi} f''(0) - \frac{im}{\pi} f'(0) + \dots \end{aligned} \tag{4.89}$$

Again, the ellipses represent potentially divergent terms which contribute only to the vacuum energy. We have

$$-i \int dx \lim_{y \to x} \partial_y \left(\Delta(y - x) e^{g(x,y,x_1,y_1)} \right) = E_0 - \int dp p^2 C(p)^2 (1 - e^{ip(x_1 - y_1)}), \tag{4.90}$$

and (4.87) reduces to

$$\Delta * (p^2 C^2) + \Delta * (M^2/p^2) + 2\Delta(\Delta * pC) = 0, \tag{4.91}$$

which may be solved by expanding $C(p)$ in a suitable basis of functions (derivatives of the Dirac delta function seem to provide a convenient choice).

Now if we had included functions of four variables in the Ansatz (4.83) they would not have contributed to this equation, but they would be determined by the $a = 2$ part of (4.85). Thus the solution of (4.91) gives an exact result for the quadratic term in the vacuum WF, as well as giving the two-point factorizable contributions to the higher terms. This is also sufficient to determine the exact two-point function, whose ultraviolet behaviour gives the chiral condensate.

In a similar way, the $2n$-point contributions can be obtained from the $a = n$ part of the Schrödinger equation (4.85).

4.4 Finite temperature

A system at finite temperature may be represented by the density matrix

$$\rho(\phi_1, \phi_2) = \sum_n p_n \Psi_n(\phi_1)\Psi_n^*(\psi_2), \tag{4.92}$$

where Ψ_n are a complete set of wave-functionals corresponding to energy eigenstates labelled by n, and p_n is the probability that the system is in the state n. For a system in thermal equilibrium at temperature $T = 1/\beta k$ the occupation probabilities are given by the Boltzmann distribution:

$$p_n = \frac{e^{-\beta E_n}}{\sum_l e^{-\beta E_l}}. \tag{4.93}$$

The density matrix can also be interpreted as [44, 45]

$$\langle \phi_1 | e^{-\beta \hat{H}} | \phi_2 \rangle, \tag{4.94}$$

which is just the Schrödinger functional with $\tau = \beta$. Expectation values of operators $\mathcal{O}$ are given by

$$\langle \mathcal{O} \rangle = \mathrm{tr} \rho \mathcal{O} = \int D\phi_1 D\phi_2 \rho(\phi_1, \phi_2) \mathcal{O}(\phi_1, \phi_2). \tag{4.95}$$

Now as we saw, the two-point function, and hence the chiral condensate, is essentially determined by the function $C(p)$. So we have all the information we require to study the condensate in the massive and finite temperature cases.

For $N_f > 1$ this is known to exhibit a second-order phase transition at $T = 0$. This has been studied using bosonization techniques [46], but there remain unanswered questions [47]. Our results could be used to find exact expressions for the condensate, allowing it to be studied without relying on bosonization.

Chapter 5

AdS/CFT in the Schrödinger representation

5.1 Introduction

Supergravity in the background of Anti-de-Sitter space has generated a lot of interest recently as the result of a conjecture by Maldacena [1] that $N = 4$ Super-Yang-Mills with an SU(N) gauge group is dual to Type IIB string theory on $AdS_5 \times S^5$. More generally, string theory on $AdS \times M$, for some compact manifold M, corresponds to a conformal field theory on the boundary of AdS. This sort of correspondence is extremely useful from a practical point of view, because it relates the strong-coupling regime of each theory to the weak-coupling regime of the other. But for precisely the same reason, it is hard to test, especially beyond the classical approximation.

According Maldacena's conjecture, the large N limit of the gauge theory corresponds to the low energy limit of string theory, which is supergravity. The leading order terms in a 1/N expansion are given by classical supergravity, and it is in this approximation that the conjecture has been most studied. But it seems natural to develop techniques which go beyond tree-level. For example, one-loop effects in supergravity correspond to next-to-leading order effects in the 1/N expansion of the gauge theory, and non-renormalization theorems protect certain one-loop quantities from higher-loop and stringy corrections.

Our aim will be to understand the correpondence in the Schrödinger representation with a view to acheiving two aims—calculating loop-corrections and producing tests of Maldacena's conjecture.

Because the perturbative expansions available in the bulk and boundary theories are valid in different domains, to test the conjecture we must find quantities which can be evaluated exactly at strong coupling in at least one of the theories. Such quantities are rare, but examples are given by certain global anomalies in the Yang-Mills theory. The theory is classically invariant under local scalings of the metric, but the quantum theory breaks this symmetry in a way which, thanks to the supersymmetry, can be computed exactly.

On the supergravity side, this anomaly has a leading order term which arises from classical supergravity [12] and a $1/N^2$ correction which is a one-loop effect. The leading order term depends only on the graviton, but the one-loop calculation receives contributions from all of the Kaluza-Klein modes of supergravity, and thus provides a much more rigorous test. Furthermore, these modes also contribute divergences which renormalize the cosmological and Newton's constants on the boundary; these divergences must cancel if the boundary theory is to be finite, and this provides additional tests.

The one-loop effect which we calculate depends on the field-independent part of the partition function, and thus arises from the linearized action. But by including interaction terms in the Lagrangian we can use our technology to compute objects such as n-point functions at higher loops. We will illustrate a numerical approach to the resulting non-linear equations.

The central object of study in the AdS/CFT correspondence is a functional integral for a quantum field theory in Anti de Sitter space expressed in terms of the boundary values of the field. Whilst this can be treated by the usual semiclassical expansion it may also be interpreted as the large time limit of the Schrödinger functional and so satisfies a functional Schrödinger equation. We will show how to obtain this large time behaviour from a short time expansion using analyticity, as we did for fields in flat space.

In computing the Weyl anomalies of the boundary partition functions, we will find a discontnuous dependence on the particle mass, with non-zero values for precisely the values appearing in the Kaluza-Klein compactifications of supergravity. The coefficient of the anomaly increases with the Kaluza-Klein mass, so that large masses do not decouple, contrary to the usual case in field theory compactifications.

For example, when the boundary of the AdS spacetime is two-dimensional the Virasoro central charges are non-zero, and take integer values, $\tilde{N}$, when the mass of the scalar field is $\sqrt{\tilde{N}^2 - 1}$ and when the mass of the fermion is $\tilde{N} - 1/2$. This implies that for generic values of the mass, the boundary CFTs for the scalar or fermi fields alone are non-unitary, since in a unitary theory a vanishing central charge implies an absence of quasi-primaries.

When the boundary is four dimensional the conditions on the scalar and fermion masses for the conformal anomaly to be non-zero coincide with the mass spectra resulting from Kaluza-Klein compactification of supergravity on $AdS_5 \times S^5$, and when it is six dimensional they coincide with the mass spectra resulting from Kaluza-Klein compactification of supergravity on $AdS_7 \times S^4$.

In this chapter we will restrict our attention to fields of spin 0 and 1/2, and in the next chapter we will extend our results to fields of higher spin and sum them over all the individual supergravity multiplets so as to compare the overall result with the corresponding one in Yang-Mills.

5.2 The Schrödinger equation

Following [2] we consider the Euclidean version of AdS_{d+1} with coordinates $\{x^\mu\} \equiv \{t, x^1, .., x^d\}$ and metric

$$ds^2 = \frac{1}{(x^0)^2} \sum_{\mu=0}^{d} (dx^\mu)^2 = \frac{1}{t^2}(dt^2 + d\mathbf{x} \cdot d\mathbf{x}). \tag{5.1}$$

We will think of t, which is restricted to the range $t > 0$, as Euclidean time. The boundary, ∂M, consists of R^d at $t = 0$ conformally compactified to a sphere by adding a point corresponding to $t = \infty$ where the metric vanishes. For illustration consider a scalar field theory propagating on this space-time. We study the functional integral

$$Z[\varphi] = \int \mathcal{D}\phi \, e^{-S} \Big|_{\phi|_{\partial M} = \varphi}, \qquad S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \, (g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + V(\phi)) \tag{5.2}$$

where $\mathcal{D}\phi$ is the volume element induced by the reparametrization invariant inner product on variations of ϕ, $||\delta\phi||^2 = \int dt \, d\mathbf{x} \, \delta\phi^2 / t^{d+1}$. We will need to regulate this by restricting t to the range $\tau > t > \tau'$, so define

$$\Psi_{\tau,\tau'}[\tilde{\varphi}, \varphi] = \int \mathcal{D}\phi \, e^{-S} \Big|_{\phi(\tau) = \tilde{\varphi}, \phi(\tau') = \varphi}. \tag{5.3}$$

Since the point corresponding to $t = \infty$ is part of ∂M we set $\tilde{\varphi} = \lim_{|\mathbf{x}| \to \infty} \varphi(\mathbf{x})$ as we take the limit of large τ and small τ' to recover Z. This is usually described as a partition function, but it may also be interpreted in terms of the wave-functionals that represent states in the Schrödinger representation. First change variables from t, ϕ to $\bar{t} = \ln t$, $\bar{\phi} = \phi / t^{d/2}$ so that the volume element and kinetic term become the usual ones associated with the canonical quantization of $\bar{\phi}$. Thus $Z[\varphi]$ is the

$\tau' \to 0$, $\tau \to \infty$ limit of

$$\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = \int \mathcal{D}\bar{\phi}\, e^{-\bar{S}-S_b} \equiv \bar{Z}[\bar{\phi}_f,\bar{\phi}_i]\, e^{-S_b} \tag{5.4}$$

where

$$\bar{S} = \frac{1}{2}\int d\bar{t}\, d\mathbf{x} \left(\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + \frac{d^2}{4}\bar{\phi}^2 + t^2\nabla\phi\cdot\nabla\phi + t^{-d}\, V(\bar{\phi}\, t^{d/2}) \right), \quad S_b = \frac{d}{4}(\bar{\phi}_f^2 - \bar{\phi}_i^2) \tag{5.5}$$

and $\bar{\phi}_f$, $\bar{\phi}_i$ are the value of $\bar{\phi}$ on the surfaces $\bar{t} = \bar{t}_1 = \ln\tau$ and $\bar{t} = \bar{t}_2 = \ln\tau'$ respectively. Now $\bar{Z}[\bar{\phi}_f,\bar{\phi}_i]$ can be interpreted as the Schrödinger functional, i.e. the matrix element of the time evolution operator between eigenstates of the field operator,

$$\bar{Z}[\bar{\phi}_f,\bar{\phi}_i] = \langle\, \bar{\phi}_f \,|\, T\, \exp(-\int_{\bar{t}_2}^{\bar{t}_1} dt\, H(t)) \,|\, \bar{\phi}_i \,\rangle\,, \tag{5.6}$$

which satisfies the functional Schrödinger equation

$$\frac{\partial}{\partial\bar{t}_1}\bar{Z}[\bar{\phi}_f,\bar{\phi}_i] = -\frac{1}{2}\int d\mathbf{x} \left(-\frac{\delta^2}{\delta\bar{\phi}_f^2} + t_1^2\,\nabla\bar{\phi}_f\cdot\nabla\bar{\phi}_f + \frac{d^2}{4}\bar{\phi}_f^2 + t_1^{-d}\, V(\bar{\phi}_f\, t_1^{d/2}) \right) \bar{Z}[\bar{\phi}_f,\bar{\phi}_i], \tag{5.7}$$

with the initial condition that it tends to a the delta-functional $\delta[\bar{\phi}_f - \bar{\phi}_i]$ as $\bar{t}_1$ approaches $\bar{t}_2$. We can re-write this in terms of the boundary values of our original variables t,ϕ, i.e. τ and $\tilde{\varphi}$

$$\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = -\frac{1}{2}\int d\mathbf{x} \left(-\Omega^{-1}\frac{\delta^2}{\delta\tilde{\varphi}^2} + \Omega\,\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi} + \Omega'\, V(\tilde{\varphi}) + \mathcal{E}/\tau \right) \Psi_{\tau,\tau'}[\tilde{\varphi},\varphi], \tag{5.8}$$

where $\Omega = \tau^{1-d}$, $\Omega' = \Omega/\tau^2$. $\mathcal{E}$ arises from the action of the Laplacian on S_b, formally

$$\mathcal{E} = -\tau^d\,\frac{\delta^2}{\delta\tilde{\varphi}^2}S_b = -\frac{d}{2}\delta^d(0) \tag{5.9}$$

Clearly the coincident functional derivatives stand in need of regularization, so we introduce a short-distance cut-off. By extending the flat-space arguments of Symanzik [3] we would expect that for a renormalizable field theory wave-function renormalization and an appropriate choice of the dependence of $\mathcal{E}$ on the cut-off would ensure the finiteness of the solution to (5.8) in the limit that the cut-off is removed. Since this can involve the use of counterterms associated with the boundaries we would expect that the renormalization constants may depend on τ and τ'. However in many applications the tree-level solution is sufficent for which these considerations are unnecessary. We will see later that $\mathcal{E}$ contributes to the conformal anomaly. A similar argument yields the Schrödinger equation that gives the τ' dependence

$$-\frac{\partial}{\partial\tau'}\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = -\frac{1}{2}\int d\mathbf{x}\,\left(-\Omega^{-1}\,\frac{\delta^2}{\delta\varphi^2} + \Omega\,\nabla\varphi\cdot\nabla\varphi + \Omega'\,V(\varphi) - \mathcal{E}/\tau'\right)\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi], \tag{5.10}$$

Now just as in flat space the logarithm of $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$, $W_{\tau,\tau'}[\tilde{\varphi},\varphi]$, can be expanded in local terms. In the next section we will generalize our previous argument that the functional evaluated for scaled fields $\tilde{\varphi}(\mathbf{x}/\sqrt{\rho})$ and $\varphi(\mathbf{x}/\sqrt{\rho})$ can be analytically continued to the complex ρ-plane with the negative real axis removed. Then as before Cauchy's theorem will allow us to relate rapidly varying fields (small ρ) to slowly varying ones (large ρ), and from this we can use the behaviour for small τ to obtain that for large τ, which is what is needed in the AdS/CFT correspondence.

5.3 Analyticity of Schrödinger functional

To demonstrate the analyticity of $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$ we generalize to curved space the argument of chapter 3. We first make the dependence on $\tilde{\varphi}$, φ explicit by modifying a standard phase-space derivation of the functional integral representation of the Schrödinger functional. By the usual argument we can write $Z[\bar{\phi}_f,\bar{\phi}_i]$ as

$$\lim_{n\to\infty}\int\prod_{j=1}^{n}\mathcal{D}\bar{\phi}_j\prod_{j=1}^{n+1}\mathcal{D}\Pi_j\, exp\left(\sum_{j=1}^{n+1}(-H[\Pi_j,\bar{\phi}_{j-1}]\,\delta\bar{t}_j)+i\int d\mathbf{x}\,\Pi_j\,(\bar{\phi}_j-\bar{\phi}_{j-1})\right)\,, \tag{5.11}$$

where Π is the eigenvalue of the canonical momentum conjugate to $\bar{\phi}$, $\delta\bar{t}_j = \bar{t}_j - \bar{t}_{j-1}$ and $\bar{\phi}_{n+1} = \bar{\phi}_f$ and $\bar{\phi}_0 = \bar{\phi}_i$. $\bar{\phi}_f$ appears in this expression only in the $i\Pi_{n+1}\bar{\phi}_{n+1}$ term in the exponent, whereas $\bar{\phi}_i$ appears in $-i\Pi_1\bar{\phi}_0$, and terms proportional to $\delta\bar{t}_1$, which can be neglected as $\delta\bar{t}_1 \to 0$. So the contribution of $\bar{\phi}_i$ and $\bar{\phi}_f$ to (5.11) can be manifested by adding to the exponent $i\int d\mathbf{x}\,(\Pi_{n+1}\,\bar{\phi}_f - \Pi_1\,\bar{\phi}_i)$ and taking $\bar{\phi}_{n+1} = \bar{\phi}_0 = 0$. Thus $\bar{\phi}_i$ and $\bar{\phi}_f$ appear as sources coupled to $\dot{\bar{\phi}}$ when we integrate out the momenta, Π_j, so that we arrive at the functional integral

$$\int\mathcal{D}\bar{\phi}\,\exp\left(-\bar{S}+\int d\mathbf{x}\,\left(\bar{\phi}_f\,\dot{\bar{\phi}}(\bar{t}_1)-\bar{\phi}_i\dot{\bar{\phi}}(\bar{t}_2)\right)\right)\,\exp\left(\int d\mathbf{x}\,\Lambda\,(\bar{\phi}_i^2+\bar{\phi}_f^2)\right)\,, \tag{5.12}$$

where the boundary condition on $\bar{\phi}$ is now that it should vanish at $\bar{t} = \bar{t}_1,\, \bar{t}_2$. Λ is a regularization of $1/\epsilon$ which cancels certain divergences that arise in the evaluation of (5.12) whose origin is explained in [4]. If we now interchange the rôles of $\bar{t}$ and x^1, and think of x^1 as Euclidean time and $\bar{t}, x^2, ..x^d$ as spatial coordinates then we can give an alternative interpretation of the functional integral (ignoring the Λ factor for the time being) as the vacuum expectation value

$$\langle\, O_r\,|T\,\exp(\int dx^1\,\varphi_i\hat{R}_i(x^1))\,|\,O_r\,\rangle\,, \tag{5.13}$$

where $|\,O_r\,\rangle$ is the vacuum for the Hamiltonian, $\hat{H}_r$, associated with the quantization surfaces of constant x^1 and $\bar{t}_2 < \bar{t} < \bar{t}_1$. Unlike the previous Hamiltonian which depended on $\bar{t}$, this operator is independent of its associated 'time', x^1. φ_i is a compact notation for the sources, so that

$$\varphi_i \hat{R}_i(x^1) \equiv \int dx^2..dx^d \left(\bar{\phi}_f \, \dot{\bar{\phi}}(\bar{t}_1) - \bar{\phi}_i \dot{\bar{\phi}}(\bar{t}_2) \right) \tag{5.14}$$

Expanding (5.13) in powers of the sources and using $\hat{H}_r$ to generate the x^1-dependence of the operators $\hat{R}_i$ gives

$$\begin{aligned}
&\langle\, O_r \,| T \, \exp\left(\int dx^1 \, \varphi_i \hat{R}_i(x^1) \right) \,|\, O_r \,\rangle = \\
&\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx_n^1 \int_{-\infty}^{x_n^1} dx_{n-1}^1 .. \int_{-\infty}^{x_3^1} dx_2^1 \int_{-\infty}^{x_2^2} dx_1^1 \prod_{j=1}^{n} \varphi_{i_j}(x_j^1) \\
&\times \langle\, O_r \,|\, \hat{R}_{i_n}(0) \, e^{(x_{n-1}^1 - x_n^1)\, \hat{H}_r} \, \hat{R}_{i_{n-1}}(0) .. \hat{R}_{i_2}(0) \, e^{(x_2^1 - x_1^1)\, \hat{H}_r} \, \hat{R}_{i_1}(0) \,|\, O_r \,\rangle
\end{aligned} \tag{5.15}$$

We have taken the eigenvalue of $\hat{H}_r$ belonging to $|\, O_r \,\rangle$ to be zero. Fourier transforming the x^1-dependence of the sources as $\varphi_i(x^1) = \int dk \tilde{\varphi}_i(k) \exp(-ikx^1)$ enables the x^1 integrals to be done yielding

$$\begin{aligned}
&\sum_{n=0}^{\infty} \delta\big(\sum_{j=1}^{n} k_j \,\big) \prod_{j=1}^{n} \int dk_j \, \tilde{\varphi}_{i_j}(k_j) \\
&\times \langle\, O_r \,|\, \hat{R}_{i_n}(0) \, \frac{1}{\hat{H}_r - i \sum_1^{n-1} k_j} \, \hat{R}_{i_{n-1}}(0) .. \hat{R}_{i_2}(0) \, \frac{1}{\hat{H}_r - ik_1} \, \hat{R}_{i_1}(0) \,|\, O_r \,\rangle
\end{aligned} \tag{5.16}$$

Suppose that we had computed the Schrödinger functional for new sources obtained by scaling x^1, $\varphi_i(x^1/\sqrt{\rho}, x^2, ..x^d)$, with ρ real and positive. Then we would have obtained the same expression as (5.16) but multiplied by $\sqrt{\rho}$ and with the $\hat{H}_r$ in the denominators replaced by $\sqrt{\rho}\hat{H}_r$. We took ρ to be real and positive, but we can use this expression to continue to the complex ρ-plane. Since the eigenvalues of $\hat{H}_r$ are real we conclude that the result is analytic in the whole plane with the negative real axis removed. This assumes that we work to finite order in the sources, and that the spectral decomposition of (5.16) as a sum over eigenvalues of $\hat{H}_r$ converges, as we should expect if the Schrödinger functional is finite. The terms in Λ in (5.12) do not affect this conclusion. By repeating the argument with x^1 interchanged with each of the other coordinates in turn we conclude that for $\varphi_i^\rho(\mathbf{x}) \equiv \varphi_i(\mathbf{x}/\sqrt{\rho})$ the Schrödinger functional $Z[\bar{\phi}_f^\rho, \bar{\phi}_i^\rho]$ and consequently $\Psi_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$ are (to any finite order in the sources) analytic in ρ in the plane cut along the negative real axis. Since $\mathbf{x} \to \mathbf{x}/\sqrt{\rho}$, $t \to t/\sqrt{\rho}$ is an isometry of (5.1) it follows that

$$\Psi_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho] = \Psi_{\tau/\sqrt{\rho},\tau'/\sqrt{\rho}}[\tilde{\varphi}, \varphi] \tag{5.17}$$

and we see that analyticity in ρ corresponds to the analyticity in time associated with Wick rotation. Subject to the caveat that we work to finite order in the sources the logarithm $\Psi_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$, $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$, is just a sum of products of terms appearing in (5.16) so it too is analytic in the cut ρ-plane when it is evaluated for the scaled sources.

We can use (5.16) to justify a local expansion for $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ with a non-zero radius of convergence. Since $\hat{H}_\tau$ is the Hamiltonian for a field theory on a finite 'spatial' interval such that $\hat{\phi}$ vanishes at the ends there is a mass-gap of the order of $1/\tau$ for small τ. When the momenta k_j are sufficiently small on the scale of this mass-gap we can expand the denominators in the spectral decomposition of (5.16) in integer powers of $\sum k$, except for the contribution of the vacuum, which is of the form $1/\sum k$. The singular behaviour as $k \to 0$ must disappear when we take the logarithm to ensure cluster decomposition. We conclude that any term of finite order in the sources in $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ has a local expansion for $\tilde{\varphi}, \varphi$ that vary sufficiently slowly with $\mathbf{x}$. This will take the form $W_{\tau,\tau'}[\tilde{\varphi}, \varphi] = \int d^d\mathbf{x}(a\varphi^2 + b\tilde{\varphi}^2 + c\varphi\nabla^2\varphi + d\varphi\nabla^2\nabla^2\varphi..)$ with a, b, c,.. depending on τ, τ'. If the Fourier transforms of the sources have bounded support then $\tilde{\varphi}^\rho, \varphi^\rho$ are slowly varying for large ρ. Since each term in the expansion of $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$ has a simple dependence on ρ, we have that for large ρ the functional $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho] = \sqrt{\rho}^d \int d^d\mathbf{x}\,(a\varphi^2 + b\tilde{\varphi}^2 + c\varphi\nabla^2\varphi/\rho + d\varphi\nabla^2\nabla^2\varphi/\rho^2...$ The powers of ρ are $d/2 +$ integer so we conclude that the cut on the negative real axis in $\sqrt{\rho}^d W_{\tau,\tau'}[\tilde\varphi^\rho, \varphi^\rho]$ runs only a finite distance from the origin.

We will now exploit the analyticity to obtain the logarithm of the partition function $\log Z[\varphi] = \lim_{\tau\to\infty,\tau'\to 0} W_{\tau,\tau'}[0,\varphi]$ from the small τ behaviour, which is computable from a power series solution of the Schrödinger equation. (We take $\tilde\varphi = \lim_{\mathbf{x}\to\infty}\varphi = 0$.) For simplicity we assume continuity in τ' at the origin, and consider $W_{\tau/\sqrt{\rho},0}[\tilde\varphi,\varphi]$. In general there will need to be some τ-dependent renormalization in order that the limit as $\tau\to\infty$ exists, including wave-function renormalization, $\varphi_{\rm ren} = \sqrt{z(\tau)}\varphi$, this will be the case even for free 'massive' fields at tree-level. Given that φ is a scalar the isometry $x^\mu \to \lambda x^\mu$ implies that the functional takes the form

$$\int d^d\mathbf{x}\,\frac{\Omega}{\tau}\left(a + \frac{1}{2}\tilde\varphi\,\Gamma(-\tau^2\nabla^2)\,\tilde\varphi + \tilde\varphi\,\Xi(-\tau^2\nabla^2)\,\varphi + \frac{1}{2}\varphi\,\Upsilon(-\tau^2\nabla^2)\,\varphi + ..\right) \tag{5.18}$$

where the dots stand for terms of higher order in the fields of which the general term is of the form

$$\frac{\Omega}{\tau}\int d^d\mathbf{x}\,\Gamma_{m,n}(\tau\nabla_1,..,\tau\nabla_{n+m})\varphi(\mathbf{x}_1)..\varphi(\mathbf{x}_n)\,\tilde\varphi(\mathbf{x}_{n+1})..\tilde\varphi(\mathbf{x}_{n+m})\Big|_{\{\mathbf{x}_i=\mathbf{x}\}} \tag{5.19}$$

At short times, or equivalently, for slowly varying fields, we have the local expansions

$$\Gamma = \sum_{n=0}^{\infty} b_n\,(-\tau^2\nabla^2)^n,\quad \Xi = \sum_{n=0}^{\infty} c_n\,(-\tau^2\nabla^2)^n,\quad \Upsilon = \sum_{n=0}^{\infty} f_n\,(-\tau^2\nabla^2)^n, \tag{5.20}$$

with b_n, c_n, f_n constants. Renormalizability would imply that

$$\frac{1}{\tau^d z(\tau)}\left[\Upsilon(-\tau^2\nabla^2) + \text{polynomial in }\nabla\right] \tag{5.21}$$

is finite as $\tau\to\infty$. Suppose that for large τ the renormalization constant depends on τ as $z(\tau)\sim\tau^{2q}$ then finiteness of the limit of (5.21) requires that for large τ, $\Upsilon(-\tau^2\nabla^2)\sim(-\tau^2\,\nabla^2)^{d/2+q}\upsilon$, and our problem is to calculate υ and q. The general term in (5.19) should depend on τ as $\Gamma_{m,n}(\tau\nabla_1,..,\tau\nabla_{n+m}) \sim \tau^{d+(n+m)q}F_{m,n}(\nabla_1,..,\nabla_{n+m})$ and we need to calculate $F_{m,n}$. Now our previous arguments imply that $\Upsilon(1/\rho)$ is analytic in the complex plane with a finite cut extending from the origin along the negative real axis so we can evaluate the following integral

$$I(\lambda) = \frac{1}{2\pi i}\int_C \frac{d\rho}{\rho}\, e^{\lambda\rho}\,\Upsilon(1/\rho) \tag{5.22}$$

in two ways. We take C to be a circle centred on the origin and large enough for us to be able to use the local expansions (5.20) to give

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{f_n\lambda^n}{n!} \tag{5.23}$$

The integral may also be evaluated by collapsing the contour C onto the cut. Let this consist of a small circle about the origin, of radius η, and two lines close to the negative real axis running from the circle to the end of the cut. The contribution from the latter is suppressed if the real part of λ is large and positive. That from the circle is controlled by the large time behaviour

$$\frac{1}{2\pi i}\int_{|\rho|=\eta} d\rho\, \frac{\upsilon\, e^{\lambda\rho}}{\rho^{d/2+q+1}} = \upsilon\lambda^{d/2+q}\left(\frac{1}{2\pi i}\int_{|\rho|=|\lambda|\eta} d\rho\, \frac{e^{\rho}}{\rho^{d/2+q+1}}\right) \tag{5.24}$$

and for large λ this is $\lambda^{d/2+q}/\Gamma(d+2q+1)$. So, as the real part of λ tends to $+\infty$ we obtain

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{f_n\lambda^n}{n!} \sim \frac{\upsilon\lambda^{d/2+q}}{(d+2q)!} \tag{5.25}$$

enabling us to compute υ and q from a knowledge of the local expansion alone. Now for positive real λ, $\sum_{n=0}^{\infty} f_n\lambda^n$ is an alternating series with finite radius of convergence. By comparing terms with those of $\exp(-\text{constant}\lambda)$ it follows that this converges for all λ and so we can take λ large, even though this series is in positive powers of λ. Furthermore, as we shall see in some examples below, we obtain a good approximation to the large λ limit by truncating the series at some order, and then taking λ as large as is consistent with the truncation, i.e. so that the term of highest order in λ is a small fraction of the sum. This generalizes in an obvious way to the other terms in (5.18).

5.4 Free scalar field

To illustrate the solution of the Schrödinger equation consider the free massless theory so $V = 0$ in (5.2); the action is

$$S = \frac{1}{2}\int d^d\mathbf{x}\, dt \left(\Omega \sum_{\mu=0}^{d} (\partial_\mu \phi)^2\right) \tag{5.26}$$

with $\Omega = 1/t^{d-1}$. For this Gaussian functional integral only those terms shown explicitly in (5.18) are present. Substituting this into the Schrödinger equation (5.8) yields

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\left(\frac{\Omega}{\tau}\Gamma\right) = \frac{\Omega}{\tau^2}\Gamma^2 + \Omega\nabla^2, \qquad \frac{\partial}{\partial\tau}\left(\frac{\Omega}{\tau}\Xi\right) = \frac{\Omega}{\tau^2}\Gamma\Xi \\
&\frac{\partial}{\partial\tau}\left(\frac{\Omega}{\tau}\Upsilon\right) = \frac{\Omega}{\tau^2}\Xi^2, \qquad \frac{\partial}{\partial\tau}\left(\frac{\Omega}{\tau}a\right) = \frac{1}{2\tau}\left(\Gamma + \frac{d}{2}\right)\delta^d(\mathbf{x}-\mathbf{y})\Big|_{\mathbf{x}=\mathbf{y}}
\end{aligned} \tag{5.27}$$

These, together with the initial condition, lead to the recursive solution of the coefficients of the local expansions (5.20)

$$\begin{aligned}
&b_0 = -d = -c_0 = f_0, \quad b_1 = -1/(2+d) \\
&b_n = \frac{\sum_{q=1}^{n-1} b_q\, b_{n-q}}{2n+d}, \quad c_n = \frac{\sum_{q=1}^{n} b_q\, c_{n-q}}{2n}, \quad f_n = \frac{\sum_{q=0}^{n} c_q\, c_{n-q}}{2n-d}.
\end{aligned} \tag{5.28}$$

If we were to take d to be an even positive integer the relations for the f_n would break down, thus keeping d variable regulates the solution for f_n.

These relations are easily solved in terms of Bessel functions, as we will demonstrate explicitly when we consider fermionic fields. But when interaction terms are included the situation is not so simple. To illustrate a tractable numerical method which works at any number of loops, we will take a slightly different approach at this stage.

These relations as they stand are ideally suited to numerical evaluation of the coefficients. Interaction terms will just lead to similar recursion relations of higher order.

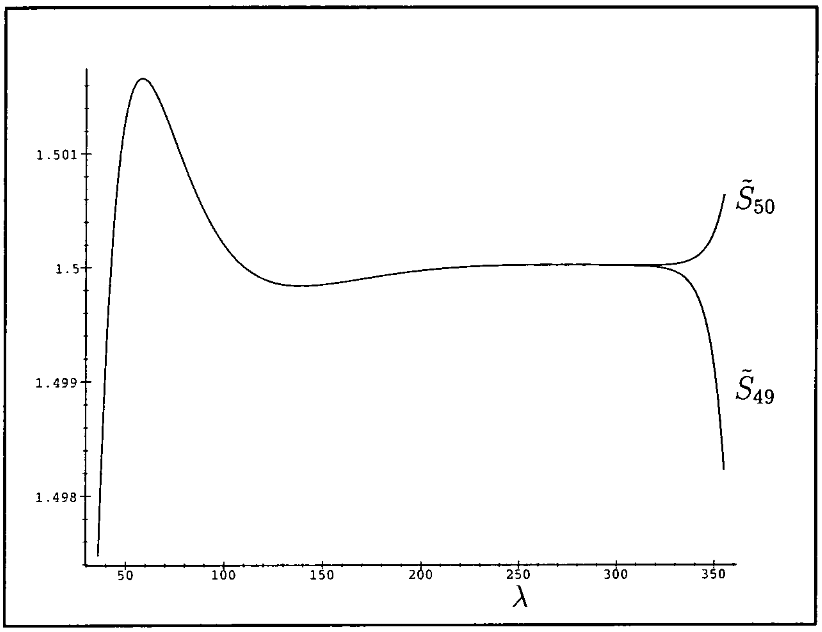

Figure 5.1: The truncated series as a function of λ.

We want to calculate v and q, which specify the two-point function. To illustrate the calculation take the example of $d = 3$. From (5.25) we have that for large λ, $\tilde{I}(\lambda) \equiv d(\log(I(\lambda))/d(\log(\lambda)) \to \tilde{d}/2 + q$. Truncating the infinite series to its first N terms, $S_N(\lambda)$, gives an approximation to this. In Figure 1 we have shown $\tilde{S}_N \equiv d(\log(S_N(\lambda))/d(\log(\lambda))$ for $N = 50$ and $N = 49$. The two curves rapidly settle down to a value of approximately 1.5 for $\lambda > 50$ but separate noticeably at $\lambda \approx 300$ above which the truncated series cease to be good approximations to $I(\lambda)$. We

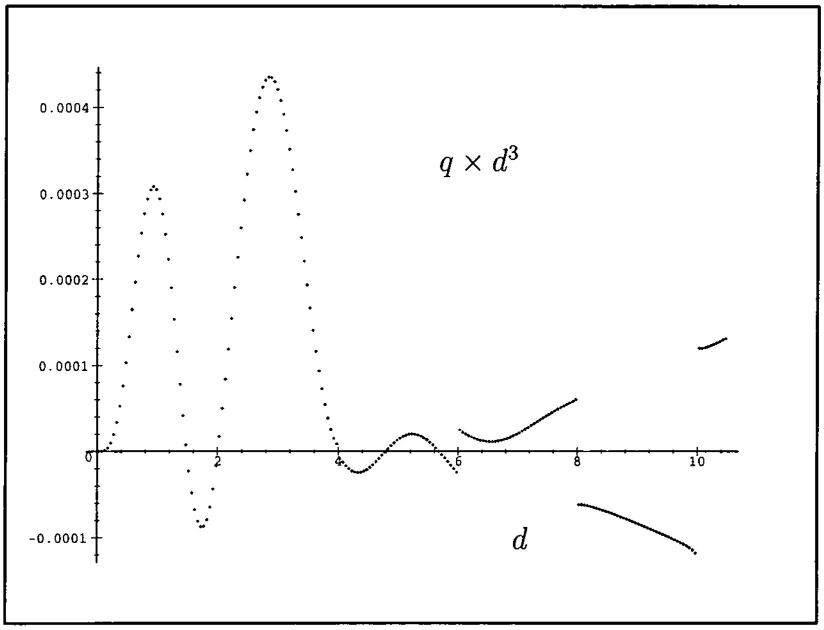

Figure 5.2: q as a function of d.

estimate $d/2 + q$ by taking $\lambda = 290$ where the separation between the two curves is about 0.5×10^{-6}, which is much less than the error $\mathcal{E}$ obtained by approximating the limiting value of $\tilde{I}$ by its value for finite λ. This gives $d/2+q \approx \tilde{S}_{50}(290) = 1.500017$. The error is obtained by studying how $I(\lambda)/\lambda^{d/2+q}$ settles down to a constant value. For small λ the approach to a constant value is controlled by exponential terms that originate from the suppression of the contribution of the cut, but for larger λ the error is dominated by power corrections to the small ρ behaviour of $\Upsilon(1/\rho)$. A plot of $(S_{50}(\lambda)/\lambda^{1.5} - S_{50}(290)/290^{1.5})\lambda^{2.6}$ reveals oscillations of roughly equal amplitude approximately equal to 12, so that the error in approximating the $\lambda \to \infty$ value of $I(\lambda)/\lambda^{d/2+q}$ is of order $12/\lambda^{2.6}$ leading to an estimate of the error $\mathcal{E} = \pm 2 \times 10^{-5}$. So we conclude that $q = 0$ to the accuracy of our calculations. Having obtained q we can estimate υ from the $\lambda = 290$ value of $S_{50}\Gamma(2.5)/\lambda^{1.5}$ as 0.999998 with an error of $\pm 10^{-5}$. We have repeated the calculation of q for various values of d. The results are shown in Figure 2. We have plotted our estimate of q (multiplied by d^3 to make the results for large d visible) for d varying in steps of 0.025 from 0.05 to 10.5. The results are consistent with $q = 0$, which is the exact result of [2]. By calculating a few values of υ we guessed that its dependence on d is given by

$$\upsilon = \frac{-((d-3)/2)!^2\, 2^{d-3}}{\sin(d\pi/2)\,(d-2)!^2} \equiv \tilde{\upsilon}(d), \tag{5.29}$$

and we test this by plotting in Fig 3 our numerical estimates of υ divided by the right hand side of (5.29) for the same range of d as before. From this, and $q = 0$ we conclude that the φ-dependence of the AdS partition function is given by

$$\log Z[\varphi] = \lim_{\tau\to\infty} \frac{1}{2\tau^d} \int d^d\mathbf{x}\, \varphi\, \tilde{\upsilon}(d)\, (-\tau^2\, \nabla^2)^{d/2}\, \varphi \tag{5.30}$$

This is a *non-local* expression, even when d is an even integer, thanks to the singularity that then appears in $\tilde{\upsilon}$, since

$$\tilde{\upsilon}(d)\,(-\nabla^2)^{d/2}\,\delta^d(\mathbf{x}-\mathbf{y}) = \frac{\tilde{c}}{|\mathbf{x}-\mathbf{y}|^{2d}}, \quad \tilde{c} = \frac{(d-1)\,(d/2)!\,(d/2-3/2)!^2}{8\,\pi^{d/2+1}\,(d-2)!} \tag{5.31}$$

so that

$$\log Z[\varphi] = \frac{\tilde{c}}{2} \int d^d\mathbf{x}\, d^d\mathbf{y}\, \frac{\varphi(\mathbf{x})\,\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2d}} \tag{5.32}$$

as in [2].

The calculation of a in (5.27) requires the introduction of a regulator into the functional Laplacian of (5.8). We will do this with a cut-off on the eigenvalues, k^2 of $-\nabla^2$, restricting them to be less than $1/(\tau^2 s)$ where s is the square of a fixed proper distance. Thus we replace $\delta^d(\mathbf{x})$ in (5.27) by

$$\theta\left(s\tau^2\nabla^2 + 1\right)\, \delta^d(\mathbf{x}) = \int_{k^2<1/(\tau^2 s)} \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.33}$$

where θ is the step-function, giving

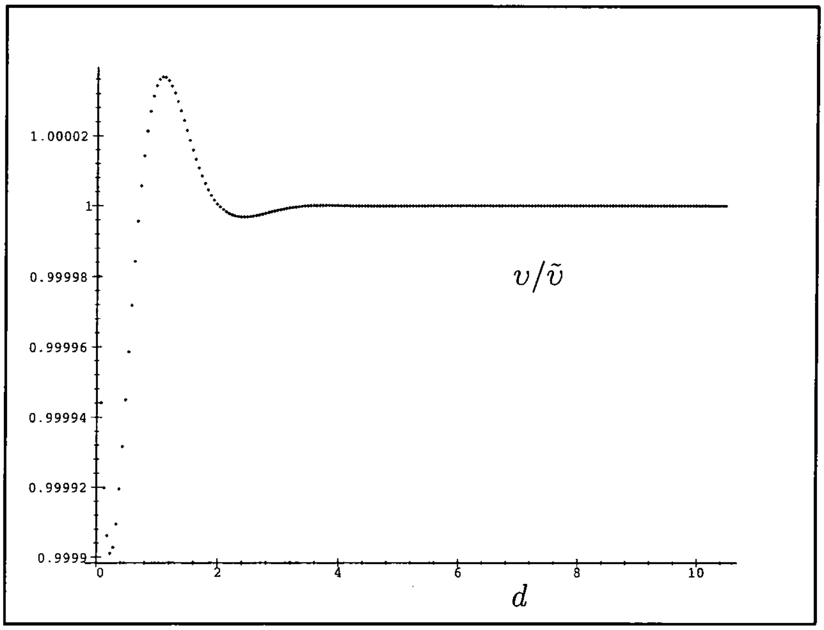

Figure 5.3: v as a function of d.

$$a(s) = -\frac{\tau^d}{2d} \int_{k^2<1/(\tau^2 s)} d^d k \left(\Gamma(\tau^2 k^2) + \frac{d}{2} \right) = -\frac{1}{2d} \int_{k^2<1/s} d^d k \left(\Gamma(k^2) + \frac{d}{2} \right). \quad (5.34)$$

The continuum limit corresponds to taking s to zero. Unfortunately when we substitute our local expansion (5.20) and (5.28) into this we obtain a series that will converge only for large values of s

$$a(s) = -\frac{V_d}{2ds^{d/2}} \sum_{n=0}^{\infty} \frac{b_n}{s^n\,(2n+d)} \quad (5.35)$$

where V_d/d is the volume of the unit ball in d-dimensions. However our previous arguments imply that $a(s)$ is an analytic function of s in the complex s-plane cut along the negative real axis, so that if $a(s) \sim a_0/s^\nu$ for small s then for large λ

$$\frac{1}{2\pi i} \int \frac{ds}{s}\, e^{\lambda s}\, a(s) = -\frac{V_d}{2d} \sum_{n=0}^{\infty} \frac{b_n\, \lambda^{n+d/2}}{(2n+d)\,(n+d/2)!} \sim \frac{a_0\, \lambda^\nu}{\nu!} \quad (5.36)$$

Numerical investigation of this suggests that $\nu = d + 1$. The ultraviolet divergence of a can be cancelled by renormalizing the cosmological constant. But when we sum over all the supergravity fields we will find that the relevant counterterm vanishes; this supports the conjecture that the boundary theory is $\mathcal{N} = 4$ Yang-Mills, which is a finite theory.

5.5 Massive scalar field

The effect of adding a mass term to the action, $V(\phi) = m^2\,\phi^2$ can be understood quite simply by a change of variables back to the massless action. If we set $\phi = t^{-r}\psi$ in the action

$$S = \frac{1}{2}\int_0^\tau dt \int d^d\mathbf{x}\, \left(t^{1-d} \sum_{\mu=0}^{d} (\partial_\mu \phi)^2 + t^{-d-1}\, m^2\, \phi^2 \right), \tag{5.37}$$

it becomes

$$\begin{aligned} S = \quad & \frac{1}{2}\int_0^\tau dt \int d^d\mathbf{x}\, \left(t^{1-d-2r} \sum_{\mu=0}^{d} (\partial_\mu \psi)^2 + t^{-d-1-2r}\, (m^2 - r^2 - rd)\, \psi^2 \right) \\ & -\frac{1}{2}\int d^d\mathbf{x}\, r\tau^{-d-2r}\psi^2(\tau, \mathbf{x}) \end{aligned} \tag{5.38}$$

so that if we take $r(r + d) = m^2$ we are left with a boundary term plus the massless action (5.26) with $\Omega = 1/t^{d+2r-1} \equiv 1/t^p$, (The inner product on variations of ψ, from which we can construct the functional integral volume element $\mathcal{D}\psi$, is $||\delta\psi||^2 = \int dt\, d\mathbf{x}\, \delta\psi^2/t^{p+2}$.) Consequently if we express $W_{\tau,0}[\tilde{\varphi}, \varphi]$ in terms of ψ it takes the form corresponding to (5.18)

$$\int d^d\mathbf{x}\, \frac{\Omega}{\tau} \left(a + \frac{1}{2}\tilde{\psi}\, \left(\Gamma(-\tau^2\nabla^2) - r\right)\, \tilde{\psi} + \tilde{\psi}\, \Xi(-\tau^2\nabla^2)\, \psi + \frac{1}{2}\psi\, \Upsilon(-\tau^2\nabla^2)\, \psi \right) \tag{5.39}$$

where, as before Γ, Ξ and Υ have the local/short-time expansions (5.20) with solutions (5.28) leading to

$$\log Z[\varphi] = \lim_{\tau\to\infty} \frac{1}{2\tau^{p+1}} \int d^d\mathbf{x}\, \psi\, \tilde{v}(p+1)\, \left((-\tau^2\,\nabla^2)^{(p+1)/2} - r\right)\psi \tag{5.40}$$

A non-trivial limit occurs for positive $p+1$, which means that r is the larger root of $r(r+d) = m^2$, giving

$$\log Z[\varphi] = \frac{c'}{2} \int d^d\mathbf{x}\, d^d\mathbf{y}\, \frac{\varphi(\mathbf{x})\,\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2d+2r}} \tag{5.41}$$

since $d+p+1 = 2(d+r)$, agreeing with [2]. Taking this limit required wave-function renormalization of φ with $z(\tau) = \tau^r$.

Having obtained $\Psi_{\tau,0}[\tilde{\varphi},\varphi]$ we can find $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$ using the self-reproducing property of the Schrödinger functional,

$$\Psi_{\tau,0}[\tilde{\varphi},\varphi] = \int \mathcal{D}\hat{\varphi}\, \Psi_{\tau,\tau'}[\tilde{\varphi},\hat{\varphi}]\, \Psi_{\tau',0}[\hat{\varphi},\varphi]. \tag{5.42}$$

If we denote the logarithm of $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$ by

$$\int d^d\mathbf{x}\, \left(a_{\tau,\tau'} + \frac{1}{2}\tilde{\varphi}\,\Gamma_{\tau,\tau'}\,\tilde{\varphi} + \tilde{\varphi}\,\Xi_{\tau,\tau'}\,\varphi + \frac{1}{2}\varphi\,\Upsilon_{\tau,\tau'}\,\varphi\right) \tag{5.43}$$

then computing the Gaussian integral leads to

$$\begin{aligned}
\Upsilon_{\tau,0} &= \Upsilon_{\tau',0} - (\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0})^{-1}\Xi^2_{\tau',0} \\
\Xi_{\tau,0} &= -\Xi_{\tau',0}\,(\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0})^{-1}\Xi_{\tau,\tau'} \\
\Gamma_{\tau,0} &= \Gamma_{\tau,\tau'} - (\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0})^{-1}\Xi^2_{\tau,\tau'}
\end{aligned} \tag{5.44}$$

hence

$$\Upsilon_{\tau,\tau'} = (\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1}\,\Xi^2_{\tau',0} - \Gamma_{\tau',0}$$

$$\Xi_{\tau,\tau'} = -(\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1}\,\Xi_{\tau,0}\,\Xi_{\tau',0}$$
$$\Gamma_{\tau,\tau'} = \Gamma_{\tau,0} + (\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1}\,\Xi^2_{\tau,0} \tag{5.45}$$

So, in terms of Γ, Ξ and Υ

$$\Upsilon_{\tau,\tau'} = \tau'^{-(p+1)}\Big(\big(\Upsilon(-\tau'^2\nabla^2) - (\tau'/\tau)^{p+1}\Upsilon(-\tau^2\nabla^2)\big)^{-1}\,\Xi(-\tau'^2\nabla^2)^2 - \Gamma(-\tau'^2\nabla^2)\Big)$$
$$\Xi_{\tau,\tau'} = (\tau\tau')^{-(p+1)}\Big(\tau^{-(p+1)}\Upsilon(-\tau^2\nabla^2) - \tau'^{-(p+1)}\Upsilon(-\tau'^2\nabla^2)\Big)^{-1}\,\Xi(-\tau^2\nabla^2)\,\Xi(-\tau'^2\nabla^2)$$
$$\Gamma_{\tau,\tau'} = \tau^{-(p+1)}\Big(\Gamma(-\tau^2\nabla^2) + \big((\tau/\tau')^{p+1}\Upsilon(-\tau'^2\nabla^2) - \Upsilon(-\tau^2\nabla^2)\big)^{-1}\,\Xi(-\tau^2\nabla^2)^2\Big) \tag{5.46}$$

5.6 Conformal anomaly for scalar fields

So far we have worked with a flat boundary metric. To discuss the conformal anomaly it will be necessary to generalize to a curved boundary. Let us make the Ansatz

$$g_{00} = 1/t^2, \qquad g_{i0} = 0, \qquad g_{ij} = e^{\rho(t)} g_{ij}(\mathbf{x}), \tag{5.47}$$

with g_{ij} the metric on the boundary. Then the Einstein equations in the bulk imply that

$$e^{\rho(t)} = (1 - \frac{Rt^2}{4d(d-1)})^2. \tag{5.48}$$

They also imply that the boundary metric is Einstein

$$R_{ij} = g_{ij} R/d \tag{5.49}$$

When we use the metric (5.47), the transformation back to canonical variables in (5.5) leads to terms in powers of Rt^2. The first of these will modify the boundary Laplacian (in fact we end up with the conformal operator $-\nabla + R/6$), and the others, of higher order in t^2, will have no effect on the anomaly, which we will express in terms of the small-t limit. But in any case, for the purposes of this thesis we will restrict our attention to Ricci-flat boundaries, for which (5.47) reduces to

$$ds^2 = \frac{1}{t^2}\left(dt^2 + \sum_{i,j} g_{ij}\, dx^i\, dx^j\right). \tag{5.50}$$

where $i, j = 1..d$ and g_{ij} is the Ricci-flat boundary metric.

The conformal anomaly measures the response of the free energy, which is the field independent part of $\log Z$, to a Weyl transformation of g_{ij}. When the boundary is two-dimensional the free energy should change by $\int d^2\mathbf{x}\, \sqrt{g} R\, c\delta\rho/(48\pi) = c\delta\rho/6$ when $\delta g_{ij} = g_{ij}\,\delta\rho$ where R is the curvature of the boundary and c the central charge of the Virasoro algebra. More generally there will be a conformal anomaly when the boundary has an even number of dimensions, $2N$ say. We will continue to keep d a continuous variable allowing it to tend to $2N$ at the end of our calculations. Now $\log Z[\varphi] = \lim_{\tau\to\infty,\tau'\to 0} W_{\tau,\tau'}[0,\varphi]$ so we need to compute $W_{\tau,\tau'}[0,\varphi]$ in the presence of the curved metric g_{ij}, which we can still do using our previous technique. It will be sufficent to find the free energy from a derivative expansion. In this section we consider the scalar field. We will discuss a free massive theory, because at one-loop the calculation is the same as for an interacting scalar theory. The Schrödinger equation takes the same form as before, (5.8), provided that Ω and Ω' acquire a factor of $\sqrt{\det g}$ and that ∇ is the covariant derivative constructed from g_{ij}, and the solution is again of the form (5.18), but with a no longer constant, but depending on g_{ij} and τ. If we set $g_{ij} = \delta_{ij} + h_{ij}(\mathbf{x})$, and treat h_{ij} as a source in the same way that we treated φ and $\tilde{\varphi}$ as sources, we can generalize our earlier discussion to argue that $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho, g^\rho_{ij}]$ is analytic in the cut ρ-plane, where $g^\rho_{ij}(\mathbf{x}) = g_{ij}(\mathbf{x}/\sqrt{\rho})$. Again, this allows us to reconstruct the large τ solution of the Schrödinger equation from the small τ solution for which we have the local expansion (5.20). By using g_{ij} the Schrödinger functional can be made invariant under reparametrizations of the space-like variables, giving

$$W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho, g^\rho_{ij}] = W_{\tau,\tau'}[\tilde{\varphi}, \varphi, \rho g_{ij}] = W_{\tau/\sqrt{\rho},\tau'/\sqrt{\rho}}[\tilde{\varphi}, \varphi, g_{ij}], \tag{5.51}$$

which firstly shows that the functional evaluated for the scaled fields is the same as the Weyl transformed functional, and secondly that this transformation can be absorbed into a rescaling of τ and τ'. This implies that $W_{\tau,\tau'}[\tilde{\varphi},\varphi]$ is analytic in the complex τ-plane cut along the negative real axis. In particular, since the part that is quadratic in $\tilde{\varphi}$ is $\tau^{-d}\int d^d\mathbf{x}\tilde{\varphi}\Gamma(-\tau^2\nabla^2)\tilde{\varphi}$, it follows that $\Gamma(-(\tau^2/\rho)\nabla^2)$ is analytic in the cut ρ-plane. This allows us to express $\Gamma(-\tau^2/\nabla^2)$ for arbitrary τ in terms of the local expansion (5.20)

$$
\begin{aligned}
\Gamma\left(-\tau^2\nabla^2\right) &= \lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_C \frac{d\rho}{\rho-1}\, e^{\lambda(\rho-1)}\,\Gamma\left(-(\tau^2/\rho)\nabla^2\right) \\
&= \lim_{\lambda\to\infty}\sum_{n=0}^{\infty} b_n \frac{1}{2\pi i}\int_C \frac{d\rho}{\rho-1}\,\frac{e^{\lambda(\rho-1)}}{\rho^n}(-\tau^2\nabla^2)^n
\end{aligned}
\tag{5.52}
$$

since the large contour, C, on which we can use the local expansion, (5.20), can be collapsed to a contribution from the cut, which is suppressed for large positive λ and the pole at $\rho = 1$ which gives us the left hand side. Expanding the denominators in powers of $1/\rho$ gives, for example

$$
\Gamma\left(-\tau^2\nabla^2\right) = \lim_{\lambda\to\infty}\sum_{n,r=0}^{\infty}(-)^r \frac{b_n\,\lambda^{n+r}\,(-\tau^2\nabla^2)^n}{(n-1)!\,r!\,(n+r)} \tag{5.53}
$$

The free energy is the $\tau\to\infty$, $\tau'\to 0$ limit of $F[\tau', g_{ij}] = \int d^d\mathbf{x}\, a_{\tau,\tau'}$. A Weyl scaling of g_{ij} can be compensated by scaling τ, and τ' so when $\delta g_{ij} = g_{ij}\,\delta\rho$ the change in F is

$$
\delta F = -\frac{\delta\rho}{2}\left(\tau\frac{\partial F}{\partial\tau} + \tau'\frac{\partial F}{\partial\tau'}\right) \tag{5.54}
$$

F satisfies equations similar to regulated versions of (5.27) (even if we include interactions), that follow from the Schrödinger equations (5.8) and (5.10). If we use the same regulator as before, cutting off the large eigenvalues of ∇^2, then

$$
\frac{\partial F}{\partial\tau} = \frac{1}{2\tau}\int d^d\mathbf{x}\,\left(\tau^{p+1}\Gamma_{\tau,\tau'} + \frac{p+1}{2}\right)\,\theta\left(s\tau^2\nabla^2+1\right)\,\delta^d(\mathbf{x}-\mathbf{y})|_{\mathbf{x}=\mathbf{y}}. \tag{5.55}
$$

and

$$
-\frac{\partial F}{\partial\tau'} = \frac{1}{2\tau'}\int d^d\mathbf{x}\,\left(\tau'^{p+1}\Upsilon_{\tau,\tau'} - \frac{p+1}{2}\right)\,\theta\left(s\tau'^2\nabla^2+1\right)\,\delta^d(\mathbf{x}-\mathbf{y})|_{\mathbf{x}=\mathbf{y}}. \tag{5.56}
$$

If we represent the step function by

$$\theta(x) = \frac{1}{2\pi i}\int_{C'} \frac{dy}{y}\, e^{iyx}, \tag{5.57}$$

with C' a contour running just below the real axis, and if f is some function of $-\nabla^2$ then

$$f(-\nabla^2)\,\theta\left(s\tau^2\nabla^2+1\right)\,\delta^d(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi i}\int_{C'} \frac{dy}{y} e^{iy}\, f\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\, e^{iys\tau^2\nabla^2}\,\delta^d(\mathbf{x}-\mathbf{y}). \tag{5.58}$$

Now $e^{it\nabla^2}\,\delta^d(\mathbf{x}-\mathbf{y}) \equiv H(t,\mathbf{x},\mathbf{y})$ satisfies the finite-dimensional Schrödinger equation

$$i\frac{\partial}{\partial z}H = -\nabla^2 H, \quad H(0,\mathbf{x},\mathbf{y}) = \delta^d(\mathbf{x}-\mathbf{y}). \tag{5.59}$$

At coincident argument H has the small z expansion in powers of derivatives of g_{ij}

$$H(z,\mathbf{x},\mathbf{x}) \sim \frac{\sqrt{g}}{(4\pi i z)^{d/2}}\sum_{n=0}^{\infty} a_n(\mathbf{x})\, z^n. \tag{5.60}$$

The $a_n(\mathbf{x})$ are scalars made out of the metric and its derivatives at $\mathbf{x}$, and $a_0(\mathbf{x}) = 1$. Thus we can express (5.58) as

$$\sum_{n=0}^{\infty}\frac{1}{(4\pi i)^{d/2}}\left(\int \frac{d^d\mathbf{x}}{\tau^d}\sqrt{g}\,a_n(\mathbf{x})\,\tau^{2n}\right)\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}\,f\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\,(sy)^{n-d/2} \tag{5.61}$$

Only a finite number of the a_m contribute. For the case of a two-dimensional boundary these are just $a_0 = 1$ and $a_1 = iR/6$, and the conformal anomaly is proportional to a_1. When the boundary has $2N$ dimensions the conformal anomaly is proportional to a_N. If we assume that f has a series expansion, $f(x) = \sum \tilde{f}_n x^n$ then we can compute the relevant integral in (5.61) as

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}\,f\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\,(sy)^{N-d/2}\\
&=\sum\tilde{f}_n\,s^{N-d/2}\left(-\frac{i}{s\tau^2}\right)^n\frac{\sin(\pi d/2)\,(-1)^{N+1}\,(N-d/2)!}{\pi\,(N-d/2-n)}\\
&=\sin(\pi d/2)\,(-1)^N\,(N-d/2)!\frac{1}{\pi}\int_s^\infty ds'\,s'^{N-1-d/2}\,f\left(-\frac{i}{s'\tau^2}\right)
\end{aligned}
\tag{5.62}
$$

where we have taken $N < d/2$. If f has a finite limit, f_{lim}, as $d \downarrow 2N$ then, for d close to $2N$ this becomes

$$
\begin{aligned}
&(d/2-N)\int_s^\infty ds'\,s'^{N-1-d/2}\,f\left(-\frac{i}{s'\tau^2}\right)\\
&=s^{N-d/2}\,f\left(-\frac{i}{s\tau^2}\right)+\int_s^\infty ds'\,s'^{N-d/2}\,\frac{d}{ds'}f\left(-\frac{i}{s'\tau^2}\right)
\end{aligned}
\tag{5.63}
$$

which tends to $f_{\text{lim}}(0)$ as $d \downarrow 2N$. Putting all this together we obtain the conformal anomaly as the large τ small τ' limit of

$$
\delta F=-\frac{\delta\rho}{4}\left(p+1+\hat{\Gamma}-\hat{\Upsilon}\right)\frac{1}{(4\pi i)^N}\int d^d\mathbf{x}\sqrt{g}\,a_N(\mathbf{x})
\tag{5.64}
$$

where

$$
\begin{aligned}
\hat{\Gamma}&=\lim_{\xi\to 0}\left(\lim_{d\downarrow 2N}\left(\Gamma(-\tau^2\xi)+\left\{(\tau/\tau')^{p+1}\Upsilon(-\tau'^2\xi)-\Upsilon(-\tau^2\xi)\right\}^{-1}\,\Xi(-\tau^2\xi)^2\right)\right),\\
\hat{\Upsilon}&=\lim_{\xi\to 0}\left(\lim_{d\downarrow 2N}\left(\left\{\Upsilon(-\tau'^2\xi)-(\tau'/\tau)^{p+1}\Upsilon(-\tau^2\xi)\right\}^{-1}\,\Xi(-\tau'^2\xi)^2-\Gamma(-\tau'^2\xi)\right)\right)
\end{aligned}
\tag{5.65}
$$

From the series expansions (5.28) we see that $\hat{\Gamma} = b_0 = -(p+1)$ and for generic values of p we have $\hat{\Upsilon} = 0$. The order of the limits is important since when p approaches an odd integer as $d \downarrow 2N$ the coefficient f_N diverges so that for $\xi \neq 0$ there is a suppression of $\left\{\Upsilon(-\tau'^2\xi)-(\tau'/\tau)^{p+1}\Upsilon(-\tau^2\xi)\right\}^{-1}$ thus $\hat{\Upsilon} = -\hat{\Gamma}$ which is just $-b_0$. Since $p+1=\sqrt{d^2+4m^2}=2\sqrt{N^2+m^2}$ we have that when $\sqrt{N^2+m^2}$ is an integer, $\tilde{N}$, the conformal anomaly is

$$\delta F = \frac{\delta\rho}{2}\frac{\tilde{N}}{(4\pi i)^N}\int d^{2N}\mathbf{x}\sqrt{g}\,a_N(\mathbf{x}) \tag{5.66}$$

otherwise it vanishes. In particular for $\mathbf{d = 2}$ we have that the central charge of the Virasoro algebra, c, equals $\tilde{N}$ when $m = \sqrt{\tilde{N}^2 - 1}$ and vanishes otherwise. For $\mathbf{d = 4}$

$$\delta F = -\frac{\delta\rho}{32\pi^2}\int d^4x\,\sqrt{g}\,a_2(\mathbf{x})\,\tilde{N} \tag{5.67}$$

or zero, where

$$a_2 = \frac{1}{180}\left(R_{ijkl}\,R^{ijkl}\right). \tag{5.68}$$

Our conventions for the curvature tensors are as in [14], i.e. $R^i{}_{jkl} = \partial_l\Gamma^i_{jk} - ..$, $R_{ij} = R^k{}_{ikj}$ and $\Box = g^{ij}\nabla_i\nabla_j$. The mass condition $m^2 = \tilde{N}^2 - N^2$, corresponds to the mass spectrum of the scalar fields of supergravity compactified on $AdS_5 \times S^5$, [26], for $d = 2N = 4$, and on $AdS_7 \times S^4$ for $d = 2N = 6$, [27]. Note that for $D = 2N \geq 4$ there are negative values of m^2 with non-vanishing conformal anomaly, these also appear in the Kaluza-Klein compactifications, and are known to be stable.

5.6.1 Generic boundary conditions

Now our calculation assumed Dirichlet boundary conditions (we chose to diagonalize ϕ on the boundary), but we could equally well have chosen Neumann conditions, or in general a combination of both, diagonalizing $\dot{\phi} - \alpha\phi$ for an arbitrary constant α.

Does this affect the value of the anomaly? We can quite easily check by performing a functional Fourier transformation on the boundary which modifies the boundary conditions to the more general ones. We start with our existing WF

$$\log\Psi = \tfrac{1}{2}\int d^dx\left(\tilde{\phi}\Gamma\tilde{\phi} + 2\tilde{\phi}\Xi\phi + \phi\Upsilon\phi\right), \tag{5.69}$$

and integrate over ϕ after adding an additional boundary term

$$\int D\phi \Psi e^{-\frac{\alpha}{2}\int \frac{d^d x}{\tau'^{d+2r}}\phi^2}. \tag{5.70}$$

This modifies the free energy by $F \to F + \tilde{F}$ where

$$e^{-2\tilde{F}} = \det\left(\tau'^{d+2r}\Upsilon - \alpha\right), \tag{5.71}$$

and under a Weyl scaling this changes as

$$\tau'\frac{\partial}{\partial \tau'}2\tilde{F} = \mathrm{tr}\frac{(d+2r)\tau'^{d+2r}\Upsilon + \tau'^{d+2r+1}(-\tau'^{d+2r-1}\Upsilon^2 - \tau'^{1-d-2r}\nabla^2)}{\tau'^{d+2r}\Upsilon - \alpha}. \tag{5.72}$$

For generic values of the scalar field mass this tends to zero. But for the special cases where the anomaly is non-zero, we have $\tau'^{d+2r}\Upsilon \sim d+2r+\frac{1}{d+2r+2}(-\tau^2\nabla^2)+\ldots$ and (5.72) becomes

$$\mathrm{tr}\frac{-2\tau^2\nabla^2}{(d+2r-\alpha)(d+2r+2)-\tau^2\nabla^2}, \tag{5.73}$$

which has a τ-independent contribution $\mathrm{tr}(-2)$ for the specific value $\alpha = d+2r$, and thus gives a mass-independent contribution to the Weyl anomaly. In the generic case the anomaly is unchanged. When we sum the anomaly over supergravity multiplets, we will find that we need to choose the generic case for the AdS/CFT correspondence to work.

5.7 Free fermion field

We can extend everything we have done so far to fermions, using the representation described in chapter 2. We will give analytic results in this case, since we can always arrange for interactions to be quadratic in fermion fields. To begin with we choose $Q = \pm\gamma^0$, but again, we will consider generic boundary conditions later.

Imposing the constraints

$$Q_{-}u = u^{\dagger}Q_{+} = 0, \tag{5.74}$$

and

$$Q_{+}v = v^{\dagger}Q_{-} = 0, \tag{5.75}$$

will cause a Dirac fermion in the AdS theory to become a chiral fermion in the boundary theory. This is in agreement with the findings of other authors, for example in [18].

Consider the AdS metric (5.1). With the choice of vielbein

$$e^{a}_{\mu} = t^{-1}\delta^{a}_{\mu}, \tag{5.76}$$

the Euclidean action is given by[1]

$$S = \int d^{d+1}x\sqrt{g}\bar{\psi}(t\gamma\cdot D - m)\psi = \int d^{d+1}x t^{-d}\bar{\psi}(\gamma\cdot\partial - \frac{\gamma^{0}d}{2t} - \frac{m}{t})\psi, \tag{5.77}$$

since according to (5.1), $\sqrt{g} = t^{-d-1}$, and the spin covariant derivative is $D_{\mu} = \partial_{\mu} - \frac{1}{t}\Sigma_{0\mu}$. Changing variables to $\phi = t^{-d/2}\psi$ and $\phi^{\dagger} = t^{-d/2}\psi^{\dagger}$ the action becomes

$$S = \int d^{d+1}x\bar{\phi}(\gamma\cdot\partial - \frac{m}{t})\phi, \tag{5.78}$$

and for $m = 0$ it coincides with the flat-space action. As in the bosonic case, if we also put $\bar{t} = \ln t$, the volume element in the corresponding path-integral becomes the usual flat-space one induced by $||\delta\phi||^{2} = \int d\bar{t}\, d\mathbf{x}\, \delta\phi^{\dagger}\delta\phi$, and the action becomes the flat-space one even for $m \neq 0$. Thus we can make use of the representation of chapter 2 for ϕ and $\phi^{\dagger}$. The integrands in (2.48) and (2.61) do not aquire a factor

[1]Gamma matrices obey $\{\gamma^{i}, \gamma^{j}\} = 2\delta^{ij}$ throughout this section.

from the metric, as this has been absorbed into the definition of the fields.

The partition function is again given by the $\tau' \to 0$, $\tau \to \infty$ limit of the Schrödinger functional, with $u = u^\dagger = \lim_{|\mathbf{x}|\to\infty} v(\mathbf{x}) = \lim_{|\mathbf{x}|\to\infty} v^\dagger(\mathbf{x}) = 0$. In path-integral form the Schrödinger functional is

$$\Psi_{\tau,\tau'}[u, u^\dagger, v, v^\dagger] = \int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-S-S_B} \tag{5.79}$$

where the boundary term is

$$\begin{aligned} S_B = \quad & \int_{x^0=\tau'} d^dx \left(\phi^\dagger Q_-\phi - \sqrt{2}\phi^\dagger Q_- v + \sqrt{2}v^\dagger Q_+\phi\right) \\ & - \int_{x^0=\tau} d^dx \left(\phi^\dagger Q_+\phi - \sqrt{2}\phi^\dagger Q_+ u + \sqrt{2}u^\dagger Q_-\phi\right). \end{aligned} \tag{5.80}$$

If ϕ and $\phi^\dagger$ are integrated over freely then we can shift them by solutions to the classical equations of motion. Choosing these solutions to satisfy the boundary conditions corresponding to (2.55) and (2.56)

$$\begin{aligned} t = \tau' : \qquad & Q_-\phi = -\sqrt{2}Q_- v, \quad \phi^\dagger Q_+ = -\sqrt{2}v^\dagger Q_+ \\ t = \tau : \qquad & Q_+\phi = \sqrt{2}u, \quad \phi^\dagger Q_- = \sqrt{2}u^\dagger Q_- \end{aligned} \tag{5.81}$$

causes the action to separate into a piece depending only on the integration variables and a piece depending only on the classical solution. Our boundary term S_B is thus determined by the conditions (2.55) and (2.56). Note that the classical action does *not* vanish, and there is therefore no need to add any additional boundary term with undetermined coefficients, as in [18]. (Other authors have discussed boundary terms for fermions [19]-[20]).

The Schrödinger equation is

$$-\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'} = \hat{H}\Psi_{\tau,\tau'}, \tag{5.82}$$

where

$$\hat{H} = \frac{1}{2}\int d^dx(u^\dagger + \frac{\delta}{\delta u})h(u + \frac{\delta}{\delta u^\dagger}), \qquad h = (\gamma^0\gamma^i\partial_i - \frac{\gamma^0 m}{\tau}). \tag{5.83}$$

Now the logarithm of the partition function is obtained as $\lim_{\tau\to\infty} W_{\tau,0}[0,0,v,v^\dagger]$, where $W_{\tau,0}[u,u^\dagger,v,v^\dagger] = \log\Psi_{\tau,0}[u,u^\dagger,v,v^\dagger]$ may be expanded in analogy with (5.18) as

$$\int d^dx\left\{f + u^\dagger\Gamma(\tau\gamma_0\gamma^i\partial_i)u + u^\dagger\Xi(\tau\gamma_0\gamma^i\partial_i)v + v^\dagger\Pi(\tau\gamma_0\gamma^i\partial_i)u + v^\dagger\Upsilon(\tau\gamma_0\gamma^i\partial_i)v\right\}. \tag{5.84}$$

Substituting (5.84) into (5.82) gives

$$\begin{cases} \dot{\Gamma} = -\frac{1}{2}(1-\Gamma)h(1+\Gamma) & \\ \dot{\Xi} = -\frac{1}{2}(1-\Gamma)h\Xi & \Gamma(0) = \Upsilon(0) = Q \\ \dot{\Pi} = \frac{1}{2}\Pi h(1+\Gamma) & \Xi(0) \sim -2Q_- \\ \dot{\Upsilon} = \frac{1}{2}\Pi h\Xi & \Pi(0) \sim 2Q_+ \\ \dot{f} = -\frac{1}{2}\mathrm{Tr}h(1+\Gamma)\delta^d(\mathbf{x})|_{\mathbf{x}=0} & \end{cases} \tag{5.85}$$

where the initial conditions are read off from (2.61). We will find that $\Xi(0)$ and $\Pi(0)$ diverge as a result of the ill-defined nature of $\lim_{\tau\to 0}\int_0^\tau \frac{m}{t}dt$, but that $\lim_{\tau\to\tau'}\Psi_{\tau,\tau'}$ is well-defined and given by (2.61) for all $\tau' \neq 0$.

As may be verified by direct substitution, the equation for Γ is solved by

$$\Gamma = (\Sigma - Q_-)(\Sigma + Q_-)^{-1}, \tag{5.86}$$

with Σ satisfying the Dirac equation $\dot{\Sigma} + h\Sigma = 0$. Making the ansatz

$$\Sigma = \sum_{n=0}^{\infty} a_n(-\tau\gamma^0\gamma^i\partial_i)^n Q_+\tau^{-m}, \tag{5.87}$$

leads (for the specific choice $Q = \gamma^0$) to the recurrence relation $a_n = \frac{a_{n-1}}{n+m(1-(-1)^n)}$.

The boundary condition is satisfied if $a_0 = 1$ and $m \geq 0$, and we can explicitly sum the series in terms of Bessel functions. In momentum space

$$
\begin{aligned}
\Sigma &= \Gamma(1/2+m)\left(\frac{p\tau}{2}\right)^{1/2-m}(I_{m-1/2}(p\tau)+PI_{m+1/2}(p\tau))Q_+\tau^m \\
&= (E+O)Q_+.
\end{aligned}
\tag{5.88}
$$

Here $P = \frac{i\gamma^0\gamma\cdot\mathbf{p}}{|\mathbf{p}|}$; the operators $\frac{1}{2}(1 \pm P)$ project onto +ve/-ve eigenvalues of the massless flat-space hamiltonian $\gamma^0\gamma^i\partial_i$. Substituting back into (5.86) we find that

$$
\begin{aligned}
\Gamma &= Q + 2Q_-OE^{-1} \\
&= Q + 2Q_-P\frac{I_{m+1/2}(p\tau)}{I_{m-1/2}(p\tau)}.
\end{aligned}
\tag{5.89}
$$

Next, as may again be verified by substitution, the equation for Π has the solution

$$
\begin{aligned}
\Pi &= 2Q_+(\Sigma + Q_-)^{-1} \\
&= 2Q_+E^{-1},
\end{aligned}
\tag{5.90}
$$

which gives the expected divergent behaviour at $\tau = 0$.

Now consider the equation for Ξ. We can rewrite Γ in the following way: $\Gamma = -(\bar{\Sigma} + Q_+)^{-1}(\bar{\Sigma} - Q_+)$ where

$$
\bar{\Sigma} = Q_+(E - O). \tag{5.91}
$$

which satisfies $\dot{\bar{\Sigma}} - \bar{\Sigma}h = 0$. This enables us to find the solution

$$
\begin{aligned}
\Xi &= -2(\bar{\Sigma} + Q_-)^{-1}Q_- \\
&= -2Q_-E^{-1}.
\end{aligned}
\tag{5.92}
$$

Finally, to solve the equation for Υ we put $\Upsilon = \Pi R \Xi + Q$ where R satisfies

$\{R, Q\} = 0$. Substituting into (5.85) gives

$$\Pi\left(2\dot{R} - (1 + 4ROE^{-1})\gamma^0\gamma^i\partial_i\right)\Xi = 0. \tag{5.93}$$

Now define $\tilde{O}$, $\tilde{E}$ such that $O \to \tilde{O}$, $E \to \tilde{E}$ as $m \to -m$. The correponding expansion coefficients $\tilde{a}_n$ are divergent as $m - 1/2$ approaches an integer, so we keep m variable, allowing it to approach such values within convergent expressions. This is the analogue of keeping d variable in the scalar case. The necessity for such regularization is due to our working in momentum space, rather than configuration space. Using the identity $E\tilde{E} - O\tilde{O} = 1$, we find that $R = -\frac{1}{2}\tilde{O}E$, so that

$$\begin{aligned} \Upsilon &= Q + 2Q_+\tilde{O}E^{-1} \\ &= Q + 2CQ_+Pp^{2m}\frac{I_{1/2-m}(p\tau)}{I_{m-1/2}(p\tau)}, \end{aligned} \tag{5.94}$$

where $C = 2^{-2m}\frac{(-1/2-m)!}{(m-1/2)!}$. This is nonlocal even when $m = n + 1/2$ for some integer n causing the Bessel functions to cancel because then C diverges and we have a similar situation to that which we encountered in the bosonic case.

The large τ behaviour of Υ is easily found:

$$\lim_{\tau\to\infty}\Upsilon = 2CQ_+Pp^{2m}. \tag{5.95}$$

Fourier transforming, we find that the partition function is given by

$$\log Z[v, v^\dagger] = \int d^dx d^dy v^\dagger(\mathbf{x})\left(K\gamma^0\frac{\gamma\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2m+1}}\right)v(\mathbf{y}), \tag{5.96}$$

where $K = -\frac{2(\frac{d-1}{2}+m)!}{(m-1/2)!\pi^{d/2}}$. Here we have imposed the constraints (5.74). This is the correct two-point function for a quasi-primary fermion field of scaling dimension $d/2 + m$.

Now all this assumed that $m \geq 0$; but if $m < 0$ the above argument holds with

m replaced by $-m$ provided we take $Q = -\gamma^0$. Thus we conclude that the scaling dimension is $d/2 + |m|$ in general.

An incidental point of interest is the relationship of our solution of the Schrödinger equation to the classical field configuration ϕ to which it corresponds. This is not quite the same as the solution Σ to the Dirac equation which we found, because the latter is divergent as $\tau \to \infty$. However, defining $\tilde{\Sigma}$ by $\Sigma \to \tilde{\Sigma}$ as $m \to -m$, we find that $P\tilde{\Sigma}$ also solves the Dirac equation. Taking a suitable linear combination of these two solutions (and allowing them to operate on the boundary value v) we can construct a unique field configuration which satisfies the appropriate boundary conditions and is finite as $\tau \to \infty$. By Fourier transforming this configuration, we found that it coincides exactly with that given in [18] (when we change back to the variables in the original action).

Now that we have found $\Psi_{\tau,0}$, we can construct $\Psi_{\tau,\tau'}$ from the self-reproducing property. The inner-product of wave-functionals follows from (2.63):

$$\begin{aligned} \langle 1|2\rangle &= \int DuDu^\dagger DvDv^\dagger \langle 1|u,u^\dagger\rangle\langle u,u^\dagger|v,v^\dagger\rangle\langle v,v^\dagger|2\rangle \\ &= \int DuDu^\dagger DvDv^\dagger \langle 1|u,u^\dagger\rangle\langle v,v^\dagger|2\rangle e^{2v^\dagger u - 2u^\dagger v}. \end{aligned} \tag{5.97}$$

It is important to note that we integrate over the *constrained* fields, (5.74), which reflect the true functional dependence of the wave-functionals. This allows us to drop the Q-dependence from Γ, etc. so we write the logarithm of $\Psi_{\tau,\tau'}$ as

$$\int d^dx \left\{ f_{\tau,\tau'} + u^\dagger\Gamma_{\tau,\tau'}u + u^\dagger\Xi_{\tau,\tau'}v - v^\dagger\Xi_{\tau,\tau'}u + v^\dagger\Upsilon_{\tau,\tau'}v \right\}, \tag{5.98}$$

and we have $\Gamma_{\tau,0} = 2OE^{-1}$, $\Xi_{\tau,0} = -2E^{-1}$, and $\Upsilon_{\tau,0} = 2\tilde{O}E^{-1}$. Note that $\Pi_{\tau,0} = -\Xi_{\tau,0}$. Then from

$$\Psi_{\tau,0}[\tilde{u},\tilde{u}^\dagger,\tilde{v},\tilde{v}^\dagger] = \int d^dx DuDu^\dagger DvDv^\dagger \Psi_{\tau,\tau'}[\tilde{u},\tilde{u}^\dagger,u,u^\dagger]\Psi_{\tau',0}[v,v^\dagger,\tilde{v},\tilde{v}^\dagger]e^{2v^\dagger u - 2u^\dagger v}, \tag{5.99}$$

we obtain

$$
\begin{aligned}
\Gamma_{\tau,0} &= \Gamma_{\tau,\tau'} + \Xi^2_{\tau,\tau'}\Upsilon^{-1}_{\tau,\tau'} - 4\Xi^2_{\tau,\tau'}\Upsilon^{-2}_{\tau,\tau'}(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau',0})^{-1} \\
\Xi_{\tau,0} &= -2\Xi_{\tau,\tau'}\Xi_{\tau',0}\Upsilon^{-1}_{\tau,\tau'}(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau',0})^{-1} \\
\Upsilon_{\tau,0} &= \Upsilon_{\tau',0} + \Xi^2_{\tau',0}(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau,0})^{-1} \qquad (5.100)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\Upsilon_{\tau,\tau'} &= -4(\Xi^2_{\tau',0}(\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1} + \Gamma_{\tau',0})^{-1} \\
\Xi_{\tau,\tau'} &= \frac{1}{2}\Xi_{\tau',0}\Pi_{\tau,0}(\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1}\Upsilon_{\tau,\tau'} \\
\Gamma_{\tau,\tau'} &= \Gamma_{\tau,0} - \frac{1}{4}\Xi^2_{\tau',0}\Xi^2_{\tau,0}(\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-2}\Upsilon_{\tau,\tau'} - (\Upsilon_{\tau',0} - \Upsilon_{\tau,0})^{-1}\Xi^2_{\tau,0}. (5.101)
\end{aligned}
$$

From this we can check that as $\tau \to \tau' \neq 0$ the Schrödinger functional $\Psi_{\tau,\tau'}$ reduces to $\langle u, u^\dagger | v, v^\dagger \rangle$ as it should.

5.8 Conformal anomaly for fermions

We can calculate the conformal anomaly for fermions in the same way as we did for scalar fields. Working with the metric (5.50) the Schrödinger equation is unchanged, except that derivatives become covariant with respect to g and the Hamiltonian density aquires a factor of $\sqrt{g}$. We need to introduce a UV regulator, which we would like to write in terms of a heat-kernel expansion, so it is convenient to re-express everything in terms of positive definite operators. Hence, for example, we rewrite the solution (5.89) as

$$
\Gamma(\tau\gamma^0\gamma^i\nabla_i) = Q + 2Q_-\tau\gamma^0\gamma^i\nabla_i\sum_{n=0}^{\infty} d_n(\tau^2 D^2)^n, \qquad (5.102)
$$

for some appropriate coefficients d_n. We have defined $D \equiv \gamma^0\gamma^i\nabla_i)$.

As before, a Weyl transformation may be implemented by scaling τ, so when

$\delta g_{ij} = g_{ij}\delta\rho$ the free energy changes by

$$\delta F = -\frac{\delta\rho}{2}\int d^d x\left(\tau\frac{\partial f_{\tau,\tau'}}{\partial\tau} + \tau'\frac{\partial f_{\tau,\tau'}}{\partial\tau'}\right). \tag{5.103}$$

The Schrödinger equations yield equations for F corresponding to regulated versions of (5.85)

$$-\frac{\partial f}{\partial\tau} = \frac{1}{2}\mathrm{tr}\left(h(1+Q+Q_-\Gamma_{\tau,\tau'})\theta(1-s\tau^2 D^2)\delta^d(\mathbf{x}-\mathbf{y})|_{\mathbf{x}=\mathbf{y}}\right), \tag{5.104}$$

and

$$\frac{\partial f}{\partial\tau'} = \frac{1}{2}\mathrm{tr}\left(h(1+Q+Q_+\Upsilon_{\tau,\tau'})\theta(1-s\tau'^2 D^2)\delta^d(\mathbf{x}-\mathbf{y})|_{\mathbf{x}=\mathbf{y}}\right). \tag{5.105}$$

Representing the step function by (5.57), we have

$$\theta(1-s\tau^2 D^2)\delta^d(\mathbf{x}-\mathbf{y})\,1 = \frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}e^{-iys\tau^2 D^2}\delta^d(\mathbf{x}-\mathbf{y})\,1, \tag{5.106}$$

where $e^{-izD^2}\delta^d(\mathbf{x}-\mathbf{y})\,1 \equiv H(z,\mathbf{x},\mathbf{y})$ satisfies

$$i\frac{\partial}{\partial z}H = D^2 H, \qquad H(0,\mathbf{x},\mathbf{y}) = \delta^d(\mathbf{x}-\mathbf{y})\,1, \tag{5.107}$$

and has the small z expansion

$$H(z,\mathbf{x},\mathbf{x}) \sim \frac{\sqrt{g}}{(4\pi i z)^{d/2}}\sum_{n=0}^{\infty}\tilde{a}_n(\mathbf{x})z^n. \tag{5.108}$$

Thus a general function $g(D^2)$ satisfies

$$\begin{aligned}\mathrm{tr}\left(g(D^2)\theta(1-s\tau^2 D^2)\delta^d(\mathbf{x}-\mathbf{y})\right) &= \frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}\,\mathrm{tr}\left(g\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\,e^{iys\tau^2 D^2}\,\delta^d(\mathbf{x}-\mathbf{y})\right)\\ = \mathrm{tr}\left(\sum_{n=0}^{\infty}\frac{1}{(4\pi i)^{d/2}}\left(\int\frac{d^d\mathbf{x}}{\tau^d}\sqrt{g}\,\tilde{a}_n(\mathbf{x})\,\tau^{2n}\right)\right.&\left.\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}\,g\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\right)(sy)^{n-d/2}\end{aligned} \tag{5.109}$$

where only terms up to $n = N$ contribute when $d = 2N$. The conformal anomaly is again proportional to $\tilde{a}_N$; in the limit $\tau \to \infty$, $\tau' \to 0$ the other terms arising from (5.104) and (5.105) either vanish or reproduce ultraviolet divergences associated with the renormalization of the cosmological constant etc. Using the same argument as in the scalar case, the term proportional to $\tilde{a}_N$ is

$$\lim_{\xi \to 0} \left(g(\xi) \frac{1}{(4\pi i)^N} \int d^d \mathbf{x} \sqrt{g}\, \tilde{a}_n(\mathbf{x}) \right) \tag{5.110}$$

For generic values of the mass m the conformal anomaly can be computed from expansions of $\Gamma_{\tau,\tau'}$ and $\Upsilon_{\tau,\tau'}$ in powers of D. These expansions both begin with terms of order D. Using these in (5.104) and (5.105) gives vanishing contributions to the anomaly. There are also contributions from $\mathrm{tr} hQ$ which cancel between (5.104) and (5.105). So for generic values of the mass the conformal anomaly is zero. But due to the divergent nature of $\Upsilon_{\tau,0}$ as $2|m| \to 2\tilde{N} - 1$ for any positive integer $\tilde{N}$, we have $\Upsilon_{\tau,\tau'} \to -4\Gamma^{-1}_{\tau',0}$, and this has a leading order term proportional to $1/D$ which combines with the D in h to give a finite contribution as $\xi \to 0$. We conclude that the conformal anomaly is zero unless $2|m|$ is an odd integer $2\tilde{N} - 1$. For a four-dimensional boundary these are precisely the values appearing in the mass spectrum of Supergravity compactified on $AdS_5 \times S^5$ [26], and for a six-dimensional boundary they coincide with the mass spectrum of Supergravity compactified on $AdS_7 \times S^4$ [27]. For these special mass values we have

$$\delta F = -\delta\rho \tilde{N} \frac{1}{(4\pi i)^N} \mathrm{tr} \int d^{2N} x \sqrt{g} Q_{-} a_N. \tag{5.111}$$

For $\mathbf{d = 2}$ and $r \to \infty$ the anomaly is proportional to $\tilde{a}_1 = -iR\,1/12$, so for a fermion with σ spinor components,

$$\delta F = \frac{\delta\rho}{96\pi} \int d^2 x \sqrt{g} R \tilde{N} \sigma, \tag{5.112}$$

from which we identify the central charge as $\tilde{N}$.

For $\mathbf{d = 4}$ we have

$$\delta F = \frac{\delta \rho}{32\pi^2} \int d^4x \sqrt{g} \mathrm{tr} \tilde{a}_2(\mathbf{x}) \tilde{N}, \tag{5.113}$$

where[2]

$$\mathrm{tr} \tilde{a}_2 = \frac{\sigma}{720} \left(-\frac{7}{2} R_{ijkl} R^{ijkl} \right). \tag{5.114}$$

5.8.1 Generic boundary conditions

We want to transfer this result to the case of more general boundary conditions where we diagonalize $\tilde{Q}_+ \psi$ and $\psi^\dagger \tilde{Q}_-$. Again we start with the existing WF

$$\log \Psi = \int d^dx \left(u^\dagger \Gamma u + 2u^\dagger \Xi v + v^\dagger \Pi u + v^\dagger \Upsilon v \right), \tag{5.115}$$

and perform an additional integration over the boundary:

$$\int D\tilde{u} D\tilde{u}^\dagger Dv Dv^\dagger \Psi[u, u^\dagger, \tilde{u}, \tilde{u}^\dagger] \langle v, v^\dagger | w, w^\dagger \rangle e^{2v^\dagger \tilde{u} - 2\tilde{u}^\dagger v}, \tag{5.116}$$

where $|w, w^\dagger\rangle$ satisfies

$$Q_-(\psi - w)|w, w^\dagger\rangle = (\psi^\dagger - w^\dagger) Q_+ |w, w^\dagger\rangle = 0. \tag{5.117}$$

Since we are only interested in the free energy, we can set $w = 0$, in which case we have (cf. (2.68))

$$\langle v, v^\dagger | 0, 0 \rangle = \langle Q | B e^{u^\dagger A^{-1} C u}, \tag{5.118}$$

and recall $A = \{\tilde{Q}_-, Q_+\}$, $C = [\tilde{Q}_-, Q_+]$.

Now $Q = \pm\gamma^0$ is in odd dimensions the unique choice without zero modes.

[2]For $Q = \gamma^0$ the result appears to be chiral. But a careful treatment of the zero modes which we have neglected shows that the actual result is as given.

All $\tilde{Q} \neq \pm\gamma^0$ (remember $\tilde{Q}^2 = 1$) satisfy $\{Q, \tilde{Q}\} = 0$[3]. Thus we find $A^{-1}C = Q + Q_+ 2\tilde{Q}Q_-$, and the integrations in (5.116) produce the additional determinants

$$\det Q_+ \Lambda Q_- \det Q_-(\Lambda^{-1} + \tilde{Q})Q_+ = \det Q_+(1 + 2\Lambda\tilde{Q})Q_+ = e^{-2\tilde{F}}, \tag{5.119}$$

where $\Upsilon = Q + Q_+ \Lambda Q_-$. The contribution to the Weyl anomaly is given by

$$\tau' \frac{\partial}{\partial \tau'} 2\tilde{F} = \mathrm{tr}\left(\tfrac{1}{2} Q_+ (4D - \Lambda D \Lambda + \tfrac{4m}{\tau}\Lambda) M (1 + \Lambda M)^{-1} Q_+\right), \tag{5.120}$$

where $M = 2Q_- \tilde{Q} Q_+$. For generic values of the mass $\Lambda \sim \frac{2\tau' D}{1-2m}$ and (5.120) does not contribute to the anomaly. But for the special values $2m + 1 = \tilde{N}$ for which we found the Weyl anomaly to be non-zero, we have $\Lambda \sim -(2m+1)\tau(-\tau' D)^{-1}$, and (5.120) produces a mass-independent contribution proportional to $\mathrm{tr}Q_+$, which has the effect of shifting the coefficient of the anomaly from $\tilde{N}$ to $\tilde{N} - 1/2 = m$.

Thus we conclude that in the special case $Q = \gamma^0$ the anomaly is proportional to $\tilde{N}$, as in (5.111), but in the generic case, it is given in $d = 2N$ dimensions by

$$\delta F = -\delta\rho(\tilde{N} - 1/2) \frac{1}{(4\pi i)^N} \mathrm{tr} \int d^{2N}x \sqrt{g} Q_- a_N. \tag{5.121}$$

We will discover that it is the generic boundary conditions which must be used to enable the Maldacena conjecture to work.

[3]We are carefully avoiding a discussion of the chiral anomaly, whose effect can be seen by chirally rotating $\tilde{Q}$.

Chapter 6

Testing the Maldacena conjecture

We now wish to apply the results of the last section to testing the Maldacena conjecture[1], which we state as follows:

Type IIB string theory compactified on $AdS_5 \times S^5$ is exactly dual to $\mathcal{N} = 4$ SU(N) super-Yang-Mills theory.

We will need to generalize the calculation of the conformal anomaly to fields of other spin, as well as dealing with subtleties like ghosts for the gauge fields.

As before we will find that the conformal anomaly vanishes except for integer values of the scaling dimension, which correspond precisely to the mass values appearing in the Kaluza-Klein compactifications of supergravity such as $AdS_5 \times S^5$ and $AdS_7 \times S^4$. (It is interesting to note that for these same values there is also a breakdown of conformal invariance in the two point functions, when their local behaviour is suitably regulated [63]; the Fourier transforms of (5.41), etc. (considered as distributions) pick up a logarithmic term.)

The spectrum of the compactification on $AdS_5 \times S^5$ of Type IIB Supergravity was obtained in [26] by perturbing classical equations of motion about a solution in which all the fields vanish except the metric and a self-dual five-form field strength. These classical equations do not follow from an action principle, but rather are chosen to be compatible with supersymmetry. (If there had been an action this calculation would have given its second functional derivative.)

Since we work with a more general background than the maximally symmetric AdS_5 we should repeat the calculation. We have done this, and constructed the Lagrangian which we need in order to go beyond the classical level and quantize the theory.

Since we do not expect the spectrum of particles to change, it is sufficient at the linearized level to construct the Lagrangian by summing the naive Lagrangians, in the appropriate background, for each of the fields in the diagonalized spectrum. For simplicity, this is the approach which we will adopt in this thesis.

A more rigorous approach, which we have described elsewhere [55], is to compactify the linearized ten-dimensional Lagrangian, and diagonalize the resulting spectrum explicitly. This involves a lot of calculational work, and the details do not warrant inclusion here. (Complications arise because some of the equations of motion used in [26] are no longer present when we impose all the gauge conditions, and in ten dimensions there is also the paradox associated with a Lagrangian description of self-dual field strengths).

The contributions to the anomaly arising from the supergravity fields at one loop will be summed over supermultiplets to give the complete conformal anomaly of the boundary theory. This should reproduce the conformal anomaly of $\mathcal{N} = 4$ Super-Yang-Mills theory, which has been calculated for free fields [61] as

$$-\frac{N^2-1}{\pi^2}(E+I), \tag{6.1}$$

where

$$E = \frac{1}{64}(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2), \qquad I = -\frac{1}{64}(R^{ijkl}R_{ijkl} - 2R^{ij}R_{ij} + R^2/3). \tag{6.2}$$

This result is protected by supersymmetry, so that it should give the correct anomaly for all N. Notice that the anomaly vanishes for Ricci-flat boundary metrics.

6.1 Tree-level gravity

To begin with we consider the tree-level conformal anomaly, which arises from the pure gravity sector. The methods we developed in the last chapter give a particularly simple derivation of the tree-level contribution. Previously this has been calculated for AdS_3 using the Chern-Simons formulation, [10]-[11]. In higher dimensions it has been computed by solving the Einstein equations perturbatively in terms of boundary data, [12]. Our method employs a somewhat different regularization procedure, so it is important to show that it is consistent with earlier calculations. The main difference is that authors of [12] effectively compute the classical free energy of the gravitational sector by finding $W_{\tau,\tau'}$ for $\tau = \infty$ and τ' a small regulator, whereas we take $\tau' = 0$ and treat τ as a large regulator. (At tree-level it is not necessary to keep τ' non-zero as we did in the earlier computations of one-loop anomalies).

Formally we need to compute the Euclidean functional integral that represents a state of pure gravity, with Einstein-Hilbert action and cosmological constant $\Lambda < 0$,

$$Z[g_{rs}] = \int \mathcal{D}G \, \exp\Big(-\int d^{d+1}x \, \sqrt{G}(R + 2\Lambda) + \text{boundary terms}\Big) \tag{6.3}$$

where we should integrate over all metrics $G_{\mu\nu}$ of a $d+1$ dimensional manifold which induce the metric g_{rs} on the boundary. This is ill-defined for a variety of reasons, such as non-renormalizability and unboundedness of the action, which we ignore in the hope that these pathologies are absent from the more fundamental theory of which this is only a part. The integral over all metrics includes reparametrizations which can be factored out using the Faddeev-Popov method. The standard ADM decomposition of the metric is

$$(G_{\mu\nu}) = \begin{pmatrix} N^2 + N_i N_j G^{ij} & N_j \\ N_i & G_{ij} \end{pmatrix} \tag{6.4}$$

with G^{ij} the inverse of the $d \times d$ matrix G_{ij}. We will fix the gauge by choosing

$$N^2 = L^2/t^2, \quad N_i = 0, \quad G_{ij} = \frac{g_{ij} + h_{ij}}{t^2} \tag{6.5}$$

with $t = x^0$. The dynamical variables are just the h_{ij}, and we take the boundary to be at $t = \tau = 0$, where $h_{ij} = 0$. The gauge conditions should be accompanied by the introduction of ghosts, but these will not contribute to the tree-level conformal anomaly. Expanding the action in powers of h_{ij}, and taking $L^2 = -d(d-1)/(2\Lambda)$ gives

$$\int d^{d+1}x\,\sqrt{G}(R + 2\Lambda) - \text{boundary terms} = \int d^{d+1}x\,\frac{\sqrt{g}L}{t^{d-1}}\Big(R(g) \\ + h_{ij}\,\tilde{G}^{ij}(g) + (\dot{h}_{ij}\,\dot{h}^{ij} - (\dot{h}^i_i)^2)/(4L^2) + h_{ij}\,\Box^{ijkl}h_{kl} + ..\Big) \tag{6.6}$$

where the boundary terms are uniquely determined by the requirement that the action should be quadratic in first derivatives of h_{ij}. (This causes the term proportional to the cosmological constant to be cancelled). The dots denote terms of higher order in h_{ij}, and indices are raised and lowered with g_{ij}. $R(g)$ is the d-dimensional curvature calculated by taking g_{ij} as metric and $\sqrt{g}(R(g) + h_{ij}\,\tilde{G}ij + h_{ij}\,\Box^{ijkl}h_{kl}$ are the first three terms in the expansion of $\sqrt{det(g+h)}\,R(g+h)$, so that $\Box$ is a second order differential operator. The terms of higher order in h each contain one or two derivatives. As in section (1) the state $Z[g_{ij}]$ is the $\tau \to \infty$ limit of the Schrödinger functional. The the tree-level contribution to $\log Z[g_{rs}]$ is thus the $\tau \to \infty$ limit of minus the action evaluated on shell for a manifold with boundaries at $t = 0$, where the induced metric is g_{ij}, and $t = \tau$ where it is $g_{ij} + h_{ij}$. If we denote this as $W^{\text{tree}}_{\tau,0}[g_{ij} + h_{ij}, g_{ij}]$ then it satisfies the Hamilton-Jacobi equation, (which is the tree-level Schrödinger equation). This is simply the statement that $W^{\text{tree}}_{\tau+\delta\tau,0}$ can be obtained from $W^{\text{tree}}_{\tau,0}$ by allowing the fields to propagate according to the equations of motion from τ to $\tau+\delta\tau$, i.e.

$$W^{\text{tree}}_{\tau+\delta\tau,0}[g_{ij}+h_{ij}+\delta\tau\dot{h}_{ij},g_{ij}] = W^{\text{tree}}_{\tau,0}[g_{ij}+h_{ij},g_{ij}] - \mathcal{L}\,\delta\tau \tag{6.7}$$

where $\mathcal{L}$ is the Lagrangian in (6.6). The momenta conjugate to h_{ij} are represented on $e^{W^{\text{tree}}_{\tau,0}}$ by functional differentiation, ie. we have

$$\pi_{ij} = -\frac{\delta W^{\text{tree}}_{\tau,0}}{\delta h_{ij}} = \frac{\sqrt{g}}{2L\tau^{d-1}}\left(\dot{h}^{ij} - g^{ij}\,\dot{h}^{r}_{r}\right) \tag{6.8}$$

which leads to

$$\begin{aligned}
-\frac{\partial W^{\text{tree}}_{\tau,0}}{\partial\tau} &= L + \int d^dx\,\dot{h}_{ij}\frac{\delta W^{\text{tree}}_{\tau,0}}{\delta h_{ij}} \\
&= \int d^dx\frac{\sqrt{g}L}{\tau^{d-1}}R(g) \\
&+ \int d^dx\left(-\frac{\tau^{d-1}L}{\sqrt{g}}\pi^{ij}G_{ijrs}\pi^{rs} + \frac{\sqrt{g}L}{\tau^{d-1}}\left(h_{ij}\,\tilde{G}^{ij}(g) + h_{ij}\,\Box^{ijkl}h_{kl}\right)\right) + ..
\end{aligned} \tag{6.9}$$

with

$$G_{ijrs} = g_{ir}g_{js} - \frac{1}{d-1}g_{ij}g_{rs}. \tag{6.10}$$

and the initial condition is that $\exp W^{\text{tree}}_{\tau,0}[\tilde{g}_{ij},g_{rs}] \sim \delta[h_{ij}]$. When the curvature tensors constructed from g_{ij} are small, and h_{ij} is slowly varying we can expand W^{tree} in powers of h and its derivatives as

$$W^{\text{tree}}_{\tau,0}[g_{ij}+h_{ij},g_{rs}] = \int\frac{d^dx\,\sqrt{g}}{\tau^d}\left(-\frac{d}{4L}h_{ij}G^{-1\,ijrs}h_{rs} + \Gamma_0 + \Gamma_1^{ij}h_{ij} + h_{ij}\Gamma_2^{ijrs}h_{rs} + ..\right) \tag{6.11}$$

Apart from a constant term in Γ_0, only the first term on the right has no derivatives of either h_{ij} or g_{ij}, and so provides the dominant behaviour as $\tau\to 0$, satisfying the initial condition (provided h^i_i is suitably treated, [13]). Substituting into (6.9) and equating powers of h_{ij} gives

$$-\frac{\partial}{\partial\tau}\left(\frac{\Gamma_0}{\tau^d}\right)=\frac{L}{\tau^{d-1}}R(g)-\frac{L}{\tau^{d+1}}\Gamma_1^{ij}G_{ijrs}\Gamma_1^{rs}, \tag{6.12}$$

The free energy is the infinite τ limit of the h_{ij} independent part of (6.11), i.e. $F[\tau, g_{ij}] = \int d^dx\,\sqrt{g}\,\Gamma_0/\tau^d$, and, as before, a Weyl scaling of the metric can be compensated by a scaling of τ, so for $\delta g_{ij} = \delta\rho\, g_{ij}$ the change in F is given by (6.12)

$$\delta F=-\frac{\delta\rho}{2}\tau\frac{\partial F}{\partial\tau}=\frac{\delta\rho}{2}\int d^dx\,\sqrt{g}L\left(\frac{1}{\tau^{d-2}}R(g)-\frac{1}{\tau^d}\Gamma_1^{ij}G_{ijrs}\Gamma_1^{rs}\right) \tag{6.13}$$

Up to terms involving two derivatives, we have from (6.9)

$$-\frac{\partial\Gamma_1^{ij}}{\partial\tau}=L\tau\tilde{G}^{ij}(g) \tag{6.14}$$

Solving this for Γ_1 and substituting into (6.13) gives the variation as an expansion in powers of τ times the curvatures constructed from g_{ij}. We want to work at finite τ, so our expansion will be valid when we take the curvatures to zero.

Now for $\mathbf{d=2}$ we have identically $\tilde{G}^{ij}=0$, so $\Gamma_1=0$ and to the order we need (6.13) reduces to

$$\delta F=\frac{\delta\rho}{2}\int d^2x\,\sqrt{g}LR(g) \tag{6.15}$$

from which we can identify the central charge as $c=24\pi L$, or, since we have chosen units such that the three-dimensional gravitational constant satisfies $16\pi G_{\text{Newton}} = 1$, we have $c=3L/(2G_{\text{Newton}})$ as in [10].

For $\mathbf{d=4}$ we obtain Γ_1 to the desired order from (6.14) as $\Gamma_1=-L\tau^2\tilde{G}^{ij}/2$, so if we substitute into (6.12) we have

$$\delta F = \frac{\delta\rho}{2}\int d^4x\,\sqrt{g}L\left(\frac{R(g)}{\tau^2} - L^2\tilde{G}^{ij}G_{ijrs}\tilde{G}^{rs}/4\right) \tag{6.16}$$

The first term represents a divergence that should be cancelled by introducing a counterterm, the second is the finite Weyl anomaly. If we reinstate the five-dimensional Newton constant this becomes

$$\delta\rho\frac{L^3}{128\pi G_{\text{Newton}}}\left(R_{ij}\,R^{ij} - R^2/3\right) \tag{6.17}$$

which agrees with [12], and correctly gives the leading order behaviour of (6.1) for large N.

6.2 Vector fields

Now we will demonstrate how the calculation in Chapter 5 of the one-loop Weyl anomaly can be applied to fields of higher spin. For all the fields in the supergravity spectrum we will express the anomaly as

$$\mathcal{A} = -\alpha N R^{ijkl}R_{ijkl}/(5760\pi^2), \tag{6.18}$$

(with $N \to N - 1/2$ in the corresponding expression for fermions). We wish to determine the values of N and α for each variety of field.

The vector field action in the metric (5.50) is given by

$$S = \int d^{d+1}x\frac{\sqrt{g}}{t^{d-3}}\left(\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}m^2g^{\mu\nu}A_\mu A_\nu/t^2\right) \tag{6.19}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For $m^2 = 0$ we have the usual gauge invariance under $A_\mu \to A_\mu + \partial_\mu\phi$.

We consider the gauge fields first. Adding a gauge-fixing term $\frac{1}{2}\sqrt{g}(g^{\mu\nu}D_\mu A_\nu)^2/t^{d-3}$ and writing out the connection terms explicitly (recall $R = d(d+1)$ in AdS space-time), we have

$$S = \int d^{d+1}x \frac{\sqrt{g}}{t^{d-3}}[\dot{A}_0^2 + A_0\Delta A_0 + (d-1)A_0^2/t^2 + g^{ij}(\dot{A}_i\dot{A}_j + A_i\Delta A_j) \\ +2A\cdot\partial A_0/t - 2A_0\partial\cdot A/t)]. \tag{6.20}$$

Here $\Delta = -\nabla\cdot\nabla$ is the d-dimensional Laplacian. The cross-terms in (6.20) cannot affect the mass spectrum, and so do not contribute to the Weyl anomaly. Ignoring these terms, (6.20) in terms of A_0 and A_i has the same form as the scalar field action, but with $d \to d-2$. Otherwise, the Schrödinger equation and the calculation of the Weyl anomaly are identical to the scalar field case. The anomaly is thus given by (6.18) with $N^2 = m^2 + (d-2)^2/4$, and α is the coefficient in the heat-kernel expansion of the Laplacian. Putting $d = 4$ in (6.20) we find $N = 2$ for A_0 and $N = 1$ for A_i, while from [62] we have $\alpha = 1$ for A_0, and $\alpha = -11$ for A_i. There are also two Faddeev-Popov ghosts b_{FP} and c_{FP}, which are massless anticommuting scalars, and thus have $\alpha = -1$ and $N = 2$. These results are summarized in table(6.3).

Now for $m^2 \neq 0$, A_0 is an auxiliary field, and integrating it out gives the determinant

$$\prod_t \det{}^{-1/2}\left(t^2\Delta + m^2\right). \tag{6.21}$$

The resulting action is simplified if we decompose A_i as follows:

$$A_i = \hat{A}_i + D_i\frac{1}{\sqrt{\Delta}}\sqrt{\Delta t^2 + m^2}\,\phi, \tag{6.22}$$

where $\hat{A}_i$ is transverse, and the Jacobian for this change of variables exactly cancels (6.21). The action can now be written as a boundary term plus

$$S = \int d^{d+1}x \frac{\sqrt{g}}{t^{d-3}}\left[g^{ij}(\dot{A}_i\dot{A}_j + A_i\Delta A_j + \frac{m^2}{t^2}A_iA_j) + m^2(\dot{\phi}^2 + \phi\mathcal{D}\phi)\right], \tag{6.23}$$

where $\mathcal{D} = m^2 + t^2\Delta + O(t^4\Delta^2)$. It is clear that the terms of higher order in $t^2\Delta$ will not affect the anomaly. So we have contributions to the anomaly from the Laplacian acting on a scalar and a transverse vector, with $N^2 = m^2 - (d-2)^2/4$. But this is the same as the Laplacian acting on an *unconstrained* vector. So the total contribution to the anomaly is given by (6.18) with $\alpha = -11$.

6.3 Antisymmetric tensor fields

The Lagrangian for the antisymmetric tensor fields which arises from the ten-dimensional field equations can be written

$$\mathcal{L} = a^{\mu\nu}\left(2k + i^*D\right)\left(2(k+4) - t^*D\right)a_{\mu\nu}, \tag{6.24}$$

where k labels the spherical harmonic on S^5 that determines the mass, and we define

$$({}^*D)a_{\lambda\mu} = \epsilon_{\lambda\mu}{}^{\nu\rho\sigma} D_\nu a_{\rho\sigma}. \tag{6.25}$$

The free energy which determines the anomaly depends on the determinant of the operator in (6.24), specifically

$$\det{}^{-1/4}\left((2k + i^*D)^2\right)\det{}^{-1/4}\left((2(k+4) - i^*D)^2\right). \tag{6.26}$$

Now by hermiticity, we have

$$\det\left((2k + i^*D)^2\right) = \det\left((2k + i^*D)^2(2k - i^*D)^2\right) = \det\left(4(k^2 - \mathrm{Max})\right)^2, \tag{6.27}$$

where Max is the Maxwell operator. So if we use the action

$$S = \int d^{d+1}x \frac{\sqrt{g}}{t^{d-5}}\left(3\partial_{[\rho}a_{\sigma\tau]}\partial^{[\rho}a^{\sigma\tau]} + \frac{m^2}{t^2}a_{\sigma\tau}a^{\sigma\tau}\right), \tag{6.28}$$

for $m^2 = k^2$ *and* $m^2 = (k+4)^2$, then this propagates twice as many modes as (6.24) but is otherwise equivalent. Now if $d = 4$ and $m^2 = k^2 = 0$, (6.28) describes a topological model, and can be gauged away completely. It is independent of the metric, and thus does not contribute to the Weyl anomaly. For $m^2 \neq 0$ we make the decomposition

$$a_{0i} = \hat{a}_i + \partial_i \frac{1}{\sqrt{\nabla}}\phi, \qquad a_{ij} = \hat{a}_{ij} + 2D_{[i}\frac{1}{\sqrt{\Delta}}\sqrt{\Delta t^2 + m^2}\,\hat{\phi}_{j]}, \tag{6.29}$$

where the hats indicate transversality. The a_{0i} part of the action becomes

$$\int \frac{d^{d+1}x\sqrt{g}}{t^{d-5}}\left(2\frac{m^2}{t^2}\phi^2 + 2\hat{a}^i\Delta\hat{a}_i + 2\frac{m^2}{t^2}\dot{\hat{a}}^i\dot{\hat{a}}_i - 4\hat{a}^i\nabla^j\dot{a}_{ij}\right). \tag{6.30}$$

Integrating out ϕ gives an uninteresting constant, while integrating out $\hat{a}_i$ gives $\det^{-1/2} t^2(\Delta + m^2/t^2)$, where the operator in this determinant is understood to act on divergenceless vectors. Again, this exactly cancels the Jacobian from the change of variables (6.29)

The remaining action then takes a similar form to the vector field one:

$$\int \frac{d^{d+1}x\sqrt{g}}{t^{d-5}}\left(\dot{\hat{a}}^{ij}\dot{\hat{a}}_{ij} + \hat{a}^{ij}(\Delta + m^2/t^2)\hat{a}_{ij} + 2\hat{a}^{ij}R_{ikjl}\hat{a}^{kl} + 2\frac{m^2}{t^2}\dot{\hat{\phi}}^i\dot{\hat{\phi}}_i + 2\hat{\phi}^i\mathcal{D}\hat{\phi}_i\right), \tag{6.31}$$

and for $d = 4$ we thus have non-zero values of the anomaly for $m^2 = N^2$. Thus we end up with a vector and an antisymmetric tensor, both divergenceless, and the heat kernels of their Laplacians sum to that of an *unconstrained* antisymmetric tensor.

The calculation of the anomaly is again in direct analogy with the scalar field case. The heat-kernel coefficients are modified by the Riemann tensor appearing in (6.31), but for Ricci-flat metrics this gives the usual gravitational coupling on the boundary, so we can read off the result in [62]. We find that $\alpha = 33$; this includes a division by 2 to compensate for propagating twice the requisite number of modes.

6.4 Rarita-Schwinger fields

For gravitino fields we start with the Lagrangian [22]

$$\mathcal{L} = \sqrt{G}(\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - m\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu - \bar{m}\bar{\psi}^\mu \psi_\mu). \tag{6.32}$$

The metric and the gamma matrices are for the moment the bulk space (AdS) ones. We make the decomposition

$$\psi_\mu = \varphi_\mu + \frac{D^T_\mu}{D \cdot D^T} D^T \cdot \psi + \frac{1}{d+1}\gamma_\mu \gamma \cdot \psi, \tag{6.33}$$

where $D^T_\mu = (\delta^\nu_\mu - \gamma_\mu\gamma^\nu/(d+1))D_\nu$ is γ-transverse so that

$$\gamma \cdot \phi = D \cdot \phi = 0. \tag{6.34}$$

Then (6.32) becomes

$$\begin{aligned} \mathcal{L} \;=\; & \sqrt{G}\Big[\bar{\phi}^\mu(\not{D} + m)\phi_\mu + \bar{\psi} \cdot D^T \Big(\frac{\frac{d-1}{d+1}\not{D} - m + \bar{m}}{D \cdot D^T}\Big) D^T \cdot \psi - \frac{d-1}{d+1}\bar{\psi} \cdot D^T \gamma \cdot \psi \\ & -\frac{d-1}{d+1}\bar{\psi} \cdot \gamma D^T \cdot \psi + \bar{\psi} \cdot \gamma \Big(\frac{d(d-1)}{(d+1)^2}\not{D} + \frac{d}{d+1}m + \frac{1}{d+1}\bar{m}\Big)\gamma \cdot \psi\Big]. \end{aligned} \tag{6.35}$$

Now if we put $\psi_1 = \sqrt{D \cdot D^T} D^T \cdot \psi$ and $\psi_2 = \gamma \cdot \psi$ then the change of variables $\psi_\mu \to (\phi_\mu, \psi_1, \psi_2)$ has a trivial Jacobian, and the coupled pair (ψ_1, ψ_2) can be diagonalized from (6.35), using $D \cdot D^T = \frac{d}{d+1}\not{D}^2 - \frac{1}{4}d(d+1)$. One of the resulting fields is a trivial auxiliary field; the other has the Lagrangian

$$\mathcal{L} = \sqrt{G}\bar{\psi}\Big(\frac{d-1}{d+1}\bar{m}\not{D} - \frac{d}{d+1}m^2 + \frac{d-1}{d+1}m\bar{m} + \frac{1}{d+1}\bar{m}^2 + \frac{d(d-1)^2}{4(d+1)}\Big)\psi. \tag{6.36}$$

If $\bar{m} \neq 0$ then we can normalize ψ and the original Lagrangian (6.32) is therefore equivalent to

$$\mathcal{L} = \sqrt{G}[\bar{\phi}^{\mu}(\not{D} + m)\phi_{\mu} + \bar{\psi}(\not{D} + M)\psi], \quad M = \frac{\bar{m}}{d-1} + m - \frac{dm^2}{(d-1)\bar{m}} + \frac{d(d-1)}{4\bar{m}}. \tag{6.37}$$

So the effect of having $\bar{m} \neq 0$ in (6.32) is the same as adding an additional spin-half fermion of mass M. Note that this conclusion is unaffected by the introduction of interaction terms which would merely couple ψ to ϕ_{μ}. Thus we may without loss of generality set $\bar{m} = 0$.

Having done so, we conclude that (ψ_1, ψ_2) produces *two* auxiliary fields, unless $M\bar{m} = 0$, in which case one of them decouples completely from the action, signalling the presence of a gauge invariance. Indeed, we can easily verify that for

$$m = \pm\frac{1}{2}(d-1), \tag{6.38}$$

(6.32) is invariant under

$$\delta\psi_{\mu} = D_{\mu}\lambda + \frac{m}{d-1}\gamma_{\mu}\lambda. \tag{6.39}$$

This gives the correct value for the mass for the gravitino appearing in the massless multiplet of supergravity on $AdS_5 \times S^5$. Using the methods of [23] it is straightforward to show that the Lagrangian in the "Feynman" gauge [24] can still be written in the Dirac form, with two Faddeev-Popov ghosts of mass $\pm\frac{1}{2}(d+1)$ and a "gauge-fixing" commuting spinor ghost of mass $5m/(d-1)$. In this case the field in the Lagrangian is unconstrained. Of course there is nothing to stop us making another choice of gauge, in which case the operator in the Lagrangian takes a slightly different form [25].

For the massive gravitinos we still have the constraints (6.34). By introducing Lagrange multiplier fields for these quantities we find that we can remove the constraints at the cost of introducing a pair of ghosts with masses $\pm\sqrt{m^2 + d}$. For the "massless" values (6.38) these coincide with the Faddeev-Popov ghosts. So for *all* values of the mass we can write the Lagrangian as[1]

$$\mathcal{L} = \sqrt{G}\bar{\phi}^{\mu}(\not{D} + m)\phi_{\mu}, \tag{6.40}$$

where ϕ satisfies (6.34). For the massless gravitino we have a single ghost of mass $5m/(d-1)$.

Now we need to make the decomposition into d dimensional fields. We use the unconstrained form of the action (6.40). Rewriting everything in terms of the boundary metric and gamma matrices, and writing out the spin-connection explicitly, we obtain

$$S = \int d^{d+1}x\sqrt{g}\left(\bar{\psi}^{\mu}(\not{D} + m/t)\psi_{\mu} - \frac{1}{t}\bar{\psi}^{i}g_{ij}\gamma^{j}\psi_{0} - \frac{1}{t}\bar{\psi}^{0}\gamma^{i}g_{ij}\psi^{j}\right). \tag{6.41}$$

Now making the decomposition $\psi_i = \phi_i + \frac{1}{d}\gamma_i\gamma^i g_{ij}\psi^j$ we are left with an irreducible d-dimensional gravitino (satisfying $\gamma\cdot\phi = 0$) and two spin half fermions ψ_0 and $\gamma^i g_{ij}\psi^j$ which, when diagonalized, cancel the ghosts corresponding to the constraints (6.34).

Thus the anomaly is non-zero for $|m| + 1/2 = N$, and given by

$$\mathcal{A} = -\alpha(N - 1/2)R^{ijkl}R_{ijkl}/(5760\pi^2), \tag{6.42}$$

where $\alpha = -219/2$ [62]. For the massless gravitino there is also the spin-$\frac{1}{2}$ ghost with $\alpha = -7/2$.

[1]Again, the Kaluza-Klein compactification of supergravity on $AdS_5 \times S^5$[57] yields gravitinos satisfying precisely the equations of motion implied by (6.40) and (6.34). This action can also be derived by diagonalizing the compactified ten-dimensional action.

6.5 Gravitons

For spin 2 fields we start with the Lagrangian (2.89) (with an optional mass term). Specializing to our five dimensional Anti-de-Sitter spacetime, this becomes[2]

$$\mathcal{L} = -\frac{1}{4}h^{(\mu\nu)}(\Box+m^2+2)h_{(\mu\nu)}-\frac{1}{2}D_\mu h^{(\mu\nu)}D^\rho h_{(\rho\nu)}+\frac{3}{25}H(\Box+\frac{5}{3}m^2-5)H-\frac{3}{10}HD^\mu D^\nu h_{(\mu\nu)}, \tag{6.43}$$

where $H = h^\mu_\mu$, and $h_{(\mu\nu)}$ is the traceless part of $h_{\mu\nu}$. We make the further decomposition

$$h_{(\mu\nu)} = \hat{h}_{(\mu\nu)} + D_{(\mu}\hat{\Lambda}_{\nu)} + D_{(\mu}D_{\nu)}\phi. \tag{6.44}$$

The action factorises:

$$\begin{aligned}
\mathcal{L}[h] &= \mathcal{L}_1[\hat{h}] + \mathcal{L}_2[\Lambda] + \mathcal{L}_3[\phi] && (6.45)\\
\mathcal{L}_1[\hat{h}] &= -\frac{1}{4}h^{(\mu\nu)}(\Box + m^2 + 2)h_{(\mu\nu)} && (6.46)\\
\mathcal{L}_2[\hat{\Lambda}] &= \frac{m^2}{8}\hat{\Lambda}^\mu(\Box - 4)\hat{\Lambda}_\mu && (6.47)\\
\mathcal{L}_3[\phi] &= \frac{1}{5}\phi\Box(\Box - 5)(\frac{3}{5}\Box - m^2)\phi - \frac{6}{25}H\Box(\Box - 5)\phi + \frac{3}{25}H(\Box - \frac{5}{3}m^2 - 5)H && (6.48)
\end{aligned}$$

We will consider the $m^2 \neq 0$ case first. In this case, the determinant which arises from integrating out $\hat{\Lambda}$ in (6.47) cancels against the Jacobian arising from the change of variables in (6.44), so this part of the action does not contribute to the anomaly. Similarly, when the scalar part of the action (6.48) is diagonalized, its determinant cancels against the scalar part of the Jacobian of (6.44).

So we are left with the action (6.45). The constraints on $\hat{h}_{(\mu\nu)}$ can be lifted at the cost of introducing a vector ghost $\tilde{V}_\mu$ of mass $m^2 - 4$.

Now we make the decomposition into four-dimensional fields. We put $h_{(00)} = \varphi$ and $h_{(0i)} = V_i$, and define $\tilde{h}_{ij} = h_{(ij)} - \varphi g_{ij}/4$ which satisfies $\tilde{h}_{ij}g^{ij} = 0$. Writing out

[2]For simplicity $\Box$ will be understood to include the gravitational couplings to the boundary metric, which for Ricci-flat boundaries are again just the usual ones.

the Christoffel symbols, etc. explicitly, (6.45) becomes

$$\int d^5x\sqrt{g}t\Big\{\frac{5}{4}\dot{\varphi}^2 + 15\frac{\varphi^2}{t^2} + \frac{5}{4}(\nabla\varphi)^2 + 2\left(\dot{V}\cdot\dot{V} + \frac{9}{t^2}V\cdot V + V\cdot(-\nabla^2 V)\right)$$
$$+\dot{\tilde{h}}_{ab}\dot{\tilde{h}}^{ab} + \frac{4}{t^2}\tilde{h}_{ab}\tilde{h}^{ab} + \tilde{h}_{ab}(-\nabla^2\tilde{h}^{ab}) + \frac{8}{t}\tilde{h}_{ab}V_{a;b} + \frac{10}{t}\nabla\varphi\cdot V\Big\}. \tag{6.49}$$

The derivative couplings $\tilde{h}_{ab}V_{a;b}$ and $\nabla\varphi\cdot V$ do not contribute to the anomaly.

Now compare this with the action for the ghost $\tilde{V}_\mu$:

$$\begin{aligned} S_g &= \int d^5x\frac{\sqrt{g}}{t}\tilde{V}^\mu(\Box - 4)\tilde{V}_\mu \\ &= \int d^5x\frac{\sqrt{g}}{t}\Big(-\dot{\tilde{V}}_0^2 + \tilde{V}_0\left(-\frac{11}{t^2}\tilde{V}_0 + \nabla^2\tilde{V}_0 + \frac{2}{t}\nabla\cdot V\right) \\ &\quad + g^{ij}\left(-\dot{\tilde{V}}_i\dot{\tilde{V}}_j + \tilde{V}_i\left(-\frac{11}{t^2}\tilde{V}_j + \nabla^2\tilde{V}_j + \frac{2}{t}\nabla_j V_0\right)\right)\Big). \end{aligned} \tag{6.50}$$

If we make a change of variables $\tilde{V}_\mu \to t\tilde{V}_\mu$ (this adjusts the mass values by 1) then we see that these ghosts cancel the vector and scalar parts of (6.49).

So we are left with a symmetric traceless tensor field on the boundary. From (6.49) we see that we have a non-vanishing anomaly for $m^2 + 4 = N^2$, and from [62] we find that $\alpha = 189$.

Now consider the case $m^2 = 0$, when we have the gauge invariance

$$\delta h_{\mu\nu} = D_\mu\Lambda_\nu + D_\nu\Lambda_\mu, \qquad \Lambda_\mu = \hat{\Lambda}_\mu + D_\mu\phi. \tag{6.51}$$

The action (6.43) can be written

$$\begin{aligned} \mathcal{L} = \quad & -\frac{1}{4}h^{(\mu\nu)}(\Box + m^2 + 2)h_{(\mu\nu)} + \frac{9}{200}D^\mu H D_\mu H \\ & -\frac{1}{2}\left(D_\mu h^{(\mu\nu)} - \frac{3}{10}D^\nu H\right)\left(D^\rho h_{(\rho\nu)} - \frac{3}{10}D_\nu H\right), \end{aligned} \tag{6.52}$$

Table 6.1: Anomaly coefficients of massive fields on AdS_5.

Field	N	α
ϕ	$\sqrt{m^2+4}$	1
ψ	$\|m\|+1/2$	7/2
A_μ	$\sqrt{m^2+1}$	-11
$A_{\mu\nu}$	m	33
ψ_μ	$\|m\|+1/2$	-219/2
$h_{\mu\nu}$	$\sqrt{m^2+4}$	189

and we fix the gauge by introducing

$$\int D\mathcal{A}\delta[D_\mu h^{(\mu\nu)} - \frac{3}{10}D^\nu H - \mathcal{A}^\nu]e^{-\frac{1}{2}\int \mathcal{A}^2}\Delta_{FP}, \tag{6.53}$$

whereupon (on integrating out $\mathcal{A}$) (6.52) becomes

$$\mathcal{L} = -\frac{1}{4}h^{(\mu\nu)}(\Box + m^2 + 2)h_{(\mu\nu)} + \frac{3}{40}H(\Box - 8)H. \tag{6.54}$$

The variation of $D_\mu h^{(\mu\nu)} - \frac{3}{10}D^\nu H$ is $(\Box - 4)\Lambda_\mu$, so the action for the Faddeev-Popov ghosts B_μ^{FP} and C_μ^{FP} is just two copies of (6.50). The decomposition into four-dimensional variables is just as before, except that we have the additional scalar H, which cancels one of the additional ghosts. So in addition to the irreducible tensor field on the boundary we have in total one uncancelled vector ghost; from (6.50) we see that it has $N = 3$. These results are summarized in tables 6.3 and 6.1.

6.6 Summing the anomaly

So let us summarize the results of the last sections. At one loop, the Weyl anomalies are given by

$$\mathcal{A} = -\alpha\, N\, R^{ijkl}R_{ijkl}/(5760\pi^2), \tag{6.55}$$

Table 6.2: Mass spectrum. The supermultiplets (irreps of U(2,2/4)) are labelled by the integer p. Note that the doubleton ($p = 1$) does not appear in the spectrum. The (a, b, c) representation of $SU(4)$ has dimension $r = (a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$, and a subscript c indicates that the representation is complex. (Spinors are four component Dirac spinors in AdS_5).

Field	$SO(4)$ rep$^{\rm n}$	$SU(4)$ rep$^{\rm n}$	Mass on S^5
$\phi^{(1)}$	$(0,0)$	$(0,p,0)$	$m^2 = p(p-4), \quad p \geq 2$
$\psi^{(1)}$	$(\frac{1}{2},0)$	$(0,p-1,1)_c$	$m = p - 3/2, \quad p \geq 2$
$A^{(1)}_{\mu\nu}$	$(1,0)$	$(0,p-1,0)_c$	$m^2 = (p-1)^2, \quad p \geq 2$
$\phi^{(2)}$	$(0,0)$	$(0,p-2,2)_c$	$m^2 = (p+1)(p-3), \quad p \geq 2$
$\phi^{(3)}$	$(0,0)$	$(0,p-2,0)_c$	$m^2 = (p+2)(p-2), \quad p \geq 2$
$\psi^{(2)}$	$(0,0)$	$(0,p-2,1)_c$	$m = p - 1/2, \quad p \geq 2$
$A^{(1)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-2,1)$	$m^2 = p(p-2), \quad p \geq 2$
$\psi^{(1)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-2,0)_c$	$m = p - 1/2, \quad p \geq 2$
$h_{\mu\nu}$	$(1,1)$	$(0,p-2,0)$	$m^2 = (p+2)(p-2), \quad p \geq 2$
$\psi^{(3)}$	$(\frac{1}{2},0)$	$(2,p-3,1)_c$	$m = p - 1/2, \quad p \geq 3$
$\psi^{(4)}$	$(\frac{1}{2},0)$	$(0,p-3,1)_c$	$m = p + 1/2, \quad p \geq 3$
$A^{(2)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-3,1)_c$	$m^2 = (p+1)(p-1), \quad p \geq 3$
$A^{(2)}_{\mu\nu}$	$(1,0)$	$(2,p-3,0)_c$	$m^2 = p^2, \quad p \geq 3$
$A^{(3)}_{\mu\nu}$	$(1,0)$	$(1,p-3,0)_c$	$m^2 = (p+1)^2, \quad p \geq 3$
$\psi^{(2)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-3,0)_c$	$m = p + 1/2, \quad p \geq 3$
$\phi^{(4)}$	$(0,0)$	$(2,p-4,2)$	$m^2 = (p+2)(p-2), \quad p \geq 4$
$\phi^{(5)}$	$(0,0)$	$(0,p-4,2)_c$	$m^2 = (p+3)(p-1), \quad p \geq 4$
$\phi^{(6)}$	$(0,0)$	$(2,p-4,2)$	$m^2 = p(p+4), \quad p \geq 4$
$\psi^{(5)}$	$(\frac{1}{2},0)$	$(2,p-4,1)_c$	$m = p + 1/2, \quad p \geq 4$
$\psi^{(6)}$	$(\frac{1}{2},0)$	$(0,p-4,1)_c$	$m = p + 3/2, \quad p \geq 4$
$A^{(3)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-4,1)$	$m^2 = (p+3)(p-1), \quad p \geq 4$

Table 6.3: Decomposition of gauge fields for the massless multiplet. Some components of the graviton have non-physical mass values, but cancel against identical Faddeev-Popov ghosts. A superscript "irr" indicates that the corresponding field is irreducible.

Original field	Gauge fixed fields	N	α
A_μ	A_i	1	-11
(**15** of $SU(4)$)	A_0	2	1
	b_{FP}, c_{FP}	2	-1
ψ_μ	ψ_i^{irr}	2	-219/2
	$\gamma^i\psi_i$	3	7/2
(**4** of $SU(4)$)	ψ_0	3	7/2
	λ_{FP}, ρ_{FP}	3	-7/2
	σ_{GF}	3	-7/2
$h_{\mu\nu}$	h_{ij}^{irr}	2	189
($SU(4)$ singlet)	h_{0i}	3	-11
	h_{00}, h_μ^μ	cancel	B_0^{FP},C_0^{FP}
	B_0^{FP},C_0^{FP}	cancel	h_{00}, $H = h_\mu^\mu$
	B_i^{FP},C_i^{FP}	3	11

for bosonic fields, and

$$\mathcal{A} = -\alpha\,(N - 1/2)\,R^{ijkl}R_{ijkl}/(5760\pi^2) \tag{6.56}$$

for fermionic ones. We list the values of N and α which we calculated for all the massive fields in table 6.1, and for the fields in the massless multiplet (which include ghosts) in table 6.3.

Now we need to add all of these results up. Although our test of the Maldacena conjecture could have been passed trivially by all the one-loop contributions vanishing separately, we have instead non-vanishing contributions from each of the infinite number of fields corresponding to the Kaluza-Klein modes on S^5, and we will observe a highly non-trivial cancellation between them.

The mass spectrum is given in table 6.2, where the supermultiplets are labelled by an integer p. The first three fields in the table form the doubleton representation for $p = 1$, but this is not present in the spectrum. The $p = 2$ or massless multiplet

Table 6.4: Coefficents of cosmological constant renormalization.

Spin	N	$\mathrm{tr}a_0$
$(0,0)$	$\sqrt{m^2+4}$	1
$(0,\frac{1}{2})$	$\lvert m\rvert+1/2$	-4
$(\frac{1}{2},\frac{1}{2})$	$\sqrt{m^2+1}$	4
$(1,0)$	m	3
$(1,\frac{1}{2})$	$\lvert m\rvert+1/2$	-12
$(1,1)$	$\sqrt{m^2+4}$	9

contains all the gauge fields. For the $p \geq 3$ multiplets the total anomaly is proportional to $\sum_{\text{fields}} \alpha N r$ where r is the dimension of the $SU(4) \sim SO(6)$ representation in which the fields transform. It can be seen from the table that r is always a polynomial of degree 4 in p, while N is linear in p. So each field contributes a polynomial of degree 5 to the sum. If we add up the contributions from all the fields, we find that the anomaly cancels within each of the $p \geq 3$ supermultiplets.

Before we discuss the massless multiplet, notice that the result so far is in accordance with the existence of a consistent truncation which essentially discards all of the $p \geq 3$ multiplets. Even if the overall anomaly was non-zero, the contribution from these multiplets should still vanish.

Using the information in table 6.3 we see that the anomaly sums to zero for this multiplet also, so we conclude that the Maldacena conjecture has successfully predicted the correct value for the Weyl anomaly of the boundary theory.

6.7 Finiteness of boundary theory

There is another non-trivial test of Maldacena's conjecture which we can perform immediately. In the calculation of the Weyl anomaly for the individual fields in Chapter 5, we encountered divergences proportional to the heat-kernel coefficients a_0 and a_1. These divergences renormalize the boundary cosmological and Newton's constants respectively.

Now the boundary theory is conjectured to be $N = 4$ super-Yang-Mills, which is a finite theory, so the renormalization counterterms should also vanish if Maldacena's conjecture is to hold. Indeed, if they do not cancel for each of the infinite number of massive supermultiplets, we have to perform an infinite number of renormalizations for the boundary theory to make sense.

This is easily tested. For each of the supergravity fields we have $a_0 = 1$, so $\mathrm{tr}a_0$ just counts the number of degrees of freedom of the field (listed in table 6.4. The divergent terms are proportional to $N\mathrm{tr}a_0$, where N is the mass-dependent integer we calculated before, and it is easy to check that the divergence cancels in each of the supermultiplets, including the massless one.

The coefficient a_1 is proportional to the Ricci tensor on the boundary, and thus vanishes for Ricci-flat boundaries. So in the present case this does not give us an additional test of the conjecture. But when we generalize the result to non-Ricci flat boundaries, we again expect to see a non-trivial cancellation.

6.8 Conclusions and outlook

To summarize. The coefficient of the square of the Riemann tensor in the Weyl anomaly of $\mathcal{N} = 4$ Super Yang-Mills theory is zero to all loops. For the Maldacena conjecture to hold the same must be true in the calculation on the AdS_5 side. We found that at one-loop the contributions of the individual species of fields was non-zero, however they sum to zero over each supermultiplet, in agreement with the conjecture. Similarly, divergences which renormalize the boundary cosmological and Newton's constants vanish within supermultiplets, providing the conjecture with further rigorous tests. To make the cancellation work required that we make a generic choice of boundary conditions for the bulk fields; if we had chosen certain special boundary conditions, the calculation would have failed, leading to pathological divergences in the boundary theory.

Our calculation was done in the field theory limit of String Theory, however we expect that the neglected string modes have contributions that vanish separately. This is because the anomaly is not a continuous function of mass, but vanishes for generic values. It is non-zero for the Kaluza-Klein spectrum of supergravity which has a special property. A standard construction of Anti-de-Sitter space is as a hyperboloid in $R^{2,4}$. This contains closed time-like curves which are removed by going to the universal cover. Curiously the spectrum of supergravity allows the corresponding fields to be single-valued on this hyperboloid. This property will not be shared by the higher string modes whose masses depend on the string-scale α'.

We have calculated one half of the Weyl anomaly. The remaining terms that depend on the Ricci tensor can be computed by a similar calculation in the more general background given by (5.47). This would provide a further test, as would the computation of higher loops in the bulk, and testing the finiteness of the boundary theory in the more general metric. All of this can be done using our techniques, without the need for further conceptual development.

It has been argued that there is also a cancellation over supermultiplets in the case of the chiral anomaly [28]. This could also easily be studied using our methods; in particular, the gravitational chiral anomaly was hinted at at the end of Chapter 5.

Although Maldacena's conjecture relates to a specific compactification of string theory, in principle there may be a similar correspondence of theories for any compactification onto anti-de-Sitter spacetime. One such correspondence which is receiving a great deal of study involves M-theory (whose low energy limit is 11-dimensional supergravity) compactified on $AdS_7 \times S^4$. This is dual to a (2,0) superconformal field theory describing N coincident M5-branes, and the correspondence provides a useful way of defining M-theory.

The M-theory effective action contains R^4 terms which correspond to stringy corrections to supergravity and modify the subleading behaviour of the anomaly [67].

(Such terms are also present for a compactification of string theory on $AdS_5 \times S^5/Z_2$ whose boundary theory is $N = 2$ supersymmetric [66].) These terms can easily be incorporated into our formalism. We would like to further understand the origin of these stringy corrections, but in any case their inclusion is important for the following reason.

The exact N-dependence of the anomaly for the boundary theory is not known, but it should reduce for $N = 1$ to the anomaly for the free tensor multiplet theory corresponding to the low energy dynamics of a single M5-brane. This provides a check on whatever answer we find. In particular, we need the contribution from the bulk supergravity theory to be finite. Because the anomaly increases with the Kaluza-Klein mass, which is unbounded above, this is only possible if we have a highly non-trivial cancellation of anomalies within each massive supermultiplet, similar to that which we observed for the four-dimensional Ricci flat boundary.

Our calculation of the anomaly works in any dimension, so there are many more compactifications which could be studied, for example the numerous compactifications of string theory on $AdS_3 \times M^7$ for some compact seven-dimensional manifold M^7. It is difficult to guess what further aspects of the intruiging relationship between string and field theories might be elucidated by this work, but it seems clear that the possibilities are enormous.

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