Quantum corrections to the classical reflection factor of the sinh-Gordon model

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Abstract

This thesis studies the quantum reflection factor of the sinh-Gordon model under boundary conditions consistent with integrability. First, we review the affine Toda field theory in Chapter One. In particular, the classical and quantum integrability of the theory are reviewed on the whole line and on the half-line as well, that is, in the presence of a boundary.

We next consider the sinh-Gordon model which is restricted to a half-line by boundary conditions maintaining integrability in Chapter Two. A perturbative calculation of the reflection factor is given to one loop order in the bulk coupling and to first order in the difference of the two parameters introduced at the boundary. The result provides a further verification of Ghoshal's formula. The calculation is consistent with a conjecture for the general dependence of the reflection factor on the boundary parameters and the bulk coupling.

In Chapter Three, quantum corrections to the classical reflection factor of the sinh-Gordon model are studied up to second order in the difference of boundary data and to one loop order in the bulk coupling.

Chapter Four deals with the quantum reflection factor for the sinh-Gordon model with general boundary conditions. The model is studied under boundary conditions which are compatible with integrability and in the framework of the conventional perturbation theory generalised to the affine Toda field theory. It is found that the general form of a subset of the related quantum corrections are hypergeometric functions.

Finally, we sum up this thesis in Chapter Five along with some conclusions and suggestions for further future studies.

Declaration

The work presented in this thesis is based on research carried out by the author between October 1996 and April 2000 in the Department of Mathematical Sciences at the University of Durham, under the supervision of Professor Ed Corrigan.

The material in this thesis has not been submitted for any other degree in this or any other University.

No claims of originality are made for the review Chapter One, excepting the derivations of the formulae (1.19), (1.36) and (1.41). The work in Chapter Two was carried out in collaboration with Ed Corrigan currently in preprint form as hep-th/0002065 and it will be published in International Journal of Modern Physics A. Chapter Four, Appendix A and Appendix B are based on original research by the author in preprint form as hep-th/0004121 and submitted to International Journal of Modern Physics A. The work in Chapter Three is not yet published.

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To my parents, my wife and my brother Mohammad Reza

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Chapter 1

Introduction

1.1 Affine Toda field theory

Affine Toda field theory [1,2] is an integrable quantum field theory in two-dimensional Minkowski space-time. In this theory r scalar fields take values in an r-dimensional Euclidean space which is spanned by the simple roots of a compact semi-simple Lie algebra g with rank r. The candidates for the Lie algebra of the theory are:

$$a_n,\ b_n,\ c_n,\ d_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

the above algebras are the classical Lie algebras that were, historically for the first time, classified by Cartan [3].

The Lagrangian density for the classical field theory is defined as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(\phi), \qquad a = 1, ..., r, \tag{1.1}$$

in which

$$V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}. \tag{1.2}$$

In the potential function m and β are a mass scale and a coupling constant of the theory, respectively. The vectors α_i, i=1,...,r are the simple roots of the Lie algebra g and α_0 is a linear combination of the simple roots, i.e.:

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i. \tag{1.3}$$

Here $n_0 = 1$ conventionally, however, the other integers are not arbitrary, in fact, they depend on the algebraic structure of the Lie algebra g [4]. From the point of view of integrability, there are infinitely many independent conserved quantities in involution which are called charges.

In terms of the Dynkin diagrams for untwisted and twisted affine Kac-Moody algebra [5], α_0 is the extra spot on Dynkin-Kac diagrams. Moreover for the untwisted algebra, the special root α_0, which is called the affine root, is equal to the minus of highest root, that is,

$$\alpha_0 = -\psi. \tag{1.4}$$

In fact, in the adjoint representation α_0 corresponds to the lowest weight because the weights of the adjoint representation are the roots. Meanwhile it is evident that in r dimensional Euclidean space α_0 is not a simple root of the affine Kac-Moody algebra $\hat{g}$. However in (r+1,1) Minkowski space-time, the simple roots of $\hat{g}$ are $\hat{\alpha}_i$, i=0,...,r, which are defined by

$$\begin{aligned} \hat{\alpha}_i &= (\alpha_i, 0, 0), \qquad 1 \leq i \leq r, \\ \hat{\alpha}_0 &= (-\psi, 0, 1). \end{aligned} \tag{1.5}$$

Then $\hat{\alpha}_0$ and $\hat{\alpha}_i$ are independent and span the root lattice of $\hat{g}$. It is necessary to mention that in the (r+1,1) space for two vectors $A = (\alpha, \mu, \nu)$, $B = (\alpha', \mu', \nu')$ the scalar product is defined by :

$$A \cdot B = \alpha \cdot \alpha' + \mu\nu' + \nu\mu'. \tag{1.6}$$

If the first term (the affine term) is excluded from the bulk potential (1.2), then the theory (in both classical and quantum versions) is conformally invariant and is called conformal Toda field theory or, briefly, Toda field theory. On the other hand, if the first term is kept, then the theory breaks conformal symmetry but its integrability is preserved.

1.2 Conformal invariance of Toda field theory

In this section it is shown that Toda field theory is invariant under a conformal transformation [6–8]. A conformal transformation is a general coordinate transformation followed by a Weyl transformation so that the metric tensor is unchanged.

Let us now illustrate the above definition in detail [9]. A general coordinate transformation

$$x^\mu \longrightarrow \bar{x}^\mu(x^\mu) \tag{1.7}$$

changes the metric tensor of the curved space-time as follows (as a result of invariance of the geometrical quantity ds^2 which is equal to $g_{\mu\nu}dx^\mu dx^\nu$) :

$$g_{\mu\nu}(x) \longrightarrow \bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}. \tag{1.8}$$

On the other hand, under a Weyl transformation the space-time is changed through a local rescaling of the metric tensor, that is:

$$g_{\mu\nu}(x) \longrightarrow e^{\phi(x)}g_{\mu\nu}(x). \tag{1.9}$$

In certain space-times it may be possible to find a coordinate transformation whose effect on the metric is equivalent to a Weyl transformation i.e.

$$\begin{aligned} g_{\mu\nu}(x) \longrightarrow \bar{g}_{\mu\nu}(\bar{x}) &= g_{\alpha\beta}(x(\bar{x}))\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \\ &= e^{\phi(\bar{x})}g_{\mu\nu}(\bar{x}). \end{aligned}$$

If this can be done then define a conformal transformation to be the above coordinate transformation followed by a compensating Weyl transformation, so that the metric is unchanged

$$g_{\mu\nu}(x) \longrightarrow \bar{g}_{\mu\nu} = e^{\phi}g_{\mu\nu}(x) \longrightarrow e^{-\phi}\bar{g}_{\mu\nu} = g_{\mu\nu}(\bar{x}).$$

If we study the effect of the conformal transformation on the scalar product of two vectors, for example U and V, then

$$U\cdot V = g_{\mu\nu}(x)U^\mu(x)V^\nu(x) \to \bar{g}_{\mu\nu}(\bar{x})\bar{U}^\mu(\bar{x})\bar{V}^\nu(\bar{x}) = U\cdot V \to e^{-\phi}\bar{g}_{\mu\nu}\bar{U}^\mu\bar{V}^\nu = e^{-\phi}U\cdot V \tag{1.10}$$

Therefore, the lengths of the vectors are scaled as

$$|U| = \sqrt{U\cdot U} \to \sqrt{e^{-\phi}U\cdot U} = e^{-\phi/2}|U|. \tag{1.11}$$

However, the angle between the two vectors is unchanged i.e.

$$\cos\theta = \frac{U\cdot V}{|U||V|} \to \frac{e^{-\phi}U\cdot V}{e^{-\phi/2}|U|e^{-\phi/2}|V|} = \cos\theta. \tag{1.12}$$

Now consider an infinitesimal conformal transformation

$$\bar{x}^\mu = x^\mu + \epsilon^\mu(x^\mu) \tag{1.13}$$

then the vector ϵ^μ is called a conformal Killing vector if it satisfies the conformal Killing equation for the metric i.e.

$$\frac{\partial \epsilon^\sigma}{\partial x^\sigma} g_{\mu\nu} = g_{\mu\alpha} \frac{\partial \epsilon^\alpha}{\partial x^\nu} + g_{\beta\nu} \frac{\partial \epsilon^\beta}{\partial x^\mu}. \tag{1.14}$$

In terms of light-cone coordinates, that is,

$$x_\pm = (x^0 \pm x^1)/2$$

and by means of expanding components of the conformal Killing equation we obtain

$$\frac{\partial \epsilon^-}{\partial x_+} = 0 \qquad \text{and} \qquad \frac{\partial \epsilon^+}{\partial x_-} = 0$$

or equivalently

$$\frac{\partial \bar{x}_+}{\partial x_-} = 0 \qquad \text{and} \qquad \frac{\partial \bar{x}_-}{\partial x_+} = 0.$$

Therefore, a conformal transformation in two-dimensional Minkowski space-time in terms of light-cone coordinates is

$$x_\pm = (x^0 \pm x^1)/2 \longrightarrow \bar{x}_\pm(x_\pm). \tag{1.15}$$

Moreover, the metric is equal to:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = {dx^0}^2 - {dx^1}^2 = 4 dx_+ dx_-. \tag{1.16}$$

Now we can get the transformation of the derivatives which are given by

$$\bar{\partial}_+ = \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial}{\partial x_+} \tag{1.17}$$

$$\bar{\partial}_- = \frac{\partial x_-}{\partial \bar{x}_-} \frac{\partial}{\partial x_-}. \tag{1.18}$$

Therefore, the second derivative of the scalar field transforms as

$$\frac{\partial^2 \phi}{\partial x_+ \partial x_-} \longrightarrow \frac{\partial^2 \phi}{\partial \bar{x}_+ \partial \bar{x}_-} = \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \frac{\partial^2 \phi}{\partial x_+ \partial x_-}. \tag{1.19}$$

Supposing that the field changes according to

$$\phi(x_+, x_-) \longrightarrow \bar{\phi}(\bar{x}_+, \bar{x}_-) = \phi(x_+, x_-) + \frac{\rho}{\beta} \ln \left(\frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \right), \tag{1.20}$$

then the bulk potential will transform as:

$$\sum_{i=1}^{r} n_i \alpha_i \, e^{\beta \alpha_i \cdot \phi} \longrightarrow \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \sum_{i=1}^{r} n_i \alpha_i \, e^{\beta \alpha_i \cdot \phi}, \tag{1.21}$$

provided the vector ρ has the following property

$$\rho \cdot \alpha_i = 1, \qquad i = 1, ..., r. \tag{1.22}$$

Given that the fundamental weights μ_i satisfy

$$2\mu_i \cdot \alpha_j / |\alpha_j|^2 = \delta_{ij}, \tag{1.23}$$

ρ takes the form

$$\rho = \sum_{i=1}^{r} \frac{2}{|\alpha_i|^2} \mu_i. \tag{1.24}$$

Finally, the equation of motion for the Toda field theory which is equal to

$$\partial_+ \partial_- \phi + \frac{m^2}{\beta} \sum_{i=1}^{r} n_i \alpha_i e^{\beta \alpha_i \cdot \phi} = 0, \tag{1.25}$$

would be invariant under conformal transformation.

1.3 Integrability of affine Toda field theory

Behind the integrability of affine Toda field theory there is a zero curvature condition or Lax pair [10–12]. In fact, the Lax pair can be expressed in terms of a two-dimensional zero curvature gauge potential A_μ where

$$A_0 = H \cdot \partial_1 \phi/2 + \sum_{i=0}^{r} m_i (\lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}, \tag{1.26}$$

$$A_1 = H \cdot \partial_0 \phi/2 + \sum_{i=0}^{r} m_i (\lambda E_{\alpha_i} + \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}. \tag{1.27}$$

Here, H is the Cartan subalgebra of g, E_{α_i} and $E_{-\alpha_i}$ are the the step operators associated with the root α_i and λ is a spectral parameter. The coefficients m_i are not arbitrary and actually they are

$${m_i}^2 = n_i {\alpha_i}^2/8. \tag{1.28}$$

Meanwhile the conjugation properties of the potentials are chosen so that

$${A_1}^\dagger(x, \lambda) = A_1(x, \frac{1}{\lambda}), \tag{1.29}$$

$${A_0}^\dagger(x, \lambda) = A_0(x, -\frac{1}{\lambda}). \tag{1.30}$$

Now applying the Lie algebra relations

$$[H, E_{\alpha_i}] = \alpha_i E_{\alpha_i}, \tag{1.31}$$

$$\begin{aligned} [E_{\alpha_i}, E_{\alpha_j}] &= \frac{2\alpha_i \cdot H}{|\alpha_i|^2} && \text{if} \quad \alpha_i = -\alpha_j \\ &= N_{\alpha_i,\alpha_j} E_{\alpha_i+\alpha_j} && \text{if} \quad \alpha_i + \alpha_j \quad \text{is a root} \\ &= 0 && \text{otherwise} \end{aligned} \tag{1.32}$$

so that

$$N_{\alpha_i,\alpha_j} = -N_{-\alpha_i,-\alpha_j} \qquad \text{and} \qquad N_{\alpha_i,\alpha_j} = -N_{\alpha_j,\alpha_i}$$

and after using the equation of motion for the affine Toda field theory which is given by

$$\partial^2\phi + \sum_{i=0}^{r} n_i \alpha_i e^{\alpha_i\cdot\phi} = 0, \tag{1.33}$$

we can obtain

$$\begin{aligned} \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] &= H\cdot\partial_0^2\phi/2 - H\cdot\partial_1^2\phi/2 \\ &+ \sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i})\alpha_i\cdot\partial_0\phi/2 e^{\alpha_i\cdot\phi/2} \\ &- \sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i})\alpha_i\cdot\partial_1\phi/2 e^{\alpha_i\cdot\phi/2} \\ &+ [H\cdot\partial_1\phi/2, \sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}] \\ &+ [\sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}, H\cdot\partial_0\phi/2] \\ &+ [\sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}, \sum_{j=0}^{r} m_j(\lambda E_{\alpha_j} + \frac{1}{\lambda}E_{-\alpha_j})e^{\alpha_j\cdot\phi/2}]. \end{aligned} \tag{1.34}$$

After simplifying

$$\begin{aligned} \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] &= -\frac{1}{2}\sum_{i=0}^{r} n_i H\cdot\alpha_i e^{\alpha_i\cdot\phi/2} \\ &+ 2\sum_{i=0}^{r}\sum_{j=0}^{r} m_i m_j [E_{\alpha_i}, E_{-\alpha_j}] e^{\alpha_i\cdot\phi/2} e^{\alpha_j\cdot\phi/2} \end{aligned} \tag{1.35}$$

and provided in the right-hand side of (1.35) the roots are the α_0 or the simple roots then the Lax pair can be derived

$$\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0. \tag{1.36}$$

Note the m and β have been removed from the equation of motion (1.33) via a rescaling of the fields and space-time coordinates. Furthermore, it is important to mention that the affine Toda equations (1.33) can be obtained from the Lax pair (1.36) provided the two components of the two-dimensional vector potential A_μ are given by (1.26) and (1.27).

Speaking more precisely the Lax pair plays a substantial role in the generation of conserved quantities. To see this in detail, it is a good idea to begin with the path ordered integral. Consider the path ordered integral of A_1 i.e.

$$T(a,b,\lambda) = P\exp\int_a^b dx A_1. \tag{1.37}$$

In fact, the right hand side of the above relation means

$$\begin{aligned} &\mathrm{Lim}\prod_{i=1}^{i=N}\left(1+h_i A_1(x_i,t)\right), \\ &h_i \to 0 \end{aligned} \tag{1.38}$$

where

$$h_1 = x_2 - x_1,\ h_2 = x_3 - x_2,\ \ldots,\ h_N = x_{N+1} - x_N$$

and

$$x_1 = a,\ \ x_{N+1} = b.$$

In order to find the time derivative of $T(a,b,\lambda)$, we calculate the following quantity

$$\begin{aligned}
&\prod_{i=1}^{N}(1+h_iA_1(x_i,t+\delta t)) - \prod_{i=1}^{N}(1+h_iA_1(x_i,t)) \\
&= \prod_{i=1}^{N}(1+h_i(A_1(x_i,t)+\delta t\partial_0 A_1(x_i,t))) - \prod_{i=1}^{N}(1+h_iA_1(x_i,t)) + O(\delta t^2) \\
&= \prod_{i=1}^{N}(1+h_i(A_1(x_i,t)+\delta t(\partial_1 A_0(x_i,t) - [A_0,A_1]))) - \prod_{i=1}^{N}(1+h_iA_1(x_i,t)) + O(\delta t^2) \\
&= \delta t\Big\{(h_1\partial_1 A_0(x_1,t) - h_1[A_0(x_1,t),A_1(x_1,t)])\prod_{i=2}^{N}(1+h_iA_1(x_i,t)) \\
&+ (1+h_1A_1(x_1,t))\,(h_2\partial_1 A_0(x_2,t) - h_2[A_0(x_2,t),A_1(x_2,t)])\prod_{i=3}^{N}(1+h_iA_1(x_i,t)) \\
&+\ldots + \prod_{i=1}^{N-1}(1+h_iA_1(x_i,t))\,(h_N\partial_1 A_0(x_N,t) - h_N[A_0(x_N,t),A_1(x_N,t)])\Big\} \\
&+O(\delta t^2) + \ldots
\end{aligned} \tag{1.39}$$

or

$$\prod_{i=1}^{N}(1+h_iA_1(x_i,t+\delta t))-\prod_{i=1}^{N}(1+h_iA_1(x_i,t))$$
$$=\delta t\Big\{(A_0(x_2,t)-A_0(x_1,t)-h_1[A_0(x_1,t),A_1(x_1,t)])\prod_{i=2}^{N}(1+h_iA_1(x_i,t))$$
$$+(1+h_1A_1(x_1,t))\,(A_0(x_3,t)-A_0(x_2,t)-h_2[A_0(x_2,t),A_1(x_2,t)])$$
$$\prod_{i=3}^{N}(1+h_iA_1(x_i,t))$$
$$+\ldots+\prod_{i=1}^{N-1}(1+h_iA_1(x_i,t))\,(A_0(x_{N+1},t)-A_0(x_N,t)-h_N[A_0(x_N,t),A_1(x_N,t)])\Big\}$$
$$+O(\delta t)^2+\ldots \tag{1.40}$$

Therefore, regarding the relation (1.40), the path ordered integral of A_1 has the time derivative

$$\frac{dT}{dt}=TA_0(b)-A_0(a)T. \tag{1.41}$$

Now introduce an additional quantity $Q(\lambda)$ defined as

$$Q(\lambda)=\mathrm{tr}T(-\infty,\infty,\lambda). \tag{1.42}$$

$Q(\lambda)$ is time independent if $A_0(\infty,t)=A_0(-\infty,t)$. This will be true if $\partial_x\phi\to 0$ and if the exponential terms are also equal. Note

$$\phi(\infty)=\phi(-\infty)+2k,\qquad \text{where}\qquad k\cdot\alpha_i\in\pi iZ \tag{1.43}$$

can be interpreted as a periodic boundary condition.

One of the magnificent aspects of the Lax pair is the existence of a gauge transformation such that the potentials lie in a fixed Cartan subalgebra of $g(h_i)$. After doing this gauge transformation, the potential A_1 will be

$$A_1\longrightarrow a_1=\lambda E_1+\sum_{s\geq 1}\lambda^{-s}h_sI_0{}^{(s)}, \tag{1.44}$$

where two members of the Cartan subalgebra are

$$E_{\pm 1}=\sum_{i=0}^{r}m_iE_{\pm\alpha_i}. \tag{1.45}$$

Note the sum in (1.44) extends over the exponents of the algebra g which are the integers n_i in (1.3). Meanwhile $I_0{}^{(s)}$ is a functional of the fields. For the new gauge potential, the zero curvature condition reduces to

$$\partial_0a_1=\partial_1a_0, \tag{1.46}$$

from which it is concluded that the integral of a_1 over the whole line is conserved. Since λ is arbitrary, it is understood that there are infinitely conserved quantities given by

$$Q_s = \int_{-\infty}^{+\infty} dx I_0{}^{(s)}. \tag{1.47}$$

In order to show that

$$\{Q(\lambda), Q(\mu)\} = 0, \tag{1.48}$$

it is shown that there is a classical r-matrix so that(for further details see [10,12,13])

$$\{T(\lambda), \otimes T(\mu)\} = [r(\lambda/\mu), T(\lambda) \otimes T(\mu)], \tag{1.49}$$

where

$$T(\lambda) \equiv T(-\infty, \infty, \lambda). \tag{1.50}$$

This is a consequence of the canonical equal-time Poisson bracket between the fields and their conjugate momenta. In fact, Olive and Turok [10] provide the form of r:

$$r(\lambda/\mu) = \frac{\mu^h + \lambda^h}{\mu^h - \lambda^h} \sum_{i=1}^{r} H_i \otimes H_i + \frac{2}{\mu^h - \lambda^h} \sum_{\alpha>0} \frac{|\alpha|^2}{2} \left(\lambda^{l(\alpha)} \mu^{h-l(\alpha)} E_\alpha \otimes E_{-\alpha} + \lambda^{h-l(\alpha)} \mu^{l(\alpha)} E_{-\alpha} \otimes E_\alpha \right), \tag{1.51}$$

where the sum runs over all positive roots of g and $l(\alpha)$ is the length of the root α, which is the sum of the integers in its expansion in terms of the simple roots and h is the Coxeter number corresponding to the Lie algebra g, equal to $\sum_{i=0}^{r} n_i$.

In fact the classically conserved charges are two dimensional Lorentz tensors which are characterized by their spin in light-cone coordinates. This means the conserved charges may be described as Q_{s+kh} so that the possible of spins are exponents of the algebra and k is an integer. The quantities $Q_{\pm 1}$ correspond to the light-cone components of the energy-momentum tensor. After quantizing the classical field theory, the quantum operators associated with conserved quantities still commute mutually provided the integrability property is conserved. Meanwhile multi-particle states of particles of definite rapidities are eigenstates of the conserved quantities. In other words, for single-particle states

$$Q_p |a> = q_p^a e^{p\theta_a} |a>, \qquad p = s + kh, \tag{1.52}$$

where θ_a is the rapidity of the particle a (for further discussions see [14]).

1.4 Quantum Toda field theory

In this section the two-dimensional exact S-matrix theory [15, 16] is studied. By introducing particle creation operators $A_a(p)$ which create particle a with momentum p, the asymptotic n-particle states can be written as

$$|A_{a_1}(p_1)A_{a_2}(p_2)...A_{a_n}(p_n) >= A_{a_1}(p_1)A_{a_2}(p_2)...A_{a_n}(p_n)|0> . \tag{1.53}$$

As usual, it is appropriate to express the momentum of a particle in two-dimensional Minkowski space-time in terms of a useful quantity, namely the rapidity of the particle:

$$p_a = m_a(\cosh\theta_a, \sinh\theta_a). \tag{1.54}$$

So in what follows it is convenient to deal with n-particle states described by the rapidities of the particles i.e.

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_n}(\theta_n) >= A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_n}(\theta_n)|0> . \tag{1.55}$$

The above state can be interpreted as an in or out state if the rapidities are arranged as $\theta_1 > \theta_2 > ... > \theta_n$ or $\theta_1 < \theta_2 < ... < \theta_n$ respectively. The S-matrix is defined as:

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_n}(\theta_n) >_{in}= S^{b_1 b_2 ... b_n}_{a_1 a_2 ... a_n}(\theta_1, \theta_2, ..., \theta_n)|A_{b_1}(\theta_1)A_{b_2}(\theta_2)...A_{b_n}(\theta_n) >_{out} . \tag{1.56}$$

In fact the n-particle scattering amplitude is determined by the matrix element of the S-matrix i.e. $S^{b_1 b_2 ... b_n}_{a_1 a_2 ... a_n}(\theta_1, \theta_2, ...\theta_n)$ for the following process where time runs from left to right:

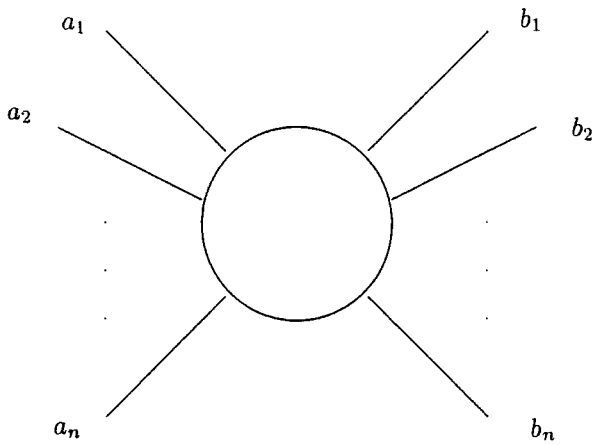

Figure 1.1: N-particle S-matrix.

One of the outstanding and interesting features of an integrable quantum field theory is the fact that the corresponding S-matrix is factorizable and this has been discussed by many people. For example, Zamolodchikov and Zamolodchikov [15] studied the factorisation via a wave function approach whereas Shankar and Witten [17] used a wave packet method.

In fact the existence of a couple of higher-spin conserved charges $\{Q_s, Q_{-s}\}$ $s > 1$ for scattering process of two-dimensional integrable quantum field theory gives rise to three properties:

I) particle production does not occur

II) the set of initial momenta is the same as the final one

III) in the scattering of n-particles, the S-matrix factorizes.

More precisely factorization means the S-matrix corresponding to n-particle scattering factorizes into a product of two particles S-matrices. If the scattering of three particles is examined then the factorization will have a sensible pictorial meaning. So, consider 3-particle process. Regarding the relative position of the incident particles, there are three completely separate scattering processes I, II and III where time runs from bottom to top:

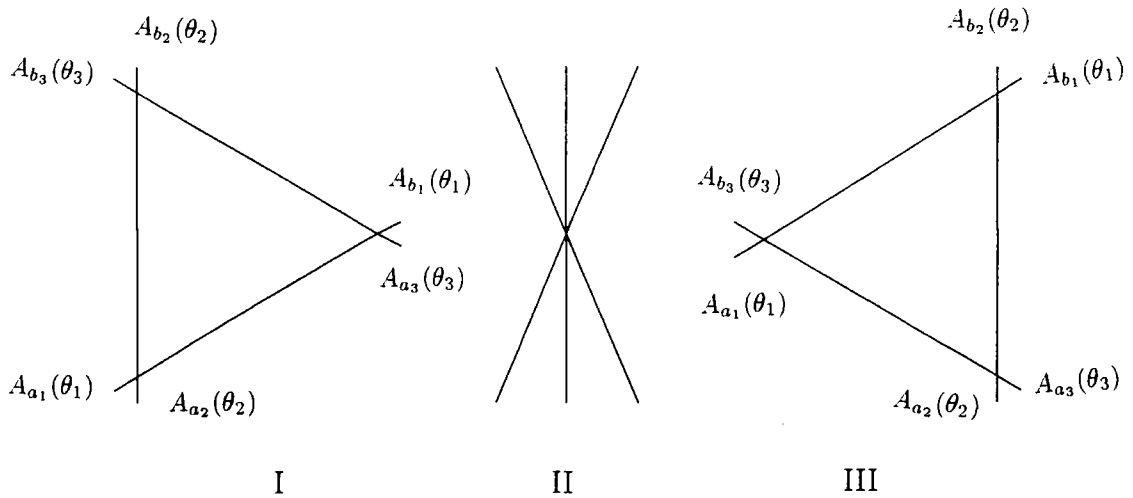

Figure 1.2: The Yang-Baxter equation.

But the middle scattering can be converted into either of the other ones by taking a suitable limit. In several papers [16–18] the action of the conserved charges on lo-

calized wave packets is studied, the emerging result is the equality of the amplitudes of processes I and III. In other words, in the S-matrix language

$$S_{a_1a_2}^{c_1c_2}(\theta_{12})S_{c_1a_3}^{b_1c_3}(\theta_{13})S_{c_2c_3}^{b_2b_3}(\theta_{23}) = S_{c_1c_2}^{b_1b_2}(\theta_{12})S_{a_1c_3}^{c_1b_3}(\theta_{13})S_{a_2a_3}^{c_2c_3}(\theta_{23}) \tag{1.57}$$

The cubic equation (1.57) is called the Yang-Baxter or factorization equation where $\theta_{ij} = \theta_i - \theta_j$ because of Lorentz invariance. The equation (1.57) does not determine the S-matrix completely. Moreover if the particles are completely distinguishable by charges of non-zero spin as for real affine Toda theory, then the above equation is an identity. The general solution of the Yang-Baxter equation is not known, nevertheless general classes of solution have been found [19].

Regarding the factorization equation, studying the two-particle scattering becomes substantial and crucial. Therefore in what follows we mention some highlights of the properties of the two particle S-matrix which is defined according to:

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2) >_{in} = S_{a_1a_2}^{b_1b_2}(\theta_1 - \theta_2)|A_{b_1}(\theta_1)A_{b_2}(\theta_2) >_{out} \tag{1.58}$$

or pictorially

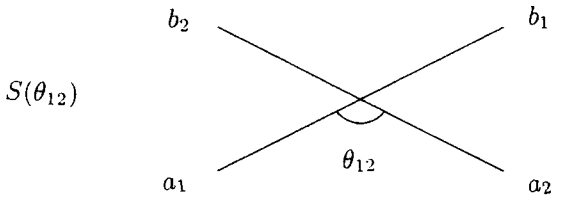

Figure 1.3: Two-particle S-matrix.

where $\theta_1 > \theta_2$ to distinguish the in and out states and time runs from bottom to top. Assuming invariance of S-matrix under charge conjugation lead to the following relation:

$$S_{a_1a_2}^{b_1b_2}(\theta_{12}) = S_{\bar{a}_1\bar{a}_2}^{\bar{b}_1\bar{b}_2}(\theta_{12}). \tag{1.59}$$

Note also that under the charge conjugation operator, A_{a_1} corresponding to particle a_1 transforms to $A_{\bar{a}_1}$ which is associated to the antiparticle $\bar{a}_1$.

In order to study some analytic properties of the S-matrix it is convenient to deal with the Mandelstam variables s and t which are equal to

$$s = (p_1 + p_2)^2 = m_{a_1}^2 + m_{a_2}^2 + 2m_{a_1}m_{a_2}\cosh\theta_{12}, \tag{1.60}$$

$$t = (p_1 - p_2)^2 = 2m_{a_1}^2 + 2m_{a_2}^2 - s. \tag{1.61}$$

For the S-matrix to describe a physical process θ must be real and positive. This corresponds to taking the S-matrix on the upper edge of the branch cut in the s-plane. Moreover θ_{12} will be equal to

$$\begin{aligned}\theta_{12} &= \cosh^{-1}\left(\frac{s-m_{a_1}^2-m_{a_2}^2}{2m_{a_1}m_{a_2}}\right)\\ &= \ln\left(\frac{s-m_{a_1}^2-m_{a_2}^2+\sqrt{(s-(m_{a_1}+m_{a_2})^2)(s-(m_{a_1}-m_{a_2})^2)}}{2m_{a_1}m_{a_2}}\right)\end{aligned} \quad (1.62)$$

So, the rapidity difference maps the s-plane into the strip $0 \leq \text{Im}\ \theta_{12} \leq \pi$ which is called the physical strip.

Analytic continuation of the S-matrix into the complex s-plane results in a function S that is meromorphic with two branch cuts on the parts of the real axis where

$$s \geq (m_{a_1}+m_{a_2})^2 \quad \text{and} \quad s \leq (m_{a_1}-m_{a_2})^2.$$

Moreover, the S-matrix has poles and physical processes associate with those poles which are inside the physical strip.

Here we note some properties of the S-matrix [16, 20]:

I) Hermitian analyticity: that is

$$S_{a_1a_2}^{b_1b_2}(\theta_{12}) = \left[S_{b_2b_1}^{a_2a_1}(-\theta_{12}^*)\right]^* \quad (1.63)$$

II) R-matrix unitarity: i.e. running the whole thing backward

$$S_{a_1a_2}^{c_1c_2}(\theta_{12})S_{c_1c_2}^{b_1b_2}(-\theta_{12}) = \delta_{a_1}^{b_1}\delta_{a_2}^{b_2} \quad (1.64)$$

III) Crossing symmetry: as a result of invariance of the S-matrix under the transformation which interchange s-channel to t-channel i.e. $s \to t$ that corresponds to $\theta \to i\pi - \theta$ so,

$$S_{a_1a_2}^{b_1b_2}(\theta_{12}) = S_{a_1\bar{b}_2}^{b_1\bar{a}_2}(i\pi-\theta_{12}) \quad (1.65)$$

Bootstrap equation

Now assume the scattering of two-particles is purely elastic. For real affine Toda field theory the S-matrix is diagonal and it has just two indices i.e. the S-matrix element may be written as $S_{a_1a_2}(\theta_{12})$. In this case the Yang-Baxter equation is satisfied identically. Now, in this special case the unitarity and crossing relations will be simple, that is,

I) Unitarity

$$S_{a_1a_2}(\theta_{12})S_{a_1a_2}(-\theta_{12}) = 1, \tag{1.66}$$

II) Crossing

$$S_{a_1\bar{a}_2}(\theta_{12}) = S_{a_1a_2}(i\pi - \theta_{12}). \tag{1.67}$$

By combining these two relations, one obtains

$$S_{a_1a_2}(\theta_{12}) = S_{a_1a_2}(\theta_{12} + 2\pi i). \tag{1.68}$$

Therefore, the S-matrix elements are periodic in terms of $i\theta$ and this is the origin of the fact that S-matrix can be written down as a product of hyperbolic building blocks which are defined as:

$$(x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})}. \tag{1.69}$$

Assuming $S_{a_1a_2}(\theta_{12})$ has a simple pole at $\theta_{12} = iU^{\bar{a}_3}_{a_1a_2}$ which is caused by formation of a bound state (time runs from left to right):

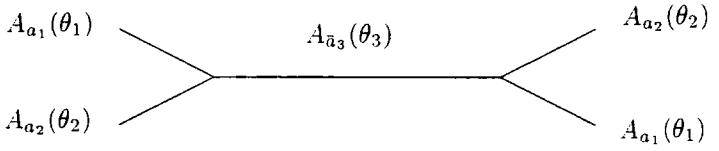

Figure 1.4: Bound state.

At the point where particles a_1, a_2 and a_3 are all on shell, the three point coupling $C^{a_1a_2a_3}$ does not vanish, meanwhile the energy of the incident particles is equal to the mass of the third particle, so

$$m^2_{a_3} = m^2_{a_1} + m^2_{a_2} + 2m_{a_1}m_{a_2}\cos U^{a_3}_{a_1a_2} \tag{1.70}$$

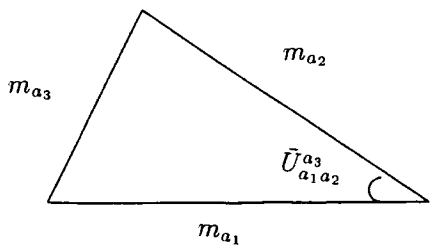

Figure 1.5: The mass triangle.

The real number U is called the fusing angle. By operation of Q_s on $|A_{a_1}(\theta_1)A_{a_2}(\theta_2)>$ and $|A_{\bar{a}_3}(\theta_3)>$ then one can get a conserved charge after the fusing of a_1, a_2 into $\bar{a}_3$ i.e.

$$Q_s|A_{a_1}(\theta_1)A_{a_2}(\theta_2)>=(q_{a_1}^{(s)}e^{s\theta_1}+q_{a_2}^{(s)}e^{s\theta_2})|A_{a_1}(\theta_1)A_{a_2}(\theta_2>, \tag{1.71}$$

$$Q_s|A_{\bar{a}_3}(\theta_3)>=q_{\bar{a}_3}^{(s)}e^{s\theta_3}|A_{\bar{a}_3}(\theta_3)>. \tag{1.72}$$

Note $s=p+kh$,$k\in Z$ i.e. the spin is given by the exponents of the Lie algebra. Now

$$q_{a_1}^{(s)}e^{s\theta_1}+q_{a_2}^{(s)}e^{s\theta_2}=q_{\bar{a}_3}^{(s)}e^{s\theta_3}, \tag{1.73}$$

or using the fusing angle and after some manipulation

$$q_{a_1}^{(s)}+q_{a_2}^{(s)}e^{isU_{a_1a_2}^{a_3}}+q_{a_3}^{(s)}e^{is(U_{a_1a_2}^{a_3}+U_{a_2a_3}^{a_1})}=0. \tag{1.74}$$

To get the relation (1.74) note that $\bar{U}=\pi-U$ and $q_{\bar{a}_3}^{(s)}=(-1)^{s+1}q_{a_3}^{(s)}$, the last relation shows particles and antiparticles are distinguished when spin charges are even.

Now consider another particle a_4 which presumed to interact with particles a_1 and a_2. There are two cases because particle a_4 may scatter either before or after the two-particle a_1 and a_2 constitute a bound state to form antiparticle $\bar{a}_3$. In general quantum field theory the above two cases may be different but in an integrable quantum field theory, they are not so much different and in fact because of factorization property of S-matrix [21]

$$S_{a_4\bar{a}_3}(\theta)=S_{a_4a_1}(\theta-i\bar{U}_{a_1a_3}^{a_2})S_{a_4a_2}(\theta+i\bar{U}_{a_2a_3}^{a_1}), \tag{1.75}$$

in which $\theta=\theta_4-\theta_{\bar{a}_3}$ i.e. the relative rapidity of $\bar{a}_3$ and a_4. The equation (1.75) is called the bootstrap equation and can be shown diagramatically

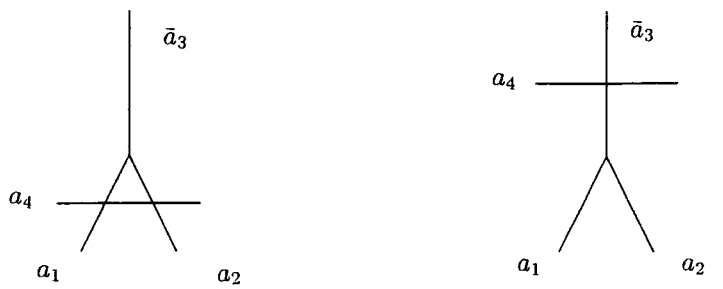

Figure 1.6: The bootstrap equation.

By means of unitarity and crossing properties of S-matrix, the bootstrap equation can be converted to:

$$S_{a_4a_1}(\theta + iU^{a_2}_{a_1a_3} + iU^{a_1}_{a_2a_3})S_{a_4a_3}(\theta + iU^{a_1}_{a_2a_3})S_{a_4a_2}(\theta) = 1, \tag{1.76}$$

which is a product counterpart of the relation (1.74). The bootstrap equation is very important but it does not determine the S-matrix element uniquely. All it may do is to provide a set of consistency equations which must be supplemented by other data.

In the real affine Toda theories whose corresponding untwisted affine Kac-Moody algebras are $\hat{a}_n$, $\hat{d}_n$ or $\hat{e}_n$, the S-matrices have poles of both even and odd order. What distinguishes the odd poles is that these correspond to bound states in the direct or crossed channel [22, 23]. At the same time, the couplings obey Dorey's rule [24, 25] which is closely related to the properties of root systems and the Coxeter element of the Weyl group. For the other Toda field theories corresponding to the non simply-laced algebras an appropriate generalization of Dorey's rule is known now [26–28].

Exact S – matrices

Toda field theory is one of the most successful examples of two-dimensional quantum field theory since it can be solved exactly. On the other hand, quantum field theory in four dimensional Minkowski space-time is not exactly solvable due to several kind of difficulties such as infinite degrees of freedom.

All the Toda field theories whose associated Lie algebras are simply laced or non-simply laced algebras have known exact quantum S-matrices [1, 22, 29–35]. S-matrices corresponding to simply laced algebra have a much higher singularity structure than the other ones.

In previous arguments it is mentioned that the Yang-Baxter equation is the necessary condition for the factorization property of the S-matrix. However in purely elastic scattering of two-particle followed by forming a bound state, the S-matrix is diagonal and in this case the Yang-Baxter equation becomes trivial. But it has been shown in the previous discussions that there is another powerful equation called the bootstrap equation. In according to the statements of this section, the building blocks of the S-matrix are of the form:

$$(x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{h})}.$$

In order to include the coupling constant dependence it is better to define the generalized building blocks as

$$\{x\} = \frac{(x+1)(x-1)}{(x+1-B)(x-1+B)}, \tag{1.77}$$

where

$$B = \frac{\beta^2/2\pi}{1+\beta^2/4\pi}, \tag{1.78}$$

and β is the coupling constant of the theory.

Now the S-matrix of $\hat{a}_n$ Toda field theory is:

$$\hat{a}_n : S_{ab} = \{a+b-1\}\{a+b-3\}...\{|a-b|+1\} = \prod_{|a-b|+1}^{a+b-1} \{p\}, \tag{1.79}$$

in which $a, b = 1, ..., r$ are the labels of two incoming particles. For the S-matrices of $\hat{e}_n$, $\hat{d}_n$ and non-simply laced theories see [1, 22, 29–35].

1.5 Affine Toda field theory on a half-line

In the recent decade much work has been done in relation to integrable field theory with a boundary [36–50]. In particular, boundary conditions which are compatible with classical or quantum integrability have been studied for affine Toda field theories corresponding to simply-laced Lie algebra, as well as non-simply laced Lie algebras.

In this section affine Toda field theory on a half-line is studied. In fact the theory on a half-line is determined by the Lagrangian density

$$\bar{\mathcal{L}} = \theta(-x)\mathcal{L} - \delta(x)\mathcal{B}, \tag{1.80}$$

in which $\mathcal{L}$ is the Lagrangian density of the theory on the whole line and $\mathcal{B}$, which is called the boundary term, is a functional of the fields but not their derivatives. Meanwhile at the boundary $x = 0$

$$\frac{\partial\phi}{\partial x} = -\frac{\partial\mathcal{B}}{\partial\phi}. \tag{1.81}$$

Actually the above relation is the boundary condition. Moreover due to some evidence [36, 37], the generic form of the boundary term is given by

$$\mathcal{B} = \frac{m}{\beta^2}\sum_{i=0}^{r} A_i e^{\frac{\beta}{2}\alpha_i\cdot\phi}, \tag{1.82}$$

where the set of coefficients A_i :i=1,...,r are real numbers and there is a constraint on the A_i for all simply-laced affine Toda theories i.e.

$$|A_i| = 2\sqrt{n_i} \qquad \text{or} \qquad A_i = 0. \tag{1.83}$$

To sum up, for the half-line theory the equation of motion along with the boundary equation is given by

$$\partial^2\phi = -\sum_{i=0}^{r} \alpha_i n_i e^{\alpha\cdot\phi} \qquad \text{when} \quad x < 0, \tag{1.84}$$

$$\frac{\partial\phi}{\partial x} = -\sum_{i=0}^{r} \frac{1}{2}\alpha_i A_i e^{\alpha_i\cdot\phi/2} \qquad \text{at} \quad x = 0. \tag{1.85}$$

Note the mass scale and the coupling constant have been eliminated from the equations (1.84) and (1.85) via a rescaling of the fields and the space-time coordinates. In third section of this chapter, it was shown that the foundation of the integrability property of the affine Toda field theory on the whole line is based on the existence of the Lax pair. In connection with the half-line theory Bowcock et.al [40] developed a generalization of the Lax pair idea and we review their work. In what follows the boundary of the affine Toda field theory is chosen at $x = a$ in order to follow their studying. Meanwhile for the purpose of construction of a modified Lax pair containing the boundary condition, it is convenient to introduce an another particular point $x = b$ which is greater than a and two overlapping regions

$$R_- : x \leq (a + b + \epsilon)/2 \qquad \text{and} \qquad R_+ : x \geq (a + b - \epsilon)/2. \tag{1.86}$$

The second region may be considered as a reflection of the first one, having the meaning that if $x \in R_+$ then

$$\phi(x) \equiv \phi(a + b - x). \tag{1.87}$$

The two regions overlap each other in a small interval around the midpoint of [a,b]. Now in the two regions, the modified two dimensional zero curvature gauge potential for the theory on a half-line may be defined as:

$$R_- : \hat{A}_0 = A_0 - \frac{1}{2}\theta(x - a)\left(\partial_x\phi + \frac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H, \qquad \hat{A}_1 = \theta(a - x)A_1, \tag{1.88}$$

$$R_+ : \hat{A}_0 = A_0 - \frac{1}{2}\theta(b - x)\left(\partial_x\phi - \frac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H, \qquad \hat{A}_1 = \theta(x - b)A_1. \tag{1.89}$$

It is evident that in the region $x < a$ the generalized Lax pair is the same as the primary one but clearly at $x = a$ the derivative of the step function θ in the zero curvature condition impose the boundary condition

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \mathcal{B}}{\partial \phi}, \qquad x = a. \tag{1.90}$$

For the region $x \geq b$, the same situation holds apart from this distinction that the boundary condition at $x = b$ has an opposite sign in order to be consistent with the reflection condition (1.87).

On the other hand, if $x \in R_-$ and $x > a$ then, $\hat{A}_1$ will be zero and in this case the zero curvature condition just means $\hat{A}_0$ has no dependence on x. In other words, this fact implies that ϕ is independent of x. The same statements can be obtained in the region $x \in R_+$ and $x < b$. Therefore by considering the reflection relation, ϕ is independent of x in every part of the interval $[a, b]$, and equal to its value at a or b. By regarding the modified gauge potentials (1.88) and (1.89), it is clear that the gauge potential $\hat{A}_0$ is different in the two regions $R_\pm$ for general boundary conditions. Nevertheless to preserve the zero curvature condition over the whole line a gauge transformation must relate the values of $\hat{A}_0$ on the overlapping region. Due to the fact that $\hat{A}_0$ is independent of $x \in [a, b]$ on both parts, although it has different value on each part, the zero curvature condition requires the existence of a gauge transformation. Moreover, such a gauge transformation which may be denoted by $\mathcal{K}$ has to satisfy the following relation

$$\frac{\partial \mathcal{K}}{\partial t} = \mathcal{K}\hat{A}_0(t, b) - \hat{A}_0(t, a)\mathcal{K}, \tag{1.91}$$

in which $\mathcal{K}$ is a group element of the group G whose corresponding Lie algebra is associated to the affine Toda field theory.

It is understood that the next step is to determine the analogue of $Q(\lambda)$ in the presence of the boundary which may be defined by the expression

$$\hat{Q}(\lambda) = tr\left(T(-\infty, a, \lambda)\mathcal{K}T^\dagger(-\infty, a, 1/\lambda)\right). \tag{1.92}$$

Indeed, $\hat{Q}(\lambda)$ provides a generating function for the conserved quantities of the half-line theory.

For the purpose of understanding the structure of $\mathcal{K}$, it is useful to make a couple of additional assumptions. Now assuming the gauge transformation $\mathcal{K}$ to be

independent of time and the fields or their derivatives as well, then (1.91) simplifies to

$$\mathcal{K}\hat{A}_0(t,b) - \hat{A}_0(t,a)\mathcal{K} = 0 \tag{1.93}$$

or considering the exact expressions for $\hat{A}_0$

$$\frac{1}{2}\left[\mathcal{K}(\lambda), \frac{\partial \mathcal{B}}{\partial \phi}\cdot H\right]_+ = -\left[\mathcal{K}(\lambda), \sum_{i=0}^{r} m_i(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}\right]_-, \tag{1.94}$$

in which the quantities which depend on the field, are evaluated at the boundary i.e. $x = a$.

Note the boundary term $\mathcal{B}$ does not depend on the spectral parameter λ. The equation (1.94) is strong enough because it can not only determine boundary term $\mathcal{B}$, but can also determine the gauge transformation $\mathcal{K}$. The several solutions of the equation (1.94) along with a list of constraints on boundary term can be found in detail in [40]. For example, two examples of $\mathcal{K}$ from those given in [40] are

$$\hat{a}_1 : \mathcal{K}(\lambda) = (\lambda^2 - \frac{1}{\lambda^2})I + \begin{pmatrix} 0 & \lambda A_1 - A_0/\lambda \\ \lambda A_0 - A_1/\lambda & 0 \end{pmatrix}, \tag{1.95}$$

$$\hat{a}_n : \mathcal{K}(\lambda) = I + 2\sum_{\alpha>0}\prod_i C_i^{l_i(\alpha)}\left[\frac{(-\lambda)^{l(\alpha)}E_\alpha}{1+C\lambda^h} + \frac{(-\frac{1}{\lambda})^{l(\alpha)}E_{-\alpha}}{1+C/\lambda^{-h}}\right]. \tag{1.96}$$

Note in the relation (1.96), $C_i = A_i/2$, $C = \prod_i C_i$ and each positive root can be written down as a sum of simple roots and $l_i(\alpha)$ represent the number of times that α_i appears in the sum and $l(\alpha) = \sum_i l_i(\alpha)$.

As far as the boundary potential is concerned, the mentioned conjecture seems to be correct for the *ade* series of models and in turn, this implies the strongly restricted boundary parameters. For all the others, the form of the boundary potential is the same but the restrictions on the parameters are less strict.

All that remains is to show consistency of $\mathcal{K}(\lambda)$ with the classical r matrix which specifies the Poisson brackets between the generating functions for the conserved charges associated to the whole line theory [10,12]. In fact [10,12,13]

$$\{T(\lambda)\otimes T(\mu)\} = [r(\lambda/\mu), T(\lambda)\otimes T(\mu)], \tag{1.97}$$

where $T(\lambda)$ represents the path-ordered exponential defined in (1.37). Meanwhile the classical r-matrix has the general form

$$r(s) = \sum_i r_i(s) g_i \otimes g_i^\dagger. \tag{1.98}$$

The quantities g_i represent the generators of the Lie algebra whose root system correspond to a particular Toda model. If the Poisson bracket between two charges of the form given by (1.92) is calculated then it will require a compatibility condition to be satisfied containing r and $\mathcal{K}$.

The consistency condition involving $\mathcal{K}$ and r has been verified [40] and in fact, it satisfies the following relation

$$[r(\lambda/\mu), \mathcal{K}^{(1)}(\lambda)\mathcal{K}^{(2)}(\mu)] = \mathcal{K}^{(1)}(\lambda)\tilde{r}(\lambda\mu)\mathcal{K}^{(2)}(\mu) - \mathcal{K}^{(2)}(\mu)\tilde{r}(\lambda\mu)\mathcal{K}^{(1)}(\lambda), \quad (1.99)$$

where

$$\mathcal{K}^{(1)}(\lambda) = \mathcal{K}(\lambda) \otimes 1, \qquad \mathcal{K}^{(2)}(\mu) = 1 \otimes \mathcal{K}(\mu) \quad (1.100)$$

and

$$\tilde{r}(s) = \sum_i r_i(s) g_i \otimes g_i. \quad (1.101)$$

At a first glance $\mathcal{K}$ is a fundamental quantity and due to the relationship between $\mathcal{K}$ and r (1.99), it may be expected that the classical r matrix should be chosen to be consistent with $\mathcal{K}$. However, it is a remarkable fact that even though $\mathcal{K}$ and r have been determined independently, (1.99) is satisfied especially since apparently powerful assumptions were made to derive the expressions for $\mathcal{K}$ in the various cases. Even in the quantum case, as will be discussed later, there is a set of reflection bootstrap equations whose solutions provide a calculation of the complete set of S-matrix factors.

1.6 Quantum Toda field theory on a half-line

In sections (1.3) and (1.4) classical and quantum integrability of the Affine Toda field theory defined over the whole-line, were studied respectively. It is mentioned that due to the classical integrability of the theory, in fact for every integrable field theory, there is an infinite set of conserved charges. In connection with quantum integrability an important and remarkable characteristic of an integrable field theory including affine Toda field theory, is a fundamental property of the S-matrix: the so called factorization equation.

On the other hand, in section (1.5) classical integrability for affine Toda field theory on a half-line is studied. In fact, in the presence of a boundary, the existence of

the conserved charges depend on choosing a particular boundary condition. In other words, in that section boundary conditions consistent with classical integrability were reviewed.

Now in this section quantum integrability of the half-line theory is studied. For the first time Cherednik [51] generalized the factorization property of S-matrix to the case where a reflecting boundary is present. In fact the basic ideas of factorizable scattering were set out by Cherednik and supplemented by Ghoshal and Zamolodchikov [39] in relation to the sine-Gordon theory, and by Fring and Koberle [52,53] as well, and by Sasaki [54] in connection with the affine Toda theories.

When a particle is moving towards the boundary located for example at $x = 0$, then it is expected that the particle reverses its direction however, in general, the particle may lose its identity as well. So in terms of one-particle state it might be expected that the final state or out state is proportional to the initial state or in state i.e.:

$$|A_a(\theta) >_{in} = R_a^b(\theta)|A_b(-\theta) >_{out}, \tag{1.102}$$

or pictorially

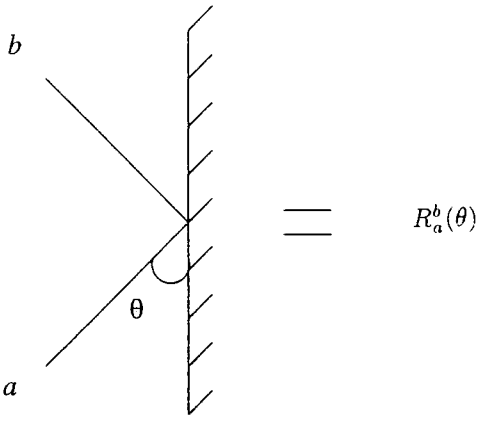

Figure 1.7: Reflection matrix.

where $R_a^b(\theta)$ can be interpreted as the amplitude of one particle reflection from the boundary and time runs from bottom to top.

It is possible to extend the above process to the n-particle scattering in presence of a boundary and define the related S-matrix as:

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_n}(\theta_n) >_{in} = R_{a_1 a_2 ... a_n}^{b_1 b_2 ... b_n}(\theta_1, ..., \theta_n)|A_{b_1}(-\theta_1)A_{b_2}(-\theta_2)...A_{b_n}(-\theta_n) >_{out} \tag{1.103}$$

First of all, the rapidities of the particles are arranged as $\theta_1 > \theta_2 > ... > \theta_n$ or $\theta_1 < \theta_2 < ... < \theta_n$ for n particle in or out states respectively. Secondly, the n-particle S-matrix can be expressed in terms of the fundamental amplitudes $S_{a_1 a_2}^{b_1 b_2}(\theta)$ and R_a^b, that is, two-particle scattering amplitude and one-particle reflection amplitude respectively. Moreover, both amplitudes have to satisfy several general conditions analogous to the conditions of the whole-line theory [39] as:

1. Boundary Yang-Baxter equation

$$R_{a_2}^{c_2}(\theta_2) S_{a_1 c_2}^{c_1 d_2}(\theta_1+\theta_2) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{b_2 b_1}(\theta_1-\theta_2) = S_{a_1 a_2}^{c_1 c_2}(\theta_1-\theta_2) R_{c_1}^{d_1}(\theta_1) S_{c_2 d_1}^{d_2 b_1}(\theta_1+\theta_2) R_{d_2}^{b_2}(\theta_2). \tag{1.104}$$

The equation (1.104) can be represented diagramatically as

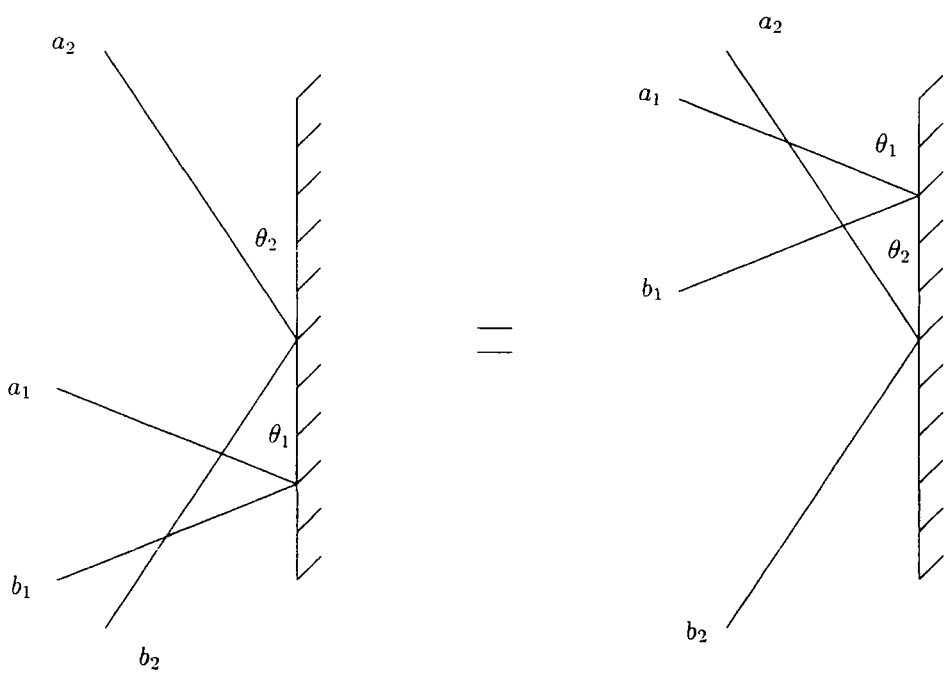

Figure 1.8: The boundary Yang-Baxter equation.

As it may be seen in the figure, it is clear that the two particles not only scatter from the boundary but also from each other. However, the order of the individual scatterings and reflections have no importance since they depend on the initial condition establishing the two-particle state. Finally, the factorization property of the scattering process in the presence of the boundary leads to the boundary Yang-Baxter equation. For affine Toda field theory which has distinguishable particles, R and S matrices are both diagonal and the boundary Yang-Baxter equation is satisfied identically. So, it is necessary to have an alternative equation i.e. boundary

bootstrap equation. The boundary Yang-Baxter equation has been introduced first by Cherednik [51].

2. Boundary group-theoretic unitarity condition

$$R_a^c(\theta)R_c^b(-\theta) = \delta_a^b \tag{1.105}$$

or pictorially

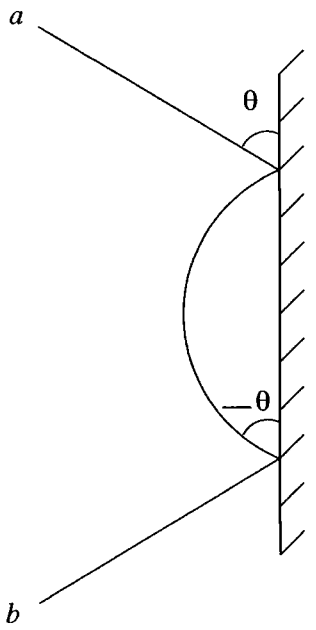

Figure 1.9: The boundary unitarity.

3. Boundary crossing-symmetry condition

$$K^{ab}(\theta) = S_{a'b'}^{ab}(2\theta)K^{b'a'}(-\theta), \tag{1.106}$$

where

$$K^{ab}(\theta) = R_{\bar{a}}^{b}(\frac{i\pi}{2} - \theta). \tag{1.107}$$

Now assuming :

I) The scattering process of the two particles to be purely elastic, in the sense that there is no particle production i.e. there is no process other than elastic two particle scattering of the form $ab \to ab$. Then, the S-matrix corresponding to two-particle scattering is diagonal and it has just two indices

$$|a,b>_{in} = S_{ab}(\theta_{ab})|a,b>_{out} . \tag{1.108}$$

II)The reflection of a particle from the boundary is purely elastic. This means when a particle moves toward $x = 0$ then, the effect of the boundary is an inversion of the momentum. So in this case, the particle maintains its identity but just reverses its direction. Moreover, the out state is proportional to the in state

$$|A_a(\theta) >_{in} = K_a(\theta)|A_a(-\theta) >_{out}, \tag{1.109}$$

where $K_a(\theta)$ is called the reflection factor. In fact, in an integrable theory with distinguishable scalar particles the relation (1.109) may be used. It is necessary to mention that multiplets of particles are distinguishable only by spin-zero charges. For real affine Toda field theory the particles are distinguishable and hence, both assumptions I and II are valid. Moreover, there must be a set of reflection factors, one for each particle, corresponding to every boundary condition consistent with integrability.

For the affine Toda field theory on the whole-line there is a consistent bootstrap principle which in turn, has the meaning that there is a consistent set of couplings between the particles due to the appearance of poles in the S-matrix at fixed imaginary relative rapidities corresponding to a bound state. Now by supposing this fact that the whole line couplings still remain relevant in the presence of a boundary, the bootstrap provides relations between the various reflection factors. Algebraically, the reflection bootstrap equation is

$$K_c(\theta_c) = K_a(\theta_a)S_{ab}(\theta_b + \theta_a)K_b(\theta_b), \tag{1.110}$$

in which

$$\theta_a = \theta_c - i\bar{U}^b_{ac} \ , \qquad \theta_b = \theta_c + i\bar{U}^a_{bc}. \tag{1.111}$$

Moreover, $\bar{U} = \pi - U$ and the coupling angles are the angles of the triangle whose side-lengths are equal to the masses of particles a, b and c. The reflection bootstrap equation may be shown pictorially as

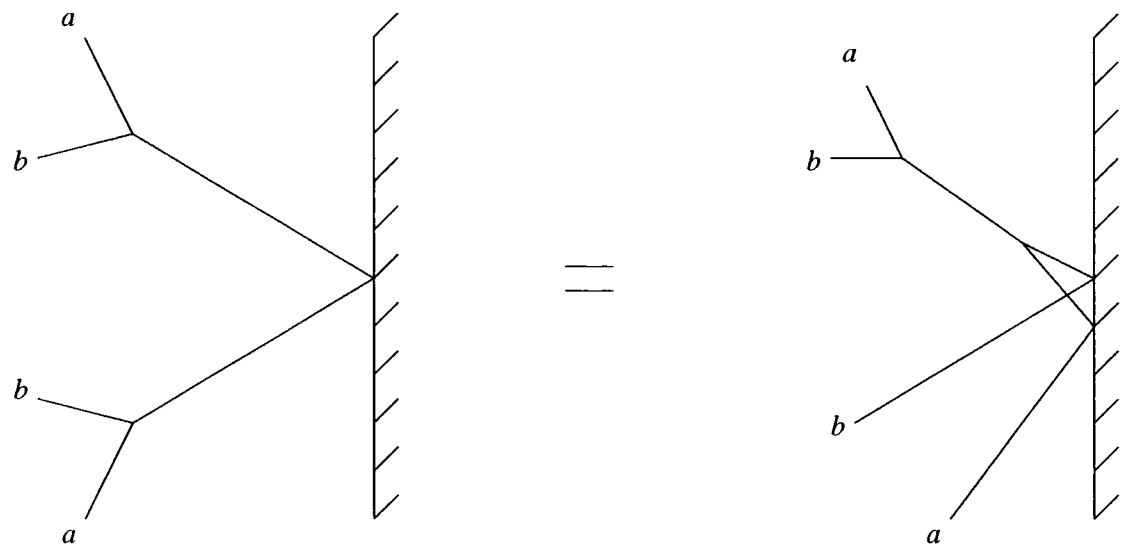

Figure 1.10: The boundary bootstrap equation.

Looking at the above figure one may think of either the two particles a, b individually reflecting from the boundary before (after) the bound state forms, or the particle c reflects from the boundary. There is also the possibility of bound states involving a particle and the boundary, with their own coupling angles and bootstrap property (see [36, 39, 55]).

Moreover, there are the crossing relations

$$S_{ab}(i\pi - \theta) = S_{a\bar{b}}(\theta) = S_{\bar{a}b}(\theta), \tag{1.112}$$

in which $\theta = \theta_a - \theta_b$ and

$$K_a(\theta - i\pi/2)K_{\bar{a}}(\theta + i\pi/2)S_{\bar{a}a}(2\theta) = 1. \tag{1.113}$$

The unitarity relations are

$$S_{ab}(\theta) = S_{ab}^{-1}(-\theta) \tag{1.114}$$

and

$$K_a(\theta) = K_a^{-1}(-\theta). \tag{1.115}$$

Although for the reflection bootstrap equation (1.110), there are many known solutions for some theories such as affine Toda field theory [54] however, their relation with the different choices of boundary condition has not been found clearly. For this

reason applying a semi-classical approximation or perturbation theory would be a substantial option. For example Kim has done some work on this basis [56–58] only for the Neumann boundary condition. Meanwhile quantum versions of the conserved quantities have been studied by Penati and Zanon [43, 49] whose calculations lead to the renormalization of the boundary parameters.

Chapter 2

First order quantum corrections to the classical reflection factor of the sinh-Gordon model

2.1 Introduction

Over the last few years after a series of papers written by Ghoshal and Zamolodchikov [39, 41], and others [52–54], much work has been done to study integrable quantum field theory with a boundary. In particular, the affine Toda field theories have offered a rich algebraic structure and remarkable properties. The classical affine Toda field theories remain integrable in the presence of certain boundary conditions restricting them to a half-line, or to an interval [36, 37, 40, 42, 43, 45–48, 59]. Indeed, Corrigan et.al [36, 37, 40] have investigated thoroughly the boundary conditions arising from boundary potentials of a particular form which preserves classical integrability. Then, Delius [60] found new boundary conditions. However, the corresponding quantum field theories on the half-line have not been studied completely. In fact, there still remains much to be studied in relation to quantum integrability on the half-line. For the models based on $a_n^{(1)}$ much is now known [36, 61–63]. The simplest affine Toda field theory, the sinh-Gordon model has been studied much more than other models in the context of integrable boundaries. This model is the only theory in the *ade* series of affine Toda field theories for which continuous boundary parameters are possible. In contrast, for most of the Toda theories corresponding

to the affine simply-laced algebras, the boundary conditions are strictly limited to a finite number in order to preserve integrability. Integrable boundary conditions in the sinh-Gordon model depend on a pair of parameters which are called boundary parameters. But, even in this case, it remains to be seen precisely how the two boundary parameters influence quantities of interest such as the reflection factors.

Firstly, Ghoshal and Zamolodchikov [39] obtained the soliton reflection factors in the sine-Gordon model with a boundary consistent with integrability. Then, Ghoshal [41] using these results calculated the reflection factors of the soliton-anti-soliton bound states (the breathers) of the model. However, apart from two special cases (Neumann and Dirichlet boundary conditions) Ghoshal and Zamolodchikov's formulae fail to provide a relationship between the reflection factors and the boundary parameters themselves. One of the interesting and difficult problems in the boundary sine(sinh)-Gordon model is to find the relation between the free parameters appearing in Ghoshal's formula and the boundary data appearing in the Lagrangian formulation of the model. Corrigan [64] was the first to notice that the lightest breather reflection factor of the sine-Gordon model is identical to the reflection factor of the sinh-Gordon model after an analytic continuation in the coupling constant.

In a recent paper Corrigan and Delius [65] studied the boundary breather states of the sinh-Gordon model on a half-line. They noticed that for certain ranges of the boundary parameters in the sinh-Gordon model there are real periodic classical finite-energy solutions called boundary breathers. The sinh-Gordon model has no such constant solutions on the whole line. They calculated the energy spectrum of the boundary states in two ways, by using the bootstrap equations then by using a WKB approximation. By comparing the results obtained by the two methods, they provided strong evidence for a conjectured relationship between the boundary parameters, the bulk coupling constant and the parameters appearing in the quantum reflection factor calculated by Ghoshal. They carried out the calculations in the special case when the boundary parameters are equal and the boundary condition preserve the $\phi \to -\phi$ symmetry of the bulk theory.

In [64] the quantum corrections up to $O(\beta^2)$ to the classical reflection factor of the sinh-Gordon model were found when the boundary parameters are equal. In this

case, the static background configuration is $\phi = 0$. If the boundary data is different and the $\phi \to -\phi$ symmetry is broken then, the lowest energy solution will not be a trivial background. The corresponding perturbation theory involves complicated coupling constants and two-point Green function as well. Nevertheless, as will be shown in this chapter, provided calculations are restricted to first order in the difference of the two boundary parameters, some of the complications disappear and we are able to calculate the correction to the reflection factor at one loop order. The result provide a further verification of Ghoshal's formula. In this chapter sinh-Gordon model is studied under boundary conditions which are compatible with integrability and in the framework of the conventional perturbation theory generalised to the affine Toda field theory.

2.2 sinh-Gordon model

The sinh-Gordon theory corresponds to the affine Toda field theory whose associated untwisted affine Kac-Moody algebra is $a_1^{(1)}$. The physical difference between sinh-Gordon theory and sine-Gordon theory is the fact that in the former the coupling constant is real but in the latter the coupling constant is imaginary. In what follows we deal with sinh-Gordon model whose bulk Lagrangian density is defined as:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{2.1}$$

where

$$V(\phi) = \frac{m^2}{\beta^2}\left(e^{-\beta\alpha\phi} + e^{\beta\alpha\phi}\right). \tag{2.2}$$

The real constants m and β provide a mass scale and a coupling constant respectively. For this model the affine root α_0 is equal to $-\alpha$ where α is the simple root of $SU(2)$ Lie algebra. Meanwhile by considering the normalization condition $\alpha^2 = 2$, which is customary in the affine Toda field theory, we find $\mathcal{L}$ takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{\beta^2}\left(e^{\sqrt{2}\beta\phi} + e^{-\sqrt{2}\beta\phi}\right). \tag{2.3}$$

Hence, the equation of motion of the theory becomes

$$\partial_\mu\partial^\mu\phi + \sqrt{2}\frac{m^2}{\beta}\left(e^{\sqrt{2}\beta\phi} - e^{-\sqrt{2}\beta\phi}\right) = 0. \tag{2.4}$$

The sinh-Gordon model is integrable classically which means there are infinitely many independent conserved quantities. On the other hand, the model is integrable after quantizing which implies the S-matrix describing the n-particles scattering factorises into a product of two-particles scattering amplitudes. The S-matrix describing the elastic scattering of two sinh-Gordon particles of relative rapidity θ is conjectured to have the form [1,15,66]

$$S(\theta) = -\frac{1}{(B)(2-B)}, \tag{2.5}$$

where the hyperbolic building blocks have been used

$$(x) = \frac{\sinh(\theta/2 + \frac{i\pi x}{4})}{\sinh(\theta/2 - \frac{i\pi x}{4})}, \tag{2.6}$$

and the quantity B is related to the coupling constant β by $B = \frac{2\beta^2}{4\pi+\beta^2}$. It is evident that the S-matrix is invariant under the following transformation

$$\beta \to 4\pi/\beta \tag{2.7}$$

and this property is known as the weak-strong coupling duality.

On the other hand, the sinh-Gordon theory on the half-line [36,37] is described by the following Lagrangian density

$$\bar{\mathcal{L}} = \theta(-x)\mathcal{L} - \delta(x)\mathcal{B}. \tag{2.8}$$

Here, $\mathcal{B}$ is a functional of the field but it does not depend on its derivative and the generic form of $\mathcal{B}$ or the boundary term is given by

$$\mathcal{B} = \frac{m}{\beta^2}\left(\sigma_0 e^{-\frac{\beta}{\sqrt{2}}\phi} + \sigma_1 e^{\frac{\beta}{\sqrt{2}}\phi}\right). \tag{2.9}$$

In the above relation, the two real coefficients σ_0 and σ_1 are arbitrary and are called [37,47] the boundary parameters. In fact, Bowcock et.al [40] obtained some results about the form of the boundary term via a generalised Lax pair when there is a boundary.

Now we show that the boundary potential satisfies the boundary condition

$$\frac{\partial\phi}{\partial x} = -\frac{\partial\mathcal{B}}{\partial\phi}. \tag{2.10}$$

To show this, the starting point is the action of the model which may be written down as

$$S = \int_{-\infty}^{+\infty} dt \left(\int_{-\infty}^{0} dx\mathcal{L} - \mathcal{B}\right) \tag{2.11}$$

and under an arbitrary change in the field, $\delta\phi$, the variation of S becomes

$$\delta S = \int_{-\infty}^{+\infty} dt \left\{ \int_{-\infty}^{0} dx \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) \right) - \frac{\partial \mathcal{B}}{\partial \phi}\delta\phi \right\} \tag{2.12}$$

or by using the equation of motion

$$\delta S = \int_{-\infty}^{+\infty} dt \left(\int_{-\infty}^{0} dx \partial_\mu(\partial^\mu \phi \delta\phi) - \frac{\partial \mathcal{B}}{\partial \phi}\delta\phi \right). \tag{2.13}$$

Finally, if Stoke's theorem is applied then the relation (2.10) will be obtained. Therefore, for the boundary sinh-Gordon model the equation of motion and the boundary condition after rescaling the mass become, respectively :

$$\partial^2\phi = -\frac{\sqrt{2}}{\beta}\left(e^{\sqrt{2}\beta\phi} - e^{-\sqrt{2}\beta\phi}\right) \qquad \text{when} \quad x < 0, \tag{2.14}$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sqrt{2}}{\beta}\left(\sigma_1 e^{\beta\phi/\sqrt{2}} - \sigma_0 e^{-\beta\phi/\sqrt{2}}\right) \qquad \text{at} \quad x = 0. \tag{2.15}$$

The constraints on the boundary parameters i.e. σ_0 and σ_1 have been discussed by Corrigan et.al [37] and by Fujii and Sasaki [47]. Meanwhile as a result of preserving integrability on the half-line, the boundary condition is desired to have the provided form.

For the boundary sinh-Gordon model, besides the two-particle S-matrix it is necessary to know the boundary S-matrix or reflection factor describing one particle reflection off the boundary.

2.3 Reflection Factor

In this section the reflection factors for the sinh-Gordon model on the half-line associated with boundary conditions which are consistent with the integrability, are discussed.

For the first time Cherednik [51] studied from an algebraic point of view the exact boundary reflection matrices corresponding to the b_n, c_n, d_n root systems by introducing the boundary Yang-Baxter equation. In fact, he generalised the factorisation property of the S-matrix to the case where a reflecting boundary is present. Then, Ghoshal and Zamolodchikov [39] did more studies on the subject in relation to the sine-Gordon theory via establishing the boundary crossing symmetry and boundary unitarity conditions. Afterwards, Fring and Koberle [52, 53] and

Sasaki [54] carried out further research on the topic in connection with the affine Toda field theory by considering the boundary bootstrap equation.

Assuming there is a boundary at $x = 0$ then, when the sinh-Gordon particle approaches to the boundary it may elastically reflect from that so, in according to the arguments of section six in chapter 1, we may write down the following relation

$$|A_a(\theta) >_{in} = K_a(\theta)|A_a(-\theta) >_{out} . \tag{2.16}$$

In other words, the equation (2.16) might be regarded as a definition relation for the reflection factor i.e. $K_a(\theta)$. In fact, Ghoshal and Zamolodchikov [39] calculated the soliton reflection factors for the sine-Gordon model by solving the boundary Yang-Baxter equation, and using general constraints implementing unitarity and a form of crossing symmetry. Then, Ghoshal [41] calculated the reflection factors of the soliton-anti-soliton bound states (the breathers). He used the boundary bootstrap equations along with the result of reference [39]. At the same time, the reflection factors of the sinh-Gordon model can be checked by means of perturbation theory and therefore, Ghoshal's formula may be checked perturbatively. Ghoshal's formula [41] for the bound state boundary scattering amplitude $R_B^{(n)}(\theta)$ can be written as (with $u = -i\theta, \lambda = 8\pi/\beta^2 - 1$)

$$R_B^{(n)}(u) = R_0^{(n)} R_1^{(n)}(u), \tag{2.17}$$

where

$$\begin{aligned} R_0^{(n)}(u) &= (-1)^{n+1} \frac{\cos(\frac{u}{2} + \frac{n\pi}{4\lambda}) \cos(\frac{u}{2} - \frac{\pi}{4} - \frac{n\pi}{4\lambda}) \sin(\frac{u}{2} + \frac{\pi}{4})}{\cos(\frac{u}{2} - \frac{n\pi}{4\lambda}) \cos(\frac{u}{2} + \frac{\pi}{4} + \frac{n\pi}{4\lambda}) \sin(\frac{u}{2} - \frac{\pi}{4})} \\ &\times \prod_{l=1}^{n-1} \frac{\sin(u + \frac{l\pi}{2\lambda}) \cos^2(\frac{u}{2} - \frac{\pi}{4} - \frac{l\pi}{4\lambda})}{\sin(u - \frac{l\pi}{2\lambda}) \cos^2(\frac{u}{2} + \frac{\pi}{4} + \frac{l\pi}{4\lambda})}, \\ &n = 1, 2, ... < \lambda. \end{aligned} \tag{2.18}$$

Ghoshal found that $R_1^{(n)}(u)$, which contains the boundary parameters η and ϑ, is different depending on whether n is even or odd:

$$R_1^{(2n)}(u) = S^{(2n)}(\eta, u) S^{(2n)}(i\vartheta, u), \tag{2.19}$$

where

$$\begin{aligned} S^{(2n)}(x, u) &= \prod_{l=1}^{n} \frac{\sin(u) - \cos(\frac{x}{\lambda} - (l - \frac{1}{2})\frac{\pi}{\lambda}) \sin(u) - \cos(\frac{x}{\lambda} + (l - \frac{1}{2})\frac{\pi}{\lambda})}{\sin(u) + \cos(\frac{x}{\lambda} - (l - \frac{1}{2})\frac{\pi}{\lambda}) \sin(u) + \cos(\frac{x}{\lambda} + (l - \frac{1}{2})\frac{\pi}{\lambda})}, \\ &n = 1, 2, ... < \frac{\lambda}{2} \end{aligned} \tag{2.20}$$

and

$$R_1^{(2n-1)}(u) = S^{(2n-1)}(\eta, u) S^{(2n-1)}(i\vartheta, u) \tag{2.21}$$

with

$$\begin{aligned} S^{(2n-1)}(x,u) &= \frac{\cos(\frac{x}{\lambda}) - \sin(u)}{\cos(\frac{x}{\lambda}) + \sin(u)} \\ &\times \prod_{l=1}^{n-1} \frac{\sin(u) - \cos(\frac{x}{\lambda} - \frac{l\pi}{\lambda})}{\sin(u) + \cos(\frac{x}{\lambda} - \frac{l\pi}{\lambda})} \frac{\sin(u) - \cos(\frac{x}{\lambda} + \frac{l\pi}{\lambda})}{\sin(u) + \cos(\frac{x}{\lambda} + \frac{l\pi}{\lambda})}, \\ & n = 1, 2, ... < \frac{\lambda + 1}{2}. \end{aligned} \tag{2.22}$$

Now looking at the Ghoshal's formula, considering the lightest breather and doing the required analytic continuation $\beta \to \sqrt{2}\, i\, \beta$ to obtain

$$R_B^{(1)}(\theta) = R_0^{(1)}(\theta) R_1^{(1)}(\theta), \tag{2.23}$$

where

$$R_0^{(1)}(\theta) = -\frac{\sin(-\frac{i\theta}{2} + \frac{\pi}{4}) \cos(\frac{i\theta}{2} + \frac{\pi B}{8}) \cos(\frac{i\theta}{2} + \frac{\pi}{4} + \frac{\pi B}{8})}{\sin(-\frac{i\theta}{2} - \frac{\pi}{4}) \cos(\frac{i\theta}{2} - \frac{\pi B}{8}) \cos(\frac{i\theta}{2} - \frac{\pi}{4} - \frac{\pi B}{8})} \tag{2.24}$$

and

$$R_1^{(1)}(\theta) = \frac{\cos(\frac{\eta B}{2}) - \sin(-\frac{i\theta}{2})}{\cos(\frac{\eta B}{2}) + \sin(-\frac{i\theta}{2})} \frac{\cos(\frac{-i\vartheta B}{2}) - \sin(-\frac{i\theta}{2})}{\cos(\frac{-i\vartheta B}{2}) + \sin(-\frac{i\theta}{2})}. \tag{2.25}$$

In Ghoshal's notation $E = \frac{B\eta}{\pi}$ and $F = \frac{iB\vartheta}{\pi}$. The lightest breather $R_B^{(1)}(\theta)$, which from now on is called the quantum reflection factor $K_q(\theta)$, is given in terms of the hyperbolic building blocks (2.6) by:

$$K_q(\theta) = \frac{(1)(2 - B/2)(1 + B/2)}{(1 - E(\sigma_0, \sigma_1, \beta))(1 + E(\sigma_0, \sigma_1, \beta))(1 - F(\sigma_0, \sigma_1, \beta))(1 + F(\sigma_0, \sigma_1, \beta))}. \tag{2.26}$$

For the Neumann boundary condition which is defined as

$$\frac{\partial \phi}{\partial x} = 0 \qquad \text{when} \qquad x = 0, \tag{2.27}$$

Ghoshal's formula reduces to

$$K_N(\theta) = \frac{(1 + B/2)}{(B/2)(1)} \tag{2.28}$$

because the Neumann condition (2.27) demands the following restrictions on the functions E and F

$$F = 0, \qquad E = 1 - B/2. \tag{2.29}$$

In fact, Kim [56, 57, 67, 68] in an attempt to generalize the idea of conventional perturbation theory [70–72] for affine Toda field theory compatible with the Neumann boundary condition, confirmed (2.29) perturbatively and verified this up to $O(\beta^2)$. Kim [56, 57, 67, 68] has obtained one loop amplitudes of the boundary reflection matrix corresponding to the Neumann boundary condition by means of a perturbative study of the exact boundary reflection matrices for simply-laced and some non simply-laced affine Toda field theories under the assumption of weak-strong coupling duality. He used two-point propagators in coordinate space instead of momentum space.

However, the exact form of the E and F functions in the general case other than Neumann boundary condition is a hard problem. Corrigan and Delius [65] investigated the boundary breather states of the sinh-Gordon model in the presence of a boundary. They calculated the energy spectrum of the states in two ways. First, by using the bootstrap equations, and then by finding a set of periodic finite-energy solutions which could be quantized by means of a WKB approximation. The marriage of the two methods yields strong evidence for a relationship between the quantum reflection factor and the boundary data. For technical reasons they found expressions for E and F in the special case where $\sigma_0 = \sigma_1 = \cos a\pi$ and a is restricted to the range $1/2 < a < 1$ as

$$E = 2a\,(1 - B/2)\,, \quad F = 0. \tag{2.30}$$

Note in the limit $a \to 1/2$, the above relation reduces to (2.29).

Regarding the important role of the propagator in this chapter and the calculations in the next chapters, it is necessary to derive the two-point Green function for the sinh-Gordon model. Corrigan found [64] this propagator. We shall review his derivation below where the classical reflection factor of the model will be one of the emerging results. Studying the perturbation expansion near the static background solution to the equation of the model is the starting point.

First of all, it is necessary to find the lowest energy static solution for the sinh-Gordon model with a given boundary condition. Using the equation of motion (2.14) and the boundary condition (2.15), the static background solution has to satisfy

$$\frac{d^2\phi_0}{dx^2} = \frac{\sqrt{2}}{\beta}\left(e^{\sqrt{2}\beta\phi_0} - e^{-\sqrt{2}\beta\phi_0}\right) \qquad \text{when} \quad x < 0, \tag{2.31}$$

$$\frac{d\phi_0}{dx} = -\frac{\sqrt{2}}{\beta}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} - \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right) \qquad \text{at} \quad x = 0. \tag{2.32}$$

Integrating the first equation then comparing the result with the boundary equation yields:

$$\frac{d\phi_0}{dx} = \frac{\sqrt{2}}{\beta}\left(e^{\beta\phi_0/\sqrt{2}} - e^{-\beta\phi_0/\sqrt{2}}\right) \qquad \text{when} \quad x < 0, \tag{2.33}$$

$$e^{\sqrt{2}\beta\phi_0} = \frac{1+\sigma_0}{1+\sigma_1} \qquad \text{at} \quad x = 0. \tag{2.34}$$

Therefore, the static solution has the form

$$e^{\beta\phi_0/\sqrt{2}} = \frac{1+e^{2(x-x_0)}}{1-e^{2(x-x_0)}}. \tag{2.35}$$

Meanwhile the boundary condition imposes a relation on the parameter x_0

$$\coth x_0 = \sqrt{\frac{1+\sigma_0}{1+\sigma_1}}. \tag{2.36}$$

Note that if $\sigma_0 \geq \sigma_1$ then $x_0 \geq 0$. Otherwise, it is necessary to adjust the solution (2.35) by shifting x_0 through $i\pi/2$ in order to guarantee that $x_0 \geq 0$. The singularity in the equation (2.35) is unimportant as long as x_0 is positive. So from now on we assume $\sigma_0 \geq \sigma_1$.

For the other models of $\hat{a}_n$, Bowcock [48] found classical solutions which satisfy integrable boundary conditions using solitons which are analytically continued from imaginary coupling theories.

After determining the static background solution, the next step is to linearize the field equation and the boundary condition in this background. So, linear perturbation near the static background

$$\phi = \phi_0 + \phi_1,$$

yields:

$$\partial^2\phi_1 + 2\phi_1\left(e^{\sqrt{2}\beta\phi_0} + e^{-\sqrt{2}\beta\phi_0}\right) = 0 \qquad \text{when} \quad x < 0, \tag{2.37}$$

$$\frac{\partial\phi_1}{\partial x} + \phi_1\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right) = 0 \qquad \text{at} \quad x = 0. \tag{2.38}$$

Now by means of substituting the relation (2.35) in the above equations and after some manipulation, one obtains the following equations :

$$\partial^2\phi_1 + 4\left(1 + \frac{2}{\sinh^2 2(x-x_0)}\right)\phi_1 = 0 \qquad \text{when} \quad x < 0, \tag{2.39}$$

$$\frac{\partial\phi_1}{\partial x} + (\sigma_1 \coth x_0 + \sigma_0 \tanh x_0)\,\phi_1 = 0 \qquad \text{at} \quad x = 0, \tag{2.40}$$

in which ϕ_1 denotes the first order correction to ϕ.

In fact, the equations (2.39) and (2.40) have been solved [37] precisely. If the eigenfunctions of the second order differential operator in (2.39) are denoted by $\phi_{k,\omega}$ which corresponds to the eigenvalue of $\omega^2 - k^2 - 4$, then the eigenfunction is given by

$$\phi_{k,\omega} = ie^{-i\omega t} r(k) \left(F(k,x)e^{ikx} - F(-k,x)e^{-ikx} \right), \tag{2.41}$$

in which r(k) is an even, real function and $F(k,x)$ is equal to

$$F(k,x) = P(k) \left(ik - 2\coth 2(x - x_0) \right), \tag{2.42}$$

where

$$P(k) = (ik)^2 - 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1). \tag{2.43}$$

It is necessary to mention that $r(k)$ can be specified via normalization of the propagator. Meanwhile after some calculation, the two-point Green function associated with the eigenfunction (2.41) can be derived

$$\begin{aligned} G(x,t;x',t') \;=\; i\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{dk}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - k^2 - 4 + i\rho} \Big(f(k,x)f(-k,x')e^{ik(x-x')} \\ + K_c f(-k,x)f(-k,x')e^{-ik(x+x')} \Big), \end{aligned} \tag{2.44}$$

in which

$$f(k,x) = \frac{ik - 2\coth 2(x - x_0)}{ik + 2}. \tag{2.45}$$

The classical reflection factor of the model is given by

$$K_c = \frac{(ik)^2 + 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0+\sigma_1)}{(ik)^2 - 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0+\sigma_1)} \frac{ik-2}{ik+2} \tag{2.46}$$

or, if the momentum of the particle is expressed in terms of its rapidity i.e. $k = 2\sinh\theta$, then

$$K_c = -\frac{(1)^2}{(1-a_0-a_1)(1+a_0+a_1)(1-a_0+a_1)(1+a_0-a_1)} \tag{2.47}$$

where ( ) denotes the hyperbolic building blocks and

$$\sigma_0 = \cos a_0\pi, \qquad\qquad \sigma_1 = \cos a_1\pi.$$

The above classical reflection factor, which has been calculated by Corrigan et.al [37] through solving the linearized wave equations (2.39) and (2.40) around the static background solution, is similar to the quantum reflection factor suggested by

Ghoshal [41]. In other words, the classical reflection factor (2.47) can be derived from the quantum reflection factor (2.26) by considering the classical limit i.e. when $\beta \to 0$. In this limit [37] $E = a_0 + a_1$ and $F = a_0 - a_1$. Meanwhile Classical reflection matrices of the other models of Affine Toda field theory corresponding to different choices of boundary conditions compatible with integrability have been found [36, 37, 45, 48].

So, from the above arguments the classical reflection factor, in general, may be defined as the coefficient of the reflected term of the free field two-point propagator calculated within the classical static background solution.

On the other hand, after calculating perturbatively the exact two-point Green function, the quantum reflection factor can be defined as the coefficient of the reflected term of the two-point correlation function i.e. $e^{-ik(x+x')}$ in the residue of the on-shell pole in the asymptotic region far away from the boundary, i.e. $x, x' \to -\infty$.

2.4 First order quantum corrections of the reflection factor

Corrigan calculated [64] the $O(\beta^2)$ correction to the classical reflection factor of the sinh-Gordon model. At the same time, he checked Ghoshal's formula for the lightest breather perturbatively in the special case when the boundary parameters σ_0 and σ_1 are equal.

Now in this chapter, we calculate the $O(\beta^2)$ quantum corrections to the reflection factor of the model corresponding to those boundary conditions which are compatible with integrability. However, in this case the boundary parameters are not equal and we shall assume that their difference $\epsilon = \sigma_0 - \sigma_1$ is small. Therefore, the problem is solved in the first order of ϵ. In order to find the quantum corrections of the reflection factor for the sinh-Gordon model from the boundary and the bulk potential, the model is considered in low order perturbation theory. In general for a model of affine Toda field theory the perturbative calculations are performed around the static background field configuration. So, the problem reduces to the standard Feynman diagrams.

Perturbation theory

First of all we need to expand the bulk potential and the boundary term and by this way three point and four point couplings may be provided and so, various types of Feynman diagrams will be clearly determined. The free parameters σ_0 and σ_1 appearing in the boundary potential and the bulk coupling constant β can be considered as expansion parameters. Looking at the Lagrangian density of the sinh-Gordon model on the half-line, it is evident that the bulk potential is given by

$$V(\phi) = \frac{2}{\beta^2}\cosh(\sqrt{2}\beta\phi). \tag{2.48}$$

Now by the expansion of the bulk potential around the background solution to the equation of motion, we can derive the three point and four point couplings corresponding to the bulk potential as :

$$C^{(3)}_{bulk} = \frac{1}{3!}V'''(\phi_0) = \frac{2\sqrt{2}}{3}\sinh(\sqrt{2}\beta\phi_0) \tag{2.49}$$

and

$$C^{(4)}_{bulk} = \frac{1}{4!}V^{(IV)}(\phi_0) = \frac{1}{3}\beta^2\cosh(\sqrt{2}\beta\phi_0). \tag{2.50}$$

On the other hand, the static background solution satisfies the relation (2.35). So, for example, the four point coupling constant of the bulk theory converts to

$$C^{(4)}_{bulk} = \frac{1}{6}\beta^2\left(\left(\frac{1+e^{2(x-x_0)}}{1-e^{2(x-x_0)}}\right)^2 + \left(\frac{1-e^{2(x-x_0)}}{1+e^{2(x-x_0)}}\right)^2\right) \tag{2.51}$$

or after some simplification

$$C^{(4)}_{bulk} = \frac{1}{3}\beta^2\left(2\coth^2 2(x-x_0) - 1\right). \tag{2.52}$$

Similarly the three point coupling constant of the bulk theory is simplified as

$$C^{(3)}_{bulk} = \frac{4\sqrt{2}}{3}\cosh 2(x-x_0)\left(\coth^2 2(x-x_0) - 1\right). \tag{2.53}$$

In a similar way we can derive the three point and four point couplings associated with the boundary term which are given by

$$C^{(3)}_{boundary} = \frac{1}{3!}B'''(\phi_0) = \frac{\sqrt{2}\beta}{12}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} - \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right), \tag{2.54}$$

$$C^{(4)}_{boundary} = \frac{1}{4!}B^{(IV)}(\phi_0) = \frac{\beta^2}{48}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right). \tag{2.55}$$

From now on we assume that the boundary parameters have only a small difference i.e. $\sigma_0 - \sigma_1 = \epsilon$. So, it is necessary to expand all the formulas and to keep all terms of them up to first order in ϵ. Let us first start with the static background solution which has an expansion at the boundary

$$e^{\beta\phi_0/\sqrt{2}} = \coth x_0 = 1 + \frac{\epsilon}{2(1+\sigma_1)} + O(\epsilon^2) \qquad \text{when} \qquad x = 0. \tag{2.56}$$

We then find the boundary three point and four point couplings as

$$C^{(3)}_{boundary} = \frac{\sqrt{2}\beta}{12}\left(-\frac{\epsilon}{1+\sigma_1}\right) + O(\epsilon^2), \tag{2.57}$$

$$C^{(4)}_{boundary} = \frac{\beta^2}{48}(2\sigma_1 + \epsilon) + O(\epsilon^2). \tag{2.58}$$

The bulk three point and four point couplings may be derived as

$$C^{(3)}_{bulk} = \frac{2\sqrt{2}}{3}\beta\frac{\epsilon}{1+\sigma_1}e^{2x} + O(\epsilon^2), \tag{2.59}$$

$$C^{(4)}_{bulk} = \frac{1}{3}\beta^2 + O(\epsilon^2). \tag{2.60}$$

In connection with the expansion of the related functions up to first order in ϵ, all that remains is to find the expansion of the function $f(k,x)$ i.e. the relation (2.45) and the classical reflection factor K (2.46) as well, both of which are involved in the two-point Green function (2.44) . In fact,

$$f(k,x) = 1 + O(\epsilon^2) \tag{2.61}$$

as a result of the following expansion

$$\coth 2(x - x_0) = -1 + O(\epsilon^2). \tag{2.62}$$

Moreover, the classical reflection factor (2.46) converts to

$$K_c = \frac{ik + 2\sigma_1 + \epsilon}{ik - 2\sigma_1 - \epsilon} + O(\epsilon^2) \tag{2.63}$$

or

$$K_c = \frac{ik + 2\sigma}{ik - 2\sigma} + \frac{2ik}{(ik - 2\sigma)^2}\epsilon + O(\epsilon^2). \tag{2.64}$$

It is convenient for future objectives to denote K as

$$K_c = K_0 + \epsilon K_1 \tag{2.65}$$

here, K_0 is the classical reflection factor when the boundary parameters σ_0 and σ_1 are equal so that the calculation of reference [64] is based on this special case. We may call K_1 the first order correction to classical reflection factor when $\sigma_0 \neq \sigma_1$.

To calculate quantum corrections to the classical reflection factor at one loop order (i.e. $O(\beta^2)$) we use perturbative methods generalised to the affine Toda field theory on a half-line [56,57,64,69]. (For earlier references on boundary perturbation theory in general see [70–72], for affine Toda perturbation theory see [22] or the review [14].) The $O(\beta^2)$ correction to K_0 has been calculated before and the purpose of this chapter is to calculate the corrections to K_1 to the same order. In general, at one loop order there are three basic types of Feynman diagrams contributing to the two-point Green function [57,58]. These diagrams can be shown as

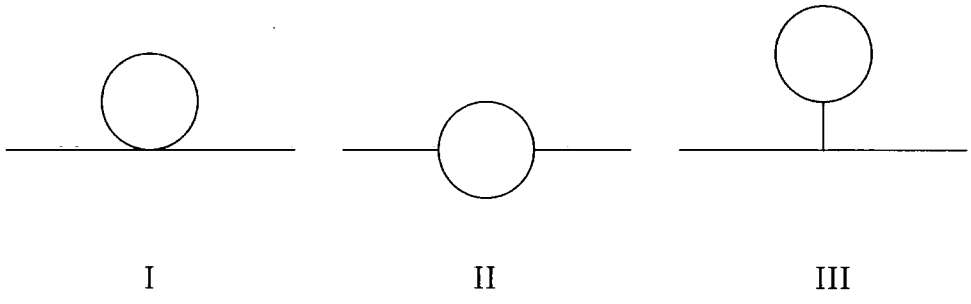

Figure 2.1: Three basic Feynman diagrams in one loop order.

However, due to doing the calculations up to first order in ϵ throughout this chapter, only the type I Feynman diagram is involved in the following computations. The reason for the above claim may be realised by looking at the three point and four point couplings. There are two contributions for the reflection factor. The first one is related to the boundary, when the interaction vertex lies on the boundary, and it can be written down as

$$-\frac{i\beta^2}{4}(2\sigma_1+\epsilon)\int_{-\infty}^{+\infty} dt'' G(x,t;0,t'')G(0,t'';0,t'')G(0,t'';x',t'), \tag{2.66}$$

in which the combinatorial factor of the related Feynman diagram are included.

The second contribution corresponds to the bulk potential which means the interaction vertex lying in the bulk region $x<0$ and is given by

$$-4i\beta^2\int_{-\infty}^{+\infty} dt''\int_{-\infty}^{0} dx'' G(x,t;x'',t'')G(x'',t'';x'',t'')G(x'',t'';x',t'), \tag{2.67}$$

where, once more the combinatorial factor has been inserted in this formula.

On the other hand, in according to the previous discussions, in our problem the two-point Green function is equal to

$$G(x,t;x',t') = i\int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-k^2-4+i\rho}\left(e^{ik(x-x')}+K_c e^{-ik(x+x')}\right), \tag{2.68}$$

where

$$K_c = K_0 + K_1 = \frac{ik+2\sigma_1}{ik-2\sigma_1} + \frac{2ik\epsilon}{(ik-2\sigma_1)^2}. \tag{2.69}$$

Let us first calculate the boundary contribution i.e. the expression (2.66) and it is convenient to start with deriving the middle propagator which may be written as

$$G(0,t'';0,t'') = i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''^2-k''^2-4+i\rho}\left(1+\frac{ik''+2\sigma_1}{ik''-2\sigma_1}+\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}\right). \tag{2.70}$$

Note the above integral is clearly divergent but the divergence is removed by the infinite renormalization of the boundary term. In other words, considering the following relation

$$1+\frac{ik''+2\sigma_1}{ik''-2\sigma_1} = 2+\frac{4\sigma}{ik''-2\sigma_1} \tag{2.71}$$

it is seen that a minimal subtraction of the divergent portion can be made by adding an appropriate counter term to the boundary, converting the integral to a finite one. Meanwhile that part of the integral which corresponds to the zeroth order of the classical reflection factor has been solved in reference [64] and therefore

$$G(0,t'';0,t'') = -a_1\frac{\cos a_1\pi}{\sin a_1\pi} + i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''^2-k''^2-4+i\rho}\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}. \tag{2.72}$$

Looking at the above integral, focusing on the energy variable and choosing the integration contour in the upper half-plane, we encounter a simple pole at $\sqrt{k''^2+4}$ and therefore, we are led to solve the following integral

$$\frac{1}{2}\int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}. \tag{2.73}$$

Clearly, the k'' integration can be performed by closing the contour into the upper half-plane however, because of the branch cut which runs from $+2i$ to infinity along the imaginary axis, actually the contour has to encircle the $+2i$ point. Note, if $\sigma_1 > 0$ there is no pole inside the contour. If $\sigma_1 < 0$ then the effect of taking σ_1 negative will be to pick up a term corresponding to the discrete boundary bound state that exists for σ_1 negative [66]. In what follows we assume that $\sigma_1 > 0$ and

integrals along the branch cut are remained to be evaluated. Hence, the integral (2.73) converts to

$$\int_2^{+\infty} \frac{dy}{2\pi} \frac{1}{\sqrt{y^2-4}} \frac{2y\epsilon}{(y+2\sigma_1)^2} \tag{2.74}$$

or after a change of variables

$$-\frac{\epsilon}{2\pi} \int_0^{\infty} du \frac{\cosh u}{(\cosh u + \sigma_1)^2} \tag{2.75}$$

and finally doing another change of variables i.e. $e^u = v$ and some manipulation we obtain (if $\sigma_1 > 0$)

$$i \int\int \frac{d\omega''}{2\pi} \frac{dk''}{2\pi} \frac{1}{\omega''^2 - k''^2 - 4 + i\rho} \frac{2ik''\epsilon}{(ik'' - 2\sigma_1)^2} = -\frac{\epsilon}{2} \frac{a_1}{\sin^3 a_1\pi} + \frac{\epsilon \cos a_1\pi}{2\pi} \frac{1}{\sin^2 a_1\pi}. \tag{2.76}$$

So until now, the boundary contribution takes the following form (Note in the beginning term of the first propagator, the transformation $k \to -k$ is needed)

$$\begin{aligned} &-i\frac{\beta^2}{4}(2\sigma_1+\epsilon)\left(-\frac{\epsilon}{2}\frac{a_1}{\sin^3 a_1\pi} + \frac{\epsilon\cos a_1\pi}{2\pi}\frac{1}{\sin^2 a_1\pi}\right) \\ &\times i\int dt'' \int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{e^{-i\omega(t-t'')}}{\omega^2-k^2-4+i\rho} e^{-ikx}\left(\frac{2ik}{ik-2\sigma_1} + \frac{2ik\epsilon}{(ik-2\sigma_1)^2}\right) \\ &\times i\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{e^{-i\omega'(t''-t')}}{\omega'^2-k'^2-4+i\rho} e^{-ik'x'}\left(\frac{2ik'}{ik'-2\sigma_1} + \frac{2ik'\epsilon}{(ik'-2\sigma_1)^2}\right). \end{aligned} \tag{2.77}$$

The integration over t'' ensures energy conservation at the interaction vertex and creates a Dirac delta function which immediately removes one of the energy variables, for example ω'. All that remains is to integrate over the momenta k and k' which can be performed by completing the contours in the upper half-plane and taking into account the poles at $\hat{k} = k = k' = \sqrt{\omega^2-4}$. However, if $\sigma_1 > 0$ it is evident that the expressions for K_0 and K_1 have no pole inside the contour. If $\sigma_1 < 0$ there is an additional pole but its contribution turns out to be exponentially decreasing in the asymptotic region $x, x' \to -\infty$.

Finally, we obtain the boundary contribution (2.66) in the form (if $\sigma_1 > 0$)

$$\begin{aligned} -i\frac{\beta^2}{4}\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \Bigg\{ &\frac{2a_1\cos^2 a_1\pi}{\sin a_1\pi}\frac{1}{(i\hat{k}-2\sigma_1)^2} \\ &+\left(\frac{a_1\cos a_1\pi}{\sin a_1\pi} + \frac{a_1\cos a_1\pi}{\sin^3 a_1\pi} - \frac{\cos^2 a_1\pi}{\pi\sin^2 a_1\pi}\right)\frac{\epsilon}{(i\hat{k}-2\sigma_1)^2} \\ &+\frac{4a_1\cos^2 a_1\pi}{\sin a_1\pi}\frac{\epsilon}{(i\hat{k}-2\sigma_1)^3}\Bigg\}, \end{aligned} \tag{2.78}$$

where $\hat{k} = 2\sinh\theta$.

Let us bring our attention to the bulk potential contribution (2.67), which by means of the preceding discussions, can be written in the expanded form:

$$-4i\beta^2\int dt''\int_{-\infty}^{0}dx''\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{i}{\omega^2-k^2-4+i\rho}e^{-i\omega(t-t'')}$$
$$\left(e^{-ik(x-x'')}+\frac{ik+2\sigma_1}{ik-2\sigma_1}e^{-ik(x+x'')}+\frac{2ik\epsilon}{(ik-2\sigma_1)^2}e^{-ik(x+x'')}\right)$$
$$\int\int\frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{i}{\omega''^2-k''^2-4+i\rho}\left(1+\frac{ik''+2\sigma_1}{ik''-2\sigma_1}e^{-2ik''x''}+\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}e^{-2ik''x''}\right)$$
$$\int\int\frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{i}{\omega'^2-k'^2-4+i\rho}e^{-i\omega'(t''-t')}$$
$$\left(e^{ik'(x''-x')}+\frac{ik'+2\sigma_1}{ik'-2\sigma_1}e^{-ik'(x''+x')}+\frac{2ik'\epsilon}{(ik'-2\sigma_1)^2}e^{-ik'(x''+x')}\right) \quad (2.79)$$

The integral over t'' yields a Dirac delta function which allows us to substitute ω' by ω . Furthermore, to calculate the integration over x'', it is convenient to use the following device

$$\int_{-\infty}^{0}dx''e^{ikx''+\tau x''}=\frac{-i}{k-i\tau}, \quad (2.80)$$

where the small positive quantity τ will be taken to zero at the final stage of the calculations.

The loop integral which corresponds to the middle propagator of (2.79), is obviously logarithmically divergent. Nevertheless, this divergence can be removed by the infinite renormalization of the mass parameter in the bulk potential. So, after making the minimal subtraction and integrating over x'' and ω'' we obtain (Note as we mentioned earlier, our job is to solve those parts of the contribution which are proportional to ϵ)

$$-\frac{i}{2}\int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}e^{-\omega(t-t')}e^{-i(kx+k'x')}\frac{i}{\omega^2-k^2-4+i\rho}\frac{i}{\omega^2-k'^2-4+i\rho}$$
$$\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\left\{\frac{2ik\epsilon}{(ik-2\sigma_1)^2}\frac{ik''+2\sigma_1}{ik''-2\sigma_1}\left(\frac{1}{-k+k'-2k''-i\tau}\right.\right.$$
$$\left.-\frac{1}{k+k'+2k''+i\tau}\frac{ik'+2\sigma_1}{ik'-2\sigma_1}\right)$$
$$+\frac{2ik'\epsilon}{(ik'-2\sigma_1)^2}\frac{ik''+2\sigma_1}{ik''-2\sigma_1}\left(\frac{1}{-k'+k-2k''-i\tau}-\frac{1}{k'+k+2k''+i\tau}\frac{ik+2\sigma_1}{ik-2\sigma_1}\right)$$
$$+\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}\left(\frac{1}{k+k'-2k''-i\tau}+\frac{1}{k-k'-2k''-i\tau}\frac{ik'+2\sigma_1}{ik'-2\sigma_1}\right.$$

$$-\frac{1}{k-k'+2k''+i\tau}\frac{ik+2\sigma_1}{ik-2\sigma_1}-\frac{1}{k+k'+2k''+i\tau}\frac{ik+2\sigma_1}{ik-2\sigma_1}\frac{ik'+2\sigma_1}{ik'-2\sigma_1}\Bigg)\Bigg\}. \quad (2.81)$$

In order to evaluate the integral over k'', we encounter the following two types of integrals

$$\int\frac{1}{\sqrt{k''^2+4}}\left(\frac{ik''+2\sigma_1}{ik''-2\sigma_1}\right)\frac{1}{(k+k'-2k''-i\tau)} \quad (2.82)$$

and

$$\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}\frac{1}{(k+k'-2k''-i\tau)}. \quad (2.83)$$

Both of them may be performed by an appropriate contour in the upper half-plane and ensuring that it runs around the branch cut located from $k''=2i$ to infinity along the imaginary axis. Note if $\sigma_1>0$ then there is no pole inside the contour however, if $\sigma_1>0$ there is an extra pole but its residue integrated over k and k' will give vanishing contribution in the limit $x,x'\to-\infty$. So, the integral (2.82) has the solution:

$$\begin{aligned}
&\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\left(\frac{ik''+2\sigma_1}{ik''-2\sigma_1}\right)\frac{1}{(k+k'-2k''-i\tau)}\\
=\;&\frac{4i\sigma_1}{(ik+ik'-4\sigma_1)}\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{1}{(ik''-2\sigma_1)}\\
&+\left(\frac{ik+ik'+4\sigma_1}{ik+ik'-4\sigma_1}\right)\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{1}{(k+k'-2k''-i\tau)}\\
=\;&-\frac{2ia_1\sigma_1}{\sin a_1\pi(ik+ik'-4\sigma_1)}\\
+\;&\frac{1}{2\pi}\left(\frac{ik+ik'+4\sigma_1}{ik+ik'-4\sigma_1}\right)\frac{1}{\sqrt{\frac{(k+k')^2}{4}+4}}\ln\left\{\frac{1+\frac{i(k+k')}{4}+\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4}}{1+\frac{i(k+k')}{4}-\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4}}\right\}
\end{aligned} \quad (2.84)$$

The integral (2.83) can be shown to be equal to

$$2\int_2^\infty\frac{idy}{2\pi}\frac{1}{\sqrt{4-y^2}}\frac{2y\epsilon}{(y+2\sigma_1)^2}\frac{1}{(k+k'-2iy-i\tau)} \quad (2.85)$$

and after changing the variable $y=2\cosh u$ then, again another change like $e^u=v$, we obtain the following result

$$\begin{aligned}
\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{2ik''\epsilon}{(ik''-2\sigma_1)^2}\frac{1}{(k+k'-2k''-i\tau)}&=\frac{\epsilon}{\pi}\frac{k+k'}{(ik+ik'-4\sigma_1)^2}\frac{a_1\pi}{\sin a_1\pi}\\
-\frac{\epsilon}{\pi}\frac{i\sigma_1}{(4\sigma_1-ik-ik')}\frac{1}{\sin^2 a_1\pi}&+\frac{\epsilon}{\pi}\frac{i\sigma_1^2}{(4\sigma_1-ik-ik')}\frac{a_1\pi}{\sin^3 a_1\pi}\\
+\frac{\epsilon}{\pi}\frac{2i(k+k')}{(ik+ik'-4\sigma_1)^2}\frac{1}{\sqrt{\frac{(k+k')^2}{4}+4}}&\ln\left\{\frac{1+\frac{i(k+k')}{4}+\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4}}{1+\frac{i(k+k')}{4}-\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4}}\right\}.
\end{aligned} \quad (2.86)$$

Let us divide the bulk contribution (2.81) in two parts, that is, one part involving the integrals (2.82) and the other part containing the integrals (2.83), we call the former B_1 and the latter B_2 . For both parts it is necessary, after performing the integration over ω'', to do the k and k' integrals via closing the contours in the upper half-plane to pick up poles $\hat{k} = k = k' = \sqrt{\omega^2 - 4}$. All other pole contributions lead to exponentially damped terms in the limit $x, x' \to -\infty$.

After some manipulation, B_1 is found to be equal to

$$B_1 = -2\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \frac{1}{(2\hat{k})^2} \frac{2i\hat{k}\epsilon}{(i\hat{k}-2\sigma_1)^2} \left\{ -\frac{i}{4} + \frac{ia_1}{\sin a_1\pi} \frac{i\hat{k}}{i\hat{k}-2\sigma_1} + \frac{1}{\pi} \frac{1}{\sqrt{\hat{k}^2+4}} \left(\frac{i\pi}{2} - \theta\right) \right\}. \quad (2.87)$$

Notice that the last term inside the braces of B_1 depends on θ and therefore, is very inconvenient for Ghoshal's formula. Fortunately, this term will be cancelled by a counterpart term in B_2. Note throughout the calculation process we used the fact that $\hat{k} = 2\sinh\theta$ in which θ is the rapidity of the particle.

After somewhat lengthier calculations, B_2 is given by

$$\begin{aligned} B_2 = -2\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \frac{1}{(2\hat{k})^2} \Bigg\{ & \frac{-2i\epsilon}{(i\hat{k}-2\sigma_1)^2} \frac{\sigma_1^2}{\pi \sin^2 a_1\pi} \\ & + \frac{2i\epsilon}{(i\hat{k}-2\sigma_1)^2} \frac{a_1\sigma_1^3}{\sin^3 a_1\pi} + \frac{2i\epsilon\hat{k}}{(i\hat{k}-2\sigma_1)^2} \frac{1}{\pi\sqrt{\hat{k}^2+4}}\theta \\ & + \frac{i\hat{k}+2\sigma_1}{i\hat{k}-2\sigma_1} \left(-\frac{i\epsilon}{2\pi \sin^2 a_1\pi} + \frac{i\epsilon a_1\sigma_1}{2\sin^3 a_1\pi} \right) \Bigg\}. \end{aligned} \quad (2.88)$$

First of all, as we mentioned before, in B_2 the term which depends explicitly on the rapidity of the particle is eliminated by the corresponding term in B_1. Secondly, we may still do further simplification to obtain the following result (after the removal of the θ term)

$$B_2 = -2\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \frac{i\epsilon}{8(i\hat{k}-2\sigma_1)^2} \left(\frac{1}{\pi \sin^2 a_1\pi} - \frac{a_1\sigma_1}{\sin^3 a_1\pi} \right). \quad (2.89)$$

Now if we add the boundary (2.78) and the bulk ((2.87) and (2.88)) contributions together, we obtain

$$\begin{aligned} -i\frac{\beta^2\epsilon}{2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \Bigg\{ & \frac{1}{(i\hat{k}-2\sigma_1)^2} \left(\frac{1}{2\pi} + \frac{a_1\sigma_1}{2\sin a_1\pi} \right) \\ & + \frac{1}{(i\hat{k}-2\sigma_1)^3} (-2a_1 \sin a_1\pi) + \frac{1}{(i\hat{k}-2\sigma_1)^2} \frac{2}{\hat{k}} \left(-\frac{i}{4} + \frac{i}{2\sqrt{\hat{k}^2+4}} \right) \Bigg\} \end{aligned} \quad (2.90)$$

from which we can deduce the correction to the quantity K_1 in (2.65). Explicitly, we have,

$$\begin{aligned}\delta K_1 &= -i\beta^2\epsilon\hat{k}\Big\{\frac{1}{(i\hat{k}-2\sigma_1)^2}\Big(\frac{1}{2\pi}+\frac{a_1\sigma_1}{2\sin a_1\pi}\Big)+\frac{1}{(i\hat{k}-2\sigma_1)^3}(-2a_1\sin a_1\pi)\\ &\quad+\frac{1}{(i\hat{k}-2\sigma_1)^2}\frac{2}{\hat{k}}\Big(-\frac{i}{4}+\frac{i}{2\sqrt{\hat{k}^2+4}}\Big)\Big\}. \qquad (2.91)\end{aligned}$$

The correction to K_0 which was calculated before in [64] is,

$$\begin{aligned}\delta K_0 &= -\frac{i\beta^2}{8}K_0(\hat{k})\sinh\theta\Big\{\Big(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\Big)\\ &\quad+2a_1\Big(\frac{1}{\cosh\theta-\sin a_1\pi}-\frac{1}{\cosh\theta+\sin a_1\pi}\Big)\Big\}. \qquad (2.92)\end{aligned}$$

This completes the collection of ingredients we need.

2.5 Comparison with Ghoshal's formula

In this section, the corrections to the classical reflection factor calculated above will be compared with the formula of Ghoshal quoted in (2.26).

Using (2.65), the relative correction to the classical reflection factor K_c is given in terms of the corrections δK_0 and δK_1 by

$$\frac{\delta K_c}{K_c} = K_0^{-1}\delta K_0+\epsilon\left(K_0^{-1}\delta K_1-K_1K_0^{-2}\delta K_0\right). \qquad (2.93)$$

Hence, using (2.91) and (2.92) we have,

$$\begin{aligned}\frac{\delta K_c}{K_c} &= -\frac{i\beta^2}{8}\sinh\theta\Big\{\Big(\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}\Big)\\ &\quad+2a_1\Big(\frac{1}{\cosh\theta-\sin a_1\pi}-\frac{1}{\cosh\theta+\sin a_1\pi}\Big)\Big\}\\ &\quad+\frac{i\beta^2\epsilon\sinh\theta}{8\sin a_1\pi}\Big\{\frac{1}{\pi}\Big(\frac{1}{\cosh\theta-\sin a_1\pi}-\frac{1}{\cosh\theta+\sin a_1\pi}\Big)\\ &\quad+a_1\cos a_1\pi\Big(\frac{1}{(\cosh\theta-\sin a_1\pi)^2}+\frac{1}{(\cosh\theta+\sin a_1\pi)^2}\Big)\Big\}. \qquad (2.94)\end{aligned}$$

On the other hand, Ghoshal's formula (2.26) for the reflection factor up to one loop order is given by:

$$K_q(\theta)\sim K_c(\theta)\left(1-\frac{i\beta^2}{8}\sinh\theta\,\mathcal{F}(\theta)\right), \qquad (2.95)$$

where

$$\begin{aligned}\mathcal{F}(\theta) &= \frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta} \\ &+\frac{e_1}{\cosh\theta+\sin(e_0\pi/2)}-\frac{e_1}{\cosh\theta-\sin(e_0\pi/2)} \\ &+\frac{f_1}{\cosh\theta+\sin(f_0\pi/2)}-\frac{f_1}{\cosh\theta-\sin(f_0\pi/2)}.\end{aligned} \tag{2.96}$$

In calculating (2.96) we have made use of the expansions of E and F to $O(\beta^2)$:

$$E\sim e_0+e_1\frac{\beta^2}{4\pi} \qquad F\sim f_0+f_1\frac{\beta^2}{4\pi}, \tag{2.97}$$

with

$$e_0=a_0+a_1 \qquad \text{and} \qquad f_0=a_0-a_1. \tag{2.98}$$

Since $K_q=K_c+\delta K_c$, we deduce that

$$\frac{\delta K_c}{K_c}=-\frac{i\beta^2}{8}\sinh\theta\ \mathcal{F}(\theta). \tag{2.99}$$

Hence, expanding to $O(\epsilon)$, we find,

$$\begin{aligned}\mathcal{F}(\theta) &= \left\{\frac{1}{\cosh\theta+1}-\frac{1}{\cosh\theta}+\frac{e_1}{\cosh\theta+\sin a_1\pi}-\frac{e_1}{\cosh\theta-\sin a_1\pi}\right. \\ &+\frac{e_1\epsilon\cos a_1\pi}{2\sin a_1\pi}\left(\frac{1}{(\cosh\theta+\sin a_1\pi)^2}+\frac{1}{(\cosh\theta-\sin a_1\pi)^2}\right) \\ &\left.+\frac{\epsilon f_1}{\sin a_1\pi\cosh^2\theta}\right\}.\end{aligned} \tag{2.100}$$

Comparing (2.94) with (2.100) we see a pleasing similarity. In fact the two formulae are identical, to $O(\epsilon)$, provided we choose e_1 and f_1 suitably. In other words, we may deduce that

$$e_1=-2a_1+\frac{\epsilon}{\pi\sin a_1\pi}\equiv-(a_0+a_1)+O(\epsilon^2) \tag{2.101}$$

and that f_1 is proportional to ϵ. Unfortunately, the calculation does not allow anything more detailed to be learned about f_1. To do better needs a correction to the reflection factor to $O(\epsilon^2)$.

2.6 Discussion

In this chapter we tested a little more deeply the expression for the sinh-Gordon particle reflection factor, that is, Ghoshal's formula and we learned additional information in connection with its dependence on the boundary parameters σ_0 and

σ_1. The result of our calculations is satisfying because it agrees with alternative derivations of the reflection factor and it also agrees with the following conjecture. Everything we have learned so far is consistent with quite simple expressions for E and F:

$$E = (a_0 + a_1)(1 - B/2) \qquad F = (a_0 - a_1)(1 - B/2), \tag{2.102}$$

where the coupling constant dependence comes into the formulae by means of the expression for B. We will discuss the conjecture (2.102) much more in the final chapter.

In order to learn more about the quantum reflection factor's parameters, for example e_1 and especially f_1 via perturbation calculations, second order quantum corrections to the classical reflection factor of the sinh-Gordon model should be carried out. In fact, we deal with this problem in the next chapter.

Chapter 3

Second order quantum corrections to the classical reflection factor of the sinh-Gordon model

3.1 Introduction

In chapter 2 we found the quantum corrections to the classical reflection factor of the sinh-Gordon model at one loop order i.e. $O(\beta^2)$ and up to the first order in the difference of the boundary parameters. The calculations provided a further verification of Ghoshal's formula. Meanwhile we derived the relation between parameters of the quantum reflection factor and the boundary parameters up to the first order difference in σ_0 and σ_1. In order to know the relation up to the higher order, it is necessary to carry out the second order quantum corrections to the classical reflection factor of the model. So, in this chapter it is intended to follow the calculations up to the second order in ϵ in which $\epsilon = \sigma_0 - \sigma_1$. Hence, it is necessary to expand the bulk and the boundary couplings. Let us first of all expand the static background solution (2.35) at the boundary up to the second order in ϵ:

$$e^{\beta\phi_0/\sqrt{2}} = \coth x_0 = 1 + \frac{\epsilon}{2(1+\sigma_1)} - \frac{1}{8}\frac{\epsilon^2}{(1+\sigma_1)^2} + ... \tag{3.1}$$

Therefore, by looking at the boundary three point (2.54) and four point (2.55) couplings, we derive the following formulae

$$C^{(3)}_{boundary} = \frac{\sqrt{2}\beta}{12}\left(-\frac{\epsilon}{1+\sigma_1}\right) + ... \tag{3.2}$$

and

$$C^{(4)}_{boundary} = \frac{\beta^2}{48}\left(2\sigma_1 + \epsilon - \frac{\sigma_1 + 2}{4(1+\sigma_1)^2}\epsilon^2\right) + ... \tag{3.3}$$

The reason for expanding the boundary three point coupling up to the first order in ϵ rather than up to the second order is the fact that in type II and III Feynman diagrams corresponding to one loop order, there are two interaction vertices. By regarding the bulk three point (2.53) and four point (2.52) couplings, the expansions of them can be obtained as

$$C^{(3)}_{bulk} = \frac{2\sqrt{2}}{3}\beta\frac{\epsilon}{1+\sigma_1}e^{2x} + ... \tag{3.4}$$

and

$$C^{(4)}_{bulk} = \frac{1}{3}\beta^2\left(1 + \frac{1}{2}\frac{\epsilon^2}{(1+\sigma_1)^2}e^{4x} + ...\right). \tag{3.5}$$

Now let us find the expansions of the function $f(k,x)$ and the classical reflection factor K_c up to the second order, both of them appear in the two-point Green function (2.44) associated with the sinh-Gordon model. In order to do the first job, it is sufficient to use the following expansion

$$\coth 2(x - x_0) = -1 - \frac{1}{8}\frac{\epsilon^2}{(1+\sigma_1)^2}e^{4x} + ..., \tag{3.6}$$

so

$$f(k,x) = 1 + \frac{1}{4}\frac{\epsilon^2}{(1+\sigma_1)^2}\frac{1}{ik+2}e^{4x} + ... \tag{3.7}$$

In addition, after some manipulation the classical reflection factor (2.46) expands as

$$K = \frac{ik + 2\sigma_1 + \epsilon - \frac{ik\epsilon^2}{4(1+\sigma_1)(ik+2)}}{ik - 2\sigma_1 - \epsilon + \frac{ik\epsilon^2}{4(1+\sigma_1)(ik-2)}} \tag{3.8}$$

or

$$\begin{aligned} K = {} & \frac{ik + 2\sigma_1}{ik - 2\sigma_1} + \frac{2ik}{(ik - 2\sigma_1)^2}\epsilon \\ & + \frac{1}{2}\frac{ik(ik^3 - 4k^2 - 6k^2\sigma_1 - 4ik\sigma_1 + 8\sigma_1^2 - 16 - 16\sigma_1)}{(1+\sigma_1)(ik-2)(ik+2)(ik-2\sigma_1)^3}\epsilon^2 \end{aligned} \tag{3.9}$$

or in a compact form

$$K = K_0 + K_1\epsilon + K_2\epsilon^2 + ... \tag{3.10}$$

Here, K_0 is the classical reflection factor when the boundary parameters σ_0 and σ_1 are equal, K_1 and K_2 are the first and second order correction to the classical reflection factor when $\sigma_0 \neq \sigma_1$, respectively.

As mentioned in the previous chapter there are three basic types of Feynman diagrams [52,53] which contribute to the two-point Green function at one loop order (see figure 1 in chapter 2). However, only the type I diagram is involved in the calculations up to the first order in ϵ . On the other hand, all types of the Feynman diagrams must be considered up to the second order in ϵ. In fact, each interaction vertex can either be located in the bulk region or at the boundary so, there are ten contributions to the classical reflection factor of the theory. It is better to begin with the type III diagram.

3.2 Type III Feynman diagram

Type III (boundary-boundary)

It is clear that the type III Feynman diagram includes four distinct cases depending on the fact that the vertices to be settled in the bulk region or at the boundary. This section deals with the calculations corresponding to the contribution of these four diagram to the reflection factor. The simplest one is type III (boundary-boundary), that is, when both the vertices are located at the boundary. The associated contribution may be given by

$$-\frac{\beta^2}{4}\frac{\epsilon^2}{(1+\sigma_1)^2}\int\int dtdt' G(x_1,t_1;0,t)G(0,t;0,t')G(0.t';0,t')G(0,t;x_2,t_2) \tag{3.11}$$

here, the related three point couplings and the combinatorial factor have been considered. It is evident that in this case the two-point Green function has the simplest form as

$$G(x,t;x',t') = i\int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-k^2-4}\left(e^{ik(x-x')}+K_0(k)e^{-ik(x+x')}\right), \tag{3.12}$$

where

$$K_0(k) = \frac{ik+2\sigma_1}{ik-2\sigma_1}. \tag{3.13}$$

In this case as we saw in chapter 2

$$G(0,t';0,t') = -a_1\frac{\cos a_1\pi}{\sin a_1\pi}. \tag{3.14}$$

We need to find an expression for the integral:

$$\int dt' G(0,t;0,t') \tag{3.15}$$

which is equal to

$$i\int dt'\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{e^{-i\omega''(t-t')}}{\omega''^2-k''^2-4}\left(1+\frac{ik''+2\sigma_1}{ik''-2\sigma_1}\right) \tag{3.16}$$

or after doing integration over t', the above relation reduces to

$$i\int \frac{dk''}{2\pi}\left(\frac{1}{-k''^2-4}\right)\left(\frac{2ik''}{ik''-2\sigma_1}\right). \tag{3.17}$$

Hence,

$$\int dt' G(0,t;0,t') = -\frac{i}{2(1+\sigma_1)}. \tag{3.18}$$

Therefore, up to now the contribution (3.11) has the form

$$-\frac{i\beta^2}{8}\frac{a_1 cos a_1\pi}{\sin a_1\pi}\frac{\epsilon^2}{(1+\cos a_1\pi)^3} i\int dt\int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{e^{-i\omega(t_1-t)}}{\omega^2-k^2-4}\left(e^{ikx_1}+K_0(k)e^{-ikx_1}\right)$$
$$\times i\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{e^{-i\omega'(t-t_2)}}{\omega'^2-k'^2-4}\left(e^{-ik'x_2}+K_0(k')e^{-ik'x_2}\right). \tag{3.19}$$

First of all, it is understood to do the change $k\to -k$ in the first term of the first propagator. Secondly, after integration over t, the result will be a Dirac delta function which immediately gives rise to the substitution $\omega=\omega'$. Finally, the momenta of the two propagator can be integrated out by taking the contours to be closed in the upper half-plane and regarding the pole at $\hat{k}=k=k'=\sqrt{\omega^2-4}$ and ignoring the other pole i.e. $-2i\sigma_1$ (when $\sigma_1<0$) due to the fact that its residue will vanish in the limit $x_1,x_2\to-\infty$. Thus the type III (boundary-boundary) contribution becomes

$$\frac{i\beta^2}{8}\frac{a_1\cos a_1\pi}{\sin a_1\pi}\frac{\epsilon^2}{(1+\cos a_1\pi)^3}\int\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}e^{-i\hat{k}(x_1+x_2)}\frac{1}{(i\hat{k}-2\sigma_1)^2}. \tag{3.20}$$

Type III (boundary-bulk)

Now let us focus our attention on the type III (boundary-bulk) when the vertex corresponding to the loop is placed inside the bulk region and the other vertex at the boundary. The contribution is

$$\frac{2\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2\int_{-\infty}^{0}dx'\int\int dtdt' G(x_1,t_1;0,t)G(0,t;x',t')$$
$$G(x',t';x',t')G(0,t;x_2,t_2)e^{2x'}. \tag{3.21}$$

As before,

$$\int dt G(x_1,t_1;0,t)G(0,t;x_2,t_2) = -\int\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}e^{-i\hat{k}(x_1+x_2)}\frac{1}{(i\hat{k}-2\sigma_1)^2}. \tag{3.22}$$

Meanwhile, it is convenient to compute one of the middle propagator as

$$\int dt' G(0,t;x',t') = i \int dt' \int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\left(\frac{e^{-i\omega'(t-t')}}{\omega'^2 - k'^2 - 4}\right)\left(\frac{2ik'}{ik' - 2\sigma_1}\right) e^{-ik'x'} \quad (3.23)$$

or after some calculations (3.23) reduces to

$$i\int \frac{dk'}{2\pi}\left(\frac{e^{-ik'x'}}{-k'^2 - 4}\right)\left(\frac{2ik'}{ik' - 2\sigma_1}\right). \quad (3.24)$$

So, we obtain

$$\int dt' G(0,t;x',t') = -\frac{i}{2(1+\sigma_1)} e^{2x'}. \quad (3.25)$$

Now let us calculate the loop propagator which has the form

$$G(x',t';x',t') = i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''^2 - k''^2 - 4}\left(1 + K_0(k'')e^{-2ik''x'}\right). \quad (3.26)$$

Clearly, the above divergent integral after a minimal subtraction, changes to a finite one and if we integrate over ω'' then, we will find

$$G(x',t';x',t') = \frac{1}{2}\int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{ik'' + 2\sigma_1}{ik'' - 2\sigma_1} e^{-2ik''x'}. \quad (3.27)$$

Hence, the remaining part of the relation (3.21) is simplified as (apart from the coefficients)

$$\begin{aligned}&\int\int dt'dx' G(0,t;x',t')G(x',t';x',t')e^{2x'} \\ &= -\frac{i}{4(1+\sigma_1)}\int_{-\infty}^{0} dx' \int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\frac{ik'' + 2\sigma_1}{ik'' - 2\sigma_1} e^{(4-2ik'')x'}\end{aligned} \quad (3.28)$$

or after performing the required integrations

$$\begin{aligned}&\int\int dt'dx' G(0,t;x',t')G(x',t';x',t')e^{2x'} \\ &= \frac{i}{16\pi(1+\cos a_1\pi)}\left(\frac{2\pi a_1 \cos a_1\pi}{(1-\cos a_1\pi)\sin a_1\pi} - \frac{1+\cos a_1\pi}{1-\cos a_1\pi}\right).\end{aligned} \quad (3.29)$$

Therefore, the contribution of the type III (boundary-bulk) diagram to the reflection factor is

$$\begin{aligned}&-\frac{i\beta^2\epsilon^2}{8\pi(1+\cos a_1\pi)^3}\left(\frac{2\pi a_1\cos a_1\pi}{(1-\cos a_1\pi)\sin a_1\pi} - \frac{1+\cos a_1\pi}{1-\cos a_1\pi}\right) \\ &\int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i\hat{k}(x_1+x_2)} \frac{1}{(i\hat{k} - 2\sigma_1)^2}.\end{aligned} \quad (3.30)$$

Type III (bulk-boundary)

Let us consider the type III (bulk-boundary) diagram when the loop vertex is at the boundary and the other vertex is inside the bulk region. So, the contribution this time is

$$\frac{2\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2\int_{-\infty}^{0}dx\int\int dtdt'G(x_1,t_1;x,t)G(x,t;0,t') \times G(0,t';0,t')G(x,t;x_2,t_2)e^{2x}. \quad (3.31)$$

In fact, we have obtained the two middle propagators in the previous diagram. Hence,

$$G(0,t';0,t') = -a_1\frac{\cos a_1\pi}{\sin a_1\pi} \quad (3.32)$$

and

$$\int dt'G(x,t;0,t') = -\frac{i}{2(1+\sigma_1)}e^{2x}. \quad (3.33)$$

So, the contribution (3.31) can be shown in detail as (after doing the integration over t)

$$\frac{i\beta^2\epsilon^2}{(1+\cos a_1\pi)^3}\frac{a_1 cos a_1\pi}{\sin a_1\pi} \times\int_{-\infty}^{0}dx\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t_1-t_2)}}{\omega^2-k^2-4}\left(e^{-ik(x_1-x)}+K_0(k)e^{-ik(x_1+x)}\right) \times\int\frac{dk'}{2\pi}\frac{i}{\omega^2-k'^2-4}\left(e^{ik'(x-x_2)}+K_0(k')e^{-ik'(x+x_2)}\right)e^{4x}. \quad (3.34)$$

The integration over x is simply performed and therefore, the above expression becomes

$$\frac{i\beta^2\epsilon^2}{(1+\cos a_1\pi)^3}\frac{a_1\cos a_1\pi}{\sin a_1\pi} \int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}e^{-i\omega(t_1-t_2)}e^{-i(kx_1+k'x_2)}\frac{i}{\omega^2-k^2-4}\frac{i}{\omega^2-k'^2-4} \times\left\{-\frac{i}{k+k'-4i}-\frac{iK_0(k')}{k-k'-4i}-\frac{iK_0(k)}{-k+k'-4i}-\frac{iK_0(k)K_0(k')}{-k-k'-4i}\right\}. \quad (3.35)$$

Clearly, the final job is to integrate over the momenta k, k'. This can be performed by completing the contours in the upper half-plane and picking up the pole at $\hat{k}=k=k'=\sqrt{\omega^2-4}$ and regarding the fact that all the other poles have no contributions

because their residues yield exponentially damped terms in the asymptotic region i.e. $x_1, x_2 \to -\infty$. Hence, the contribution of the type III (bulk-boundary) Feynman diagram to the reflection factor is

$$\frac{\beta^2\epsilon^2}{(1+\cos a_1\pi)^3}\frac{a_1\cos a_1\pi}{\sin a_1\pi}\int\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}e^{-i\hat{k}(x_1+x_2)}\frac{1}{(2\hat{k})^2} \times\left\{\frac{1}{2\hat{k}-4i}+\frac{i}{2}K_0(\hat{k})-\frac{1}{2\hat{k}+4i}K_0^2(\hat{k})\right\}. \tag{3.36}$$

Type III (bulk-bulk)

The last part of this section is devoted to the calculation in connection with the type III (bulk-bulk) Feynman diagram whose computations are much more lengthy than the previous ones. In fact, in this case the contribution is

$$16\frac{\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2\int_{-\infty}^{0}dx\int_{-\infty}^{0}dx'\int\int dtdt'G(x_1,t_1;x,t)G(x,t;x',t') G(x',t';x',t')G(x,t;x_2,t_2)e^{2x}e^{2x'}. \tag{3.37}$$

As was shown in the type III (boundary-bulk) case, the loop propagator can be simplified to obtain

$$G(x',t';x',t')=\frac{1}{2}\int\frac{dk_1}{2\pi}\frac{1}{\sqrt{k_1^2+4}}\left(\frac{ik_1+2\sigma_1}{ik_1-2\sigma_1}\right)e^{-2ik_1x'}. \tag{3.38}$$

Moreover, for the other middle propagator, it is appropriate to deal with

$$\int dt'G(x,t;x',t')=i\int\int\int dt'\frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{e^{-i\omega''(t-t')}}{\omega''^2-k''^2-4}\left(e^{ik''(x-x')}+K_0(k'')e^{-ik''(x+x')}\right) \tag{3.39}$$

or after doing integration over t'

$$\int dt'G(x,t;x',t')=\int\frac{dk''}{2\pi}\left(\frac{i}{-k''^2-4}\right)e^{-ik''x'}\left(e^{ik''x}+K_0(k'')e^{-ik''x}\right). \tag{3.40}$$

Let us rewrite the contribution (3.37) in the expanded form to see how we can find the order of the integrations in order to solve this contribution

$$16\frac{\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2\int_{-\infty}^{0}dx\int_{-\infty}^{0}dx'\int dt\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t_1-t)}}{\omega^2-k^2-4}\left(e^{-ik(x_1-x)} +K_0(k)e^{-ik(x_1+x)}\right)$$

$$\times \int \frac{dk''}{2\pi} \frac{i}{-k''^2-4} e^{-ik''x'} \left(e^{ik''x} + K_0(k'')e^{-ik''x}\right) e^{2x} e^{2x'}$$
$$\times \frac{1}{2} \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2+4}} \frac{ik_1+2\sigma_1}{ik_1-2\sigma_1} e^{-2ik_1x'}$$
$$\times \int\int \frac{d\omega'}{2\pi} \frac{dk'}{2\pi} \frac{ie^{-i\omega(t-t_2)}}{\omega'^2-k'^2-4} \left(e^{ik'(x-x_2)} + K_0(k')e^{-ik'(x+x_2)}\right). \tag{3.41}$$

Looking at the above relation after multiplying two (the first and the fourth) propagators then, one gets four pole pieces. If the calculations corresponding to one of them, for example the one which involves $e^{i(k+k')x}$ term, can be done then the remaining three pole pieces may be performed in the same manner, except that $k+k'$ is replaced by one of $k-k'$, $-k+k'$, $-k-k'$. So, it is sufficient to follow the discussion just for one pole piece. Now as far as the integration over x' is concerned, we encounter the following integral which is solved to obtain

$$\int_{-\infty}^{0} dx' e^{(2-i(k''+2k_1))x'} = \frac{i}{k''+2k_1+2i}. \tag{3.42}$$

On the other hand, the same story for the variable x gives rise to

$$\int_{-\infty}^{0} dx' e^{(2+i(k+k'))x} \left(e^{ik''x} + K_0(k'')e^{-ik''x}\right)$$
$$= -\frac{i}{k+k'+k''-2i} - \frac{i}{k+k'-k''-2i} K_0(k''). \tag{3.43}$$

Up to now, the expression which must be solved has the following form

$$16\frac{\beta^2\epsilon^2}{(1+\cos a_1\pi)^2} \int\int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i(kx_1+k'x_2)} \frac{i}{\omega^2-k^2-4} \frac{i}{\omega^2-k'^2-4}$$
$$\times \frac{1}{2} \int\int \frac{dk''}{2\pi}\frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2+4}} \left(\frac{ik_1+2\sigma_1}{ik_1-2\sigma_1}\right) \left(\frac{i}{k''^2+4}\right) \left(\frac{1}{-k''-2k_1-2i}\right)$$
$$\times \left(\frac{1}{k+k'+k''-2i} + \frac{1}{k+k'-k''-2i} K_0(k'')\right). \tag{3.44}$$

It is better, first of all, to integrate over k'' and then k_1 and the former job is actually the following integral

$$\int \frac{dk''}{2\pi} \frac{i}{k''^2+4} \frac{1}{-k''-2k_1-2i} \left(\frac{1}{k+k'+k''-2i} + \frac{1}{k+k'-k''-2i} K_0(k'')\right). \tag{3.45}$$

The above integral may be solved after doing partial fractions and afterwards closing the contours in the upper or lower half-plane in order to get rid of all extra poles other than $\pm 2i$. Therefore, after some manipulation

$$\int \frac{dk''}{2\pi} \frac{i}{k''^2+4} \frac{1}{-k''-2k_1-2i} \left(\frac{1}{k+k'+k''-2i} + \frac{1}{k+k'-k''-2i} K_0(k'')\right)$$

$$= \frac{1}{4i}\left\{\frac{1-\sigma_1}{1+\sigma_1}\frac{1}{2k_1+4i}\frac{1}{k+k'-4i} + \frac{1}{2k_1+4i}\frac{1}{k+k'-2k_1-4i} - \frac{1}{k+k'-4i}\frac{1}{k+k'-2k_1-4i}\right\}. \quad (3.46)$$

Now it is the time to do the integration over k_1. In order to perform this task, we have different kinds of integrals and we prefer to find only the one which is more lengthy than the others i.e.

$$\int \frac{dk_1}{\sqrt{k_1^2+4}}\frac{1}{k+k'-2k_1-4i}. \quad (3.47)$$

Let us close the contour in the upper half-plane and due to the branch cut which extends from $2i$ to infinity along the imaginary axis, the contour has to turn around the cut line so that the above integral reduces to

$$2\int_2^\infty \frac{dy}{\sqrt{y^2-4}}\frac{1}{k+k'-2iy-4i} \quad (3.48)$$

or after doing change of variable $y = 2\cosh x$

$$2\int_0^\infty \frac{dx}{k+k'-4i\cosh x-4i} \quad (3.49)$$

and another change; $e^x = u$

$$i\int_1^\infty \frac{du}{u^2+\frac{i}{2}(k+k'-4i)u+1}. \quad (3.50)$$

So, the following formula may be derived

$$\int \frac{dk_1}{\sqrt{k_1^2+4}}\frac{1}{k+k'-2k_1-4i} = \frac{1}{\sqrt{\frac{(k+k'-4i)^2}{4}+4}}\ln\left\{\frac{1+\frac{i}{4}(k+k'-4i)+\frac{i}{2}\sqrt{\frac{(k+k'-4i)^2}{4}+4}}{1+\frac{i}{4}(k+k'-4i)-\frac{i}{2}\sqrt{\frac{(k+k'-4i)^2}{4}+4}}\right\}. \quad (3.51)$$

Now we can write down the solution of the k_1 integration as

$$\begin{aligned}
&\frac{1}{8i}\int\frac{dk_1}{2\pi}\frac{1}{\sqrt{k_1^2+4}}\left(\frac{ik_1+2\sigma_1}{ik_1-2\sigma_1}\right)\left\{\frac{1-\sigma_1}{1+\sigma_1}\frac{1}{2k_1+4i}\frac{1}{k+k'-4i}\right.\\
&\qquad\left.+\frac{1}{2k_1+4i}\frac{1}{k+k'-2k_1-4i}-\frac{1}{k+k'-4i}\frac{1}{k+k'-2k_1-4i}\right\}\\
&=-\frac{k+k'-2i-2i\cos a_1\pi}{16\pi(1-\cos a_1\pi)(k+k')(k+k'-4i)}\\
&+\frac{a_1\cos a_1\pi}{16\sin^3 a_1\pi}\frac{2k+2k'+4i\cos a_1\pi-12i}{(k+k'-4i)(k+k'+4i\cos a_1\pi-4i)}\\
&-\frac{1}{4\pi(k+k')(k+k'-4i)}\frac{k+k'-4i\cos a_1\pi-4i}{k+k'+4i\cos a_1\pi-4i}\frac{1}{\sqrt{\frac{(k+k'-4i)^2}{4}+4}}\\
&\qquad\times\ln\left\{\frac{1+\frac{i}{4}(k+k'-4i)+\frac{i}{2}\sqrt{\frac{(k+k'-4i)^2}{4}+4}}{1+\frac{i}{4}(k+k'-4i)-\frac{i}{2}\sqrt{\frac{(k+k'-4i)^2}{4}+4}}\right\}. \quad (3.52)
\end{aligned}$$

Finally, regarding the type III (bulk-bulk) contribution, all that remains is to integrate over the momenta k, k'. These can be done by means of the contours in the upper half-plane and considering the pole at $\hat{k} = k = k' = \sqrt{\omega^2 - 4}$ and ignoring all the other extra poles because their residues will vanish when $x_1, x_2 \to -\infty$. Hence, the relation (3.37) has the solution

$$\begin{aligned}
&\frac{\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i\hat{k}(x_1+x_2)} \frac{1}{(2\hat{k})^2} \\
&\Bigg\{\Bigg(-\frac{\hat{k}-i-i\cos a_1\pi}{\pi(1-\cos a_1\pi)\hat{k}(2\hat{k}-4i)} + \frac{a_1\cos a_1\pi}{\sin^3 a_1\pi}\frac{\hat{k}+i\cos a_1\pi-3i}{(\hat{k}-2i)(\hat{k}+2i\cos a_1\pi-2i)} \\
&\qquad -\frac{1}{\pi\hat{k}(\hat{k}-2i)}\frac{\hat{k}-2i\cos a_1\pi-2i}{\hat{k}+2i\cos a_1\pi-2i}\frac{1}{\sqrt{(\hat{k}-2i)^2+4}} \\
&\qquad\qquad \times \ln\left\{\frac{1+\frac{i}{2}(\hat{k}-2i)+\frac{i}{2}\sqrt{(\hat{k}-2i)^2+4}}{1+\frac{i}{2}(\hat{k}-2i)-\frac{i}{2}\sqrt{(\hat{k}-2i)^2+4}}\right\}\Bigg) \\
&+K_0(\hat{k})\left(-\frac{i}{2\pi(1-\cos a_1\pi)^2} - \frac{ia_1\cos a_1\pi(\cos a_1\pi-3)}{2\sin^3 a_1\pi(1-\cos a_1\pi)} + \frac{i(1+\cos a_1\pi)}{12\pi(1-\cos a_1\pi)}\right) \\
&+K_0^2(\hat{k})\Bigg(\frac{\hat{k}+i+i\cos a_1\pi}{\pi(1-\cos a_1\pi)\hat{k}(2\hat{k}+4i)} - \frac{a_1\cos a_1\pi}{\sin^3 a_1\pi}\frac{\hat{k}-i\cos a_1\pi+3i}{(\hat{k}+2i)(\hat{k}-2i\cos a_1\pi+2i)} \\
&\qquad -\frac{1}{\pi\hat{k}(\hat{k}+2i)}\frac{\hat{k}+2i\cos a_1\pi+2i}{\hat{k}-2i\cos a_1\pi+2i}\frac{1}{\sqrt{(\hat{k}+2i)^2+4}} \\
&\qquad\qquad \times \ln\left\{\frac{1-\frac{i}{2}(\hat{k}+2i)+\frac{i}{2}\sqrt{(\hat{k}+2i)^2+4}}{1-\frac{i}{2}(\hat{k}+2i)-\frac{i}{2}\sqrt{(\hat{k}+2i)^2+4}}\right\}\Bigg)\Bigg\}. \qquad (3.53)
\end{aligned}$$

One of the interesting results which we find in this chapter is the fact that the *ln* terms in the above relation will be cancelled with the counter terms in the type I (bulk) Feynman diagram in the next section. So, the type III (bulk-bulk) contribution takes the simple form

$$\begin{aligned}
&\frac{\beta^2}{(1+\cos a_1\pi)^2}\epsilon^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i\hat{k}(x_1+x_2)} \frac{1}{(2\hat{k})^2} \\
&\Bigg\{\Bigg(-\frac{\hat{k}-i-i\cos a_1\pi}{\pi(1-\cos a_1\pi)\hat{k}(2\hat{k}-4i)} + \frac{a_1\cos a_1\pi}{\sin^3 a_1\pi}\frac{\hat{k}+i\cos a_1\pi-3i}{(\hat{k}-2i)(\hat{k}+2i\cos a_1\pi-2i)}\Bigg) \\
&+K_0(\hat{k})\left(-\frac{i}{2\pi(1-\cos a_1\pi)^2} - \frac{ia_1\cos a_1\pi(\cos a_1\pi-3)}{2\sin^3 a_1\pi(1-\cos a_1\pi)} + \frac{i(1+\cos a_1\pi)}{12\pi(1-\cos a_1\pi)}\right) \\
&+K_0^2(\hat{k})\Bigg(\frac{\hat{k}+i+i\cos a_1\pi}{\pi(1-\cos a_1\pi)\hat{k}(2\hat{k}+4i)} - \frac{a_1\cos a_1\pi}{\sin^3 a_1\pi}\frac{\hat{k}-i\cos a_1\pi+3i}{(\hat{k}+2i)(\hat{k}-2i\cos a_1\pi+2i)}\Bigg)\Bigg\}. \qquad (3.54)
\end{aligned}$$

3.3 Type I Feynman diagram

Type I (boundary)

In fact the type I diagram constitutes two cases depending on whether the interaction vertex is located at the boundary or inside the bulk region. From now on we call the former type I (boundary) and the latter, type I (bulk). Considering the boundary four point coupling (3.3) and the associated combinatorial factor as well, the contribution corresponding to the type I (boundary) diagram is described by

$$-\frac{i\beta^2}{4}\left(2\sigma_1+\epsilon-\frac{\sigma_1+2}{4(1+\sigma_1)^2}\epsilon^2\right)\int_{-\infty}^{\infty}dt''G(x,t;0,t'')G(0,t'';0,t'')G(0,t'';x',t'). \tag{3.55}$$

Let us find the appropriate form of the two-point Green function which will be used many times throughout this section. Now by looking at the general form of the propagator (2.44) and considering the expansions of the classical reflection factor (3.9) and the function $f(k,x)$ i.e. (3.7) as well, the required form of the two-point Green function will be

$$\begin{aligned}G(x,t;x',t') = i\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-k^2-4}\Big\{\Big(1+\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik+2)}e^{4x}\\ -\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x'}\Big)e^{ik(x-x')}\\ +\Big(K_0-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x}K_0-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x'}K_0\\ +\epsilon K_1+\epsilon^2K_2\Big)e^{-ik(x+x')}\Big\}.\end{aligned} \tag{3.56}$$

So, in our problem the middle propagator has the following form

$$\begin{aligned}G(0,t'';0,t'') = \int\int\frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{i}{\omega''-k''^2-4}\Big(1+K_0+\epsilon K_1+\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik''+2)}\\ -\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik''-2)}-\frac{\epsilon^2}{2(1+\sigma_1)^2}\frac{1}{(ik''-2)}K_0+\epsilon^2K_2\Big).\end{aligned} \tag{3.57}$$

First of all it is necessary to do a minimal subtraction in order to remove the divergence of the above integral. Secondly, we have solved the integral in chapter 2 up to first order in ϵ . Hence, what we need to do is to solve that part which is proportional to ϵ^2 which includes four terms. Actually three of them can be simply

manipulated and because of this reason we solve in detail the last one i.e.

$$i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''-k''^2-4}\epsilon^2 K_2(k'') \tag{3.58}$$

or

$$\frac{1}{2}\int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\epsilon^2\left(\frac{i}{4(\sigma_1^2-2\sigma_1+1)(k''+2i)}+\frac{i}{4(1+\sigma_1)^2(k''-2i)}\right.$$
$$+\frac{i\sigma_1(\sigma_1^2-2\sigma_1-1)}{(1+\sigma_1)(\sigma_1^3-\sigma_1^2-\sigma_1+1)}\frac{1}{(k''+2i\sigma_1)}-\frac{\sigma_1^2-2}{(1+\sigma_1)(\sigma_1-1)}\frac{1}{(k''+2i\sigma_1)^2}$$
$$\left.+\frac{4i\sigma_1}{(k''+2i\sigma_1)^3}\right). \tag{3.59}$$

Fortunately, we have already found these kinds of integrals in chapter 2 except the last one which apart from the coefficients is

$$\int \frac{dk''}{\sqrt{k''^2+4}}\frac{1}{(ik''-2\sigma_1)^3}. \tag{3.60}$$

The above integral can be converted to a complex one and regarding the branch cut, it reduces to

$$-2\int_2^\infty \frac{dy}{\sqrt{y^2-4}}\frac{1}{(y+2\sigma_1)^3} \tag{3.61}$$

or after the substitution $y=2\cosh x$

$$-\frac{1}{4}\int_0^\infty \frac{dx}{(\cosh x+\sigma_1)^3} \tag{3.62}$$

and another change $e^x=u$

$$-2\int_1^\infty du\frac{u^2}{(u^2+2\sigma_1 u+1)^3}. \tag{3.63}$$

The roots of the denominator are $-e^{\pm ia_1\pi}$ and so, after using partial fraction and some manipulation

$$\int \frac{dk''}{\sqrt{k''^2+4}}\frac{1}{(ik''-2\sigma_1)^3}=\frac{3\cos a_1\pi}{8\sin^4 a_1\pi}-\frac{1+2\cos^2 a_1\pi}{8\sin^5 a_1\pi}a_1\pi. \tag{3.64}$$

Now using the above formula and the required formulas in chapter 2, we may obtain the following result

$$i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''-k''^2-4}\epsilon^2 K_2(k'')$$
$$=\frac{\epsilon^2}{4\pi}\left(\frac{1+\cos a_1\pi+\cos^2 a_1\pi}{\sin^4 a_1\pi}-\frac{\cos a_1\pi(2+\cos a_1\pi)}{\sin^5 a_1\pi}a_1\pi\right). \tag{3.65}$$

Finally after doing the necessary simplifications in connection with (3.57), we derive the middle propagator as

$$G(0,t'';0,t'') = -a_1\frac{\cos a_1\pi}{\sin a_1\pi}+\epsilon\left(\frac{\cos a_1\pi}{2\pi\sin^2 a_1\pi}-\frac{a_1}{2\sin^3 a_1\pi}\right) +\epsilon^2\left(\frac{2+\cos^2 a_1\pi}{4\pi\sin^4 a_1\pi}-\frac{3\cos a_1\pi}{4\pi\sin^5 a_1\pi}a_1\pi\right). \tag{3.66}$$

Let us rewrite the boundary contribution (3.55) and first of all the t'' integration means $\omega'=\omega$ i.e. the energy variables of the first and the third propagators are equal. Secondly, the integration over the momenta of the two Green function can be done as before, just by substituting k,k' with $\hat{k}=\sqrt{\omega^2-4}$. So, the type I (boundary) contribution is

$$\begin{aligned}
&-\frac{i\beta^2}{4}\left(2\sigma_1+\epsilon-\frac{\sigma_1+2}{4(1+\sigma_1)^2}\epsilon^2\right)\\
&\left\{-a_1\frac{\cos a_1\pi}{\sin a_1\pi}+\epsilon\left(\frac{\cos a_1\pi}{2\pi\sin^2 a_1\pi}-\frac{a_1}{2\sin^3 a_1\pi}\right)\right.\\
&\qquad\left.+\epsilon^2\left(\frac{2+\cos^2 a_1\pi}{4\pi\sin^4 a_1\pi}-\frac{3\cos a_1\pi}{4\pi\sin^5 a_1\pi}a_1\pi\right)\right\}\\
&\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}e^{-i\hat{k}(x+x')}\frac{1}{(2\hat{k})^2}\\
&\left\{\frac{2i\hat{k}}{i\hat{k}-2\sigma_1}+\frac{2i\hat{k}}{(i\hat{k}-2\sigma_1)^2}\epsilon\right.\\
&\qquad\left.+\frac{i\hat{k}(-i\hat{k}^3+2\hat{k}^2-4i\hat{k}\sigma_1+6\hat{k}^2\sigma_1+16+16\sigma_1)}{2(1+\sigma_1)(i\hat{k}-2)(i\hat{k}+2)(-i\hat{k}+2\sigma_1)^3}\epsilon^2\right\}.
\end{aligned} \tag{3.67}$$

Type I (bulk)

Now let us examine the type I (bulk) Feynman diagram in which the interaction vertex is placed inside the bulk region. This time we have to take into account the bulk four point coupling (3.5), however, the related combinatorial factor is the same as the boundary case. The corresponding contribution may be formulated as

$$-4i\beta^2\int_{-\infty}^{\infty}dt''\int_{-\infty}^{0}dx''G(x,t;x'',t'')G(x'',t'';x'',t'') G(x'',t'';x',t')\left(1+\frac{\epsilon^2}{2(1+\sigma_1)^2}e^{4x''}\right). \tag{3.68}$$

By means of looking at the formula (3.56), the loop propagator for the two-point Green function is

$$G(x'',t'';x'',t'') = i\int\int \frac{d\omega''}{2\pi}\frac{dk''}{2\pi}\frac{1}{\omega''^2-k''^2-4}\Bigg\{1+\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik''+2)}e^{4x''} -\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik''-2)}e^{4x''} +\left(K_0-\frac{\epsilon^2}{2(1+\sigma_1)^2}\frac{1}{(ik''-2)}e^{4x''}K_0+\epsilon K_1+\epsilon^2K_2\right)e^{-2ik''x''}\Bigg\}. \quad (3.69)$$

As before, after performing a minimal subtraction, the divergence of the loop integral will be removed. Moreover doing integration over ω'', we then obtain

$$\begin{aligned} G(x'',t'';x'',t'') &= \frac{\epsilon^2}{8\pi(1+\sigma_1)^2}e^{4x''} \\ &\quad +\frac{1}{2}\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\Big(K_0-\frac{\epsilon^2}{2(1+\sigma_1)^2}\frac{1}{(ik''-2)}e^{4x''}K_0 \\ &\quad +\epsilon K_1+\epsilon^2K_2\Big)e^{-2ik''x''}. \quad (3.70) \end{aligned}$$

Therefore, until now, the bulk contribution (3.68) takes the form (after doing the t'' integration and setting $k\to -k$ in the first term of the first propagator)

$$\begin{aligned} &-4i\beta^2\int_{-\infty}^{0}dx''\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t-t')}}{\omega^2-k^2-4}\Bigg\{\Bigg(1+\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik+2)}e^{4x} \\ &\quad -\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x''}\Bigg)e^{-ik(x-x'')} \\ &\quad +\Big(K_0-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x}K_0(k)-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik-2)}e^{4x''}K_0 \\ &\quad +\epsilon K_1+\epsilon^2K_2\Big)e^{-ik(x+x')}\Bigg\} \\ &\Bigg\{\frac{\epsilon^2}{8\pi(1+\sigma_1)^2}e^{4x''}+\frac{1}{2}\int\frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}}\Big(K_0-\frac{\epsilon^2}{2(1+\sigma_1)^2}\frac{1}{(ik''-2)}e^{4x''}K_0 \\ &\quad +\epsilon K_1+\epsilon^2K_2\Big)e^{-2ik''x''}\Bigg\} \\ &\int\frac{dk'}{2\pi}\frac{i}{\omega^2-k'^2-4}\Bigg\{\Bigg(1+\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik'+2)}e^{4x''} \\ &\quad -\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik'-2)}e^{4x'}\Bigg)e^{ik'(x''-x')} \\ &\quad +\Big(K_0-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik'-2)}e^{4x''}K_0(k')-\frac{\epsilon^2}{4(1+\sigma_1)^2}\frac{1}{(ik'-2)}e^{4x'}K_0 \\ &\quad +\epsilon K_1+\epsilon^2K_2\Big)e^{-ik'(x''+x')}\Bigg\} \\ &\Bigg\{1+\frac{1}{2}\frac{\epsilon^2}{(1+\sigma_1)^2}e^{4x''}\Bigg\} \end{aligned} \quad (3.71)$$

or after integration over x''

$$
\begin{aligned}
&-2\beta^2\int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}e^{-i\omega(t-t')}e^{-i(kx+k'x')}\frac{i}{\omega^2-k^2-4}\frac{i}{\omega^2-k'^2-4}\\
&\int\frac{dk''}{2\pi}\frac{\epsilon^2}{\sqrt{k''^2+4}}\Bigg\{K_0(k'')\left(\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik+2}\frac{1}{k+k'-2k''-4i}\right.\\
&\qquad\left.+\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik'+2}\frac{1}{k+k'-2k''-4i}\right)\\
&+\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik+2}\frac{K_0(k')}{k-k'-2k''-4i}+\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik'+2}\frac{K_0(k)}{k'-k-2k''-4i}\\
&-\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik'-2}\frac{K_0(k')}{k-k'-2k''-4i}-\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik-2}\frac{K_0(k)}{k'-k-2k''-4i}\\
&+\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik'-2}\frac{K_0(k)K_0(k')}{k+k'+2k''+4i}+\frac{1}{4(1+\sigma_1)^2}\frac{1}{ik-2}\frac{K_0(k)K_0(k')}{k'+k+2k''+4i}\\
&+\frac{1}{k-k'-2k''}K_2(k')+\frac{1}{k'-k-2k''}K_2(k)\\
&-\frac{1}{k+k'+2k''}K_0(k)K_2(k')-\frac{1}{k'+k+2k''}K_0(k')K_2(k)\\
&\qquad\left.-\frac{1}{k+k'+2k''}K_1(k)K_1(k')\right)\\
&+K_1(k'')\left(\frac{1}{k-k'-2k''}K_1(k')-\frac{1}{k+k'+2k''}K_0(k)K_1(k')\right.\\
&\qquad\left.+\frac{1}{k'-k-2k''}K_1(k)-\frac{1}{k+k'+2k''}K_0(k')K_1(k)\right)\\
&-\frac{1}{2(1+\sigma_1)^2}\frac{K_0(k'')}{ik''-2}\left(\frac{1}{k+k'-2k''-4i}+\frac{1}{k-k'-2k''-4i}K_0(k')\right.\\
&\qquad\left.+\frac{1}{k'-k-2k''-4i}K_0(k)-\frac{1}{k+k'+2k''+4i}K_0(k)K_0(k')\right)\\
&+K_2(k'')\left(\frac{1}{k+k'-2k''}+\frac{1}{k-k'-2k''}K_0(k')\right.\\
&\qquad\left.+\frac{1}{k'-k-2k''}K_0(k)-\frac{1}{k+k'+2k''}K_0(k)K_0(k')\right)\\
&+\frac{1}{2(1+\sigma_1)^2}K_0(k'')\left(\frac{1}{k+k'-2k''-4i}+\frac{1}{k-k'-2k''-4i}K_0(k')\right.\\
&\qquad\left.+\frac{1}{k'-k-2k''-4i}K_0(k)-\frac{1}{k+k'+2k''+4i}K_0(k)K_0(k')\right)\Bigg\}. \qquad (3.72)
\end{aligned}
$$

Meanwhile, there is another term which should be considered

$$
\begin{aligned}
&-\frac{\beta^2\epsilon^2}{2\pi(1+\sigma_1)^2}\int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}e^{-i\omega(t-t')}e^{-i(kx+k'x')}\frac{i}{\omega^2-k^2-4}\frac{i}{\omega^2-k'^2-4}\\
&\qquad\left(\frac{1}{k+k'-4i}+\frac{1}{k-k'-4i}K_0(k')\right.\\
&\qquad\left.+\frac{1}{k'-k-4i}K_0(k)+\frac{1}{-k-k'-4i}K_0(k)K_0(k')\right). \qquad (3.73)
\end{aligned}
$$

After this stage the calculations, especially integration over k'', are too lengthy to perform. Note all the terms involving the following kind of integral

$$\int \frac{dk''}{\sqrt{k''^2+4}} \frac{1}{k+k'-2k''}$$

which is proportional to θ (after integrations over k and k'), will be cancelled and this is one of the interesting facts that may be found. On the other hand, there are some other terms which contain this type of integral

$$\int \frac{dk''}{\sqrt{k''^2+4}} \frac{1}{k+k'-2k''-4i}$$

and apart from one of them, all the others cancel with the counterpart terms in the type III (bulk-bulk) Feynman diagram as we mentioned in the previous section. Meanwhile further simplification can be made by using the values of k and k' given by their poles and simplifying the integrand. So the type I (bulk) Feynman diagram has the following contribution

$$
\begin{aligned}
&-\frac{i\beta^2\epsilon^2}{\pi}\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}e^{-i\hat{k}(x+x')}\frac{1}{(2\hat{k})^2}\\
&\Bigg\{-\frac{\cos a_1\pi}{(1+\cos a_1\pi)^2}\frac{a_1\pi}{\sin a_1\pi}\frac{1}{i\hat{k}+2}\frac{1}{i\hat{k}-2\cos a_1\pi+2}+\frac{1}{\sin^2 a_1\pi}\frac{1}{\hat{k}^2+4}K_0(\hat{k})\\
&-\frac{2\cos a_1\pi}{\sin^2 a_1\pi}\frac{a_1\pi}{\sin a_1\pi}\frac{1}{\hat{k}^2+4}K_0(\hat{k})-\frac{\cos a_1\pi}{(1+\cos a_1\pi)^2}\frac{a_1\pi}{\sin a_1\pi}\frac{1}{i\hat{k}-2}\frac{K_0^2(\hat{k})}{i\hat{k}+2\cos a_1\pi-2}\\
&+\left(-\frac{1}{4}+\frac{a_1\pi}{\sin a_1\pi}\frac{2i\hat{k}}{i\hat{k}-2\cos a_1\pi}+\frac{\pi}{\sqrt{\hat{k}^2+4}}\right)K_2(\hat{k})\\
&-\left(\frac{1}{2\sin^2 a_1\pi}-\frac{a_1\pi\cos a_1\pi}{2\sin^3 a_1\pi}\right)\frac{2i\hat{k}}{i\hat{k}-2\cos a_1\pi}K_1(\hat{k})+\frac{a_1\pi}{\sin a_1\pi}\frac{i\hat{k}}{i\hat{k}+2\cos a_1\pi}K_1^2(\hat{k})\\
&+\frac{a_1\pi\cos a_1\pi(2\cos a_1\pi-3)}{2(1+\cos a_1\pi)\sin^3 a_1\pi}\frac{1}{i\hat{k}-2\cos a_1\pi+2}-\frac{i}{4\sin^2 a_1\pi}\frac{1}{\hat{k}}\\
&+\left(\frac{a_1\pi\cos a_1\pi(2\cos a_1\pi-3)}{2\sin^5 a_1\pi}+\frac{7}{12\sin^2 a_1\pi}+\frac{\cos a_1\pi}{2\sin^4 a_1\pi}\right)K_0(\hat{k})\\
&+\frac{a_1\pi\cos a_1\pi(3-2\cos a_1\pi)}{2(1+\cos a_1\pi)\sin^3 a_1\pi}\frac{1}{i\hat{k}+2\cos a_1\pi-2}K_0^2(\hat{k})+\frac{i}{4\sin^2 a_1\pi}\frac{1}{\hat{k}}K_0^2(\hat{k})\\
&+\frac{i}{4}\left(\frac{a_1\pi}{\sin a_1\pi}-1\right)\frac{(1+\cos^2 a_1\pi)\hat{k}-4i\cos a_1\pi}{\sin^4 a_1\pi}\frac{1}{\hat{k}^2+4}\\
&\frac{1}{4}\left(\frac{a_1\pi}{\sin a_1\pi}-1\right)\frac{\cos a_1\pi}{\sin^4 a_1\pi}K_0(\hat{k})\\
&-\frac{i}{4}\left(\frac{a_1\pi}{\sin a_1\pi}-1\right)\frac{(1+\cos^2 a_1\pi)\hat{k}+4i\cos a_1\pi}{\sin^4 a_1\pi}\frac{1}{\hat{k}^2+4}K_0^2(\hat{k})\\
&-\frac{a_1\pi}{2\sin a_1\pi}\Big(K_2(\hat{k})+K_0(\hat{k})K_2(0)+K_0^2(\hat{k})K_2(-\hat{k})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+i\frac{\cos^2 a_1\pi - 2}{\sin^2 a_1\pi}\left(\frac{1}{2\sin^2 a_1\pi} - \frac{a_1\pi\cos a_1\pi}{2\sin^3 a_1\pi}\right)\\
&\qquad\left(\frac{1}{2(\hat{k}+2i\cos a_1\pi)} - \frac{i}{2\cos a_1\pi}K_0(\hat{k}) - \frac{1}{2(\hat{k}-2i\cos a_1\pi)}K_0^2(\hat{k})\right)\\
&-4\cos a_1\pi\left(\frac{1}{2\sin^2 a_1\pi} - \frac{a_1\pi\cos a_1\pi}{2\sin^3 a_1\pi}\right)\\
&\qquad\left(\frac{1}{2(\hat{k}+2i\cos a_1\pi)^2} - \frac{1}{4\cos^2 a_1\pi}K_0(\hat{k}) + \frac{1}{2(\hat{k}-2i\cos a_1\pi)^2}K_0^2(\hat{k})\right)\\
&-4i\cos a_1\pi\left(\frac{3\cos a_1\pi}{8\sin^4 a_1\pi} - \frac{a_1\pi(1+2\cos^2 a_1\pi)}{8\sin^5 a_1\pi}\right)\\
&\qquad\left(\frac{1}{2(\hat{k}+2i\cos a_1\pi)} - \frac{i}{2\cos a_1\pi}K_0(\hat{k}) - \frac{1}{2(\hat{k}-2i\cos a_1\pi)}K_0^2(\hat{k})\right)\\
&+\pi/2K_0(\hat{k})K_2(0)\\
&-\frac{i}{2(1+\cos a_1\pi)^2}\left(\frac{i\hat{k}+2\cos a_1\pi+2}{i\hat{k}-2\cos a_1\pi+2}\frac{1}{\sqrt{(\hat{k}-2i)^2+4}}\right.\\
&\qquad\times\ln\left(\frac{1+\frac{i}{2}(\hat{k}-2i)+\frac{i}{2}\sqrt{(\hat{k}-2i)^2+4}}{1+\frac{i}{2}(\hat{k}-2i)-\frac{i}{2}\sqrt{(\hat{k}-2i)^2+4}}\right)\\
&\qquad+K_0^2(\hat{k})\frac{-i\hat{k}+2\cos a_1\pi+2}{-i\hat{k}-2\cos a_1\pi+2}\frac{1}{\sqrt{(\hat{k}+2i)^2+4}}\\
&\qquad\left.\left.\times\ln\left(\frac{1-\frac{i}{2}(\hat{k}+2i)+\frac{i}{2}\sqrt{(\hat{k}+2i)^2+4}}{1-\frac{i}{2}(\hat{k}+2i)-\frac{i}{2}\sqrt{(\hat{k}+2i)^2+4}}\right)\right)\right\}. \qquad (3.74)
\end{aligned}
$$

3.4 Type II Feynman diagram

Type II (boundary-boundary)

This section deals with the type II diagrams and in this case we encounter three distinguishable contributions as a result of the fact that the interaction vertices can be placed at the boundary or inside the bulk region. It is instructive to start with the simplest one i.e. the type II (boundary-boundary) contribution which may be expressed in terms of the following integral

$$
-\frac{\beta^2\epsilon^2}{4(1+\sigma_1)^2}\int\int dtdt' G(x_1,t_1;0,t)G(0,t;0,t')G(0,t;0,t')G(0,t';x_2,t_2), \qquad (3.75)
$$

where, as for the type III diagram, the propagator is given by

$$
G(x_1,t_1;x_2,t_2) = i\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{e^{-i\omega_1(t_1-t_2)}}{\omega_1^2-k_1^2-4}\left(e^{ik_1(x_1-x_2)}+K_0(k_1)e^{-ik_1(x_1+x_2)}\right). \qquad (3.76)
$$

Therefore, (3.75) will be equal to

$$-\frac{\beta^2\epsilon^2}{4(1+\sigma_1)^2}\int\int dtdt'\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{ie^{-i\omega_1(t_1-t)}}{\omega_1^2-k_1^2-4}e^{-ik_1x_1}\Big(1+K_0(k_1)\Big)$$
$$\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t-t')}}{\omega^2-k^2-4}\Big(1+K_0(k)\Big)$$
$$\int\int\frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{ie^{-i\omega'(t-t')}}{\omega'^2-k'^2-4}\Big(1+K_0(k')\Big)$$
$$\int\int\frac{d\omega_2}{2\pi}\frac{dk_2}{2\pi}\frac{ie^{-i\omega_2(t'-t_2)}}{\omega_2^2-k_2^2-4}e^{-ik_2x_2}\Big(1+K_0(k_2)\Big). \quad (3.77)$$

Let us rewrite the t and t' integrations along with their results i.e.

$$\int\frac{dt}{2\pi}e^{-it(\omega+\omega'-\omega_1)}=\delta(\omega+\omega'-\omega_1) \quad (3.78)$$

and

$$\int\frac{dt'}{2\pi}e^{-it'(\omega+\omega'-\omega_2)}=\delta(\omega+\omega'-\omega_2). \quad (3.79)$$

In other words, the above Dirac delta functions implies $\omega_2=\omega_1$ and, at the same time, $\omega'=\omega_1-\omega$. So considering these two substitutions in (3.77), we obtain

$$-\frac{\beta^2\epsilon^2}{4(1+\sigma_1)^2}\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{ie^{-i\omega_1(t_1-t_2)}}{\omega_1^2-k_1^2-4}e^{-ik_1x_1}\Big(1+K_0(k_1)\Big)$$
$$\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{i}{\omega^2-k^2-4}\Big(1+K_0(k)\Big)$$
$$\int\frac{dk'}{2\pi}\frac{i}{(\omega_1-\omega)^2-k'^2-4}\Big(1+K_0(k')\Big)$$
$$\int\frac{dk_2}{2\pi}\frac{i}{\omega_1^2-k_2^2-4}e^{-ik_2x_2}\Big(1+K_0(k_2)\Big). \quad (3.80)$$

Integrations over k_1 and k_2 can be simply done in the final stage just by substituting $k_1=k_2=\hat{k}_1=\hat{k}_2=\sqrt{\omega_1^2-4}$. So the crucial part of the calculations is

$$\int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}\frac{i}{\omega^2-k^2-4}\frac{i}{(\omega_1-\omega)^2-k'^2-4}\frac{2ik}{ik-2\sigma_1}\frac{2ik'}{ik'-2\sigma_1} \quad (3.81)$$

First of all it is better to perform the ω integration. Otherwise, we will encounter more difficult integrals. Meanwhile, integration over ω may be achieved by closing the contour in the upper half-plane and collecting two poles at $\sqrt{k^2+4}$ and $\omega_1+\sqrt{k'^2+4}$. Hence, (3.81) reduces to

$$\int\int\frac{dk}{2\pi}\frac{dk'}{2\pi}\frac{-4kk'}{(ik-2\sigma_1)(ik'-2\sigma_1)}\left(\frac{1}{2\sqrt{k^2+4}}\frac{i}{(\omega_1-\sqrt{k^2+4})^2-k'^2-4}\right.$$
$$\left.+\frac{1}{2\sqrt{k'^2+4}}\frac{i}{(\omega_1+\sqrt{k'^2+4})^2-k^2-4}\right) \quad (3.82)$$

Now this time in order to integrate over k', as before, the required contour would be in the upper or lower half-plane depending on whether σ_1 is greater or less than zero respectively and considering the pole at $\hat{k}' = \sqrt{(\omega_1 - \sqrt{k^2+4})^2 - 4}$. So, (3.82) becomes

$$\int \frac{dk}{2\pi} \frac{-k}{\sqrt{k^2+4}} \frac{1}{(ik - 2\sigma_1)(i\hat{k}' - 2\sigma_1)} + \omega_1 \to -\omega_1. \tag{3.83}$$

Therefore, up to now the type II (boundary-boundary) contribution takes the form

$$\frac{\beta^2\epsilon^2}{4(1+\sigma_1)^2} \int \frac{d\omega_1}{2\pi} e^{-i\omega_1(t_1-t_2)} e^{-i\hat{k}_1(x_1+x_2)} \frac{1}{(2\hat{k}_1)^2} \frac{(2i\hat{k}_1)^2}{(i\hat{k}_1 - 2\sigma_1)^2}$$
$$\int \frac{dk}{2\pi} \frac{-k}{\sqrt{k^2+4}} \frac{1}{(ik - 2\sigma_1)(ik' - 2\sigma_1)} + \omega_1 \to -\omega_1 \tag{3.84}$$

Type II (boundary-bulk)

Now let us consider the type II (boundary-bulk) Feynman diagram and it is clear that this diagram has the same contribution as the type II (bulk-boundary) one, due to the symmetry which is involved in the diagrams. This time the contribution may be formulated as

$$\frac{2\beta^2\epsilon^2}{(1+\sigma_1)^2} \int\int dt dt' \int_{-\infty}^{0} dx G(x_1,t_1;x,t) G(x,t;0,t') G(x,t;0,t') G(0,t';x_2,t_2) e^{2x} \tag{3.85}$$

or using (3.76) for the propagators

$$\frac{2\beta^2\epsilon^2}{(1+\sigma_1)^2} \int\int dt dt' \int_{-\infty}^{0} dx \int\int \frac{d\omega_1}{2\pi} \frac{dk_1}{2\pi} \frac{ie^{-i\omega_1(t_1-t)}}{\omega_1^2 - k_1^2 - 4} e^{-ik_1x_1} \left(e^{ik_1x} + K_0(k_1) e^{-ik_1x}\right)$$
$$\int\int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - 4} e^{-ikx} e^{2x} \left(1 + K_0(k)\right)$$
$$\int\int \frac{d\omega'}{2\pi} \frac{dk'}{2\pi} \frac{ie^{-i\omega'(t-t')}}{\omega'^2 - k'^2 - 4} e^{-ik'x} \left(1 + K_0(k')\right)$$
$$\int\int \frac{d\omega_2}{2\pi} \frac{dk_2}{2\pi} \frac{ie^{-i\omega_2(t'-t_2)}}{\omega_2^2 - k_2^2 - 4} e^{-ik_2x_2} \left(1 + K_0(k_2)\right). \tag{3.86}$$

As it was shown in the previous diagram, integrations over t and t' means $\omega_2 = \omega_1$, $\omega' = \omega_1 - \omega$. Moreover the integration over x can be done to obtain

$$-\frac{2i\beta^2\epsilon^2}{(1+\sigma_1)^2} \int\int \frac{d\omega_1}{2\pi} \frac{dk_1}{2\pi} \frac{ie^{-i\omega_1(t_1-t_2)}}{\omega_1^2 - k_1^2 - 4} e^{-ik_1x_1}$$
$$\int\int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{i}{\omega^2 - k^2 - 4} \left(1 + K_0(k)\right) \left(\frac{1}{k_1 - k - k' - 2i}\right.$$

$$+\frac{K_0(k_1)}{-k_1-k-k'-2i}\Big)$$
$$\int\frac{dk'}{2\pi}\frac{i}{(\omega_1-\omega)^2-k'^2-4}\Big(1+K_0(k')\Big)$$
$$\int\frac{dk_2}{2\pi}\frac{i}{\omega_1^2-k_2^2-4}e^{-ik_2x_2}\Big(1+K_0(k_2)\Big). \tag{3.87}$$

After this stage it is sufficient to use the same procedure that we followed in the previous diagram and after some manipulation the type II (boundary-bulk) contribution will be

$$-\frac{2i\beta^2\epsilon^2}{(1+\sigma_1)^2}\int\frac{d\omega_1}{2\pi}e^{-i\omega_1(t_1-t_2)}e^{-i\hat{k}_1(x_1+x_2)}\frac{1}{(2\hat{k}_1)^2}\frac{(2i\hat{k}_1)^2}{(i\hat{k}_1-2\sigma_1)^2}$$
$$\int\frac{dk}{2\pi}\frac{-k}{\sqrt{k^2+4}}\frac{1}{(ik-2\sigma_1)(ik'-2\sigma_1)}\Big(\frac{1}{\hat{k}_1-k-\hat{k}'-2i}$$
$$+\frac{K_0(\hat{k}_1)}{-\hat{k}_1-k-k'-2i}\Big)$$
$$+\omega_1\rightarrow-\omega_1. \tag{3.88}$$

Type II (bulk-bulk)

In the last part of this section, the type II (bulk-bulk) Feynman diagram is studied whose calculation is much more lengthy than the previous ones. In order to find the contribution of this diagram to the reflection factor we have to find this integral

$$\frac{16\beta^2\epsilon^2}{(1+\sigma_1)^2}\int\int dtdt'\int_{-\infty}^{0}\int_{-\infty}^{0}dxdx'G(x_1,t_1;x,t)G(x,t;x',t')G(x,t;x',t')$$
$$G(x',t';x_2,t_2)e^{2x}e^{2x'} \tag{3.89}$$

or in its expanded form

$$\frac{16\beta^2\epsilon^2}{(1+\sigma_1)^2}\int\int dtdt'\int_{-\infty}^{0}\int_{-\infty}^{0}dxdx'\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{ie^{-i\omega_1(t_1-t)}}{\omega_1^2-k_1^2-4}\Big(e^{ik_1(x_1-x)}$$
$$+K_0(k_1)e^{-ik_1(x_1+x)}\Big)$$
$$\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{ie^{-i\omega(t-t')}}{\omega^2-k^2-4}\Big(e^{ik(x-x')}+K_0(k)e^{-ik(x+x')}\Big)e^{2x}$$
$$\int\int\frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{ie^{-i\omega'(t-t')}}{\omega'^2-k'^2-4}\Big(e^{ik'(x-x')}+K_0(k')e^{-ik'(x+x')}\Big)e^{2x'}$$
$$\int\int\frac{d\omega_2}{2\pi}\frac{dk_2}{2\pi}\frac{ie^{-i\omega_2(t'-t_2)}}{\omega_2^2-k_2^2-4}\Big(e^{ik_2(x'-x_2)}+K_0(k_2)e^{-ik_2(x'+x_2)}\Big). \tag{3.90}$$

Now, as before, the integrations over t and t' generate two Dirac delta function which allows these substitutions; $\omega_1 = \omega_2$ and $\omega' = \omega_1 - \omega$. Besides, integrations over x and x' can be done easily but they will give rise to lots of terms and it is convenient to adopt a shorthand notation via defining a function which will be clear later. Hence, (3.90) reduces to

$$\frac{16\beta^2\epsilon^2}{(1+\sigma_1)^2}\int\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{-i\omega_1(t_1-t_2)}e^{-i(k_1x_1+k_2x_2)}\frac{i}{\omega_1^2-k_1^2-4}\frac{i}{\omega_1^2-k_2^2-4}$$
$$\int\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}\frac{i}{\omega^2-k^2-4}\frac{i}{(\omega_1-\omega)^2-k'^2-4}\Big\{f(k_1,k_2,k,k')$$
$$+K_0(k_1)f(-k_1,k_2,k,k')+K_0(k_2)f(k_1,-k_2,k,k')$$
$$+K_0(k_1)K_0(k_2)f(-k_1,-k_2,k,k')\Big\}, \tag{3.91}$$

where

$$f(k_1,k_2,k,k') = \frac{i}{k_1+k+k'-2i}\frac{i}{k_2-k-k'-2i}$$
$$+\frac{i}{k_1-k+k'-2i}\frac{iK_0(k)}{k_2-k-k'-2i}$$
$$+\frac{i}{k_1+k-k'-2i}\frac{iK_0(k')}{k_2-k-k'-2i}$$
$$+\frac{iK_0(k)}{k_1-k-k'-2i}\frac{iK_0(k')}{k_2-k-k'-2i}. \tag{3.92}$$

Integration over ω may be performed as before, i.e. exactly what we have done in the type II (boundary-boundary) case. However, if we then integrate over k', this time, in addition to a pole at $k' = \hat{k}' = \sqrt{(\omega_1 - \sqrt{k^2+4})^2 - 4}$ in the upper half-plane, we will have extra poles due to the functions $f(\pm k_1, \pm k_2, k, k')$. So, we have to make sure that the residues of these poles are considered as well. Hence, after some calculations, (3.91) is converted to

$$\frac{16\beta^2\epsilon^2}{(1+\cos a_1\pi)^2}\int\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{-i\omega_1(t_1-t_2)}e^{-i(k_1x_1+k_2x_2)}\frac{i}{\omega_1^2-k_1^2-4}\frac{i}{\omega_1^2-k_2^2-4}$$
$$\int\frac{dk}{2\pi}\frac{1}{4\sqrt{k^2+4}}\frac{1}{\hat{k}'}\Big\{f(k_1,k_2,k,\hat{k}')+K_0(k_1)f(-k_1,k_2,k,\hat{k}')$$
$$+K_0(k_2)f(k_1,-k_2,k,\hat{k}')+K_0(k_1)K_0(k_2)f(-k_1,-k_2,k,\hat{k}')$$
$$+g(k_1,k_2,k,\omega_1)+K_0(k_1)g(-k_1,k_2,k,\omega_1)$$
$$+K_0(k_2)g(k_1,-k_2,k,\omega_1)+K_0(k_1)K_0(k_2)g(-k_1,-k_2,k,\omega_1)$$

$$+h(k_1,k_2,k,\omega_1)+K_0(k_1)h(-k_1,k_2,k,\omega_1)$$
$$+K_0(k_2)h(k_1,-k_2,k,\omega_1)+K_0(k_1)K_0(k_2)h(-k_1,-k_2,k,\omega_1)\Big\} \quad (3.93)$$

where

$$g(k_1,k_2,k,\omega_1)=\frac{2}{(\omega_1-\sqrt{k^2+4})^2-(2i-k_1-k)^2-4}\frac{1}{k_1+k_2-4i} \quad (3.94)$$

and

$$h(k_1,k_2,k,\omega_1)=\frac{2}{(\omega_1-\sqrt{k^2+4})^2-(2i-k_1+k)^2-4}\frac{1}{k_1+k_2-2k-4i}. \quad (3.95)$$

Note if we substitute the functions f, g and h inside the bracket in (3.93), doing the k_1 and k_2 integrations and after some simplification, the type II (bulk-bulk) contribution will have the form

$$\begin{aligned}
&\frac{4\beta^2\epsilon^2}{(1+\cos a_1\pi)^2}\int\frac{d\omega_1}{2\pi}e^{-i\omega_1(t_1-t_2)}e^{-i\hat{k}_1(x_1+x_2)}\frac{1}{(2\hat{k}_1)^2}\\
&\int\frac{dk}{2\pi}\frac{1}{\sqrt{k^2+4}}\frac{1}{\hat{k}'}\Big\{\frac{1}{(\hat{k}_1-2i)(\hat{k}'+k-\hat{k}_1+2i)}+\frac{K_0(k)}{(\hat{k}_1-k-2i)(\hat{k}'+k-\hat{k}_1+2i)}\\
&-\frac{K_0(\hat{k}')}{(\hat{k}'-k-\hat{k}_1+2i)(\hat{k}'+k-\hat{k}_1+2i)}-\frac{K_0(k)K_0(\hat{k}')}{(\hat{k}'+k-\hat{k}_1+2i)(\hat{k}'+k-\hat{k}_1+2i)}\\
&+\frac{iK_0(\hat{k}_1)(\hat{k}'+2i)}{(\hat{k}'-k+\hat{k}_1+2i)(\hat{k}'+k-\hat{k}_1+2i)}\\
&-\frac{2K_0(\hat{k}_1)K_0(k)(k+\hat{k}'+2i)}{(\hat{k}'+k+\hat{k}_1+2i)(\hat{k}'+k-\hat{k}_1+2i)(k+2i)}\\
&-\frac{2K_0(\hat{k}_1)K_0(\hat{k}')}{(\hat{k}'-k+\hat{k}_1+2i)(\hat{k}'+k-\hat{k}_1+2i)}-\frac{2K_0(\hat{k}_1)K_0(k)K_0(\hat{k}')}{(\hat{k}'+k-\hat{k}_1+2i)(\hat{k}'+k+\hat{k}_1+2i)}\\
&+\frac{K_0(\hat{k}_1)K_0(\hat{k}_2)}{(\hat{k}_1+2i)(\hat{k}'+k+\hat{k}_1+2i)}-\frac{K_0(\hat{k}_1)K_0(\hat{k}_2)K_0(k)}{(\hat{k}_1+k+2i)(\hat{k}'+k+\hat{k}_1+2i)}\\
&-\frac{K_0(\hat{k}_1)K_0(\hat{k}_2)K_0(\hat{k}')}{(\hat{k}'-k+\hat{k}_1+2i)(\hat{k}'+k+\hat{k}_1+2i)}-\frac{K_0(\hat{k}_1)K_0(\hat{k}_2)K_0(k)K_0(\hat{k}')}{(\hat{k}'+k+\hat{k}_1+2i)(\hat{k}'+k+\hat{k}_1+2i)}\Big\}\\
&+\omega_1\to-\omega_1. \qquad (3.96)
\end{aligned}$$

3.5 Discussion

In this chapter we tried to find second order quantum corrections to the classical reflection factor of the sinh-Gordon model at one loop order. In fact, we evaluated

ten contributions including two for the type I, four for the type II and four for the type III diagram. We completely did the calculations relating to the types I and III Feynman diagrams. However, in connection with type II diagram the calculations still are in progress. In this case the two middle Green functions are exactly identical. Then the middle momenta are not related to each other in a simple way and the computations become more intricate. Note the contribution of the type II (boundary-bulk) is the same as the type II (bulk-boundary) because of the symmetry involved in these diagrams.

It is understood that if the second order calculations are finished then, the Ghoshal's formula will be checked much more deeply than the first order calculations. Meanwhile the conjecture (2.102) in chapter 2 could be verified perturbatively at higher order.

Chapter 4

On the quantum reflection factor for the sinh-Gordon model with general boundary conditions

4.1 Introduction

In recent years there has been considerable interest [22,56,57,63,64,69] in perturbative affine Toda field theory. The motivation behind this fact is that the boundary S-matrices of the models are largely unknown. The most progress has been made for $a_1^{(1)}$ affine Toda field theory for which the general form of the boundary S-matrix has been found by Ghoshal [41]. In fact, the boundary bootstrap equations yield the boundary S-matrices up to some unknown parameters. The perturbation method not only provides an additional check of the results which come from the bootstrap technique, but also it could make a connection between the unknown parameters of the boundary S-matrices and the boundary parameters which are involved in the Lagrangian formulation of the theories.

In chapter 2 we obtained the quantum correction to the classical reflection factor of the sinh-Gordon model at one loop order when the boundary parameters are not equal. However, the calculations were restricted to first order in the difference of the two-boundary parameters. Then, by comparison of our result with Ghoshal's formula we conjectured a relation between the parameters of Ghoshal's formula and the boundary data up to the first order. This chapter extends the results of chapter

2, by calculating the quantum reflection factor for any value of the boundary parameters. It is found that most parts of the one loop quantum corrections to the classical reflection factor of the sinh-Gordon model which are calculated partially, may be expressed in terms of the hypergeometric functions. This result and how it could relate to Ghoshal's formula is discussed in the conclusions. The model is considered at low order perturbation theory and under integrable boundary conditions.

4.2 Low order perturbation theory

As we mentioned in chapter 2 in general, for a model of affine Toda field theory, the perturbation theory is studied around the static background solution to the equation of motion of the model. Therefore, the standard Feynman Rules may be used. In connection with the sinh-Gordon model, it was shown in chapter 2 that the classical static solution is given by

$$e^{\beta\phi_0/\sqrt{2}} = \frac{1+e^{2(x-x_0)}}{1-e^{2(x-x_0)}}.$$

Meanwhile, it was seen that after linear perturbation of the field equation and the boundary condition in this background, the two-point Green function corresponding to the model has the following form

$$\begin{aligned} G(x,t;x',t') &= \int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{i}{\omega^2-k^2-4+i\rho}e^{-i\omega(t-t')}\Big(f(k,x)f(-k,x')e^{ik(x-x')} \\ &\quad + K(k)f(-k,x)f(-k,x')e^{-ik(x+x')}\Big), \end{aligned} \tag{4.1}$$

where

$$f(k,x) = \frac{ik-2\coth 2(x-x_0)}{ik+2} \tag{4.2}$$

and the classical reflection factor is

$$K = \left(\frac{(ik)^2+2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}{(ik)^2-2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}\right)\left(\frac{ik-2}{ik+2}\right). \tag{4.3}$$

In the expression (4.1) for the propagator, the classical reflection factor appears as the coefficient of the reflection part of the free field two-point function calculated within the classical static background. Now following the idea introduced by Kim [56] and developed by Corrigan [64] to calculate perturbative corrections to the two-point function and then to identify corrections to the classical reflection factor by picking out the coefficient of $e^{-ik(x+x')}$ as $x, x' \to -\infty$.

In order to calculate the one loop quantum correction to the classical reflection factor, we use the standard perturbation theory which is generalised [56, 57, 64, 69] to the affine Toda field theory on the half-line. In general, at $(O(\beta^2))$ there are three basic kinds of Feynman diagrams which contribute to the two point propagator of affine Toda field theory. These are shown in figure 1 in chapter 2. These will be computed in configuration space noting that each vertex may either be situated at the boundary or within the bulk. In effect, there are ten contributions to be calculated.

It is evident that four point couplings are involved in type I diagram and they may be formulated by means of expanding the bulk and the boundary potential (see chapter 2). So,

$$C^{(4)}_{bulk} = \frac{1}{3}\beta^2 \cosh(\sqrt{2}\beta\phi_0) \tag{4.4}$$

and

$$C^{(4)}_{boundary} = \frac{\beta^2}{48}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} + \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right). \tag{4.5}$$

Clearly, the three point couplings corresponding to the bulk potential and the boundary are included in types II and III diagrams and actually they are

$$C^{(3)}_{bulk} = \frac{2\sqrt{2}}{3}\beta \sinh(\sqrt{2}\beta\phi_0) \tag{4.6}$$

and

$$C^{(3)}_{boundary} = \frac{\sqrt{2}\beta}{12}\left(\sigma_1 e^{\beta\phi_0/\sqrt{2}} - \sigma_0 e^{-\beta\phi_0/\sqrt{2}}\right). \tag{4.7}$$

By inspection of the forms of the three point and four point couplings which we have found in chapter 2, it is clear that all types of these diagrams are involved in our problem.

In fact, when the boundary parameters are not equal then, the calculations corresponding to the one loop order in the sinh-Gordon model are lengthy and intricate. In the following sections we try to compute the contributions of types I and III diagrams to the reflection factor. The remaining diagrams will be treated elsewhere. Meanwhile, it is instructive to start with type III.

4.3 Type III diagram (boundary-boundary)

In this section we shall calculate the contribution of the type III diagram to the reflection factor when both vertices are located at the boundary $x = 0$. So, we are led to the following integral

$$-\frac{\beta^2}{4}(\sigma_1 \coth x_0 - \sigma_0 \tanh x_0)^2 \times \int\int dt dt' G(x_1, t_1; 0, t) G(0, t; 0, t') G(0, t'; 0, t') G(0, t; x_2, t_2) \tag{4.8}$$

in which the combinatorial factor has been taken into account.

Let us start by looking at the loop propagator $G(0, t'; 0, t')$, which is equal to

$$G(0,t';0,t') = i\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{1}{\omega'^2 - k'^2 - 4 + i\rho}\Big(f(k',0)f(-k',0) + K'(k')f(-k',0)f(-k',0)\Big), \tag{4.9}$$

where

$$f(k',0) = \frac{ik' + 2\coth 2x_0}{ik' + 2} \tag{4.10}$$

and $K'(k')$ is the classical reflection factor (4.3). After some manipulation, we obtain

$$G(0,t';0,t') = i\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{1}{\omega'^2 - k'^2 - 4 + i\rho} \times \frac{2ik'\,(ik' - 2\coth 2x_0)}{(ik')^2 - 2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1)}. \tag{4.11}$$

The above integral is clearly divergent however, the divergence can be removed by the infinite renormalization of the boundary term. In other words, considering the following relation

$$\frac{2ik'\,(ik' - 2\coth 2x_0)}{(ik')^2 - 2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1)} = 2 + 4\frac{ik'\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \coth 2x_0\right) - (\sigma_0 + \sigma_1)}{(ik')^2 - 2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1)}, \tag{4.12}$$

it is seen that a minimal subtraction of the divergent part can be made by adding an appropriate counter term to the boundary, replace the logarithmically divergent integral by the finite part. Hence,

$$G(0,t';0,t') = 4i\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{1}{\omega'^2 - k'^2 - 4 + i\rho} \times \frac{ik'\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} - \coth 2x_0\right) - (\sigma_0 + \sigma_1)}{(ik')^2 - 2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0 + \sigma_1)}. \tag{4.13}$$

The integration over ω' may be performed by closing the contour into the upper half-plane and collecting a pole at $\omega' = \sqrt{k'^2+4}$ so that

$$G(0,t';0,t') = 2\int \frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\frac{ik'\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}-\coth 2x_0\right)-(\sigma_0+\sigma_1)}{(ik')^2-2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}. \tag{4.14}$$

In order to integrate over k', as before, one chooses a contour in the upper half-plane, however due to the branch cut the contour has to turn around the cut line. Moreover we assume that the roots of the denominator of the integrand i.e. $2\cos\frac{(a_0\pm a_1)\pi}{2}$ are positive, otherwise we may close the contour in the lower half-plane. Therefore (4.14) is converted to

$$4\int_2^\infty \frac{dy}{2\pi}\frac{1}{\sqrt{y^2-4}}\frac{y\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}-\coth 2x_0\right)+(\sigma_0+\sigma_1)}{y^2+2y\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)} \tag{4.15}$$

or after changing of variable

$$2\int_0^\infty \frac{du}{2\pi}\frac{2\cosh u\left(\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}-\coth 2x_0\right)+(\sigma_0+\sigma_1)}{2\cosh^2 u+2\cosh u\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+(\sigma_0+\sigma_1)} \tag{4.16}$$

or

$$\begin{aligned}&\frac{1}{2\pi}\frac{4\cos\frac{(a_0+a_1)\pi}{2}\sin^2\frac{(a_0+a_1)\pi}{2}}{\cos^2\frac{(a_0-a_1)\pi}{2}-\cos^2\frac{(a_0+a_1)\pi}{2}}\int_0^\infty \frac{du}{2\cosh u+2\cos\frac{(a_0+a_1)\pi}{2}}\\ &+\frac{1}{2\pi}\frac{4\cos\frac{(a_0-a_1)\pi}{2}\sin^2\frac{(a_0-a_1)\pi}{2}}{\cos^2\frac{(a_0+a_1)\pi}{2}-\cos^2\frac{(a_0-a_1)\pi}{2}}\int_0^\infty \frac{du}{2\cosh u+2\cos\frac{(a_0-a_1)\pi}{2}}.\end{aligned} \tag{4.17}$$

Finally the above integrals can be solved to get the following result

$$G(0,t';0,t') = -\frac{a_0}{2}\cot a_0\pi - \frac{a_1}{2}\cot a_1\pi. \tag{4.18}$$

Now it is convenient to calculate the time integral of the other middle propagator in (4.8) which is equal to

$$\begin{aligned}\int dt' G(0,t;0,t') &= \int\int\int dt'\frac{d\omega}{2\pi}\frac{dk}{2\pi}e^{-i\omega(t-t')}\frac{i}{\omega^2-k^2-4+i\rho}\\ &\times\frac{2ik\,(ik-2\coth 2x_0)}{(ik)^2-2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}.\end{aligned} \tag{4.19}$$

Clearly, in the boundary-boundary contribution (4.8), it is seen that the t' dependence is involved only in the above propagator i.e. $G(0,t;0,t')$, so the integration

over t' produces a Dirac delta function which give rise to substitute zero for ω and hence

$$\int dt' G(0,t;0,t') = \int \frac{dk}{2\pi}\left(\frac{i}{-k^2-4}\right)\left(\frac{2ik\,(ik-2\coth 2x_0)}{(ik)^2-2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}\right). \tag{4.20}$$

As we mentioned before, throughout this chapter we assume that the roots of $P(k) = (ik)^2 - 2ik\sqrt{1+\sigma_0}\sqrt{1+\sigma_1} + 2(\sigma_0+\sigma_1)$ which are equal to $2\cos\frac{(a_0\pm a_1)\pi}{2}$ are positive, so the $P(k)$ has no pole in the upper half-plane. Obviously if the roots are negative then we can choose the contour in the lower half-plane in which no pole is inserted. Therefore, (4.20) after integrating over k yields

$$\int dt' G(0,t;0,t') = \frac{i(1+\coth 2x_0)}{2+2\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+(\sigma_0+\sigma_1)} \tag{4.21}$$

and by substituting $\sigma_0 = \cos a_0\pi$ and $\sigma_1 = \cos a_1\pi$, we obtain

$$\int dt' G(0,t;0,t') = -\frac{i}{4\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}. \tag{4.22}$$

Up to now, the boundary-boundary contribution has the form

$$\begin{aligned}
&-\frac{i\beta^2}{32}\frac{(\sigma_1\coth x_0-\sigma_0\tanh x_0)^2(a_0\cot a_0\pi+a_1\cot a_1\pi)}{\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}\\
&\times\int dt\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{ie^{-i\omega_1(t_1-t)}}{\omega_1^2-k_1^2-4+i\rho}\Big(f(k_1,x_1)f(-k_1,0)e^{ik_1x_1}\\
&\qquad +K_1(k_1)f(-k_1,x_1)f(-k_1,0)e^{-ik_1x_1}\Big)\\
&\times\int\int\frac{d\omega_2}{2\pi}\frac{dk_2}{2\pi}\frac{ie^{-i\omega_2(t-t_2)}}{\omega_2^2-k_2^2-4+i\rho}\Big(f(k_2,x_2)f(-k_2,0)e^{-ik_2x_2}\\
&\qquad +K_2(k_2)f(-k_2,x_2)f(-k_2,0)e^{-ik_2x_2}\Big).
\end{aligned} \tag{4.23}$$

First of all, it is necessary to perform the transformation $k_1 \to -k_1$ in the first term of the first propagator. Secondly, integration over t ensures energy conservation at the interaction vertex and generates a Dirac delta function because of which we can set $\omega_1 = \omega_2$. Moreover, it is better to define a new function as

$$A(k,x) = f(-k,x)f(k,0) + K(k)f(-k,x)f(-k,0) \tag{4.24}$$

or, in an expanded form,

$$\begin{aligned}
A(k,x) \;=\; &\frac{ik+2\coth 2x_0}{ik+2}\frac{ik+2\coth 2(x-x_0)}{ik-2}\\
&+\frac{(ik+2\cos\frac{(a_0+a_1)\pi}{2})(ik+2\cos\frac{(a_0-a_1)\pi}{2})}{(ik-2\cos\frac{(a_0+a_1)\pi}{2})(ik-2\cos\frac{(a_0-a_1)\pi}{2})}\\
&\times\frac{ik-2\coth 2x_0}{ik-2}\frac{ik+2\coth 2(x-x_0)}{ik+2},
\end{aligned} \tag{4.25}$$

then the expression (4.23) reduces to

$$-\frac{i\beta^2}{32}\frac{(\sigma_1\coth x_0-\sigma_0\tanh x_0)^2(a_0\cot a_0\pi+a_1\cot a_1\pi)}{\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}$$
$$\times\int\!\!\int\!\!\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{-i\omega_1(t_1-t_2)}\frac{i}{\omega_1^2-k_1^2-4+i\rho}\frac{i}{\omega_1^2-k_2^2-4+i\rho}e^{-ik_1x_1}e^{-ik_2x_2}$$
$$\times A(k_1,x_1)A(k_2,x_2). \tag{4.26}$$

Obviously, what we need to do next is to integrate over the momenta k_1 and k_2 and this task may be achieved by closing the contours in the upper half-plane and considering the poles at $\hat{k}_1=k_1=k_2=\sqrt{\omega_1^2-4}$. Note, the additional poles due to functions $A(k_1,x_1)$ and $A(k_2,x_2)$ are not important because their contributions will be exponentially damped as x_1, x_2 go to $-\infty$. Therefore, the boundary-boundary contribution is

$$-\frac{i\beta^2}{32}\frac{(\sigma_1\coth x_0-\sigma_0\tanh x_0)^2(a_0\cot a_0\pi+a_1\cot a_1\pi)}{\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}$$
$$\times\int\frac{d\omega_1}{2\pi}e^{-i\omega_1(t_1-t_2)}e^{-i\hat{k}_1(x_1+x_2)}\frac{1}{(2\hat{k}_1)^2}A(\hat{k}_1,x_1)\,A(\hat{k}_2,x_2). \tag{4.27}$$

Now recall the definition of the quantum reflection factor as the coefficient of $e^{-ik(x+x')}$ in the two-point Green function in the residue of the on-shell pole in the asymptotic region $x,x'\to-\infty$. Thus, the correction to the reflection factor from the type III (boundary-boundary) piece is

$$-\frac{i\beta^2}{32}\frac{(\sigma_1\coth x_0-\sigma_0\tanh x_0)^2(a_0\cot a_0\pi+a_1\cot a_1\pi)}{\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}$$
$$\times\frac{1}{2\hat{k}_1}\left(\frac{(i\hat{k}_1+2\coth 2x_0)^2}{(i\hat{k}_1+2)^2}+2K(\hat{k}_1)\frac{(i\hat{k}_1+2\coth 2x_0)(i\hat{k}_1-2\coth 2x_0)}{(i\hat{k}_1+2)^2}\right.$$
$$\left.+K^2(\hat{k}_1)\frac{(i\hat{k}_1-2\coth 2x_0)^2}{(i\hat{k}_1+2)^2}\right). \tag{4.28}$$

4.4 Type III (boundary-bulk)

This section deals with the determination of the contribution of the type III Feynman diagram to the classical reflection factor when one of the vertices corresponding to the loop is situated at the boundary and the other vertex is inside the bulk region. It is evident that in this case we have to take into account the bulk three

point coupling $C^{(3)}_{bulk}$ in the corresponding vertex as well as the boundary three point coupling $C^{(3)}_{boundary}$ in the other vertex. Meanwhile, the combinatorial factor associated with the related Feynman diagram must be considered as a coefficient factor. Therefore, the contribution of the type III (boundary-bulk) to the reflection factor may be written as

$$-2\beta^2(\sigma_1 \coth x_0 - \sigma_0 \tanh x_0) \int\int\int dt dt' dx G(x_1,t_1;x,t)G(x,t;0,t') \times G(0,t';0,t')G(x,t;x_2,t_2)\sinh(\sqrt{2}\beta\phi_0). \quad (4.29)$$

The propagator $G(0,t';0,t')$ corresponding to the loop has been found in the previous section and is given by

$$G(0,t';0,t') = -\frac{a_0}{2}\cot a_0\pi - \frac{a_1}{2}\cot a_1\pi. \quad (4.30)$$

The calculation of the other middle propagator i.e. $G(x,t;0,t')$ is the next step and clearly, the t' dependence in (4.29) is included only in this propagator. Hence it is convenient to compute the following relation

$$\int dt' G(x,t;0,t') = \int dt' \int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi} e^{-i\omega(t-t')} \frac{i}{\omega^2 - k^2 - 4 + i\rho}\Big(f(k,x)f(-k,0)e^{ikx} + K(k)f(-k,x)f(-k,0)e^{-ikx}\Big). \quad (4.31)$$

Integrating over t' gives us a Dirac delta function which simplifies the integral (4.31) to

$$\int dt' G(x,t;0,t') = \int \frac{dk}{2\pi}\frac{i}{-k^2 - 4 + i\rho}\Big(f(k,x)f(-k,0)e^{ikx} + K(k)f(-k,x)f(-k,0)e^{-ikx}\Big). \quad (4.32)$$

Now if we split the above integral in two parts, the first part after setting $k \to -k$ is equal to

$$i\int \frac{dk}{2\pi}\frac{ik + 2\coth 2(x-x_0)}{ik-2}\frac{ik + 2\coth 2x_0}{ik+2}\left(\frac{1}{-k^2-4}\right)e^{-ikx} \quad (4.33)$$

and the residue theorem gives

$$\frac{d}{dk}\left(\frac{(ik + 2\coth 2(x-x_0))(ik - 2\coth 2x_0)}{(ik-2)^2}e^{-ikx}\right)|_{k=2i} \quad (4.34)$$

or after simplification, the first part has such a form

$$\frac{ie^{2x}}{8}\{-1 + \coth 2x_0 \coth 2(x-x_0) - 2x + 2x\coth 2x_0 + 2x\coth 2(x-x_0) - 2x\coth 2x_0 \coth 2(x-x_0)\}. \quad (4.35)$$

In (4.32) the second part is given by

$$i\int \frac{dk}{2\pi}\frac{(ik+2\cos\frac{(a_0+a_1)\pi}{2})(ik+2\cos\frac{(a_0-a_1)\pi}{2})}{(ik-2\cos\frac{(a_0+a_1)\pi}{2})(ik-2\cos\frac{(a_0-a_1)\pi}{2})} \times \frac{(ik+2\coth 2(x-x_0))(ik-2\coth 2x_0)}{(ik+2)(ik-2)}\left(\frac{e^{-ikx}}{-k^2-4}\right). \quad (4.36)$$

Assuming $\cos\frac{(a_0\pm a_1)\pi}{2}$ is greater than zero, then the above integral can be evaluated to obtain first

$$\frac{d}{dk}\left(\frac{(ik+2\cos\frac{(a_0+a_1)\pi}{2})(ik+2\cos\frac{(a_0-a_1)\pi}{2})}{(ik-2\cos\frac{(a_0+a_1)\pi}{2})(ik-2\cos\frac{(a_0-a_1)\pi}{2})} \frac{(ik+2\coth 2(x-x_0))(ik-2\coth 2x_0)}{(ik-2)^2}e^{-ikx}\right)|_{k=2i} \quad (4.37)$$

or after simplifying

$$\begin{aligned}
&\frac{ie^{2x}}{8}\left\{\left(\frac{\tan^2\frac{(a_0+a_1)\pi}{4}}{2\cos^2\frac{(a_0-a_1)\pi}{4}}+\frac{\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0+a_1)\pi}{4}}\right)(1+\coth 2x_0)(-1+\coth 2(x-x_0))\right.\\
&+\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}(-2+\coth 2(x-x_0)-\coth 2x_0)\\
&+2x\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}(1+\coth 2x_0)(-1+\coth 2(x-x_0))\\
&+\left(-\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}-\frac{\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0-a_1)\pi}{4}}\right.\\
&\left.\left.-\frac{\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0+a_1)\pi}{4}}\right)(1+\coth 2x_0)(-1+\coth 2(x-x_0))\right\}. \quad (4.38)
\end{aligned}$$

Now adding the first part (4.35) and the second part (4.38), then rearranging the terms we find the following result

$$\int dt' G(x,t;0,t') = \frac{ie^{2x}}{8}\left(c_0+c_1\coth 2(x-x_0)+d_0x+d_1x\coth 2(x-x_0)\right), \quad (4.39)$$

where

$$\begin{aligned}
c_0=-c_1 \;=\; &-1-\left(\frac{\tan^2\frac{(a_0+a_1)\pi}{4}}{2\cos^2\frac{(a_0-a_1)\pi}{4}}+\frac{\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0+a_1)\pi}{4}}\right)(1+\coth 2x_0)\\
&+\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}(-2-\coth 2x_0)\\
&+\left(\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}+\frac{\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0-a_1)\pi}{4}}\right.\\
&\left.+\frac{\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}}{2\cos^2\frac{(a_0+a_1)\pi}{4}}\right)(1+\coth 2x_0) \quad (4.40)
\end{aligned}$$

and

$$d_0 = -d_1 = -2\tan^2\frac{(a_0+a_1)\pi}{4}\tan^2\frac{(a_0-a_1)\pi}{4}(1+\coth 2x_0)-2+2\coth 2x_0. \quad (4.41)$$

However, the above coefficients may be simplified much more to obtain first

$$d_0 = d_1 = 0 \quad (4.42)$$

and

$$c_0 = -c_1 = -\left(1+\tan^2\frac{(a_0+a_1)\pi}{4}\right)\left(1+\tan^2\frac{(a_0-a_1)\pi}{4}\right). \quad (4.43)$$

In order to check the result (4.39), if we set $x = 0$ in this equation then it becomes

$$\int dt' G(0,t;0,t') = \frac{i}{8}(c_0 - c_1\coth 2x_0) \quad (4.44)$$

or using (4.43), we obtain

$$\int dt' G(0,t;0,t') = -\frac{i}{4\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}} \quad (4.45)$$

which is equal to (4.22) in the previous section.

Up to now the type III (boundary-bulk) contribution has the following form, of course, after integrating over t:

$$\begin{aligned}
&\beta^2 c_0(\sigma_1\coth x_0 - \sigma_0\tanh x_0)(a_0\cot a_0\pi + a_1\cot a_1\pi)\\
&\times\int_{-\infty}^{0}dx\int\int\frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi}e^{-i\omega_1(t_1-t_2)}\frac{i}{\omega_1^2-k_1^2-4+i\rho}\Big(f(k_1,x_1)f(-k_1,x)e^{ik_1(x_1-x)}\\
&\qquad\qquad +K_1(k_1)f(-k_1,x_1)f(-k_1,x)e^{-ik_1(x+x_1)}\Big)\\
&\times\left\{\frac{ie^{2x}}{8}(1-\coth 2(x-x_0))\sinh(\sqrt{2}\beta\phi_0)\right\}\\
&\times\int\frac{dk_2}{2\pi}\frac{i}{\omega_1^2-k_2^2-4+i\rho}\Big(f(k_2,x)f(-k_2,x_2)e^{ik_2(x-x_2)}\\
&\qquad\qquad +K_2(k_2)f(-k_2,x)f(-k_2,x_2)e^{-ik_2(x+x_2)}\Big),
\end{aligned} \quad (4.46)$$

where

$$\sinh(\sqrt{2}\beta\phi) = 2\cosh 2(x-x_0)\left(\coth^2 2(x-x_0)-1\right). \quad (4.47)$$

By multiplying the two propagator in (4.46) by each other, it is clear that one obtains four pole pieces and, as far as the integration over x is concerned, if we can do the integration over x on one of them then obviously the other three pole pieces

could be done in the same manner. Hence in what follows it is sufficient to treat only one of them and, meanwhile, keeping those terms which are functions of x, we are led to the following complicated integral

$$\int_{-\infty}^{0} dx f(-k_1,x)f(k_2,x)\Big(1-\coth 2(x-x_0)\Big)\sinh(\sqrt{2}\beta\phi_0)\,\exp\{2+i(k_2-k_1)x\}. \tag{4.48}$$

After some substitutions and collecting together powers of $\coth 2(x-x_0)$ we obtain

$$\frac{1}{(ik_1-2)(ik_2+2)}\int_{-\infty}^{0} dx\,\exp\{2+i(k_2-k_1)x\}\sinh(\sqrt{2}\beta\phi_0)\Big(-k_1k_2 \\ +(2ik_2-2ik_1+k_1k_2)\coth 2(x-x_0)(2ik_1-2ik_2-4)\coth^2 2(x-x_0) \\ +4\coth^3 2(x-x_0)\Big). \tag{4.49}$$

It is clear that to solve the above integral, it is necessary to manipulate the following integrals

$$\int_{-\infty}^{0} dx\,\exp\{2+i(k_2-k_1)x\}\sinh(\sqrt{2}\beta\phi_0)\coth^n 2(x-x_0), \tag{4.50}$$

where, $n=0,1,2,3$.

In fact in Appendix A, we have found the integrals (4.50) and the solutions of them are expressed in terms of hypergeometric functions. So using the formulae in Appendix A and simplifying, we find that $\mathcal{J}$ in (4.49) can be rewritten

$$\begin{aligned}
\mathcal{F}(k_1,k_2) &= -\frac{3k_1k_2+4ik_2-4ik_1+8}{3}\frac{1}{\sinh 2x_0}-\frac{3k_1k_2+6ik_2-6ik_1+13}{6}\frac{\cosh 2x_0}{\sinh^2 2x_0} \\
&- \frac{ik_2-ik_1+2}{3}\frac{\cosh^2 2x_0+1}{\sinh^3 2x_0}-\frac{1}{6}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0} \\
&- \frac{12ik_1k_2-16k_2+16k_1+40i-(k_2-k_1)(9k_1k_2+16ik_2-16ik_1+34)}{3(k_2-k_1-4i)} \\
&\qquad\times e^{-2x_0}F\left(1,\frac{i}{4}(k_2-k_1)+1,\frac{i}{4}(k_2-k_1)+2,e^{-4x_0}\right) \\
&- \frac{12ik_1k_2-48k_2+48k_1+136i-(k_2-k_1)(6k_1k_2+28ik_2-28ik_1+84)}{3(k_2-k_1-8i)} \\
&\qquad\times e^{-6x_0}F\left(2,\frac{i}{4}(k_2-k_1)+2,\frac{i}{4}(k_2-k_1)+3,e^{-4x_0}\right) \\
&+ \frac{32k_2-32k_1-192i+(k_2-k_1)(16ik_2-16ik_1+104)}{3(k_2-k_1-12i)} \\
&\qquad\times e^{-10x_0}F\left(3,\frac{i}{4}(k_2-k_1)+3,\frac{i}{4}(k_2-k_1)+4,e^{-4x_0}\right) \\
&+ \frac{16k_2-16k_1-32i}{(k_2-k_1-16i)}
\end{aligned}$$

$$\times\, e^{-14x_0}\, F\left(4, \frac{i}{4}(k_2-k_1)+4, \frac{i}{4}(k_2-k_1)+5, e^{-4x_0}\right). \qquad (4.51)$$

Now regarding (4.46), after doing the transformation $k_1 \to -k_1$ in the first term of the first propagator, all that remains is to integrate over the momenta k_1 and k_2 and this can be achieved by closing the contours in the upper half-plane and considering poles at $\hat{k}_1 = k_1 = k_2 = \sqrt{\omega_1^2 - 4}$. The extra poles in the four functions $\mathcal{F}(\pm k_1, \pm k_2)$ are not important because their contributions will be discounted when x_1 and x_2 go to $-\infty$.

Let us write down the type III (boundary-bulk) contribution to the reflection factor

$$\begin{aligned}
&\frac{\beta^2}{2}\,\frac{\tan\frac{(a_0+a_1)\pi}{4}\tan\frac{(a_0-a_1)\pi}{4}}{\cos\frac{a_0\pi}{2}\cos\frac{a_1\pi}{2}}\,(a_0\cot a_0\pi + a_1\cot a_1\pi)\\
&\int\frac{d\omega_1}{2\pi}\,e^{-i\omega_1(t_1-t_2)}\,e^{-i\hat{k}_1(x_1+x_2)}\,\frac{1}{(2\hat{k}_1)^2}\,\frac{i\hat{k}_1+2\coth 2(x_1-x_0)}{i\hat{k}_1-2}\,\frac{i\hat{k}_1+2\coth 2(x_2-x_0)}{i\hat{k}_1-2}\\
&\times\left\{\frac{i}{(i\hat{k}_1+2)^2}\mathcal{F}(-\hat{k}_1,\hat{k}_1) - \frac{i}{(i\hat{k}_1+2)(i\hat{k}_1-2)}K_1(\hat{k}_1)\mathcal{F}(\hat{k}_1,\hat{k}_1)\right.\\
&\left.-\frac{i}{(i\hat{k}_1+2)(i\hat{k}_1-2)}K_1(\hat{k}_1)\mathcal{F}(-\hat{k}_1,-\hat{k}_1) + \frac{i}{(i\hat{k}_1-2)^2}K_1^2(\hat{k}_1)\mathcal{F}(\hat{k}_1,-\hat{k}_1)\right\}. \qquad (4.52)
\end{aligned}$$

Now looking at the function $\mathcal{F}(k_1,k_2)$ given by (4.51), let us show the detailed forms of the $\mathcal{F}(-\hat{k}_1,\hat{k}_1)$, $\mathcal{F}(\hat{k}_1,\hat{k}_1)$,$\mathcal{F}(-\hat{k}_1,-\hat{k}_1)$ and $\mathcal{F}(\hat{k}_1,-\hat{k}_1)$. In fact,

$$\begin{aligned}
\mathcal{F}(\hat{k}_1,\hat{k}_1) \;=\; &-\frac{3\hat{k}_1^2+8}{3}\,\frac{1}{\sinh 2x_0} + \frac{3\hat{k}_1^2+13}{6}\,\frac{\cosh 2x_0}{\sinh^2 2x_0}\\
&-\frac{2}{3}\,\frac{\cosh^2 2x_0+1}{\sinh^3 2x_0} - \frac{1}{6}\,\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\\
&+\frac{3\hat{k}_1^2+10}{3}\,e^{-2x_0}\,F(1,1,2,e^{-4x_0})\\
&+\frac{3\hat{k}_1^2+34}{6}\,e^{-6x_0}\,F(2,2,3,e^{-4x_0})\\
&+\frac{16}{3}\,e^{-10x_0}\,F(3,3,4,e^{-4x_0})\\
&+2\,e^{-14x_0}\,F(4,4,5,e^{-4x_0}). \qquad (4.53)
\end{aligned}$$

Note the above expression can be simplified using Mathematica:

$$\begin{aligned}
\mathcal{F}(\hat{k}_1,\hat{k}_1) \;=\; &-\frac{3\hat{k}_1^2+16}{6}\,\frac{1}{\sinh 2x_0} + \frac{3\hat{k}_1^2+15}{6}\,\frac{\cosh 2x_0}{\sinh^2 2x_0}\\
&+\frac{1}{3}\,\frac{2\cosh^2 2x_0-5}{\sinh^3 2x_0} - \frac{1}{6}\,\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}. \qquad (4.54)
\end{aligned}$$

It can be easily verified that

$$\mathcal{F}(-\hat{k}_1,-\hat{k}_1)=\mathcal{F}(\hat{k}_1,\hat{k}_1) \tag{4.55}$$

and

$$\begin{aligned}\mathcal{F}(-\hat{k}_1,\hat{k}_1) &= \frac{3\hat{k}_1^2-8i\hat{k}_1-8}{3}\frac{1}{\sinh 2x_0}+\frac{3\hat{k}_1^2-12i\hat{k}_1-13}{6}\frac{\cosh 2x_0}{\sinh^2 2x_0}\\
&-\frac{2i\hat{k}_1+2}{3}\frac{\cosh^2 2x_0}{\sinh^3 2x_0}-\frac{1}{6}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\\
&+\frac{-9\hat{k}_1^3+38i\hat{k}_1^2+50\hat{k}_1-20i}{3(\hat{k}_1-2i)}e^{-2x_0}F(1,\frac{i}{2}\hat{k}_1+1,\frac{i}{2}\hat{k}_1+2,e^{-4x_0})\\
&+\frac{-6\hat{k}_1^3+62i\hat{k}_1^2+132\hat{k}_1-68i}{3(\hat{k}_1-4i)}e^{-6x_0}F(2,\frac{i}{2}\hat{k}_1+2,\frac{i}{2}\hat{k}_1+3,e^{-4x_0})\\
&+\frac{32i\hat{k}_1^2+136\hat{k}_1-96i}{3(\hat{k}_1-6i)}e^{-10x_0}F(3,\frac{i}{2}\hat{k}_1+3,\frac{i}{2}\hat{k}_1+4,e^{-4x_0})\\
&+\frac{16\hat{k}_1-16i}{(\hat{k}_1-8i)}e^{-14x_0}F(4,\frac{i}{2}\hat{k}_1+4,\frac{i}{2}\hat{k}_1+5,e^{-4x_0}).\end{aligned} \tag{4.56}$$

As before, the above formula can be simplified using Mathematica:

$$\begin{aligned}\mathcal{F}(-\hat{k}_1,\hat{k}_1) &= \frac{3\hat{k}_1^2-8i\hat{k}_1-8}{3}\frac{1}{\sinh 2x_0}-\frac{6i\hat{k}_1+11}{6}\frac{\cosh 2x_0}{\sinh^2 2x_0}\\
&-\frac{3\hat{k}_1^2(\cosh^2 2x_0-1)-i\hat{k}_1(5\cosh^2 2x_0-6)-(2\cosh^2 2x_0-3)}{3\sinh^3 2x_0}\\
&-\frac{1}{6}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}.\end{aligned} \tag{4.57}$$

Finally $\mathcal{F}(\hat{k}_1,-\hat{k}_1)$ can be obtained from $\mathcal{F}(-\hat{k}_1,\hat{k}_1)$ after setting $\hat{k}_1\to-\hat{k}_1$.

4.5 Type III(bulk-boundary)

In this section we study the quantum correction to the classical reflection factor due to the contribution of the type III Feynman diagram, when the vertex associated with the loop is located in the bulk region and the other vertex coincides with the boundary. The associated contribution is given by

$$\begin{aligned}\mathcal{C}=-2\beta^2(\sigma_1\coth x_0-\sigma_0\tanh x_0)\int\int\int dtdt'dx'G(x_1,t_1;0,t)G(0,t;x',t')\\
\times G(x',t';x',t')G(0,t;x_2,t_2)\sinh(\sqrt{2}\beta\phi_0)\end{aligned} \tag{4.58}$$

in which, as before, $\sinh(\sqrt{2}\beta\phi_0)$ is, apart from the related coefficient, the three point coupling.

The following relation which is some part of the contribution (4.58), can be derived independently from the remaining part

$$\int dt G(x_1,t_1;0,t)G(0,t;x_2,t_2) \tag{4.59}$$

or

$$\begin{aligned}\int dt \int\int \frac{d\omega_1}{2\pi}\frac{dk_1}{2\pi} e^{-i\omega_1(t_1-t)} \frac{i}{\omega_1^2-k_1^2-4+i\rho}\Big(f(k_1,x_1)f(-k_1,0)e^{ik_1x_1} \\ +K_1(k_1)f(-k_1,x_1)f(-k_1,0)e^{-ik_1x_1}\Big) \\ \times \int\int \frac{d\omega_2}{2\pi}\frac{dk_2}{2\pi} e^{-i\omega_2(t-t_2)} \frac{i}{\omega_2^2-k_2^2-4+i\rho}\Big(f(k_2,0)f(-k_2,x_2)e^{-ik_2x_2} \\ +K_2(k_2)f(-k_2,0)f(-k_2,x_2)e^{-ik_2x_2}\Big).\end{aligned} \tag{4.60}$$

First of all, it is necessary to set $k_1 \to -k_1$ in the first term of the first propagator. Secondly, integration over t leads to the substitution of $\omega_2 = \omega_1$ and finally integration over the momenta k_1 and k_2 , as before, may be done immediately by closing the contour in the upper half-plane and looking at the poles at $\hat{k}_1 = k_1 = k_2 = \sqrt{\omega_1^2-4}$ and ignoring all the other poles as their contributions vanish rapidly as $x_1, x_2 \to -\infty$. So we obtain (after taking the limit $x_1, x_2 \to -\infty$)

$$\begin{aligned}C_1 &= \int dt G(x_1,t_1;0,t)G(0,t;x_2,t_2) \\ &= \int \frac{d\omega_1}{2\pi} e^{-i\omega_1(t_1-t_2)} e^{-i\hat{k}_1(x_1+x_2)} \frac{1}{(2\hat{k}_1)^2} A(\hat{k}_1,x_1)\, A(\hat{k}_1,x_2),\end{aligned} \tag{4.61}$$

where

$$A(\hat{k}_1,x_1) = f(-\hat{k}_1,x_1)f(\hat{k}_1,0) + K(\hat{k}_1)f(-\hat{k}_1,x_1)f(-\hat{k}_1,0). \tag{4.62}$$

So, our next job is to calculate the integral which is the remaining part of the contribution

$$\int\int dt'dx' G(0,t;x',t')G(x',t';x't')\sinh(\sqrt{2}\beta\phi_0). \tag{4.63}$$

Obviously, this part will be appeared as a constant and it must be multiplied by (4.61). Clearly, the time variable t' appears only in one of the propagator i.e. in $G(0,t;x',t')$. On the other hand, this propagator along with integration over t' has been obtained in previous section. Hence,

$$\int dt' G(0,t;x',t') = \frac{i}{8} e^{2x'} c_0 \left(1-\coth 2(x-x_0)\right), \tag{4.64}$$

where

$$c_0 = -\left(1+\tan^2\frac{(a_0+a_1)\pi}{4}\right)\left(1+\tan^2\frac{(a_0-a_1)\pi}{4}\right). \tag{4.65}$$

Therefore, (4.63) reduces to

$$\frac{ic_0}{8}\int dx'\, e^{2x'}\,\sinh(\sqrt{2}\beta\phi_0)(1-\coth 2(x'-x_0)G(x',t';x',t'), \tag{4.66}$$

where

$$G(x',t';x',t') = \int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi}\frac{i}{\omega'^2-k'^2-4+i\rho}\Big(f(k',x')f(-k',x') \\ +K(k')(f(-k',x')(f(-k',x')e^{-2ik'x'}\Big) \tag{4.67}$$

or after integration over ω' which can be performed by completing the contour into the upper half-plane and picking up the pole $\omega' = \sqrt{k'^2+4}$ and therefore

$$G(x',t';x',t') = \frac{1}{2}\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\Big(f(k',x')f(-k',x') \\ +K(k')(f(-k',x')(f(-k',x')e^{-2ik'x'}\Big). \tag{4.68}$$

In fact, the above integrand has two parts, the first part can be easily manipulated but the other part which includes the exponential term is hard to calculate and we prefer to leave the computation of that part for later. Let us rewrite the first part of the loop propagator

$$\frac{1}{2}\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\frac{ik'-2\coth 2(x'-x_0)}{ik'+2}\frac{ik'+2\coth 2(x'-x_0)}{ik'-2}. \tag{4.69}$$

The above integral is logarithmically divergent. Nevertheless, this divergence can be removed by an infinite renormalisation of the mass parameter in the bulk potential. Then, doing the integration over k' we obtain

$$2\left(1-\coth^2 2(x-x_0)\right)\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\frac{1}{(ik'+2)(ik'-2)} = -\frac{\left(1-\coth^2 2(x-x_0)\right)}{2\pi}. \tag{4.70}$$

To sum up, (4.63) reduces to

$$-\frac{ic_0}{16\pi}\int_{-\infty}^{0}dx'\,e^{2x'}\left(1-\coth^2 2(x-x_0)\right)\left(1-\coth 2(x-x_0)\right)\sinh(\sqrt{2}\beta\phi_0) \\ +\frac{ic_0}{16}\int_{-\infty}^{0}dx'\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\left(1-\coth 2(x-x_0)\right)\sinh(\sqrt{2}\beta\phi_0)\,e^{2x'} \\ \left\{\frac{(ik')^2+2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}{(ik')^2-2ik'\sqrt{1+\sigma_0}\sqrt{1+\sigma_1}+2(\sigma_0+\sigma_1)}\frac{(ik'+2\coth 2(x-x_0))^2}{(ik'+2)(ik'-2)}e^{-2ik'x'}\right\} \tag{4.71}$$

The above relation has two parts and the first part which is a single integral can be performed by means of the formulae in Appendix A and we write down only the solution of this part which is expressed in terms of hypergeometric functions, that is,

$$\begin{aligned}
C_2 &= -\frac{ic_0}{16\pi}\int_{-\infty}^{0} dx'\, e^{2x'}\left(1-\coth^2 2(x'-x_0)\right)\left(1-\coth 2(x'-x_0)\right)\sinh(\sqrt{2}\beta\phi_0) \\
&= \frac{i}{16\pi}\left(1+\tan^2\frac{(a_0+a_1)\pi}{4}\right)\left(1+\tan^2\frac{(a_0-a_1)\pi}{4}\right) \\
&\quad\times\left\{\frac{1}{3}\frac{1}{\sinh 2x_0}-\frac{1}{24}\frac{\cosh 2x_0}{\sinh^2 2x_0}-\frac{1}{6}\frac{\cosh^2 2x_0+1}{\sinh^3 2x_0}-\frac{1}{24}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\right. \\
&\quad -\frac{1}{6}e^{-2x_0}F(1,1,2,e^{-4x_0}) \\
&\quad +\frac{11}{12}e^{-6x_0}F(2,2,3,e^{-4x_0}) \\
&\quad +\frac{4}{3}e^{-10x_0}F(3,3,4,e^{-4x_0}) \\
&\quad \left.+\frac{1}{2}e^{-14x_0}F(4,4,5,e^{-4x_0})\right\}. \qquad (4.72)
\end{aligned}$$

By means of Mathematica the above expression can be simplified to

$$\begin{aligned}
C_2 &= \frac{i}{16\pi}\left(1+\tan^2\frac{(a_0+a_1)\pi}{4}\right)\left(1+\tan^2\frac{(a_0-a_1)\pi}{4}\right) \\
&\quad\times\left\{\frac{1}{3}\frac{1}{\sinh 2x_0}+\frac{1}{24}\frac{\cosh 2x_0}{\sinh^2 2x_0}-\frac{1}{12}\frac{4\cosh^2 2x_0-1}{\sinh^3 2x_0}\right. \\
&\quad \left.-\frac{1}{24}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\right\}. \qquad (4.73)
\end{aligned}$$

So, in connection with the type III (bulk-boundary) contribution, the remaining integral is

$$\begin{aligned}
C_3 &= \frac{ic_0}{16}\int_{-\infty}^{0} dx'\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\left(1-\coth 2(x'-x_0)\right)\sinh(\sqrt{2}\beta\phi_0)\,e^{2x'} \\
&\times\frac{\left(ik'+2\cos\frac{(a_0+a_1)\pi}{2}\right)\left(ik'+2\cos\frac{(a_0-a_1)\pi}{2}\right)}{\left(ik'-2\cos\frac{(a_0+a_1)\pi}{2}\right)\left(ik'-2\cos\frac{(a_0-a_1)\pi}{2}\right)}\frac{\left(ik'+2\coth 2(x'-x_0)\right)^2}{(ik'+2)(ik'-2)}\,e^{-2ik'x'} \qquad (4.74)
\end{aligned}$$

As we mentioned before it is more convenient to integrate over x' then afterwards over k' since to integrate over k' first is a difficult problem. Let us do partial fraction decomposition for the rational function in (4.74). Obviously we will have four elementary partial fraction including

$$\frac{1}{\left(ik'-2\cos\frac{(a_0+a_1)\pi}{2}\right)},\ \frac{1}{\left(ik'-2\cos\frac{(a_0-a_1)\pi}{2}\right)},\ \frac{1}{ik'+2},\ \frac{1}{ik'-2}.$$

Now in what follows we perform the calculations in detail for one of them, for example, $\frac{1}{\left(ik'-2\cos\frac{(a_0+a_1)\pi}{2}\right)}$ due to the fact that for all the others the computations are similar except that $\cos\frac{(a_0+a_1)\pi}{2}$ is replaced by one of $\cos\frac{(a_0-a_1)\pi}{2}$, -1, 1, respectively. What we need to do is to calculate the following:

$$
\begin{aligned}
&-\frac{\left(\tan^2\frac{(a_0+a_1)\pi}{4}-\cot^2\frac{(a_0+a_1)\pi}{4}\right)}{\cos^2\frac{(a_0+a_1)\pi}{4}\cos^2\frac{(a_0-a_1)\pi}{4}}\cot\frac{a_0\pi}{2}\cot\frac{a_1\pi}{2}\\
&\times\frac{i}{64}\int_{-\infty}^{0}dx'\int\frac{dk'}{2\pi}\frac{1}{\sqrt{k'^2+4}}\Big(1-\coth 2(x'-x_0)\Big)\sinh(\sqrt{2}\beta\phi_0)\,e^{(2-2ik')x'}\\
&\times\left(4\coth^2 2(x'-x_0)+8\cos\frac{(a_0+a_1)\pi}{2}\coth 2(x'-x_0)+4\cos^2\frac{(a_0+a_1)\pi}{2}\right)\\
&\times\left(\frac{1}{ik'-2\cos\frac{(a_0+a_1)\pi}{2}}\right).
\end{aligned}
\tag{4.75}
$$

The integration over x' may be done by using the formulae in Appendix A and gives

$$
\begin{aligned}
&\int_{-\infty}^{0}dx'\,e^{(2-2ik')x'}\Big(1-\coth 2(x'-x_0)\Big)\sinh(\sqrt{2}\beta\phi_0)\left(4\coth^2 2(x'-x_0)\right.\\
&\qquad\left.+8\cos\frac{(a_0+a_1)\pi}{2}\coth 2(x'-x_0)+4\cos^2\frac{(a_0+a_1)\pi}{2}\right)\\
&=\left(4\cos^2\frac{(a_0+a_1)\pi}{2}-\frac{16}{3}\cos\frac{(a_0+a_1)\pi}{2}+\frac{8}{3}\right)\frac{1}{\sinh 2x_0}\\
&\quad+\left(2\cos^2\frac{(a_0+a_1)\pi}{2}-4\cos\frac{(a_0+a_1)\pi}{2}+\frac{13}{6}\right)\frac{\cosh 2x_0}{\sinh^2 2x_0}\\
&\quad+\left(-\frac{4}{3}\cos\frac{(a_0+a_1)\pi}{2}+\frac{2}{3}\right)\frac{\cosh^2 2x_0+1}{\sinh^3 2x_0}\\
&\quad+\frac{1}{6}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\\
&\quad+\frac{(A_1k'+B_1)}{(k'+2i)}e^{-2x_0}F(1,-\frac{i}{2}k'+1,-\frac{i}{2}k'+2,e^{-4x_0})\\
&\quad+\frac{(A_2k'+B_2)}{(k'+4i)}e^{-6x_0}F(2,-\frac{i}{2}k'+2,-\frac{i}{2}k'+3,e^{-4x_0})\\
&\quad+\frac{(A_3k'+B_3)}{(k'+6i)}e^{-10x_0}F(3,-\frac{i}{2}k'+3,-\frac{i}{2}k'+4,e^{-4x_0})\\
&\quad+\frac{(A_4k'+B_4)}{(k'+8i)}e^{-14x_0}F(4,-\frac{i}{2}k'+4,-\frac{i}{2}k'+5,e^{-4x_0}),
\end{aligned}
\tag{4.76}
$$

where the coefficients A_n, B_n, $n=1,2,3,4$ are constants which in fact only depend on $\cos\frac{(a_0+a_1)\pi}{2}$. Now the final calculation is to integrate over k' and it is evident that in order to do that, we have to convert the hypergeometric functions to infinite series. Note that it was not possible to find a simplification of (4.76) as in previous

cases. Considering the equation (A.10) in Appendix A, we conclude that

$$\frac{F(1,-\frac{i}{2}k'+1,-\frac{i}{2}k'+2,e^{-4x_0})}{k'+2i}=\sum_{n=0}^{\infty}\frac{e^{-4nx_0}}{k'+i(2+2n)}. \tag{4.77}$$

If we differentiate both sides of the above relation with respect to x_0, then the following identity may be derived

$$\frac{F(2,-\frac{i}{2}k'+2,-\frac{i}{2}k'+3,e^{-4x_0})}{k'+4i}=\sum_{n=1}^{\infty}\frac{ne^{-4(n-1)x_0}}{k'+i(2+2n)}. \tag{4.78}$$

In the same way, one obtains

$$\frac{F(3,-\frac{i}{2}k'+3,-\frac{i}{2}k'+4,e^{-4x_0})}{k'+6i}=\frac{1}{2!}\sum_{n=2}^{\infty}\frac{n(n-1)e^{-4(n-2)x_0}}{k'+i(2+2n)} \tag{4.79}$$

and

$$\frac{F(4,-\frac{i}{2}k'+4,-\frac{i}{2}k'+5,e^{-4x_0})}{k'+8i}=\frac{1}{3!}\sum_{n=3}^{\infty}\frac{n(n-1)(n-2)e^{-4(n-3)x_0}}{k'+i(2+2n)}. \tag{4.80}$$

Now if we substitute (4.77), (4.78), (4.79) and (4.80) in (4.76), all that remains in connection with the contribution (4.75) is the integration over k'. Obviously we encounter these kind of integrals

$$\int_{-\infty}^{\infty}\frac{dk'}{\sqrt{k'^2+4}}\left(\frac{1}{ik'-2\cos\frac{(a_0+a_1)\pi}{2}}\right)\left(\frac{Ak'+B}{k'+i(2+2n)}\right) \tag{4.81}$$

and the k' integration may be performed by closing the contour into the upper half-plane and onto the branch cut which stretches from $k'=2i$ to infinity along the imaginary axis (y-axis). In fact, integrals along the branch cut remain to be evaluated and after changing of variable $y=2\cosh u$, then another change as $e^u=v$ and doing some manipulation, we obtain the required formula

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{dk'}{\sqrt{k'^2+4}}\left(\frac{1}{ik'-2\cos\frac{(a_0+a_1)\pi}{2}}\right)\left(\frac{Ak'+B}{k'+i(2+2n)}\right)\\&=\frac{\left(2A\cos\frac{(a_0+a_1)\pi}{2}+iB\right)}{\left(2+2n-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\\&-\frac{((2+2n)A+iB)}{\left(2+2n-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\frac{1}{2\sqrt{n^2+2n}}\ln\left\{\frac{n+1-\sqrt{n^2+2n}}{n+1+\sqrt{n^2+2n}}\right\}.\end{aligned} \tag{4.82}$$

Note (4.82) is valid when $n\neq 0$, on the other hand if $n=0$ then one may find

$$\int_{-\infty}^{\infty}\frac{dk'}{\sqrt{k'^2+4}}\left(\frac{1}{ik'-2\cos\frac{(a_0+a_1)\pi}{2}}\right)\left(\frac{Ak'+B}{k'+2i}\right)$$

$$= \frac{\left(2A\cos\frac{(a_0+a_1)\pi}{2}+iB\right)}{\left(2-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}} - \frac{2A+iB}{\left(2-2\cos\frac{(a_0+a_1)\pi}{2}\right)}. \tag{4.83}$$

Now we are in a position to write down (4.75) or, in fact, the integral (4.74)

$$\begin{aligned}
\mathcal{C}_3 \;=\; & \frac{i}{128\pi}\frac{\left(\tan^2\frac{(a_0+a_1)\pi}{4}-\cot^2\frac{(a_0+a_1)\pi}{4}\right)}{\cos^2\frac{(a_0+a_1)\pi}{4}\cos^2\frac{(a_0-a_1)\pi}{4}}\cot\frac{a_0\pi}{2}\cot\frac{a_1\pi}{2} \\
& \times\Bigg\{\Bigg(\Bigg(\Bigg(4\cos^2\frac{(a_0+a_1)\pi}{2}-\frac{16}{3}\cos\frac{(a_0+a_1)\pi}{2}+\frac{8}{3}\Bigg)\frac{1}{\sinh 2x_0} \\
& +\left(2\cos^2\frac{(a_0+a_1)\pi}{2}-4\cos\frac{(a_0+a_1)\pi}{2}+\frac{13}{6}\right)\frac{\cosh 2x_0}{\sinh^2 2x_0} \\
& +\left(-\frac{4}{3}\cos\frac{(a_0+a_1)\pi}{2}+\frac{2}{3}\right)\frac{\cosh^2 2x_0+1}{\sinh^3 2x_0} \\
& +\frac{1}{6}\frac{\cosh^3 2x_0+5\cosh 2x_0}{\sinh^4 2x_0}\Bigg)\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}} \\
& +\frac{1}{12}e^{-2x_0}\left(\frac{2A_1+iB_1}{2-2\cos\frac{(a_0+a_1)\pi}{2}}-\frac{2\cos\frac{(a_0+a_1)\pi}{2}A_1+iB_1}{2-2\cos\frac{(a_0+a_1)\pi}{2}}\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\right) \\
& -\sum_{n=1}^{\infty}\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\frac{e^{-(2+4n)x_0}}{\left(2+2n-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\Bigg[\left(2\cos\frac{(a_0+a_1)\pi}{2}A_1+iB_1\right) \\
& +\frac{n}{1!}\left(2\cos\frac{(a_0+a_1)\pi}{2}A_2+iB_2\right)+\frac{n(n-1)}{2!}\left(2\cos\frac{(a_0+a_1)\pi}{2}A_3+iB_3\right) \\
& +\frac{n(n-1)(n-2)}{3!}\left(2\cos\frac{(a_0+a_1)\pi}{2}A_4+iB_4\right)\Bigg] \\
& +\sum_{n=1}^{\infty}\frac{e^{-(2+4n)x_0}}{\left(2+2n-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\frac{1}{2\sqrt{n^2+2n}}\ln\left\{\frac{n+1-\sqrt{n^2+2n}}{n+1+\sqrt{n^2+2n}}\right\} \\
& \Bigg(\left((2+2n)A_1+iB_1\right)+\frac{n}{1!}\left((2+2n)A_2+iB_2\right) \\
& +\frac{n(n-1)}{2!}\left((2+2n)A_3+iB_3\right)+\frac{n(n-1)(n-2)}{3!}\left((2+2n)A_4+iB_4\right)\Bigg)\Bigg\} \\
& +\text{ other pole pieces.}
\end{aligned} \tag{4.84}$$

Note, in the above expression all the series are convergent. As we mentioned before, (4.84) must be considered (after adding to (4.73))as a coefficient factor of (4.61) in order to constitute the type III (bulk-boundary) contribution i.e.:

$$\mathcal{C} = \mathcal{C}_1\left(\mathcal{C}_2+\mathcal{C}_3\right). \tag{4.85}$$

4.6 Type I diagram

In this section we calculate the contribution of the type I Feynman diagram to the classical reflection factor when the vertex is placed inside the bulk region. In fact, when the vertex is located at the boundary then, the corresponding contribution has been found [73] and is given by

$$-\frac{i\beta^2}{8}(\sigma_1 \coth x_0 + \sigma_0 \tanh x_0)(a_0 \cot a_0\pi + a_1 \cot a_1\pi)$$
$$\int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i\hat{k}(x_1+x_2)} \frac{i\hat{k} + 2\coth 2(x_1-x_0)}{P(\hat{k})} \frac{i\hat{k} + 2\coth 2(x_2-x_0)}{P(\hat{k})} \quad (4.86)$$

Clearly, in this case the bulk four point coupling should be considered in the interaction vertex. Moreover, as before, the combinatorial factor associated with this diagram will appear as a coefficient. Hence the contribution has the form

$$-4i\beta^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx G(x_1,t_1;x,t)G(x,t;x,t)G(x,t;x_2,t_2)\cosh(\sqrt{2}\beta\phi_0), \quad (4.87)$$

where

$$\cosh(\sqrt{2}\beta\phi_0) = \left(2\coth^2 2(x-x_0) - 1\right). \quad (4.88)$$

In the previous section, we simplified the middle propagator to obtain

$$G(x,t;x,t) = -\frac{\left(1-\coth^2 2(x-x_0)\right)}{2\pi} + \frac{1}{2}\int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2+4}} K(k'') f(-k'',x) f(-k'',x)\, e^{-2ik''x}. \quad (4.89)$$

Also the integral part of the loop Green function is hard to evaluate and we found out that it is better to do this integration during the final stage. Now let us rewrite the contribution (4.87) in the expanded form

$$-4i\beta^2 \int dt \int_{-\infty}^{0} dx \int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi} e^{-i\omega(t_1-t)} \frac{i}{\omega^2-k^2-4+i\rho}$$
$$\Big(f(k,x_1)f(-k,x)e^{ik(x_1-x)} + K(k)f(-k,x_1)f(-k,x)e^{-ik(x_1+x)}\Big)\cosh(\sqrt{2}\beta\phi_0)$$
$$\left\{-\frac{\left(1-\coth^2 2(x-x_0)\right)}{2\pi} + \frac{1}{2}\int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}} K(k'')f(-k'',x)f(-k'',x)\, e^{-2ik''x}\right\}$$
$$\int\int \frac{d\omega'}{2\pi}\frac{dk'}{2\pi} e^{-i\omega'(t-t_2)} \frac{i}{\omega'^2-k'^2-4+i\rho}\Big(f(k',x)f(-k',x_2)\, e^{ik'(x-x_2)}$$
$$+K'(k')f(-k',x)f(-k',x_2)\, e^{-ik'(x+x_2)}\Big), \quad (4.90)$$

where, as before

$$f(k,x) = \frac{ik - 2\coth 2(x - x_0)}{ik + 2} \tag{4.91}$$

and

$$K(k) = \frac{(ik + 2\cos\frac{(a_0+a_1)\pi}{2})(ik + 2\cos\frac{(a_0-a_1)\pi}{2})}{(ik - 2\cos\frac{(a_0+a_1)\pi}{2})(ik - 2\cos\frac{(a_0-a_1)\pi}{2})}\frac{ik-2}{ik+2}. \tag{4.92}$$

Looking at (4.90), one can predict that the calculations will be lengthy and intricate. The starting point is to do the t integration which allows the substitution $\omega = \omega'$. Secondly, it is necessary to perform a transformation $k \to -k$ in the first term of the first propagator. Moreover, if we multiply the first and the third propagator with each other, then definitely we will have four pole pieces and fortunately if we do the calculation for one of them (for example the first one), then the calculations corresponding to the other three pole pieces may be treated similarly obtaining the same contributions, except that $k + k'$ is replaced by one of $k - k'$, $-k + k'$ and $-k - k'$. Because of this in what follows we follow the problem only for one pole piece. Hence our job is, in fact, the following integral

$$\mathcal{D} = -4i\beta^2 \int_{-\infty}^{0} dx \int\int \frac{d\omega}{2\pi}\frac{dk}{2\pi} e^{-i\omega(t_1-t_2)} \frac{i}{\omega^2 - k^2 - 4 + i\rho}$$

$$\cosh(\sqrt{2}\beta\phi_0) f(-k,x_1) f(k,x)\, e^{-ik(x_1-x)}$$

$$\left(-\frac{\left(1-\coth^2 2(x-x_0)\right)}{2\pi} + \frac{1}{2}\int \frac{dk''}{2\pi}\frac{1}{\sqrt{k''^2+4}} K(k'') f(-k'',x) f(-k'',x)\, e^{-2ik''x}\right)$$

$$\int \frac{dk'}{2\pi}\frac{i}{\omega^2 - k'^2 - 4 + i\rho} f(k',x) f(-k',x_2)\, e^{ik'(x-x_2)}. \tag{4.93}$$

In fact, the above contribution has two parts. The first part, in which the integral of the middle momentum (k'') is not involved, can be calculated by means of the formulae in Appendix B and we call this part $\mathcal{D}_1$. Let us write down the solution of this part. This contribution is expressed in terms of hypergeometric functions as:

$$\begin{aligned}\mathcal{D}_1 \;=\; & \frac{i\beta^2}{\pi}\int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i\hat{k}(x_1+x_2)} f(-\hat{k},x_1) f(-\hat{k},x_2)\frac{1}{\hat{k}^2} \\ & \times\left\{\frac{4i - 4\hat{k} - i\hat{k}^2}{\hat{k}-2i} e^{-4x_0} F\left(2, \frac{i}{2}\hat{k}+1, \frac{i}{2}\hat{k}+2, e^{-4x_0}\right)\right. \\ & \left. + \frac{48i - 40\hat{k} - 8i\hat{k}^2}{\hat{k}-4i} e^{-8x_0} F\left(3, \frac{i}{2}\hat{k}+2, \frac{i}{2}\hat{k}+3, e^{-4x_0}\right)\right.\end{aligned}$$

$$+\frac{176i-96\hat{k}-8i\hat{k}^2}{\hat{k}-6i}\,e^{-12x_0}\,F\left(4,\frac{i}{2}\hat{k}+3,\frac{i}{2}\hat{k}+4,e^{-4x_0}\right)$$
$$+\frac{256i-64\hat{k}}{\hat{k}-8i}\,e^{-16x_0}\,F\left(5,\frac{i}{2}\hat{k}+4,\frac{i}{2}\hat{k}+5,e^{-4x_0}\right)$$
$$+\frac{128i}{\hat{k}-10i}\,e^{-20x_0}\,F\left(6,\frac{i}{2}\hat{k}+5,\frac{i}{2}\hat{k}+6,e^{-4x_0}\right)\Bigg\} \tag{4.94}$$

Note that it was not possible to find a further simplification of (4.94).

Now it is better for the second part, which we call $\mathcal{D}_2$, to integrate first over x then over k''. Meanwhile, before starting the integration, it is useful to note that if we do the partial fraction decomposition for $K''(k'')f(-k'',x)f(-k'',x)$, then we will have four elementary partial fractions as

$$\frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)},\ \frac{1}{\left(ik''-2\cos\frac{(a_0-a_1)\pi}{2}\right)},\ \frac{1}{ik''+2},\ \frac{1}{ik''-2}.$$

As before in the remaining section we continue the computations in detail for one of them (for example, $\frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)}$) because the calculations corresponding to the other three elementary partial fractions can be done in the same manner just by the substitution of $\cos\frac{(a_0+a_1)\pi}{2}$ by one of $\cos\frac{(a_0-a_1)\pi}{2}$, -1, 1, respectively. So, our problem reduces to this integral

$$i\beta^2\cot\frac{a_0\pi}{2}\cot\frac{a_1\pi}{2}\left(\tan^2\frac{(a_0+a_1)\pi}{4}-\cot^2\frac{(a_0+a_1)\pi}{4}\right)$$
$$\times\int_{-\infty}^{0}dx\int\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\,e^{-i\omega(t_1-t_2)}\,\frac{i}{\omega^2-k^2-4+i\rho}$$

$$\times\cosh(\sqrt{2}\beta\phi_0)f(-k,x_1)f(k,x)\,e^{ik(x_1-x)}$$

$$\times\frac{1}{2}\int\frac{dk''}{2\pi}\,\frac{1}{\sqrt{k''^2+4}}\,\frac{e^{-2ik''x}}{ik''-2\cos\frac{(a_0+a_1)\pi}{2}}\left(2\coth 2(x-x_0)+2\cos\frac{(a_0+a_1)\pi}{2}\right)^2$$
$$\times\int\frac{dk'}{2\pi}\,\frac{i}{\omega^2-k'^2-4+i\rho}f(k',x)f(-k',x_2)\,e^{ik'(x-x_2)}. \tag{4.95}$$

Now as far as integration over x is concerned, we are led to

$$\int_{-\infty}^{0}dx\,e^{i(k+k'-2k'')x}\cosh(\sqrt{2}\beta\phi_0)f(k,x)f(k',x)\left(2\coth 2(x-x_0)+2\cos\frac{(a_0+a_1)\pi}{2}\right)^2 \tag{4.96}$$

or

$$\int_{-\infty}^{0} dx\, e^{i(k+k'-2k'')x} \frac{ik - 2\coth 2(x-x_0)}{ik+2} \frac{ik' - 2\coth 2(x-x_0)}{ik'+2}$$
$$\left(2\coth^2 2(x-x_0) - 1\right)\left(2\coth 2(x-x_0) + 2\cos\frac{(a_0+a_1)\pi}{2}\right)^2 \quad (4.97)$$

and the above integral can be evaluated by means of the formulae given in Appendix B as

$$\int_{-\infty}^{0} dx\, e^{i(k+k'-2k'')x} \frac{ik - 2\coth 2(x-x_0)}{ik+2} \frac{ik' - 2\coth 2(x-x_0)}{ik'+2}$$
$$\left(2\coth^2 2(x-x_0) - 1\right)\left(2\coth 2(x-x_0) + 2\cos\frac{(a_0+a_1)\pi}{2}\right)^2$$
$$= \frac{1}{(ik+2)(ik'+2)}\Bigg\{\frac{(A'_0kk' + B'_0(k+k') + C'_0)}{(k+k'-2k'')}$$
$$+\frac{(A'_1kk' + B'_1(k+k') + C'_1)}{(k+k'-2k'')}$$
$$\times F(1, \frac{i}{4}(k+k'-2k''), \frac{i}{4}(k+k'-2k'')+1, e^{-4x_0})$$
$$+\frac{(A'_2kk' + B'_2(k+k') + C'_2)}{(k+k'-2k''-4i)}$$
$$\times e^{-4x_0} F(2, \frac{i}{4}(k+k'-2k'')+1, \frac{i}{4}(k+k'-2k'')+2, e^{-4x_0})$$
$$+\frac{(A'_3kk' + B'_3(k+k') + C'_3)}{(k+k'-2k''-8i)}$$
$$\times e^{-8x_0} F(3, \frac{i}{4}(k+k'-2k'')+2, \frac{i}{4}(k+k'-2k'')+3, e^{-4x_0})$$
$$+\frac{(A'_4kk' + B'_4(k+k') + C'_4)}{(k+k'-2k''-12i)}$$
$$\times e^{-12x_0} F(4, \frac{i}{4}(k+k'-2k'')+3, \frac{i}{4}(k+k'-2k'')+4, e^{-4x_0})$$
$$+\frac{(B'_5(k+k') + C'_5)}{(k+k'-2k''-16i)}$$
$$\times e^{-16x_0} F(5, \frac{i}{4}(k+k'-2k'')+4, \frac{i}{4}(k+k'-2k'')+5, e^{-4x_0})$$
$$+\frac{C'_6}{(k+k'-2k''-20i)}$$
$$\times e^{-20x_0} F(6, \frac{i}{4}(k+k'-2k'')+5, \frac{i}{4}(k+k'-2k'')+6, e^{-4x_0})\Bigg\}, \quad (4.98)$$

where the coefficients $A'_n, B'_n, C'_n; n = 0, 1, .., 6$ are constants and depend only on $\cos\frac{(a_0+a_1)\pi}{2}$. Now the subsequent calculation is to integrate over k'' and it is clear that to do this, it is necessary to convert the hypergeometric function to an infinite

series. Looking at (B.10) in Appendix B, we may write down

$$\frac{F(1,\frac{i}{4}(k+k'-2k''),\frac{i}{4}(k+k'-2k'')+1,e^{-4x_0})}{k+k'-2k''}=\sum_{n=0}^{\infty}\frac{e^{-4nx_0}}{k+k'-2k''-4ni} \tag{4.99}$$

and by differentiating both sides of the above relation with respect to x_0, then we obtain

$$\frac{F(2,\frac{i}{4}(k+k'-2k'')+1,\frac{i}{4}(k+k'-2k'')+2,e^{-4x_0})}{k+k'-2k''-4i}=\sum_{n=1}^{\infty}\frac{ne^{-4(n-1)x_0}}{k+k'-2k''-4ni}. \tag{4.100}$$

Similarly one can derive the infinite series forms of the other hypergeometric functions as

$$\frac{F(3,\frac{i}{4}(k+k'-2k'')+2,\frac{i}{4}(k+k'-2k'')+3,e^{-4x_0})}{k+k'-2k''-8i}=\frac{1}{2!}\sum_{n=2}^{\infty}\frac{n(n-1)e^{-4(n-2)x_0}}{k+k'-2k''-4ni}, \tag{4.101}$$

$$\frac{F(4,\frac{i}{4}(k+k'-2k'')+3,\frac{i}{4}(k+k'-2k'')+4,e^{-4x_0})}{k+k'-2k''-12i}=\frac{1}{3!}\sum_{n=3}^{\infty}\frac{n(n-1)(n-2)e^{-4(n-3)x_0}}{k+k'-2k''-4ni}, \tag{4.102}$$

$$\frac{F(5,\frac{i}{4}(k+k'-2k'')+4,\frac{i}{4}(k+k'-2k'')+5,e^{-4x_0})}{k+k'-2k''-16i}=\frac{1}{4!}\sum_{n=4}^{\infty}\frac{n(n-1)(n-2)(n-3)e^{-4(n-4)x_0}}{k+k'-2k''-4ni} \tag{4.103}$$

and

$$\frac{F(6,\frac{i}{4}(k+k'-2k'')+5,\frac{i}{4}(k+k'-2k'')+6,e^{-4x_0})}{k+k'-2k''-20i}=\frac{1}{5!}\sum_{n=5}^{\infty}\frac{n(n-1)(n-2)(n-3)(n-4)e^{-4(n-5)x_0}}{k+k'-2k''-4ni}. \tag{4.104}$$

Let us substitute (4.99), (4.100), (4.101), (4.102), (4.103) and (4.104) in (4.98) and obviously what remains in connection with the contribution (4.95) are the integrations over k'', k' and k. As before in the previous sections, in order to integrate over the momenta k and k', it is sufficient to close the contours in the upper half-plane and pick up poles at $\hat{k}=k=k'=\sqrt{\omega^2-4}$ as all the other poles' contributions will be exponentially damped when $x,x'\to-\infty$. Meanwhile the integration over k'' is of the form:

$$\int\frac{dk''}{\sqrt{k''^2+4}}\frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\frac{1}{(k+k'-2k''-4ni)} \tag{4.105}$$

which is immediately split into two integrals

$$\frac{1}{\left(k+k'+4i\cos\frac{(a_0+a_1)\pi}{2}-4ni\right)}\left\{\int\frac{dk''}{\sqrt{k''^2+4}}\,\frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\right.$$
$$\left.-2i\int\frac{dk''}{\sqrt{k''^2+4}}\,\frac{1}{(k+k'-2k''-4ni)}\right\} \tag{4.106}$$

or

$$\frac{1}{\left(k+k'+4i\cos\frac{(a_0+a_1)\pi}{2}-4ni\right)}\left\{-\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\right.$$
$$\left.-2i\int\frac{dk''}{\sqrt{k''^2+4}}\,\frac{1}{(k+k'-2k''-4ni)}\right\}. \tag{4.107}$$

Now to manipulate the remaining integral, let us choose the contour in the upper half-plane, taking care of the branch cut which runs from $k''=2i$ to infinity along the imaginary axis. Clearly this integral reduces to the integral along the branch cut i.e.

$$\int\frac{dk''}{\sqrt{k''^2+4}}\,\frac{1}{(k+k'-2k''-4ni)}=2\int_2^\infty\frac{idy}{\sqrt{4-y^2}}\,\frac{1}{(k+k'-2iy-4ni)} \tag{4.108}$$

or after changing the variable $y=2\cosh x$, the above integral becomes

$$2\int_0^\infty\frac{dx}{(k+k'-4i\cosh x-4ni)} \tag{4.109}$$

and another change $e^x=u$ yields

$$i\int_1^\infty\frac{du}{u^2+\left(2n+\frac{i(k+k')}{2}\right)u+1} \tag{4.110}$$

and finally after finding the solution of the above integral, we obtain

$$\int\frac{dk''}{\sqrt{k''^2+4}}\,\frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)}\,\frac{1}{(k+k'-2k''-4ni)}$$
$$=\frac{1}{\left(k+k'+4i\cos\frac{(a_0+a_1)\pi}{2}-4ni\right)}\left(-\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\right.$$
$$-\frac{2i}{\sqrt{\frac{(k+k')^2}{4}+4-4n^2-2ni(k+k')}}$$
$$\left.\times\ln\left\{\frac{1+\frac{i(k+k')}{4}+n+\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4-4n^2-2ni(k+k')}}{1+\frac{i(k+k')}{4}+n-\frac{i}{2}\sqrt{\frac{(k+k')^2}{4}+4-4n^2-2ni(k+k')}}\right\}\right). \tag{4.111}$$

When $n=0$, (4.111) is simplified much more, especially after doing the integration over k and k' and using the fact that $\hat{k}=k=k'=2\sinh\theta$. So the following

formula can be obtained

$$\int \frac{dk''}{\sqrt{k''^2+4}} \frac{1}{\left(ik''-2\cos\frac{(a_0+a_1)\pi}{2}\right)} \frac{1}{(2\hat{k}-2k'')}$$
$$= \frac{1}{\left(2\hat{k}+4i\cos\frac{(a_0+a_1)\pi}{2}\right)} \left(-\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}} + \frac{2}{\sqrt{\hat{k}^2+4}}(\frac{\pi}{2}-i\theta) \right). \quad (4.112)$$

Now, let us write down the solution of (4.95) or more generally the contribution $\mathcal{D}_2$:

$$\begin{aligned}
\mathcal{D}_2 \;=\; & \frac{i\beta^2}{16\pi}\cot\frac{a_0\pi}{2}\cot\frac{a_1\pi}{2}\left(\tan^2\frac{(a_0+a_1)\pi}{4}-\cot^2\frac{(a_0+a_1)\pi}{4}\right) \\
& \int\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}e^{-i\hat{k}(x_1+x_2)}\left(\frac{1}{\hat{k}}\right)^2 f(-\hat{k},x_1)f(-\hat{k},x_2) \\
& \left\{\frac{2i\left(\cos\frac{(a_0+a_1)\pi}{2}-1\right)^2}{\hat{k}+2i\cos\frac{(a_0+a_1)\pi}{2}}\left(\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}-\frac{2}{\sqrt{\hat{k}^2+4}}(\frac{\pi}{2}-i\theta)\right)\right. \\
& -\frac{1}{(i\hat{k}+2)^2}\sum_{n=1}^{\infty}\frac{e^{-4nx_0}}{2\hat{k}+4i\cos\frac{(a_0+a_1)\pi}{2}-4ni}\left(\frac{\frac{(a_0+a_1)\pi}{2}}{\sin\frac{(a_0+a_1)\pi}{2}}\right. \\
& \left.\left.+\frac{2i}{\sqrt{\hat{k}^2+4-4n^2-4ni\hat{k}}}\ln\left\{\frac{1+\frac{i}{2}\hat{k}+n+\frac{i}{2}\sqrt{\hat{k}^2+4-4n^2-4ni\hat{k}}}{1+\frac{i}{2}\hat{k}+n-\frac{i}{2}\sqrt{\hat{k}^2+4-4n^2-4ni\hat{k}}}\right\}\right)\right. \\
& \left((A_1'\hat{k}^2+B_1'\hat{k}+C_1')+n(A_2'\hat{k}^2+B_2'\hat{k}+C_2')\right. \\
& +\frac{n(n-1)}{2!}(A_3'\hat{k}^2+B_3'\hat{k}+C_3')+\frac{n(n-1)(n-2)}{3!}(A_4'\hat{k}^2+B_4'\hat{k}+C_4') \\
& \left.\left.+\frac{n(n-1)(n-2)(n-3)}{4!}(B_5'\hat{k}+C_5')+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}C_6'\right)\right\} \\
& + \text{other pole pieces.} \qquad (4.113)
\end{aligned}$$

Firstly, in order to check the above solution, if we set $a_0 = a_1$ and consider the other pole pieces then, we can derive the formula (3.10) in reference [64]. As we mentioned before, the calculation of this reference is based on the case when the boundary parameters are equal. Secondly, in this solution, we verified that the term which depends explicitly on the rapidity of the particle (θ) is cancelled by counterpart terms in the other pole pieces. It is evident that if we add the expressions (4.94) and (4.113) then, the contribution (4.93) will be obtained i.e. $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$.

4.7 Discussion

Affine Toda field theory on the whole line is an exactly solvable theory for which the S-matrices have been formulated. However, when a boundary is present then

the boundary S-matrices of the theory i.e. the reflection factors, have not been clearly found. The bootstrap technique does not uniquely determine the reflection factors. Fortunately perturbation theory provides the link between the expressions for the reflection factors which come from the bootstrap equations and the boundary parameters. Nevertheless, this method normally involves complicated calculations.

In this chapter the quantum reflection factor for the $a_1^{(1)}$ affine Toda field theory or sinh-Gordon model with integrable boundary conditions has been studied in low order perturbation theory when $\sigma_0 \neq \sigma_1$. It is found that at one loop order the quantum corrections to the classical reflection factor of the model can be expressed in terms of hypergeometric functions for most of the related Feynman diagrams. Although there is still some work to do to calculate the contributions of the remaining diagrams, it is however understood that the provided procedure and some formalisms may be followed for them.

The calculations corresponding to the type II Feynman diagram which are not carried out in this chapter, are more difficult than the others. In this case the two middle propagators are exactly the same and this fact influences the difficulty of the computations. However some formulae that have been presented here, could be helpful for the remaining diagram. For example, consider the contribution of the type II (boundary-bulk) diagram:

$$-2\beta^2(\sigma_1 \coth x_0 - \sigma_0 \tanh x_0) \int\int\int dt dt' dx G(x_1,t_1;x,t)G(x,t;0,t') \times G(x,t;0,t')G(0,t';x_2,t_2)\sinh(\sqrt{2}\beta\phi_0). \quad (4.114)$$

Now as far as the integration over x is concerned we should obtain the following integrals

$$\int_{-\infty}^{0} dx e^{i(k+k'-k_1)x}\sinh(\sqrt{2}\beta\phi_0)\coth^n 2(x-x_0), \quad (4.115)$$

where n=0,1,2,3. It is better to solve:

$$\int_{-\infty}^{0} dx\, \exp\{\tau + i(k+k'-k_1)x\}\sinh(\sqrt{2}\beta\phi_0)\coth^n 2(x-x_0), \quad (4.116)$$

in which τ is a small positive quantity and will be taken to zero later. In fact, the relation (4.116) is very similar to the formula (A.1) in Appendix A. So, following the same procedure that have been followed in Appendix A, one can find the solution

of (4.115) when $n = 0$ as:

$$\int_{-\infty}^{0} dx \; e^{i(k+k'-k_1)x} \sinh(\sqrt{2}\beta\phi_0)$$
$$= \frac{1}{\sinh 2x_0} \frac{2(k+k'-k_1)}{k+k'-k_1-2i} \times e^{-2x_0} F(1, \frac{i}{4}(k+k'-k_1)+\frac{1}{2}, \frac{i}{4}(k+k'-k_1)+\frac{3}{2}, e^{-4x_0}). \quad (4.117)$$

Then, the solutions of (4.115) for $n = 1, 2, 3$ can be derived exactly in according to the Appendix A terms. But, this is not the whole of the story. As we mentioned before, in type II diagram double Green functions cause the middle momenta to be linked to each other in a complicated way and the calculations become more intricate. Actually this diagram must be studied in three cases depending on the interaction vertices being located in the bulk region or at the boundary. Moreover because of the symmetry, the contribution of the type II (boundary-bulk) diagram is the same as the type II (bulk-boundary) one.

When the boundary parameters are equal only the type I diagram is involved in the theory. As we mentioned before, in this special case [64] the quantum corrections to the classical reflection of the model have been found and Ghoshal's formula for the lightest breather is checked perturbatively to $O(\beta^2)$. In our case, we realised that the contribution of the type I (bulk) reduces to the special case. Taking the expressions (4.94) and (4.113), if we put $\sigma_0 = \sigma_1$ then, we obtain the same result as reference [64] and this is a check on our calculations. Moreover, when $\sigma_0 \neq \sigma_1$ the following expressions for E and F in Ghoshal's formula (2.26) have been conjectured [74] to be:

$$E = (a_0 + a_1)(1 - B/2) \qquad F = (a_0 - a_1)(1 - B/2). \quad (4.118)$$

So, it will be interesting to check the above conjecture after finding the contributions of the remaining diagrams and adding the results all together. This will lead to a deeper understanding of the quantum integrability of the theory. However, it is necessary to find simplifications of the contributions when they add among themselves in order to get Ghoshal's formula.

Chapter 5

Conclusions and future work

In this thesis we studied the boundary sinh-Gordon model with integrable boundary conditions. First we calculated first order quantum corrections to the classical reflection factor of the sinh-Gordon model in chapter Two. In fact, Ghoshal found the general form of the quantum reflection factor of the sinh-Gordon model. However, apart from two special cases (Neumann and Dirichlet boundary conditions) Ghoshal's formula fails to provide a complete relationship between the reflection factor and the boundary data. Up to first order in the difference of the boundary parameters σ_0 and σ_1, we perturbatively verified Ghoshal's formula. Meanwhile, we conjectured expressions for the unknown functions E and F in Ghoshal's formula (2.26):

$$E = (a_0 + a_1)(1 - B/2) \qquad F = (a_0 - a_1)(1 - B/2), \tag{5.1}$$

where the coupling constant dependence comes into the formulae by means of the expression for B. Similar expressions for these parameters have been arrived at via other arguments by Zamolodchikov [75].

If (5.1) is correct then the reflection factor is invariant under the interchange $a_0 \leftrightarrow a_1$. So, this invariance reconstructs the $\mathbf{Z_2}$ bulk symmetry which apparently was broken by the boundary condition and replaced by a symmetry under the simultaneous interchange of ϕ with $-\phi$ and a_0 with a_1. The reflection factor is also invariant if a_0 and/or a_1 is replaced by its negative which provide the definitions of σ_0 and σ_1. It is consistent with what is known at the special value of the coupling constant, known as the 'free-fermion' point in the sine-Gordon model, where

$B = -2$ and the S-matrix is unity. There, the restrictions on the parameters in the reflection factor can be solved exactly and are in agreement with (5.1) [64].

Note that with the expressions (5.1) the quantum reflection factor (2.26) has a weak-strong coupling symmetry which matches the symmetry of the S-matrix under $\beta \to 4\pi/\beta$. In other words, considering

$$(a_0^*, a_1^*, \beta^*) = \frac{4\pi}{\beta^2}(a_0, a_1, \beta) \tag{5.2}$$

defines a new triple of coupling constants with the property that

$$K_q(\theta, a_0, a_1, \beta) = K_q(\theta, a_0^*, a_1^*, \beta^*). \tag{5.3}$$

If (5.1) is correct, which implies the duality symmetry (5.2), then we are faced with other puzzles. For example, it is known that the supersymmetric version of the sinh-Gordon model is only integrable when restricted to a half-line with some very special boundary conditions (either $a_0 = a_1 = 0, \pi$) (see [76]), and this restriction would appear to be incompatible with a weak-strong coupling symmetry without modifying (5.1).

It is also known [36,40] that the other affine Toda field theories constructed from data in the *ade* series, when restricted to a half-line, allow only a finite number of possible boundary conditions. In fact, the $a_1^{(1)}$ or sinh-Gordon model is apparently the only example within this series which allows continuous boundary parameters. Expressions for the associated reflection factors for the other models are largely unknown but it will be interesting to discover if they too can permit a duality symmetry in the presence of a boundary to match the symmetry of their bulk S-matrices.

In order to find the unknown parameters in Ghoshal's formula up to higher order (second order), we tried to calculate the second order quantum corrections to the classical reflection factor of the sinh-Gordon model in chapter 3. Actually in this case there are ten contributions. We calculated six of them precisely. However, in connection with the type II contributions the calculations are still in progress. If the remaining contributions are found then, the conjecture (5.1) will be verified perturbatively at higher order.

In chapter 4 we studied the quantum reflection factor for the sinh-Gordon model with general boundary conditions. For general boundary conditions the lowest en-

ergy solution to the equation of the field no longer will be a trivial background $\phi = 0$. The calculations become more lengthy and intricate since the perturbation theory involves complicated coupling constants and a complicated propagator as well. We found that at one loop order the quantum correction for the reflection factor of the theory can be expressed in terms of hypergeometric functions for most of the related Feynman diagrams.

The boundary conditions which preserve classical integrability have been classified before by Corrigan et.al [36, 37, 40] for affine Toda field theories. However, quantum integrability is hardly explored although there has been some progress in the $a_n^{(1)}$ class of models. So, there still remains much to be studied in this area. Meanwhile, it will be interesting to investigate the weak-strong coupling duality in the quantum reflection factors of the models of affine Toda field theories.

It is better to discuss the $a_n^{(1)}$ models in more details as much is now known about them. By looking at (1.85) it is evident that for the $a_2^{(1)}$ affine Toda field theory there are only nine possible boundary conditions which lead to a classically integrable theory. However, for the corresponding quantum field theory of the $a_2^{(1)}$ model, Gandenberger found [61] three different quantum reflection factors with Neumann or (+++) boundary conditions. This fact shows that not all boundary conditions, which were found to be classically integrable, are also quantum integrable. Gandenberger also noticed that the three quantum reflection factors of the $a_2^{(1)}$ theory are not self-dual under the weak-strong coupling duality.

Delius and Gandenberger [62], by generalising the results in [61], determined the exact quantum reflection factors for $a_n^{(1)}$ affine Toda field theory on the half-line with integrable boundary conditions. They also noticed that the Neumann boundary condition is dual to the (++...++) boundary condition. This duality had been observed earlier in the sinh-Gordon model [64] and the $a_2^{(1)}$ theory [61].

An interesting problem is to carry out the next order calculations in the bulk coupling ($O(\beta^4)$) for the sinh-Gordon theory. It is predicted that the two loop calculations will be more difficult than the one-loop one. It is understood that at two loop order, one can make use of the following expansions of the E and F functions in the quantum reflection factor (2.26) as,

$$E = e_0 + \frac{\beta^2}{4\pi} e_1 + \left(\frac{\beta^2}{4\pi}\right)^2 e_2 + O(\beta^6), \tag{5.4}$$

$$F = f_0 + \frac{\beta^2}{4\pi} f_1 + \left(\frac{\beta^2}{4\pi}\right)^2 f_2 + O(\beta^6). \tag{5.5}$$

This problem has been solved in the Neumann boundary condition [77] in which $a_0 = a_1 = \pi/2$ however, even in this special case the obtained result does not match with the physical properties expected. In other words, the obtained result for the reflection factor fails to satisfy unitarity and periodic properties. As the author in [77] mentions, it is necessary to overview the renormalisations of the theory and in fact, their influence on the duality of the model. Meanwhile solving the problem to higher order for general boundary conditions will be interesting. This will lead to a deeper understanding of the quantum integrability of the theory.

Appendix A

In this Appendix we obtain such integrals

$$S_n = \int_{-\infty}^{0} dx\, e^{(2+ik)x} \sinh(\sqrt{2}\beta\phi_0) \coth^n 2(x - x_0), \tag{A.1}$$

in which $n = 0, 1, 2, 3$, ϕ_0 is the background solution to the equation of field so that $\sinh(\sqrt{2}\beta\phi_0)$ is proportional to the bulk three point coupling which is given by

$$\sinh(\sqrt{2}\beta\phi_0) = 2\cosh 2(x - x_0)\left(\coth^2 2(x - x_0) - 1\right). \tag{A.2}$$

Let us start with the simplest case when $n = 0$:

$$S_0 = \int_{-\infty}^{0} dx\, e^{(2+ik)x} \sinh(\sqrt{2}\beta\phi_0). \tag{A.3}$$

Using (A.2), we have

$$S_0 = -\int_{-\infty}^{0} e^{(2+ik)x}\, d\left(\frac{1}{\sinh 2(x - x_0)}\right) \tag{A.4}$$

or after integration by parts

$$S_0 = \frac{1}{\sinh 2x_0} + 2(2 + ik)\int_{-\infty}^{0} dx\, e^{(2+ik)x} \frac{1}{e^{2(x-x_0)} - e^{-2(x-x_0)}}. \tag{A.5}$$

Now, according to the arguments in section three of chapter 2, x_0 is greater or equal to zero and, in contrast, x is less than zero so $0 < e^{4(x-x_0)} < 1$ and hence

$$\frac{1}{e^{2(x-x_0)} - e^{-2(x-x_0)}} = -e^{2(x-x_0)} \sum_{n=0}^{\infty} e^{4n(x-x_0)}, \tag{A.6}$$

substituting (A.6) in (A.5), we obtain

$$S_0 = \frac{1}{\sinh 2x_0} - 2(2 + ik)e^{-2x_0}\int_{-\infty}^{0} dx\, e^{(4+ik)x} \sum_{n=0}^{\infty} e^{4n(x-x_0)}. \tag{A.7}$$

Clearly, the series $\sum_{n=0}^{\infty} e^{4n(x-x_0)}$ is uniformly convergent so the above relation may be written down as

$$S_0 = \frac{1}{\sinh 2x_0} - 2(2+ik)e^{-2x_0}\sum_{n=0}^{\infty} e^{-4nx_0}\int_{-\infty}^{0} dx\, e^{(4+4n+ik)x}. \tag{A.8}$$

After integration over x, we obtain

$$S_0 = \frac{1}{\sinh 2x_0} + 2i(2+ik)e^{-2x_0}\sum_{n=0}^{\infty}\frac{e^{-4nx_0}}{k-(4+4n)i}. \tag{A.9}$$

On the other hand, the above infinite series is a hypergeometric function. That is

$$\sum_{n=0}^{\infty}\frac{e^{-4nx_0}}{k-i(4+4n)} = \frac{F(1,\frac{i}{4}k+1,\frac{i}{4}k+2,e^{-4x_0})}{k-4i}. \tag{A.10}$$

Therefore, we get the following relation

$$\begin{aligned} S_0 \;=\; & \frac{1}{\sinh 2x_0} \\ & -2\frac{k-2i}{k-4i}\,e^{-2x_0}\,F(1,\frac{i}{4}k+1,\frac{i}{4}k+2,e^{-4x_0}). \end{aligned} \tag{A.11}$$

Note, the hypergeometric function is defined as [78]

$$F(a,b,c,z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}z^n \qquad c\neq 0,-1,-2,..., \tag{A.12}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1) \qquad n=1,2,3,..., \tag{A.13}$$

in which $\Gamma(a)$ is the gamma function defined by

$$\Gamma(a) = \int_0^{\infty} dt\, e^{-t}\, t^{a-1} \qquad \text{Re } a>0. \tag{A.14}$$

The above series defines a function which is analytic when $|z|<1$ and meanwhile the derivative of the hypergeometric function is given by

$$\frac{d}{dz}F(a,b,c,z) = \frac{ab}{c}F(a+1,b+1,c+1,z) \tag{A.15}$$

or in general

$$\frac{d^n}{dz^n}F(a,b,c,z) = \frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n,z). \tag{A.16}$$

Let us calculate (A.1) when $n=1$ i.e.

$$S_1 = \int_{-\infty}^{0} dx\, e^{(2+i(k_2-k_1))x}\sinh(\sqrt{2}\beta\phi_0)\coth 2(x-x_0) \tag{A.17}$$

or using (A.2) for the bulk three point coupling

$$S_1 = 2\int_{-\infty}^{0} dx\, e^{(2+i(k_2-k_1))x} \cosh 2(x-x_0) \coth 2(x-x_0) \left(\coth^2 2(x-x_0) - 1\right). \tag{A.18}$$

On the other hand, if we differentiate the left hand side of (A.11) with respect to x_0, which is given by

$$\begin{aligned}\frac{\partial S_0}{\partial x_0} = -\int_{-\infty}^{0} dx\, e^{(2+ik)x} \Big\{ & 4\sinh 2(x-x_0)\left(\coth^2 2(x-x_0) - 1\right) \\ & +8\cosh 2(x-x_0)\coth 2(x-x_0)\left(1-\coth^2 2(x-x_0)\right)\Big\}\end{aligned} \tag{A.19}$$

and by comparing the above formula with (A.18) then, the following equation may be derived

$$S_1 = \frac{1}{4}\frac{\partial S_0}{\partial x_0} + \int_{-\infty}^{0} dx\, e^{(2+ik)x} \frac{1}{\sinh 2(x-x_0)}. \tag{A.20}$$

The second term in the above relation can be manipulated as before. Moreover, it is evident that we need to differentiate the right hand side of (A.11) which is equal to

$$\begin{aligned}\frac{\partial S_0}{\partial x_0} = & -\frac{2\cosh 2x_0}{\sinh^2 2x_0} \\ & +4\frac{k-2i}{k-4i} e^{-2x_0} F(1, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\ & +8\frac{k-2i}{k-8i} e^{-6x_0} F(2, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}).\end{aligned} \tag{A.21}$$

Finally by substituting the above relation in (A.20), doing the computation of second term in (A.20) and after simplifying we obtain

$$\begin{aligned}S_1 = & -\frac{1}{2}\frac{\cosh 2x_0}{\sinh^2 2x_0} \\ & +\frac{k}{k-4i} e^{-2x_0} F(1, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\ & +2\frac{k-2i}{k-8i} e^{-6x_0} F(2, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}).\end{aligned} \tag{A.22}$$

In the same way, we may derive (A.1) when n is equal to 2 or 3 however, gradually the calculations become lengthy and we only write down the results, that is,

$$\begin{aligned}S_2 = & \frac{2}{3}\frac{1}{\sinh 2x_0} + \frac{1}{6}\frac{\cosh^2 2x_0 + 1}{\sinh^3 2x_0} \\ & -\frac{1}{3}\frac{5k-8i}{k-4i} e^{-2x_0} F(1, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\ & -\frac{4}{3}\frac{2k-3i}{k-8i} e^{-6x_0} F(2, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}) \\ & -\frac{8}{3}\frac{k-2i}{k-12i} e^{-10x_0} F(3, \frac{i}{4}k+3, \frac{i}{4}k+4, e^{-4x_0})\end{aligned} \tag{A.23}$$

and

$$
\begin{aligned}
S_3 \;=\; & -\frac{13}{24}\frac{\cosh 2x_0}{\sinh^2 2x_0} - \frac{1}{24}\frac{\cosh^3 2x_0 + 5\cosh 2x_0}{\sinh^4 2x_0} \\
& + \frac{1}{6}\frac{7k-4i}{k-4i}\, e^{-2x_0}\, F(1, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\
& + \frac{1}{3}\frac{13k-22i}{k-8i}\, e^{-6x_0}\, F(2, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}) \\
& + \frac{1}{3}\frac{18k-32i}{k-12i}\, e^{-10x_0}\, F(3, \frac{i}{4}k+3, \frac{i}{4}k+4, e^{-4x_0}) \\
& + 4\frac{k-2i}{k-16i}\, e^{-14x_0}\, F(4, \frac{i}{4}k+4, \frac{i}{4}k+5, e^{-4x_0}). \qquad \text{(A.24)}
\end{aligned}
$$

Appendix B

In this Appendix we find the following integrals

$$C_n = \int_{-\infty}^{0} dx\, e^{ikx} \cosh(\sqrt{2}\beta\phi_0) \coth^n 2(x - x_0). \tag{B.1}$$

Here, $\cosh(\sqrt{2}\beta\phi_0)$ is proportional to the bulk four point coupling and is given by

$$\cosh(\sqrt{2}\beta\phi_0) = \left(2\coth^2 2(x - x_0) - 1\right). \tag{B.2}$$

So, we are led to calculate such integrals

$$I_n = \int_{-\infty}^{0} dx\, e^{ikx} \coth^n 2(x - x_0), \tag{B.3}$$

where $n = 1, 2, ..., 6$. It is better to find the solution of the above integrals when $n = 1$. Considering the following inequality (see Appendix A)

$$0 < e^{4(x-x_0)} < 1$$

and therefore, in what follows we will use the expanded form of $\coth 2(x - x_0)$ as

$$\coth 2(x - x_0) = 1 - 2\sum_{n=0}^{\infty} e^{4n(x-x_0)}. \tag{B.4}$$

It turns out to be simple if we consider this integral

$$\int_{-\infty}^{0} dx\, e^{(\tau+ik)x} \coth 2(x - x_0), \tag{B.5}$$

where τ is a positive constant quantity which will be taken to zero at the end of the calculation. Moreover by using (B.4) then, (B.5) becomes

$$\int_{-\infty}^{0} dx\, e^{(\tau+ik)x} - 2\int_{-\infty}^{0} dx \sum_{n=0}^{\infty} e^{4n(x-x_0)}\, e^{(\tau+ik)x}. \tag{B.6}$$

Regarding the uniform convergence of the series $\sum_{n=0}^{\infty} e^{4n(x-x_0)}$, we may write the above relation as

$$\int_{-\infty}^{0} dx\, e^{(\tau+ik)x} - 2\sum_{n=0}^{\infty} e^{-4nx_0} \int_{-\infty}^{0} dx\, e^{(\tau+4n+ik)x} \tag{B.7}$$

or after integrating over x

$$-\frac{i}{k-i\tau} + 2\sum_{n=0}^{\infty} \frac{i}{k-(\tau+4n)i}\, e^{-4nx_0}. \tag{B.8}$$

Now we are in a position to write down the desired result, that is,

$$I_1 = -\frac{i}{k} + 2i\sum_{n=0}^{\infty} \frac{e^{-4nx_0}}{k-4ni}. \tag{B.9}$$

On the other hand, the above series is equal to a hypergeometric function

$$\sum_{n=0}^{\infty} \frac{e^{-4nx_0}}{k-4ni} = \frac{1}{k} F(1, \frac{i}{4}k, \frac{i}{4}k+1, e^{-4x_0}) \tag{B.10}$$

and finally we find this formula

$$I_1 = -\frac{i}{k} + \frac{2i}{k} F(1, \frac{i}{4}k, \frac{i}{4}k+1, e^{-4x_0}). \tag{B.11}$$

Now let us compute (B.1) when $n = 2$

$$I_2 = \int_{-\infty}^{0} dx\, e^{ikx} \coth^2 2(x-x_0). \tag{B.12}$$

In order to solve the above integral, it is sufficient to differentiate both sides of (B.11) with respect to x_0 to obtain

$$I_2 = -\frac{i}{k} - \frac{4i}{k-4i}\, e^{-4x_0}\, F(2, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}). \tag{B.13}$$

We can follow a similar method to get higher order forms of (B.1) which we need in chapter 4, so it is appropriate to write down all of them i.e.

$$\begin{aligned} I_3 \;=\; & -\frac{i}{k} + \frac{2i}{k} F(1, \frac{i}{4}k, \frac{i}{4}k+1, e^{-4x_0}) \\ & \frac{4i}{k-4i}\, e^{-4x_0}\, F(2, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\ & \frac{8i}{k-8i}\, e^{-8x_0}\, F(3, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}), \end{aligned} \tag{B.14}$$

$$\begin{aligned} I_4 \;=\; & -\frac{i}{k} - \frac{8i}{k-4i} F(2, \frac{i}{4}k+1, \frac{i}{4}k+2, e^{-4x_0}) \\ & -\frac{16i}{k-8i}\, e^{-8x_0}\, F(3, \frac{i}{4}k+2, \frac{i}{4}k+3, e^{-4x_0}) \\ & -\frac{16i}{k-12i}\, e^{-12x_0}\, F(4, \frac{i}{4}k+3, \frac{i}{4}k+4, e^{-4x_0}), \end{aligned} \tag{B.15}$$

$$
\begin{aligned}
I_5 \;=\; & -\frac{i}{k} + \frac{2i}{k} F(1, \tfrac{i}{4}k, \tfrac{i}{4}k+1, e^{-4x_0}) \\
& \frac{8i}{k-4i}\, e^{-4x_0} F(2, \tfrac{i}{4}k+1, \tfrac{i}{4}k+2, e^{-4x_0}) \\
& \frac{32i}{k-8i}\, e^{-8x_0} F(3, \tfrac{i}{4}k+2, \tfrac{i}{4}k+3, e^{-4x_0}) \\
& \frac{48i}{k-12i}\, e^{-12x_0} F(4, \tfrac{i}{4}k+3, \tfrac{i}{4}k+4, e^{-4x_0}) \\
& \frac{32i}{k-16i}\, e^{-16x_0} F(5, \tfrac{i}{4}k+4, \tfrac{i}{4}k+5, e^{-4x_0})
\end{aligned} \tag{B.16}
$$

and

$$
\begin{aligned}
I_6 \;=\; & -\frac{i}{k} - \frac{12i}{k-4i}\, e^{-4x_0} F(2, \tfrac{i}{4}k+1, \tfrac{i}{4}k+2, e^{-4x_0}) \\
& -\frac{48i}{k-8i}\, e^{-8x_0} F(3, \tfrac{i}{4}k+2, \tfrac{i}{4}k+3, e^{-4x_0}) \\
& -\frac{112i}{k-12i}\, e^{-12x_0} F(4, \tfrac{i}{4}k+3, \tfrac{i}{4}k+4, e^{-4x_0}) \\
& -\frac{128i}{k-16i}\, e^{-16x_0} F(5, \tfrac{i}{4}k+4, \tfrac{i}{4}k+5, e^{-4x_0}) \\
& -\frac{64i}{k-20i}\, e^{-20x_0} F(6, \tfrac{i}{4}k+5, \tfrac{i}{4}k+6, e^{-4x_0}).
\end{aligned} \tag{B.17}
$$

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